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SIR CHANDRASEKHARA VENKATARAMAN, KT.,
M.A., D.SC., F.R.S.,
NOBEL PRIZEMAN FOR PHYSICS, 1930
AND HUGHES MEDALLIST OF THE ROYAL SOCIETY.
(HON. MEMBER OF THE INDIAN MATHEMATICAL SOCIETY).

ON A FUNCTION ANALOGOUS TO G′ (k).

BY S. SIVASANKARANARAYANA PILLAI,

Annamalai University.

Let G′ (k) be the minimum number of positive kth powers of primes required to represent every number from a certain point onwards. It is very difficult to prove the existence of G′ (k). We require here a combination of weapons used in proving Goldbach's theorem and Waring's problem. If Goldbach's theorem is true, G′ (1) = 3. From Hardy-Littlewood's provisional results we know $3 \leqslant G'(1) \leqslant 4$. For higher values of k, little is known. The object of this note is only fo find a lower-limit for some particular forms of k. It is obvious $G'(k) \geqslant G(k)$.

THEOREM I. $G'(2) \geqslant 7$.

Proof.

$$x \equiv 2 \pmod{8}$$
$$x \equiv 3 \pmod{9}$$
$$x \equiv 15 \pmod{49}$$
$$x \equiv 27 \pmod{121}.$$

This system of congruences is possible; let $x_1 > 30$ be any one solution.

Then since

x_1 is divisible by 3 and by no higher power of 3

$x_1 - 8$ ... ... 7 ... ... of 7

$x_1 - 16$ ... ... 11 ... ... of 11

none of x_1, $x_1 - 8$, $x_1 - 16$ can be expressed as the sum of 2 squares. Obviously they cannot be squares. Again, since except 2, every prime is odd and square of every number is $\equiv 1 \pmod{8}$, $x_1 - 4 = 8m + 6$ requires at least 6 squares. So in any case x_1 requires more than 6 prime squares.

Hence $G'(2) \geqslant 7$

266 is the smallest number which requires 7 prime squares.

(If $f(2)$ is the minimum number of odd squares required to represent every number from a certain point onwards, then from the above proof it follows $f(2) \geqslant 10$).

II. $G'(2^r) \geqslant 2^{r+2} + 2$ when $r \geqslant 2$.

If $p_1, p_2, \ldots, p_s$ are primes of the form $4r + 3$ in order, we can find x such that

$$\begin{aligned} x &\equiv 2 \ (\text{mod. } 2^{r+2}) \\ &\equiv 2^{2^r} + p_1 \ (\text{mod. } p_1^2) \\ &\equiv 2 . 2^{2^r} + p_2 \ (\text{mod. } p_2^2) \\ &\ldots \quad \ldots \quad \ldots \\ &\equiv s . 2^{2^r} + p_s \ (\text{mod. } p_s^2) \\ &\ldots \quad \ldots \quad \ldots \\ &\equiv (2^{r+2})\, 2^{2^r} + p_{2r+2} \ (\text{mod. } p^2_{2r+2}). \end{aligned}$$

Now obviously, none of

$$x, \ x - 2^{2^r}, \ x - 2 . 2^{2^r}, \ \ldots \ \ldots, \ x - (2^{r+2})\, 2^{2^r}$$

can be expressed as the sum of two squares, nor are they squares. So if $T(x)$ is the least number of prime $2^{r\text{th}}$ powers required, then $T(x) \geqslant 2^{r+2} + 2$ for $2^{r\text{th}}$ power of an odd number is of the form $2^{r+2} m + 1$.

$$\therefore \quad G'(2^r) \geqslant 2^{r+2} + 2.$$

III. If $k = p - 1$ where p is an odd prime.

Then $$G'(k) \geqslant p + 2.$$

Proof. We can find x so that

$$\begin{aligned} x &\equiv 2 \ \text{mod. } p) \\ x &\equiv p^k + p_1 \ (\text{mod. } p_1^2) \\ x &\equiv 2p^k + p_2 \ (\text{mod. } p_2^2) \\ &\ldots \quad \ldots \quad \ldots \\ x &\equiv p . p^k + p_p \ (\text{mod. } p_p^2) \end{aligned}$$

where $p_1, p_2 \ldots, p_p$ are primes of the form $4r + 3$ in order, different from p.

Now as in II, we can show x requires at least $p + 2$ kth powers of primes since $a^k - 1 \equiv 0$ (mod. p) if $a \not\equiv 0$ (mod. p).

Similarly we can show if $k = \phi(p^r)$ where p is an odd prime, then

$$G'(k) \geqslant p^r + 2.$$

The corresponding known results for $G(k)$ are:—

$$G(2^r) \geqslant 2^{r+2} \text{ and } G'(\phi(p^r)) \geqslant p^r.$$

k	$=1$	2	3	4	5	6	7	8	9	10
$G(k)$	$\geqslant 1$	4	4	16	6	9	8	32	13	12
$G'(k)$	$\geqslant 3$	7	4	18	6	11	8	34	18	13

ON SOME DIOPHANTINE EQUATIONS

By S. Sivasankaranarayana Pillai, M.Sc.
Annamalai University.

1. We make use of the following two well-known lemmas :—

Lemma (1.1). *If* $(x, y) = 1$, *then* $x \pm y$ *and*

$$x^{2r} \mp x^{2r-1} y + x^{2r-2} y^2 \mp x^{2r-3} y^3 + \dots + y^{2r}$$

cannot have any common factor except the divisiors of $r + 1$.

For if possible let m be their common factor.

Then m is also a factor of

$$(x^{2r} \mp x^{2r-1} y + x^{2r-2} y^2 \mp \dots \mp xy^{2r-1} + y^{2r})$$
$$-x^{2r-1}(x \pm y) \pm 2x^{2r-2} y (x \pm y)$$
$$-x^{2r-3} y^2 (x \pm y) \pm \dots \pm 2\, ry^{2r-1} (x \pm y)$$

which is equal to $(2r + 1)\, y^{2r}$

But m is prime to y

$\therefore\ 2r + 1 \equiv 0 \pmod m$

Lemma (1.2). *If p is an odd prime and* $(x, y) = 1$, $\dfrac{x^p + y^p}{x + y}$ *is divisible by p only when $x + y$ is a multiple of p and then* $\dfrac{x^p + y^p}{x + y}$ *is not divisible by* p^2.

Let $x + y = sp - a$

Then $x = sp - (y + a) = sp - t$ say

Then $\dfrac{x^p + y^p}{x + y} = (sp - t)^{p-1} - (sp - t)^{p-2}\, y + \dots + y^{p-1}$

$\equiv t^{p-1} + t^{p-2}\, y + t^{p-3}\, y^2 + \dots + y^{p-1} \pmod p$

If $a \neq 0$ and y is prime to p then t is prime to p

Then $t^{p-1} \equiv 1$

$$y^{p-1} \equiv 1 \pmod p$$

$\therefore \quad t^{p-1} - y^{p-1} \equiv 0 \pmod p$

$\therefore \quad t^{p-2} + t^{p-3}y + \ldots + y^{p-2} \equiv 0$ since $t \not\equiv y$

But if $\dfrac{x^p + y^p}{x + y} \equiv 0 \pmod p$, then

$t^{p-1} + t^{p-2}y + \ldots + y^{p-1} \equiv 0$

It follows from the above results that $y^{p-1} \equiv 0$ and hence that

$$y \equiv 0 \pmod p.$$

But if $y \equiv 0 \pmod p$, since $(x, y) = 1$,

$(x^p + y^p)/(x + y) \not\equiv 0 \pmod p$.

If $a = 0$, $x = sp - y$.

$$\therefore \quad \frac{x^p + y^p}{x + y} = (sp - y)^{p-1} - (sp - y)^{p-2}y + \ldots + y^{p-1}$$

$$= p \cdot y^{p-1} - sp \cdot y^{p-2} \times \frac{p(p-1)}{2} + Mp^2$$

$$= p \cdot (y^{p-1} + M_1 p)$$

But since $x = sp - y$ and $(x, y) = 1$,

$$y \not\equiv 0 \pmod p.$$

Hence the proposition.

(2) A similar theorem and proof hold for $\dfrac{x^p - y^p}{x - y}$

2. In this section *I assume the truth of* **Fermat's last theorem,** namely, $x^n + y^n \neq z^n$ when $n > 2$.

(2.1) $x^{2m} - y^{2m} \neq 2z^m$, $(x, y) = 1$, $m > 2$

If it were possible, $2z^m = (x^m - y^m)(x^m + y^m)$

But $x^m - y^m$ and $x^m + y^m$ can have no common factor except 2.

$\therefore \quad x^m - y^m = a^m$ or $x^m + y^m = b^m$ both of which are impossible.

This was proved by Liouville

(2.2) $x^{pm} - y^{pm} \neq p \cdot z^m$, $(x, y) = 1$, $m > 2$, p is an odd prime

Suppose it is possible. Then

$$p \cdot z^m = (x^m)^p - (y^m)^p = (x^m - y^m)\,(x^{m(p-1)} + \dots + y^{m(p-1)})$$

But by lemmas (1.1) and (1.2)

$$x^m - y^m = a^m \text{ which is impossible}$$

(2.3) $x^{rm} - y^{rm} \neq rz^m$ where r is any positive integer, $(x, y) = 1$, $m \geqslant 3$.

Assume that the equation is possible.

Let $r = ps$ where p is an odd prime.

Then $\quad psz^m = (x^{ms})^p - (y^{ms})^p$

$$= (x^{ms} - y^{ms})\,\{x^{ms(p-1)} + \dots + y^{ms(p-1)}\}$$

Hence by lemma's (1.1) and (1·2)

$$x^{ms} - y^{ms} = s \cdot a^m.$$

This is of the original form. So proceeding we shall arrive at either $x^m - y^m = t^m$ if r is odd or $x^{2^h m} - y^{2^h m} = 2^h\, l^m$ if r is even, where $h \geqslant 1$.

But the first case is impossible

As for the second, we get

$$2^h\, l^m = (x^{2^{h-1}m} - y^{2^{h-1}m})\,(x^{2^{h-1}m} + y^{2^{h-1}m})$$

But if $h \nless 2$, the second factor cannot be divisible by any higher power of 2 than the first

But obviously x and y are odd

$$\therefore \quad x^{2^{h-1}m} - y^{2^{h-1}m} = 2^{h-1}\, q^m.$$

Proceeding thus we get

$$x^m - y^m = u^m \text{ which is impossible.}$$

Hence the theorem is proved.

(2.4) $x^{rm} + y^{rm} \neq rz^m$, $(x, y) = 1$, $m \geqslant 3$, r is odd.

Proof as for (2.3)

Corollaries from (2.3) and (2.4)

(1) $s\,(rs + y^{rn}) \neq t^{rn}$ where r is any positive integer.

(2) $s\,(rs - y^{rn}) \neq t^{rn}$ where r is odd.

(2.5) $x^{2m} + y^{2m} \neq z^2$, $(x, y) = 1$, $m \geqslant 3$

Suppose it is possible.

Then either x on y is even; let y be even.

$$x^{2m} = (z - y^m)(z + y^m)$$

$\therefore$ $z - y^m = a^{2m}$ and $z + y^m = b^{2m}$ for $(z - y^m)$ and $z + y^m$ are relative primes

$\therefore$ $b^{2m} - a^{2m} = 2 \cdot y^m$, Now $(b, a) = 1$

Hence by (1), this is impossible.

This was proved by Lebesque.

(2.6) $x^{2rm} + y^{2rm} \neq r \cdot z^2$ where r is odd $(x, y) = 1$.

Proceeding as in the proof for (2.3) for every prime factor r, we shall arrive at the equation $b^{2m} - a^{2m} = 2y^m$, $(a, b) = 1$, which is impossible by (2.5).

(2.7) $x^{trm} + y^{trm} \neq rz^t$ where r is odd, $(x, y) = 1$, $m \geqslant 3$.

Proof as in (b)

(2.8) A triangular number is not a $2r^{th}$ power when $r \geqslant 3$, that is

$$x(x+1) \neq 2 \cdot y^{2r}$$

For if possible, let $x(x+1) = 2 \cdot y^{2r}$

Now $(x, x+1) = 1$

$\therefore$ x or $x + 1 = 2a^{2r}$ and $x + 1$ or $x = b^{2r}$.

$\therefore$ $b^{2r} - 2a^{2r} = \pm 1$.

Let $b^{2r} - 2a^{2r} = 1$,

Then $2a^{2r} = (b^r - 1)(b^r + 1)$

$\therefore$ $b^r \pm 1 = t^r$ which is impossible when $r \geqslant 2$ and $b, t \geqslant 1$

Suppose

$b^{2r} - 2a^{2r} = -1$.

$a^{4r} - 2a^{2r} + 1 = a^{4r} - b^{2r}$

Put $X = a^{2r} + 1$, $b = Yz$, $a^2 = Y$

Then $X^2 = Y^{2r} - z^{2r}$ which is impossible from (5) when $r \geqslant 3$.

Hence the theorem is proved.

Fermat stated and Euler, Legendre and Lucas proved without any assumption that a triangular number can be neither a cube nor a biquadrate.

