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CONTENTS.

A paper should contain a short and clear summary of the new results obtained and the relations in which they stand to results already known. Contributors are requested to bear in mind that, at the present stage of mathematical research, hardly any paper is likely to be so completely original as to be independent of earlier work in the same direction; and that readers are often helped to appreciate the importance of a new investigation by seeing its connection with earlier results.

The principal results of a paper should, when possible, be enunciated separately and explicitly in the form of definite theorems.

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and since $k/n_k \to 0$, the unit circle is a natural boundary of $\Sigma\, a_n z^n$, which implies the existence of an ordered configuration. 17

6. *Lemma I.** *If* p *is an integer,* ε *a primitive* p*th root of unity, and* Π *a closed set of points on the unit circle such that*

$$\Pi_p = \Pi + \varepsilon \Pi + \varepsilon^2 \Pi + \ldots + \varepsilon^{p-1} \Pi$$

completely covers the circumference of the unit circle, then there are two points belonging to the set Π *such that their ratio is a* p*th root of unity, and in particular to every frontier point* P *of* Π *there corresponds another point* P_1 *of* Π *such that* P/P_1 *is a* p*th root of unity.*

The Lemma is trivial when Π represents the entire unit circumference. If not, the set will contain a frontier point† P. Since Π_p completely covers the unit circumference it follows that one of the points P, εP, ε^2P, ... ε^{p-1}P other than P is a limit point of the set Π. Since Π is a closed set that limit point must belong to Π, and this proves the lemma.

7. **Theorem 1.** *If* p *and* q *are mutually prime integers, and* Π *is a closed set of points on the unit circumference such that each of the sets* Π_p, Π_q *completely covers the unit circle, then* Π *contains an ordered configuration of four points,*

In what follows two points are said to be connected by p or q relation when the pth or the qth power of their ratio is equal to unity.

If Π consists of the entire unit circumference the problem is trivial. In other cases, I shall now prove that there is a sub-set Π' of Π which has the following properties :—

(*a*) Π' is a closed set,

(*b*) Π'_p and Π'_q each separately covers the entire unit circumference,

and (*c*) there are two frontier points P_1 and P_2 which belong to Π' and are connected either by p relation or by q relation.

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* This represents the case $r = 1$, and the proof is Dr. Ostrowski's.

† A point of a closed set is a frontier point if it belongs to the set, and is such that in any interval of the unit circumference enclosing the point there is a point not belonging to the set.

The theorem is then deduced thus; let P_1 and P_2 be connected by q relation. From Lemma 1 we know that there are two other points of II′ which are connected with the frontier points P_1 and P_2 by p relation, and hence it follows that there is an *ordered* configuration contained in II′ and therefore in II.

8. Now to prove the existence of II′, we proceed as follows: The set II has the properties (*a*) and (*b*), and if it satisfies (*c*) also, we can then take II itself for II′. Otherwise it is clear from the Lemma* that to each frontier point of II there correspond interior points of the set connected by p and by q relations. The existence of interior points implies that there are intervals all of whose points belong to II†. Let these intervals, finite or infinite in number, be $I_1, I_2, \ldots I_n, \ldots$, and we may suppose that the lengths of these intervals are non-increasing. Let $A_1 A_2$ be the interval I_1, and let A_1 be connected with the interior point B of the interval I_m by p relation, and be connected with the interior point C of the interval I_n by q relation. Let $B_1 B_2$ be the interval I_m, and $C_1 C_2$ the interval I_n. A_1A_2, B_1B_2, C_1C_2 need not necessarily be different intervals. Let the lengths of the intervals A_1A_2, BB_2 and CC_2 be d_1, d_2 and d_3 respectively. We can suppose without loss of generality that $d_2 \geq d_3$. Since C is an interior point of its interval, and since A_1A_2 is the longest interval we can find a point A in the interval A_1A_2 such that $A_1A = d_3$. Then it is easily seen that the removal of the interior of A_1A will not affect the properties (*a*) and (*b*) of the set. Denoting by II′ the remaining set we shall have the frontier points A and C_2 of this set are connected by q relation. This completes the proof.

9. I shall now show by an example that the order of p and q is not irrelevant in deriving the ordered configuration. If p and q be mutually prime, $p < q$, and if II represents the set of points in an arc of the unit circle of the length $\theta = 2\pi/p$, then it is easy to see that II

* An interior point of a set is an interior point of a segment all whose points belong to the set.

† It can be proved from the mere fact that II is a closed set and II_p covers the entire unit circumference that the *inner content* of the set is not less than $\frac{2\pi}{p}$ and this, in its turn, implies that whole segments are contained in II.

has the properties (*a*) and (*b*). The two extreme points of the set are the only two points of the set which are connected by p relation, and hence it is seen that we cannot have four points α, β, γ and δ belonging to the set II and forming an ordered configuration in which q precedes p: *i.e.*, where

$$\left(\frac{\alpha}{\beta}\right)^p = \left(\frac{\alpha}{\gamma}\right)^q = \left(\frac{\beta}{\delta}\right)^q = 1.$$

10. I shall now discuss the case for three numbers p, q, r.

THEOREM II. *It is possible to find three mutually prime numbers p, q, and r and a closed set* II *such that*

(1) Π_p, Π_q, Π_r *each completely covers the unit circumference,*

(2) II *does not contain eight points forming an ordered configuration with respect to p, q and r.*

The theorem will be proved if it is shown that we can construct a set II for which (1) and (2) simultaneously hold. I shall take $p = 43$, $q = 47$, $r = 53$. Now, let α denote the point $e^{2\pi\alpha i/2021}$, and let the set II consist of the points $0 \leqslant \alpha \leqslant 39$, $1543 \leqslant \alpha \leqslant 1551$. It is easy to verify, that each of the sets Π_{43}, Π_{47}, Π_{53} completely covers the unit circumference.*

Further, it is not difficult to verify that

$$\alpha = 0,\ 1551,\ 39,\ 1543$$

are the only four points of the set which are connected with other points

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* To prove that Π_{43} completely covers the unit circumference, we note that since $e^{2\pi i/43} = e^{2\pi i \times 47/2021}$, it is sufficient to prove that the α of the set II reduced to modulus 47, runs through all the values between 0 and 47. Since the total variation of α in II is equal to 47, and since $39 \equiv 1543 \pmod{47}$ the result follows;

In the case of Π_{47}, it is sufficient to prove that the α of the set II reduced to modulus $2021/47 = 43$, runs through all the values between 0 and 43. That such is the case obvious, since $39 \equiv 644 \pmod{43}$.

In the case of Π_{53} the result follows at once from the fact that the set contains an arc of length $2\pi . 39/2021 > \frac{2\pi}{53}$.

of the set by '43' relation*. In a similar way it is seen that the set

$$0 \leqslant \alpha \leqslant 3,\ 1548 \leqslant \alpha \leqslant 1551,\ 38 \leqslant \alpha \leqslant 39,\ 1543 \leqslant \alpha \leqslant 1544$$

contains all the points of the set II to which there correspond other points of the set connected by '47' relation. Similarly the set

$$0 \leqslant \alpha \leqslant \frac{46}{53},\ 38\frac{7}{53} \leqslant \alpha \leqslant 39,\ 17\frac{38}{53} \leqslant \alpha \leqslant 25\frac{38}{53},\ 1543 \leqslant \alpha \leqslant 1541$$

contains all the points of the set II to which there correspond other points of the set connected by '53' relation. From these, we see at once that

(α) To an interior point of the set II there corresponds no other point of the set connected by '43' relation.

(β) Only interior points of the set II are connected with the frontier points by '47' or '53' relation.

Therefore if it is possible to derive an *ordered* configuration from the set II, we must start with two points of II connected by '43' relation. In that case $\alpha = 1543$ or $\alpha = 1551$ is one of the points with which we should start.

Now $\alpha = 38$ is the only point that is connected with $\alpha = 1543$ by '47' relation, and there is no point of the set II which is connected with $\alpha = 38$ by '53' relation. Writing this in the form

$$1543,\quad (47),\quad 38,\quad (53)\quad \times$$

we can make the three following similar statements :—

$$1543,\quad (53)\quad 17\frac{38}{53},\quad (47)\quad \times$$

$$551,\quad (47),\quad 3,\quad (53)\quad \times$$

$$1551.\quad (53),\quad 25\frac{38}{53},\quad (47)\quad \times$$

---

* Indeed it follows from Lemma 1. that each of the frontier points is connected with a certain other point by '43' relation, and since the total length of the interval is exactly $2\pi/43$, it is clear that no interior point can be connected with another point of the set by '43' relation.

From these statements it is seen that we cannot derive an *ordered* configuration by starting from one of the points 1543, 1551, and hence it is seen that II does not contain an *ordered* configuration :

THEOREM III.—*If* II *is a closed set and if* II_p, II_q, II_r *each completely covers the unit circle, then after a suitable arrangement of* p, q, *and* r, *we can find six points belonging to the set such that with two other points they form an ordered configuration.*

I shall give a sketch of the proof. By the method used for proving Theorem 1 we can show that in the present case there is a sub-set II′ such that

(*a*) II′ is a closed set,

(*b*) II_p, II_q, II_r each completely covers the unit circumference,

(*c*) a certain pair of the frontier points of II′ are connected by one of the three relations.

Thus if α_1 and α_2 be two frontier points of II′ connected by p relation, then from Lemma 1 it follows that there are points β_1, β_2, β_3, β_4 belonging to the set II′ such that

$$\left(\frac{\alpha_1}{\beta_1}\right)^q = \left(\frac{\alpha_2}{\beta_2}\right)^q = \left(\frac{\alpha_1}{\beta_3}\right)^r = \left(\frac{\alpha_2}{\beta_4}\right)^r = 1.$$

Now if we add to the set II two points γ and δ such that

$$\left(\frac{\beta_1}{\gamma}\right)^r = 1, \left(\frac{\beta_2}{\delta}\right)^r = 1$$

then the eight points α_1, α_2, β_1, β_2, β_3, β_4, γ, δ form an *ordered* configuration.

SOLUTIONS OF CONGRUENCES BY MEANS OF DIVERGENT SERIES*

BY BALAKRAM.

Summary:—It is shown in this note that if the solution, in the form of an infinite power series, of the algebraic equation $f(x) = 0$ with integral co-efficients, is known, we can derive the solutions (if any) of the congruence $f(x) = 0$ (mod. k^n) when we have solved it for the special case $n = 1$.

