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§ 39.] *Instability of the Pear-shaped Figure of Equilibrium.* 97

In this let us note that a'_i vanishes for $i \geq m$ (15.7)

By the definition of h_i (38.13), h_i vanishes for $i \geq m$. (38.17)

Further
$$b'_i = \frac{d^2 b_i}{d\rho^2} \frac{d(P^3 Q_i)}{d\rho} - \frac{db_i}{d\rho} \frac{d^2(P^3 Q_i)}{d\rho^2},$$

where
$$\frac{\cos^3 m\psi}{H} = \sum b_l \cos l\psi.$$

Hence we can show exactly as for the case of a'_i that, for l large enough, $b'_l \equiv 0$. (38.18)

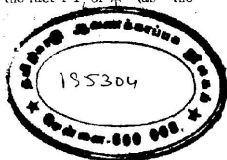
Thus we see that the above expression for τ_2 is really a finite expression and that it can be written in the form

$$\frac{F(\cos \psi)}{H^3}$$

where $F(z)$ is a rational integral expression of degree $3m$ in z .

§ 39. Before we proceed further and find the expression for A_3 , we must make a few remarks on the various implicit assumptions that we have made in calculating τ_2 . The first thing that strikes us is that we have handled infinite series as if they were finite (though in some cases they were really so). We have often rearranged them in whatever way was most convenient for our purpose. Next in importance is the fact that we have continually substituted a Fourier Series for a function and after multiplication by $\log R^2(uv)$ we have integrated it term by term. Frequently we have differentiated under the integral sign, or inverted the order of integration, or taken the limit of an integral for the integral of the limit. All these assumptions require justification.

Of these the last three are more or less obvious when we consider the forms of the functions with which they deal; for all the functions that we use are finite and continuous and have continuous differential co-efficients. Further the limits of integration are finite and constant. We shall therefore confine ourselves to the justification of the first two assumptions mentioned above. This is also easy because the series used are all absolutely and uniformly convergent as may be seen from formulae (15.4), (15.5), (15.6) and others similar to them. From these forms it is obvious that all the series used are series of positive terms except for the cosine factor and also the factor T or $\frac{l}{r}$ (as the



case may be) when that is negative. Thus the series used are of the form $\sum a_n t^n$ where $0 < t < 1$ and a_n is bounded. Hence all the series used are absolutely and uniformly convergent, and remain so even after multiplying by $\log R^2(uv)$ and integrating.* Hence our implicit assumptions are justified.

§ 40. Let us now calculate the constant A_3 on which the calculation of the angular momentum of the pear-shaped figure depends. We know from (23.4) that

$$\begin{aligned} A_3 &= \text{co-efficient of } \alpha^3 \text{ in } \frac{1}{\pi} \int W \cos m\psi \, d\psi \\ &= \frac{1}{\pi} \int W_3 \cos m\psi \, d\psi \quad (33.1) \\ &= \frac{1}{\pi} \int [2(\tau_1, \tau) + (\tau, \tau, \tau)] \cos m\psi \, d\psi. \quad (34.3) \end{aligned}$$

Hence remembering that the equation of the Elliptic Cylinder at the point of Bifurcation is $T_m = 0$ we get from (38.14)

$$\begin{aligned} A_3 &= \frac{\pi}{3P^3} \frac{P}{2m} (b'_m + h_m) \quad (40.1) \\ &\equiv \frac{\pi}{6mP^2} \left[\frac{d^2 J}{d\rho^2} \cdot \frac{d(P^3 Q)}{d\rho} - \frac{dJ}{d\rho} \cdot \frac{d^2(P^3 Q)}{d\rho^2} \right] \\ &\quad + \frac{\pi \Delta}{mP^2} \sum_{(i)} \frac{a'_i \cdot a'_{im}}{4i T_{2i}} \text{ from (38.9) and (38.13)} \end{aligned}$$

where we have put

$$J = b_m = \frac{1}{\pi} \int_0^{2\pi} \frac{\cos^4 m\psi}{H} \, d\psi.$$

Now it is obvious from the form of J whose integrand is a positive function that

$$J, -\frac{dJ}{d\rho} \text{ and } \frac{d^2 J}{d\rho^2}$$

are all positive. We have also shown in § 15 that a'_i vanishes for $i > m$ and that for $i < m$ it is always positive. In § 36 we have

* This and similar questions are more fully discussed in Part V dealing with the convergence of the series used in the paper.

mentioned how a'_{il} is also zero for sufficiently large l (e.g. $l > 3m$) though we have not actually obtained its expression. However as we want a'_{im} , let us calculate it.

We have from (35.8)

$$a'_{im} = a_{im} \frac{d(\text{PQP}_{2i})}{d\rho} - \text{PQP}_{2i} \frac{da_{im}}{d\rho}$$

where
$$\sum_l a_{il} \cos l\psi = \frac{\cos 2i\psi \cos m\psi}{H}$$

so that
$$a_{im} = \frac{1}{\pi} \int_0^{2\pi} \frac{\cos 2i\psi \cos^2 m\psi}{H} d\psi = a \quad (14.5)$$

Thus we have
$$a'_{im} \equiv a_i \frac{d(\text{PQP}_{2i})}{d\rho} - \text{PQP}_{2i} \frac{da_i}{d\rho}.$$

Now
$$\frac{d}{d\rho} [\text{PQP}_{2i}] = \frac{d}{d\rho} \left[\frac{e^{-m\alpha} (e^{m\alpha} + e^{-m\alpha}) (e^{2i\alpha} + e^{-2i\alpha})}{\Delta} \right]$$

But
$$\frac{d}{d\rho} \left[\frac{e^{2k\alpha}}{\Delta} \right] = \frac{e^{2k\alpha}}{2a^2b^3} \left[2k - \frac{a^2 + b^2}{ab} \right] = \frac{t^{-k}}{2a^2b^2} \left[2k - \frac{a^2 + b^2}{ab} \right]$$

Hence

$$\begin{aligned} \frac{d(\text{PQP}_{2i})}{d\rho} &= \frac{1}{2a^2b^3} \left[it^{-i} + (i-m)t^{m-i} - it^i - (m+i)t^{m+i} \right] \\ &\quad - \frac{a^2 + b^2}{2a^3b^3} \left[t^{-i} + t^{m-i} + t^i + t^{m+i} \right]. \end{aligned}$$

Therefore using (15.5) and (15.6) we get

$$\begin{aligned} a_{im} &= \frac{1}{ab} \left[t^i + \frac{1}{2} t^{m+1} + \frac{1}{2} t^{l-m-1} \right] \times \\ &\quad \left\{ \frac{1}{a^2b^2} \left\{ i(t^{-i} - t^i) - (m-i)t^{m-i} - (m+i)t^{m+i} \right\} \right. \\ &\quad \left. - \frac{a^2 + b^2}{2a^3b^3} \left\{ (t^{-i} + t^i)(1 + t^m) \right\} \right\} \\ &\quad + \frac{1}{ab} \cdot \frac{1}{2a^3b^3} \left[(t^{-i} + t^i)(1 + t^m) \right] \\ &\quad \left\{ t^i(a^2 + 2iab + b^2) + \frac{1}{2} t^{m+i}(a^2 + 2m + iab + b^2) \right. \\ &\quad \left. + \frac{1}{2} t^{l-m-1}(a^2 + 2|m-i|ab + b^2) \right\} \end{aligned}$$

In this the co-efficient of $\frac{1}{2a^4b^2} (t^{-i} + t^i) (1 + t^m)$ is

$$\begin{aligned}
 & - (a^2 + b^2) \left[t^i + \frac{1}{2} t^{m+i} + \frac{1}{2} t^{|m-i|} \right] \\
 & \left\{ t^i (a^2 + 2iab + b^2) + \frac{1}{2} t^{m+1} (a^2 + 2m + i ab + b^2) \right. \\
 & \quad \left. + \frac{1}{2} t^{|m-i|} (a^2 + 2|m-i| ab + b^2) \right\} \\
 & = 2ab \left[it^i + \frac{1}{2}(m+i)t^{m+i} + \frac{1}{2}|m-i|t^{|m-i|} \right]
 \end{aligned}$$

Therefore we get the following expression for $2a^3b^3a'_{im}$ for $i < m$:—

$$\begin{aligned}
 & [2t^i + t^{m+i} + t^{m-i}] [i(t^{-i} - t^i) - (m-i)t^{m-i} - (m+i)t^{m+i}] \\
 & + (t^{-i} + t^i)(1 + t^m) [2it^i + (m+i)t^{m+i} + (m-i)t^{m-i}].
 \end{aligned}$$

The two second factors in [] can be written

$$\begin{aligned}
 & i(t^{-i} - t^i)(1 + t^m) - mt^m(t^i + t^{-i}); \\
 & 2it^i - it^m(t^{-i} - t^i) + mt^m(t^{-i} + t^i).
 \end{aligned}$$

Hence the co-efficient of

$$\begin{aligned}
 & mt^m(t^{-i} + t^i) \text{ in } 2a^3b^3a'_{im} \text{ is} \\
 & (t^{-i} + t^i + t^{m-i} + t^{m+i}) - (2t^i + t^{m+i} + t^{m-i}) = t^{-i} - t^i.
 \end{aligned}$$

The co-efficient of

$$\begin{aligned}
 & i(t^{-i} - t^i) \text{ in } 2a^3b^3a'_{im} \text{ is} \\
 & (1 + t^m) [2t^i + t^{m+i} + t^{m-i}] - t^m [t^{-i} + t^i + t^{m-i} + t^{m+i}] \\
 & = 2t^i + t^{m+i} + t^{m-i} - t^m(t^{-i} - t^i) = 2t^i(1 + t^m).
 \end{aligned}$$

The rest of $2a^3b^3a'_{im}$ is equal to $2it^i(t^{-i} + t^i)(1 + t^m)$.

Hence

$$\begin{aligned}
 2a^3b^3a'_{im} &= 2it^i(1 + t^m)(t^{-i} + t^i) + 2t^i(1 + t^m) \times i(t^{-i} - t^i) \\
 & \quad + mt^m(t^{-2i} - t^{2i}) \\
 &= 4i(1 + t^m) + mt^m(t^{-2i} - t^{2i}).
 \end{aligned} \tag{40.2}$$

Thus we see that a'_{im} is positive for all i .

Again

$$\begin{aligned}
 \frac{d(P^3Q)}{d\rho} &= \frac{d}{d\rho} \left[\frac{e^{-m\alpha} (e^{3m\alpha} + 3e^{m\alpha} + 3e^{-m\alpha} + e^{-8m\alpha})}{\Delta} \right] \\
 \text{and } \frac{d}{d\rho} \left[\frac{e^{\frac{2k\alpha}{\Delta}}}{\Delta} \right] &= \frac{e^{\frac{2k\alpha}{\Delta}}}{2\Delta^2} \left[2k - \frac{2\rho + q}{\Delta} \right] = \frac{t^{-k}}{2a^2b^2} \left[2k - \frac{a^2 + b^2}{ab} \right]
 \end{aligned}$$

$$\therefore \frac{d(P^3 Q)}{d\rho} = \frac{1}{a^2 b^2} \left[m t^{-m} - 3 m t^m - 2 m t^{2m} \right] - \frac{a^2 + b^2}{2 a^3 b^3} \left[t^{-m} + 3 + 3 t^m + t^{2m} \right]. \quad (40.3)$$

Therefore in our case where $m = 3$, $t = \frac{1}{2}$ if we take $ab = 1$ so that $a^2 = 3$, $b^2 = \frac{1}{3}$, we get

$$\frac{d(P^3 Q)}{d\rho} = \left[3.8 - 9 \cdot \frac{1}{8} - 6 \cdot \frac{1}{64} \right] - \frac{10}{6} \left[8 + 3 + \frac{1}{64} \right] = \frac{243}{64}. \quad (40.4)$$

Again

$$\frac{d^2(P^3 Q)}{d\rho^2} = \frac{d^2}{d\rho^2} \left[\frac{e^{2m\alpha} + 3 + 3e^{-2m\alpha} + e^{-4m\alpha}}{\Delta} \right]$$

$$\text{and } \frac{d^2}{d\rho^2} \left[\frac{e^{2k\alpha}}{\Delta} \right] = \frac{t^{-k}}{4a^5 b^5} \left[\frac{3(a^4 + b^4) - 6kab(a^2 + b^2)}{+ (4k^2 + 2)a^3 b^2} \right]$$

$$\therefore 4a^5 b^5 \frac{d^2(P^3 Q)}{d\rho^2} = 3(a^4 + b^4) t^{-m} (1 + t^m)^3 + 6m t^{-m} ab(a^2 + b^2) [-1 + 3t^{2m} + 2t^{3m}] + a^2 b^2 t^{-m} [(4m^2 + 2) + 6t^m + 3(4m^2 + 2)t^{2m} + (16m^2 + 2)t^{3m}].$$

