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PROGRESS REPORT.

The following gentlemen have been elected members of the Society :—

(1) Mr. Mahadev Gangadhar Lele, Teacher, New English School, 75, Shanwar Peth, Poona City (*at concessional rate*) :

(2) Mr. S.B. Bondale, M.A., Assistant Professor of Mathematics, Fergusson College, Poona City (*at concessional rate*).

2. The following book has been purchased for the Library—

Principia Mathematica Part III—by Messrs. A. N. Whitehead and B. Russell, Camb. Univ. Press, 1913. 21*s*.

POONA,
31*st July* 1913.

D. D. KAPADIA,
Hon : Joint Secretary.

On Tetrahedral Coordinates.

By A. C. L. Wilkinson, M.A., F.R.A.S.

(*Continued from page 55.*)

§ 15. *Properties of a tetrahedron.*

I propose to solve analytically the problem of determining all tetrahedra for which the three shortest distances between pairs of opposite edges intersect in a point; while, however, this is the direct object of the following sections a number of other results that arise out of the work will be noticed.

We commence by finding the equations of the shortest distances:

Take the common perpendicular to AD and BC. If

$$\frac{\alpha-\alpha'}{l}=\frac{\beta-\beta'}{m}=\frac{\gamma-\gamma'}{n}=\frac{\delta-\delta'}{p},$$

is this perpendicular, the conditions of perpendicularity between this straight line and the lines

$$\frac{\alpha-1}{1}=\frac{\beta}{0}=\frac{\gamma}{0}=\frac{\delta}{-1}, \text{ and } \frac{\alpha}{0}=\frac{\beta-1}{1}=\frac{\gamma}{-1}=\frac{\delta}{0},$$

are seen to be

$$m\text{AB}^2+n\text{AC}^2+(p-l)\text{AD}^2-m\text{BD}^2-n\text{CD}^2=0,$$
$$l\text{AB}^2-l\text{AC}^2-(m-n)\text{BC}^2+p\text{BD}^2-p\text{CD}^2=0,$$
$$l+m+n+p=0$$

The plane through BC containing the shortest distance is

$$p\alpha-l\delta=0,$$

since this passes through the point (l,m,n,p) at infinity.

Eliminating $l:m:n:p$, this plane is

$$\begin{vmatrix} -\delta & 0 & 0 & \alpha \\ \text{AB}^2-\text{AC}^2 & -\text{BC}^2 & \text{BC}^2 & \text{BD}^2-\text{CD}^2 \\ -\text{AD}^2 & \text{AB}^2-\text{BD}^2 & \text{AC}^2-\text{CD}^2 & \text{AD}^2 \\ 1 & 1 & 1 & 1 \end{vmatrix}=0.$$

On expansion this will be found to be

$$\delta[f(\text{CD})+f(\text{BD})]=\alpha[f(\text{AB})+f(\text{BC})] \quad \dots \quad \dots \quad (1)$$

The plane through AD is similarly

$$\beta[f(\text{BD})+f(\text{AB})]=\gamma[f(\text{CD})+f(\text{AC})] \quad \dots \quad \dots \quad (2)$$

The shortest distance between AC and BD is

$$\left.\begin{array}{l}\delta[f(AD)+f(CD)]=\beta[f(AB)+f(BC)]\\ \alpha[f(AB)+f(AD)]=\gamma[f(BC)+f(CD)]\end{array}\right\}\quad\ldots\quad(3)$$

and that between AB, and CD is

$$\left.\begin{array}{l}\delta[f(AD)+f(BD)]=\gamma[f(AC)+f(BC)]\\ \alpha[f(AC)+f(AD)]=\beta[f(BC)+f(BD)]\end{array}\right\}\quad\ldots\quad(4)$$

§ 16. *The condition a shortest distance intersects a perpendicular from a vertex on the opposite face.*

If the shortest distance between AC, BD intersects the perpendicular from A on BCD, we have

$$\delta[f(AD)+f(CD)]=\beta[f(AB)+f(BC)]$$
$$\alpha[f(AB)+f(AD)]=\gamma[f)BC)+f(CD)]$$

and
$$\frac{\alpha-1}{a}=\frac{\beta}{f(CD)}=\frac{\gamma}{f(BD)}=\frac{\delta}{f(BC)},$$

are consistent.

$$\therefore\quad f(BC)[f(AD)+f(CD)]=f(CD)[f(AB)+f(BC)],$$

or
$$f(BC)f(AB)=f(CD)f(AB).$$

The symmetry of this result shows that this shortest distance also intersects the other three perpendiculars from B,C,D on the opposite faces.

