Conference Number

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A Quarterly Dedicated to the service of Students and Teachers of Mathematics in India.

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and others.

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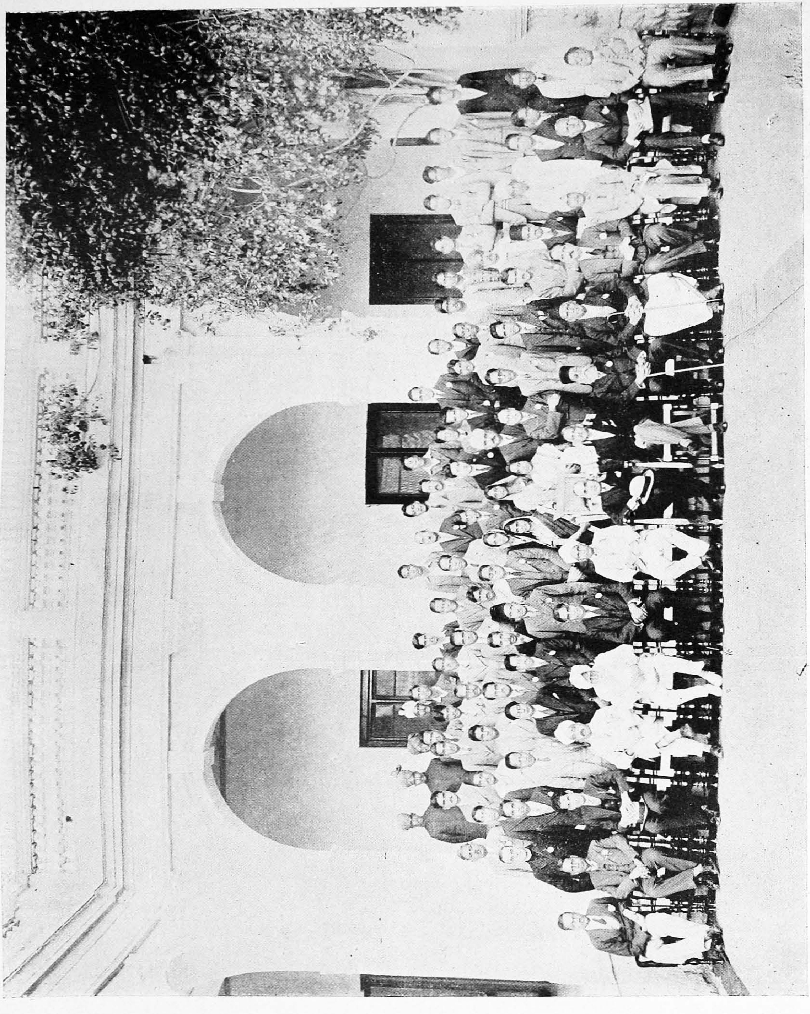

THE INDIAN MATHEMATICAL SOCIETY

NINTH CONFERENCE, DELHI.

18th December, 1935.

Top Row : Peon Peon Peon Peon.

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THE MATHEMATICS STUDENT

Volume IV] MARCH 1936 [Number 1

The Ninth Conference

OF THE

Indian Mathematical Society

The Indian Mathematical Society was invited by the University of Delhi and held its Ninth Conference, on the 18, 19, 20 and 21st December at the Old Viceregal Lodge.

On this occasion, the Society was fortunate in securing the gracious patronage of the *Hon'ble Sir Girja Shankar Bajpai,* Member of Education, Health and Lands, Government of India, who inaugurated the Conference and of the ***Hon'ble** Mr. J. N. G. Johnson,* Chief Commissioner, Delhi. Another happy feature was that the Founder of the Society *Mr. V. Ramaswamy Aiyar* (who was 65 years of age) attended the Conference and also contributed two original papers.* Owing to the illness of the President Mr. H. G. Gharpurey, and at his request, the Conference was presided over by Dr. R. Vaidyanathaswami whose Presidential address was devoted to a review of recent Physics from the point of view of a Pure Mathematician.

There were three public discourses, the first by *Dr. G. S. Mahajani,* Principal, Fergusson College, Poona on 'Science and Fatalism', the second by *Dr. R. Vaidyanathaswamy* on 'The Place of Mathematics in the Present Day Culture' and the third by *Prof. A. C. Banerji* of the University of Allahabad on 'Cosmography' illustrated with lantern slides.

An excursion was arranged to the observatory known as Jantar Mantar (Jai Singh's observatory with masonry instruments) where Rao Bahadur Chotey Lal kindly took the delegates round and explained its mysteries in his inimitable style, and to historic relics of Delhi such as Humanyun's Tomb and the Qutab Minar.

With the kind permission of the authorities of the Delhi Broadcasting Station a talk on the life and work of *Srinivasa Ramanujam* was given by the President on the evening of the 21st December.

An interesting and useful discussion on 'The Teaching of Mathematics in Schools and Colleges' was also held, in which delegates and Teachers of many local Schools participated.

About 40 papers were contributed to the Conference.

* Ramaswami Iyer died in January 1936 soon after his return to Chittor.

The Social side of the Conference activities included two Garden Parties arranged by *Rai Bahadur Ram Kishore,* Vice-Chancellor, University of Delhi and by *Khan Bahadur Dr. Sir Mohammad Abdur Rahman,* besides a large number of occasions on which the Reception Committee was at home to the delegates.

The Conference was attended by a large number of delegates from different parts of the country.

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Daily Programme

Wednesday, 18th December 1935

4 p.m. (Delhi Time) — Welcome Address by Rai Bahadur Ram Kishore, Chairman of the Reception Committee.
Inauguration of the Conference by the Hon'ble Sir Girja Shankar Bajpai.
Group Photo: Delegates and Members.

4-30 p.m. — Garden Party by Rai Bahadur Ram Kishore, in the University Gardens, Old Viceregal Lodge.

Thursday, 19th December 1935

11 a.m.—1 p.m. — Inaugural Address by Dr. Vaidyanathaswami.
Reading of the Secretary's Report.

1—2 p.m. — Lunch to Delegates.

2—4 p.m. — Business Meeting of the Society.

4-30—6 p.m. — Reception Committee "At Home" to the Delegates in the University Gardens, Old Viceregal Lodge.

6-30—7-30 p.m. — Public Lecture on "Science & Fatalism," by Dr. G. S. Mahajani, St. Stephen's College Hall, Kashmere Gate, Delhi.

Friday, 20th December 1935

10 a.m.—1 p.m. — Reading of Papers.

1—2 p.m. — Lunch to Delegates.

2—6 p.m. — Excursion to Humayun's Tomb and Qutab Minar.
Tea by the Reception Committee in the Qutab Gardens.

6-30—7-30 p.m.	Public Lecture on "The Place of Mathematics in Present-day Culture," by Dr. Vaidyanathaswamy, in Hindu College Hall, Kashmere Gate, Delhi.

Saturday, 21st December 1935

10 a.m.—1 p.m.	Reading of Papers.
1—2 p.m.	Lunch to Delegates.
2—4 p.m.	Discussion on "The Teaching of Mathematics in Schools and Colleges".
4-30 p.m.	Garden Party by Khan Bahadur Dr. Sir Mohammad Abdur Rahman, at his residence 26, Ferozeshah Road, New Delhi.
6-30—7-30 p.m.	Public Lecture on "Cosmography" illustrated by lantern slides, by Prof. A. C. Banerji, in St. Stephen's College Hall, Kashmere Gate, Delhi.

Address of Welcome by Rai Bahadur Ram Kishore, Chairman of the Reception Committee.

Sir Girja Shankar, Delegates, Ladies and Gentlemen,

It is my pleasant duty to extend to you a cordial welcome to our historic city. To those of you who are not accustomed to the rigour of a Northern winter, Delhi may appear a little inhospitable, and we regret that it was not possible to select a more agreeable time of the year for this conference; but I trust the warmth of scientific discussions will help you to overlook the inclemency of our climate. The Reception Committee are aware of their own limitations and I hope you will forgive any shortcomings in their arrangements.

I am afraid, Delhi cannot boast in its history of any first-rate mathematicians who can be said to rank with Brahmagupta and Bhaskaracharya of old, or with Srinivasa Ramanujan and Sir Shah Mohammad Sulaiman of modern days. This city, however, is not without some title to mathematical fame. One has only to look around to discover in our ancient architectural monuments almost baffling results of mathematical knowledge of the highest order. The names of those whose mathematical genius contributed to the designing of such titanic monuments as the Qutab Minar and Humayun's Tomb may be lost to us, but we cannot pass over the results of their labour. Ustad Hamid, who designed the Jami Masjid as well as the Red Fort and its palaces, is immortalized in these buildings by the testimony they bear to his deep knowledge of Applied Mathematics. Even in more recent years the name of the late Ram Chandra who is distinguished for his original work on Maxima and Minima, published in 1850, which was so highly praised by well-known mathematicians and that of Shams-ul-Ulema Munshi Zakaullah whose mathematical works were prescribed as text-books by Allahabad University fifty years ago may be mentioned in support of Delhi's claim to a place, however small, in the history of Mathematics. The observatory constructed under the supervision of Maharaj Sawai Singh and known as "Jantar Mantar" which I hope you will see during your stay here is also a monument of great astronomical knowledge.

Every Indian who is aware of ancient India's great achievements in Mathematics will be deeply interested in the country's progress in this branch of knowledge. India, as all know, was renowned even in the far-off days of antiquity as a most distinguished leader in the science of Mathematics. " Ganit ", or Mathematics, reached in the ancient Indian centres of learning a remarkable height, which some think it difficult to surpass. The formulæ and calculi which they evolved for astronomical calculations, their vast knowledge of the theory and properties of numbers, geometry, trigonometry and conics may even today prove fruitful fields for research.

Anybody who knows anything of matters educational cannot be ignorant of the value of Mathematics in the sphere of intellectual discipline. It is indispensable for a first class education, because it helps to cultivate habits of clear and precise thinking and sound reasoning, and trains our concrete mentalities to deal with practical facts no less than with abstractions and generalities.

Mathematics has been universally regarded as essential in all schemes of education. A science which cannot state its methods and conclusions in mathematical terms is rightly held to be only half a science. According to a well-known author, " what is physical is subject to the laws of Mathematics, and what is spiritual to the laws of God, and the laws of Mathematics are but the expression of the thoughts of God ". In ancient India this conception was commonplace, as is illustrated beyond cavil and doubt by the theory of Brahma's Day, as divided into Chaturyugs, and Manvantaras.

Mathematics is playing a role of increasing importance in higher education. Many students pursuing a cultural course find it necessary for their enlightenment to study Mathematics to a greater extent than they did formerly. It is indispensable to the proper study of Physics, Engineering and Actuarial Sciences. Chemistry and Biology also are rapidly becoming exact sciences. All chemical phenomena are being interpreted in terms of Physical Chemistry, for the mastery of which a knowledge of Calculus is absolutely necessary. Mathematical treatment of various topics in Economics, such as the 'Theory of Value', 'Monopoly', 'Competition', 'Foreign Exchange', 'Population and Mortality Curves' is attaining a position of more and more importance, and mathematical methods are rapidly being applied in Logic and Psychology. The same is true of Pedagogy, Psycho-Physics and even Sociology, according to the authority of those who know these subjects.

The teaching of Mathematics contributes a great deal to those qualities and habits of the mind which make for intellectual greatness and no less for the building of character, the need for which as a part of education is universally recognised. In order that India may take her place in the front rank of the countries of the world, it is necessary that our Indian mathematicians should not only learn and assimilate what other countries are achieving in the field, but that they should strive to make their own contributions to the stock of existing knowledge and occupy prominent places in at least some branches of the subject. I have noted with great satisfaction, if I may say so without impertinence, that your society has not only undertaken to make striking achievements in the higher and more abstruse aspects of this subject but also devotes itself to the improvement of the methods of teaching Mathematics in schools and colleges, to its application to the social and other sciences, and to the bringing together of students and teachers of the subject in this country, through the medium of your journal 'The Mathematics Student'.

In these days of progress, when the stock of existing knowledge is so vast, and when new knowledge is being added almost every day, it is impossible for any one worker, however brilliant he may be, to master all the branches of Mathematics. He has to confine himself to only a few branches of the subject; but in order to avoid the narrow-mindedness which specialization may produce it is desirable that research workers should have opportunities of coming into contact with men who are working in similar, though not identical fields, so that they may get fresh ideas and new points of view or revive their interest in other branches of the subject. This object can be achieved in the best possible manner by holding conferences such as this. I have every hope that this Conference will afford an incentive to the students of Mathematics in this Province to undertake original work and to contribute materially to the advancement of the subject.

I am very thankful to you all for accepting our invitation and coming to Delhi for this Conference. Let me also express our deep obligation to the Hon'ble Sir Girja Shankar Bajpai who has so kindly found time, in the midst of his multifarious activities, to open our Conference. As an eminent scholar and thinker and as the Member for Education, I am sure he will be interested in this subject which forms an important part of every educational curriculum.

May I now request you, Sir, to open formally the Conference?

Sir Girja Shankar Bajpai's Speech declaring the Conference open.

In opening the Conference, Sir Girja Shankar observed:

"I extend to you a welcome of which the warmth is tempered only by some trepidation. You are all eminent in a science in which my last academic performance was marked by complete failure, for I did not score a single mark out of a substantial total of 1,200. The blame for the incongruity of my presence here in inauguratory prominence, however, lies at other doors than mine; though inept in many ways, I am not presumptuous.

"After the confession that I have made, you will not be surprised at the brevity and jejuneness of my contribution to your proceedings. Mathematics has made giant strides since I lost my barren touch with it. New gospels and new prophets have appeared on the horizon; relativity and Einstein are single but the most striking examples of each.