Therefore using (40.3) we get for our case (*i.e.* $m = 3$)

$$\begin{aligned} 4 \frac{d^2(P^3 Q)}{d\rho^2} &= \frac{82}{3} \times 8 \times \left(\frac{9}{8}\right)^3 + 60 \times 8 \left[-1 + \frac{3}{64} + \frac{1}{256}\right] \\ &\quad + 8 \left[38 + \frac{3}{4} + \frac{3 \times 38}{64} + \frac{146}{512}\right] \\ &= \frac{41 \times 243}{32} - 480 + \frac{15 \times 13}{8} + 310 + 8 \frac{24 \times 19 + 73}{256} = \frac{729}{4} \end{aligned}$$

or

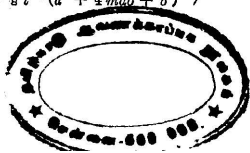
$$\frac{d^2(P^3 Q)}{d\rho^2} = \frac{729}{16}. \quad (40.5)$$

Again

$$J = \frac{1}{\pi} \int_0^{2\pi} \frac{\cos^4 m\psi}{H} d\psi = \frac{2}{ab} \left[\frac{8}{3} + \frac{1}{2} t^m + \frac{1}{8} t^{3m} \right]$$

$$\text{and } -\frac{dJ}{d\rho} = \frac{1}{\pi} \int_0^{2\pi} \frac{\cos^4 m\psi}{H^2} d\psi$$

$$= \frac{1}{a^3 b^3} \left(\frac{8}{3} (a^2 + b^2) + \frac{1}{2} t^m (a^2 + 2mab + b^2) + \frac{1}{8} t^{3m} (a^2 + 4mab + b^2) \right)$$



$$\frac{d^2 J}{d\rho^2} = \frac{1}{2a^5 b^5} \left\{ \begin{aligned} & \frac{3}{8} (3a^4 + 2a^2 b^2 + 3b^4) \\ & + \frac{1}{2} t^m \{ 3(a^4 + b^4) + 6mab(a^2 + b^2) + (4m^2 + 2)a^2 b^2 \} \\ & + \frac{1}{8} t^{2m} \{ 3(a^4 + b^4) + 12mab(a^2 + b^2) + (16m^2 + 2)a^2 b^2 \} \end{aligned} \right\}$$

Hence putting in the numerical values given by (40.3), we get

$$\begin{aligned} -\frac{dJ}{d\rho} &= \frac{3}{8} \times \frac{10}{3} + \frac{1}{16} \left(\frac{10}{3} + 6 \right) + \frac{1}{8} \cdot \frac{1}{64} \left(\frac{10}{3} + 12 \right) = \frac{5}{4} + \frac{7}{12} + \frac{23}{768} \\ &= \frac{1431}{768} = \frac{477}{256}, \end{aligned} \quad (40.6)$$

$$\begin{aligned} 2 \frac{d^2 J}{d\rho^2} &= \frac{3}{8} \left(27 + 2 + \frac{1}{3} \right) + \frac{1}{16} \left\{ 3 \left(9 + \frac{1}{9} \right) + 18 \left(\frac{10}{3} \right) + 38 \right\} \\ &\quad + \frac{1}{8} \cdot \frac{1}{64} \left\{ 3 \cdot \frac{82}{9} + 36 \left(\frac{10}{3} \right) + 146 \right\} \\ &= 11 + \frac{1}{16} \left(125 + \frac{1}{3} \right) + \frac{1}{512} \left(293 + \frac{1}{3} \right) \\ &= 19 \frac{13}{32} = \frac{621}{32}. \end{aligned}$$

or
$$\frac{d^2 J}{d\rho^2} = \frac{621}{64}.$$

We have now to calculate

$$\sum \frac{a'_i a'_{im}}{4im T_{2i}}.$$

But $a^3 b^3 a'_i = (1 + t^m) [m(1 + t^{2i-m}) - i(1 + t^m)] \quad (15.9)$

$$2a^3 b^3 a'_{im} = 4i(1 + t^m) + mt^m(t^{-2i} - t^{2i}). \quad (40.2)$$

$$T_{2i} = \frac{1}{2}(1 - t^2) - \frac{1}{2i}(1 + t^{2i}) \equiv \frac{1}{m}(1 + t^m) - \frac{1}{2i}(1 + t^{2i})$$

or $2im T_{2i} = 2i(1 + t^m) - m(1 + t^{2i}).$

In our case, $m = 3$ and therefore $i = 1$ or 2 . Hence using (40.3), we have

$$a'_1 = \frac{9}{8} \left(3 \times 3 - \frac{9}{8} \right) = \frac{9}{8} \times \frac{63}{8} = \frac{567}{64}$$

$$2a'_1 m = 4 \times \frac{9}{8} + \frac{3}{8} \left(4 - \frac{1}{4} \right) = \frac{9}{2} + \frac{45}{32} = \frac{189}{32}.$$

$$6T_2 = 2 \times \frac{9}{8} - 3 \left(1 + \frac{1}{4} \right) = -\frac{3}{2}.$$

$$a'_2 = \frac{9}{8} \left[3 \times \frac{3}{2} - 2 \times \frac{9}{8} \right] = \frac{9}{8} \left(\frac{9}{2} - \frac{9}{4} \right) = \frac{81}{32}.$$

$$2a'_{2m} = 8 \times \frac{9}{8} + \frac{3}{8} \left(16 - \frac{1}{16} \right) = 15 - \frac{3}{128} = \frac{1917}{128}.$$

$$12T_4 = 4 \times \frac{9}{8} - 3 \left(1 + \frac{1}{16} \right) = \frac{9}{2} - \frac{51}{16} = \frac{21}{16}.$$

Therefore we have

$$\begin{aligned} \left(\frac{a'_1 a'_{1m}}{6T_2} + \frac{a'_2 a'_{2m}}{12T_4} \right) &= -\frac{2}{3} \times \frac{189}{64} \times \frac{567}{64} + \frac{16}{21} \times \frac{1917}{256} \times \frac{81}{32} \\ &= \frac{81 \times 27}{512} \left\{ -\frac{7 \times 7}{3 \times 4} + \frac{71}{21} \right\} \\ &= \frac{81 \times 27}{512} \times \frac{852 - 1029}{21 \times 12} = -\frac{177 \times 243}{512 \times 28}. \end{aligned} \quad (40.8)$$

Now from (40.1) we have

$$\frac{P^2}{\pi} A_3 = \frac{1}{6m} \left[\frac{d^2 J}{d\rho^2} \cdot \frac{d(P^3 Q)}{d\rho} - \frac{dJ}{d\rho} \cdot \frac{d^2(P^3 Q)}{d\rho^2} \right] + \frac{\Delta}{2} \sum \frac{a'_i a'_{im}}{2im T_{2i}}.$$

Hence using (40.3, 40.4, 40.5, 40.6, 40.7, 40.8), we get

$$\begin{aligned} \frac{P^2}{\pi} A_3 &= \frac{1}{18} \left[\frac{621}{64} \times \frac{243}{64} + \frac{477}{256} \times \frac{729}{16} \right] - \frac{177 \times 243}{512 \times 28 \times 2} \\ &= \frac{243}{512 \times 4} \left[\frac{1}{18} \cdot \frac{621}{2} + \frac{1431}{18 \times 2} - \frac{177}{14} \right] \\ &= \frac{243}{512 \times 4} \times \frac{621}{14}, \end{aligned}$$

$$\text{or } A_3 = \pi \times \frac{243}{512 \times 4} \times \frac{621}{14} \times \frac{8}{81} = \pi \frac{621 \times 3}{512 \times 7}.$$

Now when m is odd we have by (27.3)

$$\eta = -\frac{A_3}{B} a^2 + \dots;$$

$$\text{also by § 27, } B = -\frac{1 + t^{m-2}}{ab} = -\frac{3}{2} \quad (\text{for } m=3)$$

$$\frac{\eta}{\pi} = \frac{207 \times 3}{7 \times 512} \alpha^2 + \dots$$

Since

$$\frac{\omega^2}{2\pi} = \frac{\Omega}{\pi} = \frac{1}{\pi} (\Omega^0 + \eta)$$

$$\frac{\omega_0^2}{2\pi} = \frac{2ab}{(a+b)^2} = \frac{3}{8},$$

we get

$$\begin{aligned} \frac{\omega^2}{2\pi} &= \frac{\omega_0^2}{2\pi} \left(1 + \frac{207 \times 3}{7 \times 256} \times \frac{8}{3} \alpha^2 + \dots \right) \\ &= \frac{\omega_0^2}{2\pi} \left(1 + \frac{207}{224} \alpha^2 \right) \end{aligned}$$

up to the second order.

Thus in the case we are most interested in, *viz.*, $m = 3$ the constant, A_3 is positive and the increment of the angular velocity is also positive, *i.e.*, we must increase the angular velocity to pass on to the pear-shaped figures. Consequently in this case the pear-shaped figure F has an angular momentum less than that of the Elliptic Cylinder from which it is derived. *Thus we see that at least in the initial stages the series of Pear-shaped figures of Equilibrium is an unstable series.*

PART V.

The Discussion of the Convergence of the series used.

§ 41. So far we have assumed that the series we took for ζ , *viz.*

$$(1) \quad \zeta = \zeta_1 \alpha + \zeta_2 \alpha^2 + \zeta_3 \alpha^3 + \dots$$

or
$$(2) \quad \zeta = \sum_{r+s>0} \zeta_{rs} \alpha^r \eta^s$$

as the case may be, is absolutely and uniformly convergent so that there exist numbers l_i or l_{ij} such that

$$|\zeta_i| < l_i, \text{ or } |\zeta_{ij}| < l_{ij};$$

and that the series

$$\sum l_r \alpha^r, \text{ or } \sum l_{rs} \alpha^r \eta^s$$

is convergent for sufficiently small values of α or α, η . Owing to this assumption we could substitute this assumed series for ζ in the absolutely convergent series $U_2 + U_3 + U_4 + \dots$ and rearrange it in ascending powers of α or α, η . Further we have very often differentiated or integrated the series occurring in the calculations assuming that the process was valid. We have now to justify these assumptions.

Consider first the case when the new figure F is in equilibrium owing to an angular velocity which is the same as that of the Elliptic Cylinder from which it is derived. Then the figure of comparison is this very cylinder and therefore η does not occur in our equations. We therefore solve for ξ by means of the series

$$\xi = \xi_1 \alpha + \xi_2 \alpha^2 + \xi_3 \alpha^3 + \dots$$

in which the ξ_r are found successively. The fundamental equation from which ξ is to be got by successive approximations is

$$\frac{2ab}{(a+b)^2} H \xi' + \frac{1}{2\pi} \int \log R^2(\rho\rho') H' \xi' d\psi' = \frac{\Delta}{\pi} W + \text{constant},$$

where $W = U_2 + U_3 + U_4 + \dots$

In this equation, we substitute

$$\xi = \xi_1 \alpha + \xi_2 \alpha^2 + \xi_3 \alpha^3 + \dots$$

$$W = W_1 \alpha + W_2 \alpha^2 + W_3 \alpha^3 + \dots$$

and equating co-efficients of equal powers of α , we get the equations

$$\frac{2ab}{(a+b)^2} H \xi_1 + \frac{1}{2\pi} \int \log R^2(\rho\rho') H' \xi_1' d\psi' = \frac{\Delta}{\pi} W_1 + \text{constant}.$$

Further we have shown by actual calculation that ξ_1 , ξ_2 and ξ_3 are of the form

$$\frac{F(\cos \psi)}{[H(\rho\psi)]^n} \quad (41.1)$$

where $F(z)$ is an integral expression in z of degree nm . If we write

$$F(z) = \sum_{r=0}^{n_m} A_{n,r} z^r$$

we can verify that for the above three functions the co-efficients $A_{n,r}$ satisfy the equation

$$|A_{n,r}| < L H_{\lambda,r} x^r,$$

where L, x are fixed numbers less than unity, $\lambda = c(3nm - 7)$, c being some positive integer and

$$H_{\lambda,r} = \frac{\lambda(\lambda+1)(\lambda+2)\dots(\lambda+r-1)}{r!} = \frac{(\lambda+r-1)!}{r!(\lambda-1)!}. \quad (41.2)$$

Thus we see that the functions ξ_1 , ξ_2 , ξ_3 satisfy the inequality

$$|\xi_n| < \frac{1}{[H(\rho\psi)]^n} \cdot \frac{L}{(1-x)^\lambda} \leq \frac{L\rho^{-n}}{(1-x)^\lambda} \quad (41.3)$$

since

$$H(\rho\psi) = \rho + q \sin^2 \psi \geq \rho.$$

In order to justify our assumptions in the case when ξ is a function of α only, viz., that $|\xi_n| < l_n$ and that $\sum l_n \alpha^n$ is convergent for values of α , small enough, it is easy to see that we need only prove that the inequality (41.3) is also true for all values of $n > 3$. But before proceeding to do so let us first prove some theorems which we shall have to use very frequently.