3. In this section, no assumption is made.

(3.1) $x^{2m} - 1 \neq 2y^m$ when $m \geqslant 2$.

$$2y^m = (x^m - 1)(x^m + 1)$$

But, if at all, 2 alone can be a common factor of the two factors on the right-side.

$$\therefore \quad x^m - 1 = a^m \text{ or } x^m + 1 = b^m$$

both of which are obviously impossible. This was proved by Kempner.

(3.2) $x^{pm} - 1 \neq p.y^m$ whre p is an odd prime.

Proceeding as in the case of (2.2) we shall arrive at the equation, $t^m - 1 = u^m$ which is impossible.

(3.3) $x^{rm} - 1 \neq r.y^m$ where r is any positive integer. Proceeding as for the proof of (2.3), we arrive at $t^m - 1 = y^m$ which is impossible. These two are generalisations of (1).

(3.4) $x^{rm} + 1 \neq r.y^m$ where r is an odd positive integer.

Proof similar to that of (2.4).

Corollaries.

(1) $s(rs + 1) \neq t^{rn}$ when r is any positive integer.

(2) $s(rs - 1) = t^{rn}$ when r is odd,

ON THE FEET OF CONCURRENT NORMALS OF A CONIC.

By R. Vaidyanathaswamy.

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The main object of this paper is to investigate the curious and mutually correlated properties of three triads of conics connected with a triangle ABC. These are: (1) the three rectangular hyperbolas which pass through the in- and ex-centres I, I_1, I_2, I_3, and have concurrent normals at these points, (2) the three parabolas which are inscribed to ABC so as to have concurrent normals at the points of contact with the sides, and (3) the three parabolas circumscribed to ABC so as to have concurrent normals at these points. It is shewn that the centres of the three rectangular hyperbolas (1), and the foci of the three inscribed parabolas (2) are the same three points p, q, r on the circum-circle ABC, and that these may be obtained parametrically as the roots of the binary Jacobian of the triad ABC, and the pair of circular points. Several other remarkable properties connected with the points $p\ q\ r$ are obtained.

The method adopted is synthetic and relies on apolarity. The characterisation through apolarity of *the pedal tetrads* of a central conic (that is, tetrads of points at which normals are concurrent) was obtained by H. Wiener* in 1903. His result is practically equivalent to the theorem proved here (§ 1.(3)) that *the pedal tetrads of a central*

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* *Encyc. des Sc. Mathematiques*: III (3) 1, pp. 133—135. The German *Enc. Math. Wiss* (*Dingeldey*, Kegelschnitte und Kegelschnittsysteme, III C (1) does not mention this result).

Two n-ads of points on a conic given parametrically as roots of the two n-ics

$$a_0 t^n + \binom{n}{1} a_1 t^{n-1} + \dots + a_n,$$

$$b_0 t^n + \binom{n}{1} b_1 t^{n-1} + \dots + b_n,$$

are *apolar*, if

$$a_0 b_n - \binom{n}{1} a_1 b_{n-1} + \binom{n}{2} a_2 b_{n-2} - \dots + (-1)^n a_n b_0 = 0.$$

conic are all the tetrads apolar to the inscribed rectangles of the conic. The use of apolarity results in several new properties of pedal tetrads and triads.

The main theorem which has been used in the investigations is an important one relating to one-parameter families of similar conics. It is as follows:

Let Γ be the envelope of a one-parameter system of similar conics (base points, if any exist, being considered to be part of the envelope). Then the conics of the system whose normals at their points of contact with Γ are concurrent, are those which are congruent to their consecutive conics in the system.

1. Pedal tetrads on a central conic.

The feet of the normals drawn from a point P to a central conic S form a set of four points which may be conveniently termed 'the pedal tetrad of P'. If two points a, b of a pedal tetrad $abcd$ are known, the remaining points c, d are in general determined uniquely; hence, if the co-ordinates of points on the conic are expressed as rational functions of a parameter t, it follows that pedal tetrads are given as roots of a net of quartics of the form

$$\lambda_1 f_1(t) + \lambda_2 f_2(t) + \lambda_3 f_3(t), \qquad \dots \quad (1)$$

where $\lambda_1 \lambda_2 \lambda_3$ are arbitrary parameters, and $f_i(t) = 0$ $(i = 1, 2, 3)$ represent three pedal tetrads.

Now, it is known that a general net of tetrads $abcd$ contains three *neutral pairs*, namely, pairs ab which do not determine the remaining points c, d uniquely; for the net of pedal tetrads, the neutral pairs are clearly the extremities of the binormals, namely, the extremities a_1a_2, b_1b_2 of the two axes, and the intersections c_1c_2 with the line at infinity. Now any two neutral pairs must together constitute a tetrad of the net; thus $b_1b_2c_1c_2$, $c_1c_2a_1a_2$, $a_1a_2b_1b_2$ are pedal tetrads. Since these are linearly independent, it follows from (1) that the net of pedal tetrads is given by quartics of the form:*

$$\lambda_1 Q(t) R(t) + \lambda_2 R(t) P(t) + \lambda_3 P(t) Q(t); \qquad \dots \quad (2)$$

$$P(t) = (t-a_1)(t-a_2);\ Q(t) = (t-b_1)(t-b_2);\ R(t) = (t-c_1)(t-c_2)$$

* For the reduction of a general net of binary quartics to the canonical orm (2), see my paper on 'The Special pencil of binary quartics' *Pro. Edin. Math. Soc.* Ser 2. Vol. 1. 1928. pp. 104—110.

It follows immediately from this that *the net of pedal tetrads on S is cut out by the Apollonian hyperbolas, namely, the rectangular hyperbolas which pass through the centre, and have asymptotes parallel to the axes of S.*

The form (2) leads immediately to the apolar property of the pedal tetrads. For, remembering that the three pairs a_1a_2, b_1b_2, c_1c_2 separate each other harmonically, it easily follows that the tetrad $(a_1a_2a_1a_2)$ is apolar to each of the three tetrads $(b_1b_2c_1c_2)$, $(a_1a_2b_1b_2)$. $(a_1a_2c_1c_2)$. Thus the system (2) is the net apolar to the pencil of tetrads $\mu_1 \{P(t)\}^2 + \mu_2 \{Q(t)\}^2$. These latter however determine the inscribed rectangles of the conics. We thus have:

The pedal tetrads on S are all the tetrads apolar to the inscribed rectangles of S. ... (3)

Wiener's result (*l.c*) that the pedal tetrads are mixed second polars of the sextette $a_1a_2\ b_1b_2\ c_1c_2$, is equivalent to this. For, the pencil of inscribed rectangles is the syzygetic pencil whose Jacobian is $(a_1a_2b_1b_2c_1c_2)$, and it is known that a syzygetic pencil is always apolar to its Jacobian.* Hence the system (2) apolar to the inscribed rectangles must be the system of second polars of $(a_1a_2b_1b_2c_1c_2)$.

We may easily infer from the property of the neutral pairs, the truth of Joachimsthal's theorem, that *if pqrs be a pedal tetrad, the circle qrs cuts the conic again in the diametrically opposite point p′ of p.* For, if p_1, p_2 be the reflections of p in the axes a_1a_2. b_1b_2, then $(a_1a_2pp_1)$, and $(b_1b_2pp_2)$ are pedal tetrads. The circles $a_1a_2p_1$, $b_1b_2p_2$ both pass through p', and therefore any circle coaxal with them must cut the conic in three points other than p', which form a pedal tetrad with p. This proves Joachimsthal's theorem, which may also be stated in the following alternative form:

If p is any point on a conic whose axes are a_1a_2, b_1b_2, and if any circle co-axal with the circles a_1a_2p, b_1b_2p meets the conic again in $q'r's'$, then the normals at p and at the diametrically opposite points of $q'r's'$ are concurrent.

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† This corresponds to the theorem that $(f,t)^4 = 0$, where f is a binary quartic, and t its sextic covariant; for a proof of this, see Grace and Young: *Algebra of Invariants*, p. 94.

Lastly we must mention *the kinematical property* of the pedal tetrads. An infinitesimal movement of a plane upon itself is an infinitesimal rotation about an instantaneous centre P. Hence if we give an infinitesimal displacement to a conic S in its own plane, it will intersect itself in a pedal tetrad, namely, the feet of the normals for the corresponding instantaneous centre P. This is clear since it is only the feet of these normals which begin to move *along* S. We may also see this analytically by taking the equation of S referred to its axes, in the form:

$$S: \qquad ax^2 + by^2 - 1 = 0,$$

so that the Apollonian hyperbolas H are of the form:

$$H: \qquad Axy + Bx + Cy = 0.$$

Now if we subject S to the infinitesimal rigid movement given by:

$$x = x' + \delta_1 y' + \delta_2$$
$$y = -\delta_1 x' + y' + \delta_3,$$

($\delta_1, \delta_2, \delta_3$ being small quantities whose squares are negligible), then it becomes immediately evident by substitution, that the displaced conic S' is of the form

$$S': \qquad S + \delta H = 0,$$

where δ is small. Thus S' intersects S in a pedal tetrad.

The kinematical property may be visualised in the following manner. Imagine a rigid wire in the form of a conic S, which is passed through four small rings *pqrs* fixed to a plane. In general the wire will feel taut; but if *pqrs* form a pedal tetrad, the wire will feel *slightly* loose since it would now admit an infinitesimal movement.

This property leads to the following theorem:

The necessary and sufficient condition that four points pqrs may form a pedal tetrad on a central conic S (which is not *a circle) is that the consecutive conic through pqrs be congruent to S.* ... (5)

2. Pedal triads on a central conic S.

If the normals at the points *p q r* on S are concurrent, they form a *pedal triad.* Thus every pedal triad is part of a unique pedal tetrad: hence if two points *p, q* of a pedal triad *pqr*, are known, the third point *r* can be determined in *two* ways. Thus the pedal triads form a quadratic two-dimensional manifold contained in the linear three-dimensional manifold constituted by *all* triads. Now a quadric surface in three dimensions has two systems of generating lines, such that two lines do or do not intersect according as they belong to different systems, or to the same system. Hence the ∞^2 pedal triads can be arranged into two families of pencils Γ_1, Γ_2, such that two pencils do or do not have a common member, according as they belong to different families or to the same family. If s is a fixed point on S, then the pedal triads pqr which constitute with s a pedal tetrad, belong to a pencil which we may call Γ_s. It is clear that two pencils Γ_s, Γ_t can have no common member if $s \neq t$; hence for different points s, the pencils Γ_s belong to the same family, which we may take to be Γ_1.

We shall obtain the second family of pencils Γ_2, by means of the apolarity-specification (3) of the pedal tetrads. Let pqr be a pedal triad, and $pqrs$ the unique pedal tetrad containing it. If t be any point on the conic different from s, the pencil of inscribed rectangles must contain a member apolar to $pqrt$; this member being apolar both to $pqrt$, and (by (5)) to $pqrs$ must be apolar to pqr. We are thus led to the apolarity definition of pedal triads; namely,

A triad pqr is a pedal triad, if and only if, it is apolar to some inscribed rectangle.

Each inscribed rectangle has a pencil of apolar triads, all of which are, by this theorem, pedal triads. It is easy to see that these constitute the second family Γ_2, of pencils of pedal triads.

We now proceed to the *geometrical* specification of the two families of pencils Γ_1, Γ_2. Consider a pencil Γ_s; the triads of this pencil must constitute triangles which are all self-polar with respect to a unique conic C_s. If s' be the diametrically opposite points of s, and s_1, s_2 the reflections of s in the axes, we have seen already that $a_1a_2s_1$, $b_1b_2s_2$, c_1s_2s' are three particular triads of Γ_s. Since these must be self-polar triangles of C_s, it follows that s' is the centre of C_s, and that s_1, s_2 are the poles of the axes with respect to C_s. Hence Σ_s, the polar reciprocal of the given conic S with respect to C_s, is a parabola which touches the axes, and is therefore *inpolar* to S; but Σ_s is also inscribed in each of the ∞^1

triangles constituted by the triads of Γ_s. Hence, by a well-known theorem* Σ_s is both *inpolar* and *outpolar* to S. We have thus reached the result:

The pencils of pedal triads, belonging to the family Γ_1 are constituted by the vertices of triangles inscribed in S, and circumscribed to a parabola which touches the axes, and is doubly apolar to S. ... (7)

Next consider a pencil of the second family Γ_2; it has been shewn that it is the pencil of triads pqr apolar to an inscribed rectangle $\alpha_1\alpha_2\alpha_3\alpha_4$. Hence by a well-known theorem, the triangles pqr are all self-polar triangles of the unique conic S_1 which touches the tangents at $\alpha_1\alpha_2\alpha_3\alpha_4$, and is inpolar to S^*. It is clear that S_1 is coaxal with S, and that the conic S' (the polar reciprocal of S with respect to S_1), which is inscribed in the ∞^1 triangles pqr, is also coaxal with S. We thus have

The pencils of pedal triads, belonging to the family Γ_2, are constituted by the vertices of triangles inscribed in S and circumscribed to a conic S' coaxal with S.

It is also easy to see from considerations of duality, that the normals to S' at its three points of contact with the sides of any of the triangles of (8), are also concurrent. A similar result does not hold for the parabolas of (7), therein shewing the algebraic distinctness of the two families of pencils.

