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CONTENTS.

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A. NARASINGA RAO,
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By Cor. (2) of the same para, one (and since f is non-cyclic, only one) factor of the operative product $f_1 \otimes f_p$ is a mth root of identity. Hence every f_p occurs on the right-side of one and only one of these π equations. Further f_p, f_q could become identical only in two cases, namely when ω_t is E and when m is even and ω_t is one of the group-involutions. *Hence the number of distinct* m*th roots of identity i*

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when m *is odd and* $\frac{m^2+4}{2}$ *when* m *is even.**

To find the number of *distinct special* mth *roots*, we must discard from the distinct mth roots all those which are tth roots where t is a

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* Clifford has given a short consideration to this problem in his paper on 'Transformation of Elliptic Functions' (*Mathematical Papers*, page, 209), but has arrived at an erroneous result. He says:

" Suppose then, the relation between U and V to be such that a polygon of n sides may be inscribed in U and circumscribed to V. Let $x x_1 \ldots x_{n-1}$ be the vertices of such a polygon; then if $x = \text{sn}\, u$, we must have $x_1 = \text{sn}\,(u + \gamma)$, $x_2 = \text{sn}\,(u + 2\gamma) \ldots$ and therefore $x = \text{sn}\,(u + n\gamma)$. Therefore $n\gamma$ is a period of the elliptic function; and the number of conics of the series $U - \sigma V$ which can be inscribed in n-gons inscribed in U is equal to the number of periods whose nth parts are not congruent, that is for n, a prime number it is $n + 1$ "

The number $n + 1$ is a clear mistake; it is neither n^2 the number of nth roots, nor $\frac{n^2+1}{2}$ the number of distinct nth roots for n an odd prime. The mistake arises from the overlooked fact that γ and $-\gamma$ correspond to the same conic. The correct statement would therefore be; *If* Ω *be a period of the elliptic function, the required number of conics is the number of distinct incongruent numbers of the form* $\frac{\Omega}{n}$, *two numbers of this form not being considered distinct if they are congruent to numbers which differ only in sign.* When the points $\frac{\Omega}{n}$ are marked in the fundamental parallelogram, this means that two such points lying within the parallelogram should be considered identical if they are symmetrically situated with respect to the centre and two points on a side, symmetrically situated with respect to the extremities should be similarly considered identical. Thus if $a\omega_1 + b\omega_2$ $\left(a, b = \frac{1}{n}, \frac{2}{n} \ldots \frac{n}{n}\right)$ are the n^2 points, then if n is odd $\frac{(n-1)^2}{2}$ points within the parallelogram and $n - 1$ points on the sides would be redundant giving $\frac{n^2+1}{2}$ distinct points; and if n is even $\frac{(n-1)^2-1}{2}$ points within the parallelogram and $n - 2$ points on the sides would be redundant giving $\frac{n^2+4}{2}$ distinct points.

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factor of m. Thus, let $t_1 t_2 t_3$ ... be the factors of m and let the greatest common factor (which is itself one of the t's) of $t_1\ t_2$ ... t_k be denoted by $t_{1\,2\,3 \ldots k}$. Then the number of distinct special mth roots would be given by

$$\psi\,(m) = \frac{m^2+1}{2} - \Sigma\,\frac{t_1^2+1}{2} + \Sigma\,\frac{t_{12}^2+1}{2} - \Sigma\,\frac{t_{123}^2+1}{2} \ldots \ldots$$

$$= \tfrac{1}{2}\,(m^2 - \Sigma\, t_1^2 + \Sigma\, t_{12}^2 - \Sigma\, t_{123}^2 \ldots \ldots)$$

if m is odd

and by $$\psi\,(m) = \frac{m^2+1}{2} - \Sigma\,\frac{t_1^2+1}{2} + \Sigma\,\frac{t_{12}^2+1}{2} - \ldots \ldots$$

$$+ \tfrac{3}{2}\,(1 - {}_\lambda C_1 + {}_\lambda C_2 \ldots \ldots)$$

which is also equal to $\tfrac{1}{2}\,(m^2 - \Sigma\, t_1^2 + \Sigma\, t_{12}^2 \ldots \ldots)$ if m is even and λ is the number of its even factors.

The series $m^2 - \Sigma\, t_1^2 + \Sigma\, t_{12}^2 - \ldots$ when evaluated by means of a theorem in combinatory analysis* is found to be equal to $m^2\, \Pi \left(1 - \frac{1}{p^2}\right)$ where the product extends over the prime factors p of m. Thus $\psi\,(m)$ the number of special mth roots of identity, is equal to $\tfrac{1}{2}\, m^2\, \Pi \left(1 - \frac{1}{p^2}\right)$. Geometrically, this number would correspond to the number of conics of a four-point system which could be inscribed in proper m-gons inscribed in a given conic of the system.

It will be noticed that the above value for the number of the distinct special mth roots fails in one case, *viz.*, for $m = 2$, the reason for the failure being that $\lambda = 0$.

The distinct special m-th. roots of identity count twice among the special roots of identity. For, the number of special m-th roots, not necessarily distinct, of identity would be equal to the series

$$m^2 - \Sigma\, t_1^2 + \Sigma\, t_{12}^2 - \ldots$$

---

* The theorem referred to is: If $\phi_r\,(m)$ be the number of ways in which r factors of m (m itself not counting as a factor) can be chosen so as to have no common factor, then the value of the series $\phi_1\,(m) - \phi_2\,(m) + \phi_3\,(m) - \ldots \ldots$ is equal to zero if m has a repeated prime factor and is equal to $(-1)^{\mu-1}$ if m is the product of μ distinct primes. For proof, see "A Note in Combinatory Analysis:" *Journal of the Indian Mathematical Society*, Vol. X, (1918), page 414.

i.e., to $2\,\psi(m)$. Further, among the m^2 roots of identity, identity itself counts only once; for if it counted $f(m)$ times, then $f(mn) = f(m)\,f(n)$ shewing that $f(m)$ is a power of m and $f(2) = 1$ shewing that $f(m) = 1$. Hence, among the m^2 roots of identity any special tth root of identity counts exactly twice ($t = m$ or any factor of m except 2). We have thus the result :

> *The* m^2 m*th roots of identity coalesce in pairs, with the exception of identity itself when* m *is odd and with the exception of identity and the group-involutions when* m *is even.*

These conditions shew us the form of the polynomial $F_m\,(abc)$ where $F_m\,(abc) = 0$, is the condition that the transformation

$$a\,P_xP_y + b\,Q_xQ_y + c\,R_xR_y$$

may be cyclic of period m. Thus $F_m)\ a, b, c)$ will be a polynomial of order $\dfrac{m^2-1}{2}$ if m is odd and $\dfrac{m^2-4}{2}$ if m is even, *not vanishing for any of the values* $(1, \pm 1, \pm 1)$, Further, if $t_1 t_2$ ... are the factors of m not including 1 or 2, then

$$F_m \equiv (f_m f_{t_1} f_{t_2} \ldots \ldots)^2$$

where f_k is an irreducible polynomial of order $\psi(k)$. If $\phi(abc) = 0$ be any curve-family of transformations, the intersections of ϕ with F give all the mth roots of identity in the family, not including identity and the group involutions. Thus, for instance, the number of distinct special mth roots of identity in a family with given latent points is $\frac{1}{2}\,\psi(m)$ ($m \neq 2$ of course); for in this case

$$\phi(abc) = \lambda a + \mu b + \nu c \qquad (\lambda + \mu + \nu = 0).$$

15. Other Properties.

The Geometry of inscribed-circumscribed polygons of two conics suggests immediately various properties of cyclic transformations. Thus,

THEOREM I : *If a transformation F possesses a single cyclic m-group of points, it is cyclic of period m.*

For, the branch-points of F^m comprise the points of the m-group in addition to the branch-points of F, Hence $F^m = E$, *i.e.*, F is cyclic of period m and admits an infinity of cylic m-groups. As a corollary, if we transform repeatedly a branch-point (latent point) of F by F itself, we would finally reach a different branch or latent point (latent or branch-point) according as m is even or odd.

THEOREM II: *Pairs of opposite points of every cyclic group of a transformation of even period belong to one (the same for all groups) of the group-involutions.*

For if $F^{2m} = E$, then F^m is of period 2 and therefore is an involution, determined by one of the group-quadratics P.

P may be called the *relevant* quadratic of F. It would be obvious that if F is of period $2m$, then both the λ-transformation, and the derived transformation of F with respect to its relevant quadratic, are of period m.

The pencil of the vertices of m-gons inscribed in one conic and circumscribed to another (which we may term a cyclic pencil of order m) is difficult to study algebraically on account of its many-sided reducibility. We may notice this much, that the Jacobian (and therefore also the apolar $(2m - 2)$-ic of a cyclic pencil of order m is the product of $\frac{m-1}{2}$ standard quartics if m is odd; and if m is even it is the product of the relevant quadratic of the corresponding cyclic transformation and $\left(\frac{m}{2} - 1\right)$ standard quartics.

Ex. (10). A line a cuts three other lines b, c, d in B, C, D. A is any point on a, $A_1A_2A_3$ the harmonic conjugates of A with respect to CD, DB, BC, respectively and X, Y, Z the harmonic conjugates of B, C, D with respect to AA_1, AA_2, AA_3 respectively, shew that quadrilaterals can be inscribed in the conic of the four-line system $(abcd)$ which touches a at A and circumscribed to the conic of the system which touches a at X, Y or Z.

Ex. (11) If F, ϕ_1, ϕ_2 have common branch-points and if $(F\phi_1\phi_2)$ is a closed triad, shew that the relation between ϕ_1, ϕ_2 is a symmetric (2,2) correspondence which is completely determined by F and which is of period m, if F is of period m. Hence shew that if S, Σ, S_1, $S_2 \dots S_m$ are conics of a four-point system such that m-gons could be inscribed in S and circumscribed to Σ and triangles could be inscribed in S with their sides touching severally the conics $S_k S_{k+1} \Sigma$ $(k = 1, 2 \dots, m - 1)$, then triangles could be inscribed in S with their sides touching severally the conics $S_m S_1 \Sigma$.

Ex. (12). Concerning the nth roots of identity with given branch points, shew that the algebraical product of two roots is equal to the operative product of two and only two other roots, if n is odd. If n is even, shew that there exist root-pairs whose algebraic product is not equal to the operative product of two roots and also root-pairs whose algebraical product is the operative product of more than one pair of roots.

Ex. (13). Any pencil of binary cubics is a cyclic pencil; a pencil of binary quartics is a cyclic pencil only if it contains two perfect squares.

Closed n-ads of Transformations.

16. If $F_1 F_2 \ldots F_n$ be n transformations with common branch points and if F_n be an algebraic factor of the continued operative product $F_1 \otimes F_2 \otimes F_3 \otimes \ldots \otimes F_{n-1}$ then the transformations $F_1 F_2 \ldots F_n$ are symmetrically related and may be said to form a closed n-ad.

