THE JOURNAL

OF THE

Indian Mathematical Society

Vol. XIII.] APRIL 1921. [No. 2

SOME APPLICATIONS OF HEAWOOD'S THEOREM*

This note deals with some applications of Heawood's Theorem, *viz.* "If $f(z) = 0$ is a cubic representing the vertices of a triangle, then $f'(z) = 0$ gives the foci of the maximum inscribed ellipse." The theorem has been discussed in this Journal by Prof. Naraniengar in Vol. IV, p. 96, and Vol. V, p. 14.

Notation. α, β, γ are the vectors to the angular points of a $\triangle$ ABC; z the vector to any point P is $(x + iy)$; and l, m, n are real qnantities.

The equation $\dfrac{l}{z-\alpha} + \dfrac{m}{z-\beta} + \dfrac{n}{z-\gamma} = 0$ has roots z_1, z_2 which are geometrically related to the $\triangle$ ABC and are denoted by the points S and H.

P is the point ζ, given by $\zeta = \dfrac{l\alpha + m\beta + n\gamma}{l+m+n}$.

1. The points S and H are isogonal conjugates, (Vide: Prof. Naraniengar's Note in Vol. V, p. 14 of the Journal),

The equation on reduction is

$$z^2(l+m+n) - z\{(l+m+n)\Sigma\alpha - \Sigma l\alpha\} + \Sigma l \cdot \beta\gamma = 0.$$

Taking the origin at A, $z_1 z_2 \cdot \Sigma l = l\,\beta\gamma$.

Hence AS, AH, have the same bisectors as AB, AC. Similarly for BS, BH. The product of the moduli are equal. Hence S, H are isogonal.

* *Quarterly Journal of Mathematics*, Vol. 33, p. 84.

$$| z_1 | . | z_2 | . \Sigma l = l , | \beta | . | \gamma | .$$

$$\therefore \quad AS . AH = \frac{l}{l + m + n} . AB . AC.$$

$$\therefore \quad \frac{AS . AH}{AB . AC} + ... + . = 1 \ [(Q. 254. (2)];$$

also $\frac{AS . BS . CS . AH . BA . CH}{a^2 - b^2 - c^2} = \frac{l - m - n}{(\Sigma l)^3} = \frac{1}{27}$.

When $l = m = n$, S, H are the foci of the maximum inscribed ellipse and we have Question 511.

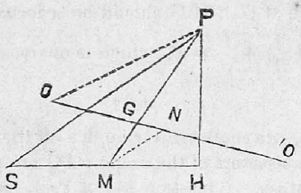
2. Δs ABC, PSH, have the same centroid, for

$$z_1 + z_2 = \frac{\Sigma l . \Sigma \alpha - \Sigma l \alpha}{\Sigma l} = \Sigma \alpha - \zeta.$$

$$\therefore \quad z_1 + z_2 + \zeta = \alpha + \beta + \gamma.$$

3. If M be the mid-pt. of SH, and N the centre of the nine-pt. circle, O the circumcentre, $MN = \frac{1}{2}OP$.

This follows from the similarity of the Δs OGP, MGN.



4. $OS . OH = R . OP$, where R is the circumradius.

Take the origin at O, the circumcentre. Then

$$\therefore \quad | \alpha | = | \beta | = | \gamma | = R.$$

$$z_1 . z_2 = \frac{\alpha \beta \gamma}{l + m + n} . \left(\frac{l}{\alpha} + \frac{m}{\beta} + \frac{n}{\gamma} \right) .$$

Put $\alpha = R . e^{i\theta_1}$; $\beta = R . e^{i\theta_2}$; $\gamma = R . e^{i\theta_3}$.

$$\therefore \quad z_1 . z_2 = \frac{1}{R^2} . \frac{\alpha . \beta . \gamma}{l + m + n} . \ (l . R e^{-i\theta_1} + ... + ...)$$

The modulus of $\frac{\Sigma l . R e^{-i\theta_1}}{\Sigma l}$ is the same as that of $\frac{\Sigma l . R e^{i\theta_1}}{\Sigma l}$, which latter is OP (the two points being reflections about x-axis).

$$\therefore \quad | z_1 | . | z_2 | = \frac{1}{R^2} . | \alpha | . | \beta | . | \gamma | . OP.$$

$$\therefore \qquad \text{OS . OH} = \text{R . OP},$$

and $$\text{OP} = 2\,.\,\text{MN} = 2d,$$

where d is the distance of the centre of the inconic from the nine-point centre.

Hence $$\text{OS . OH} = 2\text{R}\,.\,d. \quad \text{(See Vol. III, p. 68)}.$$

Similarly we can establish the result

$$(l+m+n)\,\frac{\text{KS . KH}}{\text{KA . KB . KC}} = \left[\Sigma\frac{l^2}{\text{KA}^2} + \Sigma\frac{mm(\text{KB}^2+\text{KC}^2-\text{BC})}{\text{KB}^2\,.\,\text{KC}^2}\right],$$

K being any point in the plane.

5. If AG, BG, CG are produced to meet the circumcircle in A′B′C′, show that G is a focus of the maximum inscribed ellipse of the $\triangle$ A′B′C′. (Question 351 of Prof. Naraniengar).

The foci of the max. inscribed ellipse of the $\triangle$ A′B′C′ are given by

$$\frac{1}{z-\alpha'} + \frac{1}{z-\beta'} + \frac{1}{z-\gamma'} = 0, \ (l = m = n).$$

where α', β', γ' are the vectors to the points A′, B′, C′.

Choose the origin at G. If G should be a focus, $z = 0$ is a root of the above ; and $\frac{1}{\alpha'} + \frac{1}{\beta'} + \frac{1}{\gamma'} = 0$, which is obviously true since G is the centroid of ABC.

6. If BCP, BCQ are equilateral triangles on the base BC of a triangle ABC, show that the bisectors of the angle PAQ are parallel to the axes of the max. inscribed ellipse. (Question 444 of Prof. Swaminarayan).

Take A as origin the points P, Q as δ, δ' and S, H the foci of the max. inscribed ellipse.

The equation giving S, H is

$$3z^2 - 2z(\beta + \gamma) + \beta\gamma = 0. \qquad \ldots \qquad \ldots \qquad \text{(i)}$$

If θ, β, γ form an equilateral triangle,

$$\theta^2 + \beta^2 + \gamma^2 - \theta\beta - \theta\gamma - \beta\gamma = 0.$$

(See Hardy's *Pure Mathematics*).

Hence δ, δ' are the roots of the above equation.

$$\therefore \ \delta + \delta' = \beta + \gamma,$$
$$\delta\delta' = \beta^2 + \gamma^2 - \beta\gamma.$$

The direction SH is that of the vector $(z_1 - z_2)$, $z_1\,z_2$ being roots of (i)

$$\therefore \ z_1 - z_2 = \tfrac{1}{3}\sqrt{\beta^2 - \beta\gamma + \gamma^2 -} \ \ldots \text{ from (1)}.$$

$$\therefore \ \delta\delta' = 9\,(z_1 - z_2)^2.$$

Hence one of the bisectors of the angle PAQ, is parallel to SH.

That is, the bisectors of $\angle$ PAQ are parallel to the axes of the ellipse.

Also the product of the moduli being equal,

$$AP \,.\, AQ = 9 \,.\, SH^2.$$

7. $\overline{AS} \cdot \overline{BS} \cdot \overline{CH} + \overline{AH} \cdot \overline{BH} \cdot \overline{CH} = 2\overline{AG} \cdot \overline{BG} \cdot \overline{CG}$.

(Question 478 of Mr. M. T. Naraniengar).

S, H are the roots of

$$3z^2 - 2z(\alpha + \beta + \gamma) + \alpha\beta + \beta\gamma + \gamma\alpha = 0.$$

Take the origin at the centroid G, then $\alpha + \beta + \gamma = 0$.

$\therefore$ S, H, are the points $+ z$, $- z$ given by

$$3z^2 + \alpha\beta + \beta\gamma + \gamma\alpha = 0.$$

Now $\overline{\alpha + z} \cdot \overline{\beta + z} \cdot \overline{\gamma + z} + \overline{\alpha - z} \cdot \overline{\beta - z} \cdot \overline{\gamma - z}$

$= (\alpha\beta\gamma + z \cdot \Sigma\beta\gamma + z^2 \cdot \Sigma\alpha + z^3) + (\alpha\beta\gamma - z \cdot \Sigma\beta\gamma + z^2 - z^3)$

$= 2 \cdot \alpha \cdot \beta \cdot \gamma$., on simplification.

Hence the result.

N. Durai Rajan.

THE THEORY OF RATIONAL TRANSFORMATION.

By R. Vythyanathaswami.

1. The general Rational Transformation is of the form

$$x' = \mathrm{R}(x) \equiv \frac{f(x)}{\phi(x)},$$

$$f(x) = \sum_{r=0}^{r=n} a_r\, x^{n-r},\quad \phi(x) = \sum_{r=0}^{r=n} b_r\, x^{n-r},$$

n being the *order* of the transformation. The roots of $x = \mathrm{R}(x)$, $(n+1)$ in number, are the *fixed points*. The transformation R involves $(2n + 1)$ effective constants; that is, n constants other than the fixed points.

The pencil $[f(x) - \lambda\, \phi(x)]$ where λ is a parameter is called the pencil associated with R or briefly the pencil of R. Two transformations having the same pencil will be said to be *congruent*. It is obvious that any transformation congruent to R is of the form SR where S is homographic.

The Rational Transformation R effects an (1, 1) correspondence between the number x' and the members of the pencil of R, the x' corresponding to any assigned member being the transform by R of any of the roots of the member, — this transform being the value of the parameter λ corresponding to the member.

2. If $\alpha_0\ \alpha_1 \ldots \alpha_n$ are the fixed points (supposed distinct) of R (x), the equation $x' = \mathrm{R}(x)$ can be written in the form

$$\sum_{r=0}^{r=n} k^r\, \frac{x' - \alpha_r}{x - \alpha_r} = 0. \quad \ldots \quad \ldots \quad \ldots \quad (1).$$

The numbers k_r will be called the *parameters* of the transformation.

Note.—If all the parameters are finite and their sum is zero, the transformation is the identical transformation $x' = x$.

The value of $\dfrac{dx'}{dx}$ at a fixed point is called a *multiplier*. When the fixed points are all distinct, the multiplier at α_r is seen to be

$$m_r = 1 - \frac{\Sigma(k)}{k_r}.$$

The multipliers $m_0\ m_1 \ldots m_n$ thus satisfy the identical relation

$$\Sigma \frac{1}{1-m_r} = 1.$$

Note.—When the transformation is homographic, the identical relation becomes $m_0 m_1 = 1$, which is obviously true.

When the fixed points $\alpha_0\ \alpha_1 \ldots \alpha_{p-1}$ coalesce at α, equation (1) becomes

$$k\,\frac{x'-a}{x-a} + (x'-x)\sum_{1}^{p-1}\frac{k_r}{(x-a)^{r+1}} + \sum_{p}^{n}\frac{k_r\,(x'-a_r)}{(x-a_r)} = 0.$$

k may be called the principal parameter at a and $k_1, k_2 \ldots k_{p-1}$, the secondary parameters. The multiplier at a is unity and at a_p is

$$-\frac{(k + k_{p+1} + k_{p+2}\ldots)}{k_p}.$$

The multipliers are thus functions of the principal parameters only.

In this case there is no identical relation between the multipliers.

Note (1).—If R and R′ have a common fixed point α with multipliers m, m' respectively, then α is a fixed point of RR′ (and also R′R) with the multiplier mm^1.

