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A paper should contain a short and clear summary of the new results obtained and the relations in which they stand to results already known. Contributors are requested to bear in mind that, at the present stage of mathematical research, hardly any paper is likely to be so completely original as to be independent of earlier work in the same direction ; and that readers are often helped to appreciate the importance of a new investigation by seeing its connection with earlier results.

The principal results of a paper should, when possible, be enunciated separately and explicitly in the form of definite theorems.

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Contributors will be supplied, if so desired, with extra copies of their contributions at net cost.

All contributions should be written legibly on one side only of the paper and all diagrams should be neatly and accurately drawn on separate slips.

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3. Cayley's Condition.

If the three points (x_i, y_i, z_i) $(i = 1, 2, 3)$ be collinear, we have the necessary condition due to Cayley, *viz.*

$$x_1x_2x_3 + y_1y_2y_3 + z_1z_2z_3 = 0.$$

In virtue of the importance of this result we just add an easy method of verification;* consider the cubic

$$x^3 + y^3 + z^3 + 3kxyz = 0.$$

Any straight line $\frac{x - a}{l} = \frac{y - b}{m} = \frac{z - c}{n} = r$,

will meet it in points given by

$$\phi(r) = \Sigma (a + lr)^3 + 3k \,\Pi\, (a + lr)$$

i.e., $$Ar^3 + 3Br^2 + 3Cr + D = 0,$$

where
$$A = l^3 + m^3 + n^3 + 3klmn$$
$$B = \Sigma (al^2) + k \Sigma (amn)$$
$$C = \Sigma (a^2l) + k \Sigma (abn)$$
$$D = a^3 + b^3 + c^3 + 3\,kabc.$$

Now
$$x_1 x_2 x_3 = (a + lr_1)(a + lr_2)(a + lr_3)$$
$$= \frac{(an - cl)^3 - (bl - am)^3}{A}$$

$\therefore$
$$\Sigma x_1 x_2 x_3 = 0,$$

which is Cayley's result.

It is quite easy to show that Cayley's condition is not sufficient. That is, if

$$\Sigma x_1 x_2 x_3 = 0,$$

it does not necessarily follow that the three points are collinear. In fact, consider the line $xx_1x_2 + yy_1y_2 + zz_1z_2 = 0$ which passes through (x_3, y_3, z_3) if we suppose Cayley's condition to be satisfied. This intersects the cubic in two points other than (x_3, y_3, z_3) which are not collinear with (x_1, y_1, z_1) and (x_2, y_2, z_2) but yet satisfy the condition in question. Thus the condition although necessary is not sufficient and we can even specify that the order of insufficiency is two, inasmuch as there are two points which satisfy the condition but do not ensure collinearity.

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* Kindly suggested by Prof. M T. Naraniengar.

It follows therefore that the full geometrical import of Cayley's condition must be wider in scope than mere collinearity. We derive the geometrical significance of the condition from the following considerations. We know that if P (x_1, y_1, z_1) and Q (x_2, y_2, z_2) are two points in the plane of the cubic

$$x^3 + y^3 + z^3 + 6\,mxyz = 0 \quad \ldots \quad \ldots \quad (i)$$

the equation of the apolar line of P and Q with respect to the cubic is

$$x\,x_1\,x_2 + y\,y_1\,y_2 + z\,z_1\,z_2 + 2m\,\{\ x\,(y_1\,z_2 + y_2\,z_1) + \ldots\ \} = 0. \ (ii)$$

Now if R (x_3, y_3, z_3) is any point on (ii) the relation between the co-ordinates is symmetrical and the triangle PQR is apolar to the cubic.

When P and Q are fixed points, we have an apolar line associated with each cubic of the syzygetic family (i), the apolar lines forming a pencil. When P and Q are points on a syzygetic cubic, the vertex of the pencil is the third point where PQ meets that cubic again.

Let us consider P and Q, two points on the cubic (i). The apolar line of P and Q with respect to the equi-anharmonic cubic

$$x^3 + y^3 + z^3 = 0$$

is

$$x_1\,x_2\,x + y_1\,y_2\,y + z_1\,z_2\,z = 0; \quad \ldots \quad (iii)$$

the triangle PQR where R is any point on (iii) is apolar to $x^3 + y^3 + z^3 = 0$. The line (iii) meets the cubic (i) in three points, one of which is the vertex of the apolar lines and is therefore the point where PQ meets the cubic (i) again. Hence the triangle formed by P, Q and any one of these points is apolar with respect to $x^3 + y^3 + z^3 = 0$. The geometrical significance of three points on a cubic satisfying Cayley's condition is therefore that "*the triangle formed by these points is apolar with respect to the equi-anharmonic cubic of the first species with its Hessian consisting of three real straight lines.*"

The property is an invariant one and as such, if it holds for the cubic (i) it also holds for the cubic in the general form.

4. Other linear conditions.

In addition to Cayley's condition, we have also a condition due to Hilton* *viz.*,

$$x_3\,(y_1z_2 + y_2z_1) + y_3\,(z_1x_2 + z_2x_1) + z_3\,(x_1y_2 + x_2y_1) = 0,$$

* See Hilton: *Plane Algebraic Curves*, p. 238; Ex. 13. We give a proof of this in § 5.

which is also a necessary condition but not a sufficient one. This fact can be established by considering the intersections of the cubic by the line

$$x(y_1z_2 + y_2z_1) + \ldots\ldots + \ldots\ldots = 0;$$

we find two points in addition to (x_3, y_3, z_3) which satisfy the condition but are not collinear with (x_1, y_1, z_1) and (x_2, y_2, z_2).

We now proceed to establish a theorem, *viz.*:

THEOREM: *Cayley's and Hilton's conditions taken together are necessary and sufficient for the collinearity of the three points.*

For, if the join of A (x_1, y_1, z_1) and B (x_2, y_2, z_2) be not collinear with C (x_3, y_3, z_3) let it meet the cubic again at a point C′ (x_3', y_3', z'_3). Since A,B,C′ are collinear we have the two conditions

$$x_1x_2x'_3 + y_1y_2y'_3 + z_1z_2z'_3 = 0$$

$$(y_1z_2 + y_2z_1)\, x'_3 + \quad \ldots \quad + \quad \ldots \quad = 0$$

but since C also satisfies both the conditions we deduce that C and C- both lie on the lines

$$xx_1x_2 + \ldots\ldots + \ldots\ldots = 0;$$

$$(y_1z_2 + y_2z_1)x + \qquad = 0;$$

i.e. C and C′ are one and the same point. Thus the two conditions are together necessary and sufficient.

It is to be observed that the conditions due to Hilton and Cayley are linear in any one set of letters (x_1, y_1, z_1), (x_2, y_2, z_2) and (x_3, y_3, z_3). We now proceed to determine other such conditions which can be called *linear conditions*. We have the general condition of collinearity of any three points in the plane

$$x_3\,(y_1z_2 - y_2z_1) + \ldots\ldots + \ldots\ldots = 0$$

which is certainly a necessary condition. Combining this with Hilton's condition, *viz.*,

$$x_3\,(y_1z_2 + y_2z_1) + \ldots\ldots + \ldots\ldots = 0$$

which is also a necessary condition, we obtain two other necessary conditions by addition and subtraction in the form

$$x_3y_1z_2 + y_3z_1x_2 + z_3x_1y_2 = 0,$$

$$x_3y_2z_1 + y_3z_2x_1 + z_3x_2y_1 = 0;$$

while each of these conditions taken separately is merely necessary, we can establish, by a proof analogous to that used above that these two taken together are necessary as well as sufficient. These two conditions are further worthy of notice inasmuch as they give us remarkably simple expressions for the co-ordinates of the collinear point on the cubic. In fact, we have from the two equations above*

$$\frac{x_3}{(y_1 z_1 x_2^2 - y_2 z_2 x_1^2)} = \frac{y_3}{(z_1 x_1 y_2^2 - z_2 x_2 y_1^2)} = \frac{z_3}{(x_1 y_1 z_2^2 - x_2 y_2 z_1^2)}.$$

We can go still further and state the elegant result that of the three conditions

$$\left.\begin{aligned} x_1 x_2 x_3 + y_1 y_2 y_3 + z_1 z_2 z_3 &= 0 \\ y_1 z_2 x_3 + y_3 z_1 x_2 + y_2 z_3 x_1 &= 0 \\ x_3 y_2 z_1 + y_3 z_2 x_1 + z_3 x_2 y_1 &= 0 \end{aligned}\right\}$$

any two taken together constitute the necessary and sufficient conditions for the collinearity of the three points in question.

5. Necessary conditions of higher degree.

We have hitherto considered only conditions which are linear in terms of the co-ordinates of any one of the three points. We next proceed to determine other necessary conditions which are of higher degree, incidentally indicating a general method of deducing necessary conditions.

Let the cubic

$$x^3 + y^3 + z^3 + 6\,mxyz = 0$$

be cut by the line

$$\alpha x + \beta y + \gamma z = 0$$

in the points (x_1, y_1, z_1); (x_2, y_2, z_2) and (x_3, y_3, z_3).

Let us join with these two another arbitrary relation

$$px + qy + rz = 0.$$

The result of the elimination† of $x : y : z$ from these three expressions is

$$(\beta r - \gamma q)^3 + (\gamma p - \alpha r)^3 + (\alpha q - \beta p)^3 + 6m(\beta r - \gamma q)(\gamma p - \alpha r)(\alpha q - \beta p) = 0,$$

---

* These are the expressions for the collineal, otherwise deduced by Sylvester in his *Collected Mathematical Papers*: Vol. 3, pp. 354-5.

† This method of elimination is due to **Schlafi**: See his memoir "*Ueber die Resultante eines Systemes meherer algebraischer Gleichungen*" which has been expounded by **Cayley**. See his *Coll. Papers*, Vol. 2, pp. 454—64.

which is a ternary cubic in (p, q, r), say

$$(\mathrm{A, B, C, F, G, H, I, J, K, L})(p, q, r)^3 = 0,$$

where

$$\mathrm{A} = \beta^3 - \gamma^3;\ \mathrm{B} = \gamma^3 - \alpha^3;\ \mathrm{C} = \alpha^3 - \beta^3$$

$$\mathrm{F} = \gamma(\beta\gamma + 2k\,\alpha^2); \quad - \quad \mathrm{I} = \beta(\beta\gamma + 2k\,\alpha^2);$$

$$\mathrm{G} = \alpha(\gamma\alpha + 2k\,\beta^2); \quad - \quad \mathrm{J} = \gamma(\gamma\alpha + 2k\,\beta^2);$$

$$\mathrm{H} = \beta(\alpha\beta + 2k\,\gamma^2); \quad - \quad \mathrm{K} = \alpha(\alpha\beta + 2k\,\gamma^2);$$

and $\mathrm{L} = 0.$

But this resultant can also be put in the form

$$(px_1 + qy_1 + rz_1)(px_2 + qy_2 + rz_2)(px_3 + qy_3 + rz_3) = 0,$$

so that we have the relations which express A, B and C in terms of the fundamental symmetrical functions of of the system of roots $(x_1\ y_1\ z_1)$, *viz.*,

$$\mathrm{A} = x_1 x_2 x_3$$

$$\mathrm{B} = y_1 y_2 y_3$$

$$\mathrm{C} = z_1 z_2 z_3.$$

$$3\,\mathrm{F} = y_1 y_2 z_3 + y_2 y_3 z_1 + y_3 y_1 z_2;$$

$$3\,\mathrm{I} = y_1 z_2 z_3 + y_2 z_3 z_3 + y_3 z_2 z_2$$

$$3\,\mathrm{G} = z_1 z_2 x_3 + z_2 z_3 x_1 + z_3 z_1 x_2;$$

$$3\,\mathrm{J} = z_1 x_2 x_3 + z_2 x_3 x_1 + z_3 x_1 x_2$$

$$3\,\mathrm{H} = x_1 x_2 y_3 + x_2 x_3 y_1 + x_3 x_1 y_2;$$

$$3\,\mathrm{K} = x_1 y_2 y_3 + x_2 y_3 y_1 + x_3 y_1 y_2$$

$$6\,\mathrm{L} = \Sigma\, x_3(y_1 z_2 + y_2 z_1).$$

Thus A, B, C, ... L are expressed in terms of (x_i, y_i, z_i) as well as of (α, β, γ) and by eliminating $\alpha : \beta : \gamma$ between any three expressions thus obtained, we shall have a necessary condition involving (x_i, y_i, z_i).