§ 42. THEOREM I. *The co-efficients in the Fourier Series for the function*

$$\frac{\cos^p \psi}{H(\rho\psi)} \equiv \frac{\cos^p \psi}{\rho + q \sin^2 \psi} = \frac{\cos^p \psi}{a^2 \sin^2 \psi + b^2 \cos^2 \psi},$$

where p is a positive integer, are all positive.

To prove this we need only remark that by a well-known formula

$$\begin{aligned} 2^{p-1} \cos^p \psi &= \cos p\psi + p \cos(p-2)\psi + \dots \\ &= \sum_{r=0}^{2r \leq p} C_r \cos(p-2r)\psi \text{ say,} \end{aligned} \quad (42.1)$$

where the co-efficients C_r are all positive.

Also if
$$\frac{\cos^p \psi}{H(\rho\psi)} = \sum_l a_l \cos l\psi,$$

then
$$a_l = \frac{1}{\pi} \int_0^{2\pi} \frac{\cos^p \psi \cos l\psi}{H(\rho\psi)} d\psi. \quad (42.2)$$

Hence using the formula (15.4), viz.

$$\begin{aligned} \int_0^{2\pi} \frac{\cos n\psi}{H(\rho\psi)} d\psi &= 0 \text{ if } n \text{ is odd} \\ &= \frac{2\pi}{ab} \left(\frac{a-b}{a+b} \right)^{\frac{n}{2}} = \frac{2\pi}{\Delta} t^{\frac{n}{2}} \text{ if } n \text{ is even,} \end{aligned}$$

and also (42.1), we see that $a_l \equiv 0$ if l and p are not of the same parity and that when they are of the same parity

$$a_l = \frac{2\pi}{\Delta} \sum_{r=0}^{2r \leq p} C_r \left[t^{\frac{p+l}{2}-r} + t^{\left| \frac{p-l}{2}-r \right|} \right]. \quad (42.3)$$

Thus as C_r is positive from (42.1) a_l is always positive or zero,

THEOREM II. When k is positive the expressions

$$\frac{d^n}{d\rho^n} \left[t^k \right] \text{ and } \frac{d^n}{d\rho^n} \left[\frac{t^k}{\Delta} \right]$$

consist of terms which are all of the same sign, viz. $(-)^n$.

To prove this we note that

$$a = \sqrt{\rho + q}; \quad b = \sqrt{\rho}; \quad t = \frac{a-b}{a+b}.$$

Hence

$$\frac{dt}{d\rho} \equiv \left(\frac{1}{2a} \frac{\partial}{\partial a} + \frac{1}{2b} \frac{\partial}{\partial b} \right) \left(\frac{a-b}{a+b} \right) = -\frac{t}{ab} = -\frac{t}{\Delta}. \quad (42.4)$$

Hence

$$\frac{d^n}{d\rho^n} \left[t^k \right] = -k \frac{d^{n-1}}{d\rho^{n-1}} \left(\frac{t^k}{\Delta} \right).$$

Therefore we need only consider the second expression.

Now it is easy to prove by induction that

$$\frac{d^p}{d\rho^p} \left[\frac{t^k}{\Delta} \right] = \frac{(-)^p}{2^p \cdot \Delta^{2p+1}} \sum_{r=0}^p A_r^{(p)} a^r b^r (a^{2p-2r} + b^{2p-2r}) \quad (42.5)$$

where the $A_r^{(p)}$ are integral expressions in k and p , which are positive when k is positive.

In fact assuming (42.5) we get on differentiation the following relations between $A_r^{(p+1)}$ and $A_r^{(p)}$ by using (42.4):—

$$\left. \begin{aligned} A_0^{(p+1)} &= (2p+1) A_0^{(p)}; \\ A_1^{(p+1)} &= k A_0^{(p)} + 2p A_1^{(p)} \\ &\dots \dots \dots \\ A_r^{(p+1)} &= (r-1) A_{r-2}^{(p)} + k A_{r-1}^{(p)} + (2p+1-r) A_r^{(p)} \\ &\qquad\qquad\qquad \text{for } r = 2, 3 \dots p. \\ &\dots \dots \dots \\ A_{p+1}^{(p+1)} &= k A_p^{(p)} + 2p A_{p-1}^{(p)}. \end{aligned} \right\} (42.6)$$

These relations show that when k is positive all the A_r 's are positive. Hence the theorem.

$$\text{THEOREM III. If } P_r = t^{\frac{r}{2}} + t^{-\frac{r}{2}}, \quad Q_r = \frac{t^{\frac{r}{2}}}{\Delta}, \quad (5)$$

then

$$\overline{\text{Lim}} \frac{d^i P_r}{d\rho^i} < r(i-1)! \rho^{-i}; \overline{\text{Lim}} \frac{d^j Q_r}{d\rho^j} < j! \rho^{-(j+1)}.$$

We have shown in the last theorem that

$$\begin{aligned} \frac{d^p}{d\rho^p} \left[\frac{t^k}{\Delta} \right] &= \frac{(-)^p \cdot t^k}{2^p \cdot \Delta^{2p+1}} \sum_{r=0}^p A_r^{(p)} a^r b^r (a^{2p-2r} + b^{2p-2r}) \\ &= \phi(k, a, b) \text{ say.} \end{aligned}$$

The formulæ (42.6) show that when k is positive the co-efficients A_r are all positive but when k is negative some of them may be negative, but in any case $\phi(-k, a, b)$ can be got from $\phi(k, a, b)$ by substituting $-k$ for k . Thus

$$\frac{d^p}{d\rho^p} \left[\frac{t^{-k}}{\Delta} \right] = \phi(-k, a, b).$$

Also if $A_r^{(p)}(+k)$ represents the coefficient A_r when k is positive it is clear that

$$|A_r^{(p)}| \leq A_r^{(p)}(+k)$$

the left-hand side representing the co-efficient when k is not fixed to be positive.

Hence it is easy to see that

$$|\phi(-k, a, b)| t^{+k} \text{ and } |\phi(+k, a, b)| t^{-k} \text{ are both}$$

$$\leq \frac{1}{2^p \cdot \Delta^{2p+1}} \sum_{r=0}^p A_r^{(p)}(+k) a^r b^r (a^{2p-2r} + b^{2p-2r}).$$

In other words, we need only find the upper limit of $\phi(k, a, b) t^{-k}$ where k is positive.

Suppose now k is positive but not integral. Then if $(k+f)$ is equal to an integer where f is positive, then clearly

$$\phi(k, a, b) t^{-k} < \phi(k+f, a, b) t^{-(k+f)}.$$

Hence we can calculate this upper limit when k is a positive integer. But in this case we have by (15.4)

$$\frac{1}{2\pi} \int_0^{2\pi} \frac{\cos 2k\psi}{H(\rho\psi)} d\psi \equiv \frac{1}{2\pi} \int_0^{2\pi} \frac{\cos 2k\psi}{\rho + q \sin^2 \psi} d\psi = \frac{t^k}{\Delta}.$$

$$\begin{aligned} \text{Hence } \overline{\text{Lim}} \frac{d^p}{d\rho^p} \left[\frac{t^k}{\Delta} \right] &= \overline{\text{Lim}} \frac{d^p}{d\rho^p} \left[\frac{1}{2\pi} \int_0^{2\pi} \frac{\cos 2k\psi \, d\psi}{\rho + q \sin^2 \psi} \right] \\ &= \overline{\text{Lim}} (-) \frac{p!}{2\pi} \int_0^{2\pi} \frac{\cos 2k\psi \, d\psi}{(\rho + q \sin^2 \psi)^{p+1}} \\ &< p! \, \rho^{-(p+1)}. \end{aligned}$$

In other words, from what we have proved above it follows that

$$\overline{\text{Lim}} \frac{d^p}{d\rho^p} \left[\frac{t^k}{\Delta} \right] < \rho^{-(p+1)} p!$$

whether k be positive or negative, integral or fractional.

$$\begin{aligned} \text{Now } \frac{d^i P_r}{d\rho^i} &= \frac{d^{i-1}}{d\rho^{i-1}} \frac{d}{d\rho} \left[t^{\frac{r}{2}} + t^{-\frac{r}{2}} \right] \\ &= \frac{d^{i-1}}{d\rho^{i-1}} \left[-\frac{r}{2} \left(\frac{t^{\frac{r}{2}}}{\Delta} - \frac{t^{-\frac{r}{2}}}{\Delta} \right) \right] \text{ using (42.4)} \end{aligned}$$

$$\begin{aligned} \therefore \overline{\text{Lim}} \frac{d^i P_r}{d\rho^i} &\leq \frac{r}{2} \cdot \left\{ \overline{\text{Lim}} \frac{d^{i-1}}{d\rho^{i-1}} \left[\frac{t^{\frac{r}{2}}}{\Delta} - \frac{t^{-\frac{r}{2}}}{\Delta} \right] \right\} \\ &< \frac{r}{2} \cdot 2 \cdot (i-1)! \, \rho^{-i} \end{aligned}$$

$$\text{or } \overline{\text{Lim}} \frac{d^i P_r}{d\rho^i} < r (i-1)! \, \rho^{-i}. \quad (42.7)$$

$$\begin{aligned} \text{Similarly } \overline{\text{Lim}} \frac{d^j Q_r}{d\rho^j} &\equiv \overline{\text{Lim}} \frac{d^j}{d\rho^j} \left[\frac{t^{\frac{r}{2}}}{\Delta} \right] \\ &< j! \, \rho^{-(j+1)}. \end{aligned}$$

THEOREM IV. *The co-efficients in the Fourier Series for the function $\frac{\cos^p \psi}{[H(\rho\psi)]^\mu}$ where p and μ are positive integers, are all of positive sign.*

We remark that

$$\frac{\cos^p \psi}{[H(\rho\psi)]^\mu} \equiv \frac{\cos^p \psi}{(\rho + q \sin^2 \psi)^\mu} = \frac{(-)^{\mu-1}}{(\mu-1)!} \frac{\partial^{\mu-1}}{\partial \rho^{\mu-1}} \cdot \frac{\cos^p \psi}{\rho + q \sin^2 \psi}.$$

But by Theorem I (42.3) if

$$\frac{\cos^p \psi}{H(\rho\psi)} = \sum_i a_i \cos l\psi$$

a_i is either zero or equal to the sum of a finite number of terms of the form (Positive constant) $\times \frac{t^k}{\Delta}$ where k is positive. Also by Theorem II, all the terms of the expression $\frac{d^n}{d\rho^n} \left[\frac{t^k}{\Delta} \right]$ where k is positive have the same sign namely $(-)^n$. Hence the proposition is proved.

THEOREM V. *The product of two convergent series whose coefficients satisfy conditions of the form (41.2) is such that its coefficients also satisfy similar conditions.*

$$\text{Let } S_1 = \sum_{r=0}^{\infty} A_r x^r \text{ and } S_2 = \sum_{s=0}^{\infty} B_s x^s$$

be such that $|x|$ being less than 1

$$|A_r| < L H_{m,r} z^r \text{ and } |B_s| < L H_{n,s} z^s,$$

where z, L are fixed numbers less than unity

$$\text{and } H_{\lambda,p} = \frac{\lambda(\lambda+1)(\lambda+2)\dots(\lambda+p-1)}{p!} = \frac{(\lambda+p-1)!}{p!(\lambda-1)!}.$$

$$\text{Then } S_1 S_2 = \sum_{k=0}^{\infty} C_k x^k, \text{ where } C_k = \sum_{p=0}^k A_p B_{k-p}.$$

$$\text{Hence } |C_k| \leq \sum_{p=0}^k |A_p| |B_{k-p}| < L^2 z^k \sum_{p=0}^k H_{m,p} \cdot H_{n,k-p}$$

$$\therefore |C_k| < L^2 z^k H_{m+n,k} < L z^k H_{m+n,k}$$

since $L < 1$ by hypothesis. Let us note that if either of these series is finite, the theorem is all the more true.

