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A paper should contain a short and clear summary of the new results obtained and the relations in which they stand to results already known. Contributors are requested to bear in mind that, at the present stage of mathematical research, hardly any paper is likely to be so completely original as to be independent of earlier work in the same direction; and that readers are often helped to appreciate the importance of a new investigation by seeing its connection with earlier results.

The principal results of a paper should, when possible, be enunciated separately and explicitly in the form of definite theorems.

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ON THE HISTORY OF FINITE ABSTRACT GROUPS.

BY G. A. MILLER.

The prototype of the abstract group is the special substitution group which is often called permutation group. One of the most fundamental theorems of abstract group theory is that there is one and only one cyclic group of order g, where g represents an arbitrary positive integer. It was stated by A. Cayley,* but since no actual definition of the term abstract group was then known, it is clear that no satisfactory proof thereof could be given at that time. A considerable number of other fundamental theorems of abstract group theory were stated long before a satisfactory general definition of the term 'abstract group' was formulated; for instance, A. Cayley determined the five possible abstract groups of order eight before this time. What is still more interesting is the fact that A. Cayley gave the proof which is now commonly found in text-books, of the fundamental fact that every finite abstract group can be represented as a regular substitution group before a general definition of the term abstract group existed.

When postulates upon which definitions relating to finite abstract groups could be based were finally formulated, they were so selected as to be in accord with the laws of combinations of substitutions, and hence they affected the development of abstract group theory very little except that they enhanced greatly the elegance with which its elements could be presented. The remarkable simplicity of these postulates and of the definitions of the term abstract group based thereon tended also to exhibit the wide application of the subject, but the legitimacy of the operations in abstract groups had never been questioned since it was seen that they are allowable because they are in accord with the laws of combinations of substitutions. In a certain sense abstract group theory is not as general as the older substitution group theory since there is a (1,1) correspondence between the regular substitution groups and the possible abstract groups, while the regular

* A. Cayley: *Philosophical Magazine*, Vol. 7 (1854), p. 40.

substitution groups constitute a sub-class of all the possible substitution groups.

While the substitution group may be regarded as the prototype of the abstract group it should be noted that various important developments in abstract group theory have not had their source in the theory of substitutions One of the most remarkable of these relates to the groups of movements of the regular polygons and of the regular polyhedrons. The facts that two group operators of order 2 always generate a dihedral group and that the order of this group is always equal to twice the order of the product of these two operators is so elementary and so closely related to obvious properties of regular polygons that it is difficult to determine who first noted it. A definite proof thereof could obviously not be given before a definition of the term 'abstract group' was formulated. L. Kronecker seems to have been the first to give a satisfactory definition of an abstract abelian group (1870) and twelve years later H. Weber gave such a definition of a general abstract group. In his first paper on group theory, to which we referred above, A. Cayley said that "a set of symbols, all of them different, and such that the product of any two of them (no matter in what order), or the product of any one of them into itself, belongs to the set, is said to be a *group*." This is evidently an incomplete definition.

Since A. Cayley assumed implicitly the ordinary postulates relating to the combinational laws of abstract group operators his results were usually correct. A singular exception appears in his article which was published in the first volume of the *American Journal of Mathematics*, 1878, page 51, where it is incorrectly stated that there are three abstract groups of order 6, and illustrative examples of what were supposed to be three distinct groups of this order were given. Since this oversight relates to such a fundamental and elementary question it throws light on the backward state of the theory of abstract groups at that time. Almost twenty years earlier A. Cayley had stated results which seem to imply that he understood that two group operators of order 2 always generate the dihedral group whose order is twice the order of the product of these two operators. In view of its very elementary nature it is likely that this fundamental theorem of abstract groups was known by many mathematicians at a much earlier date. A. L. Cauchy had determined at an earlier date that the cyclic substitution group of order g has $\phi(g)$

generators, wher $\phi(g)$ represents the totient of g. Some of the most fundamental advances in abstract group theory, as well as in other mathematical subjects were made by almost intangible steps and hence it is difficult to give due credit.

The multiplication table of A Cayley which may be associated with every finite group, and which is commonly regarded as one of the earliest illustrations of a general method relating to finite abstract groups, is in reality nothing more than a regular substitution group. The law of combinations of the operators of an abstract group may therefore be regarded as being represented by a regular substitution group. The group of isomorphisms of a given group G is an abstract group but the interchanges of the elements of G corresponding to the various possible automorphisms of G raise exactly the same questions as regards transitivity and intransitivity, primitivity and imprimitivity, etc.. as were raised in the earlier developments of substitution groups. While there is therefore no clear line of division between the theory of abstract groups and the older theory of substitution groups, yet the classification has served a useful purpose, just as the division of pure mathematics into Algebra, Analysis, and Geometry has been convenient even if there is no clear line of division between these various subjects.

The term general group is sometimes used for an abstract group. This is done, for instance, in the *Encyclopédie des Sciences Mathématiques*. It should, however, be noted that the letters of a substitution group are mere symbols which may represent various concrete objects and hence it is difficult to see how anything could be more general than a substitution group. On the other hand, if the attention is centered on the laws according to which the substitutions combine it is true that, for the moment at least, one has a different and more general view of the group than if one thinks of the symbols by which the group is represented. This difference of view is, however, only superficial since the laws according to which the substitutions combine when translated into symbols are again substitutions. Hence the statement that the substitution group is the prototype of the abstract group should perhaps be extended by adding that most of the problems of the abstract groups, in turn, give rise to problems in substitution groups.

The developments in abstract group theory which seem at first to be most remote from the theory of substitutions and which had their source in a different field are those which relate to generational relations. The first important steps along this line, besides the very elementary ones noted above, were due to W. R. Hamilton, and were based upon the fact that every finite group may be regarded as a special algebra, or calculus. In 1843 he discovered the well-known quaternion group, whose eight operators may be associated with the four quaternion units, and about ten years later he directed attention to the very simple abstract generational definitions of the group of movements of the regular polyhedrons. He observed that if A_1, A_2 are two group operators of orders 2 and 3 respectively such that their product is of order 3, then they generate 12 operators which combine in exactly the same way as the 12 movements of space which interchange various parts of a regular tetrahedron but leave invariant this solid as a whole. On the other hand, if the product of A_1 and A_2 is of order 4, they generate the group of order 24 which leaves invariant a particular cube, as well as a particular octahedron. Finally, if this product is of order 5 the group generated by A_1 and A_2 is of order 60 and leaves invariant a particular icosahedron, or a particular duodecahedron.

It should be noted that while the laws according to which W. R. Hamilton operated in these investigations are laws of operation in the modern abstract group theory, the postulates of this theory had not then been formulated and his work naturally lacked clearness and definiteness. It tends to show, however, how natural the laws of operation in abstract group theory really are since they are so intimately connected with the transformations of the solids which played a fundamental role already in the history of Greek mathematics. It is thus seen that while the theory of substitution groups constitutes the prototype of abstract group theory this theory was also profoundly influenced by developments which were, not directly connected with the theory of substitution groups. On the other hand, this theory can be used to advantage in proving that the groups in question are actually defined by the relations noted above.

The very interesting results of Hamilton were extended along various lines. Quite recently all the possible groups which can be generated by two operators of order 3 whose product is also of order 3 have been determined. It has been found that these groups constitute a very elementary infinite category such that each group is composed of an Abelian invariant sub-group of index 3, and that all the operators of the group which do not appear in this invariant sub-group are of order 3. Moreover, wherever this sub-group is non-cyclic it can be generated by two operators.* While two operators of order 2 whose product is also of order 2 generate the single non-cyclic group of order 4, two operators of order 3 whose product is of order 3 may generate any one of an infinite system of finite groups. The groups which can be generated by two operators of order $n > 3$ whose product is of order n have not yet been determined and seem to present great difficulties even for $n = 4$ While many very elegant results have been obtained by the consideration of generational relations along the line followed by W. R. Hamilton, these results appear as yet to be somewhat fragmentary.

We have thus far practically confined our attention to advances in abstract group theory which were not made directly from the stand-point of the theory of substitut on groups and which lacked definiteness because the necessary postulates were lacking. Such indefiniteness is very common in mathematical literature. For instance, H. Cardan tried to prove, a quarter of a century after the publication of his noted *Ars Mgna* (1545), that it was incorrect to say that the product of $-a$ and $-b$ is $+ab$, and C. Clarius, in speaking of the law of signs as regards a negative multiplier in his *Algebra* (1608) said that "one must attribute it to the weakness of the human spirit that it cannot comprehend why it is true." As a matter of fact one cannot prove such things without formulating the appropriate postulates. It can, however, be proved that the common methods are *allowable* in the sense that the procedure can be associated with obvious geometric operations, but even this was not done with respect to the fundamental point in question until about the beginning of the nineteenth century. †

A number of fundamental advances in abstract group theory were made by men who seem to have confined their attention to substitution groups while developing methods which apply also to abstract groups. For instance, the arrangement of all the substitutions of a group in a rectangular form such that the first row is composed of all the substitu-

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* G. A Miller: *Proceedings of the National Academy of Sciences*, Vol. 13 (1927). p. 170.

† F. Klein : *Elementarmathematik vom höheren standpunkt aus*, (1924), p. 29.

tions of a sub-group was used by P. Abbati at about the beginning of the nineteenth century By means of this arrangement it can readily be proved that the order of an abstract group is divisible by the order of each of its sub-groups. This theorem is commonly attributed to Lagrange, who failed, however, to give a complete proof thereof. In fact, neither Lagrange nor Abbati seem to have thought of this theorem as extending beyond the domain of substitution groups. The work of E. Galois, relating to invariant sub-groups and the corresponding quotient groups was also apparently developed without any view to an abstract theory. In giving credit for the proof of theorems in abstract group theory to people whose attention was confined to substitution groups while proving these theorems, one is evidently in danger of conveying an impression which is not quite accurate. Such pious fraud can scarcely be avoided in historical matters relating to modern mathematics as was pointed out by F. Klein in the introduction to his excellent recent work entitled *Vorlesungen über die Entwicklung der Mathematik im* 19 *Jahrhundert*, (1926).

ON THE ANALYTIC CONTINUATION OF FUNCTIONS REPRESENTED BY A CLASS OF DIRICHLET'S SERIES.*

By P. L. Srivastava.

Part I.