Further, § **14**, the perpendiculars from A,C on the opposite faces intersect, as also the perpendiculars from B,D.

Also, by § 11(1), we must have $AB^2+CD^2=BC^2+AD^3$, which is the condition of perpendicularity of the edges AC, BD.

Conversely, any one of these relations implies all the others.

§ 17. *The condition that two shortest distances may intersect.*

Suppose the shortest distances between AD, BC and AB, CD intersect. By § 15 (1) (2) (4), we must have

$$\frac{f(BC)+f(BD)}{f(AC)+f(AD)}\cdot\frac{f(CD)+f(AC)}{f(BD)+f(AB)}\cdot\frac{f(BD)+f(AD)}{f(BC)+f(AC)}\cdot\frac{f(AB)+f(AC)}{f(CD)+f(BD)}=1.$$

Writing this

$$\frac{[f(BD)]^2+f(BD)[f(BC)+f(AD)]+f(BC)f(AD)}{[f(BD)]^2+f(BD)[f(AB)+f(CD)]+f(AB)f(CD)}$$
$$=\frac{[f(AC)]^2+f(AC)[f(BC)+f(AD)]+f(BC).f(AD)}{[f(AC)]^2+f(AC)[f(AB)+f(CD)]+f(AB).f(CD)},$$

we obtain either
$$f(BD)=f(AC)\quad\ldots\quad\ldots\quad\ldots\quad(1)$$

or
$$\frac{f(BD)f(AC)-f(AD)f(BC)}{f(BD)f(AC)-f(AB)f(CD)}=\frac{f(BD)+f(AC)+f(AD)+f(BC)}{f(BD)+f(AC)+f(AB)+f(CD)}.\quad(2)$$

The first of these relations is capable of a simple geometrical interpretation, which I gave in Q. 356, J. I. M. S; for, it gives

$$2AD^2.BC^2+2AB^2.CD^2.-2BD^2.AC^2-(AB^2+AD^2-BD^2)(BC^2+CD^2-BD^2)$$

$$=2AD^2.BC^2+2AB^2.CD^2-2BD^2.AC^2-(AB^2+BC^2-AC^2)(AD^2+CD^2-AC^2),$$

whence cos BADcos BCD = cos ABCcos ADC; and conversely, if this relation holds good the corresponding shortest distances intersect.

The condition (2) reduces, by § 23 (1) and simplification of the right hand side, to

$$\frac{AD^2+BC^2-BD^2-AC^2}{AB^2+CD^2-BD^2-AC^2}=\frac{4AB^2.CD^2-(AD^2+BC^2-BD^2-AC^2)^2}{4AD^2.BC^2-(AB^2+CD^2-BD^2-AC^2)^2} \quad (3)$$

It is easy to find a tetrahedron for which (3) is satisfied but not (1). For, consider the tetrahedron in which AB = AD, BC = CD, (3) is satisfied and also $f(AB)=f(AD)$, $f(BC)=f(CD)$.

Hence $f(CD)f(AB)=f(AD)f(BC)$ and all the properties of § 16 hold good; also, the shortest distance between AC,BD bisects BD; these results are easily seen geometrically.

§ 18. If the shortest distance between two opposite edges AC and BD passes through the centroid, we have

$$f(AD)+f(CD)=f(AB)+f(BC)$$

$$f(AB)+f(AD)=f(BC)+f(CD),$$

whence $f(AB)=f(CD)$, and $f(BC)=f(AD)$;

and the shortest distance becomes $\delta=\beta$, $\alpha=\gamma$; this passes through (1, 0, 1, 0) and (0, 1, 0 1) which are the middle points of AC and BD. Now, by considering the circumscribing parallelopiped, a tetrahedron for which the line joining the middle points of AC, BD is perpendicular to AC and BD must have the faces through the other edges rectangles.

Thus AB = CD and BC = AD. Hence:

If in a tetrahedron $f(AB)=f(CD)$ and $f(BC)=f(AD)$, then AB = CD and BC = AD.

§ 19. Considering further the common perpendicular to AD and BC, its intersections with BC, AD respectively are

$$0,\ f(CD)+f(AC),\ f(AB)+f(BD),\ 0$$

and $f(BD)+(CD),\ 0,\ 0,\ f(AB)+f(AC)$

where all these quantities should be divided by $f(AB)+f(BD)+f(AC)+f(CD)$, which is the same for both points.

The coordinates of the middle point are, therefore,

$f(BD)+f(CD)$, $f(CD)+f(AC)$, $f(AB)+f(BD)$, $f(AB)+f(AC)$.