THEOREM VI. *If in the last theorem one of the series, say S_2 , is finite, we can get a better inequality for the product $S_1 S_2$.*

For if $|x| < 1$ and

$$S_1 = \sum_{r=0}^{\infty} A_r x^r; S_2 = \sum_{s=0}^{nm} B_s x^s$$

$$\text{where } |A_r| < L H_{M,r} z^r; |B_s| < L H_{c(3nm-7),s} z^s,$$

then since by making z sufficiently near to, but less than, unity, we can certainly make

$$\sum_{s=0}^{nm} z^s \cdot H_{c(3nm-7), s} < \frac{1}{2^n} \cdot \frac{1}{(1-z)^{c(3nm-7)}},$$

we see that the product $(S_1 S_2)$ satisfies the inequality

$$S_1 S_2 < \frac{1}{2^n} \frac{L}{(1-z)^{M+c(3nm-7)}}. \quad (42.9)$$

We have of course the property that the co-efficient C_k of x^k in $S_1 S_2$ satisfies the inequality

$$|C_k| < L z^k H_{M+c(3nm-7), k}$$

or $|C_k| < \text{the co-efficient of } x^k \text{ in}$

$$\frac{L}{(1-xz)^{M+c(3nm-7)}},$$

or the corresponding term of

$$\frac{L}{(1-z)^{M+c(3nm-7)}}; \quad (42.10)$$

THEOREM VII. *The product of any two Fourier Cosine Series whose co-efficients satisfy conditions of the form (41.2) and are therefore absolutely and uniformly convergent, when expressed as a Fourier Series is also absolutely and uniformly convergent. Further any such cosine series when multiplied by*

$$-\frac{\log R^2(uv)}{\Delta(v)}, \text{ or by } -\log R^2(uv) \cdot \frac{H(v\Psi')}{\Delta(v)}$$

and integrated term by term, gives rise to another Fourier Cosine Series which is also such. Also this last series got after integration can be differentiated any number of times with respect to u and v and will still remain absolutely and uniformly convergent.

$$\text{Let } S_1 = \sum_{r=0}^{\infty} a_r \cos r\Psi; \quad S_2 = \sum_{s=0}^{\infty} b_s \cos s\Psi$$

be such that

$$|a_r| < L H_{m,r} z^r; \quad |b_s| < L H_{n,s} z^s. \quad (42.11)$$

Then if we put $a_{-r} = a_r$; $b_{-s} = b_s$, we get

$$2S_1 = \sum_{r=-\infty}^{+\infty} a_r e^{ir\psi}; \quad 2S_2 = \sum_{s=-\infty}^{+\infty} b_s e^{is\psi}$$

except that the terms corresponding to $r = 0$, $s = 0$ have a factor 2 in addition.

$$\therefore 4S_1S_2 = \sum_{r=-\infty}^{+\infty} \sum_{s=-\infty}^{+\infty} a_r b_s e^{i(r+s)\psi}.$$

Since $a_{-r} b_{-s} = a_r b_s$, we see at once that the product can be written down as a Fourier Cosine Series. Let then

$$4S_1S_2 = 2 \sum_{k=0}^{\infty} c_k \cos k\psi.$$

Then it is easy to see that

$$c_k = \sum_{p=0}^k a_p b_{k-p} + \sum_{p=0}^{\infty} (a_p b_{k+p} + b_p a_{k+p}).$$

Now the series S_1 and S_2 are by our hypotheses (42.11) absolutely convergent and uniformly convergent with respect to ψ ($0 \leq \psi \leq 2\pi$).

Hence the series $\sum_{r=0}^{\infty} a_r$ and $\sum_{s=0}^{\infty} b_s$

are absolutely convergent. Therefore the product

$$\sum_{r=-\infty}^{+\infty} a_r \times \sum_{s=-\infty}^{+\infty} b_s \equiv \sum_{r=-\infty}^{+\infty} \sum_{s=-\infty}^{+\infty} a_r b_s$$

is absolutely convergent. We may therefore rearrange it in any way we like without altering its sum. If we now group the terms $a_r b_s$ for which either $(r + s)$ or $|r - s|$ is equal to k we get c_k . Thus the series of which the general term is c_k is absolutely convergent. That is to say the Fourier Series for the product S_1S_2 is absolutely and uniformly

convergent. If one of the series is finite the theorem is all the more true. Hence noting that the product of $H(v\psi') \equiv v + q \sin^2 \psi'$ and the series S'_1 is also an absolutely and uniformly convergent Fourier

Cosine Series, we need only consider the product of S'_1 by $-\frac{1}{\Delta(v)} \log R^2(uv)$ integrated term by term.

But

$$\begin{aligned} & \int -\log R^2(uv) \cdot \frac{1}{\Delta(v)} S'_1 d\psi' \\ &= \frac{1}{\Delta(v)} \int -\log R^2(uv) \left[\sum_{r=0}^{\infty} a_r \cos r\psi' \right] d\psi' \\ &= \text{constant} + 2\pi \sum_{r=1}^{\infty} \frac{a_r}{r} P_r(u) Q_r(v) \cos r\psi \quad (42.12) \end{aligned}$$

from the formula (2.6) using as before the notation (14.2). But since $u < v \leq \rho$, we have

$$P_r(u) Q_r(v) = \left(e^{r\alpha_u} + e^{-r\alpha_u} \right) \frac{e^{-r\alpha_v}}{\Delta(v)} < \frac{2}{\Delta(v)}.$$

Hence we see that the series (42.12) is also absolutely and uniformly convergent.

Further

$$\begin{aligned} & \frac{\partial^{i+j}}{\partial u^i \partial v^j} \int \log R^2(uv) \cdot \frac{1}{\Delta(v)} S'_1 d\psi' \\ &= \text{constant} + 2\pi \sum_{r=1}^{\infty} \frac{a_r}{r} \cdot \frac{d^i P_r(u)}{du^i} \cdot \frac{d^j Q_r(v)}{dv^j} \cos r\psi. \quad (42.13) \end{aligned}$$

But we have shown in Theorem III that

$$\overline{\text{Lim}} \quad \frac{d^i P_r(u)}{du^i} < r(i-1)! u^{-i}$$

and

$$\overline{\text{Lim}} \quad \frac{d^j Q_r(v)}{dv^j} < j! v^{-(j+1)}.$$

Hence we see that the series (42.13) also is absolutely and uniformly convergent though its sum, which depends on the order of the differentiations, may be large.

§ 43. Let us now proceed to prove the inequality (41.3) for $n > 3$. As we shall frequently have to compare two Fourier Series the co-efficients of one of which are in absolute value less than those of another, it will be convenient, for the sake of brevity, to express this by some symbol. Let us denote by

$$F(x) << G(x) \quad (43.1)$$

the relation that when the functions $F(x)$, $G(x)$ are expressed as Fourier Series the co-efficient of any term in the series for $F(x)$ is in absolute value less than or equal to the co-efficient of the corresponding term in the Fourier Series for $G(x)$.

Then the functions ξ_1, ξ_2, ξ_3 which have been found, satisfy the relation

$$\xi_r \leq \frac{\sum_{p=0}^{rm} |A_{r,p}| \cos^p \psi}{[H(\rho\psi)]^r}, \quad [r = 1, 2, 3] \quad (43.2)$$

where

$$|A_{r,p}| < L z^p \cdot H_{c(3rm-7),p}$$

L, z are fixed numbers less than unity, and

$$H_{\lambda,p} = \frac{\lambda(\lambda+1)\dots(\lambda+p-1)}{p!} = \frac{(\lambda+p-1)!}{p!(\lambda-1)!}. \quad (43.3)$$

To prove the inequality (41.3) for $n > 3$ and hence the convergence of the series used, consider the sets of functions

$$W_r, H\xi_r, \xi_r; \quad [r = 1, 2, 3 \dots]$$

we shall show that each one of these satisfies relations of the form (43.2) and inequalities of the form (41.3).

Assuming that the relation (43.2) holds for $\xi_1, \xi_2 \dots \xi_{n-1}$, we shall show that similar relations hold also for $W_n, H\xi_n, \xi_n$. As it is true for $n = 1, 2, 3$, it will then be seen to be universally true.

In order to find ξ_n we must first find W_n . By the rule given in (33.6) in order to get W_n , we first find the co-efficient of α^n in

$$(\xi_1 \alpha + \dots + \xi_{n-1} \alpha^{n-1})^2 + \dots + (\xi_1 \alpha + \dots + \xi_{n-1} \alpha^{n-1})^n$$

and for each product $\xi_{p_1} \xi_{p_2} \xi_{p_3} \dots \xi_{p_r}$

we substitute the corresponding parenthesis

$$(\xi_{p_1}, \xi_{p_2}, \dots, \xi_{p_s})$$

where $p_1 p_2 \dots p_s$ are one or other of the numbers 1, 2, 3, ..., $(n-1)$

such that $p_1 + p_2 + \dots + p_s = n$.

Again by its definition (33.3) the parenthesis is equal to

$$\lim_{\substack{v = \rho \\ u \rightarrow \rho - 0}} \left[\sum_{i=0}^{s-1} \frac{\omega_i}{s!} \frac{\partial^{i+j}}{\partial u^i \partial v^j} \int K \varpi'_i d\psi' \right]$$

where the quantities involved are defined in article 33.

Now if we assume that $\xi_1 \xi_2 \dots \xi_{n-1}$ satisfy conditions (43.2), (43.3) we see by the continued use of Theorems V and VIII that the product ϖ of $(s-i)$ of them also satisfies similar conditions. That is to say

$$\varpi_i < < \frac{\sum_{p=0}^{\mu m} |A_{\mu, p}| \cos^p \psi}{[H(\rho \psi)]^\mu}$$

where μ = sum of the suffixes p_r of the ξ_{p_r} that are factors of ϖ and

$$|A_{\mu, p}| < L H_{c(3\mu m - 7\gamma), p} z^p$$

where $\gamma = s - i$ = the number of factors in ω_i .

Consider now the term

$$\lim_{\substack{v = \rho \\ u \rightarrow \rho - 0}} \frac{\partial^{i+j}}{\partial u^i \partial v^j} \int K \varpi'_i d\psi'.$$

This is

$$< < \sum_{p=0}^{\mu m} A_{\mu, p} | \lim \frac{\partial^{i+j}}{\partial u^i \partial v^j} \int K \frac{\cos^p \psi'}{[H(\rho \psi')]^\mu} d\psi'. \quad (43.4)$$

$$\text{Now } \lim_{v=\rho} \frac{\partial^j}{\partial v^j} \int X(v) Y(uv) Z(\rho) d\psi'$$

$$= \lim_{v=\rho} \int \left[X(\rho) \frac{\partial^j Y}{\partial v^j} + j \left\{ \frac{\partial X(v)}{\partial v} \right\}_{v=\rho} \cdot \frac{\partial^{j-1} Y(uv)}{\partial v^{j-1}} \right.$$

$$\left. + \text{ terms of higher order of differentiation for } X \right] Z(\rho) d\psi.$$

In this put $X(v) = H(v\psi') = v + q \sin^2 \psi'$, etc.