1. Let †

$$(1.1)\qquad \sum_{p}^{\infty} \phi(n)\, e^{-sf(n)}, \qquad (p, \text{ a positive integer}),$$

where $s = \sigma + it$, be a Dirichlet's series. Then the main object of this paper is to study the analytic continuation of functions represented by a class of Dirichlet's Series for which a relation of the type

$$(1.2)\quad H(s) = \sum_{p}^{\infty} \phi(n)\, e^{-sf(n)} = G(s) + \int_{p}^{\infty} \phi(x)\, e^{-sf(x)} dx = G(s) + J(s),$$

where $G(s)$ is an integral function of s, is true; that is, for which the function $H(s)$ has no other finite singularities than those of $J(s)$.

Now Dr. Cramér, observing that (1.2) is true of the series $\sum_{1}^{\infty} n^{-s}$, showed ‡ that it is also true of a general class of series

$$\sum_{1}^{\infty} \phi(n)\, n^{-s},$$

which is characterised by the following two conditions:—

A. $\phi(x)$ *possesses derivatives of every order*, $\qquad (x \geqslant 1)$;

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* This paper forms part of my thesis for the degree of Doctor of Philosophy at Oxford My thanks are due to Prof. G. H. Hardy for his kind assistance and supervision, without which this work would not have been possible.

† We shall take this to be the standard form throughout.

‡ '*Sur une classe de series de Dirichlet*,' Thèse pour le doctorat, Uppsala, 1917, Theorem I.

B. *There exists a real constant k, such that for sufficiently large values of x,*

$$\phi^{\mu}(x) = O(x^{k-\mu c}), \qquad (\mu = 0, 1, 2, 3 \ldots),$$

c being a positive number $\leqslant 1$.

This is the fundamental result of Cramér's thesis, and has been the starting point of my work.

Now the question that I have attempted to answer here is whether we can extend Cramér's method of studying the ordinary Dirichlet's series $\sum_{1}^{\infty} \phi(n)\, n^{-s}$ to Dirichlet's series of types other than $\log n$. Thus I prove a theorem from which Dr. Cramér's theorem follows as a special case.

THEOREM I.—*Suppose in (1.1) the function* $\phi(x)$ *satisfies, for* $x \geqslant p$, *conditions A and B. Let the function* $f(x)$ *satisfy the following conditions* :—

C. *for* $x \geqslant p$, $f(x)$ *is a positive and steadily increasing function of a real variable x, and possesses* differential co-efficients of every order ;*

D.† $f(x) \sim \log x, \quad f^{\mu}(x) = O(x^{-\mu})$* $\qquad (\mu = 1, 2, 3, \ldots)$;
then the relation (1.2) holds.

Both the Dirichlet's series and the corresponding integral are absolutely and uniformly convergent in the half-plane $\sigma > 1 + k$. What we have to prove is that the difference between the analytic functions represented by them is an integral function of *s*.

To do this, we use the Euler-Maclaurin Sum-formula,‡ *viz.*
If $F(x)$ possesses continuous derivatives of the first $(2h + 1)$ orders,

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* These conditions are fulfilled, in particular when $f(x)$ is an L-function, satisfying $f(x) \sim \log x$. For the definition of an L-function, see G. H. Hardy, *Orders of Infinity*,' Cam. Math. Tracts, No. 12, p. 17.

† In such equalities, it will always be understood that $x \to \infty$.

‡ Lindelöf, '*Le calcul des résidus*' pp. 78 *et seq.*

for $p \leqslant x \leqslant N$, h being a positive integer, then

$$(1.3) \qquad \sum_{p}^{N} F(\nu) - \int_p^N F(x)\,dx = \tfrac{1}{2}\{F(p) + F(N)\}$$

$$+ \sum_{\nu=1}^{h} (-1)^{\nu} \frac{B_\nu}{(2\nu)!} \{F^{(2\nu-1)}(p) - F^{(2\nu-1)}(N)\} + R(h, N),$$

where $R(h, N)$ is given by either of the expressions

$$(-1)^h \int_p^N P_{2h}(x)\, F^{(2h)}(x)\,dx; \quad (-1)^{h+1} \int_p^N P_{(2h+1)}(x)\, F^{(2h+1)}(x)\,dx,$$

and where B_ν's denote Bernoulli's numbers, and

$$P_{(2h)}(x) = \sum_{n=1}^{\infty} \frac{\cos 2n\pi x}{2^{(2h-1)}(\pi n)^{2h}}, \quad P_{(2h+1)}(x) = \sum_{n=1}^{\infty} \frac{\sin 2n\pi x}{2^{2h}(\pi n)^{(2h+1)}}.$$

The series for $P_r(x)$, $r > 1$, are absolutely and uniformly convergent and hence bounded, for every value of x.

Now suppose $\qquad F(x) = \phi(x)\, e^{-sf(x)}$

in the formula (1.3), and let us assume, for the moment, that s is bounded and $\sigma > 1 + k$, so that N can be made to tend to infinity on the left-side of (1 3). Now let us examine how the terms involving N on the right side of (1.3) behave, as $N \rightarrow \infty$.

Denote the factor $e^{-sf(x)}$ by $\psi(x)$ so that, by Leibnitz's formula we have

$$F^{r}(x) = \sum_{\mu=0}^{r} {}^{r}C_{\mu}\, \psi^{(r-\mu)}(x)\, \phi^{(\mu)}(x).$$

Now, by actual differentiation, we can easily see that

$$(1.4) \quad \psi^{(r-\mu)}(x) = e^{-sf(x)} \sum A (f^{(m)})^{k_1} (f^{(n)})^{k_2} (f^{(p)})^{k_3} \ldots,$$

such that for each term of the series,

$$mk_1 + nk_2 + pk_3 + \ldots = r - \mu,$$

and the co-efficient A is independent of x, and involves powers of s only. By virtue of condition D, we have, therefore

$$|\psi^{r-\mu}(x)| = e^{-\sigma f(x)}\, O\left(\frac{1}{x^{r-\mu}}\right), ^*$$

s being bounded.

Now making use of condition B, we have

$$F^{r}(x) = \sum_{\mu=0}^{r} {}^{r}C_{\mu}\, O\left(x^{k-\sigma-r+\mu(1-c)}\right),$$

for sufficiently large values of x, so that, when $\sigma > 1 + k$,

$$F^{(r)}(x) \to 0, \text{ as } x \to \infty, \qquad (r = 0, 1, 2, \ldots)$$

Also $$F^{(r)}(p) = e^{-sf(p)}\, u_{(r)}(s),$$

where $u_r(s)$ denotes a polynomial in s of degree r.

Writing

$$J(s) = \int_p^\infty \phi(x)\, e^{-sf(x)}\, dx,$$

we have

(1.5) $$G(s) = H(s) - J(s)$$

$$= \tfrac{1}{2}\phi(p)\, e^{-sf(p)} + \sum_{\nu=1}^{h} (-1)^{\nu} \cdot \frac{B_\nu}{(2\nu)!} \left(e^{-sf(p)}\, u_{2\nu-1}(s)\right) + R(h, \infty),$$

$$= R(h, \infty) + e^{-sf(p)} \left\{ \tfrac{1}{2}\phi(p) + v_{(2h-1)}(s) \right\},$$

where $v_{(2h-1)}(s)$ also denotes a polynomial in s of degree $(2h - 1)$

The equation (1.5) has been obtained on the assumption that s is bounded and $\sigma > 1 + k$, but now we proceed to show that the right-hand side of it represents an integral function of s, so that G (s) is an integral function of s.

Now to do this, it is evidently sufficient to prove that $R = R(h, \infty)$ represents an integral function of s.

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* The constant implied in the symbol O is independent of x only.

Denoting by q either of the numbers $2h$ or $(2h+1)$, we have

$$R = \pm \int_p^{\infty} P_q(x)\, F^q(x)\, dx, \qquad P_q(x) = O(1), \text{ for } q > 1.$$

[For our purpose, it is not necessary to specify the sign before the last integral.]

Now, as before,

$$F^{(q)}(x) = \sum_{\mu=0}^{\infty} {}^qC_\mu\, \psi^{(q-\mu)}(x)\, \phi^{(\mu)}(x),$$

where

$$\psi^{(q-\mu)}(x) = e^{-sf(x)} \sum_{r=0}^{q-\mu} E_r\, s^r,$$

where

$$E_r = O\left(\frac{1}{x^{q-\mu}}\right),$$

for every value of r, the constant implied by O being independent of both x and s. This will be evident, if we write the right side of (1.4) as a polynominal in s, so that

$$F^q(x) = \sum_{\mu=0}^{q} {}^qC_\mu\, \phi^\mu(x)\, e^{-sf(x)} \sum_{r=0}^{q-\mu} E_r\, s^r,$$

and

$$\text{(1.6)} \qquad R = \pm \sum_{\mu=0}^{q} {}^qC_\mu \sum_{r=0}^{q-\mu} s^r \int_p^{\infty} P_q(x)\, \phi^{(\mu)}(x)\, e^{-sf(x)}\, E_r\, dx,$$

$$= \pm \sum_{\mu=0}^{q} {}^qC_\mu \sum_{r=0}^{q-\mu} s^r \int_p^{\infty} O\left(x^{k-\mu c-\sigma-q+\mu}\right) dx,$$

so that each integral on the right side of (1.6) is absolutely and uniformly convergent throughout the region

$$\text{(1.7)} \qquad \sigma \geqslant 1 + k - qc + \varepsilon, \qquad \text{for every } \varepsilon > 0.$$

Hence the right side of (1.6) is regular and uniform* all over the region (1.7), and, as q is arbitrary, it is an integral function of s.

* Goursat : "*Functions of a Complex Variable*," § 95, p 227.

This completes the proof of Theorem I.

As an application of the theorem, consider the series

$$H(s) = \sum_{3}^{\infty} e^{Ai(\log n)^{\alpha}} (n \log n)^{-s}, \qquad A \neq 0,\ \alpha > 1,$$

to which Cramér's theorem is not applicable.

By Theorem I, we have

$$H(s) = G(s) + J(s),$$

where $G(s)$ is an integral function of s, and

$$(1.8)\quad J(s) = \int_{3}^{\infty} e^{Ai(\log x)^{\alpha}} (x \log x)^{-s} dx = \int_{\log 3}^{\infty} e^{Aiy^{\alpha}-(s-1)y}\, y^{-s} dy.$$

Now I wish to show that $J(s)$ and $H(s)$ are integral functions of s.

