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# Indian Mathematical Society.

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## PROGRESS REPORT.

The following gentlemen have been elected members of the Society—

1. *Mr. Charles Saldanha, M.A.*—1st Assistant in Mathematics, Deccan College, Poona:

2. *Mr. Vamon D. Alurkar, M.A.*—2nd Assistant in Mathematics, Deccan Gollege, Poona:

3. *Mr. Gopaldas, T. Thadhani, B.Sc.*—Demonstrator in Science, Deccan College, Poona, (at concessional rate).

2. Under article VIII (*e*) of the Constitution, the Committee have been pleased to appoint Dewan Bahadur R. Ramachandra Rao, B.A., Collector, Nellore, to fill the office of President until the next biennial election.

POONA,
*30th Nov., 1915.*

D. D. KAPADIA,
*Hony. Joint Secretary.*

# The Folium of Descartes.

By M. T. Naraniengar.

1. Let the equation of the curve be

$$x^3+y^3=3\,axy;$$

then the tangent at P $(x_1, y_1)$ meets the curve at a point T $(x_3, y_3)$ such that

$$x_3=\frac{x_1y_1^3}{x_1^3-y_1^3}\,;\quad y_3=\frac{y_1x_1^3}{y_1^3-x_1^3}.$$

If the tangent at Q $(x_2, y_2)$ should meet the curve at the same point, we must, similarly, have

$$x_3=\frac{x_2y_2^3}{x_2^3-y_2^3}\,;\quad y_3=\frac{y_2x_2^3}{y_2^3-x_2^3}.$$

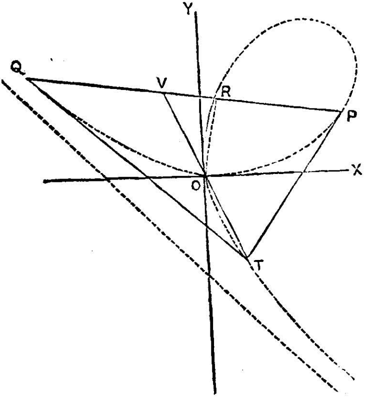
Hence, we deduce

$$\frac{y_3}{x_3}=-\frac{x_1^2}{y_1^2}=-\frac{x_2^2}{y_2^2}.$$

In other words

$$x_1:\,y_1=-x_2:\,y_2\,;$$

or in the figure, OP, OQ are equally inclined to the axis of $x$.




2. *To prove that TO produced bisects PQ at V such that OV=TO.* For, we have

$$x_3=\frac{x_1(y_1/x_1)^3}{1-(y_1/x_1)^3}=\frac{x_1(-y_2/x_2)^3}{1+(y_2/x_2)^3}=\frac{-x_1y_2^3}{3ax_2y_2}$$

$$=-x_1+x_1x_2^2/3ay_2.$$

**Similarly,**

$$x_3=-x_2+x_2x_1^2/3ay_1.$$

Hence, by addition,

$$2x_3=-(x_1+x_2.)$$

In the same way

$$2y_3=-(y_1+y_2).$$

Thus, the middle point of PQ is $(-x_3, -y_3)$ which proves the result.

3. *To find the equation of PQ in terms of the co-ordinates of T.*

The equation of PQ is written

$$\frac{Y+y}{X+x}=\frac{y_1-y_2}{x_1-x_2},$$

if the point T is taken as $(x,y)$, since PQ passes through V or $(-x, -y)$.

Now $\qquad y_1-y_2 : x_1+x_2=y_1 : x_1$ by § 1.

$$\therefore \qquad y_1-y_2=-2xy_1/x_1=-2y_1^4/(x_1^3-y_1^3)$$
$$=+2y_1y_2^3/(x_2^3+y_2^3)$$
$$=\ \ 2y_1y_2^2/3ax_2.$$

Similarly

$$x_1-x_2=2x_1x_2^2/3ay_2.$$

$$\therefore \qquad y_1-y_2 : x_1-x_2=y_1y_2^3 : x_1x_2^3=-x^2 : y^2 \text{ by § 1.}$$

Hence, the equation of PQ is

$$\frac{Y+y}{X+x}+\frac{x^2}{y^2}=0;$$

or

$$Xx^2+Yy^2+3axy=0.$$

4. *If PQ cuts the curve in R, then OR and OT are equally inclined to the axis of x.*

This follows easily from the preceding. For, the homogeneous equation

$$xy(X^3+Y^3)+XY(Xx^2+Yy^2)=0$$

should represent the three lines OP, OQ and OR.

Putting $Y=\mu X$ and $y=mx$, we have

$$m(\mu^3+1)+\mu(1+m^2\mu)=0,$$

or

$$(\mu+m)(\mu^2m+1)=0,$$

which shows that OR corresponds to the value $-m$, since OP, OQ correspond to the values $\pm 1/\sqrt{-m}$ of $\mu$. Hence the result stated.

5. *Parametric expressions for the co-ordinates of T, P, Q, R.*

Putting $y=mx$, the co-ordinates of T are easily obtained in the form

$$\left\{\frac{3am}{m^3+1}, \frac{3am^2}{m^3+1}\right\}$$

Calling T the point '$m$,' we may write the points P, Q, R as $+(\sqrt{-m})$, $(-\sqrt{-m})$ and $(-m)$ respectively.

*Cor.* 1. The chord of contact of the point '$m$' may therefore be expressed in the form

$$X+Y\,m^2+3\,a\,m=0.$$

*Cor.* 2. The tangents at T and R meet the curve at the point $(-m^2)$ and the equation of TR is

$$X-Y\,m^4-3\,am^2=0.$$

*Cor.* 3. The envelope of PQ or TR is the rectangular hyperbola

$$4\,XY=9a^2.$$

6. *The trilinear equation of the Folium referred to OPQ as the triangle of reference.*

Denote OPQ by ABC. Then, remembering that OP, OQ and OV, OR are equally inclined to the axes which are tangents at O, we note that

(i) A is a node the tangents at which are $\beta^2-\gamma^2=0$.

(ii) If BC cuts the curve at D and E is the middle point of BC, AD is the symmedian through A or $c\beta-b\gamma=0$.

Hence, the equation of the curve is written

$$a\alpha(\beta^2-\gamma^2)+4\beta\gamma(c\beta-b\gamma)=0. \qquad \text{(Basset, § 91.)}$$

*Note.*—It must be borne in mind that $a, b, c$ are not independent of one another. For, from the parametric expressions for P and Q in § 5 (where $t$ replaces $a$ to avoid confusion of symbols), we find

$$a=PQ=\sqrt{\left\{\left(\frac{6t\mu}{1-\mu^6}\right)^2+\left(\frac{6t\mu^5}{1-\mu^6}\right)^2\right\}}=\frac{6t\mu}{1-\mu^6}\sqrt{1+\mu^8},$$

$$b=OQ=\frac{3t\mu}{1-\mu^3}\sqrt{1+\mu^2},$$

$$c=OP=\frac{3t\mu}{1+\mu}\sqrt{1+\mu^2},$$

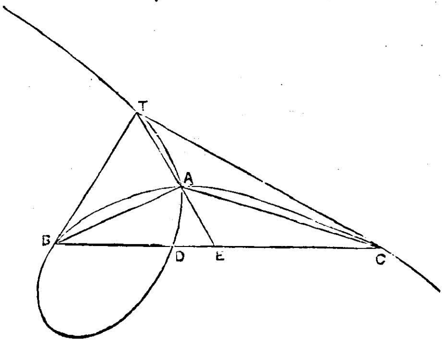
$\mu$ standing for tan POX or cot $\frac{1}{2}$ A.

Hence, the relation connecting $a, b, c$ is found by eliminating $\mu$ from the above; thus

$$\mu^2=\left(\frac{b-c}{b+c}\right)=\cot^2\tfrac{1}{2}A=\frac{s^2(s-a)^2}{(s-b)^2(s-c)^2}.$$

The equation of the Folium is, therefore,

$$\alpha(\beta^2-\gamma^2)\sqrt{\left(\frac{1+\mu^6}{1+\mu^2}\right)}+2\beta\gamma\{\beta(1-\mu^3)-\gamma(1+\mu^3)\}=0.$$




7. *If any chord $pDq$ be drawn through $D$, then the isogonal conjugates of $Ap$, $Aq$, are harmonically conjugate to $Aq$, $Ap$ respectively.*

For, the equation of $pDq$ is of the form

$$c\beta-b\gamma=k\alpha,$$

and where this cuts the cubic

$$a(\beta^2-\gamma^2)+4k\beta\gamma=0.$$

That is, the lines $Ap$, $Aq$ are of the form

$$\beta=\lambda\gamma,\ \beta=-\gamma/\lambda\ ;$$

whence the result stated follows.

8. *If $AB'$, $AC'$ be two isogonal chords, and $B'C'$ meet $BC$ in $F$ and the curve in $D'$, than $A(FB'DC')$* and *$A(FBD'C)$ are harmonic.*

Let the equation to B′C′ be $\alpha=m\beta+n\gamma$ so that AF is $m\beta+n\gamma=0$. Eliminating $\alpha$ between this and the equation of the curve, we get

$$a(m\beta+n\gamma)(\beta^2-\gamma^2)+4\beta\gamma(c\beta-b\gamma)=0$$

for the lines AB′, AD′, AC′.

Now, if this cubic has a factor $m\beta-n\gamma$, the remaining quadratic factor should be of the form

$$(\beta^2+\gamma^2+\lambda\beta\gamma)$$

as is seen by division. In other words, if AD′ be the harmonic conjugate of AF, the equation of AB′, AC′ is

$$\beta^2+\gamma^2+\lambda\beta\gamma=0,$$

that is AB′, AC′ are isogonally conjugate. This proves the first result.

Again, since the tangents at B′, C′ meet on the curve by § 1, we may consider AB′C′ as the triangle of reference and AB, AC as a pair of isogonals. Hence from the preceding it follows that the pencil A(FBD′C) is also harmonic.

9. *If through B or C a chord pq be drawn parallel to the median AE, then Ap, Aq are isogonal.*

For, the equation to the chord through B parallel to AE is

$$a\alpha+b\beta+c\gamma-(b\beta-c\gamma)=0,$$

or $$a\alpha+2c\gamma=0;$$

and this cuts the curve in points lying on

$$-2c\gamma(\beta^2-\gamma^2)+4\beta\gamma(c\beta-b\gamma)=0$$

*i.e.* $$\gamma(c\beta^2+c\gamma^2-2b\beta\gamma)=0.$$

Hence the equation to A$p$, A$q$ is

$$c(\beta^2+\gamma^2)-2b\beta\gamma=0,$$

which shows that they are isogonal to each other.

10. The equation of the curve is written

$$\beta^2(a\alpha+4c\gamma)=\gamma^2(a\alpha+4b\beta),$$

*i.e.* $$\beta^2.v=\gamma^2.w,$$

where $v$, $w$ denote the tangents at B, C respectively.

Hence: *The curve is the locus of points such that the ratio of the perpendiculars on TB, TC varies as the duplicate ratio of those on AB, AC.*

11. *If any chord Tpq be drawn through T then Ap, Aq are harmonically conjugate to AB, AC.*

Any line through T is $v=k^2w$ say.

Putting $v=k^4w$ in the equation $\beta^2v=\gamma^2w$, we have

$$k^2\beta^2=\gamma^2, \text{ or } k\beta=\pm\gamma,$$

for the lines A$p$, A$q$. Hence the result stated.

12. *Other Transformations*:

The equation $x^3+y^3=3\,axy$ is transformed into

$$x(x^2+3y^2)=a(x^2-y^2)$$

by turning the axes through 45°.

Putting $x=\rho\cosh u$, $y=\rho\sinh u$ in the latter, we get

$$\rho\cosh 3\,u=a,$$

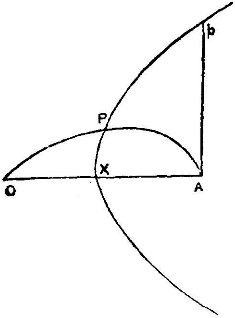
since $x^2-y^2=\rho^2$.

13. *Geometrical Interpretation of the equation $\rho\cosh 3u=a$.*

Let the rectangular hyperbola

$$x^2-y^2=\rho^2$$

be drawn, where $\overline{O}X=\rho$. Then, if $OA=a=\rho \cosh 3u$, the curvilinear area $OXp=\frac{1}{2}\rho^2.3u$; so that P or $(\rho \cosh u, \rho \sinh u)$ is such that area $OXP=\frac{1}{3}$ area $OXp$.