Then we get the following formula:—

$$\begin{aligned} & \lim_{\substack{v=\rho \\ u \rightarrow \rho-0}} \frac{\partial^{i+j}}{\partial u^i \partial v^j} \int \frac{-\log R^2(uv)}{2\Delta(v)} \cdot \frac{\cos^p \psi'}{[H(\rho\psi')]^\mu} \cdot H(v\psi') d\psi' \\ &= \lim_{\partial u^i} \frac{\partial^i}{\partial u^i} \left\{ \frac{\partial^j}{\partial v^j} \int \frac{-\log R^2(uv)}{2\Delta(v)} \frac{\cos^p \psi'}{[H(\rho\psi')]^{\mu-i}} d\psi' \right. \\ & \quad \left. + j \frac{\partial^{j-1}}{\partial v^{j-1}} \int \frac{-\log R^2(uv)}{2\Delta(v)} \frac{\cos^p \psi'}{[H(\rho\psi')]^\mu} d\psi' \right\} \quad (43.5) \\ &= \frac{(-)^{\mu-2}}{(\mu-2)!} \lim \frac{\partial^i}{\partial u^i} \left[\frac{\partial^j}{\partial v^j} \cdot \frac{\partial^{\mu-2}}{\partial \rho^{\mu-2}} - \frac{j}{\mu-1} \cdot \frac{\partial^{j-1}}{\partial v^{j-1}} \cdot \frac{\partial^{\mu-1}}{\partial \rho^{\mu-1}} \right] \times \\ & \quad \int \frac{-\log R^2(uv)}{2\Delta(v)} \frac{\cos^p \psi'}{H(\rho\psi')} d\psi'. \end{aligned}$$

Let us now put

$$\frac{\cos^p \psi}{H(\rho\psi)} = \sum_l a_l \cos l\psi$$

and use (2.6), that is, the formula

$$\int \frac{-\log R^2(uv)}{2\Delta(v)} \cos l\psi' d\psi' = \frac{\pi}{l} P_l(u) Q_l(v) \cos l\psi.$$

Hence noting that a_l is a function of ρ , we get

$$\begin{aligned} & \lim \frac{\partial^{i+j}}{\partial u^i \partial v^j} \int \frac{-\log R^2(uv)}{2\Delta(v)} \cdot H(v\psi') \cdot \frac{\cos^p \psi'}{[H(\rho\psi')]^\mu} d\psi' \\ &= \frac{\pi (-)^{\mu-2}}{(\mu-2)!} \sum_l \frac{\cos l\psi}{l} \cdot \frac{d^i P_l}{d\rho^i} \left[\frac{d^j Q_l}{d\rho^j} \cdot \frac{d^{\mu-2} a_l}{d\rho^{\mu-2}} \right. \\ & \quad \left. - \frac{j}{\mu-1} \cdot \frac{d^{j-1} Q_l}{d\rho^{j-1}} \cdot \frac{d^{\mu-1} a_l}{d\rho^{\mu-1}} \right]. \quad (43.6) \end{aligned}$$

Now by Theorem III

$$\overline{\text{Lim}} \frac{d^i P_l}{d\rho^i} < l(i-1)! \rho^{-i}; \quad \overline{\text{Lim}} \frac{d^j Q_l}{d\rho^j} < j! \rho^{-(j+1)}.$$

Also by Theorems I and II

$$\frac{d^k a_l}{d\rho^k} \text{ consists of terms which are all of the sign } (-)^k$$

$$\text{and } \sum_l \frac{d^{\mu-r} a_l}{d\rho^{\mu-r}} \cos l\psi \equiv (-)^{\mu-r} (\mu-r)! \frac{\cos^p \psi}{[H(\rho\psi)]^{\mu-r+1}}.$$

Hence the left-hand side of equation (43.6)

$$\begin{aligned} &< < \pi (i-1)! \rho^{-i} \left[j! \rho^{-(j+1)} \frac{\cos^p \psi}{[H(\rho\psi)]^{\mu-1}} \right. \\ &\quad \left. + j! \rho^{-j} \frac{\cos^p \psi}{[H(\rho\psi)]^{\mu}} \right] \\ &< < 4\pi (i-1)! j! \rho^{-(i+j+1)} \frac{\cos^p \psi}{[H(\rho\psi)]^{\mu}}, \end{aligned}$$

noting that $H(\rho\psi) \equiv \rho + q \sin^2 \psi < \rho + q = 3$ when $\Delta = ab = 1$ and $m = 3$.

Hence going back to relation (43.4) we see that, the left-hand side of that relation

$$< < 4\pi (i-1)! j! \rho^{-(i+j+1)} \frac{\sum_{p=0}^{\mu m} |A_{\mu, p}| \cos^p \psi}{[H(\rho\psi)]^{\mu}}. \quad (43.8)$$

Again

$\omega_i =$ the product of i of the ζ_{pr} 's forming the parenthesis

$$< < \sum_{p=0}^{\mu' m} \frac{|A_{\mu', p}| \cos^p \psi}{[H(\rho\psi)]^{\mu'}}$$

where $\mu' =$ sum of the suffixes p_r of the ζ_{p_r} that are factors of ω_i and since the ζ_{p_r} satisfy the conditions (43.2), (43.3) we have by the theorems proved in article 42

$$|A_{\mu', p}| < L H_{c(8\mu' m - 7i), p} e^p.$$

Hence we see that

$$\lim_{\substack{v=\rho \\ u \rightarrow \rho-0}} \frac{\omega_i}{s!} \frac{\partial^{i+j}}{\partial u^i \partial v^j} \int K \omega'_i d\psi' \\ < < \frac{4\pi (i-1)! j!}{s!} \rho^{-(i+j+1)} \cdot \frac{\sum_{p=0}^{nm} |A_{n,p}| \cos^p \psi}{[H(\rho\psi)]^n}$$

$$\text{where } |A_{n,p}| < LH_{c(3nm-7s),p} z^p. \quad (43.9)$$

But the parenthesis

$$(\xi_{p_1}, \xi_{p_2}, \dots, \xi_{p_s}) = \lim_{\substack{v=\rho \\ u \rightarrow \rho-0}} \sum_{i=0}^{s-1} C_{s,i} \frac{\omega_i}{s!} \frac{\partial^{s-1}}{\partial u^i \partial v^j} \int K \omega'_i d\psi'.$$

Hence the parenthesis satisfies the condition

$$(\xi_{p_1}, \dots, \xi_{p_s}) \leq \frac{\sum_{p=0}^{nm} |B_{n,p}| \cos^p \psi}{[H(\rho\psi)]^n}, \quad (43.10)$$

$$\text{where } |B_{n,p}| < 2^s \rho^{-s} L z^p \cdot H_{c(3nm-7s),s,p}.$$

Hence using the conclusions of Theorem VI of the last article we see that

$$|(\xi_{p_1}, \xi_{p_2}, \dots, \xi_{p_s})| < \frac{1}{2^n} \frac{1}{[H(\rho\psi)]^n} \cdot \frac{L}{(1-z)^{c(3nm-6s)}}. \quad (43.11)$$

since by making z sufficiently near to but still less than unity we can always make $2^s \rho^{-s} (1-z)^{cs}$ less than unity.

Consider now the number of such parentheses in W_n . From the manner (33.6) in which W_n is to be found explicitly in terms of the $\xi_1 \xi_2 \dots \xi_{n-1}$, it is easy to see that the number of parentheses of the type $(\xi_{p_1}, \xi_{p_2}, \dots, \xi_{p_s})$ [wherein there are s elements such that $p_1 + p_2 + \dots + p_s = n$ while the p 's may have any of the values $1, 2, \dots, n-1$] is the same as the co-efficient of α^n in

$$(\alpha + \alpha^2 + \alpha^3 + \dots + \alpha^{n-1})^s, \text{ or in } \frac{\alpha^s}{(1-\alpha)^s}.$$

Hence the number of such parentheses is

$$H_{s, n-s} \equiv \frac{(n-1)!}{(n-s)!(s-1)!} = {}_{n-1}C_{s-1} < 2^{n-1}.$$

Also
$$W_n = \sum_{s=2}^n \sum_{p_1, p_2, \dots, p_s} (\xi_{p_1}, \xi_{p_2}, \dots, \xi_{p_s}).$$

Hence we see that as each parenthesis satisfies a condition of the form (43.10) and the inequality (43.11), W_n also will satisfy the relation

$$W_n < \frac{\sum_{p=0}^{nm} |C_{n,p}| \cos^p \psi}{[H(\rho\psi)]^n} \quad (43.12)$$

Also for the corresponding inequality we have

$$\begin{aligned} W_n &< \sum_{s=2}^n H_{s, n-s} \frac{1}{2^n} \frac{L}{[H(\rho\psi)]^n} \frac{1}{(1-z)^{c(3nm-6s)}} \\ &< \frac{L}{[H(\rho\psi)]^n} \frac{1}{(1-z)^{3cnm}} \frac{(1-z)^{12c}}{1-(1-z)^{6c}}. \end{aligned}$$

or
$$W_n < \frac{L}{[H(\rho\psi)]^n} \frac{1}{(1-z)^{c(3nm-10)}} \quad (43.13)$$

whence we see on comparing (43.12) and (43.13) that

$$|C_{n,p}| < Lz^p \cdot H_{c(3nm-10), p}.$$

§ 44. We have now to show that $H\xi_n$, ξ_n also satisfy similar conditions, but this is not difficult. Consider the fundamental equation_n satisfied by the particular form of ξ_r explained in § 13. It is

$$\begin{aligned} \frac{2ab}{(a+b)^2} H\xi_n + \frac{1}{2\pi} \int \log R^2(\rho\rho') H'\xi'_n d\psi' \\ = \frac{\Delta}{\pi} [W_n + A_m \cos m\psi] + \text{constant}, \end{aligned} \quad (44.1)$$

where $-A_m$ is the co-efficient of $\cos m\psi$ in the Fourier Series for W_n . Let then the Fourier Series of $(W_n + A_m \cos m\psi)$ be

$$W_n + A_m \cos m\psi = \sum_k' c_{n,k} \cos k\psi \quad (44.2)$$

the accent denoting that the term in $\cos m\psi$ is absent. Then we can satisfy the equation (44.1) by taking

$$H\xi_n = \sum_k b_{nk} \cos k\psi$$

if the co-efficients b_{nk} are determined by the equation

$$\left[\frac{2ab}{(a+b)^2} - \frac{1+t^k}{k} \right] \int H\xi_n \cos k\psi d\psi \\ = \frac{\Delta}{\pi} \int (W_n + A_n \cos m\psi) \cos k\psi d\psi;$$

or

$$T_k b_{nk} = \frac{\Delta}{\pi} c_{nk}.$$

In other words,

$$H\xi_n = \frac{\Delta}{\pi} \sum_k' \frac{c_{nk}}{T_k} \cos k\psi. \quad (44.3)$$

Now if as before we write

$$\frac{\cos^p \psi}{H(\rho\psi)} = \sum_l a_{pl} \cos l\psi \\ \frac{\cos^p \psi}{[H(\rho\psi)]^n} = \frac{(-)^{n-1}}{(n-1)!} \sum_l \frac{d^{n-1} a_{pl}}{d\rho^{n-1}} \cos l\psi. \quad (44.4)$$

Hence from (43.12)

$$W_n < \frac{(-)^{n-1}}{(n-1)!} \sum_l \cos l\psi \sum_{p=0}^{nm} |C_{n,p}| \frac{d^{n-1} a_{pl}}{d\rho^{n-1}}.$$

$$\text{Hence } |c_{n,k}| < \frac{1}{(n-1)!} \sum_{p=0}^{nm} |c_{n,p}| \frac{d^{n-1} a_{pk}}{d\rho^{n-1}}.$$

and as, for $k \neq m$, $\left| \frac{1}{T_k} \right|$ is less than the greater of the two

$$\left| \frac{1}{T_{m-1}} \right|, \quad \left| \frac{1}{T_{m+1}} \right|$$

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NOTES AND QUESTIONS.

Notes and Questions.

The Harmonic Locus and its analogues in Hyperspace.

1. It is well-known that if

$$\Sigma = al^2 + \dots 2fmn + \dots = 0$$

be the line equation of a conic, its point equation is

$$(bc - f^2)x^2 + \dots + 2(gh - af)yz \dots = 0 \quad \dots (1)$$

Let

$$\Sigma_1 = a_1 l^2 + \dots \text{ and } \Sigma_2 = a_2 l^2 + \dots$$

be two conics, so that

$$\Sigma_1 + \lambda \Sigma_2 = 0 \quad \dots \quad \dots (\lambda)$$

represents a conic belonging to the four-line system they determine. The point equation of the conic (λ) is seen from (1) to be a quadratic in λ and may be written

$$F_1 + \lambda H + \lambda^2 F_2 = 0 \quad \dots \quad \dots (2)$$

By taking $\lambda = 0$ and $\lambda = \infty$, it is clear* that $F_1 = 0$ and $F_2 = 0$ are respectively the point equations of Σ_1 and Σ_2 .

We define

$$H = (b_1c_2 + b_2c_1 - 2f_1f_2)x^2 + \\ + 2(g_1h_2 + g_2h_1 - a_1f_2 - a_2f_1)yz + \dots = 0 \quad \dots (3)$$

to be the harmonic locus of Σ_1 and Σ_2 .