It is also shown that every quadratic congruence

$$ax^2 + bx + c \equiv 0 \pmod{k^n}$$

is merely a variation of the famous problem of finding a number of n digits in the k-ary scale, such that its square ends in the same n digits in the same order. For the ordinary 10-ary scale solutions of this problem for small values of n are known. A general formula, valid for all values of n and for all scales of notation, is given in § 10, and the result worked out for $n \leqslant 100$ in the 10-ary and for $n \leqslant 60$ in the 6-ary scale.

§ **1.** The essence of all methods of solving the congruence

(1.1) $$f(x) = A_m x^m + A_{m-1} x^{m-1} + \ldots + A_1 x + A_0 \equiv 0 \pmod{k^n}$$

the A's being polynomials in k is to write

$$x = a_0 + a_1 k + a_2 k^2 + \ldots + a_{n-1} k^{n-1}$$

and to obtain the co-efficients a by solving a succession of congruences, the first being (1.1) itself for $n = 1$, and the rest linear congruences. The non-existence of a solution of (1.1) is indicated by the non-existence of solutions of the subsidiary congruences; but if (1.1) has any solutions, they are all found by this process. The calculations can in some cases be simplified if we note that the above process is algebraically the

* Read at the Fourth Conference of the Indian Mathematical Society under the title "Solutions of Congruences modulus k^n."

same as that adopted in obtaining a series solution of the *equation*

$$f(x) = 0.$$

The statement that the infinite series

$$\sigma = \sum_{0}^{\infty} a_r k^r$$

when convergent, is a solution of $f(x) = 0$ is equivalent to the statement that n being any integer whatever,

$$(1.2) \qquad f\left(\sum_{0}^{n-1} a_r k^r\right) = k^n (b_0 + b_1 k + \ldots + b_j k^j) \qquad \ldots$$

is an identity, true for all values of k, including those for which the series is divergent, the b's being independent of k and j being a finite integer for the type of equation we are considering. If therefore the coefficients a_0, a_1, etc. are integers, the b's are also integers. Hence if k is an integer,

$$\sigma_n = \sum_{1}^{n-1} a_r k^r$$

is a solution of the congruence (1.1).

If the a's are not integers, we distinguish two cases, the first in which the denominators are all prime to k, and the second in which this condition is not fulfilled.

2. *First case*:—Let $a_r = p_r / q_r$, where q_r is prime to p_r, and also to k. We can always find an integer Q_r such that

$$q_r Q_r \equiv 1 \pmod{k^n}$$
$$= 1 + \rho k^n$$

giving

$$1/q_r = Q_r - \rho k^n / q_r$$

and

$$a_r = p_r Q_r - \rho p_r k^n / q_r$$
$$= \alpha_r + \rho_r k^n.$$

where α_r is an integer, and ρ_r an integer or a fraction with denominator prime to k. Treating all the a's in this manner, we get

$$\sigma_n = \sum_{0}^{n-1} \alpha_r k^r + k^n \sum \rho_r k^r = \sigma'_n + \lambda k^n, \text{ say}.$$

The relation (1.2) remains unaltered in form if for σ_n we write $\sigma_n - \lambda k^n$ or σ'_n. As the co-efficients α are now integers, the b's are integers, and σ'_n is the required solution.

3. We can avoid finding the α's till the end by making use of a fairly common notation, amounting to a definition, whereby the solution X of the congruenee

$$AX \equiv 1 \pmod{N}$$

is written as

(3.1) $$1/A \quad (\text{mod. } N)$$

The definition is confined to integers A prime to N, there being no value for X when A and N have a common factor.

The fraction B/A is defined as

(3.2) $$B/A \equiv B.(1/A) \equiv BX \equiv Y \quad (\text{mod. } N)$$

which gives the congruence

$$B \equiv AY \quad (\text{mod. } N)$$

Notice that the last mentioned congruence is not equivalent to the B/A as defined, unless A is prime to N. For instance

$$\left.\begin{aligned} 6 &\equiv 14.4 \\ &\equiv 14.9 \end{aligned}\right\} \quad (\text{mod. } 10)$$

so that 6/14, which is not one of the fractions defined by us, may be treated as the equivalent of 4 or 9. We may extend our definition by considering a fraction as only another way of writing a congruence whether A is prime to N or not. With this extension, a fraction may be the equivalent of more than one integer, if the modulus is not prime. The matter need not be pursued in detail; but it may be remarked that, subject to limitations which are obvious, fractions, as defined in (3.1) and (3.2) obey most of the ordinary laws of arithmetical combinations. We may leave fractions of the type we are considering untouched in σ, and treat σ_n as the required solution.

4. *Second case* :—If when a_r is reduced to its lowest terms, the denominator has a factor in common, with k we can by multiplying the numerator and the denominator by a suitable integer, write

$$a_r = p_r/(q_r k^t)' = \alpha_r/k^t$$

where q_r is prime to p_r and to k. If α_r is a fraction, it may be replaced by its integral equivalent as explained above. We therefore simplify the argument without loss of generality by taking α_r to be an integer. Substituting in σ and re-arranging the terms so as to obtain a series in ascending powers of k, we have

$$\sigma = \Sigma\, c_r k^r$$

where the c's are integers. If σ contains negative powers of k, the congruence has no solutions. If it contains no negative powers, we shall prove that

$$\omega_n = \sum_0^{n-1} c_r k^r$$

is the required solution.

Let μ be an integer such that all the terms of ω_n are included in σ_μ so that

$$\sigma_\mu = \omega_n + \mathrm{M}\,.\,k^n$$

where M is an integer or a fraction with denominator prime to k

$$f(\sigma_\mu) = k^\mu\,(b_0 + b_1\,k + \ldots\ \ldots + b_j\,k^j).$$

Though σ_μ is an integer, the b's are not necessarily so, but every fractional b will have its denominator a power of k, as the only way in which fractions can appear is through multiplication of the a's with each other. Let ν be the highest negative power of k in the b's. Then

$$f(\sigma_\mu) = k^{\mu-\nu}\,\mathrm{B}$$

where B is an integer. The integer ν depends on μ, but it is not greater than mT, where m is the degree of the congruence and T is the highest value of t in σ_μ. If μ and ν increase equally fast, $\mu - \nu$ will not exceed a fixed integer; but if so r and t will also increase equally fast, and the highest exponent of k in $\sigma = \sum_0^n c_r k^r$ will be less than a fixed integer independent of n. There may be no solutions for $n > n_0$ in this case; or a finite integer may be a solution for all values of n. If $\mu - \nu$ increases indefinitely, we take μ large enough to make $\mu - \nu \geqslant n$. Thus σ_μ is a solution; consequently

$$\omega_n \equiv \sigma_\mu \pmod{k^n}$$

is a solution.

5. It may happen that the series does not diverge sufficiently rapidly for the particular pseudo-parameter k. This difficulty can generally be removed by re-writing the co-efficients A_r in the congruence (1.1) so as to make them formal functions, not of k but of k^s. The series for σ now assumes the form

$$\sigma = \sum_0 a_r k^{rs}$$

and negative powers of k either do not arise or are cancelled much more quickly than before. If the process of expanding x in $f(x) = 0$ involves the extraction of roots, negative powers of k appear in the a's only through the binomial co-efficients

$$\frac{1}{r!}\left(\frac{g}{hk^s}\right)\left(\frac{g}{hk^s} - 1\right) \dots \dots \left(\frac{g}{hk^s} - r + 1\right)$$

where h is prime to k. These are immediately put in an integral form when the pseudo-parameter is k^{s+1}.

6. We have proved that σ_n or $\omega_n \equiv \sigma_\mu$ is a solution of the congruence in all cases when solutions exist. As every number congruent to a solution is also a solution, we may take σ itself to be a *formal* solution though it is an infinite number. The distinction between σ_n and ω_n now disappears, and we have the

THEOREM: If $\sigma = \sum_0^\infty a_r k^r$ *when convergent, is a solution of the equation*

$$f(x) = 0$$

with integral co-efficients; then it is, for every value of n, *a solution of*

$$f(x) \equiv 0 \pmod{k^n}$$

where k is an integer, provided that if the a's are fractions with denominators not prime to k, σ can be re-written as

$$\sum_{0}^{\infty} c_r k^r$$

where the c's are integers or fractions with denominators prime to k.

7. The initial difficulty in dealing with a given congruence is to introduce k into it as a pseudo-parameter. Each congruence needs individual treatment, the first step being the determination of the leading co-efficient. Writing the congruence as

$$F(x) \equiv A \pmod{k^n}$$

and obtaining somehow the term a_0, such that

$$F(a_0) \equiv A \pmod{k}$$
$$\equiv A + Bk,$$

we combine the two, and get

$$(A + Bk)\,F(x) \equiv A \,.\, F(a_0)$$

or

$$F(x) \equiv (1 + Bk/A)^{-1} \,.\, F(a_0)$$

giving the subsidiary *equation*

$$\phi(x) = F(x) - (1 + Bk/A)^{-1}\, F(a_0) = 0.$$

8. The process of finding the equivalent value of any fraction involved in the final result is apt to be tedious, unless the calculation is treated as the solution of a linear congruence by the method explained above. For instance, to find the equivalent value (x) of $\frac{1}{3}$, modulus 11^n, we write

(8.1) $$3x \equiv 1 \pmod{11^n}$$

The solution for $n = 1$ is 4,

giving $$3.4 = 1 + 11 = 1 + k, \text{ say.}$$

Combining the two,

$$(1 + k)\, x \equiv 4 \pmod{k^n}$$

The power-series solution of the related equation is

$$x = 4 - 4k + 4k^2 - 4k^3 + \ldots$$

Hence the required solution of (8.1) is

$$x \equiv 4 - 4k + 4k^2 - 4k^3 + 4k^4 - \ldots$$
$$\equiv 4 + 7k + 3k^2 + 7k^3 + 3k^4 + \ldots$$

the negative term being removed by utilizing the fact that $-4 = 7 - k$. It is not necessary to write the steps at length, and we may condense the above into the following :—

$$\frac{1}{3} \equiv \frac{4}{12} \equiv \frac{4}{1+k} \equiv 4(1-k+k^2-k^3+k^4) \equiv 4+7k+3k^2+7k^3+4k^4+\ldots$$

Simplifications will suggest themselves to the reader in individual cases.