We observe immediately that

$$\therefore\ 6\mathrm{L} = \Sigma\, x_3(y_1 z_2 + y_2 z_1) = 0,$$

which is no other than Hilton's condition.

Further $\quad \mathrm{A} + \mathrm{B} + \mathrm{C} = \Sigma(\beta^3 - \gamma^3) = 0.$

$$\therefore\ \Sigma\, x_1 x_2 x_3 = 0$$

which is Cayley's condition.

We have further, since

$$F = \gamma(\beta\gamma + 2k\alpha^2) \text{ and } - I = \beta(\beta\gamma + 2k\alpha^2),$$

$$\frac{F}{I} = -\frac{\gamma}{\beta}.$$

Similarly $$\frac{G}{J} = -\frac{\alpha}{\gamma},$$

and $$\frac{H}{K} = -\frac{\beta}{\alpha};$$

so that $$\frac{F}{I} \cdot \frac{G}{J} \cdot \frac{H}{K} = -1,$$

i.e., $$FGH + IJK = 0.$$

This is therefore a necessary condition which is of degree three in the co-ordinates of any one set, since F, G, H, I, J, K are all linear by themselves. We can similarly deduce conditions of degrees six, nine, etc., but we shall take

$$FGH + IJK = 0,$$

as typical of the higher degree conditions. That this is a necessary condition is obvious from its very method of derivation. We now proceed to show that it is really a condition independent of the linear conditions deduced in § 5 and that it is not also a sufficient condition. The condition can be written in the form

$$27(FGH + IJK) = 0:$$

and putting $l = x_1 x_2$; $m = y_1 y_2$; $n = z_1 z_2$

$$l' = y_1 z_2 + y_2 z_1; \; m' = z_1 x_2 + z_2 x_1; \; n' = x_1 y_2 + x_2 y_1$$

this can be written in the form

$$\begin{aligned} 2(lmn + l'm'n')x_3y_3z_3 &+ m(m'n' + ll')z_3^2x_3 \\ + m(l'm' + nn')z_3x_3^2 &+ n(n'l' + mm')x_3^2y_3 \\ + n(m'n' + ll')x_3y_3^2 &+ l(l'm' + nn')y_3^2z_3 \\ &+ l(n'l' + mm')y_3z_3^2 = 0. \end{aligned}$$

Using the conditions

$$A + B + C = 0, \quad L = 0$$

i.e., $$lx_3 + my_3 + nz_3 = 0$$

and $$l'x_3 + m'y_3 + n'z_3 = 0$$

this reduces on rearrangement to the symmetrical form

$$\begin{aligned} &lmn'z_3^3 + mnl'x_3^3 + nlm'y_3^3 \\ &\quad + (mn'^2 + nn'^2 + ll'^2 - 2lmn - 2l'm'n')x_3y_3z_3 = 0. \end{aligned}$$

Making x_3, y_3, z_3 the current co-ordinates, this represents a non-singular cubic and hence the condition cannot reduce to the cube of a linear expression as otherwise the corresponding cubic would have a triple point.

Thus the condition

$$FGH + IJK = 0$$

is really a condition independent of the linear conditions. This same conclusion can also be reached by examining the cubic

$$\Sigma\, mnl'x^3 + \{ \Sigma\, ll'^2 - 2lmn - 2l'm'n' \}\, xyz = 0$$

for a triple point at (x_3, y_3, z_3), *i.e.*, taking Sylvester's form for the collineal at

$$(y_1z_1x_2^2 - y_2z_2x_1^2, \ldots\ldots, \ldots\ldots).$$

That this condition is insufficient follows very easily. The cubic considered above intersects the original cubic in nine points of which (x_3, y_3, z_3) alone is collinear with $(x_1 \cdot y_1, z_1)$ and (x_2, y_2, z_2), whereas the other eight points while they satisfy the condition

$$FGH + IJK = 0$$

are non-collinear with them. We can thus state that the order of insufficiency is 8 in this case. Similarly any condition of a higher degree can be proved to be insufficient. In fact, we can state that for a condition

$$f_n(x\ y_i\ z_i)_{i=1,2,3} = 0$$

the order of insufficiency is $3n - 1$; for making (x_3, y_3, z_3) the current co-ordinates the equation would represent an n-*ic* which intersects the original cubic in $3n$-points of which (x_3, y_3, z_3) is one. Thus any one of the remaining $(3n - 1)$ points taken with the points (x_1, y_1, z_1) and (x_2, y_2, z_2) would satisfy the above condition but would not be collinear with those points.

Definition of Number*.

By S. V. Ramamurty, M.A., I.C.S.

1. In his "*Introduction to Mathematical Philosophy*," Bertrand Russell defines number as the class of similar classes. This definition was first given by Frege in 1884 but was practically ignored till it was rediscovered by Russell. I am not aware of any other definition of number by a mathematician.

Taking a particular number, 2 is defined as the class of couples. The class (or collection) of Mr. and Mrs. A, the class of Mr. and Mrs. B, the class of Mr. and Mrs. C and so on are similar classes as there is a one-to-one relationship between members of any two classes. The class of these similar classes is the number 2. As criticised by Professor Alexander in his "*Space, Time and Deity*," this definition tells us little about what number is—as little as saying that man is the class of men tells us about men—and is a definition by extension. Russell himself points out that a definition by intension is logically more fundamental than one by extension. It is indeed permissible to hold with Alexander that the similarity of classes is itself the result of each possessing a common characteristic, *viz*., a common number. Take a couple. The number 2 is already involved in that collection. It is immaterial whether in the world there are other couples. But the existence of other couples is necessary for defining 2 as Russell does. I submit that Russell's analysis of the notion of number is not deep enough.

2. Professor Alexander seeks to give a definition of number by intension by stating it to be the plan of a whole of parts. This definition is not definite enough. There is not necessarily only one relation between a whole and its parts. A number may be one scheme of relationship but not the only one. Take two men standing together. The plan of the whole of that collection of parts may be stated in different ways. We may say the whole is formed by the men standing back to back. This definition tells us as little as to what number is as Russell's.

I consider that the notion of number needs to be analysed further so as to yield a definition which is deeper than Russell's and more definite than Alexander's.

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* Presented to the Fourth Session of the Indian Mathematical Conference, Poona, April 1924.

3. In my paper on "*Time, Space, Matter and Mind*" (Vol. XV, No. 1 of this Journal) I quoted from Bergson's *Introduction to Metaphysics* that "quantity is incipient quality" and used that dictum in building up my general hypothesis as to the relation of time, space, matter and mind. But what is at the basis of quantity itself? I regard the world as evolved out of the undifferentiable primeval stuff into time, then space, then matter and then mind. Take the first evolution of an atom of time—the smallest particle of time that we cognize. Any aggregation less than that does not develop the quality of time and is not cognized as time. The emanation of that quality involves the emanation of an incipient quantity. We can say that quality is incipient quantity. Imagine that we watch the first creation of the world. An atom of time A forms. Then an atom of time B forms. Then an atom of time C. We call each an atom of time not by comparing each of them with one of them, but by comparing each atom of time with the notion of timeness in our mind.

Compare a collection of horses. Let us watch the aggregate grow and imagine that first the legs and then the head and then the full body of a horse appear. In the course of the growth, there is no horse when there are only legs, none when there are legs and the head only and so on. A point in the growth is reached when a horse evolves and we cognize it by comparing the evolved product with the notion of horseness in our mind. So too again we say there is another horse when again the evolved product agrees with the notion of horseness in our mind. The two horses may be one big and one small, one an Arab and one a Waler. But as against the notion of horseness, we attach the number 2 to that collection. So too a collection of a man and a dog is a collection of 2 animals—the quality evolved in each instance being animalness. A horse and a book are a collection of 2 things—each evolving the quality of thingness. Thus number is attached to an aggregate of perceptions of quality irrespective of the variety of the quality. The quality of time is the primary quality evolved in the Universe. It is so on the Einsteinian theory where all entities are made of space-time and space is made up of three independent directions any one of which may furnish a time line.

Take the number 2; it is attached to 2 men, a man and a dog, a horse and a book and also 2 atoms of time. Of these classes, the class of 2 atoms of time is the most primary class, as time is the most primary of all entities. Number is independent of the particular quality of the entities. All that it needs is that there should be an entity. The least an entity can be is to be an atom of time. Hence the number 2 (and so too any other natural number) is attached to a piece of time just as it is attached to a group of animals. What is the mode of this attachment ?

Russell begins by saying that most philosophers have fallen into the error of confusing a trio of men for an instance of the number 3 (*vide* Chapter II of his book on *Mathematical Philosophy*, already quoted). Having defined number as a class of classes, he states that we naturally think that the class of couples is something different from the number 2; but adds :

" There is no doubt about the class of couples. It is indubitable and not difficult to define whereas the number 2, in any other sense, is a metaphysical entity about which we can never feel sure that it exists or that we have tracked it down. It is therefore more prudent to content ourselves with the class of couples, which we are sure of, than to hunt for a problematical number 2 which must always remain elusive."

Thus Russell while denying that number is a class is content to say it is a class—of classes—and this on the ground of prudence.

Further, as already pointed out, to call number a class of (similar) classes gives no information as to what it is.

The same arguments that Russell gives for identifying a number with a class of similar classes apply to identifying number with a class, provided we can find a suitable class. Such a class I submit is an aggregate of time atoms. All entities evolve from time atoms. A number of time atoms is involved in the same number of any other entities. If we say the number is that aggregate of time, it is involved in the other aggregates also and hence is attached to them too. There is no need either in logic or from prudence to regard 2 as separate from the piece of time to which it is attached. I define a number or rather, I define a natural number, with which alone we have been dealing, thus:—

" A number is a piece of time."

Compare: Russell's definition: 'A number is a class of similar classes;' and Alexander's definition : 'A number is the plan of a whole of parts.'

As against Russell's, I regard the class of time atoms to be the most primary of the similar classes; and as against Alexander's, I may say that a piece of time is a whole of parts and its plan is involved in wholes of more complex entities.

4. Let M be a natural number; that is, a piece of time.

We define M in M horses as a relational number by stating that M horses have a one-to-one relationship with M.

Russell treats + M, — M, M/N, *i*M as relational numbers when M is a natural number. This view can be accepted on my definition too. I extend it to M in M horses. So also dinitive and trinitive numbers can be treated as relational numbers.

The primary identity

$$1 = 1$$

is true not because we identify 1 with 1, but both with the notion of oneness in our mind. Hence the primary identity can be fully written out in the form

$$1 = 1, \text{ relatively to } 1;$$

or, in the form

$$1 = 1 \parallel 1.$$

Some new formulæ connected with the working of a Superannuation Fund

BY M. VAIDYANATHAN, M.A., L.T.

[In the paper on "*An Actuarial Study of a Superannuation Fund for the Teachers of the Madras Presidency*" read at the Fourth Indian Mathematical Conference, the writer exhaustively dealt with the fundamental principles that govern the working of such funds, the methods of tabulation and graduation to be adopted, the commutation columns necessary for calculating the present values of future liabilities due to withdrawal, mortality and retirement, and lastly various points of purely actuarial interest arising out of valuation of such funds. The mathematical formulæ which form Part II of the paper are extracted below.]

In this paper, x denotes the age of entry, and s_x the average salary between ages x and $(x + 1)$.

Also l_x represents those in service at age x

d_x	,,	dying in service between x and $(x+1)$		
w_x	,,	withdrawing	,,	,,
r_x	,,	retiring	,,	,, .

SECTION A.

Present value of future contributions.

Assuming that contributions to the fund are a definite percentage of salaries, the present value of 1 per cent of future salaries

$$= \cdot 01 \int_x^w v^t \, l_{x+t} \; s_{x+t} \, dt$$

where w is the limiting age of the service, and l_{x+t} represents those in active service at the end of time t out of l_x persons entering service. This is on the assumption that l_x and s_x are continuous functions of x. But we shall suppose that deaths, withdrawals and retirements are uniformly distributed throughout the whole year and that the salaries are received at the middle of the year. Then the present value, equated to 1 of the present salary s_x, of every one of l_x persons

$$= \frac{\cdot 01}{l_x s_x} \left\{ v^{\frac{1}{2}} l_{x+\frac{1}{2}} s_x + v^{\frac{3}{2}} l_{x+\frac{3}{2}} s_{x+1} + \dots\dots\dots\dots \right\}$$

$$= \frac{\cdot 01}{v_x l_x s_x} \left\{ v^{x+\frac{1}{2}} l_{x+\frac{1}{2}} s_x + v^{x+\frac{3}{2}} l_{x+\frac{3}{2}} s_{x+1} + \dots\dots\dots \right\}$$

where $l_{x+\frac{1}{2}}$ can be taken to be $\frac{1}{2}(l_x + l_{x+1})$, the summation being extended to the limiting age of service.