"Newton, according to some worshippers of new creed, is a back number. Judged by mere human standards, there is something to be said for this view. Newton discovered gravitation but lost a sweetheart; Einstein has found both relativity and a wife. However, I profess to be no judge of mathematical values, or, indeed, of the emotional values of mathematicians. The muses of the square and the cube have an ideology of their own.

Lest these observations seem somewhat flippant, let me say that they are designed merely to lighten the strain of your impending labours. A spicy *hors d'oeuvre* may be no unwelcome preparation for the strong meat that is to follow. In any case, let me assure you that this genial persiflage is no sign of irreverence. Even to those of us who have no aptitude for its higher flights, mathematical training is a most valuable mental discipline. The Greeks realized this and Plato enjoined a course of it as a universal educational desideratum. There is no finer corrective to loose and inconsequent thinking.

"What the experimental sciences owe to it is no mystery even to a layman like myself. Someone, with a turn for epigram, described mathematics as the shorthand of the sciences; that is not even a half-truth. Shorthand is mere mechanics; a calligraphical summarization. Mathematics, in its higher form, imparts to observed experimental phenomena a philosophical unity; it integrates, it generalizes and it predicts. Mathematics, thus, is not a handmaid but a leader among the sciences, demanding for its fruitful pursuit the highest qualities of the mind: clarity, directness, vigour, imagination.

"India is the birthplace of this science. The Indian mind is rich in the qualities needed for its productive study. Indians like the late lamented Ramanujan have shown that the fathers of the old are not incapable of mastering and furthering the new mathematics. Your society, gentlemen, includes some of the best mathematicians in the land. Its co-operative activity constitutes India's continuous and most substantial contribution to mathematical studies. We in Delhi are deeply conscious of the honour that you have done us in choosing our city for this year's annual session. We wish you every success in your work and such recreation in these historic but also modern surroundings as your own inclination may permit and, let me add, the importunity of your hosts may impose."

Secretary's Report for the Period 1933-35.

LADIES AND GENTLEMEN,

I have the honour to submit the following report on the working of the Society during the period that has elapsed since the Silver Jubilee Conference at Bombay in December 1932.

First of all it is my painful duty to record the heavy loss which the Society has sustained in the death of Rao Bahadur P. V. Seshu Aiyar, one of our most enthusiastic and active Ex-Presidents. All the members who attended the Bombay Conference must be remembering him well.

Turning to routine matters, I have to report two important changes. The first is in the mode of publishing our Journal which used to be published six times a year. In accordance with the decisions arrived at in the business meeting of the members of the Society held at Bombay, it has now been split into two parts each of which appears four times a year, under a different title. The first part which retains its old appellation, *The Journal of the Indian Mathematical Society*, publishes only original papers and caters for the needs of workers in the field of advanced Mathematics. It is under the fostering care of Dr. Vaidyanathaswami of the Madras University. The other part under the title *The Mathematical Student* is edited by Prof. A. Narasinga Rao of the Annamalai University and meets the requirements of humbler workers whose main activities centre round teaching of Mathematics in Colleges.

The second noteworthy change introduced since the last Conference is the alteration in the rules governing the compounding fee for the life membership. This also was in deference to the wishes of the majority of the members assembled at Bombay. The rules have been modified so that the compounding fee will gradually diminish as the period of membership increases and many members have already taken advantage of the new rules. Their immediate effect of this, however, has been to cut down the annual income; but the managing committee hopes that with the willing co-operation of all the members, the deficit will be made up by enlisting new members.

The circulating library is flourishing under the paternal care of Prof. V. B. Naik and his colleague Prof. Shrintre of Fergusson College, Poona.

The membership of the society does not show any substantial variation worth reporting.

Another notable fact is that the Society has been able to enlist the sympathy of two other Indian Universities viz. The Nagpur University and the Osmania University. The Universities of Bombay and Madras have for a long time in the past, given financial support to the activities of the Society by making substantial annual grants to our Society and Annamalai University joined them later on. During the period under report the managing committee has succeeded in securing grants from Nagpur University and Osmania University of Hyderabad Deccan.

My last but not the least duty is to record, on behalf of the managing committee, our thanks to the Vice-Chancellor of Delhi University for giving us this opportunity of assembling here under the auspices of the University by his kind invitation.

The managing committee also thanks the various Indian Universities that have rendered financial help in the past and have contributed materially to the success of this Conference by sending Delegates.

S. B. BELEKAR,
Hon. Secretary,
Indian Mathematical Society.

Business Meeting of the Society.

Chairman: DR. R. VAIDYANATHASWAMY.

1. After the Chairman's introductory remarks, Prof. A. Narasinga Rao moved a condolence resolution expressing the Society's appreciation of the great services rendered by Rao Bahadur P. V. Seshu Aiyar and its sense of loss at his demise. The mover recounted how Seshu Aiyar had helped the Society as Assistant Secretary, Joint Secretary and later as its President, and the powerful influence he exerted in raising the standard of Mathematical teaching in High Schools and Colleges in South India by stressing the need for an adequate acquaintance on the part of teachers with the fundamentals of Mathematics and of the requirements of rigour.

The resolution was passed, all the members standing.

2. Prof. Narasinga Rao next moved another condolence resolution relating to the death of K. J. Sanjana, an active member of the society from its very inception. Prof. Sanjana, said the mover, should have become the President of the Society but he declined the honour owing to failing eyesight. Though he became totally blind he continued to contribute problems to the Journal till his very death.

The resolution was passed, all the members standing.

3. The Treasurer Prof. L. N. Subramaniam explained the financial position of the Society and stated that several members had taken advantage of the reduced rates of life composition to become life members. In order that the financial position of the Society might be on a sound basis, it was necessary, he declared, to increase the membership of the Society and appealed to all present to help in enrolling members.

4. Several suggestions were offered and discussed relating to the Journal, the circulation of periodicals from the Library the issue of books, and the printing of a Catalogue.

Discussion on the Teaching of Mathematics.

Saturday, 21st December 1935. 2-30 P.M.

Chairman:—DR. R. VAIDYANATHASWAMY.

B. N. PRASAD (Allahabad) initiating the discussion remarked that Mathematics was dreaded by many students because it was taught in a dry and mechanical manner. The remedy lay in making it more interesting by means of anecdotes (the reason for 31 days in July and August for example in the case of Junior pupils), illustrations and by giving it a historical background, so that it would not look like a set of rules imposed from without.

The knowledge of fundamentals, on the part of many teachers was very poor. It was very necessary that teachers should have clear ideas about topics like limits and generally as to what was meant by "rigour."

Our syllabuses and courses of study were out of date and based too much on the English model; we were losing much by not more freely using continental methods.

A. NARASINGA Rao (Annamalainagar) felt that in keeping up the teacher's interest in his subject and giving him the modern outlook, books like those by Felix Klein, particularly his "Elementary Mathematics from an advanced standpoint" of which an English translation was available, should be very helpful. It was a pity such books were not as freely used as they deserved to be.

Standards of rigour at different stages differed in how much was taken for granted as basic and the rest proved as logical deductions therefrom. The teacher should be clear about the standpoint at each stage and the pupil should know what was being proved (according to his standpoint) and what was being assumed without proof.

RAM BEHARI (Delhi) was for a closer correlation between Arithmetic and Algebra than was maintained at present. As regards rigour, the question before them was whether the logical principles behind each working rule should be taught or whether a process could be taught even if its rationale could not be explained.

A. C. BANERJI (Allahabad) maintained that the aim in Elementary Mathematics should be mainly utilitarian and could become academic only at the University stage.

Proceeding to criticise the teaching of the several subjects, he said that in Arithmetic too much attention was devoted to vulgar fractions as compared to decimals; in Algebra the conventional teaching was soulless and insipid, while in geometry there was a great confusion of ideas.

As regards examinations, the question papers should aim at encouraging sound work and breadth of reading ; there should be some questions set beyond what was necessary for testing the average student.

SETH (Delhi) feared that a stress on Philosophical questions and the abstruse foundations would scare away students from the subject.

SITARAM (Lahore) argued that the underlying principles must be explained and not merely the working rule, say for extracting square roots.

D. S. KOTHARI (Delhi) deplored the dearth of good text-books.

G. D. RAJPAL (Delhi) remarked that in teaching loci in geometry many students did not appreciate why two proofs were necessary, one to show that every point on the locus satisfied the given conditions and the other to show that every point satisfying the conditions lay on the locus.

S. K. ABHAYANKAR opined that in subjects like mensuration, formulae had to be given without proof. Notions which would be considered wrong from an advanced point of view had to be taught, all the same, and were right so far as school teaching was concerned. The teacher should not become the slave of rigour.

RAMCHANDER (Delhi) complained that the encouragement given to Indian publications was producing a harvest of hastily and badly written, ill-arranged text-books with a lot of misprints and miscalculations in them.

B. RAMAMURTHI (Annamalainagar) said that very few teachers had a real interest in the subject or sought to keep in living touch with it. In western countries eminent Mathematicians like Prof. Hardy, Prof. Whittaker delivered lectures intended for school teachers; such fruitful contact was rare here and the blame lay mostly with College teachers. Discussions on the teaching of particular topics like " logarithms " etc. should be held as in England in which the highest grade of University teachers should participate. There was need for the teacher's horizon to be enlarged.

The Chairman, DR. VAIDYANATHASWAMI, in winding up the discussion thanked all those who had participated in it and said that such discussions would be more useful if made systematic and confined to selected items.

Public Lecture on Cosmography

BY

Prof. A. C. BANERJI, Allahabad University.

Saturday, 21st December 1935. 6-30 P.M.

From time immemorial the depths of the Universe have fascinated the poets and astronomers alike. Each expedition into remote space has led to new discoveries. The immense development of instrumental methods in recent years has vastly enlarged the possibilities of continuous progress in Astronomy. An astronomer now feels like Saul " who went out to seek his father's asses and found a kingdom ".

January 7, 1610 was a fateful day for mankind. On that day Galileo sat in the evening to see for the first time Jupiter and its satellites through the telescope which he himself had made. It was the date from which the telescopic Astronomy may be said to have begun. Galileo had taught the Copernican theory that the planets move round the Sun and he could now demonstrate that Jupiter resembles the Sun of Copernicus set in the centre of a miniature solar system. In doing so he literally took his life in his hands. Only ten years before, Giordano Bruno, a disciple of Copernicus had been burned at the stake in Rome. Galileo himself was denounced by the Holy Inquisition in 1633, and, under threat of torture, was forced to retract his teachings. The telescope of Galileo was quite a toy in comparison with a modern telescope.

The largest telescope at present is located at Mount Wilson in California. Its reflecting mirror has a diameter of 100 inches; (tube-43 ft. in length and 10 ft. in diameter). It admits 250,000 times as much light as the unaided eye. A 200-inch telescope will shortly be erected at Mount Palovar in California which will give a million times as much light as the unaided eye. Burnett and Pease have already made tentative designs of the same. It is of the fork type.

I shall now try to say something about the geography of the Universe about us, or rather its cosmography. In order to make a trigonometrical survey of the Heavens, I would request the audience here to take a trip with me in the depths of space. A navigation of the Heavens is full of immense interest. But we have to cruise the Universe with maximum speed which we possibly can have i.e., with the velocity of light viz. about 186,285 miles per second.

The idea of flying to the Moon is not new. In a second and a half we shall reach the Moon which is at a distance of 240,000 miles from us—we shall see deserts and planes (Mare), craters, mountain chains, and peaks—we shall find no life, no vegetation, and no atmosphere. About one hundred years ago, a New York newspaper perpetrated the "grat moon hoax". It published a series of wholly fraudulent articles claiming to describe the Moon as seen through a giant new telescope in Africa. They described trees of amazing growth, weird animals, and flying men. These articles increased the circulation of a little known newspaper so much that it claimed to have the largest circulation of any paper in the world. This shows how credulous mankind is!

In eight minutes we shall reach the Sun after travelling through 91 million miles. It has a surface temperature of 5000 degrees C, and at the centre it may have a temperature of 50,000,000 degrees C. You will have to be transformed into a "Silicaceous animal" in order that you may not get burnt to ashes at the surface. Solar prominences or huge fountains of flame sprout out like "Jack the giant-killer's bean stalk" with speeds of thousands of miles a minute.

Faculae are bright streaks and patches on the surface of the Sun, and are probably clouds of vapours or gases. Sun spots are relatively dark spots on the Sun's surface and are short-lived. The change of positions of sun spots readily demonstrates that the Sun is rotating from East to West.

In our journey we come across the planets Venus and Mars. Venus is so densely surrounded by atmosphere that if we take an infra-red photograph of the planet, we find that even infra-red light which has greater penetrating power than ultra-violet light fails to find a solid surface. Mars has also a distinct atmosphere and extends to a considerable height as is shewn by the fact that ultra-violet image is distinctly larger than infrared one.

Many astronomers have seen markings on the surface of Mars. Schiaparelli and Lowell think that they are straight markings or canals which have been artificially constructed and have cases connected with them. Whereas Bernard and Antoniadi think that these so-called straight markings are nothing but disconnected dark patches on a dimly lighted object which appear connected by straight lines to the eye which is struggling to study outlines in faint light. Certain seasonal changes are undoubtedly observed on Mars. Polar show caps shrink considerably in summer and become much bigger in winter. Recent spectral observations show that there is hardly any oxygen in Martian atmosphere. Vegetation may grow on Mars—but no animal life of the type found on this Earth is possible there.

Jupiter is the largest planet of the Solar System. Ultra-violet and violet photographs of this planet give many details and show that Jupiter has got an atmosphere. Clouds of carbon dioxide float in atmosphere. Belts on its surface are temporary markings showing they are atmospheric in nature and have vigorous circulation. It has nine satellites.

