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CONTENTS.

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4. Again,

$$M'_n(x) = M_{n-1}(x) - a_{n+1} M_{n+1}(x).$$

$$a_{n+1} M'_{n+2}(x) = a_{n+1} M_{n+1}(x) - a_{n+1} a_{n+3} M_{n+3}(x)$$

$$a_{n+1} a_{n+3} M'_{n+4}(x) = a_{n+1} a_{n+3} M_{n+3}(x) - a_{n+1} a_{n+3} a_{n+5} M_{n+5}(x)$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

Adding,

$$M'_n(x) + a_{n+1} M'_{n+2}(x) + a_{n+1} a_{n+3} M'_{n+4}(x) + \ldots \ldots = M_{n-1}(x).$$

Integrating,

$$M_n(x) + a_{n+1} M_{n+2}(x) + a_{n+1} a_{n+3} M_{n+4}(x) + \ldots \ldots = \int M_{n-1}(x)\, dx + C.$$

This is true for all values of n, including zero, if we take $M_{-1}(x) = 0$. In that case when $n = 0$, we have

$$M_0(x) + a_1 M_2(x) + a_1 a_3 M_4(x) + \ldots \ldots = 1. \qquad \ldots \text{(III-}a)$$

Here the constant is 1 as, on the left-hand side, the only constant term is the 1 in $M_0(x)$.

The first member of the above equality may be transformed into

$$M_0(x) + (\Sigma a_2 - a_2) M_2(x) + (\Sigma a_4 \Sigma a_2 - a_4 \Sigma a_2) M_4(x)$$
$$+ (\Sigma a_6 \Sigma a_4 \Sigma a_2 - a_6 \Sigma a_4 \Sigma a_2) M_6(x) + \ldots \ldots$$
$$= \{ M_0(x) - a_2 M_2(x) \} + \Sigma a_2 . \{ M_2(x) - a_4 M_4(x) \}$$
$$+ \Sigma a_4 \Sigma a_2 \{ M_4(x) - a_6 M_6(x) \} + \ldots$$
$$= M'_1(x) + \Sigma a_2 M'_3(x) + \Sigma a_4 \Sigma a_2 M'_5(x) + \ldots \ldots$$

or $\qquad M'_1(x) + {}_2\alpha_2 M'_3(x) + {}_4\alpha_2 M'_5(x) + \ldots \ldots = 1$

Integrating and changing sides

$$x = M_1(x) + {}_2\alpha_2 M_3(x) + {}_4\alpha_2 M_5(x) + \ldots \ldots \ldots \qquad \text{(III-}b)$$

there being no constant of integration as there is no term independent of x on the right-hand side. A similar procedure with (III-b) gives

$$\frac{x^2}{2!} = M_2(x) + {}_3\alpha_3 M_4(x) + {}_5\alpha_3 M_6(x) + \ldots \ldots \ldots \qquad \text{(III-}c)$$

and so on. These lead to the following general result by induction, *viz.* :—

$$\frac{x^n}{n!} = M_n(x) + {}_{n+1}a_{n+1}\, M_{n+2}(x) + {}_{n+3}a_{n+1}\, M_{n+4}(x) + \ldots \qquad \text{(III)}$$

From (III), it is seen that any function of x which is expressible in positive integral powers of x can always be transformed into a series involving the M-functions. The equations (III) lead to several interesting results, some already known and obtained by direct methods.

Illustrations.

(i) Let $a_n = n^2$ so that,

$M_0(x) = \operatorname{sech} x$

and

$$M_n(x) = \operatorname{sech} x \frac{(\tanh x)^n}{n!}.$$

In this case (III . a) leads to the obvious identity

$$\cosh x = \frac{1}{\sqrt{1 - \tanh^2 x}}$$

the right-hand side being expressed as a power series. (III-b) leads to

$$\frac{\tanh^{-1} x}{\sqrt{1 - x^2}} = x + \Sigma\, 2^2 . \frac{x^3}{3!} + \Sigma\, 4^2\ \Sigma\, 2^2 . \frac{x^5}{5!} + \ldots\ \ldots$$

(ii) Let $a_n = 1$, making

$$M_0(x) = \frac{1}{x} J_1(2x),$$

$$M_n(x) = \frac{n+1}{x} J_{n+1}(2x).$$

Then, (III . a) gives*

$$J_1(2x) + 3\, J_3(2x) + 5\, J_5(2x) + \ldots\ \ldots = x$$

or, changing the variable,

$$2\, J_1(x) + 6\, J_3(x) + 10\, J_5(x) + \ldots\ \ldots = x.$$

(iii) Let $a_1 = 2$ and $a_2 = a_3 = \ldots\ \ldots = a_n = \ldots = 1$, making

$M_0(x) = J_0(2x)$

and $M_n(x) = J_n(2x)$.

Then (III . a) gives

$$1 = J_0(x) + 2\, J_2(x) + 2\, J_4(x) + \ldots\ \ldots^*$$

* *Vide*, Gray and Matthews: *Treatise on Bessel's Functions*, p. 19.

when $a_n = 1,\ {}_{2n}\alpha_2 = {}_{n+1}C_1$ and III . (*b*) gives

$$\frac{x^2}{2} = \left\{ J_2(2x) + 2^2 J_4(2x) + 3^2 J_6(2x) + \dots \dots \right\}^*$$

when $a_n = 1,\ {}_k\alpha_l = {}^{\frac{k+l}{2}}C_{l-1}$;

The general result (III) gives

$$\frac{x^n}{n!} = \frac{n+1}{x} J_{n+1}(2x) + (n+1)\frac{n+5}{x} J_{n+3}(2x)$$
$$+ \frac{(n+2)(n+1)}{2!} \cdot \frac{n+5}{x} J_{n+5}(2x) + \dots \dots$$

or, $$\frac{x^{n+1}}{(n+1)!} = J_{n+1}(2x) + (n+3) J_{n+3}(2x) + \frac{(n+2)(n+5)}{2!} J_{n+5}(2x)$$
$$+ \frac{(n+3)(n+2)}{3!} . (n+7) J_{n+7}(2x) + \dots \dots,$$

a known relation.*

5. Now,

$$\tfrac{1}{2} D \{ M_n(x) \}^2 = M_n(x) M_{n-1}(x) - a_{n+1} M_n(x) M_{n+1}(x).$$
$$\tfrac{1}{2} D \{ M_{n+1}(x) \}^2 = M_{n+1}(x) M_n(x) - a_{n+2} M_{n+1}(x) M_{n+2}(x).$$
$$\tfrac{1}{2} D \{ M_{n+2}(x) \}^2 = M_{n+2}(x) M_{n+1}(x) - a_{n+3} M_{n+2}(x) M_{n+3}(x).$$
$$\dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots$$

Here multiply the first row by a_n, the second by $a_n\, a_{n+1}$, the third by $a_n\, a_{n+1}\, a_{n+2}$ and so on. Adding,

$$\tfrac{1}{2} D \left[a_n \{ M_n(x) \}^2 + a_n a_{n+1} \{ M_{n+1}(x) \}^2 + a_n a_{n+1} a_{n+2} \{ M_{n+2}(x) \}^2 + \dots \dots \right] = a_n M_n(x) M_{n-1}(x)$$

Now, if we assume that all the M-functions and the series on the left-hand side to be absolutely convergent, we get, on integration,

$$\tfrac{1}{2} \left[a_n \{ M_n(x) \}^2 + a_n a_{n+1} \{ M_{n+1}(x) \}^2 + \dots \dots \right] = \int a_n M_n(x) M_{n-1}(x)\, dx.$$

---

* *Ibid*, p. 19.

Here, put $n = 1$. Then

$$\tfrac{1}{2}\left[a_1 \{ M_1(x) \}^2 + a_1 a_2 \{ M_2(x) \}^2 + a_1 a_2 a_3 \{ M_3(x) \}^2 + \ldots \right]$$
$$= \int a_1 M_0(x) M_1(x)\, dx.$$
$$= -\tfrac{1}{2} \{ M_0(x) \}^2 + C.$$

$$\therefore \{ M_0(x) \}^2 + a_1 \{ M_1(x) \}^2 + a_1 a_2 \{ M_2(x) \}^2 + \ldots = \text{constant}$$

which in this case is 1 since the only constant on left-hand side is the 1 in the first term. Hence,

$$\{ M_0(x) \}^2 + a_1 \{ M_1(x) \}^2 + a_1 a_2 \{ M_2(x) \}^2 + \ldots = 1. \qquad \text{(IV)}$$

Illustrations.

Let $a_n = 1$. In that case (IV) gives

$$x^2 = J_1^2(2x) + 2^2 . J_2^2(2x) + 3^2 J_3^2(2x) + \ldots \ldots^{*}$$

If, however, $a_1 = 2$ and $a_2 = a_3 = \ldots = a_n = \ldots = 1$,

$$1 = J_0^2(2x) + 2 J_1^2(2x) + 2 J_2^2(2x) + \ldots \ldots$$
$$= J_0^2(x) + 2 J_1^2(x) + 2 J_2^2(x) + \ldots \ldots \ldots^{*}$$

When $a = n^2$, it leads to an obvious identity.

6. Addition-theorem for $M_0(x)$.

Assuming that the conditions for applying Taylor's Theorem are satisfied,

$$M_0(x+y)$$
$$= M_0(x) + y \cdot \frac{d}{dx} M_0(x) + \frac{y^2}{2!} \cdot \frac{d^2}{dx^2} M_0(x) + \ldots \quad \ldots$$
$$= M_0(x) + y \{ - a_1 M_1(x) \} + \frac{y^2}{2!} \cdot \left\{ a_1 a_2 M_2(x) - a_1 M_0(x) \right\}$$
$$+ \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$
$$+ (-1)^{2n-1} \frac{y^{2n-1}}{(2n-1)!} \Big\{ a_1 a_2 \ldots a_{2n-1} M_{2n-1}(x)$$
$$- a_1 a_2 \ldots a_{2n-3} \cdot {}_{2n-2}A_{2n-2} M_{2n-3}(x)$$
$$+ a_1 a_2 \ldots a_{2n-5} \cdot {}_{2n-4}A_{2n-3} M_{2n-5}(x)$$
$$- . + \ldots \quad \ldots \quad \ldots$$
$$+ (-1)^{n-1} a_1 \cdot {}_2A_n M_1(x) \Big\}$$

* *Vide* Gray and Matthews, p. 29.

$$+ (-1)^{2n} \frac{y^{2n}}{2n!} \Big\{ a_1 a_2 \dots a_{2n} \, M_{2n}(x)$$
$$- a_1 a_2 \dots a_{2n-2} \, {}_{2n-1}A_{2n-1} \,.\, M_{2n-2}(x)$$
$$+ a_1 a_2 \dots a_{2n-4} \, {}_{2n-3}A_{2n-2} \,.\, M_{2n-4}(x)$$
$$- . + \dots \quad \dots \quad \dots \quad \dots$$
$$+ (-1)^n \, {}_1A_n \, M_0(x) \Big\}$$
$$+ \dots \quad \dots \quad \dots \quad \dots \quad \dots$$
$$= M_0(x) \Big\{ 1 - a_1 \cdot \frac{y^2}{2!} + {}_1A_2 \frac{y^4}{4!} . + \dots + (-1)^n \, {}_1A_n \frac{y^{2n}}{2n!} + \dots \Big\}$$
$$- a_1 M_1(x) \Big\{ y - {}_2A_2 \frac{y^3}{3!} + {}_2A_3 \frac{y^5}{5!} - . \dots \quad \dots \Big\}$$
$$+ a_1 a_2 M_2(x) \Big\{ \frac{y^2}{2!} - {}_3A_3 \frac{y^4}{4!} + {}_3A_4 \frac{y^5}{5!} . - \dots \quad \dots \Big\}$$
$$- \dots + \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots$$
$$= M_0(x) M_0(y) - a_1 M_1(x) M_1(y) + a_1 a_2 M_2(x) M_2(y) - . + \dots \quad \dots \qquad \text{(V)}$$

It may not be out of place if we add another method of deriving the above formula with the aid of operators. Now, from equation (III) we get

$$1 = M_0(x) + \Sigma a_1 M_2(x) + \Sigma a_3 \Sigma a_1 M_4(x) + \dots \dots$$
$$x = M_1(x) + \Sigma a_2 M_3(x) + \Sigma a_4 \Sigma a_2 M_5(x) + \dots \dots$$
$$\frac{x^2}{2!} = M_2(x) + \Sigma a_3 M_4(x) + \Sigma a_5 \Sigma a_3 M_6(x) + \dots \dots$$
$$\dots \quad \dots \quad \dots \quad \dots \quad \dots$$

Multiply the first row by 1, the second by y, the third by y^2 and and add. We get

$$e^{xy} = M_0(x) + M_1(x) . y + M_2(x) \{ y^2 + \Sigma a_1 \} + M_3(x) \{ y^3 + y \Sigma a_2 \} + \dots \dots$$

Here change y into the operator $\frac{d}{dy}$.