The parametric equation of a closed quartette or in geometrical terms of a Poncelet quartette of a given base conic may be inferred from the equation of a closed triad. Thus let the equation of which the parameters of a closed quartette are the roots be written in the form

$$f(x)(x-e)+\psi(x)=0.$$

where $\psi(x)$ is a quartic the co-efficients of which are rational functions of three parameters (the parameters of the base conic and the line-conics being e and the roots of $f(x)$ respectively). Now e, f_1, f_2, f_3 form a Poncelet quartette if and only if f_1, f_2, f_3 form a Poncelet triad. Hence $\psi(x)$ is such that whenever $f(x)(x-e)+\psi(x)$ is divisible by $x-e$, the quotient is of the form $f(x)+\lambda(x-e)(x-\mu)^2$ where λ, μ are independent of e. Hence, whenever $\psi(x)$ is divisible by $x-e$, the quotient is of the form $\lambda(x-e)(x-\mu)^2$, where since λ, μ are independent of e, they may be considered to be two of the parameters in $\psi(x)$. Thus $\psi(x)$ would necessarily be of the form $\lambda(x-\mu)^2(x-\nu)^2$ and a Poncelet quartette would be the roots of an equation of the form

$$f(x)(x-e)+\lambda(x-\mu)^2(x-\nu)^2=0.$$

In other words *any quartette belonging to a one-square pencil of quartics containing* $f(x)(x-e)$ *is a Poncelet quartette.* As a special case, *any standard quartette of* $f(x)(x-e)$ *is a Poncelet quartette.*

Generally, assume that for all values of n up to and include n, the parametric equation of a Poncelet $2n$-ad is

$$f(x)(x-e)\phi^2_{n-2} + \phi^2_n = 0. \quad \ldots \quad \ldots \quad (1)$$

and of a Poncelet $(2n+1)$—ad is

$$f(x)\phi^2_{n-1} + (x-e)\phi^2_n = 0. \quad \ldots \quad \ldots \quad (2)$$

where ϕ_k is a polynomial of order k with arbitrary co-efficient. Let now the equation of which the parameters of a Poncelet $(2n+2)$-ad are the roots be put in the form

$$f(x)(x-e)\psi_{2n-2} + \psi_{2n+2} = 0.$$

As before, whenever this expression is divisible by $x-e$, the quotient should be of the form (2) from which it would be evident that ψ_{2n-2} and ψ_{2n+2} are perfect squares of polynomials with arbitrary co-efficients. Thus the form (1) is valid for the value $n+1$ of n and therefore always. Similarly the form (2) is always valid.

The Cayley-condition for inscribed-circumscribed Polygons.

17. The condition that n-gons could be inscribed in S and circumscribed to S′ is that the repeated n-ad (S′S′ ...) be a Poncelet n-ad of S. Let the conic xS + S′ be represented parametrically by x, so that the parameters of S, S′ are ∞, 0. The equations to Poncelet $(2m+1)$-ads and $2m$-ads become

$$f(x)\phi^2_{m-1} - \phi^2_m = 0. \quad \ldots \quad \ldots \quad (1)$$

$$f(x)\phi^2_{m-2} - \phi^2_m = 0. \quad \ldots \quad \ldots \quad (2)$$

where

$$f(x) \equiv \Delta x^3 + \Theta x^2 + \Theta' x + \Delta' = 0$$

represents the line-conics.

When $n = 2m+1$, the required condition is, that it should be possible to choose the parameters in (1) in such wise that 0 is a $(2m+1)$-ple root of (1), *i.e.* of the equation

$$\sqrt{f(x)}\,\phi_{m-1} - \phi_m = 0.$$

Hence if
$$\sqrt{f(x)} = \Sigma A_r x^r$$

and
$$\phi_{m-1} = \rho_0 + \rho_1 x + \rho_2 x^2 \ldots \ldots + \rho_{m-1} x^{m-1},$$

we must have

$$\begin{array}{llll} A_2\,\rho_{m-1} + A_3\,\rho_{m-2} & \cdots & + A_{m+1}\,\rho_0 = 0 \\ A_3\,\rho_{m-1} + A_4\,\rho_{m-3} & \cdots & + A_{m+2}\,\rho_0 = 0 \\ \cdots \quad \cdots & \cdots & \cdots \\ \cdots \quad \cdots & \cdots & \cdots \\ A_{m+1}\,\rho_{m-1} \quad \cdots & \cdots & + A_{2m}\,\rho_0 \quad = 0. \end{array}$$

Eliminating the ρ's we have the necesrary and sufficient condition

$$\begin{vmatrix} A_2 & A_3 & \cdots & A_{m+1} \\ A_3 & A_4 & \cdots & A_{m+2} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ A_{m+1} & A_{m+2} & \cdots & A_{2m} \end{vmatrix} = 0.$$

Similarly for $n = 2m$, the condition would be

$$\begin{vmatrix} A_3 & A_4 & \cdots & A_{m+1} \\ A_4 & A_5 & \cdots & A_{m+2} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ A_{m+1} & A_{m+2} & \cdots & A_{2m-1} \end{vmatrix} = 0.$$

Ex. (14). The four square roots of any transformation form a closed quartette. Hence the $2n$-th roots of any transformation form a closed set.

Ex. (15). Any the same powers of a closed n-ad of transformations also form a closed n-ad.

Ex. (16). Shew that any quartette of conics of a four-point system is a Poncelet quartette of precisely four conics of the system. If the quartette is a standard quartette of $f(x)\,(x - e)$, shew that three of these conics coincide with e. If the fourth conic be λ', and if the quartette is the set of square roots of the conic λ when e is taken as the base conic shew that λ, λ' are connected by the relation:

$$\frac{2(\lambda - e)}{\left(\lambda \frac{d}{de} + t\frac{d}{dt}\right) f(e)} + \frac{\lambda' - e}{\left(\lambda' \frac{d}{de} + t\frac{d}{dt}\right) f(e)} = 0.$$

where t is the unit variable of homogeneity.

Bertrand's Second Theorem

By Dr. S. R. U. Savoor.

In Vol. 84 of the *Comptes Rendus de l'Academie des Sciences*, M. J. Bertrand proposed the problem of finding the Law of Attractive Force that led to the description of a conical (elliptical) orbit by one attracting particle round another. In the same volume, M. Darboux and M. Halphen solved the problem in two entirely different ways. In Vol. 87 of the *Comptes Rendus*, M. Bertrand proved a more general theorem, *viz* :—

" The only laws of attractive force expressible as functions of the distance, which always give rise to closed orbits whatever the initial circumstances may be (within a certain range) are

$$F = \frac{k^2}{r^2} \text{ and } F = k^2 r."$$

The proof of this second theorem of M. Bertrand is given in Tisserand's *Mecanique Celeste*, Tome I, but is rather long. Prof. H. C. Plummer has given a shorter proof in his *Introductory Treatise on Dynamical Astronomy* but the argument is not clear. Indeed Plummer's fundamental assumption itself requires justification. The object of this paper is both to clear the difficulties in Plummer's proof and to indicate the method of integration in series which is typical of those employed in the theories of perturbations and the more difficult parts of celestial mechanics.

The differential equation for a central orbit is

$$\frac{d^2 u}{d\theta^2} + u = \frac{F}{h^2 u^2} = \frac{1}{h^2} U(u), \text{ say} \quad \dots \quad \dots \quad (1)$$

Suppose the solution for u contains a small parameter e so small that we can neglect e^4 and higher powers of e.

Let

$$u = u_0 + u_1 e + u_2 e^2 + u_3 e^3 + \dots \dots + u_n e^n + \dots \quad (2)$$

be the solution of (1).

Let $u = u_0$ correspond to one of the circular orbits so that

$$u_0 = \frac{1}{h^2} U(u_0).$$

If e be small, the path under consideration is a circular orbit slightly disturbed.

Substitute (2) in equation (1) and expand by Taylor's Theorem omitting terms involving powers of e higher than e^3. Then we get

$$\frac{d^2}{d\theta^2}(u_0 + u_1 e + u_2 e^2 + u_3 e^3 + \ldots) + (u_0 + u_1 e + u_2 e^2 + u_3 e^3 + \ldots)$$

$$= \frac{1}{h^2} U(u_0 + u_1 e + u_2 e^2 + u_3 e^3 + \ldots)$$

$$= \frac{1}{h^2}\Big[U(u_0) + (u_1 e + u_2 e^2 + u_3 e^3 \ldots) U'(u_0) + \tfrac{1}{2}(u_1 e + u_2 e^2 + u_3 e^3 \ldots)^2 U''(u_0) + (\tfrac{1}{6} u_1 e + u_2 e^2 + u_3 e^3 \ldots)^3 U'''(u_0) + \ldots\Big]$$

$$= \frac{1}{h^2}\left\{\begin{array}{l} U + (u_1 e + u_2 e^2 + u_3 e^3) U' \\ \quad + \tfrac{1}{2}(u_1 e + u_2 e^2)^2 U'' + \tfrac{1}{6} u_1^3 e^3 U''' \end{array}\right\}$$

up to the third order in e, U, U′, U″, U‴ denoting respectively the constants $U(u_0)$, $U'(u_0)$, $U''(u_0)$ and $U'''(u_0)$.

Therefore equating the co-efficients of the various powers of e, we get the following equations for u_0, u_1, u_2, u_3.

$$\left.\begin{array}{ll} \text{(A)} & + u_0 = \dfrac{1}{h^2} U(u_0) = \dfrac{1}{h^2} U \\[2ex] \text{(B)} & \dfrac{d^2 u_1}{d\theta^2} + u_1 = \dfrac{1}{h^2} u_1 U'(u_0) = \dfrac{1}{h^2} u_1 U'. \\[2ex] \text{(C)} & \dfrac{d^2 u_2}{d\theta^2} + u_2 = \dfrac{1}{h^2}\Big[u_2 U' + \tfrac{1}{2} u_1^2 U''\Big]. \\[2ex] \text{(D)} & \dfrac{d^2 u_3}{d\theta^2} + u_3 = \dfrac{1}{h^2}\Big[u_3 U' + u_1 u_2 U'' + \tfrac{1}{6} u_1^3 U'''\Big]. \end{array}\right\} \ldots \quad (3)$$

Equation A of this set is merely the equation for a circular orbit. The radius for a circular orbit is got from the equation

$$\text{(A}'\text{)} \qquad u_0 = \frac{1}{h^2} U(u_0) \text{ or } \frac{1}{h^2} = \frac{u_0}{U}.$$

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Using (A′) for the solution of (B), we have

$$\frac{d^2u_1}{d\Theta^2} + u_1\left[1 - \frac{u_0\,U'}{U}\right] = 0.$$

If we write

$$m^2 = 1 - \frac{u_0\,U'}{U} \qquad \ldots \qquad \ldots \quad (4)$$

we get

$$u_1 = a_1 \cos m\,\Theta, \qquad \ldots \qquad \ldots \quad (5)$$

as the simplest solution.