Note (2).—The multipliers of R are the same as those of S^{-1} RS where S is homographic. For, let a be a fixed point and m the corresponding multiplier of R. Let $S(a + p) = a + \lambda p$ where p is infinitesimal. Then $S^{-1}(a)$ is a fixed point of S^{-1} RS and the corresponding multiplier is m, because

$$S^{-}\ RS\left(a + \frac{p}{\lambda}\right) = S^{-1}R(\alpha+\rho) = S^{-1}(a+mp) = \alpha + m\frac{p}{\lambda}.$$

Note (3).—*The multiplier at an infinite fixed point.*

Let R have a fixed point at infinity so that

$$R(x) = \frac{a_0 x^n + a_1 x^{n-1} + \ldots}{b_1 x^{n-1} + b_2 x^{n-2} + \ldots}.$$

Let $S(x) = \frac{1}{x}$. Then $S^{-1}\,RS(x) = \frac{b_1 x + b_2 x^2 + \ldots}{a_0 + a_1 x + \ldots}$.

The multiplier of R at ∞ = the multiplier of S^{-1} RS at $0 = \frac{b_1}{a_0}$.

Note (4).—The multiplier at a finite fixed point a of

$$R(x) \equiv \frac{f(x)}{\phi(x)} \text{ is } \{f'(a) - a\phi'(a)\}/\phi(a).$$

For the polynomial transformation $R(x) \equiv f(x)$, the multiplier at the fixed point α is $f'(a)$ and at the fixed point ∞ is zero.

3. *The Derived or Polar Transformation.*

There is one type of Rational Transformation—which we term the Derived or Polar Transformation—which is completely determined by its fixed points. The derived transformation with respect to a polynomial $\theta(x)$ of the $(n+1)$th degree is

$$x' = R(x) = -\frac{d\theta}{dt} \Big/ \frac{d\theta}{dx}$$

= the root of the nth polar of $\theta(y)$ with respect to x; t being the usual unit variable. The fixed points of the transformation are given by $\theta(x) = 0$.

NOTE.—All the parameters of the derived transformation are equal; and all the multipliers are also equal each being $-n$, where n is the order of the transformation.

For, the equation $x' \frac{d\theta}{dx} + \frac{d\theta}{dt} = 0$ reduces to

$$\Sigma \frac{x' - a_r}{x - a_r} = 0, \text{ } a\text{'s being the roots of } \theta.$$

In the case when θ has multiple roots, the order of the derived transformation is equal to the number of distinct roots.

For, if a is an m-ple root of θ, $(x - a)^{m-1}$ is a common factor of $\frac{d\theta}{dx}$ and $\frac{d\theta}{dt}$. It will be noticed, that when this common factor is cancelled, α is only a simple fixed point of the reduced form of the transformation. Further, the parameter of the transformation at an m-ple fixed point is seen to be proportional to m and the corresponding multiplier is

$$\left(1 - \frac{n+1}{m}\right).$$

NOTE (1).—A rational transformation with distinct fixed points and parameters proportional to positive integers is a derived transformation.

NOTE (2).—The derived transformation of order 1 is the involution.

Those of orders 2 and 3 may be reduced to

$$x' = \frac{1}{x^2} \text{ and } x' = \frac{\lambda x^3 - 1}{x(x^2 - \lambda)}.$$

The Focal Rational Transformation.

When $\theta(x)$ possesses an apolar quadratic $\gamma(x)$, the derived transformation is *focal*, the roots of $\gamma(x)$ being termed the foci. The focal transformation transforms each focus into the other and transforms no point not a focus into a focus. Taking the foci as 0 and ∞, the transformation reduces to the form $x' = \frac{k}{x^n}$.

4. *Certain Families of Rational Transformation* ($n > 2$).

When the pencil of R is given, *three* of the fixed points of R may be chosen arbitrarily, but the rest are determinate ; so that the co-efficients of $\theta(x)$ are linear functions of *four* parameters. Conversely the pencil of a transformation with given fixed points can not be chosen arbitrarily.

Theorem :If $\theta(x) = 0$ gives the fixed points of R, then $\theta(x)$ is apolar to F_{2n-2} which is the unique form of the $(2n-2)$th degree apolar to every member of the pencil of R. We have seen that $\theta(x)$ belongs to a four-parameter system of polynomials. To determine this system, suppose $f(x)$ to be a member of the pencil of R and two of its roots α, β (say) to be fixed points of R. Then $R(\alpha) = \alpha$ and $R(\beta) = \beta$. But $R(\alpha) = R(\beta)$ since α, β are the roots of $f(x)$. Hence α, β are each carried into more than one point by R, so that $(x - \alpha)(x - \beta)$ is a common factor of the numerator and denominator of R. Hence since the numerator and denominator are members of the pencil of R, they must contain the common factor $f(x)$, so that $R(x)$ reduces to the singular form $k\,\frac{f(x)}{f(x)}$.

Hence whenever *two* of the roots of $\theta(x)$ are roots of $f(x)$ a member of the pencil of R, $\theta(x)$ is of the form $f(x) \,.\, (x - k)$ where k may be arbitrary. In other words the four-parameter system of polynomials to which $\theta(x)$ belongs, is that four-parameter system which has the pencil of R for a singular pencil.

Now the four-parameter system of $(n+1)$ -ics apolar to F_{2n-2} is also a system having the pencil of R for a singular pencil.

Hence $\theta(x)$ is apolar to F_{2n-2}.

Theorem : If $R_1(x)$ is a particular transformation whose fixed points are given by $\theta(x) = 0$ and the pencil of R_1 be the pencil determined by two n-ics which are written in the forms

$$\theta(x) \sum \frac{b_r}{x - \alpha_r} \text{ and } \theta(x) \sum \frac{b'_r}{x - \alpha_r},$$

where α's are the roots of θ, then the pencil of any other transformation whose fixed points are α's is the pencil determined by two n-ics of the forms

$$\theta(x) \sum \frac{k_r\, b_r}{x - \alpha_r} \text{ and } \theta(x) \sum \frac{k_r\, b'_r}{x - \alpha_r}.$$

Dem.—Since the pencil $\theta(x)\sum\frac{b_r+\lambda b'_r}{x-\alpha_r}$ is the pencil of R_1, its apolar, $(2n-2)$ -ic viz. F_{2n-2} is apolar to $\theta(x)$.

Thus F_{2n-2} is of the form $\sum A_r\ (x-\alpha_r)^{2n-2}$.

The condition that this is apolar to $\theta(x)\sum\frac{b_r+\lambda b'_r}{x-\alpha_r}$ for all values of λ is

$$\sum A_r\,(b_r+\lambda b'_r)\,(x-\alpha_r)^{n-2}\prod(\alpha_r-\alpha_s)\equiv 0.$$

This equation is unaltered if we write $k_r b_r$ for b_r, $k_r b'_r$ for b'_r and A_r/k_r for A_r.

Thus the pencil $\theta(x)\sum k_r\frac{b_r+\lambda b'_r}{x-\alpha_r}$ is the apolar pencil of some F_{2n-2} apolar to $\theta(x)$; that is, it is the pencil of some rational transformation whose fixed points are given by $\theta(x)$.

Cor. Considering R_1 as the derived transformation defined by $\theta(x)$. Its pencil is the pencil of first polars of $\theta(x)$ and is determined by $\theta(x)\sum\frac{1}{(x-\alpha_r)}$ and $\theta(x)\sum\frac{\alpha_r}{x-\alpha_r}$. Hence the pencil of any transformation whose fixed points are roots of $\theta(x)$ is the pencil determined by two n-ics of the form $\theta(x)\sum\frac{k_r}{x-\alpha_r}$ and $\theta(x)\sum\frac{k_r\alpha_r}{x-\alpha_r}$ where the k_r's are arbitrary.

Geometrical Interpretation.

Represent every n-ic as a point in n dimensions. The n-ics which are perfect nth powers will then correspond to points on a twisted n-ic curve Γ. An n-ic $(x-\alpha_1)(x-\alpha_2)\ \ldots\ldots\ (x-\alpha_n)$ corresponds to the point of intersection of the osculating $(n-1)$ -dimensional regions at $P_1P_2\ \ldots\ldots\ P_n$ where P_r is the point on Γ corresponding to $(x-\alpha_r)^n$. By Para (1), the general Rational transformation is an (1, 1) correspondence between Γ and an arbitrary straight line. The fixed points are the points (α_r) on Γ the osculating planes at which intersect the straight line in the corresponding point. If the points (α_r) are given, the possible straight lines form an n-ply infinite family such that any line of the family is the transform of a fixed line L of the family by some collineation whose fixed $(n+1)$ -hedron Δ' is the one formed by the osculating planes at (α_r). This complex of ∞^n straight lines is evidently the generalisation of the tetrahedral complex in three dimensions.

As in the corollory above the line L may be taken to be the line which represents the pencil of first polars.

Theorem: The line L which represents the pencil of first polars is ths locus of poles with respect to Δ' of osculating planes of Γ.

The equation in plane co-ordinates of the twisted n-tic with respect Δ' may be taken to be

$$L_r = \frac{\lambda_r}{t - \alpha_r} \quad (r = 0, 1 \ldots\ldots n)$$

t representing the parameter of the current point on the curve. The equation to L which is the locus of poles with respect to Δ' of osculating planes of Γ, is then

$$X_r = \frac{t - \alpha_r}{\lambda_r}.$$

If the osculating plane at t_1 on the curve passes through the point t on the line

$$\Sigma \frac{\lambda_r}{t_1 - \alpha_r} \cdot \frac{t - \alpha_r}{\lambda_r} = \Sigma \frac{t - \alpha_r}{t_1 - \alpha_r} = 0.$$

Thus the transformation whose pencil is represented by the line is a derived transformation, so that the line represents the pencil of first polars.

The line L may be called the polar line of Δ with the respect to Γ.

5. *Argand Representations.*

(1) Transformations with real multipliers (and therefore real parameters).

Let $k_0\ k_1 \ldots k_n$ be the (real) parameters corresponding to the fixed points $\alpha_0\ \alpha_1 \ldots \alpha_n$ respectively of the transformation R. With the point z as centre describe a circle in the Argand diagram and let α'_r be the inverse of α_r with respect to this circle.

The inverse with respect to the same circle of the mean centre of $\alpha'_0\ \alpha'_1 \ldots \alpha'_n$ for multiples $k_0\ k_1 \ldots k_n$ is the point R (z).

Dem. If T represent the operation of reflection in the real axis, then

$$(z - \alpha'_r) = \mathrm{T}\left(\frac{1}{z - \alpha_r}\right).$$

If M is the mean centre

$$(z - \mathrm{M}) = \frac{\Sigma k_r(z - \alpha'_r)}{\Sigma k_r}.$$

Therefore, if z' is the inverse of M,

$$(z - z') = \mathrm{T}\left(\frac{1}{z - \mathrm{M}}\right) = \Sigma k_r \,.\, \mathrm{T}\,.\left[\Sigma k_r \frac{1}{(z - a'_r)}\right]$$

$$= \Sigma k_r \,.\, \mathrm{T}\left[\frac{1}{\Sigma k_r \mathrm{T}(z' - a_r)}\right] = \frac{\Sigma k_r}{\Sigma \dfrac{k_r}{z - a_r}},$$

so that $$\Sigma k_r \frac{z' - a_r}{z - a_r} = 0.$$

This construction is suitable for all derived transformations.

(2) The above construction may be extended to any transformation whatever, provided we define the mean centre for multiples $m_0\, m_1 \ldots m_r$ of points $z_0 \ldots z_r$ in the Argand plane to be the point $z = \dfrac{\Sigma m_r z_r}{\Sigma m_r}$, even in case the m's are complex.

6. *Powers of a Rational Transformation.*

The rth power of a Rational Transformation R of order n is a rational transformation of order n^r, which may be denoted by R^r. The $n^r + 1$ fixed points of R^r include obviously the fixed points of R and the fixed points of R^d where d is any divisor of R. Those fixed points of R^r which are not fixed points of R^d are the *special* fixed points of R^r. The special fixed points are $n^{(r)}$ in number, where

$$n^{(r)} = n^r - \Sigma n^{r/p_1} + \Sigma n^{r/p_1 p_2} - \text{etc.}$$

$p_1, p_2, \ldots$ being the prime factors of r. (For proof: see my 'Note in Combinatory Analysis,' J. I. M. S.)