Then $$e^{x \frac{d}{dy}} = M_0(x) + M_1(x) \frac{d}{dy} + M_2(x) \Big\{ \frac{d^2}{dy^2} + \Sigma a_1 \Big\} + \dots \dots$$

Now, apply this equality between operators to $M_0(y)$.

Now, $$e^{x \frac{d}{dy}} \{ M_0(y) \} = M_0(x + y).$$

On the right-hand side, we have

$$M_0(x)\,M_0(y) + M_1(x)\,\frac{d}{dy}\,(M_0(y)) + M_2(x)\left(\frac{d^2}{dy^2} + \Sigma\, a_1\right) M_0(y) + \dots\dots$$

$$= M_0(x)\,M_0(y) - a_1\,M_1(x)\,M_1(y) + a_1 a_2\,M_2(x) M_2(y) - . + \dots\dots$$

which is the same as (V).

In particular, put $y = x$. Then,

$$M_0(2x) = \{M_0(x)\}^2 - a_1\{M_1(x)\}^2 + a_1 a_2\{M_2(x)\}^2 - . + \dots\dots$$

Put $y = -x$,

$$1 = \{M_0(x)\}^2 + a_1\{M_1(x)\}^2 + a_1 a_2\{M_2(x)\}^2 + \dots\dots$$

which is the same as (IV).

Adding,

$$\frac{M_0(2x)+1}{2} = \{M_0(x)\}^2 + a_1 a_2\{M_2(x)\}^2 + a_1 a_2 a_3 a_4\{M_4(x)\}^2 + \dots\dots$$

and subtracting,

$$\frac{M_0(2x)-1}{2} = a_1\Big[\{M_1(x)\}^2 + a_2 a_3\{M_3(x)\}^2 + a_2 a_3 a_4 a_5\{M_5(x)\}^2 + \dots\dots\Big]$$

Illustrations.

Let $a_n = n^2$. Then (V) gives

$$\text{sech}\,(x+y) = \text{sech}\,x\ \text{sech}\,y\ \{1 - \tanh x\ \tanh y + \tanh^2 x\ \tanh^2 y - + \dots\dots\}$$

$$= \frac{\text{sech}\,x\ \text{sech}\,y}{1 + \tanh x\ \tanh y}$$

which gives the well-known formula

$$\cosh(x+y) = \cosh x\ \cosh y + \sinh x\ \sinh y.$$

Let $a_n = 1$. Then,

$$\frac{J_1(2x+2y)}{x+y} = \frac{J_1(2x)\,J_1(2y)}{xy} - 4\,.\,\frac{J_2(2x)\,J_2(2y)}{xy} + \frac{3^2\,J_3(2x)\,J_3(2y)}{xy} - . + \dots$$

tnat is

$$J_1(x+y). = \frac{2(x+y)}{xy}\left\{ J_1(x)\,J_1(y) - 2^2\,J_2(x)\,J_2(y) + 3^2\,J_3(x)\,J_3(y) - . + \ldots\ldots \right\}$$

Let $\qquad a_1 = 2,\ a_2 = a_3 = \ldots = 1.$

Then (V) gives finally

$$J_0(x+y) = J_0(x)\,J_0(y) - 2\,J_1(x)\,J_1(y) + 2\,J_2(x)\,J_2(y) - . + \ldots\ldots$$

a well-known addition formula for J_0.*

Let $\qquad a_{2n-1} = (2n-1)^2 k^2,\ a_{2n} = (2n)^2.$

Then $\qquad M_0(x) = dn\,(x, k),\ M_{2n-1}(x) = \dfrac{\text{sn}^{2n-1}(x, k)\ \text{cn}\,(x, k)}{(2n-1)\,!}$

and $\qquad M_{2n}(x) = \dfrac{\text{sn}^{2n}(x, k)\ \text{dn}\,(x, k)}{(2n)\,!}$

Now, (V) gives

$$\begin{aligned}\text{dn}\,(x+y) &= \text{dn}\,x\ \text{dn}\,y - 1^2\,k^2\,\frac{\text{sn}\,(x)\ \text{cn}\,x\ \text{sn}\,y\ \text{cn}\,y}{(1\,!)^2} \\ &\quad + 1^2 . 2^2 . k^2 . \frac{\text{sn}^2 x\ \text{dn}\,x\ \text{sn}^2 y\ \text{dn}\,y}{(2\,!)^2} - . + \ldots\ldots \\ &= \text{dn}\,x\ \text{dn}\,y\,\{ 1 + k^2\,\text{sn}^2 x\,\text{sn}^2 y + k^4\,\text{sn}^4 x\ \text{sn}^4 y + \ldots \} \\ &\quad - k^2\ \text{sn}\,x\ \text{sn}\,y\ \text{cn}\,x\ \text{cn}\,y\ \{ 1 + k^2\ \text{sn}^2 x\,\text{sn}^2 y + \ldots \} \\ &= \frac{\text{dn}\,x\ \text{dn}\,y - k^2\ \text{sn}\,x\ \text{sn}\,y\ \text{cn}\,x\ \text{cn}\,y}{1 - k^2\ \text{sn}^2 x\ \text{sn}^2 y},\end{aligned}$$

which is the well-known addition formula for $\text{dn}\,x$.†

If we take $a_{2n-1} = (2n-1)^2$ and $a_{2n} = (2n)^2\,k^2$, then

$$M_0(x) = \text{cn}\,(x, k),\ M_{2n-1}(x) = \frac{\text{sn}^{2n-1} x\ \text{dn}\,x}{(2n-1)\,!}$$

and $\qquad M_{2n}(x) = \dfrac{\text{sn}^{2n} x\ \text{cn}\,x}{(2n)\,!}$

In this case V leads to

$$\text{cn}\,(x+y) = \frac{\text{cn}\,x\ \text{cn}\,y - \text{sn}\,x\ \text{sn}\,y\ \text{dn}\,x\ \text{dn}\,y}{1 - k^2\ \text{sn}^2 x\ \text{sn}^2 y}.\ †$$

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* Gray and Matthews, p 24.

† Dixon: *Elliptic Functions*, 29.

7. By differentiating (V) with respect to x and cancelling $1 + \frac{dy}{dx}$, we have

$$
\begin{aligned}
-a_1 M_1(x+y) = & -a_1 M_1(x) M_0(y) \\
& -a_1 \{ M_0(x) - a_2 M_2(x) \} M_1(y) \\
& + a_1 a_2 \{ M_1(x) - a_3 M_3(x) \} M_2(y) \\
& - . + \ldots \ldots \\
= & -a_1 [\{ M_1(x) M_0(y) + M_0(x) M_1(y) \} \\
& \quad - a_2 \{ M_2(x) M_1(y) + M_1(x) M_2(y) \} \\
& \quad + a_2 a_3 \{ M_3(x) M_2(y) + M_2(x) M_3(y) \} \\
& \quad - . + \ldots \ldots]
\end{aligned}
$$

or

$$
\begin{aligned}
M_1(x+y) = & \{ M_0(x) M_1(y) + M_0(y) M_1(x) \} \\
& - a_2 \{ M_1(x) M_2(y) + M_2(x) M_1(y) \} \\
& + a_2 a_3 \{ M_2(x) M_3(y) + M_3(x) M_2(y) \} \\
& - . + \ldots \ldots] \qquad \ldots \qquad \ldots \qquad \text{(VI)}
\end{aligned}
$$

Illustrations.

Let $a_1 = 2,\ a_2 = a_3 = \ldots \ldots = 1$.

Then (VI) leads to

$$
\begin{aligned}
J_1(x+y) = & J_0(x) J_1(y) + J_1(x) J_0(y) \\
& - \{ J_1(x) J_2(y) + J_2(x) J_1(y) \} \\
& + \{ J_2(x) J_3(y) + J_3(x) J_2(y) \} \\
& - . + \ldots \ldots
\end{aligned}
$$

Put $y = x$

$$J_1(2x) = 2 \{ J_0(x) J_1(x) - J_1(x) J_2(x) + J_2(x) J_3(x) - . + \ldots \}$$

Let $a_1 = a_2 = \ldots \ldots = 1$. Then

$$
\begin{aligned}
\frac{2}{x+y} J_2(2x+2y) = & \left\{ \frac{J_1(2x) . 2 J_2(2y) + 2 J_2(2x) J_1(2y)}{xy} \right\} \\
& - \left\{ \frac{2 J_2(2x) . 3 J_3(2y) + 3 J_3(2x) 2 J_2(2y)}{xy} \right\} \\
& + . - \ldots . + \ldots
\end{aligned}
$$

or

$$
\begin{aligned}
J_2(x+y) . \frac{4xy}{x+y} = & 2 \{ J_1(x) J_2(y) + J_2(x) J_1(y) \} \\
& - 2 . 3 . \{ J_2(x) J_3(y) + J_3(x) J_2(y) \} \\
& + 3 . 4 . \{ J_3(x) J_4(y) + J_4(x) J_3(y) \} \\
& - . + \ldots \ldots
\end{aligned}
$$

Put $y = x$. Then

$$x \, J_2 (2x) = 1 . 2 . J_1 (x) J_2 (x) - 2 . 3 . J_2 (x) J_3 (x) + 3 . 4 . J_3 (x) J_4 (x) - . + \dots \dots \}$$

8. We conclude with finding the conditions that the M-function must satisfy so that

$$\frac{\{ M_n (x) \}^2}{M_{n-1} (x) M_{n+1} (x)}$$

may be independent of x for all values of n.