In other words, the point P of trisection of the curvilinear area $OXp$ traces the Folium

$$\rho \cosh 3u=a. \qquad \ldots \qquad \ldots \qquad (1)$$

14. The tangent at '$u_1$' may be written

$$a/\rho=2\cosh(u+2u_1)-\cosh(u-4u_1) \ldots \qquad \ldots \qquad (2)$$

This meets (1) at points given by

$$\cosh 3u=2\cosh(u+2u_1)-\cosh(u-4u_1)$$

*i.e.*

$$\cosh 3u+\cosh(u-4u_1)=2\cosh(u+2u_1)$$

or

$$2\cosh 2(u-u_1).\cosh(u+2u_1)-2\cosh(u+2u_1)=0.$$

$$\therefore \quad \cosh(u+2u_1)=0.$$

$$\therefore \quad u+2u_1=i(2n+1)\pi/2. \qquad \ldots \qquad \ldots \qquad (3)$$

If the tangent at '$u_2$' should meet the curve at the same point as determined by (3)

$$u+2u_2=i(2n'+1)\pi/2$$

$$\therefore \quad u_1-u_2=i(n'-n)\pi/2$$

or

$$\cosh(u_1-u_2)=0 \text{ or } \pm 1,$$

which agrees with the result of § 1.

Again, the tangent from a point $(\rho_o, u_0)$ is such that

$$a/\rho_0=2\cosh(u_0+2u)-\cosh(u_0-4u) \ \ldots \qquad \ldots \qquad (4)$$

$u$ being the point of contact.

Hence, the four points of contact $u_1, u_2, u_3, u_4$ satisfy equation (4), and are such that

$$\Sigma(\tanh u_1)=0,$$

which is identical with the property that the sum of the tangents of vectorial angles of the four points of contact is zero.

In particular if $(\rho_o, u_o)$ be on the curve

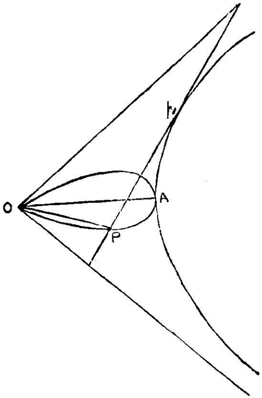
$$2 \tanh u_o + \tanh u_1 + \tanh u_2 = 0.$$

15. The line $\rho \cosh (u - u_1) = a$, which is a tangent to the hyperbola $\rho = a$, meets the curve in point given by

$$\cosh 3u = \cosh (u - u_1)$$

whence $u = -\frac{1}{2} u_1$.

In cartesians this means that the tangent at ($\sec\alpha$, $\tan\alpha$) to the hyperbola $x^2 - y^2 = a^2$ cuts the Folium at the point whose vectorial angle is $(-\frac{1}{2}\alpha)$.




Hence the following simple construction for the Folium :—

Take the tangent at the point $p$ ($\sec\alpha$, $\tan\alpha$); make $AOP = -\frac{1}{2}\alpha$. Then the locus of $P$ is the Folium

$$\rho \cosh 3u = a.$$

# SHORT NOTES.

## On the product $\prod_{n=0}^{n=\infty}\left[1+\left(\frac{x}{a+nd}\right)^3\right]$.

I. Let

$$\phi(\alpha,\beta)=\left\{1+\left(\frac{\alpha+\beta}{1+\alpha}\right)^3\right\}\left\{1+\left(\frac{\alpha+\beta}{2+\alpha}\right)^3\right\} \qquad \ldots (1)$$

It is easy to see that

$$\left\{1+\left(\frac{\alpha+\beta}{n+\alpha}\right)^3\right\}\left\{1+\left(\frac{\alpha+\beta}{n+\beta}\right)^3\right\}=\frac{\left(1+\frac{\alpha+2\beta}{n}\right)\left(1+\frac{\beta+2\alpha}{n}\right)}{\left(1+\frac{\alpha}{n}\right)^3\left(1+\frac{\beta}{n}\right)^3}\times$$

$$\left[1-\left\{\frac{(\alpha-\beta)+i(\alpha+\beta)\sqrt{3}}{2n}\right\}^2\right]=$$

$$\left[1-\left\{\frac{(\alpha-\beta)-i(\alpha+\beta)\sqrt{3}}{2n}\right\}^2\right]; \qquad \ldots (2)$$

$$\prod_{n=1}^{n=\infty}\left\{\frac{\left(1+\frac{\alpha+2\beta}{n}\right)\left(1+\frac{\beta+2\alpha}{n}\right)}{\left(1+\frac{\alpha}{n}\right)^3\left(1+\frac{\beta}{n}\right)^3}\right\}=$$

$$\frac{\{\Gamma(1+\alpha)\Gamma(1+\beta)\}^3}{\Gamma(1+\alpha+2\beta)\Gamma(1+\beta+2\alpha)}; \qquad \ldots (3)$$

and

$$\prod_{n=1}^{n=\infty}\left[1-\left\{\frac{(\alpha-\beta)+i(\alpha+\beta)\sqrt{3}}{2n}\right\}^2\right]$$

$$\left[1-\left\{\frac{(\alpha-\beta)-i(\alpha+\beta)\sqrt{3}}{2n}\right\}^2\right]=$$

$$\frac{\cosh\pi(\alpha+\beta)\sqrt{3}-\cos\pi(\alpha-\beta)}{2\pi^2(\alpha^2+\alpha\beta+\beta^2)}. \qquad \ldots (4)$$

It follows from (1)–(4) that

$$\phi(\alpha,\beta)\,\phi(\beta,\alpha)=\frac{\{\Gamma(1+\alpha)\Gamma(1+\beta)\}^3}{\Gamma(1+\alpha+2\beta)\Gamma(1+\beta+2\alpha)}\times$$

$$\left\{\frac{\cosh\pi(\alpha+\beta)\sqrt{3}-\cos\pi(\alpha-\beta)}{2\pi^2(\alpha^2+\alpha\beta+\beta^2)}\right\}. \qquad \ldots (5)$$

But it is evident that, if $\alpha-\beta$ be any integer, then $\phi(\alpha,\beta)/\phi(\beta,\alpha)$ can be expressed in finite terms. From this and (5) it follows that

$\phi(\alpha, \beta)$ can be expressed in finite terms, if $\alpha-\beta$ be any integer. That is to say

$$\left\{1+\left(\frac{x}{a}\right)^3\right\}\left\{1+\left(\frac{x}{a+d}\right)^3\right\}\left\{1+\left(\frac{x}{a+2d}\right)^3\right\}\cdots\cdots$$

can be expressed in finite terms if $x-2a$ be a multiple of $d$.

2. Suppose now that $\alpha=\beta$ in (5). We obtain

$$\left\{1+\left(\frac{2\alpha}{1+\alpha}\right)^3\right\}\left\{1+\left(\frac{2\alpha}{2+\alpha}\right)^3\right\}\left\{1+\left(\frac{2\alpha}{3+\alpha}\right)^3\right\}\cdots$$
$$=\frac{\{\Gamma(1+\alpha)\}^3}{\Gamma(1+3\alpha)}\cdot\frac{\sinh \pi\alpha\sqrt{3}}{\pi\alpha\sqrt{3}}. \quad (6)$$

Similarly putting $\beta=\alpha+1$ in (5), we obtain

$$\left\{1+\left(\frac{2\alpha+1}{1+\alpha}\right)^3\right\}\left\{1+\left(\frac{2\alpha+1}{2+\alpha}\right)^3\right\}\cdots\cdots\cdots$$
$$=\frac{\{\Gamma(1+\alpha)\}^3}{\Gamma(2+3\alpha)}\cdot\frac{\cosh \pi(\frac{1}{2}+\alpha)\sqrt{3}}{\pi}. \quad \cdots (7)$$

Again, since

$$\left\{1+\left(\frac{\alpha}{n}\right)^3\right\}\left\{1+3\left(\frac{\alpha}{2n+\alpha}\right)^2\right\}=\frac{\left(1+\frac{\alpha}{n}\right)\left(1+\frac{\alpha^2}{n^2}+\frac{\alpha^4}{n^4}\right)}{\left(1+\frac{\alpha}{2n}\right)^2},$$

it is easy to see that

$$\left[\left(1+\frac{\alpha^3}{1^3}\right)\left(1+\frac{\alpha^3}{2^3}\right)\cdots\right]\left[\left\{1+3\left(\frac{\alpha}{2+\alpha}\right)^2\right\}\left\{1+3\left(\frac{\alpha}{4+\alpha}\right)^2\right\}\cdots\right]$$
$$=\frac{\Gamma(\frac{1}{2}\alpha)}{\Gamma\{\frac{1}{2}(1+\alpha)\}}\left(\frac{\cosh \pi\alpha\sqrt{3}-\cos \pi\alpha}{2^{\alpha+2}\pi\alpha\sqrt{\pi}}\right). \quad (8)$$

3. It is known that, if the real part of $\alpha$ is positive, then

$$\log\Gamma(\alpha)=(\alpha-\tfrac{1}{2})\log\alpha-\alpha+\tfrac{1}{2}\log 2\pi+2\int_0^\infty \frac{\tan^{-1}(x/\alpha)}{c^{2\pi x}-1}dx. \quad (9)$$

From this we can show that, if the real part of $\alpha$ is positive, then

$$\tfrac{1}{2}\log 2\pi\alpha+\frac{\pi\alpha}{\sqrt{3}}+\log\left\{\left(1+\frac{\alpha^3}{1^3}\right)\left(1+\frac{\alpha^3}{2^3}\right)\left(1+\frac{\alpha^3}{3^3}\right)\cdots\right\}$$
$$=\log\left(\frac{\cosh \pi\alpha\sqrt{3}-\cos \pi\alpha}{\pi\alpha}\right)+2\int_0^\infty \frac{\tan^{-1}(x/\alpha)^3}{e^{2\pi x}-1}dx. \quad (10)$$

From this and the previous section it follows that

$$\int_0^\infty \frac{\tan^{-1} x^3}{2\pi n x_{-1}}dx$$

can be expressed in finite terms if $n$ is a positive integer. Thus for example,

$$\int_0^\infty \frac{\tan^{-1} x^3}{e^{2\pi x}-1}\,dx = \tfrac{1}{4}\log 2\pi - \frac{\pi}{4\sqrt{3}} - \tfrac{1}{2}\log\left(1+e^{-\pi\sqrt{3}}\right); \qquad (11)$$

$$\int_0^\infty \frac{\tan^{-1} x^3}{e^{4\pi x}-1}\,dx = \tfrac{1}{8}\log 12\pi - \frac{\pi}{4\sqrt{3}} - \tfrac{1}{4}\log\left(1-e^{-2\pi\sqrt{3}}\right); \qquad (12)$$

and so on

4. It is also easy to see that

$$\frac{1^2}{1^3+n^3} - \frac{2^2}{2^3+n^3} + \frac{3^2}{3^3+n^3} - \frac{4^2}{4^3+n^3} + \ldots\ldots\ldots$$

$$= \frac{1}{3}\left(\frac{1}{1+n} - \frac{1}{2+n} + \frac{1}{3+n} - \frac{1}{4+n} + \ldots\ldots\right)$$

$$+ \frac{4}{3}\left\{\frac{2-n}{(2-n)^2+3n^2} - \frac{4-n}{(4-n)^2+3n^2} + \frac{6-n}{(6-n)^2+3n^2} - \ldots\right\} \ldots \qquad (13)$$

Since

$$\frac{\pi}{4\cosh\frac{1}{2}\pi x} = \frac{1}{1^2+x^2} - \frac{3}{3^2+x^2} + \frac{5}{5^2+x^2} - \ldots\ldots$$

it is clear that the left hand side of (13) can be expressed in finite terms if $n$ is any odd integer. For example

$$\frac{1^2}{1^3+1} - \frac{2^2}{2^3+1} + \frac{3^2}{3^3+1} - \frac{4^2}{4^3+1} + \ldots = \frac{1}{3}\left(1 - \log 2 + \pi \operatorname{sech}\tfrac{1}{2}\pi\sqrt{3}\right). \qquad (14)$$

The corresponding integral in this case is

$$\int_0^\infty \frac{x^5}{\sinh \pi x}\cdot\frac{dx}{n^6+x^6} = \frac{2}{\pi}\int_0^\infty \left\{\frac{1}{2x^2} + \sum_{\nu=1}^{\nu=\infty} \frac{(-1)^\nu}{\nu^2+x^2}\right\}\cdot\frac{x^6\,dx}{n^6+x^6}$$

$$= \frac{1}{3}\left(\frac{1}{n} - \frac{1}{n+1} + \frac{1}{n+2} - \frac{1}{n+3} + \ldots\ldots\right)$$

$$- \frac{4}{3}\left\{\frac{n+2}{(n+2)^2+3n^2} - \frac{n+4}{(n+4)^2+3n^2} + \frac{n+6}{(n+6)^2+3n^2} - \ldots\right\}; \qquad (15)$$

and so the integral on the left hand side of (15) can be expressed in finite terms if $n$ is any odd integer. For example,

$$\int_0^\infty \frac{x^5}{\sinh \pi x}\cdot\frac{dx}{1+x^6} = \frac{1}{3}\left(\log 2 - 1 + \pi \operatorname{sech}\tfrac{1}{2}\pi\sqrt{3}\right). \qquad \ldots (16)$$

S. Ramanujan.

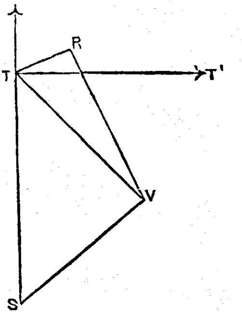
## Planetary Aberration.