The equation of the plane through the middle points of the three shortest distances is

$$\begin{vmatrix} \alpha & \beta, & \gamma, & \delta \\ f(BD)+f(CD), & f(CD)+f(AC), \text{ etc.} & . & . \\ f(BC)+f(BD), & f(AC)+f(AD), \text{ etc.} & . & . \\ f(BC)+f(CD), & f(AD)+f(CD), \text{ etc.} & . & . \end{vmatrix} = 0,$$

which reduces to

$$\begin{vmatrix} \alpha, & \beta, & \gamma, & \delta \\ f(CD), & f(CD), & f(AB), & f(AB) \\ f(BC), & f(AD), & f(AD), & f(BC) \\ f(BD), & f(AC), & f(BD), & f(AC) \end{vmatrix} = 0.$$

This passes through the isogonal conjugate of the centroid.

The condition this plane may pass through the centroid (1,1,1,1) reduces to

$$[f(CD)-f(AB)]\,[f(BC)-f(AD)]\,[f(BD)-f(AC)]=0,$$

whence a pair of shortest distances must intersect §17(1)—*vide*: Q. 357.

If the condition $f(CD)-f(AB)=0$ is satisfied, the above plane reduces to $\alpha+\beta-\gamma-\delta=0$. Hence the theorem :

If the shortest distances between BC, AD and AC, BD intersect, the plane through the middle points of the shortest distances (i) *passes through the centroid,* (ii) *is parallel to the edges AB and CD and* (iii) *bisects the other four edges of the tetrahedron.*

Suppose the three middle points of the shortest distances are collinear. The above determinant must be indeterminate, whence

either $f(AC)=f(BD)$ and $f(AB)=f(CD)$,

or $f(AC)=f(BD)=0$.

The case $f(AC)=f(BD)$ and $f(AB)=f(CD)$ has been considered in § 18, and the greatest distance between AD and BC contains both the centroid and the middle points of the other two shortest distances.

If $f(AC)=f(BD)=0$, the pairs of edges through AC, BD respectively are at right angles.

§ 20. *The tetrahedra in which the three shortest distances between pairs of opposite edges intersect.*

The conditions of intersection are of two kinds :—

(1). $f(BD)=f(AC)$.

(2). $$\frac{AD^2+BC^2-BD^2-AC^2}{AB^2+CD^2-BD^2-AC^2}=\frac{f(BD)+f(AC)+f(AD)+f(BC)}{f(BD)+f(AC)+f(AB)+f(CD)},$$

or $$\phi\,(AC)=0.$$

We have four cases to consider :

Case (i). $f(AC)=f(BD)$, $f(AB)=f(CD)$, $f(AD)=f(BC)$.

This is the well known tetrahedron for which the opposite edges are equal and the shortest distances bisect the edges to which they are perpendicular.

Case (ii). $f(AC)=f(BD)$, $f(AB)=f(CD)$, $\phi\,(AD)=0$.

From § 18, we have, $AB=CD$ and $AC=BD$, as a result of the first two relations, the third condition $\phi\,(AD)=0$ gives, by § 17 (3)

$$\frac{AB^2+CD^2-AD^2-BC^2}{AC^2+BD^2-AD^2-BC^2}=\frac{4\ AC^2BD^2-(AB^2+CD^2-AD^2-BC^2)^2}{4\ AB^2CD^2-(AC^2+BD^2-AD^2-BC^2)^2}$$

which reduces, in virtue of $AB=CD$, $AC=BD$, to

$$\frac{2\ AB^2-AD^2-BC^2}{2\ AC^2-AD^2-BC^2}=\frac{4\ AC^4-(2\ AB^2-AD^2-BC^2)^2}{4\ AB^4-(2\ AC^2-AD^2-BC^2)^2};$$

whence $$\frac{2\ AB^2-AD^2-BC^2}{2\ (AC^2-AB^2)}=$$

$$\frac{4\ AC^4-(2\ AB^2-AD^2-BC^2)^2}{4\ (AB^4-AC^4)+4\ (AB^2+AC^2-AD^2-BC^2)(AB^2-AC^2)}.$$

Thus either $AB=AC$,

or $$2\ (AD^2+BC^2-2\ AB^2)(2\ AB^2+2\ AC^2-AD^2-BC^2)$$
$$=4\ AC^4-(2\ AB^2-AD^2-BC^2)^2,$$

which gives

$$-(AD^2+BC^2)^2+4\ (AB^2+AC^2)(AD^2+BC^2)-4\ (AB^2+AC^2)^2=0,$$

or $$AD^2+BC^2=2\ AB^2+2\ AC^2.$$

This relation is impossible, for, if X is the middle point of BC, we have

$$AD^2+BC^2=2\ (AB^2+AC^2)=4\ AX^2+BC^2,$$

whence $$AD=2\ AX.$$

Also, since $AB=CD$, $AC=BD$, the triangles ABC, DCB are equal and therefore $AX=DX$. Therefore $AD=2AX=2DX=AX+DX$, whence AXD would be a straight line and the tetrahedron degenerates.