Now,

$$(\alpha) \quad \frac{M_{n-1} (x) (n-1)!}{x^{n-1}} = 1 - \Sigma a_n . \frac{x^2}{n (n+1)} + \Sigma a_n \Sigma a_{n+1} . \frac{x^4}{n (n+1) (n+2) (n+3)} - . + \dots \dots$$

$$(\beta) \quad \frac{M_n (x) . n!}{x^n} = 1 - \Sigma a_{n+1} . \frac{x^2}{(n+1) (n+2)} + \Sigma a_{n+1} \Sigma a_{n+2} \frac{x^4}{(n+1) (n+2) (n+3) (n+4)} - . + \dots \dots$$

$$(\gamma) \quad \frac{M_{n+1} (x) (n+1)!}{x^{n+1}} = 1 - \Sigma a_{n+2} \frac{x^2}{(n+2) (n+3)} + \Sigma a_{n+2} \Sigma a_{n+3} . \frac{x^4}{(n+2) (n+3) (n+4) (n+5)} - . + \dots \dots$$

Now, comparing the co-efficients of x^2 in the product of (α) and (γ) and in (β) we have

$$\frac{\Sigma a_n}{n (n+1)} - \frac{\Sigma a_{n+1}}{(n+1) (n+2)} = \frac{\Sigma a_{n+1}}{(n+1) (n+2)} - \frac{\Sigma a_{n+2}}{(n+2) (n+3)}.$$

This means that $\frac{\Sigma a_n}{n (n+1)}$ must be a linear function of n, say $C (n+2) + D$ so that $\Sigma a_n = Cn (n+1) (n+2) + Dn (n+1)$.

$$\therefore \quad a_n = 3 Cn (n+1) + 2 Dn,$$

$$= pn^2 + qn.$$

Now, it is known that if $a_n = pn^2 + qn$, then

$$M_n(x) = (\operatorname{sech} x\sqrt{p})^{-\frac{q}{p}+1} \cdot \frac{(\tanh x\sqrt{p})^n}{n!\,.(\sqrt{p})^n}.$$

and it can be easily seen that such M-functions satisfy the condition proposed in our problem. There is then no need to equate coefficients of other powers of x. Hence

$$a_n = pn^2 + qn \qquad \ldots \qquad \ldots \quad \text{(VII)}$$

is a sufficient condition that

$$\frac{\{M_n(x)\}^2}{M_{n-1}(x)\, M_{n+1}(x)}$$

be independent of x *for all values of* n. This is directly verifiable for the simple cases when (1) $p = 1,\ q = 0$ and (2) $p = 0,\ q = 1$

In conclusion, I wish to express my indebtedness to Mr. M Bhimasena Rao who was the first to define these functions and who directed my attention to their study.

On certain expansions of Elliptic Cylinder Functions

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1. In solving physical problems, such as for instance, diffraction by elliptic or hyperbolic cylinders, it is necessary to obtain solutions of Mathieu's equations which vanish at infinity. The necessity of this may be seen thus :—

If a set of plane waves defined by the function ψ_0 be incident on an elliptic or a hyperbolic cylinder, the problem is to determine ψ which, besides being solutions of the wave equation

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + k^2 U = 0 \quad \ldots \quad \ldots \quad (1)$$

will satisfy also the conditions

$$\psi_0 + \psi = 0 \quad \ldots \quad \ldots \quad (2)$$

on the boundary of the screen, and

$$\psi = 0 \quad \ldots \quad \ldots \quad (3)$$

at infinity.

In order that the problem may be solved, we have to use elliptic co-ordinates, given by

$$x + iy = h \cosh(\xi + i\eta)$$

The curves obtained by taking ξ = constant, give a system of ellipses and cutting them orthogonally, we have a system of confocal hyperbolas given by taking η = constant, the foci of the confocal system being $x = \pm h,\ y = 0$ and the range of values of ξ and η being given by

$$\xi \geqslant 0, \quad -\pi < \eta \leqslant \pi.$$

If we denote the radius vector of the point (ξ, η) by r, then

$$r = h\sqrt{\cosh^2 \xi - \sin^2 \eta} = \frac{h}{2}\sqrt{e^{2\xi} + 2\cos 2\eta + e^{-2\xi}}$$

and for large values of ξ, we may neglect $\sin^2 \eta$ in comparison with $\cosh^2 \xi$ and keep $r = h \cosh \xi$ or $\frac{1}{2} he^{\xi}$.

Hence it is evident that for satisfying (3), it is necessary to obtain forms of the solutions which will vanish for large values of r or ξ.

Various writers have worked at this problem during the last quarter of a century, notably Sieger* and John Dougall.† They all have expressed the solutions of the equation in a series of Bessel's Harmonics. In this paper I propose to investigate the forms in which the solutions can be expressed in a series of Bessel's functions, the method used being different from that of the above mentioned writers. It will be seen further that by the help of the method, it is possible to obtain various other forms of the solutions. The detailed discussion of these will form the subject-matter of a future paper.

2. By means of the transformation for elliptic cylindrical co-ordinates, the wave equation (1) resolves itself into finding the solutions of the two equations:

$$\frac{\partial^2 G}{\partial \eta^2} + (A + 16q \cos 2\eta) G = 0 \quad \dots \quad \dots \quad (4)$$

$$\frac{\partial^2 F}{\partial \xi^2} - (A + 16q \cosh 2\xi) F = 0 \quad \dots \quad \dots \quad (5)$$

where $U = G(\eta) . F(\xi)$ and $32\, q = - h^2 k^2$, and A is arbitrary constant.

It will be seen that the solutions of the equation (5) can be obtained by writing $i\xi$ for η in, the solutions of (4); for this reason the solutions of (4) have been investigated first by Mathieu and later on by various other writers and it is called Mathieu's Equation. For a given set of values of A, the solutions of the equation have been obtained. They are:—

(i) Solutions of the first kind (*periodic*).

$$\left.\begin{array}{r} ce_0(\eta, q),\ ce_1(\eta, q), \dots \dots ce_m(\eta, q), \dots \dots \\ se_1(\eta, q), \dots \dots se_m(\eta, q), \dots \dots \end{array}\right\} \quad \dots \quad (6)$$

(ii) Solutions of the second kind (*non-periodic*)

$$\left.\begin{array}{r} i\eta_0(\eta, q),\ i\eta_1(\eta, q), \dots \dots i\eta_m(\eta, q), \dots \dots \\ j\eta_1(\eta, q), \dots \dots j\eta_m(\eta, q), \dots \dots \end{array}\right\} \quad \dots \quad (7)$$

* Sieger: *Annalen der Physik*, Bd. 27.

† J. Dougall: *Proc. Edin. Math. Soc.*, Vol XXXIV, pp. 191—196.

3. The periodic solutions (6) can be obtained by various methods and expressed in a series of sines and cosines of multiples of η. Following Heine, they may be classified into four groups, *viz.*

$$\left.\begin{aligned}
&\text{(i)} \quad ce_{2r}(\eta, q) = \alpha_{0,r} + \sum_{n=1}^{\infty} \alpha_{n,r} \cos 2n\eta \\
&\text{(ii)} \quad ce_{2r+1}(\eta, q) = \sum_{n=0}^{\infty} \beta_{n,r} \cos (2n+1)\eta, \\
&\text{(iii)} \quad se_{2r+1}(\eta, q) = \sum_{n=0}^{\infty} \gamma_{n,r} \sin (2n+1)\eta, \\
&\text{(iv)} \quad se_{2r}(\eta, q) = \sum_{n=1}^{\infty} \delta_{n,r} \sin 2n\eta.
\end{aligned}\right\} \quad \ldots \quad (8)$$

with the following recurrence formulae for finding the co-efficients, *viz.*

$$\text{(i)} \quad \alpha_{n+1,r} + \alpha_{n-1,r} = \frac{4n^2 - A}{8q} \alpha_{n,r}$$

$$\text{(ii)} \quad \beta_{n+1,r} + \beta_{n-1,r} = \frac{(2n+1)^2 - A}{8q} \beta_{n,r}$$

$$\text{(iii)} \quad \gamma_{n+1,r} + \gamma_{n-1,r} = \frac{(2n+1)^2 - A}{8q} \gamma_{n,r}$$

$$\text{(iv)} \quad \delta_{n+1,r} + \delta_{n-1,r} = \frac{4n^2 - A}{8q} \delta_{n-1,r}$$

The above series converge only for definite values of the parameter A of the differential equation and have for other values, no meaning. Now for convergence, it is necessary that $\lim\limits_{n \to \infty} \alpha_n = 0$, and three similar relations should hold. As the recurrence formulae will give expressions for α_n, β_n, etc. in a series of rational and integral functions of A, the above conditions for convergence will furnish equations for determining A. Instead of proceeding as above, the special values of A for which the solutions exist have been found out by other means.*

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* E. L. Mathieu : *Liouville's Jour.* XIII.
Whittaker : *Fifth International Congress of Math., 1912.*
E. Linsay Ince : *Proc Edin Math Soc.*, Vol. XXXIII, 1914-15.
S. C. Dhar : *American Jour. of Math.*, Vol. XLV, 1923.

Assuming that the special values of A for which the solutions exist have been found out and substituted in the recurrence formulae, we will, following the method of Dougall,* find the form of $\alpha_{n,r}$ only.

4. The special value of the parameter A, for which the solution $ce_{2r}(\eta, q)$ has been constructed, is

$$(2r)^2 + \frac{32q^2}{4r^2-1} - \frac{2^7(20r^2+7)q^4}{(4r^2-1)^3(4r^2-4)} - \ldots \text{ etc.} \qquad \ldots \quad (9)$$

which we may denote by $(2R)^2$.

Further, if we write $2q = \lambda^2$, the recurrence-formula (1) becomes

$$\alpha_{n+1,r} + \alpha_{n-1,r} = \frac{n^2 - R^2}{\lambda^2}\alpha_{n,r} \qquad \ldots \quad (10)$$

Hence by Dougall's formulae,† we obtain

$$\frac{\alpha_{n,r}}{\alpha_{0,r}} = \frac{\lambda^{2n}}{\Pi(n+R)\,\Pi(n-R)}\left\{1 - \lambda^4 A_n^{(1)} + \lambda^8 A_n^{(2)} + \lambda^{12} A_n^{(3)} + \ldots \ldots\right\} \qquad \ldots \quad (11)$$

where

$$A_n^{(q)} = \sum_{p_1=0}^{\infty}\sum_{p_2=2}^{\infty}\sum_{p_3=2}^{\infty}\cdots\sum_{p_q=2}^{\infty} a_{n+p_1}\, a_{n+p_1+p_2} \cdots a_{n+p_1+p_2\cdots+p_q},$$

$$a_n = \frac{1}{\{(n+1)^2 - R^2\}\{(n+2)^2 - R^2\}}.$$

For approximate evaluation of the series for small values of q, these forms are suitable. We will, following Dougall, denote the right-hand side expression by $\phi_{2r}(n)$. Hence

$$ce_{2r}(\eta, q) = \alpha_{0,r}\sum_{n=0}^{\infty}\phi_{2r}(n)\cos 2n\eta. \qquad \ldots \quad (12)$$

5. Prof. Whittaker‡ has obtained certain homogeneous integral equations connected with the elliptic cylinder functions and has shown

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* Dougall : *loc. cit.*

† J. Dougall : *loc. cit.*

‡ Whittaker ; *loc. cit.*

how these integral equations may be utilised to construct the cylinder functions. In a paper * published in the *Journal of the Department of Science*, Calcutta University, it was proved that a general form of the integral equation connected with these elliptic cylinder functions exists and that Whittaker's types are but particular cases of the general form and correspond to special forms of the kernel

It was shown that

If $G(\eta)$ be a periodic solution of the differential equation (4) and if further $U(\xi, \eta)$ be an integral of the equation

$$\frac{\partial^2 U}{\partial \xi^2} + \frac{\partial^2 U}{\partial \eta^2} + h^2 k^2 (\cosh^2 \xi - \cos^2 \eta)\, U = 0. \quad \ldots \quad (13)$$

then $G(\eta)$ satisfies the homogeneous integral equation

$$G(\eta) = \lambda \int_{-\pi}^{\pi} U(i\theta, \eta)\, G(\theta)\, d\theta. \quad \ldots \quad (14)$$

the kernel $U(i\theta, \eta)$ being symmetrical in θ and η and also periodic.