The following method of obtaining the expression for planetary aberration is somewhat simpler than the one given by Ball, and is from elementary considerations.

Let S be the sun, T the earth moving in the direction TT′ perpendicular to ST with velocity $v$. The planet V is moving in the direction VV′ tangential to its orbit with velocity $v'$. If $ST=r$, $SV=r'$, then $v\sqrt{r}=v'\sqrt{r'}$.

Let $S\hat{T}V=E$, $S\hat{V}T=P$.

Impress on the earth a velocity equal and opposite to that of V.




Then the velocities of the earth along TT′ and ST are

$$v+v'\cos(E+P),\ v'\sin(E+P)$$

Let TR represent the direction and magnitude of the relative velocity of the earth, and VR the velocity of light on the same scale.

Then the planetary aberration $\epsilon=T\hat{V}R$.

Now, $\dfrac{\sin\epsilon}{TR}=\dfrac{\sin(90^\circ-E+\theta)}{\mu}$, where $\theta=R\hat{T}T'$.

$$\therefore\ \mu\sin\epsilon=TR\cos(E-\theta)$$
$$=\{v+v'\cos(E+P)\}\cos E+v'\sin(E+P)\sin E$$
$$=v\cos E+v'\cos P.$$

If $v_o$ denote the velocity of a planet at distance $r_o$ from the sun, we have

$$\epsilon=\frac{v_o\sqrt{r_o}}{\mu}\left\{\frac{\cos E}{\sqrt{r}}+\frac{\cos P}{\sqrt{r'}}\right\}.$$

P. R. Krishnaswami.

## Note on Planetary Aberration.

[The following note was suggested by Professor Turner's '*Notes on Aberration*' in the *Monthly Notices* of the Royal Astronomical Society, April 1909.]

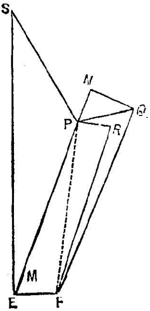
1. Let S be the sun and P, E simultaneous positions of a planet and the earth; and let a ray of light which leaves the planet in the position P reach the earth when it goes to F. Also, let Q be the position of the planet when the earth is at F; so that the angles SEF, SPQ are nearly right angles. Then within the time taken by the earth to go from E to F, the planet goes from P to Q and the planetary ray travels from P to F. That is, if $t$ be this time

$$EF = et, \; PQ = vt, \; PF = V't, \quad \ldots \quad \ldots \quad (1)$$

where $e, v, V'$ are the velocities of the earth, the planet and the planetary ray respectively.

Now, the velocity $V'$ of the planetary ray is that due to the actual velocity $V$ of light combined with the velocity $v$ of the planet. Hence, we have

$$QF = Vt. \quad \ldots \quad \ldots \quad (2)$$




Complete the parallelogram PEFR. Then the ray which is actually travelling along PF with velocity $V'$ appears to an observer on the earth to be proceeding in the direction RF.

In other words, the planet's *apparent* direction FR, is while its *true* direction is FQ; and the angle $\alpha$ between these directions is the value of the aberration of the planet at the instant of observation and is identical with the angle between the *true* positions of the planet at E and F—what amounts to the same thing, its *apparent* position at F and *apparent* position $t$ seconds later.

2. The value of $t$ is found in terms of the actual distance of the planet from the earth at the instant of observation *viz.*, QF by the formula (2), and we may write

$$t = (a^2 + b^2 - 2ab \cos \theta)^{\frac{1}{2}} / V \quad \ldots \quad \ldots \quad (3)$$

where $a, b, \theta$ denote SF, SQ, angle FSQ respectively.

3. *To find a formula for* $\alpha$.

With the usual convention of signs $\alpha$ is positive or negative according as it is measured contraclockwise or clockwise. Thus in the figure $\alpha$ should be regarded as negative. Drop the perpendiculars QN, FM on EP, then we have

$$-FQ \sin \alpha = QN - FM$$
$$= PQ \sin NPQ - EF \sin MEF$$
$$= PQ \cos SPN - EF \cos SEP.$$

*i.e.* $Vt \sin \alpha = vt \cos SPE + et \cos SEP$, by (1) and (2).

Hence $V \sin \alpha = v. \cos SPE + e. \cos SEP.$

[*Vide* : Ball's *Spherical Astron.*, § 90.]

M. T. NARANIENGAR.

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## The Face of the Sky for January and February.

### Phases of the Moon.

| | January. | | | | February. | | | |
|---|---|---|---|---|---|---|---|---|
| | D. | H. | M. | | D. | H. | M. | |
| New Moon ... ... | 4 | 11 | 15 | P.M. | 3 | 10 | 36 | A.M. |
| First Quarter ... | 11 | 10 | 8 | P.M. | 10 | 4 | 56 | P.M. |
| Full Moon ... | 20 | 2 | 59 | A.M. | 18 | 8 | 59 | P.M. |
| Last Quarter ... | 27 | 7 | 5 | P.M. | 26 | 3 | 54 | A.M. |

### Planets.

Mercury is an evening star in January. It is in inferior conjunction early in February and thereafter a morning star for the rest of the month.

Venus continues to be an evening star in these months. It is near the boundary between Capricornus and Aquarius in January and in Pisces in February.

Mars is in Leo in these months and is in opposition in the middle of February.

Jupiter is in Pisces in these months and is approaching the Sun. Saturn is still in Taurus in these months and is also approaching the Sun.

V. RAMESAM.

# SOLUTIONS.

## Question 451.

(K. J. Sanjana, M.A.):—Shew how to find pairs of isosceles triangles of rational areas such that the perimeters are in one given ratio and the areas in another given ratio.

Example: the perimeters are as 25:9 and the areas are equal.

*Remarks by the Proposer.*

We easily obtain the following among other solutions:—

193 (*bis*), 190, and 401 (*bis*), 798;
9·15834061 (*bis*), 9·19965000, and
25·12937805 (*bis*), 25·25757512;
9·15429061 (*bis*), 9·17535000, and
25·12127805 (*bis*), 25·24137512.

[The last pair of triangles is given by Mahavirácharya's rule; see sloka 137, Ch. VII, Rao Bahadur Rangácharya's edition of *Ganita-sára-sangraha.* This rule generally gives extremely large values for the sides when expressed in integers.]

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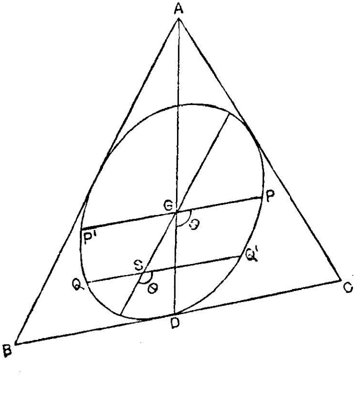
## Question 539.

(S. Krishnaswami Aiyangar):—The focal chords of the maximum inscribed ellipse of the triangle of reference parallel to the sides are the roots of the equation

$$x^3-2x^2\Delta^{\frac{1}{2}}\tan^{\frac{1}{2}}\lambda\cot w+x\Delta\tan\lambda\operatorname{cosec}^2 w-2R^2\Delta^{\frac{1}{2}}\tan^{\frac{3}{2}}\lambda=0$$

where $\lambda$ is the minor semi-steiner angle, $w$ is the Brocard angle and R the circumradius.



*Solution by N. Sankara Aiyar, M.A.*

Taking the equation of the conic in polar co-ordinates as usual, we get that the length of a focal chord is

$$\frac{l}{1-e\cos\theta}+\frac{l}{1+e\cos\theta}=\frac{2l}{1-e^2\cos^2\theta}.$$

Now, the semi-diameter of the conic parallel to this chord is given by

$$\frac{1}{r^2}=\frac{\cos^2\theta}{\alpha^2}+\frac{\sin^2\theta}{\beta^2}.$$

$$\therefore\ QQ'=\frac{2\beta^2}{\alpha(1-e^2\cos^2\theta)}=\frac{2\beta^2\alpha}{\alpha^2\sin^2\theta+\beta^2\cos^2\theta}$$

$$=\frac{2\beta^2\alpha}{\alpha^2\beta^2/r^2}=\frac{2r^2}{\alpha}=\frac{2OD^2}{\alpha}$$

$$=\frac{2a^2}{12}+\frac{3}{\Delta^{\frac{1}{2}}\cot\frac{1}{2}l}.\text{(J.I.M.S., Vol. V, p. 112)}$$

$$=\frac{a^2}{2\Delta^{\frac{1}{2}}\cot\frac{1}{2}\lambda}$$

$$\therefore\ x_1=\frac{a^2}{2}\Delta^{-\frac{1}{2}}\tan\tfrac{1}{2}\lambda.$$

$$\therefore\ x_1+x_2+x_3=\frac{a^2+b^2+c^2}{2}.\Delta^{-\frac{1}{2}}\tan\tfrac{1}{2}\lambda.$$

$$=2\Delta\cot w\,\Delta^{-\frac{1}{2}}\tan\tfrac{1}{2}\lambda$$

$$=2\Delta^{\frac{1}{2}}\cot w\tan\tfrac{1}{2}\lambda.$$

$$\Sigma x_2x_3=\sum\frac{b^2c^2\Delta^{-1}}{4}\tan\lambda$$

$$=\Sigma\Delta\operatorname{cosec}^2A\tan\lambda$$

$$=\Delta\tan\lambda\operatorname{cosec}^2w$$

$$x_1x_2x_3=\frac{a^2b^2c^2\Delta^{-\frac{3}{2}}}{8}\tan\tfrac{3}{2}\lambda.$$

$$=2R^2\Delta^2.\Delta^{-\frac{3}{2}}\tan\tfrac{3}{2}\lambda$$

$$=2R^2\Delta^{\frac{1}{2}}\tan\tfrac{3}{2}\lambda.$$

Hence the required equation is

$$x^3-2x^2\Delta^{\frac{1}{2}}\tan\tfrac{1}{2}\lambda\cot w+x\Delta\tan\lambda\operatorname{cosec}^2w-2R^2\Delta^{\frac{1}{2}}\tan\tfrac{3}{2}\lambda=0.$$

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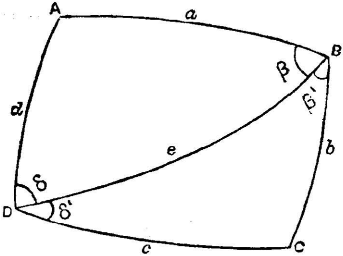
## Question 577.

(K. Appukuttan Erady, M.A.) :—If S, S′ are the semi-sums of opposite angles of a spherical quadrilateral, prove that

$$\cos(S-S')\sin\frac{a}{2}\sin\frac{b}{2}\sin\frac{c}{2}\sin\frac{d}{2}-\cos(S+S')\cos\frac{a}{2}\cos\frac{b}{2}\cos\frac{c}{2}\cos\frac{d}{2}$$
$$=\tfrac{1}{4}\Sigma\cos a,$$

$a, b, c, d$ being the sides of the quadrilateral.