Theorems (7) and (8) give the solution of the problem: Given a conic S, it is required to specify the conics S', such that (1) there exist triangles inscribed in S and circumscribed to S' and (2) the normals to S at the vertices of every such triangle are concurrent. It will be noticed that the parabolas of (7) are the polar reciprocals with respect to S of its Apollonian hyperbolas. Hence the solution of the *dual* problem "Given a conic S, it is required to specify the conics S', such that (1) there exist triangles circumscribed to S and inscribed in S', and (2) the normals to S at its contacts with the sides of every such triangle are concurrent" is; "S' is either coaxal to S, or an Apollonian hyperbola, doubly apolar, to S."†

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* Grace and Young; *Algebra of Invariants*, p. 309.

† This problem has not been considered before. The particular conics S known, are those corresponding to pencils of the first family, namely, the Apollonian hyperbolas, and the Steiner parabolas of (7): the co-axal type S' of (8) is new, to the best of my knowledge.

Each pedal triad belongs to one pencil of each family. Hence we have the theorem :

Any one of the following is a necessary and sufficient condition that pqr be a pedal triad on S :

(1) *the triangle pqr is circumscribed to (or self-polar with respect to) a conic co-axal with S ;**

(2) *the conic which is inscribed to pqr and touches the axes is a parabola ;*

(3) *the tangents at pqr form a triangle inscribed in an Apollonian hyperbola.* ... (9)

3. Pedal triads on a parabola.

The theorems proved will become specialised for the parabola. The inscribed rectangles of the parabola S have all two coincident vertices at ∞. The tetrads apolar to all inscribed rectangles will therefore be composed of ∞, and triads of finite points apolar to $(V \infty \infty)$, where V is the vertex. Thus,

The pedal triads on a parabola are all the triads apolar to the triad formed by the vertex and the repeated point at infinity ... (10)

The points on the parabola S are in homographic correspondence with the diameters through them, the point at infinity thereby corresponding to the line at infinity. Hence if ABC be a pedal triad on S, and a line cuts the diameters through them in a, b, c, the axis in g, and the line at infinity in ∞, it follows from (10) that abc is apolar to $(g \infty \infty)$, or g is the centroid of abc. Hence

The pedal triads on the parabola S are all the triads whose centroid is on the axis of S. ... (11)

* It may be of interest to point out in this connection that Burnside's condition

$$\sin(\beta+\gamma)+\sin(\gamma+\alpha)+\sin(\alpha+\beta)=0,$$

for the concurrence of normals at the points whose eccentric angles are α, β, γ on S, amounts precisely to the fact that $\alpha\beta\gamma$ is a self-polar triangle of some conic co-axal with S. See my note on ' On Burnside's Condition for the Concurrence of Three Normals to a Conic " : *J.I.M.S*, Vol. XVIII, pp. 207-208.

Joachimsthal's theorem apparently specialises into the well-known result: ***the circumcircle of a pedal triad passes through the vertex.***

The *quadratic* system of pedal triads, which was studied for the case of the central conic, splits up into two distinct linear systems for the case of the parabola, namely the system $(\infty\alpha\beta)$, where $\alpha\beta$ are arbitrary, and the system apolar to $(V\infty\infty)$. The pencils of the first family are easily seen to reduce to systems $(\infty\alpha\beta)$, where $\alpha\beta$ are extremities of a family of parallel chords; the conic-envelopes answering to the Steiner parabolas of (7) are therefore point-pairs comprising the point at infinity on the parabola, and *any* other point at infinity. The pencils of the second family, being the apolar pencils of inscribed rectangles, become now pencils containing the triad $(\infty\infty\infty)$. If ABC be a pedal triad, it is immediately evident that the triangles of the pencil determined by (ABC, $\infty\infty\infty$) are all circumscribed to a coaxal parabola S′. The focus of S′ must lie on the circle ABC, and must therefore be either the vertex V, or its other intersection V′ with the axis. It is clear that the focus cannot be at V for *every* pedal triad ABC; hence it must be at V′. Hence we have the following result corresponding to (8);

If the circumcircle of a pedal triad ABC of S intersects the axis in a point V′ other than the vertex, the parabola S′ which is inscribed to ABC and has its focus at V′ is co-axal with S. Further, the vertices pqr of any triangle inscribed in S and circumscribed to S′ form a pedal triad on S, and the contacts of S′ with qr, rp, pq form a pedal triad on S′. ... (12)

Lastly we deduce one further result of great importance from the apolarity relation (10). Let ABC be a pedal triad on S. There is then a unique conic S′ which is inscribed to each of the triangles ABC, $V\infty\infty$, inscribed in S. The conic S′ is obviously a parabola which touches the axis at V. Because, by (10), the triads ABC, $V\infty\infty$ are apolar, it follows that S′ is doubly apolar to S, and therefore admits inscribed triangles, which are self-polar in regard to S. Hence if the tangent at V to S meets S′ again in V′, the polar of V′ in regard to S must be the axis. This shews that S′ has its vertex at V, and its axis perpendicular to the axis of S. Thus the axis of S is the tangent at the vertex of an inscribed parabola, and therefore a pedal line of ABC. We have therefore:

A condition that ABC is a pedal triad on S, is that the axis of S be a pedal line of ABC. ... (13)*

From this and (11), it follows at once, that there are *three* circum-parabolas whose normals at A, B, C are concurrent, their axes being the three pedal lines which pass through the centroid G of ABC.

4. Singly infinite systems of Similar conics.

Two similar conics have to satisfy only one further condition to be congruent. Hence a singly infinite system of similar conics will contain an infinity of sets of congruent members. Therefore there will occur in the system a finite number of conics—S_1, S_2, ... , say—, which are congruent with their consecutive conic. Now the intersections of any conic S of the system with its consecutive conic, are the points of contact of S with the envelope of the system From the kinematical property (5), it then follows that S_1, S_2, ... are all the conics of the system, whose normals at the points of contact with the envelope are concurrent.

This applies in particular to singly infinite systems of rectangular hyperbolas or parabolas. If the conditions defining the system are those of passing through r base-points, and touching $4-r$ base-lines, the conics S_1, S_2, ... will comprise all the members of the system, whose normals at the base-points and at the points of contact with the base-lines are concurrent. Suppose, for instance, we have four points $II_1I_2I_3$, It will not in general be possible to draw a conic through $II_1I_2I_3$ so as to have the normals at these points concurrent. But if $II_1I_2I_3$ form an orthocentric tetrad, all conics through them would be rectangular hyperbolas, and therefore similar to one another. Consequently, a certain number of these would be congruent to their consecutive hyperbolas, and by (5), it is precisely these that would have the normals at the base points concurrent.

5. Three particular systems of similar conics.

Let I, I_1, I_2, I_3 be the in- and ex-centres of ABC, so that $II_1I_2I_3$ is an orthocentric tetrad. The three systems of similar conics which we propose to study in exemplification of the general theorem, are:

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* This result appears to be new. For some further properties of pedal triads on a parabola deduced from apolarity, see my paper 'Linear Systems of the third order on the Conic'. *Jour. Ind. Math. Soc.*, Vol. 13, 1921, pp. 147—150.

(α) The system of rectangular hyperbolas through $II_1I_2I_3$; the centre-locus of this system is the circumcircle of ABC;

(β) The system of parabolas inscribed to ABC; the locus of the focus for this system is the circum-circle of ABC;

(γ) The system of parabolas circumscribed to ABC.

Each of these systems is a *rational* system, that is to say, the co-efficients of the general conic of each system are polynomials in a parameter t. If we use point co-ordinates throughout, it is clear that the system (α) is a *linear* system (meaning that the co-efficients of the general conic are linear in t), while (β) and (γ) are quadratic systems. Hence if $\Delta(t)$ represent the discriminant of the general conic of the system, then $\Delta(t)$ will be a cubic in t for the system (α), and a sextic for the other two systems.

Further each of these systems contains *five* special members, which it will be convenient to denote throughout by the same symbols $\delta_1\delta_2\delta_3\varepsilon_1\varepsilon_2$. Here $\varepsilon_1\varepsilon_2$ shall represent the two circular parabolas which occur in each system. Further $\delta_1\delta_2\delta_3$ represent for the system (α), the three line-pairs (II_1, I_2I_3), (II_2, I_3I_1), (II_3, I_1I_2); for the system (β) they represent the three point-pairs (or squared lines), each composed of a vertex of ABC and the point at infinity on the opposite side; lastly, for the system (γ), $\delta_1\delta_2\delta_3$ shall represent the three line-pairs, each composed of a side of the triangle ABC, and the parallel line through the opposite vertex.

To economise symbols, it will be convenient to suppose that $\delta_1\delta_2\delta_3\varepsilon_1\varepsilon_2$ also represent in each case the *values* of the parameter t for the corresponding special members. Let us write now:

$$\delta(t) = (t-\delta_1)(t-\delta_2)(t-\delta_3)\,;\;\; \varepsilon(t) = (t-\varepsilon_1)(t-\varepsilon_2).$$

It is easy to see by direct inspection, or otherwise, that the systems (β) (γ) contain no singular conics other than $\delta_1\delta_2\delta_3$. Since $\Delta(t)$ is a sextic for these systems, we obtain the result:

The discriminant $\Delta(t)$ is a multiple of $\delta(t)$ for the system (α), and a multiple of $\{\delta(t)\}^2$ for the other two systems. ... (14)

6. A correspondence between the members of the three systems.

A point P on the circumcircle of ABC can be associated firstly with that rectangular hyperbola through $II_1I_2I_3$, whose centre is P; and secondly, with that inscribed parabola of the system (β), whose focus is P. Further, a parabola circumscribed to ABC is determined uniquely by the direction of its axis; hence the point P can be associated thirdly, with that circumparabola of the system (γ), whose axis is perpendicular to the pedal line of P with respect to ABC (that is, to say, parallel to the axis of the inscribed parabola, whose focus is P).*

We have thus established a homographic correspondence between the conics of the three systems and the points on the circle ABC. We may represent points on this circle by a parameter t; and we may suppose the several schemes of parametric representation to be so chosen, that three corresponding conics of the three systems and the corresponding point P on the circle, have all four, the same value for t

It remains to shew, that in this scheme, the five special members $\delta_1\delta_2\delta_3\varepsilon_1\varepsilon_2$ correspond. Firstly, the focus or centre of a circular parabola ε_1 or ε_2 is the corresponding circular point ω_1 or ω_2. Further, the pedal line of a circular point ω_1 or ω_2 is the line at infinity, and to a higher approximation, the limiting direction of the pedal line of P tends to ω_2, when P tends to ω_1 on the circle. It follows therefore that the two circum-parabolas which correspond to the points $\omega_1\omega_2$ on the circle, are respectively the two circular parabolas ε_2, ε_1. Thus the circular parabolas of each of the three systems correspond to the circular points on the circle.

Secondly, the centres of the line-pairs δ_1, δ_2, δ_3 of the system (α) are the points A, B, C; and it is clear that the foci of the three point-pairs δ_1, δ_2, δ_3 of the system (β) are also A, B, C, (the focus of a parabola being defined as the *finite* meet of its circular tangents). Finally, the axes of the three line-pairs $\delta_1\delta_2\delta_3$ of the system (γ) are evidently parallel to BC, CA, AB, that is to say, perpendicular to

* The direction of the axis of the inscribed parabola whose focus is P is the direction of the isogonal conjugate of P. This direction is perpendicular to the direction of the pedal line of P, so that points on the circle are in homographic correspondence with the directions of their pedal lines.

the pedal lines of A, B, C. Thus the members $\delta_1 \delta_2 \delta_3$ of each of the three systems correspond to the same three points ABC on the circle.

It follows therefore that in our scheme of common parametric representation, the points ABC, and the circular points on the circle are given parametrically by $\delta(t) = 0$, $\varepsilon(t) = 0$, respectively.

7. The Invariant Condition for the congruence of Rectangular Hyperbolas and Parabolas.

Let x, y, z be trilinear co-ordinates, with respect to the triangle ABC of reference, satisfying the identical relation:

$$x \sin A + y \sin B + z \sin C = M.$$

The tangential equation of the circular points is then:

$$l^2 + m^2 + n^2 - 2\,mn \cos A - 2\,nl \cos B - 2\,lm \cos C = 0.$$

If Δ, θ, θ' be the invariants of the general conic $(abcfgh)(xyz)^2$ with the pair of circular points, the squares of the semi-axes R of the conic are given by the quadratic:

$$R^4\theta^3 + R^2M^2\,\Delta\,\theta\theta' + M^4\,\Delta^2 = 0\,*$$

The ratio of the two roots of this equation is determined by the value of

$$\frac{(M^2\,\Delta\,\theta\theta')^2}{\theta^6} \times \frac{\theta^3}{M^4\Delta^2} = \frac{\theta'^2}{\theta}.$$

Accordingly two conics are similar when they have equal invariants θ'^2/θ. The condition of congruence of two conics known to be similar, is clearly the equality of their invariants Δ^2/θ^3, or the equivalent invariants Δ/θ'^3. Now $\theta = 0$ for parabolas, and $\theta' = 0$ for rectangular hyperbolas. Therefore the condition for the congruence of two parabolas is the equality of their invariants Δ/θ'^3, and for the congruence of two rectangular hyperbolas, the equality of their invariants Δ^2/θ^3. We have thus the theorem:

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* Salmon's *Conic Sections*, p. 392.