The integral for $J(s)$ is absolutely convergent if $\sigma > 1$. We may, therefore, suppose, for the moment, that s is real and greater than 1. Put $y = re^{i\theta}$, and suppose $A > 0$. Now consider the integral

$$I \equiv \int_{0}^{\beta} e^{Ai(re^{i\theta})^{\alpha}-(s-1)re^{i\theta}} (re^{i\theta})^{-s}\, re^{i\theta}\, i\, d\theta,$$

where β is a positive number such that $\beta \leqslant \frac{\pi}{2}$, $\alpha\beta < \pi$.

Then

$$|I| \leqslant \int_{0}^{\beta} \exp\left\{-Ar^{\alpha}\sin\alpha\theta - (s-1)r\cos\theta\right\} r^{(1-s)}\, d\theta$$

$$\leqslant \beta\, r^{(1-s)}\, e^{-(s-1)r\cos\beta},$$

which vanishes when r becomes infinite.

It follows that the integral (1.8) may be taken along the circle $|y| = \log 3$ (as intercepted by the vectors $\theta = 0$, and $\theta = \beta$), and then along the radius vector $\theta = \beta$, in the plane of the complex variable $y = re^{i\theta}$, instead of along the real axis. That is,

(1.9)

$$J(s) = \int_{\log 3}^{\infty} \exp\left\{Ai(re^{i\beta})^{\alpha} - (s-1)re^{i\beta}\right\} (re^{i\beta})^{-s} e^{i\beta}\, dr + Q(s),$$

where $Q(s)$ is the contribution of the integral (1.8) taken along the circle

$$|y| = \log 3, \quad \theta = 0, \quad \theta = \beta.$$

Now the second term on the right-side of (1.9) represents* an integral function of s. Also the integral

$$\int_{\log 3}^{\infty} \left| \exp\left\{ A i r^{\alpha} e^{i\alpha\beta} - (s-1) r e^{i\beta} \right\} (r e^{i\beta})^{-s} e^{i\beta} \right| dr$$

$$= \int_{\log 3}^{\infty} r^{-\sigma} \exp\left\{ -A r^{\alpha} \sin \alpha\beta - (\sigma - 1) r \cos\beta + t r \sin\beta + \beta t \right\} dr,$$

which is uniformly convergent throughout any finite domain of values of s, since $\alpha > 1$, and $\sin \alpha\beta > 0$. Therefore $J(s)$ is an integral function of s, and so is $H(s)$.

The modification required for the case $A < 0$ is obvious.

When $0 < \alpha \leqslant 1$, the function $H(s)$ is regular and uniform in the plane of s, when it has been cut by a line drawn from the point $s = 1 + Ai$, or $s = 1$,† according as $\alpha = 1$, or < 1, parallel to the negative real axis.

2. We may, by the way, add a word about the order of $G(s)$.

THEOREM II. *The function $\mu(\sigma)$‡ relative to $G(s)$ in Theorem I is always finite, and when it is different from zero, it is a positive, decreasing, convex, and continuous function of σ. We have*

$$\mu(\sigma) = 0, \quad \sigma \geqslant 1 + k,$$

and

$$0 \leqslant \mu(\sigma) \leqslant \frac{(k + 1 - \sigma)}{c} \qquad \sigma \leqslant 1 + k.$$

This is the analogue of Cramér's Theorem II, (2), § and may be established in the same way.

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* Goursat : *loc. cit.* § 95.

† which is the sole finite singularity.

‡ Hardy and Riesz : *The general theory of Dirichlet's series*, p. 14, *et seq.*

§ *loc. cit.*, p 15.

3. The next question is whether it is possible to apply Cramér's method to series such as

$$\sum^{\infty} \phi(n)\, e^{-s(\log n)^{\alpha}},\ (1 \geqslant \alpha > 0),\ \sum^{\infty} \phi(n)\, e^{-sn^{\beta}},\ (1 \geqslant \beta > 0),$$

by imposing suitable restrictions on $\phi(n)$. The answer is that the application of the Euler-Maclaurin Sum-formula, which is the essence of Cramér's method, is successful, if the type of the Dirichlet's series, besides satisfying some suitable conditions, is $O(\log n)$. In Theorem I, the type $f(n) \sim \log n$, and so, we next consider the case

when $$f(n) = O\left((\log n)^{\alpha}\right), \qquad 0 < \alpha < 1.$$

By putting $F(x)$ equal to

$$e^{-s(\log x)^{\alpha}},\ (\alpha > 1),\ \text{ or } e^{-sx^{\beta}} \qquad (1 > \beta > 0),$$

in the formula (1.3) one can easily observe that the method fails to prove that $R(h, \infty)$ represents an integral function of s. Nevertheless it is true that the differences

$$\sum_{1}^{\infty} e^{-s(\log n)^{\alpha}} - \int_{1}^{\infty} e^{-s(\log x)^{\alpha}}\, dx,$$

and $$\sum_{1}^{\infty} e^{-sn^{\beta}} - \int_{1}^{\infty} e^{-sx^{\beta}}\, dx,$$

represent integral functions of s, as we shall see in Part II.

THEOREM III. *In (1.1) let $\phi(x)$ and $f(x)$ satisfy the following conditions:—*

(i) *$f(x)$ is a positive and steadily increasing function of a real variable x ($\geqslant p$), and tends to infinity with (x);*

(ii) *for $x \geqslant p$, $\phi(x)$ as well as $f(x)$* possess continuous derivatives of the first q orders;*

(iii) (*a*) $f(x) = O\left((\log x)^{\alpha}\right)$, *where α is real and* $0 < \alpha < 1$,

(*b*)* $f^{\mu}(x) = O(x^{-\mu})$, for $\mu = 1, 2, 3 \ldots q$;

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* Conditions satisfied, in particular, when $f(x)$ is an L-function, satisfying the condition (iii) (*a*).

(iv) (*a*) $\phi'(x) = O(e^{kf(x)})$, *where k is some real constant,*

and (*b*) $\phi^{\mu}(x) = O(x^{k'-\mu c})$, *for* $\mu = 1, 2, 3, \ldots q$,

where c is a positive number such that $0 < c \leqslant 1$, k' *is a constant less than c, and q is the smallest positive integer greater than* 1 *such that*

$$qc \geqslant 1 + k' + \delta > 1 + k', \qquad (\delta > 0).$$

Then

$$\lim_{N \to \infty} \left\{ \sum_{p}^{N} \phi(n) e^{-sf(n)} - \int_{p}^{N} \phi(x) e^{-sf(x)} dx \right\}$$

exists for all values of s in D, where D denotes any finite region in the plane of s for all points of which $\sigma \geqslant k + \varepsilon > k$, *for every* $\varepsilon > 0$, *and defines a function which is an integral function of s.*

If, however, the Dirichlet's series be convergent for some values of s, then the relation (1.2) holds.

To prove the theorem, we put

$$F(x) = \phi(x) e^{-sf(x)} = \phi(x)\psi(x), \text{ say,}$$

in the formula (1.3), and suppose that s lies in D. Now to determine the limit of the left-hand side of (1.3) as $N \to \infty$, we have to consider the behaviour of the terms involving N on the right-hand side

Now, when x becomes infinite $F(x) = O(e^{(k-\sigma) f(x)})$,

which tends to 0, when $\sigma > k$, by virtue of condition (1). Also for $r = 1, 2, \ldots, q$, we have, as in § 1,

$$F^{r}(x) = \sum_{\mu=1} {}^{r}C_{\mu}\, \psi^{(r-\mu)}(x)\, \phi^{\mu}(x) + \psi^{(r)}(x)\, \phi(x),$$

$$= \sum_{\mu=1}^{r} {}^{r}C_{\mu}\, O(e^{-\sigma f(x)} x^{k'-r+\mu(1-c)}) + O(e^{(k-\sigma) f(x)} x^{-r}),$$

s being bounded. So that for $r = 1, 2, \ldots q$, $F^{r}(x) \to 0$, as $x \to \infty$, by virtue of (iii) (*a*), and since $k' < c$.

Therefore, as in (1.5) we have, for s lying in D,

$$(3.1) \qquad \lim_{N\to\infty} \left\{ \sum_{p}^{N} \phi(n)\, e^{-sf(n)} - \int_{p}^{N} \phi(x)\, e^{-sf(x)}\, dx \right\}$$

$$= e^{-sf(p)} \left\{ \tfrac{1}{2}\phi(p) + v_{2h-1}(s) \right\} + R(h, \infty),$$

where $v_{2h-1}(s)$ denotes a polynomial in s of degree $(2h-1)$, and h is $q/2$ or $(q-1)/2$ according as q is even or odd.

Now if we only show that $R = R(h, \infty)$ represents an integral function of s, then it will follow that the limit of the left-hand side of (3 1), as $N \to \infty$, exists for all values of s in D, and defines a function which is an integral function of s. We have

$$R = \pm \int_{p}^{\infty} P_q(x)\, F^q(x)\, dx, \qquad P_q(x) = O(1), \text{ if } q > 1.$$

Now

$$F^q(x) = \sum_{\mu=1}^{q} {}^qC_\mu\, \psi^{q-\mu}(x)\, \phi^{\mu}(x) + \psi^{(q)}(x)\, \phi(x),$$

here $\qquad \psi^{q-\mu}(x) = \left(\sum_{r=0}^{q-\mu} E_r s^r \right) e^{-sf(x)},$

$$\psi^q(x) = e^{-sf(x)} \sum_{r=0}^{q} D_r s^r,$$

where for all values of r concerned.

$$E_r = O(x^{\mu-q}), \qquad D_r = O(x^{-q}).$$

Therefore

$$(3.2) \qquad R = \pm \left(\sum_{\mu=1}^{q} {}^qC_\mu \sum_{r=0}^{q-\mu} s^r \int_{p}^{\infty} P_q(x)\, \phi^{\mu}(x)\, e^{-sf(x)}\, E_r\, dx \right.$$

$$\left. + \sum_{r=0}^{q} s^r \int_{p}^{\infty} P_q(x)\, \phi(x)\, e^{-sf(x)}\, D_r\, dx \right).$$

That is,

$$(3.3)\quad R = \pm \left(\sum_{\mu=1}^{q} {}^{q}C_{\mu} \sum_{r=0}^{q-\mu} s^r \int_p^\infty O\left(e^{-\sigma f(x)}\, x^{k'-q+\mu(1-c)}\right) dx \right.$$

$$\left. + \sum_{r=0}^{q} s^r \int_p^\infty O\left(e^{(k-\sigma) f(x)}\, x^{-q}\right) dx \right),$$

which shows that each integral on the right side of (3.3) is absolutely and uniformly convergent for all bounded values of s, by virtue of condition (iii) (a), and since $q > 1$, and $qc \geqslant 1 + k' + \delta > 1 + k'$; and hence it follows that R represents an integral function of s.