*Solution by J. C. Swaminarayan, M.A.*




From the above figure, we can easily see that

$$\left.\begin{aligned}
\sin\frac{A+\beta+\delta}{2}&=\frac{\cos a+\cos d+1+\cos e}{4\cos\frac{a}{2}\cos\frac{d}{2}\cos\frac{e}{2}}\\
\sin\frac{C+\beta'+\delta'}{2}&=\frac{\cos b+\cos c+1+\cos e}{4\cos\frac{b}{2}\cos\frac{c}{2}\cos\frac{e}{2}}\\
\cos\frac{A+\beta+\delta}{2}&=-\frac{\sin\frac{a}{2}\sin\frac{d}{2}\sin A}{\cos\frac{e}{2}}\\
\cos\frac{C+\beta'+\delta'}{2}&=-\frac{\sin\frac{b}{2}\sin\frac{c}{2}\sin C}{\cos\frac{e}{2}}
\end{aligned}\right\}\quad\ldots\quad\ldots\quad(1)$$

and

$$\left.\begin{aligned}
\sin\frac{A-\beta-\delta}{2}&=\frac{\cos a+\cos d-1-\cos e}{4\cos\frac{e}{2}\sin\frac{a}{2}\sin\frac{d}{2}}\\
\sin\frac{C-\beta'-\delta'}{2}&=\frac{\cos b+\cos c-1-\cos e}{4\cos\frac{e}{2}\sin\frac{b}{2}\sin\frac{c}{2}}\\
\cos\frac{A-\beta-\delta}{2}&=\frac{\cos\frac{a}{2}\cos\frac{d}{2}\sin A}{\cos\frac{e}{2}}\\
\cos\frac{C-\beta'-\delta'}{2}&=\frac{\cos\frac{b}{2}\cos\frac{c}{2}\sin C}{\cos\frac{e}{2}}.
\end{aligned}\right\}\quad\ldots\quad\ldots\quad(2)$$

Hence

$$16\cos(S-S')\prod \sin\frac{a}{2}=\frac{\sin A \sin C \prod \sin a}{\cos^2\frac{e}{2}}$$

$$-\frac{\{\cos b+\cos c-1-\cos e\}\{\cos a+\cos d-1-\cos e\}}{\cos^2\frac{e}{2}}$$

and

$$16\cos(S+S')\prod \pi \cos\frac{a}{2}=\frac{\sin A \sin C \prod \sin a}{\cos^3\frac{e}{2}}$$

$$-\frac{\{\cos b+\cos c+1+\cos e\}\{\cos a+\cos d+1+\cos e\}}{\cos^2\frac{e}{2}}$$

$$\therefore \cos(S-S')\prod \sin\frac{a}{2}-\cos(S+S')\prod \cos\frac{a}{2}$$

$$=\frac{1}{16}\,\frac{2(1+\cos e)(\cos a+\cos b+\cos c+\cos d)}{\cos^2\frac{e}{2}}$$

$$=\frac{1}{4}\,\Sigma\cos a.$$

## Question 588.

(T. P. Trivedi, M.A., LL.B.) :—If $S_n$ denote the series $1-\frac{1}{3^3}+\frac{1}{5^3}+$ $+\ldots$to $n$ terms, prove that

$$\sum_1^{\infty}\frac{(-1)^{n-1}S_n}{(2n+1)^3}=\frac{\pi^6}{15\cdot 2^{11}}.$$

*Solution by A. Narasinga Rao, B.A.*

Consider the identity

$$\sum_1^{\infty}\frac{S_n}{(2n+1)^3}=\sum\frac{1}{(2n+1)^3}\cdot\frac{(-1)^{n+r-1}}{(2r+1)^3}$$

the summation extending to all $+ve$ integral values of $n$ and $r$.

Now $\sec z=\frac{2^2}{\pi^1}\Sigma_1+\frac{2^4}{\pi^3}z^2\Sigma_3+\ldots+\frac{2^{2n+2}}{\pi^{2n+1}}z^{2n}\Sigma_{2n+1}+\ldots$ (i)

where $\Sigma_{2n+1}=\frac{1}{1^{2n+1}}-\frac{1}{3^{2n+1}}+\frac{1}{5^{2n+1}}-\ldots$(Hobson's *Trig.*, p. 339.)

Comparing the co-effs. of $z^2$ on both sides of (i), we have $\Sigma_3=\frac{\pi^3}{2^5}$.

But $$(\Sigma_3)^3 = \left(\frac{1}{1^6}+\frac{1}{3^6}+\frac{1}{5^6}\dots\right)+2\sum\frac{(-)^{n+r}}{(2n+1)^3(2r+1)^3}.$$

$$\sum\frac{S_n}{(2n+1)^3}=-\frac{1}{2}\left\{\frac{\pi^6}{2^{10}}-\frac{\pi^6}{945}\cdot\frac{63}{64}\right\}=\frac{\pi^6}{15\cdot 2^{11}}.$$

Comparing the general terms on both sides of (i), we have

$$\Sigma_{2n+1}=\frac{\pi^{2n+1}}{2^{2n+2}}\begin{vmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ -\frac{1}{2\,!} & 0 & 1 & \dots & 0 & 0 \\ 0 & -\frac{1}{2\,!} & 0 & \dots & 0 & 0 \\ \frac{1}{4\,!} & 0 & -\frac{1}{2\,!} & \dots & 0 & 0 \\ \dots & \dots & \dots & & \dots & \\ \dots & \dots & \dots & & \dots & \\ \dots & \dots & \dots & & 0 & 1 \\ \frac{(-)^n}{(2n)\,!} & 0 & \frac{(-)^{n-1}}{(2n-2)\,!} & \dots & -\frac{1}{2\,!} & 0 \end{vmatrix}$$

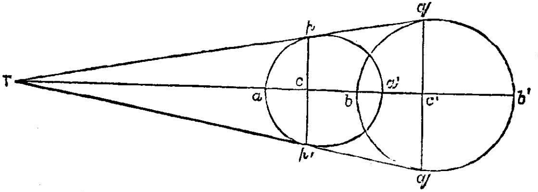
[J. I. M. S., Vol. VII, p, 52.]

We may therefore find by the same method the values of

$$\sum\frac{(-)^{m+n}}{(2m+1)^{2r+1}}\ \frac{1}{(2n+1)^{2s+1}}=(\Sigma_{2r+1})(\Sigma_{2s+1})-\sum\frac{1}{(2m+1)^{2(r+s+1)}}$$

## Question 599.

(V. V. S. Narayan) :—If two circles cut orthogonally find the focus of a point such that the tangents from it to the two circles form a harmonic pencil.



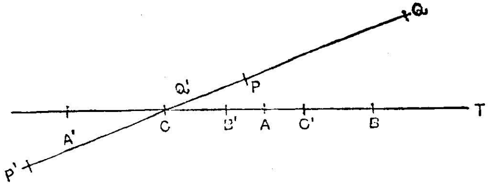

*Geometrical solution by N.B. Pendse, M.A., LL.B.*

Draw the common tangents to the two circles. Let them meet at T. Let T$aba'b'$ be the line of centres and let $pcp'$ and $qc'q'$ be the chords of contact of the circles.

Reciprocate the curve with respect to the point T.

Then T will be the common focus of two hyperbolas. Inverse points of $c$ & $c'$ will be the centres of the two hyperbolas. Inverse points of $a, a', b, b'$ will be the extremities of the major axes of the two hyperbolas.

These two hyperbolas have parallel asymptotes. Therefore their




major axes and minor axes and all parallel diameters are in the ratio of CT : C'T.

Through C the centre draw any straight line and let it be cut by the two hyperbolas in P, P', Q, Q.' Then

$$\frac{CQ \cdot CQ'}{CB \cdot CB'} = \frac{CP^2}{CA^2}$$

But $CB \cdot CB' = CA^2$, as A'B'AB is harmonic range.

$$\therefore \qquad CQ \cdot CQ^1 = CP^2.$$

Hence any straight line through the centre is cut harmonically by the two hyperbolas.

Therefore in the two hyperbolas: any straight line passing through C or C' is cut harmonically by the two hyperbolas.

Reciprocating, tangents drawn to the two circles from any point on the chord of contact $pcp'$ or $qc'q'$ form a harmonic pencil.

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### Question 609 (i).

(M. Bhimasena Rao) :—Show that :

$$\frac{1}{e^{2\pi}-1} + \frac{2^{4n+1}}{e^{4\pi}-1} + \frac{3^{4n+1}}{e^{6\pi}-1} + \dots = \frac{B_{2n+1}}{4[2n+1]}.$$

*Solution by K. B. Madhava.*

The series is obviously the integral $\int_0^\infty \frac{x^{4n+1}}{e^{2\pi x}-1} dx.$

But $\quad B_r = \frac{(2r)!}{2^{2r-1} \cdot \pi^{2r}} \cdot \sum_1^\infty \frac{1}{n^{2r}}.$

$$\therefore \quad \frac{B_r}{4r} = \frac{\Gamma(2r)}{(2\pi)^{2r}} \cdot \sum_1^\infty \frac{1}{n^{2r}}$$

$$= \int_0^\infty x^{2r-1} \left\{ e^{-2\pi x} + e^{-4\pi x} + e^{\;\;6\pi} + \dots \right\} dx.$$

$$=\int_0^\infty \frac{x^{2r-1}}{e^{2\pi x}-1}\,dx.$$

Hence putting $r=2n+1$,

$$\frac{1}{e^{2\pi}-1}+\frac{2^{4n+1}}{e^{4\pi}-1}+\frac{3^{4n+1}}{e^{6\pi}-1}+\ldots=\frac{B_{2n+1}}{4(2n+1)}.$$

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**Question 619.**

(S. KRISHNASWAMI AIYANGAR) :—If

$$S_n=\frac{1}{(2n-1)^2(2n-3)^2}+\frac{1}{(2n-5)^2}\left\{\frac{1}{(2n-1)^2}+\frac{1}{(2n-3)^2}\right\}$$
$$+\frac{1}{(2n-7)^2}\left\{\frac{1}{(2n-1)^2}+\frac{1}{(2n-3)^2}+\frac{1}{(2n-5)^2}\right\}+\ldots\ldots,$$

shew that

$$\frac{S_2}{5^2}+\frac{S_3}{7^2}+\frac{S_4}{9^2}+\ldots=\frac{\pi^6}{2^6\underline{|6}}$$

*Solution by T. P. Trivedi, M.A., LL.B.*

Writing down a few terms of the series, we get

$$\frac{1}{1^2\cdot3^2\cdot5^2}+\frac{1}{7^2}\left\{\frac{1}{5^2}\cdot\frac{1}{3^2}+\frac{1}{1^2}\left(\frac{1}{5^2}+\frac{1}{3^2}\right)\right\}$$
$$+\frac{1}{9^2}\left\{\frac{1}{7^2}\cdot\frac{1}{5^2}+\frac{1}{3^2}\left(\frac{1}{5^2}+\frac{1}{7^2}\right)+\frac{1}{1^2}\left(\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}\right)\right\}+\text{etc},$$

which is clearly equivalent to the product three at a time of the terms of the series $\frac{1}{1^2},\frac{1}{3^2},\frac{1}{5^2},\frac{1}{7^2}$ etc., to infinity.

Now

$$\cos\theta=\left(1-\frac{2^2\theta^2}{\pi^2}\right)\left(1-\frac{2^2\theta^2}{3^2\pi^2}\right)\left(1-\frac{2^2\theta^2}{5^2\pi^2}\right)\ldots;$$

Hence the series multiplied by $\frac{2^6}{\pi^6}$ is equal to the coefficient of $-\theta^6$ in the expansion of $\cos\theta$, and the series is thus equal to $\frac{\pi^6}{2^6\cdot6}$

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**Question 620.**

(S. KRISHNASWAMI AIYANGAR) :—Find the value of

$$1-\frac{3^5x^2}{2\,!}+\frac{5^5x^4}{4\,!}-\frac{7^5x^6}{6\,!}+\ldots$$

*Solution by T. P. Trivedi, M.A., LL.B., A. Narasinga Rao, B.A., and N. Sankara Aiyar, M. A.*

We have

$$x \cos x = x - \frac{x^3}{\underline{|2}} + \frac{x^5}{\underline{|4}} - \frac{x^7}{\underline{|6}} + \dots$$

The series evidently is $\left(x\frac{d}{dx}\right)^5 (x \cos x)$ which when expanded becomes

$$\cos x(15x^4 - 90x^2 + 1) - \sin x(x^5 - 65x^3 + 31x).$$

*Remarks by K. B. Madhava.*

By applying formula (3) on p. 184 of the Journal for 1913, we find $\phi(n) = (1+n)^5$, $\Delta\phi(0) = 1$, $\Delta^2\phi(0) = 31$, $\Delta^3\phi(0) = 180$, $\Delta^4(0) = 390$, $\Delta^5\phi(0) = 360, \Delta^6\phi(0) = 120, \Delta^7\phi(0) = \Delta^8\phi(0) = \dots = 0$.

Hence the series is equal to

$$\cos x(15x^4 - 90x^2 + 1) - \sin x(x^5 - 65x^3 + 31x).$$

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## Question 621.

(R. Vythynathaswamy) :—If C is the chord of an arc $\lambda$ of a closed curve, shew that $\int(C\, ds)$ taken over the contour is equal to $\lambda$ times the perimeter of the locus of centroids of arcs of the curve equal to $\lambda$.

*Solution by A. Narasinga Rao, B.A.*

Let AB, $A_1B_1$ be consecutive positions of the chord so that

$$\text{arc } AB = \text{arc } A_1B_1 = \lambda.$$
$$\therefore\ AA_1 = BB_1 = ds.$$

Let H be the centroid of the arc $A_1$ B. Join H to the mid-points of $AA_1$, $BB_1$ and divide the joining lines in the ratio of $ds : A_1B_1$ at $\alpha$, $\beta$ respectively. Then $\alpha$, $\beta$ are the centroids of the arcs AB, $A_1B_1$.

Also, we have ultimately

$$\alpha\beta : \text{chord } AB = ds\,;\ \text{arc } AB$$

*i.e.*

$$\alpha\beta : C = ds : \lambda.$$

Thus $\int C\, ds = \lambda \int \alpha\beta$

$= \lambda$ times the corresponding perimeter of the locus of $\alpha$.

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## Question 622.

(R. Vythynathaswamy) :—A moving quadrilateral has its four corners on a fixed circle. Shew that at any instant the sum of the angular velocities of the four pedal lines is equal to the sum of the angular velocities of the four corners about the centre.