Use now (A′). (4) and (5) for solving (C) and we get

$$\frac{d^2u_2}{d\Theta^2} + u_2 = \frac{u_0}{U}\left[u_2\,U' + \frac{U''}{2}\,a_1^2 \cos^2 m\Theta\right]$$

or

$$\frac{d^2u_2}{d\Theta^2} + m^2u_2 = \frac{1}{4}\,\frac{u_0\,U''}{U}\,a_1^2\,(1 + \cos 2m\Theta).$$

Hence

$$u_2 = A_2 \cos m\Theta + a_0 + a_2 \cos 2m\Theta.$$

where

$$\left.\begin{aligned} a_0 &= \frac{1}{4}\,\frac{u_0\,U''\,a_1^2}{m^2\,U} \\ a_2 &= -\frac{1}{12}\,\frac{u_0\,U''\,a_1^2}{m^2\,U} \end{aligned}\right\} \qquad \ldots \qquad \ldots \quad (6)$$

Now let us use (A′), (4), (5) and (7) to solve equation (D); we get

$$\frac{d^2u_3}{d\Theta^2} + u_3 = \frac{u_0}{U}\left[u_3\,U' + u_1u_2\,U'' + \tfrac{1}{6}\,u_1^3\,U'''\right]$$

or

$$\frac{d^2u_3}{d\Theta^2} + m^2\Theta = \frac{u_0}{U}\left[a_1 \cos m\Theta\,(a_0 + a_2 \cos 2m\Theta)\,U'' + \tfrac{1}{6}\,a_1^3 \cos^3 m\Theta\;U'''\right]$$

using the simplest (particular) solutions on the right-hand side because what we want is the simplest particular integral for u_3.

The equation for u_3 is

$$\frac{d^2u_3}{d\Theta^2} + m^2u_3 = \frac{u_0}{U}\left[\{\tfrac{1}{2}\,U''\,(2a_0a_1 + a_1a_2) + \tfrac{1}{8}\,a_1^3\,U'''\}\cos m\Theta + (\tfrac{1}{2}U''\,a_1a_2 + \frac{1}{24}\,a_1^3\,U''')\cos 3m\Theta\right]$$

$$= L \cos m\Theta + M \cos 3m\Theta$$

where $$L = \frac{u_0}{U}\left[\tfrac{1}{2} U''(2a_0a_1 + a_1a_2) + \tfrac{1}{8} a_1^3 U'''\right] \qquad \dots \quad (8)$$

Now the C. F. of the equation

$$\frac{d^2u_3}{d\theta^2} + m^2u_3 = 0$$

is of the form $$u_3 = A_3 \cos m\theta.$$

Hence the term $L \cos m\theta$ on the right-hand side of the equation for u_3 will introduce into the particular integral the term

$$\frac{L\theta}{2m} \sin m\theta$$

In this case the solution $u = u_0 + u_1 e + u_2 e^2 + u_3 e^3 + \dots \dots$ will no longer be an approximation. In fact as $\theta \to \infty$ the term $\frac{L\theta}{2m} \sin m\theta$ will tend to $+\infty$ if $m\theta = 2p\pi + \alpha \quad \left(0 < \alpha < \frac{\pi}{2}\right)$.

Hence however small e may be, u will become infinite, *i.e.*, the radius vector will be constantly diminishing down to zero. In other words, the curve will no longer be a closed orbit but will appoximate to some sort of spiral. Hence a necessary condition that the orbit should be a closed one is that

$$L \equiv 0.^* \dots \qquad \dots \qquad \dots \quad (9)$$

Using equations (6), (8) and (9), we get

$$0 \equiv L = \frac{u_0}{U}\left[\tfrac{1}{2} U''(2a_0a_1 + a_1a_2) + \tfrac{1}{8} a_1^3 U'''\right]$$

$$= \frac{u_0}{U}\left[\tfrac{1}{2} U''\left(2a_1 \cdot \frac{1}{4}\,\frac{u_0 U'' a_1^2}{m^2 U} - \frac{1}{12}\,\frac{u_0 U'' a_1^2}{m^2 U} \cdot a_1\right) + \tfrac{1}{8} a_1^3 U'''\right]$$

Thus we get

$$\frac{5}{24}\,\frac{u_0 U''^2}{U} + \frac{1}{8} m^2 U''' = 0$$

or using (4)

$$5u\, U''^2 + 3U'''(U - u\,U') = 0. \qquad \dots \quad (10)$$

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* Equations (6) and (9) are exactly those which Plummer uses and the rest of the proof is exactly the same as in Plummer's book. However it is given for the sake of completeness.

In a closed orbit the number of apses must be finite and hence m must be a rational number.

Since
$$m^2 = 1 - \frac{u\,U'}{U}$$

this gives the solution
$$U = ku^{1-m^2} \quad \ldots \quad \ldots \quad (11)$$

$\therefore$ Substituting this in equation (10) we get
$$5m^4 (1 - m^2)^2 + 3m^2 (1 - m^4) [1 - (1 - m^2)] = 0$$

or
$$2m^4 (1 - m^2) (4 - m^2) = 0$$

$\therefore$
$$m^2 = 1 \text{ or } 4.$$

Hence from (11)
$$U = k \text{ or } ku^{-3}.$$

Hence from (1) we get
$$F \equiv u^2 U = ku^2 \text{ or } ku^{-1}$$

i e
$$F = \frac{k}{r^2} \text{ or } kr.$$

Pseudo-Foci of Sphero-Conics

BY S. L. MALURKAR.

In plane geometry the foci of a conic are defined as the intersections of tangents from the circular points at infinity. For a sphero-conic, they are the intersections of the common tangents to the imaginary circle at infinity and the given conic. As far as many metrical properties are concerned these points are very useful; but there are certain other theorems where we cannot attach any significance to these points, *e.g.*, those concerning the auxiliary circle. The auxiliary circle has been defined to be a circle which has double contact with the given conic; it also serves to define the parametric angle (J. I. M. S., Vol. II, p. 170, "On Sphero-Conics by Prof. M. T. Naraniengar.") Apparently we have no analogue to the theorem 'The foot of the perpendicular from a focus to a tangent lies on the auxiliary circle.' The analogy is found to exist if we make use of two points on the axis of the sphero-conic other than the foci. These points may be called the 'pseudo-foci' of the sphero-conic. Further, it is posssible to define (in analogy to plane confocal conics) the 'mutual circle' of two sphero-conics if their 'pseudo-foci' are common.

In any sphero-conic defined by $SP + S'P = 2\alpha$ and $SS' = 2\gamma$, C being the centre of SS′, there exist two pseudo-foci H and H′ on the axis SS′ such that $HC = CH' = \text{arc tan}\left(\frac{\sin\gamma}{\cos\alpha}\right)$; these points are the intersections of tangents from the two imaginary points ω and ω' common to the circle at infinity and circles with their centres at C.

Taking a triquadrantal triangle ABC for reference and denoting by x, y, z the sines of perpendiculars from a point P, the equation of a conic whose centres are the angular points is

$$ax^2 + by^2 + cz^2 = 0.$$

The equation of the circle at infinity is

$$x^2 + y^2 + z^2 = 0.$$

Hence their common tangents are given by

$$[a(b+c)x^2 + b(c+a)y^2 + c(a+b)z^2]^2$$
$$= 4abc(ax^2 + by^2 + cz^2)(x^2 + y^2 + z^2).$$

So the foci on BC are such that

$$\sin CS = -\sin CS' = \sqrt{\frac{c(b-a)}{a(b-c)}} = \sin \gamma.$$

Again $$SP + S'P = 2\alpha.$$

$$\sin \alpha = \sqrt{\frac{c}{c-b}}; \quad \cos \alpha = \sqrt{\frac{b}{b-c}}.$$

The points common to circles with centres at C and the imaginary circle at infinity are given by the tangential equation

$$\lambda^2 + \mu^2 = 0.$$

So the equation of the tangents from these points is

$$[-abx^2 + aby^2 + c(a-b)z^2]^2 + 4a^2b^2x^2y^2 = 0.$$

Hence $$\sin H'C = \sin CH = \sqrt{\frac{c(b-a)}{ab+bc-ca}}.$$

$$\therefore \quad \tan CH = \sqrt{\frac{c(b-a)}{ab}} = \frac{\sin \gamma}{\cos \alpha}.$$

It may be useful to note that for any triangle of reference the equation of points common to circles centre (p, q, r) is

$$\lambda^2 + \mu^2 + \nu^2 - 2\mu\nu\cos A - 2\nu\lambda\cos B$$
$$- 2\lambda\mu\cos C - l(p\lambda + q\mu + r\nu)^2 = 0,$$

where $$l = \frac{1 - \cos^2 A - \cos^2 B - \cos^2 C - 2\cos A\cos B\cos C}{p^2\sin^2 A + \ldots + 2pq(\cos A + \cos B\cos C) + \ldots}.$$

The equation of a conic with centre C may be simplified to

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} - z^2 = 0.$$

where $\tan \alpha = a$, and $\tan \beta = b$, α and β being the semi-axes along BC and AC. Then it is easily seen that $\tan^2 CH = a^2 - b^2$.

Hence it follows that the equation of conics with the same pseudo-foci can be expressed in the form

$$\frac{x^2}{b^2+\theta} + \frac{y^2}{a^2+\theta} - z^2 = 0.$$

Suppose now that two lines PT, PT′, are such that, when produced to meet the great circle for which C is the pole, they cut off a constant arc of length $\pi/2$.

It is easily seen that ($\sin\theta$, $\cos\theta$ 0), and ($\cos\theta$, $-\sin\theta$ 0), are conjugate points for the circle $x^2+y^2=0$. Hence lines from P drawn to these points form harmonic conjugates with lines from P to the imaginary points ω, ω'. If two lines

$$\lambda x + \mu y + \upsilon z = 0$$
$$\lambda' x + \mu' y + \upsilon' z = 0$$

intercept on the great circle AB an arc $\pi/2$,

then $$\lambda\lambda' + \mu\mu' = 0.$$

This is also the condition that perpendiculars from C on these lines may be at right angles to each other. Hence, *the perpendiculars from C on PT, PT′ are perpendicular, if PT, PT′ separate Pω, Pω′ harmonically.**

From a point P let tangents be drawn to two conics which have the same pseudo-foci. *If the perpendiculars drawn from the centre on these tangents are at right angles to each other, the locus of P is a concentric circle.* For, let the conics be

$$\frac{x^2}{b^2}+\frac{y^2}{a^2}-z^2=0.$$

$$\frac{x^2}{b^2+\theta}+\frac{y^2}{a^2+\theta}-z^2=0.$$

Let the tangents from P (one to each conic), be :

$$\lambda x + \mu y + \nu z = 0;\ \lambda' x + \mu' y + \nu' z = 0.$$

If the perpendiculars from (0, 0, 1) to these tangents be at right angles to each other,

$\lambda\lambda' + \mu\mu' = 0$; eliminating $\lambda, \lambda', \mu, \mu' + \nu, \nu'$

from these equations it is seen that

$$x^2+y^2-(a^2+b^2+\theta)\,z^2=0.$$

Hence the locus of P is a circle of radius r; where $\tan^2 r = a^2+b^2+\theta$ $=\tan^2$ (semi-major axis of one conic) $+\tan^2$ (semi-minor axis of the other).