The second part of the theorem follows at once.

It may be remarked that if

$$c \geqslant 1 + k' + \delta > 1 + k'$$

and $\phi(x) = O(x^{-\gamma})$, where $\gamma > 0$, q need not be taken greater than unity. In this case, the formula (1.3) takes the form

$$\sum_{p}^{N} \phi(n)\, e^{-sf(n)} - \int_p^N \phi(x)\, e^{-sf(x)}\, dx = \tfrac{1}{2}\left(F(p) + F(N)\right)$$

$$- \int_p^N P_1(x)\, F'(x)\, dx,$$

where $\quad F(x) = \phi(x)\, e^{-sf(x)}, \quad P_1(x) = [x] - x + \tfrac{1}{2},$

$[x]$ denoting the integral part of x. Hence for all bounded values of s.

$$(3.4)\quad \lim_{N\to\infty} \left\{ \sum_{p}^{N} \phi(n)\, e^{-sf(n)} - \int_p^N \phi(x)\, e^{-sf(x)}\, dx \right\}$$

$$= \tfrac{1}{2}\, \phi(p)\, e^{-sf(p)} - \int_p^\infty P_1(x)\, F'(x)\, dx.$$

Now $\qquad P_1(x) = O(1),$

$$F'(x) = -s\, \phi(x)\, f'(x)\, e^{-sf(x)} + \phi'(x)\, e^{-sf(x)},$$

$$= (-s)\, O\left(e^{-\sigma f(x)}\, x^{-1-\gamma}\right) + O\left(e^{\sigma f(x)}\, x^{k'-c}\right),$$

which shows at once that the integral on the right side of (3.4) represents an integral function of s.

We may apply the theorem to the following series :—

$$\sum_1^\infty e^{-s\log(n)^\alpha} \quad (0 < \alpha < 1): \sum^\infty \frac{1}{n^{1+\alpha i}(\log n)^{s}}, \quad \alpha \neq 0;$$

$$\sum_1^\infty \frac{e^{Ai(\log n)^\beta - s(\log n)^\alpha}}{n}, \quad A \neq 0, \quad 0 < \alpha < 1, \quad \beta > 0;$$

$$\sum_1^\infty \frac{e^{Ain^\beta - s(\log n)^\alpha}}{n^{1-\beta}}, \quad A \neq 0, \quad 0 < \beta < 1, \quad 0 < \alpha < 1.$$

4. No meaning has so far been given to a divergent Dirichlet's series such as

$$\sum_1^\infty e^{-s(\log n)^\alpha}, \qquad 0 < \alpha < 1.$$

Theorem III enables us, however, to give a meaning to

$$\sum_1^\infty e^{-s(\log n)^\alpha} - \int_1^\infty e^{-s(\log x)^\alpha} dx,$$

by defining it as

$$\lim_{N\to\infty} \left\{ \sum_1^N e^{-s(\log n)^\alpha} - \int_1^N e^{-s(\log x)^\alpha} dx \right\}.$$

This limit, as we now know, exists for $\sigma > 0$ and s bounded and defines a function which is an integral function of s. But even this kind of information is denied to us in the case of such a series as

$$\sum_1^\infty n^k e^{-s(\log n)^\alpha} \qquad k > 0,\ 0 < \alpha < 1;$$

for condition (iv) (a) of Theorem III, which is essential for the application of the Euler-Maclaurin Sum-formula, is not fulfilled. We have, therefore, to proceed otherwise in order to give a meaning to

$$\sum_{1}^{\infty} n^{k}\, e^{-s(\log n)^{\alpha}} - \int_{1}^{\infty} x^{k}\, e^{-s(\log x)^{\alpha}}\, dx.$$

This is attempted by means of the following theorem, which applies to a larger class of divergent series than the preceding one. In these circumstances, it is but natural to expect that our conditions will have to be made more stringent.

In the next section, I shall show how Theorem IV enables us to give a meaning to a divergent series such as

$$\sum_{1}^{\infty} e^{-s(\log n)^{\alpha}},\ 0 < \alpha < 1;\ \text{or}\ \sum_{3}^{\infty} e^{-s \log\log n}$$

THEOREM IV. *Let the series (1.1) be a divergent Dirichlet's series, where $\phi(x)$ satisfies, for $x \geqslant p$, conditions A and B of § 1, with this additional restriction that $k > -1$, and $f(x)$ satisfies the following two conditions :—*

(i) *$f(x)$ is a positive, and monotonic function, tending to infinity with x, and possesses* derivatives of every order, $(x \geqslant p)$;*

(ii) $f(x) = O\left((\log x)^{\alpha}\right),\ 0 < \alpha < 1,$

$$f^{\mu}(x)^{*} = O(x^{-\mu}),\ (\mu = 1, 2, 3, \ldots);$$

then
$$\lim_{z \to 0} \left(H(s, z) - J(s, z)\right)$$

exists for all bounded values of s, and defines a function which is an integral function of s, where

$$H(s, z) = \sum_{p}^{\infty}{}' n^{-z}\, \phi(n)\, e^{-sf(n)},$$

$$J(s, z) = \int_{p}^{\infty} x^{-z}\, \phi(x)\, e^{-sf(x)}\, dx,$$

for $R(z) > 1 + k > 0$, and all bounded values of s.

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* Conditions satisfied, in particular, when f is an L-function which is of the same order as $(\log x)^{\alpha}$, $0 < \alpha < 1$.

To establish the theorem, it is sufficient to show that

$$H(s, z) - J(s, z) = G(s, z)$$

is an integral function of both the variables s an z.

Now put

$$F(x) = x^{-z}\,\phi(x)\,e^{-sf(x)},$$

in the formula (1.3) and suppose, for the moment, that

$$R(z) \geqslant 1 + k + \delta > 1 + k > 0, \qquad (\delta > 0).$$

Then $\quad F(x) = O(x^{-R(z)+k}\,e^{-\sigma f(x)}) \to 0$, as $x \to \infty$,
s being bounded.

Also let

$$F(x) = \theta(x)\,\psi(x),$$

where

$$\theta(x) = x^{-z}\,\phi(x), \qquad \psi(x) = e^{-sf(x)}.$$

Then

$$F^{r}(x) = \sum_{\mu=0}^{r} {}^{r}C_{\mu}\;\theta^{(r-\mu)}(x)\,\psi^{(\mu)}(x).$$

Now

$$\theta^{r-\mu}(x) = \sum_{\mu'=0}^{r-\mu} {}^{r-\mu}C_{\mu'}\,(-z)(-z-1)\ldots(-z-\nu+1)\,x^{-z-\nu}.\phi^{\mu'}(x).$$

$$= \sum_{\mu'=0}^{r-\mu} {}^{r-\mu}C_{\mu'}(-z)(-z-1)\ldots(-z-\nu+1)\,O(x^{-R(z)-r+\mu+k+\mu'(1-c)}).$$

where

$$\nu = r - \mu - \mu'.$$

Also

$$\psi^{\mu}(x) = e^{-sf(x)} \sum_{u=0}^{\mu} E_u\,s^{u},$$

where for each u, $E_u = O(x^{-\mu})$.

It follows, therefore, that $F^{r}(x) \to 0$, as $x \to \infty$, $(r = 1, 2, 3, \ldots)$, $R(z) \geqslant 1 + k + \delta > 1 + k$, and s being bounded.

Also

$$F^{r}(p) = \sum_{\mu=0}^{r} {}^{r}C_{\mu}\,U_{r-\mu}(z)\,p^{-z}\,e^{-sf(p)}\,V_{\mu}(s),$$

where U and V are polynomials whose degrees are indicated by their suffixes. So $F^r(p)$ is an integral function of both z and s, $r=1, 2, 3, \ldots$

We have, therefore, proved that, for $R(z) \geqslant 1+k+\delta > 1+k$, and s bounded,

(4.1) $$H(s, z) - J(s, z) = \tfrac{1}{2} p^{-z} \phi(p) e^{-sf(p)}$$

$$+ \sum_{\nu=1}^{h} (-1)^{\nu} \frac{B_\nu}{(2\nu)!} \left(p^{-z} e^{-sf(p)} \sum_{\mu=0}^{(2\nu-1)} {}^{2\nu-1}C_\mu \, U_{(2\nu-1-\mu)}(z) V_\mu(s) \right)$$

$$+ R(h, \infty) = \text{an integral function of both } z \text{ and } s + R(h, \infty).$$

Now it remains to show that $R = R(h, \infty)$ represents an integral function of both s and z. Let q denote either of the numbers $2h$ and $(2h+1)$. Then

$$R = \pm \int_p^\infty P_q(x) F^q(x)\, dx, \qquad P_q(x) = O(1), \text{ if } q > 1.$$

Now

$$F^q(x) = \sum_{\mu=0}^{q} {}^qC_\mu \left(\sum_{\mu'=0}^{q-\mu} W_{(q-\mu-\mu')} \, x^{(-z-q+\mu+\mu')} \phi^{\mu'}(x) \right)$$

$$\times \left(e^{-sf(x)} \sum_{u=0}^{\mu} E_u s^u \right),$$

where W is a polynomial of the degree indicated by its suffix.

Therefore,

$$R = \pm \sum_{\mu=0}^{q} {}^qC_\mu \left\{ \sum_{\mu'=0}^{q-\mu} \sum_{u=0}^{\mu} W_{(q-\mu-\mu')}(z) s^u \right.$$

$$\left. \int_p^\infty x^{(-z-q+\mu+\mu')} \phi^{\mu'}(x) e^{-sf(x)} E_u P_q(x)\, dx \right\},$$

$$= \pm \sum_{\mu=0}^{q} {}^qC_\mu \left\{ \sum_{\mu'=0}^{q-\mu} \sum_{u=0}^{\mu} W_{(q-\mu-\mu')}(z) s^u \right.$$

$$\left. \int_p^\infty O\left(x^{-R(z)-q+\mu'(1-c)+k} e^{-\sigma f(x)} \right) dx \right\}$$

so that each integral involved is absolutely and uniformly convergent for all bounded values of s, and for

$$R(z) \geqslant 1 + k - qc + \varepsilon > 1 + k - qc, \quad (\varepsilon > 0),$$

q being an arbitrary positive integer. It follows, therefore, that the above expression for R represents an integral function of both s and z.