*Solution by A. Narasinga Rao, B.A.*

Let ABCD be one of the positions of the moving quadrilateral, and let $A_1$ $B_1$ the points in which the $\perp^s$ from A and B on CD meet the circle ABCD. The pedal lines of A, B *w.r.t.* $\triangle$ formed by the other three points are parallel respectively to $BA_1$ and $A_1B$.

Hence the angle between these two

$$=\text{angle between } BA_1 \text{ and } A_1B$$
$$=\text{angle which AB subtends at the centre O.}$$

Let $\alpha'$, $\alpha$ be the angles which the pedal line of A, and radius OA make with a fixed line ; with similar notations for the points B, C,D and their pedal lines, by what has just been proved,

$$\alpha'-\beta'=\alpha-\beta\,;$$

or $$\alpha'-\alpha=\beta'-\beta=\gamma'-\gamma=\delta'-\delta'=c \text{ (say).}$$

$\therefore$ $$\alpha'+\beta'+'\gamma+\delta'=\alpha+\beta+\gamma+\delta+4c.$$

$\therefore$ $$\frac{d}{dt}(\alpha'+\beta'+\gamma'+\delta')=\frac{d}{dt}(\alpha+\beta+\gamma+\delta).$$

Hence the sum of the angular vels. of the pedal lines is equal to the sum of the angular vels. of the corners about the centre.

---

## Question 627.

(K. V. Anantanarayana Sastri, B.A.) ;—Four spheres of radii $a$, $b$, $c$, $d$, intersect at right angles. Shew that the volume of the tetrahedron formed by their centres is

$$\tfrac{1}{6}\, abcd\,(a^{-2}+b^{-2}+c^{-2}+d^{-2})^{\frac{1}{2}}.$$

*Solution (1) by R. J. Pocock, T. P. Trivedi, M.A., L.L.B., and T. P. Krishnaswami.*

If A, B, C, D are the centres of the spheres $AB=\sqrt{a^2+b^2}$, &c.,

$$\cos CAB=\frac{a^2}{\sqrt{a^2+b^2}\sqrt{a^2+c^2}},\ \&c.\,;$$

also volume of tetrahedron

$$=\tfrac{1}{6}.AB.AC.AD.\left\{1-\Sigma\cos^2 CAB+2\cos CAB.\cos BAD.\cos DAC\right\}^{\frac{1}{2}}$$

$$=\frac{1}{6}\left[(a^2+b^2)(a^2+c^2)(a^2+d^2)\left\{1-\sum\frac{a^4}{(a^2+b^2)(a^2+c^2)}+\frac{2a^6}{(a^2+b^2)(a^2+c^2)(a^2+d^2)}\right\}\right]^{\frac{1}{2}}$$

$$=\tfrac{1}{6}\,abcd(a^{-2}+b^{-2}+c^{-2}+d^{-2})^{\frac{1}{2}}, \text{ after reduction.}$$

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## Question 631.

(S. Narayana Aiyar, M.A.) :—Shew that if $\phi\,(a, b, c, d)$ denotes

$$1+\frac{x}{1}\frac{c-d}{d}\frac{b+x-1}{a+x-1}+\frac{x(x-1)}{1.2}\frac{(c-d)(c-d-1)}{d(d+1)}\frac{(b+x-1)(b+x-2)}{(a+x-1)(a+x-2)}+\dots,$$

then
$$\frac{\phi(a,\,b,\,c,\,d)}{\phi(c,\,d,\,a\,\,b)}=\frac{\Gamma(x+b)\Gamma(x+c)\Gamma(a)\Gamma(d)}{\Gamma(x+a)\Gamma(x+d)\Gamma(b)\Gamma(c)}.$$

*Solution by T. P. Trivedi, M.A., LL.B.*

From Question 576, we get

$$e^y=\frac{1+y\,\frac{a}{b}+\frac{y^2}{1\cdot 2}\frac{a(a+1)}{b(b+1)}+\dots\dots}{1+y\,\frac{a-b}{b}+\frac{y^2}{1.2}\,\frac{(a-b)(a-b-1)}{b(b+1)}+\dots}$$

and also

$$=\frac{1+y\,\frac{c}{d}+\frac{y^2}{1\cdot 2}\frac{c(c+1)}{d(d+1)}+\dots}{1+y\frac{c-d}{d}+\frac{y^3}{1.2}\,\frac{(c-d)(c-d-1)}{d(d+1)}+\dots}$$

Eliminating $e^y$, we get

$$\left\{1+y\frac{c-d}{d}+\frac{y^2}{1.2}\frac{(c-d)(c-d-1)}{d(d+1)}+\dots\right\}\left\{1+y\frac{a}{b}+\frac{y^2}{1.2}\frac{a(a+1)}{b(b+1)}+\dots\dots\right\}$$
$$=\left\{1+y\,\frac{a-b}{b}+\frac{y^2}{1.2}\,\frac{(a-b)(a-b-1)}{b(b+1)}+\dots\right\}$$
$$\left\{1+y\,\frac{c}{d}+\frac{y^2}{1.2}\,\frac{c(c+1)}{d(d+1)}+\dots\dots\right\}.$$

Equating the coefficient of $y^x$ on both sides, we get

$$\frac{1}{\lfloor x}\,\frac{a(a+1)\dots(a+x-1)}{b(b+1)\dots(b+x-1)}+\frac{c-d}{d}\cdot\frac{1}{\lfloor x-1}\,\frac{a(a+1)\dots(a+x-2)}{b(b+1)\dots(b+x-2)}$$
$$+\frac{(c-d)(c-d-1)}{d(d+1)}\,\frac{1}{\lfloor x-2}\cdot\frac{1}{1.2}\,\frac{a(a+1)\dots(a+x-3)}{b(b+1)\dots(b+x-3)}+\dots$$
$$=\frac{1}{\lfloor x}\frac{c(c+1)\dots(c+x-1)}{d(d+1)\dots(d+x-1)}+\frac{a-b}{b}\cdot\frac{1}{\lfloor x-1}\,\frac{c(c+1)\dots(c+x-2)}{d(d+1)\dots(d+x-2)}$$
$$+-+\dots;$$

whence it is evident that

$$\frac{1}{\lfloor x}\,\frac{a(a+1)\dots(a+x-1)}{b(b+1)\dots(b+x-1)}\,\phi\,(a,b,c,d)$$
$$=\frac{1}{\lfloor x}\,\frac{c(c+1)\dots(c+x-1)}{d(d+1)\dots(d+x-1)}\,\phi\,(c,d,a,b).$$

Thus

$$\frac{\phi(a,b,c,d)}{\phi(c,d,a,b)}=\frac{b(b+1)\ldots(b+x-1)\,c(c+1)\ldots(c+x-1)}{a(a+1)\ldots(a+x-1)d(d+1)\ldots(d+x-1)},$$

$$=\frac{\Gamma(x+b)\Gamma(x+c)\Gamma(a)\Gamma(d)}{\Gamma(x+a)\Gamma(x+d)\Gamma(b)\Gamma(c)}$$

## Question 632.

(R. N. Apte, M. A., F. R. A. S.) :—If $r_1$, $r_2 \ldots r_n$ be the distances of a point P in the plane of a regular polygon from the vertices, find $\Sigma(r_1^2 r_2^2 r_3^2)$ in terms of $n$, $a$ the radius of the circum-circle of the polygon and $c$ the distance of P from the centre of the circle, the summation extending over all possible groups of vertices.

*Solution by T. P. Trivedi, M.A., LL.B., R. Vythynathaswamy, and N. Sankara Aiyar, M.A.*

If $A_1A_2A_3\ldots A_n$ be the polygon, O the centre of the circum-circle and the angle $POA_1$ be $\theta$, supposing P to lie between $A_1$ and $A_n$, the angle $POA_2=\theta+\frac{2\pi}{n}$ and so on. It will be at once seen from a figure

that $$PA_1^2=OP^2+OA_1^2-2OP\ OA_1\cos\theta$$

or $$r_1^2=c^2+a^2-2ac\cos\theta.$$

Similarly

$$r_2^2=c^2+a^2-2ac\cos\left(\theta+\frac{2\pi}{n}\right),\text{ etc.}$$

We are thus to sum up terms of the type

$$\left\{c^2+a^2-2ac\cos\theta\right\}\left\{c^2+a^2-2ac\cos\left(\theta+\frac{2\pi}{n}\right)\right\}$$
$$\times\left\{c^2+a^2-2ac\cos\left(\theta+\frac{4\pi}{n}\right)\right\}$$

that is,

$$(c^2+a^2)^3-2ac(a^2+c^2)^2\left\{\cos\theta+\cos\left(\theta+\frac{2\pi}{n}\right)+\cos\left(\theta+\frac{4\pi}{n}\right)\right\}$$
$$+4a^2c^2(a^2+c^2)\left\{\cos\theta\cos\left(\theta+\frac{2\pi}{n}\right)+\ldots+\ldots\right\}$$
$$-8a^3c^3\cos\theta\cos\left(\theta+\frac{2\pi}{n}\right)\cos\left(\theta+\frac{4\pi}{n}\right).$$

As the number of such terms is ${}_nC_3$, it will be found that the co-efficient of $-2ac(a^2+c^2)^2$ is

$$\frac{3\,{}_nC_3}{n}\Sigma\cos\theta,\text{ or }\frac{(n-1)(n-2)}{2}\Sigma\cos\theta;$$

the co-efficient of $4a^2c^2(a^2+c^2)$ is $\frac{3_nC_3}{_nC_2}\Sigma\cos\theta\cos\left(\theta+\frac{2\pi}{n}\right)$,

or $$(n-2)\Sigma\cos\theta\cos\left(\theta+\frac{2\pi}{n}\right);$$

and the co-efficient of $-8a^2c^2$ is

$$_nC_3\Sigma\cos\theta\cos\left(\theta+\frac{2\pi}{n}\right)\cos\left(\theta+\frac{4\pi}{n}\right).$$

Moreover, from the expansion of $\cos n\theta$, it is seen that $\cos\theta$, $\cos\left(\theta+\frac{2\pi}{n}\right)$, etc., are the roots of the equation

$$(2\cos\theta)^n-\frac{n}{1}(2\cos\theta)^{n-2}+\frac{n(n-3)}{1\cdot 2}(2\cos\theta)^{n-4}-\ldots 2\cos n\theta=0.$$

Hence $\Sigma\cos\theta=0$; $\Sigma\cos\theta\cos\left(\theta+\frac{2\pi}{n}\right)=-\frac{n}{4}$;

and $\Sigma\cos\theta\cos\left(\theta+\frac{2\pi}{n}\right)\cos\left(\theta+\frac{4\pi}{n}\right)=0$; so that the summation is

$$\tfrac{1}{6}n(n-1)(n-2)(a^2+c^2)^3-n(n-2)a^2c^2(a^2+c^2)$$

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## Question 633.

(K. Appukuttan Erady, M.A.):—A particle is projected from a point on the boundary of a circular pond of radius $a$ with the velocity of projection $\sqrt{(2gc)}$. Prove that if all directions of projection are equally probable, the chance of its falling into the pond is

$$\tfrac{1}{2}-\frac{1}{\pi}\int_0^{\frac{\pi}{2}}\sqrt{\left(1-\frac{a}{c}\cos\phi\right)}\,d\phi.$$

*Solution by T. P. Trivedi and K. J. Sanjana.*

With the usual axes the equation of the circular boundary may be written $r=2a\cos\phi$, $\phi$ lying between $\frac{1}{2}\pi$ and $-\frac{1}{2}\pi$. If the angle of projection is $\theta$ the range of the particle is $2c\sin 2\theta$, and for the favourable cases we must have $c\sin 2\theta \not> a\cos\phi$. Putting $\sin\psi$ for $a\cos\phi\div c$, this requires

$$\sin 2\theta \not> \sin\psi$$

*i.e.*, $$2\theta<\psi \text{ and } >0, \text{ or } >\pi-\psi \text{ and } <\pi.$$

Hence the chance is seen to be

$$\int_0^{\frac{\pi}{2}}\left[\int_0^{\frac{1}{2}\psi}\cos\theta d\theta+\int_{\frac{1}{2}(\pi-\psi)}^{\frac{1}{2}\pi}\cos\theta d\theta\right]d\phi\div 2\pi$$

or
$$\frac{1}{\pi}\int_0^{\frac{\pi}{2}} (\sin \tfrac{1}{2}\psi + 1 - \cos \tfrac{1}{2}\psi)\, d\phi.$$

As $\psi$ is evidently positive and $<\frac{1}{2}\pi$, we get

$$\frac{1}{\pi}\int_0^{\frac{\pi}{2}} d\phi - \frac{1}{\pi}\int_0^{\frac{\pi}{2}} (1-\sin\psi)^{\frac{1}{2}} d\phi$$

$$= \tfrac{1}{2} - \frac{1}{\pi}\int_0^{\frac{\pi}{2}} \left(1 - \frac{a}{c}\cos\phi\right) d\phi.$$

When $a=c$, we get $\frac{1}{2}-\frac{2}{\pi}(\sqrt{2}-1)$, which is the result of Ex. 3. Ch. XIV, of Todhunter's *Int. Calculus.*

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## Question 634.