6. The result stated above may be easily proved thus :—Suppose $U(\xi, \eta)$ is an integral of the above differential equation other than those given by (6) and (7). Then if $\phi(\theta)$ be analytic in $(-\pi, \pi)$, the integral defined by

$$G(\eta) = \int_{-\pi}^{\pi} U(i\theta, \eta)\, \phi(\theta)\, d\theta$$

will satisfy the differential equation (4), if

$$\int_{-\pi}^{\pi} \left(\frac{\partial^2}{\partial \eta^2} + A + 16q \cos 2\eta\right) . U(i\theta, \eta) . \phi(\theta)\, d\theta = 0,$$

i.e. if $\displaystyle\int_{-\pi}^{\pi} \left(\frac{\partial^2 U}{\partial \eta^2} + 16q \cos 2\eta . U\right) . \phi(\theta)\, d\theta +$

$$A \int_{-\pi}^{\pi} U(i\theta, \eta)\, \phi(\theta)\, d\theta = 0.$$

But from (13) we obtain on substituting $\xi = i\theta$, and $32q = -h^2k^2$

$$\frac{\partial^2 U}{\partial \eta^2} + 16q \cos 2\eta . U = \frac{\partial^2 U}{\partial \theta^2} + 16q \cos 2\theta . U.$$

* S. C. Dhar : "On some integral equations connected with elliptic cylinder functions"- *Jour. Dept. of Science* (Cal. University) Vol. III.

Hence the integral will satisfy the differential equation (4), if

$$\int_{-\pi}^{\pi} \phi(\theta) . \left(\frac{\partial^2 U}{\partial \theta^2} + 16q \cos 2\theta . U \right) d\theta +$$

$$A \int_{-\pi}^{\pi} U(i\theta, \eta) \, \phi(\theta) \, d\theta = 0$$

which on integrating by parts, reduces to

$$\left[\frac{\partial U}{\partial \theta} . \phi - U \frac{\partial \phi}{\partial \theta} \right]_{-\pi}^{\pi} +$$

$$\int_{-\pi}^{\pi} U(i\theta, \eta) \left\{ \frac{\partial^2 \phi}{\partial \theta^2} + (A + 16q \cos 2\theta) . \phi \right\} d\theta = 0.$$

Now, if U and ϕ be such that

$$U(i\pi, \eta) = U(-i\pi, \eta)$$

$$\left[\frac{\partial}{\partial \theta} U(i\theta, \eta) \right]_{\theta = \pi} = \left[\frac{\partial}{\partial \theta} U(i\theta, \eta) \right]_{\theta = -\pi}$$

$$\phi(\pi) = \phi(-\pi)$$

$$\left[\frac{\partial \phi}{\partial \theta} \right]_{\theta = \pi} = \left[\frac{\partial \phi}{\partial \theta} \right]_{\theta = -\pi}$$

and further if

$$\frac{\partial^2 \phi}{\partial \theta^2} + (A + 16q \cos 2\theta) \, \phi = 0$$

then both the integral and the integrated parts va ish and the integral gives solutions of (4). In particular if U and ϕ be *periodic* and ϕ satisfies the above differential equation, then the above conditions are satisfied. ϕ is thus a periodic elliptic cylinder function of θ formed with the same constants A and q as $G(\eta)$ itself. As there does not exist more than one distinct periodic solution of (4) with the same constants A and q, ϕ θ) must be (save for a constant multiplier, $G(\eta)$ with η replaced by θ, that is, $\phi(\theta) = \lambda \, G(\theta)$. Hence the periodic solution of (G) must satisfy the integral equation

$$G(\eta) = \lambda \int_{-\pi}^{\pi} U(i\theta, \eta) \, G(\theta) \, d\theta.$$

the kernel $U(i\theta, \eta)$ being *symmetrical* in θ and η and *periodic.*

Now, various forms of $U(i\theta, \eta)$ and the integrals associated with them have been given in the paper* cited above and also in another

* *Jour. Dept. of Science*, Cal. University.

paper* published in the *Tohoku Math. Jour.* By means of these kernels and the integral equation just established, we shall try to obtain expressions for the integrals of (4) or (5) in a series of Bessel's Functions. Some of these were obtained by other writers by different methods, while others are new.

6. (i) If we take the kernel as $\cos(kh\cos\theta\cos\eta)$, then we shall obtain the integral equation

$$ce_{2r}(\eta, q) = \lambda_r \int_{-\pi}^{\pi} \cos(kh\cos\theta\cos\eta)\, ce_{2r}(\theta, q)\, d\theta$$

$$= \lambda_r \sum_{n=0}^{\infty} a_{n,r} \int_{-\pi}^{\pi} \cos(kh\cos\theta\cos\eta)\cos 2n\theta\, d\theta.$$

The value of λ_r can be obtained at once by writing $\eta = \frac{\pi}{2}$ in the above equation, *i e.*

$$ce_{2r}\left(\frac{\pi}{2}, q\right) = \lambda_r \sum_{n=0}^{\infty} a_{n,r} \int_{-\pi}^{\pi} \cos 2n\theta\, d\theta$$

$$= \lambda_r \cdot a_{0,r} \cdot 2\pi.$$

$$\therefore \quad \lambda_r = \frac{1}{2\pi} \cdot \frac{ce_{2r}\left(\frac{\pi}{2}, q\right)}{a_{0,r}}$$

Hence

$$ce_{2r}(\eta, q) = ce_{2r}\left(\frac{\pi}{2}, q\right) \cdot \frac{1}{2\pi} \sum_{n=0}^{\infty} \frac{a_{n,r}}{a_{0,r}} \times$$

$$\int_{-\pi}^{\pi} \cos(kh\cos\theta\cos\eta)\cos 2n\theta\, d\theta,$$

But by a modification of Bessel's integral, we have

$$J_{2n}(z) = \frac{(-1)^n}{2\pi} \int_{-\pi}^{\pi} \cos(z\cos\theta)\cos 2n\theta\, d\theta.$$

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* S. C. Dhar : *Tohoku Math. Jour.*, Vol. 24, 1924.

Therefore

$$ce_{2r}(\eta, q) = ce_{2r}\left(\frac{\pi}{2}, q\right) \sum_{n=0}^{\infty} (-1)^n \frac{\alpha_{n,r}}{\alpha_{0,r}} J_{2n}(kh \cos \eta)$$

$$= ce_{2r}\left(\frac{\pi}{2}, q\right) \sum_{n=0}^{\infty} (-1)^n \phi_{2r}(n) J_{2n}(kh \cos \eta) \ldots \quad (15)$$

or $$ce_{2r}(i\xi, q) = ce_{2r}\left(\frac{\pi}{2}, q\right) \sum_{n=0}^{\infty} (-1)^n \phi_{2r}(n) J_{2n}(kh \cosh \xi)$$

(ii) If we take $U(i\theta, \eta) = \cosh(kh \sin\theta \sin\eta)$, we get

$$ce_{2r}(\eta, q) = \lambda'_r \int_{-\pi}^{\pi} \cosh(kh \sin\theta \sin\eta)\, ce_{2r}(\theta, q)\, d\theta$$

$$= \lambda'_r \sum_{n=0}^{\infty} \alpha_{n,r} \int_{-\pi}^{\pi} \cosh(kh \sin\theta \sin\eta) \cos 2n\theta \, d\theta.$$

To obtain λ'_r, put $\eta = 0$ in the above relation and we get

$$ce_{2r}(0, q) = \lambda'_r \sum_{n=0}^{\infty} \alpha_{n,r} \int_{-\pi}^{\pi} \cos 2n\theta \, d\theta = \lambda'_r \alpha_{0,r} \cdot 2\pi$$

$$\therefore \qquad \lambda'_r = \frac{1}{2\pi} \cdot \frac{ce_{2r}(0, q)}{\alpha_{0,r}}.$$

Hence

$$ce_{2r}(\eta, q) = ce_{2r}(0, q) \sum_{n=0}^{\infty} \frac{\alpha_{n,r}}{\alpha_{0,r}} \frac{1}{2\pi} \int_{-\pi}^{\pi} \cosh(kh \sin\theta \sin\eta) \cos 2n\theta \, d\theta$$

$$= ce_{2r}(0, q) \sum_{n=0}^{\infty} \frac{\alpha_{n,r}}{\alpha_{0,r}} \cdot \frac{1}{2\pi} \int_{-\pi}^{\pi} \cos(ikh \sin\theta \sin\eta) \cos 2n\theta \, d\theta.$$

But $$J_{2n}(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \cos(z \sin\theta) \cos 2n\theta \, d\theta,$$

$$\therefore \quad ce_{2r}(\eta, q) = ce_{2r}(0, q) \cdot \sum_{n=0}^{\infty} \frac{a_{n,r}}{a_{0,r}} \cdot J_{2n}(ikh \sin \eta)$$

$$= ce_{2r}(0, q) \sum_{n=0}^{\infty} \phi_{2r}(n) \cdot J_{2n}(ikh \sin \eta)$$

Hence also

$$ce_{2r}(i\xi, q) = ce_{2r}(0, q) \sum_{n=0}^{\infty} \phi_{2r}(n)\, J_{2n}(-kh \sin \xi)$$

$$= ce_{2r}(0, q) \cdot \sum_{n=0}^{\infty} \phi_{2r}(n)\, J_{2n}(kh \sin \xi). \qquad \ldots \quad (16)$$

(iii) Further if we take $U(i\theta, \eta) = J_0\left(kh \sqrt{\tfrac{1}{2}(\cos 2\theta + \cos 2\eta)}\right)$, then

$$ce_{2r}(\eta, q) = \mu_r \int_{-\pi}^{\pi} J_0\left(kh \sqrt{\tfrac{1}{2}(\cos 2\theta + \cos 2\eta)}\right) ce_{2r}(\theta, q)\, d\theta$$

$$= \mu_r \sum_{n=0}^{\infty} a_{n,r} \int_{-\pi}^{\pi} J_0\left(kh \sqrt{\tfrac{1}{2}(\cos 2\theta + \cos 2\eta)}\right) \cos 2n\theta\, d\theta.$$

Now, putting $\eta = 0$ in the above relation, we obtain

$$ce_{2r}(0, q) = \mu_r \sum_{n=0}^{\infty} a_{n,r} \int_{-\pi}^{\pi} J_0(kh \cos \theta) \cos 2n\theta\, d\theta.$$

To find an expression for μ_r we proceed as follows :

We know that

$$J_0(z) = \frac{1}{2\pi} \cdot \int_{-\pi}^{\pi} \cos(z \sin \phi)\, d\phi,$$

or

$$J_0(kh \cos \theta) = \frac{1}{2\pi} \cdot \int_{-\pi}^{\pi} \cos(kh \cos \theta \sin \phi)\, d\phi.$$