In a rational system of rectangular hyperbolas or parabolas, there will occur sets of congruent members, represented parametrically by equations of the respective forms:

$$\{\Delta(t)\}^2 + \lambda\{\theta(t)\}^3 = 0;$$

$$\{\Delta(t)\} + \lambda\{\theta'(t)\}^3 = 0;$$

where $\Delta(t)$, $\theta(t)$, $\theta'(t)$ are the invariants of the rectangular hyperbola or parabola whose parameter is t, with the circular points. ... (15)

8. Congruent sets of conics in the three systems.

To apply this result to our three systems of conics, we have first to examine $\theta(t)$, $\theta'(t)$. It is clear that θ can vanish for a rectangular hyperbola only when it is a circular parabola, and θ' can vanish for a parabola only when it is a circular parabola. Further θ is quadratic, and θ' linear, in the co-efficients of the conic. Hence for the system (α), $\theta(t)$ must be a multiple of $\varepsilon(t)$, and for the systems (β), (γ), $\theta'(t)$ must be a multiple of $\varepsilon(t)$. We have already seen in (14) that $\Delta(t)$ is a multiple of $\delta(t)$ for the system (α), and a multiple of $\{\delta(t)\}^2$ for the other two systems. Substituting these values of $\Delta(t)$, $\theta(t)$ $\theta'(t)$, in (15), we reach the theorem:

The sets of six conics represented parametrically by an equation of the form:

$$\{\delta(t)\}^2 + \lambda\{\varepsilon(t)\}^3 = 0,$$

are congruent sets in each of the three systems (α), (β), (γ). ... (16)

Further, since the conics of different systems corresponding to the same value of the parameter t, are related through the corresponding point on the circle ABC, we have:

*The centres of six congruent rectangular hyperbolas through $II_1I_2I_3$ are also the foci of six congruent inscribed parabolas of ABC. In addition, the six circum-parabolas of ABC, whose axes are parallel to the axes of six congruent inscribed parabolas, are also congruent.** ... ... (17)

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* This suggests that the ratio of the latera recta of two parabolas with parallel axes, of which one is circumscribed, and the other inscribed to a

The conics which are congruent to their consecutive conics are given, in each of the three systems, by the Jacobian of $\{\delta(t)\}^2$ and $\{\varepsilon(t)\}^3$; this Jacobian is a multiple of $\delta(t)\ \{\varepsilon(t)\}^2\ \delta'(t)$, where $\delta'(t)$ is the Jacobian cubic of $\delta(t)$ and $\varepsilon(t)$. Of the ten roots of this Jacobian, it is clear that only the three roots of $\delta'(t) = 0$ will correspond to conics properly congruent with their consecutive conics. It follows from the kinematical property ((5) and §4) that these three conics in each of the systems, are precisely those, for which the normals at the base elements (points or lines) are concurrent. The points on the circle ABC, whose parameters are given by $\delta'(t) = 0$, are the Jacobian points p, q, r of the triad ABC and the pair of circular points on the circle. It is clear that they will play a fundamental role in the properties we are studying. We have thus reached the result:

There are three rectangular hyperbolas through. $I I_1 I_2 I_3$, whose normals at these points are concurrent, namely those whose centres are p, q, r; there are three inscribed parabolas of ABC, whose normals at their contacts with the sides are concurrent, namely, those whose foci are p, q, r; and lastly there are three circumparabolas of ABC whose normals at these points are concurrent, namely those whose axes are perpendicular to the pedal lines of p, q, r. It also follows readily from (12), that the axes of these two triads of parabolas are not merely parallel, but identical. ... (18)

9. Properties of the points p, q, r.

Let $p'q'r'$ be the points on the circle diametrically opposite to pqr. The axes of the circumscribed parabolas with concurrent normals at ABC, pass by (11) through the centroid G of ABC; by (18) these are co-axal with the inscribed parabolas whose foci are p, q, r. Also by (13), the axes of the former must be pedal lines of ABC. It therefore follows that Gp, Gq, Gr are the pedal lines which pass through G, and are the common axes of the three circumscribed, and the three inscribed para-

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triangle ABC, may be simply numerical. This is easily verified. As a matter of fact, the conditiou $\theta^2 = 4\,\Delta\theta'$ for the two parabolas:

$$y^2 + 2g'x + 2fy + c, \quad y^2 + 2gx,$$

gives immediately $g = 4g'$.

bolas with concurrent normals. Hence Gp should be perpendicular to the pedal line of p, and can therefore only be the pedal line of p'.

We thus have:

The points $p'q'r'$ may be defined as the points whose pedal lines pass through the centroid G; the points $p\,q\,r$ may be defined as the points which lie in the pedal lines of their diametrically opposite points. ... (19)

If Gp, Gq, Gr cut the circle again in P, Q, R, it follows from (12) that P, Q, R are the vertices of the three circumparabolas with concurrent normals.

Two rectangular hyperbolas through $II_1I_2I_3$, which are such that the axes of each are parallel to the asymptotes of the other, must clearly have their centres at diametrically opposite points of the circle. Since $II_1I_2I_3$ form a pedal tetrad on the hyperbola H whose centre is p, it follows that the hyperbola $II_1I_2I_3\,p$ is an Apollonian hyperbola of H, and has therefore asymptotes parallel to the axes of H; therefore its centre must be p'. Thus:

The hyperbolas of system (a) whose centres are $p'q'r'$ pass respectively through p, q, r, and are respectively Apollonian hyperbolas of those whose centres are p, q, r. ... (20)

For obtaining further properties relating to p, q, r, we must have recourse to the theory of the pedal angle of inscribed triangles of a circle.* If ABC, $A'B'C'$ are inscribed in the same circle, the angle θ between the pedal lines with respect to ABC, $A'B'C'$ of a point T on the circle, is evidently independent of T, and is called *the pedal angle* between the inscribed triangles ABC, $A'B'C'$. The triangles are said to be *pedo-parallel* or *pedo-perpendicular*, according as their pedal angle is equal to to 0 or $\pi/2$ (mod. π). The pedal lines with respect to ABC, of the vertices of $A'B'C'$ make the same angle θ' with their opposite sides, where θ' is the complement of the pedal angle between ABC, $A'B'C'$. In particular, if $A'B'C'$ is pedo-parallel (pedo-perpendicular) to ABC, the pedal line of each of its vertices is perpendicular (parallel) to its

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* This theory is developed in my paper 'The (2.1) correspondence' *Proc. Camb. Phil. Soc.*, Vol. 23, 1926, pp. 233—261, particularly, pp. 253—255.

oppoosite side, and conversely. If the pedal lines of A′, B′, C′ with respect to ABC, are concurrent, then A′B′C′ is pedo-parallel to ABC. and conversely. The pedo-parallel triangles of ABC are all the triangles apolar to the unique equilateral triangle pedo-parallel to ABC. The triangle ABC is pedo-perpendicular to the triangle whose vertices are the diametrically opposite points of ABC.

Now the pedal lines of $p'q'r'$ are concurrent in G; hence $p'q'r'$ is pedo-parallel to ABC. Since pqr are the diametrically opposite points of $p'q'r'$, it follows that pqr is pedo-perpendicular to $p'q'r'$, and therefore to ABC. Hence the pedal line of p is parallel to qr; but it was seen that it is perpendicular to G p. Thus qr is perpendicular to GP and therefore :

The centroid G is the orthocentre of the triangle pqr. ... (21)
From a known property of the orthocentre, the line qr may be obtained as the perpendicular-bisector of GP.

Now, through any point there pass three pedal lines of ABC. Hence there are two points $p_1 p_2$ on the circle, other than p', whose pedal lines pass through p. Since the pedal lines of $p_1 p_2 p'$ are concurrent in p, $p_1 p_2 p'$ is pedo-parallel to ABC; therefore $p_1 p_2 p$ is pedo-perpendicular to ABC. Thus the pedal lines of $p_1 p_2$ are respectively parallel to pp_2, pp_1; but by hypothesis, these pedal lines pass through p. Therefore pp_2, pp_1 *are* the pedal lines of p_1, p_2. We thus reach the result:

> *The points p, q, r may also be defined as the meets of the pedal lines of the three pairs of points* possessing the property that the pedal line of each passes through the other.* ... (22)

Since $p_1 p_2$ is parallel to the pedal line of p, and therefore perpendicular to Gp. the knowledge of *one* point on it will suffice for its construction. Produce $p_1 p_2$ both ways to $p'_2\, p'_1$ so that $p'_1 p_1 = p_1 p_2 = p_2 p'_2$: draw lines through p'_1, p'_2 parallel respectively to pp_1, pp_2, meeting in H. If Hp meets $p_1 p_2$ in H′, it is then easy to see that Hp = 2pH′. Now, it is a well-known property of the pedal line of a point, that it bisects its join with the orthocentre. Therefore, since pp_2 is the pedal line of p_1, the orthocentre of ABC must lie on Hp'_2; similarly it lies on Hp_1', and

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* It may be easily shewn that there are only *three* such non-trivial pairs on the circle,

therefore must be H. Therefore if we join the orthocentre H to p, and produce Hp to half its length, we reach a point in p_1p_2. Now the pedal line of p bisects Hp and is perpendicular to Gp. Hence:

The line p_1p_2 is the reflection in p of its pedal line. ... (23)

10. The meets of the concurrent normals.

Since the normals at $I I_1 I_2 I_3$ to the hyperbola whose centre is p, are concurrent, the variation of this hyperbola is for the moment a rigid rotation, round the point of concurrence of the normals as instantaneous centre. Since the centre of this hyperbola is moving along the tangent at p to the circle, the instantaneous centre must lie in pp'. Further the point of concurrence of the normals lies on the Apollonian hyperbola, which, by (20), passes through p, and has its centre at p'. Thus the point of concurrence of the normals is the reflection of p in p'.

Applying the same argument to the inscribed parabola, whose focus is p, we see that the point of concurrence of the normals lies in pp'. Since the directrix of an inscribed parabola passes continually through the orthocentre H, of ABC, the instantaneous centre must also lie in the parallel through H to Gp. Since G divides the join of H and the circum-centre in the ratio 2 : 1, we have finally the theorem:

The points of concurrence of the normals for the three rectangular hyperbolas, whose centres are pqr, are the reflections of pqr in their diametrically opposite points; the points of concurrence of the normals for the three inscribed parabolas whose foci are pqr, are the reflections in pqr of their diametrically opposite points. ... (24)

The result for the meets of the normals in the case of the circum-parabolas is not equally simple. The following construction for these may be verified:

Bisect pq, pr in r'', q'', and let $q''r''$ meet Gp in g. Take a point λ in $q''r''$ on the same side of Gp as p', such that

$$g\lambda = 2\,pp'.$$

Then λ is the point of concurrence of the normals at ABC to the circum-parabola whose axis is Gp.

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NOTES AND QUESTIONS.

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Notes and Questions.

On the Quasi-normals of the Central Conic.

In a note on 'The Conic of minimum eccentricity through four points,'* Mr. Dorairajan stated that the normals at A, B, C, D to the conic of minimum eccentricity through these points are concurrent. That this is erroneous was pointed out by Mr. A Narasinga Rao, who stated and proved the correct result, namely, that there existed a value of θ, such that θ-normals at A, B, C, D to the conic of minimum eccentricity through these points are concurrent (the θ-normal at A being defined as the line through A making an angle $\pi/2 + \theta$ with the positive direction of the tangent). If we say that the feet of concurrent normals constitute 'a pedal tetrad,' and that the feet of concurrent θ-normals constitute 'a quasi-pedal tetrad of deviation θ,' Mr. Narasinga Rao's theorem may be stated thus:

> *Any four (non-concyclic) points A, B, C, D form a quasi-pedal tetrad on the conic of minimum eccentricity through them.*

Some properties connected with θ-normals are given as exercises in Casey's *Analytical Geometry*, p. 538. In this paper I propose to indicate general geometrical properties of the quasi-pedal tetrads, from the point of view of apolarity, and their connection with similar conics.

I. *The Infinitesimal Similarity-transformation.*

It is well-known that the general similarity-transformation of the plane can be reduced to a rotation through an angle θ about a point O, followed by a uniform dilatation of parameter $1 + k$ about O. We shall call O, the *centre*, and the quantity k/θ, the ***pitch***, of the similarity-transformation.

For an infinitesimal similarity-transformation with the centre O, k and θ are both infinitesimal, while the pitch $p = k/\theta$ is finite. If the co-ordinates of O referred to rectangular axes, be α, β, the equations

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* *J. I. M S.* Vol. VII (1915), p. 135.

to the infinitesimal transformation are easily seen to be:

$$\left.\begin{aligned} x' &= x + \delta\theta \, \{ p(x-\alpha) - (y-\beta) \} \\ y' &= y + \delta\theta \, \{ p(y-\beta) + (x-\alpha) \}. \end{aligned}\right\} \quad \ldots \quad (1)$$

It is clear that if a point T is transformed into a near point T′ by (1), then TT′ makes an angle $\cot^{-1} p$ with OT. Hence the normal to the path of T makes the angle $-\tan^{-1} p$ with TO (the positive direction of the normal being that obtained by rotating TT′ through a right angle in the positive direction, that is, counterclockwise). In other words, we have the theorem :

In an infinitesimal similarity-transformation, with centre O and pitch p, any point T moves in such a way that TO is the θ-normal of its path, where θ = $\tan^{-1} p$. ... (2)

As a corollary, it follows that four θ-normals can be drawn from any point O to a given central conic S, the feet of these being the intersections of S with the conic S′ obtained by transforming S by the infinitesimal similarity-transformation, with the centre O and pitch $p = \tan\theta$. Since an infinitesimal rigid movement is an infinitesimal similarity-transformation of pitch zero, it follows as a particular case, that the pedal tetrads on S are its intersections with consecutive congruent conics S′.

