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PROGRESS REPORT.

The matter of holding a meeting of our society during the next Xmas holidays is now engaging the attention of the Committee. The Committee however regret that the support accorded to this movement by the General Body is not sufficiently enthusiastic. The final decision of the Committee in the matter will be shortly notified to the General Body.

2. (*a*) The following book has been purchased for the library: *The Theory of Functions of Real Variables*, Vol II: by Dr. James Pierpont, L L. D. etc.; published by Messrs. Ginn & Co.; 1912, Boston.

(*b*) The following volumes have been presented to the library by Mr. Balakram, M. A., I. C. S.—

Nature, Vols 70 to 87, from May 1904 to 1911.

(*c*) The following have been received as presents to the library from the publishers.—

1. *Aristarchus of Samos*—(A history of greek Astronomy up to Aristarchus &c.)—by Sir Thomas Heath; Oxford Univ. Press. 18*s* net, 1913, Oxford;

2. *Elementary Mechanices*—By G. Goodwill; Oxford Univ. Press. 4 *s*. 6 *d*. Oxford, 1913.

3. *A text-book of Elementary Trigonometry*—By Dr. R. S. Heath, Oxford Univ. Press, 3*s* 6*d*. Oxford, 1913.

4. *Elements of Trigonometry*—By J. C. Swaminarayan, Rs. 1-8-0, Ahmedabad, 1913.

5. *Calcutta University Calendar*, Part III for 1913.

BOMBAY,
30*th Sept*. 1913.

D. D. KAPADIA,
Hon: Joint Secretary.

Some Thoughts on Modern Mathematical Research[1]

By Professor G. A. Miller.

MATHEMATICS has a large household and there are always rumours of prospective additions despite her age and her supposed austerity. Without aiming to give a complete list of the names of the members of this household we may recall here a few of the most prominent ones. Among those which antedate the beginning of the christian era are surveying, spherical astronomy, general mechanics and mathematical optics. Among the most thriving younger members are celestial mechanics, thermodynamics, mathematical electricity and molecular physics.

Usually a large household serves as one of the strongest incentives to activity, and mathematics has always responded heartily to this incentive. As the most efficient continued service calls for unusual force and ingenuity, mathematics has had to provide for her own development and proper nourishment in addition to providing as liberally as possible for her household. This double object must be kept prominently before our eyes if we would comprehend the present mathematical activities and tendencies.

There is another important incentive to mathematical activity which should be mentioned in this connection. Mathematics has been very hospitable to a large number of other sciences and as a consequence some of these sciences have become such frequent visitors that it is often difficult to distinguish them from the regular members of the household. Among these visitors are economics, dynamical geology dynamical meteorology and the statistical parts of various biological sciences. Visitors usually expect the best that can be provided for them, and the efforts to please them frequently lead to a more careful study of available resources than those which are put forth in providing for the regular household.

We have thus far spoken only of what might be called the materialistic incentives for mathematical development. While these have always been very significant, it is doubtful whether they have been the most powerful. Symmetry, harmony and elegance of form have always appealed powerfully to dame mathematics; and a keen curiosity, fanned into an intense flame by little bits of apparently in-

[1] Read before the Illinois Society, April, 1912.

coherent information, has inspired some of the most arduous and prolonged researches. Incentives of this kind have led to the *mathematics of the invisible*, relating to refinements which are essentially foreign to counting and measuring. The first important refinement of this type relates to the concept of the irrational, introduced by the ancient Greeks. As an instance of a comparatively recent development along this line we may mention the work based upon Dedekind's definition of an infinite aggregate as one in which a part is similar or equivalent to the whole.[2]

Mathematics is commonly divided into two parts called pure and applied, respectively. It should be observed that there are various degrees of purity and it is very difficult to say where mathematics becomes sufficiently impure to be called applied. The engineer or the physicist may reduce his problem to a differential equation, the student of differential equations may reduce his troubles to a question of function theory or geometry, and the workers in the latter fields find that many of their difficulties reduce themselves to questions in number theory[3] or in higher algebra. Just as the student of applied mathematics can not have too thorough a training in the pure mathematics upon which the applications are based so the student of some parts of the so-called pure mathematics cannot get too thorough a training in the basic subjects of this field.

As mathematics is such an old science and as there is such a close relation between various fields, it might be supposed that fields of research would lie in remote and almost inaccessible parts of this subject. It must be confessed that this view is not without some foundation, but these are days of rapid transportation and the student starts early on his mathematical journey. The question as regards the extent of explored country which should be studied before entering unexplored regions is a very perplexing one. A lifetime would not suffice to become acquainted with all the known fields, and there are those who are so much attracted by the explored regions that they do not find time or courage to enter into the unknown.

In 1840 C. G. J. Jacobi used an illustration, in a letter[4] to his brother, which may serve to emphasize an important point. He states that at various times he had tried to persuade a young man to begin

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2 "Encyclopédie des sciences mathematiques," Vol. I., part 1, 1904, p. 2.

3 "Der Urquell aller Mathematik sind die ganzen Zahlen, Minkowski, Diophantische Approximation," 1907, preface.

4 "Briefwechsel zwischen C.G.J. Jacobi und M. H. Jacobi," 1907, p. 64.

research in mathematics, but this young man always excused himself on the ground that he did not yet know enough. In answer to this statement Jacobi asked this man the following question: Suppose your family would wish you to marry, would you then also reply that you did not see how you could marry now, as you had not yet become acquainted with *all* the young ladies?

In connection with this remark by Jacobi we may recall a remark by another promiment German mathematician who also compared the choice of a subject of research with marriage. In the "Festschrift zur Feier des 100 Geburtstages Eduard Kummers," 1910, page 17, Professor Hensel states that Kummer declined, as a matter of principle, to assign to students a subject for a doctor's thesis, saying that this would seem to him as if a young man would ask him to recommend to him a pretty young lady whom he should marry.

While it may not be profitable to follow these analogies into details, it should be stated that the extent to which a subject has been developed does not necessarily affect adversely its desirability as a field of research. The greater the extent of the development the more frontier regions will become exposed. The main question is whether the new regions which lie just beyond the frontier are fertile or barren. This question is much more important than the one which relates to the distance that must be traveled to reach these new fields. Moreover, it should be remembered that mathematics is n-dimensional, n being an arbitrary positive integer, and hence she is not limited, in her progress, to the directions suggested by our experiences.

If we agree with Minkowski that the integers are the source of all mathematics[5] we should remember that the numbers which have gained a place among the integers of the mathmatician have increased wonderfully during recent times. According to the views of the people who preceded Gauss, and according to the elementary mathematics of the present day, the integers may be represented by points situated on a straight line and separated by definite fixed distance. On the other hand, the modern mathematician does not only fill up the straight line with algebraic integers, placing them so closely together that between any two of them there is another, but he fiills up the whole plane equally closely with these integers. If our knowledge of mathematics

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[5] This view was expressed earliar by Kronecker, who was the main founder of the school of mathematicians who aim to make the concept of the positive integers the only foundation of mathematics. Cf. Klein und Schimmack "Der mathematische Unterricht an den hoeheren Schulen," 1907, p. 175.

had increased during the last two centuries as greatly as the number of integers of the mathematician, we should be much beyond our present stage. The astronomers may beled to the conclusion that the universe is probably finite from the study of the number of stars revealed by telescopes of various powers, but the mathematician finds nothing which seems to contradict the view that his sphere of action is infinite.

From what precedes one would expect that the number of fields of mathematical research appears unlimited and this may serve to furnish a partial explanation of the fact that it seems impossible to give a complete definition of the term mathematics. If the above view is correct we have no reason to expect that a complete definition of this term will ever be possible, although it seems possible that a satisfactory definition of the developed parts may be forthcoming.[6]

Among the various fields of research those which surround a standing problem are perhaps most suitable for a popular exposition, but it should not be inferred that these are necessarily the most important points of attack for the young investigator. On the contrary, one of the chief differences between the great mathematician and the pooe one is that the former can direct his students into fields which are likely to become well known in the near future, while the latter can only direct them to the well-known standing problems of the past, whose approaches have been tramped down solid by the feet of the mediocre, who are often even too stupid to realize their limitations. The best students can work their way through this hard crust, but the paddle of the weaker ones will only serve to increase its thickness if it happens to make any impression whatever.

It would not be difficult to furnish a long list of standing mathematical problems of more or less historic interest. Probably all would agree that the most popular one at the present time is Fermat's greater theorem. In fact, this theorm has become so popular that it requires courage to mention it before a strictly mathematical audience, but it does not appear to be out of place before a more general audience like this.

The ancient Egyptians knew that $3^2+4^2=5^2$, and the Hindus knew several other such triplets of integers at least as early as the fourth century before the christian era.[7] These triplets constitute positve integral solutions of the equation

$$x^2+y^2=z^2.$$

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[6] Bocher discussed some of the proposed definitions in the *Bulletin of the American Mathematical Society*, Vol. (1904), p. 115.

[7] Lietzmann, "Der Pythagoreische Lehrsatz,' 1912, p. 52.

Pythagoras gave a general rule by means of which one can find any desired number of such solutions, and hence these triplets are often called Pythagorean numbers. Another such rule was given by Plato, while Euclid and Diophantus generalized and extended these rules.

Fermat, a noted French mathematician of the seventeenth century, wrote on the margin of a page of his copy of Diophantus the theorem that it is impossible to find any positive integral solution of the equation

$$x^n + y^n = z^n \qquad (n > 2).$$

He added that he had discovered a wonderful proof of this theorem, but that the margin of the page did not afford enough room to add it.[8] This theorem has since become known as Fermat's greater theorem, and has a most interesting and important history, which we proceed to sketch.

About a century after Fermat had noted this theorem Euler (1707—1783) proved it for all the cases when n is a multiple of either 3 or 4, and. during the following century, Dirichlet (1805-1859) and Legendre (1752-1833) proved it for all the cases when n is a multiple of 5. The most important step towards a general proof was taken by Kummer (1810-1893), who applied to this problem the modern theory of algebraic numbers and was thus able to prove its truth for all multiples of primes which do not exceed 100 and also for all the multples of many larger primes.

The fact that such eminent mathematicians as Fermat, Euler, Dirichlet. Legendre and Kummer were greatly interested in this prblem was sufficient to secure for it considerable prominence in mathematical literature, and several mathematicians, including Dickson of Chicago succeeded in extending materially some of the results indicated above. The circle of those taking an active interest in the problem was suddenly greatly enlarged, a few years ago, when it become known that a prize of 100,000 Marks (about \$ 25,000) was awaiting the one who could present the first complete solution. This amount was put in trust of the Göttingen Gesellschaft der Wissenschaften by the will of a deceased German mathematician named Wolfskehl, and it is to remain open for about a century, until 2007, unless some one should successfully solve the problem at an earlier date.

It is too early to determine wether the balance of the effects of this prize will tend towards real progress. One desirable feature is the fact that the interest on the money is being used from year to year

[8] Fermat's words are as follows: "Cujus reidemonstrationem mirabilem sane detexi. Hanc marginis exiguitas non caperet."

to further important mathematical enterprises. A certain amount of of this has already been given to A. Wieferich for results of importance towards the solution of Fermat's problem, and other amounts were employed to secure at Gottingen courses of lectures by Poincaré and Lorentz.

What appears as a bad effect of this offered prize is the fact that many people with very meager mathematical training and still less ability are wasting their time and money by working out and publishing supposed proofs. The number of these is already much beyond 1,000 and no one can foresee the extent to which this kind of literature will grow, especially if the complete solution will not be attained during the century. A great part of this waste would be eliminated if those who would like to test their ability along this line could be induced to read, before they offer their work for publication, the discussion of more than 100 supposed proofs whose errors are pointed out in a German mathematical magazine called *Archiv der Mathematik und Physik*, published by B. G. Teubner, of Leipzig. A very useful pamphlet dealing with this question is entitled, "Ueber das letzte Fermatische Theorem, von B. Lind," and was also published by B. G. Teubner, in 1910.

