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CONTENTS.

A paper should contain a short and clear summary of the new results obtained and the relations in which they stand to results already known. Contributors are requested to bear in mind that, at the present stage of mathematical research, hardly any paper is likely to be so completely original as to be independent of earlier work in the same direction; and that readers are often helped to appreciate the importance of a new investigation by seeing its connection with earlier results.

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THEOREM V. *If L lies on the polar line of L′,*

and $$(L'_1 L'_2 L'_3 L'_4)\, L = P,$$

then $$(ABCP)\, L'_1 = L.$$

Proof.—Let Ω be a collineation of the form

$$x' = ax,\ y' = by,\ z' = cz,$$

such that $\Omega\,(L) = L'_1$. Then by Theorem IV

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 0.$$

If L, L'_1 be $(\alpha\beta\gamma)$, $(\lambda\mu\nu)$ respectively

then $$a = \frac{\lambda}{\alpha}, \text{ etc.}$$

Hence P is

$$\left(\frac{\lambda^2}{\alpha},\ \frac{\mu^2}{\beta},\ \frac{\nu^2}{\gamma}\right) \text{ or } (\lambda a,\ \mu b,\ \nu c),$$

that is to say, $P = \Omega\,(L'_1)$. We have now to show

$$(ABC\ \Omega\,(L'_1))\ L'_1 = L$$

This is seen directly from the equations of the transformation (ABCP), namely,

$$x' = x\left(-\frac{\alpha x}{\lambda^2} + \frac{\beta y}{\mu^2} + \frac{\gamma z}{\nu^2}\right) \text{ etc.,}$$

by using the fact that

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = \frac{\alpha}{\lambda} + \frac{\beta}{\mu} + \frac{\gamma}{\nu} = 0.$$

It follows also from Th. IV that L'_1 lies on the polar line of P.

This theorem also shows indirectly that if (ABCP) Q = R, then RQ, QP can be similar pairs only if R lies on the polar line of Q.

5. Point Representation of Conics of the Net.

We will now proceed to interpret the above results with respect to the conics of the special net with a common self-polar triangle. It was stated in § 2, that invariant relations between conics of the net are equivalent to invariant relations between the same conics considered as elements of the ternary domain of the net and the three squired lines. We may work out this idea by instituting a correspondence between conics of the net and points in a plane.

Represent the three squared lines of the net, by three points α, β, γ in a second plane π. Then a one to one correspondence is established between points in the π plane and the members of the net. A point on the side, say $\beta\gamma$, of the triangle $\alpha\beta\gamma$ will represent a pair of straight lines through the vertex A of the common self-polar triangle ABC and the sides of the triangle $\alpha\beta\gamma$ represent the totality of pairs of lines through the vertices of the triangle ABC.

To fix our ideas we can suppose that a conic of the net whose equation is $ax^2 + by^2 + cz^2 = 0$ is represented in the π-plane by the point whose homogeneous co-ordinates referred to the triangle $\alpha\beta\gamma$ are (a, b, c).

6. The Pencil and Range of Conics of the Net.

All conics of the net passing through a fixed point, also pass through three other fixed points and form a pencil. Since any two conics determine the pencil and since every conic will have the equation expressed as a linear combination of the equations of the two conics, the representative points in the π-plane will be a straight line passing through the representative points of the two conics. The points of intersection of this line with the sides of the triangle $\alpha\beta\gamma$ represent the three linepairs of the pencil.

All conics of the net touching a fixed line also touch three other fixed straight lines. The conditions that a conic

$$ax^2 + by^2 + cz^2 = 0$$

should touch the line $lx + my + nz = 0$ is

$$\frac{l^2}{a} + \frac{m^2}{b} + \frac{n^2}{c} = 0.$$

Hence the locus of representative points in the π-plane of the conics of the range is a circum-conic of the fundamental triangle $\alpha\beta\gamma$.

Since two lines in the π-plane intersect in only one point, a unique conic of the net passes through two given points, which are not specialised in position. Also, since two circum-conics intersect in only one other point, only one conic of the net touches the sides of two harmonic quadrilaterals. But because a line intersects a conic in two points, there are two conics of the net that pass through a harmonic quadrangle and touch a harmonic quadrilateral. If the conics of a pencil have double contact, the line in the π-plane corresponding to the pencil, will pass through a vertex of $\alpha\beta\gamma$. A quadrangle in π, having $\alpha\beta\gamma$ for its harmonic triangle, corresponds to four conics of the net, every two of which have double contact.

7. The ϕ-and the F-Conic of two Conics S_1, S_2.*

Given two conics S_1, S_2, the ϕ-conic of $S_1\ S_2$ is the conic touching the eight tangents at the common points of S_1 and S_2. The F-conic is a conic passing through the eight points of contact of common tangents to S_1 and S_2. These conics are also members of the net (c.f. Introduction). Representing these conics by points s_1, s_2 ϕ, and f in the π-plane, the quadratic transformations studied in § 4 lead, as we proceed to show, to constructions for the points ϕ and f, given s_1 and s_2.

ϕ is the point of intersection of the two circum-conics of the triangle $\alpha\beta\gamma$ which touch the line s_1, s_2 at s_1 and s_2.

For it is a member of the four-line system determined by the tangents to S_1 at the common points, in other words, by the common tangents of S_1 and a consecutive conic of the system with the same common points. Similarly it is a conic of the system determined by the common tangents to S_2 and a consecutive conic of the system having the same common points.

Secondly, f is the pole of the line $s_1\ s_2$ with respect to the conic through $\alpha\beta\gamma$ $s_1\ s_2$. For, the F-conic passes through the points of contact of the common tangents and hence is a member of the system of conics through the common points of S_1 and a consecutive conic having the same common tangents, and also of the system through the

* *Vide* A. Narasinga Rao: "The Harmonic locus and its analogues in hyperspace," *J. I. M. S.* Vol. XVI, where analogous theorems for space of any number of dimensions are discussed. Some of the results stated in this and the subsequent paragraphs will also be found therein.

common points of S_2 and a consecutive conic with the same common tangents, and hence, it is a point of intersection of tangents to the circum-conic $\alpha\beta\gamma\, s_1\, s_2$ at s_1 and s_2, namely, it is the pole of $s_1\, s_2$ with respect to the conic $\alpha\beta\gamma\, s_1\, s_2$.

8. The Totality of Pairs of Conics with a given F- or ϕ-Conic.

From the geometrical construction for ϕ (§ 7) it is evident that the points s_1, s_2 are quadratic transforms of one another with respect to the quadrangle $\alpha\beta\gamma\phi$. Given the point ϕ, we have an infinity of pairs of points s_1, s_2 which are quadratic transforms of one another with respect to $\alpha\beta\gamma\phi$. So there are an infinity of pairs of conics S_1, S_2 in the net that have a given conic for their ϕ-conic.

From the dualistic properties of the ϕ- and F-conic, it is evident, that the polar lines of s_1 and s_2 with respect to the triangle $\alpha\beta\gamma$ are quadratic transforms of one another with respect to the quadrilateral formed by the sides of the triangle and the polar line of f. Since also, given f there are an infinity of pairs of points s_1, s_2 satisfying the above condition, there are an infinity of pairs of conics having a given F-conic.

9. Reciprocation.

Given two conics S_1 and Ω represented by the equations

$$S_1 \equiv ax^2 + by^2 + cz^2 = 0$$

$$\Omega \equiv \lambda x^2 + \mu y^2 + \nu z^2 = 0$$

the reciprocal conic S_2 of S_1 with respect to Ω is

$$S_2 \equiv \frac{\lambda^2}{a} x^2 + \frac{\mu^2}{b} y^2 + \frac{\nu^2}{c} z^2 = 0$$

Therefore it follows that the points s_1, s_2 and ω in the π-plane are such that s_1, s_2 are quadratic transforms of one another with respect to the harmonic quadrangle determined by ω. The transformation is the same whatever corner of the quadrangle is taken initially. Hence there are four conics which reciprocate two given conics into one another. These four, as shown in § 6, have double contact with each other and have the sides of ABC for chords of contact. These four conics reciprocate any one of them into itself*.

If there are three conics S_1, S_2, S_3, which are such that any two of them are reciprocals of one another with respect to the third, their equations will be of the form

$$ax^2 + by^2 + cz^2 = 0$$

$$ax^2 + b\omega y^2 + c\omega^2 z^2 = 0$$

$$ax^2 + b\omega^2 y^2 + c\omega z^2 = 0.$$

Hence the corresponding points $s_1 s_2 s_3$ in the π-plane will be in sextaple perspective with the triangle $\alpha\beta\gamma$. The three points are also such that the polar line and the polar conic of any one of them intersect at the other two points. (cf. § 3).

10. In- and out-Polar Conics.

A conic S_2 is said to be out-polar to S_1 when there exists an inscribed triangle of S_2 which is also self-polar with respect to S_1. Also S_3 is in-polar to S_1 when there exists a circumscribed triangle of S_3 which is also self-polar with respect to S_1. The condition for this is known to be the vanishing of the θ invariants of the two conics. Suppose

$$S_r \equiv a_r x^2 + b_r y^2 + c_r z^2 = 0 \qquad (r = 1, 2, 3).$$

The first condition is

$$\frac{a_2}{a_1} + \frac{b_2}{b_1} + \frac{c_2}{c_1} = 0.$$

and for the second the condition is

$$\frac{a_1}{a_3} + \frac{b_1}{b_3} + \frac{c_1}{c_3} = 0.$$

So all conics of the net which are out polar to S_1 correspond to points on the polar line of the corresponding point S_1 with respect to

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* In Milne's Homogeneous Co-ordinates, page98, Ex. 14, it is stated that one of these four conics reciprocate a second into a third of them. This is an error.

$\alpha\beta\gamma$ in the π-plane, and all conics inpolar to S_1 correspond to points on the polar conic of s_1 with respect to $\alpha\beta\gamma$ in the π-plane. The two points of intersection of the polar line and the polar conic represent conics which are both in- and out-polar or doubly apolar to S_1. The three points are in sextaple perspective with $\alpha\beta\gamma$ and possess symmetric properties. Each of these conics reciprocates the other two into each other.