This circle may be called the mutual circle of the two conics.

* Mr. M. Bhimasena Rao has pointed out this result to me. At his suggestion the enunciations have been re-drafted.

Particular Cases.

(i) Let one of the conics degenerate into two points coinciding with the pseudo-foci. The mutual circle is then the auxiliary circle, that is, if PT be a tangent to the conic from any point P on the auxiliary circle, PH and PT are such that perpendiculars on them from the centre are at right-angles to each other. This is a geometrical property of the auxiliary circle, apart from its utility for defining the eccentric angle. Of course, the harmonic property holds provided the proper tangent is taken.

(ii) Next, let the conics become coincident. Then from the result

$$\tan^2 r = \tan^2 (\text{semi major axis}) + \tan^2 (\text{semi minor axis}),$$

we see that the mutual circle becomes the director circle of the conic. This property of central perpendiculars is given by Prof. M. T. Naraniengar.*

(iii) Lastly, let both the conics degenerate to two points; the mutual circle is the circle on the join of the two points as diameter. Hence it follows that the central perpendiculars on the joins of a point on a circle to the extremities of a diameter are at right angles to each other. It is also easy to prove that if we replace the diameter by any chord the angle contained between the central perpendiculars is constant. This corresponds to the property in plane conics that the angle in a segment is constant.

Again corresponding to the property that the opposite angles of a cyclic quadrilateral are supplementary, in a spherical cyclic quadrilateral, the angle contained between the central perpendiculars on one pair of adjacent sides is supplementary to that contained between the other pair.

To conclude I have to thank Mr. M. Bhimasena Rao to whom various suggestions are due.

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* *Loc. cit*, p. 172.

On the History of Logarithms

BY G. A. MILLER.

The history of logarithms constitutes an especially interesting chapter of the history of mathematics for various reasons. In the first place it clearly contradicts, in as far as it furnishes any evidence, the commonly expressed view that the modern child should study a mathematical subject by following the path pursued by the human race in acquiring a knowledge thereof. On the contrary, it points to the fact that it is a part of the business of the teacher to lead the child along paths which avoid as many as possible of the difficulties encountered by the human race in acquiring a knowledge of the subject in question. The history of logarithms also exhibits the intimate relation between various mathematical developments, for Napier, just as we do, said that the $\log \sin 90^\circ = 0$, but in Napier's time the value of $\sin 90^\circ$, the *sinus totus*, was assumed to be equal to the radius of the circle with respect to which this function was considered, and Napier used a circle whose radius is 10,000,000. Hence Napier said in his famous *Mirifici logarithmorum canonis descriptio*, (Edinburgh, 1614), that $\log 10{,}000{,}000 = 0$, while we say $\log 1 = 0$. If our modern definitions of the trigonometric functions had been in use then, Napier would doubtless have constructed a table of real logarithms of the sines of various angles instead of constructing a table which we should probably not regard now as an actual logarithmic table, for reasons which we shall advance in what follows.

If a student of elementary mathematics who has made no special study of its history were asked to name the most fundamental properties of logarithms he would probably reply that these properties are expressed by the following well-known laws: the logarithm of the product of two numbers is the sum of the logarithms of these numbers, the logarithm of the quotient of two numbers is the logarithm of the numerator diminished by that of the denominator, the logarithm of a power of a number is equal to the product of the logarithm of this number and the index of this power, and the logarithm of a root of a number is equal to the logarithm of this number divided by the index of this root. The numbers which Napier called logarithms in the work to which we referred in the preceding paragraph do not satisfy a single one of these four fundamental laws and hence the question arises why we should now call his tables logarithmic tables, as is commonly done in our general histories of mathematics. In fact, these histories usually state that Napier published the *first* table of logarithms in the work in question. The fact that Napier called certain numbers which appear in these tables "logarithms" is obviously not a sufficient reason for our continuing to do so if we thereby convey the idea to the average reader that these numbers obey laws to which they are not subject, or if we thereby complicate the already too difficult history of mathematics.

If a mathematical history employs technical terms with meanings which differ from those commonly used, it is evidently desirable that the exact sense in which these terms are used should be clearly explained. It is obviously possible to extend the meaning of the technical term *logarithms* so as to include both the numbers which we now commonly call logarithms as well as those which Napier called by this name. For this purpose, it is only necessary to say that if the numbers of an arithmetic series are put into a (1, 1) correspondence with those of a geometric series as follows:—

$$\begin{array}{ccccccccc} \ldots & -3d & -2d & -d & 0 & d & 2d & 3d & \ldots \\ \ldots & -ar^{-3} & ar^{-2} & ar^{-1} & a & ar & ar^2 & ar^3 & \ldots \end{array}$$

then the terms of the arithmetic series are the logarithms of the corresponding numbers of the geometric series. The logarithms thus defined obviously satisfy the following laws: the logarithm of the product of two numbers is equal to the sum of their logarithms — log 1, the logarithm of a quotient is equal to the logarithm of the numerator diminished by the logarithm of the denominator + log 1, etc. In particular, if $a = 1$ these laws reduce to the four fundamental laws noted above. There seems, however, to be no good mathematical reason for thus extending the meaning of the term logarithm at the present time and the historical reason for doing so is insignificant in comparison with thec omplications introduced thereby. It seems much wiser simply to say that the numbers which Napier called logarithms were not logarithms in our sense of the word and that therefore Napier did not publish the first table of logarithms. If this were done, all questions relating to leading a base into Napier's logarithms would be removed, and the history of logarithms would be greatly simplified thereby without losing anything of real value.

The preceding paragraph illustrates the fact that the history of mathematics demands a type of mathematical knowledge which is usually partly new to those who first enter upon a study of this history, and the acquisition of this new mathematical knowledge constitutes the most attractive feature at least for many readers of the history of our subject. Unfortunately our text-books on the general history of mathematics seldom emphasize sufficiently this important element thereof. From problem 40 of the well-known work by Ahmes it seems to result that the ancient Egyptians already knew that if all the terms of a given arithmetic series are multiplied by the same number the resulting series is again arithmetic, and hence this early source of historical information suggests the question, under what transformations does the general arithmetic series remain arithmetic? The more difficult mathematical question of relating a geometric and an arithmetic series in such a way as to simplify the ordinary processes of calculation, illustrated in the preceding paragraph, is clearly raised by a study of Napier's work relating to the subject of logarithms. One of the most encouraging results of such a study is likely to be a realization of how fruitful very imperfect work along important lines may become. In the light of our present mathematical advancement it is difficult to comprehend why Napier did not associate 0 and 1 in his table, and thus make it very much more useful. J. Bürgi also failed along the same line and hence his noted *Progress Tabulen* (1620) should not now be regarded as "antilogarithms with the base 1.0001." In fact, they are not antilogrithms at all in the modern sense of this term since 0 corresponds to 10^8 in these tables.

One of the most difficult questions relating to the history of logarithms is the unravelling of various apparently conflicting statements with respect to the use of a base of a logrithmic system. In the valuable account of the history of logarithms found in J. Tropfke's *Geschichte der Elementar-Mathematik*, volume 2, 1921, it is stated on page 168 that the conception of logarithms as exponents of a fixed base did not prevail until about the middle of the eighteenth century. The fact that logarithms may be regarded as such exponents had however been recognized in special cases before this time, and the laws of calculating with numbers having a common base but different exponents were well-known even before the time of Napier, so that J. Tropfke could truly say, on page 172 of the volume cited above, that theoretically the science of logarithmic calculation was known since the time of Stifel. It should be explicitly noted that by logarithmic calculation is here meant our modern logarithmic calculation with respect to a fixed base and a variable exponent, so that the more simple theoretical logarithmic calculation was known long before the practical logarithmic calculation by means of a table of logarithms was established. The latter, however, did not at first follow the path of the theory of exponents but the more complicated one of the relation between the corresponding terms of an arithmetic and a geometric series, as noted above.

When $a = 1$ but $d \neq 1$ we can obviously replace n by r_1^d so as to reduce this relation between a geometric and an arithmetic series directly to the theory of exponents, and this step was taken, at the suggestion of Napier, by H. Briggs and other immediate followers of Napier, but these followers continued for a long time to consider the corresponding arithmetic and geometric series merely as such series instead of explicitly regarding the second series also as powers of a certain base. Hence their logarithms were logarithms to a certain base (usually the base 10), but for a long time they failed to bring out this fact explicitly. Hence it is correct to say that Briggs constructed a logarithmic table with the base 10, but it is not true that in his publication he explicitly recognized such a base. Such an explicit recognition came much later and did not become well established until L. Euler threw his influence in this direction. The fact that many of the early workers along this line used a base implicitly long before it was used explicitly has led to many statements which appear at first to be contradictory. In fact, all the tables which deserve the name of logarithmic tables relate either explicitly or implicitly, to a definite number as a base. The choice of the base 10, that is, of letting $r = 10$ and $d = 1$ in the given relation between an arithmetic and a geometric series, is theoretically of much less importance than letting $a = 1$ in this relation. In fact, the early tables with the base 10 included explicitly the characteristics of the numbers. It was after the beginning of the eighteenth century (1705) that the characteristics were first omitted, in accord with our modern practice*

The main object of the present article is to make a plea in favour of discontinuing the practice of calling the tables which Napier and Bürgi constructed "logarithmic" tables. since this practice seems to make the history of logarithms unnecessarily difficult and to serve no useful purpose whatsoever. Both Napier and Bürgi deserve much credit with respect to the invention of practical logarithms but both of them missed a fundamental element of simplicity relating thereto which had been clearly exhibited, from the theoretical standpoint, in earlier works. Perhaps our regard for them will become more sincere in view of the fact that they exerted such a wholesome influence notwithstanding their lack of deep insight or full appropriation of what others had prepared

* J. Tropfke: *Geschichte der Elementar-Mathematik*, Volume 2, 1921, page 189.

for them. They worked for the improvement of mathematical calculation with great energy and their accomplishments have brought them immortality in spite of their serious shortcomings from the modern point of view. We have already referred to the fact that Napier would doubtless have prepared more useful tables if in his day more suitable definitions of the trigonometric functions had been in use. It may be added that if the equation had in his day replaced proportion as largely as it does in our times he would doubtless have constructed a more permanently useful system of tables, since the annoying log 1, resulting from the use of his tables now as logarithms, is eliminated in the examples which he gives for finding certain terms in a proportion by means by them. The shortcomings in question are thus considerably mitigated by the state of other mathematical developments near the beginning of the seventeenth century and they reflect this state.