$n^{(r)}$ is divisible by r. (Extension of Fermat's Theorem, *Ibid*).

Hence the $n^{(r)}$ special fixed points can be divided into sets of r points each set being carried into itself cyclically and therefore *primitively* by R. Such a set may be called a *primitive* set of R and the corresponding polynomial a *primitive* polynomial. Any polynomial carried into itself by R is either primitive or the product of primitive polynomials.

The number of primitive r-ics of R is $\dfrac{n^{(r)}}{r}$.

Theorem: The multipliers of R^r corresponding to the points of a primitive r-set, are equal to the same rational function of the co-efficients of the corresponding primitive r-ic.

Let $\alpha_1\, \alpha_2 \ldots \alpha_r$ be a primitive r-set so that $\alpha_2 = R(\alpha_1), \ldots \alpha_k = R(\alpha_{k-1})$ and $\alpha_1 = R(\alpha_r)$.

Then $R(\alpha_1 + d\alpha_1) = \alpha_2 + d\alpha_2, \ldots R(\alpha_k + d\alpha_k) = \alpha_{k+1} + d(\alpha_{k+1})$ $[k = 2 \ldots (r-1)]$ and $R(\alpha_r + d\alpha_r) = \alpha_1 + d\alpha'_1$.

The multiplier of R^r at α_1

$$= \frac{d\alpha'_1}{d\alpha_1} = \frac{d\alpha'_1}{d\alpha_r} \cdot \frac{d\alpha_r}{d\alpha_{r-1}} \cdots \frac{d\alpha_2}{d\alpha_1}$$

$$= \alpha_1 \cdot \left(\frac{f'(\alpha_r)}{f(\alpha_r)} - \frac{\phi'(\alpha_r)}{\phi.\alpha_r)} \right) \times \alpha_r \left(\frac{f'(\alpha_{r-1})}{f(\alpha_{r-1})} - \frac{\phi'(\alpha_{r-1})}{\phi(\alpha_{r-1})} \right) \times \ldots$$

$$\left(\text{where } R(x) \equiv \frac{f(x)}{\phi(x)} \right)$$

$$= \alpha_1\, \alpha_2 \ldots \alpha_r \prod \left(\frac{f'(\alpha_k)}{f(\alpha_k)} - \frac{\phi'(\alpha_k)}{\phi(\alpha_k)} \right)$$

$=$ a symmetric function of $\alpha_1\, \alpha_2 \ldots \alpha_r$,

which proves the theorem.

The multipliers of R^r can be divided into the following sets :—

(1) The multipliers at the special fixed points of R^r. There are $\frac{n^{(r)}}{r}$ such multipliers, each multiplier corresponding to a primitive r-ic in virtue of the theorem proved. Call these $M_{r_1}, M_{r_2} \ldots$

Each fixed point, not a special fixed point of R^r, must be a special fixed point of one and only one power of R, say R^d, where d is a divisor of r.

Hence the other multipliers can be classified into—

(2) The multipliers of R^r at the special fixed points of R^d. These multipliers are the $\left(\frac{r}{d}\right)^{\text{th}}$ powers of the multipliers of R^d at the same points. That is, in our previous notation $\left(M^d_k\right)^{r/d}$ $\left(k = 1, 2, 3 \ldots \frac{n^{(d)}}{d}\right)$ each multiplier corresponding to a primitive d-set.

(3) The multipliers at the fixed points of R. These are the rth powers of the multipliers of R.

Thus the problem of finding all the multipliers of R^r reduces to that of finding the multipliers at the special fixed points.

Ex. (1) Deduce from simple considerations a proof of $\sum n^{(d)} = n^r$, where the values of d are the divisors of r (unity and r included).

Ex. (2) To find $R^2(x)$, where

$$a_1 \frac{R(x) - \alpha_1}{x - \alpha_1} + a_2 \frac{R(x) - \alpha_2}{x - \alpha_2} + a_3 \frac{R(x) - \alpha_3}{x - \alpha_3} = 0.$$

Let $m_1\ m_2\ m_3$ be the multipliers of R and $(\beta\gamma)$ the primitive pair.

The multipliers of R^2 at β, γ are equal to the same number m and at $\alpha_1\ \alpha_2\ \alpha_3$ are $m_1^2\ m_2^2\ m_3^2$, m is thus determined by the equation

$$\frac{1}{1 - m_1^2} + \frac{1}{1 - m_2^2} + \frac{1}{1 - m_3^2} + \frac{2}{1 - m} = 1.$$

This determines m as a function of $m_1\ m_2\ m_3$, where

$$m_1 = \frac{-(a_2 + a_3)}{a_1},\ \text{etc.}$$

Hence $R^2(x)$ is given by

$$\sum_{r=1}^{3} \frac{1}{1 - m_r^2} \cdot \frac{(R^2(x) - \alpha_r)}{x - \alpha_r} + \frac{1}{1 - m} \left[\frac{R^2(x) - \beta}{x - \beta} + \frac{R^2(x) - \gamma}{x - \gamma} \right] = 0.$$

7. *Powers of the Derived Transformation.*

If $\theta(x) = 0$, represent the fixed points of a derived transformation, then any primitive r-ic is a covariant of $\theta(x)$ but not a rational covariant. However *the product of all primitive r-ics is a rational covariant of* $\theta(x)$ (which we may call the rth primitive covariant). To prove this assume this to be true for all values of r up to a given one r. The fixed points of the rth power of the derived transformation is a rational covariant of $\theta(x)$, say L_x. If d is a divisor of r, the dth primitive covariant (which by hypothesis is rational) is a factor of L_x. On removing all such factors, the part which remains in L_x is a rational covariant and is the rth primitive covariant. Thus the rth primitive covariant is a rational covariant—shewing that the theorem is true for r. Since obviously the theorem is true for $r = 2$, the induction is complete.

Ex. (1) The primitive quadratic of a derived quadratic transformation is the Hessian of $\theta(x)$.

For since a cubic is apolar to its Hessian, either root of the Hessian must be the second polar of the other.

Ex. (2) The primitive quadratics of the derived cubic transformation are the apolar quadratic factors of the sextic covariant of $\theta(x)$.

Let $\alpha\beta\gamma\delta$ be the roots of $\theta(x)$ and (pq) the pair harmonic to $(\alpha\beta)$ such that $(\alpha\beta pq)$ is apolar to $(\alpha\beta\gamma\delta)$, (pq) is thus a covariant of $(\alpha\beta)$ and

$(\gamma\delta)$ and must therefore be the common harmonic pair of $(\alpha\beta)$ and $(\gamma\delta)$. Thus $(pq\alpha\beta)$ and $(pq\gamma\delta)$ are both apolar to $(\alpha\beta\gamma\delta)$ and therefere $(pppq)$ and $(pqqq)$ are both apolar to $(\alpha\beta\gamma\delta)$. Thus either of p, q is the third polar of the other, shewing that (pq) is a primitive pair.

The following are examples of this theorem.

Ex. (3) To find pairs of points on the rational space-quartic, such that the osculating plane at each passes through the other.

Let $f(t) = 0$, represent parametrically the super-osculation points. Then if the osculating plane at the point t cuts the curve again in the point t', t' is the derived cubic transformation of t determined by $f(t)$. Thus the pairs sought are given by the quadratic factors of the sextic covariant of $f(t)$.

Ex, (4) To find pairs of points on an ellipse the osculating circle at each of which passes through the other.

Let $\theta(t)$ represent parametrically the axial extremities of the ellipse. If $f(t) = 0$ represent four concyclic points on the curve, then obviously f is apolar to θ. Thus if the osculating circle at t cuts tne ellipse in t', t' is the derived transformation of t determined by $\theta(t)$. The pairs sought are therefore the quadratic factors of the sextic covariant of $\theta(t)$, *i.e.*, the points at ∞ on the ellipse and the extremities of the equiconjugate diameter.

8. *Reducible Pencils.*

A pencil Γ_n of n-ics is said to be *reducibls* over a pencil Γ_p of p-ics (where p is a factor of n other than 1 or n) if every member of Γ_n is the product of members of Γ_p.

The Jacobian of Γ_n is then the product of the Jacobian of Γ_p and of $\left(\frac{2n}{p} - 2\right)$ members of Γ_p. Conversley it is easy to see that Γ_n is reducible over Γ_p whenever the Jacobian of Γ_n is such a product.

From the definition of reducibility it follows that Γ_n is certainly irreducible if n ia a prime.

Ex. (1) A pencil of quartics is reducible if it contains two perfect squares.

For if Γ_4 is reducible over Γ_2, the Jacobian of Γ_4 is the product of the Jacobian of Γ_2 and of two members P_1, P_2 of Γ_2. Obviously P_1^{2}, P_2 are members of Γ_4.

Ex. (2) A pencil Γ_{pq} is reducible if it contains two perfect qth powers.

Let $P_1{}^q$, $P_2{}^q$ be members of Γ_{pq}, P_1, P_2 being of order p. $P_1{}^q - \lambda P_2{}^q$ is resoluble into q factors of the form $P_1 - \mu P_2$ and therefore Γ_{pq} is reducible over the pencil (P_1, P_2).

Ex. (3) An example of a pencil of n-ics which is reducible over a pencil of order *p for every divisor p of n*, is the pencil of vertices of n-gons inscribed in one conic, circumscribed to another.

Ex. (4) A pencil of quartics if reducible over more than one pencil of quadratics, is reducible over *three* such pencils and is a standard pencil.

For if Γ_4 is reducible over two distinct pencils, it must contain three perfect squares P^2, Q^2, R^2 (since it cannot contain four). Thus Γ_4 is reducible over each of the three pencils (P, Q), (Q, R), (R, P). Obviously Γ_4 is a standard pencil.

9. *Reducible Transformations.*

A rational transformation (or function) $R(x)$ of order n is said to be reducible if $R(x) \equiv R_p\{R_q(x)\}$ where R_p, R_q are rational functions of orders p, q respectively ($p, q \neq 1$). If R is reducible, any transformation congruent to R is also reducible; for if $R = R_pR_q$ then $SR = SR_p . R_q$ where S is a homography. Thus the reducibility or irreducibility of R is determined purely by the pencil of R.

Theorem : The necessary and sufficient condition for the reducibility of R is the reducibility of the pencil of R.

Firstly, let $R = \dfrac{f}{\phi}$ and the pencil $f + \lambda\phi$ be reducible over a pencil of order p $(n = pq)$. Then

$$f \equiv a_0P_1{}^q + a_1P_1{}^{q-1}P_2 + \ldots\ldots + a_qP_2{}^q$$

$$\phi \equiv b_0P_1{}^q + b_1P_1{}^{q-1}P_2 + \ldots\ldots + b_qP_2{}^q$$

where P_1, P_2 are p-ics.

Hence $R \equiv R_1R_2$,

$$\text{where } R_2(x) = \frac{P_1(x)}{P_2(x)} \quad R_1(x) = \frac{a_0x^q + a_1x^{q-1} \ldots + a_q}{b_0x^q + b_1x^{q-1} \ldots + b_q}.$$

Secondly, let $R \equiv R_1 R_2$, $R = \frac{f}{\phi}$, $R_1 = \frac{f_1}{\phi_1}$, $R_2 = \frac{f_2}{\phi_2}$.

Then the equation $f + \lambda\phi = 0$ is equivalent to $f_1(R_2) + \lambda\phi_1(R_2) = 0$ and therefore the pencil of R is reducible over the pencil of R_2.

Note.—A rational function of prime order is certainly irreducible. When a rational function R is reducible, the reduction can be performed in a triply infinite number of ways: For if $R = R_1 R_2$, then R is also equal to $R_1 S \,.\, S^{-1} R_2$ where S is homographic.