(E. H. Neville, B.Sc.) :—On the section of a surface by a plane through the normal ON at a point O is taken a point S at a small distance $f$ from O, and the plane parallel to the tangent plane at O and distant $1/\lambda$ from O meets the normals at O and S to the surface in points L and P. Shew that $LP^2$ is approximately $(f^2/\lambda^2)\{(k_n-\lambda)^2 + a_g\}$, where $k_n$ and $a_g$ are the normal curvature and geodesic torsion of the surface along OS. Deduce that if S traces a small contour round O, $\lambda$ can be chosen so that P traces a curve which is almost a circle with centre L only if the contour traced by S approximates either to a circle with centre O or to an ellipse whose axes are the principal tangents to the surface, and that $\lambda$ in the first case must be the mean curvature of the surface, in the second case may have either of two values.

*Solution by the Proposer.*

Let the normal at S to the plane section meet the plane through L in Q. Since SQ and SP are both perpendicular to the tangent at S to the plane section, SQ is the projection of SP on the normal plane and the angle QSP is $f\ a_g$; also the angle between SQ and ON is $f\ k_n$. Hence LQ is $\pm f\ \{(k_n/\lambda)-1\}$ and QP is $f\ (a_g/\lambda)$, and the first result follows at once.

Again, if D is half the sum of the principal curvatures at O and I is half the difference between them, and if $h$ is the angle between the normal plane OSN and the first principal plane,

$$k_n = D - I \cos 2h, \quad a_g = I \sin 2h$$

and therefore

$$(k_n - \lambda)^2 + a_g^2 = \{ (D-\lambda)^2 + I^2 \} - 2 (D-\lambda) I \cos 2h.$$

For LP to be independent of $h$, we must have either $f$ a constant and $\lambda$ equal to D, or $f^2$ of the form $1/(p-q \cos 2h)$; in the first case S describes a curve approximately circular, in the second case the locus of S is approximately a central conic with axes along the principal tangents and $\lambda$ is determined by the quadratic equation

$$q (D-\lambda)^2 - 2p\, I (D-\lambda) + q\, I^2 = 0$$

which has real roots if and only if the conic is an ellipse. Then in fact if $a$, $b$ are the semi-axes of the ellipse and $\alpha$, $\beta$ the principal curvatures of the surface the two values of $\lambda$ are $(a\alpha - b\beta)/(a-b)$ and $(a\alpha + b\beta)/(a+b)$, and the possible positions of L form a pair of points harmonic with respect to the principal centres.

The importance of the example lies in its application to optics. Rays which emanate from a single point form after any number of reflections and refractions a set of normals to a family of parallel surfaces, the wave-fronts. If rays forming a small pencil enter the final medium through a circular aperture, the section of the pencil near the aperture is approximately an ellipse, and if the axes of this ellipse are approximately the principal tangents to the wave surface, a condition often secured by symmetry, there are usually two planes normal to the pencil on which the cross section of the pencil is a circle. The two circles obtainable generally differ in size, and the smaller of them is the best approximation possible without further reflection or refraction to an image of the point from which the rays originated. This circle is known as the circle of least confusion; mathematically it always exists, but physically it may be virtual if the rays are diverging when they enter the final medium.

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## Question 635.

(S. Krishnaswami Aiyangar, B.A.):—If $\sigma_1$, $\sigma_2$, $\sigma_3$ represent KA, KB, KC where K is the symmedian point and $m_1$, $m_2$, $m_3$ the medians of a triangle ABC, shew that

1. $\Sigma \{ a(b^2+c^2)\sigma_1 m_1 \} = \frac{3}{4}abc(a^2+b^2+c^2) + \frac{3abc}{2} \cdot \frac{a^4+b^4+c^4}{a^2+b^2+c^2}.$

2. $\Sigma \left\{ a(b^2+c^2) \frac{\sigma_1}{m_1} \right\} = 4abc.$

3. $\Sigma \left\{ bc \frac{m_1}{\sigma_1} \right\} = \frac{2}{3}(a^2+b^2+c^2).$

*Solution by K. B. Madhava, N. Sankara Aiyar and others.*

The actual areals of K are $\left(\frac{a^2}{S}, \frac{b^2}{S}, \frac{c^2}{S}\right)$, where $S = a^2 + b^2 + c^2$.

$$\therefore \quad AK^2 = \sigma_1{}^2 = \frac{b^2c^2}{S^2}(2S - 3a^2) \text{ from Askwith, § 261.}$$

Also, $AG^2 = \frac{1}{9}(2b^2 + 2c^2 - a^2)$ so that $m_1{}^2 = \frac{1}{4}(2S - 3a^2)$.

$$\therefore \quad \sigma_1 m_1 = \frac{bc}{2S}(2S - 3a^2).$$

$$\therefore \quad \Sigma a(b^2 + c^2)\sigma_1 m_1 = 2abc.\, S - \frac{3abc}{S}(a^2b^2 + b^2c^2 + c^2a^2)$$

$$= \tfrac{1}{2}abc(a^2 + b^2 + c^2) + \frac{3abc}{2} \cdot \frac{a^4 + b^4 + c^4}{a^2 + b^2 + c^2}.$$

Also $$\frac{\sigma_1}{m_1} = \frac{2bc}{S}.$$

$$\therefore \quad a(b^2 + c^2)\frac{\sigma_1}{m_1} = \frac{2abc}{S}(2a^2 + 2b^2 + 2c^2) = 4abc.$$

Lastly $$\frac{m_1}{\sigma_1} = \frac{S}{2bc}.$$

$$\therefore \quad \Sigma ba\frac{m_1}{\sigma_1} = \frac{3}{2}(a^2 + b^2 + c^2).$$

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## Question 638.

(S. B. Belekar) :—If $\lambda = e^{i\theta}$, shew that the point $a\frac{(\lambda^2 + 1)}{2\lambda}, \frac{b(\lambda^2 - 1)}{2i\lambda}$,

lies on $$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

If $\lambda_1, \lambda_2, \lambda_3$ be three points as above, the circle passing through them is

$$x^2 + y^2 - \frac{(a^2 - b^2)}{4aN}\{N^2 + M + (LN + 1)\}x$$

$$- \frac{(a^2 - b^2)}{4biN}\{N^2 + M - (LN + 1)\}y$$

$$+ \frac{(MN + L)(a^2 - b^2) - 2(a^2 + b^2)N}{4N} = 0,$$

where $L = \Sigma\lambda_1$, $M = \Sigma\lambda_1\lambda_2$, $N = \lambda_1\lambda_2\lambda_3$.

*Solution by R. Srinivasan, M.A., T. P. Trivedi and K. J. Sanjana.*

As $\lambda = \cos\alpha + i\sin\alpha$, $\lambda^{-1} = \cos\alpha - i\sin\alpha$, $\cos\alpha = \frac{1}{2}\left(\lambda + \frac{1}{\lambda}\right)$, and $\sin\alpha = \frac{1}{2i}\left(\lambda - \frac{1}{\lambda}\right)$. Thus the point mentioned is $a\cos\alpha$, $b\sin\alpha$, which is of course on the given ellipse.

The equation of the circle through the three points $\alpha_1, \alpha_2, \alpha_3$ on this ellipse is given by Salmon in the form

$$x^2+y^2-\frac{2(a^2-b^2)}{a}\cos\frac{\alpha_2+\alpha_3}{2}\cos\frac{\alpha_3+\alpha_1}{2}\cos\frac{\alpha_1+\alpha_2}{2}.x$$

$$-\frac{2(b^2-a^2)}{b}\sin\frac{\alpha_2+\alpha_3}{2}\sin\frac{\alpha_3+\alpha_1}{2}\sin\frac{\alpha_1+\alpha_2}{2}.y$$

$$-\frac{a^2+b^2}{2}+\frac{a^2-b^2}{2}\left\{\cos(\alpha_1+\alpha_2)+\cos(\alpha_2+\alpha_3)\cos(\alpha_3+\alpha_1)\right\}=0.$$

Now $\Pi\cos\frac{\alpha_2+\alpha_3}{2}=\frac{1}{4}\left\{\Sigma\cos\alpha_1+\cos(\Sigma\alpha_1)\right\}$

$$=\tfrac{1}{4}\left\{\tfrac{1}{2}\Sigma\left(\lambda_1+\frac{1}{\lambda_1}\right)+\tfrac{1}{2}\left(\lambda_1\lambda_2\lambda_3+\frac{1}{\lambda_1\lambda_2\lambda_3}\right)\right\}$$

$$=\tfrac{1}{8}\left\{L+\frac{M}{N}+N+\frac{1}{N}\right\}=\frac{1}{8N}(N^2+M+LN+1);$$

and $\Pi\sin\frac{\alpha_2+\alpha_3}{2}=\frac{1}{4}\left\{\Sigma\sin\alpha_1-\sin(\Sigma\alpha_1)\right\}$

$$=\frac{1}{8i}\left\{L-\frac{M}{N}-N+\frac{1}{N}\right\}$$

$$=\frac{1}{8Ni}\left\{LN+1-(N^2+M)\right\},$$

also $\Sigma\cos(\alpha_1+\alpha_2)=\frac{1}{2}\Sigma\left(\lambda_1\lambda_2+\frac{1}{\lambda_1\lambda_2}\right)$

$$=\tfrac{1}{2}\left\{\Sigma(\lambda_1\lambda_2)+(\Sigma\lambda_1)\div\lambda_1\lambda_2\lambda_3\right\}=\frac{MN+L}{2N}.$$

Thus the required equation is

$$x^2+y^2-\frac{a^2-b^2}{4aN}\left\{LN+1+(N^2+M)\right\}$$

$$-\frac{b^2-a^2}{4biN}\left\{LN+1-(N^2+M)\right\}$$

$$+\frac{1}{4N}\left\{(a^2-b^2)(MN+L)-2(a^2+b^2)N\right\}=0.$$

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## Question 640.

(SELECTED) :—*The Magic Number Cards.*

"This trick consists of six cards with certain of the numbers from 1 to 63 printed on them. The magician asks you to think of a number between 1 and 63 and then presents these cards one at a time with the request that you indicate whether your number is on the card. This done he at once declares the number which you thought of."

Explain the trick and point out any easy extensions of it.

*Solution by K. J. Sanjana and V. Anantharaman.*

The result follows at once from the fact that in the scale of 2 there are only two digits, 1 and 0, so that any number is the sum of a certain number of powers of 2 (including $1\equiv 2^0$). With six cards standing for $2^0, 2^1, 2^2, 2^3, 2^4, 2^5$, any number up to the sum of all these (*i.e.* $63\equiv 111111$ in the scale of 2) may be presented; if in any number some of these powers are not required, the cards corresponding to those powers will not be used. Thus

$23(=1+2+2^2+2^4)$ requires the first, second, third and fifth;

$24(=2^3+2^4)$ requires the fourth and fifth;

25 requires the first, fourth and fifth; and so on.

It is evident that $n$ cards may be used giving numbers up to $2^n-1$ inclusive. We can also use the scale of 3, which will however require some powers to be subtracted; otherwise the digit 2 may be shown by repeating the number for the power which has this coefficient. Thus

$23(=2\cdot 3^2+3+2\cdot 3^0)$ requires two entries on the first, one on the second and two on the third;

$24(=2\cdot 3^2+2\cdot 3)$ requires two on the second and two on the third;

$25(=2\cdot 3^2+2\cdot 3+3^0)$, two on the third, two on the second and one on the first; and so on.

In books on Algebra these properties are generally enunciated thus: "A weight of N lbs. can be balanced by a number of weights of the series 1 lb., 2 lbs., $2^2$ lbs......., not more than one weight of each kind being used; and by a number of weights of the series 1 lb., 3 lbs., $3^2$ lbs.,......, not more than one weight, each kind being used but *in either scalepan* as may be necessary."