Now, let $D_x = v^x l_x$; $D^s_x = v^x l_x s_x$

$$D^{(2)s}_x = v^{x+\frac{1}{2}} l_{x+\frac{1}{2}} s_x ; \sum_x^w D^{(2)s}_x = N^{(2)s}_x .$$

Then, present value $= \cdot 01 \left[N^{(2)s}_x \Big/ D^s_x \right]$.

Hence the tables necessary for the calculation of the present value of all future contributions are $l_{x+\frac{1}{2}}$, $D^{(2)}_x$, $D^{(2)s}_x$, D_x and D^s_x.

SECTION B.

Present value of the benefit due to mortality or withdrawal at a uniform rate.

If the benefit on death is to be a fixed percentage of the contributions, that is, of the salaries as well, then, on the supposition of the uniform distribution of deaths, on an average

d_x persons get a fixed percentage of $(\frac{1}{2}s_x)$ in the middle of the year x,

d_{x+1} .... $(s_x + \frac{1}{2} s_{x+1})$... of the year $(x + 1)$,

d_{x+2} .... $(s_x + s_{x+1} + \frac{1}{2} s_{x+2})$... of the year $(x + 2)$

and so on.

Hence the total present value equated to the denominator D^s_x of the benefit of 1 per cent of salaries paid up to death

$$= \frac{\cdot 01}{D^s_x} \left\{ \tfrac{1}{2} s_x d_x v^{x+\frac{1}{2}} + \left(s_x + \tfrac{1}{2} s_{x+1} \right) d_{x+1} v^{x+\frac{3}{2}} + \dots\dots \right\}$$

$$= \frac{\cdot 01}{D^s_x} \left\{ s_x \left(d_x v^{x+\frac{1}{2}} + d_{x+1} v^{x+\frac{3}{2}} + \dots\dots \right) \right.$$

$$+ s_{x+1} (d_{x+1} v^{x+\frac{3}{2}} + d_{x+2} v^{x+\frac{5}{2}} + \dots\dots) + \dots\dots$$

$$\left. - \tfrac{1}{2} (s_x d_x v^{x+\frac{1}{2}} + s_{x+1} d_{x+1} v^{x+\frac{3}{2}} + \dots\dots) \right\}.$$

Assume $d_x v^{x+\frac{1}{2}} = \frac{d_x v^{x+1}}{v^{\frac{1}{2}}} = \sqrt{1+i}\, d\, C_x$,

where 'd' signifies 'death' and

$$\sum_x^w \sqrt{1+i}\, {}^dC_x = {}^dM_x;$$

also introducing salaries, let

$$\sqrt{1+i}\, {}^dC_x \cdot s_x = {}^dM'^s_x;$$

$${}^dM_x \cdot s_x = {}^dM^s_x; \ \sum_x^w {}^dM^s_x = {}^dR^s_x.$$

Then the required present value is equal to

$$\frac{.01}{D^s_x}\left\{ {}^dR^s_x - \tfrac{1}{2}\, {}^dM'^s_x \right\}^* \quad \ldots \quad \ldots (b)$$

If a definite percentage of salaries be granted to withdrawals, then the present value of the benefit due to withdrawals is

$$\frac{\cdot 01}{D^s_x}\left\{ {}^wR^s_x - \tfrac{1}{2}\, {}^wM'^s_x \right\}. \quad \ldots \quad \ldots (b')$$

where 'w' signifies 'withdrawal.'

SECTION C.

Present value of the benefit at varying rates.

In the previous section, a *uniform* percentage of contributions was considered as benefit to deaths or withdrawals. But if it increases by a constant difference at the ends of fixed periods of service, say, as per the following scheme :—

for the first 5 years of service, $k\%$ of salaries is returned ;

from the 6th to the 10th, $(k+l)\%$ .... ;

from the 11th to the 15th, $(k+2l)\%$ ....

and so on, increasing by $l\%$ at the end of every quinquennium, then the total value of this benefit equated to the denominator D^s_x is

---

* This formula, though identical with the one given by G. King (*vide* J. I. A., Vol. XLI) has the advantage of being free from too many notations.

$$\frac{\cdot 01}{D^s_x}\Big[k\Big\{\tfrac{1}{2}s_x d_x v^{x+\frac{1}{2}} + (s_x+\tfrac{1}{2}s_{x+1})d_{x+1}v^{x+\frac{3}{2}} + \ldots \text{ to 5 terms}\Big\}$$

$$+ (k+l)\Big\{(s_x+s_{x+1}+\ldots+\tfrac{1}{2}s_{x+5})d_{x+5}v^{x+\frac{11}{2}} + \ldots \text{ to 5 terms}\Big\}$$

$$+ (k+2l)\Big\{(s_x+s_{x+1}+\ldots+\tfrac{1}{2}s_{x+10})d_{x+10}v^{x+\frac{21}{2}} + \ldots \text{ to 5 terms}\Big\}$$

$$+ \text{etc.}\Big]$$

$$=\frac{\cdot 01}{D^s_x}\Big[k\{s_x(M_x - M_{x+5}) + s_{x+1}(M_{x+1} - M_{x+5}) + \ldots\ldots\}$$

$$+ (k+l)\{\sum_x^5 s_x(M_{x+5} - M_{x+10}) + s_{x+6}(M_{x+6} - M_{x+10}) + \ldots\ldots\}$$

$$+ (k+2l)\{\sum_x^{10} s_x(M_{x+10} - M_{x+15}) + \ldots\ldots\}$$

$$-\tfrac{1}{2}\{k(\sum_x^4 c_x s_x) + (k+l)\sum_{x+5}^{x+9} c_x s_x + \ldots\ldots\}\Big],$$

$$=\frac{\cdot 01}{D^s_x}\Big[kR^s_x + l\sum_{x+5} R^s_x + l\sum s_x . \sum_{t=1} M_{x+5t}$$

$$- M_{x+5}\sum S_{x+5} - M_{x+10}\sum S_{x+10} - M_{x+15}\sum S_{x+15} - \text{etc}).$$

$$-\tfrac{1}{2}\{kM'^s_x + l\sum_{t=1} M'^s_{x+5t}\}\Big], \qquad \ldots \qquad \ldots (c)$$

where $\sum_x^m s_x$ stands for $\sum_x^{x+m} s_x$

in every case the summations being taken to the limiting age of the table.

If we can tabulate ΣM_{x+5t}, $\Sigma M'^s_{x+5t}$ and ΣR^s_{x+5t} for values of t and also $\sum_x^m S_x$, the evaluation of this formula becomes very simple. As we proceed by quinary intervals, the number of terms to be summed up gets small.

SECTION D.

Present value of the benefit at compound interest.

In the case of some funds, a definite percentage of contributions accumulated at compound interest is granted on death or withdrawal. For convenience, assume that the rate of valuation is the same as the rate of interest at which the benefit is calculated.

Now let $\Sigma d_x = l'_x$ which is different from l_x which is subject to three decrements of death, withdrawal and retirement.

Then $(l'_x - \frac{1}{2} d_x)$ persons will receive each in the middle of the first year a salary of s_x, whose present value

$$= v^{\frac{1}{2}}(l'_x - \tfrac{1}{2} d_x) . s_x.$$

Similarly, $(l'_{x+1} - \frac{1}{2} d_{x+1})$ will receive each in the middle of the second year a salary of s_{x+1}, whose present value

$$= v^{\frac{3}{2}}(l'_{x+1} - \tfrac{1}{2} d_{x+1})\ s_{x+1};$$

and so on.

Hence the present value of 1 per cent. of the salaries accumulated at compound interest paid to those who die in the course of their service equated to 1 of the present salary s_x

$$= \frac{\cdot 01}{D^s_x} [v^{x+\frac{1}{2}}(l' - \tfrac{1}{2} d_x)\ s_x + v^{x+\frac{3}{2}}(l'_{x+1} - \tfrac{1}{2} d_{x+1}) s_{x+1} + \ldots\ldots]$$

Now, let

$$v^{x+\frac{1}{2}}(l'_x - \tfrac{1}{2} d_x) . s_x = {}^{di}M^s_x$$

(this is King's notation with 'bar' over M omitted); also let

$$\Sigma\, {}^{di}M^s_x = {}^{di}R^s_x.$$

Then the present value

$$= \frac{\cdot 01}{D^s_x}\, {}^{di}R^s_x \quad \ldots \quad \ldots \quad \ldots (d)$$

A similar formula holds good in the case of withdrawals; but instead of the d_x column, we have to take up the w_x column.

SECTION E.

Present value of pensions.

Then we pass on to calculate the present value of pensions. We are not considering the pension as an annuity but as something guaranteed to a retiring employee over and above the contributions paid by him towards the fund. We allow retiring pensions, after a minimum of ten years service, due to serious disability. The pensions allowed are a definite percentage of the past salaries multiplied by the number of years of service (not exceeding 25). To get the present value of the future pensions :—

(1) We shall suppose that pensions are allowed even after 1 year of service, and they are 1 % of the past salaries multiplied by the number of years of completed service (without any maximum). Then on the hypo-

thesis of the uniform distribution of retirements throughout the year, the present value of the benefit due to retirements after the first year

$$= \cdot01\, v^{\frac{3}{2}}\,.\, r_{x+1}(S_x + \tfrac{1}{2}S_{x+1}).$$

The value due to retirements after the second year

$$= \cdot01\, v^{\frac{5}{2}} \times 2 \times r_{x+2}(S_x + S_{x+1} + \tfrac{1}{2}S_{x+2});$$

and so on.

Hence the total present value equated to the denominator D^s_x.

$$= \frac{\cdot01}{D^s_x}\sqrt{1+i}\,\{\, S_x(C_{x+1} + 2\,.\,C_{x+2} + 3\,.\,C_{x+3} + \ldots\ldots)$$
$$+ S_{x+1}(C_{x+1} + 2\,.\,C_{x+2} + 3\,.\,C_{x+3} \quad + \ldots\ldots)$$
$$+ S_{x+2}(2\,.\,C_{x+2} + 3\,.\,C_{x+3} + 4\,.\,{}_{x+4} + \ldots\ldots) + \text{etc.}$$
$$- \tfrac{1}{2}(C^s_{x+1} + 2\,C^s_{x+2} + 3\,C^s_{x+3} + \ldots\ldots)\}.$$

But since

$$\sqrt{1+i}\,\{\, C_{x+1} + 2\,C_{x+2} + 3\,C_{x+3} \ldots\ldots \} = R_x - M_x\,(= R_{x+1}),$$
$$\sqrt{1+i}\,\{\, 2.\,C_{x+2} + 3\,.\,C_{x+3} + \ldots\ldots\ldots \} = R_{x+2} + M_{x+2},$$
$$\sqrt{1+i}\,\{\, 3\,.\,C_{x+3} + 4\,.\,C_{x+4} + \ldots\ldots\ldots \} = R_{x+3} + 2\,.\,M_{x+3},$$

and so on; also

$$\sqrt{1+i}\,\{\, C^s_{x+1} + 2\,C^s_{x+2} \quad + \ldots\ldots\ldots \} = R'^s_{x+1} \text{ (say)}$$

where $$R' = \Sigma\, M';$$

it follows that the total present value

$$= \frac{\cdot01}{D^s_x}\,\{\, S_x(R_x - M_x) + S_{x+1}\,R_{x+1} + S_{x+2}(R_{x+2} + M_{x+2})$$
$$+ \ldots - \tfrac{1}{2}R'^s_{x+1}\}$$
$$= \frac{\cdot01}{D^s_x}\left[\Sigma\, R_x\, S_x - M^s_x + \sum_0 t\,M^s_{x+1+t} - \tfrac{1}{2}R'^s_{x+1}\right].$$

Assume $\Sigma\, R_x\, S_x = X'^s_x$; also $\sum\limits_0 t\, M^s_{x+1+t} = X^s_{x+2} = X^s_{x+1} - R^s_{x+1}$

where $$X = \Sigma\, R = \Sigma\,(\Sigma\, M).$$

Hence the present value

$$= \frac{\cdot01}{D^s_x}\left[X'^s_x - M^s_x + X^s_{x+1} - R^s_{x+1} - \tfrac{1}{2}R'^s_{x+1}\right]$$
$$= \frac{\cdot01}{D^s_x}\left[X'^s_x - R^s_x + X^s_{x+1} - \tfrac{1}{2}R'^s_{x+1}\right] \quad \ldots \quad \ldots \quad (e)$$

Assume this to be $\frac{\cdot 01}{D^s_x} \cdot P^s_x$.