If we call R_2 the pre-factor and R_1 the post-factor of R, then all the possible pre-factors are congruent to one another and the pencils of the post-factors are linearly transformable into one another.

Ex. (1) A reducible quartic transformation can be expressed in infinity ways as the product of two congruent transformations.

For S can be chosen in infinity ways so as to make the pencil of $R_1 S$ identical with that of R_2.

Hence there is an infinity of quadratic transformations R_1 such that $R_1{}^2$ is congruent to a given reducible quartic transformation.

[This is an attempt to study the general Rational Transformation of one variable in some of its most characteristic aspects. More questions have been raised than solved; and the positive results obtained are few. In spite of the fragmentary character of the article, it is believed that it will be of some value to readers interested in the subject.]

THE GROUP-THEORY ELEMENT OF THE HISTORY OF MATHEMATICS.

BY PROFESSOR G. A. MILLER.

(*Continued from p.* 12.)

FOR about a century mathematicians studied these special groups with only occasional glimpses into their deeper meanings and wider applications. E. Galois, A. L. Cauchy, A. Cayley, and W. R. Hamilton made references to these deeper meanings, especially as regards an abstract theory, but none of these men formulated the abstract laws governing this theory. About 1870, an eminent triumvirate of mathematicians, C. Jordan, S. Lie and F. Klein, began to exhibit the applications of the group concept to new fields. In his "Traité des Substitutions" (1870) and in an article on the groups of movements (1868), C. Jordan made fundamental geometric applications, which were greatly extended by F. Klein. About the same time S. Lie founded a new theory of continuous groups of transformations and made extensive applications of these groups in the theory of differential equations and in other mathematical subjects.

It may be of interest to note that during the first, or implicit period, of the development of our subject, groups involving an infinite number of elements exercised the greatest influence. During the second, or specialization period, the attention was centered on groups of a finite number of elements, while during the third, or generalization period, groups involving an infinite number of elements again moved to the foreground, but groups of finite order continued to receive considerable attention. Two types of groups of infinite order were studied during this period, *viz.*, those in which the transformations were continuous and those in which the transformations were discontinuous.

The fundamental abstract notions involved in group theory are so elementary that they can be easily understood by those who are not professional mathematicians. Hence it is the more interesting that these notions were not explicitly formulated before 1870. In formulating these for the special case where the elements obey the commutative law when they are combined, L. Kronecker expressed himself as follows : "The extremely simple principles upon which the method of Gauss is founded, find applications not only in the place named but also in others, and, indeed, already in the elementary parts of the theory of numbers. This circumstance points to the fact, about which it is easy to convince oneself, that the said principles belong to a more general and more abstract sphere of ideas. Hence it appears appropriate to free their development from all non-essential limitations so that one will be spared the trouble of repeating the same method of reaching a conclusion in the different instances of its use. The advantage of this appears even in the development itself, and the presentation gains at the same time in simplicity, and, by the clear exhibition of the essentials only, also in distinctness when it is given in the most general permissible way."[1]

The student of the history of science may be especially interested in the fact that the formulation of a definition of an abstract group came so late in the development of this subject. For a full century mathematicians were dealing with special substitution groups before making a serious effort to develop an abstract theory embodying the fundamental principles of these groups as a special case. It was not until such an abstract theory was being developed that mathematicians began to see that the group concept had been a dominant factor in some of the most important early mathematical work and hence it became an important means not only for suggesting further advances but also for securing an insight into the large body of earlier mathematical developments.

A few statements found in well-known text-books may serve to illustrate the attitude of leading mathematicians at the beginning of the present century as regards the theory of groups. In the preface of his "Géometrie," 1905, E. Borel says:

> The new foundation (of elementary geometry) has been laid in the nineteenth century by the works of leading mathematicians. It consists of the recognition that elementary geometry is equivalent to the investigation of the group of movements. Such a view is in accord with the characteristic tendency of modern scientists to replace static investigations of the phenomena by dynamic; or, to speak in more general terms, the thought of development penetrates more and more our observations.

In his "Lehrbuch der Algebra" (kleine Ausgabe), 1912, page 180, H. Weber notes that:

> There are chiefly two large general concepts which dominate modern algebra. The existence and importance of these concepts could be observed only after algebra was completed to a certain extent, and had become the property of the mathematicians. Only then could be observed the combining and guiding principles. These are the concepts of groups and of domains (koerper) which we now proceed to explain. The more general of these is the concept of group.

In his "Berührungstransformationen" 1914, page 11, H. Liebmann makes the following statement:

> The rules and concept development of group theory may be compared with the organizing laws of nature according to which crystals arise. If it is allowed to

---

[1] L. Kronecker, Berlin *Monatsberichte*, 1831, p. 882.

continue the figure of speech it may be added that the remaining mother liquor is a rich fostering soil on which luxuriant organized life unfolds itself.

These quotations may suffice to indicate in a general way to what extent group-theory influenced the trend of mathematical progress since the beginning of the third period of its development. The infinite number of finite groups, each of which exhibits special laws of operations, which had been discovered during the second period of the development of this subject, showed that this theory can never be completely mastered in its details. There are, however, large categories of groups which have many properties in common and whose common operational laws throw light on other mathematical developments.

Comparatively little progress has been made in the study of those abstract properties which all groups have in common, yet it is just these common properties which were popularized by the mathematical literature of the last quarter of the nineteenth century. While they are so simple that the ancients did not consider it necessary to mention them explicitly it was found that they furnish a point of view which offers many advantages. For instance, few mathematical terms are more useful than the term equivalent, and one of the services which group-theory has rendered is to give this term a flexible yet perfectly definite meaning by noting that the equivalence of two objects implies that one can be transformed into the other by the operations of a certain group.

Hence the term equivalent is relative to the group under consideration. For instance, in Euclidean geometry two figures are equivalent if they can be made to coincide by operations of the group composed of displacements and symmetries. The distance between any two points is an absolute invariant under this group. On the other hand, in elementary geometry two figures are equivalent when they can be transformed into each other by the operators of the group composed of the similarity transformations which includes the preceding group as an invariant sub-group. In elementary geometry all circles are equivalent, and all squares are equivalent, but this is not true in Euclidean geometry.

Euclid's "Elements" could have been enriched not only by the explicit use of groups of infinite order but also by the introduction of groups of finite order. In particular, the five regular solids which play an important rôle in Greek mathematics and in Greek philosophy represent three interesting groups of finite order. In the words of E. Picard:

A regular polyhedron, say an icosahedron, is on the one hand the solid that all the world knows; it is also, for the analyst, a group of finite order, corresponding to the divers ways of making the polyhedron coincide with itself. The investigation of all the types of groups of motion of finite order interests not only the geometers, but also the crystallographers; it goes back essentially to the study of groups of ternary linear substitution of determinant unity, and leads to the thirty-two systems of symmetry of the crystallography for the particular complex.

While it seems impossible to establish the reasons why Euclid did not make explicit use of groups of finite and of infinite order in his "Elements," the fact that Aristotle frequently expressed the view that mathematics has to do with the *immovable* objects except such as relate to astronomy, is suggestive. While movements were used to illustrate the demonstrations of theorems the Greek philosophers seemed to hold the view that geometry itself was essentially a static subject. It is difficult to overestimate the great influence which this view had on the later history of mathematics.

If Euclid had emphasized in his "Elements" the dynamic rather than the static elements of mathematics, it is likely that his work would have exerted a more vigorous influence. The cube of Euclid, for instance, is of great interest but it is not so inspiring as the cube composed of the twenty-four movements of space which leave Euclid's cube invariant. These movements affect all space and convey big and far-reaching notions. Moreover, they suggest many questions as regards sub-groups and abstract laws of operation. In particular, this group of order 24 is completely defined by the fact that it contains two operators of orders 2 and 3 respectively whose product is of order 4.

While a group-theory of the third century B. C. is conceivable, it could not have been the group-theory of the nineteenth century since the latter century had a much richer mathematical heritage. The rapid strides of group-theory during the last century were largely due to the utilization of old results, as is always the case in generalizations by abstraction. The soil had been prepared by the labors of earlier centuries and it was only necessary to sow on it the new seed to secure the bountiful harvest with which the labors of many workers in this field were rewarded, especially during the last decades of the ninteenth century.

When group-theory appeared explicitly, it naturally took a form which was in accord with the spirit of the times. Substitution groups constitute a type of combinatory analysis and arose about the time when the Combinatorial School flourished in Germany under the leadership of C. F. Hindenburg (1741-1808). Abstract group-theory is a type of postulational mathematics and its early development during the middle of the preceding century was in the van of the postulational activity which was so prominent during the second half of the nineteenth century. Continuous and geometric group-theory are mainly applied group-theory and their rapid development during the last quarter of the preceding century is in accord with the spirit of this age when the fear of mathematical isolation through over-specialization tended to make the study of applications especially popular.

SOLUTIONS.

Question 1102.

(M. BHIMASENA RAO) :—P is the inverse of the incentre of a triangle ABC with respect to the circum-circle of ABC. Show that the isogonal conjugate of P with respect to ABC lies on the common diameter of the in-circle and the nine-point circle of ABC.

Solution by N. Sundaram Aiyar and C. N. Sreenivasa Iyengar.

Let O be the circumcentre, I the incentre and N the nine-point centre of $\triangle$ ABC. Then OI . OP = R^2 (R being the circum-radius). The trilinear co-ordinates of O and I are (R cos A, R cos B, R cos C) and (r, r, r) respectively. Let α, β, γ be the co-ordinates of P.

Then obviously

$$\frac{r - R\cos A}{OI} = \frac{\alpha - R\cos A}{OP} = \frac{\sqrt{(r - R\cos A)(\alpha - R\cos A)}}{R}.$$

$$\therefore \quad \frac{\alpha - R\cos A}{r - R\cos A} = \frac{R^2}{OI^2} = \frac{R}{R - 2r}.$$

$$\therefore \quad \alpha = \frac{R^2\cos A - 2R\gamma\cos A + R\gamma - R^2\cos A}{R - 2r}$$

$$= \frac{R\gamma}{R - 2\gamma} \cdot (1 - 2\cos A).$$

The co-ordinates of P are thus proportional to $(1-2\cos A)$, $(1-2\cos B)$ and $(1-2\cos C)$.

The co-ordinates of its isogonal conjugate are proportional to

$$\frac{1}{1 - 2\cos A}, \ \frac{1}{1 - 2\cos B}, \ \frac{1}{1 - 2\cos C}.$$

To prove that this point is on a line with I and N whose co-ordinates are proportional to (1, 1, 1) and [cos (B — C), cos (C — A), cos (A — B)], respectively, we are to show that

$$\begin{vmatrix} 1 & 1 & 1 \\ \cos(B - C) & \cos(C - A) & \cos(A - B) \\ \dfrac{1}{1 - 2\cos A} & \dfrac{1}{1 - 2\cos B} & \dfrac{1}{1 - 2\cos C} \end{vmatrix} = 0.$$

Now $\Sigma(1 - 2\cos A)\,[\cos \overline{C - A}\,(1 - 2\cos C)$

$= \Sigma(1-2\cos A)\,[\cos \overline{C - A} - \cos \overline{A - B} - (\cos \overline{2C - A}\,(\cos \overline{A - 2B})]$

$= \Sigma(\cos \overline{C - A} - \cos \overline{A - B}) - \Sigma(\cos C - \cos B) - \Sigma[\cos \overline{C - 2A}$

$- \cos(2A - B) - \Sigma(\cos \overline{2C - A} - \cos \overline{A - 2B}$

$+ \Sigma(\cos 2C - \cos 2B) + \Sigma(\cos \overline{2C - 2A} - \cos \overline{2A - 2B}) = 0.$

Hence the result.

Similar solution by M. K. Kewalramani.

Question 1104.