Therefore

$$\int_{-\pi}^{\pi} J_0 (kh \cos \theta) \cos 2n\theta \, d\theta$$

$$= \frac{1}{2\pi} \cdot \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \cos(kh \cos \theta \sin \phi) \cos 2n\theta \, d\theta d\phi$$

$$= \frac{1}{2\pi} \cdot \int_{-\pi}^{\pi} d\phi \int_{-\pi}^{\pi} \cos (kh \cos \theta \sin \phi) \cos 2n\theta \, d\theta$$

$$= \int_{-\pi}^{\pi} (-1)^n \cdot J_{2n} (kh \sin \phi) \, d\phi$$

$$\therefore \qquad ce_{2r} (0, q) = \mu_r \int_{-\pi}^{\pi} \Sigma (-1)^n \alpha_{n, r} J_{2n} (kh \sin \phi) \, d\phi$$

$$= \mu_r \cdot \frac{\alpha_{0, r}}{ce_{2r} \left(\frac{\pi}{2}, q \right)} \int_{-\pi}^{\pi} ce_{2r} \left(\frac{\pi}{2} - \phi, q \right) d\phi. \text{ from (15)}$$

i.e.

$$\mu_r = \frac{1}{\alpha_{0, r}} \cdot \frac{ce_{2r} (0, q) \cdot ce_{2r} \left(\frac{\pi}{2}, q \right)}{\int_{-\pi}^{\pi} ce_{2r} \left(\frac{\pi}{2} - \phi, q \right) d\phi}. \qquad \ldots \quad (17)$$

Now, putting $i\xi$ for η, we obtain

$$ce_{2r} (i\xi, q) = \mu_r \sum_{n=0}^{\infty} \alpha_{n, r} \int_{-\pi}^{\pi} J_0 (kh \sqrt{\tfrac{1}{2} (\cos 2\theta + \cosh 2\xi)}) \cos m\theta \, d\theta$$

But by Neumann's addition theorem,

$$J_0 (kh \sqrt{\tfrac{1}{2} (\cos 2\theta + \cosh 2\xi)})$$

$$= J_0 \left(\frac{kh}{2} \sqrt{e^{2\xi} + 2 \cos 2\theta + e^{-2\xi}} \right)$$

$$= \sum_{m=0}^{\infty} (-1)^m \varepsilon_m J_m (\tfrac{1}{2} kh e^{\xi}) J_m (\tfrac{1}{2} kh e^{-\xi}) \cos 2m\theta,$$

where ε_m is the Neumann factor which is equal to 2 when $m \neq 0$ and is equal to 1, when m is zero.

Hence

$$ce_{2r}(i\xi, q) = \mu_r \sum_{n\ 0}^{\infty} \sum_{m\ 0}^{\infty} \alpha_{n,r} (-1)^m \varepsilon_m J_m (\tfrac{1}{2} kh e^{-\xi}) J_m (\tfrac{1}{2} kh e^{\xi})$$

$$\int_{-\pi}^{\pi} \cos 2n\theta \cos 2m\theta \, d\theta.$$

But $\int_{-\pi}^{\pi} \cos 2m\theta \cos 2n\theta \, d\theta = 0$, if $m \neq n$ and equal to π, when $m = n$.

Therefore

$$ce_{2r}(i\xi, q) = \mu_r . 2\pi \sum_{n=0}^{\infty} (-1)^n \alpha_{n,r} J_n \left(\frac{kh}{2} e^{\xi}\right) J_n \left(\frac{kh}{2} e^{-\xi}\right)$$

$$= \frac{ce_{2r}(0, q) . ce_{2r}\left(\frac{\pi}{2}, q\right)}{\frac{1}{2\pi} \int_{-\pi}^{\pi} ce_{2r}\left(\frac{\pi}{2} - \phi, q\right) d\phi} \times$$

$$\sum_{n=0}^{\infty} (-1)^n \phi_{2r}(n) J_n \left(\frac{kh}{2} e^{\xi}\right) J_n \left(\frac{kh}{2} e^{-\xi}\right) \dots \quad (18)$$

(iv) Again by taking

$$U(i\theta, \eta) = J_2 [kh \sqrt{\tfrac{1}{2}(\cos 2\theta + \cos 2\eta)}] \cos 2\phi,$$

we obtain

$$ce_{2r}(\eta, q) = \mu'_r \int_{-\pi}^{\pi} J_2 . \cos 2\phi . ce_{2r}(\theta, q) \, d\theta.$$

$$= \mu'_r \sum_{n=0}^{\infty} \int_{-\pi}^{\pi} \alpha_{n,r} J_2 \cos 2\phi . \cos 2n\theta \, d\theta$$

$$\therefore \quad ce_{2r}(i\xi, q) = \mu'_r \sum_{n=0}^{\infty} \alpha_{n,r} \int_{-\pi}^{\pi} J_2 (\tfrac{1}{2} kh \sqrt{e^{2\xi} + 2\cos 2\theta + e^{-2\xi}}) \times$$

$$\cos 2\phi \cos 2n\theta \, d\theta. \quad \dots \quad (19)$$

$$\text{Now}\quad J_2\left[\tfrac{1}{2}kh\sqrt{e^{2\xi}+2\cos 2\eta+e^{-2\xi}}\right]\cos 2\phi$$

$$=\frac{J_2\left[\tfrac{1}{2}kh\sqrt{e^{2\xi}+2\cos 2\eta+e^{-2\xi}}\right]}{\tfrac{1}{4}h^2(e^{2\xi}+2\cos 2\eta+e^{-2\xi})}\,r^2\cos 2\phi$$

$$=\frac{J_2(kr)}{r^2}\times h^2(\cosh 2\xi\cos^2\eta-\sinh^2\xi)$$

Now, by Gegenbauer's Addition Theorem,* we have

$$\frac{J_2(kr)}{(kr)^2}$$

$$=2^2\sum_{m=0}^{\infty}(-1)^m(m+2)\frac{J_{2+m}(\tfrac{1}{2}khe^{\xi})\,J_{2+m}(\tfrac{1}{2}khe^{-\xi})}{(\tfrac{1}{2}kh)^2}C_m^2(\cos 2\eta). \quad \dots \quad (20)$$

where $C_m^2(\cos 2\eta)$ denotes the co-efficient of α^m in the expansion of $(1-2\alpha\cos 2\eta+\alpha^2)^{-2}$ in ascending powers of α. This co-efficient is evidently seen to be equal to

$$(m+1).1.2\cos 2m\eta+m.2.2\cos 2(m-2)\eta$$
$$+(m-1).3.2\cos 2(m-4)\eta+\quad\dots\quad(21)$$

Hence from (19) and (20), we have

$$ce_{2r}(i\xi,q)=\mu'_r\sum_{m=0}^{\infty}(-1)^m(2+m)J_{2+m}(\tfrac{1}{2}khe^{\xi})J_{2+m}(\tfrac{1}{2}khe^{-\xi})$$
$$\times\{\alpha''_{m,r}\cosh 2\xi-\alpha'_{m,r}\sinh^2\xi\}\quad\dots\quad(22)$$

where $$\alpha''_{m,r}=\sum_{n=0}^{\infty}\alpha_{n,r}\int_{-\pi}^{\pi}\cos^2\theta\cos 2n\theta\cdot C_m^2(\cos 2\theta)\,d\theta$$

and $$\alpha'_{m,r}=\sum_{n=0}^{\infty}\alpha_{n,r}\int_{-\pi}^{\pi}\cos 2n\theta\, C_m^2(\cos 2\theta)\,d\theta$$

Now $$\alpha''_{m,r}=\sum_{n=0}^{\infty}\alpha_{n,r}\int_{-\pi}^{\pi}\{\cos 2(n+1)\theta+2\cos 2n\theta$$
$$+\cos 2(n-1)\theta\}\,C_m^2(\cos 2\theta)\,d\theta$$

* Watson: *Theory of Bessel's Functions*, p. 363.

$$= \tfrac{1}{2}\pi \{ (m+1).1 (\alpha_{m+1,r} + 2\alpha_{m,r} + \alpha_{m-1,r})$$
$$+ m.2 (\alpha_{m-1,r} + 2\alpha_{m-2,r} + \alpha_{m-3,r})$$
$$+ (m-1).3 (\alpha_{m-3,r} + 2\alpha_{m-4,r} + \alpha_{m-5,r})$$
$$+ \quad \ldots \quad \ldots \quad \ldots \} \quad \ldots \quad (23)$$

which is obtained by using the relation (21) and integrating.

Similarly

$$\alpha'_{m,r} = \tfrac{1}{2}\pi \{ (m+1) \; 1\, \alpha_{m,r} + m.2.\, \alpha_{m-2,r} + (m-1).3,\, \alpha_{m-4,r} + \ldots \} \quad \ldots \quad (24)$$

Similar expansions of $ce_{2r}(i\xi, q)$ by using $J_3 \cos 3\phi$, $J_4 \cos 4\phi$ ... etc. for the kernels can be obtained.

(v) Again, if we take the kernel in the form

$$U(i\xi, \eta) = Y_0 [kh \sqrt{\tfrac{1}{2}(\cos 2\xi + \cos 2\eta)}]$$

then

$$ce_{2r}(i\xi, q) = \lambda'_r \int_{-\pi}^{\pi} Y_0 (\tfrac{1}{2}kh \sqrt{e^{2\xi} + 2\cos 2\theta + e^{-2\xi}}\,) \, ce_{2r}(\theta, q)\, d\theta$$

Here also by the addition theorem, we get

$$Y_0 (\tfrac{1}{2} kh \sqrt{(e^{2\xi} + 2\cos 2\theta + e^{-2\xi})}\,)$$

$$= \sum_{m=0}^{\infty} (-)^m \varepsilon_m Y_n (\tfrac{1}{2} kh\, e^{\xi}) J_n (\tfrac{1}{2} kh\, e^{-\xi}) \cos 2m\theta.$$

Hence substituting the above expression for Y_0 and also for $ce_{2r}(\theta, q)$ and integrating, we get

$$ce_{2r}(i\xi, q) = \lambda'_r . 2\pi \sum_{n=0}^{\infty} (-1)^n \varepsilon_n \alpha_{n,r}\, Y \left(\frac{kh}{2} e^{\xi} \right) J_n \left(\frac{kh}{2} e^{-\xi} \right) . (25)$$

It should be noticed in passing that the above expression (25) was obtained by Sieger as the expansion for the solution of the second kind of Mathieu's equation, which cannot be the case as is evident from above.

7. It will be now easy to show that the expressions (15) to (25), which we have obtained for the integrals of the first kind, vanish when ξ is made infinitely large.

For when x is very large, the asymptotic expansions for J_n and Y_n are given by the formulae

$$J_n(x) = \sqrt{\frac{2}{\pi x}} \cdot \cos\left\{\frac{(2n+1)\pi}{4} - x\right\}$$

$$Y_n(x) = \sqrt{\frac{2}{\pi x}}\left(\log 2 - r\right)\cos\left\{\frac{(2n+1)\pi}{4} - x\right\}$$

$$- \sqrt{\frac{\pi}{2x}} \sin\left\{\frac{(2n+1)\pi}{4} - x\right\}$$

Hence when ξ is made very great,

$$J_n\left(\tfrac{1}{2}kh\,e^{\xi}\right) = 0 \text{ and } Y_n\left(\tfrac{1}{2}kh\,e^{\xi}\right) = 0.$$

Further when x is very small, we have

$$J_n(x) = \frac{x^n}{2^n \cdot n!}$$

Hence, when ξ is very large,

(i) $J_0\left(\frac{kh}{2}e^{-\xi}\right) = 1,\ J_n\left(\frac{kh}{2}e^{-\xi}\right) = 0,\ [n > 0].$

(ii) $J_n(kh\cosh\xi) = 0,\ n \geqslant 0,\ J_n(kh\sinh\xi) = 0.$

(iii) $\cosh 2\xi \,.\, J_2\left(\tfrac{1}{2}kh\,e^{-\xi}\right) = \tfrac{1}{2}\,\frac{k^2h^2}{4.8},$

$\cosh 2\xi\ J_{2+n}\left(\frac{kh}{2}e^{-\xi}\right) = 0, \qquad [n \geqslant 1]$

Also $\sinh^2\xi \,.\, J_2\left(\tfrac{1}{2}kh\,e^{-\xi}\right) = \tfrac{1}{4}\,.\,\frac{k^2h^2}{4.8},$

$\sinh^2\xi\ J_{2+n}\left(\frac{kh}{2}e^{-\xi}\right) = 0, \qquad [n \geqslant 1].$

Therefore, on examining the expressions (15), (16), (18), (22) and (25), it will be found that they all vanish when ξ is infinitely large.