Hence from Theorem V, I of § 4 we have the theorems :—

" If two conics S_1, S_2 are such that S_1 is out-polar to S_2, then the ϕ conic of S_1, S_2 is the reciprocal of S_1 with respect to S_2 ".

"If S_1, S_2, S_3 are three conics such that S_1 is out-polar to the ϕ-conic of S_2, S_3, then S_1, S_2, S_3 have the symmetric property that each is out-polar to the ϕ-conic of the other two ".

11. Metrical properties of the net.*

Since circles are conics through the two circular points and since only one conic of the net can pass through two given points, we see that there is only one circle in the whole net. This is the well-known auto-polar circle. In trilinear co-ordinates the equation of this circle will be

$$x^2 \sin 2A + y^2 \sin 2B + z^2 \sin 2C = 0.$$

and hence the corresponding point in the π-plane is (sin 2A, sin 2B, sin 2C).

Parabolas are conics touching the line at infinity and hence if

$$\Omega \equiv \lambda^2 + \mu^2 + \nu^2 - 2\mu\nu\cos A - 2\nu\lambda\cos B - 2\lambda\mu\cos C = 0$$

be the tangential equation to the circular points, the condition that

$$\alpha' x^2 + \beta' y^2 + \gamma' z^2 = 0$$

is a parabola is

$$\beta'\gamma' \sin^2 A + \gamma'\alpha' \sin^2 B + \beta'\alpha' \sin^2 C = 0$$

Hence the locus of the corresponding point in the π-plane is the circum-conic

$$P_L \equiv \frac{\sin^2 A}{x} + \frac{\sin^2 B}{y} + \frac{\sin^2 C}{z} = 0.$$

* Karl Meister ; *loc. cit.*

Since the circular points are conjugate points with respect to rectangular hyperbolas, the condition that the conic

$$lx^2 + my^2 + nz^2 = 0$$

is a rectangular hyperbola is

$$l + m + n = 0$$

and so the locus of the representative point in the π-plane is the line

$$R_L \equiv x + y + z = 0.$$

The intersection of P_L and R_L gives two points representing two conics which are both parabolas and rectangular hyperbolas—termed the circular parabolas. They are the two parabolas which touch the line at infinity at the circular points. The F-conic of these two is a conic through the circular points and is a member of the net. So it is the unique circle of the net. This is otherwise evident from the fact that the point (sin 2A, sin 2B, sin 2C) is the pole of R_L with respect to P_L.

Since the locus of a general range intersects P_L and R_L in one and two points respectively, there are only a parabola and two rectangular hyperbolas in any range. But in a pencil, there are two parabolas and only one rectangular hyperbola as the locus for the pencil intersects P_L and R_L in two and one point respectively.

12. Similar Conics.

Two conics are similar if the angles between their asymptotes are equal. So for all similar conics, the angle between the asymptotes is constant. If

$$lx^2 + my^2 + nz^2 = 0$$

is the equation of a conic, the angle θ between the asymptotes is given by

$$\tan\theta = \frac{2\sqrt{\{-mn\sin^2 A - nl\sin^2 B - lm\sin^2 C\}}}{l + m + n}.*$$

Hence for all conics similar to a given conic

$$(l + m + n)^2 + k(mn\sin^2 A + nl\sin^2 B + lm\sin^2 C) = 0.$$

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* Salmon : *Conic Sections*, page 392, Ex. 7.

The two terms on the left-side equated to zero give as we have seen in § 11, the loci R_L and P_L corresponding to rectangular hyperbolas and parabolas. Hence the locus of the points in π-plane representing similar conics is a conic having double contact with P_L, the chord of contact being R_L and passing through the point representing the given conic. This is in conformity with the fact that conics similar to parabolas and rectangular hyperbolas are all parabolas and rectangular hyperbolas respectively.

13. Conics with equal areas.

The squares of semi axes of a conic

$$lx^2 + my^2 + nz^2 = 0$$

are given by the quadratic equation

$$R^4 \theta^3 + R^2M^2 \Delta \theta \theta' + M^4 \Delta^2 = 0^*$$

(θ, θ', Δ being the invariants of the given conic and the circular points) from which we deduce that for all conics with equal areas we must have

$$lmn/(mn \sin^2 A + nl \sin^2 B + lm \sin^2 C)^{3/2} = \text{constant.}$$

Hence the locus of the representative points of conics with equal areas is a sextic curve whose equation is

$$(mn \sin^2 A + nl \sin^2 B + lm \sin^2 C)^3 + kl^2m^2n^2 = 0.$$

In every pencil of conics there are sets of two similar conics and sets of six equal-area conics. In every range there are sets of four similar conics and sets of 12 equal-area-conics Conics that are both similar and equal in area, namely, congruent, occur in sets of 12 in the net. †

14. Conics with a common focus.

The foci of a conic are the points of intersection of tangents to the conic from the circular points and the system of conjugate lines through the focus form an orthogonal involution pencil. Given a focus there is only one conic of the net in general having the point for focus. If a

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* Salmon : *Conic Sections*, page 392, Ex. 11.

† For further information, see Karl Meister, *loc. cit.*

point is to be the focus of more than one conic, a pair of lines at right-angles through it should be a pair of conjugate lines for all the conics. The only points that satisfy this condition are the feet of the perpendiculars from the vertices on to the opposite sides of the common self-polar triangle. A focus being fixed is equivalent to being given two tangents from that focus, namely, the circular lines through it, and hence all conics that have that point for common focus form a range. There are three such ranges, one for the foot ot each perpendicular.

THEOREM. *The foci of a conic of the net are isogonal conjugates with respect to the pedal triangle of the triangle of reference.*

For, if DEF be the pedal triangle of ABC and DT, DT′ are tangents from D to the conic, then D(TAT′C) = — 1 and so the angle between the tangents is bisected by AD and BC. Also, if D be joined to the foci S, S′, then $T\hat{D}S = T'\hat{D}S'$. Hence DS, DS′ are equally inclined to AD and so also FS, FS′ to CF and ES, ES′ to BE. If one of the foci be D, E or F, the other can be anywhere on EF, FD, or DE. Hence in the case of the three ranges of conics with a common focus, the other focus lies on the opposite side of the pedal triangle.

15. Co-axial net of conics.

If the pair of circular points is a member of the net, then the line at infinity must be a squared line and the intersections of the other two squared lines with the line at infinity must be harmonic conjugates with respect to the circular points and so the other two squared lines are lines at right angles through the common centre of all the conics and are conjugate lines with respect to all the conics. Hence all conics of this net form a system of co-axial conics. This is a special case of the general net, where the three squared lines are the two axes and the line at infinity. So the properties of the general net must necessarily hold for this net also. But since the pair of circular points is a member of the net, conics through one of them necessarily pass through the other and hence there are an ∞^1 concentric circles in this net.

As in the general net, conics through a fixed point necessarily pass through three other fixed points and form a pencil. If, by a representation similar to that of the general case, we let α be the point corresponding to the squared line at infinity, the system of concentric circles will be represented by a straight line αk through α. All double contact conics having the line at infinity for the common chord will now be represented by lines through α. Since all four-line systems of conics include point pairs on the line at infinity, the corresponding locus in the π-plane will be a circumconic and the locus of double contact conics through the point-pairs will be a tangent to this circum-conic at α. When the point pairs take the special position of circular points, the above circum-conic F represents a system of confocal conics. Since a straight line intersects the confocal locus in two points, through every point there are two confocals of the system. Also every straight line is touched by only one confocal conic. The director circle of a conic S is the F-conic of S and the circular points. So, the director circle is represented by the point of intersection of the concentric-circle locus αk with the tangent at the representative point of the conic to the locus F_L corresponding to confocal conics of S.

It is a known property that confocal conics S, S′ intersect at right angles. The orthoptic circle of S and S′ must necessarily pass through the points of intersection of S and S′ and so is a member of the pencil determined by S and S′. So the orthoptic circle of S, S′ is represented by the point of intersection of the concentric circle locus αk and the line SS′.

Pairs of points which are conjugate with respect to the circular points are degenerate rectangular hyperbolas and all rectangular hyperbolas form a pencil. Hence the rectangular hyperhola locus R_L is a straight line through α. The parabola locus P_L becomes the pair of lines $\alpha\beta$, $\alpha\gamma$. A similar conic locus has double contact with the parabola locus P_L with the rectangular hyperbola locus R_L for chord of contact. As P_L is a line pair, the similar-conic locus also breaks up into a pair of lines through α. One line corresponds to the conics having the major and minor axes in the same direction as the given conic and the other corresponding to conics having the major and minor axes at right angles to those of the given conic.

ON THE TWO SYSTEMS OF GENERATING REGIONS ON A QUADRIC IN SPACE OF EVEN ORDER.

R. VAIDYANATHASWAMY.

A quadric in space of three dimensions contains infinitely many straight lines, which fall into two distinct systems such that two lines intersect only when they belong to different systems. That an analogous property holds for the generating planes of a quadric in space of five dimensions was proved from line-geometric considerations by Cayley* in 1873, and also independently by the writer.† Using the method of stereographic projection, Veronese ‡ shewed in 1882, that the generating regions of a quadric in space of odd dimension (or even order) § also fall into two distinct systems characterised by intersection-properties. The properties of linear spaces which lie on a quadric were studied more elaborately by Segre ‖ in 1884.

The existence of two systems of generating regions was also proved independently of earlier work, by the present writer in a paper read before the Third Conference of the Indian Mathematical Society (1920), by an algebraic method depending on the properties of orthogonal matrices. As no algebraic proof appears to have been published, an improved version of the original proof is presented here in this paper.

The group of collineations preserving the two systems of generating regions, which is considered here, has also been treated in a different manner by Segre (*loc. cit.*). For the properties of orthogonal matrices used here, reference may be made to

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* *Collected Papers*, Vol. 9, p. 19.

† Quadric in Five Dimensions, *J. I. M. S.*, June 1920.

‡ Behandlung der projectivischen Verhältnisse der Räume von verschiedenen Dimensionen durch das Prinzip des Projicierens und Schneidens, *Math. Ann.* Band 19. p. 161.

§ The *order* of a linear space is the number of elements (points) which are necessary to define it; it is therefore greater than the dimensional number by *one*.

‖ Studio sulle quadriche in uno spazio lineare ad un numero qualunque di dimensioni. *Mem. Acc. Torino* (2) 36, 1884, p. 3.