Thus we have only the tetrahedron for which

$$AB=CD=AC=BD.$$

Writing $AB^2=x$, $BC^2=y$, $AD^2=z$, and taking account of the above equality, we find

$$f(AB)=f(AC)=f(CD)=f(BD)=yz=\lambda \text{ say,}$$
$$f(AD)=4\,xz-2\,yz-z^2=u \text{ say,}$$
$$f(BC)=4\,xy-2\,yz-y^2=v \text{ say.}$$

The three shortest distances are

$$\left.\begin{array}{l}\delta=\alpha\\ \beta=\gamma\end{array}\right\};\quad \left.\begin{array}{l}\delta(u+\lambda)=\beta(v+\lambda)\\ \alpha(u+\lambda)=\gamma(v+\lambda)\end{array}\right\};\quad \left.\begin{array}{l}\delta(u+\lambda)=\gamma(v+\lambda)\\ \alpha(u+\lambda)=\beta(v+\lambda)\end{array}\right\}$$

These all intersect in the point

$$(v+\lambda,\ u+\lambda,\ u+\lambda,\ v+\lambda).$$

Case (iii). $\phi(AB)=\phi(AC)=\phi(BC)=0.$

Writing $l=AB^2+CD^2,\ m=AC^2+BD^2,\ n=AD^2+BC^2$

$l'=2AB.CD,\ m'=2AC.BD,\ n'=2AD.BC,$

the conditions are

$$\frac{l-n}{m'^2-(l-n)^2}=\frac{m-n}{l'^2-(m-n)^2}$$

$$\frac{m-l}{n'^2-(m-l)^2}=\frac{n-l}{m'^2-(n-l)^2}$$

$$\frac{n-m}{l'^2-(n-m)^2}=\frac{l-m}{n'^2-(l-m)^2}.$$

None of the denominators can be zero, since the condition a pair of opposite edges may be parallel, say AB parallel to CD, gives

$$1=-\tfrac{1}{2}\cdot\frac{AC^2-AD^2-BC^2+BD^2}{AB.CD}$$

Conversly, if any denominator were zero, a pair of opposite edges would be parallel and the tetrahedron would degenerate.

Multiply the above conditions together and we get

$$(\alpha-\beta)(\beta-\gamma)(\gamma-\alpha)=0,$$

whence $\alpha=\beta=\gamma$, and we have the tetrahedron for which

$$AB^2+CD^2=AC^2+BD^2=AD^2+BC^2.$$

In this tetrahedron the opposite edges are perpendicular and from § **11** (**1**), we have

$$f(AC)f(BD)=f(AD)f(BC)=f(AB)f(CD);$$

whence from § 16 the four perpendiculars from the vertices intersect in a point, the ortho-centre, and the shortest distances intersect all the perpendiculars; thus the shortest distances between opposite edges must also meet in the orthocentre.

In this tetrahedron the centroid bisects the line joining the circumcentre to the orthocentre.

Denote AB^2, AC^2, AD^2 by x, y, z; then

CD^2, BD^2, BC^2 are $k-x$, $k-y$, $k-z$

whence $k = AB^2 + CD^2 = AC^2 + BD^2 = AD^2 + BC^2$.

Calculating $f(AB)$, etc., we find

$$f(AB) = (x+y+z-k)(x+k-y-z)$$
$$f(CD) = (k-x-y+z)(k-x+y-z).$$

Write $x+y+z-k = \lambda$, $k+x-y-z = \mu$,

$k-x+y-z = \gamma$, $k-x-y+z = \rho$;

we have $f(AB) = \lambda\mu$, $f(AC) = \lambda\nu$, $f(AD) = \lambda\rho$,

$f(CD) = \nu\rho$, $f(BD) = \mu\rho$, $f(BC) = \mu\nu$.

The equations of the shortest distances become on omitting factors as $\mu + \nu_0$ which cannot be zero since it is $2\,BC^2$,

$$\left.\begin{matrix}\delta\rho = \alpha\lambda \\ \beta\mu = \gamma\nu\end{matrix}\right\};\quad \left.\begin{matrix}\delta\rho = \beta\mu \\ \alpha\lambda = \gamma\nu\end{matrix}\right\};\quad \left.\begin{matrix}\delta\rho = \gamma\nu \\ \alpha\lambda = \beta\mu\end{matrix}\right\};$$

these intersect in the point $\left(\frac{1}{\lambda}, \frac{1}{\mu}, \frac{1}{\nu}, \frac{1}{\rho}\right)$, which must be the orthocentre as above proved.