### Question 642.

(S. RAMANUJAN):—Shew that

$$(1)\ \sum_{0}^{\infty}\left(1+\frac{1}{3}+\frac{1}{5}+-\frac{1}{2n+1}\right)5^{-n}(2n+1)^{-1}$$

$$=\frac{\pi^2}{4\sqrt{5}}-\frac{\sqrt{5}}{24}\log(2+\sqrt{5})^2.$$

$$(2)\ \sum_{0}^{\infty}\left(1+\frac{1}{3}+\dots+\frac{1}{2n+1}\right)9^{-n}(2n+1)^{-1}$$

$$=\frac{\pi^2}{8}-\frac{3}{8}(\log 2)^2$$

*Solution and Remarks by M. Bhimasena Rao.*

$$\text{Let}\quad f(x)\equiv\sum_{n=0}^{\infty}\left(1+\tfrac{1}{3}+\tfrac{1}{5}+\cdots\frac{1}{2n+1}\right)\frac{x^{2n+1}}{2n+1}.$$

$$=\int_0^x \frac{1}{2}\log\frac{1+x}{1-x}\cdot\frac{dx}{(1-x^2)x}$$

$$=\int_0^x \frac{1}{4x}\log\frac{1+x}{1-x}\,d\log\frac{1+x}{1-x}$$

$$=\int_1^y \frac{1}{4}\cdot\frac{1+y}{1-y}\log\frac{1}{y}\,d\log\frac{1}{y},\ \text{where } y=\frac{1-x}{1+x}$$

$$=\int_1^y \frac{1}{4}\log y\,\frac{dy}{y}\left(1+\frac{2y}{1-y}\right)$$

$$=\frac{(\log y)^2}{8}+\frac{1}{2}\int_1^y\frac{\log y}{1-y}\cdot dy$$

$$=\frac{(\log y)^2}{8}-\frac{1}{2}\log y.\log(1-y)+\frac{1}{2}\int_1^y\frac{\log(1-y)}{y}dy$$

$$=\frac{(\log y)^2}{8}-\tfrac{1}{2}\log y\log(1-y)-\frac{1}{2}\int\sum_1^y\frac{y^n}{x^2}$$

$$=\frac{1}{8}\log y\log\frac{y}{(1-y)^4}-\tfrac{1}{2}\phi(y)+\frac{\pi^2}{12}\qquad\dots\qquad\dots\qquad(1)$$

where $\phi(y)$ stands for $\sum\frac{y^n}{n^2}$.

It may be shown that

(i) $\phi(\tfrac{1}{2})=\dfrac{\pi^2}{12}-\tfrac{1}{2}(\log 2)^2$

(ii) $\phi\left(\dfrac{\sqrt{5}-1}{2}\right)=\dfrac{\pi^2}{10}-\left(\log\dfrac{\sqrt{5}-1}{2}\right)^2$

(iii) $\phi\left(\frac{3-\sqrt{5}}{2}\right)=\frac{\pi^2}{15}-\left(\log\frac{\sqrt{5}-1}{2}\right)^2.$

[Bromwich : *Infinite Series*, p. 487, Ex. 21.]

For these values of $y$, the corresponding values of $x$ are $\frac{1}{3}, \frac{1}{\sqrt{5}+2}, \frac{1}{\sqrt{5}}$, since $x=\frac{1-y}{1+y}$.

$$\therefore\quad f\left(\frac{1}{3}\right)=-\frac{3}{8}(\log 2)^2+\frac{1}{4}(\log 2)^2-\frac{\pi^2}{24}+\frac{\pi^2}{12}=\frac{\pi^2}{24}-\frac{(\log 2)^2}{8}\quad\ldots\ (2)$$

$$f\left(\frac{1}{\sqrt{5}+2}\right)=\frac{1}{8}\log\frac{\sqrt{5}-1}{2}\log\frac{\frac{\sqrt{5}-1}{2}}{\left(1\ \frac{\sqrt{5}-1}{2}\right)^4}$$

$$-\frac{\pi^2}{20}+\frac{1}{2}\left(\log\frac{\sqrt{5}-1}{2}\right)^2+\frac{\pi^2}{12}$$

$$=\frac{\pi^2}{30}-\frac{1}{24}\left\{\log(2+\sqrt{5})\right\}^2\quad\ldots\quad\ldots\ (3)$$

since $\quad\left(\frac{\sqrt{5}-1}{2}\right)^2=\frac{3-\sqrt{5}}{2}, \left(\frac{\sqrt{5}-1}{2}\right)^3=\sqrt{5}-2=(2+\sqrt{5})^{-1}.$

$$f\left(\frac{1}{\sqrt{5}}\right)=\frac{1}{8}\log\frac{3-\sqrt{5}}{2}\log\frac{\frac{3-\sqrt{5}}{2}}{\left(\frac{\sqrt{5}-1}{2}\right)^4}$$

$$-\frac{\pi^2}{30}+\frac{1}{2}\left(\log\frac{\sqrt{5}-1}{2}\right)^2+\frac{\pi^2}{12}$$

$$=\frac{1}{8}\log\left(\frac{\sqrt{5}-1}{2}\right)^2\log\left(\frac{\sqrt{5}-1}{2}\right)^{-2}$$

$$+\frac{1}{2}\left(\log\frac{\sqrt{5}-1}{2}\right)^2+\frac{\pi^2}{20}=\frac{\pi^2}{20}\quad\ldots\ (4)$$

Multiplying (2) by 3 we get the second result, but multiplying (4) by $\sqrt{5}$ we see that the right hand side of the first result should be $\frac{\pi^2}{4\sqrt{5}}$ only. This may be verified, both sides being equal to 1·1034553....

The result corresponding to (3) is

$$\sum_{n=0}^{\infty}\left(1+\frac{1}{3}+\frac{1}{5}+\ldots\frac{1}{2n+1}\right)(9+4\sqrt{5})^{-n}(2n+1)^{-1}$$

$$=\pi^2\frac{\sqrt{5}+2}{30}-\frac{\sqrt{5}+2}{24}\left\{\log(2+\sqrt{5})\right\}^2.$$

## Question 643.

(K. V. ANANTHANARAYANA SASTRY, B.A.)—Prove that

$$\frac{1}{2n+2}+\frac{1}{2}\cdot\frac{1}{2n+4}+\frac{1\cdot3}{2\cdot4}\cdot\frac{1}{2n+6}+\frac{1\cdot3\cdot5}{2\cdot4\cdot6}\cdot\frac{1}{2n+8}+\ldots=\frac{2\cdot4\cdot6\ldots2n}{3\cdot5\cdot7\ldots(2n+1)}$$

*Solution* (1) *by K. B. Madhava and K. J. Sanjana, M. A.*;

(2) *by R. Srinivasan, M.A and T. P. Trivedi, M.A., LL.B.*

(1) We can verify by term-by-term integration that

$$f(x, y)=\int_0^1 t^{x-1}(1-t)^y dt$$

$$=\frac{1}{x}-y.\frac{1}{x+1}+\frac{y(y-1)}{1\cdot2}\cdot\frac{1}{x+2}-\ldots$$

and that the latter converges if $y+1$ is positive (see Bromwich, p. 165, Ex. 31); so that putting for the present purpose $y=-\frac{1}{2}$ $x=n+1$, we have

$$\frac{1}{n+1}+\frac{1}{2}\cdot\frac{1}{n+2}+\frac{1\cdot3}{2\cdot4}\cdot\frac{1}{n+3}+\frac{1\cdot3\cdot5}{2\cdot4\cdot6}\cdot\frac{1}{n+4}+\ldots=\frac{\Gamma(n+1)\Gamma(\frac{1}{2})}{\Gamma(n+\frac{3}{2})}$$

$$=2.\frac{2\cdot4\cdot6\ldots2n}{3\cdot5\cdot7\ldots(2n+1)}$$

and the result is established.

(2) We have $(1-x)^{-\frac{1}{2}}=1+\frac{1}{2}.x+\frac{1\cdot3}{2\cdot4}x^2+\ldots$

$$\therefore \quad x^n(1-x)^{-\frac{1}{2}}=x^n+\frac{1}{2}.x^{n+1}+\frac{1\cdot3}{2\cdot4}.x^{n+2}+\ldots$$

If we integrate both sides, we get

$$\frac{x^{n+1}}{n+1}+\frac{1}{2}\cdot\frac{x^{n-2}}{n+2}+\frac{1\cdot3}{2\cdot4}\cdot\frac{x^{n+8}}{n+3}+\ldots=\int_0^x\frac{x^n}{\sqrt{1-x}}dx.$$

Putting $x=1$, twice the left side expression in the question

$$=\int_0^1\frac{x^n}{\sqrt{1-x}}dx;$$

Put $x=\sin^2\theta$; this integral

$$=\int_0^{\frac{\pi}{2}} 2\sin^{2n+1}\theta d\theta$$

$$=2.\frac{\Gamma(n+1)\Gamma\left(\frac{1}{2}\right)}{2\Gamma\left(n+\frac{3}{2}\right)}$$

$$=\frac{1\cdot2\cdot3\ldots\ldots n}{\frac{1}{2}\frac{3}{2}\frac{5}{2}\ldots\ldots\left(n+\frac{1}{2}\right)}$$

$$=2.\frac{2\cdot4\cdot6\ldots\ldots2n}{1\cdot3\cdot5\ldots\ldots(2n+1)}$$

### Question 644.

(K. V. Ananthanarayana Sastry, B.A.) :—Show that the length of the fourth positive pedal of a loop of the Lemniscate of Bernoulli is given by

$$18a\int_0^1 \frac{x^8dx}{\sqrt{(1-x^4)}}.$$

*Solution by K. B. Madhava; T. P. Trivedi, M.A., LL.B. and R. Srinivasan, M.A.*

The fourth positive pedal of the Lemniscate is

$$r^{2/9}=a^{2/9}\cos\tfrac{2}{9}\theta \text{ (Edwards: P. 166, Ex. 1)}$$

so that

$$\frac{ds}{d\theta}=a\cos^{7/2}\frac{2}{9}\theta;$$

and therefore the length of the loop is given by $9a\int_0^{\frac{\pi}{2}}\cos^{\frac{7}{2}}\phi\, d\phi.$

This is easily seen to be equal to

$$18\,a\int_0^1\frac{x^8dx}{\sqrt{1-x^4}}, \text{ by the substitution } \cos\phi=x^2.$$

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### Question 646.

(R. Srinivasan, M.A.) :—If $xy+yz+zx=1$, show that

$$\Sigma\{(1+x^2)^{\frac{1}{2}}-x\}\{(1+y^2)^{\frac{1}{2}}-y\}=1.$$

*Solution (1) by J. C. Swaminarayan, M.A. and K. B. Madhava, (2) by K. J. Sanjana, C. Bhaskaraiya and H. V. Venkataramniengar.*

(1) Here

$$1+x^2=x^2+xy+yz+zx=(x+y)(x+z).$$

$$\therefore\ \Sigma\{(1+x^2)^{\frac{1}{2}}-x\}\{(1+y^2)^{\frac{1}{2}}-y\}$$

$$=\Sigma(1+x^2)^{\frac{1}{2}}(1+y^2)^{\frac{1}{2}}-\Sigma(1+x^2)^{\frac{1}{2}}(x+y)+1.$$

$$=\Sigma(x+y)\{(x+z)(y+z)\}^{\frac{1}{2}}-\Sigma(x+y)(1+z^2)^{\frac{1}{2}}+1$$

$$=\Sigma(x+y)(1+z^2)^{\frac{1}{2}}-\Sigma(x+y)(1+z^2)^{\frac{1}{2}}+1$$

$$=1$$

(2) If $x=\tan\alpha$, $y=\tan\beta$, $z=\tan\gamma$ and $\alpha+\beta+\gamma=\frac{\pi}{2}$, then

$xy+yz+zx=1.$

In this case $\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)+\left(\frac{\pi}{4}-\frac{\beta}{2}\right)+\left(\frac{\pi}{4}-\frac{\gamma}{2}\right)=\frac{\pi}{2}$

and $\sum \tan\left(\frac{\pi}{4}-\frac{x}{2}\right)\tan\left(\frac{\pi}{4}-\frac{\beta}{2}\right)=1$ ... ... ... (2)

But $\tan\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)=\dfrac{1-\cos\left(\frac{\pi}{2}-\alpha\right)}{\sin\left(\frac{\pi}{2}-\alpha\right)}=\sec\alpha-\tan\alpha=(1+x^2)^{\frac{1}{2}}-x;$

from which the result follows at once.

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## Question 647.

(K. Appukuttan Erady, M.A.) :—The curves $\alpha=$constant, and $\beta=$ constant, where $\alpha$, $\beta$ are conjugate functions of $(x, y)$ pass through a point. Prove that the curvatures of the two curves at the point are

$$-\frac{\partial h}{\partial \alpha} \text{ and} -\frac{\partial h}{\partial \beta}$$

where
$$h=\left\{\frac{\partial(\alpha,\beta)}{\partial(x,y)}\right\}^{\frac{1}{2}}.$$

*Solution by J. C. Swaminarayan, M.A.*

We know that $\alpha_x=\beta_y$, $\beta_x=-\alpha_y$,

$\alpha_{xx}+\alpha_{yy}=0$, $\beta_{yx}+\beta_{yy}=0$, $x\alpha=y\beta$ and $x\beta=y_\alpha$.