9. We give a few numerical examples to illustrate the method, putting down only the important steps.

(i) $$x^2 \equiv 2 \pmod{k^5} \text{ where } k = 7.$$

Here a_0 has two values 4 and -4, and in both cases

$$(\pm 4)^2 = 2(1 + k).$$

Combining it with the given congruence, we have

$$(1 + k)\, x^2 \equiv 16$$

or

$$\pm x \equiv 4(1 + k)^{-\frac{1}{2}}$$
$$\equiv 4 - 2k + \frac{3k^2}{2} - \frac{5k^3}{4} + \frac{35k^4}{32} - \ldots$$
$$\equiv 4 - 2k + \frac{3k^2}{2} - \frac{5k^3}{4} \pmod{k^5} \ldots$$

Now $$\frac{3}{2}k^2 \equiv k^2 + \frac{4k^2}{8} \equiv k^2 + \frac{4k^2}{1+k} \equiv k^2 + 4k^2(1 - k + k^2 - \ldots)$$
$$\equiv 5k^2 - 4k^3 + 4k^4$$

and $$-\frac{5}{4}k^3 \equiv -k^3 - \frac{2k^3}{8} \equiv -k^3 - \frac{2k^3}{1+k} \equiv -k^3 - 2k^3(1 - k)$$
$$\equiv -3k^3 + 2k^4$$

$$\therefore \quad \pm x \equiv 4 - 2k + 5k^2 - 7k^3 + 6k^4$$
$$\equiv 4 + 5k + 4k^2 + 5k^4$$
$$\equiv 12240 \quad (\text{mod. } 7^5)$$

(ii) $$x^3 \equiv 2 \quad (\text{mod. } k^n) \text{ where } k = 3.$$

Here $a_0 = 2$, giving

$$(1 + k)\, x^3 \equiv 2^3$$

or $$x \equiv 2\left(1 - \frac{k}{3} + \frac{2k^2}{9} - \frac{14k^3}{81} + \dots\right)$$
$$\equiv 2\left(1 - 1 + 2 - \frac{14}{3} + \dots\right)$$
$$\equiv 4 - \frac{28}{3} + \dots$$

The negative powers of 3 cannot be got rid of. Hence there is no solution for $n > 1$.

(iii) $$x^3 \equiv 10, \quad (\text{mod. } 3^n)$$
$$\equiv 1 + k^2.$$

To minimise the formal appearance of negative powers of 3 in the expansion, write the equation as

$$x^3 \equiv 1 + 3k$$

giving $$x \equiv (1 + 3k)^{\frac{1}{3}}$$
$$\equiv 1 + k - k^2 + \frac{5}{3}k^3 - \dots$$

The negative powers of 3 appear because the denominator is $r!$; but on account of the multiplication by k^r, the series, on the whole, is in ascending powers of k. It is however not rapidly divergent. To obtain a better series, solve the equation for $n = 4$. This gives $x = 13$, an unexpectedly convenient number, as

$$13^3 = 10 + 3^7 = 10 + 3 \,.\, k^6,$$

and we get

$$x^3 \equiv 13^3\,(1 + 3k^6/10)^{-\frac{1}{3}}$$

or $$x \equiv 13 \left[1 - \frac{k^6}{10} + \frac{2k^{12}}{10^2} - \frac{14k^{17}}{10^3} + \ldots\right]$$

$$\equiv 13 + 23k^6 + 59 . k^{11} + 8 . k^{17} + 2k^{22} \pmod{k^{23}}$$

We leave it to the reader to fill in the details.

10.. The general quadratic congruence is

(9.1) $$ax^2 + bx + c \equiv 0 \pmod{k^n}$$

equivalent to

(9.2) $$\left.\begin{array}{c}(2ax + b)^2 \equiv b^2 - 4ac \\ \text{or} \quad \xi^2 \equiv \Delta\end{array}\right\} \pmod{4ak^n}$$

If ξ_0 is any solution, we have

$$\xi^2 \equiv \Delta \equiv \xi_0^2$$

or

(9.3) $$(\xi / \xi_0)^2 \equiv 1.$$

If $\varepsilon_1, \varepsilon_2, \ldots \varepsilon_p$ be the various roots of

$$\varepsilon^2 \equiv 1,$$

(9.4) the solutions of (9.2) are $\varepsilon_1 \xi_0, \varepsilon_2 \xi_0, \ldots \varepsilon_p \xi_0$. Congruence (9.4) is therefore fundamental in the theory of quadratic congruences. We may without loss of generality, take $a = 1$. As ε is odd, let

$$\varepsilon = 2\eta - 1.$$

(9.4) reduces to

$$4\eta^2 - 4\eta \equiv 0 \pmod{4k^n}$$

or

(9.5) $$\eta^2 - \eta \equiv 0 \pmod{k^n}$$

which is the mathematical expression of the famous problem : "*Find a number η of* n *digits in the k-nary scale, such that its square ends in the same* n *digits in the same order.*

If k is a prime, there are only two solutions, both trivial, *i.e.*, $\eta = 0$ and 1, preceded by the requisite number of cyphers. If k is not a prime' non-trivial solutions exist. These we proceed to find.

11. Starting with (9.4),

(10.1) $$\varepsilon^2 \equiv 1 \pmod{4k^n}$$

let ε^2_0 be a solution of

$$\varepsilon^2 \equiv 1 \pmod{4k^m}$$

where $m < n$, and let

(10.2) $$\varepsilon^2_0 = 1 + 4\beta k^m.$$

Combining (10.1) and (10.2)

(10.3) $$\left.\begin{aligned} \varepsilon &\equiv \pm\, \varepsilon_0 (1 + 4\beta k^m)^{-\frac{1}{2}} \\ &\equiv \pm\, \varepsilon_0 \sum_0 \frac{(2r)!}{(r!)^2} \cdot (-\beta k^m)^r \end{aligned}\right\} \pmod{2k^n}.$$

As ε_0 and $-\varepsilon_0$ are both solutions of (10.2) and ε, $-\varepsilon$ of (10.1) we may associate ε with ε_0 and $-\varepsilon$ with $-\varepsilon_0$. The sign in (10.3) may thus be taken to be positive.

(10.3) reduces to

$$\eta \equiv \eta_0 - (2\eta_0 - 1)\, \beta k^m \sum_0 \frac{(2r+1)!}{r!\,(r+1)!} (-\beta k^m)^r \quad [\text{mod. } k^n]$$

where

(10.4) $$2\eta_0 - 1 = \varepsilon_0$$

The value corresponding to $-\varepsilon_0$ is $k^n - \eta + 1$.

For numerical calculation m must be chosen so as to make β small without making m too small.

12. For $k = 10$, solutions by other methods are given in various publications, the largest n being $n = 20$, one of the solutions being $5^{20} \times 81199 + 1$.*

To obtain a solution from (10.4) the most convenient value of m is 4, giving

$$\eta_0 = \cdot 0625, \text{ as one of the values.}$$

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* *Vide* Dickson's *History of the Theory of Numbers*, Vol. I, Chap. XX, p 454.

$$\eta^2{}_0 = 390625$$

and $$\beta = 39.$$

Substituting in (10.4)

$$\eta \equiv 625 - 39 \times 1249 \times 10^4 \sum_0 \frac{(2r+1)!}{r!\,(r+1)!} (-39 \,.\, 10^4)^r.$$

Four terms are sufficient to give 20 digits.

One solution for $n = 100$ is

η_1 = 3953, 0073, 1910, 8169, 8029, 3850, 9890, 0621, 6650, 9580 8638, 1100, 0557, 4234, 2323, 0896, 1090, 0410, 6619, 9773, 9225 6259, 9182, 1289, 0625.

The other solution is $10^{100} + 1 - \eta_1$. The lowest p digits of these two numbers give the two solutions for $n = p$.

13. One solution for $n = 60$ in the 6-ary scale is

4303, 3204, 2521, 0204, 1331, 0422, 4033, 4135, 5525, 0555 2131, 4155, 1522, 2135, 0213.

ON THE NET OF CONICS HAVING A COMMON SELF-POLAR TRIANGLE*

By S. AUDINARAYANAN, M.A.

1. Introduction The totality of conic-loci

$$S \equiv ax^2 + by^2 + cz^2 + 2fyz + 2gxz + 2hxy = 0$$

whose co-efficients satisfy r independent linear equations, constitute a *linear* ∞^{5-r} system, N_{5-r}. In particular, if $r = 3$, we have a linear ∞^2 system or a *net* N_2. We define in a dual manner the net $\overline{N}_2$ of conic-envelopes. It is clear that the conics of a net which satisfy a further linear condition will belong to a pencil; *e.g.*, the conics of N_2 ($\overline{N}_2$) which pass through a given point (touch a given line) will equally pass through three other associated points (touch three other associated lines).

The Jacobian of N_2 (namely the locus of vertices of line pairs of N_2) is a cubic curve J_3; similarly, the Jacobian of $\overline{N}_2$ is a class cubic $\overline{J}_3$ which is the envelope of joins of point pairs of $\overline{N}_2$. The envelope of the line pairs themselves of N_2, is also a class cubic $\overline{C}_3$ the *Cayleyan* of N_2; similarly the Cayleyan of $\overline{N}_2$ (the locus of point pairs of $\overline{N}_2$) is an order cubic C_3.†

It is clear that the three conics which define a net N_2 may have one, two, or three common points, but not four. Such points will be common to all the conics of the net and will give rise to a double point of the Jacobian J_3. If the net contains on the other hand, a squared line L^2, then L will have to be a bi-tangent of the Cayleyan.‡ A curve of the

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* Read at the 5th Conference of the Indian Mathematical Society, Bangalore 1926.

I am greatly indebted to Dr R. Vaidyanathaswamy, for kindly guidance and encouragement in the preparation of this paper.

† *Ency. der Math. Wiss.* III C_3 (1), page 135.

‡ L will also be part of the Jacobian.

third-class may have one, two or three bi-tangents; accordingly a net may have squared lines up to three in number. This brings forward two of the simplest types of nets, *viz.* those which have three base points a, b, c and those which have three squared lines. The Jacobian of the former is composed of the sides of abc and the Cayleyan is the three points a, b, c, and a similar remark holds true for the other type.

Some metrical properties of the second type of net, *i.e.*, the net of conics with three squared lines, have heen given by Karl Meister in his paper "Uber die Systeme welche durch Kegelschnitte mit einem gemeinsamen Polardreieck bez. durch Flächen zweiten Grades mit einem gemeinsamen Polar—tetrae der gebildet werden" published in *Zeitschrift fur Math. und Phy.* Band 31. page 321. All conics of the net can be expressed in the form

$$ax^2 + by^2 + cz^2 = 0$$

when the three squared lines are taken as the three base conics and the triangle formed by the three lines as the triangle of reference. It is seen from the form of the equation that all conics of the net have the triangle formed by the three squared lines for a common self-polar triangle. The dual net also possesses the same common self-polar triangle. So, given two conics, in general position, the net is uniquely determined, as the two conics have a single self-polar triangle.