(1) Cullis: *Matrices and Determinoids*, Vol. 2.

(2) *Ency. des Sciences Math.*, Bilinear Forms.

The 'generator matrices' of the paper are identical with the 'completely extravagant' matrices of Cullis

I. *The equations to a generating region.*

From the theory of quadratic forms, we know that the equation to any non-singular quadric in S_{2n}, a space of order $2n$ can be reduced to

$$x_1^2 + x_2^2 + \dots \dots x_{2n}^2 = 0.$$

By the theorem of Veronese,* the generating regions of this quadric are of order n. Obviously any prime (region of order $2n - 1$) through a generating region must touch the quadric. Hence if the equations

$$\sum_k a_{rk} x_k = 0 \qquad (r = 1, 2 \dots n) \qquad \dots \quad (1)$$

represent a generating region, we must have'

$$\sum_k \left(\sum_r \lambda_r a_{rk}\right)^2 \equiv 0$$

identically in the λ's.

This gives two sets of equations,

$$\sum_k a_{rk}^2 = 0, \quad \sum_k a_{rk} a_{sk} = 0 \quad \dots \qquad \dots \quad (2)$$

The matrix

$$A = \begin{vmatrix} a_{11} & a_{12} & \cdot & \cdot & a_{12n} \\ \cdot & & & & \\ \cdot & & & & \\ \cdot & & & & \\ a_{n1} & \cdot & \cdot & \cdot & a_{n2n} \end{vmatrix}$$

may be called the matrix of the generating region defined by equations (1). Since the equations (1) are all independent, A must be undegenerate, *i.e.*, has rank n. The equations (1) may be written in the compact matrix form

$$A . \begin{vmatrix} x_1 \\ \cdot \\ \cdot \\ \cdot \\ x_{2n} \end{vmatrix} = 0.$$

* See Whitehead: *Universal Algebra*, p. 147.

More generally, any undegenerate matrix with n rows and $2n$ columns may be called a 'generator matrix,' if its elements satisfy equations (2). Such a matrix corresponds to a definite generating region of the quadric $\Sigma\ x^2 = 0$.

II. *Properties of a generator matrix A.*

(1) A generator matrix A may be pre-multiplied by an undegenerate square matrix of order n, without affecting its essential properties. For such pre-multiplication replaces the rows of A by independent linear combinations of themselves and therefore neither affects the region represented by A, nor the validity of equations (2).

(2) If $\bar{A}$ denotes the conjugate matrix of A, then $A\bar{A} = 0$. For this is the matrix equivalent of equations (2).

The equation $A \begin{vmatrix} x_1 \\ \vdots \\ x_{2n} \end{vmatrix} = 0$, regarded as a set of linear equations in the x's has n linearly independent solutions. The fact that $A\bar{A} = 0$ shews that the rows of A themselves constitute n linearly independent solutions of

$$A \begin{vmatrix} x_1 \\ \vdots \\ x_{2n} \end{vmatrix} = 0.$$

(3) Let $p_1 p_2 \cdots p_n q_1 \cdots q_n$ be any permutation of the first $2n$ natural numbers and let $A_{p_1 p_2 \cdots p_n}$ denote the determinant formed by the p_1th p_2th ... p_nth columns of A in order. The determinants $A_{p_1 p_2 \cdots p_n}$, $A_{q_1 q_2 \cdots q_n}$ will be called *conjugate minor determinants* of A. The determinants $A_{123 \ldots n}$, $A_{n+1, n+2 \ldots, 2n}$ will be called the *leading minor determinants* of A. If the leading minor determinants are δ, δ', A may be written in the form $[\delta, \delta']$. It should be noticed that premultiplication by an undegenerate square matrix of order n does not alter the mutual ratios of the minor determinants of A.

THEOREM. *Two conjugate minor determinants of a generator matrix* A *either both vanish or both do not vanish.*

Suppose for instance that A is $[\delta, \delta']$ and that the leading minor determinant δ does not vanish. On pre-multiplying by $i\delta^{-1}$, A becomes

$$[i,\ S] = \begin{vmatrix} i & & & & S_{11} & S_{12} & \cdot & \cdot & S_{1n} \\ & i & & & \cdot & & & & \\ & & i & & \cdot & & & & \\ & & & \cdot & \cdot & & & & \\ & & & \cdot\ i & S_{n1} & \cdot & \cdot & \cdot & S_{nn} \end{vmatrix}$$

Since $A\bar{A} = 0$, we have $[i,\ S] \begin{bmatrix} i \\ \bar{S} \end{bmatrix} = 0$ or $S\bar{S} = 1$,

shewing that [S] is an orthogonal determinant. Hence [S] and therefore $[\delta']$ does not vanish. Thus the conjugate of a non-vanishing minor determinant cannot vanish.

A canonical form for A.

(4) Let ω represent any permutaion of the first $2n$ natural numbers and let ωA represent the result of performing the permutation ω upon the columns of A.

Any generator matrix A *can always be reduced to the form* $\omega\ [i, S]$ *where* $[S]$ *is orthogonal.*

For A is undegenerate and has therefore at least one non-vanishing minor determinant δ. Hence there exists a permutation ω^{-1} such that

$$\omega^{-1} A = [\delta,\ \delta'].$$

But since δ does not vanish $[\delta, \delta'] = [i,\ S]$ as in II(3), S being orthogonal. Thus

$$A = \omega\ [i,\ S]$$

The number of such canonical forms for A is clearly equal to the number of its non-vanishing minor determinants.

(5) *If* ω *is any permutation* $p_1\, p_2 \ldots p_n\, q_1 \ldots q_n$ *of the first* $2n$ *natural numbers and* $E_\omega = \pm 1$ *according as* ω *is an even or odd permutation,*

$$E_\omega A_{p_1\, p_2 \ldots p_n} / A_{q_1\, q_2 \ldots q_n}$$

is an invariant for all permutations ω.

Clearly it is sufficient to prove the theorem when A is of the form $[i,\ S]$. Also since any permutation is the product of transpositions, it is sufficient to prove the theorem for interchanges of two columns of $[i,\ S]$. Now if the interchanged columns both belong to $[i]$ or both to $[S]$ the theorem is obvious. Suppose then that the pth ($p \ngtr n$) column of $[i, S]$ is interchanged with the $(n + q)$th The ratio of the leading minor determinants becomes thereby changed from $i^n / |S|$ to $i^{n-1} S_{qp} / iS_{qp} |S|$ (since the affected minor of any element of an orthogonal matrix S is equal to the product of the element by $|S|$). Thus the ratio suffers merely an alteration of sign. Since E_ω for a transposition $= -1$, this proves the theorem for transpositions and therefore for any permutation. Since the invariant of the theorem is a ratio of minor determinants it is unaltered by pre-multiplication of A (II (3)). Thus the invariant refers not merely to the matrix A, but to the generating region represented by it.

We notice that the value of the invariant can differ, if at all, only in sign from $i^n / |S|$, *i.e.*, from i^n since $|S| = \pm 1$. *We have therefore two systems of generating regions, the positive system and the negative system, comprising respectively the generating regions whose invariants are* $+ i^n$ *and* $- i^n$.

THEOREM. *If* $\omega\,[i, S]$ *is a canonical form for* A, *the generating region* A *belongs to the positive or negative system according as* $E_\omega\,|S| = \pm 1$.

For if ω is $p_1\, p_2 \cdots p_n\, q_1\, q_2 \cdots q_n$

$$\frac{E_\omega A_{p_1\, p_2\, \dots\, p_n}}{A_{q_1\, q_2\, \dots\, q_n}} = \frac{E_\omega i^n}{|S|}$$

$$= i^n E_\omega\, |S| \text{ (since } |S| = \pm 1)$$

$$= \pm i^n \text{ according as } E_\omega\, |S| = \pm 1.$$

(6) *Any two generator matrices A, B have at least one pair of corresponding non-vanishing minor determinants.*

It is clearly sufficient to prove the theorem for ωA, ωB where ω is some permutation of columns. Now we can choose ω so that ωA is $[i, S]$. Let $\omega B = [b, c]$.

Since $[S]$ is undegenerate, it has a non-vanishing first minor, say the minor of $S_{p_1 q_1}$. Again, this first minor has itself a non-vanishing first minor, say that of $S_{p_2 q_2}$ ($p_1 \neq p_2$, $q_1 \neq q_2$); and so on. Hence if

S_r is the determinant which results on substituting respectively the p_1th, p_2th ... p_rth columns of $[i]$ for the q_1th ... q_rth columns of $[S]$, then $S_r \neq 0$. We thus have the series of non-vanishing minor determinants $S, S_1, S_2 \ldots S_n$ which are formed by the progressive substitution of columns of $[i]$ for the columns of $[S]$.

Let $c\, c_1\, c_2 \ldots c_n$ be the series of determinants formed in exactly the same way from $[b, c]$, *i.e.*, c_r is the result of substituting respectively the p_1th p_2th ... p_rth columns of $[b]$ for the q_1th ... q_rth columns of $[c]$. If the theorem were untrue, every determinant of the series $c\, c_1 \ldots c_n$ should vanish. If this could happen, then (1) because c_1 vanishes, the p_1th column of $[b]$ is a linear combination of columns of $[c]$ (2) because c_2 vanishes the p_2th column of $[b]$ is a linear combination of the p_1th column of $[b]$ and of columns of $[c]$; therefore in view of the previous statement, the p_2th column of $[b]$ is a linear combination of columns of $[c]$ (3). Proceeding in the same way we would find finally that all the columns of $[b)$ are linear combinations of columns of $[c]$ (4). Lastly, since c itself vanishes, one at least of the columns of $[c]$ is a linear combination of the rest. We would thus be led to the conclusion that the columns of $[b, c]$ are all linear combinations of at most $n-1$ of them. This would make the rank of $[b, c]$ to be $n-1$ at most thus contradicting the fact that the rank of a generator matrix is necessarily n.

Hence one at least of the series $c\, c_1 \ldots c_n$ say c_r does not vanish. This proves the theorem, for S_r and c_r are non-vanishing corresponding minor determinants of $[i, S]$ and $[b, c]$.

The following alternative form of the theorem is more instructive:

There is at least one set of n co-ordinate variables for which the equations to any two given generating regions can be simultaneously solved.