It will now be shown* that this definition is equivalent to the more usual one, namely: that (3) is the locus of points from which the tangents to Σ_1 and Σ_2 form two harmonic pairs.

2. Let P be any fixed point in the plane. With each of the conics (λ) we may associate a quadratic whose roots are the parameters of the two lines of the pencil, vertex P, which touch (λ) . Since five tangents determine a conic, one root determines the quadratic and hence these quadratics belong to a system of two terms

$$Q_1 + \lambda Q_2 = 0 \quad \dots \quad \dots [\lambda]$$

If the tangents from P to the conic (λ_1) separate harmonically the tangents from it to (λ_2) , the associated quadratics $[\lambda_1]$ and $[\lambda_2]$ are

* This result is well-known; but the proof differs from that given in text-books where the locus is obtained directly and identified with the equation (3).
Vide: Askwith: *Analytical Geometry of the Conic Sections*, p. 425.



apolar. As the apolarity invariant is bi-linear in the co-efficients of the two forms, this amounts to an involutric correspondence between λ_1 and λ_2 , the double members of which are the self-apolar members, i.e. the perfect squares of the pencil and correspond to the two conics which pass through P. But the double members of an involution separate harmonically every pair of corresponding elements. Hence the parameters of the two conics of the pencil $\Sigma_1 + \lambda \Sigma_2$ which pass through P [the roots of the equation (2)] form a dyad apolar to the dyad $(0, \infty)$.

$$\therefore \lambda_1 + \lambda_2 = 0$$

so that the co-ordinates of P must satisfy $H = 0$.*

3. The results of our investigation may also be stated thus:—

In any involution among a four-line system of conics, the locus of intersection of corresponding members is a conic. This conic is the harmonic locus of the double members of the involution. ... (4)

From this result we easily deduce that:—

(i) *The harmonic locus F has double contact with three of the conics (λ), viz., those which correspond in the involution to the three degenerate conics of the system. The chords of contact form the sides of the common self-polar triangle of Σ_1 and Σ_2 †. ... (5)*

(ii) *If in the involution whose double members are Σ_1 and Σ_2 two degenerate members of the system correspond, F belong to the same pencil as two co-incident line pairs and hence breaks up into a pair of lines.* ... (6)

4. Point representation of conics of the net

$$ax^2 + by^2 + cz^2 = 0.$$

The conic $ax^2 + by^2 + cz^2 = 0$ for which the triangle of reference is self-polar may be associated with the point whose homogeneous co-

* For a different proof, and a critical treatment by symbolic methods, vide: Prof. E. H. Neville. [A Note on the Harmonic Conic. *Messenger of Mathematics*, March 1925]. This note was suggested by a study of Prof. Neville's paper, but while his proof follows the lines of the corresponding analytical investigation without however going through the details of calculation, the one given here is based on the idea of correspondence which has proved so fruitful in geometrical research. The scope of the paper is also much wider than Prof. Neville's.

† Gabbat "On the generalisation of the Theory of Circles associated with a Triangle." *Proc. Camb. Phil. Soc.*, Vol. 21, p. 324.

ordinates are (a, b, c) . Since the reciprocal of the above conic with respect to

$$x^2 + y^2 + z^2 = 0$$

$$\text{is } bcx^2 + cay^2 + abz^2 = 0$$

such a reciprocation is equivalent to the quadratic transformation T given by

$$a' : b' : c' = bc : ca : ab. \quad \dots \quad \dots \quad (7)$$

A four point system of conics will be represented by a straight line, and a four line system by a conic through the vertices ABC of the reference triangle.

Representing Σ_1 and Σ_2 by their associated points σ_1 and σ_2 , pairs of conics (λ_1) and (λ_2) of § 2 correspond to point-pairs in involution on the conic ABC $\sigma_1 \sigma_2$ the double points being σ_1 and σ_2 . Since F belongs to the same point pencil as such a pair of conics, it corresponds to a point collinear with such pairs of points.

The Harmonic locus of a pair $\Sigma_1 \Sigma_2$ corresponds to the pole of the chord $\sigma_1 \sigma_2$ with respect to the conic ABC $\sigma_1 \sigma_2$ (8)

Again, the harmonic conjugate of the point H with respect to such a pair of points collinear with it, lies on its polar line $\Sigma_1 \Sigma_2$. Hence the following:—

The harmonic conjugate of H with respect to the two conics of the system $\Sigma_1 + \lambda \Sigma_2$ which intersect on it, is a conic belonging to the point pencil determined by Σ_1 and Σ_2 (9)

Let us denote by H the operation of replacing a pair of points $\Sigma_1 \Sigma_2$ by their pole *w.r.t.* the conic ABC $\Sigma_1 \Sigma_2$. From the reciprocal character of the F and ϕ covariants, we easily deduce that:—

The Harmonic Envelope or ϕ conic of two conics $S_1 S_2$ is obtained by subjecting the representative points successively to the operations THT .

5. Definitions.

Let us denote by S_n a flat space of n -dimensions, and by S_r ($r < n$) any flat sub-region of r dimensions contained in S_n . Thus S_0 is a point while S_1 and S_2 stand respectively for a line and a plane in S_n .

A *simplex* in S_n is a set of $(n + 1)$ points $A_0 A_1 A_2 \dots A_n$ which do not all lie in a sub-space of S_n . The *elements* of the simplex are:—the $n + 1$ vertices $A_0 A_1 A_2 \dots A_n$ which are the elements of dimension zero, the $\frac{n(n + 1)}{2}$ lines $A_i A_j$ which are the elements of dimension 1, etc. ...

and finally the $(n + 1)$ primes* $A_0 A_1 A_2 \dots A_{r-1} A_{r+1} \dots A_n$ which are the elements of dimension $(n - 1)$.

A simplex is said to be *self-polar* with respect to a quadric if the polar region of each vertex is the prime determined by the other n vertices. It is obvious that the polar region of an S_r determined by any $(r + 1)$ vertices of a self-polar simplex is the S_{n-r-1} passing through the others.

A quadric is said to be *harmonically r -scribed* to another if there exist a simplex self-polar with respect to the first quadric and whose r -dimensional elements touch the second quadric.

6. Concomitants of two quadric envelopes and their geometrical significance.

$$\text{Let} \quad A \equiv \sum a_{rs} u_r u_s = 0$$

and

$$B \equiv \sum b_{rs} u_r u_s = 0$$

$$r, s = 0, 1, \dots, n.$$

be the "prime" equations of two quadrics in S_n . The discriminant of $(A + \lambda B)$ is

$$| a_{rs} + \lambda b_{rs} | \quad \dots \quad \dots \quad (11)$$

which is of degree $(n + 1)$ in λ , and may be written

$$\lambda^{n+1} \Delta_B + \theta_1 \lambda^n + \theta_2 \lambda^{n-1} + \dots + \theta_n \lambda + \Delta_A = 0, \dots \quad (12)$$

where Δ_A, Δ_B are the discriminants and the θ 's certain joint invariants† of the two forms A and B . It will now be shown that $\theta_{r+1} = 0$ is the condition that A is harmonically r -scribed to B .

If B touches the r -dimensional region determined by the $r + 1$ points‡

$$x_r = (x_{r0} x_{r1} \dots x_{rn}) \quad r = 0, \dots, r$$

* A term due to Prof. H. F. Baker denoting a flat space of $n-1$ dimensions situated in S_n .

† It is obvious $\theta_r = \frac{1}{r!} \Omega^r | b_{rs} |$ where Ω is the polar operator

$$\sum_{rs} a_{rs} \frac{\partial}{\partial b_{rs}}.$$

Its invariant character follows from the co-gredience of a_{rs} and b_{rs} .

‡ Bertini—*Introduzioni alla Geometria proiettiva degli Iperspazi*, p. 149 where the corresponding condition is given for the point equation of the quadric.

then

$$\begin{vmatrix} b_{00} & \dots & b_{0n} & x_{00} & \dots & x_{r0} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ b_{n0} & \dots & b_{nn} & x_{n0} & \dots & x_{rn} \\ x_{00} & \dots & x_{0n} & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ t_{r0} & \dots & t_{rn} & 0 & \dots & 0 \end{vmatrix} = 0 \quad \dots (13)$$

In particular, if the r -dimensional elements of the reference simplex touch B , the co-axial* minor determinants of order $(n-r)$ in the discriminant $|b_{rs}|$ all vanish. $\dots \dots \dots (14)$

If, in addition, the reference simplex be self-polar with respect to A we have

$$A \equiv a_0 u_0^2 + a_1 u_1^2 + \dots a_n u_n^2$$

and the discriminant (11) is now

$$C \equiv |c_{rs}| \begin{cases} c_{rr} = a_r + \lambda b_{rr} \\ c_{rs} = \lambda b_{rs} \end{cases} \quad \left[r, s = 0, 1, \dots n \right]$$

which, it should be noted, has the same co-axial minors as $|b_{rs}|$.

The co-efficient of λ^{n-r} in the corresponding equation (12) is now the sum of a number of co-axial minor determinants of order $(n-r)$ in C , and hence vanishes in virtue of (14). Hence $\theta_{r+1} = 0$.

This result may be expressed in another form more in keeping with spirit of our present line of thought.

A quadric A of an envelope pencil is harmonically r -scribed to another member B , if the $n+1$ singular quadrics of the pencil form an $(n+1)$ -ad apolar to $A^r B^{n+1-r}$.

7. It is clear from (13) that the point equation of $A + \lambda B$ is

$$\begin{vmatrix} a_{00} + \lambda b_{00} & \dots & a_{0n} + \lambda b_{0n} & x_1 \\ \dots & \dots & \dots & \dots \\ a_{n0} + \lambda b_{n0} & \dots & a_{nn} + \lambda b_{nn} & x_n \\ x_1 & \dots & x_n & 0 \end{vmatrix} = 0 \quad \dots (15)$$

$$\text{or } \lambda_n F_B + \lambda^{n-1} H_1 + \lambda^{n-2} H_2 + \dots \lambda H_{n-1} + F_A = 0 \quad \dots (16)$$

* The axis of a determinant is its leading diagonal. Two determinants are co-axial if they have the same axis.

Here $F_A = 0$ and $F_B = 0$ are the point equations of Σ_1 and Σ_2 while the remaining co-efficients H are certain covariant quadric loci which correspond to the harmonic locus in a plane.

We shall refer to H_r as the r -th harmonic locus of A and B . It is obviously also the $(n - r)$ th harmonic locus of B and A .

Equation (15) may be interpreted as the incidence condition of the quadric (λ) and the point x and shows that these are n quadrics of the system through any assigned point x the parameters λ of which are the roots of (16.) If the point lies on $H_r = 0$ the sum of the products of the roots taken r at a time vanishes, i.e. the parameters form an n -ad apolar to $0^r \infty^{n-r}$. Hence analogous to (4) we have that

*If in any involution of order n and freedom $n - 1$ * among the members of an envelope pencil $A + \lambda B$ the n -ple elements consist of A taken r times and B $n - r$ times, the locus of the common points of quadrics of a set of the involution, is the r -th harmonic locus of A and B .* ... (17)

8. The above locus passes through the point $(1, 0, 0 \dots)$ if, on substituting these values for $x_1 \dots x_n$ the first member in (15) has no term in λ^{n-r} . The determinant now reduces to

$$|a_{rs} + \lambda b_{rs}| \quad [r, s = 1, 2, \dots n. \dots (18)$$

But

$$\sum a_{rs} u_r u_s = 0$$

$$\sum b_{rs} u_r u_s = 0 \quad r, s = 1, 2, \dots n$$

are respectively the equations of the tangent cones from $(1, 0, 0 \dots)$ to A and B and, if their discriminant (18) lacks a term in λ^{n-r} , the first cone is harmonically r -scribed to the second. Hence

the r th harmonic locus of A and B is the locus of points the tangent cone from which to A is harmonically r -scribed to the tangent cone to B .†

9. Point representation of Quadrics of the net

$$\alpha_0 x_0^2 + \alpha_1 x_1^2 + \dots \alpha_n x_n^2 = 0. \ddagger$$

* Sets of n -quadrics of which any $n - 1$ determine the n th.

† For the corresponding theorem in 3 dimensions, *vide* Salmon-Rogers, p. 219 or Baker, *Principles of Geometry*, Vol. III, p. 90, Ex. 4.

‡ The idea of this representation and the results of this paragraph and § 4 are due to Dr. R. Vaidyanathaswami.