A possible good effect of the offered prize is that it may give rise to new developments and to new methods of attack. As the most successful partial solution of the problem was due to the modern theory of algebraic numbers, one would naturally expect that further progress would be most likely to result from a further extension of this theory, or, possibly, from a still more powerful future theory of numbers. If such extensions will result from this offer they will go far to offset the bad effect noted above, and they may leave a decided surplus of good. Such a standing problem may also tend to lessen mathematial idolatry which is one of the most serious barriers to real progress. Wo should welcome everything which tends to elevate the truth above our idols formed by men, institutions or books.

In view of the fact that the offered prize is about $25,000 and that lack of marginal space in his copy of Diophantus was the reson given by Fermat for not communicating his proof, one might be tempted to wish that one could send credit for a dime back through the ages to Fermat and thus secure this coveted prize and the wonderful proof, if it actully existed. This might, however, result more seriously than one would at first suppose; for if Fermat had bought on credit a dime's worth of paper even during the year of his death, 1665, and if this bill had been drawing compound interest at the rate of six per cent. since that time, the bill would now amount to more than seven times as much as the prize. It would therefore require more than $150,000, in addition to the amount of the prize, to settle this bill now.

While it is very desirable to be familiar with such standing problems as Fermat's theorem, they should generally be used by the young investigator as an indirect rather than as a direct object of research. Unity of purpose can probably not be secured in any better way than by keeping in close touch with the masters of the past,[9] and this unity of purpose is almost essential to secure real effective work in the immense field of mathematical endeavour. As a class of problems which are much more suitable for direct objects of research on the part of those who are not in close contact with a master in his field, we may mention the numerous prize subjects which are announced from year to year by foreign academies.

Among the learned societies which announce such subjects the Paris Academy of Sciences is probably most widely known, but there are many others of note. The subjects announced annually by these societies cover a wide range of mathematical interests, but they are frequently beyond the reach of the young investigator.[10] It is very easy to obtain these subjects, since they generally appear in the "notes" of many mathematical journals. In our country the *Bulletin of the American Mathematical Society* is rendering very useful service along this and many other lines. While some of these subjects are very general, there are others which indicate clearly the particular difficulties which must be overcome before further progress in certain directions seems possible and hence these subjects deserve careful study, especially on the part of the younger investigators.

As long as one is completely guided, in selecting subjects for research, by the standing problems or by the subjects announced by learned bodies and those proposed individually by prominent investigators, one is on safe ground. Real progress along any of these lines is welcomed by our best journals, as such progress can easily be measured, and it fits into a general trend of thought which is easily accessible in view of the many developed avenues of approach. Notwithstanding

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9 Darboux, *Bulletin des Sciences Mathematiques*, Vol. 32 (1908), p. 107.

10 For solutions of such problems in pure mathematics by Americans, see Bulletin of the American Mathematical Society, Vol. 7 (1901), p. 190; Vol. 16 (1910), p. 267.

these advantages, the real investigator should reach the time when he can select his own problems without advice or authority; when he feels free to look at the whole situation from a higher point of view and to assume the responsibility of an independent choice, irrespective of the fact that an independent choice may entail distrust and misgivings on the part of many who would have supported him nobly if he had remained on their plane.

In looking at the whole situation from this higher point of view many new and perplexing questions confront us. Why should the developments of the past have followed certain routes? What is the probability that the development of the territory lying between two such routes will exhibit new points of contact and greater unity in the whole development? What should be some guiding principles in selecting one rather than another subject of investigation? What explanation can we give for the fact that some regions bear evidences of great activity in the past but are now practically deserted, while others maintained or increased their relative popularity through all times?

One of the most important tests that can be applied to a particular mathematical theory is whether it serves as a unifying and clarifying principal of wide applications. Whether these applications relate to pure mathematics only, or to related fields, seems less important. In fact, the subjects of application may have to be developed. If this is the case, it is so much the better, provided always that the realm of thought whose relations are exhibited by the theory is extensive and that the relations are of such a striking character as to appeal to a large number of mathematical intellect of the present or of the future. Some isolated facts may be of great interest, but as long as they are isolated they have little or no real mathematical interest. One object of mathematics is to enable us to deal with infinite sets with the same ease and confidence as if they were individuals. In this way only can our finite mind treat systematically some of the infinite sets of objects of mathematical thought.

In comparatively recent years the sprit of organization has made itself felt among mathematicians with rapidly increasing power, and it has already led to many important results. Beginning with small informal organizations in which the social element was often most prominent, there have resulted large societies, national and even international, with formal organizations and with extensive publications. In reference to one of these early organizations, the mathematical society of Spitalfields in London, which lasted for more than a century

(1717-1845), it is said that each member was expected to come to the meetings with his pipe, his mug and his problem.[11]

The modern mathematical society is dominated by a different spirit. It generally supports at least one organ for publication, and scholarly publicity develops scholarly cooperation as well as scholarly ambitions. This cooperation has led to movements which could not have been undertaken by a few individuals. One may recall here the *Revue Semestrielle*, published under the auspices of the Amsterdam Mathematical Society; the extensive movement to examine and compare methods and courses of mathematical instruction in various countries, inaugurated at the fourth international congress, held at Rome in 1908; and, especially the great mathematical encyclopedia; whose start was largely influenced by the support of the deutschen Mathematiker-Vereinigung as expressed at the Vienna meeting in 1894. The French edition of the latter work, which is now in the course of publication, is expected to include thirty-four large volumes, besides those which are to be devoted to questions of the philosophy, the teaching and the history of mathematics.

These encyclopedias and other large works of reference are doing much to expedite travel in the mathematical field. In fact, it would probably not be exaggerating if we should say that by these encyclopedias alone the distances, in time and effort, between many points of the mathematical field have been cut in two. In this connection, it may be fitting to recall, with a deep sense of obligation, the great work which is being done by the Royal Society of London—not only for mathematics, but also for a large number of other sciences—in providing bibliographical aids on a large scale. If the increase in knowledge will always be attended by a corresponding increase in means to learn readily what is known, even the young investigator of the future will have no reason to regret the extent of the developments. On the contary, these should make his task easier, since they furnish such a great richness of analogies and of tried methods of attack.

The last to or three decades have witnessed a great extention of mathematical research activity. As a result of this we have a large number of new mathematical societies. A few of the most recent ones are as follows: Calcutta Mathematical Society (1908), Manchester Mathematical Ssciety (1908), Scandinavian Congress of Mathematicians

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11 "Es wards von jedem erwartet. dass er seine Pfeife, seinen Krug und sein Problem mitbringe." Cantor, "Vorlesungen ueber Geschichte der Mathematik," Vol. 4, 1908, p. 59.

(1909), Swiss Mathematical Society (1910), Spanish Mathematical Society (1911), and the Russian Congress of Mathematicians (1912). In Japan a new mathematical periodical, called *Tohoku Mathematical Journal*, was started in 1911, and a few years earlier the *Journal of the Indian Mathematical Society* was started at Madras, India. The Calcutta Mathematical Society and the Spanish Mathematical Society have also started new periodicals during the last two or three years.

While three has been a very rapid spread of mathematical activity during recent years, it must be admitted that the greater part of the work which is being done in the new centers is quite elementary from the standpoint of research. The city of Paris continues to hold its preeminent mathematical position amoung the cities of the world; and Germany, France and Italy continue to lead all other countries in regard to the quality and the quantity of research in pure mathematics.

Although America is not yet doing her share of mathematical research of a high order, we have undoubtedly reached a position of respectability along this line, and it should be easier to make further progress. Moreover, our material facilities are increasing relatively more rapidly than those of the countries which are ahead of us, and hence many of our younger men start under very favourable conditions Unfortunately there is not yet among us a sufficiently high appreciation of scholarly attainments and scientific distinction. The honest and outspoken investigator is not always encouraged as he ought to be and the best positions do not always seek the best man. I coupled *outspoken* with *investigator* advisedly, since research of high order implies liberty and scorns shams, especially shams relating to scholarship. Even along these lines there seems to be encouraging progress, and this progress may reasonably be expected to increase with the passing of those who belong to the past in spirit and attainments. What appears to be a very serious element in our situation is the fact that the American university professor does not yet seek and safeguard his freedom with the zeal of his European colleague. It is too commonly assumed that loyalty implies lying.

The investigators in pure mathematics form a small army of about two thousand men and a few women.[12] The question naturally arises :

[12] Between five and ten per cent. of the members of the American Mathematical Society are women, but the per cent. of women in the leading foreign mathematical societies is much smaller. Less than one per cent. of the members of the national mathematical societies of France, Germany and Spain are women, according to recent lists of members. The per cent. of important mathematical contri-

what is this little army trying to accomplish. A direct answer is that they are trying to find and to construct paths and roads of thought, which connect with or belong to a network of thought-roads commonly known as mathematics. Some are engaged in constructing trails through what appears an almost impassable region while others are widening and smoothing roads which have been travelled for centuries There are others who are engaged in driving piles in the hope of securing a solid foundation through regions where quicksand and mire have combined to obstruct progress.

A characteristic property of mathematics is that by means of certain postulates its thought-roads have been proved to be safe; and they always lead to some prominent objective points. Hence they primarily serve to economize thought. The number of objects of mathematical thought is infinite, and these roads enable a finite mind to secure an intellectual penetration into some parts of this infinitude of objects. It should also be observed that mathematics consists of a *connected* network of thought-roads, and mathematical progress means that other such connected or connecting roads are being established, which either lead to new objective points of interest or exhibit new connections between known roads.

The network of thought-roads called mathematics furnishes a very interesting chapter in the intellectual history of the world, and in recent years an increasing number of investigators have entered the field of mathematical history. The results are very encouraging. In fact, there are very few other parts of mathematics where the progress during the last twenty years has been as great as in this history. This progress is partly reflected by special courses in this subject in the leading universities of the world. While the earliest such course seems to have been given only about forty years ago, a considerable number of universities are now offering regular courses in this subject, and these courses have the great advantage that they establish another point of helpful contact between mathematics and other fields.

Mathematical thought-roads may be distinguished by the facts that by means of certain assumptions they have been *proved* to lead safely

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butions by women does not appear to be larger, as a rule, than that of their representation in the leading societies. The list of about three hundred collaborators on the great new German and French mathematical encyclopedias does not seem to include any woman. Possibly women do not prize sufficiently intellectual freedom to become good mathematical investigators. Some of them exhibit excellent ability as mathematical students.

to certain objective points of interest, and each of them connects, at least in one point, with a network of other such roads which were called mathematics, $\mu\alpha\theta'\eta\mu\alpha\tau\alpha$ by the ancient Greeks. The mathematical investigator of the present day is pushing these thought roads into domains which were totally unknown to the older mathematicians. Whether it will ever be possible to penetrate all scientific knowledge in this way and thus to unify all the advanced scientific subjects of study under the general term of mathematics, as was the case with the ancient Greeks,[13] is a question of deep interest.

The scientific world has devoted much attention to the collection and the classification of facts relative to material things, and has secured already an immensely valuable store of such knowledge. As the number of these facts increases, stronger and stronger means of intellectual penetration are needed. In many cases mathematics has already provided such means in a large measure ; and, judging from the past, one may reasonably expect that the demand for such means will continue to increase as long as scientific knowledge continues to grow. On the other hand, the domain of logic has been widely extended through the work of Russell, Poincaré and others; and Russell's conclusion that any false proposition implies all other propositions whether true or false is of great general interest.

During the last two or three centuries there has been a most remarkable increase in facilities for publication. Not only have academies and societies started journals for the use of their members, but numerous journals inviting suitable contributions from the public have arisen. The oldest of the latter type is the *Journal des Scavans*, which was started at Paris in 1665, while the *Transactions of the Royal Society of London*, started in the same year, should probably be regarded as the oldest of the former type. These journals have done an inestimable amount of good for the growth of knowledge and the spread of the spirit of investigation. At the present time more than 2,000 articles which are supposed to be contributions to knowledge in pure mathematics appear annually in such periodicals. In addition to these there is a growing annual list of books.