Topologically, this amounts to asserting that we can find at least one co-ordinate region of the nth order (*i.e.*, a region determined by n of the reference points) which intersects neither of two given generating regions.

Corollary:

Any two generator matrices A, B can be reduced to $\omega\,[i, S]$, $\omega\,[i, S']$, ω denoting the *same* permutation of columns in the two cases.

For there exists a pair of corresponding non-vanishing minor determinants of A, B. Hence there exists a permutation ω^{-1} such that ω^{-1} A and ω^{-1} B have both non-vanishing leading minor determinants. Thus

$$\omega^{-1} A = [i, S], \quad \omega^{-1} B = [i, S'],$$

which proves the theorem.

III. *Order of the region of intersection of two generating regions.*

Let r be the order of the region of intersection of two generating regions given by the generator matrices A, B. Obviously if N is the rank of the matrix $\begin{bmatrix} A \\ B \end{bmatrix}$, then $r = 2n - N$.

By II (6), cor., A and B can be reduced to the forms

$$\omega\,[i, S], \quad \omega\,[i, S'],$$

ω denoting the same permutation in the two cases.

$$\begin{aligned} \text{Hence } N &= \text{Rank} \begin{bmatrix} A \\ B \end{bmatrix} = \text{Rank } \omega \begin{vmatrix} i & S \\ i & S' \end{vmatrix} \\ &= \text{Rank} \begin{vmatrix} i & S \\ i & S' \end{vmatrix} = \text{Rank} \begin{vmatrix} 0 & S-S' \\ i & S' \end{vmatrix} \\ &= n + \text{Rank } | S - S' | \\ &= n + \text{Rank } | S_1 - 1 | \text{ where } S_1 = SS'^{-1} \\ &= n + n - k \end{aligned}$$

where k is the number of invariant factors of $| S_1 - \lambda |$ to the base $(\lambda - 1)$.

Thus $r = k$.

Now $S_1 = SS'^{-1}$ is an orthogonal matrix. A classical theorem states that the invariant factors with even index to the base $\lambda \pm 1$ of

$|S_1 - \lambda|$ where S_1 is orthogonal, occur in pairs of the type $(\lambda \pm 1)^{2r}$, $(\lambda \pm 1)^{2r}$. Hence, provisionally using $=$ to denote equality modulo 2, we have

k = Number of invariant factors to base $(\lambda - 1)$,

= Number of invariant factors with odd index,

= Sum of the indices of such invariant factors,

= Sum of the indices of all the invariant factors,

= Multiplicity of the characteristic root 1 of S_1.

Now the characteristic roots of an orthogonal matrix S_1 comprise $+1, -1$ and pairs of the type $t, \frac{1}{t}$; and the product of all the characteristic roots $= |S_1|$. Hence the multiplicity of the characteristic root 1 of $S_1 = n$ or $n - 1$ (mod 2) according as $|S_1| = \pm 1$.

Thus k and therefore $r = n$ or $n - 1$ (mod. 2) according as

$$|S_1| = \pm 1.$$

Now if the generating regions A, B belong to the same system

$$E_\omega |S| = E_\omega |S'| \qquad \text{by (II } (\xi));$$

so that $\qquad |S| = |S'|$ and therefore $|S_1| = 1$.

If A, B belong to different systems

$$E_\omega |S| = -E_\omega |S'|,$$

so that $\qquad |S| = -|S'|$ and therefore $|S_1| = -1$.

We therefore reach the theorem that *the order of the region of intersection of two generating regions A, B is equal to n or n — 1 (mod. 2) according as A, B belong to the same or different systems.*

IV. We now proceed to specify, in the group of automorphic collineations of the quadric $\Sigma x^2 = 0$, that precise sub-group which preserves each system of generating regions. The most general non-singular collineation, which transforms the quadric $\Sigma x^2 = 0$ into itself is an orthogonal collineation, namely, a collineation of the form

$$\begin{vmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ x_{2n} \end{vmatrix} = [S] \begin{vmatrix} x'_1 \\ \cdot \\ \cdot \\ \cdot \\ x'_{2n} \end{vmatrix}, \qquad [S] = \begin{vmatrix} S_{11} & \dots & S_{12n} \\ \cdot & & \\ \cdot & & \\ \cdot & & \\ S_{2n1} & & S_{2n2n} \end{vmatrix}$$

where $\sum_q S_{pq}^2 = 1, \quad \sum_b S_{pq} S_{p'q} = 0 \qquad (p \neq p').$

The totality of the matrices S forms a continuous group, *the orthogonal group*. Among the orthogonal matrices S, we can distinguish two types, the positive and the negative orthogonal matrices, namely, those the determinants of which are equal to $+1$ and -1 respectively. The positive orthogonal matrices form a group by themselves, *the positive orthogonal group*, which is clearly a self-conjugate sub-group of the orthogonal group. We shall shew now that *the collineations corresponding to the positive orthogonal group preserve the two systems of generating regions of the quadric* $\Sigma x^2 = 0$.

To prove this it will be sufficient to shew that each of a set of independent infinitesimal operations which generate the positive orthogonal group transforms any generating region into a neighbouring generating region *of the same system*.

Now an infinitesimal positive orthogonal matrix is of the form $\lambda_{pq} |$ where $\lambda_{pp} = 1$ and $\lambda_{pq} = -\lambda_{qp} =$ a small quantity. Hence a set of independent generating operations is the set Δ_{pq} where Δ_{pq} is a matrix of the type $| \lambda_{pq} |$, but with all non-diagonal elements vanishing with the exception of λ_{pq} and λ_{qp}.

The effect of operating on a generator matrix $A = | a_{pq} |$, by one of the operations Δ, say Δ_{12} (in which $\lambda_{12} = -\lambda_{21} = \delta$) is the generator matrix.

$$A \Delta_{12} = A' = \begin{vmatrix} a_{11} - \delta a_{12} & a_{12} + \delta a_{11} & a_{13} & a_{14} & \dots \\ \cdot & \cdot & & & \\ \cdot & \cdot & & & \\ \cdot & \cdot & & & \\ a_{r1} - \delta a_{r2} & a_{r2} + \delta a_{r1} & a_{r3} & a_{r4} & \dots \\ \cdot & \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \cdot & \end{vmatrix}$$

To shew that A, A′ belong to the same system, we have to shew that they have equal invariants, *i e.*, if $p_1 p_2 \dots p_n\, q_1 q_2 \dots q_n$ is any permutation of the first $2n$ natural numbers, we have to shew

$$A'_{p_1 p_2 \dots p_n} / A'_{q_1 q_2 \dots q_n} = A_{p_1 p_2 \dots p_n} / A_{q_1 q_2 \dots q_n} \quad \dots \quad (1)$$

To prove this, suppose first that both the numbers 1, 2 occur in one of the groups $(p_1 \dots p_n)$, $(q_1 \dots q_n)$, say, the first. Then

$$A'_{q_1 \dots q_n} = A_{q_1 \dots q_n}$$

and obviously $A'_{p_1 \dots p_n}$ differs from $A_{p_1 \dots p_n}$ by quantities of the second order. Thus the above equality is true to the first order of small quantities. Suppose next that one of the numbers 1, 2 occurs in each of the groups $(p_1 \dots p_n)$, $(q_1 \dots q_n)$, so that we may put $p_1 = 1$, $q_1 = 2$. Then

$$A'_{1 p_2 \dots p_n} / A'_{2 q_2 \dots q_n} = \frac{A_{1 p_2 \dots p_n} - \delta A_{2 p_2 \dots p_n}}{A_{2 q_2 \dots q_n} + \delta A_{1 q_2 \dots q_n}} \quad \dots \quad (2)$$

Now, the permutation $2 p_2 \dots p_n \mid q_1 \dots q_n$ is obtained by performing a single transposition on the permutation $1 p_2 \dots p_n\, 2 q_1 \dots q_n$. Hence

$$A_{1 p_2 \dots p_n} A_{1 q_2 \dots q_n} + A_{2 p_2 \dots p_n} A_{2 q_2 \dots q_n} = 0.$$

But this is the condition that the quantity on the right of (2) may differ from its finite part by quantities of the second and higher orders only. Hence in all cases equation (1) is true to the first order. Therefore, reasoning backwards, the two systems of generating regions are preserved by the infinitesimal generating operations and therefore by any operation of the positive orthogonal group.

If we adjoin to the positive group a single negative orthogonal matrix, we generate the whole orthogonal group; in other words there is no sub-group of the orthogonal group which contains the positive group. Hence *all negative orthogonal matrices must interchange the two systems of generating regions of the quadric.*

In a space S_{2n+1} of odd order, the distinction between positive and negative orthogonal matrices ceases to be significant. Thus if S be a positive orthogonal matrix of order $2n+1$, and $-S$ the matrix obtained by changing the signs of all the elements of S, then $-S$ is a negative matrix and yet represents exactly the same collineation as S. This may be taken to indicate the non-existence of two systems of generating regions on a quadric in S_{2n+1}.

V. The equation to a non-singular quadric in S_{2n} can also be reduced to the form

$$X_1Y_1 + X_2Y_2 + \dots \quad + X_nY_n = 0.$$

The equations

$$X_1 = X_2 = \dots = X_n = 0$$

represent a generating region A. The equations

$$Y_1 = X_2 = \dots = X_n = 0 \text{ and } Y_1 = Y_2 = X_3 = \dots = X_n = 0$$

represent two other generating regions B, C of which C belongs to the same system as A, while B belongs to the other system. More generally if in the equations to A, r X's are replaced by Y's with the same index, the resulting generating region belongs to the same system as A if r is even and to the other system if r is odd.

The equations

$$\begin{vmatrix} X_1 \\ \cdot \\ \cdot \\ \cdot \\ X_n \end{vmatrix} = M \begin{vmatrix} Y_1 \\ Y_2 \\ \cdot \\ \cdot \\ Y_n \end{vmatrix}$$

where M is a square matrix of order n represent a generating region only if M is skew-symmetric. In this case the generating region belongs to the same system as A; for on putting $X_1 = X_2 \dots = X_n = 0$, we have the equations

$$M \begin{vmatrix} Y_1 \\ Y_2 \\ \cdot \\ Y_n \end{vmatrix} = 0,$$

which, since the rank of M, a skew-symmetric matrix, is an even number $2r$, represent a region of order $n - 2r$.