College of Science,
Nagpur, *April*, 1926.

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NOTES AND QUESTIONS.

Notes and Questions.

A New Proof of Von Staudt's Theorem.

1. So many proofs* of this remarkable theorem have been given, that one more may not be out of place. The proof given here has been suggested by Niels Nielsen's *Traite Elementaire des Nombres de Bernoulli* (1923) pp. 210—215. It will be noticed, however, that the gist of the proof, which is contained in § 3 differs altogether from Nielsen's, both in length and in method.

2. Let $S_n(p)$ denote $1^n + 2^n + \dots + p^n$ *where* p *is a prime*, and n a positive integer. Then it is well-known that

$$S_n(p) \equiv \begin{cases} -1 \\ 0 \end{cases} \pmod p$$

according as $p-1$ is, or is not, a divisor of n.

Now let ω denote the product of all the primes up to $2n+1$ ($2n+1$ included if it is a prime). Then if p be any one of these primes

$$\begin{aligned} S_{2n}(\omega) = \; & 1^{2n} + 2^{2n} + \dots \; p^{2n} \\ & + (p+1)^{2n} + (p+2)^{2n} + \dots \; (2p)^{2n} \\ & + (2p+1)^{2n} + (2p+2)^{2n} + \dots \; (3p)^{2n} \\ & + \dots \quad \dots \quad \dots \quad \dots \\ & (\omega - p + 1)^{2n} + (\omega - p + 2)^{2n} + \dots \; \omega^{2n} \end{aligned}$$

There are $\frac{\omega}{p}$ rows, and the expression in each row is obviously $\equiv S_{2n}(p) \pmod p$;

3. *Case i.* Let $p-1$ be a factor of $2n$.

Then $S_{2n}(p) \equiv -1 \pmod p$ and hence $S_{2n}(\omega) \equiv -\frac{\omega}{p} \pmod p$.

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* Sechlafli : *Quarterly Journal of Mathematics* (1864).
Schwering : ***Mathematische Annalen***, Vol. 52 (1899).
Kluyver : ***Ibid.*** Vol. 53 (1900).
Shovelton : *Messenger of Maths*, Vol. 44, (1914), p. 24.

$$\therefore \quad \frac{S_{2n}(\omega)}{\omega} = \frac{-\frac{\omega}{p} + k.p}{p.\frac{\omega}{p}} = -\frac{1}{p} + \frac{k}{\frac{\omega}{p}}.$$

Case ii. If $p-1$ be not a factor of $2n$, then $S_{2n}(p) \equiv 0 \pmod p$.

$$\therefore \quad \frac{S_{2n}(\omega)}{\omega} = \frac{k.p}{p.\frac{\omega}{p}} = \frac{0}{p} + \frac{k}{\frac{\omega}{p}}.$$

It follows at once that

$$\frac{S_{2n}(\omega)}{\omega} = \text{an integer} - \left(\frac{1}{2} + \frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \ldots + \frac{1}{\lambda_t}\right)$$

where $\lambda_1, \lambda_2 \ldots \lambda_t$ are all the primes from 0 to $2n+1$ such that each of these *minus* 1, is a factor of $2n$.

4. If B_n is the nth Bernoullian number we will now prove Von Staudt's theorem that

$$(-1)^n B_n = \text{an integer} + \left(\frac{1}{2} + \frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \ldots + \frac{1}{\lambda_t}\right) \quad .. \quad (1)$$

The theorem holds for $n=1$ since $-B_1 = -\frac{1}{6} = -1 + \frac{1}{2} + \frac{1}{3}$. Also

$$\frac{S_{2n}(\omega)}{\omega} = \frac{\omega^{2n}}{2n+1} + \frac{\omega^{2n-1}}{2} + \sum_{s=1}^{n} \frac{(-1)^{s-1}}{2n+1} \cdot \binom{2n+1}{2s} B_s \cdot \omega^{2n-2s}. \quad (2)$$

Supposing Staudt's theorem has been proved for $B_1, B_2 \ldots \ldots B_{n-1}$ then by (1) $B_s\,\omega$ is an integer for $s = 1, 2, 3 \ldots (n-1)$ and (2) becomes :

$$\text{an integer} - \left(\frac{1}{2} + \frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \ldots + \frac{1}{\lambda_t}\right) = \text{an integer} + (-1)^{n-1} B_n.$$

Thus the theorem has been proved by induction.

Lahore,
March, 1926.

S. D. CHOWLA.

On the Criteria for the Nature of the Roots of a Cubic Equation.

The usual discussion of the nature of the roots of the general cubic is based on the equation of the squared differences of its roots. It may, therefore, be of interest to see how, as in the case of the quadratic, these criteria can be obtained directly from the known expressions for the roots.

When the second term is removed, the general cubic equation

$$ax^3 + 3bx^2 + 3cx + d = 0. \quad \ldots \quad \ldots \quad (1)$$

takes the form

$$z^3 + 3Hz + G = 0. \quad \ldots \quad \ldots \quad (2)$$

where $\quad H = ac - b^2$ and $G = a^2d - 3abc + 2b^3. \quad \ldots \quad (3)$

The roots of (2), are (Burnside and Panton: *Theory of Equations*, Vol. I pp. 108 and 109).

$$\left[\sqrt[3]{p} + \frac{-H}{\sqrt[3]{p}}\right], \left[\omega\sqrt[3]{p} + \omega^2\frac{-H}{\sqrt[3]{p}}\right], \left[\omega^2\sqrt[3]{p} + \omega\frac{-H}{\sqrt[3]{p}}\right],$$

where $\quad \sqrt[3]{p}\sqrt[3]{q} = -H,$

$$p = \tfrac{1}{2}\left(-G + \sqrt{G^2 + 4H^3}\right),$$

and $\quad q = \tfrac{1}{2}\left(-G - \sqrt{G^2 + 4H^3}\right).$

For the sake of convenience let us write u and v for $\sqrt[3]{p}$ and $\sqrt[3]{q}$ respectively. Then the three roots are :—

(i) $\quad u + v = u + v.$

(ii) $\quad \omega u + \omega^2 v = -\dfrac{u+v}{2} + \dfrac{(u-v)\sqrt{-3}}{2},$

(iii) $\quad \omega^2 u + \omega v = -\dfrac{u+v}{2} - \dfrac{(u-v)\sqrt{-3}}{2},$

Now the criteria for the nature of the roots of equation (2) may be obtained as follows :—

(*a*) *If the roots are all real,* $u + v$ must be real and $u - v$ wholly imaginary so that u and v, and consequently u^3 and v^3, must be conjugate imaginary numbers.

$\therefore$ *$G^2 + 4H^3$ must be negative.*

(*b*) *If two of the roots are imaginary*, both $u + v$ and $u - v$ must be real so that u and v, and consequently u^3 and v^3, must be real.

$\therefore$ *$G^2 + 4H^3$ must be positive.*

(*c*) *If two of the roots are equal*, u must be equal to v, so that

$$u^3 = v^3.$$

$\therefore$ *$G^2 + 4H^3$ must be zero.*

(*d*) *If all the roots are equal*, then $u = v$, $u + v = -\dfrac{u+v}{2}$

so that $u = 0$ and $v = 0$ that is $u^3 = 0$ and $v^3 = 0$

$\therefore$ *G and H both vanish.*

The criteria for the nature of the roots of equation (1) are necessarily those of equation (2) and are obtained by substituting for H and G the values given in (3).

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Solutions.

Question 865.

(M. K. KEWALRAMANI, M.A.):—Evaluate

$$\int_0^\infty \cos\left(x^{\frac{3}{5}} + ax^{-\frac{2}{5}}\right) dx.$$

Solution by K. R. Rama Iyer.

Consider the integral

$$I = \int_0^\infty \sin\left(\lambda^2 y^2 + \frac{\mu^2}{y^2}\right) dy.$$

Let $$z = \lambda y - \frac{\mu}{y}.$$

Then $$y = \frac{1}{2\lambda}\left\{z + \sqrt{z^2 + 4\lambda\mu}\right\}$$

The negative sign is not taken for $\sqrt{z^2 + 4\lambda\mu}$ because y is positive throughout the range from 0 to ∞.

Therefore $$dy = \frac{1}{2\lambda}\left\{1 + \frac{z}{\sqrt{z^2 + 4\lambda\mu}}\right\}.$$

Hence $$I = \frac{1}{2\lambda}\int_{-\infty}^{\infty} \sin(z^2 + 2\lambda\mu)\, dz$$

$$+ \frac{1}{2\lambda}\int_{-\infty}^{\infty} \sin(z^2 + 2\lambda\mu)\frac{z}{\sqrt{z^2 + 4\lambda\mu}}\, dz.$$

By splitting the second integral into $\int_{-\infty}^{0}$ and $\int_0^\infty$ we see that it is equal to zero.

$$\therefore \quad I = \frac{1}{2\lambda}\int_{-\infty}^{\infty} \sin(z^2 + 2\lambda\mu)\, dz.$$

$$= \frac{1}{\lambda}\int_0^\infty \{\sin z^2 \cos 2\lambda\mu + \cos z^2 \sin 2\lambda\mu\}\, dz.$$

$$= \sqrt{\left(\frac{\pi}{2}\right)} \cdot \frac{1}{\lambda}\left(\cos 2\lambda\mu + \sin 2\lambda\mu\right)$$

In I put $\lambda^2 = a$ and $\mu^2 = b$, so that

$$\mathrm{I} = \int_0^\infty \sin\left(ay^2 + \frac{b}{y^2}\right) dy = \sqrt{\frac{\pi}{a}} \cdot \sin\left(\frac{\pi}{4} + 2\sqrt{ab}\right)$$

$$\therefore \quad \frac{d^3\mathrm{I}}{db^3} = -\int_0^\infty \frac{1}{y^6} \cos\left[ay^2 + \frac{b}{y^2}\right] dy$$

$$= \sqrt{\frac{\pi}{a}} \frac{d^3}{db^3} \sin\left[\frac{\pi}{4} + 2\sqrt{ab}\right].$$

Let $\qquad y = x^{-\frac{1}{5}}$ so that $-\frac{5}{y^6} dy = dx.$

$$\therefore \quad \frac{d^3\mathrm{I}}{db^3} = \frac{1}{5} \int_\infty^0 \cos\left(bx^{\frac{2}{5}} + ax^{-\frac{2}{5}}\right) dx$$

Putting $b = 1$ the given integral is seen to be

$$\left[\frac{d^3\mathrm{I}}{db^3}\right]_{b=1} = 5\sqrt{\frac{\pi}{4}} \left[\frac{d^3}{db^3} \sin\left(\frac{\pi}{4} + 2\sqrt{ab}\right)\right]_{b=1}.$$

Question 931.