(2) Now we shall consider the present value of the same benefit on the supposition that a minimum of 10 years' completed service is necessary for retirement. It is the same series as in (*e*) but whose first term is

$$\frac{\cdot 01}{D_x} \sqrt{1+i} \left\{ 10\, r_{x+10} v^{x+2\frac{1}{2}} (S_x + S_{x+1} + \ldots + S_{x+9} + \tfrac{1}{2} S_{x+10}) \right\}.$$

Hence the present value

$$= \frac{\cdot 01}{D^s_x} \left\{ P^s_{x+9} + 9 \sum_x^8 s_x \left(\sqrt{1+i}\, \Sigma\, r_{x+10}\, v^{x+\frac{2}{2}}\right) \right\}.$$

$$= \frac{\cdot 01}{D^s_x} \left\{ P^s_{x+9} + 9 \sum_x^8 s_x \times M_{x-10} \right\}. \quad \ldots \quad \ldots \quad (e')$$

(3) Now we pass on to the present value of the same benefit where a minimum of 10 years completed service and a maximum of 25 years are fixed for pension. In this case even if the period of service exceeds 25, for purposes of pension the multiplying factor is taken to be 25. The present value is the same as (*e′*) diminished by

$$\frac{\cdot 01}{D^s_x} \left\{ P^s_{x+25} + \sum_x^{24} S_x .\ (M_{x+26}) \right\}.$$

Hence the total present value

$$= \frac{\cdot 01}{D^s_x} \left\{ P^s_{x+9} - P^s_{x+25} + 9\, M_{x+10} \sum_x^8 S_x - M_{x+26} \sum_x^{24} S_x \right\}. \quad \ldots \quad (e'')$$

N.B.—At the inception of a fund, actuarial advice will be necessary to calculate the amount of pensions that could be granted consistently with the 'solvency' of the fund. Pension as an annuity has long ceased to be attractive, and there is clamour everywhere that it must be a definite lump sum at the time of retirement. Hence the formulæ worked out here; these when applied to the 'Life and Service' table give the actuary the exact pensions possible after allowing for benefits to withdrawals and mortality.

Variation of wind with height— Co-efficient of turbulence.*

By R. N. Apte.

[G. T. Taylor's recent investigation of the variation of wind with height (shortly stated: *geostrophic* wind) is taken as the standard of comparison, though in the equatorial belt the cyclostrophic terms cannot be neglected.

Taylor regards k the turbulence co-efficient as independent of height and obtains the relation for the surface wind S

$$S/G = \cos\alpha - \sin\alpha. \qquad \ldots \qquad \ldots (1)$$

which is amply supported by observations.

Harold Jeffreys is led to conclude that k the turbulence co-efficient depends upon the height. He proposes the law $k = k_0\ (1+\lambda z)^2$ and suggests that (1) must be really

$$S/G = \cos\alpha - h \sin\alpha. \qquad \ldots \qquad \ldots (2)$$

As Taylor's equation (1) is amply supported by direct observational evidence and as k also is to depend upon height, any law proposed for the dependence of k upon height must be quite consistent with equation (1): the present writer thinks it useful to consider the law

$$k = k_0\,(1 + \lambda z)^{2-\frac{2}{2n+1}}$$

for positive integral values, $n = 1, 2$, etc., and proves that the results obtained are consistent with Taylor's equation (1).

An interesting case, $k = -k_0 z^{\frac{4}{3}}$ is examined.]

1. A great deal of light has recently been thrown upon the question of the variation of wind with height by G. T. Taylor's investigations of eddy-motion in the atmosphere. Taylor relies upon the *geostrophic wind* as representing the undisturbed wind in the upper air and, as remarked by Sir Napier Shaw, no other standard of reference seems possible. The *cyclostrophic term* is neglected though a consideration of it must be made with respect to the winds in the equatorial belt.

* Presented to the Fourth Conference of the Indian Mathematical Society, Poona, April 1924.

2. As regards eddy-motion in the atmosphere, Taylor showed (*Phil. Transactions*, R. S., Vol. 215, p. 14) that if u and v be the components of velocity parallel to the horizontal axes of x and y, the rates of gain of momentum parallel to these axes, in unit of volume at height z are

$$k\rho\left(\frac{d^2u}{dz^2}\right),\quad k\rho\left(\frac{d^2v}{dz^2}\right), \qquad \ldots (1)$$

k being the constant used in temperature variations

$$\rho\sigma\frac{\delta\theta}{\delta t}=\frac{\delta}{\delta z}\left(k\rho\sigma\frac{\delta\theta}{\delta z}\right). \qquad \ldots \qquad \ldots (2)$$

If the motion be steady the resultant gain or loss of momentum must balance the other forces acting on the element of volume considered. Thus k is a new meteorological quantity of great importance.

3. Taylor regards k as independent of height. Starting with the equations of motion of an incompressible viscous fluid given in Lamb's *Hydro-dynamics*, p. 338, he gets for horizontal motion equations equivalent to

$$2\omega v + k\frac{d^2u}{dz^2} = 0 \qquad \ldots \qquad \ldots (3)$$

$$-2\omega u + k\frac{d^2v}{dz^2} = 2\omega G \qquad \ldots \qquad \ldots (4)$$

where $\omega = \Omega \sin \lambda$, Ω is the earth's angular velocity of rotation, λ is the latitude and G the velocity of the geostrophic wind defined by

$$2\omega\rho G = -\frac{\partial p}{\partial y}.$$

The co-ordinate axes of x, y, z are such that z is measured vertically upwards and the axis is taken along the surface isobars. Now regarding k as independent of height in (3) and (4), he solves them obtaining

$$u = G - A_2 e^{-Bz} \cos Bz + A_4 e^{-Bz} \sin Bz \qquad \ldots (5)$$

$$v = A_2 e^{-Bz} \sin Bz - A_4 e^{-Bz} \cos Bz. \qquad \ldots (6)$$

The constants are found from boundary conditions and if S is the surface wind

$$S = G(\cos\alpha - \sin\alpha). \qquad \ldots \qquad \ldots (7)$$

Taylor's strong point is that this equation (7) is confirmed by observations and he holds that k, the co-efficient of turbulence, is independent of height.

4. Harold Jeffrey (*Proceedings* R. S. Vol. 96, p. 233) however, thinks that k is not independent of height. He has analysed a number of observations and examined the relations between S/G and α and the

possibility of a systematic contortion of isobars and is led to conclude that there must be a variation in the vertical distribution ot turbulence.

He proposes the law $k = k_0 (1 + \lambda z)^2$ where k_0 and λ are constants and z is the height. He solves equations (3) and (4) with this hypothesis and gets

$$\frac{S}{G} = \cos \alpha - \sin \alpha \cot \beta, \quad \ldots \quad \ldots (8)$$

where β is the argument of $\frac{m(m-1)}{m+1}$, and $W = G + A (1+\lambda z)^m$.

Here, however, it must be admitted that the real ratio is

$$\frac{S}{G} = \cos \alpha - h \sin \alpha \quad \ldots \quad \ldots (9)$$

and that the mean value of h is about 1, so as to justify Taylor's equation (7) though the value of h varies over a wide range with individual observations.

As a rule, the wind determined by pilot baloon ascents, is found to agree fairly closely with the geostrophic wind at altitudes of 3000 feet and more. The question whether k is independent of height or not must be ultimately settled by observations which are not yet available. Direct quantitative evidence can be the complete test of the theory.

5. The present writer thinks it useful to try the laws

$$k = k_0 (1 + \lambda z)^{2 - \frac{2}{2n+1}}$$

where n is a positive integer. Harold Jeffrey's law is a particular case of this when n is ∞. The equations (3) and (4) combined into one single equation now give an equation in Ricatti's form which in this case is finite. But there would be a great difficulty in determining one of the constants.

6. We may, therefore, take successively

$$n = 1, 2, \&c. \ldots\ldots$$

Take the law $k = k_0 (1 + \lambda z)^{2-\frac{2}{3}}$.

Equations (3) and (4) are

$$2\omega v + k \frac{d^2 u}{dz^2} = 0.$$

$$-2\omega u + k \frac{d^2 u}{dz^2} = 2\omega G\,;$$

write $u + iv = w$. Then these equations combine into the equation

$$k \frac{d^2 w}{dz^2} - 2i\omega w = 2i\omega G.$$

$$\therefore \quad k\frac{d^2V}{dz^2} - 2i\omega V = 0, \text{ where } V = w - G.$$

$$\therefore \quad k\frac{d^2V}{dz^2} - (1+i)^2\omega V = 0. \quad \ldots \quad \ldots \quad (10)$$

Now $$k = k_0(1+\lambda z)^{\frac{4}{3}}.$$

$$\therefore \quad (1+\lambda z)^{\frac{4}{3}}\frac{d^2V}{dz^2} - c^2V = 0 \quad \ldots (11)$$

where $$c = \frac{(1+i)\omega^{\frac{1}{2}}}{\lambda k_0^{\frac{1}{2}}}; \quad \ldots \quad \ldots \quad \ldots (11')$$

put $$1 + \lambda z = \xi = t^3, \quad \ldots \quad \ldots (12)$$

and the solution of (11) is

$$w - G = V = A(1-3ct)e^{3ct} + B(1+3ct)e^{-3ct} \quad \ldots (13)$$

The constants A and B are to be determined from boundary conditions. Since the velocity shall not tend to infinity with height, A = 0.

$$\therefore \quad w = G + B(1+3ct)e^{-3ct}. \ldots \quad \ldots (14)$$

If α denote the angle between the wind at the ground ($z = 0$) and the tangent to the isobar, we have at the surface

$$w = Se^{i\alpha}. \quad \ldots \quad \ldots (15)$$

$$\therefore \quad B = Se^{i\alpha} - G. \quad \ldots \quad \ldots (16)$$

Now another boundary condition is that the skin-friction at the surface is equal to $k_1\rho S^2$ per unit of area where k_1 is constant and that it acts in the direction opposite to the velocity. It must be equal to $k\rho$ times the rate of speed in the air near the surface.

$$\therefore \quad k_1\rho S^2 e^{i\alpha} = -\int_0^\infty k\rho\frac{d^2w}{dz^2}\,dz$$

$$= -3Bk_0\rho\left(c^2 + \frac{4c}{3} + \frac{8}{9} + \frac{8}{27c}\right)e^{-3c}, \quad \ldots (17)$$

taking the real part of c to be positive.

From (11′) writing $c = R(1+i)$ and putting

$$Be^{-i\alpha} = S - Ge^{-i\alpha} = Z_1, \quad e^{-3R(1+i)} = Z_2,$$

and $$\frac{2R}{3} + \frac{4}{9} + \frac{2}{27R} + i\left(R^2 + \frac{2R}{3} - \frac{2}{27R}\right) = Z_3,$$

where Z_1, Z_2, Z_3 are complex numbers, (17) may be written

$$k_1\rho S^2 = (\text{a real quantity}).\ Z_1Z_2Z_3. \quad \ldots \quad \ldots (18)$$

Let θ_1, θ_2, θ_3 be respectively the amplitudes of Z_1, Z_2, Z_3 thus Taylor's equation (7) will be true if

$$\tan \theta_1 = -1 \quad ie., \quad \frac{G \sin \alpha}{S - G \cos \alpha} = -1$$

resulting in $\qquad S = G (\cos \alpha - \sin \alpha).$

Now $\qquad \tan \theta_2 = -\tan 3R$

and $\qquad \tan \theta_3 = \dfrac{27R^3 + 18R^2 - 2}{18R^2 + 12R + 2},$

and subject to the condition mentioned above, we can adjust the constants k_0 and λ and therefore R, so that

$$\tan \theta_2 = \frac{1 - \tan \theta_3}{1 + \tan \theta_3}$$

or

$$\tan 3R = \frac{27R^3 - 12R - 4}{3R (3R + 2)^2}; \quad \ldots \quad \ldots (19)$$

and when this condition is fulfilled

$$\tan \theta_1 = -1.$$

Thus with this adjustment of the constants the law $k = k_0 (1 + \lambda z)^{\frac{4}{5}}$ satisfies Taylor's equation (7), which is confirmed by observations, and also the condition required by Harold Jeffrey that the co-efficient of turbulence k must depend upon height.