The reader may easily prove the following theorem :—

The equations to a generating region of the quadric $\Sigma X_r Y_r = 0$ *can always be solved for at least one set of* n *co-ordinate variables with different suffixes ; and the generating region belongs to one or the other system according as the number of Y's occurring in this set is even or odd.*

It is not possible to widen the statement in the first part of the theorem, for there are always generating regions the equations to which cannot be solved for more than one set of co-ordinate variables with different suffixes. The region $X_1 = X_2 = \ldots = X_n = 0$ is an obvious instance.

THE THEORY OF ENVELOPES OF PLANE CURVES

BY C. N. SRINIVASIENGAR.

§ 1. The envelope of a family of plane curves $f(x, y, c) = 0$ is defined in many text-books as the locus of the ultimate intersections of $f(x, y, c) = 0$ and $f(x, y, c + \delta c) = 0$. This definition has one serious disadvantage; for, it is easily proved that if the family possess any double points, these will appear in the c-discriminant.* According to this

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* Since there are some important points to be noted in connection with this theorem, let us examine the proof at length. Let (x, y) be any point of intersection of the two curves $f_c = 0$ and $f_{c+\delta c} = 0$. Then we have $\frac{\partial f(x, y, c)}{\partial c} = 0$. Let now (x_1, y_1) be a double point on the curve $f_c = 0$, so that $f(x_1, y_1, c) = 0$; $\frac{\partial f}{\partial x_1} = 0$; $\frac{\partial f}{\partial y_1} = 0$. Let us suppose that it is possible to find two infinitesimals δx_1 and δy_1 such that $(x_1 + \delta x_1, y_1 + \delta y_1)$ is a point on the curve $f(x, y, c + \delta c) = 0$. Therefore,

$$f(x_1, y_1, c) + \delta x_1 . \frac{\partial f(x_1, y_1, c)}{\partial x_1} + \delta y_1 . \frac{\partial f(x_1, y_1, c)}{\partial y_1} + \delta c . \frac{\partial f(x_1, y_1, c)}{\partial c} = 0 \qquad \ldots \quad (a)$$

Therefore $\frac{\partial f(x_1, y_1, c)}{\partial c} = 0$, showing that (x_1, y_1) is a point common to the two consecutive curves. *Conversely* if (x_1, y_1) is a double point on $f_c = 0$ and is common to both the curves $f_c = 0$, and $f_{c+\delta c} = 0$, we conclude from equation (a) that $(x_1 + \delta x_1, y_1 + \delta y_1)$ is a point on $f_{c+\delta c} = 0$ *for all infinitesimal values of* $\delta x_1, \delta y_1$. In other words, either (x_1, y_1) itself (which is only a trivial case), or $(x_1 + h, y_1 + k)$ where h and k are two definitely known infinitesimals is a double point on $f_{c+\delta c} = 0$. In other words, *it is necessary to assume that the family of plane curves* $f(x, y, c) = 0$ *possesses a locus of double points, the locus being in the form of a continuous curve*. Now in general $f(x, y, c) = 0$ possesses double points only for a finite number of values of c, given by the equation $\phi(c) = 0$, which is the result of eliminating x and y between the three equations $f(x, y, c) = 0$; $\frac{\partial f}{\partial x} = 0$; $\frac{\partial f}{\partial y} = 0$. And these isolated double points are not necessarily common to the curves $f_c = 0$ and $f_{c+\delta c} = 0$.

definition therefore the locus of double points forms part of the envelope It is usual, however, to exclude this locus from the envelope for two reasons: first, the locus of the double points does not appeal to us as a curve 'touching' every member of the system. (A remark on the definition of tangent will be made presently). The second, and more important reason is, that the locus of the double points does not usually satisfy the differential equation of the system of plane-curves.

To avoid this difficulty, the following definition of envelope has been given: "A non-singular point on the curve $f(x, y, c) = 0$ whose distance from the consecutive curve $f(x, y, c + \delta c) = 0$ is of the *second order of smallness at least*, is called a characteristic point of the curve 'c.' Such a point is an ordinary point at which (it can be proved) $\frac{\partial f}{\partial c} = 0$, and is an *isolated characteristic point*. The locus of the isolated characteristic points is called the envelope."*

This definition is very artificially constructed so as to exclude the locus of the singular points of the system. A more natural definition will be obtained by referring to the etymology of the word 'envelope,' which signifies tangency. Now, in the theory of plane-curves, a straight line is said to be a tangent to a curve at a point, if the line meets the curve there in $k + 1$ points, the point being supposed to be a multiple point of order k. The values $k = 1$, and $k = 2$, give the familiar cases; and two curves are said to touch each other at a point if a line through the point touches them both.

§ **2.** *Let us then define the envelope of a system of plane-curves as a curve which touches every member of the system at some point.*

Goursat, in his *Mathematical Analysis*, Vol. I (English translation by Hedrick), starts with this definition, but his method of obtaining the equation of the envelope from this definition, is defective. He assumes that the co-ordinates of the point of contact of the envelope with the curve $f(x, y, c) = 0$ can be expressed in the form

$$x = \phi_1(c); \quad y = \phi_2(c),$$

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* De la Vallée Poussin: ***Cours d'Analyse Infinitesimale***. Tome II, p. 384 (1922 edition).

R. H. Fowler: ***Elementary Differential Geometry of Plane Curves***. (Cambridge Tracts, No 20). Chapter V.

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NOTES AND QUESTIONS.

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Notes and Questions.

On Conics through four given points.

1. The totality of conics through four given points A, B, C, D constitute an one-parameter family whose members can be placed in $1-1$ correspondence with the points on a line (or a rational curve) or dually with lines through a point. There are three projectively specialised members, *viz.*, the line pairs. In relation to the circular points at infinity I, J, we have two *parabolas* which touch the line IJ and a unique *rectangular hyperbola* which has I and J for conjugate points.

By isogonal transformation with respect to the triangle formed by three of the points say ABC, the family reduces to the pencil of lines through D' the isogonal conjugate of D. The three degenerate conics correspond to the three lines through the vertices. Now the points at infinity on a conic correspond to the intersections T_1 T_2 of its isogonal transform with the circum-circle. It follows from this that the asymptotic angle of a conic is the angle of intersection of its transform with the circumcircle. As particular cases we note that the two parabolas correspond to the tangents from D' to the circle ABC and the rectangular hyperbola to the diameter through D'.* We shall denote the rectangular hyperbola by R and the conic whose isogonal transform is the line through D' perpendicular to the diameter, by M. It will be shown later that M is the conic of minimum eccentricity through the four given points.

2. If p be the length of the perpendicular from the circum-centre O on T_1T_2 the asymptotic angle of the corresponding conic is, by the preceeding paragraph, given by

$$\cos\theta = \frac{p}{R} \quad \dots \quad \dots \quad \dots \quad (1)$$

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* This is obvious otherwise since, when the conic is a rectangular hyperbola, it passes through the orthocentre and hence its isogonal conjugate should pass through the circum-centre of ABC.

R being the circum-radius. Since we can draw two lines through D′ at a given distance p from O, these being equally inclined to D′O, we deduce from the correspondence between the pencil of lines and the conics of the system that:

There are two conics of a 4-point system which have the same asymptotic angle; the parameters of these two form a dyad apolar to (separating harmonically) those of R and M.*

These two conics will coincide in the case of R or M. In the case of M the asymptotic angle passes through a minimum value. This may also be deduced from (1) since p is a maximum when the line through D′ is perpendicular to the diameter through it.

Again, the eccentricity e is connected with the angle θ by the relation

$$\frac{p}{R} = \cos\theta = \frac{2}{e^2} - 1 \qquad \ldots \qquad \ldots \qquad (2)$$

so that the minimum value of e corresponds to the maximum value of p. Thus the conic of minimum eccentricity is M.

When the four points A, B, C, D are such that each is outside the triangle formed by the other three, the system of conics contains both ellipses and hyperbolas. For the former, only one of the eccentricities, namely, that associated with the real foci is real; for the latter, both the eccentricities are real, one being less than $\sqrt{2}$ and the other greater, and the two are equal when the hyperbola is rectangular. There is therefore properly speaking no *conic of maximum eccentricity*. The rectangular hyperbola corresponds to a double point and not a true maximum for the eccentricity locus.

Since the two tangents from D′ to the circum-circle of ABC are equally inclined to D′O, we deduce that:

* The asymptotes form four angles which are equal in pairs. If for two conics these four angles be the same, they are said to have the same asymptotic angle, independently of whichever quadrant the curve—supposed real—may happen to be occupying. Two such conics are ***similar*** and have a real or an imaginary ratio of similitude. Thus two conjugate hyperbolas are similar, their ratio of similitude being $\sqrt{-1}$.

The parameters of the rectangular hyperbola R and the conic of minimum eccentricity M separate harmonically those of the two parabolas of the system.

3. If with each conic of the system we associate its centre, the conics are placed in correspondence with the points on the centre locus which is a conic. The two points at infinity on this conic correspond to the two parabolas (centre at infinity). The points corresponding to (*i.e.*, the centres of) R and M separate harmonically the two points at infinity and are therefore the extremities of a diameter. They are therefore equidistant from the centre of the centre locus which is the centroid of the points ABCD. (*Vide* Q. 1369, first part). Conics with the same asymptotic angle correspond to pairs of points conjugate to R and M; hence they are extremities of the series of parallel chords* bisected by RM.

Cartesian Investigation.

4. Take the co-ordinate axes, parallel to the axes of the two parabolas through the 4 points and the origin at the centroid of the points. The equations of the two parabolas are

$$P \equiv x^2 + 2qy + m = 0$$

and

$$Q \equiv y^2 + 2px + l = 0$$

and any conic of the system is given by

$$P - \lambda Q = 0 \qquad \ldots \qquad \ldots \quad (3)$$

The centre of this conic is the point $\left(p\lambda, \frac{q}{\lambda}\right)$ and the centre locus is thus the conic $xy = pq$. The conic (3) is an ellipse, parabola or hyperbola according as λ is negative zero or positive. Hence if the centre locus be a hyperbola, points on one branch correspond to hyperbolas of the system, and points on the other branch to ellipses.