The great extent of the fields of mathematics and the rapid growth of this literature have made it very desirable to secure means of judging more easily the relative merit of various publications. Along this

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[13] The term mathematics was first used with its present restricted meaning by the Peripatetic School. Cantor, "Vorlesungen über Geschichte der Mathematik," Vol. I. (1907), p. 216.

line our facilities are still very meager and many serious difficulties present themselves. In America we have the book reviews and the indirect means provided by the meetings of various societies and by such publications as the "American Men of Science."

The most importani aid to judge contemporaneous work is furnished by a German publications known as the *Jahrbuch uber die Fortschritte der Mathematik*. In this work there appear annually about 1,000 pages of reviews of books and articles published two or three years earlier. These reviews are prepared by about 60 different mathematicians who are supposed to be well prepared to pass judgment on the particular books and articles which they nndertake to review. While these reviews are of very unequal merit, they are rendering a service of the greatest value.

The main object of such reviews is to enable the true student to learn easily what progress others are making, especially in his own field and in those closely related thereto. They serve, however, another very laudable purpose in the case that they are reliable. We have the pretender and the unscrupulous always with us, and it is almost as important to limit their field of operation, as to encourage the true investigator. "Companions in zealous research" should be fearless in the pursuit of truth and in the disclosure of falsehood, since these qualities are essential to the atmosphere which is favourable to research.

While the mathematical investigator is generally so engrossed by the immediate objects in view that he seldom finds time to think of his services to humanity as a whole, yet such thoughts naturally come to him more or less frequently, especially since his direct objects of research seldom are well suited for subjects of general conversation. If these thoughts do come to him they should bring with them great inspiration. Who can estimate the amount of good mathematics has done and is doing now? If all knowledge of mathematics could suddenly be taken away from us there would be a state of chaos, and if all those things whose development depended upon mathematical principles could be removed, our lives and thoughts would be pauperized immeasurably. This removal would sweep away not only our modern houses and bridges, our commerce and landmarks, but also most of our concepts of the physical universe.

Some may be tempted to say that the useful parts of mathematics are very elementary and have little contact with modern research. In answer we may observe that it is very questionable whether the ratio of the developed mathematics to that which is finding direct application to things which relate to material advanges is greater now than it was at the time of the ancient Greeks. The last two centuries have witnessed a wonderful advance in the pure mathematics which is commonly used.[14] While the advance in the extent of the developed fields has also been rapid, it has probably not been relatively more rapid. Hence the mathematical investigator of to-day can pursue his work with the greatest confidence as regards his services to the general uplift both in thought and in material betterment of the the human race. All of his real advances may reasonably be expected to be enduring elements of a structure whose permanence is even more assured that that of granite pillars.

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[14] In 1726, arithmetic and geometry were studied during the senior year in Harvard College. Natural philosophy and physics were still taught before arithmetic and geometry. Cajori, "The Teaching and History of Mathematics in the United States," 1890, p. 22.

SHORT NOTES.

Some properties of Polynomials.

1. Let A and B be two polynomials of the same degree n. Then, as in the process of finding the H. C. F. of A and B, we may write

$$\left.\begin{array}{l} R_1 = A - Q_1 . B \\ R_2 = B - Q_2 . R_1 \\ R_3 = R_n - Q_3 . R_2 \\ \dots \quad \dots \quad \dots \quad \dots \\ R_n = R_{k-2} - Q_n . R_{n-1} \ ; \end{array}\right\} \quad \dots \quad \dots \quad (1)$$

that is,
$$A/B = Q_1 + \frac{1}{Q_2 +}\,\frac{1}{Q_3 +}\cdots\frac{1}{+Q_n}; \qquad \dots \qquad (2)$$

where Q_1 is a constant, and $Q_2, Q_3, \dots Q_n$ are each of the first degree. If A and B have a H.C.F. of degree k, it is evident that R_{n-k} will be the last of the series of remainders, and the continued fraction (2) will stop at Q_{n-k}.

2. Any remainder R_p is of degree $(n-p)$, and can be expressed in the form $F_{p-1}A + G_{p-1}B$, where F_{p-1} and G_{p-1} are integral functions of degree $(p-1)$. Expressions for any R, F or G, can be readily found in terms of A, B and the Q's by mathematical induction.* The following method is, however, somewhat instructive :

Assume that

$$\begin{aligned} R_p &= r_0 + r_1 x + \quad \dots \quad r_{n-p} x^{n-p} \\ F_{p-1} &= \lambda_0 + \lambda_1 x + \dots \quad \lambda_{p-1} x^{p-1} \\ G_{p-1} &= \mu_0 + \mu_1 x + \dots \quad \mu_{p-1} x^{p-1} \ ; \end{aligned}$$

and let $A \equiv f(x) = a_0 + a_1 x + \quad \dots \quad a_n x^n$, $B \equiv g\,(x) = b_0 + b_1 x + \dots b_n x^n$.

Take β any root of $B = 0$, and put $x = \beta$ in the identity

$$R_p = F_{p-1} A + G_{p-1} \mathbf{B}. \qquad \dots \qquad \dots \qquad (3)$$

We have

$$r_0 + r_1\beta + r_2\beta^2 + \quad \dots \quad r_{n-p}\beta^{n-p} = (\lambda_0 + \lambda_1\beta + \quad \dots \quad \lambda_{p-1}\beta^{p-1})\, f\,(\beta).$$

Multiply by $\beta, \beta^2, \dots \beta^{n-1}$ in succession ; and similarly deal with the other roots of $B = 0$; and add the resulting equations. Then we obtain the following :

---

* Thus

$$-\frac{R_p}{R_{p-1}} = Q_p \frac{1}{+Q_{p-1}}\,\frac{1}{+Q_{p-2}}\cdots\frac{1}{+Q_1}\,\frac{A}{-B},$$

for all values of p up to n.

$$\left.\begin{array}{l} r_0s_0+r_1s_1+r_2s_2+\ \dots\ r_{n-p}s_{n-p}-\lambda_0\sigma_0-\lambda_1\sigma_1-\lambda_2\sigma_2\ \dots\ \lambda_{p-1}\sigma_{p-1}=0 \\ r_0s_1+r_1s_2+r_2s_3+\ \dots\ r_{n-p}s_{n-p+1}-\lambda_0\sigma_1-\lambda_1\sigma_2-\lambda_2\sigma_3-\dots\lambda_{p-1}\sigma_p=0 \\ \quad\dots\qquad\dots\qquad\dots\qquad\dots \\ r_0s_{n-1}+r_1s_{n-0}+r_2l_{n+1}+\ \dots\ r_{n-p}s_{2n-p-1}-\lambda_0\sigma_{n-1}-\lambda_1\sigma_{n-2} \\ \qquad\qquad -\lambda_2\sigma_{n+1}\dots\lambda_{p-1}\sigma_{n+p-2}=0 \end{array}\right\}(4)$$

where $s_k=\Sigma(\beta^k),\sigma_k=\Sigma\{\beta^kf(\beta)\}=(a_0s_k+a_1s_{k+1}+\ \dots\ a_ns_{n+k})$.

From the above linear equations the ratios of the r's the λ's are readily found, as we are not concerned with their absolute values.

3. It is thus clear that corresponding to any two polynomials A & B of the n^{th} degree there exist definite expressions F_{p-1}, G_{p-1} of the $(p-1)^{th}$ degree, such that $(F_{p-1}A+G_{p-1}B)$ is of degree $(n-p)$.

Now, if A and B have a H. C. F. of degree $(n-p)$, it must be R_p found as above. The method, however, admits of slight modifications in as much as R_p may be replaced by any group of $(n-p+1)$ consecutive powers of x. Thus, we may assume that

$$x^q(r_0+r_1x+\dots r_{n-p}x^{n-p})=F_{p-1}A+G_{p-1}B$$

and proceed to find the values of the r's, as before. The only modifications in the equations (4) are that the suffixes of the s's and σ's are throughout increased by q.

Hence it appears that the values of $(r_0, r_1\dots\dots r_{n-p})$ as determined by (4) remain unchanged when the suffixes of s's and σ's are increased by any the same quantity $q \not> 2p-1$; and the $(n-p)$ conditions for a H. C. F. of degree $(n-p)$ are accordingly written down.

4. The fundamental property of equation (3) admits of generalization as follows :

(i) It is always possible to determine F_{p-1}, G_{p-1} so that the expression $(F_{p-1}A+G_{p-1}B)$ contains $(n-p+1)$ distinct terms. For, the method of § 2 applies with equal force in this case, in as much as the $(n-p+1)$ distinct terms involve $(n-p+1)$ undetermined co-efficients.

(ii) If A_m, B_n are functions of degrees m, n respectively, two other functions F_{n-p-1}, G_{m-p-1} can be found such that

$$A_m.F_{n-p-1}+B_n.G_{m-p-1}=R_p,$$

where suffixes indicate the degrees of the respective functions. This result is also true if R_p is replaced by any expression containing $(p+1)$ distinct terms.

5. Suppose A, B consist of μ given powers of x each, *viz.*, x^n, x^p, $x^q\dots$ Then functions F, G always exist such that

$$A.F+B.G=C,$$

where C contains one or more of these μ powers. This is merely a particular case of § 4 (i). Thus, if C_p denote an expression containing $(p+1)$ of the μ powers, we can write

$$A.F_{n-\mu+1}+B.G_{n-\mu+1}=C_{\mu-2}$$
$$A.F_{n-\mu+2}+B.G_{n-\mu+2}=C_{\mu-3}$$
$$A.F_{n-p-1}+B.G_{n-p-1}=C_p$$
$$A.F_{n-1}+B.G_{n-1}=C_o.$$

M. T. NARANIENGAR.

The series $\sum_{1}^{n=\infty}\left\{\frac{n^p}{\underline{|n}}\right\}$

Consider the expansion

$$e^{e^x}=1+e^x+\frac{e^{2x}}{\underline{|2}}+\frac{e^{3x}}{\underline{|3}}+\ldots ad.\ inf.$$

The coefficient of x^p on the right hand side is, evidently,

$$\frac{1}{\underline{|p}}\left\{1+\frac{2^p}{\underline{|2}}+\frac{3^q}{\underline{|3}}+\ldots\right\}=\frac{a_p}{\underline{|p}}\text{ say.}$$

$$\therefore\qquad e^{e^x}=a_0+\frac{a_1x}{\underline{|1}}+\frac{a_2x^2}{\underline{|2}}+\qquad\ldots\qquad\ldots\qquad\ldots\qquad(1)$$

Differentiating both sides of (1), we have

$$e^xe^{e^x}=a_1+\frac{a_2x}{\underline{|1}}+\frac{a_3x^2}{\underline{|2}}+\qquad\ldots\qquad\ldots\qquad\ldots\qquad(2)$$

Expanding the left hand side by means of (1), and equating the coefficients of x^p on both sides, we get

$$\frac{a_p}{\underline{|p}}+\frac{a_{p-1}}{\underline{|p-1}\,\underline{|1}}+\frac{a_{p-2}}{\underline{|p-2}\,\underline{|2}}+\ \ldots\ \frac{a_0}{\underline{|p}}=\frac{a_{p+1}}{\underline{|p}}.$$

or

$$a_p+p\cdot a_{p-1}+\frac{p\cdot\overline{p-1}}{1\cdot 2}a_{p-2}+\ldots=a_{p+1}.\qquad\ldots\qquad\ldots\qquad(3)$$

Now, putting $p=0,1,2,\ldots$ in succession in (3), and remembering that $a_o=e$, we find

$$a_1=a_0=e$$
$$a_2=a_1+a_0=2e$$
$$a_3=a_2+2a_1+a_0=5e$$
$$a_4=a_3+3a_2+3a_1+a_0=15e$$
$$a_5=a_4+4a_3+6a_2+4a_1+a_0=52e.$$

&c., &c., &c.