With its ring system Saturn perhaps forms the finest object in the sky. The rings are supposed to be fragments of a former shattered moon of Saturn. If a small body rotates about a big body and if the radius of its orbit happens to become less than 2·45 times that of the large body, then the small body will be shattered into tiny bits. This critical distance is called the Roche's limit. The outermost ring of Saturn has got a radius equal to 2·30 times the radius of the planet. It has got also nine satellites. After unthinkable ages our Moon will also be drawn down to within about 10,000 miles of our Earth and the former will be shattered into fragments which will then form a system of tiny satellites revolving round the Earth. Asteriods are also shattered remnants of a primeval planet which came within the danger zone of the Sun.

Pluto is the outermost planet of the Solar System and is rightly called its gate-keeper. Its distance from the Sun is $3{\cdot}7 \times 10^9$ miles. We shall reach Pluto with the speed of light in less than six hours. It was discovered on the 5th of March 1930, at the Lowell Observatory.

Most of the planets are accompanied by retinues of satellites proportional in number to the size and dignity of the planet—Jupiter and Saturn each have nine, whilst Uranus which comes next in size has four.

In cruising through the Solar System we also come across celestial bodies called con ets which have fantastic shapes. Comets are minute bodies which are kept together by gravitational pull : (Donati's comet, Coggia's comet, Halley's comet). Meteors are supposed to be broken up fragments of comets when they came within the danger zone of the Sun.

Now we leave the Solar System and as we proceed on our journey we meet nothing but dust and cosmic radiation for four years and a quarter till we reach the nearest star. Our Solar System may be compared to a *city with its suburbs*. Outside the suburbs we get wide tracts of uninhabited land till we come to the nearest town. Our nearest star is our neighbouring *town*. The distance of the nearest star is a million times as great as the radius of orbit of the planet nearest to the Sun. To measure such great distances and still greater distances, it is necessary to use a convenient unit of the distance. One convenient unit is the light-year i.e., the distance covered by light in one year moving with the velocity of 186,285 miles per sec. The light-year is about six billion miles or more accurately about $5{\cdot}80 \times 10^{12}$ miles. There is another convenient unit of distance called parsec i.e., the distance of a star whose parallax is one sec. It is about 3·23 light-years. Proxima centauri, the nearest star is about 25 billion miles from us, i.e., about 4·27 light-years from us. It takes light about 4·27 years to traverse this immense gap.

A wireless signal travels with the same speed as light. Its speed is about a million times that of sound. When a speaker broadcasts from Delhi, his voice takes longer as a sound wave to travel to the back of the Hall than it does to travel as an electric wave to Allahabad or Calcutta. But even the nearest star is at such a great distance that the inhabitants of Proxima centauri would hear a terrestrial concert four years and a quarter after it has been broadcasted from the Earth. There are even distant and more distant stars which a terrestrial wireless wave would not have yet reached had it begun its journey in the palmy days of Mahabharat or Mahenjodaru or before the pyramids were built or even before Man appeared on Earth: A little later on i.e., in $4\frac{1}{2}$ years' time we come to the twin stars and centauri. In eight years' time we come to Sirius. The larger of the pair Sirius A, is *apparently* the brightest star in the Heavens. It emits twenty-nine times the light of the Sun. Its more wonderful companion Sirius B is a dwarf star. Its diameter is 3 times that of the Earth and its weight $\frac{3}{4}$ times that of the Sun. It consists of very dense matter, the average density being about 60,000 times that of water. One cubic inch of material in Sirius B or the amount which can be put in an ordinary match box will contain about a ton of matter. In fifteen years' time we shall reach another big star called Altair in Constellation Aquilae. Now let me explain how the distances of these stars are measured. The year 1838 provides another landmark in the history of Astronomy. It was the year when the distance of a star (61 cygni) was first measured accurately by Bessel. The apparent swinging motion of the stars which results from the Earth's orbital motion is called the "parallactic motion". The amount of the parallactic motion of any star enables us to calculate its distance from us. The width of the Earth's orbit i.e., the distance of 185 million miles makes a base line from each side of which a star is observed at an interval of six months. Half the angular displacement with reference to very faint and remote stars which appear fixed in position is now measured and the distance of the near star is calculated. By this method we can measure the distances of only such stars as lie within a radius of 300 light-years or so.

After leaving Altair we cruise for about 135 years, and then reach a rich field of luminaries called the Hyades. In 323 years we reach a similar group of luminaries called the Pleiades. The Pleiades present a very beautiful spectacle, and the poet sang—

> "Many a night I saw the Pleiades, rising through the mellow shade
> Glitter like a swarm of fire-flies tangled in a silver braid."

Its length is 10 light-years from end to end. The Pleiades contains only blue and white stars while Hyades possesses stars of all colours. Clusters like the Pleiades and the Hyades are called galactic clusters which can be compared to our *districts* and *divisions*. We continue to meet these clusters till we have journeyed for 4000 years or so. It has been already pointed out that the distances of very remote stars cannot be measured by the parallactic method. More recently the distances of some of remotest objects have been determined by a fairly trustworthy method which will be mentioned later on. As we explore the sky we come across Variable stars of different types. The first type is that of eclipsing binaries. When one component of the binary system is eclipsed by the other its light is seen to diminish in amount at regular intervals and subsequently to return to its original strength. There are also irregular Variables, and we find also long period Variable stars. Temporary stars or novae occasionally blaze up with startling rapidity and gradually fade out again. Professor Bickerton believes that the appearance of a temporary star may be accounted for by the partial or grazing impact of two dark stars. The result would be the formation of a third body composed of the grazing portions of the stars. Due to the energy of motion of the two stars the third body would be of an explosive and a temporary character. Recently Dr. Stromberg of Mt. Wilson observatory concludes from the study of the spectrum of the Crab Nebula in Constellation Taurus that it is the later stage of a nova which exploded about 900 years ago. The Chinese Astronomers recorded a new star at the same spot, in the year 1054 A. D. It is quite possible that some of the diffuse nebulae are later stages of novae that exploded. It is also possible that a nova may be the origin of cosmic ray radiation. On the 14th of December 1934, a new Nova, nova Hercules, was first seen by an English amateur astronomer, Mr. Prentice—a lawyer. Prof. Kolhoerster has found that when he pointed his cosmic ray counters directly at Nova Hercules during its recent eruption the cosmic ray intensity increased as the star grew brighter and brighter.

There is another most important type of variables which we come across. These are Cepheid variables. In deriving distances of very remote stars and nebulae main reliance is placed in the cepheid variables. It has already been pointed out that parallactic method fails absolutely in determining the distance of the very remote objects. The brightness of cepheid variables fluctuates periodically and the period may vary from a few hours to a few weeks. It has been practically established that the Cepheids which have the same period are nearly all alike in their properties viz., intrinsic luminosity, radius, spectral type etc. The observed relation between the period and brightness is governed by what is called the period luminosity law. Miss Leavitt of Harvard first discovered in 1912 that the intrinsic luminosity of a cepheid varies more or less directly as the period. For example a cepheid having a period of 40 hours has got a luminosity 250 times that of the Sun, whereas a cepheid having a period of ten days has got a luminosity of 1600 times that of the Sun. We thus know the intrinsic brightness of a cepheid. Knowing the absolute and apparent magnitudes a simple application of the inverse square law gives its distance. If two cepheids A and B have the same intrinsic brightness and A looks four times as bright as B, then B must be at a distance which is twice the distance of A and so on. So we can get relative distances of Cepheids. Now the absolute distances of many of the nearer Cepheids have been found by the parallactic method. The nearest Cepheids are about 60 parsecs away, so the absolute distance of a remote cepheid can be determined. The Cepheid variables are "the standard candles" of the Heavens. If Cepheid variables are observed in a distant object like the Andromeda nebula then its distance can easily be determined. Applying a similar method, distances of nebulae may also be determined by observation of novae in them. There is a third method called the method of spectroscopic parallaxes by which the distances of remote objects may be determined. The cause of Cepheid variation can be explained by the pulsation theory as proposed by Shapley and Eddington. According to this theory there is a periodic expansion and contraction of the star under the combined influence of gravitation and elasticity of gases within it. Jeans suggests that the Cepheids are rotating stars which are on the point of dividing into two parts or actually in process of doing so. A cepheid will be subject to oscillations making it alternately more or less elongated and very slowly increasing until it breaks into two.

In 10,000 years' time we are amidst clusters of another kind. These are Globular Clusters, and are much richer in stars than the Galactic Clusters. Globular Clusters abound in Cepheid variables and are generally found near the rim of the Milky Way. In size Globular Clusters are in average ten times as big as the Galactic Clusters. About one hundred Globular Clusters are known. Both Galactic Clusters and Globular Clusters which we generally observe are galactic objects, i.e., are members of our galaxy. A

Globular Cluster can be compared to a *province*. Within our own galaxy or milky way we come across certain types of nebulae which are called galactic nebulae. Galactic nebulae may be sub-divided into three main groups :—

(1) Planetary nebulae

(2) Diffuse nebulae.

(3) Dark nebulae.

Planetary nebulae have nothing of planetary nature in them, but they appear to have finite discs when seen through a telescope. Probably their central portions are very massive stars of the type—white dwarfs.

Diffuse nebulae are irregular in shape, and their general appearance is like that of "huge glowing wisps of gas stretching from star to star". Variation of density, opacity and luminosity are responsible for all sorts of fantastic shapes. Diffuse nebulae may also be compared to *provinces* and some of them have a diameter of 100 light-years.

The dark nebulae do not shine and they obscure the stars which lie behind them. All these clusters and galactic nebulae are comprised within a much bigger stellar organisation which is our own galactic system—the Milky Way.

The form of our galactic system is like that of a bun or a double-convex lens. This form was first discovered by Kapteyn. Our galactic system is a super-galaxy and we can compare it to a *country*. The diameter of the galactic system is estimated at 240,000 light-years, whilst its thickness has been placed at about 20,000 light-years. We can regard our galaxy as an "Island Universe" in the vast depths of Space. Our local galaxy is the biggest of all the super-galaxies. It contains about four hundred thousand million stars.

As soon as we leave our Milky Way we reach two star clouds—Lesser Magellanic and Larger Magellanic clouds—just outside the boundary of our galaxy. These are remarkable groups of stars which were first discovered by the famous Spanish explorer Ferdinand Magellan who observed them in the southern hemisphere near the celestial pole in his first circumnavigation of the world. The distances are 85,000 and 95,000 light-years, respectively, from us.

As we cruise along in the depths of Space we come across many extra-galactic systems or nebulae. Most of them have spiral forms. In about 800,000 years we shall reach the most conspicuous of all these nebulae—the Great Nebula M. 31 Andromeda. It is a giant amongst external galaxies. Its size is colossal. It is a super-galaxy like the Milky Way and has a diameter of 40,000 light-years. If the size of some of these giant nebulae be reduced to that of Asia, then the Earth would become a microscopic body which would be hardly visible even under a most powerful microscope. Many of the spiral nebulae are rotating masses of gases. Recent investigation by Dr. Oort and others show that our galactic system is rotating like a cart-wheel with the difference that the inner part rotates more rapidly than the outer part. Our Sun is about 37,000 light-years away from the centre of the gigantic wheel, its hub lying in the direction of a massive star cloud where the constellations of Scorpio, Ophiuchus and Sagittarius meet together. Near about the Sun the gigantic wheel makes one revolution in 224 million years. The Sun and the stars in the neighbourhood reach a velocity of 200 miles per second. The mass of the galaxy is equal to about 2×10^{11} times that of the Sun. The total number of stars in our galactic system is of the order 4×10^{11}.

Several systems like our galactic systems have been found in the constellations of Coma and Virgo. Several super-galaxies possibly form a still larger organisation which Shapley calls a Meta-galaxy. A meta-galaxy can be compared to a *continent*. It is "all-comprehensive, but still incomprehensible". We have to travel for many hundreds of millions of light-years till we can explore the distant regions of a meta-galaxy.

PAPERS CONTRIBUTED TO THE DELHI CONFERENCE

M. Bhimasena Rao and M. Venkatarama Iyer : (Bangalore), *Tetracyclic Co-ordinates.*

Chowla I. : *The Singular Series in Warings Problem.*

S. Chowla, (Waltair) and S. S. Pillai, Annamalainagar : "*The Number of Representations of a Large Number as a sum of n non-negative nth powers*".

S. C. Dhar, (Nagpur) : *On Operational Representation of Whittaker's and other confluent Hyper-geometric Functions.*

——— ——— *On the Uniformization of Algebraic Curves of genus Four.*

V. Ganapathy Iyer, (Madras) : *A note of the values of an analytic function near an essential singularity.*

Govind Ram. *On the Resistance of Triangular Conducting Plates.*

D. S. Kothari, (Delhi) : *Quantum Statistics and Structure of Planets.*

Gupta Hansraj (Hoshiarpur) : *Partitions of n in which the least element that occurs is* $< [n/6]$.

——— ——— *Order of Class-numbers of Binary Quadratic Forms of a negative discriminant* $(-d)$.

——— ——— *Linear Quotient Sequences.*

A. A. Krishnaswami Ayyangar, (Mysore) : *A Comparative Study of Continued Fractions of the type*

$$b_0 + \frac{b_1 |}{| a_1} + \frac{b_2 |}{a_2}$$

where $| b_1 | = | b_2 | = \dots 1$ *and* a_1, a_2, *are positive integers.*

K. Nagabhushanam, (Waltair) : *An application of Lie's theorem to the Equations of Motion.*

A. Narasinga Rao, (Annamalainagar) : *On a certain Cremona-Transformation in Circle-Space connected with the theory of the Miquel Clifford Configuration.*

S. Pankajam (Miss), (Madras) : *Arithmetico-Logical Principle of Duality.*

B. Ramamurthi, (Annamalainagar) : *A generalisation of the null pencil of binary quartics.*

——— ——— *On the lines of Striction of a Quadric.*

——— ——— *On certain Manifolds related to a rational norm curve.*

——— ——— *On Double-binary (3, 3) forms.*

S. S. Pillay (Annamalainagar): *On Warings Problem.*

——— ——— *On Numbers containing a fixed number of Prime Factors.*

——— ——— *On Square-free Numbers.*

V. Ramaswamy Aiyar, (Chittoor): *Properties of the Durairajan Point and the Associated Conic of a Quadrangle.*
On the Fermat Point of a Triangle for given Multiples.