For the sake of increasing the usefulness of the preceding brief expository remarks we may note here their bearing on some statements which appear in the recent *History of Mathematics* by D. E. Smith. Since this is now the most extensive general history of our subject in our language, it is obviously desirable to aim at adding to its value by noting on suitable occasions some necessary modifications in order that readers may be enabled to incorporate them in their private copies if they wish to do so. Both on page 433 of volume 1 and on page 523 of volume 2, it is implied that Bürgi's tables represent exponents with respect to a certain base. This is not in accord with what was noted above as well as with what is obvious from the reproduction of a small part thereof which appears on the latter page. On page 514 of the second volume it is stated that Briggs introduced (1624) the word "mantissa" and that Gauss suggested using it for the fractional part of all decimals. As a matter of fact Briggs never used this term in his extant writings and Gauss simply used the term with the meaning which Wallis had given to it earlier. It was Euler who established its present usage as a logarithmic term. The quotation from Lord Moulton which appears on page 514 of the same volume is in complete discord not only with the history of the invention of logarithms but also with that of the general development of mathematics. It is therefore far from the truth to say that "Lord Moulton expressed the fact very clearly"

Some Properties of M-Functions*

By M. Venkatarama Ayyar, M.A., L.T.

1. Introductory.—Let a_n represent an arbitrary function of the positive integral variable n. As usual, $\Sigma\, a_n$ stands for $a_1 + a_2 + \dots + a_n$. But $\Sigma\, a_n\, \Sigma\, a_{n+1}$ is not to be taken to mean the product of $\Sigma\, a_n$ and $\Sigma\, a_{n+1}$ but is defined by the reduction formula

$$\begin{aligned}\Sigma\, a_n\, \Sigma\, a_{n+1} &= a_n\, \Sigma a_{n+1} + \Sigma\, a_{n-1}\, \Sigma\, a_n \\ &= a_n\, \Sigma\, a_{n+1} + a_{n-1}\, \Sigma\, a_n + a_{n-2}\, \Sigma\, a_{n-1} \\ &\qquad + \dots \dots + a_1\, \Sigma\, a_2\end{aligned}$$

Similarly,

$$\begin{aligned}\Sigma\, a_n\, \Sigma\, a_{n+1}\, \Sigma\, a_{n+2} \dots \Sigma\, a_{n+r} &= a_n\, \Sigma\, a_{n+1}\, \Sigma\, a_{n+2} \dots \Sigma\, a_{n+r} \\ &+ a_{n-1}\, \Sigma\, a_n\, \Sigma\, a_{n+1} \dots \Sigma\, a_{n+r-1} \\ &+ a_{n-2}\, \Sigma\, a_{n-1}\, \Sigma\, a_n \dots \Sigma\, a_{n+r-2} \\ &\dots \quad \dots \quad \dots \quad \dots \quad \dots \\ &+ a_1\, \Sigma\, a_2\, \Sigma\, a_3 \quad \dots \Sigma\, a_{r+1}.\end{aligned}$$

Again let ${}_rA_s\ (s \not< r)$ denote

$$\Sigma\, a_r\, \Sigma\, a_{r+1}\, \Sigma\, a_{r+2} \dots \Sigma\, a_s$$

with the conventions that

$${}_rA_{r-1} = 1$$

and

$${}_rA_{r-2} = {}_rA_{r-3} = \dots = 0$$

and that omission of the prefix r means taking $r = 1$.

Further let $\quad {}_k\alpha_{k-2l} = \Sigma\, a_k\, \Sigma\, a_{k-2}\, \Sigma\, a_{k-4} \dots \Sigma\, a_{k-2l}$

with the same formula of reduction for the group of 'Σ's. With this notation, the M-functions whose properties are studied in this paper, are defined by power-series thus :—

$$M_n(x) \equiv \frac{x^n}{n!} - {}_{n+1}A_{n+1}\frac{x^{n+2}}{(n+2)!} + {}_{n+1}A_{n+2}\frac{x^{n+4}}{(n+4)!} - \dots \dots$$

for n = zero or any positive integer.

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* Read at the 13th Indian Science Congress at Bombay.

For special forms of a_n, it has been shown that these expansions become analogous to those of Bessel's functions, Secant x, e^x, Exponential, Legendre's Elliptic and other functions.*

2. It has been shown elsewhere † that

$$\frac{d}{dx} \mathrm{M}_0(x) = -a_1 \mathrm{M}_1(x)$$

$$\frac{d}{dx} \mathrm{M}_n(x) = \mathrm{M}_{n-1}(x) - a_{n+1} \mathrm{M}_{n+1}(x)$$

By successive differentiation, we get

$$\frac{d^2}{dx^2} \mathrm{M}_0(x) = (-1)^2 \{ a_1 a_2 \mathrm{M}_2(x) - a_1 \mathrm{M}_0(x) \}$$

$$\frac{d^3}{dx^3} \mathrm{M}_0(x) = (-1)^3 \{ a_1 a_2 a_3 \mathrm{M}_3(x) - a_1 \mathrm{M}_1(x) . \Sigma a_2 \}$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

$$\begin{aligned} \frac{d^n}{dx^n} \mathrm{M}_0(x) = (-1)^n \{ & a_1 a_2 \ldots a_n \mathrm{M}_n(x) \\ & - a_1 a_2 \ldots a_{n-2} \mathrm{M}_{n-2}(x) \, {}_{n-1}\mathrm{A}_{n-1} \\ & + a_1 a_2 \ldots a_{n-4} \mathrm{M}_{n-4}(x) \, {}_{n-3}\mathrm{A}_{n-2} \\ & \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \\ & (-1)^{r-1} . a_1 a_2 \ldots a_{n-2r+2} \mathrm{M}_{n-2r+2}(x) \, {}_{n-2r+3}\mathrm{A}_{n-r+1} \\ & + \ldots \} . \qquad \ldots \quad (\mathrm{I}) \end{aligned}$$

As illustrations we take

(i) $a_n = n^2$ so that $\mathrm{M}_0(x) = \operatorname{sech} x$ and $\mathrm{M}_n(x) = \operatorname{sech} x \dfrac{\tanh^n x}{n!}$.

Then, from (I), we get

$$\frac{d^n}{dx^n} \operatorname{sech} x = (-1)^n \operatorname{sech} x \{ n! \tanh^n x - (n-2)! \tanh^{n-2} x . \Sigma (n-1)^2 + (n-4)! \tanh^{n-4} x . \Sigma (n-3)^2 \Sigma (n-2)^2 - + \ldots \} \ddagger$$

a result easily verified.

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* These results with the notations used here were proved in various papers on Special Determinants by Mr. M. Bhimasena Rao and the late Mr. C. Krishnamachari published in this *Journal*, Vol XIV.

† *Vide* the paper on 'Determinants involving specified numbers :' J.I.M.S., Vol. XIV, p. 128.

‡ The Σ's occurring inside the brackets have been tabulated by Messrs. M. Bhimasena Rao and the late C. Krishnamachari in the pages of this *Journal*, Vol. XIV and the *Proc. Lond. Math. Society*, 1923.

(ii) $a_n = n$, $M_0(x) = e^{-\frac{x^2}{2}}$ and $M_n(x) = \frac{x^n}{n!} e^{-\frac{x^2}{2}}$,

we get from (I),

$$\frac{d^n}{dx^n}\left[e^{-\frac{x^2}{2}}\right] = (-1)^n \Big\{ n!\,.\frac{x^n}{n!} e^{-\frac{x^2}{2}} - (n-2)!\frac{x^{n-2}}{n-2!} e^{-\frac{x^2}{2}}\,.\,\Sigma(n-1)$$

$$+ (n-4)!\,\frac{x^{n-4}}{n-4!} e^{-\frac{x^2}{2}}\,.\,\Sigma(n-3)\,\Sigma n-2) - \ldots \Big\}$$

$$= (-1)^n e^{-\frac{x^2}{2}} \Big\{ x^n - \Sigma(n-1)\, x^{n-2}$$

$$+ \Sigma(n-3)\,\Sigma(n-2)\, x^{n-4} - \ldots \ldots \Big\}.$$

3. Let us denote $\frac{d^k}{dx^k} M_n(x)$ shortly by $M_n^k(x)$. Then, from (I),

$$M_0^n(x) = (-1)^n \{ a_1 a_2 \ldots a_n M_n(x) - a_1 a_2 \ldots a_{n-2} M_{n-2}(x)\,.\,\Sigma a_{n-1}$$

$$+ a_1 a_2 \ldots a_{n-4} M_{n-4}(x)\; {}_{n-3}A_{n-2} - \ldots \ldots \}$$

$$M_0^{n-2}(x) = (-1)^{n-2} \{ a_1 a_2 \ldots a_{n-2} M_{n-2}(x)$$

$$- a_1 a_2 \ldots a_{n-4} M_{n-4}(x)\,.\,\Sigma a_{n-3}$$

$$+ a_1 a_2 \ldots a_{n-6} M_{n-6}(x)\; {}_{n-5}A_{n-4} - \ldots \ldots \}$$

... ...

Now, multiply both sides of the second equation by ${}_{n-1}\alpha_{n-1}$, of the third by ${}_{n-1}\alpha_{n-3}$, of tbe fourth by ${}_{n-1}\alpha_{n-5}$ and so on.

Then, on adding, all the terms on the right-hand side vanish, leaving only $(-1)^n a_1 a_2 \ldots a_n M_n(x)$.

Hence we have

$$(-1)^n a_1 a_2 \ldots a_n M_n(x)$$

$$= M_0^n(x) + {}_{n-1}\alpha_{n-1} M_0^{n-2}(x) + {}_{n-1}\alpha_{n-3} M_0^{n-4}(x) + \ldots \ldots \quad \text{(II)}$$

which expresses $M_n(x)$ in terms of $M_0(x)$ and its derivatives.

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NOTES AND QUESTIONS.

Notes and Questions.

The Pedal as a Roulette.