The most important property of a net of this type is *that it is closed for all co-variant processes which derive conics from conics;* for, since the pole-and-polar relation is a co-variant one, any conic co-variant of a number of conics with a common self-polar triangle must also have the same self-polar triangle. To express this in another way, we observe that every conic $ax^2 + by^2 + cz^2 = 0$, of the net is carried into itself by the collineation group G, consisting of the identical collineation and the three perspectivities of the type

$$x' : y' : z' = -x : y : z.$$

It is obvious further that no conics other than those of the form

$$ax^2 + by^2 + cz^2 = 0$$

can be invariant with respect to G; thus the net can be defined as the totality of invariant-conics of the group G and any covariant of such conics must also be transformed into itself by G and therefore belongs to the net. Thus for example, the F and ϕ — conics of two given conics of the net are also members of the net; or again if one conic is reciprocated with respect to the other, the resulting conic is a co-variant and is hence a member of the characteristic net defined by the first two conics.

It follows from the general property of a net, that conics of the net that pass through a fixed point P_1, pass also through three other fixed points P_2, P_3, P_4 and form a pencil. For a net with a common self-polar triangle xyz, it is clear that the quadrangles of the type $P_1P_2P_3P_4$ have xyz as their harmonic triangle. Thus :—

"A pencil of conics in the net contains three line pairs having the vertices coincident with those of the common self-polar triangle xyz and harmonically separating the corresponding sides."

Dually,

"A range of conics in the net contains three point pairs on the sides of the common self-polar triangle xyz harmonically separating the corresponding vertices."

2. The geometry of the ternary domain with reference to three fixed points :—

Since any conic of the net can be expressed by the equation

$$\lambda S_1 + \mu S_2 + \nu S_3 = 0$$

where $S_1 = 0$, $S_2 = 0$, $S_3 = 0$ are the equations to any three linearly independent members of the net, the totality of conics belonging to the net constitute a ternary domain. For our special net, we naturally take for S_1, S_2, S_3, the three singular members, namely, the three squared lines.

We enunciate now the following important principle :—

"Any invariant relation (in the usual sense) between the conics Σ_1, Σ_2, ... of the net is equivalent to an invariant relation between the elements Σ_1, Σ_2, .. S_1, S_2, S_3 (involving the three last symmetrically) of the ternary domain of the net."

It is necessary to remark that the truth of the principle depends essentially on the fact that S_1, S_2, S_3 are members of the net characterised by an invariant property (that of being squared lines). We shall not attempt a rigorous proof of the principle but shall assume it on an inductive basis.

The structure and properties of this special net will therefore be intimately related to the structure and properties of a ternary domain with reference to three of its elements. We may consider the ternary domain to be a plane and the fixed elements to correspond to three fixed points A, B, C. We shall in the next two paragraphs engage in a preliminary study of (1) the polar and apolar relations, (2) the quadratic transformations defined by three points ABC of a plane.

3. Apolarity relations defined by three points A, B, C.

Taking ABC as the triangle of reference, the equation of a general class-cubic is

$$\begin{aligned} a_{111}\, l^3 + a_{112}\, l^2 m + a_{122}\, l m^2 + a_{113}\, l^2 n + a_{133}\, l n^3 \\ + a_{223}\, m^2 n + a_{233}\, m n^2 + a_{222}\, m^3 + a_{333}\, n^3 \\ + a_{123}\, lmn = 0 \qquad \ldots \qquad \ldots \qquad \ldots \quad (1) \end{aligned}$$

Considering the sides of the triangle of reference ABC as a cubic curve, the cubic (1) is apolar to the triangle ABC, if the co-efficient a_{123} vanishes. If, also the class-cubic degenerates into three points P, Q, R, then the triangle PQR is termed an apolar triangle of ABC. It may be shown that the apolar relation between ABC, PQR is a symmetric one, even though we have defined it with reference to the sides of the first and the vertices of the second.

In the light of this definition of apolarity, we may define the polar conic of P (λ, μ, ν) as the locus of points Q (x, y, z) which are such that the triad PQQ is apolar to the triangle ABC. By equating the co-efficient of lmn in $(l\lambda + m\mu + n\nu)\ (lx + my + nz)^2$ to zero, the equation to the polar conic of P comes out to be

$$\lambda yz + \mu zx + \nu xy = 0.$$

Also the polar line of P can be defined as the locus of points Q such that PPQ is apolar to ABC. By equating the co-efficient of lmn in $(l\lambda + m\mu + n\nu)^2\ (lx + my + nz)$ to zero, its equation is seen to be

$$\mu\nu x + \nu\lambda y + \lambda\mu z = 0.$$

They are the polar transformations of a point in the domain.

If the polar line and the polar conic of a point P intersect in two points S, S′, the triad PSS′ is in sextuple perspective* with the triangle ABC. If we take P as (1, 1, 1); then S, S′ satisfy the equations

$$x + y + z = 0, \quad yz + zx + xy = 0,$$

so that they are

$$(1, \omega, \omega^2) \text{ and } (1, \omega^2, \omega).$$

There are ∞^2 such triads PSS′, as each triad is definitely fixed when one of the points is given. The three points PSS′ are symmetrically related, *i.e.*, the polar line and the polar-conic of any one of the points pass through the two others.

The apolar triangle PQR has a simple geometrical relation to ABC, namely, any two of the points are conjugate points with respect to the polar conic of the third. This is the immediate consequence of the vanishing of a_{123}.

4. The Quadratic Transformations related to ABC.

A quadratic transformation defined by the four base point $(P_1P_2P_3P_4)$ transforms every point L of the plane into a unique point L′, which is the point of concurrence of the polars of L with respect to the conics through P_1, P_2, P_3, P_4. We shall represent L′ by the symbol $[(P_1P_2P_3P_4)\,L]$. The base points $P_1P_2P_3P_4$ are the self-corresponding points in the transformation.

The isogonal transformation arises as a particular case of this general transformation, when all the conics are rectangular hyperbolas, or in other words, when the four base points form an orthocentric system.

As L traces out a straight line, L′ describes a conic circumscribing the harmonic triangle of $P_1P_2P_3P_4$. If the line joining pairs of points such as LL′ pass through a fixed point K, then the locus of L, L′ is a cubic curve called the *Jumelaire cubic* of K, which passes through P_1, P_2, P_3, P_4, K, the vertices of the harmonic triangle, the points of intersection of the sides of this triangle with the lines joining K to the

* H. Schroter: *Math. Ann.* 2 (1870) p. 653—662.

opposite vertices and the transform of K. The joins of K to $P_1P_2P_3P_4$ are tangents, to the curve there.*

In the ternary domain with three fixed points ABC, there are two types of quadratic transformations which are of importance. For the first type, the base points $P_1P_2P_3P_4$ have ABC for harmonic triangle. For the second type, ABC and a fourth point P, are the base points. As ABC are three fixed points, any fourth point P determines uniquely the base points for both the types of transformations. If P_1 be (λ, μ, ν) referred to ABC as the triangle of reference and L be (xyz), then it is easily shown that (P_1ABC) L has co-ordinates:—

$$x\left(-\frac{x}{\lambda}+\frac{y}{\mu}+\frac{z}{\nu}\right),\ y\left(\frac{x}{\lambda}-\frac{y}{\mu}+\frac{z}{\nu}\right),\ z\left(\frac{x}{\lambda}+\frac{y}{\mu}-\frac{z}{\nu}\right),$$

and $(P_1P_2P_3P_4)$ L† has co-ordinates $\frac{\lambda^2}{x}, \frac{\mu^2}{y}, \frac{\nu^2}{z}$.

THEOREM I. *If $(ABCP)\ L = L'$, then the polar line of P is the mixed polar of L, L'.*

Proof.—If Q is any point on the polar line of P, then PPQ is apolar to ABC and the polar conic of Q passes through ABCP. Since L, L′ are quadratic transforms of each other with respect to ABCP, they are conjugate points with respect to the polar conic of Q. Hence by § 3, LL′ Q is apolar to ABC and so L, Q are conjugate points with respect to the polar conic of L′; in other words, the L and the polar line of P are pole-and-polar with respect to the polar conic of L′. Also L′ and the polar line of P are pole-and-polar with respect to the polar conic of L.

Cor. (i) If two circum-conics be drawn touching a line at two points L, L′, then the point of intersection P of the two conics is such that (ABCP) L = L′.

Cor. (ii). If L, M, N be three points such that L lies on the polar line of a point P which is in turn, such that (ABCP) M = N, then the three points are symmetric with regard to this property.

This follows from the fact that LMN is apolar to ABC.

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* Cf. Malet: *Etude geometrique des transformations birationalles et des courbes planes*, p. 96.

† Here $P_1P_2P_3P_4$ are $\pm\lambda, \pm\mu \pm\nu$.

THEOREM II. *If $(P_1P_2P_3P_4)\,L = L'$ where ABC is the harmonic triangle of $P_1P_2P_3P_4$,* then*

$$(P_1P_2P_3P_4) \text{ polar line of } L = \text{polar conic of } L'$$

both the polar line and conic being taken with respect to the triangle ABC.

Proof.—Let $P_1P_2P_3P_4$ be $(\pm\lambda, \pm\mu, \pm\nu)$ and L, L′ be (a, b, c) and (a', b', c') respectively.

Then $$a' = \frac{\lambda^2}{a},\ b' = \frac{\mu^2}{b},\ c' = \frac{\nu^2}{c}.$$

Let X (x_1, y_1, z_1) a point on the polar line of L get transformed into X′ (x_1', y_1', z_1').

Then $$x'_1 = \frac{\lambda^2}{x_1},\ y_1' = \frac{\mu^2}{y_1},\ z_1' = \frac{\nu^2}{z_1}$$

and $$\frac{x_1}{a} + \frac{y_1}{b} + \frac{z_1}{c} = 0.$$

$$\therefore \quad \frac{\lambda^2}{a}\frac{1}{x_1'} + \frac{\mu^2}{b}\frac{1}{y_1'} + \frac{\nu^2}{c}\frac{1}{z_1'} = 0$$

i.e. $$\frac{a'}{x_1'} + \frac{b'}{y_1'} + \frac{c'}{z_1'} = 0$$

Hence X′ lies on the polar conic of L′.

Any collineation in the plane for which ABC are the three fixed points, is given by equations of the form

$$\Omega: \quad \begin{aligned} \rho x_1 &= ax \\ \rho y_1 &= by \\ \rho z_1 &= cz. \end{aligned}$$

If there exists a collineation which has ABC for fixed points, and which carries P into R and Q into S then PR, QS will be termed *similar pairs*, with respect to ABC.