Ram Behari, (Delhi): *Generalisations of the Theorems of Malus-Dupin, Beltrami and Ribaucour in rectilinear Congruences.*

——— ——— *Some ruled Surfaces through a line of a rectilinear Congruence.*

B. R. Seth, (Delhi): *On the general solution of a class of physical problems.*

M. R. Siddiqi, (Hyderabad, Deccan): *The Theory of the non-linear integral Equation.*

C. N. Srinivasiengar, (Bangalore): *The Asymptotic Curves of the Cubic and Quartic Scrolls.*

R. Vaithyanathaswami, (Madras): *Differential Line Geometry.*

T. Venkatarayudu, (Madras): *On unique factorization into prime ideals in a residue class Ring.*

VOTE OF THANKS

At the end of the opening session, M.R.Ry. V. Ramaswamy Aiyar, Founder of the Society, thanked Sir Girja Shanker Bajpai for opening the Conference and the Vice-Chancellor and the authorities of the Delhi University for their kind invitation to the Society to meet at Delhi. He also appealed to the Government through Sir Bajpai to help the Society by annual grants so that it might carry on its work undeterred by financial worries.

On the last day of the session, in St. Stephen's Hall, Mr. Ramaswami Aiyar gave expression to the gratitude of the Society and of the delegates to Rai Bahadur Ram Kishore, Chairman, Mr. S. N. Mukerjee, Vice-Chairman, and Dr. Ram Behari, Secretary and the other members of the Reception Committee for the innumerable ways in which they had helped to make the Conference a success; to Rai Bahadur Ram Kishore and Sir Mahomed Abdur Rahman for their munificent hospitality; to the authorities of the St. Stephen's College and the Hindu College for giving facilities for the delivery of popular lectures; to Mr. Lala Chote Lal for kindly explaining the mysteries of Janter Mantar, to Dr. Mahajani, Dr. Vaidyanathaswami and Prof. Banerjee for their interesting public lectures, and lastly to the Volunteers who had rendered help during the whole session.

The Conference ended with the vote of thanks.

List of Delegates and others who participated in the Conference.

Abhyankar, S. K., M.A., Victoria College, Gwalior.

Balwani, T. J., M.A., D.J., Sind College, Karachi.

Banerjee, M.A. (Cantab.), I.E.S., University of Allahabad, Allahabad.

Chandi, P. T., M. Sc. (London), St. John's College, Agra.

Chaturvedi, C. R., M.A., St. John's College, Agra.

Chowla, I., Waltair.

Chowla, S., M.A., Ph.D. (Cantab.), Andhra University, Waltair.

Dhar, S. C., P.R.S., D.Sc. (Edin.), College of Science, Nagpur.

Ghosh, B. Miss, M.A., Professor, Lahore College for Women, Lahore.

Hansraj Gupta, M.A., Ph.D., Government Intermediate College, Hoshiarpur.

Hari Shankar, M.A., Professor, Anglo-Arabic College, Delhi.

Hem Raj, M.A., Principal, Dayal Singh College, Lahore.

Inamdar, G. R., M.A., Victoria College, Gwalior.

Jeo Krishna Dayal, Miss, M.A., 2, Fane Road, Lahore.

Kewalramani, M. K., M.A., N.E.D., Engineering College, Karachi.

Khosla, B. R., M.A., Principal, Government Intermediate College, Pasrur.

Kothandaraman, B.A. (Hons.), Asstt., Dept. of Education, Govt. of India.

Kothari, D. S., M.Sc., Ph.D. (Cantab.), Reader in Physics, University of Delhi.

Kureishy, M.A., Provost, Muslim University, Aligarh.

Laroia, B. D., M.A., Ph.D. (Lond.), Reader in Chemistry, University of Delhi.

Madan Mohan, M.A., Divisional College, Meerut.

Mahajani, G. S., M.A. Ph. D. (Cantab), Principal, Fergusson College, Poona.

Metha, H. M., B.A., Office of the Polling Agent, W. Kathiawar Agency, Kathiawar.

Mukerjee, S. N., M.A. (Cantab), Treasurer, University of Delhi and Principal, St. Stephen's College, Delhi.

Nagabhushanam, M.A., Andhra University, Waltair.

Narasinga Rao, A., M.A., L.T., Annamalai University, Annamalainagar.

Pandey, K. D., M.A., King Edward College, Amraoti.

Prasad, B. N., M. Sc., Ph. D. (Liverpool), D. Sc. (Paris), University of Allahabad, Allahabad.

Racine, C., D. Sc. (Paris), Professor, St. Joseph's College, Trichinopoly.

Ramamurthi, B., M.A., Annamalai University, Annamalainagar.

Ram Behari, M.A. (Cantab), Ph.D., Reader in Mathematics, University of Delhi.

Ramaswami Iyer, V., M.A., Retired Deputy Collector, Chittoor.

Siddigi, M. R., M.A. (Cantab.), Ph.D. (Gottingen), Osmania University, Hyderabad, Deccan.

Sita Ram Gupta, M.A., Govt. College, Lahore.

Subramaniam, L. N., M.A., Madras Christian College, Madras.

Sukumar Ranjan Das, M.A., Ph.D., Professor, Commercial College, Delhi.

Vaidyanathaswami, R., M.A., D.Sc., F.R.S.E. (Edin.), University of Madras, Madras.

MATHEMATICS AND MODERN PHYSICS

BY

DR. R. VAIDYANATHASWAMY

Inaugural Address.

The most striking feature of our era is the series of record-breaking developments in physical science and the highly mathematical and abstract character of the theories designed to explain ultimate physical facts. This imposing mathematical superstructure of modern physics has also been parallelled by a digging at the foundations—by a critical examination of the bases of physical knowledge and of the validity and scope of the mathematical method. The question of the nature of mathematics which had been asked and answered several times before in the history of thought, has again come into prominence in this connection, and the philosophical investigations which attempt an answer to the question are not the least of the achievements of our age. Modern attempts to characterise mathematics may be divided into three categories. Firstly there is the logistic outlook which makes out mathematics to be a branch of logic and places special stress on its reasoning and method. Secondly, there is the formalistic view which is akin to the logistic, and has gained wide acceptance and popularity; this sees the peculiar feature of mathematics in the *form* of mathematical truths and goes so far as to deny that mathematics has any definable content. Lastly there is the intuitionistic view-point which stresses on the constructive character of mathematics and holds that mathematics is not entirely formal but has a significant intuitional content. The divergence of the intuitionistic approach to mathematics from the other two comes out most clearly in the treatment of infinite aggregates and of the continuum. It would appear in fact that the formalistic view which stresses on an aspect of mathematical science that strikes one in the eye, as it were, is only a first approximation to the nature of mathematics, and that a deeper insight is contained in the intuitionistic approach. To the Indian temperament which in its highest philosophical flights never lets go its hold on the Real and the Concrete, intuitionism will probably appeal more than formalism. The earlier definitions of mathematics—like 'Mathematics is the science of space'—or 'the science of number'—and the attempt to trace the origin of the number-idea to the experience of temporal succession—all seem to express the feeling that the subject-matter of mathematics belongs to the primary intuitions which lie behind coherent experience. Even though the formal element is an important characteristic of mathematics and is the source of its generality and power, it does not express the central truth about mathematics; this, I think, can be inferred from the disastrous effects produced by stressing on the formal aspect, in secondary and college education. When we reflect on the fact that far-reaching mathematical theories have been inspired by and developed in close relation to the needs and demands of Physical Science, and that reciprocally, general mathematical ideas have been the agents to provoke physical insights and point the way to physical truths, the formalism of mathematics appears to be but a pose and an appearance.

In this address I propose to study two general mathematical ideas—'Group' and 'Eigen-value'—in relation to our experience of the physical world, and to sketch their adventures, and the transformations they have undergone in Modern Physical Theory.

The idea of Group and Invariant.

Though the ideas 'group' and 'invariant' have received explicit formulation pretty late in mathematical history, they are among the primordial ideas of mathematics. They have their origin in our experience of movement and furnish apparently the principle by which we have intellectually constructed the objective world in defiance of the appearances of perspective. Thus the totality of possible movements of a rigid body constitute a 'group'; this is built up from the translatory movements and the rotations about a fixed point; the shape which we intellectually attribute to a body (as contrasted with the shape which it actually presents to us in perspective) is an 'invariant' of the group of movements. Generally we may say that the invariant idea is the mathematical expression of the distinction between the 'subjective' and 'objective'; the appearances from a particular position, condition or point of view of the subject are 'subjective': the substratum of these appearances which is independent of the condition or point of view of the experiencing subject is what one means by 'objective truth' and is precisely the content of mutual communication and social thought.

In Euclid's Geometry the idea of invariant does not occur explicitly, but as it deals with the properties of figures, independent of position and orientation in space, it is in substance an invariant theory of the group of movements. After Descartes' discovery of algebraic geometry, it was found that in studying a figure by the use of co-ordinates, constants were introduced which were irrelevant to the intrinsic properties of the figure, but represented its relation to the axes of co-ordinates. The elimination of such constants from the work was effected by systematic invariantive methods developed by Cayley, Sylvester, Aronhold, Clebsch, Gordan etc. Complete generality of method can be obtained in mathematics only by working with invariantive concepts. The Non-Euclidean Geometries which were discovered from the investigation of the parallel axiom were shewn by Cayley to result by taking as the group of movements, the group of linear transformations which carried a non-degenerate quadric surface into itself. This paved the way for Klein's formulation of the group-theoretic principle for classifying geometries—which is one of the great mathematical generalisations achieved by the 19th century.

The group of rotations of a rigid body round a fixed point is the fundamental kinematical group and reappears constantly in Physics. Mathematically it is represented as the group of orthogonal transformations and furnishes the basis for both the Quaternion Calculus and the Vector-Calculus. The 'inner' or scalar product of two vectors A, B is the work done by the force represented by the vector A in the displacement represented by B. The 'outer' or vector-product of the same vectors is the linear velocity of the point whose position-vector is A due to the angular velocity represented by B. In Physics we have usually to do with 'Vector-fields' i.e. with vectors varying continuously in magnitude and direction issuing from each point of a region. The typical case is the small deformation of a continuous medium. Using rectangular axes we may denote by $V(x_1, x_2, x_3)$ the small vector-displacement suffered by the particle of the medium situated at (x_1, x_2, x_3).

It is then easily seen that the relative displacement of the immediate neighbourhood of the point is described by the 9 quantities $V_{ij} = \frac{\partial V}{\partial x}$, where V_1, V_2, V_3 are the components of V. We may call V_{ij} the deformation-tensor and split it up into two parts, one skew-symmetrical and the other symmetrical. The skew-symmetric part (called the rotation or curl of the vector-field V) gives the angular rotation to which the neighbouring part of the medium has been subjected, while the symmetric part gives the distortion due to homogeneous strain. The quantity $\frac{\partial V_1}{\partial x_1} + \frac{\partial V_2}{\partial x_2} + \frac{\partial V_3}{\partial x_3}$ called the 'divergence' of the vector-field V is the volume-dilation due to the distortion. From the physical meaning, it is clear that the quantities 'divergence,' 'curl,' 'distortion,' are invariantively related to the vector-field. If the medium is elastic, the distortion calls forth a stress-tensor, which is also symmetric and the generalised form of Hooke's law would shew that for small strains the components of the stress-tensor are linear functions of those of the 'strain' or distortion-tensor. If V^i has no curl, the corresponding deformation is a pure strain and V^i is a gradient.

While a vector may be regarded as a quantity and direction associated with each point, a tensor may be regarded as a set of quantities associated with several directions at each point. While the vector-calculus restricts itself to rectangular axes, the tensor-calculus uses oblique axes in which there is a difference between the 'covariant' and 'contravariant' components of a vector, with the result that a tensor-symbol carries two sets of indices, one upper and the other lower. In Physics it is often necessary to use not cartesian but generalised co-ordinates. It was Gauss who first studied a surface by means of curvilinear co-ordinates and discovered the fact the Gaussian curvature which is the product of the principal curvatures at each point of a surface remained unaltered when the surface is deformed in any manner without stretching or tearing. This was extended to arbitrary manifolds by Riemann through the discovery of a tensor—called the Riemann-Christoffel tensor—which gave a complete account of the curvature properties of a manifold at each point. The tensor-method in differential geometry was established as a systematic invariantive calculus for arbitrary manifolds by Ricci, and has proved indispensable to the Theory of Relativity.

Closely connected with the mechanics of continuous media is the field-idea which has been introduced into Physics by Faraday and Maxwell. The Inverse-square law of distance was originally stated both for electric charges and magnetic bodies as a law of action at a distance. But Faraday in his experiments conceived of the intervening medium as transmitting the electric force through elastic stress and as being the seat of the energy. Maxwell by a brilliant mathematical analysis was able to give a complete account of the electro-magnetic field on these lines and to identify the light-wave with the propagation of electromagnetic disturbance.