(M. Bhimasena Rao):—A circle cuts the sides of the triangle of reference at angles α, β, γ. Show that it cuts the nine-point circle at the angle θ given by the equation

$$\cos\theta(a\cos\alpha + b\cos\beta + c\cos\gamma)$$
$$= \Sigma\, a\cos A\sin^2\alpha + \Sigma a\cos\beta\cos\gamma + \ldots\,.$$

If $\theta = \alpha + \beta + \gamma$, show that either

(i) $\sin A\sin\alpha + \sin B\sin\beta + \sin C\sin\gamma = 0$,

or (ii) $\cos A\sin\alpha + \cos B\sin\beta + \cos C\sin\gamma + \sin(\alpha+\beta+\gamma) = 0$.

Interpret these results geometrically.

Solution by N. Sundaram Aiyar.

The trilinear co-ordinates of the centre of the circle of radius r, which cuts the sides of the Δ of reference at angles, α, β, γ are

$$r\cos\alpha,\ r\cos\beta,\ r\cos\gamma,$$

where $$r\,(a\cos\alpha + b\cos\beta + c\cos\gamma) = 2\,\Delta.$$

If d be the distance of this centre from the nine-point centre, we have

$$d^2 = r^2 + \frac{R^2}{4} - 2r\,.\,\frac{R}{2}\cos\theta. \quad \ldots \quad \ldots \quad (1)$$

But the nine-point centre is the point

$$\frac{R}{2}\cos(B - C),\ \frac{R}{2}\cos(C - A),\ \frac{R}{2}\cos(A - B).$$

$$\therefore\ d^2 = \frac{abc}{4\,\Delta^2}\ \Sigma a\cos A\left(r\cos\alpha - \frac{R}{2}\cos\overline{B - C}\right)^2$$

$$= \frac{-abc}{4\,\Delta^2}\,.\,\Sigma a\left(r\cos\beta - \frac{R}{2}\cos\overline{C - A}\right)\left(r\cos\gamma - \frac{R}{2}\cos\overline{A - B}\right)$$

$$= \tfrac{1}{2}\,.\,\frac{abc}{4\,\Delta^2}\,[\Sigma a\cos A\,.\,r^2(1 - \sin^2\alpha) - rR\Sigma a\cos\alpha\cos A\cos\overline{B - C}$$

$$+\frac{R^2}{4}\sum a\cos A\cos^2\overline{B-C}-r^2\sum a\cos\beta\cos\gamma$$

$$-\frac{R^2}{4}\sum a\cos\overline{C-A}\cos\overline{A-B}+\frac{rR}{4}\sum(a\cos\beta\cos\overline{A-B}$$

$$+a\cos\gamma\cos\overline{C-A})]$$

$$=\frac{R}{2\Delta}[r^2.\frac{2\Delta}{R}-r^2(\sum a\cos A\sin^2\alpha+\sum a\cos\beta\cos\gamma)$$

$$+\frac{rR}{2}\sum\cos\alpha\,(c\cos\overline{C-A}+b\cos\overline{A-B}-2a\cos A\cos\overline{B-C})$$

$$+\frac{R^2}{4}\sum(a\cos A\cos^2\overline{B-C}-a\cos\overline{C-A}\cos\overline{A-B})].$$

But $c\cos\overline{C-A}+b\cos\overline{A-B}-2a\cos A\cos\overline{B-C}$

$$=R(\sin\overline{2C-A}+\sin A+\sin A+\sin\overline{2B-A}-\sin\overline{2A+B-C}$$

$$-\sin(2A-B+C)=2R\sin A=a.$$

Also $\sum\{a\cos A\cos^2\overline{B-C}-a\cos\overline{C-A}\cos(A-B)\}$

$$=\sum\frac{a\cos A}{2}+\frac{\sum a\cos A\cos\overline{2B-2C}}{2}-\sum R\sin A\,(\cos\overline{C-B}+\cos\overline{C+B-2A})$$

$$=\frac{\Delta}{R}+\frac{R}{4}\sum(\sin\overline{2A+2B-2C}+\sin\overline{2A-2B+2C})$$

$$-\frac{R}{2}\sum(\sin\overline{A+C-B}+\sin\overline{A-C+B}+\sin\overline{C+B-A}+\sin\overline{3A-B-C})$$

$$=\frac{\Delta}{R}+2R\sin 2A\sin 2B\sin 2C-2R\sin A\sin B\sin C-2R\cos\pi\sin 2A$$

$$=0.$$

$$\therefore\quad d^2=r^2-\frac{Rr^2}{2\Delta}(\sum a\cos A\sin^2\alpha+\sum a\cos\beta\cos\gamma)$$

$$+\frac{rR^2}{4\Delta}\sum a\cos\alpha=r^2-rR\cos\theta+\frac{R^2}{4}.$$

$$\therefore\quad\cos\theta=\frac{r}{2\Delta}(\sum a\cos A\sin^2\alpha+\sum a\cos\beta\cos\gamma)$$

or $$\cos\theta\sum(a\cos\alpha)=\sum a\cos A\sin^2\alpha+\sum a\cos\beta\cos\gamma. \qquad (2)$$

If $\theta=\alpha+\beta+\gamma$, this result becomes

$$(\sum a\sin\alpha)(\sum\cos A\sin\alpha)+\sum a\sin^2\alpha\cos(\beta+\gamma)$$

$$+\sum a\sin\alpha\cos\alpha\sin(\beta+\gamma)=0,$$

or $$(\sum a\sin\alpha)(\sum\cos A\sin\alpha+\sin\overline{\alpha+\beta+\gamma})=0.$$

Hence either

(i) $\sin A\sin\alpha+\sin B\sin\beta+\sin C\sin\gamma=0,$

or (ii) $\cos A\sin\alpha+\cos B\sin\beta+\cos C\sin\gamma+\sin(\alpha+\beta+\gamma)=0.$

Question 1109.

(SALDHANA AND MADHAVA).—Prove that

$$e^{\lambda_1}\left[f(x)\right] = \left(1+\frac{1}{x}\right)f(x+1) \qquad \text{(Bombay, B.A.)}$$

where $\lambda_1 = \left(\frac{d}{dx}+\frac{1}{x}\right)$;

and extend the result as follows :—

$$e^{\lambda_2}\left[\tfrac{1}{2}f(x)\right] = \left(\frac{1}{2}+\frac{1}{x}+\frac{1}{2x^2}\right)f(x+1)$$

$$e^{\lambda_3}\left[\tfrac{1}{3}f(x)\right] = \left(\frac{1}{3}+\frac{1}{x}+\frac{1}{x^2}+\frac{1}{3x^3}\right)f(x+1)$$

where $\lambda_2 = \frac{d}{dx}+\frac{2}{x}$ and $\lambda_3 = \frac{d}{dx}+\frac{3}{x}$.

Solution by A. V. Subba Rau; K. J. Sanjana and several others.

The three results to be proved are particular cases of

$$e^{\lambda r}\left[f(x)\right] = \left(1+\frac{1}{x}\right)^r f(x+1) \text{ where } \lambda_r = \frac{d}{dx}+\frac{r}{x}$$

r being any positive integer.

Let f, f_1, f_2, f_3 ...... etc., be $f(x)$ and its derived functions with regard to x.

Then $\lambda_r f = f_1 + \frac{r}{x}f$,

$$\lambda_r^2 f = f_2 + \frac{r}{x}f_1 + \frac{r}{x}f_1 - \frac{r}{x^2}f + \frac{r^2}{x^2}f$$

$$= f_2 + 2\,.\,\frac{r}{x}f_1 + \frac{r(r-1)}{x^2}f,$$

$$\lambda_r^3 f = \lambda_r f_2 + 2\,.\,\lambda_r\left[\frac{r}{x}f_1\right] + r(r-1)\,r_r\left[\frac{1}{x^2}f\right]$$

$$= f_3 + \frac{r}{x}f_3 + \frac{2r}{x}f_2 - \frac{2r}{x^2}f_1 + \frac{2r^2}{x^2}f_1$$

$$+ \frac{r(r-1)}{x^2}f_1 - \frac{2r(r-1)}{x^3}f + \frac{r^2(r-1)}{x^3}f$$

$$= f_3 + 3\,.\,\frac{r}{x}f_2 + 3\,.\,\frac{r(r-1)}{x^2}f_1 + \frac{r(r-1)(r-2)}{x^3}f.$$

In general, we get

$$\lambda_r^{\,n} \; f = f_n + {}_nC_1 \frac{r}{x} f_{n-1} + {}_nC_2 \frac{r(r-1)}{x^2} f_{n-2} + \quad \dots \quad \dots \quad \dots \quad (1)$$

the last term being

(i) $\;{}_nC_n \; \dfrac{r(r-1)(r-2)\;\dots\;(r-n+1)}{x^n} f$, if $n < r$;

(ii) $\;\dfrac{r!}{xr} \; f$, if $n = r$;

(iii) $\;{}_nC_r \; \dfrac{r\,!}{x^r} f_{n-r}$, if $n > r$.

To prove this expansion of $\lambda_r^n f$ in terms of f_n, f_{n-1}, f_{n-2}, etc., we assume it in the case of n and deduce it in the case of $n+1$ as follows :—

Now $\lambda_r^{\,n+1} \; f = \left(\dfrac{d}{dx} + \dfrac{r}{x}\right) \lambda_r^{\,n} \; f.$

and the general term in $\lambda_r^{\,n+1} \; f$

$$= {}_nC_{s-1}\, r\,(r-1)\,(r-2)\,(r-s+2)\, f_{n-s+1} \left(\frac{d}{dx} + \frac{r}{x}\right) \frac{1}{x^{-1}}$$

$$+ \; {}_nC_s \; \frac{r(r-1)\,(r-2)\,(r-s+1)}{x^s} f_{n-s+1}$$

$$= {}_nC_{s-1} \quad \frac{r(r-1)\,(r-2)\,(r-s+2)\,(r-\overline{s-1})}{x^s} f_{n-s+1}$$

$$+ \; {}_nC_s \; \frac{r(r-1)\,(r-2)\;\dots\,(r-s+1)}{x^s} f_{n-s+1}$$

$$= ({}_nC_{s-1} + {}_nC_s) \frac{r(r-1)\,(r-2)\;\dots\;(r-s+1)}{x^s} f_{n-s+1}$$

$$= (n+1)\, C_s \; \frac{r(r-1)\;\dots\;(r-s+1)}{x^s} f_{n+1-s}\,.$$

Hence by Mathematical Induction the expansion of $\lambda_r^{\,n} f$ as given in (1) above, is true.

Now $$e^{\lambda_r} \; f = \sum \frac{\lambda_r^{\,n}}{n\,!} \; f,$$

and $\displaystyle\sum \frac{f_n}{n\,!}\,, \; \sum \frac{f_{n-1}}{n-1}\,, \; \sum \frac{f_{n-2}}{n-2!}$ etc. are all equal and each is equal to $f\,(x+1)$, by Taylor's Theorem.

$$\therefore\ e^{\lambda r}\left[f(x)\right] = \left(1 + \frac{{}_rC_1}{x} + \frac{{}_rC_2}{x^2} + \frac{{}_rC_3}{x^3} + \ldots \frac{{}_rC_r}{x^r}\right) f(x+1)$$

$$= \left(1 + \frac{1}{x}\right)^r f(x+1).$$

By putting $r = 1, 2$ and 3 the three results required to be proved are got.

Similar solution by O. N. Sreenivasa Aiyangar, V. M. Gaitonde, N. Sundaram and others.

Question 1110.

(A. NARASINGA RAO) :—If for a curve there exists a functional relation connecting the area of every escribed triangle, and that of the triangle formed by the points of contact, such a curve must be a straight line or a parabola.

Solution by N. Sundaram.

Since the functional relation is true for all inscribed Δs, let us take the inscribed triangle formed by three consecutive points on the curve. Let P, Q, R be the points forming the inscribed Δ of area Δ_1. Let C be the corresponding centre of curvature, and ρ the radius of curvature. Then the area of the triangle LNM formed by the tangents at P, Q, R to the circle of curvature will differ from that formed by tangents at P, Q, R to the curve by an infinitesimal of lower order than Δ_1. Let Δ_2 be the area of the escribed triangle.