The circumcentre is (k_1, k_2, k_3, k_4), where by § 11

$$k_2 = BD^2(\lambda\nu + \nu\rho + \lambda\rho) - CD^2.\lambda\rho - AD^2.\nu\rho$$
$$= \tfrac{1}{2}(\mu+\rho)(\lambda\nu + \nu\rho + \lambda\rho) - \tfrac{1}{2}(\nu+\rho)\lambda\rho - \tfrac{1}{2}(\lambda+\rho)\nu\rho$$
$$= \tfrac{1}{2}\lambda\mu\nu\rho\left(\frac{1}{\lambda} - \frac{1}{\mu} + \frac{1}{\nu} + \frac{1}{\rho}\right).$$

Thus the circumcentre, orthocentre and centroid are

$$\left(-\frac{1}{\lambda} + \frac{1}{\mu} + \frac{1}{\nu} + \frac{1}{\rho}, \text{ etc }\right), \left(\frac{1}{\lambda}, \frac{1}{\mu}, \frac{1}{\nu}, \frac{1}{\rho}\right), (1, 1, 1, 1)$$

and the centroid bisects the line joining the circumcentre to the orthocentre.

Case (iv). $f(AC) = f(BD)$, $\phi(AB) = 0$, $\phi(AD) = 0$.

We may write the two latter conditions

$$\frac{AD^2 + BC^2 - AB^2 - CD^2}{f(AD) + f(BC) + f(AB) + f(CD)} = \frac{AC^2 + BD^2 - AB^2 - CD^2}{f(AC) + f(BD) + f(AB) + f(DC)}$$

$$= -\frac{AB^2 + CD^2 - AD^2 - BC^2}{f(AB) + f(CD) + f(AD) + f(BC)} = -\frac{AC^2 + BD^2 - AD^2 - CB^2}{f(AC) + f(BD) + f(AD) + f(BC)},$$

excluding the case where one of the numerators (and therefore all) vanishes. This being Case (iii) already investigated, we have

$$f(AC)+f(BD)-f(AD)-f(BC=-\{f(AC)+f(BD)+f(AD)+f(BC)\}$$

$$\therefore \qquad f(AC)+f(BD)=0,$$

and with the first condition, we get

$$f(AC)=f(BD)=0.$$

The conditions now become

$$f(AC)=0,\ f(BD)=0,$$

and
$$\frac{AB^2+CD^2-AC^2-BD^2}{f(AB)+f(CD)}+\frac{AD^2+BC^2-AC^2-BD^2}{f(AD)+f(BC)}=0.$$

We shall show that either $AB^2+CD^2=AC^2+BD^2=AD^2+BC^2$, or this case reduces to (i).

For, since $f(AC)=f(BD)=0$, we have from § 11 (5).

$$f(CD)\{BD^2+AC^3-BC^2-AD^2\}=f(AD)\{BC^2+CD^2-BD^2\}$$

$$f(BC)\{AB^2+CD^2-AC^2-BD^2\}=f(AB)\{BD^2-BC^2-CD^2\}.$$

By use of these relations the above condition reduces to

$$f(AB)\{AD^2-AC^2-CD^2\}+f(AD)\{AB^2-AC^2-BC^2\}=0;$$

also from § 11

$$f(AB)\{BC^2-BD^2-CD^2\}-f(AD)\{CD^2-BD^2-BC^2\}=0$$

whence
$$(BC^2-BD^2-CD^2)(AB^2-AC^2-BC^2)+$$
$$(AD^2-AC^2-CD^2)(CD^2-BD^2-BC^2)=0. \qquad \dots \quad (1)$$

This is only one of four possible ways of expressing the condition. We have in virtue of $f(AC)=f(BD)=0$, four relations, viz.,

$$f(CD)[AB^2+AC^2-BC^2]=f(AD)[AC^2+BC^2-AB^2]$$
$$f(AD)[BD^2+BC^2-CD^2]=f(AB)[BD^2+CD^2-BC^2]$$
$$f(AB)[AC^2+CD^2-AD^2]=f(BC)[AC^2+AD^2-CD^2]$$
$$f(BC)[BD^2+AD^2-AB^2]=f(CD)[BD^2+AB^2-AD^2].$$

Denote the coefficients of $f(CD)$, $f(AD)$, $f(AB)$, $f(BC)$ on the left-hand side by $x_1\ x_2\ x_3\ x_4$ and by $y_1,\ y_2,\ y_3,\ y_4$, those on the righthand side; then from conditions (1) we get

$$x_1x_2+y_1y_2=x_2x_3+y_2y_3=x_3x_4+y_3y_4=x_4x_1+y_4y_1=0,$$

whence we derive

$$x_1y_3-x_3y_1=0,\ x_2y_4-x_4y_2=0.$$

But these are equivalent to

$$f(AB)-f(CD)=0,\ f(AD)-f(BC)=0.$$

Thus, unless $AB^2+CD^2=AC^2+BD^2=AD^2+BC^2$, we are led to Case (i).