8. It is to be noted that even if the value of $\tan \theta_1$ is not taken as -1 and modified by observations to be $-\tan \beta$, a known quantity, we can still adjust the constants k_0 and λ so that the equation (18) is true.

9. A similar treatment may be given to the law of § 5 when $n = 2$.

For equation (11) in this case, we have

$$(1 + \lambda z)^{\frac{8}{5}} \frac{d^2 V}{dz^2} - c^2 V = 0 \quad \ldots \quad \ldots (20)$$

Put $1 + \lambda z = \xi = t^5$, and the solution is

$$w - G = V = A \left(1 - 3ct + \frac{25}{3} c^2 t^2\right) e^{5ct}$$

$$+ B \left(1 + 5ct + \frac{25}{3} c^2 t^2\right) e.^{-5ct}$$

As before, from the condition that the velocity cannot tend to infinity with height, A must be equal to 0.

The condition of the skin-friction also may be similarly applied and we get an equation corresponding to (17).

10. It is interesting to note that if we take $k = -k_0 z^{\frac{4}{3}}$, equations (3) and (4) give

$$k_0 z^{\frac{4}{3}} \frac{d^2V}{dz^2} + 2i\omega V = 0.$$

which may be written as

$$k_0 z^{\frac{4}{3}} \frac{d^2V}{dz^2} - (1-i)^2 \omega V = 0. \quad \ldots \quad \ldots \quad \ldots (21)$$

Put $z = x^3$, the solution is $w = G + A(1-3cx)^{+3cx} + B(1+3cx).^{-3cx}$

As before A must be zero.

$$\therefore \quad w = G + B(1+3cx),^{-3cx} \quad \ldots \quad \ldots \quad \ldots (22)$$

where $x = z^{\frac{1}{3}}$ and $c = \dfrac{(1-i)\,\omega^{\frac{1}{2}}}{k_0^{\frac{1}{2}}}$.

Applying the condition of skin-friction as before, we get

$$k_0 S^2 e^{i\alpha} = \frac{8k_0}{3c} B = \frac{8k_0}{3c}(Se^{i\alpha} - G).$$

$$\therefore \quad k_1 S^2 c = 8k_0 (S - Ge^{-i\alpha})$$

and equating the ratios of the real and imaginary parts on the two sides, we get

$$\frac{G \sin \alpha}{S - G \cos \alpha} = -1.$$

$$\therefore \quad \frac{S}{G} = \cos \alpha - \sin \alpha,$$

which is exactly Taylor's equation; but a physical interpretation of this case is difficult as k becomes zero at the surface.

Notes and Questions.

Asymptotic Expansion of Certain Series.

[The force at a point on the circumference of a circle having on it a number of equally spaced gravitating bodies of the same magnitude involves series of types

$$\Sigma \csc\left(a+\frac{m\pi}{n}\right) \text{ and } \Sigma \csc\left(a+\frac{m\pi}{n}\right)\cot\left(a+\frac{m\pi}{n}\right)$$

where $a>0$ and m takes all integral values from 0 to $n-1$; the force exerted on one of them by the others involves series of types

$$\Sigma\ \csc\frac{m\pi}{n} \text{ and } \Sigma\ \csc\left(\frac{m\pi}{n}\right)\cot\left(\frac{m\pi}{n}\right).$$

Various methods of summing these series for large values of n have been suggested*. The following method of summation may be new; it is also applicable to cases where some gravitating bodies are absent. The summation would then involve values of m from m_1 to m_2.]

1. A slightly generalised series may be taken and summed up:

If $0<R\left\{a+\frac{m_1 r\pi}{n}\right\}<R\left\{a+\frac{m_2 r\pi}{n}\right\}<\pi$, it is required to sum up

$$\sum_{n=m_1}^{m_2} \csc\left(a+\frac{mr\pi}{n}\right).$$

We start with the well-known summation formula † that: if $f(x)$ is bounded everywhere in the region

$$x_1<R(x)<x_2$$

where x_1 and x_2 are two integers,

$$f(x_1)+f(x_1+1)+f(x_1+2)\ldots f(x_2)=\tfrac{1}{2}\{f(x_1)+f(x_2)\}+\int_{x_1}^{x_2} f(x)dx$$
$$+2\int_0^\infty \frac{I[f(x_1-it)+f(x_2+it)]\,dt}{e^{2\pi t}-1},$$

where $R(x)$ and $I(x)$ denote the real and imaginary parts of x.

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* See G. A. Schott: *The Electromagnetic Theory of Radiation*, p. 216. G. A. Schott and G. N. Watson; *Quarterly Journal*, Vol. 47 (1916), pp. 311, *et seq*. A. Sommerfeld: *Ann. Der Physik* 53 (1917), p. 511.

† The form is slightly contracted from Whittaker and Watson: *Modern Analysis*, 2nd edition: p. 145, Q. 7.

Now $\displaystyle\int_{m_1}^{m_2} \csc\left(a+\frac{mr\pi}{n}\right)dm = \frac{n}{r\pi}\log\left\{\frac{\tan(a+m_1r\pi/n)}{\tan(a+m_2r\pi/n)}\right\}.$

$$I\left[\csc\left(a+\overline{m_1-it}\,\frac{r\pi}{n}\right)\right]+I\left[\csc\left(a+\overline{m_2+it}\,\frac{r\pi}{n}\right)\right]$$

$$=\frac{\cos\left(a+\frac{m_1r\pi}{n}\right)\sinh\frac{\pi rt}{n}}{\left[\cosh\frac{\pi rt}{n}\right]^2-\left[\cos\left(a+\frac{m_1r\pi}{n}\right)\right]^2}$$

$$-\frac{\cos\left(a+m_2\frac{r\pi}{n}\right)\sinh\frac{\pi rt}{n}}{\left[\cosh\frac{\pi rt}{n}\right]^2-\left[\cos\left(a+m_2\frac{r\pi}{n}\right)\right]^2}$$

$$=\sum_{s=0}^{s-1}\sinh\frac{\pi rt}{n}\left[(\cos\theta_1)^{2s+1}-(\cos\theta_2)^{2s+1}\right]\left(\cosh\frac{\pi rt}{n}\right)^{-2s-2}$$

$$+\frac{\sinh\frac{\pi rt}{n}\left(\cosh\frac{\pi rt}{n}\right)^{-2s}(\cos\theta_1)^{2s+1}}{\left[\cosh\frac{\pi rt}{n}\right]^2-\left[\cos\theta_1\right]^2}$$

$$-\frac{\sinh\frac{\pi rt}{n}\cdot\left(\cosh\frac{\pi rt}{n}\right)^{-2s}\left(\cos\theta_2\right)^{2s+1}}{\left[\cosh\frac{\pi rt}{n}\right]^2-\left[\cos\theta_2\right]^2},$$

where $\theta_1 = a + m_1\frac{r\pi}{n}$; $\theta_2 = a + m_2\frac{r\pi}{n}$.

Hence it follows that

$$\sum_{m_1}^{m_2}\csc\left(a+mr\frac{\pi}{n}\right)=\left\{\frac{\sin\left(a+\overline{m_1+m_2}\,\frac{r\pi}{2n}\right)\cos\left(\overline{m_1-m_2}\,\frac{r\pi}{2n}\right)}{\cos(m_1-m_2)\frac{r\pi}{n}-\cos\left(2a+\overline{m_1+m_2}\,\frac{r\pi}{n}\right)}\right\}$$

$$+\frac{n}{\pi r}\log\left[\frac{\tan\theta_2}{\tan\theta_1}\right]+2\sum_{s=0}^{s-1}A_s\left[(\cos\theta_1)^{2s+1}-(\cos\theta_2)^{2s+1}\right]$$

$$+R_s \qquad \ldots \qquad \ldots \qquad \ldots \qquad \ldots \qquad \ldots \text{(I)}$$

where $\displaystyle A_s=\int_0^\infty\frac{\sinh\frac{\pi rt}{n}\left[\cosh\frac{\pi rt}{n}\right]^{-2s-2}}{e^{2\pi t}-1}dt;$

and it is easily seen that

$$| R_s | \leqslant \left| \phi_s \left\{ \int_0^\infty \frac{\left(\sinh \frac{\pi r t}{n} \right) \left(\cosh \frac{\pi r t}{n} \right)^{-2s-2} dt}{e^{2\pi t} - 1} \right\} \left\{ (\cos \theta_1)^{2s+1} - (\cos \theta_2)^{2s+1} \right\} \right|$$

where $\phi_s \rightarrow 1$ as $s \rightarrow \infty$.

The evaluation of A_s is given below in § 3.

2. Again

$$\sum_{m_1}^{m_2} \csc \left(a + \frac{m \pi r}{n} \right) \cot \left(a + \frac{m \pi r}{n} \right)$$

$$= \tfrac{1}{2} [\csc \theta_1 \cot \theta_1 + \csc \theta_2 \cot \theta_2] + \frac{n}{\pi r} [\csc \theta_1 - \csc \theta_2]$$

$$+ 2 \int_0^\infty \frac{\sinh \frac{\pi r t}{n} \sin \theta_1 \left(\cos^2 \theta_1 + \cosh^2 \frac{\pi r t}{n} \right) dt}{\left(\cosh^2 \frac{\pi r t}{n} - \cos^2 \theta_1 \right)^2}$$

$$- 2 \int_0^\infty \frac{\sinh \frac{\pi r t}{n} \sin \theta_2 \left[\cos^2 \theta_2 + \cosh^2 \frac{\pi r t}{n} \right] dt}{\left(\cosh^2 \frac{\pi r t}{n} - \cos^2 \theta_2 \right)^2}$$

$$= \tfrac{1}{2} [\csc \theta_1 \cot \theta_1 + \csc \theta_2 \cot \theta_2] + \frac{n}{\pi r} [\csc \theta_1 + \csc \theta_2]$$

$$+ 2 \sum_{s=0} (s+1) (\cos \theta_1)^{2s} \sin \theta_1 (A_s + A_{s+1} \cos^2 \theta_1)$$

$$- 2 \sum_{s=0} (s+1) (\cos \theta_2)^{2s} \sin \theta_2 (A_s + A_{s+1} \cos^2 \theta_2) + R'_s. \quad \text{(II)}$$

The series (I) and (II) are seen to be convergent.

3. The next step would be to determine

$$\frac{1}{2\pi} \int_0^\infty \frac{\sin bt . (\cosh bt)^{-m} dt}{e^t - 1}, \text{ where } b = \frac{r}{2n}.$$

If $$\operatorname{sech} bt = \sum_0^\infty (-)^v . E_{2v} . \frac{(bt)^{2v}}{2v!},$$

where $| bt | < \dfrac{\pi}{2}$,

then for all positive values of bt

$$\operatorname{sech} bt = \sum_{0}^{\mu-1} (-)^{v} E_{2v} \cdot \frac{(bt)^{2v}}{2v\,!} + \theta(-)^{\mu} E_{2\mu} \frac{(bt)^{2\mu}}{2\mu\,!},$$

where $0 < |\theta| < 1$*.