It results then, that the quasi-pedal tetrads ABCD on S are its intersections with consecutive conics S′ similar to S; this implies that the eccentricity of S considered as a member of the system of conics passing through A, B, C, D, is a minimum, which proves Mr. Narasinga Rao's result.

II. *Geometrical specification of the quasi-pedal tetrads.*

Taking the equation to the central conic S, in the form :

$$S :: ax^2 + by^2 = 1,$$

the conic S′ obtained by transforming S by the infinitesimal transformation (1) is

$$S' : (ax^2 + by^2 - 1) - 2\delta \{ (b-a)xy + ax\beta - by\alpha + p(ax^2 + by^2) - p(a\alpha x + b\beta y) \} = 0 \quad \ldots \quad \ldots \quad (3)$$

Hence the feet of the θ-normals from (α, β) on S, are the intersections of S with the rectangular hyperbola

$$(b-a)xy + ax\beta - by\alpha - p(a\alpha x + b\beta y - 1) = 0 \quad \ldots \quad (4)$$

Hence :

> *The quasi-pedal tetrads on S are its intersections with the Tesch hyperbolas,* namely, the rectangular hyperbolas (4), whose asymptotes are parallel to the axes of S.* ... (5)

Also, the fact that the asymptotes of the Tesch hyperbola are parallel to the axes of S implies that the line at infinity cuts them harmonically so that the ϕ-conic of S and any Tesch hyperbola is a parabola. That such parabolas have the equiconjugate diameters of S for conjugate lines, may be proved directly by forming the equations, and is also derived below as an immediate consequence of the apolarity property of the quasi-pedal tetrads (III). We may thus state the theorem :

> *The ϕ-conic of S and any Tesch hyperbola is a parabola which has the equiconjugate diameters of S for conjugate lines. Conversely the quasi-pedal tetrads may also be specified geometrically as the points of contact on S, of its common tangents with parabolas, which have the equiconjugate diameters as conjugate lines.* ... ... (6)

In the case of the *pedal* tetrads, the Tesch hyperbolas (4) become the Apollonian hyperbolas and pass, as is well known, through the centre of S, and through the point (α, β). To obtain a corresponding result for the quasi-pedal case, we observe that the quasi-pedal tetrad of deviation θ, corresponding to the point (α, β), is by (3), the set of intersections of S with the conic :

$$S_1 : p\,(ax^2 + by^2) + (b - a)\,xy + ax\beta - by\alpha - p\,(a\alpha x + b\beta y) = 0.$$

It is clear that S_1 passes through the centre of S and also through the point (α, β); it is also easy to shew that S_1 is the locus of points T whose polar makes the angle $\pi/2 \pm \theta$ with the join of (α, β) to T. Now the asymptotic directions of S_1 are given by :

$$p\,(ax^2 + by^2) + (b - a)\,xy = 0.$$

These separate harmonically the directions of the equiconjugate diameters, and do not depend on α, β, but solely on θ. Hence :

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* Casey, *loc. cit.*

The quasi-pedal tetrads of given deviation Θ from a net; they are cut out on S by the net of conics which pass through the centre C of S, and have asymptotes parallel to

$$ax^2 + by^2 + (b - a) \cot \Theta \, xy = 0 \qquad \ldots \quad (7)$$

We shall now relate these asymptotic directions geometrically to the deviation Θ. To do this, we observe that the conic S has three pairs of diameters every two of which separate each other harmonically; namely, the axial pair CA, CA$'$, the pair of equi-conjugate diameters Ca, Ca', and the pair of asymptotes Cλ, Cλ'. Consider the syzygetic pencil of tetrads of diameters (Ct_1, Ct_2, Ct_3, Ct_4) determined by these three pairs; each such tetrad can be divided into two pairs separating Cλ, Cλ' harmonically, that is to say, into two pairs of *conjugate* diameters (Ct_1, Ct_2) and (Ct_3, Ct_4). Further (Ct_1, Ct_3) and (Ct_2, Ct_4) must each be harmonic to CA, CA,$'$ so that Ct_3, Ct_4 are the respective reflections in either axis, of Ct_1, Ct_2; lastly (Ct_1, Ct_4) and (Ct_2, Ct_3) must each be harmonic to the equi-conjugate diameters Ca, Ca'. The inclination β between Ct_1, Ct_2 is thus the same as the inclination between Ct_4, Ct_3; putting $\Theta = \pi/2 - \beta$, it follows at once that the diameters Ct_1 and Ct_4 are Θ-normals to the conic at each of their extremities (that is to say, Θ-binormals), while Ct_2 and Ct_3 are similarly ($-\Theta$) binormals.

Now if Ct, Ct' are the two diameters given by

$$ax^2 + by^2 + (b - a) \cot \Theta \, xy = 0,$$

it is clear from (7) that Ct and *any* line parallel to Ct,$'$ constitute a conic of the system, which cuts out quasi-pedal tetrads of deviation Θ; this can only mean, that Ct and similarly Ct' are the Θ-binormals to the conic from its centre. We thus have the result;

If Ct, Ct$'$ are the two Θ-binormals to the conic S from its centre, the conics which pass through C and have asymptotes parallel to Ct, Ct$'$, cut out quasi-pedal tetrads of deviation Θ on S. The two Θ-binormals Ct, Ct$'$ separate the equi-conjugate diameters harmonically, and have the property that each is the diameter conjugate to the reflection of the other in an axis. ... (8)

III. *Apolarity relations of the quasi-pedal tetrads.*

Wiener* has obtained the result that the pedal tetrads on S are mixed second polars with respect to the sextic representing parametrically the four axial extremities and the points at infinity on S. Now this sextic is the Jacobian of the syzygetic pencil of inscribed rectangles, and it is well known (and may be easily proved) that the Jacobian sextic of a syzygetic pencil is also its *apolar* sextic. Hence it follows from Wiener's theorem that the pedal tetrads are all the tetrads apolar to the inscribed rectangles of S. We shall extend this result, and show that:

The quasi-pedal tetrads on S are all the tetrads apolar to a particular inscribed rectangle, namely, that whose vertices are the extremities of the equi-conjugate diameters. ... (9)

For, we have seen that the quasi-pedal tetrads of given deviation θ, are cut by the conics, which pass through the centre, and have their asymptotes parallel to the θ-binormals Ct, Ct', from the centre. In other words, the net of quasi-pedal tetrads of deviation θ, has for its neutral pairs, the extremities of the θ-binormals Ct, Ct', and the point-pair $\infty_1 \infty_2$ at infinity on S. Now the extremities of any two conjugate diameters are mutually harmonic; hence they form a tetrad apolar to the tretrad composed of $\infty_1 \infty_2$ and *any* other pair. Thus the equi-conjugate tetrad is apolar to the two tetrads composed of ∞_1, ∞_2 and the extremities of Ct, Ct' respectively. Also by (8) Ct, Ct', separate the equi-conjugate diameters harmonically: this is easily seen to imply the apolarity of the equi-conjugate tetrad to the four extremities of Ct, Ct'. It is thus proved that the equi-conjugate rectangle is apolar to three linearly independent members of the net of quasi-pedal tetrads of deviation θ, for every value of θ. This proves the theorem.

Let now $pqrs$ be a quasi-pedal tetrad, and σ any conic which touches the tangents at $pqrs$. From the apolarity of $pqrs$ to the equi-conjugate rectangle, it follows by Meyer's Transference-principle,† that σ is inpolar to the unique conic S' which is circumscribed to the equi-conjugate rectangle and is out-polar to S. But since the equi-conjugate diameters are conjugate lines with respect to S, they form a conic out-

* *Ency. des Sciences Math.*, III (3), 1, p. 133.

† Grace and Young: *Algebra of Invariants*, page 317.

polar to S. Thus S′ is identical with the pair of equi-conjugate diameters, which are therefore conjugate lines of every conic σ. In particular, the parabolas of (6) have the equi-conjugate diameters for conjugate lines, as was stated there.

It is well-known that the necessary (but not sufficient) condition that four points whose eccentric angles are $\alpha\beta\gamma\delta$, may form a pedal tetrad on S, is $\Sigma\,\alpha$ = an odd multiple of π. We shall extend this result by shewing that this relation is both the necessary and sufficient condition for the four points, to form a *quasi-pedal* tetrad.* Consider generally the system of tetrads of points, the *sum* of whose eccentric angles is equal, mod. 2π, to a given angle θ. It is clear that three points of any tetrad of the system determine the fourth uniquely; hence the system is a linear system, and consists of all tetrads apolar to a definte set of four points, whose eccentric angles x are determined by the equation:

$$4x = \theta;\ \text{or}\ x = \frac{\theta}{4},\ \frac{\theta}{4} + \frac{\pi}{2},\ \frac{\theta}{4} + \pi,\ \frac{\theta}{4} + \frac{3\pi}{2}.$$

That is to say, the four points x are the extremities of two conjugate diameters. Now the two important pairs of conjugate diameters are the equi-conjugate pair and the axial pair. The eccentric angle of an extremity of one of the equi-conjugate diameters is a quarter of an odd multiple of π; hence the sum of the eccentric angles of a quasi-pedal tetrad is an odd multiple of π. Similarly the eccentric angle of an axial extremity is a quarter of an even multiple of π; also the system of concyclic tetrads is a linear system whose quadruple points are the axial extremities (since at these, the osculating circle has four-point contact). Hence if the sum of the eccentric angles of four points is an even multiple of π, they are concyclic. We have thus:

> *If the sum of the eccentric angles of four points is given, they are apolar to the extremities of a pair of conjugate diameters; in particular, if the given sum is an odd or even multiple of π, the points form a quasi-pedal tetrad, or concyclic tetrad respectively.* ... (10)

It is also easy to shew generally, that the tetrads apolar to the extremities of a pair of conjugate diameters Ck, Ck', (that is to say, tetrads the

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* Casey, *l. c.*

sum of whose eccentric angles is a constant) are cut out on S by the ∞^3 rectangular hyperbolas, whose asymptotes are parallel to the bisectors of Ck, Ck'.

It also follows from (10), that if ABCD form a quasi-pedal tetrad, and D′ is the diametrically opposite point of D, then ABCD′ are concyclic. This is the extension of Joachimsthal's theorem. (Casey. *l. c.*).

IV. *Metrical Relations in an Algebraic System of Conics.*

Consider an algebraic system of ∞^1 conics, with the characteristics m, n (*i.e.* such that m conics of the system pass through an arbitrary point, and n conics touch an arbitrary line). The intersections of a conic of the system with an arbitrary line must be in (m, m) correspondence, and must therefore coincide in $2m$ cases. Since however only n conics of the system touch the straight line in question, the remaining number $2m - n$ must be the number of point-pairs in the system. By similar reasoning $2n - m$ must be the number of line-pairs in the system.

Now, the asymptotic directions of a conic of the system are in (m, m) correspondence, and will therefore include a given angle in $2m$ cases. *Hence the system contains sets of 2m similar conics* The system of such sets is evidently a linear system g^1_{2m} within the family of conics. Now the family contains m rectangular hyperbolas; since the only conics similar to a rectangular hyperbola are rectangular hyperbolas, it follows that the m rectangular hyperbolas each counted twice constitute a set of g^1_{2m}. Also the family contains n parabolas, and since the only conics similar to a parabola are either parabolas or squared lines, it follows that the n parabolas together with the $2m - n$ point-pairs of the family constitute a second set of the g^1_{2m}. These two sets completely determine the g^1_{2m}.

Again, among the pairs of a symmetric (m, m) correspondence, there are m which belong to a given involution. *Hence the system contains sets of m conics with parallel axes.* These sets form a second linear system g^1_m. The two sets of m conics of the system, which pass through each circular point, obviously belong to g^1_m, and therefore determine it completely.

Now, let the genus of the system of conics be p; then any linear system g^1_r has $2r - 2 + 2p$ Jacobian elements.* Also among the $4m - 2 + 2p$ Jacobian elements of g^1_{2m}, the m rectangular hyperbolas would be included. Discarding these, it follows that there are $3m-2+2p$ conics in the system, which are similar to their consecutive conic, and are therefore of stationary eccentricity, Hence:

In a family of conics of genus p, and characteristics (m, n), there exist $3m-2+2p$ conics which touch their envelope in a quasi-pedal tetrad; there also exist $2m-2+2p$ conics which touch their envelope in a concyclic tetrad. ... (11)

In illustration of this theorem, we may consider (1) the system of conics through 4 points, and (2) the system touching four lines.

Conics through four points PQRS.