THEOREM III. *If* $(P_1P_2P_3P_4)\,L_1 = L_2$

and $(P_1P_2P_3P_4)\,L_3 = L_4$

then L_1L_3, L_4L_2 *are similar pairs,*

* We shall denote the quadrangle determined by a point R, and for which ABC is a harmonic triangle, by the symbol $(R_1R_2R_3R_4)$.

Proof. Let the co-ordinates of P, L_1, L_3 be

$$(\lambda \mu \nu), (\alpha \beta \gamma), (\alpha' \beta' \gamma').$$

Then those of

L_2, L_4, are $\left(\frac{\lambda^2}{\alpha}, \frac{\mu^2}{\beta}, \frac{\nu^2}{\gamma}\right)$ and $\left(\frac{\lambda^2}{\alpha'}, \frac{\mu^2}{\beta'}, \frac{\nu^2}{\gamma'}\right)$ respectively.

Suppose the collineation defined by the equations

$$x_1 : y_1 : z_1 = ax : by : cz$$

carries L_1 to L_3. Then $\alpha' : \beta' : \gamma' = a\alpha : b\beta : c\gamma$.

So, $$\alpha : \beta : \gamma = \frac{\alpha'}{a} : \frac{\beta'}{b} : \frac{\gamma'}{c}$$

and $$\frac{\lambda^2}{\alpha} : \frac{\mu^2}{\beta} : \frac{\nu^2}{\gamma} : = \frac{a\lambda^2}{\alpha'} : \frac{b\mu^2}{\beta'} : \frac{c\nu^2}{\gamma'}$$

which shows that the same collineation carries L_4 to L_2.

THEOREM IV.

(1) *If* Ω *(a collineation with fixed points ABC) carries one point (not on the side of the triangle) into a point on its polar line, then it carries every point into a point on its polar line.*

(2) *If* Ω *carries one point (not on the side of the triangle) into a point on its polar-conic, then it carries every point into a point on its polar conic.*

Let Ω be defined by the equations

$$x' : y' : z' = ax : by : cz.$$

Then the condition for the first is $a + b + c = 0$ and for the second

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 0.$$

If Ω is of the first type, then Ω^{-1} is of the second type.

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NOTES AND QUESTIONS.

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M. BHIMASENA RAO; G. A. SRINIVASAN, M.A., L.T.

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Notes and Questions.

The Integral Part of a Number as a Definite Integral.

1. In what follows the integral part of a number x is denoted by $[x]$, so that $[x]$ is an integer satisfying the inequalities

$$x-1 < [x] \leqslant x.$$

It is easily seen that

$$[-x] = -[x]-1 = -[x+1]$$

if x is not an integer, but

$$[-x] = -[x]$$

if x is an integer.

The function $x-[x]$ will be denoted by $\{x\}$. It is zero or a positive number less than unity. It is a periodic function of x, with period unity, with discontinuities at points where x is an integer.

2. The reader is no doubt familiar with expansion of $\{x\}$ in infinite trigonometric series. Two such have been brought to my notice by Prof. A. Narsinga Rao. One, given as an example in text-books, is

$$\{x\} = 8V^2 - U,$$

where U and V are the Fourier Series :—

$$U = \frac{1}{\pi}\sum_{1}^{\infty}\frac{\sin 2\pi n x}{n}$$

and

$$V = \frac{1}{\pi}\sum_{1}^{\infty}\frac{\sin \pi(2n-1)x}{n}$$

The other, due to Pringsheim, which was first published in 1886* is

$$\{x\} = \tfrac{1}{2}\sum_{0}^{x}\sin^2 \pi x \cos^{2n} \pi x - U$$

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* *Mathematische Annalen.* Bd. 26. p. 193.

U. beingt he Fourier Series defined above. The first infinite series in this formula vanishes when x is an integer, and for other values of x it is

$$\tfrac{1}{2}\sum_{0}^{\infty}(\cos^{2n}\pi x - \cos^{2n+2}\pi x) = \tfrac{1}{2} - \operatorname*{Lt}_{p\to\infty} \cos^{2p}\pi x = \tfrac{1}{2}.$$

The ordinary Fourier series for $\{x\}$, namely $\{x\} = \frac{1}{2} - U$, fails for integral values of x.

3. A formal, but unsatisfactory, mode of expressing $\{x\}$ is

$$\{x\} = \frac{1}{\pi}\tan^{-1}(\tan \pi x),$$

$$= \frac{1}{\pi}\int_0^{\tan \pi x} \frac{dt}{1+t^2}$$

provided the definition of $\tan^{-1} c$ is such that

$$0 \leqslant \tan^{-1} c < \pi,$$

and the path of the integral is chosen suitably.

4. The above formulae are primarily for $\{x\}$, and $[x]$ is derived from them by means of the relation

$$[x] = x - \{x\}$$

The use of Cauchy's Theorem gives $[x]$ directly, but unfortunately every formula that I have constructed fails like the ordinary Fourier series, for integral values of x.

The object is to express $[x]$ as an infinite integral

$$[x] = \int_0^x f(x, y)\, dy.$$

One form of $f(x, y)$ is found below, and others can be readily constructed. Integrals can be found equal to the reciprocal of $[x]$, or other functions of $[x]$.

By Cauchy's Theorem

$$k = \frac{\pi}{4i}\int \frac{z^2}{\sin^2 \pi z}\, dz$$

taken round a contour enclosing the point $z = k$, a positive or negative integer but no other zero of $\sin \pi z$. Choose the contour to be a rectangle whose sides are the lines parallel to the y-axis through $(x, 0)$ and $(x-1, 0)$ and the lines $y = \pm \beta$. If x is not an integer the only zero of $\sin \pi z$

within this rectangle is $[x]$ as defined in (1) and hence $k = [x]$. It is easy to see that the integrals along the sides parallel to the x-axis vanish if β becomes infinite. Along the other sides x is constant and $dz = i dy$. Hence we have

$$[x] = \frac{\pi}{4} \int_{-\infty}^{+\infty} \left(\frac{(x+iy)^2}{\sin^2 \pi (x+iy)^2} - \frac{(x-1+iy)^2}{\sin^2 \pi (x-1+iy)} \right) dy$$

$$= \frac{\pi}{4} \int_{-\infty}^{+\infty} \frac{2x + 2iy - 1}{\sin^2 \pi (x+iy)} dy$$

$$= \frac{\pi}{4} \int_{-\infty}^{\infty} \frac{4 \sin^2 \pi (x - iy).(2x + 2iy - 1)}{(\cosh 2\pi y - \cos 2\pi x)^2} dy$$

$$= 2\pi \times \text{the integral}$$

$$\int_0^\infty \frac{(\sin^2 \pi x \cosh^2 \pi y - \cos^2 \pi x \sinh^2 \pi y)(2x-1) + y \sin 2\pi x \sinh 2\pi y}{(\cosh 2\pi y - \cos 2\pi x)^2} dy$$

which is our required integral. Denoting the above integral by $\phi(x)$ it is easily verified that

$$\phi(-x) = -\phi(x+1)$$

as it should. When x is an integer $\phi(x)$ is infinite, and is *not* equal to $[x]$.

By taking $2x - 1 = \xi$, and $2y = \eta$, the above integral may be reduced to the more symmetric form

$$\left[\frac{\xi + 1}{2}\right] = \frac{\pi}{2} \int_0^\infty \frac{\xi (1 + \cos \pi\xi \cosh \pi\eta) - \eta \sin \pi\xi \sinh \pi\eta}{(\cos \pi\xi + \cosh \pi\eta)^2} dy.$$

By integrating other functions of z we obtain similar integrals involving $[x]$. Thus from $\int dz/z \sin \pi z$ we may obtain an integral for $(-1)^{[x]}/[x]$.

5. It is not easy to trace to its original cause, the disappearance of the fraction part of x when the formidable-looking integrals in § 4 are evaluated. The particular case worked out above in detail, can, I believe, be linked up with the formula

$$\tan \pi x = \tan (\pi \{ x \}),$$

as may be seen by writing

$$\int \frac{z^2 dz}{\sin^2 \pi x} = -z^2 \cot \pi z + 2 \int z \cot \pi z dz$$

and evaluating the integral on the right-hand after reducing it to its real form. The evaluation requires some care, and we leave it as an exercise for the reader. I have not been able to link up the integral for the reciprocal of $[x]$ with any simpler results.

BALAKRAM.

On a form of the Identities between Line-Co-ordinates.

If in space of n dimensions, we represent points by homogeneous co-ordinates $x_0, x_1, \dots x_n$, then the homogeneous line-co-ordinates of the line which joins the points $(x_0, x_1, \dots x_n)$, $(y_0, y_1, \dots y_n)$ are the numbers l_{rs} given by

$$l_{rs} = x_r y_s - x_s y_r \qquad (r, s = 0, 1, 2, \dots n).$$

The $\frac{(n+1)n}{2}$ numbers l_{rs} are linearly independent but are connected by lineo-linear relations of the type

$$L_{pqrs} = l_{pq} l_{rs} + l_{qr} l_{ps} + l_{rp} l_{qs} = 0 \qquad \dots \quad (1)$$

The $\frac{(n+1)n(n-1)(n-2)}{1.2.3.4}$ *relations of type (1) are all linearly independent.*

To prove this, suppose if possible that a relation of the form

$$\Sigma\, \lambda_{pqrs}\, L_{pqrs} = 0, \qquad \dots \qquad \dots \quad (2)$$

is *identically* true in the quantities l, the numbers λ being independent of the l's. Then on differentiating (2) with respect to any l, we will reach a linear identity between the l's, whose co-efficients are the quantities λ_{pqrs}. Since the l's are linearly independent, it follows that the λ's must all vanish, or the relations of the type (1) are all linearly independent. These relations will however be connected by certain syzygies*—as for instance

$$l_{tu}\, L_{pqrs} + l_{us}\, L_{pqrt} + l_{st}\, L_{pqru} = 0.$$

The object of this note is to draw attention to the fact that *the relations (1) imply, and are all implied by, the single statement that the*

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* For these syzygies, see Ency. Math. Wissen, III, 7, Segre: *Mehrdimensionale Räume*, page 792, foot-note 80.

rank of the skew-symmetric matrix $| l_{rs} |$, *of order* $n + 1$, *is equal to two.*

To prove the first part, we have to shew that the relations (1) imply the vanishing of every third order minor determinant of $| l_{rs} |$. Now the general minor of the third order of this matrix is of the form

$$\Delta_3 = \begin{vmatrix} l_{pp'} & l_{pq'} & l_{pr'} \\ l_{qp'} & l_{qq'} & l_{qr'} \\ l_{rp'} & l_{rq'} & l_{rr'} \end{vmatrix} = \begin{vmatrix} x_p & -y_p \\ x_q & -y_q \\ x_r & -y_r \end{vmatrix} \times \begin{vmatrix} y_{p'} & y_{q'} & y_{r'} \\ x_{p'} & x_{q'} & x_{r'} \end{vmatrix}$$

Since the matrix-product on the right is one in which short rows are active, it follows from a well-known theorem that every Δ_3 vanishes or the matrix $| l_{rs} |$ is of rank two.