2. These results are also obtained by writing

$$n^p = A_1 n + A_2 n(n-1) + A_3 n(n-1)(n-2) + \ldots \text{to } p \text{ terms.}$$

Thus

$$a_p = \sum \frac{n^p}{\underline{|n}} = \sum \frac{A_1}{\underline{|n-1}} + \sum \frac{A_2}{\underline{|n-2}} + \ldots = (A_1 + A_2 + A_3 + \ldots A_p)e.$$

3. The values of a_p are readily found from the sums of the diagonal numbers in the following table, where the number corresponding to the q^{th} row and r^{th} column, say $N_{q,r} = q.N_{q-1,r} + N_{q,r-1}$.

1	1	1	1	1	1
1	3	7	15	31	63
1	6	25	90	301	966
1	10	65	350	1701	7770
1	15	140	1050	6951	42525
1	21	266	2646	22827	179487

Thus $a_1 = e$, $a_2 = (1+1)e - 2e$, $a_3 = (1+3+1)e = 5e$, and so on.

S. Krishnaswami Iyengar.

Curvature in Areal Coordinates.

The following is substantially taken from Cesáro : *Lezioni di Geometria Intrinseca*, Chapter VII.

The radius of curvature of a curve referred to rectangular axes is given by

$$\frac{1}{\rho} = \frac{d^2y}{ds^2}\frac{dx}{ds} - \frac{d^2x}{ds^2}\frac{dy}{ds}.$$

Transforming to oblique axes by the substitution

$$x = X + Y\cos\omega,\quad y = Y\sin\omega,$$

we find

$$\frac{1}{\rho} = \sin\omega\left(\frac{d^2Y}{ds^2}\frac{dX}{ds} - \frac{d^2X}{ds^2}\frac{dY}{ds}\right)$$

Thus, for areals, writing $\omega = B$, $\frac{2\triangle}{a}\alpha = Y \sin B$, $\frac{2\triangle}{c}\gamma = X \sin B$, we find

$$\frac{1}{\rho} = 2\triangle\left(\frac{d^2\alpha}{ds^2}\frac{d\gamma}{ds} - \frac{d^2\gamma}{ds^2}\frac{d\alpha}{ds}\right) \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

But, it is easily seen that

$$ds^2 = -(a^2 d\beta d\gamma + b^2 d\gamma d\alpha + c^2 d\alpha d\beta) \quad \ldots \quad \ldots \quad (2)$$

If $f(\alpha, \beta, \gamma) = 0$ is the curve whose radius of curvature is required, we have

$$d\alpha + d\beta + d\gamma = 0,$$

$$\frac{\partial f}{\partial \alpha}d\alpha + \frac{\partial f}{\partial \beta}d\beta + \frac{\partial f}{\partial \gamma}d\gamma = 0;$$

whence $$\frac{d\alpha}{\frac{\partial f}{\partial \beta} - \frac{\partial f}{\partial \gamma}} = \frac{d\beta}{\frac{\partial f}{\partial \gamma} - \frac{\partial f}{\partial \alpha}} = \frac{d\gamma}{\frac{\partial f}{\partial \alpha} - \frac{\partial f}{\partial \beta}} = dt \text{ say.}$$

Thus $$\frac{d\alpha}{dt} = \frac{\partial f}{\partial \beta} - \frac{\partial f}{\partial \gamma}, \quad \frac{d\beta}{dt} = \frac{\partial f}{\partial \gamma} - \frac{\partial f}{\partial \alpha}, \quad \frac{d\gamma}{dt} = \frac{\partial f}{\partial \alpha} - \frac{\partial f}{\partial \beta},$$

and from (2) we have

$$\left(\frac{ds}{dt}\right)^2 = a^2\left(\frac{\partial f}{\partial \alpha}\right)^2 + b^2\left(\frac{\partial f}{\partial \beta}\right)^2 + c^2\left(\frac{\partial f}{\partial \gamma}\right)^2 - 2bc \cos A \frac{\partial f}{\partial \beta}\frac{\partial f}{\partial \gamma}$$
$$- 2ca \cos B \frac{\partial f}{\partial \alpha}\frac{\partial f}{\partial \gamma} - 2ab \cos C \frac{\partial f}{\partial \alpha}\frac{\partial f}{\partial \beta} \ldots \quad (3)$$

Transforming the expression (1), we have

$$\left(\frac{ds}{dt}\right)^3 \cdot \frac{1}{\rho} = 2\triangle\left(\frac{d^2\alpha}{dt^2}\frac{d\gamma}{dt} - \frac{\partial^2\gamma}{\partial t^2}\frac{\partial\alpha}{dt}\right) = 2\triangle . W, \text{ where}$$

$$W = \frac{d^2\alpha}{dt^2}\left(\frac{\partial f}{\partial \alpha} - \frac{\partial f}{\partial \beta}\right) - \frac{\partial^2\gamma}{dt^2}\left(\frac{\partial f}{\partial \beta} - \frac{\partial f}{\partial \gamma}\right)$$

$$= \frac{d^2\alpha}{dt^2}\frac{\partial f}{\partial \alpha} + \frac{d^2\beta}{dt^2}\frac{\partial f}{\partial \beta} + \frac{d^2\gamma}{dt^2}\frac{\partial f}{\partial \gamma}, \text{ (by use of } \alpha + \beta + \gamma = 1.)$$

$$= \sum \frac{\partial f}{\partial \alpha}\frac{}{dt}\left(\frac{\partial f}{\partial \beta} - \frac{\partial f}{\partial \gamma}\right)$$

$$= -\sum\left(\frac{\partial f}{\partial \beta} - \frac{\partial f}{\delta \gamma}\right)\frac{d}{dt}\left(\frac{\partial f}{\partial \alpha}\right), \text{ since} \sum \frac{\partial f}{\partial \alpha}\left(\frac{\partial f}{\partial \beta} - \frac{\partial f}{\partial \gamma}\right) \equiv 0,$$

$$= -\sum\left(\frac{\partial f}{\partial \beta} - \frac{\partial f}{\partial \gamma}\right)\left[\frac{\partial^2 f}{\partial \alpha^2}\left(\frac{\partial f}{\partial \beta} - \frac{\partial f}{\partial \gamma}\right)\right.$$
$$\left. + \frac{\partial^2 f}{\partial \alpha \partial \beta}\left(\frac{\partial f}{\partial \gamma} - \frac{\partial f}{\partial \alpha}\right) + \frac{\partial^2 f}{\partial \alpha \partial \gamma}\left(\frac{\partial f}{\partial \alpha} - \frac{\partial f}{\partial \beta}\right)\right] \ldots \quad (4)$$

I shall now identify this expression with a remarkable expression given in Cesáro.

We may suppose $f(\alpha, \beta, \gamma)$ homogeneous, and thus by **Euler's** theorems

$$\alpha\frac{\partial f}{\partial \alpha}+\beta\frac{\partial f}{\partial \beta}+\gamma\frac{\partial f}{\partial \gamma}=nf=0,$$

$$\alpha\frac{\partial^2 f}{\partial \alpha^2}+\beta\frac{\partial^2 f}{\partial \alpha\partial \beta}+\gamma\frac{\partial^2 f}{\partial \alpha\partial \gamma}=(n-1)\frac{\partial f}{\partial \alpha},$$

and similar expressions for $(n-1)\frac{\partial f}{\partial \beta}$, $(n-1)\frac{\partial f}{\partial \gamma}$.

Consider now

$$\frac{1}{(n-1)^2}\begin{vmatrix} \frac{\partial^2 f}{\partial \alpha^2}, & \frac{\partial^2 f}{\partial \alpha\partial \beta}, & \frac{\partial^2 f}{\partial \alpha\partial \gamma} \\ \frac{\partial^2 f}{\partial \alpha\partial \beta}, & \frac{\partial^2 f}{\partial \beta^2}, & \frac{\partial^2 f}{\partial \beta\partial \gamma} \\ \frac{\partial^2 f}{\partial \alpha\partial \gamma}, & \frac{\partial^2 f}{\partial \beta\partial \gamma}, & \frac{\partial^2 f}{\partial \gamma^2} \end{vmatrix}=K.$$

Multiply the rows by α, β, γ and add **the first two rows to the** third. By use of Euler's identities we have

$$K=\frac{1}{(n-1)\gamma}\begin{vmatrix} \frac{\partial^2 f}{\partial \alpha^2}, & \frac{\partial^2 f}{\partial \alpha\partial \beta}, & \frac{\partial^2 f}{\partial \alpha\partial \gamma} \\ \frac{\partial^2 f}{\partial \alpha\partial \beta}, & \frac{\partial^2 f}{\partial \beta^2}, & \frac{\partial^2 f}{\partial \beta\partial \gamma} \\ \frac{\partial f}{\partial \alpha}, & \frac{\partial f}{\partial \beta}, & \frac{\partial f}{\partial \gamma} \end{vmatrix} \quad \ldots \quad (5)$$

Multiply the columns by α, β, γ and add the first **two columns to** the third.

$$\therefore K=\frac{1}{\gamma^2}\begin{vmatrix} \frac{\partial^2 f}{\partial \alpha^2}, & \frac{\partial^2 f}{\partial \alpha\partial \beta}, & \frac{\partial f}{\partial \alpha} \\ \frac{\partial f}{\partial \alpha\partial \beta}, & \frac{\partial^2 f}{\partial \beta^2}, & \frac{\partial f}{\partial \beta} \\ \frac{\partial f}{\partial \alpha}, & \frac{\partial f}{\partial \beta}, & 0 \end{vmatrix}$$

Again in (5) multiply the columns by α, β, γ and add the **first and** third to the second.

$$\therefore K=\frac{1}{\beta\gamma}\begin{vmatrix} \frac{\partial^2 f}{\partial \alpha^2}, & \frac{\partial f}{\partial \alpha}, & \frac{\partial^2 f}{\partial \alpha\partial \gamma} \\ \frac{\partial^2 f}{\partial \alpha\partial \beta}, & \frac{\partial f}{\partial \beta}, & \frac{\partial f}{\partial \beta\partial \gamma} \\ \frac{\partial f}{\partial \alpha}, & 0, & \frac{\partial f}{\partial \gamma} \end{vmatrix}$$

We thus have six expressions for K which we write

$$K=\frac{-\frac{\partial^2 f}{\partial\alpha^2}\left(\frac{\partial f}{\partial\beta}\right)^2-\frac{\partial^2 f}{\partial\beta^2}\left(\frac{\partial f}{\partial\alpha}\right)^2+2\frac{\partial^2 f}{\partial\alpha\partial\beta}\frac{\partial f}{\partial\alpha}\frac{\partial f}{\partial\beta}}{\gamma^2}=\text{etc., etc.}$$

$$=\frac{2\left\{\frac{\partial^2 f}{\partial\alpha^2}\frac{\partial f}{\partial\beta}\frac{\partial f}{\partial\gamma}+\frac{\partial^2 f}{\partial\beta\partial\gamma}\frac{\partial^2 f}{\partial\alpha^2}-\frac{\partial^2 f}{\partial\alpha\partial\gamma}\frac{\partial f}{\partial\alpha}\frac{\partial f}{\partial\alpha}-\frac{\partial^2 f}{\partial\alpha\partial\beta}\frac{\partial f}{\partial\alpha}\frac{\partial f}{\partial\gamma}\right\}}{2\beta\gamma}$$

$$=\frac{\text{etc.}}{2\gamma\alpha}=\frac{\text{etc.}}{2\alpha\beta};$$

adding all the numerators and noting that the sum of the denominators is $(\alpha+\beta+\gamma)^2=1$, we see that

$$K=-\sum\frac{\partial^2 f}{\partial\alpha^2}\left(\frac{\partial f}{\partial\beta}-\frac{\partial f}{\partial\gamma}\right)^2-2\sum\frac{\partial^2 f}{\partial\alpha\partial\beta}\left(\frac{\partial f}{\partial\beta}-\frac{\partial f}{\partial\gamma}\right)\left(\frac{\partial f}{\partial\gamma}-\frac{\partial f}{\partial\alpha}\right)$$

$=W$, from (4).