It is obvious from a figure that

$$\Delta_1 = \tfrac{1}{2}.\ 2\rho \sin\frac{\theta}{2}\,.\,2\rho \sin\frac{\phi}{2}\sin\frac{\theta+\phi}{2} = 2\rho^2 \sin\frac{\theta}{2}\,.\,\sin\frac{\phi}{2}\sin\frac{\theta+\phi}{2},$$

and $$\Delta_2 = \rho^2 \tan\frac{\theta}{2}\,.\,\tan\frac{\phi}{2}\,.\,\tan\frac{\theta+\phi}{2},$$

where θ, ϕ stand for the angles PCQ, QCR.

Thus since θ, ϕ are small, we have as the functional relation

$$\Delta_1 = 2\Delta_2.$$

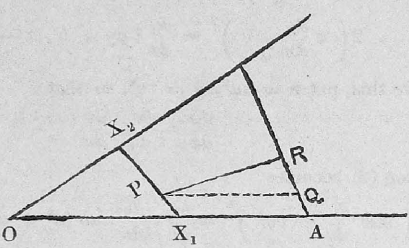
Hence either $\Delta_1 = \Delta_2 = 0$, which is obviously the case when the curve is a straight line; or the curve is a parabola which satisfies the relation $\Delta_1 = 2\,\Delta_2$.

For if $(at_1^2,\ 2at_1)$, $(at_2^2,\ 2at_2)$, $(at_3^2,\ 2at_3)$ be the vertices of the inscribed Δ, $[at_1t_2,\ a(t_1+t_2)\ ;\ at_1t_3,\ a(t_1+t_3)\ ;\ a\,t_2t_3,\ a(t_2+t_3)]$ are the vertices of the corresponding escribed triangle.

$$\therefore\ \Delta_1 = \tfrac{1}{2}a\,.\,2a \begin{vmatrix} t_1^2 & t_1 & 1 \\ t_2^2 & t_2 & 1 \\ t_3^2 & t_3 & 1 \end{vmatrix} = a^2(t_1 - t_2)(t_1 - t_3)(t_2 - t_3)$$

$$\text{and } \Delta_2 = \tfrac{1}{2}a^2 \begin{vmatrix} t_1t_2 & t_1+t_2 & 1 \\ t_2t_3 & t_2+t_3 & 1 \\ t_3t_1 & t_3+t_1 & 1 \end{vmatrix} = \tfrac{1}{2}a^2(t_1-t_2)(t_2-t_3)\times(t_3-t_1)$$

$\therefore$ $\Delta_1 = 2\,\Delta_2$, in a parabola.



[We can prove that the only form of the curve other than a straight line is a parabola, as under :

Let OA and OB be two fixed tangents to the curve at A and B. Let X_1X_2 be a variable tangent to the curve at P. It will be shown that P can trace only a parabola.

Let OA and OB be the axes of co-ordinates, and the equations to AB and X_1X_2 be $\dfrac{X}{a}+\dfrac{Y}{b}=1$ and $\dfrac{X}{x_1}+\dfrac{Y}{x_2}=1$, respectively. Let P be (x,y).

Now Δ APB $= 2\,\Delta\, OX_1X_2$ as above.

$\therefore$ $\frac{1}{2}AB\,.\,p = x_1x_2 \sin A\hat{O}B$, where p is the perpendicular PR on AB.
PQ $= p_1$ (say) is drawn parallel to OA to cut AB at Q.

$\therefore$ $\frac{1}{2}AB\,.\,p_1 \sin A = x_1x_2 \sin A\hat{O}B$.

But $$\frac{AB}{\sin A\hat{O}B} = \frac{b}{\sin A},$$

$$\therefore\quad bp_1 = 2x_1x_2. \qquad \ldots \qquad \ldots \qquad \ldots \qquad (1)$$

And length PQ or p_1 is $a\left(1-\dfrac{y}{b}\right)-x$.

Writing $\frac{X}{x_1} + \frac{Y}{x_2} = 1$ in the form $Y - y = \frac{dy}{dx}(X - x)$ and remembering that $\frac{x}{x_1} + \frac{y}{x_2} = 1$, we have

$$x_1 = x - \frac{y}{y'}, \text{ and } x_2 = -y'\left(x - \frac{y}{y'}\right).$$

Substituting in (1) for x_1, x_2 and p_1, we have

$$-2\left(x - \frac{y}{y'}\right)^2 y' = b\left\{a\left(1 - \frac{y}{b}\right) - x\right\},$$

or

$$2\left(x\frac{dy}{dx} - y\right)^2 = \frac{dy}{dx}\{ay + b.(x - a)\}. \qquad \ldots \qquad (2)$$

To solve this, put $x = au^2$, $y = bv^2$, so that

$$\frac{dy}{dx} = \frac{bv}{au} \cdot \frac{dv}{du}.$$

Equation (2) becomes

$$\left(buv\frac{dv}{du} - bv^2\right)^2 = \frac{bv}{au} \cdot \frac{dv}{du} \cdot ab\,(u^2 + v^2 - 1);$$

or

$$2uv\left(u\frac{dv}{du} - v\right)^2 = \frac{dv}{du}\,(u^2 + v^2 - 1), \quad \ldots \qquad \ldots \qquad (2')$$

which is satisfied by $v + u = 1$.

Hence the original equation is satisfied by

$$\sqrt{\frac{x}{a}} + \sqrt{\frac{y}{b}} = 1. \qquad \ldots \qquad \ldots \qquad (3)$$

Since the differential equation is of the first order, (3) should be a solution got for a particular value of the single arbitrary constant which the complete primitive can have. But since our curve is to touch the axes at distances a and b respectively and (3) does so, it is the locus of P.]

Question 1112.

(C. KRISHNAMACHARI, M.A.) :—If V is the normal velocity drawn outwards to a closed surface S in a liquid, show that

$$\iint \rho V dS + \iiint \frac{d\rho}{dt}\, dx\, dy\, dz = 0,$$

where the volume integral extends throughout the volume enclosed by S, and the surface integral over the surface S.

Hence establish the equation of continuity.

Solution by Martyn M. Thomas and S. R. Ranganatham.

Let ρ be the density at any point (x, y, z) of the fluid inside the given closed surface S, so that ρ is a function of x, y, z.

Then $\iiint \rho\, dx\, dy\, dz$ being the mass of the fluid within S we have

$$\frac{\partial}{\partial t}\iiint \rho\, dx\, dy\, dz.\ \delta t = \text{increase in mass inside the surface in time } \delta t$$

$= \text{excess of flow in, over flow out, across the surface S in time } \delta t$

$$= \Sigma\left[x\, dy\, dz\,.\,\rho - \left(x + \frac{\partial x}{\partial t}\right)\delta t\,.\,dy\, dz\, \rho\right]$$

+ two similar terms, the summation being extended to points all over the bounding surface S

$= -\iint [u\rho dydz + v\rho dzdx + w\rho dxdy]\,\delta t$, where u, v, w are the components of the normal velocity V on the surface parallel to co-ordinate axes

$= -\iint (u\rho\,.\,ldS + v\rho\,.\,mdS + w\rho\,.\,ndS)\,\delta t$, where l, m, n are the direction cosines of the normal to S

$$= -\iint (l\rho u + m\rho v + n\rho w)\ dS.\,\delta t \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

$$= -\iint [\rho\,(l^2V) + \rho(m^2V) + \rho(n^2V)]\, dS\,.\,\delta t = -\iint (\rho V)\ dS.\,\delta t.$$

$$\therefore\ \left\{\frac{\partial}{\partial t}\iiint \rho\, dxdydz + \iint \rho V dS\right\}\delta t = 0.$$

Hence
$$\iiint \frac{\partial \rho}{\partial t}\, dxdydz + \iint \rho V dS = 0.$$

To deduce the equation of continuity :

Green's Theorem states that

$$\iint (lf + mg + nh)\, dS = \iiint \left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}\right) dxdy\, dz,$$

where f, g, h are functions of (x, y, z), which, with their first derivatives, are finite and continuous through a region bounded by a closed surface S.

Hence from the equality (1) obtained above, we have

$$\frac{\partial}{\partial t}\left[\iiint \rho dxdydz\right]\delta t = -\iint [l\,(\rho u) + m(\rho v) + n(\rho w)]\ dS.\,\delta t$$

$$= -\iiint \left[\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z}\right] dxdydz \,.\, \delta t,$$

using Green's Theorem.

$$\therefore \iiint \left[\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z}\right] dxdydz = 0. \quad \ldots (2)$$

Since equation (2) is true for all ranges of integration within the fluid, we have $\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0$, at every point of the fluid ; and this is the equation of continuity, in Cartesian co-ordinates.

Question 1113.

(SELECTED):—An ellipse, of eccentricity $\sin 2\alpha$. passes through the focus of a parabola of latus-rectum $4a$ and has its foci on the curve. Show that the major axis envelopes the parabola

$$y^2 = 4a(1 - \tan^2\alpha)(x - a)\tan^2\alpha) ;$$

and that the minor axis envelopes the semi-cubical parabola

$$27a \sec^2\alpha\; y^2 = 4(x - a - a\sec^2\alpha)^3.$$

Solution by A. A. Krishnaswami Aiyangar and Martyn M. Thomas.

Let S be the focus of the parabola and P, P′ the foci of the ellipse. Denote P, P′ by the parameters t, t' respectively.

Since $$\frac{\text{SP}+\text{SP}'}{\text{PP}'} = \frac{1}{e} = \frac{1}{\sin 2\alpha},$$

$$\frac{2 + t^2 + t'^2}{(t-t')\sqrt{[(t+t')^2+4]}} = \frac{1}{\sin 2\alpha} = \frac{1+\tan^2\alpha}{2\tan\alpha}.$$

$$\therefore \quad \tan\alpha = \frac{t-t'}{\sqrt{(t+t')^2+4}} \text{ or } \frac{\sqrt{(t+t')^2+4}}{t-t'}.$$

$$\therefore \quad tt' = \left(\frac{t+t'}{2}\right)^2 (1-\tan^2\alpha) - \tan^2\alpha,$$

taking only the first value. ... ... (i)

The equation of PP′ is

$$x + att' = y.\frac{t+t'}{2},$$

i.e., $$x + a(1-\tan^2\alpha)\left(\frac{t+t'}{2}\right)^2 - a\tan^2\alpha = y\frac{t+t'}{2},$$

after substituting from (i) ;

$$i.e., \quad x - a\tan^2\alpha - y.\frac{t+t'}{2} + a\,(1-\tan^2\alpha)\left(\frac{t+t'}{2}\right)^2 = 0,$$

whose envelope is easily seen to be

$$y^2 = 4a\,(1-\tan^2\alpha)\,(x - a\tan^2\alpha).$$

If we take the second value of $\tan\alpha$, we can get similarly the envelope $y^2 = 4a\,(1-\cot^2\alpha)\,(x - a\cot^2\alpha)$, which is not given in the question.

Again, to find the envelope of the minor-axis, we may write its equation in the form

$$y + (x-2a)\frac{t+t'}{2} = \frac{a}{2}.\frac{t+t'}{2}.(t^2+t'^2)$$

$$= a\sec^2\left(\frac{t+t'}{2}\right)^3 + a\tan^2\alpha.\frac{t+t'}{2} \text{ on substitution from (i) ;}$$

$$i.e., \quad a\sec^2\alpha\left(\frac{t+t'}{2}\right)^3 - (x-2a-a\tan^2\alpha)\frac{t+t'}{2} - y = 0.$$

Considering this relation as an equation in the variable $\frac{t+t'}{2}$, we easily find its envelope to be

$$27a\sec^2\alpha\, y^2 = 4\,(x-2a-a\tan^2\alpha)^3$$
$$= 4\,(x-a-a\sec^2\alpha)^3.$$

Similary, we can get another envelope of the form

$$27a\,\operatorname{cosec}^2\alpha y^2 = 4\,(x-a-a\operatorname{cosec}^2\alpha)^3.$$

It can be easily verified that the locus of the centre of the ellipse is the parabola $\quad y^2 = 4a\cos^2\alpha\,(x - a\tan^2\alpha)$.