The centre locus of the system is a conic Γ which is circumscribed to ABC the harmonic triangle of PQRS, and passes through the middle points of the six sides of PQRS. The centre of Γ is the centroid G of PQRS, and its asymptotes are parallel to the axes of the two parabolas of the system. The ϕ-conic of the two parabolas of the system is a conic concentric with Γ having ABC for a self-polar triangle. If h, m are the centres of the rectangular hyperbola, and the conic of minimum eccentricity in the system, it follows that hm is a diameter of Γ, the extremities of chords of Γ conjugate to hm are the centres of pairs of similar conics of the system; for, from our general result, the pairs of similar conics in the system belong to an involution, whose double elements are the rectangular hyberbola H and the conic M of minimum eccentricity. Also pairs of conics of the system, belonging to the involution determined by the two parabolas, have the property that their ϕ-conic is a parabola; in particular the ϕ-conic of H and M is a parabola, as has been already proved in (6). (Note that H is a Tesch hyperbola of M).

We may note, that if A, B be the centres of two conics of the system, then the centre of their ϕ-conic is the pole of AB with respect

* See Severi: *Vorlesungen uber Algebraische Geometrie*, p. 106.

to Γ. We may also note that the asymptotes of any conic of the system are parallel to conjugate diameters of Γ; in particular, the asymptotes of H are parallel to the axes of Γ, and those of M to the equi-conjugate diameters of Γ.

It can also be shewn that if an asymptote of the conic whose centre is λ cuts the centre-locus again in λ_1, then the correspondence (λ, λ_1) is a (1, 2) *polar* correspondence.

Conics touching four lines p q r s.

The characteristics of this system are (2. 1). Hence the system contains pairs of conics with parallel axes, and tetrads of similar conics. Let ABC be the harmonic triangle of *p q r s*, and let γ be the auto-polar circle of ABC. Also let H_1, H_2 be the two rectangular hyperbolas, and π the unique parabola of the system.

There are two conics $\gamma_1\gamma_2$ of the system whose contacts with *p q r s* lie on γ. γ is thus the F-conic of γ_1, γ_2. Pairs of conics of the system separating γ_1, γ_2 harmonically have parallel axes.

Tetrads of similar conics form a pencil determined by $(H_1H_1H_2H_2)$ and the tetrad constituted by π and the three point-pairs. Hence, by a theorem proved elsewhere,* it is possible to circumscribe an infinity of quadrilaterals to the parabola π, so that their four vertices lie on any four similar conics of the system.

We may note in addition (1) that pairs of conics of the system whose director-circles are orthogonal belong to an involution whose double members are H_1, H_2, (2) that the F-conic of every such pair is a rectangular hyperbola belonging to the pencil (H_1, H_2).

The centre-locus of the system is a straight line, while the locus of the focus is a circular cubic.

Further research appears to be called for, for the elucidation of the metrical relations in these *simplest types of systems of conics.*

R. VAIDYANATHASWAMY.

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* The 'Algebraic (2,2) Correspondence,' *J.I.M.S.*, 1926, p. 34.

The Linear Indeterminate Equation.

1. The following method of solving the linear indeterminate equation does not involve the principle of continued fractions and is, perhaps, better than the tentative elementary methods given in the common text-books, *e.g.*, Ross's Algebra.

2. Suppose the equation $ax + by + c = 0$ ... (i)
where a, b, c are integers not having any common factor, is reduced by a series of rational operations to the form

$$x - k + b(a'x + b'y) = 0. \qquad \text{... (ii)}$$

Then, we get, on comparing the co-efficients of x, y and the absolute term in the two equations,

$$\frac{a}{a'b+1} = \frac{b}{bb'} = \frac{c}{-k}.$$

Hence, $$ab' - a'b = 1$$

and $$k = -b'c.$$

Evidently, $$x = k,\ y = -\frac{a'k}{b'} = a'c$$

satisfy equation (ii) and therefore also (i). Thus, the original indeterminate equation (i) can be solved in integers, as soon as we reduce it to the form (ii) in which a', b', k are integers.

Obviously, the equation (i) has no integral solutions when a, b have a common factor prime to c; therefore, a and b must be relatively prime.

3. The process of reduction of (i) to (ii) is closely parallel either (I) to that of finding the arithmetical G. C. M., *viz.*, 1 of a and b (which are relatively prime) or (II) to that of dividing a successively by b (when $a > b$) and the numerically least residues obtained in the course of the division, until unity is reached.

For example, to solve $79y - 32x - 7 = 0$ in integers.

The work may be arranged as follows :

Method I.

$$\begin{array}{l} 32)\ 79\ (2 \\ \quad\ \ 64 \\ \quad\ \ \overline{15)\ 32\ (2} \\ \qquad\qquad\ 30 \\ \qquad\qquad\ \overline{\ 2)\ 15\ (7} \\ \qquad\qquad\qquad\ \ 14 \\ \qquad\qquad\qquad\ \ \overline{\ \ 1\ } \end{array}$$

$$79y - 32x - 7 = 0$$

i.e. $15y - 32(x - 2y) - 7 = 0 \quad \ldots \quad (1)$

$\therefore \quad 30y - 32(2x - 4y) - 14 = 0$

i.e. $-2y - 32(2x - 5y) - 14 = 0 \quad \ldots \quad (2)$

Multiplying (2) by 7 and adding to (1),

$$y - 105 - 32(15x - 37y) = 0$$

whence,

$$y = 105, \quad x = 259.$$

The general solution is of the form

$$x = 259 + 79m, \quad y = 105 + 32m$$

where m is any integer.

Putting $m = -3$, we get the least solution $x = 22$, $y = 9$.

Method II.

$$\begin{array}{l} 32)\ 79\ (2 \\ \quad\ \ 64 \\ \quad\ \ \overline{15)\ 79\ (5} \\ \qquad\qquad\ 75 \\ \qquad\qquad\ \overline{\ 4)\ 79\ (20} \\ \qquad\qquad\qquad\ \ 80 \\ \qquad\qquad\qquad\ \ \overline{-1\ } \end{array}$$

$79y - 32x - 7 = 0 \quad \ldots \quad (1)$

i.e. $15y - 32(x - 2y) - 7 = 0 \quad \ldots \quad (2)$

Subtracting 5 times (2) from (1),

$4y - 32(10y - 4x) + 28 = 0 \quad \ldots \quad (3)$

Subtracting (1) from 20 times (3)

$$y + 567 - 32(200y - 81x) = 0$$

whence,

$$y = -567, \quad x = -1400.$$

The general solution is of the form

$$x = -1400 + 79m, \quad y = -567 + 32m$$

where m is any integer.

Putting $m = 18$, we get the least solution $x = 22$, $y = 9$, the same as before.

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Nature of the roots of the biquadratic.

The following discussion of the nature of the roots of the biquadratic* was suggested by the corresponding discussion for the cubic by L. G. Weld in his *Short Course in the Theory of Determinants*, page 113. The essential point in the discussion is that any two real or conjugate complex numbers are of the form $u \pm \sqrt{v^2}$.

Let the biquadratic

$$a_0x^4 + 4a_1x^3 + 6a_2x^2 + 4a_3x + a_4 = 0 \qquad \ldots \quad (1)$$

be transformed by the substitution $a_0x + a_1 = z$ to the form

$$z^4 + 6\,Hz^2 + 4\,Gz + a_0^2\,I - 3\,H^2 = 0 \qquad \ldots \quad (2)$$

(with the usual notation).

Let the roots of the biquadratics (1) and (2) be x_r, z_r, $(r = 1, 2, 3, 4)$ respectively. Since the roots of a biquadratic with real co-efficients are either all real, all complex or two real and two complex, therefore the roots of equation (2) are of the form $\alpha \pm \sqrt{\beta^2}$, $-\alpha \pm \sqrt{\gamma^2}$.

Now

$$256\,\Delta = a_0^6\,\Pi\,(x_1 - x_2)^2 = a_0^{-6}\,\Pi\,(z_1 - z_2)^2$$
$$= 16a_0^{-6}\beta^2\gamma^2\,[(4\alpha^2 - \beta^2 - \gamma^2)^2 - 4\beta^2\gamma^2]^2 \qquad \ldots \quad (3)$$

where $\Delta = I^3 - 27\,J^2$. Also

$$\Sigma\, z_1z_2 = -(2\alpha^2 + \beta^2 + \gamma^2) = 6H \qquad \ldots \quad (4)$$

$$z_1z_2z_3z_4 = (\alpha^2 - \beta^2)(\alpha^2 - \gamma^2) = a_0^2\,I - 3H^2 \qquad \ldots \quad (5)$$

Subtraction from (5) of the square of half of (4) gives

$$-2\alpha^2(\beta^2 + \gamma^2) - \left(\frac{\beta^2 - \gamma^2}{2}\right)^2 = a_0^2\,I - 12\,H^2. \qquad \ldots \quad (6)$$

I. All complex roots :—If all the roots are complex, β^2 and γ^2 are both negative, so Δ is positive.

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* *Vide* also Burnside and Panton : *Theory of Equations*, Vol. I, p. 145.

II. All real roots :—If all the roots be real, β^2 and γ^2 are both positive and so Δ is positive. Moreover the equations (4) and (6) show that

$$H < 0 \text{ and } a_0^2 I - 12 H^2 < 0.$$

III. Two complex and two real roots :—In this case either

$$\beta^2 > 0 \text{ and } \gamma^2 < 0 \text{ or } \beta^2 < 0 \text{ and } \gamma^2 > 0 \text{ and so } \Delta < 0.$$

IV. (*a*) Two equal roots :—If two roots are equal, then β^2 or γ^2 vanishes, and so $\Delta = 0$.

(*b*) Three roots equal :—If three roots are equal, $\beta^2 = 0$ and $\gamma^2 = 4\alpha^2$. This gives $\Delta = 0$ as the first condition. Also equation (4) gives $\alpha^2 = -H$, and substituting in equation (5), we obtain $I = 0$. Since $\Delta = I^3 - 27 J^2$, the two independent equations obtained are

$$I = 0 \quad J = 0.$$

(*c*) All roots equal :—If all the roots be equal, it is obvious that $\alpha = 0$, $\beta = 0$, $\gamma = 0$ and so

$$\Delta = 0 \quad H = 0, \quad G = 0, \quad a_0^2 I - 3H^2 = 0$$

These give three independent conditions, *viz.*

$$H = 0. \quad I = 0, \quad J = 0$$

as

$$a_0^3 J = a_0^2 HI - G^2 - 4H^3.$$

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Solutions.

Question 770.

(S. RAMANUJAN):—If δ_n denote the number of divisors of n

$$(e.g., \ \delta_1 = 1, \ \delta_2 = 2, \ \delta_3 = 2, \ \delta_4 = 3, \ldots)$$

shew that

$$\text{(i)} \quad \delta_1 - \frac{1}{3}\delta_3 + \frac{1}{5}\delta_5 - \frac{1}{7}\delta_7 + \frac{1}{9}\delta_9 - \ldots$$

is a convergent series; and that

$$\text{(ii)} \quad \delta_1 - \frac{1}{2}\delta_2 + \frac{1}{3}\delta_3 - \frac{1}{4}\delta_4 + \frac{1}{5}\delta_5 - \ldots$$

is a divergent series in the *strict* sense (*i.e.* not oscillating).

Solution by Prof. G. N. Watson.*

The key to these problems is to be found in Dirichlet's well-known formula†

$$(1) \quad \delta_1 + \delta_2 + \delta_3 + \ldots + \delta_N = N \log N + (2\gamma - 1) N + \varepsilon(N),$$

where $$\varepsilon(N) = O(\sqrt{N}),$$

and the analogous formulae

$$(2) \quad \sum_{2m+1 \leq N} \delta_{2m+1} = \frac{1}{4}\left\{ N \log N + (2\gamma + 2\log 2 - 1) N \right\} + \varepsilon_0(N)$$

$$(3) \quad \sum_{4m+1 \leq N} \delta_{4m+1} = \frac{1}{8}\left\{ N \log N + (2\gamma + 2\log 2 - 1) N \right\} + \varepsilon_1(N)$$

$$(4) \quad \sum_{4m+3 \leq N} \delta_{4m+3} = \frac{1}{8}\left\{ N \log N + (2\gamma + 2\log 2 - 1) N \right\} + \varepsilon_3(N)$$

where, in (2), m takes the values $0, 1, 2, 3, \ldots \frac{1}{2}(N-1)$ or $\frac{1}{2}(N-2)$, and similarly in (3) and (4); and

$$\varepsilon_0(N) = O(\sqrt{N}), \ \varepsilon_1(N) = O(\sqrt{N}), \ \varepsilon_3(N) = O(\sqrt{N}).$$

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* *Vide* Volume XVII, p. *166*, of this *Journal* for another Solution—Ed.

† *Werke*, II, p. 249.

These formulae were stated by Ramanujan in a slightly different form in his paper "Some formulae in the analytic theory of numbers"* and (1) was the first formula which he quoted in his paper "Highly composite numbers"† Elementary proofs of (2), (3) and (4) have been published by Estermann.‡

Let
$$\sum_{m=1}^{N} \delta_m = D(N), \qquad \sum_{2m+1 \le N} \delta_{2m+1} = D_0(N),$$
$$\sum_{4m+1 \le N} \delta_{4m+1} = D_1(N), \qquad \sum_{4m+3 \le N} \delta_{4m+3} = D_3(N).$$

It is evident that $\delta_N = O(\sqrt{N})$, and so the terms in both of the series (i) and (ii) tend to zero. It is consequently sufficient to consider the behaviour of an even number of terms of each series.