It can be similarly deduced by induction or otherwise that if

$$\sinh bt\,(\cosh bt.)^{-m} = \sum_{1}^{\infty} (-)^{v} . C_{2v} \frac{(bt)^{2v-1}}{(2v-1)\,!}, \text{ where } |bt| < \frac{\pi}{2},$$

it is always equal to

$$\sum_{1}^{\mu-1} (-)^{v} . C_{2v} . \frac{(bt)^{2v-1}}{(2v-1)\,!} + \theta(-)^{\mu} C_{2\mu} \frac{(bt)^{2\mu-1}}{(2r-1)!},$$

also that, if C_{2v} has the value given in this equation,

$\sinh bt\,(\cosh bt)^{-m-2}$

$$= \sum_{1}^{\mu-1} (-)^{v-1} \left[\frac{C_{2v+2} + (1-m)^2 C_{2r}}{m\,(m+1)}\right] \frac{(bt)^{2v-1}}{(2v-1)\,!} + R_{\mu},$$

$$\text{where } R_{\mu} = \theta'(-)^{\mu-1} \left[\frac{C_{2\mu+2} + (1-m)^2 C_{2\mu}}{m\,(m+1)}\right] \frac{(bt)^{2\mu-1}}{(2\mu-1)\,!}.$$

By this method if we know the co-efficients C_{2v} for a value $-m$, we can find for $\overline{-m+2}$ the new co-efficients and so on.

Knowing the value of C_{2v} for the value $-m$, substituting the series for $\sinh bt\,(\cosh bt)^{-m}$ in the integral

$$\int_{0}^{\infty} \frac{\sinh bt\,(\cosh bt)^{-m}\,dt}{e^{t}-1},$$

we get it equal to

$$\sum_{1}^{\mu-1} (-)^{v-1} C_{2v}\,\zeta(2v) . b^{2v-1} + \theta(-)^{\mu-1} C_{2\mu}\,\zeta(2\mu) b^{2\mu-1},$$

$$0 < \theta < 1,$$

where $$\zeta(x) = \frac{1}{1^{x}} + \frac{1}{2^{x}} + \frac{1}{3^{x}} + \ldots\ldots$$

The series for the integral is asymptotic for small values of b and a large number of gravitating bodies.

$$\text{Hence } A_{s} = \sum_{1}^{\mu-1} (-)^{v-1} C_{2v}(-2s-2) . \zeta(2v)\, b^{2v-1} + R_{\mu}$$

where $C_{2v}(-2s-2)$ means that C_{2v} is taken when $m = 2s+2$.

* See G. N. Watson: *Quarterly Journal of Math.*, Vol. 47, pp. 302 et. seq.

Values of C_{2v} for $\sinh x\ (\cosh x)^{-m} = \Sigma\, C_{2v}\ (-)^v \dfrac{x^{2v-1}}{(2v-1)!}$.

v.	C_{2v}.
1	1
2	$3m - 1$
3	$15m^2 + 1$
4	$21m(5m^2 + 5m + 3) - 1$
5	$445m^4 + 2520m^3 + 3150m^2 + 1322m + 1$.

These values of C_{2v} are generally enough.

S. L. MALURKAR.

Cross-Ratio Invariants.

[A single infinity of points denoted by a variable y will be brought into (1, 1) corespondence with another denoted by a variable x by means of the homographic relation

$$F(y) \equiv y(a_1 + b_1x) + a_0 + b_0x = 0,$$

or what is the same thing by

$$\phi(x) \equiv x(b_0 + b_1y) + a_0 + a_1y = 0.$$

If x_1, x_2, x_3, x_4 be four points of the x-set, corresponding to y_1, y_2, y_3, y_4 of the y-set, then it is known that the cross-ratio

$$(x_1x_2x_3x_4) = (y_1y_2y_3y_4).$$

This note obtains a simple generalisation of this well-known result.]

1. Consider the equation

$$F(y) \equiv y^n(a_n+b_nx) + y^{n-1}(a_{n-1} + b_{n-1}x) + \dots + y(a_1 + b_1x) + a_0 + b_0x = 0.$$

or, what is the same thing,

$$\phi(x) \equiv x(b_0 + b_1y + \dots\dots + b_ny^n) + (a_0 + a_1y + \dots\dots + a_ny^n) = 0.$$

For any particular value of y, $\phi(x) = 0$ gives only one value of x; while for any particular value of x, the same relation $F(y) = 0$ gives n values of y. That is, corresponding to a single infinity of points of the x-set, we obtain a single infinity of groups of points of the y-set, each group containing n points.

2 In this correspondence between n-point groups of the y-set and single points of the x-set, let us suppose four single points x_1, x_2, x_3, x_4 correspond respectively to the four groups $(\alpha_1\alpha_2\dots\alpha_n)$, $(\beta_1\beta_2\dots\beta_n)$, $(\gamma_1\gamma_2\dots\gamma_n)$ $(\delta_1\delta_2\dots\delta_n)$.

We shall write

$$[\bar{\alpha}\,\beta\,\bar{\gamma}\,\delta]_n \equiv (\alpha_l\,\beta_1\,\gamma_k\,\delta_1)(\alpha_l\,\beta_2\,\gamma_k\,\delta_2)\ (\alpha_l\,\beta_3\,\gamma_k\,\delta_3)\dots\dots(\alpha_l\,\beta_n\,\gamma_k\,\delta_n)$$

and $[\alpha\,\bar{\beta}\,\gamma\,\bar{\delta}]_n \equiv (\alpha_1\,\beta_l\,\gamma_1\,\delta_k)\ (\alpha_2\,\beta_l\,\gamma_2\,\delta_k)\ (\alpha_3\,\beta_l\,\gamma_3\,\delta_k)\dots\dots(\alpha_n\,\beta_l\,\gamma_n\,\delta_k)$,

l and k being any integers between 1 and n.

$[\bar{\alpha}\,\beta\,\bar{\gamma}\,\delta]_n$ and $[\alpha\,\bar{\beta}\,\gamma\,\bar{\delta}]_n$ may be called the *cross-ratio invariants* of the four n-point groups ; in the course of the following theorem, it will be evident that their values are independent of the integers l and k used in defining them.

The theorem of this note is

$$[\bar{\alpha}\,\beta\,\bar{\gamma}\,\delta]_n = [\alpha\,\bar{\beta}\,\gamma\,\bar{\delta}]_n = (x_1 x_2 x_3 x_4).$$

Proof: By definition

$$[\bar{\alpha}\,\beta\,\bar{\gamma}\,\delta]_n = \frac{(\alpha_l - \beta_1)\,(\gamma_k - \delta_1)}{(\alpha_l - \delta_1)\,(\gamma_k - \beta_1)} \times \frac{(\alpha_l - \beta_2)\,(\gamma_k - \delta_2)}{(\alpha_l - \delta_2)\,(\gamma_k - \beta_2)} \times \ldots \times \frac{(\alpha_l - \beta_n)\,(\gamma_k - \delta_n)}{(\alpha_l - \delta_n)\,(\gamma_k - \beta_n)}$$

$$= \frac{(\alpha_l - \beta_1)\,(\alpha_l - \beta_2)\ldots(\alpha_l - \beta_n)}{(\alpha_l - \delta_1)\,(\alpha_l - \delta_2)\ldots(\alpha_l - \delta_n)} \times \frac{(\gamma_k - \delta_1)\,(\gamma_k - \delta_2)\ldots(\gamma_k - \delta_n)}{(\gamma_k - \beta_1)\,(\gamma_k - \beta_2)\ldots(\gamma_k - \beta_n)}.$$

Now $F(y) = 0$ has roots $\beta_1, \beta_2 \ldots \beta_n$ when $x = x_2$ and $\delta_1, \delta_2 \ldots \delta_n$ when $x = x_4$. So

$$[\bar{\alpha}\,\beta\,\bar{\gamma}\,\delta]_n = \frac{F(\alpha_l)_{x=x_2}}{F(\alpha_l)_{x=x_4}} \times \frac{F(\gamma_k)_{x=x_4}}{F(\gamma_k)_{x=x_2}}$$

$$= \frac{\phi(x_2)_{y=\alpha_l} \times \phi(x_4)_{y=\gamma_k}}{\phi(x_4)_{y=\alpha_l} \times \phi(x_2)_{y=\gamma_k}}.$$

Now $\phi(x) = 0$ has root x_1 when $y = \alpha_l$ and has root x_3 when $y = \gamma_k$ whatever l or k may be.

Hence

$$[\bar{\alpha}\,\beta\,\bar{\gamma}\,\delta]_n = \frac{(x_2 - x_1)}{(x_2 - x_3)} \times \frac{(x_4 - x_3)}{(x_4 - x_1)} = \frac{(x_1 - x_2)\,(x_3 - x_4)}{(x_1 - x_4)\,(x_3 - x_2)} = (x_1\,x_2\,x_3\,x_4).$$

Similarly $\quad [\alpha\,\bar{\beta}\,\gamma\,\bar{\delta}]_n = (x_1\,x_2\,x_3\,x_4).$

Hence if a single infinity of points be in (1, 1) correspondence with a single infinity of groups of points each containing n-points, then the cross-ratio of any four points is equal to the *cross-ratio invariants* of the four corresponding n-point groups. As an illustration :

If four conics of a pencil be intersected by any fifth conic in the four tetrads of points, P_r, Q_r, R_r, S_r, ($r = 1, 2, 3, 4$), then the product

$$(PQ_1RS_1)\,(PQ_2RS_2)\,(PQ_3RS_3)\,(PQ_4RS_4)$$
$$= (P_1QR_1S)\,(P_2QR_2S)\,(P_3QR_3S)\,(P_4QR_4S)$$

P, Q, R, S denoting any of the points P_r, Q_r, R_r, S_r ; also, the product is independent of the fifth conic, and equal to the cross-ratio of the four given conics.

R. GOPALASWAMI.

Solutions.

Question 714.

(K. B. MADHAVA) :—Find the co-efficient of x^n in

$$\frac{x}{1+x} - \frac{x^2}{1+x^2} + \frac{x^3}{1+x^3} - \frac{x^4}{1+x^4} + \ldots\ldots$$

Solution by S. Audinarayanan.

On expanding each term and re-arranging in the form of a double series, it is easy to derive the co-efficient of x^n in the following manner :

Let $\quad n = (\alpha_1 \times \beta_1)$, or $(\alpha_2 \times \beta_2)$ ...... or $(\alpha_s \times \beta_s)$;

where α and β are the several conjugate factors of n and $\alpha_k = \beta_k$ say, when n is a perfect square. Then the co-efficient of x^n in the above series is $\Sigma(\lambda_r)$, where λ_r is

$+2$ when α and β are either both odd or both even,

-2, when of α and β one is odd and the other even,

and $\quad +1$, when $\alpha_k = \beta_k = \surd n$.

Question 771.

(K. B. MADHAVA) :—In his solution to Q. 629, Mr. Bhimasena Rao uses Bromwich, p. 447, Ex. 4, (after the method of Dirichlet's integrals), to show that if

$$\psi(t) = \sum_{-\infty}^{\infty} e^{-n^2\pi t}$$

we shall have

$$\psi(t) = t^{-\frac{1}{2}}\,\psi\left(\frac{1}{t}\right).$$

This result is due to Jacobi. See: *Ges. Werke.* II, p. 188. Verify this by integrating

$$\int \frac{e^{-z^2\pi t}}{e^{2\pi i z} - 1}\, dz$$

round a rectangle whose vertices are $\pm(R + \frac{1}{2}) \pm i$ and making $R \to \infty$ in the usual manner.

Show also by the same method, that if $t > 0$

$$\sum_{n=-\infty}^{\infty} e^{-n^2\pi t - 2n\pi t a} = t^{-\frac{1}{2}}\, e^{\pi a^2 t}\left\{1 + 2\sum_{1}^{\infty} e^{-\frac{n^2\pi}{t}} \cos 2n\pi a\right\},$$

also due to Jacobi.

Solution by S. Audinarayanan.

This is given in Whittaker and Watson: *Modern Analysis*, p. 124. We shall establish as a Lemma the result that if $\lambda > 0$

$$\int_{-\infty}^{\infty} e^{-\lambda x^2} \cos(2\lambda ax)dx = e^{-\lambda a^2} \int_{-\infty}^{\infty} e^{-\lambda x^2}\, dx$$

$$= \lambda^{-\frac{1}{2}} e^{-\lambda a^2} \int_{-\infty}^{\infty} e^{-x^2}\, dx.$$

(W. W., p. 114)

Integrate $\int e^{-\lambda z^2}\, dz$ round the rectangle $[-R, R, R+ai, -R+ai]$, and make $R \rightarrow \infty$.