If θ be the angle between the asymptotes of the conic (3), then

$$\tan\theta = \frac{2\sqrt{\lambda}\sin\omega}{1-\lambda} \qquad \ldots \qquad \ldots \quad (4)$$

† Vide *Encyclopädie der Mathematischen Wissenschaften*, III 2, p. 99; Dingeldey : Kegelschnitte und Kegelschnitt System.

ω being the angle between the axes.

$$\therefore \qquad \lambda + \frac{1}{\lambda} = 4 \sin^2 \omega \cot^2 \theta + 2.$$

Hence for a given value of θ there are two values of λ $\left(\lambda \text{ and } \frac{1}{\lambda}\right)$ and therefore there are two conics of the system with a given asymptotic angle. These two coincide when $\lambda = \pm 1$ which gives the minima and maxima values of $\cot^2\theta$.

The asymptotic angles corresponding to the conics $\lambda = \pm 1$, are $\theta = 90^\circ$ and $\theta = \tan^{-1}(i \sin \omega)$, the first corresponding to R, and the second to M. The eccentricity e may be calculated from $e = \sec \theta/2$ and is therefore given by

$$\frac{(2 - e^2)^2}{e^2 - 1} = \frac{(1 - \lambda)^2}{\lambda \sin^2 \omega}$$

the extreme values of e corresponding to $\lambda = \pm 1$ are found by substitution to be $e = \sqrt{2}$ and

$$e = \sqrt{2/(1 + \sec \omega)}\ *$$

5. When ABCD are the feet of the normals from a point to a conic, this conic is the one of minimum eccentricity M through the four points, and the Appolonius hyperbola is the R of the system. It is not however true as is stated in my note on the "Four Point Conic of Minimum Eccentricity," *J.I.M.S.* Vol. VII, page 135, that the conic of minimum eccentricity is such that the normals at the four poinls are concurrent. In fact any four points arbitrarily chosen in a plane will not be the feet of normals from a point to a conic, the condition for this being that the conic M of the four point system should be in-polar (*i.e.* should admit of triangles self-polar to itself and inscribed in) R. If these points fulfil this condition, the conic through them for which the normals at the points are concurrent is the conic M.

N. DURAIRAJAN.

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* *Radford's Problems*, page 101, Ex. 9.

Wolstenholme's Problems, 2nd Edition, Ex. 1210.

An Elementary Treatment of the Modular Equation of the Third Order.

1. If

$$K = \int_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{1 - k^2 \sin^2 \phi}}, \quad L = \int_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{1 - l^2 \sin^2 \phi}} \quad (k, l < 1)$$

and $\dfrac{L'}{L} = 3\,\dfrac{K'}{K}$, then it is known that $\sqrt{kl} + \sqrt{k'l'} = 1$.

As far as I am aware, all the known proofs of this result, make use of the general theory of Elliptic Functions. Mr. Venkatarama Aiyar in his paper on "Some Applications of Modular Equations"* has indicated an extremely simple proof, but the latter rests on the assumption that

$$\sqrt{\frac{2K}{\pi}} = 1 + 2q + 2q^4 + 2q^9 + \dots \dots$$

where $q = e^{-\pi \frac{K'}{K}}$, a result weich belongs to the general theory. The outline of an elementary proof is given below.

2. It is easily verified (*vide* Cayley: *Elliptic Functions*) that the transformation

$$\text{(i)} \qquad y = \frac{x\{1 + 2\alpha + \alpha^2 x^2\}}{1 + \alpha(\alpha + 2)x^2} \quad \dots \quad \dots \quad \text{(i)}$$

makes

$$\frac{dy}{\sqrt{(1 - y^2)(1 - l^2 y^2)}} = \pm \frac{(2\alpha + 1)\,dx}{\sqrt{(1 - x^2)(1 - k^2 x^2)}}$$

where

$$l^2 = \alpha\left(\frac{2 + \alpha}{2\alpha + 1}\right)^3, \quad k^2 = \frac{\alpha^3(2 + \alpha)}{2\alpha + 1}. \quad \dots \quad \text{(ii)}$$

so that

$$\sqrt{kl} + \sqrt{k'l'} = 1.$$

3. If $0 < \alpha < 1$, the graph of (i) is a curve starting from the origin, with its ordinates steadily increasing from $x = 0$ to $x = 1$.

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* *J. I. M. S.*, Vol. XV, p. 256.

Also when $x = 1$, $y = 1$ (there is a maximum at $x = 1$). It follows that

$$\int_0^1 \frac{dy}{\sqrt{(1-y^2)(1-l^2 y^2)}} = (2\alpha + 1) \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-k^2 x^2)}}$$

i.e., $$L = (2\alpha + 1)\, K \text{ when } 0 < \alpha < 1.$$

4. Next, consider the graph of

$$\text{A}: \qquad y = \frac{x(1 + 2\beta + \beta^2 x^2)}{1 + \beta(\beta + 2)x^2},$$

where $\beta = -\alpha - 1$ and $0 < \alpha < 1$.

This transformation makes

$$\frac{dy}{\sqrt{(1-y^2)(1-l_1^2 y^2)}} = \pm \frac{(2\beta + 1)\, dx}{\sqrt{(1-x^2)(1-k_1^2 x^2)}},$$

$$= \mp \frac{(2\alpha + 1)\, dx}{\sqrt{(1-x^2)(1-k_1^2 x^2)}}$$

where $l_1^2 = \beta\left(\frac{2+\beta}{2\beta+1}\right)^3$, $k_1^2 = \frac{\beta^3(2+\beta)}{2\beta+1}$

Since $\beta = -\alpha - 1$, we get

$$\left.\begin{aligned} l_1^2 &= -(\alpha+1)\left(\frac{1-\alpha}{-2\alpha-1}\right)^3 = (\alpha+1)\left(\frac{1-\alpha}{2\alpha+1}\right)^3 \\ \text{and} \qquad k_1^2 &= \frac{-(\alpha+1)^3(1-\alpha)}{-2\alpha-1} = \frac{(1-\alpha)}{2\alpha+1}\cdot(\alpha+1)^3 \end{aligned}\right\} \quad \ldots \text{(iii)}$$

From (ii) and (iii), $k^2 + k_1^2 = 1$, $l^2 + l_1^2 = 1$

$$\therefore \qquad k_1^2 = k'^2, \text{ and } l_1^2 = l'^2.$$

5. *Graph of A.*

When $y = 0$, $x = 0$; from $x = 0$ to $x = \frac{1}{\alpha+1}$, y steadily decreases, and y reaches a minimum value -1 at $x = \frac{1}{\alpha+1}$. As x

increases from $x = \frac{1}{\alpha + 1}$ to $x = \frac{1 + 2\alpha}{(\alpha + 1)^2}$, y steadily increases from the value -1 to 0. As x increases from the value $\frac{1 + 2\alpha}{(\alpha + 1)^2}$ to 1, y increases steadily from 0 to 1.

Hence, from § 4,

$$\int_0^1 \frac{dy}{\sqrt{(1 - y^2)(1 - l_1^2 y^2)}} = \int_0^1 \frac{dy}{\sqrt{(1 - y^2)(1 - l'^2 y^2)}}$$

$$= (2\alpha + 1) \int_0^{\frac{1}{1+\alpha}} \frac{dx}{\sqrt{(1 - x^2)(1 - k'^2 x^2)}}$$

$$= (2\alpha + 1) \int_{\frac{1}{1+\alpha}}^{\frac{1+2\alpha}{(1+\alpha)^2}} \frac{dx}{\sqrt{(1 - x^2)(1 - k'^2 x^2)}}$$

$$= (2\alpha + 1) \int_{\frac{1+2\alpha}{(1+\alpha)^2}}^{1} \frac{dx}{\sqrt{(1 - x^2)(1 - k'^2 x^2)}}$$

Again, since $0 < \alpha < 1$.

$$0 < \frac{1}{1 + \alpha} < \frac{1 + 2\alpha}{(1 + \alpha)^2} < 1.$$

Hence $$\int_0^1 \frac{dy}{\sqrt{(1 - y^2)(1 - l'^2 y^2)}} = \frac{2\alpha + 1}{3} \times$$

$$\left[\int_0^{\frac{1}{1+\alpha}} + \int_{\frac{1}{1+\alpha}}^{\frac{2\alpha+1}{(\alpha+1)^2}} + \int_{\frac{2\alpha+1}{(1+\alpha)^2}}^{1} \left\{ \frac{dx}{\sqrt{(1 - x^2)(1 - k'^2 x^2)}} \right\} \right]$$

$$= \frac{2\alpha + 1}{3} \int_0^1 \frac{dx}{\sqrt{(1 - x^2)(1 - k'^2 x^2)}}$$

i.e., $$L' = \frac{2\alpha + 1}{3} . K'.$$

From this and the final result of § 3, we get by division

$$\frac{3L'}{L} = \frac{K'}{K} \text{ when } \sqrt{kl} + \sqrt{k'l'} = 1.$$

6. Since $k < 1$, and $k^2 = \dfrac{\alpha^3(2+\alpha)}{2\alpha+1}$

we must have $$0 < \alpha < 1$$

or $$-2 < \alpha < -1$$

for real values of k: also from (ii) and (iii), when

$$0 < \alpha < 1, < k' \geqslant k.$$

Hence, it will be easily seen that

$$\frac{L'}{L} = 3\frac{K'}{K}\,{}^{*}\ (l' > l)$$

and $$\sqrt{kl} + \sqrt{k'l'} = 1.$$

It is obvious also that the proof given is quite general, since a value of α between 0 and 1 can always be found when k or l is given.

7. The modular equations of orders higher than the third can similarly be obtained without the use of Elliptic Functions, but the treatment becomes more complicated.

S. D. Chowla.

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* As examples, take $\alpha = -\frac{3}{2}$, then $k^2 = -\dfrac{\frac{27}{8}\cdot\frac{1}{2}}{-2} = \dfrac{27}{32}$; $k'^2 = \dfrac{5}{32}$, $l^2 = -\dfrac{\frac{3}{2}\cdot\frac{1}{8}}{-8} = \dfrac{3}{128}$, $l'^2 = \dfrac{125}{128}$ and $\sqrt{kl} + \sqrt{k'l'} = 1$. Also $\dfrac{L'}{L} = 3\dfrac{K'}{K}$.