We have therefore the formula

$$\rho=\frac{(n-1)^2\left\{\sum a^2\left(\frac{\delta f}{\partial\alpha}\right)^2-2\sum bo\cos A\frac{\partial f}{\partial\beta}\frac{\partial f}{\partial\gamma}\right\}^{\frac{3}{2}}}{2\Delta\begin{vmatrix}\frac{\partial^2 f}{\partial\alpha^2}, & \frac{\partial^2 f}{\partial\alpha\partial\beta}, & \frac{\delta^2 f}{\partial\alpha\delta\gamma}\\ \frac{\partial^2 f}{\partial\alpha\partial\beta}, & \frac{\partial^2 f}{\partial\beta^2}, & \frac{\partial^2 f}{\partial\beta\partial\gamma}\\ \frac{\partial^2 f}{\partial\alpha\partial\gamma}, & \frac{\partial^2 f}{\partial\beta\partial\gamma}, & \frac{\partial^2 f}{\partial\gamma^2}\end{vmatrix}} \qquad (6)$$

Noting that the expression in the numerator is the denominator that occurs in the expression for the perpendicular from any point on

$$x\frac{\partial f}{\partial\alpha}+y\frac{\partial f}{\partial\beta}+z\frac{\partial f}{\partial\gamma}=0,$$

which is the tangent at $(\alpha,\beta,\gamma,)$, we may write from § 5

$$\rho=\frac{4(n-1)^3\Delta^2\frac{\partial f}{\partial\alpha}\frac{\partial f}{\partial\beta}\frac{\partial f}{\partial\gamma}}{p_1p_2p_3\begin{vmatrix}\frac{\partial^2 f}{\partial\alpha^2}, & \frac{\partial^2 f}{\partial\alpha\partial\beta}, & \frac{\partial^2 f}{\partial\alpha\partial\gamma}\\ \frac{\partial^2 f}{\partial\alpha\partial\beta}, & \frac{\partial^2 f}{\partial\beta^3}, & \frac{\partial^2 f}{\partial\beta\partial\gamma}\\ \frac{\partial^2 f}{\partial\alpha\partial\gamma}, & \frac{\partial^2 f}{\partial\beta\partial\gamma}, & \frac{\partial^2 f}{\partial\gamma^3}\end{vmatrix}}$$

where p_1, p_2, p_3 are the perpendiculars from the vertices of the triangle of reference on the tangent at the point.

2. Consider the conics inscribed, circumscribed, or self-polar to the triange of reference, $u\alpha^n+v\beta^n+w\gamma^n=0$, where $n=\frac{1}{2},-1,2$ respectively. The formula becomes

$$\rho=\frac{4\Delta^2\alpha\beta\gamma}{(n-1)\ p_1p_2p_3}.$$

Thus if three conics touch at the same point of which one is inscribed in, one is circumscribed to, and one self-polar to the triangle of reference, their radii of curvature are respectively $-2\rho, -\frac{1}{2}\rho, \rho$, where $\rho=\dfrac{4\Delta^2\alpha\beta\gamma}{p_1p_2p_3}$ is the same for all. This theorem is due to Janet.

Again consider the conic $\alpha^2-k\beta\gamma=0$; we find

$$\rho=-\frac{4\Delta^2\alpha\beta\gamma}{p_1p_2p_3}.$$

If we use formala (6) we find that the ratio of the radii of curvature at B, C is $\rho_b : \rho_c=c^3 : b^3$, which is a well-known theorem.

If the tangents at any two points P, P′ of a conic intersect in T then

$$\rho : \rho'=\mathrm{TP}^3 : \mathrm{TP}'^3.$$

Consider any conic inscribed in a triangle, *viz* :

$$\sqrt{\lambda\alpha}+\sqrt{\mu\beta}+\sqrt{\nu\gamma}=0.$$

At the point of contact with $\alpha=0$, we have $\rho=\dfrac{\lambda\mu\nu a^3}{\Delta(\lambda+\mu)^3}$; and in particular for the conic touching the sides of a triangle at their middle points, $\rho_1 : \rho_2 : \rho_3=a^3 : b^3 : c^3$, giving the ratios of the radii of curvature at the points of contact.

Other examples of this formula will be found in Questions 380 and 392 of the J. I. M. S.

A. C. L. Wilkinson.

Some Theorems in Summation.

1. The following general theorems may be of use to those who take some special interest in the summation of series:—

(1) $\phi(0)+x\phi(1)+x^2\phi(2)+x^3\phi(3)+x^4\phi(4)+\ldots\ldots$ to infinity

$$=\frac{1}{1-x}\left\{\phi(0)+\frac{x}{1-x}\Delta\phi(0)+\left(\frac{x}{1-x}\right)^2\Delta^2\phi(0)+\ldots\right\}$$

(2) $\phi(0)+\frac{x}{\lfloor 1}\phi(1)+\frac{x^2}{\lfloor 2}\phi(2)+\frac{x^3}{\lfloor 3}\phi(3)+\ldots\ldots$

$$=e^x\left\{\phi(0)+\frac{x}{\lfloor 1}\Delta\phi(0)+\frac{x^2}{\lfloor 2}\Delta^2\phi(0)+\frac{x^3}{\lfloor 3}\Delta^3\phi(0)\ldots\right\}$$

(3) $\phi(0)-\frac{x^2}{\lfloor 2}\phi(2)+\frac{x^4}{\lfloor 4}\phi(4)-\frac{x^6}{\lfloor 6}\phi(6)+\frac{x^8}{\lfloor 8}\phi(8)-\ldots$

$$=\cos x\left\{\phi(0)-\frac{x^2}{\lfloor 2}\Delta^2\phi(0)+\frac{x^4}{\lfloor 4}\Delta^4\phi(0)-\frac{x^6}{\lfloor 6}\Delta^6\phi(0)\ldots\right\}$$

$$-\sin x\left\{x\Delta\phi(0)-\frac{x^3}{\lfloor 3}\Delta^3\phi(0)+\frac{x^5}{\lfloor 5}\Delta^5\phi(0)\ldots\right\}$$

(4) $x\phi(1)-\frac{x^3}{\lfloor 3}\phi(3)+\frac{x^5}{\lfloor 5}\phi(5)-\frac{x^7}{\lfloor 7}\phi(7)\ldots$

$$=\sin x\left\{\phi(0)-\frac{x^2}{\lfloor 2}\Delta^2\phi(0)+\frac{x^4}{\lfloor 4}\Delta^4\phi(0)\ldots\right\}$$

$$+\cos x\left\{x\Delta\phi(0)-\frac{x^3}{\lfloor 3}\Delta^3\phi(0)+\frac{x^5}{\lfloor 5}\Delta^5\phi(0)\ldots\right\}$$

(5) $\phi(0)+\frac{x^2}{\lfloor 2}\phi(2)+\frac{x^4}{\lfloor 4}\phi(4)+\frac{x^6}{\lfloor 6}\phi(6)+\ldots.$

$$=\cosh x\left\{\phi(0)+\frac{x^2}{\lfloor 2}\Delta^2\phi(0)+\frac{x^4}{\lfloor 4}\Delta^4\phi(0)+\ldots\right.$$

$$\left.+\sinh x\left\{x\Delta\phi(0)+\frac{x^3}{\lfloor 3}\Delta^3\phi(0)+\frac{x^5}{\lfloor 5}\Delta^5\phi(0)+\ldots\right\}\right.$$

(6) $x\phi(1)+\frac{x^3}{\lfloor 3}\phi(3)+\frac{x^5}{\lfloor 5}\phi(5)+\frac{x^7}{\lfloor 7}\phi(7)+\ldots$

$$=\sinh x\left\{\phi(0)+\frac{x^2}{\lfloor 2}\Delta^2\phi(0)+\frac{x^4}{\lfloor 4}\Delta^4\phi(0)+\ldots\right\}$$

$$+\cosh x\left\{x\Delta\phi(0)+\frac{x^3}{\lfloor 3}\Delta^3\phi(0)+\frac{x^5}{\lfloor 5}\Delta^5\phi(0)\ldots\right\}$$

2. Similar theorems may be derived by changing the sign of x on both the sides or by integrating both the sides between fixed limits.

The right hand side is rapidly convergent when $\phi(n)$ is a rational integral algebraical function of n.

Ex. Sum the series

$$1+\frac{x^2}{\lfloor 2}(1+2^4)+\frac{x^4}{\lfloor 4}(1+4^4)+\frac{x^6}{\lfloor 6}(1+6^4)+\ldots\ldots\ldots\text{to infinity.}$$

Here $\phi(n)$ is $1+n^4$. Therefore

$$\Delta\phi(n)=4n^3+6n^2+4n+1$$
$$\Delta^2\phi(n)=12n^2+24n+14$$
$$\Delta^3\phi(n)=48n+12$$
$$\Delta^4\phi(n)=48,$$

and all higher differences vanish.

Hence by theorem (5) the result is

$$\cosh x\left\{1+\frac{x^2}{\lfloor 2}.14+\frac{x^4}{\lfloor 4}.48\right\}+\sinh x\left\{\ x+\frac{x^3}{\lfloor 3}.12\right\}$$

3. The proofs of the theorems are simple and can be inferred by the reader. The above theorems are true for all functions which are not infinite when the numerical values 0, 1, 2, 3...are substituted.

4. The following theorem is due to Mr. S. Ramanujan, the Mathematics Research Student of the Madras University :

$$\int_0^\infty x^{n-1}\left\{\ \phi(0)-\frac{x}{\lfloor 1}\phi(1)+\frac{x^2}{\lfloor 2}\phi(2)-\frac{x^3}{\lfloor 3}\phi(3)\ldots\ldots\ \right\}dx$$
$$=\Gamma(n)\phi(-n),\text{ when } n \text{ is positive.}$$

This may be demonstrated as under :

Making use of theorem (2) above, with the sign of x changed, we have

$$\phi(0)-\frac{x}{\lfloor 1}\phi(1)+\frac{x^2}{\lfloor 2}\phi(2)-\frac{x^3}{\lfloor 3}\phi(3)+\ldots\ldots$$
$$=e^{-x}\left\{\ \phi(0)-\frac{x}{\lfloor 1}\Delta\phi(0)+\frac{x^2}{\lfloor 2}\Delta^2\phi(0)-\frac{x^3}{\lfloor 3}\Delta^3\phi(0)\ldots\right\}$$

Hence the integral reduces to

$$\int_0^\infty x^{n-1}e^{-x}\left\{\phi(0)-\frac{x}{\lfloor 1}\Delta\phi(0)+\frac{x^2}{\lfloor 2}\Delta^2\phi(0)-\frac{x^3}{\lfloor 3}\Delta^3\phi(0)+\ldots\right\}dx.$$

But $\int_0^\infty x^{n-1}e^{-x}\,dx$ is the Eulerian intergral which is $\Gamma(n)$ when n is positive.

Hence, integrating each term separately between the limits, we have the integral equal to

$$\Gamma(n)\phi(0)-\frac{\Gamma(n+1)}{\lfloor 1}\Delta\phi(0)+\frac{\Gamma(n+2)}{\lfloor 2}\Delta^2\phi(0)-\frac{\Gamma(n+3)}{\lfloor 3}\Delta^3\phi(0)+\ldots$$
$$=\Gamma(n).\left\{\ \phi(0)-\frac{n}{\lfloor 1}\Delta\phi(0)+\frac{n(n+1)}{\lfloor 2}\Delta^2\phi(0)-\ldots\right\}$$
$$=\Gamma(n).\phi(-n).$$

5. The above theorem of Mr. Ramanujan is very useful as it is possible, with its aid, to evaluate many definite integrals which are not integrable by the known methods.

This theorem is rigorously true if we stop at the penultimate step of the above demonstration. But in the form as given by Mr. S. Ramanujan *viz.*, $\Gamma(n)\phi(-n)$, we have to proceed with some caution as the transition from the penultimate step to the last step will not be strictly true in the case of some periodic functions. (*Vide*: Boole's *Finite Differences*).

7th September 1913. S. NARAYANA AIYAR.