Since the integrand is analytic inside the contour $\int e^{-\lambda z^2}\, dz = 0$. Hence we can write

$$\int_{-R}^{R} e^{-\lambda z^2}\, dz + \int_{R}^{R+ai} e^{-\lambda z^2}\, dz + \int_{R+ai}^{-R+ai} e^{-\lambda z^2}\, dz + \int_{-R+ai}^{-R} e^{-\lambda z^2}\, dz = 0.$$

Put $z = R + iy$ in the second integral and it transforms into

$$\left| \int_0^a e^{-\lambda(R+iy)^2}\, i\, dy \right| = \left| i \int_0^a e^{-\lambda(R^2+y^2)-2R\lambda iy}\, dy. \right|$$

$$< \int_0^a e^{-\lambda(R^2-y^2)}\, dy$$

$$< ae^{-\lambda(R^2-a^2)} \rightarrow 0, \text{ as } R \rightarrow 0.$$

Similarly $\displaystyle\int_{-R+ai}^{-R} e^{-\lambda z^2} dz \rightarrow 0$, as $R \rightarrow \infty$.

The first integral is $\displaystyle\int_{-\infty}^{\infty} e^{-\lambda x^2}\, dx.$

Put $z = (x + ai)$ in the third. It becomes

$$\lim \int_{R}^{-R} e^{-\lambda(x+ai)^2}\, dx = e^{a^2\lambda} \int_{\infty}^{-\infty} e^{-\lambda x^2} . e^{-2aix\lambda}\, dx$$

$$= e^{a^2\lambda} \int_{\infty}^{-\infty} e^{-\lambda x^2} (\cos 2a\lambda x - i \sin 2a\lambda x)$$

Hence we have

$$\int_{-\infty}^{\infty} e^{-\lambda x^2}\, dx + e^{a^2\lambda} \int_{\infty}^{-\infty} e^{-\lambda x^2} (\cos 2a\lambda x - i \sin 2a\lambda x)\, dx = 0.$$

Equating real and imaginary parts

$$\int_{-\infty}^{\infty} e^{-\lambda x^2} \cos(2a\lambda x)\, dx = e^{-\lambda a^2} \int_{-\infty}^{\infty} e^{-\lambda x^2}\, dx$$

$$= \lambda^{-\frac{1}{2}} e^{-\lambda a^2} \int_{-\infty}^{\infty} e^{-x^2}\, dx$$

and
$$\int_{-\infty}^{\infty} e^{-\lambda x^2} \sin(2a\lambda x)\, dx = 0.$$

Now we shall integrate $\int \frac{e^{-z^2 \pi t}}{e^{2\pi t z} - 1}\, dz$ along the given contour, *i.e.*, the rectangle with vertices at $\pm (R + \frac{1}{2}) \pm i$.

The poles of the integrand inside the contour are at

$$\pm 1, \pm 2, \pm 3, \ldots\ldots \pm R.$$

The residue at $z = r$ is given by the co-efficient of $\frac{1}{s}$ on putting $z = r + s$ and expanding. The residue is

$$\frac{1}{2\pi i} e^{-2\pi t}.$$

Hence the sum of the residues at poles inside the contour

$$= \frac{1}{2\pi i} \sum_{-R}^{R} e^{-n^2 \pi t}.$$

Therefore

$$\int_{-(R+\frac{1}{2})-i}^{(R+\frac{1}{2})-i} \frac{e^{-z^2\pi t}}{e^{2\pi t z} - 1}\, dz + \int_{(R+\frac{1}{2})-i}^{(R+\frac{1}{2})+i} \frac{e^{-z^2\pi t}}{e^{2\pi t z} - 1}\, dz + \int_{R+\frac{1}{2}+i}^{-(R+\frac{1}{2})+i} \frac{e^{-z^2\pi t}}{e^{2\pi t z} - 1}\, dz$$

$$+ \int_{-(R+\frac{1}{2})+i}^{-(R+\frac{1}{2})-i} \frac{e^{-z^2\pi t}}{e^{2\pi t z} - 1}\, dz = 2\pi i \frac{1}{2\pi i} \sum_{-R}^{R} e^{-n^2\pi t}.$$

When $R \rightarrow \infty$, the absolute values of the second and the fourth integrals tend to zero and we have

$$\int_{-\infty-i}^{\infty-i} \frac{e^{-z^2\pi t}}{e^{2\pi t z} - 1}\, dz + \int_{\infty+i}^{-\infty+i} \frac{e^{-z^2\pi t}}{e^{2\pi i z} - 1}\, dz$$

$$= \sum_{-\infty}^{\infty} e^{-n^2\pi t} = \psi(t).$$

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Expanding the integrands in powers of $e^{-2\pi iz}$, $e^{2\pi iz}$ respectively, we have

$$\psi(t) = \int_{-\infty-i}^{\infty-i} e^{-z^2\pi t}\,[e^{-2\pi iz} + e^{-4\pi iz} + e^{-6\pi iz} + \ldots\ldots]\,dz$$

$$+ \int_{-\infty+i}^{\infty+i} e^{-z^2\pi t}\,(1 + e^{2\pi iz} + e^{4\pi iz} + e^{6\pi iz} + \ldots\ldots)\,dz.$$

Put $(x - i)$ and $(x + i)$ respectively for z in the two integrals, then we have,

$$\psi(t) = \int_{-\infty}^{\infty} \Big[e^{-(x^2+i)^2\pi t}$$

$$+ 2\sum_1^{\infty}\Big\{ e^{-x^2\pi t + \pi t - 2n\pi} \cos 2\pi t\left(1 - \frac{n}{t}\right)_x \Big\}\Big]\,dx.$$

By the Lemma we have

$$\psi(t) = \frac{e^{\pi t - \pi t}}{\sqrt{\pi t}} \int_{-\infty}^{\infty} e^{-x^2}\,dx + \frac{2}{\sqrt{\pi t}} \sum_1^{\infty} e^{\pi t - 2n\pi - \left(1 - \frac{n}{t}\right)^2 \pi t} \int_{-\infty}^{\infty} e^{-x^2}\,dx$$

$$= \frac{1}{\sqrt{\pi t}} \Big[1 + 2\sum_1^{\infty} e^{-\frac{n^2\pi}{t}} \Big] \int_{-\infty}^{\infty} e^{-x^2}\,dx$$

$$= \frac{1}{\sqrt{\pi t}} \Big\{ \sum_{-\infty}^{\infty} e^{-\frac{n^2\pi}{t}} \Big\} \int_{-\infty}^{\infty} e^{-x^2}\,dx$$

$$= \frac{1}{\sqrt{\pi t}} . \psi\left(\frac{1}{t}\right) \int_{-\infty}^{\infty} e^{-x^2}\,dx.$$

But $\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$, which can also be seen by putting $t = 1$.

$$\therefore\ \psi(t) = t^{-\frac{1}{2}}\,\psi\left(\frac{1}{t}\right).$$

The second part of the result comes out easily, if we have $z + a$ in place of z. Then

$$\sum_{-\infty}^{\infty} e^{-(z+a)^2\pi t} = \int_{-\infty-i}^{\infty-i} e^{-(z+a)^2\pi t}\,[e^{-2\pi i(z+a)} + e^{-4\pi i(z+a)} + \ldots]\,dz$$

$$+ \int_{-\infty+i}^{\infty+i} e^{-(z+a)^2\pi t}\,[1 + e^{2\pi i(z+a)} + e^{4\pi i(z+a)} + \ldots\ldots]\,dz$$

$$= \int_{-\infty}^{\infty} [e^{-(x+a+i)^2\pi t} + 2\sum_{1}^{\infty} \{ e^{-(x-i+a)^2\pi t - 2n\pi i(x+a-i)}$$

$$+ e^{-(x+a+i)^2 + 2n\pi i(x+a+i)} \}] \, dx$$

$$= \frac{1}{\sqrt{\pi t}} \int_{-\infty}^{\infty} e^{-x^2} \, dx$$

$$+ 2 \int_{-\infty}^{\infty} \sum_{1}^{\infty} e^{-(x+a)^2 + \pi t - 2n\pi} \cos 2\pi t \left(1 - \frac{n}{t}\right) x \cos 2n\pi a \, . \, dx$$

$$= \frac{1}{\sqrt{\pi t}} \int_{-\infty}^{\infty} e^{-x^2} \, dx . \left[1 + 2\sum_{1}^{\infty} e^{-\frac{n^2\pi}{t}} \cos 2n\pi a \right]$$

by the Lemma

$$= t^{-\frac{1}{2}} [1 + 2\sum_{1}^{\infty} e^{-\frac{n^2\pi}{t}} \cos 2n\pi a] .$$

$$\therefore \sum_{1}^{\infty} e^{-n^2\pi t - 2n\pi t a} = t^{-\frac{1}{2}} e^{a^2\pi t} (1 + 2\sum_{1}^{\infty} e^{-\frac{n^2\pi}{t}} \cos 2n\pi a).$$

Question 1197.

(N. B. MITRA) :—If $(2n + 1)$ is a prime number p, prove that the numerator of $2^{4n} + p\,(1 + \frac{1}{2} + \frac{1}{3} + . + 1/n) - 1 \equiv 0 \pmod{p^2}$.

Solution by I. Totadri Iyengar.

$$(p-1)! = (p-1)(p-2)(p-3) \ldots (p-n) . n . (n-1) \ldots 2.1$$

$$= n! . (p-1)(p-2) \ldots\ldots (p-n)$$

$$\equiv n! . [(-1)^n] \, [n! - n!\, p\,(1 + \tfrac{1}{2} + \ldots + \frac{1}{n})] \bmod p^2$$

and hence $\quad (p-1)! \equiv$ numerator of $(-1)^n (n!)^2$

$$\times [1 - (1 + \tfrac{1}{2} + \ldots\ldots + 1/n)\, p], \bmod p^2.$$

But in my solution to Q. 1247 (see : Vol. XV, No. 4), it has been proved that

$$(p-1)! = 2n! \equiv (-1)^n\, 2^{4n} (n!)^2, \bmod p^2.$$

$$\therefore (-1)^n (n!)^2 \, [1 - (1 + \tfrac{1}{2} + \ldots\ldots + \frac{1}{n})\, p] \equiv (-1)^n\, 2^{4n} (n!)^2, \bmod p^2$$

$$\therefore 2^{4n} + (1 + \tfrac{1}{2} + \ldots\ldots + \frac{1}{n})\, p - 1 \equiv 0, \bmod p^2$$

on removing the common factors and taking the quantities to one side.

Question 1260.

(A. T. THOMAS):—If $a_1 = \sin x$, $a_2 = \sin\sin x$, and so on, where $0 < x < \pi$, prove that the infinite series $\Sigma\, a_n$ and $\Sigma\, a_n^2$ are divergent but that $\Sigma\, a_n^3$ is convergent; and, in fact, that if n be large $a_n \sim \surd(3/n)$.

Solution by A. Narasinga Rao.

We have here a sequence of numbers a_n decreasing steadily but never negative. Such a sequence necessarily tends to a limit which may be finite or zero. To determine this limit λ, we have to solve the equation $\sin\lambda = \lambda$. One root is $\lambda = 0$ and since λ increases more rapidly than $\sin\lambda$ there cannot be any other real root. Hence the sequence tends to zero and it remains to fix the order of a_n.

Since in the limit $\qquad a_{n+1} = \sin a_n, \qquad \ldots \qquad \ldots (1)$

it follows that

$$\log\frac{a_n}{a_{n+1}} = -\log\left(1 - \frac{a_n^2}{3!} + \frac{a_n^4}{5!} - \ldots\ldots\right) \qquad \ldots (2)$$

$$= \frac{a_n^2}{6} + 0\,(a_n^4).$$

If a_n be of order n^{-r}, we have $\dfrac{a_n}{a_{n+1}} = \left(1 + \dfrac{1}{n}\right)^r$,

and so $$\log\frac{a_n}{a_{n+1}} = \frac{r}{n} + 0\left(\frac{1}{n^2}\right). \qquad \ldots \qquad \ldots (3)$$

Substituting in (2), we have the principal part of the first member of (2) equal to (r/n) while that of the second member is of order n^{-2r} and as these must be equal $r = \frac{1}{2}$. We may therefore take

$$a_n = \frac{\alpha}{\surd n} + \frac{\beta}{n} + 0\left(\frac{1}{n}\right). \qquad \ldots \qquad \ldots (4)$$

Substituting for a_n and a_{n+1} in

$$\frac{a_{n+1}}{a_n} = 1 - \frac{a_n^2}{2!} + \frac{a_n^4}{5!} \qquad \ldots \qquad \ldots (5)$$

we have

$$\frac{\dfrac{\alpha}{\surd n+1} + \dfrac{\beta}{n+1} + 0\left(\dfrac{1}{n}\right)}{\dfrac{\alpha}{\surd n} + \dfrac{\beta}{n} + 0\left(\dfrac{1}{n}\right)} = 1 - \frac{\alpha^2}{6n} - \frac{\alpha\beta}{3n^{\frac{3}{2}}} + 0\left(\frac{1}{n^2}\right). \qquad \ldots (6)$$

The left side of (6) when expanded in ascending powers of $n^{-\frac{1}{2}}$ is found to be

$$1 - \frac{1}{2n} + 0\left(\frac{1}{n^{\frac{3}{2}}}\right)$$

and equating infinitesimals of the same order on both sides of (6), we have $\alpha = \sqrt{3}$.