(C. Krishnamachary) :—Sum to n terms and discuss the convergence of the infinite series :—

$$\frac{y}{x+y} + \frac{x(y+z)}{(x+y)(x+y+z)} + \frac{x^2(y+2z)}{(x+y)(x+y+z)(x+y+2z)} + \dots \dots$$

Solution by S. Audinarayanan.

The $(r+1)$th term of the series is

$$\frac{x^r(y+rz)}{(x+y)\dots(x+y+rz)} = \frac{x^r y}{(x+y)\dots(x+y+rz)}$$

$$+ \left[\frac{x^r}{(x+y)\dots(x+y+\overline{r-1}z)} - \frac{x^r}{(x+y+z)\dots(x+y+rz)}\right]$$

Hence the sum of the series to n terms is

$$y\left[\frac{1}{x+y} + \frac{x}{(x+y)(x+y+z)} + \dots \text{ to } n \text{ terms}\right]$$

$$+\,x\left[\frac{1}{x+y}+\frac{x}{(x+y)(x+y+z)}\quad+\ \dots \text{ to } (n-1) \text{ terms}\right]$$

$$-\left[\frac{x}{x+y+z}+\frac{x^3}{(x+y+z)(x+y+2z)}+\ \dots \text{ to } (n-1) \text{ terms}\right]$$

$$=(x+y)\left[\frac{1}{x+y}+\frac{x}{(x+y)(v+y+z)}\quad+\ \dots \text{ to } (n-1) \text{ terms}\right]$$

$$+\frac{yx^{n-1}}{(x+y)\dots(x+y+\overline{n-1}z)}$$

$$-\frac{x}{x+y+z}-\frac{x^2}{(x+y+z)(x+y+2z)}\ \dots \text{ to } (n-1 \text{ terms})$$

$$=1+\frac{yx^{n-1}}{(x+y)\dots(x+y+\overline{n-1}z)}$$

$$-\frac{x^{n-1}}{(x+y+z)\dots(x+y+\overline{n-1}z)}$$

$$=1-\frac{x^n}{(x+y)(x+y+z)\dots(x+y+\overline{n-1}z)}.$$

when $n \to \infty$ this sum will tend to a limit if the product

$$\left(1+\frac{y}{x}\right)\left(1+\frac{y}{x}+\frac{z}{x}\right)\dots\left(1+\frac{y}{x}+\frac{\overline{n-1}z}{x}\right)$$

converges to a finite limit or to zero. This happens when

$$\frac{y}{x}+\left(\frac{y}{x}+\frac{z}{x}\right)+\dots+\left(\frac{y}{x}+\overline{n-1}\,\frac{z}{x}\right)=n\,\frac{y}{x}+\frac{n(n-1)}{2}\,\frac{z}{x}$$

tends to a limit or diverges to infinity.

Hence for any finite values of x, y and z, z and x having the same sign, the series converges to the limit unity.

Question 1002.

(N. P. PANDYA) :—Find the lowest prime numbers which can be arranged into a magic square of nine cells. Also find a magic square of nine cells, all the numbers in it being perfect squares.

Partial Solutions by N. B. Mitra, and Mehta Kanhyalal.

The 9 numbers forming a magic square of order 3 must always be capable of being arranged into three sets of 3 numbers in A. P. with the same common difference, the first terms of the three progressions being themselves in A. P.

If x be the least number, the three A. P.'s may be taken as

(1) $x, x + y, x + 2y$

(2) $x + z, x + y + z, x + 2y + z$

(3) $x + 2z, x + y + 2z, x + 2y + 2z$.

It is evident that y and z are interchangeable.

If all the terms are to be prime, x must be odd and y and z even.

Let $x = 1$, then $y \equiv 0, 2$ or 4, (mod. 6.)

If $y \equiv 4$, then $x + 2y \equiv 9$ (mod. 6) and is therefore not a prime.

If $y \equiv 2$, then $x + y \equiv 3$ (mod. 6) and is not prime unless $y = 2$.

If $y = 2$, the three sets will consist of three consecutive odd numbers beginning with $1, 1 + z, 1 + 2z$ respectively. But the only sets of 3 consecutive odd numbers which are primes are 1, 3, 5 and 3, 5, 7; hence this case must be rejected. if the 9 numbers are to be different from each other.

Hence $y \equiv 0$ (mod. 6) and the least value of y is 6.

The same reasoning will show that $z \equiv 0$ (mod. 6).

Wo have to find z when $x = 1$, $y = 6$.

To mod. 30, $z \equiv 0, 6, 12, 18$ or 24.

Now $x + 2y + 2z$, $x + 2z$, $x + y + z$ or $x + z \equiv 25$ (mod. 30) if $z \equiv 6, 12, 18$ or 24 (mod. 30). Hence the only admissible value of z is 0 (mod. 30); *i.e.*, the least value of z is 30.

The values $x = 1$, $y = 6$, $z = 30$ give us the following magic square:—

43	1	67
61	37	13
7	73	31

The following is another magic square with the next lowest prime numbers :—

101	5	71
29	59	89
47	113	17

Question 1005.

(K. B. MADHAVA) :—Show that when $|x| < 1$ and the real part of z is positive and less than 1,

$$\Gamma(1-z) = \underset{x \to 1}{\text{Lt}} (1-x)^{1-z} f(x, z)$$

where
$$f(x, z) = \sum_{n=1}^{\infty} \frac{x^n}{n^z}.$$

Solutions by S. R. Ranganathan, Sadanand and V. Thiruvenkatachari.

According to Cesaro's Theorem (*Cf.* Bromwich : *Theory of Infinite Series*, p. 132) ;

If $\frac{b_n}{a_n}$ approaches a definite limit, finite or infinite, then

$$\lim_{x \to 1} \left[\frac{\Sigma\, b_n\, x^n}{\Sigma\, a_n\, x_n} \right] = \lim_{n \to \infty} \frac{b_n}{a_n}.$$

Now, $(1-x)^{1-z} f(x, z)$ can be written as

$$\sum \left[\frac{x^n}{n^z} \right] \Big/ \sum \frac{(1-z)(2-z)\ldots(n-z)}{n!} x^n,$$

so that,

$$\lim_{x \to 1} (1-x)^{1-z} f(x, z) = \lim_{n \to \infty} \frac{n^{-z} . n!}{(1-z)(2-z)\ldots(n-z)}$$

$$= \Gamma(1-z) \text{ (Bromwich : Art. 42).}$$

Question 1021.

(HEMRAJ) :—A′, B′, C′, D′ are the ortho-centres of the triangles BCD, CDA, etc., of a quadrilateral ABCD. Shew that Δ ABC = Δ A′B′C′; but if ABCD is cyclic, these two triangles are similar. Shew also that AA′, BB′, CC′, DD′ bisect each other.

Solutions by F. H. V. Gulasekharam, V. M. Gaitonde and N. Durairajan.

Take for axes the asymptotes of the unique rectangular hyperbola through A, B, C, D.

Let $\left[ct_r, \frac{c}{t_r}\right]$ $(r = 1, 2, 3, 4)$ be the co-ordinates of the points A, B, C, D.

Then if $t_1 t_2 t_3 t_4 = k$, the co-ordinates of the points A′, B′, C′, D, are respectively

$$\left(-\frac{ct_1}{k}, \frac{-ck}{t_1}\right), \quad \left(-\frac{ct_2}{k}, \frac{-ck}{t_2}\right),$$

$$\left(-\frac{ct_3}{k}, \frac{-ck}{t_3}\right), \quad \left(-\frac{ct_4}{k}, \frac{-ck}{t_4}\right).$$

from which it is obvious that Δ A′B′C′ = Δ ABC in area.

Again if ABCD is cyclic, $t_1 t_2 t_3 t_4 = 1$: so that $k = 1$.

Hence A, A′ ; B, B′ ; C, C′ ; D, D′ are the extremities of diameters of the hyperbola.

Hence AA′, BB′, CC′, DD′, bisect each other at the centre of the rect. hyperbola and AB, BC, CD, DA are parallel respectively to A′B′, B′C′, C′D′, D′A′, as may be easily verified.

Partial Solution by N. B. Mitra.

Question 1027.

(SELECTED) :—If p is a prime number equal to $2n + 1$, show that

$$(2n)\,! \equiv (-1)^n\, 2^{4n}\, (n\,!)^3 \ \text{mod.}\ p^2).$$

Remarks by S. Audinarayanan.

This result has been proved incidentally in the Solution to Question 1247, J. I. M. S., Vol. XV, page 60, and is given as result (B).

Questions 1084, 1085 and 1086.

1084. (HEMRAJ) :—Sum up the following series :—

$$\left[\frac{1}{1^3}+\frac{1}{2^3}\right]+\frac{1}{2}\left[\frac{1}{3^3}+\frac{1}{4^3}\right]+\frac{1.3}{2.4}\left[\frac{1}{5^3}+\frac{1}{6^3}\right]$$
$$+\frac{1.3.5}{2.4.6}\left[\frac{1}{7^3}+\frac{1}{8^3}\right]+\dots\dots$$

1085. (HEMRAJ) :—If

$$S=\sum_0^\infty \frac{(2n)!}{2^{2n}(n!)^2}\,\frac{1}{(2n+2)^4}$$

and

$$\sigma=\sum_0^\infty \frac{(2n)!}{2^{2n}(n!)^2}\,\frac{1}{(2n+1)^4},$$

find the value of $S+\frac{2}{\pi}\sigma$

1086. (HEMRAJ) :—Evaluate

$$\int_0^1\left[\frac{1}{x}\int_0^x \frac{\sin^{-1}x}{x}\,dx\right]dx.$$

Solutions by V. Tiruvenkatachariar and S. D. S. Chowla.

The series in Question 1084 is the sum of

$$\frac{1}{1^3}+\frac{1}{2}\,\frac{1}{3^3}+\frac{1.3}{2.4}\,\frac{1}{5^3}+\quad\dots\quad\dots\quad(1)$$

and

$$\frac{1}{2^3}+\frac{1}{2}\,\frac{1}{4^3}+\frac{1.3}{2.4}\,\frac{1}{6^3}+\quad\dots\quad\dots\quad(2)$$

which are respectively equivalent to the integrals

$$\int_0^1\frac{dx}{x}\int_0^x\frac{\sin^{-1}x}{x}\,dx\quad\dots\quad\dots\quad(3)$$

and

$$\int_0^1\frac{dx}{x}\int_0^x\frac{dx}{x}\left(1-\sqrt{1-x^2}\right)\quad\dots\quad(4)$$

Integrating by parts (3) becomes

$$\left[\log x \int_0^1 \frac{\sin^{-1} x}{x}\, dx\right]_0^1 - \int \frac{\log x}{x} \sin^{-1} x \, dx$$

The first part vanishes when the limits tend to 0 and 1 and the second again integrated by parts reduces to

$$\frac{1}{2}\int_0^1 \frac{(\log x)^2}{\sqrt{1-x^2}}\, dx = \frac{1}{2}\int_0^{\frac{\pi}{2}} (\log \sin \theta)^2\, d\theta = \frac{\pi}{4}\left\{ \log 2)^2 + \frac{\pi^2}{12} \right\}$$

[*vide* Bromwich: *Infinite Series*, p. 476] the value required in Q. 1086.