§ 21. We may note however, that the condition

$$(BC^2-CD^2-BD^2)(AB^2-AC^2-BC^2)+ (AD^2-AC^2-CD^2)(CD^2-BD^2-BC^2)=0$$

in conjunction with $f(BD)=0$, is impossible.

For, write

$$BD^2+CD^2-BC^2=2a$$
$$BC^2+BD^2-CD^2=2b$$
$$AC^2+BC^2-AB^2=c$$
$$CD^2+AC^2-AD^2=d$$
$$AC^2=x,\ CD^2=y;$$

and we find

$$BD^2=a+b,\ BC^2+y=a+b,$$
$$AB^2=x+y-a+b+c,\ AD^2=x+y-d.$$

Substitute these values in $f(BD)=0$, and we obtain

$$2y(a+b)-2a^2=bd+ac.$$

Thus, if $bd+ac=0$, we have $y(a+b)=a^2$, which gives

$$4CD^2.BD^2=(BD^2+CD^2-BC^2)^2$$

or

$$BD\pm CD=BC,$$

whence BCD would be a straight line.

SHORT NOTES.

Curvature and Torsion of Curves on Surfaces.

The equation to a surface, referred to the lines of curvature and normal to the surface at a point as axes of coordinates, is

$$2\,z = x^2/\rho_1 + y^2/\rho_2 + \phi(x,y) \qquad \ldots \qquad \ldots \qquad \ldots \qquad (1)$$

where ρ_1, ρ_2 are the principal radii of curvature, and ϕ involves third and higher degree terms in x and y.

Let θ be the angle which the tangent at the origin to a tortuous curve through the origin makes with the axis of x, and α the angle between the principal normal to the curve and the axis of z. Then, evidently,

$$l_1 = x' = \cos\theta,\ m_1 = y' = \sin\theta,\ n_1 = z' = 0,$$
$$l_2 = \rho x'' = \sin\theta\sin\alpha,\ m_2 = \rho y'' = -\cos\theta\sin\alpha,\ n_2 = \rho z'' = \cos\alpha,$$
$$l_3 = \sin\theta\cos\alpha,\ m_3 = -\cos\theta\cos\alpha,\ n_3 = -\sin\alpha,$$

corresponding to the direction cosines of the tangent, the principal normal and the binormal; dashes denote differentiation with respect to the arc of the curve.

Differentiating (1) twice we have, at the origin,

$$z'' = \frac{x'^2}{\rho_1} + \frac{y'^2}{\rho_2} = \frac{\cos^2\theta}{\rho_1} + \frac{\sin^2\theta}{\rho_2};$$

that is, $$\frac{\cos\alpha}{\rho} = \frac{\cos^2\theta}{\rho_1} + \frac{\sin^2\theta}{\rho_2}. \qquad \ldots \qquad \ldots \qquad \ldots \qquad (2)$$

Again, a point on the curve near the origin is denoted by $(x'ds, y'ds, z'ds)$ or $(\cos\theta\,ds, \sin\theta\,ds, 0)$. The principal normal at this point has for direction cosines $(l_2 + l_2'\,ds)$ &c.; and the normal to the surface is $(-\cos\theta\,ds/\rho_1, -\sin\theta\,ds/\rho_2, 1)$ approximately.

If $(\alpha + d\alpha)$ denote the angle between these normals

$$\cos(\alpha + d\alpha) = -l_2\cos\theta\,ds/\rho_1 - m_2\sin\theta\,ds/\rho_2 + n_2 + n_2'\,ds.$$

Proceeding to the limit and remembering that $n_2 = \cos\alpha$, and $n_2' = -n_1/\rho - n_3/\sigma$, by Frenets Formulae (C), we get

$$\sin\alpha\frac{d\alpha}{ds} = \frac{l_2\cos\theta}{\rho_1} + \frac{m_2\sin\theta}{\rho_2} + \frac{\sin\theta}{\sigma}.$$

Hence $$\frac{1}{\sigma} = \sin\theta\cos\theta\left(\frac{1}{\rho_2} - \frac{1}{\rho_1}\right) + \frac{d\alpha}{ds}. \qquad \ldots \qquad \ldots \qquad (3)$$

19th October, 1912.