By hypothesis, $\alpha+i\beta=f(x+iy)$

$\therefore \quad \alpha_x+i\beta_x=f'(x+iy)$

and $\quad f'(x+iy)\{x_\alpha+iy_\alpha\}=1.$

$\therefore \quad (\alpha_x-i\alpha_y)(x_\alpha+iy_\alpha)=1$

$\therefore \quad \alpha_x.\,x_\alpha+\alpha_y\cdot y_\alpha=1$ and $\alpha_y.\,x_\alpha-\alpha_x\cdot y_\alpha=0.$

$\therefore \quad x_\alpha=\dfrac{\alpha_x}{\alpha_x^2+\alpha_y^2}$ and $y_\alpha=\dfrac{\alpha_y}{\alpha_x^2+\alpha_y^2}.$

$$\therefore \quad -\frac{\partial h}{\partial\alpha}=-\frac{\partial}{\partial\alpha}\{\alpha_x\beta_y-\alpha_y\beta_x\}=-\frac{\partial}{\partial\alpha}\{\alpha_x^2+\alpha_y^2\}$$

$$=\frac{-[\alpha_{xx}(\alpha_x)^2+2\alpha_x\alpha_y\alpha_{xy}+\alpha_{yy}(\alpha_y)^2]}{(\alpha_x^2+\alpha_y^2)^{\frac{3}{2}}}$$

$$=\frac{\alpha_{xx}\alpha_y^2-2\alpha_x\alpha_z\alpha_{xy}+\alpha_{yy}\alpha_y^2}{(\alpha_x^2+\alpha_y^2)^{\frac{3}{2}}}$$

$=$curvature at the point of intersection on the curve. $\alpha=$constant.

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### Question 648.

(K. Appukuttan Erady, M.A.) :—Tangents TP, TQ are drawn to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, and the parallelogram TP SQ is completed. Prove that the area of the locus of S, as T completes a circuit along $\frac{x^2}{\lambda^2}+\frac{y^2}{\mu^2}=1$, is

$$\pi \lambda \mu+2\pi ab\left(\frac{a^2}{\lambda^2}+\frac{b^2}{\mu^2}-2\right).$$

*Solution by J. C. Swaminarayan, M.A.*

If TS is produced, it will pass through the centre C of the ellipse. If $\alpha+\beta$, $\alpha-\beta$ are the eccentric angles of P, Q

$$\frac{a\cos\alpha}{\cos\beta}=\lambda\cos\phi,\quad \frac{b\sin\alpha}{\cos\beta}=\mu\sin\phi,$$

since T lies on the ellipse $\frac{x^2}{\lambda^2}+\frac{y^2}{\mu^2}=1$.

Hence $\sec^2\beta=\frac{\lambda^2}{a^2}\cos^2\phi+\frac{\mu^2}{b^2}\sin^2\phi$.

The coordinates of S are $\left(\frac{a\cos\alpha}{\cos\beta}\cos 2\beta,\ \frac{b\sin\alpha}{\cos\beta}\cos 2\beta\right)$

Thus the polar co-ordinates of S will be

$$\left[\left\{(\lambda^2\cos^2\phi+\mu^2\sin^2\phi)^{\frac{1}{2}}\left(\frac{2}{\frac{\lambda^2}{a^2}\cos^2\phi+\frac{\mu^2}{b^2}\sin^2\phi}-1\right),\ \tan^{-1}\left(\frac{\mu}{\lambda}\tan\phi\right)\right\}\right]$$

Hence the area traced out by S in a complete circuit

$$=\int_0^{\frac{\pi}{2}} r^2 d\theta=2\lambda\mu\int_0^{\frac{\pi}{2}}\left\{\frac{2}{\frac{\lambda^2}{a^2}\cos^2\phi+\frac{\mu^2}{b^2}\sin^2\phi}-2\right\}^2 d\phi$$

$$=2\lambda\mu\int_0^{\frac{\pi}{2}}\left\{\frac{4}{\left(\frac{\lambda^2}{a^2}\cos^2\phi+\frac{\mu^2}{b^2}\sin^2\phi\right)^2}-\frac{4}{\left(\frac{\lambda^2}{a^2}\cos^2\phi+\frac{\mu^2}{b^2}\sin^2\phi\right)}+1\right\}d\phi$$

$$=2\lambda\mu\left[4.\frac{\pi a^3b^3}{\lambda^3\mu^3}\left(\frac{\lambda^2}{a^2}+\frac{\mu^2}{b^2}\right)-4.\frac{\pi ab}{2\lambda\mu}+\frac{\pi}{2}\right]$$

$$=\pi\lambda\mu+2\pi ab\left(\frac{a^2}{\lambda^2}+\frac{b^2}{\pi^2}-2\right)$$

*Additional Solutions by A. Narasinga Rao and T. P. Krishnanswami.*

# QUESTIONS FOR SOLUTION.

**706**. (N. Sankara Aiyar, M.A.) :—If $n$ is any number, show that

$$\frac{1}{(n+1)(n+2)}+\frac{1}{2}\,\frac{1}{(n+2)(n+3)}+\frac{1\cdot3}{2\cdot4}\,\frac{1}{(n+3)(n+4)}+\ldots$$
$$=2^{2n+2}\frac{\Gamma(n+1)\ \Gamma(n+2)}{\Gamma\quad(2n+4)}$$

**707**. (D. Krishnamurti) :—Evaluate the integral

$$\int_0^a ydy\int_0^y (a^3-x^3)^{\frac{1}{3}}dx.$$

**708**. (A. Narasinga Rao) :—Find the maximum and minimum values of $x+2y^2+3z^3+4w^4$, when $4x+3y^2+2z^3+w^4=62$.

**709**. (R. Srinivasan, M.A.) :—The common tangent to the nine points circle of a triangle ABC and the ex-circle opposite to A meets BC in D ; E and F are corresponding points for CA and AB. Show that AD, BE, CF are concurrent.

**710**. (R. Srinivasan, M.A) :—Investigate the motion of a rough circular hoop which rolls and describes a circle on a horizontal table keeping its plane inclined at a constant angle to the table.

**711**. (R. Vythynathaswami) :—A pack of $mn$ cards is dealt round face upwards in a circle so as to constitute $m$ heaps of $n$ cards each. The heaps are then placed one upon another in order beginning with the $r$th. heap ; and the cards are again dealt round starting as before. If the same operation is repeated after each dealing, investigate the number of shuffles necessary to bring the cards back to the original positions. Also, if $n>m$, find, after how many arrangements, each heap consists of cards which were in different heaps initially.

**712**. (J. C. Swaminarayan, M.A.) :—If $a>b$ and

$$f(a,b)=\int_0^\pi \log\,(a+b\cos\theta)d\theta, \text{ prove that}$$

$$f(a,b)=\tfrac{1}{2}f\left(a^2-\frac{b^2}{2},\ \frac{b^2}{2}\right)=\pi\log\left(\frac{a+\sqrt{a^2-b^2}}{2}\right).$$

**713**. (*Communicated by* K. V. Anantanarayana Sastri, B.A.) :— Find the value of the infinite continued fraction

$$\frac{1}{1+}\frac{1}{3+}\frac{16}{5+}\frac{81}{7+}\cdots\frac{n^4}{2n+1}+\cdots\cdots$$

**714**. (K. B. MADHAVA) :—Find the co-efficient of $x^n$ in

$$\frac{x}{1+x}-\frac{x^2}{1+x^2}+\frac{x^3}{1+x^3}-\frac{x^4}{1+x^4}+\ldots\ldots$$

**715**. (MARTYN M. THOMAS, B.A.) :—Prove that

$$\int_0^{\frac{\pi}{2}} (\sin x \log \sin x)(\cos x \log \cos x)dx=\frac{12-\pi^2}{48}.$$

**716**. (S. KRISHNASWAMI AIYANGAR, B.A.) :—Shew how to find the sum of

$$1+\frac{a}{1!}+\frac{2^r a^2}{2!}+\frac{3^r a^3}{3!}+\ldots\frac{p^r a^p}{p!}+\ldots$$

**717**. (C. KRISHNAMACHARY) :—Shew that

$$\phi(x+h)-\phi(x+3h)+\phi(x+5h)-\ldots$$
$$=\phi(x-h)-\phi(x-3h)+\phi(x-5h)-\ldots$$

**718**. (S. NARAYANA AIYAR, M.A.) :—If

$$A_n=\frac{(a-b)(a-bx)(a-bx^2)\ldots\ldots\ldots(a-bx^{n-1})}{(1-x)(1-x^2)(1-x^3)\ldots\ldots\ldots(1-x^n)}$$

and
$$B_n=\frac{(b-a)(b-ax)(b-ax^2)\ldots\ldots\ldots(b-ax^{n-1})}{(1-x)(1-x^2)(1-x^3)\ldots\ldots\ldots(1-x^n)},$$

then show that

(1) $A_n+A_{n-1}\cdot B_1+A_{n-2}\cdot B_2+\ldots\ldots+A_2B_{n-2}+A_1B_{n-1}+B_n=0.$

(2) $A_1\cdot B_n+2A_2\cdot B_{n-1}+3A_3\cdot B_{n-2}+4A_4\cdot B_{n-3}+\ldots\ldots(n+1)\cdot A_{n+1}$
$$=\frac{a^{n+1}-b^{n+1}}{1-x^{n+1}}.$$

**719**. (J. C. SWAMINARAYAN, M.A.) :—Determine the limit, when $n\to\infty$, of

$$\frac{e^n\{1\cdot3\cdot5\ldots\ldots(2n-1)\}}{(2n+1)^n}.$$

**720**. (A. C. L. WILKINSON, M.A., F.R.A.S.) :—In a paper in the J. I. M. S. Vol. IV, p. 162, on "Approximations in Curve Tracing", Mr. M. T. Naraniengar discusses two examples of cubic asymptotes. Prove the following results;

(i) For the curve $x^2(y-x)^3=ay^4$, the family of cubics

$$(y-x)^3=36^2-8xy+xy^2-13x+17y+c$$

has contact of the 7th order at infinity, the cubic having contact of the 8th order consisting of the line at infinity repeated three times.

(ii) For the curve $x(y-x)^3=ay^2(3x-y)$, the cubic

$$(y-x)^3=3ax^2y-ax^2-3ay+a^2x+2a^3$$

has contact of the 9th order at infinity.

**721.** (A. C. L. Wilkinson, M.A., F. R. A. S.):—A straight line AB of constant length slides between two rectangular axes, and circles are drawn touching the axes and the line AB; show that the locus of the points of contact of these circles with AB consists of four trinodal quartics whose equations referred to the triangle formed by the nodes is

$$\alpha^2\beta^2+\beta^2\gamma^2+\gamma^2\alpha^2-4\alpha\beta\gamma(\alpha+\beta+\gamma)=0.$$

**722.** (S. Ramanujan):—Solve completely $x^2=a+y$; $y^2=a+z$; $z^2=a+u$; $u^2=a+x$; and deduce that if

$$x=\sqrt{5+\sqrt{5+\sqrt{5-\sqrt{5+x}}}},$$

then $$x=\tfrac{1}{2}\left(2+\sqrt{5}+\sqrt{15-6\sqrt{5}}\right);$$

and if $$x=\sqrt{5+\sqrt{5-\sqrt{5-\sqrt{5+x}}}},$$

then $$x=\tfrac{1}{4}\left(\sqrt{5}-2+\sqrt{13-4\sqrt{5}}+\sqrt{50+12\sqrt{5}-2\sqrt{65-20\sqrt{5}}}\right)$$

**723.** (S. Ramanujan):—If $[x]$ denotes the greatest integer in $x$ and $n$ is any positive integer show that

(i) $$\left[\frac{n}{3}\right]+\left[\frac{n+2}{6}\right]+\left[\frac{n+4}{6}\right]=\left[\frac{n}{2}\right]+\left[\frac{n+3}{6}\right]$$

(ii) $$\left[\frac{1}{2}+\sqrt{n+\tfrac{1}{2}}\right]=\left[\frac{1}{2}+\sqrt{n+\tfrac{1}{4}}\right]$$

(iii) $$\left[\sqrt{n}+\sqrt{n+1}\right]=\left[\sqrt{4n+2}\right]$$

**724.** (S. Ramanujan):—Show that

(i) $$\tan^{-1}\frac{1}{2n+1}+\tan^{-1}\frac{1}{2n+3}+\tan^{-1}\frac{1}{2n+5}+\ldots\text{to } n \text{ terms}$$
$$=\tan^{-1}\frac{1}{1+2\cdot1^2}+\tan^{-1}\frac{1}{1+2\cdot2^2}+\tan^{-1}\frac{1}{1+2\cdot3^2}+\ldots\text{to } n \text{ terms}.$$

(ii) $$\tan^{-1}\frac{1}{(2n+1)\sqrt{3}}+\tan^{-1}\frac{1}{(2n+3)\sqrt{3}}+\tan^{-1}\frac{1}{(2n+5)\sqrt{3}}+\ldots\text{to } n \text{ terms}$$
$$=\tan^{-1}\frac{1}{(\sqrt{3})^3}+\tan^{-1}\frac{1}{(3\sqrt{3})^3}+\tan^{-1}\frac{1}{(5\sqrt{5})^3}+\ldots\text{to } n \text{ terms}.$$

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