The integral (4) may be similarly reduced to

$$\frac{1}{2}\int_0^{\frac{\pi}{2}} (\log \sin \theta)^2 \sin \theta \, d\theta = \frac{1}{2}(\log 2 - 1)^2 + 1 - \frac{\pi^2}{12}$$

[Bromwich: *Infinite Series*, page 476]. By adding the values of the integrals (3) and (4) just found, we have the sum of the series in Q. 1084.

Again σ and S in Q. 1085 are given by

$$\sigma = \int_0^1 \frac{dx}{x}\int_0^x \frac{dx}{x}\int_0^x \frac{\sin^{-1} x}{x}\, dx = -\tfrac{1}{6}\int_0^{\frac{\pi}{2}} (\log \sin \theta)^3\, d\theta$$

$$\text{and } S = \int_0^1 \frac{dx}{x}\int_0^x \frac{dx}{x}\int_0^x (1-\sqrt{1-x^2})\frac{dx}{x}$$

$$= -\tfrac{1}{6}\int_0^{\frac{\pi}{2}} (\log \sin \theta)^3 \sin \theta \, d\theta.$$

To find these, we start with

$$\int_0^{\frac{\pi}{2}} \sin^{2\alpha-1} d\theta = \frac{\sqrt{\pi}}{2}\,\frac{\Gamma(\alpha)}{\Gamma(\alpha+\frac{1}{2})}$$

and differentiate it thrice with respect to α under the integral sign and obtain

$$8\int_0^{\frac{\pi}{2}} \sin^{2\alpha-1}\theta \,.(\log \sin \theta)^3\, d\theta$$

$$= \frac{\sqrt{\pi}}{2}\,\frac{\Gamma(\alpha)}{\Gamma(\alpha+\frac{1}{2})}\{u^3 + 3uu' + u''\} \quad \ldots \quad (5)$$

where $u = \dfrac{\Gamma'(\alpha)}{\Gamma(\alpha)} - \dfrac{\Gamma'(\alpha+\frac{1}{2})}{\Gamma(\alpha+\frac{1}{2})}$, $u' = \dfrac{du}{d\alpha}$ and $u'' = \dfrac{d^2u}{d\alpha^2}$.

Putting $\alpha = \frac{1}{2}$ and $\alpha = 1$ respectively in (5), we get the values of σ and S. Thus we have

$$\sigma = \frac{\pi}{48} \{ 4 \log^3 2 + \pi^2 \log 2 + 6 S_3 \},$$

$$S = \frac{1}{48} \{ 8 (1 - \log 2)^3 + 6 (1 - \log 2)(4 - \tfrac{1}{3} \pi^2) - 12 S_3 + 16 \}$$

$$\therefore \quad S + \frac{2}{\pi} \sigma = 1 - \log 2 + \tfrac{1}{2} \log^2 2 + \frac{\pi^2}{24} (2 \log 2 - 1).$$

Solution to Question 1086 by G. R. Narayana Iyer.

Question 1290.

(K. J. Sanjana and I. B. Mukherji) :—Examine whether the following two problems are identical or not, and solve the first of them :—

" Find the size of a cube which will stop up a tube of uniform bore, the section of which is a regular hexagon whose sides are given."

Todhunter.

" Show that in general, a plane which cuts the six faces of a cube cuts them in a hexagon whose opposite sides are parallel, and show how to make the section a regular hexagon."

Davison.

Solution and Comments by A. Narasinga Rao.

Let us discuss the general problem of when a given solid S can stop up a tube of specified cross-section.

Let the bounding surfaces of S be projected orthogonally on a plane which we shall take to be horizontal. The projections form regions which cover a total region Σ (parts of it possibly more often than others). Now if S is blocking up a uniform tube held vertically, the cross-section of the tube must be Σ, for otherwise, either there exist vertical paths from one side of S to the other so that the block is ineffective or the solid cuts out of the tube. This condition, though necessary, is not sufficient to ensure the stopping up of the tube. Such a condition is the nonexistence of paths leading from one side of S to the other and nowhere meeting either the tube or the solid.

The existence or otherwise of such paths is not affected by any continuous deformation of S which introduces no new connectivities in S and which does not create or destroy gaps between the tube and the solid. One such deformation is a gradual flattening of S in which each point on the surface moves vertically up or down. By such a flattening we may reduce S to a continuous many-sheeted surface—somewhat like a Reimann's surface—covering the region Σ and the paths we have been discussing are deformed into contours on the surface leading from the top side of the first sheet to the bottom side of the lowest one and nowhere meeting the boundary of Σ.

If S be a convex solid, Σ is a simply connected region, and the flattened S is a two sheeted surface, each sheet co-extensive with Σ. The two sheets are connected along the boundary of Σ and hence if this be cut out as an impassable barrier the sheets become distinct and there is no path leading from one to the other. Hence

Any convex solid S will block up a tube whose cross-section Σ is congruent with the projection of the surface of S on some plane.

Now if a cube of side a be held with one diagonal vertical, the projection of its faces form a regular hexagon of side $\frac{\sqrt{2}}{\sqrt{3}}a$ and hence it can be used to stop a tube whose cross-section is such a hexagon, It is easy to prove, by similar reasoning that we may stop up a tube of square section by a cube or a tetrahedron; of regular hexagonal section by a cube, an octahedron or an icosahedron and so on.

The two questions are thus *not identical.* The effectiveness or otherwise of a solid in stopping up a tube of uniform bore depends not on the section of the blocking solid S, but on the nature of its projection on a plane. As a matter of fact the hexagonal section of the cube is of side $\frac{a}{\sqrt{2}}$ while the tube which the cube will block is, as shown above, one whose cross-section is of side $\frac{\sqrt{2}}{\sqrt{3}}a$.

Questions for Solution

Proposers of Questions are requested, whenever possible, to send their own solutions along with their questions.

1437. (K. ANANDA RAU) :—If p is a prime number, prove that the necessary and sufficient condition that $p+2$ is also a prime number is

$$\frac{(p-1)!+1}{p} \equiv -\frac{p+3}{4} \quad (\text{mod. } p+2)$$

when p is of the form $(4m+1)$,

and
$$\frac{(p-1)!+1}{p} \equiv \frac{p+1}{4} \quad (\text{mod. } p+2)$$

when p is of the form $(4m+3)$.

1438. (S. D. S. CHOWLA) :—If $2q+1$ is a prime and

$$2p \equiv 2q \ (\text{mod. } 4q+2)$$

then
$$^{2p}C_{2q} + {}^{2p}C_{4q} + {}^{2p}C_{6q} + \dots \equiv 0 \quad (\text{mod. } 2q+1)$$

where the last term has for lower index either $2p-2$ itself or some even number less than it.

Ex. Taking $2p = 14$, $2q+1 = 5$, $q = 2$,

$$^{14}C_4 + {}^{14}C_8 + {}^{14}C_{12} \equiv 0 \quad (\text{mod. } 5)$$

$$^{20}C_6 + {}^{20}C_{12} + {}^{20}C_{18} \equiv 0 \quad (\text{mod. } 7)$$

$$^{24}C_4 + {}^{24}C_8 + \dots {}^{24}C_{20} \equiv 0 \quad (\text{mod. } 5)$$

1439. (K. C. SHAH. M.A.) :—If $f(r)$ denotes the greatest product of n positive integers whose sum is r, shew that

$$\sum_{r=kn+p}^{r=ln+q} f(r) = l^{n-q}(l+1)^{q+1} - k^{n-p+1}(k+1)^{p} - \sum_{t=k+1}^{l} t^{n}$$

where k, l, p, q are all positive integers, p and q being less than n, and the summations extend to all the integers between the limits.

1440. (N. DURAIRAJAN):—Show how to construct a triangle of given species with its vertices A, B, C at given distances from a fixed point O in the plane.

Find the relations between the distances OA, OB and OC in order that a real solution may exist, whatever the shape of the triangle.

1441. (K. J. SANJANA, M.A.):—A chord of length λ of a hyperbola lies along the straight line L, and l is any point on the line; f, f' are the foci of the curve and Σ, Σ' its asymptotes. If from the point l tangents T and T′ are drawn to the hyperbola, prove that

$$\sin(\mathrm{LT}) \sin(\mathrm{LT}') \sin^2(\Sigma\,\Sigma') \,.\, \overline{lf} \,.\, \overline{lf}' = \sin^2(\mathrm{L}\,\Sigma) \sin^2(\mathrm{L}\,\Sigma') \,.\, \lambda^2.$$

The property holds evidently for the ellipse and its imaginary asymptotes; examine what it indicates in the case of the parabola.

[*Notation*: $\overline{xy}$ denotes the distance between the points x and y, and (XY) the angle between the straight lines X and Y].

1442. (S. AUDINARAYANAN):—Prove the following generalization of Question 1240 :—

S, S_1, S_2, ... S_r are conics of a four-point system such that l-gons, m-gons, n-gons, etc., are inscribable in S, circumscribed to S_1, S_2, ... S_r respectively. A polygon of $(r + 1)$ sides is inscribed in S, r of its sides touching the conics S_1, S_2, ... S_r respectively. Show that the envelope of the $(r + 1)$th side consists of 2^{r-1} conics of the system each of which has k-gons inscribed in S circumscribed to itself, k being the lowest common multiple of l, m, n ... ...

1443. (A. NARASINGA RAO):—Determine the conditions under which any given family of plane curves $f(x, y, c) = 0$ admit of being described simultaneously by a mechanism of rigidly-connected tracing points.

If a triangular lamina moves in its own plane and its three vertices describe three circles, show that the circles must have either a common centre or equal radii.

LIST OF JOURNALS RECEIVED

during the months of July and August 1926.

Journals.

1 American Mathematical Monthly, **33**, 3 & 4 (2 copies each).
2 American Journal of Mathematics, **48**, 2 (2 copies each).
3 Annales de L'Ecole Normale Supérieure (1926), 2, 3, 4, 5.
4 Astrophysical Journal, **63**, 3 & 4.
5 Bulletin of the American Mathematical Society, **32**, 3.
6 Bulletin des Sciences Mathematiques, Tome **2**.
7 Bulletin of the Calcutta Mathematical Society, **17**, 1.
8 Journal fur die Reine und Angewandte Math. **153**, 4.
9 Journal of the Science Association, Vizayanagaram, **2**, 4.
10 Japanese Journal of Mathematics, **11**, 4.
11 Mathematische Annallen, **96**, 1 & 2.
12 Mathematical Gazette, **13**, 182, 183.
13 Messenger of Mathematics, **81**, 5 & 6.
14 Monthly Notices of the Royal Astronomical Society, **55**, 11. 12.
15 Nature, **117**. 2949, 2951, 2952, 2953, 2954, 2955, 2956, 2957.
16 Philosophical Magazine, **1**, 5 & 6, **2**, 7 & 8.
17 Popular Astronomy, **34**, 5, 6 (2 copies each).
18 Proceedings of the Royal Society, A111, A758, A759.
19 Proceedings of the Cambridge Philosophical Society, **23**, 2.
20 Tohoku Mathematical Journal, **26**, 3 & 4 ; **27**, 2 & 3.
21 Transactions of the American Math. Society, **28**, 1.

Pamphlets.

1 Freezing points of very dilute Solutions of Electrolytes.
2 Heat of Sublimation of Carbon Dioxide.
3 Selenium Compounds as Spray Materials.
[Theses submitted for the Degree of Doctor of Philosophy in the University of Illinois, America.]

Numbers in black type refer to the volume, and those in ordinary type to the number of the issue.

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