To prove the second part, we have to shew that the elements of any skew-symmetric matrix $| l_{rs} |$, of rank two, satisfy the relations (1). Since the matrix is of rank two, any three of its rows are linearly related; in partcular, there is a linear relation between the three rows:

$$\begin{matrix} l_{p0} & l_{p1} & \dots & l_{pn} \\ l_{q0} & l_{q1} & \dots & l_{qn} \\ l_{r0} & l_{r1} & \dots & l_{rn}, \end{matrix}$$

of the form

$$\lambda\, l_{ps} + \mu\, l_{qs} + \nu\, l_{rs} = 0 \qquad (s = 0, 1, 2, \dots n).$$

To determine λ, μ, ν two of these $n + 1$ relations suffice. Taking the two relations for which $s = p, q$ respectively, we immediately find

$$\lambda : \mu : \nu = l_{qr} : l_{rp} : l_{pq}.$$

Thus we have

$$l_{qr}\, l_{ps} + l_{rp}\, l_{qs} + l_{pq}\, l_{rs} = 0;$$

these are however precisely the relations (1). This proves our theorem.

It will be interesting to know whether, and how a corresponding statement can be made for general region-coordinates.

R. VAIDYANATHASWAMY.

Solutions.

Question 873.

(H. R. KAPADIA) :—Prove that

$$\text{(i)}\quad \tfrac{1}{2}\cot^{-1}\left(\frac{2\sqrt[3]{4}+1}{\sqrt{3}}\right)+\tfrac{1}{3}\tan^{-1}\left(\frac{\sqrt[3]{4}+1}{\sqrt{3}}\right)=\frac{\pi}{6};$$

$$\text{and (ii)}\quad \tfrac{1}{2}\tan^{-1}\left(\frac{\sqrt[3]{2}+1}{\sqrt{3}}\right)-\tfrac{1}{3}\tan^{-1}\left(\frac{2\sqrt[3]{2}+1}{\sqrt{3}}\right)=\frac{\pi}{36}.$$

Solution by P. R. Venkatakrishna Iyer and others.

(i) The first member of (i)

$$=\tfrac{1}{6}\left\{3\tan^{-1}\frac{\sqrt{3}}{2\sqrt[3]{4}+1}+2\tan^{-1}\frac{\sqrt[3]{4}+1}{\sqrt{3}}\right\}$$

$$=\tfrac{1}{6}\left\{\tan^{-1}\frac{\dfrac{3\sqrt{3}}{2\sqrt[3]{4}+4}-\dfrac{3\sqrt{3}}{(2\sqrt[3]{4}+1)^3}}{1-3\,.\,\dfrac{3}{(2\sqrt[3]{4}+1)^2}}+\tan^{-1}\frac{2\,.\,\dfrac{\sqrt[3]{4}+1}{\sqrt{3}}}{1-\left(\dfrac{\sqrt[3]{4}+1}{\sqrt{3}}\right)^2}\right\}$$

This works out to

$$\tfrac{1}{6}\left\{\tan^{-1}\frac{\sqrt{3}\,(1+\sqrt[3]{4})}{\sqrt[3]{2}+\sqrt[3]{4}-1}+\tan^{-1}\frac{\sqrt{3}\,(\sqrt[3]{4}+1)}{1-\sqrt[3]{4}-\sqrt[3]{2}}\right\}$$

$$=\tfrac{1}{6}\left\{\tan^{-1}0\right\}=\frac{\pi}{6}.$$

That $\tan^{-1}0=\pi$ in this case is determined by the angles in the left-hand side. In this case

$$\tfrac{1}{2}\cot^{-1}\left(\frac{2\sqrt[3]{4}+1}{\sqrt{3}}\right)=11^\circ\ 16'$$

and

$$\tfrac{1}{3}\tan^{-1}\left(\frac{\sqrt[3]{4}+1}{\sqrt{3}}\right)=18^\circ\ 44'$$

taking the angles between 0 and π in each case.

(ii) The first member of (ii) may be written

$$= \tfrac{1}{6} \left\{ 3 \tan^{-1} \frac{\sqrt[3]{2}+1}{\sqrt{3}} - 2 \tan^{-1} \left(\frac{2\sqrt[3]{2}+1}{\sqrt{3}} \right) \right\}$$

$$= \tfrac{1}{6} \left\{ \tan^{-1} \frac{\sqrt{3}\,(\sqrt[3]{2}+1-(\sqrt[3]{2}+1)^3/\sqrt{27}}{1-(\sqrt[3]{2}+1)^2} \right.$$

$$\left. - \tan^{-1} \frac{2\,(3\sqrt[3]{2}+1)/\sqrt{3}}{1-(2\sqrt[3]{2}+1)^2/3} \right\}$$

After simplification this becomes

$$= \tfrac{1}{6} \tan^{-1} \frac{1}{\sqrt{3}} = \frac{1}{6} \cdot \frac{\pi}{6} = \frac{\pi}{36}.$$

The determination $\tan^{-1} \frac{1}{\sqrt{3}} = \frac{\pi}{6}$ is fixed by the angles on the left-hand side.

Question 928

(M. K. KEWALRAMANI) :—Show that the infinite product

$$\left(1-\frac{1}{2^{12}}\right)\left(1-\frac{1}{3^{12}}\right)\left(1-\frac{1}{4^{12}}\right) \ldots \quad \ldots$$

$$= \frac{\sinh \pi \cosh^2 \sqrt{3}\,\pi/2}{24\,\pi^5} (\cosh \pi - \cos \sqrt{\_}\,\pi).$$

Solutions by N. Sankara Aiyar, S. Audinaryan, S. R. Ranganathan, and N. B. Mitra.

Now $\sinh \pi = \pi \prod_1^\infty \left(1+\frac{1}{r^2}\right)$;

and $\cosh^2 \frac{\sqrt{3}\,\pi}{2} = \tfrac{1}{2}(\cosh \sqrt{3}\,\pi + 1) = \tfrac{1}{2}(\cosh \sqrt{3}\,\pi - \cos \pi)$.

But $\cosh x - \cos y = \cos xi - \cos y$,

$$= 2 \sin \frac{y+xi}{2} \sin \frac{y-xi}{2},$$

$$= \frac{y^2+x^2}{2} \prod_1^\infty \left\{ 1 - \frac{y^2-x^2}{2r^2\pi^2} + \frac{(y^2+x^2)^2}{16r^4\pi^4} \right\}.$$

Hence $\cosh\sqrt{3}\,\pi - \cos\pi = 2\pi^2 \prod_1^\infty \left\{1 - \frac{1}{r^2} + \frac{1}{r_4}\right\}$,

and $\cosh\pi - \cos\sqrt{3}\,\pi = 2\pi^2 \prod_1^\infty \left\{1 + \frac{1}{r^2} + \frac{1}{r^4}\right\}$,

Also $\frac{1}{2} = \prod_2^\infty \left(1 - \frac{1}{r^2}\right)$.

The expression on the right side in the question is therefore equal to

$$\frac{2\pi \prod_1^\infty \left(1 + \frac{1}{r^2}\right) \prod_2^\infty \left(1 - \frac{1}{r^2}\right) \times 4\pi^4 \times 3 \times \prod_2^\infty \left\{1 + \frac{1}{r^4} + \frac{1}{r^8}\right\}}{24\pi^5}$$

$$= \prod_2^\infty \left\{ \left(1 - \frac{1}{r^4}\right)\left(1 + \frac{1}{r^4} + \frac{1}{r^8}\right)\right\} = \prod_2^\infty \left(1 - \frac{1}{r^{12}}\right).$$

Question 1003.

(S. R. RANGANATHAN):—If $\phi_n(x)$ stands for the nth Bernoulli's Polynomial, show that

$$\phi_n\left(\tfrac{1}{4}\right) = 2^{n-1}\left[\phi_n\left(\tfrac{1}{8}\right) - \phi_n\left(\tfrac{3}{8}\right)\right]$$

or

$$= \frac{2^{3n-2} + 2^{2n-2} - 2^{n-1}}{2^{3n-1} + 2^{n-1} - 1}\left[\phi_n\left(\tfrac{1}{8}\right) + \phi_n\left(\tfrac{3}{8}\right)\right]$$

according as n, which is greater than 1, is odd or even.

Solution by G. A. Srinivasan.

This is a particular case of Ex. 4 on p. 240 of Bromwich's *Infinite Series*, (1st edition), *viz* :—

" Prove that if n is odd and k is an integer,

$$\sum_{r=0}^{k-1} \phi_n\left(x + \frac{r}{k}\right) = \frac{\phi_n(kx)}{k^{n-1}},$$

and obtain the corresponding result when n is even."

These results are easily obtained as follows :—

The expression

$$S \equiv \sum_{r=0}^{k-1} \phi_r\left(x + \frac{r}{k}\right)$$

is the co-efficient of $\frac{t^n}{n!}$ in the expansion of

$$\frac{t}{e^t - 1}\left\{ \sum_{r=0}^{k-1} e^{(x+r/k)t} - k \right\}$$

i.e. of
$$\frac{t}{e^t - 1}\left\{ e^{xt}\frac{(e^t - 1)}{e^{t/k} - 1} - k \right\}$$

i.e. of
$$\frac{t\,(e^{kx \cdot t/k} - 1)}{e^{t/k} - 1} + \frac{t}{e^{t/k} - 1} - \frac{kt}{e^t - 1}.$$

Thus we get

$$S = \frac{\phi_n(kx)}{k^{n-1}} \qquad \dots \qquad \dots \quad (1)$$

when n is odd integer other than 1, and

$$S' = \frac{\phi_n(kx)}{k^{n-1}} + (-1)^{\frac{n}{2}} B_{\frac{n}{2}} \cdot \frac{k^n - 1}{k^{n-1}} \qquad \dots \quad (2)$$

when n is even.