Hence
$$H(s, z) - J(s, z) = G(s, z),$$

is an integral function of s and z, and so that

$$\lim_{z \to 0} (H(s, z) - J(s, z))$$

exists for all bounded values of s, and defines a function which is an integral function of s.

5. Now suppose that a limit or a finite number of limits of $J(s, z)$ (considered in the last section) exist, as z approaches zero, then the preceding theorem enables us to define the divergent Dirichlet's series $\sum_{p}^{\infty} \phi(n) e^{-sf(n)}$ by means of

$$\lim_{z \to 0} H(s, z) = \lim_{z \to 0} J(s, z) + G(s),$$

where $G(s)$ is an integral function of s. Where more than one limit of $J(s, z)$, as z tends to zero, exists, we shall define our divergent series by means of any one of those limits added to $G(s)$. To illustrate this principle, I add a few examples here.

(*a*) Consider the divergent series

$$\sum_{1}^{\infty} e^{-s(\log n)^{\alpha}}, \qquad 1 < \alpha < 1.$$

Then
$$H(s, z) = \sum_{1}^{\infty} n^{-z} e^{-s(\log n)^{\alpha}},$$

the series being convergent for $R(z) > 1$, and for all bounded values of s.

By Theorem IV, we have

$$H(s, z) = J(s, z) + G(s, z),$$

where G is an integral function of both the variables s and z, and

$$J(s, z) = \int_1^\infty x^{-z} e^{-s(\log x)^\alpha}\, dx, \qquad (R(z) > 1,\ s \text{ bounded}),$$

$$= \int_0^\infty e^{-(z-1)y - sy^\alpha}\, dy, \qquad (\text{putting } y = \log x).$$

Now expand e^{-sy^α} and integrate term by term. Also suppose for the moment that z is real and > 1. Then

$$(5.1) \qquad J(s, z) = \int_0^\infty e^{-(z-1)y} \left(\sum_0^\infty \frac{(-s)^n}{\Gamma(1+n)} y^{n\alpha} \right) dy,$$

$$= \sum_0^\infty \frac{(-s)^n}{\Gamma(1+n)} \frac{\Gamma(1+n\alpha)}{(z-1)^{1+n\alpha}}.$$

Now draw a cut along the line $(1, -\infty)$ and interpret $(z-1)^{1+n\alpha}$ to mean exp. $(1+n\alpha)\log(z-1)$, where the logarithm has its principal value Then if $z \neq 1$, and lies in the cut plane, the series on the right-side of (5.1) represents an integral function of s, and a regular and uniform function of z. So that

$$(5.2) \qquad \lim_{z \to 0} H(s, z) = \sum_{n=0}^\infty \frac{(-s)^n}{\Gamma(1+n)} \frac{\Gamma(1+n\alpha)}{e^{\pm \pi i(1+n\alpha)}} + G(s),$$

where G is an integral function of s, and the sign $+$ or $-$ is taken in $e^{\pm \pi i(1+n\alpha)}$ according as z approaches the point $z = 0$ from above or below the cut.

Now we define our divergent series (a) by means of either of the functions (which are integral functions of s) appearing on the right-hand side of (5.2).

Similarly, we define the divergent series

$$\sum_2^\infty n^k (\log n)^\beta e^{-s(\log n)^\alpha}, \quad (\alpha, \beta, k \text{ all real}, 0 < \alpha < 1, \beta > -1, k > -1),$$

by means of either of the functions

$$\sum_0^\infty \frac{(-s)^n}{\Gamma(1+n)} \frac{\Gamma(1+n\alpha+\beta)}{(1+k)^{1+n\alpha+\beta}} e^{\mp \pi i(1+n\alpha+\beta)} + G(s),$$

where G (s) is an integral function of s, The definition of the divergent series, considered above, is rightly open to the objection that it has an ambiguity in it, and therefore hardly seems likely to be very profitable. But I have to point out that there exist cases where no ambiguity arises, and at least in such cases, the definition adopted is not only very useful, but also very natural. I give two such examples here.

(b) $\sum_{1}^{\infty} e^{Ai\,n^{\beta} - s\,(\log n)^{\alpha}}\, n^{k}$, $A \neq 0,\ 0 < \beta < 1,\ 0 < \alpha < 1,\ k$ real.

It is easy to prove that if $k+1=\beta$, the series is convergent (conditionally) if $\sigma > 0$; if $k+1 < \beta$, the series is convergent for all values of s, and represents an integral function of s; and if $k+1 > \beta$, the series does not converge for any value of s. What I wish to prove is that when $k+1=\beta$, the function represented by the series, convergent for $\sigma > 0$, is an integral function of s; and that when $k+1 > \beta$, the series, though divergent, may be defined by an integral function of s. Suppose, firstly, that $k+1=\beta$. Then, by Theorem III,

$$\text{(5.3)} \qquad H(s) = \sum_{1}^{\infty} e^{Ai\,n^{\beta} - s\,(\log n)^{\alpha}}\, n^{\beta-1}, \qquad (\sigma > 0),$$

$$= G(s) + \int_{1}^{\infty} e^{Ai\,x^{\beta} - s\,(\log x)^{\alpha}}\, x^{\beta-1}\, dx,$$

$$= G(s) + \int_{1}^{\infty} e^{Ai\,y - s\,(1/\beta)^{\alpha}\,(\log y)^{\alpha}}\, dy,$$

where $G(s)$ is an integral function of s, and the integral is conditionally convergent, if $\sigma > 0$.

Now suppose for a moment that s is real and positive, and let

$$\text{(5.4)} \qquad J(s) = \int_{1}^{\infty} e^{Ai\,y - sd\,(\log y)^{\alpha}}\, dy, \qquad \left(d = \frac{1}{\beta^{\alpha}}\right).$$

Put $y = re^{i\theta}$, and suppose $A > 0$.* Now consider the integral

$$I \equiv \int_{0}^{\gamma} e^{Ai\,re^{i\theta} - sd\,(\log r + i\theta)^{\alpha}}\, re^{\theta}\, i\, d\theta,$$

* The modification required for the case $A < 0$ will be obvious.

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NOTES AND QUESTIONS.

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THE INDIAN MATHEMATICAL SOCIETY.

PROGRESS REPORT.

In reply to the General Circular No. 1 of 1927, dated the 15th November 1927, sixty-four members of the Society sent me their voting papers. All of them have voted for the recommendations of the present Managing Committee. Hence I declare the following gentlemen duly elected members of the Managing Committee for 1928-29, which will be constituted as under :—

President.

V. Ramaswamy Aiyar, Esq., M.A.

Other Members.

1. Dr. R. Vaidyanathaswamy, M.A., D.Sc , F.R.S.E.
2. Prof. N. M. Shah, M.A., *(Cantab)*.
3. Prof. K. Ananda Rao, M.A. *(Cantab.)*.
4. Mr. Balak Ram, M.A., I.C.S.
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6. Prof. K. R. Ramaswami Aiyangar, M.A.
7. Prof. G. S. Chowla, Rai Bahadur, B.A. *(Cantab.)*, M.A.
8. Prof. S. B. Belekar, M.A.
9. Prof. T. V. Mone, M.A.

Prof. V. B. Naik, M.A. and Mr. S. Narayana Aiyar, M.A. continue as the Hon. Librarian and the Hon. Treasurer respectively.

The New Managing Committee will elect its own Secretaries.

POONA,
15th December, 1927.

N. M. SHAH,
Hon. Joint Secretary.

Notes and Questions.

Some Properties of Polar Curves.*

Introduction. The results stated in this note were obtained incidentally in an attempt to realise the significance of the following result† due to Mr. Youngman :

The conic passing through the node of a cubic, and the points of contact of tangents from any point on the line of flexes, circumscribes the triangle of inflexional tangents.

It appeared in the course of the investigation that the original cubic and the degenerate one consisting of the inflexional tangents, had the same first polar with respect to any point on the line of flexes, whence the theorem follows from the property that the first polar of P passes through the singular points and the points of contact of tangents from P to the curve. It is shown in this note that for every point P in the plane there exists an infinite system of curves having the same first polar as a given curve, and that the first polar may be obtained as the locus of singular points of this system. A number of results are deduced which may be regarded as generalisations of Mr. Youngman's theorem in various directions, to the case when P is not on the line of flexes, to quartics, to curves of higher degree, etc. The treatment is extended to higher polars towards the end of the paper.

1. Let $F(x, y, z)$ be any homogeneous ternary n-ic, and $x_1\ y_1\ z_1$ a set of variables co-gredient with x, y, z It is then known that the polar operator

$$\Omega = x_1 \frac{\partial}{\partial x} + y_1 \frac{\partial}{\partial y} + z_1 \frac{\partial}{\partial z}$$

converts F into a co-variant in the two sets of variables. If x, y, z be interpreted as the homogeneous co-ordinates of points on a certain plane, $F=0$ represents a certain curve and $\Omega(F) = 0$ is its first polar with respect to the point $P(x_1\ y_1\ z_1)$. It is known that this first polar passes through

(i) the points of contact of tangents from P to the curve F... (1)

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* I am indebted to Mr. G. A. Srinivasan for some valuable suggestions incorporated in this paper.

† Question 18078 : *Educational Times*.

and (ii) the singular points (if any) of F. ... ... (2)

Consider now the first polars of P with respect to the family of curves

$$F + \lambda L_1 L_2 \dots L_n = 0 \quad \dots \quad \dots \quad (3)$$

where F is of order n and L_r is linear in the variables. These form another family

$$\Omega(F) + \Sigma L_1 \dots L_{r-1} L_{r+1} \dots L_n \Omega(L_r) = 0$$

but if P lies on each of the lines $L_r = 0$, the second term vanishes and the system reduces to the single curve $\Omega(F) = 0$. Combining this result with (2) we get the following new definition :—

The first polar of a point P with respect to a curve $F = 0$ is the locus of singular points of the family $F + \lambda L_1 L_2 \dots L_n = 0$ where $L_r = 0$ is the equation of any line through P.* ... ... (4)

It follows from (3) that the first polar is also the locus of the points of contact of tangents from P to curves of the above family. ... (5)

We now proceed to deduce some interesting corollaries from this result.