1. Let Γ, Γ' be two planes which slide in contact with each other. At any instant the movement of one relatively to the other may be regarded as one of rotation about an instantaneous centre I and if the positions of I at various instants be marked on the two planes, we obtain two curves C and C′; the movement may then be reproduced by making the curves C and C′ roll in contact each carrying its plane with it. The paths traced out on either plane by the fixed points of the other form a system of curves determinable entirely in terms of the constants of the movement, *viz.*, the curves C, C′ and one of their positions of contact. In general the nature of the dependence of the paths, on the two *centrodes** C and C′ is very obscure and not easily expressible in terms of well-known constructions on the metrical plane. There is an exception however, when C and C′ are congruent curves rolling with corresponding points in contact. It is shown that in this case the family of paths on either plane is the family of pedals of the corresponding centrode C with respect to various points in the plane each enlarged to twice its size with the pedal origin as centre of similitude. This result is used towards the end of the paper to obtain a three-bar linkage for tracing nodal bicircular quartics.

2. The two congruent curves C, C′ establish, in general, a 1 — 1 correspondence of points on the two planes Γ, Γ', namely, that in which corresponding points come together when C and C′ are superposed.† Let OO′ be such a set of corresponding points. (See figure 1).

The two curves being congruent, if the point of contact P be self-corresponding in one position, it will be so throughout the movement and in any position the tangent PT at P is a line about which the two curves are situated symmetrically and O and O′ are thus reflexions of each other in PT.

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* This is the orthodox definition of the centrodes and connects them with the movement. It may be of interest to know that they may also be defined in terms of the paths, the centrode on each plane being the cuspidal locus of the paths traced on that plane by fixed points of the other.

† If there be more than one way of superposing C on C′ there will be more than one correspondence between the planes. For the purpose of the subsequent argument it is indifferent which is chosen provided it is strictly adhered to.

If OO′ meets the tangent in T, it follows that since O is a fixed point on Γ, T traces the pedal of C with respect to O, and O′ traces a

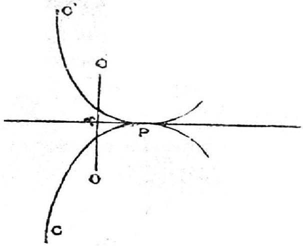

Fig. 1.

curve similar to the pedal but of double its dimensions. Hence we have the Theorem :

*When a curve C′ rolls symmetrically on another congruent with it the path of any carried point O′ is a curve similar to the pedal of C with respect to the corresponding point O and of double its dimensions.**

3. From the known properties of pedal curves† we deduce by means of the above theorem that when a curve of class n rolls symmetrically on a congruent curve

(i) The locus of any carried point is a n-circular $2n$-ic with nodes of order n at each of the circular points and at the point O of the fixed curve which corresponds to the carried point O′ of the moving curve.

(ii) The order of the roulette diminishes by unity whenever the curve touches the line at infinity and by *two* when the tracing point is a simple focus.

As particular cases we have

(iii) If a circle, roll on an equal circle, the locus of any carried point is a Limacon (pedal of a circle) which becomes a cardioid when the tracing point is on the circumference. The path of the centre is however a circle.

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* Cayley—Note on the problem of pedal curves, *Phil. Mag.*, Vol. XXVI or *Collected Papers*, Vol. V, page 113. Also *Ency. der Math., Wissenschaften* IV. i, page 219.

† Hilton—*Plane Algebraic Curves*, pp. 165—168 or *Journal of the Indian Math. Soc.*, Dec. 1920.

(iv) If a parabola roll symmetrically on an equal parabola, the locus of any carried point is a nodal circular cubic; the path of the focus is however a straight line.

(v) If any central conic roll symmetrically on another congruent with it, the locus of any carried point is a nodal bicircular quartic. The two foci however describe circles.

4. **A linkage for a nodal bicircular quartic.**

Let us examine the last case in detail. It is clear from Fig. 2 that the two carried foci F′ and S′ describe circles whose centres are S and F respectively and whose radii are each equal to the diameter $2a$ of the auxiliary circle.

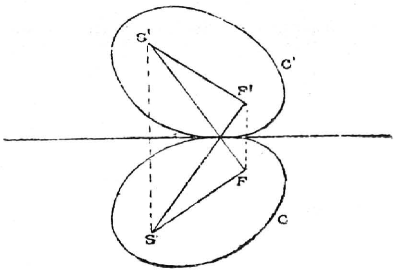

Fig. 2.

As the motion of two fixed points on a lamina determines the movement of the lamina as a whole, it follows that this relative motion of rolling can be secured by the link work SF′S′F where S and F are fixed points in one plane and S′F′ in the other and

$$SF' = S'F = 2a$$
$$SF = S'F' = 2ae$$

Hence we see that in such a *contra parallelogram* the relative motion of a pair of opposite sides is of the type we have been discussing* and that if any side be held fixed, points on the opposite side or those rigidly connected with it describe nodal bicircular quartics which are the pedals

* This result is well-known and has been applied to machinery under the name of "the parallelogram of Reulleux."

of ellipses or hyperbolas according as one of the shorter or longer sides is held fixed.*

5. **Three-bar motion.**

The 3-bar motion of which the linkage described above is a particular case seems to have been first studied by S. Roberts † who showed that when one side of a jointed quadrilateral was held fixed, any point on, or rigidly connected with, the opposite side traces a tri-circular sextic. Roberts noted moreover that this sextic breaks up into a circle and a bicircular quartic when the quadrilateral $AB'CD$ (supposed non-reentrant (Fig. 3) is symmetrical about one of its diagonals, say AC, *i.e.*, $AB' = AD$ and $B'C = CD$. We have seen in the contra-parallelogram ABCD another case in which the sextic breaks up into

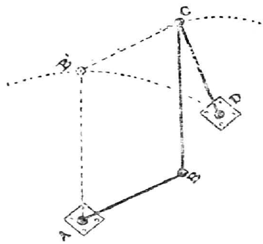

Fig. 3.

a bicircular quartic and a residual curve which must necessarily be a circle. These two results may be deduced each from the other by means of a beautiful result due to Kempe ‡ and Sylvester that in 3-bar motion the nature of the curves described is not affected by interchanging any two of the bars; for by interchanging the unequal bars AB' and $B'C$ in Roberts quadrilateral with diagonal symmetry we obtain the contra-parallelogram ABCD and *vice-versa*. Sylvester claims to have shown that there are no other cases in which the sextic breaks into a circle and a quartic. §

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* Every bicircular quartic may be derived as the pedal of a conic. ***Ency. der. Math. Wissenschaften*** Band III Teil 2, p. 564.

† ***Proc. of the London Math. Soc.***, First Series., Vol. II.

‡ Kempe: ***How to draw a straight line.*** pp. 20—22.

§ Sylvester: ***Collected Math Papers***, Vol. III, p. 32, foot-note.

6. The break-up of the Three-bar sextic.

I shall conclude this note by pointing out how the break-up of the sextic in the two cases cited above may be expected from *a-priori* considerations.

In the case discussed by Roberts, the middle rod B′C moves with its extremities on two circles one of which passes through the centre D of the other, the length of the rod being equal to the radius of the latter circle. Let us start with the rod lying along the line of centres, one end B′ at the centre D and the other C on the circumference of the second circle The rod is now capable of two distinct movements with its extremities moving along their respective circles either a pure rotation about B′ or a movement in which both B′ and C move along the two circumferences. Any carried point traces out a circle in the former case and a bicircular quartic in the latter and these two together constitute the sextic locus. The circular loci may be of any radius but are all concentric *

In the case of the contra parallelogram ABCD with one side fixed, the middle rod BC moves with its extremities on two equal circles, the distance between the centres being equal to the length of the rod. Let the rod be lying along the common diameter with its extremities B, C on the two circles. The initial movements of B and C are thus along parallel lines but they may be in the same or opposite senses. In the first case the motion is one of translation, the rod remaining always parallel to the line of centres, and every connected point describes a circle of the same radius as the two given circles. If B and C move in opposite senses, the motion is no longer one of translation and any carried point traces a bicircular quartic and the two together make up the sextic locus. The circular loci may have any centre but are all of the same radius.

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* S. Roberts: The Pedals of Conics Section; ***Proc. London Math. Soc.***, First Series, Vol. III, p. 88.

Note on the relations between the sides and angles of a triangle.

The following method of deriving the formulæ necessary for solving triangles appears to be new and may be found suitable for the class-room.

Let any line make an angle θ with the side BC of a triangle ABC, so that CA, AB make angles

$$\theta + \pi - C \text{ and } \theta + \pi - C + \pi - A$$

with it. Now projecting the broken line BC, CA, AB on the line we have

$$BC\cos\theta + CA\cos(\theta + \pi - C) + AB\cos(\theta + 2\pi - C - A) = 0$$

or

$$a\cos\theta - b\cos(\theta - C) - c\cos(\theta + B) = 0. \quad \ldots \quad (1)$$

which is our fundamental formula.

Setting $\theta = 0$ in (1) we get

$$a = b\cos C + c\cos B$$

The relations

$$\left.\begin{aligned} b &= c\cos A + a\cos C \\ c &= a\cos B + b\cos A \end{aligned}\right\} \quad \ldots \quad (2)$$

and

may be inferred from symmetry or obtained by taking $\theta = C$ and $\theta = -B$ respectively.

Substituting $\theta = \frac{\pi}{2}$ in (1) we have

$$-b\sin C + c\sin B = 0$$

or

$$\frac{\sin B}{b} = \frac{\sin C}{c}.$$

Thus we get the sine rule

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} \quad \ldots \quad \ldots \quad (3)$$

Putting $\theta = \frac{C - B}{2}$ in (1) we get

$$a\cos\frac{B-C}{2} = (b+c)\cos\frac{B+C}{2} = (b+c)\sin\frac{A}{2}. \quad \ldots \quad (4)$$

Similarly putting $\theta = \frac{\pi}{2} - \frac{B-C}{2}$ in (1) and simplifying we get

$$a\sin\frac{B-C}{2} = (b-c)\cos\frac{A}{2}. \quad \ldots \quad (5)$$

Dividing (5) by (4) gives

$$\tan\frac{B-C}{2}=\frac{b-c}{b+c}\cdot\cot\frac{A}{2}. \qquad \ldots \qquad \ldots \quad (6)$$

Squaring (4) and (5) and adding we have

$$a^2=(b+c)^2\sin^2\frac{A}{2}+(b-c)^2\cos^2\frac{A}{2}$$

from which we easily derive

$$\sin\frac{A}{2}=\sqrt{\frac{(s-b)(s-c)}{bc}} \qquad \ldots \qquad \ldots \quad (7)$$

$$\cos\frac{A}{2}=\sqrt{\frac{s(s-a)}{bc}} \qquad \ldots \qquad \ldots \quad (8)$$

and therefore

$$\tan\frac{A}{2}=\sqrt{\frac{(s-b)(s-c)}{s(s-a)}} \qquad \ldots \qquad \ldots \quad (9)$$

By eliminating C between (2) and (3) we have

$$b^2=(a-c\cos B)^2+c^2\sin^2 B$$

$$=a^2+c^2-2ac\cos B \qquad \ldots \qquad \ldots \quad (10)$$

B. B. BAGI.

Solutions.