As in § 4, we represent the quadric $\sum \alpha_r x_r^2 = 0$ by the point whose homogeneous co-ordinates are $(\alpha_1 \alpha_2 \dots \alpha_n)$. In this scheme the singular quadrics of the net correspond to points on the reference simplex, a quadric r -ply specialised* being represented by a point on an S_{n-r} of the simplex

Associated with the quadrics $A \equiv \sum a_r x_r^2 = 0$ and $B \equiv \sum b_r x_r^2 = 0$ we have the two points

$$A, \left(\frac{1}{a_1} \dots \frac{1}{a_n} \right) \text{ and } B, \left(\frac{1}{b_1} \dots \frac{1}{b_n} \right)$$

while to the several quadrics of the system $A + \lambda B$ correspond the points $x_r = \frac{1}{a_r + \lambda b_r}$ of a Norm curve† passing through A, B , and the vertices of the simplex. This curve is fixed by these $n + 3$ points.

If O be any fixed points, sets of n points on the Norm Curve which lie on the same prime with O , determine an involution of points on the curve which is of order n and freedom $n - 1$. The united or n -ple points of this involution are the n points whose osculating primes pass through O .

It is now obvious from (17) that the r th harmonic locus H_r of the two quadrics A and B corresponds in our representation to the intersection of the osculating S_{n-r} at A and the osculating S_r at B .

10. Covariant complexes associated with a pair of quadrics.

The covariants H_r discussed hitherto are the point loci in S_n analogous to the harmonic locus in a plane. We have besides certain Quadratic regional complexes‡ in S_n which enjoy similar properties.

It will be obvious from (13) that the quadric

$$A + \lambda B \equiv \sum a_{rs} + \lambda b_{rs} = 0$$

* i.e., with a "singular region" or "a region of vertices" of dimension r , Bertini—p. 139. This happens when the rank of the matrix of the quadratic form is $n - r + 1$. Boscher: *Introduction to Higher Algebra*, p. 130.

† The Rational Norm curve or twisted n -ic in S_n is a curve of order n not contained in an S_{n-1} . It is given parametrically by equations of the form $x_r = f_r(t)$, $r = 0, \dots, n$, the f 's being any set of linearly independent polynomials of order n in t .

‡ The words *complex* and *congruence* are used in this paper to denote the totality of S_r 's whose co-ordinates satisfy one and two equations respectively.

touches the S_r through the points $x_r (x_{r0} \dots x_{rn})$, $r = 0 \dots r$

$$\text{if } \begin{vmatrix} a_{00} + \lambda b_{00} & \dots & a_{n0} + \lambda b_{n0} & x_{01} & \dots & x_{r1} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a_{n0} + \lambda b_{n0} & \dots & a_{nn} + \lambda b_{nn} & x_{0n} & \dots & x_{rn} \\ x_{01} & \dots & x_{0n} & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ x_{r1} & \dots & x_{rn} & 0 & \dots & 0 \end{vmatrix} = 0$$

This is a quadratic in the co-ordinates* of S_r and of order $n - r$ in λ and hence may be written

$$\lambda^{n-r} B^r + \lambda^{n-r-1} H_1^r + \lambda^{n-r-2} H_2^r + \dots \lambda H_{n-r-1}^r + A^r = 0$$

The equations $A^r = 0$ and $B^r = 0$ are the r -dimensional regional equations of A and B respectively while the other co-efficients are the complexes of S_r 's referred to above. It will be noticed that there are altogether $\frac{1}{2}(n^2 - 3n + 2)$ such complexes H_k^r [$r, k = 1, 2, \dots, n-2$], $r + k \leq n-1$, besides the $(n-1)$ loci H_k or complexes of S_0 discussed in §§ 7 and 8.

It can be shown by methods similar to those employed already that

(1) $H_k^r = 0$ is the complex of S_r such that of the two quadric cones having S_r for singular region and enveloping A and B , the first is harmonically k -scribed to the second.

(2) $H_k^r = 0$ is the totality of S_r touching $(n-r)$ quadrics of the pencil whose parameters λ form a set apolar to $O^k \propto \omega^{n-r-k}$.

For $n = 3$ and $r = 1$, this reduces to the theorem that the harmonic complex [or the complex of lines from which the tangent planes to one quadric separate harmonically the tangent planes to another] consists of an infinity of congruences whose local surfaces break up into the pairs of quadrics $(A + \lambda B)$ and $(A - \lambda B)$.†

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* The regional co-ordinates of S_r are the determinants of the matrix

$$\begin{vmatrix} x_{00} & x_{01} & \dots & x_{0n} \\ \dots & \dots & \dots & \dots \\ x_{r0} & x_{r1} & \dots & x_{rn} \end{vmatrix} \quad x, y, \dots, t \text{ being any simplex in } S_r$$

† Jessop : *Treatise on the Line Complex*, p. 134.

"Two Elementary Definite Integrals."

1. This note is suggested by the somewhat incomplete solution of Q. 1304, which appears in the April issue.* The second part of the original question had probably a misprint (x for x^2), since we can prove that

$$\int_0^{\infty} \frac{\sin^{2k} x}{x^2} dx = f(k-1) = \frac{1.3.5 \dots 2k-3}{2.4.6 \dots (2k-2)} \pi.$$

But it can be shown that the two results are only particular cases of quite general definite integrals.

2. These are:—

$$\text{I. } \int_{-\infty}^{\infty} \frac{\phi(x)}{x} dx = \int_0^{\pi} \phi(x) \cot \frac{x}{2} dx = u, \text{ say}$$

where $\phi(x)$ = any odd trigonometric function of x

$$\text{i.e. } \phi(2n\pi - x) = -\phi(x);$$

$$\text{and II. } \int_{-\infty}^{\infty} \frac{\psi(x)}{x^2} dx = \int_0^{\pi} \psi(x) \cdot \operatorname{cosec}^2 x dx = v, \text{ say}$$

where $\psi(x)$ is any even trigonometric function with period π .

Proof:—

$$u = \int_{-\infty}^{+\infty} = 2 \left[\int_0^{\pi} + \int_{\pi}^{2\pi} + \int_{2\pi}^{3\pi} + \dots \right].$$

Now transform the integrals within square brackets by putting

$$x = z, \quad 2\pi - z, \quad 2\pi + z, \quad 4\pi - z, \dots \text{ respectively.}$$

* Question (1304). (N. R. Jaya Rao, M.A.):—If $\frac{\pi}{2} \frac{1.3.5 \dots (2k-1)}{2.4.6 \dots 2k} = f(k)$ then show that

$$\int_0^{\infty} \frac{\sin^{2k+1} x}{x} dx = f(k)$$

$$\text{and } \int_0^{\infty} \frac{\sin^{2k} x}{x} dx = f(k-1).$$

Then we get, in virtue of the restriction on ϕ ,

$$\begin{aligned} u &= 2 \int_0^\pi \phi(z) \left[\frac{1}{z} + \frac{1}{z-2\pi} + \frac{1}{z+2\pi} \right. \\ &\quad \left. + \frac{1}{z-4\pi} + \frac{1}{z+4\pi} + \dots \right] dz \\ &= \int_0^\pi \phi(z) \cdot \sum_{-\infty}^{\infty} \frac{1}{\frac{z}{2} + r\pi} \cdot dz. \\ &= \int_0^\pi \phi(z) \cot \frac{z}{2} dz. \end{aligned}$$

Next consider,

$$v = 2 \left[\int_0^\pi + \int_\pi^{2\pi} + \int_{2\pi}^{3\pi} + \dots \right] = v_1 + v_2$$

where $v_1 = v_2 = \int_0^\pi + \int_\pi^{2\pi} + \dots$

We transform these in two different ways.

Transform v_1 by putting

$$x = \pi - z, \quad 2\pi - z, \quad 3\pi - z, \quad \dots \text{ \&c.}$$

and transform v_2 by putting

$$x = z, \quad \pi + z, \quad 2\pi + z, \quad \dots$$

We get

$$v_1 = \int_0^\pi \psi(z) \left[\frac{1}{(z-\pi)^2} + \frac{1}{(z-2\pi)^2} + \frac{1}{(z-3\pi)^2} + \dots \right] dz$$

and $v_2 = \int_0^\pi \psi(z) \left[\frac{1}{z^2} + \frac{1}{(z+\pi)^2} + \frac{1}{(z+2\pi)^2} + \dots \right] dz$

$$\begin{aligned} \therefore v &= v_1 + v_2 = \int_0^\pi \psi(z) \cdot \sum_{-\infty}^{\infty} \frac{1}{(z+r\pi)^2} dz \\ &= \int_0^\pi \psi(z) \operatorname{cosec}^2 z dz. \end{aligned}$$

3. The two results in Q. 1304, are obtained by taking

$$\phi(x) = \sin^{2k+1} x$$

and

$$\psi(x) = \sin^{2k} x.$$

Many examples can be constructed by taking

$$\phi(x) = \sin x \times \text{any function of } \cos x$$

and

$$\psi(x) = \tan^p x \times \text{any function of } (\sin^2 x)$$

where p is any even integer including 0.

4. As regards the "mis-print" integral, *viz.*

$$\int_0^\infty \frac{\sin^{2k} x}{x} dx,$$

I need only refer to a paper by Mr. A. N. Singh, in the *Bulletin* of the Calcutta Mathematical Society, where it is shown that

$$\int_0^\infty \frac{\sin^{2k} x}{x} dx$$

diverges.

5. We find various results in Wolstenholme's *Mathematical Examples* which are merely particular cases of I and II.

Thus, particular cases of (I) are :—

$$(1) \int_{-\infty}^{\infty} \frac{\sin^{2n+1} x}{x} dx = \int_0^\pi \sin^{2n+1} x \cdot \cot \frac{x}{2} dx = \frac{1.3.5 \dots (2n-1)}{2.4 \dots 2n} \pi$$

$$(2) \int_0^\infty \frac{dx}{x} \log \left(\frac{1 + 2n \sin x + n^2}{1 - 2n \sin x + n^2} \right) dx = 2\pi \tan^{-1} n. \quad (n < 1)$$

$$(3) \int_0^\infty \frac{\log(1 + n \sin^2 x)}{x \sin x} dx = \pi (\sqrt{1+n} - 1).$$

$$(4) \int_0^\infty \frac{\tan^{-1}(m \tan x)}{x} dx = \frac{\pi}{2} \log(1+m)$$

$$(5) \int_0^\infty \frac{\tan^{-1}(m \sin x)}{x} dx = \frac{\pi}{2} \log(\sqrt{1+m^2} + m).$$

The following are particular cases of (II):—

$$(1) \int_{-\infty}^{\infty} \frac{\sin^{2n} x}{x^2} dx = \int_0^\pi \sin^{2n-2} x dx = \dots \&c.$$

$$(2) \int_{-\infty}^{\infty} \frac{\log(1 + m \sin^2 x)}{x^2} dx = 2\pi (\sqrt{1+m} - 1). \quad (m < 1)$$

$$(3) \int_0^\infty \frac{1}{x^2} \log \left(\frac{1 + 2n \cos ax + n^2}{1 + 2n \cos bx + n^2} \right) dx \\ = \pi (b-a) \frac{n}{1+n}. \quad (n < 1)$$

G. S. MAHAJANI.

Solutions.

Question 1298.

(I. TOTADRI IYENGAR):—The function $\phi(n)$ denotes the number of integers less than n and prime to it, n being any positive integer. The function $\psi(x)$ stands for the infinite series

$$\phi(1)x - \phi(3)x^3 + \phi(5)x^5 - \phi(7)x^7 + \dots$$

Show that the infinite series

$$\psi(\tan \theta) - \psi(\tan^3 \theta) + \psi(\tan^5 \theta) - \dots = \frac{\sin 4\theta}{4}.$$

Solution by S. D. Chowla.