The Face of the Sky for September and October 1913.

Sidereal time at 8 p.m.

	September.				October.		
	H.	M.	S.		H.	M.	S.
1	18	40	34	...	20	38	51
8	19	8	10	...	21	6	27
15	19	35	46	...	21	34	3
22	20	3	22	...	22	1	39
29	20	30	58	...	22	29	65

From this table the constellations visible during the evenings of September and October can be ascertained by a reference to the positions as given in a star-atlas.

The Sun

Enters the autumnal equinox on September 23 at 7-35 a m.

Phases of the Moon.

		September.					October.			
		D.	H.	M.			D.	H.	M.	
New Moon	...	1	2	8	A.M.	...	...		...	
First Quarter	...	7	6	36	P.M.	...	6	0	4	A.M.
Full Moon	...	15	6	16	,,	...	14	4	11	,,
Last Quarter	...	23	6	30	,,	...	21	1	26	P.M.
New Moon	...	30	10	27	A.M.	...	28	7	11	A.M.

Eclipses.

There will be Solar eclipses on August 31 and September 30 invisible in India. A lunar eclipse on September 15 is visible in India.

First contact	...	2-50 P.M.
Middle	...	4-32 „
Last contact	...	6-35 „

The Planets.

Mercury is in superior conjunction with the sun on September 16 and is in conjunction with the moon on October 31.

Venus continues to be a morning star. It is in conjunction with the moon on September 28 and October 28.

Mars is in quadrature to the sun on October 2 and is in conjunction with the moon on September 24 at 1-52 A.M. and on October 22 at 6-37 A.M.

Juno is in conjunction with the moon on September 15 at 1-30 A.M.

Jupiter is stationary on September 4. It is in quadrature to the sun on October 3, and in conjunction with the moon on September 9 and on October 6 at 11-51 P.M.

Saturn is stationary on September 30. It is in conjunction with the moon on September 22 and on October 20 at 2-49 A.M.

Uranus is stationary on October 14 and in quadrature to the sun on October 27. It is in conjunction with the moon on October 8.

Neptune is in conjunction with the moon on September 25 at 5-37 A.M. and on October 23 at 1-24 A.M.

V. RAMESAM.

SOLUTIONS.

Question 380.

(A. C. L. WILLKINSON, M. A.) :—At the extremity A of the major (or minor) axis of a conic S, the circle of curvature C is drawn. **PQ is** any chord of the conic S which touches C, and T is its pole with respect to the conic S. A conic S′ is described through PQT touching the conic S at A. Show that the tangents at P, Q to S′ also touch the circle C, and that the radius of curvature of the conic S′ at A is half the radius of the circle C.

Solution by J. C. Swaminarayan M. A.

Let the equation of the conic S referred to A as origin be

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{2x}{a}=0 \qquad \dots \qquad \dots \qquad (1)$$

A being the vertex $(-a, 0)$.

The equation to C is easily found to be

$$\frac{x^2}{b^2}+\frac{y^2}{b^2}-\frac{2x}{a}=0. \qquad \dots \qquad \dots \qquad (2)$$

If T be the point $(x'\ y')$ the equation of S′ will be

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{2x}{a}+\lambda x\left\{\frac{xx'}{a^2}+\frac{yy'}{b^2}-\frac{x+x'}{a}\right\}=0$$

where $$\frac{x'^2}{a^2}+\frac{y'^2}{b^2}-\frac{2x'}{a}+\lambda x'\left\{\frac{x'^2}{a^2}+\frac{y'^2}{b^2}-\frac{2x'}{a}\right\}=0$$

i.e. $\lambda x'+1=0.$

Hence the equation of S′ is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{2x}{a}+x\left\{\ x\left(\frac{1}{ax'}-\frac{1}{a^2}\right)-\frac{yy'}{b^2x'}+\frac{1}{a}\right\}=0$$

i.e., $$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{2x}{a}+x\ \left\{lx+my+\frac{1}{a}\right\}=0 \ \dots \qquad \dots \ (3)$$

where l, m are arbitrary, $lx+my+\frac{1}{a}=0$ is the equation of PQ.

Since PQ touches the circle C we must have

$$m^2b^4-2lb^2-1=0$$

or $$b^2m^2=2l+\frac{1}{b^2}. \qquad \dots \qquad \dots \qquad \dots \qquad \dots \qquad (4)$$

The circle of curvature of S′ at A is seen to be

$$\frac{x^2}{b^2}+\frac{y^2}{b^2}-\frac{x}{a}=0.$$

Hence the radius of curvature of S′ is half the radius of circle C.

Tangents at P, Q to S′ will meet in (ξ, η) if the polar of (ξ, η) with respect to S′, namely

$$x\left\{\frac{1}{a\xi}-\frac{m\eta}{\xi}-2\left(\frac{1}{a^2}+l\right)\right\}-y\left\{\frac{2\eta}{b^2\xi}+m\right\}+\frac{1}{a}=0,$$

is identical with PQ.

$$\therefore\quad \left.\begin{aligned}\eta&=-b^2m\xi\\ \frac{1}{a\xi}&=l+\frac{2}{a^2}-\frac{1}{b^2}\end{aligned}\right\}\quad \cdots\quad \cdots\quad (5)$$

The equation of the tangents from (ξ, η) to S′, when simplified assumes the form

$$\left(\frac{1}{b^2}-\frac{1}{a^2}\right)\left\{x^2\left(\frac{1}{a^2}+l\right)+\frac{y^2}{b^2}+mxy-\frac{x}{a}\right\}=\frac{1}{4}\left(lx+my+\frac{1}{a}\right)^2. \quad (6)$$

Similarly the equation of the tangents from (ξ, η) to circle C is

$$\left(\frac{x^2}{b^2}+\frac{y^2}{b^2}-\frac{2x}{a}\right)\left(\frac{\xi^2}{b^2}+\frac{\eta^2}{b^2}-\frac{2\xi}{a}\right)=\left(\frac{x\xi}{b^2}+\frac{y\eta}{b^2}-\frac{x+\xi}{a}\right)^2,$$

which with the help of (4) and (5) at once becomes

$$\left[\frac{x^2}{b^2}+\frac{y^2}{b^2}-\frac{2x}{a}\right]\left(\frac{1}{b^2}-\frac{1}{a^2}\right)=\frac{1}{4}\left[2x\left(\frac{1}{b^2}-\frac{1}{a^2}\right)-\left(lx+my+\frac{1}{a}\right)\right]^2$$

$$\therefore\left[\frac{x^2}{b^2}+\frac{y^2}{b^2}-\frac{2x}{a}-x^2\left(\frac{1}{b^2}-\frac{1}{a^2}\right)+x\left(lx+my+\frac{1}{a}\right)\right]\left(\frac{1}{b^2}-\frac{1}{a^2}\right)$$

$$=\frac{1}{4}\left(lx+my+\frac{1}{a}\right)^2,$$

$$\therefore\left(\frac{1}{b^2}-\frac{1}{a^2}\right)\left[x^2\left(\frac{1}{a^2}+l\right)\frac{y^2}{b^2}+mxy-\frac{x}{a}\right]=\frac{1}{4}\left(lx+my+\frac{1}{a}\right)^2. \quad (7)$$

Since the equations (6) and (7) are identical, tangents at P, Q to S′ touch also the circle C.

Question 421.

(A. C. L. Wilkinson, M.A., F.R.A.S.) :—An ellipsoid of revolution is inscribed in a tetrahedron ABCD with its focus at the centroid G of the tetrahedron. Prove that the plane parallel to BCD which touches the ellipsoid divides GA in the ratio of three times the square of the triangle BCD to four times the sum of the squares of the triangles BCD, CDA, DAB, ABC.

Solution (1) *by N. Durairajan*, (2) *by A.K. Erady, M.A.*

(1) Let K be the semi-minor axis of the ellipse which is revolved about the major axis.

The quadriplanar coordinates of G are

$$(\tfrac{1}{4}\,p_1\ \tfrac{1}{4}\,p_2,\ \tfrac{1}{4}\,p_3,\ \tfrac{1}{4}\,p_4.)$$

If $\alpha, \beta, \gamma, \sigma$ be the other focus,

$$\tfrac{1}{4} p_1 \alpha = k^2 = \tfrac{1}{4} p_2 \beta = \tfrac{1}{4} p_3 \gamma = \tfrac{1}{4} p_4 \delta.$$

But α = perpendicular from G on the tangent plane parallel to BCD.

If this tangent plane meets GA in F,

$$\mathrm{GF} : \mathrm{GA} = \alpha : \tfrac{3}{4} p_1$$

$$\therefore \quad \frac{\mathrm{GF}}{\mathrm{GA}} = \frac{\alpha}{\frac{3}{4} p_1} = \frac{16k^2}{3p_1^2}.$$

Now, $a\alpha + b\beta + c\delta + d\delta = 3\mathrm{V} = ap_1 = \ldots\ldots$

where a, b, c, d are the areas of the faces.

$$\therefore \quad 3\mathrm{V} = \frac{4k^2}{p_1} \times a + \ldots + \ldots + \ldots$$

$$= \frac{4k^2}{3\mathrm{V}} (a^2 + b^2 + c^2 + d^2).$$

$$\therefore \quad k^2(a^2 + b^2 + c^2 + d^2) = \frac{9}{4} \mathrm{V}^2 \quad \ldots \quad \ldots \quad (1)$$

$$\therefore \quad \frac{\mathrm{GF}}{\mathrm{GA}} = \frac{16k^2}{3p_1^2} = \frac{16k^2a^2}{3a^2p_1^2} = \frac{16k^2a^2}{27\mathrm{V}^2}$$

$$= \frac{16a^2}{27\mathrm{V}^2} \times \frac{9}{4} \frac{\mathrm{V}^2}{a^2 + b^2 + c^2 + d^2}$$

$$= \frac{4}{3} \frac{a^2}{(a^2 + b^2 + c^2 + d^2)}.$$

Thus $\mathrm{GF} : \mathrm{GA} = 4a^2 : 3(a^2 + b^2 + c^2 + d^2)$.

(2) Take a system of rectangular axes with G as origin and with the unequal axis of the ellipsoid as the axis of x. Let a_1, a_2, a_3 be the projections of the area a on the co-ordinate planes.

Then the equation to the plane BCD can be put in the form

$$a_1 x + a_2 y + a_3 z + \tfrac{3}{4} \mathrm{V} = 0.$$

Let the equation to the ellipsoid be

$$\frac{(x - he)^2}{h^2} + \frac{y^2 + z^2}{k^2} = 1.$$

Then the plane BCD can also be written in the form ... (1)

$$a_1(x - he) + a_2 y + a_3 z = -\left(\frac{3\mathrm{V}}{4} + he\, a_1\right) \ldots. \quad \ldots \quad (2)$$

The condition that (2) shall touch (1) is

$$h^2a_1^2 + k^2(a_2^2 + a_3^2) = (\tfrac{3}{4}\mathrm{V} + eh\, a_1)^2.$$

i.e., $$h^2a^2 = \frac{9\mathrm{V}^2}{4} + \frac{3\mathrm{V}}{2e} h\, a_1 \; (a_1^2 + a_2^2 + a_3^2 = a^2)\,; \quad \ldots \quad (3)$$

Similarly three other equations are obtained in order that the other three planes may touch the ellipsoid. Adding up these four equations we have

$$k^2\Sigma(a^2)=\frac{9}{4}V^2, \text{ since } a_1+b_1+c_1+d_1=0. \qquad \dots \qquad \dots \qquad (4)$$

Now the plane parallel to (2) and touching (1) is evidently

$$a_1 (x-he)+a_2 y+a_3 z=\tfrac{3}{4}V+he\ a_1,$$

i.e.,

$$a_1 x+a_2 y+a_3 z=\tfrac{3}{4} V+2\ he\ a_1.$$

If this plane cuts GA at the point F $(x_0\, y_0\, z_0)$, and A is $(x_1\, y_1\, z_1)$, then

$$\frac{GF}{GA}=\frac{x_0}{x_1}=\frac{y_0}{y_1}=\frac{z_0}{z_1}=\frac{\frac{3}{4} V+2\ he\ a_1}{a_1x_1+a_2y_1+a_3z_1},$$

But $a_1\ x_1+a_2y_1+a_3z_1+\frac{3}{4}\ V=p_1a=3\ V$, where p_1 is $\perp^r$ from A on BCD.