2. To take the simplest case, let $F = 0$ be a conic, and let L_1 and L_2 coincide with a chord PQ_1Q_2 through P. Of the various conics of the system (3), *i e.*, having double contact with the conic at Q_1 and Q_2, one of the singular members is the pair of tangents at Q_1 and Q_2 and their intersection is the singular point of this curve. Hence we have the familiar result that the polar of P is the locus of such intersections. By (2) it is also the chord of contact of tangents from P.

By taking two distinct chords PQ_1Q_2 and PR_1R_2 we see that the polar line of P passes through the intersections (Q_1R_1, Q_2R_2) and (Q_1R_2, Q_2R_1).

3. Next let $F = 0$ denote a cubic and P any point on its line of flexes. One of the cubics of the family $F + \lambda L^3 = 0$ (L denoting the line of flexes) is the triad of inflexional tangents, and by (4) the first polar

* These curves form an $n+1$ *parameter* family (λ being one parameter and the other n being implied, one in each of the equations $L_r = 0$, and is the most general family of curves, having the same first polar as $F=0$.

of P passes through the singular points of this composite curve, *i.e.*, through their intersections. Hence we have the theorem :

If tangents be drawn to a cubic from any point on the line of flexes, their six points of contact and the three vertices of the triangle of inflexional tangents lie on a conic.

If the cubic be rational, the points of contact diminish by two, but the polar conic now passes also through the node of the given cubic. Hence we have Mr. Youngman's* result stated in the introduction.

4. Let XYZ be any three collinear points on a cubic and P any point in its plane. Let the lines PX, PY, and PZ meet the cubic again in X_1 X_2, Y_1 Y_2 and Z_1 Z_2 respectively. These six points will lie on a conic C.† Let us call this the residual conic of the line XYZ with respect to the point P. Any line XYZ and its residual conic together form a cubic which, with the original cubic, determines a pencil of type (3) and hence both these cubics (the original and the degenerate one) have the same polar conic with respect to P.

Hence we have the following theorem :

The polar conic of any point P with respect to a cubic passes through the following sets of points :—

(i) *The six points of contact of tangents from P to the cubic.*

(ii) *The points of contact of tangents from P to the* ∞^2 *system of residual conics of P.*

(iii) *The pairs of intersections of any line with its residual conic.*

5. Similar considerations apply to the general quartic. We define the residual cubic of a line with respect to P in a similar manner, and we have :—

If P be any point on the plane, the polar cubic of P passes through the following points :—

(i) the 12 points of contact of tangents from P to the quartic.

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* For a proof by the method of parameters, *vide* Question **1285**, J. I. M. S., Vol. XV, page *32*.

† This is most easily proved by the theory of residuation. *Vide* Hilton : *Plane Algebraic Curves*, P. **189**, Ex. **8**.

(ii) the points of contact of tangents from P to the residual cubic of every line in the plane with respect to P.

(iii) the intersections of any line with its residual cubic.

6. Lastly, let F degenerate into n straight lines $T_1\, T_2 \ldots T_n$ and take $L_1 = L_2 = \ldots = L_n = L$. Any curve having n-tactic contact with each of these lines (T) where they are cut by the line L has its equation of the form

$$T_1\, T_2 \ldots T_n + \lambda L^n = 0$$

Now the points of contact of tangents from a point P on the line of flexes $L = 0$ lie on the first polar of P which is the same for all the members of the pencil. But one of these members consists of the n lines T. The first polar passes through their nC_2 intersections which are the nodes of the composite curve.

We have thus a direct generalisation of Mr. Youngman's result to curves of order n.

If a curve of order n have n-point contact with n straight lines where they are cut by a line L *there exists a curve of order n — 1 passing through the $n(n-1)$ points of contact of tangents from a point on* L *and the $\frac{1}{2}n\,(n-1)$ intersections of the straight lines.*

7. To extend this investigation to polar curves of higher order, we notice that if a curve has P for a $(n+1-r)$-ple point, its rth polar with respect to the point P vanishes identically. This may be verified easily by choosing P for one of the vertices of the reference triangle, when Ω becomes equivalent to differentiation with respect to one of the variables. Now if $G\,(x,\, y,\, z) = 0$ represents the most general curve of this sort, every member of the family

$$F\,(x,\, y,\, z) + \lambda\, G\,(x,\, y,\, z) = 0$$

has the same rth polar with respect to P. If any curve of this family has an $(r+1)$-ple point at Q, the rth polar of P passes through Q and may be obtained as the locus of such points. This result may be expressed more symmetrically thus:—

If two curves of order n, one having an $(n+1-r)$-ple point at P and the other an $(r+1)$-ple point at Q intersect on the given n-ic $F = 0$, then the rth polar of P with respect to F passes through Q and the $(n-r)$th polar of Q passes through P.

A. Narasinga Rao.

On the Inversion of Multiplicative Arithmetic Functions.

I. One usually understands by 'the principle of inversion of Arithmetic Functions,' the fact that the relation

$$\sum_{d/n} f(d) = \phi(n)^*$$

implies the reciprocal relation,

$$\sum_{d/n} \phi\left(\frac{n}{d}\right) \mu(d) = f(n),$$

where $\mu(N)$ is the Mertens function, which vanishes when N has any squared factor, and is equal to $(-1)^\nu$ when N is the product of ν different primes. The function μ has the property:

(1)
$$\sum_{d/n} \mu(d) = 1, \text{ if } n = 1$$
$$= 0, \text{ otherwise.}$$

It is on this property that the inversion-formula depends. But, as a matter of fact, there holds a property analogous to (1) for *any* multiplicative † function $f(N)$. That is to say,

(2) *Given any multiplicative arithmetic function $f(N)$, we can find a unique multiplicative arithmetic function $f^{-1}(N)$, with the property*

$$\sum_{d/n} f\left(\frac{n}{d}\right) f^{-1}(d) = 1, \text{ if } n = 1$$
$$= 0, \text{ if } n \neq 1.$$

We call the function f^{-1}, the *inverse* of the function f. If $E(N) = 1$ for all values of N, then $E(N)$ is a multiplicative function, and (1) shews that its inverse function E^{-1} is precisely the Mertens function μ.

The theorem (2) is a fundamental one in the whole theory of Arithmetic functions. It is rather surprising that it does not seem to have

* The symbol d/n indicates that the summation is for all divisors d of n.

† An arithmetic function f is said to be *multiplicative* if $f(MN) = f(M) f(N)$, whenever M, N are mutually prime. If $f(M) f(N) = f(MN)$ for *all* integers M, N I shall say that f is a ***linear*** function. For a multiplicative function f we have to take $f(1) = 1$ since $f(m) = f(m) . f(1)$.

been utilised before (at any rate, so far as one can see from Dickson's History). The existence of the inverse of any multiplicative function throws a great deal of light on many known theorems on Arithmetic Functions.

I proceed to demonstrate the theorem (2).

II. *The Composition of Arithmetic Functions.*

By the composition of two arithmetic functions f, ϕ, we shall mean the process of forming the function F defined by :

$$F(n) = \sum_{d/n} f(d)\, \phi\left(\frac{n}{d}\right).$$

We shall denote F by $f.\phi$, and call it the *composite* of f and ϕ. It is immediately seen that the process of composition is both commutative and associative. Further, if the multiplicative function E_0 (N) is defined by $E_0(N) = 1$ or 0, according as N is or is not equal to 1, then, obviously

$$f.E_0 = f. \tag{3}$$

It is a well-known theorem, and may be easily proved, that when f and ϕ are both multiplicative, their composite $f.\phi$ is also multiplicative.

III. If $N = p_1^{\alpha} p_2^{\beta}$ ...... and if ϕ is any multiplicative function, then

$$\phi(N) = \phi(p_1^{\alpha})\, \phi(p_2^{\beta}) \ldots \ldots$$

Hence if the values of a multiplicative function are known for all prime-power values of the argument N, then its value for every other value of N is determined. In other words, from arbitrary values associated with prime-power values of N, we can build up a unique multiplicative function.

Suppose, now, that ϕ is a multiplicative function, and that we are able to determine numbers $\phi^{-1}(p^{\alpha})$ for every prime p, and every index α, such that

$$\sum_{d/p^{\alpha}} \phi\left(\frac{p^{\alpha}}{d}\right) \phi^{-1}(d) = 1, \text{ for } \alpha = 0 \tag{4}$$

$$= 0, \text{ for } \alpha \neq 0.$$

Let ϕ^{-1} (N) be the multiplicative function constructed from the values $\phi^{-1}(p^{\alpha})$. Since ϕ, and ϕ^{-1} thus defined, are both multiplicative, their composite F is also multiplicative. Hence,

$$F(N) = F(p_1^{\alpha}) \cdot F(p_1^{\beta}) \ldots .$$

But from (4), each $F(p^{\alpha}) = 1$ or 0, according as α is or is not equal to zero. Hence,

$$F(N) = \sum_{d/N} \phi\left(\frac{N}{d}\right) \phi^{-1}(d) = 1, \text{ if } N = 1$$
$$= 0, \text{ if } N \neq 1.$$

This proves theorem (2), provided we can determine numbers $\phi^{-1}(p^{\alpha})$ satisfying equations of the type (4).

IV. The determination of the numbers $\phi^{-1}(p^{\alpha})$ is a matter of straightforward solution of linear equations. Thus, for given p, α, the equations (4) are:

$$\begin{aligned} &\phi(p) + \phi^{-1}(p) &&= 0 \\ &\phi(p^2) + \phi(p)\,\phi^{-1}(p) + \phi^{-1}(p^2) &&= 0 \\ &\quad\cdot \qquad\qquad \cdot \qquad\qquad \cdot \\ &\quad\cdot \qquad\qquad \cdot \qquad\qquad \cdot \\ &\phi(p^{\alpha}) + \phi(p^{\alpha-1})\,\phi^{-1}(p) + \ldots + \phi^{-1}(p^{\alpha}) &&= 0 \end{aligned}$$

Solving, we have:

$$(-1)^{\alpha}\,\phi^{-1}(p^{\alpha}) = \begin{vmatrix} \phi(p) & 1 & 0 & 0 & \ldots & 0 \\ \phi(p^2) & \phi(p) & 1 & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & & \ldots & \ldots \\ \phi(p^{\alpha-1}) & \phi(p^{\alpha-2}) & \ldots & & \phi(p) & 1 \\ \phi(p^{\alpha}) & \phi(p^{\alpha-1}) & \ldots & \ldots & \ldots & \phi(p) \end{vmatrix}$$

This completes the demonstration of Theorem (2).