The given series is

$$\begin{aligned} & [\phi(1)\tan\theta - \phi(3)\tan^3\theta + \phi(5)\tan^5\theta - \dots + \dots] \\ & - [\phi(1)\tan^3\theta - \phi(3)\tan^9\theta + \phi(5)\tan^{15}\theta - \dots + \dots] \\ & + [\phi(1)\tan^5\theta - \phi(3)\tan^{15}\theta + \phi(5)\tan^{25}\theta - \dots + \dots] \\ & - + - + \dots \end{aligned}$$

$$= \frac{\phi(1)\tan\theta}{1+\tan^2\theta} - \frac{\phi(3)\tan^3\theta}{1+\tan^6\theta} + \frac{\phi(5)\tan^5\theta}{1+\tan^{10}\theta} - \dots$$

on adding by columns

$$\begin{aligned} & = \phi(1)\tan\theta - [\phi(1) + \phi(3)]\tan^3\theta \\ & \quad + - \dots + (-1)^{n-1} \sum \phi(d_r) \tan^{2n-1}\theta \dots \end{aligned} \quad \dots (1)$$

$$= \sum_{n=1}^{\infty} \left[(-1)^{n-1} \left\{ \sum_{(d)} \phi(d_r) \right\} \tan^{2n-1}\theta \right]$$

where $d_1, d_2, \dots, d_r, \dots$ are all the divisors of $2n-1$, and the inner Σ in (1) extends over all the divisors of $2n-1$.

But $\sum \phi(d_r) = 2n-1$, as is well-known.

\therefore the series in (1) $= \tan\theta - 3\tan^3\theta + 5\tan^5\theta$

$$\begin{aligned} & \quad - + \dots + (-1)^{n-1} \tan^{2n-1}\theta \dots \\ & = \tan\theta (1 - \tan^2\theta) (1 + \tan^2\theta)^{-2} = \frac{1}{4} \sin 4\theta. \end{aligned}$$

Note.—It is obvious that this result is only valid if θ lies between

$$r\pi \text{ and } r\pi \pm \frac{\pi}{4},$$

since
$$x - 3x^3 + 5x^5 - 7x^7 + \dots = \frac{x(1-x^2)}{(1+x^2)^2}$$

is valid only if $|x| < 1$.

Question 1306.

(N. B. MITRA):—Prove that if p is a prime, then the period of $\frac{1}{p}$ (when converted into a recurring decimal) will run to an even sub-multiple of $(p-1)$ figures, if and only if $p \equiv \pm 3^n \pmod{40}$, where n is any positive integer or zero; otherwise, it will run to its full period of $(p-1)$ figures or to some odd sub-multiple of $(p-1)$ figures,

$$(p \not\equiv 2 \text{ or } 5).$$

Solution by I. Totadri Iyengar.

Let f be the number of figures in the period of $\frac{1}{p}$.

Then evidently, f is the haupt exponent of 10, modulus p , that is,

$$10^f \equiv 1, \text{ mod. } p.$$

If f be an even sub-multiple of $(p-1)$, that is $(p-1)$ divided by an even number

then $\frac{p-1}{2f}$ is an integer.

$$\therefore 10^{\frac{p-1}{2}} = (10^f)^{\frac{p-1}{2f}} \equiv 1, \text{ mod. } p.$$

Hence 10 is a quadratic residue of p and in Legendre's notation

$$\left(\frac{10}{p}\right) = 1.$$

Conversely, if $\left(\frac{10}{p}\right) = 1$, that is $10^{\frac{p-1}{2}} \equiv 1 \text{ mod. } p$, then

f must be a sub-multiple of $\frac{p-1}{2}$ and hence an even sub-multiple of $(p-1)$.

The only forms, therefore, for p for which the period of $\frac{1}{p}$ runs to an even sub-multiple of $(p-1)$ figures are those for which

$$\left(\frac{10}{p}\right) = 1;$$

for if $\left(\frac{10}{p}\right) \neq 1,$

then $10^{\frac{p-1}{2}} \not\equiv 1 \text{ mod. } p,$

that is, f is a sub-multiple of $(p-1)$ but not of $\frac{p-1}{2},$

that is, f is an odd sub-multiple of $(p-1)$; and as a particular case, if 10 is a primitive root of p ,

$$f = p - 1.$$

Therefore we shall consider the forms for p such that

$$\left(\frac{10}{p}\right) = 1.$$

$$\text{Now } \left(\frac{10}{p}\right) = \left(\frac{2}{p}\right) \left(\frac{5}{p}\right) = (-1)^{\frac{p^2-1}{8}} \left(\frac{5}{p}\right) = 1. \quad (i)$$

Also by Legendre's Law of Quadratic Reciprocity

$$\begin{aligned} \left(\frac{5}{p}\right) \left(\frac{p}{5}\right) &= (-1)^{\frac{(5-1)(p-1)}{4}} \\ &= (-1)^{p-1} = 1 \end{aligned} \quad (ii)$$

since $p \neq 2$ and is an odd prime.

From (i) and (ii)

$$\left(\frac{p}{5}\right) = (-1)^{\frac{p^2-1}{8}}$$

$$\text{i.e. } p^2 \equiv (-1)^{\frac{p^2-1}{8}} \pmod{5}.$$

Two cases now arise:—

$$\left. \begin{array}{l} \text{If } p \equiv \pm 1 \pmod{8}, \\ p^2 \equiv 1 \pmod{5} \end{array} \right\} \dots (iii) \quad \left. \begin{array}{l} \text{If } p \equiv \pm 3 \pmod{8}, \\ p^2 \equiv -1 \pmod{5} \end{array} \right\} \dots (iv)$$

The two conditions in (iii) are simultaneously satisfied

when $p \equiv \pm 1$, or $\pm 9 \pmod{40}$;

and similarly $p \equiv \pm 3$, or $\pm 27 \pmod{40}$ satisfies (iv).

Hence the forms for p are

$$p \equiv \pm 1, \pm 3, \pm 9, \text{ or } \pm 27 \pmod{40}.$$

Since $3^4 \equiv 81 \equiv 1 \pmod{40}$,

3^n (for integral or zero values of n) is congruent with any one of the four residues 1, 3, 3^2 or 3^3 .

All the forms of p are thus included in the form

$$p \equiv \pm 3^n \pmod{40}.$$

As already shown, if p is not of this form f is either equal to $p -$ or an odd sub-multiple of $(p - 1)$.

Hence the above theorem.

Questions for Solution.

1407. (B. B. BAGI):—Prove that the following polynomials are divisible by $\sigma \equiv 1 + x + x^2 + \dots x^{n-1}$:

(a) $\Sigma_1, \Sigma_2 \dots \Sigma_{n-1}$ and $\Sigma_n + 1$

where Σ is the sum of products of

$$\sigma - 1, \sigma - x, \sigma - x^2 \dots \sigma - x^{n-1}$$

taken τ at a time.

$$(b) \quad (\sigma - 1)^n + (\sigma - x)^n + \dots + (\sigma - x^{n-1})^n$$

where m is prime to n .

$$(c) \quad (\sigma - 1)^m + (\sigma - x)^m + \dots + (\sigma - x^{n-1})^m - (-1)^m$$

where m is a multiple of n .

$$1^m + x^m + x^{2m} + x^{3m} + \dots + x^{(n-2)m} + (-1)^m (1 + x + \dots + x^{n-2})^m$$

where m is prime to n .

In (a), Σ_{n-1} is also divisible by $\sigma - x^{n-1}$.

In (d) if m is of the form $np + 1$, the polynomial is divisible by σ^2 .

1408. (B. B. BAGI) :—

If $f_1(x) \equiv (x - a_1)(x - b_1)(x - c_1) \dots (x - l_1)$

$$f_0(x) \equiv (x - a_0)(x - b_0)(x - c_0) \dots (x - l_0)$$

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$$f_n(x) \equiv (x - a_n)(x - b_n)(x - c_n) \dots (x - l_n)$$

and

$$a_1, a_2, \dots, a_n, b_1, b_2, \dots, b_n, c_1, c_2, \dots, c_n$$

are arranged in order of magnitude, then the equation

$$\frac{L_1^2}{f_1(x)} + \frac{L_2^2}{f_2(x)} + \frac{L_3^2}{f_3(x)} + \dots + \frac{L_n^2}{f_n(x)} = 0$$

has all its roots real, all the quantities involved in the equation being real.

1409. (K. J. SANJANA, M.A.):—Solve the differential equation

$$yy_2 + (n-1)y_1^2 + \frac{1}{n}y^2 = 0.$$

Prove also that when r is an even positive integer, y_r can be expressed as a finite series of powers of $\sec x$, or $\operatorname{cosec} x$.

1410. (HANSRAJ GUPTA):—If $\sum_{n=0}^{\infty} a_n x^n$ be a convergent recurring

series of the k -th order, show that its sum is equal to $D_1 \div D_2$

$$\text{where } D_1 = \begin{vmatrix} 0 & s_1 x^{k-1} & s_2 x^{k-2} & \dots & s_k \\ a_0 & a_1 & a_2 & \dots & a_k \\ a_1 & a_2 & a_3 & \dots & a_{k+1} \\ . & . & . & \dots & . \\ . & . & . & \dots & . \\ a_{k-1} & a_k & a_{k+1} & \dots & a_{2k-1} \end{vmatrix}$$

$$D_2 = \begin{vmatrix} x^k & x^{k-1} & x^{k-2} & \dots & x & 1 \\ a_0 & a_1 & a_2 & \dots & a_{k-1} & a_k \\ a_1 & a_2 & a_3 & \dots & a_k & a_{k+1} \\ . & . & . & \dots & . & . \\ . & . & . & \dots & . & . \\ a_{k-1} & a_k & a_{k+1} & \dots & a_{2k-2} & a_{2k-1} \end{vmatrix}$$

$$\text{and } s_r = \sum_{n=0}^{r-1} a_n x^n.$$

1411. (K. J. SANJANA, M.A.):—(1) The bisector of the angle A of a triangle ABC meets BC in K; Q is the mid. point of BC, and X, R are respectively the feet of the perpendiculars on it from the in-centre I and the vertex A. Prove that QX is the mean proportional between QK and QR.

(2) The triangle PQR is right-angled at R and any point K is taken in QR; QX is taken along QR equal to the mean proportional between QK and QR, and the straight line through K parallel to the bisector of the angle P meets the perpendicular to QR at X in I. If N is the mid. point of PQ, prove that $NI = NQ = IX$. (The lower sign is to be taken when X lies in RQ produced.)

(3) Employ (1) and (2) to establish the tangency of the nine-point circle of a triangle with the in-circle and ex-circles of the triangle.



LIST OF JOURNALS RECEIVED IN THE LIBRARY

(during September and October 1925.)

- 1 American Journal of Mathematics, **147**, 3.*
- 2 American Mathematical Monthly, **32**, 7.
- 3 Annales de l'observatoire de Paris **1** to **10**, **11**, 1, 2 **12** to **18**
and **20** to **31**.
- 4 Annals of Mathematics, **26**, 4.
- 5 Astrophysical Journal, **61**, 5 and **62**, 1.
- 6 Bulletin of the American Math. Society, **31**, 8 and General
Index for Vols. **21** to **30** (2 copies).
- 7 Bulletin of the Calcutta Math. Society, **16**, 1 (2 copies)
- 8 Bulletin des Sciences Mathematiques: Fevrier, Mars, Avril, Mai
and Aout, 1925.
- 9 Contribucion al Estudio de Las Ciencias Fisicas y Mate-
maticas, Serie Técnica **3** Entrega **2a**, 65.
- 10 Japanese Journal of Mathematics, **1**, 4 and **2**, 1.
- 11 Jahrbuch über die Fortschritte der Mathematic, **1** to **34** and
43 to **47**, also **36**, 3.
- 12 Messenger of Mathematics, **55**, 2, 3 (2 copies each).
- 13 Monthly Notices of the Royal Astronomical Society, **85**, 7.
- 14 Nature, **116**, 2913, 2914, 2915.
- 15 Phil. Mag. and Journal of Science, **50**, 297 (Sept. 1925).
Also July 1911, July 1913, Jan. 1914, July 1915, Nov. 1916,
Dec. 1916, April 1921, Aug. 1922, Nov. 1922, Dec. 1922.
- 16 Popular Astronomy, **33**, 7 (2 copies).
- 17 Proc. of the Camb. Phil. Soc., **22**, 5. [Presentation copy].
- 18 Proc. of the London Math. Soc. **24**, 4. (3 copies).
- 19 Proc. of the Royal Society, **198**, A 748.
- 20 Rendiconti del Circolo Matematico di Palermo, **49**, 1.
- 21 Tohoku Mathematical Journal, **25**, 3, 4.
- 22 Trans. of the Amer. Math. Soc., **27**, 3.
- 23 Trans. of the Camb. Phil. Soc., **23**, 7.

Books and Pamphlets.

- 1 Sydney University Reprints, Series XI.
- 2 University of Madras: The Calendar for 1925-26, Vol. II.

* Numbers in thick black type refer to the Volume, and in ordinary type to the number of the issue.

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