For the electrostatic field produced by any distribution of electric charges with volume-density ρ (say), the vector E of electric intensity is the gradient of the electric potential ϕ and the divergence of the field-vector is the density of charge ρ. According to Faraday's theory the ether is subjected to a tension of magnitude $\frac{1}{2}$ E^2 along the lines of force and a pressure of the same magnitude perpendicular thereto. It is easy to verify that this distribution of stress will exactly produce the ponderomotive force-density ρE on the charges. The magnetic field H produced by stationary currents satisfies the field laws curl H = current-density; div H = 0, and the ponderomotive force here also, which is [SH] where S is the current, can be derived from a symmetrical stress tensor, corresponding to a tension $\frac{1}{2}$ H^2 along the lines of force and a pressure of the same amount perpendicular thereto.

In the case of the variable electro-magnetic field, Faraday's law of induction states that the electromotive force generated in any closed conducting circuit is equal to the rate of variation of the magnetic induction across any surface bounded by the circuit. Maxwell by a brilliant intuition lighted on the law correlative to this—namely the law of the displacement current. These two relations are the starting point of the electro-magnetic theory. Both the electric and the magnetic potential for the moving field satisfy the equation of retarded potentials $\Delta^2\phi - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = 0$, shewing that electro-magnetic disturbances are propagated with the velocity of light. That electromagnetic waves are of essentially the same nature as light-waves has received both direct and indirect experimental confirmation.

Just as the Laplacian expression $\Delta^2\phi$ is invariant for rotation of rectangular axes, so the corresponding expression $\Delta^2\phi - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2}$ in the case of the retarded potentials is invariant for the Lorentz group —namely the group of transformations which carry $x^2+y^2+z^2-c^2t^2$ into itself. It was in fact recognised by Lorentz that the whole system of electromagnetic laws for the ether was invariant for the Lorentz-group. Lorentz himself interpreted the Lorentz transformation to mean that a rod moving with a velocity v in the direction of its length contracted in the ratio $\left(1-\frac{u^2}{c^2}\right)^{\frac{1}{2}}$. This contraction explained also the null result of the Michaelson-Morely experiment.

It was reserved for Einstein to discover the simple physical meaning of the Lorentz transformation and thereby found the special theory of Relativity. This theory is based on two postulates (1) The formal invariance of the laws of nature for a uniform translation of the observer and (2) The constancy of the velocity of light or rather the independence of the velocity of light on the motion of the source. This latter postulate is a plausible one in view of the null result of experiments designed to detect the influence of the motion of the earth on optical phenomena. On account of the first postulate, the law of inertia that isolated bodies are in uniform translational motion should hold for all the observers in question. Hence if two observers in uniform translational motion v relative to each other have coincident position and axes, the relation between their co-ordinate-systems $(xyzt)$, $(x'y'z't')$ must be linear, and since the velocity of light is measured to be the same by both, we see by considering the wave-fronts that $x^2+y^2+z^2-c^2t^2=f(v)(x'^2+y'^2+z'^2-c^2t'^2)$. It is easy to shew that $f(v)=1$ and Lorentz equations result. The special theory of Relativity is thus the invariant-theory of the Lorentz group. It is remarkable that the Lorentz-transformations which are fundamental for electromagnetic theory should be formally similar to the rotations about a fixed point in four dimensions.

According to Einstein, the Lorentz contraction should not be causally interpreted but is a relation of the rod to the frame of reference. The transformation equations imply also a time-dilation of the same amount as the Lorentz contraction. Among the consequences of the Lorentz transformation, may be mentioned the non-commutative law of the compounding of velocities and the fact that the velocity of light plays the part of an infinite velocity. The notable result in mechanics is the essential identity of mass and energy—a fact substantiated in the case of fast-moving electrons; the mass of a moving particle is greater than its rest-mass on account of the inertia of its kinetic energy. The special theory of Relativity asserts that neither space nor time is objectively real, but only their four-dimensional combination, space-time, which is split up into space and time by each observer according to his motion. On account of its treating the space and time variables on the same basis, distinct physical laws are fused together in a single tensor law in Relativity. Thus the laws of conservation of energy and momentum are united into a single tensor-law; the scalar and vector potentials are united into single 4-potential. In fact for an observer moving relatively to an electrostatic field, the observed electric intensity is no longer the gradient of a potential i.e. neither the scalar nor the vector potential is objectively real, but only, the 4-potential; and the field equations are united into a symmetrical form etc. It may be mentioned finally that classical mechanics is the limiting case for $c=\infty$ of Relativistic mechanics.

The special theory of Relativity which is concerned with the invariant formulation of physical laws for observers with uniform relative velocity, is valid only in the absence of gravitation. The gravitational field is unlike fields of physical origin in that gravitational mass is identical with inertial mass. In other words it imposes the same acceleration on every mass, being similar in this respect to fictitions or 'co-ordinate-forces' like centrifugal forces. Einstein's principle of equivalence assumes that like centrifugal forces, gravitation can be transformed away in the neighbourhood of any world-point by choosing a suitably accelerated frame of reference. It is further assumed that the results of the mathematical transformation represent the actual physical experience in the new frame. The expression for gravitation is then found by the requirement of the formal invariance of the laws of nature for arbitrary frames of reference. The interval ds^2 between two neighbouring events must therefore be invariant for all frames of reference, and since its expression in an inertial frame is of the form used in the special theory, its expression in general co-ordinates is a quadratic differential form $g_{ik}\, dx^i\, dx^k$, where the g_{ik}'s are functions of the co-ordinates. This form establishes a close connection between metrik and gravitation which might be expressed either as a geometrisation of physics or alternatively as the reduction of geometry to a branch of physics. The fact that g_{ik}'s are not constants indicates the departure from the first law of motion or the non-inertial character of the frame of reference; that is to say, it indicates the presence of gravitation and the g_{ik}'s may be regarded as the gravitational potentials; they are not scalars like the Newtonian Potential, but the components of a tensor. The free path of a particle is given in special theory by the variational equation $\delta \int ds = 0$; since ds is invariant, it is given by the same equation in any accelerated frame of reference. Thus the free paths of material particles are the 'geodesics' of the metrical groundform which represents also the gravitational potential. Hence the christoffel symbols are analogous to the gravitational force. In the same way it is seen that the light rays are the null geodesics of the metrical groundform. It is easy to deduce from the transformation theory both the variability of the velocity of light, and the curvature of light-rays in a gravitational field. Since the values of the gravitational potentials g_{ik} have to be obtained at each point by measurement with material rods and clocks, it follows that the geometry of the world is a consequence of the behaviour of material rods and clocks. Thus the geometry of the four-dimensional world (and therefore, of space) is not given a priori, but is determined by matter—thus substantiating a view expressed speculatively by Riemann.

Two forms of the law of gravitation have been put forward by Einstein, based on various formal and physical considerations; the first of these is a special case of the second $G_{ik} = \lambda g_{ik}$ which as has been shewn by Eddington, simply expresses the isotropy of the four-dimensional-world. The law means in fact that the spherical radius of curvature or the scalar curvature of every three-dimensional section of the World in empty space is an absolute constant. In other words, since length is not absolute but a result of measurement, the length of a specified material structure has a constant ratio to the radius of curvature of the world at the place and in the direction in which it lies. The perihelion advance of Mercury has been satisfactorily explained by the theory and the predicted bending of rays in the neighbourhood of the sun has actually been verified to within limits of experimental error. The position as regards the amount of the predicted shift towards the red in the solar spectrum is not however equally satisfactory.

The solution of the gravitational equations has led to investigations by Einstein and De Sitter on the form of the world as a whole. These are of course highly speculative in character; because in any differential analysis of a manifold we can never be certain that our co-ordinate representation will continue to be valid beyond a limited domain. Since the effect of the General Theory of Relativity has been to unify and interrelate diverse physical concepts, it has been felt that there must be some inner connection between the electromagnetic field and the gravitational or metrical field. Now Maxwell's equations of the electromagnetic field remain invariant not only for Lorentz transformations but also for conformal transformations. This fact led Weyl to propose a theory by which the electromagnetic four-vector at each world-point served to determine the *gauge* or the unit of interval there. While this interpretation has been discarded by Weyl himself subsequently, the idea of conformal geometry has gained ground and established itself under the name of 'Projective differential Geometry' and found important application in the more modern elaborations of the Theory of Relativity. The idea that a measuring-rod is itself part of the world which it measures and that the gravitational law exhibits the world as self-gauging has enabled Eddington to give a more acceptable interpretation of Weyl's theory and to extend it as well. Further theories of unification of the field depending on the concept of parallel transference have been proposed by Einstein himself and Levi-Civita. However, no physical support for such extensions has been available.

General Relativity and in particular, the principle of equivalence for arbitrarily accelerated frames of reference have not won universal acceptance among physicists. Alternative theories of gravitation between bodies in relative motion, based on a finite velocity of transmission of the gravitational field, have been put forward by Weber, Riemann, Levy, Gerber and others. These are generally speaking based on the elementary idea of force and not on Poisson's differential equation which should, properly speaking, serve as the starting point of a theory of gravitation propagated with finite velocity; indeed according to modern conceptions, the idea of force has an inferior physical status to Energy and Momentum which are the primary physical quantities subject to transport in a field. Since Force is the space-derivate of the Potential, we may say if we choose, that the Potential is one step further removed from the plane of manifestation or appearance, and therefore one step nearer than Force to the causal or truth side of phenomena. Levy's and Gerber's theories obtain the correct value for the perihelion movement of Mercury. But in point of theoretical cogency and correct deduction through accepted physical principles, they appear to be all equally unsound. The grounds for this conclusion may be easily gathered from the discussion and criticisms of Gerber's theory (which is typical of this class) by Seeliger, Lane and others in the volumes of the *Annalen der Physik*, 1917. In the first place these theories make out the gravitational potential of A at B (moving relatively to A) to depend on the relative position and the relative velocity at the *same* instant, whereas the potential ought to be of the 'retarded' form, and depend on the position at an earlier instant, if gravitation is to be propagated with finite velocity. Secondly the gravitational field of the Sun, with respect to axes fixed in the Sun is a stationary field and it is only changes in this field which are propagated with finite velocity. It is therefore difficult to see how planets moving in this stationary field can betray the effects of a finite velocity of propagation. Sir Sulaiman's 'Mathematical Theory of a New Relativity' published in the Proceedings of the U. P. Academy of Sciences Aug. 1934, belongs to the same class of theories and is subject to precisely the same objections.

The Theory of Eigen-values

The concept of Eigen-value is not as primitive as the idea of group and invariant, but as it is related fundamentally to the notions of 'vector' and 'scalar', its importance for Physics was recognised even before the New Mechanics. With the advent of the new Quantum-Mechanics, and its view of the nature of measurement, the content and scope of the idea of Eigen-value has been greatly enlarged, and it appears now as one of the deepest ideas in The whole range of Physics.

To explain the idea of Eigen-value, we may take the case of a linear homogeneous strain. If O is a point of the medium and P any other point of the medium, the vector OP regarded as composed of the particles of the medium will on account of the strain take up a different position OP′. Thus the strain changes the vectors issuing from O and may be regarded as an operator on these vectors; it is further a *linear distributive* operator, since it changes k.OP into k.OP′ and the sum of two vectors into the sum of the corresponding transformed vectors. Now we know that there are three directions—namely the principal axes of the strain or of the strain-ellipsoid—which are unchanged by the strain; in other words there are three mutually perpendicular vectors which are simply multiplied by a number—namely the corresponding coefficient of strain—without being displaced by the strain-operator. These three vectors are *the Eigen-vectors* of the strain-operator and the corresponding coefficients of strain are its *Eigen-values*. Thus *the whole theory of Eigen-values may be regarded as the generalisation of the idea behind the theorem that an ellipsoid has always three mutually perpendicular principal axes*, or in more general form, *two conics have in general a unique common self-polar triangle*. It is very remarkable that this fundamental theorem of elementary geometry is capable of the wide generalisation and scope given to it in Physics. To give a more precise definition if Eigen-value, let L be any linear distributive operator which acts on a class of entities X, which may be either vectors or analogous to vectors, and which may have to satisfy other conditions besides. If there exists a number λ and an entity X, such that L acting on X produces simply λX, then λ is an Eigen-value of L and X the corresponding Eigen-vector. Now, if X is a vector in a finite or infinite number of dimensions, it is evident that the general form the linear operator acting upon it can be exhibited as a matrix. It is for this reason that one form of the New Mechanics is matrix-mechanics.

Again if two homogeneous strains, L, L′ have the same principal axes of strain, then it is easy to see that in applying them successively, the same resulting strain is obtained in whichever order we apply them; in other words two homogeneous strains commute with each other if they have the same principal axes of strain. It can be shewn that they do not commute in any other case. The commutation-rules of the new mechanics are a generalisation of this fact.

This illustration of Eigen-value is equally valid for vectors in any finite or infinite number of dimensions. A function of a real variable x defined for a given range may be regarded as a vector in a space of infinite dimensions, whose components are the values of the function at different points of the range. The linear distributive operator $\frac{1}{i}\frac{d}{dx}$ acting on e^{ivx} multiplies the function by v. Thus every number v is an Eigen-value of the oprator and the corresponding Eigen-function or Eigen-vector is e^{ivx}. If however we limit ourselves to functions of period 2π, we must say that every integer $\pm n$ is an Eigen-value of the operator with the corresponding Eigen-function $e^{\pm inx}$. Now the three principal axes of an ellipsoid are not coplanar, and therefore any vector can be expressed linearly in terms of them. The corresponding theorem here is that any function of period 2π, can under certain differentiability conditions, be expanded in terms of the series of Eigen-functions $e^{\pm inx}$; we have in other words the Fourier series:

$$f(x) = \sum_{-\infty}^{\infty} a_n e^{inx}.$$

In the three-dimensional case the length of a vector V was $\sqrt{V_1^2 + V_2^2 + V_3^2}$, and any two Eigen-vectors of the strain-operator were orthogonal. We can ensure the analogues of these properties for our present case, if we define the integral

$$\int_0^{2\pi} f(x)\,\overline{f(x)}\,dx,$$

as the square of the length of the vector corresponding to the periodic function $f(x)$. We have then to say that the vectors corresponding to the periodic functions $f(x)$, $\phi(x)$ are orthogonal if $\int_0^{2\pi} f(x)\,\overline{\phi}(x)\,dx = 0$. We then see that the vectors corresponding to distinct Eigen-functions e^{inx}, e^{imx} are orthogonal, while the length of each Eigen-function considered as a vector, is $\sqrt{2\pi}$. Pythagoras' theorem states that the square of the length of a vector is the sum of the squares of its orthogonal components. Its analogue and generalisation is Parseval's equation:

$$\int_0^{2\pi} f(x)\,\overline{f(x)}\,dx = 2\pi \sum_{-\infty}^{\infty} a_n \bar{a}_n,$$

which holds provided f is a continuous periodic function. This instance must suffice here to indicate the possibilities of the Eigen-value idea.