V. (1) *The inverse of the composite of any number of multiplicative functions is the composite of their inverses*; in symbols,

$$(f \cdot \phi \ldots)^{-1} = f^{-1} \cdot \phi^{-1} \ldots$$

For, since composition is associative,

$$\begin{aligned} f \cdot \phi \ldots f^{-1} \cdot \phi^{-1} \ldots &= f \cdot f^{-1} \cdot \phi \cdot \phi^{-1} \cdot \ldots \\ &= E_0 \cdot E_0 \cdot E_0 \ldots \\ &= E_0. \end{aligned}$$

(2) *If I is a linear* function, $I^{-1} = I \times E^{-1}$,† where E^{-1} is the same as Mertens' function μ.*

For $$I \cdot (I \times E^{-1}) = \sum_{d/n} I(d)\, I\left(\frac{n}{d}\right) E^{-1}\left(\frac{n}{d}\right)$$

$$= I(n) \times \Sigma\, E^{-1}\left(\frac{n}{d}\right), \text{ since I is linear;}$$

$$= I \times E_0 = E_0.$$

More generally, the same proof will shew that $I \times f$ and $I \times f^{-1}$ are inverse functions.

VI. In a functional equation of the type

(5) $$f \cdot \phi = f_1 \cdot \phi_1,$$

any term can be transposed from one side to the other, provided it is replaced by its inverse. In other words, we can *divide out* both sides of (5) by one of the functions involved. It is this property of the inverse function, which renders it so useful.

To prove this, we have only to compose both sides of (5) with the inverse of, say, f. We have

$$f^{-1} \cdot f \cdot \phi = f^{-1} \cdot f_1 \cdot \phi_1$$

i.e., $$E_0 \cdot \phi = f^{-1} \cdot f_1 \cdot \phi_1$$

Since $E_0 \cdot \phi = \phi$, it follows that

$$\phi = f^{-1} \cdot f_1 \cdot \phi_1.$$

The usual form of the Inversion principle is an instance of this theorem. For,

$$\sum_{d/n} f(d) = F(n),$$

is equivalent to $f \cdot E = F$.

Hence by the present theorem,

$$f = F \cdot E^{-1}$$

that is, $$f(n) = \sum_{d/n} F\left(\frac{n}{d}\right) \mu(d).$$

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* *Vide* previous footnote.

† The functional symbol $I \times f$ denotes the function $I(N) \times f(N)$.

VII. *Some Special Functions.*

Let $E(N) = 1$ for all values of N, and

$$E^r = E.E \ldots (r \text{ factors}),$$

$$E^{-r} = (E^r)^{-1} = E^{-1}.E^{-1} \ldots (r \text{ factors}).$$

Then, if $N = p_1^{\alpha} p_2^{\beta} \ldots \ldots$,

(6) $$E^{-r}(N) = \prod_{\alpha} {}_rC_{\alpha}.$$

(7) $$E_r(N) = \prod_{\alpha} (-1)^{\alpha} {}_{r+\alpha-1}C_{\alpha}$$

Let $\varepsilon_k(N) = 1$ if N is a perfect kth power, and equal to 0 otherwise. Then

(8) $\varepsilon_k^{-1}(N) = \mu(N^{\frac{1}{k}})$ if N is a kth power and equal to 0 otherwise.

Let $E_k(N) = k$, if $N \neq 1$, and $= 1$, if $N = 1$. Then

(9) $$E_k^{-1} = E_{k/k-1} \times T_{1-k},$$

where T_r is the linear function which takes the value r for every prime.

Let $\pi_k(N) = 0$, if N is divisible by a kth power, and equal to 1, otherwise.

Then

(10) $$\pi_k = E.\varepsilon_k^{-1},$$

(11) $$\pi_k^{-1}(N) = \mu\left(\frac{N}{M^k}\right),$$

where M^k is the greatest kth power which divides N.

VIII. I add two further theorems of interest on linear functions.

(1) The necessary and sufficient condition that f be a linear function is the vanishing of $f^{-1}(N)$ for all values of N having a square divisor.

(2) The necessary and sufficient condition that f be the composite of r linear functions is the vanishing of $f^{-1}(N)$ for all values of N divisible by an $(r+1)$th power.

R. VAIDYANATHASWAMY.

Reviews.

Projective Geometry, BY C. V. DURELL. Macmillan and Co., London, 1926. 8vo. Pp. x + 222. *Price* 7*s*. 6*d*.

This is something more than a mere revised edition of the author's popular *Course of Plane Geometry for Advanced Students*, Part 2, first published in 1910. By re-arrangement and omission, the author has succeeded in considerably reducing the size of the book without detriment to its usefulness, and the result is a neat and fairly complete account of the projective geometry of the conic.

One of the most welcome features of the book both in the old edition and the present one, is the frequent use made of algebraic methods to illustrate points arising in the text. Algebraic Geometry can be a very real ally to the young student of projective geometry, especially in giving a sense of reality to the imaginary and ideal elements in the projective plane: besides, a study of the same problem by the two different methods helps in bringing out the possibilities and the limitations of each.

Each chapter begins with a short historical note, confined however to the barest outlines. The student would have been very glad to have these supplemented by an appendix containing a connected account of the more important lines of advance in the subject. Modern Projective Geometry has a very interesting and almost continuous history from the time that Poncelet concieved his *Traite des proprieties projectives des figures* (1822) in the Russian prison at Saratoff, (Poncelet was an officer in Napoleon's army during the disastrous campaign against Russia) right down to the comparatively recent work of Segre and Klein. This should make very interesting reading, and would have provided an opportunity to refer to developments in the subject which could not be included in the book.

One wonders why an author who freely makes use of analytical ideas and takes the trouble of pointing out that projection is equivalent to a linear transformation, should not go a step further and emphasize the distinction between a projective property and a metrical one and the linear groups which correspond to the two. Any property of a geometrical configuration either has a special relation to two definite points in the plane, (which are called the Circular Points) or it has none; in the former case it is a metrical property and in the latter a projective one. The notion is an extremely simple one, and leads immediately to Klein's fundamental concept of Geometry as the invariant theory of a particular transformation group, that corresponding to elementary geometry being the linear group which carries the circular points into themselves. This leads again to the concept of the Absolute and of Non-Euclidean Geometry—subjects which lie next door to projective geometry. One does not expect that these topics can be treated at any length in a small book, but one feels that the reader who claims to have undergone a course in projective geometry cannot afford to be entirely ignorant of what constitutes the peculiar features of his subject.

Text-book writing has developed into an art, and one of its canons appears to be that each book should be complete in itself to the extent of ignoring all topics that cannot receive adequate treatment in the text. From the point of view of the student, one would prefer to have a book with loose threads sticking out here and there, one which gives occasional glimpses of the outlying regions surrounding each subject and shows mathematics as it is,—a living and growing body of knowledge.

A Concise Geometrical Conics, BY C. V. DURELL. Macmillan and Co., London, 1927. 8vo. Pp. xii + 100. *Price* 4*s*.

This forms a companion volume to the author's *Projective Geometry* and contains a brief account of those properties which are best treated without the use of projective methods. The author hopes however that the volume may be of service to those who have not the time to study projective methods. An appendix deals with the proofs of theorems which lend themselves to a peculiarly simple treatment by the use of projection with a view to make the book self-contained. A brief treatment of the metrical properties of the conic to supplement a course in projective geometry, has long been a necessity, and the author has done a service in producing such a book.

The printing and general appearance of both the books maintain the high standard of excellence that one has learnt to associate with the publishers.

A. N. R.

Solutions.

Question 1355.

(MARTYN M. THOMAS):—Show that

$$\int \left[\frac{\partial (u, v)}{\partial (x, y)} \cdot \frac{xdy - ydx}{uv}\right] = 2 \log \frac{v}{u},$$

where $u \equiv ax^2 + 2hxy + by^2, \quad v \equiv Ax^2 + 2Hxy + By^2.$

Solution by G. A. Srinivasan.

Denote by u and v any two homogeneous functions of x and y, each of the nth degree. Then, by Euler's Theorem, we have

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = nu, \quad x \frac{\partial v}{\partial x} + y \frac{\partial v}{\partial y} = nv.$$

Also $$\frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy = du, \quad \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy = dv.$$

From these we obtain

$$n (udv - vdu) = \frac{\partial (u, v)}{\partial (x, y)} (xdy - ydx).$$

Hence $$\int \left[\frac{\partial (u, v)}{\partial (x, y)} \frac{xdy - ydx}{uv}\right] = \int n \left(\frac{dv}{v} - \frac{du}{v}\right)$$

$$= n \log \frac{v}{u}.$$

The result in the question is for the case $n = 2$.

Additional Solution by S. D. Chowla.

Question 1356.

(R. VAIDYANATHASWAMY) :—Prove the following construction for finding the point of contact of the pedal line of a point P with its envelope : Let O be the ortho-centre and PQ the chord parallel to the pedal line of P. Produce QP to Q′ so that PQ′ = PQ. The required point of contact is the intersection of the pedal line with OQ′.

Is this construction known ?

Solution by A. Narasinga Rao.

Let us find out first the point of contact of PQ with its envelope. As P moves round the circum-circle to a neighbouring position P_1 the pedal line, and hence also PQ, turns in the opposite direction through an angle equal to that subtended by PP_1 at the circumference. Hence the line P_1Q_1 drawn parallel to the pedal line of P_1 meets PQ ultimately at a point Q' in QP produced such that $Q'P = PQ$, *i.e.* Q' is the required point of contact.

Now the pedal line of any point P bisects OP; hence if all distances from O in our figure were contracted in the ratio of 2 : 1, we shall be replacing the chords PQ and P_1Q_1 by the pedals of the points P and P_1 and the point Q' by the contact-point of the pedal with its envelope. The point of contact is, therefore, the mid-point of OQ' or the intersection of OQ' with the pedal.

Question 1371.

(B. B. Bagi) :—If at the mid-points of the sides of an acute-angled triangle ABC, perpendiculars to the sides are erected of lengths $\frac{1}{2}r_1$, $\frac{1}{2}r_2$, $\frac{1}{2}r_3$, all drawn outwards, then the triangle whose vertices are the extremities of these perpendiculars is similar to the triangle of ex-centres.