Question 882.

(S. KRISHNASWAMY AIYANGAR) :—Prove that

$$\sum_{n=1}^{\infty} (-1)^{n-1} S_n^2 \frac{(3 \pm \sqrt{7})^n}{n!} = 0$$

where $$S_n = 1 + 2 + 3 + \ldots n$$

Salutions by Sadanand M. K. Kewalramani, R. D. Karve, K. R. Rama Iyer, N. Sankara Aiyar and others.

It can be easily verified that

$$\frac{S_n^2}{n!} = \frac{n^2 (n+1)^2}{4n!} = \frac{1}{4}\left[\frac{1}{(n-4)!} + \frac{8}{(n-3)!} + \frac{14}{(n-2)!} + \frac{4}{(n-1)!}\right]$$

Therefore the sum of the series

$$\Sigma (-1)^{n-1} S_n^3 \frac{x^n}{n!} \frac{1}{4} = (-x^4 + 8x^3 - 14x^2 + 4x)\, e^{-x}$$

$$= -\frac{xe^{-x}}{4} (x-2)(x-3+\sqrt{7})(x-3-\sqrt{7})$$

Hence the series vanishes when $x = 2,\ 3 \div \sqrt{7}$, or $3 - \sqrt{7}$.

Question 1273.

(S. AUDINARAYANAN) :—Fill up the vacant cells in the following square with integers in G. P. such that the continued product of the numbers along each row and column and along the diagonals is the same.

1			
			2

Solutions by P. Krishnamachar and I. Totadri Iyengar

This is analogous to and can be made to depend upon the construction of magic squares in which the sums of the elements in each row, etc., are equal; Here 1 and 2 are given to be two elements of a G. P. The common ratio will be 2 and the several elements are therefore powers of 2. Hence the indices of 2 in each row, etc., must, when added, give the same result. The required square is the following :-

1	2^{11}	2^{7}	2^{12}
2^{14}	2^{5}	2^{9}	2^{2}
2^{13}	2^{6}	2^{10}	2
2^{3}	2^{8}	2^{4}	2^{15}

Question 1293.

(MARTYN M. THOMAS, M.A.) :—If in bipolar co-ordinates the equation of a family of curves be $f(r\,r') = c$, prove that the differential equation of the orthogonal trajectories is

$$r\frac{\partial f}{\partial r}d\,\theta = r'\frac{\partial f}{\partial r'}d\theta'.$$

Solutions by S. M. Shah, Miss Y. Bhate, V. A. Mahalingam, A. Mahadevan, K. C. Shah, M. K. Kewalramani, I. B. Mukherjee, K. N. Srikantha Sastri and several others.

The differential equation of the given family of curves is

$$\frac{\partial f}{\partial r}\frac{dr}{ds} + \frac{\partial f}{\partial r'}\frac{dr'}{ds} = 0 \quad \ldots \quad \ldots \quad (1)$$

Now if ϕ be the angle which the radius vector to a point P makes with the positive direction (s-increasing) of the tangent at P, we have

$$\cos\phi = \frac{dr}{ds}, \quad \sin\phi = \frac{rd\theta}{ds}$$

ϕ being measured *from* the radius vector in the direction of θ increasing.

If $r_1\ \theta_1$ be a point on the orthogonal trajectory, we have with a similar notation

$$\cos\phi_1 = \frac{dr_1}{ds_1}, \quad \sin\phi_1 = \frac{r_1 d\theta_1}{ds_1}.$$

Also for the same point $r = r_1$, $\theta = \theta_1$ and $\phi = \phi_1 \pm \frac{\pi}{2}$

so that

$$\frac{dr}{ds} = \cos\phi = \cos\left(\phi_1 \pm \frac{\pi}{2}\right) = \mp \sin\phi_1 = \mp r_1 \frac{d\theta_1}{ds_1} \ldots \quad (2)$$

Similarly

$$\frac{dr'}{ds} = \cos\phi' = \cos\left(\phi'_1 \mp \frac{\pi}{2}\right) = \pm \sin\phi'_1 = \pm r'_1 \frac{d\theta'_1}{ds_1}. \quad (3)$$

the upper or lower signs being taken in each case, the difference of signs between (2) and (3) being due to θ and θ' being measured positively in opposite directions.

From (1), (2) and (3) we have the orthogonal trajectories given by

$$\frac{df}{dr_1} \cdot r_1 \frac{d\theta_1}{ds_1} = \frac{df}{dr'_1} r'_1 \frac{d\theta'_1}{as_1}, \text{ that is } r_1 \frac{df}{dr_1} d\theta_1 = r'_1 \frac{df}{dr'_1}, d\theta'_1$$

$\frac{\partial f}{\partial r_1}$ and $\frac{\partial f}{\partial r'_1}$, denoting the result of replacing r, r' by r_1, r'_1 after differentiation.

Question 1358.

(V. Ramaswami Aiyar):—Four mutually orthocentric points ABCD are the centres of four mutually orthogonal circles. If any point P invert with respect to these circles into P′Q′R′S′, then the set (P′Q′R′S′) will invert with respect to each of these circles into the same set of points (PQRS).

If G, G′ be the centroids of the sets (PQRS) and (P′Q′R′S′), show that G, G′ are inverses of one another with respect to the nine-point circle of the system A, B, C, D.

Examine what happens to the theorem if P were on one of the circles, say, the circle with centre A.

Solutions (1) *by S. Audinarayanan and* (2) *by A. Narasinga Rao and R. Vaidyanathaswamy.*

(1) The first part of the question may be proved by inverting the four orthogonal circles into two perpendicular lines, the axes of co-ordinates and the two circles $r^2 - a^2 = 0$ and $r^2 + a^2 = 0$. The two sets of points (PQRS) and (P′Q′R′S′) are those whose polar co-ordinates are

$$\left\{ (\pm r, \theta), \pm \left(\frac{a^2}{r}, -\theta\right) \right\} \text{ and } \left\{ \left(\pm \frac{a^2}{r}, \theta\right), (\pm r, -\theta) \right\}$$

and each of the inversions interchanges the two sets.

To prove the second part we begin with the following

Lemma :—

The mid-points of the diagonals of a cyclic quadrilateral are inverse points with respect to the circle on the third diagonal as diameter.

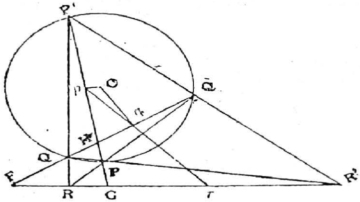

Fig. 1.

Let PQP′Q′ be a cyclic quadrilateral and RR′ its third diagonal. Let O be the centre of the circle, p, q, r the mid-points of the diagonals PP′, QQ′ and RR′, and FGH the triangle formed by the diagonals.

The range (Q FH) Q′, is harmonic.

Hence
$$qF \cdot qH = qQ^2.$$
$$\therefore \quad qH, HF = qQ^2 - qH^2 = OQ^2 - OH^2$$

Likewise
$$pH \cdot HG = pP^2 - pH^2 = OP^2 - OH^2$$
$\therefore$ $pH \cdot HG = qH \cdot HF$ so that p, q, F, G are concyclic.
$$\therefore \quad rp \cdot rq = rF r \cdot G = rR^2 = rR'^2$$

which proves the lemma.

Now let LMN be the mid. points of BC, CA, AB and α, β, γ of

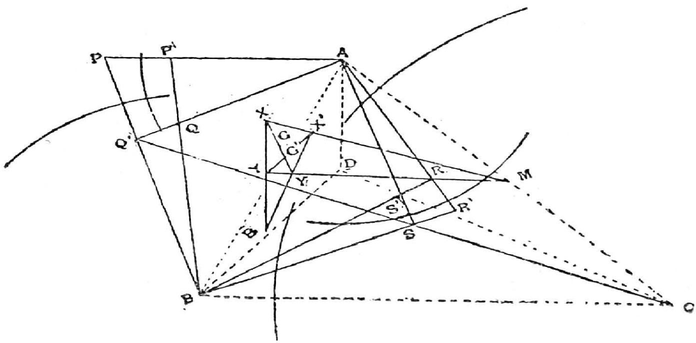

Fig. 2.

AD, BD, CD respectively. Let PP′, QQ′, RR′, SS′ be pairs of inverse points in the circle centre A. Also let X, X′ Y, Y′ be the mid.-points of PR, P′R′, QS, Q′S′ respectively. Applying the lemma to the cyclic quadrilateral PP′ RR′ we deduce that the mid-points of PR, P′R′, *viz.*, X and X′ are inverse points with respect to the circle on the third diagonal AC as diameter. Similarly since QQ′ SS′ are concyclic, the points YY′ are inverse with respect to circle on AC as diameter. Therefore XX′ YY′ are concyclic.

Now in the quadrilateral P′QR′S the mid.-points of the three diagonals are X′, Y, and β and so are collinear. Similarly X,Y′ and β. Hence we have the following collinearities :—

$$XX' M$$
$$YY' M$$
$$XY' \beta$$
$$X'Y \beta$$

whence it is clear that the third diagonal of XX′ YY′ is Mβ.

Hence applying the lemma to the cyclic quadrilateral XX′YY′ we see that the mid-points of XY and X′Y′ are inverse with respect to the circle on Mβ as diameter, *i.e.*, G, G′ are inverse with respect to the nine-point cirle of ABCD.

When P is a point on the circle centre A, P′ coincides with P; and since the circles are orthogonal, QRS also lie on the same circle and the two systems of points co-incide. We have then the result given by the proposer in Question 1122 which with a change of lettering runs thus :—

" If the joins of four concyclic points PQRS taken in opposite pairs intersect in BCD, prove that the nine-points circle of BCD passes through the centroid of the points PQRS."

(2) Let $O_1O_2O_3O_4$ be a set of four operators which are commutative and satisfy the relations

$$O_1^2 = O_2^2 = O_3^2 = O_4^2 = O_1O_2O_3O_4 = 1. \qquad \ldots \quad (1)$$

From (1) it follows that

$$O_1 \quad = O_1O_2^2O_3^2O_4^2 = O_1O_2O_3O_4 . O_2O_3O_4 = O_2O_3O_4$$
$$O_1O_2 = O_1O_2O_3^2O_4^2 = O_1O_2O_3O_4 . O_3O_4 \quad = \quad O_3O_4,$$

etc.