K. A. Erady, M. A.

Squaring the Circle.

Let PQR be a circle with centre O, of which a diameter is PR. Bisect PO at H and let T be the point of trisection of OR nearer R. Draw TQ perpendicular to PR and place the chord RS = TQ.

Join PS, and draw OM and TN $\|^l$ to RS. Place a chord PK = PM, and draw the tangent PL = MN. Join RL, RK and KL. Cut off RC = RH. Draw CD $\|^l$ to KL meeting RL at D.

Then the square on RD will be equal to the ⊙ PQR approximately.

For $$RS^2 = \tfrac{5}{36}d^2,$$

where d is the diameter of the circle.

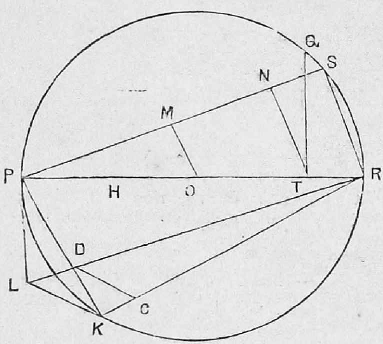
∴ $$PS^2 = \tfrac{31}{36}d^2.$$

But PL and PK are equal to MN and PM.

∴ $$PK^2 = \tfrac{31}{144}d^2, \text{ and } PL^2 = \tfrac{31}{324}d^2$$

Hence $$RK^2 = PR^2 - PK^2 = \tfrac{113}{144}d^2,$$

and $$RL^2 = PR^2 + PL^2 = \tfrac{355}{324}d^2$$



But $$\frac{RK}{RL} = \frac{RC}{RD} = \frac{}{2}\sqrt{\frac{113}{355}}$$

and $$RC = \tfrac{3}{4}d.$$

∴ $$RD = \frac{d}{2}\sqrt{\frac{355}{113}} = r\sqrt{\pi}, \text{ very nearly.}$$

Note.—If the area of the circle be 140,000 sq. miles, then RD is greater than the true length by about an inch.

12th May, 1913. S. RAMANUJAN.

The Sine and Cosine Series.

Euler arrived at the well known expansions of the sine and cosine by the development of Demoivre's theorem

$$\cos.n\theta+\sqrt{-1}\sin n\theta=(\cos\theta+\sqrt{-1}\sin\theta)^n.$$

In the following note I shall derive the expansions from the formula of sines given in our ancient *Surya-siddhanta* without the use of imaginary quantities.

I shall first deduce the peculiar *furmula* of sines given in the *Surya-siddhanta.* The usual formulæ for the sum and difference of two angles are given in the *Siddhánta-śiromaṇi* under the name of *samás-bhávana* and *antar-bhávana.* They are

$$\sin(A+B)=\sin A.\cos B+\cos A.\sin B \quad \ldots \quad \ldots \quad (1)$$

$$\sin(A-B)=\sin A.\cos B-\cos A.\sin B \quad \ldots \quad \ldots \quad (2)$$

From equation (2) we get

$$\cos A.\sin B=\sin A.\cos B-\sin(A-B),$$

Substituting this in equation (1) we get,

$$\begin{aligned}\sin(A+B)&=2\sin A.\cos B-\sin(A-B).\\&=2\sin A(1-\text{vers }B)-\sin(A-B).\\&=2\sin A-2\text{ vers }B.\sin A-\sin(A-B).\\&=\sin A+\{\sin A-\sin(A-B\}-2\text{ vers }B\sin A.\end{aligned}$$

Writing $n\theta$ for A and θ for B in the above equations, we get the following general formula

$$\sin(n+1)\theta=\sin n\theta+\{\sin n\theta-\sin(n-1)\theta\}-\sin n\theta.\,2\text{ vers }\theta. \quad (3)$$

This is the very formula employed in the *Surya-siddhanta* for calculating the sines of 24 multiple arcs contained in a quadrant. Although this formula may not be of use when great accuracy is desired, yet it is unrivalled in simplicity, as it employs a single multiplier throughout the calculation. It has excited the wonder of renowned western mathematicians, and has no less taxed their talents, as the following quotation shows—

Delambre, as quoted in Warren's *Kála-sankalita*, says :—'That process is extremely curious. One finds nothing like it in the trigonometry of Ptolemy and in order to find some vestige of it, one must, after having vainly poured over all the authors on trigonometry, come to Briggt who knew that divisor, which he seems to have found out by the facs in comparing the second differences obtained by other means, for Briggs himself was not aware that it was the square of the cord of the differential arc δA.'