In the New Quantum Mechanics the state of a microscopic system is represented as a complex vector with an infinite number of components, $(x_1, x_2 \ldots)$ of which the length defined by $\sum x_1 \bar{x}_1$ is equal to 1; it is also represented alternatively by a wave-function, which from what is said above, may be considered as equivalent to a vector. The introduction of the imaginary is essential for brevity and elegance, since periodic functions are best treated with the help of the imaginary unit. On the other hand *a physical quantity* connected with the system is conceived as a linear operator on the vectors, that is to say as a rotation and strain of the infinite-dimensional space in which the state-vectors are represented. (As the vectors here are complex, our usual pictures will not exactly hold.) The distinction between *states* of a microscopic system and *physical quantities* or *observables* connected with the system is related to the view of measurement in the new mechanics; according to this view the *state* of a microscopic system is something abstract and theoretical with which we never come into direct contact by experiment.

On the other hand what we directly measure are the physical quantities. The *states* of a system must be entities of the sort that would exhibit the properties of interference and linear superposability with arbitrary phase-difference that belong to the light-wave; hence they must be represented mathematically by complex vectors in a finite or infinite-dimensional space. From the empirical standpoint, the physical quantity which we measure in the laboratory *is* the process of measurement by which we claim to determine it. Now, the measurement of a microscopic system can only be performed by disturbing it; hence the process of measurement, and therefore the physical quantity must be conceived mathematically as a linear homogeneous strain on the vector-body of its possible states. To interpret the number obtained by measurement, we note that for a microscopic system, the result of measurement must be the average value of the physical quantity for the constitutive microscopic systems. There are strong reasons for thinking that in the case of a microscopic system, the measured value of a physical quantity A for a state V would be the proportional elongation of the state-vector V, measured along itself, due to the strain corresponding to A. Thus if the strain A has the principal coefficients (Eigen-values) $\lambda_1\lambda_2 \ldots$, and if the components of V referred to the principal axes of A be $V_1, V_2, \ldots$, then the components of the strained vector (that is, of the new state to which V is altered by the process of measurement) are $\lambda_1\overline{V}_1, \lambda_2\overline{V}_2, \ldots$. The proportional elongation measured along itself of the state-vector V is then given by:

$$\frac{\sum \lambda_1 V_1 \overline{V}_1}{\sum V_1 \overline{V}_1}.$$

This then is the value of the physical quantity A for the state V. Assuming the possibility of direct measurement of microscopic systems, it is clear that two measurements of a physical quantity carried out at successive instants will not in general lead to the same result; for the first measurement would give the value of A for the state V, while the second gives the value of A for the state V′ into which V is changed by the first measurement. If, however, V lies along a principal axis of strain, then it is unchanged in direction by the measurement, and it is clear that successive measurements will agree, being equal in fact to the corresponding coefficient of strain. Thus the possible *certain* values of a physical quantity A are its Eigen-values (namely the Eigen-values of the corresponding homogeneous strain, or the principal coefficients of strain); and it takes these values for its *Eigen states*, namely the state-vectors which lie along a principal axis of strain. The value of A for an arbitrary state V is, as our expression shews, the mean value of its Eigen-values for multiples equal to the squared lengths of the corresponding principal components of V. We may express this by saying that for an arbitrary state V, the quantity A takes its Eigen-value λ_i with the probability $\frac{V_i \overline{V}_i}{\sum V_1 \overline{V}_1}$.

The most important physical quantity associated with the system is the *energy*. According to Planck, Energy has a granular structure and exists in quanta hv associated with vibrations of frequency v. If we suppose the microscopic system to consist of an infinity of harmonic oscillators $a_n e^{iv_n t} = x_n$, we can take $(x_1, x_2, \ldots)$ as the state-vector. The linear operator $\frac{h}{i}\frac{d}{dt}$ acting on the n^{th} component x_n simply multiplies it by $E_n = h\nu_n$. Hence the physical quantity *Energy* can be identified with the linear operator $\frac{h}{i}\frac{d}{dt}$, and the discrete values or *Eigen-values* E_n of the energy correspond to the stationary states of the system, since the probability of the value E_n for the energy, being $|a_n|^2$, is independent of the time. The energy operator $\frac{h}{i}\frac{d}{dt}$ can be simply related to the rate of self-strain of the system when left to itself. For if x is the state-vector, $\frac{dx}{dt}$ is the rate of self-strain of the system; since the length $\sum x_1\bar{x}_1 = \sum a_1^2$ of the state-vector is independent of the time, the self-strain must be conceived as a *rotation* (or rather as a *unitary transformation* which is the extension of rotation to complex vectors) of the state-vectors. By Planck's hypothesis this rotatory self-strain of the system can be written in the form $\frac{i}{h}$ E, where E is the energy-operator. Since the energy operator is thus related to the self-strain of the system, it follows that the change in any physical quantity A associated with the system can be expressed in terms of the energy-operator. Thus :

$$\frac{i}{h} E\, dt. (A.\ X) = d\,(A.\ X) = dA.\ X + A.\ dx$$

$$= dA.\ X + A.\ \frac{i}{h}\ E.X$$

or

$$\frac{h}{i}\frac{dA}{dt} = EA - AE.$$

In particular the quantity is independent of the time, if and only if it commutes with the Energy.

ON QUADRATIC EQUATIONS

BY

A. NARASINGA RAO, *Annamalainagar.*

1. Introduction

My young readers, for whom most of this paper is intended, must have studied and solved many individual quadratic equations. In this paper, we shall try to take a bird's-eye-view of the whole tribe of quadratic equations and study their structure in relation to those properties of one or more quadratics which have already attracted our attention.

We have, firstly, to distinguish between a quadratic expression

$$A(x) \equiv a_0x^2 - 2a_1x + a_2 \qquad \dots \quad (1\cdot1)$$

and the quadratic equation $A(x) = 0$. On multiplying by k throughout, the former is altered while the equation remains essentially the same. Quadratic expressions depend on three constants a_0, a_1, a_2 while the equations form a two dimensional manifold depending on the two ratios $a_0 : a_1 : a_2$. It is the latter that form the subject of our study.

2. Representation by points on a plane

The characteristic property of a quadratic equation is that it has two roots. Since they form a two dimensional aggregate, we seek to represent each quadratic by a point on a surface, and at first one thinks of the two roots t_1 t_2 as the two Cartesian co-ordinates on a plane. This has, however, the serious disadvantage that the two points $(t_1\ t_2)$ and $(t_2\ t_1)$ represent the same quadratic. A representation free from this defect is obtained by taking $(a_0\ a_1\ a_2)$ as the *homogeneous* co-ordinates of the equation, since it is only the ratios which count. The vertices of the triangle of reference (1, 0, 0), (0, 1, 0) (0, 0, 1) correspond to the quadratics whose roots are 0, 0 ; 0, ∞, and ∞, ∞. The representation is thus on a projective plane π.

3. The Fundamental Conic Ω

Among quadratic equations, those with equal roots form a subclass represented by a locus. This is a conic Ω whose equation is $xz = y^2$, since the discriminant $a_0a_2 - a_1^2$ should vanish for equal roots. The point $(1, t, t^2)$ which lies on this locus represents the equation $x^2 \mp 2xt + t^2 = 0$ whose roots are (t, t) and may be called 'the point t'. The conic Ω divides the projective plane into two distinct regions. Points in one of these regions such as (1, 0, 1) make $a_0\ a_2 - a_1^2$ positive and correspond to quadratics with imaginary roots and points in the other such as (1, 0, -1), to quadratics with real roots.

4. Pencils of Quadratics

The point $(a_0 + \lambda b_0,\ a_1 + \lambda b_1,\ a_2 + \lambda b_2)$ will, as λ varies, describe a straight line. Hence the quadratics

$$A(x) + \lambda B(x) \equiv (a_0 + \lambda b_0)x^2 - 2(a_1 + \lambda b_1)x + (a_2 + \lambda b_2) = 0$$

forming a *pencil of quadratic equations* correspond to a line.

Now every straight line meets a conic in two points real or imaginary. Hence we infer that

in any pencil of quadratic equations there are two members with repeated roots. ... (4·1)

In fact $A(x) + \lambda B(x) = 0$ will be a perfect square for two values of λ, namely those for which the discriminant

$$(a_0 + \lambda b_0)(a_2 + \lambda b_2) - (a_1 + \lambda b_1)^2 = \lambda^2(a_0 a_2 - a_1^2) + \lambda(a_0 b_2 - 2a_1 b_1 + b_0 a_2) + \lambda^2(b_0 b_2 - b_1^2)$$

vanishes. If the sum of the two values of λ is zero, that is if

$$a_0 b_2 - 2a_1 b_1 + b_0 a_2 = 0 \qquad \ldots \quad (4\cdot 2)$$

we say the two quadratics $A(x)$ and $B(x)$ are *apolar*. The two points corresponding to $A(x)$ and $B(x)$ then separate harmonically the two intersections of their join with Ω. Hence

two points which are conjugate with respect to Ω correspond to apolar quadratics. ... (4·3)

Again, consider all quadratics having one root fixed, say t. They are given by $(x - t)\ (x - \lambda) = (x^2 - xt) + \lambda(t - x) = 0$ where λ is variable and form a pencil. The representative point

$$\left(1,\ \frac{\lambda + t}{2},\ \lambda\, t\ \right)$$

traces the straight line $xt^2 - 2ty + z = 0$ which is seen to be the tangent to Ω at "t." Hence,

all quadratics with a common root t correspond to points on the tangent to Ω at "t" ... (4·4)

From any point P may be drawn two tangents to Ω. Let the points of contact be "t_1" and "t_2". Using the above result we see that

if P be any point, the two roots of the corresponding quadratic are the parameters of the points of contact of tangents from P to Ω. ... (4·5)

Lastly, since any point on the chord $t_1\ t_2$ is conjugate, with respect to Ω, to the intersection of the tangents at t_1 and t_2, we have from (4·3) and (4·5)

the quadratic $(x - t_1)\ (x - t_2)$ is apolar to all quadratics of the pencil $(x - t_1)^2 + \lambda\ (x - t_2)^2 = 0$; that is, two quadratics are apolar if the roots of the one separate harmonically the roots of the other. ... (4·6)

The last part of (4·6) follows from the fact that two numbers separating $t_1\, t_2$ harmonically are of the form $\frac{t_1 + \lambda t_2}{1+\lambda}$ and $\frac{t_2 - \lambda t_2}{1-\lambda}$ which are roots of $(x-t_1)^2 - \lambda(x-t_2)^2 = 0$.

5. Systems of the Second Degree

If a_0, a_1, a_2 satisfy an equation of the second degree, the representative point traces a conic and the quadratic equations are said to belong to a *second degree system*. We have already met one important second degree system, namely equations with repeated roots, corresponding to the conic Ω. A more general system of the second degree is that of all equations, for which the difference between the roots is a constant, say k; for the required condition is

$$\frac{2\sqrt{(a_1^2 - a_0\, a_2)}}{a_0} = k, \quad \text{that is} \quad 4(a_1^2 - a_0\, a_2) = a_0^2\, k^2 \qquad \ldots(5\cdot 1).$$

Hence the corresponding conic $4(y^2 - xz) = x^2k^2$ is one which has double contact with Ω where $x = 0$ cuts it. Ω itself corresponds to $k = 0$.

Another example is that of all equations in which the ratio of the roots is a given constant k, for the required condition is

$$4\,k\,a_1^2 - a_0\, a_2(k+1)^2 = 0.$$

We may now interpret known results in the geometry of conics in terms of systems of quadratics. Thus take the theorem that if there exists one triangle inscribed in one conic S and circumscribed to another conic, say the fundamental conic Ω, there exist an infinite number of such triangles. Taking the points of contact with Ω to be $t_1\, t_2\, t_3$ and using (4·5) we have the corresponding result:

If a second degree system of equations contain one triad of the type $(x-t_1)(x-t_2) = 0$, $(x-t_2)(x-t_3) = 0$, $(x-t_3)(x-t_1) = 0$, it contains an infinite number of such triads. ...(5·2)

6. Oriented Quadratics: Laguerre Representation

So far we have made no distinction between the two orders $t_1\, t_2$ and $t_2\, t_1$ of the roots. If we agree to make this distinction, the equation is said to be *oriented*, so that each non-oriented equation gives rise to two oriented equations. We may represent these two equations in the Cartesian plane by the two points (t_1, t_2) and (t_2, t_1) which are reflexions of each other in $y = x$. All oriented quadratics with equal roots are thus represented by the straight line $y = x$, and thus form a linear system. Quadratics of which the first root is given $t_1 = \alpha$ say, are represented by lines $x=\alpha$ parallel to the y axis, and those for which the second root is given by lines parallel to the x-axis. Quadratics for which the difference between the roots is k, now break into two systems corresponding to $x-y=k$ and $y-x=k$, according as the first root or the second is the greater. The representation specified above may be called the *Laguerre* representation of oriented quadratics. Instead of the two roots, we may also take $x=\frac{1}{2}(t_1+t_2)$ $y=\frac{1}{2}(t_1-t_2)$ which would correspond to a rotation of the axes through 45°.