First take the series (i). We have

$$\frac{\delta_1}{1} - \frac{\delta_3}{3} + \frac{\delta_5}{5} - \frac{\delta_7}{7} + \dots + \frac{\delta_{4N-3}}{4N-3} - \frac{\delta_{4N-1}}{4N-1}$$
$$= \left[\frac{\delta_1}{1} + \frac{\delta_5}{5} + \frac{\delta_9}{9} + \dots + \frac{\delta_{4N-3}}{4N-3}\right]$$
$$- \left[\frac{\delta_3}{3} + \frac{\delta_7}{7} + \frac{\delta_{11}}{11} + \dots + \frac{\delta_{4N-1}}{4N-1}\right]$$
$$= \left[\frac{D_1(4)}{1} + \frac{D_1(8) - D_1(4)}{5} + \frac{D_1(12) - D_1(8)}{9} + \dots + \frac{D_1(4N) - D_1(4N-4)}{4N-3}\right]$$
$$- \left[\frac{D_3(4)}{3} + \frac{D_3(8) - D_3(4)}{7} + \frac{D_3(12) - D_3(8)}{11} + \dots + \frac{D_3(4N) - D_3(4N-4)}{4N-1}\right]$$

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* *Messenger of Math.* XLV (1916), pp 81—84. I have to thank Prof. Hardy for supplying me with this reference to Ramanujan's statement and with the reference to Estermann's paper; I had previously constructed the formulae in a different manner.

† *Proc. London Math. Soc.*, (2) XIV (1915), pp. 347—409.

‡ *Journal London Math. Soc.*, III. (1928), pp. 247—250.

$$= 4 \sum_{m=1}^{N-1} \frac{D_1(4m)}{(4m-3)(4m+1)} + \frac{D_1(4N)}{4N-3}$$

$$- 4 \sum_{m=1}^{N-1} \frac{D_3(4m)}{(4m-1)(4m+3)} - \frac{D_3(4N)}{4N-1}$$

$$= \tfrac{1}{2} \sum_{m=1}^{N-1} \frac{4m \log 16m + 4m(2\gamma - 1) + 8\varepsilon_1(4m)}{(4m-3)(4m+1)}$$

$$+ \frac{4N \log 16N + 4N(2\gamma - 1) + 8\varepsilon_1(4N)}{8(4N-3)}$$

$$- \tfrac{1}{2} \sum_{m=1}^{N-1} \frac{4m \log 16m + 4m(2\gamma - 1) + 8\varepsilon_3(4m)}{(4m-1)(4m+3)}$$

$$- \frac{4N \log 16N + 4N(2\gamma - 1) + 8\varepsilon_3(4N)}{8(4N-1)}$$

$$= \sum_{m=1}^{N-1} \frac{32m^2 \{\log 16m + 2\gamma - 1\}}{(4m-3)(4m-1)(4m+1)(4m+3)}$$

$$+ \sum_{m=1}^{N-1} \frac{4\varepsilon_1(4m)}{(4m-3)(4m+1)} - \sum_{m=1}^{N-1} \frac{4\varepsilon_3(4m)}{(4m-1)(4m+3)}$$

$$+ \frac{N\{\log 16N + 2\gamma - 1\}}{(4N-3)(4N-1)} + \frac{\varepsilon_1(4N)}{4N-3} - \frac{\varepsilon_3(4N)}{4N-1}.$$

When $N \to \infty$, the three series in the last expression converge, and the three isolated terms tend to zero. Hence the series (i) is convergent.

Next take the series (ii). We have

$$\frac{\delta_1}{1} - \frac{\delta_2}{2} + \frac{\delta_3}{3} - \frac{\delta_4}{4} + \ldots + \frac{\delta_{2N-1}}{2N-1} - \frac{\delta_{2N}}{2N}$$

$$= 2\left[\frac{\delta_1}{1} + \frac{\delta_3}{3} + \frac{\delta_5}{5} + \ldots + \frac{\delta_{2N-1}}{2N-1}\right]$$

$$- \left[\frac{\delta_1}{1} + \frac{\delta_2}{2} + \frac{\delta_3}{3} + \ldots + \frac{\delta_{2N}}{2N}\right]$$

$$= 2\left[\frac{D_0(1)}{1} + \frac{D_0(3) - D_0(1)}{3} + \frac{D_0(5) - D_0(3)}{5} + \ldots + \frac{D_0(2N-1) - D_0(2N-3)}{2N-1}\right]$$

$$- \left[\frac{D(1)}{1} + \frac{D(2) - D(1)}{2} + \frac{D(3) - D(2)}{3} + \ldots + \frac{D(2N) - D(2N-1)}{2N}\right]$$

$$= 4\sum_{m=1}^{N-1} \frac{D_0(2m-1)}{(2m-1)(2m+1)} + \frac{2D_0(2N-1)}{2N-1} - \sum_{m=1}^{2N-1} \frac{D(m)}{m(m+1)} - \frac{D(2N)}{2N}$$

$$= \sum_{m=1}^{N-1} \frac{\log(8m-4) + 2\gamma - 1}{2m+1} + \sum_{m=1}^{N-1} \frac{4\varepsilon_0(2m-1)}{(2m-1)(2m+1)} + \tfrac{1}{2}\{\log(8N-4) + 2\gamma - 1\} + \frac{2\varepsilon_0(2N-1)}{2N-1}$$

$$-\sum_{m=1}^{2N-1} \frac{\log m + 2\gamma - 1}{m+1} - \sum_{m=1}^{2N-1} \frac{\varepsilon(m)}{m(m+1)} - \{\log 2N + 2\gamma - 1\} - \frac{\varepsilon(2N)}{2N}$$

$$= \sum_{m=1}^{N-1} \left\{\frac{\log(8m-4) + 2\gamma - 1}{2m+1} - \frac{\log(2m) + 2\gamma - 1}{2m+1} - \frac{\log(2m+1) + 2\gamma - 1}{2m+2}\right\}$$

$$+ \tfrac{1}{2}\{\log(8N-4) - 2\log 2N - 4\gamma + 2\}$$

$$+ \sum_{m=1}^{N-1} \frac{4\varepsilon_0(2m-1)}{(2m-1)(2m+1)} - \sum_{m=1}^{2N-1} \frac{\varepsilon(m)}{m(m+1)}$$

$$+ \frac{2\varepsilon_0(2N-1)}{2N-1} - \frac{\varepsilon(2N)}{2N},$$

the last step being obtained by grouping the terms of

$$\sum_{m=1}^{2N-1} \frac{\log m + 2\gamma - 1}{m+1} \text{ in pairs with those of } \sum_{m=1}^{N-1} \frac{\log(8m-4) + 2\gamma - 1}{2m+1}$$

the second and third with the first, the fourth and fifth with the second and so on.

Now

$$\frac{\log(8m-4) + 2\gamma - 1}{2m+1} - \frac{\log(2m) + 2\gamma - 1}{2m+1} - \frac{\log(2m+1) + 2\gamma - 1}{2m+2}$$

$$= \frac{1}{2m+1} \log\left(1 - \frac{1}{2m}\right) + \frac{\log 4}{(2m+1)(2m+2)}$$

$$- \frac{\log(2m+1) + 2\gamma - 1 - \log 4}{2m+2}$$

and the first two terms are the general terms of convergent series while the third is the general term of a series which diverges to $-\infty$. Since also

$$\tfrac{1}{2}\{\log(8N-4) - 2\log 2N - 4\gamma + 2\} \to -\infty$$

as $N \to \infty$, while

$$\sum_{m=1}^{\infty} \frac{4\varepsilon_0(2m-1)}{(2m-1)(2m+1)} \text{ and } \sum_{m=1}^{\infty} \frac{\varepsilon(m)}{m(m+1)}$$

are convergent and

$$\frac{2\varepsilon_0(2N-1)}{2N-1} \to 0, \quad \frac{\varepsilon(2N)}{2N} \to 0,$$

it is evident from the transformed form of

$$\frac{\delta_1}{1} - \frac{\delta_2}{2} + \frac{\delta_3}{3} - \frac{\delta_4}{4} + \dots + \frac{\delta_{2N-1}}{2N-1} - \frac{\delta_{2N}}{2N}$$

that it tends to $-\infty$ as $N \to \infty$. Consequently the series (ii) diverges to $-\infty$.

Questions 1369, 1480, 1481, 1485, 1495 and 1505.

1369. (R. GOPALASWAMY):—A_1, B_1, C_1, D_1 are the isogonal conjugates of each of the points A, B, C, D with respect to the triangles formed by the other three. If for these triangles (S_1, ρ_1) (S_2, ρ_2) (S_3, ρ_3) and (S_4, ρ_4) are the circum-centres and circum-radii respectively, then

$$\frac{A_1 S_1}{\rho_1} = \frac{B_1 S_2}{\rho_2} = \frac{C_1 S_3}{\rho_3} = \frac{D_1 S_4}{\rho_4}$$

1480. (N. DURAIRAJAN) :—A, B, C, D are four points in a plane. A', B', C', D' are respectively the isogonal conjugates of A, B, C, D with respect to the triangles BCD, CDA, DAB and ABC. Shew that there exists a point M such that A', B', C', D', are the inverses of M with respect to the circles BCD, CDA, DAB and ABC.

1481. (M. BHIMASENA RAO):—Shew that (i) the six circles of similitude of the circum-circles of the four triangles BCD, CDA, DAB, ABC, taken two by two, are concurrent at a point M, and (ii) if O is the centre of the rectangular hyperbola ABCD, then M is the Fregier point of O with respect to the conic ABCDO.

1485. (A. A. KRISHNASWAMY AIYANGAR):—Denoting the point M in Question 1480 as the M-point of the quadrangle ABCD, if A_1, B_1, C_1, D_1 are the circum-centres of the triangles BCD, CDA, DAB, ABC, shew that the quadrangles ABCD and $A_1B_1C_1D_1$ have the same M-point and that $MA.MA_1 = MB.MB_1 = MC.MC_1 = MD.MD_1$.

1495. (A. A. KRISHNASWAMY AIYANGAR):—A_1, B_1, C_1, D_1 are the circum-centres of the triangles BCD, CDA, DAB, ABC; A_2, B_2, C_2, D_2 are the circum-centres of $B_1C_1D_1$, $C_1D_1A_1$, $D_1A_1B_1$ and $A_1B_1C_1$; and O, O_1, and O_2 are the centres of rectangular hyperbolas circumscribing the quadrangles ABCD, $A_1B_1C_1D_1$ and $A_2B_2C_2D_2$. Shew that the quadrangles ABCD and $A_2B_2C_2D_2$ are homothetic, the homothetic centre being the M-point defined in Question 1485. If MO_1 cuts the conic ABCDO at Q, R, shew that O_1 is the middle-point of QR and that $OO_2.MO_2 = O_1O_2^2$.

1505. (A. A. KRISHNASWAMY AIYANGAR):—Given in position the perpendicular bisectors of the four sides of a quadrilateral, construct the quadrilateral.

Solution and Remarks by T. R. Raghavasastri.

In the following proof, the points have been named in a slightly different way from that of the first three questions. A_1, B_1, C_1, D_1 are the circum-centres of BCD, CDA ... and A_2, B_2, C_2, D_2 are derived in the same way from $A_1B_1C_1D_1$. T is the mean centre of ABCD. O, O_1 and O_2 are the centres of rectangular hyperbolas ABCD, $A_1B_1C_1D_1$ and $A_2B_2C_2D_2$. M is the inverse of D for the circle $A_1B_1C_1$.

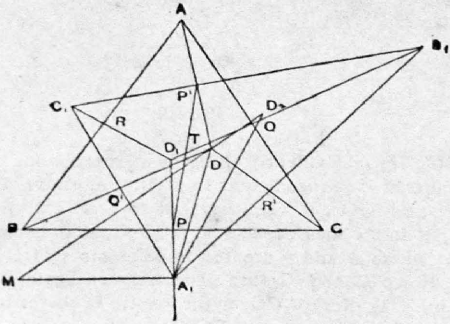

FIG. 1.

From Fig. 1, evidently D and D_1 are isogonal conjugates in $A_1B_1C_1$. For $\angle C_1A_1D_1 = \angle DBC = \angle DA_1B_1$ as A_1 is the circum-centre of BCD. Similarly for the others. Hence A, B, C, D are the isogonal conjugates of A_1, B_1, C_1, D_1 for the triangles $B_1C_1D_1$, $C_1D_1A_1$, etc.

Also PP$'$, QQ$'$ and RR$'$ are concurrent at T as P, P$'$, Q, Q$'$, etc. are the middle-points of the opposite joins of ABCD. Therefore the circle P$'$Q$'$R$'$ is the symmetrique of PQR about T. But the former circle is the pedal circle of D in $A_1B_1C_1$ and as D and D_1 are isogonal conjugates for $A_1B_1C_1$ it contains O_1 the centre of the rectangular hyperbola through $A_1B_1C_1D_1$. Also the circle PQR being the nine-points circle of ABC contains O. By considering similarly another pair of corresponding triangles it can be shown that the pedal circle PR$'$Q is a symmetrique of P$'$Q$'$R the nine-points circle of ABD about T. It follows therefore that O and O$'$ are equidistant from T as they are the intersections of two symmetrical pairs of circles.