Hence the group generated by the four operators is of order 8 and consists of

$$1 = O_1O_2O_3O_4 \quad \ldots \quad \ldots \quad (2)$$

$$O_1 = O_2O_3O_4, \quad O_2 = O_1O_3O_4, \quad O_3 = O_1O_2O_4, \quad O_4 = O_1O_2O_3 \quad (3)$$

$$O_1O_2 = O_3O_4, \quad O_1O_3 = O_2O_4, \quad O_1O_4 = O_2O_3. \quad \ldots \quad (4)$$

These may be divided into two sets of four, the one consisting of those in (2) and (4) which are expressible only as the products of an even number of operations—and the other of those which are expressible only as the products of an odd number of operations. Any operation of the group carries each set into itself or interchanges the two sets according as itself belongs to the first or second set.

We have an example of such a group if the four operators are inversions in four mutually orthogoual circles. The commutative property is readily perceived by inverting two circles into two straight lines in which case inversion reduces to reflexion, and such reflexions are commutative when the two straight lines are at right angles. It is also easy to prove that successive inversion with respect to the four circles carries each point into itself so that $O_1O_2O_3O_4 \equiv 1$.

It follows then that such an inversion group carries each point P into two sets PQRS and P′Q′R′S′ each of which is carried into itself or into the other set by any of the four inversions.

A second example of such a group is that generated by the four collineations

$$\left.\begin{array}{llllll} x_1' : x_2' : x_3' : x_4' = & -x_1 : x_2 & : x_3 & : x_4 & \ldots O_1 \\ \ldots \quad \ldots \quad \ldots = & x_1 & : -x_2 : x_3 & : x_4 & \ldots O_2 \\ \ldots \quad \ldots \quad \ldots = & x_1 & : x_2 & : -x_3 : x_4 & \ldots O_3 \\ \ldots \quad \ldots \quad \ldots = & x_1 & : x_2 & : x_3 & : -x_4 \ldots O_4. \end{array}\right\} \quad \ldots \quad (5)$$

All the operators of this group have

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = 0 \quad \ldots \quad \ldots \quad (6)$$

for an invariant quadric and carry any point $x_1x_2x_3x_4$ into twc tetrahedra which are carried either each into itself or each into the other by every collineation of the group. This is the well-known configuration of three desmic tetrahedra (the third being the reference tetrahedron) of which any two are in quadruple perspective with the vertices of the third as the 4 centres of perspective.

The isomorphism of these two geometrical groups is brought into evidence by taking $x_1x_2x_3x_4$ as the tetracyclic co-ordinates of a circle in a plane with the identical relation (6). Since the reference tetrahedron is self-polar its vertices correspond to four orthogonal circles—the four inversion circles—and the collineations in (5) since they satisfy (6) are *point transformations* which carry each point into its inverse with respect to the corresponding circle.

Similar considerations apply to a group of any even nnmber n of commutative operators $O_1O_2 \dots O_n$ satisfying,

$$O_1^2 = O_2^2 = \dots O_n^2 = O_1O_2 \dots O_n = 1$$

and may be utilised to prove a corresponding property for inversion with respect to $n+2$ orthogonal hyperspheres in n-dimensional space, n being any *even* integer.

To prove the second part of the question, let the plane of the circle be regarded as the Argand plane of a complex variable $z = (x + iy)$ and let us denote by $\bar{z}$ the conjugate variable $(x - iy)$. These are images of each other in the real axis, and each determines the other uniquely so that any real point in the plane may be specified either by its z or its $\bar{z}$.

The equation of a real circle is given by a bilinear form of the type

$$\gamma z\bar{z} - \bar{\alpha} z - \alpha \bar{z} - \beta = 0 \qquad \dots \qquad \dots \qquad (7)$$

where β, γ are real and α, $\bar{\alpha}$ conjugate complex numbers. Two points ξ, η are inverse points* with respect to this circle if

$$\gamma \xi \eta - \bar{\alpha} \xi - \alpha \bar{\eta} - \beta = 0$$

that is

$$\xi = \frac{\alpha \bar{\eta} + \beta}{\alpha \bar{\eta} - \bar{\alpha}} \qquad \dots \qquad \dots \qquad (8)$$

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* *Vide* Coolidge : *Treatise on the Circle and the Sphere*, Chap. VIII.
or Vivanti : *Fonctions polyedriques*, pp. 43-44.

Conversely, it is easy to prove that the transformation

$$\xi = \frac{\alpha \bar{\eta} + \beta}{\gamma \bar{\eta} + \delta} \quad \ldots \quad \ldots \quad (9)$$

does not in general have more than two fixed points but that when it has more, then $\alpha + \bar{\delta}$ (or $\bar{\alpha} + \delta$) $= 0$ and β, γ are real. Then every point on the circle (7) is a fixed point, ξ and η are inverse with respect to the circle.

Consider now real tetrads of points like PQRS. The complex numbers ξ corresponding to such a tetrad are the roots of a quartic in ξ and since each tetrad is uniquely determined by any one of its members, these quartics belong to a pencil

$$f(\xi) + \lambda\, \phi(\xi) = 0 \quad \ldots \quad \ldots \quad (10)$$

The centroid of the tetrad (10) is given by

$$g = \frac{\text{sum of the roots}}{4} = \frac{a\lambda + b}{c\lambda + d}, \text{ say.} \quad \ldots \quad (11)$$

To get the inverse tetrad (P′Q′R′S′) we invert (PQRS) with respect to one of the given circles, say (7), *i.e.*, eliminate ξ between (8) and (10) and thus obtain

$$f_1(\bar{\eta}) + \lambda\, \phi_1(\bar{\eta}) = 0 \quad \ldots \quad \ldots \quad (12)$$

If g' be the centroid of this set P′Q′R′S′, we have

$$\bar{g}' = \frac{\bar{\eta}_1 + \bar{\eta}_2 + \bar{\eta}_3 + \bar{\eta}_4}{4} = \frac{a_1\lambda + b_1}{c_1\lambda + d}, \text{ say} \quad \ldots \quad (13)$$

On eliminating λ between (11) and (13) we have a relation of the type

$$\bar{g}' = \frac{pg + q}{rg + s} \quad \ldots \quad \ldots \quad (14)$$

connecting the two centroids g and g'.

Now (14) has an infinite number of fixed points as the two tetrads coincide whenever one and therefore all the points PQRS lie on one of the inversion circles. Hence g and g' are inverse with respect to some circle Γ. In particuiar, if P be at one of the intersections L, M of two of the circles, the two tetrads coalesce into (LLMM) and hence the mid-point of LM lies on C but as the circles are orthogonal and have their centres at A, B, C, D, this mid-point is the intersection of a pair of opposite sides of the quadrangle ABCD, Γ circumscribes the harmouic triangle and is thus identified with the nine-point circle of ABCD.

Questions for Solution.

Proposers of Questions are requested, whenever possible, to send their own solutions along with their questions.

1431. **(Corrected)**: (V. RAMASWAMI AIYAR):—"If two triangles be such that the sides of either are pedal lines with respect to the other, show that the six sides of the triangles touch a conic whose centre is the middle point of the join of the circum-centres of the triangles.

1434. (V. RAMASWAMI AIYAR):—O is the circum-centre of a *spherical* triangle ABC; and Σ is the circle passing through the reflexions of O in the sides BC, CA, AB. If PQ be the diameter of the in-circle, or any ex-circle, of ABC which passes through O; and P', Q' be the reflexions of O in P, Q; show that the circle described on $P'Q'$ as diameter touches the circle Σ.

More generally, if the axis PQ of a sphero-conic inscribed in ABC passes through O, and P', Q' be the reflexions of O in P, Q, then the circle described on $P'Q'$ as diameter touches the circle Σ.

1435. (K. V. VEDANTAM):—If L, M, N are the mid-points of the sides of a triangle and D, E, F are the points of contact of the in-circle with the sides BC, CA, AB and O_1, O_2, O_3 are the ortho-centres of the triangles AMN, BNL, CLM and O'_1, O'_2, O'_3 are the ortho-centres of the triangles AEF, BFD, CDE.

(i) $O_1O'_1$, $O_2O'_2$, $O_3O'_3$, are concurrent at the Feuerbach point.

(ii) $$\frac{O_1O'_1}{R\cos A} = \frac{O_2O'_2}{R\cos B} = \frac{O_3O'_3}{R\cos C} = \frac{SI}{R}.$$

1436. (B. B. BAGI):—A straight line OAB through a given point O meets two given planes P_1, P_2 in A and B. If $\{OO', AB\} = -1$ the point O' traces a plane (P_1P_2).

If there are three given planes $P_1P_2P_3$ then (P_2P_3), (P_3P_1), (P_1P_2), meet P_1, P_2, P_3 in straight lines lying in a plane $(P_1P_2P_3)$. If there are four planes $P_1P_2P_3P_4$ then the four straight lines in which $(P_2P_3P_4)$, etc., meet P_1 etc., lie in a plane $(P_1P_2P_3P_4)$ and so on.

Obtain a similar theorem in plane geometry.

By drawing all planes to pass through a given goint and cutting them by a sphere with its centre at that point, obtain a theorem in spherical Trigonometry.

If the planes $P_1P_2P_3P_4$ form a tetrahedron whose pairs of opposite edges are at right-angles and the point O is the *orthocentre*, prove that the plane $(P_1P_2P_3P_4)$ is perpendicular to the join of the centroid and the circum-centre of the tetrahedron.

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LIST OF JOURNALS RECEIVED

during the months of May and June 1926.

Journals.

1 Acta Mathematica, **47**, 3 and 4, and **48** complete.
2 American Mathematical Monthly, **1** & **2** (2 copies).
3 Annales de L'Ecole Normale Supérieure, Jan. 1926.
4 Astrophysical Journal, January 1926.
5 Bulletin of the American Mathematical Soc. **33**, 1.
6 Bulletin des Sciences Mathematiques, **2**, 3.
7 Crelle's Journal, January 1926.
8 Japanese Journal of Mathematics, **2**, 3.
9 Journal de Mathe. pures et appliques, **5**, 2.
10 Liouville's Journal, **5**, 1.
11 Mathematische Annallen, **96**, 5.
12 Messenger of Mathematics, **55**, 10 (2 copies).
13 Monthly Notices of the Royal Astronomical Society, **86**, 4.
14 Nature, **117**. 2945, 2947, 2948.
15 Philosophical Magazine, **1**, 4.
16 Popular Astronomy, **34**, 4, 5 (2 copies each).
17 Proceedings of the Royal Society, **110**, 756.
18 Proceedings of the Edinburgh Mathematical Society, **44**, 1.

Pamphlets.

1 Bulletin de la classes des Sciences Physiques et Mathematiques **1**, 3, 4.
2 Die Bausteine des Gouv. Kiew.
3 Die Lagerstatten.
4 Das Uferland des Flusses Sherew.
5 Nieuw Archief voor Wiskunde Deel, XV.
6 Sur la preparation Electrochimique de L'Arsente de Plomb.

Numbers in black type refer to the volume, and those in ordinary type to the specific number of the issue.

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