Similarly taking $\alpha = \dfrac{\sqrt{3}-1}{2}$ we get Legendre's famous result: $\dfrac{K'}{K} = \sqrt{3}$ then $k = \sin 15^\circ$.

Solutions.

Question 1299.

(T. TOTADRI IYENGAR) :—Let the number of combinations of n things r at a time be symbolised by $\binom{n}{r}$. Prove that

$$\binom{n}{1} + 5\binom{n}{3} + 5^2\binom{n}{5} + 5^3\binom{n}{7} + \ldots$$

$$= 2^{n-1} + 2^2\Bigg[\left\{\binom{n-2}{1} + \binom{n-3}{1} + \binom{n-4}{1} + \ldots\right\}$$

$$+ 5\left\{\binom{n-2}{3} + \binom{n-3}{3} + \binom{n-4}{3} + \ldots\right\}$$

$$+ 5^2\left\{\binom{n-2}{5} + \binom{n-3}{5} + \binom{n-4}{5} + \ldots\right\}$$

$$+ \ldots \quad \ldots \quad + \quad \ldots \quad \ldots\Bigg]$$

Remarks by A A. Krishnaswamy Iyengar and A. Mahadevan.

This question is obviously wrong, as the substitution of any numerical value (say 5) in both sides will show.

The correct identity is given below.

Put $$f(n) = \binom{n}{1} + 5\binom{n}{3} + 5^2\binom{n}{5} + \ldots$$

and $$\psi(n) = \binom{n}{2} + 5\binom{n}{4} + 5^2\binom{n}{6} + \ldots$$

Then $$1 + 5\,\psi(n) + \sqrt{5}\,f(n) = (1+\sqrt{5})^n$$

$$= (1+\sqrt{5})\,\{1 + 5\,\psi(n-1) + \sqrt{5}\,f(n-1)\}$$

Equating the rational and the irrational parts on both sides, we get, on simplification

$$\psi(n) = \psi(n-1) + f(n-1)$$

$$f(n) = 1 + f(n-1) + 5\,\psi(n-1)$$

$$\therefore \quad f(n) - \psi(n) = 1) + 4\,\psi(n-1)$$

or $\quad f(n) = \psi(n) + 1 + 4\,\psi(n-1) \qquad \ldots \qquad \ldots \qquad (1)$

$$= \psi(n) + 1 + 2^2 \Big[\Big\{ \binom{n-2}{1} + \binom{n-3}{1} + \binom{n-4}{1} + \ldots \Big\}$$

$$+ 5 \Big\{ \binom{n-2}{3} + \binom{n-3}{3} + \binom{n-4}{3} + \ldots \Big\}$$

$$+ 5^2 \Big\{ \binom{n-2}{5} + \binom{n-3}{5} + \binom{n-4}{5} + \ldots \Big\}$$

$$+ \quad \ldots \quad + \quad \ldots \quad \Big]$$

since $\binom{n+1}{p+1} = \binom{n}{p} + \binom{n-1}{p} + \binom{n-2}{p} + \quad \ldots \quad + 1$

Now $\quad \psi(n) + 1 = 1 + \binom{n}{2} + 5\binom{n}{4} + 5^2\binom{n}{6} + \ldots$

$$= 2^{n-1} + 4\binom{n}{4} + (5^2 - 1)\binom{n}{6} + \ldots$$

Thus the assumption $\psi(n) + 1 = 2^{n-1}$ which is involved in the question is wrong.

It is interesting to express $\psi(n)$ as a series involving the f-functions.

Assuming for the sake of uniformity

$$\psi(0) = 0 = \psi(1),$$

we may write down changing n to $n-1$, $n-2$, ... in succession, in (1)

$$f(n) = 1 + \psi(n) + 4\,\psi(n-1)$$

$$f(n-1) = 1 + \psi(n-1) + 4\,\psi(n-2)$$

$$\ldots \qquad \ldots \qquad \ldots \qquad \ldots$$

$$f(1) = 1 + \psi(1) + 4\,\psi(0).$$

Multiplying these equations in order by $1, -4, +4^2, \ldots (-4)^{n-1}$ and adding, we get

$$f(n) - 4\,f(n-1) + 4^2\,f(n-2) - + \ldots + (-4)^{n-1}\,f(1)$$
$$= \{1 - 4 + 4^2 - \ldots + (-4)^{n-1}\} + \psi(n)$$
$$= \frac{1-(-1)^n\,4^n}{5} + \psi(n).$$

Remarks by the proposer.

I regret that the question as given by me was incorrect. Mr. A. A. Krishnaswamy Ayyangar has given the question an orientation, which is extremely interesting, though different from what I had intended the question to take I proceed to give the solution which I had in mind.

Sylvester (*Mess. of Mathematics*, Vol. VIII, pp. 187—9) gives the equality

$$1 + \binom{n-1}{1} + \binom{n-2}{2} + \binom{n-3}{3} + \ldots$$
$$= \frac{1}{\sqrt5}\{(\tfrac12 + \tfrac12\sqrt5)^{n+1} - (\tfrac12 - \tfrac12\sqrt5)^{n+1}\} \quad \ldots \quad (A)$$

i.e., $$1 + \binom{n-1}{1} + \binom{n-2}{2} + \binom{n-3}{3} + \ldots$$
$$= \frac{1}{2^n}\left\{\binom{n+1}{1} + 5\,.\binom{n+1}{3} + 5^2\binom{n+1}{5} + \ldots\right\}$$
$$= f(n+1)\ \text{(say)} \quad \ldots \quad \ldots \quad \ldots \quad \ldots (B)$$

Then
$$f(n+1) + f(n) = 1 + \binom{n}{1} + \binom{n-1}{2} + \binom{n-2}{3} + \ldots$$

by adding and making use of the relation $\binom{n}{r} + \binom{n}{r-1} = \binom{n+1}{r}$.

$$\therefore \quad f(n+1) + f(n) = f(n+2) \qquad \text{from (A).}$$

Hence we get the set of equalities

$$f(n) = f(n-1) + f(n-2)$$
$$f(n-1) = f(n-2) + (f(n-3)$$
$$\ldots \quad \ldots \quad \ldots$$
$$f(3) = f(2) + f(1)$$
$$f(2) = f(1) \text{ by definition.}$$

$$\therefore \quad f(n) = f(1) + [f(1) + f(2) + \ldots + f(n-2)]$$

On replacing the values f from (B) and multiplying by 2^{n-1}, we obtain

$$\binom{n}{1} + 5\binom{n}{3} + 5^2\binom{n}{5} + 5^3\binom{n}{7} + \ldots$$

$$= 2^{n-1} + \left\{2^{n-1}\binom{1}{1}\right\} + \left\{2^{n-2}\binom{2}{1}\right\} + 2^{n-3}\left\{\binom{3}{1} + 5\,.\binom{3}{3}\right\}$$

$$+ \ldots + 2^2\left\{\binom{n-2}{1} + 5\binom{n-2}{3} + 5^2\binom{n-2}{5} + \ldots\right\}$$

Re-arranging the left-hand side, the result intended was

$$\binom{n}{1} + 5\binom{n}{3} + 5^2\binom{n}{5} + 5^3\binom{n}{7} + \ldots$$

$$= 2^{n-1} + \left\{2^2\binom{n-2}{1} + 2^3\binom{n-3}{1} + 2^4\binom{n-4}{1} + \ldots\right\}$$

$$+ 5\left\{2^2\binom{n-2}{3} + 2^3\binom{n-3}{3} + 2^4\binom{n-4}{3} + \ldots\right\}$$

$$+ 5^2\left\{2^2\binom{n-2}{5} + 2^3\binom{n-3}{5} + 2^4\binom{n-4}{5} + \ldots\right\}$$

$$+ 5^3\left\{2^2\binom{n-2}{7} + 2^3\binom{n-3}{7} + 2^4\binom{n-4}{7} + \ldots\right\}$$

$$+ \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

Question 1315.

(K. J. SANJANA) :—The trilinear normal co-ordinates of a point P with regard to the triangle of reference ABC being proportional to

$$\sec (B - C), \ \sec (C - A), \ \sec (A - B),$$

prove that

$$\cot PBC \cot PCA \cot PAB + \cot PCB \cot PBA \cot PAC$$

$$= \frac{12 \cot \omega + 20 \cot A \cot B \cot C}{2 + \cot A \cot B \cot C \cot \omega + \cot^2 A \cot^2 B \cot^2 C}.$$

Solutions by M. V. Ramakrishnan, V. A. Mahalingam and S. N. Kumaraswamy.

P being a point in the plane of a triangle ABC, it is easy to show that

$$\cot P\hat{B}C \cot P\hat{C}A \cot P\hat{A}B + \cot P\hat{C}B \cot P\hat{B}A \cot P\hat{A}C$$

$$= \Sigma \cot P\hat{B}C + \Sigma \cot P\hat{C}B$$

since $\quad P\hat{B}C + P\hat{C}A + P\hat{A}B = \pi - P\hat{C}B - P\hat{A}C - P\hat{B}A$

and $\quad \sin P\hat{A}B \sin P\hat{B}C \sin P\hat{C}A = \sin P\hat{A}C \sin P\hat{C}B \sin P\hat{B}A.$

Hence $\cot P\hat{B}C \cot P\hat{C}A \cot P\hat{A}B + \cot P\hat{C}B \cot P\hat{B}A \cot P\hat{A}C$

$$= \frac{\Sigma\, a \cos(B-C)}{2\,\Delta} . \Sigma\, a \sec(B-C)$$

$$= \frac{4\,R \sin A \sin B \sin C}{2\,\Delta} . \frac{abc}{4\,R^2} . \frac{3 + 8 \cos A \cos B \cos C}{\Pi \cos(B-C)}$$

(Hobson's *Trigonometry*, p. 59, Ex. 27)

$$= \frac{4\,(3 + 8 \cos A \cos B \cos C) \sin A \sin B \sin C}{\Pi \cos(B-C)}$$

$$= \frac{4\,(3 + 8 \cos A \cos B \cos C) \sin A \sin B \sin C}{\sin^2 A \sin^2 B \sin^2 C\, \Pi\, (\cot B \cot C + 1)}$$

$$= \frac{4\,(3 \operatorname{cosec} A \operatorname{cosec} B \operatorname{cosec} C + 8 \cot A \cot B \cot C)}{\Pi\, (\cot B \cot C + 1)}$$

$$= \frac{4\,(3 \cot \omega - 3 \cot A \cot B \cot C + 8 \cot A \cot B \cot C)}{\cot^2 A \cot^2 B \cot^2 C + \cot A \cot B \cot C \cot \omega + 2}$$

since $\quad \Sigma \cot A = \cot A \cot B \cot C + \operatorname{cosec} A \operatorname{cosec} B \operatorname{cosec} C$

$$\Sigma \cot A = \cot \omega$$

and $\quad \Sigma \cot B \cot C = 1)$

$\therefore$ the given expression

$$= \frac{12 \cot \omega + 20 \cot A \cot B \cot C}{2 + \cot A \cot B \cot C \cot \omega + \cot^2 A \cot^2 B \cot^2 C}.$$

Question 1332.