Hence

$$\frac{GF}{GA}=\frac{\frac{3}{4}V+2e\ h\ a_1}{\frac{9}{4}\ V}=\frac{16\ k^2a^2}{27\ V^2}, \text{ from (3)}$$

$$=\frac{16\ a^2{\cdot}k^2\Sigma(a^2)}{27\ V^2{\cdot}\Sigma(a^2)}$$

$$=\frac{4\ a^2}{3(a^2+b^2+c^2+d^2)},$$

where a, b, c, d are the areas of the four faces.

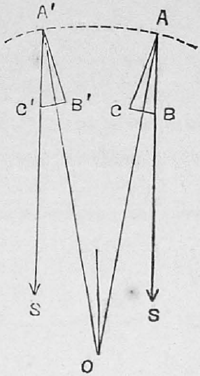
N.B.—The result given in the question is not correct as can be easily verified in the case of a sphere inscribed in a regular tetrahedron, which is a particular case of the above question.

Question 434.

A railway passenger seated in one corner of the carriage looks out of the window at the further end and observes that a star near the horizon is traversing these windows in the direction of the train's motion and that it is obscured by the partition between the corner windows on his own side of the carriage and the middle window, while the train is moving through the seventh part of a mile. Prove that the train is on the curve, the concavity of which is directed towards the star and which if it be circular, has a radius of nearly three miles, the breadth of the carriage being 7 ft., and the breadth of the partition 4 inches.

Solution by N. Durairajan.

Let A denote the passenger and BC the partition. In the triangle ABC the altitude is 7 ft., and the base BC is 4 inches. Since the altitude is large compared with the base, the circular measure of the angle BAC is $\frac{4}{84}=\frac{1}{21}$ nearly.



Now, since the star is near the horizon, it may be supposed to be in the plane ABC. If O be the centre of the arc AA′ described by the train, it must be the intersection of the normals at A and A′; that is the altitudes of the triangles ABC, A′B′C′ meet at O. Also the star which is at an infinite distance is obscured by the partition BC while the train passes from A to A′, where AA′ = 1/7 mile. Hence AB, A′C′ point to the star and are parallel; and the angle AOA′ = OAS + OA′S = OAS + OAC = BAC, since OAC = OA′C′.

Thus the circular measure of AOA′ = *c.m.* of BAC = 1/21 nearly.

∴ rad. OA of the arc AA′ = 21 × arc AA′

= 21 × 1/7 = 3 miles nearly.

Question 449.

(S. Krishnaswami Aiyangar) :—Establish the result

$$\frac{1}{1^4}-\frac{1}{3^4}-\frac{1}{5^4}+\frac{1}{7^4}+\frac{1}{9^4}-\ldots=\frac{11\pi^4}{768\sqrt{2}}.$$

Solution by T. P. Trivedi and K. J. Sanjana.

It is known that

$$\cos\theta-\frac{1}{3}\cos 3\theta+\ldots\ldots=\frac{\pi}{4}.$$

Integrating with respect to θ, we get

$$\sin\theta-\frac{1}{3^2}\sin 3\theta+\frac{1}{5^2}\sin 5\theta\ldots\ldots=\frac{\pi}{4}\theta;$$

no constant is added as the left side vanishes with θ.

Integrating again,

$$-\cos\theta+\frac{1}{3^3}\cos 3\theta-\frac{1}{5^3}\cos 5\theta+\ldots\ldots=\frac{\pi\theta^2}{8}+C;$$

putting $\theta=0$, $$C=-\frac{1}{1^3}+\frac{1}{3^3}-\frac{1}{5^3}+\ldots\ldots\ldots=-\frac{\pi^3}{32};$$

so that . $$\cos\theta-\frac{1}{3^3}\cos 3\theta+\frac{1}{5^3}\cos 5\theta-\ldots\ldots=\frac{\pi^3}{32}-\frac{\pi\theta^2}{8}.$$

Finally, integrate again, we have

$$\sin\theta-\frac{1}{3^4}\sin 3\theta+\frac{1}{5^4}\sin\ 5\theta-\ldots\ldots=\frac{\pi^3\theta}{32}-\frac{\pi\theta^3}{24}.$$

Putting $\theta=\frac{\pi}{4}$ in these four equalities and simplifying, we obtain

$$1+\frac{1}{3}-\frac{1}{5}-\frac{1}{7}+\frac{1}{9}\ldots\ldots\ldots\ldots\ldots\ldots=\frac{\pi}{2\sqrt{2^3}}$$

$$1-\frac{1}{3^2}-\frac{1}{5^2}+\frac{1}{7^2}+\frac{1}{3^2}\ldots\ldots\ldots\ldots=\frac{3\pi^3}{8\sqrt{2^3}}$$

$$1+\frac{1}{3^3}-\frac{1}{5^3}-\frac{1}{7^3}+\frac{1}{9^3}\ldots\ldots\ldots\ldots=\frac{3\pi^3}{64\sqrt{2^3}}$$

$$1-\frac{1}{3^4}-\frac{1}{5^4}+\frac{1}{7^4}+\frac{1}{9^4}\ldots\ldots\ldots\ldots=\frac{11\pi^4}{768\sqrt{2}}$$

Remarks by J. C. Swaminarayan, M. A.

This is a particular case of the identity

$$\cos x+\frac{\cos 3x}{3^4}+\frac{\cos 5x}{5^4}+\ldots\ldots=\frac{\pi}{48}\left(\frac{\pi}{2}-x\right)\left(\pi^2+2\pi x-2x^2\right)$$

proposed by me in the *Ed. Times*, Jan. 1912, Q. 17237. If $x=\frac{\pi}{4}$, the required result follows.

Question 450.

(K. Appukuttan Erady, M.A.):—If A, B, C......are the minors of a, b, c......in the determinant

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$$

and if $$\frac{al+hm+gn}{l}=\frac{hl+bm+fn}{m}=\frac{gl+fm+cn}{n},$$

prove that $$\frac{Al+Hm+Gn}{l}=\frac{Hl+Bm+Fn}{m}=\frac{Gl+Fm+Cn}{n}.$$

Hence, prove that the cone $(a\,b\,c\,f\,g\,h)\,(x\,y\,z)$ and its reciprocal are coaxal.

Solution by J. C. Swaminarayan, M.A.

By hypothesis,

$$l=\frac{1}{\lambda}(al+hm+gn),\quad m=\frac{1}{\lambda}(hl+bm+fn),\ n=\frac{1}{\lambda}(gl+fm+cn),$$

where λ is the value of each of the given ratios.

$$\therefore\ Al+Hm+Gn=\frac{1}{\lambda}\left\{(Aa+Hh+Gg)l+(Ah+Hb+Gf)m+(Ag+Hf+Gc)n\right\}=\frac{\triangle}{\lambda}l.$$

$$\therefore\quad \frac{Al+Hm+Gn}{l}=\frac{Hl+Bm+Fn}{m}=\frac{Gl+Fm+Cn}{n}=\frac{\triangle}{\lambda},$$ whence the result.

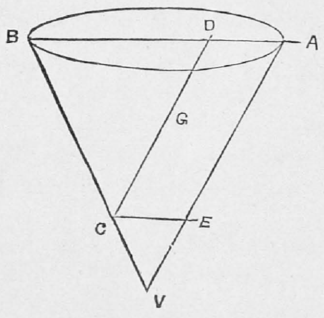
(ABCFGH)(xyz) is the reciprocal of the cone $(abcfgh)$ (xyz). The equations that give the axes for each of them as interdependent. Hence the cone and its reciprocal are coaxal.

Question 453.

(S. P. Singaravelu Modeliar, B.A., L.T.) :—A hollow right cone is filled which a homogeneous liquid and held with axis vertical and vertex down. Find the parabolic section of the cone on which the thrust is a maximum.

Solution by T. P. Trivedi, M.A., L.L.B.

Let VAB be any cone, and CD the axis of a parabolic section, parallel to the generator VA.



Let 2α be the semi-vertical angle of the cone.

Draw CE parallel to the base AB. The latus-rectum of the parabolic section whose vertex is C is CE^2/CV. Let $CV=l'$ and $VB=l$; then the latus rectum $=4\ l' \sin^2\alpha$.

Again, the ordinate of the parabola at D is $\sqrt{4\ l'\ \sin^2\alpha.\ CD}=\sqrt{4\ l'\ \sin^2 a\ (l-l')}$, since $CD=CB=l-l'$.

$\therefore$ The area of the parabolic section

$$=\frac{2}{3}.\ 4\ \sin\alpha\sqrt{l'(l-l')}\ (l-l').$$

Take G in CD so that $DG=\frac{2}{5}$ DC ; then G is the *c.g.* of the parabolic area. The depth of G below the free surface is

$$DG\cos\alpha=\frac{2}{5}\ (l-l')\cos\alpha.$$

$\therefore$ The thrust on the parabolic area

$$=\frac{16}{15}\,g\,\rho\,w\sin\alpha\,\cos\alpha\ l'^{\frac{1}{2}}(l-l')^{\frac{5}{2}}.$$

For this to be a maximum or minimum, we require

$$\tfrac{1}{2}\,l'^{-\frac{1}{2}}(l-l')^{\frac{5}{2}}-\frac{5}{2}l'^{\frac{1}{3}}\,(l-l')^{\frac{3}{2}}=0,\text{ or }\infty.$$

This gives (1) $l'=0$ or l, which clearly corresponds to a minimum thrust ; or (2) $l-l'-5l'=0$, *i.e.*, $l'=\frac{1}{6}\,l$, which gives the maximum thrust.

Question 454.

(A. A. KRISHNASWAMI AIYANGAR) :—If $s_r=1^r+2^r+3^r+\ldots\ldots+x^r$ prove that

$$sC_1s-_3+({}_{r-1}C_1-{}_rC_2)s_{r-2}+$$
$$({}_{r-2}C_r-{}_{r-1}C^2+rC_3)s_{r-3}+\ldots\ldots\ldots\text{for } r \text{ terms}=\frac{(n^{r+1}-n}{(n-1)}.$$

Solution by N. P. Pandya aud J. C. Swaminarayanan.

$$(1+x)^r+(2+x)^r+\ldots\ldots+(n+x)^r=s_r+{}_rC_1s_{r-1}x+{}_rC_2s_{r-2}x^2+\ldots\ldots+{}_rC_rs\phi.$$

Put $x=-1$, then

$$s_r-n^r=s_r-{}_rC_1s_{r-1}+{}_rC_2s_{r-2}-\ldots\ldots=(-1)^r{}_rC_2s_0.$$

$$\therefore\quad {}_rC_1s_{r-1}-{}_rC_2s_{r-2}+{}_rC_3s_{r-3}-{}_rC_4s_{r-4}+\ldots\ldots-(-1){}_rC_rs_o=n^r.$$

Similarly

$$ {}_{r-1}C_1s_{r-1}-{}_{r-1}C_2s_{r-3}+{}_{r-1}C_3s_{r-4}+\ldots\ldots(-1)^{r-1}{}_{r-1}C_{r-1}S_c=n^{r-3}$$
$$ {}_{r=3}C_1s_{r-9}+{}_{r-2}C_2s_{r-4}+\ldots\ldots-(-1)^{r-2}{}_{r-2}C_{r-2}s_o=n^{r-2}$$

... ...

... ...

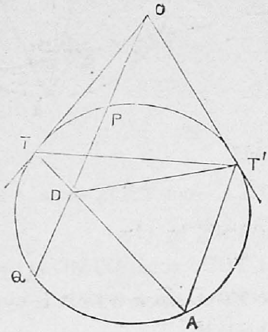
$$-(-1)^1{}_1C_1s_o=n^2.$$

Adding up the vertical columns, we get the required result.