Moreover from Fig. 2 the *asymptotes of the hyperbolas ABCD and $A_1B_1C_1D_1$ are parallel.* For, let N be the middle-point of B_1 C_1. The asymptotes of ABCD are parallel to the bisectors of the angle $OP'D$ and those of $A_1B_1C_1D_1$ are parallel to the bisectors of angle O_1NB_1. Evidently NB_1 is perpendicular to AD. It can be shown that OP' is perpendicular to O_1N. For, since T is the middle point of PP' and OO_1, $P'O$ is parallel to O_1P. If S is the middle-point of NP, it is the mean centre of BCB_1C_1 and hence is the middle-point of $\beta\gamma$, where β and γ are the middle-points of B B_1 and CC_1 respectively. Since C and C_1 are isogonal conjugates for the Δ $A_1B_1D_1$, γ the middle-point of CC_1 is the centre of the pedal circle $PQR'O_1$. Similarly β is the centre of $PQ'RO_1$. Hence $\beta P = \beta O_1$ and $\gamma P = \gamma O_1$. Hence the two Δs $\beta\gamma P$ and $\beta\gamma O_1$ are identical and since SP and SO_1 are corresponding medians, they are equal. So $SO_1 = SP = SN$. Hence $\angle NO_1P$ is a rt. $\angle$. The asymptotes of the two rectangular hyperbolas are parallel.

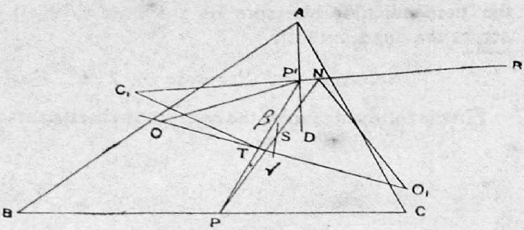

FIG. 2.

To prove the properties of the M point.

Let $\alpha, \beta, \gamma, \delta$ be the circum-radii of BCD, CDA, etc. and $\alpha_1, \beta_1, \gamma_1, \delta_1$ those of $B_1C_1D_1$, $C_1D_1A_1$, etc , and $\alpha_2, \beta_2, \gamma_2, \delta_2$ those of $B_2C_2D_2$ ... Since M is the inverse of D for the circle $A_1B_1C_1$, its tri-polar co-ordinates are in the ratio $DA_1 : DB_1 ; DC_1$. (*i.e.*) $\alpha : \beta : \gamma$ in the Δ A_1 B_1 C_1. Similarly if M' is the inverse of C for the circle $A_1B_1D_1$ its tri-polar co-ordinates in $A_1B_1D_1$ will be $\alpha : \beta : \delta$. *M and M' coincide*, for, they lie on the Appollonian circle dividing A_1B_1 in the ratio $\alpha : \beta$ and also on the circle containing an angle $A_1C_1B_1$ — ACB by an easy property of triangular co-ordinates of inverse points (*Vide* Gallatly's *Modern Geometry of the Triangle*).

Hence it follows that M is the inverse of A, B, C, D for the circles $B_1C_1D_1$, $C_1D_1A_1$, $D_1A_1B_1$ and $A_1B_1C_1$ respectively, and its distances from A_1, B_1, C_1, D_1 are in the ratio $\alpha : \beta : \gamma : \delta :$ ... (Qn. 1480).

Since D_2 is the circum-centre of $A_1B_1C_1$, D_2, D, M are in a line and

$D_2A_1^2 = D_2D \cdot D_2M$. Hence the two Δs D_2DA_1 and D_2MA_1 are similar; therefore $\frac{MA_1}{DA_1} = \frac{MD_2}{D_2A_1}$. Since MA_1 varies as DA_1 we have MD_2 varying as D_2A_1 or α_1. Hence M is the M-point also for the quadrangle $A_1B_1C_1D_1$.

Therefore

$$MA \cdot MA_1 : MB \cdot MB_1 : MC \cdot MC_1 : MD \cdot MD_1$$
$$= \alpha\alpha_1 : \beta\beta_1 : \gamma\gamma_1 : \delta\delta_1$$

But it is easily seen that $\alpha\alpha_1 = \beta\beta_1 = \gamma\gamma_1 = \delta\delta_1$; for δ is twice the radius of the circle $P'Q'R'$

$$= \frac{Q'R'}{\sin Q'P'R'} = \frac{A_1D \sin A_1}{\sin D_1} = \frac{\alpha\alpha_1}{\delta_1}$$

[since $\angle Q'P'R' = A_1C_1D + A_1B_1D = D_1C_1B_1 + D_1B_1C_1 = \pi - D_1$ and since A_1 and D_1 stand on the same base B_1C_1].

Hence $\alpha\alpha_1 = \delta\delta_1$; similarly each equals $\gamma\gamma_1 = \beta\beta_1$ ... (Qn. 1485).

Qn. 1369 follows at once. Since $D_2D \cdot D_2M = \delta_1^2$ and as D_2M varies as δ_1 therefore

$$\frac{D_2D}{\delta_1} = \frac{C_2C}{\gamma_1} = \frac{B_2B}{\beta_1} = \frac{A_2A}{\alpha_1}.$$

Also M is a point on the six circles of similitude of the 4 circles BCD, CDA, DAB and ABC as its distances from the circum-centres are as the radii of the circles. This is (i) of Qn. 1481.

To prove the second part of the same question, we may note that the rectangular hyperbola $A_1B_1C_1D_1$ is the isogonal transform of each of the diameters D_2M, C_2M, B_2M and A_2M in the triangles $A_1B_1C_1$, $B_1A_1D_1$, $C_1A_1D_1$ and $B_1C_1D_1$ respectively. Hence its centre O_1 is the focus of the parabolas with those diameters as directrices and with respect to which the above four triangles are respectively self-conjugate. (*Vide* Gallatly, *loc. cit.*). By reciprocating with respect to O_1 therefore, the conic $A_1B_1C_1D_1O_1$ reciprocates into a parabola touching the reciprocals of A_1, B_1, C_1 and D_1 and the four parabolas with O_1 as focus reciprocate into polar circles of the four triangles formed by the reciprocals of A_1, B_1, C_1 and D_1, the directrices of those parabolas reciprocating into the orthocentres of those triangles. Hence M the point of concurrency of the directrices reciprocates into the line of orthocentres which is evidently the directrix of the parabola, the reciprocal of the conic $A_1B_1C_1D_1\,O_1$. Hence in the original figure M is the Fregier point of O_1 for the conic $A_1B_1C_1D_1O_1$—a theorem which holds good for any of the quadrangles.

From what has been proved already, it is evident that ABCD and $A_2B_2C_2D_2$ are homothetic for the centre M, as $\frac{MD}{MD_2}$ is a constant.

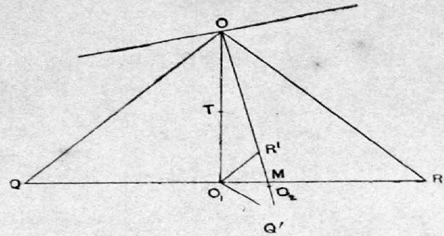

FIG. 3.

Taking the conic ABCDO as $ax^2 + 2hxy + by^2 + 2gx + 2fy = 0$ and the rectangular hyperbola ABCD as $xy = c^2$, we at once see that the centre locus of conics passing through A, B, C, D can be equally well got by omitting c^2 from our equations. This shows that T the centre of mean position of ABCD and the centre of the centre locus is also the centre of mean position of O, O and Q, R where Q and R are the points where the lines $xy = 0$ meet the conic ABCDO. Hence T is the middle point of OO_1 where O_1 is the middle-point of QR. But QR also evidently passes through M the Fregier point of O. Also from what has been seen at the outset, O_1 is the centre of the rectangular hyperbola $A_1B_1C_1D_1$ whose asymptotes have been shown to be parallel to $xy = 0$ the asymptotes of the hyperbola ABCD. Hence O_1R', O_1Q' drawn parallel to OQ and OR are the asymptotes. But O_2 the centre of the hyperbola $A_2B_2C_2D_2$ must lie on OM, since M is the homothetic centre of ABCD and $A_2B_2C_2D_2$; and as O_1 is the middle-point of the chord QR, so also must O_2 be the middle-point of $Q'R'$. So from the figure $(Q'R', MO) = -1$.

Hence $$O_2Q'^2 = O_2O_1{}^2 = O_2M\,.\,O_2O,$$

which proves Qn. 1495.

Qn. 1505 can be put thus; given the quadrangle $A_1B_1C_1D_1$ in Fig. 1. construct ABCD. The construction is obvious. We have to find the isogonal conjugates A, B, C, D, of A_1, B_1, C_1, D_1 in th respective Δs formed by the latter.

Questions for Solution.

Proposers of Questions are requested to send their own solutions along with their questions.

1578. (V. RAMASWAMI AIYAR, M.A.):—Let H_1 be a rectangular hyperbola, centre Ω, circumscribing a given triangle ABC. Let T, T′ be the isogonal conjugates of the points at infinity on H_1. If H_2 be a hyperbola having ΩT, $\Omega T'$ for asymptotes and TT′ for a tangent, show that triangles $\alpha\beta\gamma$ can be inscribed in H_1 so as to circumscribe H_2; and that for every such triangle $\alpha\beta\gamma$, the pedal circles of α, β, γ with respect to ABC touch one another at Ω.

1579. (A. A. KRISHNASWAMI AIYANGAR, M.A.):—Show that the minimum number of possible distinct quadratic surds in the expansion

$$(a_1^{\frac{1}{2}} + a_2^{\frac{1}{2}} + \dots + a_n^{\frac{1}{2}})^2$$

$2^{\lambda} - 1$ where $\lambda - 1 < \log_2 n \leqslant \lambda$.

Find the square root of ;

(i) $222 + 26\sqrt{35} + 14\sqrt{210} + 58\sqrt{6}$.

(ii) $397 - 80\sqrt{2} - 60\sqrt{3} - 54\sqrt{6} - 100\sqrt{7} + 58\sqrt{14}$.

[These are the exceptional cases not treated of in our Algebra text-books.]

1580. (T. VIJIARAGHAVAN):—Show that if a function is continuous for all rational values of x in the interval (0,1) then it is also continuous for an unenumerably infinite number of values of x in that interval.

1581. (B. RAMAMURTI):—A, B, C, D are four points in a plane (not forming an orthocentric tetrad) co-normal for some conic passing through them. O is the centre of the rectangular hyperbola through them. Show that

$$\frac{1}{d_{12}^2 + d_{34}^2 - R^2} + \frac{1}{d_{13}^2 + d_{24}^2 - R^2} + \frac{1}{d_{14}^2 + d_{23}^2 - R^2} = 0.$$

where $d_{12} = AB$ and so on, and $R^2 = OA^2 + OB^2 + OC^2 + OD^2$.

LIST OF JOURNALS & BOOKS RECEIVED IN THE LIBRARY

during the months of October and November 1930.

1 Abhandlungen aus dem Mathematichen Seminar, Hamburg, **8**, 2.
2 Acta Mathematica, **54**, 3, 4; **55**, 1, 2, 3,
3 American Journal of Mathematics, **52**, 4.
4 American Mathematical Monthly, **37**, 7, 8.
5 Annals of Mathematics, **31**, 4.
6 Astrophysical Journal, **72**, 1.
7 Bulletin of the American Mathematical Society. **36**, 8, 9.
8 Bulletin des Sciences Mathematiques, **54**, July, Aug. and Sep.
9 Japanese Journal of Mathematics. **7**, 2..
10 Mathematical Gazette, **15**, 209 (2 copies).
11 Mathematische Annalen, **103**, 4, 5.
12 Monthly Notices of the Royal Astronomical Society, **90**, 9.
13 Nieuw Archief Voor Wiskunde, **16**, Dell.
14 Philosophical Magazine, **10**, 65.
15 Philosophical Transactions of the Royal Society of London, **229**. pp. 329—425.
16 Proceedings of the Edinburgh Mathematical Society, **2**, 1, 2.
17 Proceedings of the London Mathematical Society, **31**, 4, 5 (3 copies each).
18 Proceedings of the Physico-Mathematical Society of Japan, **12**, 7, 8.
19 Proceedings of the Royal Society of London, **129**, 809 (2 copies, **129**, 810.
20 Revista Mathematica, **5**, 4, 5, 6, 7, 8.
21 Transactions of the Royal Society of South Africa, **19**, 1.

Books.

1 Examination Papers of the University of Bombay, 1929.
2 Calendar of the University of Madras for 1930-1931, Vol. 1, Parts 1, 2.
3 Reprints from the Transactions of the Royal Society of South Africa.

(i) Note on Monge's Relation between Primary minors of a 3 by 5 Array. By Sir Thomas Muir.
(ii) Note on sums of n line minors pertaining to an n by $(n+2)$ Array. By Sir Thomas Muir.
(iii) Note on the derivatives of the eliminant of Two Binary Cubics. By Sir Thomas Muir.

Numbers in black type refer to the volumes, and those in ordinary typ the numbers of the issues.

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