(K. SATYANARAYANA) :—Prove that a triangle whose sides are parallel to $y = m_1 x$, $y = m_2 x$, $y = m_3 x$, will be acute-angled, right-angled or obtuse-angled according as

$$(1 + m_1 m_2)(1 + m_2 m_3)(1 + m_3 m_1) < = \text{ or } > 0.$$

Solution (1) *by S. D. Chowla, P. Kameswara Rao and V. A. Mahalingam;*
(2) *by G. V. Krishnaswami.*

(1) Move the lines parallel to themselves so that the origin lies *within* the new triangle formed. If the equations of the lines now are $y = m_1 x + c_1$, etc., we have by Q. 1331

(i) If the triangle be *acute angled*, the origin is in the acute angle between the three lines taken two at a time

$\therefore c_1 c_2 (1 + m_1 m_2) \; c_2 c_3 (1 + m_2 m_3) \,.\, c_3 c_1 (1 + m_1 m_3)$ is negative

$\therefore \qquad \Pi (1 + m_1 m_2)$ is < 0.

(ii) If the triangle be *right-angled*, one of the three factors $1 + m_r m_s$ must be zero.

$\therefore \qquad \Pi (1 + m_1 m_2) = 0.$

(iii) If the triangle be *obtuse-angled*, the origin lies in one obtuse-angle and in two of the acute angles formed by the sides

$\therefore c_1 c_2 (1 + m_1 m_2) \,.\, c_2 c_3 (1 + m_2 m_3) \,.\, c_3 c_1 (1 + m_1 m_3)$ is positive.

$\therefore \qquad \Pi (1 + m_1 m_2) > 0.$

Conversely, if the product $\Pi\, c_r c_s (1 + m_r m_s)$ be positive, then either all the factors are positive or two are negative and one is positive. The former case is impossible since a triangle cannot have all its angles obtue. Hence two angles are acute and one obtuse giving us case (iii).

The other two cases, *viz.*, when the product is zero or negative can be dealt with similarly.

(2) Let BC, CA, AB be parallel to the lines $y = m_1 x$, $y = m_2 x$ and $y = m_3 x$ and let $m_1 = \tan \alpha$, $m_2 = \tan \beta$, $m_3 = \tan \gamma$. Then $A = \beta - \gamma$, $B = \gamma - \alpha$, $C = 180^\circ + \alpha - \beta$.

Now the triangle ABC is acute-angled, right-angled or obtuse-angled according as the expression $\cos A \cos B \cos C > = \text{ or } < 0$

i.e. $\cos(\beta - \gamma) \cos(\gamma - \alpha) (\cos(\alpha - \beta) < = \text{ or } > 0$

i.e. $\cos^2 \alpha \cos^2 \beta \cos^2 \gamma \, (1 + m_2 m_3)(1 + m_3 m_1)(1 + m_1 m_2) < = \text{ or } > 0$

i.e. $(1 + m_2 m_3)(1 + m_3 m_1)(1 + m_1 m_2) < = \text{ or } > 0.$

Question 1397.

(MARTYN THOMAS) :—If

$$a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + \dots \dots$$

be a convergent series of the third order, show that its sum to infinity is

$$\begin{vmatrix} 0 & s_1x^2 & s_2x & s_3 \\ a_0 & a_1 & a_2 & a_3 \\ a_1 & a_2 & a_3 & a_4 \\ a_2 & a_3 & a_4 & a_5 \end{vmatrix} \div \begin{vmatrix} x^3 & x^2 & x & 1 \\ a_0 & a_1 & a_2 & a_3 \\ a_1 & a_2 & a_3 & a_4 \\ a_2 & a_3 & a_4 & a_5 \end{vmatrix}$$

where s_1, s_2, s_3 denote respectively the sums of the first 1, 2, 3 terms of the given series.

Solution by N. Sundaram Aiyar.

Let S stand for the given series and $D = 1 + px + qx^2 + rx^3$ the scale of relation.

Then clearly $S = \dfrac{a_0 + (a_1 + a_0 p)x + (a_0q + a_1 p + a_2)x^2}{1 + px + qx^2 + rx^3}$

$$= \frac{s_3 + xs_2 \cdot p + x^2s_1 \cdot q}{1 + px + qx^2 + rx^3} = \frac{N}{D} \quad \text{(say)}.$$

Now.

$$\begin{aligned} x^3 \cdot r + x^2 \cdot q + x \cdot p + (1 - D) &= 0 \quad \dots \quad (1) \\ a_0r + a_1q + a_2p + a_3 &= 0 \quad \dots \quad (2) \\ a_1r + a_2q + a_3p + a_4 &= 0 \quad \dots \quad (3) \\ a_2r + a_3q + a_4p + a_5 &= 0 \quad \dots \quad (4) \\ s_1x^2 \cdot q + s_2x \cdot p + (s_3 - N) &= 0 \quad \dots \quad (5) \end{aligned}$$

From (1), (2), (3) and (4),

$$\begin{vmatrix} x^3 & x^2 & x & 1 - D \\ a_0 & a_1 & a_2 & a_3 \\ a_1 & a_2 & a_3 & a_4 \\ a_2 & a_3 & a_4 & a_5 \end{vmatrix} = 0 = \Delta_2 + D \mid a_0 a_2 a_4 \mid$$

where Δ_2 is the second of the two determinants given in the problem,

Similarly from (5), (2), (3) and (4), we have

$$0 = \Delta_1 + N \mid a_0 a_2 a_4 \mid$$

where Δ_1 is the first of the two determinants given in the problem.

$$\therefore \quad S = \frac{N}{D} = \frac{\Delta_1}{\Delta_2}.$$

Questions for Solution.

Proposers of Questions are requested to send their own solutions along with their questions.

1474. (SELECTED) :—If p denote one of the series of primes 2, 3, 5, 7, 11 ... , then $\Sigma f(p)$ is convergent if $\sum \frac{f(p)}{\log p}$ is convergent.

1475. (M. MOHD DIN GHARIB) :—Prove that the sum of the squares on one pair of the opposite sides of a quadrilateral together with twice the square on the line joining their middle points is equal to the sum of the squares on the other pair of opposite sides with twice the square on the line joining their middle points, without joining the point of intersection of the bisecting lines to its corners.

1476. (N DURAIRAJAN) :—Given four points A,B,C,D in a plane; show that there exists a curve of class three which touches their six joins and has the line at infinity for a bi-tangent.

Prove further that if ABCD be an orthocentric set, the corresponding curve is a tricusp hypocycloid; and that if points A, B, C, D be concyclic the contact with the line at infinity will be in perpendicular directions.

1477. (S. D. CHOWLA) :—If

$$\sum_{n/d} (-1)^{\frac{n}{d}+d} d^a = \rho_a(n), \text{ and } \sum_{n/d} d^a = \tau_a(n)$$

then prove that

$$\sum_{n/d'} \left\{ \rho_a \left[\frac{n}{d'}\right] 2^d \, (2^{d+1} - 1 \right\} = \tau_a(n)$$

the summation extending over all the divisors $d' = 2^d$ of n which are powers of 2.

Ex. Take $a = 1$, $n = 12$.

Left side $= \rho_1(12)\,1 + \rho_1(6)\,2(2^2 - 1) + \rho_1(3)\,2^2(2^3 - 1)$

$= 1.(-1-12+2+6-3-4) + 2(2^2-1)(-1-6-2-3)$
$+ (1+3)2^2(2^3-1) = 28 = 1+2+3+4+6+12$

= sum on the right.

LIST OF JOURNALS RECEIVED IN THE LIBRARY

during the months of May and June 1927.

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1 Abhandlungen aus dem Mathematischen Seminar d. Hamburgishcen Universität, **5**, 3.

2 Annals of Mathematics, **28**, 2.

3 Annales de L'Ecole Normale Superieure, Jan. & **Feb.**, 1927.

4 American Mathematical Monthly, **34**, 3, 4 & 5.

5 Astrophysical Journal, **65**, 1, 2 & 3.

6 Bulletin of the American Mathematical Society, **33**, 2.

7 Bulletin of the Calcutta Mathematical Society, **18**, 1 (2 copies).

8 Bulletin des Sciences Mathematiques April 1927.

9 Contribucion al Estudio da les Ciencias Fisicas y Mathematicas (La Plata), **4**, 2.

10 Crelle's Journal, **156**, 3.

11 Japanese Journal of Mathematics, **3**, 3 & 4.

12 Mathematical Gazette, **13**, 188 (2 copies)

13 Mathematische Annalen, **97**, 3.

14 Messenger of Mathematics, **56**, 8 & 9 (2 copies).

15 Monthly Notices of the Royal Astronomical Society, **87**, 5 & 6.

16 Philosophical Magazine, **3**, 16, 17 & 18.

17 Philosophical Transactions of the Royal Society, London, **226**, 642, 643 & 644.

18 Proceedings of the Cambridge Philosophical Society, **23**, 6.

19 Proceedings of the Edinburgh Mathematical Society, (Series II), **1**, 1.

20 Proceedings of the Royal Society, London, **114**, 769.

21 Rendiconti del Cicolo Mathematico de Palermo, **51**, 1 (and supplement).

22 Revue Semestrielle des publications Mathematiques, **32**, 1 & 2.

23 Transactions of the American Mathematical Society, **29**, 2.

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