The first result in the question follows by putting $x = \frac{1}{8}$ and $k =$ in (1); thus, where n is odd

$$\phi_n(\tfrac{1}{4}) = 2^{n-1}\left[\phi_n(\tfrac{1}{8}) + \phi_n(\tfrac{5}{8})\right]$$
$$= 2^{n-1}\left[\phi_n(\tfrac{1}{8}) - \phi_n(\tfrac{3}{8})\right]$$

remembering that* when $n > 1$

$$\phi_n(1 - x) = (-1)^n\,\phi_n(x). \qquad \dots \quad (3)$$

To prove the second result, take (2) and let $k = 2$, while $x = \frac{1}{8}, \frac{1}{4}$ and $\frac{1}{2}$ successively. We obtain,

$$\phi_n\left(\frac{1}{8}\right) + \phi_n\left(\frac{5}{8}\right) = \frac{1}{2^{n-1}}\,\phi_n\left(\frac{1}{4}\right) + (-1)^{\frac{n}{2}} B_{\frac{n}{2}}\left(\frac{2^n - 1}{2^{n-1}}\right) \dots (4)$$

$$\phi_n\left(\frac{1}{4}\right) + \phi_n\left(\frac{3}{4}\right) = \frac{1}{2^{n-1}}\,\phi_n\left(\frac{1}{2}\right) + (-1)^{\frac{n}{2}} B_{\frac{n}{2}}\left(\frac{2^n - 1}{2^{n-1}}\right) \dots (5)$$

and
$$\phi_n\left(\frac{1}{2}\right) + \phi_n\left(1\right) = \frac{1}{2^{n-1}}\,\phi_n\left(1\right) + (-1)^{\frac{n}{2}} B_{\frac{n}{2}}\left(\frac{2^n - 1}{2^{n-1}}\right) \dots (6)$$

* *Vide* Bromwich: *Infinite Series*, (1st edition), p. 237.

Now, since n is even,

$$\phi_n(1) = \phi_n(0) = 0, \quad \phi_n(\tfrac{3}{4}) = \phi_n(\tfrac{1}{4}), \quad \phi_n(\tfrac{5}{8}) = \phi_n(\tfrac{3}{8}).$$

Hence (6) gives :

$$\phi_n(\tfrac{1}{2}) = (-1)^{\frac{n}{2}} B_{\frac{n}{2}} \left(\frac{2^n - 1}{2^{n-1}}\right)$$

$$= \frac{2^n}{2^{n-1}+1} \cdot \phi_n(\tfrac{1}{4}) \text{ by (5).}$$

Lastly (4) gives

$$\phi_n(\tfrac{1}{8}) + \phi_n(\tfrac{3}{8}) = \left(\frac{1}{2^{n-1}} + \frac{2^n}{2^{n-1}+1}\right) \phi_n(\tfrac{1}{4})$$

so that $$\phi_n(\tfrac{1}{4}) = \frac{2^{n-1}(2^{n-1}+1)}{2^{2n-1}+2^{n-1}+1} [\, \phi(\tfrac{1}{8}) + \phi(\tfrac{3}{8}) \,] \quad \ldots \quad (7)$$

This agrees with the given result on cancelling the common factor $2^n - 1$ from the numerator and the denominator.

Also partially solved by Sadanand.

Question 1061.

(LAKSHIMISHANKAR, N. BHATT).—The Euler line of the triangle AB_1C_1 meets the sides AB_1, AC_1, in B_2, C_2; the Euler line of AB_2C_2 meets these sides in B_3, C_3, and this process is continued indefinitely. If N_r is the nine-point centre of the rth triangle AB_rC_r thus formed, and AN_r meets B_rC_r in D_r, prove that the straight lines $N_x\,N_{x+2y}$ and $D_x\,D_{x+2y}$ are parallel, x any y being any two positive integers.

Remarks by F. H. V. Gulasekharam, M.A.

On a closer study of the figure, it will be found that $N_x\,N_{x+2y}$ and $D_x\,D_{x+2y}$ are coincident with a fixed straight line through A.

In fact, ($AN_1\,D_1\,N_3\,D_3$ ......) and ($AN_2\,D_2\,N_4\,D_4$ ......) form two sets of collinear points; for, from the solution to Q. 1009 (*J. I. M. S.* Vol. XI, p. 185), it is obvious that

(i) the different Euler lines form two systems of parallel lines;

(ii) if O_r, P_r be the circum-centre and ortho-centre of AB_rC_r then

($AP_1\,P_3\,P_5$ ......),
($AP_2\,P_4\,P_6$ ......),
($AO_1\,O_3\,O_5$ ......)
($AO_2\,O_4\,O_6$ ......)

form four sets of collinear points.

So, remembering that $O_x P_x$ and $O_{x+2y} P_{x+2y}$ are parallel, it is seen that

$$(AN_1 N_3 N_5 \ldots\ldots) \text{ and } (AN_2 N_4 N_6 \ldots\ldots)$$

form two sets of collinear points. Hence the result.

Question 1234.

(M. VAIDYANATHAN, M.A.).—2^n players of equal skill enter for a tournament. They are drawn in pairs and the winners of the first round are drawn again for the next and so on. Find

(1) the probability that that two given competitors play against each other in the course of the tournament and

(2) the probability that a given player will win.

Solution by S. Audinarayanan.

(i) A competitor A can be drawn with any of the 2^{n-1} other competitors in the first round. Hence the probability of his playing with any specified individual B in the first round is $1/(2^n - 1)$.

If A and B do not meet in the first round, the chance of each winning his set is $\frac{1}{2}$ and, having won, the chance of their meeting in the second round is $\frac{1}{(2^{n-1} - 1)}$. Hence the chance of their meeting in the second round is $\frac{1}{4(2^{n-1} - 1)}$.

Arguing thus, it is clear that the probability of A and B playing against each other at all is

$$\rho = \frac{1}{2^n - 1} + \frac{1}{4(2^{n-1} - 1)} + \frac{1}{4^2(2^{n-2} - 1)} + \cdots \frac{1}{4^{n-1} \cdot 1}.$$

(ii) If A has to be the final victor, he must win in each of the n rounds, and the chance on each occasion is $\frac{1}{2}$. Hence the required probability is $\frac{1}{2^n}$: otherwise thus: there are 2^n players and being of *equal skill*, they have equal chances of winning finally, and hence the probability of any specified player winning is $\frac{1}{2^n}$.

Question 1333.

(K. J. SANJANA) :—(X, X′), (Y, Y′), (Z, Z′) are points in the sides BC, CA, AB respectively of a triangle ABC, such that Y′Z, Z′X, X′Y are equal and respectively anti-parallel to BC, CA, AB. If circles are described about the triangles AY′Z, BZ′X, CX′Y, prove that the point isogonally conjugate to their radical centre lies on the Euler line of the triangle. Also state the corresponding property when the lines Y′Z, Z′X, X′Y of equal lengths are parallel respectively to the opposite sides of the triangle.

Solution by the Proposer and G. V. Bhagavat.

For the general construction and notation, reference may be made to the *Journal of the Indian Mathematical Society*, Vol. IX, p. 214.*

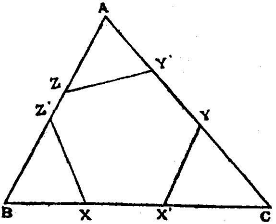
The equation of a circle in trilinear co-ordinates is of the form.†

$$\Sigma\, a\beta\gamma - \frac{1}{abc}(a\alpha + b\beta + c\gamma)(t_1^2 a\alpha + t_2^2 b\beta + t_3^2 c\gamma) = 0.$$

where t_1^2, t_2^2, t_3^2 are the powers of the vertices A, B, C with respect to the circle.

(i) In the first case denote by μ the length of each *anti-parallel* intercept. We have from the inversely similar triangles AY′Z, ABC,

$$AY' : AB = AZ : AC = Y'Z : BC,$$



so that $$AY' = c\mu / a, \quad AZ = b\mu / a$$

hence for the circle AY′Z,

$$t_1^2 = 0, \quad t_2^2 = c(c - b\mu / a), \quad t_3^2 = b(b - c\mu / a)$$

* K. J. Sanjana: Tucker circles that touch the inscribed and escribed circles of a triangle.

† Askwith: *Analytical Geometry of the Conic Sections*, p. 307, ex. Also Sanjana: *loc. cit.*,

Thus the equation of the circle AY′Z is

$$\Sigma\,(\alpha\beta\gamma) - \frac{1}{abc}\,\Sigma\,(a\alpha).\{\,bc\,(c - b\mu/a)\,\beta + cb\,(b - c\mu/a)\,\gamma\,\} = 0$$

or $$\Sigma\,(\alpha\beta\gamma) - \frac{1}{a}\,\Sigma\,(a\alpha).\{(c - b\mu/a)\,\beta + (b - c\mu/a)\,\gamma\,\} = 0 \quad \ldots \quad (1)$$

So also we get for the equations of the circles BZ′X and CX′Y

$$\Sigma\,(\alpha\beta\gamma) - \frac{1}{b}\,\Sigma\,(a\alpha).\{(a - c\mu/b)\,\gamma + (c - a\mu/b)\,\alpha\,\} = 0 \ldots \quad (2)$$

and $$\Sigma\,(\alpha\beta\gamma) - \frac{1}{c}\,\Sigma\,(a\alpha).\{(b - a\mu/c)\,\alpha + (a - b\mu/c)\,\beta\,\} = 0. \quad (3)$$

Subtracting (2) from (1) and (3) from (2), we have for the radical axes of the circles AY′Z, BZ′X and the circles BZ′X, CX′Y, the equations

$$\frac{1}{b}\,\{(a - c\mu/b)\,\gamma + (c - a\mu/b)\,\alpha\,\}$$
$$- \frac{1}{a}\,\{(c - b\mu/a)\,\beta + (b - c\mu/a)\,\gamma\,\} = 0 \quad \ldots \quad (4)$$

and $$\frac{1}{c}\,\{(b - a\mu/c)\,\alpha + (a - b\mu/c)\,\beta\,\}$$
$$- \frac{1}{b}\,\{(a - c\mu/b)\gamma + (c - a\mu/b)\,\alpha\,\} = 0, \quad \ldots \quad (5)$$

after suppressing the factor $\Sigma\,(a\alpha)$. The radical centre of the three circles can be found as the point of intersection of (4) and (5). We therefore eliminate μ from the equations (4) and (5) and get the following equation for the locus of the radical centre.

$$\begin{vmatrix} \dfrac{b\beta + c\gamma}{a^2} - \dfrac{c\gamma + a\alpha}{b^2}, & \dfrac{a\gamma + c\alpha}{b} - \dfrac{c\beta + b\gamma}{a} \\ \dfrac{c\gamma + a\alpha}{b^2} - \dfrac{a\alpha + b\beta}{c^2}, & \dfrac{b\alpha + a\beta}{c} - \dfrac{a\gamma + c\alpha}{b} \end{vmatrix} = 0 \quad \ldots \quad (A)$$

which is evidently of the second degree.