Additional Solution by M. Venkatarama Aiyar.*

Let A', B', C' be the mid-points of the sides, I, I_1, I_2, I_3 the in-and ex-centres, D, E, F the points of contact of the in-circle with the sides, and D_1, E_2, F_3 the points of contact of the ex-circles.

Let X, Y, Z be the extremities of the perpendiculars erected at A', B', C' of lengths $\frac{1}{2}r_1$, $\frac{1}{2}r_2$, $\frac{1}{2}r_3$ drawn outwards. Now, since A', B', C' are also the middle points of DD_1, EE_2 and FF_3, we see that X, Y, Z are the mid-points of DI_1, EI_2, and FI_3. But the triangles DEF and $I_1I_2I_3$ are homothetic. Hence XYZ is homothetic with each of these.

[Note.—The restriction that the triangle ABC should be acute-angled is unnecessary.—*Ed.*]

Also solved by N. P. Subramanyan and S. N. Kumarasami.

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* *Vide : J.I.M.S.*, Vol. XVI, p. *62*, for a solution employing trigonometric formulæ.

Question 1405.

(S. NARAYANA AIYAR):—If P and Q denote the expressions

$$\sin\left[\tan^{-1}\{\tan\theta\sqrt{1-k^2\sin^2\phi}\} + \tan^{-1}\{\tan\phi\sqrt{1-k^2\sin^2\theta}\}\right]$$

and $$\sin\left[\tan^{-1}\frac{k\sin\theta\cos\phi}{\sqrt{1-k^2\sin^2\theta}} + \tan^{-1}\frac{k\sin\phi\cos\theta}{\sqrt{1-k^2\sin^2\phi}}\right]$$

show that the ratio $\frac{P}{Q}$ is independent of θ and ϕ and is equal to k.

Solution by B. Achyutharama Sastry, M. V. Seshadri, V. A. Mahalingam, Kanwar Bahadur, P. C. Basu and N. P. Subramanian.

The ratio $\frac{P}{Q}$ is equal to $\frac{1}{k}$ and not equal to k as stated in the question.

Let $$x = \tan\theta\sqrt{1-k^2\sin^2\phi}.$$

$$y = \tan\phi\sqrt{1-k^2\sin^2\theta}.$$

Then $$P = \sin[\tan^{-1}x + \tan^{-1}y]$$

$$= \sin\tan^{-1}x\cos\tan^{-1}y + \cos\tan^{-1}x\sin\tan^{-1}y$$

$$= \frac{x}{\sqrt{1+x^2}} \times \frac{1}{\sqrt{1+y^2}} + \frac{y}{\sqrt{1+y^2}} \times \frac{1}{\sqrt{x^2+1}}$$

$$= \frac{x+y}{\sqrt{(1+x^2)}\sqrt{(1+y^2)}}$$

$$= (x+y)^{1}(1+\tan^2\theta - k^2\tan^2\theta\sin^2\phi)^{\frac{1}{2}}$$
$$(1+\tan^2\phi - k^2\tan^2\phi\sin^2\theta)^{\frac{1}{2}}$$

$$= \frac{(x+y)\cos\theta\cos\phi}{1-k^2\sin^2\theta\sin^2\phi}$$

Similarly,

$$Q = \frac{k\sin\theta\cos\phi(1-k^2\sin^2\phi)^{\frac{1}{2}} + k\cos\phi\sin\theta(1-k^2\sin^2\theta)^{\frac{1}{2}}}{(1-k^2\sin^2\theta + k^2\sin^2\theta\cos^2\phi)^{\frac{1}{2}}(1-k^2\sin^2\phi + k^2\sin^2\phi\cos^2\theta)^{\frac{1}{2}}}$$

$$= \frac{k\cos\theta\cos\phi(x+y)}{1-k^2\sin^2\theta\sin^2\phi}.$$

Hence the ratio of P to Q is $\frac{1}{k}$ which is independent of θ and ϕ.

Questions for Solution.

Proposers of Questions are requested to send their own solutions along with their questions.

1483. (V. RAMASWAMI AIYAR):—ABC is a triangle, circum-centre O. D, E, F are the middle points of the sides, and L, M, N the feet of the perpendiculars on them from any point P.

(i) If circles be described with centres D, E, F and radii DL, EM, FN, prove that the circle Σ, cutting these circles orthogonally touches the nine-point circle of ABC, and that the point of contact is the orthopole of OP.

(ii) Examine the theorem when P is the in-centre or an ex-centre of ABC.

(iii) Show that M'Cay's extension of Feuerbach's theorem is also contained in (i).

(iv) If the circle Σ be a pedal circle, show that the locus of P is a central cubic Γ, with centre at O, and passing through the in- and ex-centres, and such that the tangent to it at O passes through the ortho-centre.

(v) Show that the asymptotes of Γ cut at O making equal angles with one another.

1484. (M. BHIMASENA RAO):—With the same notation as in the above question, if the circle Σ be a contact circle, show that (i) the locus of P is the central self-isogonal cubic Γ' with centre at O, and passing through the in- and ex-centres, and such that the tangent to it at O passes through the symmedian point, (ii) the cubic Γ' is the locus of a point whose pedal triangle LMN is in perspective with ABC, and (iii) the asymptotes of Γ' are the perpendicular bisectors of the sides of ABC.

1485. (A. A. KRISHNASWAMI AYYANGAR):—Denoting the point M in Question 1480 as the M-point of the quadrangle ABCD, if A_1, B_1, C_1, D_1 are the circum-centres of the triangles BCD, CDA, DAB, ABC, show that the quadrangles ABCD and $A_1B_1C_1D_1$ have the same M-point, and that

$$MA \cdot MA_1 = MB \cdot MB_1 = MC \cdot MC_1 = MD \cdot MD_1.$$

1486. (S. NARAYANA AIYAR, M.A.):—Establish the following relations between the elliptic and circular functions:

If $\operatorname{sn} u = \sin a$ and $k \operatorname{sn} u = \sin A$, then

(i) $$\tan \frac{a}{n} \tan \frac{2a}{n} \ldots \ldots \tan \frac{n-1}{n} a =$$

$$\operatorname{sc} \frac{u}{n} \operatorname{sc} \frac{2u}{n} \ldots \operatorname{sc} \frac{n-1}{n} u \,.\, \operatorname{dn} \frac{u}{n} \operatorname{dn} \frac{2u}{n} \ldots \operatorname{dn} \frac{n-1}{n} u.$$

(ii) $$\tan \frac{A}{n} \tan \frac{2A}{n} \tan \frac{3A}{n} \ldots \ldots \tan \frac{n-1}{n} A =$$

$$k^{\frac{n-1}{2}} \operatorname{sd} \frac{u}{n} \operatorname{sd} \frac{2u}{n} \ldots \operatorname{sd} \frac{n-1}{n} u \,.\, \operatorname{cn} \frac{u}{n} \operatorname{cn} \frac{2u}{n} \ldots \operatorname{cn} \frac{n-1}{n} u.$$

Examine the cases when a and A are equal to $\frac{\pi}{2}$.

1487. (S. D. CHOWLA):—Prove that

(i) $$1^2 + \frac{1^2}{x^2} + \frac{1^2 (x+1)^2}{x^2 (2x)^2} + \frac{1^2 (x+1)^2 (2x+1)^2}{x^2 (2x)^2 (3x)^2} + \ldots \ldots$$

$$= \frac{1}{\pi} \sin \frac{\pi}{x} \,.\, \frac{\Gamma\left(\frac{1}{x}\right) \Gamma\left(1 - \frac{2}{x}\right)}{\Gamma\left(1 - \frac{1}{x}\right)}, \quad (x > 2)$$

(ii) $$1^2 - \frac{1^2}{x^2} + \frac{1^2 (x+1)^2}{x^2 (2x)^2} - \frac{1^2 (x+1)^2 (2x+1)^2}{x^2 (2x)^2 (3x)^2} + \ldots \ldots$$

$$= \frac{1}{2\pi} \sin \frac{\pi}{x} \,.\, \frac{\Gamma\left(\frac{1}{2x}\right) . \Gamma\left(1 - \frac{1}{x}\right)}{\Gamma\left(1 - \frac{1}{2x}\right)}, \quad (x > 1)$$

(iii) $$1 + \frac{1^3}{2^3} x + \frac{1^3 . 3^3}{2^3 . 4^3} x^2 + \frac{1^3 . 3^3 . 5^3}{2^3 . 4^3 . 6^3} x^3 + \ldots \ldots$$

$$= \left(1 + \frac{1^2}{4^2} x + \frac{1^2 . 5^2}{4^2 . 8^2} x^2 + \frac{1^2 . 5^2 . 9^2}{4^2 . 8^2 . 12^2} x^3 + \ldots\right)^2, \quad (|x| \leqslant 1).$$

and deduce Ramanujan's result

$$1 - \frac{1^3}{2^3} + \frac{1^3 . 3^3}{2^3 . 4^3} - \frac{1^3 . 3^3 . 5^3}{2^3 . 4^3 . 6^3} + \ldots \ldots = \left\{\frac{\Gamma\left(\frac{9}{8}\right)}{\Gamma\left(\frac{5}{4}\right) . \Gamma\left(\frac{7}{8}\right)}\right\}^2.$$

LIST OF BOOKS & JOURNALS RECEIVED IN THE LIBRARY

during the months of September and October 1927.

1 Abhandlungen aus dem Mathematischen Seminar der Hamburgischen Universitat, **5**, 4.

2 American Mathematical Monthly, **34**, 7 (2 copies).

3 Astrophysical Journal, **65**, 5. and **66**, 1.

4 Bulletin of the Calcutta Mathematical Society, **18**, 3 (2 copies).

5 Bulletin des Sciences Mathematiques, August 1927.

6 Crelle's Journal, **158**, 1.

7 Japanese Journal of Mathematics, **4**, 2.

8 Mathematische Annalen, **98**, 1.

9 Messenger of Mathematics, **56**, 12. (2 copies).

10 Monthly Notices of the Royal Astronomical Society, **87**, 8.

11 Mysore University Journal, **1**, 2 (2 copies).

12 Nieuw Archief Voor Wiskunde, **15**, 3.

13 Philosophical Journal, **4**, 22.

14 Proceedings of the Physico-Mathematical Society of Japan, **9**, 1—5.

15 Proceedings of the Royal Society, London, **115**, *772-73*.

16 Philosophical Transactions of the Royal Society, **227**, 648.

17 Quarterly Journal of Mathematics, **50**, 4.

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