If it is desired to carry the investigation further, we may take

$$a_n = \frac{\sqrt{3}}{\sqrt{n}} + \frac{\beta}{n} + \frac{\gamma}{n^{\frac{3}{2}}} + \ldots\ldots$$

substitute in (5) and equate the coefficients of different powers of $\frac{1}{\sqrt{n}}$ on both sides.

As $a_n \sim \sqrt{(3/n)}$, it is obvious that Σa_n and Σa_n^2 diverge but that Σa_n^3 converges.

Question 1261.

(A. T. THOMAS) :—If the numbers represented by $f(x, y)$, [where f is an integral expression with integral co-efficients of the positive integers x, y], are arranged in increasing order of magnitude and those less than N be considered, the probability that x and y are prime to each other is a number tending to $\frac{6}{\pi^2}$, as $N \rightarrow \infty$.

Solution by S. Audinarayanan.

Since f is an integral expression with integral co-efficients of positive integers x, y and since we consider numbers less than N, it is evident that the values of x and y are less than some number M. As N tends to infinity, M also tends to infinity. Hence the problem reduces itself to finding the probability of x and y being prime to each other, where x and y are less than M and $M \rightarrow \infty$.

Suppose that a prime number s is taken and x and y are not divisible by it. Then the probability of x and y being prime to each other is

$$P = p_2 \cdot p_3 \cdot p_5 \cdot \ldots p_s,$$

where the small p's are the probabilities that each of the primes 2, 3, 5, ... is not a common factor of x and y and s the highest prime $< M$.

Next, consider that x and y are divided by s. We are sure to get one of the following remainders 0, 1, 2, ... $(s - 1)$.

Now s divides x and y exactly where 0 is got as remainder. Thus the probability of s being a factor of x is $(1/s)$ and similarly for y. The probability of s being a common factor of x and y is $(1/s^2)$. So the probability of its not being a common factor is $\left(1 - \frac{1}{s^2}\right)$.

$$\therefore\ P = \left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{3^2}\right)\left(1 - \frac{1}{5^2}\right) \ldots\ldots \left(1 - \frac{1}{s^2}\right).$$

As M tends to infinity, P becomes an infinite product

Thus $P = \left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{3^2}\right)\left(1 - \frac{1}{5^2}\right)$ ...... to infinity

$$\therefore \quad \frac{1}{P} = \frac{1}{1-\frac{1}{2}^2} \cdot \frac{1}{1-\frac{1}{3}^2} \cdot \frac{1}{1-\frac{1}{5}^2} \cdots\cdots$$

$$= \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots\ldots \text{ ad inf.} = \frac{\pi^2}{6}.$$

Hence $$P = \frac{6}{\pi^2}.$$

Question 1266.

(A. A. KRISHNASWAMI IYENGAR):—Find general expressions for the sides of rational triangles, the squares of whose areas are perfect cubes. (Ex. 5, 6, 7).

Solution by N. B. Mitra.

(i) If by a rational triangle we mean a triangle whose sides only are rational, then the triangle whose sides are $n^3(p^2 - pq^3)$, $n^3(p^2 - pq^3 + q^6)$ and $n^3(2pq^3 - q^6)$, where p, q and n are rational and p greater than q^3 will have its $(\text{area})^2$ equal to a perfect cube, namely the cube of $n^4pq^2(p - q^3)$.

(ii) Ordinarily, however, by a rational triangle is meant a triangle whose sides as well as area are rational. In this case, the solution may be given thus: the triangle whose sides are

$n^3p^4(4p^2 - q^6)^2$; $n^3p^4(16p^4 + 56p^2q^6 + q^{12})$ and $16n^3p^5q^3(4p^3 + q^6)$ will have its area rational and a perfect cube, *viz.*, the cube of $2n^2p^3q(4p^2 - q^6) - p, q, n$ being rational.

Questions for Solution.

1327. (S. NARAYANA AIYAR, M.A.):—Show, from independent considerations, that the addition equations of the Jacobian elliptic functions may be thrown, in Dr. Glaisher's notation, into the forms

(1) $\tan^{-1}[\mathrm{sc}(u+v)] = \tan^{-1}\mathrm{sc}\,u.\mathrm{dn}\,v + \tan^{-1}\mathrm{sc}\,v.\mathrm{dn}\,u.$

(2) $\tan^{-1}[k\,\mathrm{sd}(u+v)] = \tan^{-1}k\,\mathrm{sd}\,u.\mathrm{cn}\,v + \tan^{-1}k\,\mathrm{sd}\,v\,\mathrm{cn}\,u.$

Hence, or otherwise, show that, if s_n denotes the sum of n quantities $u_1, u_2, u_3 \ldots\ldots u_n$, s_r the sum of any r of the quantities and s_{n-r} the sum of the remaining $n-r$ of the quantities

(3) $$\mathrm{cn}\,s_n + i\,\mathrm{sn}\,s_n = \frac{\mathrm{cn}\,s_r + i\,\mathrm{sn}\,s_r.\mathrm{dn}\,s_{n-r}}{\mathrm{cn}\,s_{n-r} - i\,\mathrm{sn}\,s_{n-r}.\mathrm{dn}\,s_r}$$

(4) $$\mathrm{dn}\,s_n + ik\,\mathrm{sn}\,s_n = \frac{\mathrm{dn}\,s_r + ik\,\mathrm{sn}\,s_r.\mathrm{cn}\,s_{n-r}}{\mathrm{dn}\,s_{n-r} - ik\,\mathrm{sn}\,s_{n-r}.\mathrm{cn}\,s_r}.$$

Examine the cases, when $\mathrm{sn}\,u = 1$ or $\frac{1}{k}$ in (1) and (2) above.

1328. (K. J. SANJANA and S. N. KUMARASWAMY):—On the major axis AA′ of an ellipse, whose minor axis is of length $2b$, a rectangle A′ADE is described, having the height AD $= b\sqrt{2}$; a straight line EG meets AA′ in G and cuts the ellipse at P and Q. If DP, DQ meet AA′, produced if necessary in F,F′, prove that A′F = A′F′ and that $(A'F'^2 + AG^2)$ is constant. When EG touches the ellipse at R, prove that DRA′ is a straight line.

1329. (A. A. KRISHNASWAMI IYENGAR, M.A.):—Prove that the greatest coefficient in the expansion of $(1 + 2x + 3x^2)^n$ is that of x^{4m}, x^{4m+2}, or x^{4m+3}, according as $n=3m$, $3m+1$, or $3m+2$.

1330. (N. DURAIRAJAN):—If I_1, I_2, $I_3 \ldots\ldots$ be the centres of the inscribed circles of the triangles ABC, I_1BC, I_2BC....; show that I_n tends to a point I on BC, such that

$$BI : IC = \text{arc } BA : \text{arc } AC.$$

1331. (K. SATYANARAYANA):—Prove that the origin lies in the acute or obtuse angle between the straight lines

$$y = m_1 x + c_1,\ y = m_2 x + c_2,$$

according as

$$c_1 c_2 (1 + m_1 m_2) \lesseqgtr 0.$$

1332. (K. SATYANARAYANA) :—Prove that a triangle whose sides are parallel to $y = m_1 x$, $y = m_2 x$, $y = m_3 x$, will be acute-angled, right-angled or obtuse-angled according as

$$(1 + m_1 m_2)(1 + m_2 m_3)(1 + m_3 m_1) \lesseqgtr 0.$$

1333. (K. J. SANJANA) :—(X, X′), (Y, Y′), (Z, Z′) are points in the sides BC, CA, AB respectively of a triangle ABC, such that Y′Z, Z′X, X′Y are equal and respectively anti-parallel to BC, CA, AB. If circles are described about the triangles AY′Z, BZ′X, CX′Y, prove that the point isogonally conjugate to their radical centre lies on the *Euler line* of the triangle. Also state the corresponding property when the lines Y′Z, Z′ X, X′Y of equal lengths are parallel respectively to the opposite sides of the triangle.

1334. (HEMRAJ) : —Triangles $X_1X_2X_3$, $X'_1X'_2X'_3$ [where $X_r = x_r y_r z_r$)] are self-conjugate for the conic $ux^2 + vy^2 + wz^2 = 0$. Conics through their vertices and those of the triangle of reference (two at a time) have a common self-conjugate triangle. Prove that

(1) $$\prod_{r=1}^{r=3} (ux_r^2 + vy_r^2 + wz_r^2) \div \prod_{r=1}^{r=3} (ux_r'^2 + vy_r'^2 + wz_r'^2)$$

$$= \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}^2 \div \begin{vmatrix} x_1' & y_1' & z_1' \\ x_2' & y_2' & z_2' \\ x_3' & y_3' & z_3' \end{vmatrix}^2$$

(2) $$\xi^2/u^3 + \eta^2/v^3 + \zeta^2/w^3 = 0,$$

where $\xi = y_1 y_2 y_3 z'_1 z'_2 z'_3 - z_1 z_2 z_3 y_1' y_2' y_3'$

$\eta = z_1 z_2 z_3 x_1' x_2' x_3' - x_1 x_2 x_3 z_1' z_2' z_3'$

and $\zeta = x_1 x_2 x_3 y_1' y_2' y'_3 - y_1 y_2 y_3 \, x_1' x_2' x_3'$.

1335. (MARTYN THOMAS) :—An endless inelastic string of uniform density is placed in a smooth tube in the form of a closed curve without singularities, and is acted on by a field of force such that unit mass experiences a force μr perpendicular to the radius vector. Show that the string will complete one revolution from rest in time $\frac{l}{\mu\sqrt{\Delta}}$, where l and Δ are the length and area of the curve.

Generalize the result when the force is μr^n, and examine the cases when $n = -1$, -2 and $\frac{3}{2}$.

LIST OF JOURNALS RECEIVED

(From the 5th of March to the 1st of May 1924)

1 Proceedings of the London Mathematical Society, Series 2, Vol. 22, Parts 5, 6.

2 The Messenger of Mathematics, Vol. 53, Nos. 5, 6 & 7.

3 Transactions of the American Mathematical Society, Vol. 24, No. 4, Dec. 1924.

4 American Journal of Mathematics, Vol. 45, No. 4.

5 The American Mathematical Monthly, Vol. 31, Nos. 1, 2.

6 The Tohoku Mathematical Journal, Vol. 23, Nos. 1, 2.

7 Nature, Vol. 113, Nos. 2832, 2833, 2834, 2835, 2836, 2837, 2838.

8 Transactions of the Cambridge Philosophical Society, Vol. 23, No. 2.

9 The Astrophysical Journal, Vol. 48, No. 5, Vol. 49, No. 1.

10 Monthly Notices of the Royal Astronomical Society, Vol. 84, No 2.

11 Acta Mathematica, Vol. 39, Nos. 3 & 4.

12 Philosophical Transactions of the Royal Society of London, Vol. 224, A. 617.

13 Philosophical Magazine, Vol. 47, No. 279.

14 Annals of Mathematics (Second Series), Vol. 24, No. 4.

15 Popular Astronomy, Vol. 32, Nos. 1, 2, 3.

16 Bulletin of the American Mathematical Society, Vol. 3, Nos. 1, 2. Jan.-Feb., 1924.

17 The Mathematical Gazette, Vol. 12, No. 169, March 1924.

18 Bulletin of the Calcutta Mathematical Society, Vol. 14, No. 4.

19 Revista De Mathematicas Ano. V, Nos. 9, 10 (57, 58).

20 Memoria, No. 10.

21 Cätâlogo y. Reglamento de la Bibliotica General de la Faculted.

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