On expanding the determinant and using the well-known formulae connecting the sides and angles of a triangle, the equation of the required locus works out to

$$\Gamma': \qquad \Sigma\,\beta\gamma\,\sin 2A\,\sin(B\text{-}C) = 0 \qquad \ldots \quad (6)$$

This conic is called the Hyperbola Γ' and is the isogonal transformation of $\Sigma\,\alpha\,\sin 2A\,\sin(B\text{-}C) = 0$, the Euler line of the triangle ABC joining the circumcentre and the orthocentre. The hyperbola passes therefore through the following seven points :—A, B, C ; the circumcentre and the orthocentre which are a pair of isogonal conjugates; the Lemoine point and the isogonal conjugate of the nine-point centre.

The hyperbola Γ' possesses the following property:— if a triangle is taken homothetically placed with respect to the triangle formed by the tangents to the circumcircle of ABC at the vertices, and the two triangles are in perspective, then Γ'' is the locus of the centre of perspective. In the present case as Y′Z, Z′X, X′Y are parallel to the tangents to the circle ABC at A, B, C respectively and the triangle formed by them is in perspective with that formed by the tangents, the connection of the radical centre with the curve was to be expected.

(ii) In the second case denote by λ the length of each *parallel* intercept; we have from the directly similar triangles AY′Z, ACB,

$$AY' = \lambda\, b/a, \quad AZ = \lambda\, c/a;$$

hence for the circle AY′Z,

$$t_1^2 = 0, \quad t_2^2 = c^2(1-\lambda/a), \quad t_3^2 = b^2(1-\lambda/a)$$

Thus the equation of this circle becomes

$$\Sigma(a\beta\gamma) - \frac{1}{a}\cdot\left(1-\frac{\lambda}{a}\right)\left(c\beta+b\gamma\right)\left(a\alpha+b\beta+c\gamma\right) = 0 \quad \ldots \quad (1')$$

So also we get for the equations of the circles BZ′X and CX′Y

$$\Sigma(a\beta\gamma) - \frac{1}{b}\cdot\left(1-\frac{\lambda}{b}\right)\left(a\gamma+c\alpha\right)\left(a\alpha+b\beta+c\gamma\right) = 0 \quad \ldots \quad (2')$$

and $$\Sigma(a\beta\gamma) - \frac{1}{c}\cdot\left(1-\frac{\lambda}{c}\right)\left(b\alpha+a\beta\right)\left(a\alpha+b\beta+c\gamma\right) = 0. \quad \ldots \quad (3')$$

Subtracting (2′) from (1′) and (3′) from (2′) we have the radical axes of the circles taken in pairs.

Eliminating λ from these equations we get as before the equation for the locus of the radical centre in the form of a determinant:

$$\begin{vmatrix} \dfrac{c\beta+b\gamma}{a^2} - \dfrac{a\gamma+c\alpha}{b^2}, & -\dfrac{c\beta+b\gamma}{a} + \dfrac{a\gamma+c\alpha}{b} \\ \dfrac{a\gamma+c\alpha}{b^2} - \dfrac{b\alpha+a\beta}{c^2}, & -\dfrac{a\gamma+c\alpha}{b} + \dfrac{b\alpha+a\beta}{c} \end{vmatrix} = 0. \quad (A')$$

which after expansion and simplification may be written

$$\Sigma\, a(b-c)\alpha^2 + 2\,\Sigma\,(b-c)(s-2a\cos^2 \tfrac{1}{2}A)\,\beta\gamma = 0.$$

This is a conic not passing through the vertices A, B, C and its isogonal conjugate is a curve of the fourth degree.

Questions for Solution

Proposers of Questions are requested to send their own solutions along with their questions.

1466. (G. A. SRINIVASAN) :—Required a proof by purely elementary methods (without the use of multiplication of series) that

$$E(x) \equiv 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

is positive for all real values of x.

1467. (C. N. SRINIVASIENGAR) :—Find the number of selections of n things from $n + r$ such that no two selections have $n - 1$ things in common.

1468. (S. D. CHOWLA) :—

Let $$n = p^{\alpha} q^{\beta} r^{\gamma} \ldots, \chi(n) = \alpha\beta\gamma \ldots$$

$[x]$ = integer *nearest* to x, then prove that

$$\sum_{n/d^6} \chi\left(\frac{n}{d^6}\right) = \left[\frac{(\alpha+3)^2}{12}\right]\left[\frac{(\beta+3)^2}{12}\right]\left[\frac{(\gamma+3)^2}{12}\right] \ldots$$

the summation extending to all the divisors d^6 of n which are perfect sixth powers.

Example. $$n = 2^3 . 3^7 . 5^{13} . 17.$$

$$\begin{aligned} \text{L. H. S.} &= \chi(2^3 . 3 \;\; 5^{13} . 17) + \chi(2^3 . 3 . 5^7 . 17) \\ &+ \chi(2^3 . 3 . 5 . 17) + \chi(2^3 . 3^7 . 5 . 17) + \chi(2^3 . 3^7 . 5^7 . 17) \\ &+ \chi(2^3 . 3^7 . 5^{13} . 17) \\ &= 39 + 21 + 3 + 21 + 147 + 273 \\ &= 504 = 3 . 8 . 21 = \left[\frac{6^2}{12}\right]\left[\frac{10^2}{12}\right]\left[\frac{16^2}{12}\right] = \text{R. H. S.} \end{aligned}$$

1469. (S. D. CHOWLA) :—Prove geometrically that

$$\frac{r^2}{2} \sin\frac{2\pi}{n} + \sum_{t=1}^{t=\infty} 2^{t-1} x_t \left\{ r - \sqrt{r^2 - x_t^2/4} \right\} = \frac{\pi r^2}{n}$$

where $$x_1 = 2r \sin \pi/n$$

$$x_{t+1} = \sqrt{2r^2 - \sqrt{4r^4 - r^2 x_t^2}}$$

r being positive, and n a positive integer.

1470. (A. NARASINGA RAO):—Given two equal circles, show that for every triangle inscribed in the one, there is a triangle inscribed in the other which has with it a common Steiner Tricusp.

1471. (R VAIDYANATHASWAMY):—$A_1\ A_2\ \ldots\ A_m$ are centres of force in a plane, of which A_r attracts accoding to the law $\frac{a_r}{\text{distance}}$ $(r = 1, 2 \ldots m)$. If the sum $a_1 + a_2 \ldots + a_m = 0$, show that the equilibrium positions are invariant for inversion.

Deduce the theorem of Bocher. that, if the roots of the polynomials $F_m(z)$, $F_n(z)$ be represented by the points $A_1\ A_2 \ldots A_m$, $B_1\ B_2 \ldots B_n$ in the Argand plane, then the roots of their Jacobian. :

$$\begin{vmatrix} F_m(z) & F_n(z) \\ \frac{dF_m}{dz} & \frac{dF_n}{dz} \end{vmatrix}$$

are the equlibrium positions of the field due to centres of force at each of $A_1\ A_2 \ldots A_m$ at'racting according to the law $\frac{n}{\text{distance}}$, and centres of force at each of $B_1\ B_2 \ldots B_n$ repelling according to the law $\frac{m}{\text{distance}}$.

1472. (R. VAIDYANATHASWAMY):—Shew that the circles which touch two non-intersecting circles Γ, Γ' fall into two algebraically distinct systems.

Shew that the two ranges of the contact-points of Γ, Γ' with the circles of either of these systems are equicross.

If C_1, C_2, C_3, C_4 be four circles of one of these systems, and if the cross-ratio of their contacts on Γ (or Γ') be the real number λ, and if z_1, z_2, z_3 be three points chosen arbitrarily within or on the circles C_1, C_2, C_3 respectively, then the locus of the point z_4 such that the cross-ratio $(z_1 z_2 z_3 z_4) = \lambda$ (the plane being considered as an Argand plane) is the interior and the periphery of the circle C_4 (Walsh's Theorem).

1473. (A. A. KRISHNASWAMY AYYANGAR):—If ABC, A′B′C′ be two triangles, inscribed in a conic and XYZ be the triangle formed by AA′, BB′ and CC′ (BB′, CC′ meeting in X and so on); prove that, if any triangle PQR inscribed in the same conic be in perspective with two of the three triangles ABC, A′B′C′, XYZ, it is also in perspective with the third triangle.

A LIST OF JOURNALS AND BOOKS RECEIVED

during the months of March and April 1927.

Journals.

1 Abhandlungen aus dem Mathematischen Seminar der Hamburgishcen Universitat, **5**, 1 & 2.

2 Acta Mathematica, **49**, 3 & 4.

3 American Journal of Mathematics, **49**, 1 (2 copies).

4 American Mathematical Monthly, **34**, 1 & 2 (2 copies)

5 Astrophysical Journal, **64**, 5.

6 Bulletin des Sciences Mathematiques, Dec. 1926 & Jan. 1927.

7 Bulletin of the American Mathematical Society, **32**, **6**.

8 Journal fur die reine und angewandte Mathematik, **156**, 2.

9 Journal of the Science Association, Vijayanagaram, **3**, 1.

10 Mathematical Gazette, **13**, 187

11 Monthly Notices of the Royal Astronomical Society, **87**, 3 & 4.

12 Nature, **119**, 2983 to 2993

13 Nieuwe Opgaven.

14 Philosophical Magazine, **3**, 14 & 15.

15 Philosophical Transaction of the Royal Society, A-640.

16 Proceedings of the Cambridge Philosophical Society, **23**, 3 & 4.

17 Proceedings of the Royal Society of London, **114**, 766—768.

18 Recueil Mathematique de la Societe Mathematique de Moscou, **31**, 3 & 4, **32**, complete. **33**, 1.

19 Rendiconti del Cicolo Mathematico di Palermo, **50**, 1—3.

20 Universidad Nacionel de la Plata—Anuario Para el Ano., 1926, No. 16.

Memoirs de l'Universite d'Etat a l'Extreme-Orient.

1 Sur la transformation de l'Equation generale d'une courbe du Second Ordre. Memoirs, Series VII, 1.

2 Essai d'un cours de haute mathematique mene par une methode laboratoire et par groupes dans l'Universitat d'Etat a l'Extreme Orient. Memoirs Ser. I, 1.

3 Resultants des travaux Sur Mars en 1924-1925. Series VII, 2.

Numbers in black type refer to the volumes, and those in ordinary type to the numbers of the issues.

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