Question 455.

(J. C. SWAMINARAYAN, M. A.) :—From an external point O, a tangent OT and a secant OPQ are drawn to a circle; D is the middle point of PQ, and TD cuts the circle again at A. If any other secant OXY is drawn, and AX, AY cut PQ in R, S, prove that RD = DS.

Solution (1) by V. Ramesam, (2) by V. V. Satyanarayan.



(1) Drawn AT′ parallel to OPQ, cutting the circle in T′ Join OT′ DT′ TT′. Then $\angle OTT' = \angle TAT' = \angle ADQ = \angle ODT'$. ∴ OTDT′ is a cyclic quldriateral.

$$\therefore \qquad \angle OT'T = \angle ODT = \angle TAT'.$$

Therefore CT′ is the other tangent to the circle, and the theorem then is the same as Question 26 by Mr. T. Rajaram Rao, a solution of which appears on page 27 of Vol. I, of this Journal.

(2) To show that RD = DS, it will suffice to prove

$$\cot \angle RAD - \cot \angle SAD = 2 \cot \angle SDA,$$

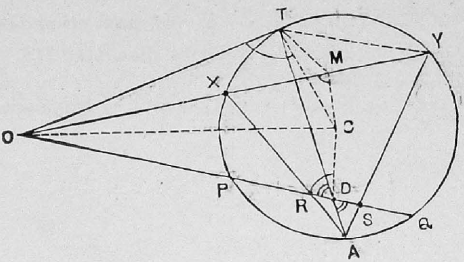
by the well-known theorem that if AM be the median of ABC, then

$$\cot B - \cot C = \cot BAM - \cot CAM = 2 \cot AMC. \quad \ldots \quad (1)$$

Let C be the centre of the circle and M the middle point of XY.

Join the necessary lines as in the figure.

It is easy to see that the quadrilaterals AXTY, OTMC, OTCD are cyclic ; the first is cyclic evidently, and the last two because, the angles OTC, OMC, ODC are right angles.



$\therefore$ cot $R\hat{A}D$ − cot $S\hat{A}D$

= cot $T\hat{Y}X$ − cot $T\hat{X}Y$, since AXTY is cyclic,

= 2 cot $T\hat{M}X$ by (1)

= 2 cot $T\hat{C}O$, since OTMC is cyclic

= 2 cot $T\hat{D}O$, since OTCD is cyclic

= 2 cot $S\hat{D}A$.

Hence AD is the median of $\triangle$RAS, so that RD = DS.

Question 456.

(SELECTED) .—If $f(x)$ is a continuous function of x, not necessarily satisfying the conditions for expansion as a Fourier series, prove that

$$\frac{1}{2\pi}\int_0^{2\pi}[f(x)]^2dx = a_o^2+\tfrac{1}{2}\sum_{n=1}^{\infty}(a_n^2+b_n^2),$$ where $a_0, a_1, a_2 \ldots b_1, b_2 \ldots\ldots$

are the Fourier constants of $f(x)$.

Shew also that $a_ox+\Sigma\frac{1}{x}\{a_n \sin nx + b_n (1-\cos nx)\} = \int_0^x f(\xi)d\xi.$

Solution by J. C. Swaminarayan, M.A.

Here $f(x) = a_o + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$.

It is easy to see that $\int_0^{2\pi}\cos rx\, dx = 0, \int_0^{2\pi}\sin rx\, dx = 0,$

$$\int_0^{2\pi} \sin rx \cos sx \, dx = 0, \int_0^{2\pi} \sin rx \sin sx \, dx = 0,$$

$$\int_0^{2\pi} \sin^2 rx \, dx = \pi, \int_0^{2\pi} \cos^2 rx \, dx = \pi.$$

$$\therefore \quad \int_0^{2\pi} [f(x)]^2 dx = \int_0^{2\pi} \{ a_o^2 + \sum_{n=1}^{\infty} (a_n^2 \cos^2 nx + b_n^2 \sin^2 nx) \} \, dx$$

$$= 2\pi a_o^2 + \pi \sum_{n=1}^{\infty} (a_n^2 + b_n^2),$$

whence the given result follows at once.

$$\int_0^x f(\xi) d\xi = \int_0^x [a_o + \Sigma(a_n \cos n\xi + b_n \sin n\xi)] d\xi$$

$$= \left[a_o \xi + \sum \left(a_n \frac{\sin n\xi}{n} - b_n \frac{\cos n\xi}{n} \right) \right]_0^x$$

$$= a_o x + \Sigma \frac{1}{n} \{ a_n \sin nx + b_n (1 - \cos nx) \} .$$

Question 457.

(S. P. / SINGARAVELU MODELIAR, B.A., L.T.): Shew that the curve $4(2x^2-1)^2 + (y^2-4)(y^2-1)^2 = 0$, is the path of a point whose motion is the resultant of two simple harmonic motions in perpendicular directions.

Solution by K. J. Sanjana M. A., and Trivedi M.A., L.L.B.

The two motions in perpendicular directions may be represented by

$$x = \sin \frac{3\theta}{2} t \text{ and } y = 2 \sin \theta t.$$

Now

$$4 (2x^2-1)^2 \equiv 4 \left(2 \sin^2 \frac{3\theta}{2} t - 1 \right)^2 = 4 \cos^2 3\theta t,$$

Again

$$\cos 3\theta t = 4 \cos^3 \theta t - 3 \cos \theta t = \cos \theta t \, (1 - 4 \sin^2 \theta t)$$

$$= (1-y^2)\sqrt{1-y^2/4}$$

$$\therefore \quad 4 \cos^2 3\theta t = (4-y^2)(y^2-1^2)^2$$

Hence $\quad 4 (2x^2-1)^2 = (4-y^2)(y^2-1)^2,$

or $\quad 4 (2x^2-1)^2 + (y^2-4)(y^2-1) = 0.$

QUESTIONS FOR SOLUTION.

481. (R. VYTHYNATHASWAMY) :—If $x^{n-1} \equiv a_0 + a_1(x-1) + a_2(x-1)(x-2) + \ldots a_{n-1}(x-1)(x-2)\ldots(x-n+1)$, shew that

$$(xd/dx)^n y = a_0 x y_1 + a_1 x^2 y_2 + \ldots a_{n-1} x^n y_n.$$

Hence shew that, if $f(x) = A_0 + A_1 x + A_2 x^2 + \ldots$

$$A_1 1^n + A_2 2^n + A_3 3^n + \ldots = a_0 f'(1) + a_1 f''(1) + a_2 f'''(1) + \ldots a_{n-1} f^n(1).$$

492. (A. NARASINGA RAO) :—If d_r be the number of divisors of the integer r (unity excluded), shew that

$$\underset{n-\leftarrow\infty}{\mathrm{Lt}} (d_1 + d_2 + \ldots d_n) \rightarrow n\log n.$$

483. (V. V. S. NARAYAN) :—Construct a triangle ABC being given the angle A, the side AB, and the distance of the middle point of AC from the symmedian point.

484. (K. APPUKUTTAN ERADY, M.A.) :—If $\phi(xyz) \equiv (abcfgh)(xyz)^2$, prove that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\phi}\cos(px+qy+rz)\,dxdydz = \frac{\pi^{\frac{3}{2}}}{\Delta^{\frac{1}{2}}} e^{-\lambda},$$

where Δ is the discriminant of ϕ, and $\lambda = (ABCFGH)(pqr)^2 \div 4\Delta$, A,B,C... being the cofactors $a,b,c,\ldots$ in Δ.

485. (S. KRISHNASWAMI AIYANGAR) :—Normals at P,Q,R,S on the ellipse $x^2/a^2 + y^2/b^2 = 1$ meet at O. If PQ envelops $x^{\frac{2}{3}} + y^{\frac{2}{3}} = (ab)^{\frac{1}{3}}$, shew that the envelope of RS is a coaxal ellipse.

486. (A. N. RAGHAVACHAR, M.A.) :—If $\alpha, \beta, \gamma, \delta$ be the roots of $x^4 + px^3 + qx^2 + rx + s = 0$, form the equation whose roots are

$$\frac{\beta\gamma(\alpha+\delta) - \alpha\delta(\beta+\gamma)}{\beta+\gamma-\alpha-\delta}, \text{ \&c., \&c.}$$

487. (N. P. PANDYA) :—If P be a point on a hyperbola whose foci are S and H, show that its asymptotes are parallel to the axes of the parabolas passing through S, H and having P for their common focus.

488. (ZERO) :—Prove that the roots of the determinant equation of the n^{th} degree

$$\begin{vmatrix} x, 1, 0, 0, 0, \ldots \\ 1, x, 1, 0, 0, \ldots \\ 0, 1, x, 1, 0, \ldots \\ 0, 0, 1, x, 1, \ldots \\ \cdot\ \ \cdot\ \ \cdot\ \ \cdot\ \ \cdot \end{vmatrix} = 0,$$

are $2\cos\frac{k\pi}{n+1}(k=1,2,3,\ldots n)$.

489. (S. RAMANUJAN) :—Shew that

$$(1+e^{-\pi\sqrt{55}})(1+e^{-3\pi\sqrt{55}})(1+e^{-5\pi\sqrt{55}})\ldots$$
$$=\frac{1+\sqrt{(3+2\sqrt{5})}}{\sqrt{2}}\left\{e^{-\pi\sqrt{55}}\right\}^{\frac{1}{24}}.$$

490. (R. SRINIVSAN, M.A.) :—Shew that

$$\sum_{n=0}^{\infty}\frac{\Gamma(n+1)}{(2n+1)\Gamma(n+\frac{3}{2})}=\sum_{n=1}^{\infty}\left\{\frac{4}{\sqrt{\pi}}(-1)^n\frac{1}{(2n+1)^2}\right\}$$

491. (M. T. NARANIENGAR) :—In any triangle prove that a circum-conic passing through the ends of a diameter of the maximum inscribed ellipse touches the ellipse..

492. (K. V. ANANTANARAYANA SASTRI, B.A.) :—Expand $\theta \tan \theta/2$ in powers of $\sin \theta$.

493. (S. P. SINGARAVELU MODELIAR) :—Sum to infinity the following series :

(1) $\frac{1}{1^2}+\frac{1}{2}\cdot\frac{1}{3^2}+\frac{1\cdot3}{2\cdot4}\cdot\frac{1}{5^2}+\frac{1\cdot3\cdot5}{2\cdot4\cdot6}\cdot\frac{1}{7^2}+\ldots$

(2) $\frac{1}{1^3}+\frac{1}{2}\cdot\frac{1}{3^3}+\frac{1\cdot3}{2\cdot4}\cdot\frac{1}{5^3}+\frac{1\cdot3\cdot5}{2\cdot4\cdot6}\cdot\frac{1}{7^3}+\ldots$

(3) $\frac{1}{2}\frac{s_1}{3}+\frac{1}{2\cdot4}\frac{s_2}{5}+\frac{1\cdot3}{2\cdot4\cdot6}\frac{s_3}{7}+\frac{1\cdot3\cdot5}{2\cdot4\cdot6\cdot8}\frac{s_4}{9}+\ldots$

where s_n = the sum of the squares of the reciprocals of the first n odd numbers.

494. S. NARAYANA AIYAR, M.A. :—Prove that

$$\int_0^{\infty}\frac{\phi(a+ix)-\phi(a-ix)}{2ix}dx=\frac{\pi}{2}\{\phi(a)-\phi(\infty)\}.$$

495. (A. C. L. WILKINSON) :—Prove that

$$\begin{vmatrix} 1, & \cos c, & \cos b, & \cos(b-c) \\ \cos c, & 1, & \cos a, & \cos(c-a) \\ \cos b, & \cos a, & 1, & \cos(a-b) \\ \cos(b-c), & \cos(c-a), & \cos(a-b), & 1 \end{vmatrix} = -\sigma^2$$

where $\sigma \equiv \sin(s-a).\sin(s-b).\sin(s-c)$.