Similar Solutions by N. B. Mitra, K. B. Madhava, Martyn M. Thomas and several others.

Question 1114.

(T. P. Trivedi, M.A., LL.B.) :—If p, q are any two integers selected at random ; prove that that the probability that the fraction (p/q) is in its lowest terms is $6/\pi^2$.

Solution by N. B. Mitra and A. Narasinga Rao.

The probability that p/q is in its lowest terms is the same as the probability that p and q may be prime to each other.

Let a be any prime number, If any integer p be taken at random, the probability that p may have a factor a may be found thus :

At first suppose $p \ngtr m$. Two cases arise.

(1) Let m be a multiple of a say $= np$; there are n integers $\ngtr m$ which contain a as a factor, *viz*., the integers $a, 2a, \ldots\ldots na$: also the total number of integers $\ngtr m$ is m. Hence the probability that p may have a factor a is $n/m = 1/a$.

(2) Next, let m lie between na and $(n-1)a$, where n is an integer. In this case the number of integers $\ngtr m$ which have a factor a is $n-1$. Hence the probability that p may contain a as a factor is $(n-1)/m$ which lies between $1/a$ and $1/a - 1/m$. But as m is taken larger and larger $1/a$ and $1/a - 1/m$ tend to equality.

Hence the probability that an integer p taken at random may contain a as a factor is ultimately $1/a$.

Similarly the probability that another integer q taken at random may contain a as a factor is $1/a$.

Hence the probability that the two integers p, q taken at random may both contain a as a factor is $1/a^2$.

Therefore the probability that p, q may not have a common factor a is $1 - 1/a^8$.

Similarly the probability that p, q may not have a common factor b (where b is another prime) is $1 - 1/b^2$.

Hence the probability that p and q may not have a common factor which is either a or b is $(1 - 1/a^2)(1 - 1/b^2)$.

And so on.

Hence the probability that p and q may be prime to each other is $P = (1 - 1/a^2)(1 - 1/b^2)(1 - 1/c^2) \ldots\ldots$ where $a, b, c \ldots\ldots$ are the succession of natural primes.

To evaluate P.

Let S denote the convergent series $1 + 1/2^2 + 1/3^2 + \ldots\ldots$ multiply by $1 - 1/2^2$. The effect will be to remove all terms from the series whose denominators contain 2 as a factor. Thus

$$S(1 - 1/2^2) = 1 + 1/3^2 + 1/5^2 + \ldots\ldots$$

Next multiply this by $1 - 1/3^2$. This will remove from the right side all terms whose denominators are multiples of 3. Thus

$$S(1-1/2^2)(1-1/3^2)=1+1/5^2+1/7^2+1/11^2+1/13^2+\ldots\ldots$$

Proceeding in this way we shall arrive at

$$S(1-1/2^2)(1-1/3^2)\ldots\ldots(1-1/k^2)=1+1/l^2+\ldots\ldots$$

where 2, 3, ... k are the successive natural primes up to k and l is the prime next to k. We may make l as large as we please and then $1/l^2+\ldots$ will tend to 0, since it is less than the residue after $(l-1)$ term of the convergent series S.

Hence finally SP = 1. But $S=\pi^2/6$. Therefore $P=6/\pi^2$.

Note by G. V. Krishnaswami.

This problem is due to the Russian Mathematician Tchebycheff,

(i) The probability that *none* of the primes from 3 onwards is a common factor of p/q is

$$P=\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{5^2}\right)\left(1-\frac{1}{7^2}\right)\left(1-\frac{1}{11^2}\right)\ldots\textit{ad inf.}$$
$$=\frac{6}{\pi^2}\div\left(1-\frac{1}{2^2}\right).$$

(ii) Suppose furthermore, we were assured that none of the two primes 2, 3 was a common factor of both p and q. The probability that the fraction might not be reduced by division by one or more of the other primes is, from the above,

$$\frac{6}{\pi^2}\div\left[\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\right].$$

(iii) Generally if we know that none of the n primes $m_1, m_2, \ldots m_n$ was a common factor of both p and q, the probability that the fraction might not be reduced by division by one or more of the other primes is

$$\frac{6}{\pi^2}\div\left[\left(1-\frac{1}{m_1^2}\right)\left(1-\frac{1}{m_2^2}\right)\left(1-\frac{1}{m_3^2}\right)\ldots\ldots\left(1-\frac{1}{m_n^2}\right)\right].$$

Question 1115.

(T. P. TRIVEDI, M.A., LL.B.) := Prove that

(i) $$\frac{2^2-1}{2^2+1}\,\frac{3^2-1}{3^2+1}\,\frac{5^2-1}{5^2+1}\ldots\textit{ad inf.}=\frac{2}{5}.$$

(ii) $$\frac{2^4-1}{2^4+1}\,\frac{3^4-1}{3^4+1}\,\frac{5^4-1}{5^4+1}\ldots\textit{ad inf.}=\frac{6}{7}.$$

Solution by N. B. Mitra, H. R. Kapadia and C. N. Sreenivasa Iyengar and others.

[N.B. : *Master S. Chowla, aged 13, has sent a solution on similar lines.*—Ed.]

Let us consider the series $\frac{1}{1^n}+\frac{1}{2^n}+\frac{1}{3^n}+\frac{1}{4^n}$ ... *ad inf.* ... (1)

We know that it is convergent if $n>1$. Let us denote it by S_n.

Multiply the series by $1-\frac{1}{2^n}$. The result will be to deprive the series of all terms whose denominators are multiples of 2.

Thus, $$S_n\left(1-\frac{1}{2^n}\right)=1+\frac{1}{3^n}+\frac{1}{5^n}+\ldots$$

Let us multiply this by $\left(1-\frac{1}{3^n}\right)$, 3 being the prime next to 2. The result will be to deprive the series on the right of all terms whose denominators are multiples of 3.

Proceeding in this way we get

$$S_n\left(1-\frac{1}{2^n}\right)\left(1-\frac{1}{3^n}\right)\left(1-\frac{1}{5^n}\right)\ldots\left(1-\frac{1}{p^n}\right)=1+\frac{1}{q^n}+\ldots,$$

where 2, 3, 5, ... p are the primes up to p and q is the prime next to p Also $1/q^n+\ldots$ is less than the residue after $q-1$ terms of the convergent series (1) and hence can be made as small as we please by making q as large as we please.

Hence we get finally,

$$\left(1-\frac{1}{2^n}\right)^{-1}\left(1-\frac{1}{3^n}\right)^{-1}\left(1-\frac{1}{5^n}\right)^{-1}\ldots\text{ to }\infty=S_n \quad\ldots\quad (a)$$

Similarly

$$\left(1-\frac{1}{2^{2n}}\right)^{-1}\left(1-\frac{1}{3^{2n}}\right)^{-1}\left(1-\frac{1}{5^{2n}}\right)^{-1}\ldots\text{ to }\infty=S_{2n} \quad\ldots\quad (b)$$

Dividing (*b*) by (*a*) we get

$$\left(1+\frac{1}{2^n}\right)^{-1}\left(1+\frac{1}{3^n}\right)^{-1}\left(1+\frac{1}{5^n}\right)^{-1}\ldots\text{ to }\infty=\frac{S_{2n}}{S_n} \quad\ldots\quad (c)$$

Dividing (c) by (*a*), wet get, after clearing

$$\frac{2^n-1}{2^n+1}\cdot\frac{3^n-1}{3^n+1}\cdot\frac{5^n-1}{5^n+1}\cdot\ldots\textit{ ad inf.}=\frac{S_{2n}}{S_n^2}\ \ldots \quad\ldots\quad (A)$$

The questions proposed are particular cases of this for $n = 2$ and 4.

We know $\quad S_2 = \frac{\pi^2}{6}, \; S_4 = \frac{\pi^4}{90}, \; S_8 = \frac{\pi^8}{9450}.$

These can be successively calculated from the well-known formula

$$S_2 \frac{\pi^{2n-2}}{(2n-1)!} - S_4 \frac{\pi^{2n-4}}{(2n-3)!} + - (-1)^n S_{2n} = \frac{n\pi^{2n}}{(2n+1)!}.$$

Hence putting $n = 2, 4$ in (A) and substituting the above values, we get the results :—

(i) $\frac{2^2-1}{2^2+1} \cdot \frac{3^2-1}{3^2+1} \cdot \ldots\ldots = \frac{6^2}{90} = \frac{2}{5}$,

(ii) $\frac{2^4-1}{2^4+1} \cdot \frac{3^4-1}{3^4+1} \cdot \ldots\ldots = \frac{90^2}{9450} = \frac{6}{7}$.

Question 1117.

(PROF. K. J. SANJANA) :—If a given function of x, y, z be transformed by the substitutions

$$r = \tfrac{1}{2} \log (x^2 + y^2 + z)^2, \; \phi = \tan^{-1} \frac{y}{x}, \quad \theta = \tan^{-1} \frac{(x^2+y^2)^{\frac{1}{2}}}{z},$$

prove that the operation

$$(x^2 + y^2 + z^2) \left(\frac{\delta^2}{\partial x^2} + \frac{\delta^2}{\delta y^2} + \frac{\delta^2}{\delta z^2} \right)$$

is transformed into

$$\left(\frac{\delta^2}{\delta r^2} + \frac{\delta^2}{\delta \theta^2} + \operatorname{cosec}^2 \theta \frac{\delta^2}{\delta \phi^2} + \frac{\delta}{\delta r} + \cot \theta \frac{\delta}{\delta \theta} \right).$$

Solution by Martyn M. Thomas and several others.

Let $\qquad x^2 + y^2 + z^2 = R^2,$

Then the given relations are equivalent to :—

$R = e^r; \qquad x = R \sin \theta \cos \phi$

$\qquad\qquad\quad y = R \sin \theta \sin \phi$

$\qquad\qquad\quad z = R \cos \theta.$

It has been proved in Edward's *Diff. Calculus*, § 532, that

$$\frac{d^2V}{dx^2} + \frac{d^2V}{dy^2} + \frac{d^2V}{dz^2} = \frac{d^2V}{dR^2} + \frac{2}{R}\frac{dV}{dR} + \frac{1}{R^2}\frac{d^2V}{d\theta^2} + \frac{\cot}{R^2\,\theta}\frac{dV}{d} + \frac{1}{R^2} \operatorname{cosec}^2 \theta \frac{d^2V}{d\phi^2}.$$

$$\therefore \quad R^2\left(\frac{d^2V}{dx^2}+\frac{d^2V}{dy^2}+\frac{d^2V}{dz^2}\right)$$

$$= R^2\frac{d^2V}{dR^2}+2R\frac{dV}{dR}+\frac{d^2V}{d\theta^2}+\cot\theta\,\frac{dV}{d\theta}+\operatorname{cosec}^2\theta\,\frac{d^2V}{d\phi^2}$$

$$= \frac{d}{dR}\left[R^2\frac{dV}{dR}\right]+ \quad ,, \quad ,, \quad ,,$$

$$= \frac{d}{e^r dr}\left[e^{2r}\frac{dV}{e^r dr}\right]+ \quad ,, \quad ,, \quad ,,$$

$$= \frac{1}{e^r}\cdot\frac{d}{dr}\left[e^r\frac{dV}{dr}\right]+ \quad ,, \quad ,, \quad ,,$$

$$= \frac{d^2V}{dr^2}+\frac{dV}{dr}+ \quad ,, \quad ,, \quad ,,$$

Thus
$$(x^2+y^2+z^2)\left(\frac{d^2}{dx^2}+\frac{d^2}{dy^2}+\frac{d^2}{dz^2}\right)V$$
$$=\left(\frac{d^2}{dr^2}+\frac{d}{dr}+\frac{d^2}{d\theta^2}+\cot\theta\,\frac{d}{d\theta}+\operatorname{cosec}^2\theta\,\frac{d^2}{d\phi^2}\right)V.$$

Question 1118.