7. Oriented Quadratics: Lie representation *

Another device for representing oriented quadratics, which may be associated with the Norwegian mathematician Sophus Lie, is the use of a redundant homogeneous co-ordinate a_3 where

$$a_3=a_0(t_1-t_2)=a_0\times \text{ the first root minus the second root.}$$

The oriented equation $A(x)=0$ is now represented by a point in 3-dimensional space whose homogeneous co-ordinates are (a_0, a_1, a_2, a_3). These are connected by the relation

$$a_3^2=a_0^2(t_1-t_2)^2=4(a_1^2-a_0a_2)$$

so that the representative point always lies on a ruled quadric—whose equation is

$$a_3^2=4(a_1^2-a_0a_2). \qquad \ldots(7\cdot1)$$

The same quadratic with the order of the roots reversed, will correspond to the point $(a_0, a_1, a_2, -a_3)$. Hence

> quadratics with equal roots correspond to the section of the quadric H by the plane $a_3=0$. More generally, equations in which the difference between the roots is k correspond to the two planes $a_3/a_0=\pm k$. ...(7·2)

Since the homogeneous co-ordinates of the oriented equation $x^2-x(t_1+t_2)+t_1t_2=0$ whose first and second roots are t_1 and t_2 are 1, $\frac{1}{2}(t_1+t_2)$, t_1t_2, t_1-t_2, it is clear that if t_1 is given but t_2 unspecified, the co-ordinates being linear in t_2 represent points on a straight line. But all representative points lie on the quadric, so all equations $(t_1\,?)$ correspond to a generator. Similarly, the equations with roots $(?\ t_2)$ give a second system of generators. It is obvious that two lines of the same system do not meet while two generators $(t_1\,?)$ and

* The two representations of oriented quadratics are associated with the names of Laguerre and Lie in analogy with circle-geometry. A circle in one dimension is, in fact, a pair of points, the centre being the mid-point and the diameter the distance between them.

(? t_2) of opposite systems have the common point corresponding to the equation (t_1 t_2). Thus

> the two systems of generators of H correspond to equations in which one or other of the roots is fixed. ...(7·3)

Since 1, t_1, t_2 and $t_1 t_2$ are proportional to linear functions of a_0, a_1, a_2, a_3, it follows that

> a plane section of the Lie quadric corresponds to a homographic relation between t_1 and t_2. In particular the tangent planes correspond to the singular homographies
>
> $$(t_1 - \alpha)(t_2 - \beta) = 0. \qquad \ldots(7{\cdot}4)$$

8. Infinite Regions

In the projective plane we are accustomed to think in terms of a "line at infinity" and the question of an analogue in the plane of the non-oriented and oriented quadratics naturally suggests itself. The problem is, however, one which requires careful formulation if the analogy is not to be merely superficial.

We conceive of the points or elements of any manifold as defined by one or more sets of numbers called the co-ordinates of the element. The co-ordinates are homogeneous if only the ratios in each set matter, and non-homogeneous if the individual values have significance. In the former case we may divide all the numbers in a set by any one of them, so that each may be supposed not greater than unity in absolute value, and there is thus no question of a co-ordinate becoming infinite. On the other hand, in non-homogeneous co-ordinates, there is no upper bound for their values. If we agree not to consider "infinity" as a number but as associated with a certain process, the co-ordinates of every element are finite but in their totality unbounded.

Suppose now that we establish a correspondence between the elements of two manifolds or of a manifold with itself, which is in general one-to-one (biunivoque) and bicontinuous. If there are no exceptions in this correspondence, there is no need to introduce ideal elements. One such example is the transformation $x' = x + a$, $y' = y + b$ in the cartesian plane. It may however happen as in the case of the projective transformation.

$$x' = \frac{lx + m_1 y + n_1}{l_3 x + m_3 y + n_3}; \qquad y' = \frac{l_2 x + m_2 y + n_2}{l_3 x + m_3 y + n_3}; \qquad \ldots(8{\cdot}1)$$

that the correspondence breaks down for all points on a certain locus Γ —the straight line $l_3 x + m_3 y + n_3 = 0$ in our case—and that as the point x, y approaches this locus its transform x' y' has one or more of its non-homogeneous co-ordinates increasing indefinitely. We have then to introduce ideal elements at infinity to correspond to points on the locus Γ. The particular locus will depend on the nature of the correspondence set up but will be independant of the co-ordinate system used.

It thus appears that ideal elements at infinity may have to be introduced when an analytical correspondence is set up by means of non-homogeneous co-ordinates between two manifolds. The nature of the locus at infinity will depend on the correspondence in question, in other words, on the transformation group whose invariant theory we are studying, or equivalently on the structure we impose on the manifold. When homogeneous co-ordinates are used, there is no question of elements at "infinity" in the sense of having infinite co-ordinates.

Now a proper quadratic equation which has two roots distinct or coincident is specified by three numbers a_0 a_1 a_2 where $a_0 \neq 0$. We take $\xi = a_1/a_0$ and $\eta = a_2/a_0$ as non-homogeneous co-ordinates on the quadratic plane. Now apply a linear transformation

$$x' = (lx + m)/(px + q) \qquad \ldots(8{\cdot}2)$$

to both the roots, since we cannot distinguish between the roots of a non-oriented equation. The point ξ', η' representing the transformed equation is connected with ξ η by relations of the type (8·1) and the correspondence breaks down only when $x \rightarrow -q/p$, when ξ' or η' or both increase indefinitely. Thus to preserve the biunivoque character of the transformation, we postulate a line at infinity on the second plane to correspond to the tangent to Ω at $-q/p$. Similarly there will be a "line at infinity" on the first plane and the closure at infinity is the same as in projective geometry.

In the case of the non-oriented equation we may do likewise, but as the roots are individually known, it would be more general to subject the two roots to two different linear of the type (8·2). In either case, each generator of the quadric H is carried over into another of the same system by a one-to-one correspondence; and the common point of two generators into the common point of their transforms. As in the former case there will be values α, β such that when $t_1 \rightarrow \alpha$ or $t_2 \rightarrow \beta$, the transforms increase indefinitely. We are therefore obliged to postulate on the Lie quadric a region at infinity consisting of two generators intersecting at a unique "point at infinity" to correspond to these generators and their intersections.

For the Laguerre representation, the two pencils of lines $\xi = t_1$ and $\eta = t_2$ take the place of the two systems of generators. The ideal elements at infinity consist of two straight lines, one of each pencil and their common "point at infinity." In fact, the Laguerre plane may be obtained by stereographic projection from the Lie quadric. The closure corresponds to that of an anchor ring.

The group of the oriented quadratic is the automorphic collination group of a ruled quadric, and is a mixed group. There will, therefore, be also transformations which involve an exchange of the two systems of generating lines.

9. Cubics : Non-Oriented

Before concluding, I shall indicate briefly how the concepts developed earlier for quadratics may be extended to cubic equations. The equation

$$A(x) \equiv a_0x^3 - 3a_1x^2 + 3a_2x - a_3 = 0 \qquad \ldots \quad (9{\cdot}1)$$

has three roots distinct or coincident provided $a_0 \neq 0$ and may be represented by the point whose tetrahedral co-ordinates are the ratios $a_0 : a_1 : a_2 : a_3$ or whose Cartesian co-ordinates are a_1/a_0; a_2/a_0; a_3/a_0. Cubics $(x-t)^3 = 0$ whose roots are coincident are represented by the points $(1, t, t^2, t^3)$ which trace out a twisted cubic Γ playing the same role as Ω in the quadratic plane. The osculating plane at any point "t" represents all equations one of whose roots is t; hence the three roots $t_1\ t_2\ t_3$ of the equation (9·1) are the parameters of the three points whose osculating planes pass through $(a_0\ a_1\ a_2\ a_3)$. Points on any tangent to Γ represent equations two of whose roots are equal, and the totality of such equations corresponds to the tangent surface of Γ.

This surface is of the fourth degree and is given by $D = 0$ where

$$\begin{aligned} -27(G^2 + 4H^3) &= a_0^2 D\,; \\ G &= a_0^2a_3 - 3a_0a_1a_2 + 2a_1^3\,; \\ H &= a_0a_2 - a_1^2 \end{aligned} \qquad \ldots \quad (9{\cdot}2)$$

so that

$$D = a_0^4(t_1 - t_2)^2(t_2 - t_3)^2(t_3 - t_1)^2 \equiv 27(3a_1^2a_2^2 + 6a_0a_1a_2a_3 - a_0^2a_3^2 - 4a_0^2a_2^3 - 4a_1^3a_3) \qquad \ldots \quad (9{\cdot}3)$$

The cubic surface $G = 0$ and the conicoid $H = 0$ both pass through Γ.

Equations defined by a pencil of cubics $A(x) + \lambda B(x) = 0$ correspond to a straight line which will be a chord of Γ if it contains two perfect cubes. A net of cubics $A(x) + \lambda B(x) + \mu C(x) = 0$ correspond to a plane.

If we subject all the roots to a homographic transformation, the points in the representative 3-space are subject to a collineation which carries Γ into itself.

10. Oriented Cubics : Partial Orientation

If we are content with a modest degree of orientation we may regard the cyclical orders $t_1t_2t_3$, $t_2t_3t_1$, $t_3t_1t_2$ as equivalent but different from the orders $t_1t_3t_2$, $t_2t_1t_3$, $t_1t_3t_2$ which are also to be considered equivalent. The expression

$$a_4 = (t_1 - t_2)(t_2 - t_3)(t_3 - t_1)a_0$$

is the same for equivalent cyclical orders but changes its sign in passing from the one to the other. We use $(a_0, a_1, a_2, a_3, a_4)$ as five homogeneous co-ordinates in space of four dimensions. These are connected by the relation

$$a_0^2a_4^2 = 27(3a_1^2a_2^2 + 6a_0a_1a_2a_3 - a_0^2a_3^2 - 4a_0^2a_2^3 - 4a_1^3a_3) \qquad \ldots \quad (10{\cdot}1)$$

There are subregions corresponding to one or two of the roots taking assigned values, the former being sections by the primes $a_0\ t^3 - 3a_1t^2 + 3a_2t - a_3 = 0$ and the latter by the planes common to two such primes. These latter are conics, since all the co-ordinates except a_4 are linear in say t_3, when t_1 and t_2 have given constant values.

11. Oriented Cubics : Complete Orientation

A more ambitious scheme of orientation would be to consider all the six different permutations of $t_1\ t_2\ t_3$ as different orientations. I sketch below three possible modes of geometric representation.

(1) The simplest is to take the corresponding point as the one with Cartesian co-ordinates $t_1\ t_2\ t_3$, these being the roots in the proper order. The sub-loci corresponding to the r^{th} root $t_r =$ const. are a series of parallel planes, and three planes belonging one of each system cut in a point. The discriminant locus breaks into three planes $(x - y)(y - z)(z - x) = 0$ and the locus corresponding to three equal roots becomes the line $x = y = z$.

(2) As a redundant co-ordinate, we may take an unsymmetric function of $t_1\ t_2\ t_3$ say $a_0(c_1t_1 + c_2t_2 + c_3t_3) = a_4$ (the c's being unequal constants) which takes, in general, six values. We have then a sextic variety in four dimensions as the field of representation. All cubics for which the first root $t_1 = \alpha$ are represented by points on a conicoid (V_2^2) and we have three systems of such conicoids corresponding to $t_r = \alpha (r = 1, 2, 3)$. It would appear that two surfaces of the same system have two points in common while two of opposite systems

have a common generator. In fact when t_1 and t_2 have assigned values, all the co-ordinates are linear in t_3 so that the variety is a ruled one, whose rulings correspond to such cubics.

(3) Lastly we may introduce three redundant co-ordinates a_4, a_5, a_6 where

$$a_4 = a_0(t_1 - t_2)\,; \quad a_5 = a_0(t_2 - t_3)\,; \quad a_6 = a_0(t_3 - t_1) \quad \ldots \quad (11{\cdot}1)$$

but as these are connected by the identity

$$a_4 + a_5 + a_6 = 0 \quad \ldots \quad (11{\cdot}2)$$

our representative locus lies really in the 5-dimensional linear space. (11.1)

It is given by the two additional equations

$$a_4a_5 + a_5a_6 + a_6a_4 = a_0^2 \sum (t_1 - t_2)\ (t_2 - t_3) = 9\ (a_0a_2 - a_1^2) \quad \ldots \quad (11{\cdot}3)$$

and

$$a_4^2\ \ a_5^2\ \ a_6^2 = a_0^2\mathrm{D} \quad \ldots \quad (11{\cdot}4)$$

where D is a quartic polynomial given by (9·3)

The one and two dimensional sub-loci corresponding to all roots equal or two roots equal are the same as in the non-oriented case and lie in the 3-space $a_4 = a_5 = a_6 = 0$. When t_1 and t_2 have given values, all the co-ordinates are linear in t_3 and the locus is a straight line. For given t_1, the first and last of the equations (11·1) give t_2 and t_3 while the middle one may be replaced by (11·2). All the relations will be found to be linear except one which is quadratic so that the locus is a conicoid in 3-dimensions. On the conicoid on which t_1 has an assigned value, the two systems of generators will correspond to $t_2 = \text{constant}$ and $t_3 = \text{constant}$ respectively.

REVIEWS

J. L. WALSH, *Interpolations and approximations by rational functions.* (American Mathematical Society Colloquium Publications Vol. XX.)