(N. Dorai Rajan) :—Show that in the cubical parabola $x^3 = a^2y$, all tangent chords are trisected by the y-axis.

Solution by V. M. Gaitonde and several others.

Tangent at any point (x, y) on the curve $X^3 = a^2Y$ is
$$X(3x^2) + Y(-a^2) + (-2a^2y) = 0.$$

Eliminating Y from these two equations, the abscissae of the points of intersection of the tangent chord with the curve are given by the cubic equation
$$X^3 - 3x^2X + 2a^2y = 0.$$

Let x correspond to the two coincident points at the point of contact P of the tangent, and x' to the other extremity Q of the tangent chord. Then
$$x + x + x' = \text{co-efficient of } X^2 = 0.$$

Let the tangent chord PQ intersect the y-axis in T.

Then, by similar triangles, $\dfrac{PT}{TQ} = \dfrac{x}{x'} = \dfrac{1}{2}$ numerically.

Hence T is a point of trisection of PQ.

Question 1120.

(N. Dorai Rajan):—Show that the curve whose (p, r) equation is $r = (a + p)$ is an involute of an involute of a circle. Find its polar equation and trace the curve.

Solution by N. B. Mitra, C. N. Sreenivasa Iyengar and several others.

If (r', p') be the pedal co-ordinates of a point on the evolute corresponding to the point (r, p) on the curve, we have

$$p'^2 = r^2 - p^2 \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

and

$$r'^2 = r^2 + \rho^2 - 2\rho p, \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

where ρ refers to the given curve and $= r\, dr/dp$.

Here the given curve is

$$r = a + p. \quad \ldots \quad \ldots \quad \ldots \quad (3)$$

$$\therefore \quad \rho = r,$$

and from (2), $\quad r'^2 = 2r\,(r - p) = 2ra.$

Also from (1), $p'^2 = a\,(r + p) = a\,(2r - a).$

$$\therefore \quad p'^2 + a^2 = r'^2.$$

Hence the pedal equation of the evolute of (3) is $r^2 = a^2 + p^2$, which is the involute of the circle $r = p$.

In other words, the evolute of $r = a + p$ is the involute of the circle $r = p$.

Thus $r = (a + p)$ is the involute of the involute of the circle $r = p$.

To obtain the polar equation of (3) :

We know that

$$\frac{ds}{d\theta} = \frac{r^2}{p} - \frac{r^2}{r-a}.$$

$$\therefore \quad \left(\frac{dr}{d\theta}\right)^2 + r^2 = \frac{r^4}{(r-a)}^2,$$

$$\therefore \quad \frac{dr}{d\theta} = \frac{r}{r-a}\sqrt{(a\,2r - a)}.$$

$$\therefore \quad \sqrt{a}\, d\theta = ar.\,(r - a) \,/\, \{\, r\,\sqrt{(2r - a)}\,\}.$$

Integrating,

$$\pm\ \theta = \text{vers} - 1\frac{a}{r} + \frac{\sqrt{2r - a}}{a},$$ which may be put into the form

$$r = \frac{a}{2}\operatorname{cosec}^2\phi\,;\ \ \theta = 2\phi + \cot\phi,$$

where ϕ is an arbitrary parameter.

QUESTIONS FOR SOLUTION

1161. (K. J. SANJANA) :—Prove that

$$\frac{n(n+1)\ldots(2n-1)}{n!}+\frac{n(n+1)\ldots 2n}{\overline{n+1}!}\cdot\frac{1}{2}+\frac{n(n+1)\ldots(2n+1)}{\overline{n+2}!}\cdot\frac{1}{2^2}+\ldots\ ad.\ inf.=2^{2n-1}.$$

1162. (K. J. SANJANA) :—Solve the following differential equations and explain their geometrical significance :—

(i) $x^2+y^2-\frac{2xy_1(1+y_1^2)}{y_2}+\frac{2y(1+y_1^2)}{y_2}=k^2;$

(ii) $x^2+y^2-\frac{2xy_1(1+y_1^2)}{y_2}+\frac{2y(1+y_1^2)}{y_2}+\frac{(1+y_1^2)^3}{y_2^2}=k^2;$

(iii) $y-mx+\frac{1+y_1^2}{y_2}+\frac{my_1(1+y_1^2)}{y_2}=0.$

Here y_1, y_2 stand for $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$, and k and m are given constants.

1163. (V. RAMASWAMI AIYAR) :—If a rectangular hyperbola passes through the incentre of a triangle and the feet of the perpendiculars drawn therefrom to the sides, prove that it cuts the inscribed circle again at the point which is diametrically opposite the Feuerbach point.

1164. [G. T. V.] :—O is the circumcentre of a triangle ABC. Points X, X′; Y, Y′; Z, Z′ are taken on BC, CA, AB respectively, such that O is the common incentre of the triangles AXX′, BYY′, CZZ′. Show that (1) their circumcircles intersect at the Euler point E of the triangle ABC; (ii) if these circles intersect again in points A′, B′, C′, then AA′, BB′, CC′ concur at the centre of the N.P. circle of the triangle ABC; (iii) E is the twin-point of O with respect to the triangle A′B′C′.

1165. (N. P. PANDYA) :—Find the lowest prime of 17 digits.

1166. (HEMRAJ) :—If n be prime and

$$\prod_{k=1}^{n-1}(x+k)=\sum_{k=0}^{n-1}A_k\,x^{n-k-1},$$

prove that

$$2A_k\equiv\left\{\sum_{r=1}^{\Sigma}(-1)^{r-1}\,n^r\;{}^{n-k+r-1}C_r\;A_{k-r}\right\},\ \left(\text{mod. } n^{r+3}\right),$$

where k and r are both odd and $k>(r+2)$.

1167. (HEMRAJ) :—If $I=(ae-4bd+3c^2)$ is a negative number ; prove that a necessary and sufficient condition that the roots of the biquadratic

$$ax^4 + 4bx^3 + 6cx^2 + 4dx + e = 0$$

with real co efficients may be concyclic in the Argand Diagram is that a root of its reducing cubic should be proportional to $\sqrt{I}$.

1168. (S. KRISHNASWAMI AIYANGAR) :—If $(x\ y)$ stands for $\text{sn}(x+y)\,\text{sn}(x-y)$, then

$$\frac{1-k(\alpha\beta)}{1+k(\alpha\beta)} \times \frac{1-k(\beta\gamma)}{1+k(\beta\gamma)} = \frac{1+k(\gamma\delta)}{1-k(\gamma\delta)} \times \frac{1+k(\delta\alpha)}{1+k(\delta\alpha)}.$$

1169. (S. KRISHNASWAMI AIYANGAR) :—

If $(x\ y) \equiv \dfrac{\text{dn}^2 x - \text{dn}^2 y}{1+k^2\text{dn}^2 x\,\text{dn}^2 y}$, show that

$$(\alpha\beta) + (\beta\gamma) + (\gamma\alpha) = k^2 (\alpha\beta)(\beta\gamma)(\gamma\alpha).$$

1170. (P. V. SESHU AIYAR) :—If y_r is the rth term in the expansion of $(p+q)^n$, show that

$$\sum_{r=1}^{n+1} (y_r \cdot r^s) = \frac{d}{dq}\left(q\frac{d}{dq}\right)\left(q\frac{d}{dq}\right) \ldots\ldots q(p+q)^n,$$

where $\dfrac{d}{dq}$ is repeated s times.

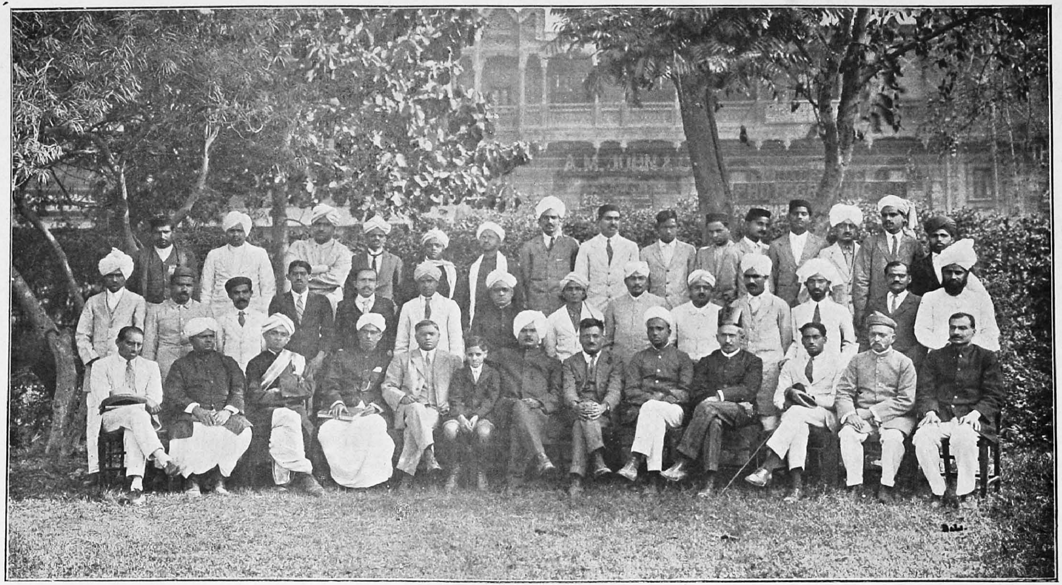
1171. (R. VAIDYANATHASWAMI) :—If $S + \lambda S'$ is a pair of lines for the values $a, \frac{1}{a}, -\frac{1}{a}$ of λ; prove that quadrilaterals can be inscribed in $S - aS'$, with their four sides touching respectively the conics

$$S + bS',\ S - bS',\ S + \frac{1}{b}S',\ S - \frac{1}{b}S'.$$

1172. (R. VAIDYANATHASWAMI) :—E is a conic of a given four-point system and S_1, S_2 are the two conics of the system which are inscribed to a self-conjugate Δ of E. Δs are inscribed in E with two of their sides touching S_1, S_2 respectively. Prove that the envelope of the third side is composed of the two conics of the system each of which has Δs inscribed in itself, circumscribed to the other.

INDIAN MATHEMATICAL SOCIETY.

3RD. CONFERENCE, MARCH 1921. LAHORE.



Govt. Photozinco: Office Poona—1921.

The Indian Mathematical Society.

THIRD CONFERENCE—MARCH 1921, LAHORE.

First Row (standing) from Left to Right.

Rulia Ram, Mulraj, S. B. Belekar, K. B. Madhawa, N. Raghunatha Aiyangar, K. S. Patrachari, Mehrchand Suri,
Bhagwan Das, Dhanpat Rai, Daulat Ram, Ishar Das, Mulk Raj, Sandhya Lal, M. A. Majid, Hukum Chand.

Second (middle) Row (standing).

Pandit Hemraj, A. B. Chandrachud, S. P. Shah, Ram Behari, M. K. Kewalramani, L. S. Vaidyanathan,
M. V. Arunachala Sastri, R. Vythianatha Swami, T. V. Mone, G. G. Pendse, Sardari Lal, Ghulam Abbas,
Mukund Lal, Moh'd Arshad Khan.

Third Row (sitting).

C. V. H. Rao, S. Narayana Aiyar, M. T. Naraniengar, P. V. Seshu Aiyar, Balak Ram, Sarvadaman Chowla, Devi Dayal,
G. S Chowla, T P. Bhaskara Sastri, D. D. Kapadia, T. S Narayana D. K. Hardikar. S. N. Das Gupta.