The topic treated in this book has its basis in one of the cardinal problems in the theory of analytic functions, namely, the explicit representation of functions known to be analytic on a given point set. One well-known method is to determine a sequence or series of known function, usually polynomials or rational functions, which converges to the given function uniformly on the point set. The Talyor series is a typical example of approximation by polynomials, the sum of its first n terms—a polynomial of degree n— converging uniformly to the function in the interior points of the circle of convergence of the Taylor series; similarly, a Laurent's series is a typical example of approximation by rational functions. But the usefulness of these is limited by the fact that a Taylor series can represent a function only in a circular region; and a Laurent's series only in a circular ring formed by concentric circles.

The earliest results for more general domains are, in the main, due to Runge (see the reference at the end of the book) who proved that a function analytic in a closed Jordan region could be uniformly approximated by a sequence of polynomials; also that a sequence of polynomials could be found which converges uniformly to a function simultaneously in a finite number of mutually exterior Jordan regions in each of which the function is analytic. He proved a similar result for approximation by rational functions. Though properties of various special types of polynomial sequences have been investigated, definite advance on the results of Runge has been made only recently, a matter of two decades. The major contribution to the development of the subject has been made by the author himself in a series of papers ranging over ten years, in various mathematical journals. In the book any general theorem is invariably accompanied by a short reference to the stages through which the result has reached its present form. Besides analytic functions, the approximation to functions continuous on given point sets is also considered in several parts of the book.

The first problem is the possibility of approximation on arbitrary point sets and this is discussed in the first two chapters. In the first chapter Runge's theorem on approximation by rational functions is extended to functions analytic on any closed point set and definite results are obtained on the freedom of choice of the poles of the approximating rational functions. Some necessary and sufficient conditions for the possibility of approximation by rational functions with assigned poles are given. In chapter 2 similar results are obtained for functions analytic on the interior points of a closed set and continuous on the boundary, the method used being that of conformal mapping of variable regions coupled with the results of the previous chapter. Some results are obtained on the approximation of functions merely continuous on arcs of Jordan curves.

In chapters 3 to 7 a detailed study is made of polynomial sequences approximating to given functions. Chapter three deals with polynomial sequences obtained by interpolation to functions analytic in regions bounded by lemniscates—curves of the form $|p(z)| = \text{constant}$, $p(z)$ being a polynomial. In chapter 4, these results are applied to the study of polynomial approximations in more general regions by the use of Green's functions coupled with Hilbert's famous theorem on uniform approximation to an arbitrary Jordan curve by means of lemniscates. The rapidity of convergence, as measured by $\mu_n = \max_{z \text{ on } c} : \sigma_n(z)$ where $\sigma_n(z) = |f(z) - p_n(z)|$, and $p_n(z)$ approximates to $f(z)$ on the point set c, is studied by the introduction of the notion of maximal convergence. Chapter 5 introduces various measures of approximation besides μ_n, such as, the line integral of $\{\sigma_n(z)\}^p, p > 0$, over the boundary of c with or without weight, and the surface integral, with or without weight, of the same expression over the area of c. The relations of these to the uniform convergence of $\sigma_n(z)$ to zero at the interior points of c are also studied. Chapter 6 gives a detailed account of approximation by least squares based on the Riesz-Fischer theory of Functions orthogonal on a given curve. Chapter 7 is devoted to the study of polynomial sequences obtained by interpolation at special sets of points such as, the nth roots of unity, $n = 1, 2, \cdots$; sets of points of uniform density; and sets determined by various extremal properties such as making a given expression a maximum or minimum.

Chapters 8, 9 deal with sequences of rational functions with assigned poles obtained by interpolation in the same way as rational function of chapters 3—7. The various problems studied in the latter connection are discussed with reference to these rational functions. It is proved that related configurations G_1 and G_2 and points sets S_1 and S_2 could be found so that

(i) S_1 is in G_1 and S_2 in G_2;

(ii) the rational functions with poles at S_2 obtained by interpolation at S_1 to a function analytic in G_1 converge uniformly to the function in the interior of G_2;

(iii) the rational functions with poles at S_1 obtained by interpolation at S_2 to a function analytic in G_2 converges uniformly to the function in the interior of G_2;

provided S_1 and S_2 satisfy certain conditions in relation to G_1 and G_2. This is called the principle of duality. As an illustration consider a circle Γ round the origin $z = 0$ and its closed compliment Λ. Let $f(z)$ be analytic in Γ and $g(z)$ in Λ. Then, the sequence of rational functions with poles of order n at infinity (polynomials) interpolated to $f(z)$ at $z = 0$ counted $n + 1$ times, $n = 1, 2, \cdots$, converges to $f(z)$ uniformly in the interior of Γ; this is the Taylor series. Similarly, rational functions with poles of order n at $z = 0$ interpolated to $g(z)$ at infinity counted $n+1$ times, $n=1, 2, \cdots$, converges to $g(z)$ in the interior of Λ; this is the Laurents' series.

Chapter 10 deals with interpolation to functions holomorphic in the unit circle. It discusses the existence of functions with a given upper bound taking prescribed values at assigned points. Several extremal problems are discussed in this connection. Chapter 11 is devoted to the existence and properties of sequences of approximating rational functions fulfilling various types of auxiliary conditions, some of these being extremal conditions.

The concluding chapter discusses the question of the existence of polynomials and rational function of given degree which approximates to a given function with least error as measured by the various measures of approximation introduced in chapter 5.

There is a well-chosen bibliography at the end on the topics treated in the book. The book will be found very useful for those who wish to carry on to further research in the same field and should serve as an excellent book of reference.

V. GANAPATI.

WILHELM BLASCHKE: *Intégralgéométrie*—(Actualités Scientifiques et Industrielles, Hermann et Cie, Paris, 1935.)

This is the first fascicle of the series "Exposés de Géométrie" which will be published under the direction of Prof. W. Blaschke.

By "geometrical probabilities" is generally meant a theory based on the determination of absolute integral invariants of certain groups. This is clearly developed, for instance, in Deltheil's *Probabilités géométriques* (Gauthier-Villars, Paris, 1926), chiefly in the second chapter. As a matter of fact, the determination of such integral invariants is the main difficulty of the theory. Prof. Blaschke proposes therefore to call it "Intégralgéométrie" rather than "Geometrische Wahrscheinlichkeiten".

In the short fascicle (22 pages) under review, Prof. Blaschke considers the following cases:

1. A group of movements in a $(n-1)$-dimensional sphere or in a $(n-1)$-dimensional space of Cayley transforming the *points* of that manifold, *plus* the general group of euclidean rotations transforming at any point of the manifold the unitary vectors.

2. The general punctual group of euclidean rotations in an n-dimensional euclidean space, *plus* the general group of euclidean rotations transforming as above at any point of the manifold the unitary vectors.

3. The most general punctual group of movements in an n-dimensional euclidean space, *plus* at any point of it the same group of rotations as above for the unitary vectors.

In each of these cases the method employed is exceedingly simple and elegant. It is the method of the "repère mobile" of Prof. Cartan (which has been explained recently in the same series of the Actualités). It starts from the "elementary rotations" which are called the ω_{ij}'s in Cartan's work and which Prof. Blaschke calls here the p^{ij}'s. The integral invariants which are required for some of the most important problems of geometrical probabilities (Cf. Deltheil's treatise) are formed by means of outer products of certain of the p^{ij}'s and of the expressions of certain volumes, when the punctual group is the general group of motion.

C. RACINE.

G. BOULIGAND, G. GIRAUD AND P. DELENS: *Le Problème de la Dérivée Oblique en Théorie du Potentiel*—(Actualités Scientifiques et Industrielles, Hermann et Cie, Editeurs, Paris, 1935; 278 pages.)

The theory of the partial differential equations of the second order—chiefly of the elliptic and hyperbolic types—are of the utmost importance in theoretical physics, and it has been very much studied during the past years. The short book under review deals with certain developments of the theory directly connected with the work of Prof. G. Giraud who has written the second part of this fascicle.

In its first part, Prof. G. Bouligand points out the nature and the different aspects of the "problème de la dérivée oblique". The problem is to determine a harmonic function which satisfies the following boundary condition: on the surface S which bounds a domain Ω, the derivative of the function along a direction l_q varying continuously as the point Q varies continuously on S, must be equal to a certain function $f(Q)$.

The most interesting result of the general theory as outlined by Prof. Bouligand is that if l_q is tangent to S at the points of a certain domain, the problem is altogether indeterminate. If, on the contrary, l_q is never tangent to S there exists under certain very general conditions one and only one solution.

This proposition shows the importance of the equipotential congruences of curves in the theory of the above problem. Dr. P. Delens, in the third part of the fascile deals with them and his main theorem is that if a congruence is equipotential, it is also isotropic (p. 70). This third part is not always very easy reading, at least for those who are not familiar with the vectorial notation but it is full of very suggestive ideas for research students.

The second part constitutes, I think, the most interesting one. Prof. G. Giraud gives in it a brief account of some of his methods. The "problème de la dérivée oblique" in the 'regular' case, namely when l_q is never tangent to S, is solved by means of a generalized Green's function. This function is itself determined by means of a Fredholm equation. But the integral which appears in this equation is an improper integral and therefore it must be interpreted as its own 'Cauchy's principal value'. This leads to a new theory of Fredholm equations which has been developed of late by Prof. G. Giraud* and which he is still busy improving. The present book may be taken as a good introduction to this new chapter of the theory of integral equations. Moreover to those engaged in teaching I point to pages 30-35 which contain a mine of very interesting problems of integral Calculus for honours students.

C. RACINE,

BOOKS RECEIVED FOR REVIEW

Actualités Scientifiques et industrielles: Hermann Fils, Paris.

No. 270. GODEUX: Les involutions cycliques appartenant à une surface algébrique, Price 12 Fr.

No. 274. BOULIGAND: Définitions moderns de la dimension, Price 12 Fr.

No. 285. MUKHERJEE, A. C.: Etude Statisque de la Fécondite Matrimoniale, Price 16 Fr.

No. 302. FAVARD: Les Théorems de la moyenne pour les polynomes, Price 15 Fr.

No. 305. MANDELBROJT: Series Lacunaires, Price 12 Fr.

No. 323. CHEVALLEY: L'Arithmétique dans les algébres de matrices, Price 10 Fr.

No. 325. CH. PLATRIER: Cinématique du solide et Théorie des vecteurs. Fr. 12.

No. 326. CH. PLATRIER: La masse en cinématique et théorie des tenseurs du second ordre. Fr. 18.

No. 327. CH. PLATRIER: Cinématique des milieux continus. Fr. 8.

No. 329. D. MENCHOFF: Les conditions de monogenitie. Fr. 15.

No. 333. EDOUARD GOURSAT: Proprietes générales de l'equation d'Euler et de Gauss. Fr. 20.

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*Cf. *Annales de l'Ecole Normale Supérieure*, tome 51, 1934 p. 251-372 and t. 53, 1936, p. 1-40.

THE INDIAN MATHEMATICAL SOCIETY

Statement of Accounts for 1934

Receipts	Rs.	A.	P.
Opening Balance			
Savings Bank a/c.	1540	2	10
Current a/c.	142	6	3
Fixed Deposit (Post-office Cash Certificates)	5600	0	0
Subscriptions from members	1397	0	0
Life Subscriptions	450	0	0
Subscription for Journal	718	3	0
Grant in aid			
Annamalai University	100	0	0
Bombay University	200	0	0
Silver Jubilee Donation	45	0	0
Interest on S. B. Acct.	49	6	5
Total ...	10,242	2	6

Expenditure	Rs.	A.	P.
Ordinary working expenses ...	398	3	0
Books and periodicals	556	2	0
Library expenses	310	0	0
Journal and Silver Jubilee Volume printing	2418	3	9
Miscellaneous	11	9	6
Closing balance			
Fixed Deposit (P. O. Cash Certificates)	5600	0	0
Savings Bank a/c.	838	3	9
Current a/c.	109	12	6
Total ...	10,242	2	6

L. N. SUBRAMANIAM,

Honorary Treasurer.

THE INDIAN MATHEMATICAL SOCIETY

Statement of Accounts for 1935

Receipts

Receipts	Rs.	A.	P.
Opening Balance			
Savings Bank a/c.	838	3	9
Current a/c.	109	12	6
Fixed Deposit (P. O. Cash Certificates)	5600	0	0
Subscriptions from members	1109	0	0
Subscription for Journal	1019	6	0
Life Subscription	180	0	0
Miscellaneous	11	14	0
Grant in aid			
Bombay University	200	0	0
Oosmania University	200	0	0
Madras University	150	0	0
Nagpur University	100	0	0
Annamalai University	100	0	0
Interest on S. B. a/c.	38	14	4
Advance by Asst. Secretary	30	0	0
Total ...	9687	2	7

Expenditure

Expenditure	Rs.	A.	P.
Return of advance to Asst. Secretary	30	0	0
Ordinary working expenses ...	375	14	0
Books and periodicals	680	4	0
Journal and other printing ...	983	8	0
Library expenses	400	0	0
Miscellaneous	6	5	0
Closing balance			
Fixed Deposit (P. O. Cash Certificates)	5600	0	0
Savings Bank A/c.	1415	10	1
Current A/c.	195	9	6
Total ...	9687	2	7

L. N. SUBRAMANIAM,

Honorary Treasurer.

The Indian Mathematical Society

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All communications intended for publication should be sent to the Editor, A. NARASINGA RAO, Annamalai University, Annamalainagar, South India. Communications relating to the receipt of the Journal, changes of address, and the purchase of current or back numbers of this journal should be sent to the Assistant Secretary, S. MAHADEVA IYER, Assistant Professor of Mathematics, Presidency College, Madras.

Remittances of membership fees and of subscriptions should be sent to the Treasurer, Prof. L. N. SUBRAMANIAM, Department of Mathematics, Christian College, Madras.