I shall work symbolically a few consecutive values of $\sin(n+1)\theta$, when n is equal to 1, 2, 3, 4 &c.

Let $a=\sin\theta$; and $x=2$ vers θ. Then according to the method of working given in the *Sūrya-siddhanta*, we have

Sines		Difference
$\sin 0^\circ = 0$		a
$\sin\theta = a$	multiplied by $x=$	$+ax$
$\sin 2\theta = a-ax$		$a-ax$
$2a-ax$	,, $x=$	$+2ax-ax^2$
$\sin 3\theta = a-3ax+ax^2$		$a-3ax+ax^2$
$3a-4ax+ax^2$	,, $x=$	$+3ax-4ax^2+ax^3$
$\sin 4\theta = a-6ax+5ax^2-ax^3$		$a-6ax+5ax^2-ax^3$
$4a-10ax+6ax^2-ax^3$		
&c.		&c.

The law of the co-efficients of the terms of the series for sines may be formulated. Let us take for example the series for $\sin 4\theta$. Here $n=4$.

4 = The number of combinations of n things taken $(n-1)$ at a time.

10 = The number of combinations of $(n+1)$ things taken $(n-3)$ at a time.

6 = The number of combinations of $(n+2)$ things taken $(n-5)$ at a time.

1 = The number of combinations of $(n+3)$ things taken $(n-7)$ at a time.

The series may therefore be written symbollically, according to the formula of combinations, thus:

$$\sin n\theta = na - \frac{(n+1)(n)(n-1)}{1\cdot 2\cdot 3}.ax + \frac{(n+2)(n+1)(n)(n-1)(n-2)}{1\cdot 2\cdot 3\cdot 4\cdot 5}.ax^2 - \ldots$$

We may, without departing from the hypothesis of n being an integer, conceive n to increase to infinity, and θ to diminish to zero; so that $n\theta$ shall be any arc β. It is also plain that when the arc θ is diminished to zero it becomes equal to its sine a; so that $n\theta = na = \beta$. Also, when $a=0$, $2a=0$, $3a=0$ &c.; and when θ is infinitely diminished twice its versed sine becomes equal to a^2, since

$$x = 2 \text{ vers } \theta = \frac{2\sin^2\theta}{1+\cos\theta} = \frac{2a^2}{2} = a^2, \text{ in the limit.}$$

Therefore the terms ax, ax^2,...become a^3, a^5,...respectively; and if we make these substitutions in the preceding formula, we get

$$\sin\beta = \beta - \frac{(na+a)(na)(na-a)}{1\cdot 2\cdot 3} + \ldots\ldots\ldots$$

Therefore, finally, we have

$$\sin\beta=\beta-\frac{\beta^3}{1.2.3}+\frac{\beta^5}{1.2.3.4.5}-\frac{\beta^7}{1\cdot2\cdot3\cdot4\cdot5\cdot6\cdot7}+\dots$$

So far as I am aware, I have not been anticipated by any mathematician in deriving the above formula in the manner indicated.

The *formula* for cosine may be similarly deduced:

For, if we make $n\theta=\frac{1}{2}\pi$; $\sin n\theta=1$ and when θ is diminished indefinitely

$$\sin(n+1)\,\theta=\cos\theta\,;\ \sin n\theta=1,$$
$$\{\sin n\theta-\sin(n-1)\,\theta\}=0,$$
$$-\sin n\theta\times2\text{ vers }\theta=x.$$

Therefore $\cos\theta=1+0-x$, in the limit.

$$\begin{array}{lll}
\cos0^\circ=1 & \text{multiplied by } x= & +x \\
 & & \overline{0-x} \\
\cos\theta=\ -x & & \\
\overline{1-x} & \quad ,, \quad x= & x-x^2 \\
\cos2\theta=\ -2x+x^2 & & \overline{-2x+x^2} \\
\overline{1-3x+x^2} & \quad ,, \quad x= & +x-3x+x^3 \\
\cos3\theta=\ -3x+4x^2-x^3 & & \overline{-3x+4x^2-x^3} \\
\overline{1-6x+5x^2-x^3} & \quad ,, \quad x= & +x-6x^2+5x^3-x^4 \\
\cos4\theta=\ -4x+10x^2-6x^3+x^4 & & \overline{-4x+10x^2-6x^3+x^4} \\
\overline{1-10x+15x^2-7x^3+x^4} & & \\
\&c. & & \&c.
\end{array}$$

The law of the co-efficients of the terms in the series for cosine is formulated thus: the co-efficient of

2nd term = combinations of n things taken $(n-2)$ things at a time,

3rd term = combinations of $(n+1)$ things taken $(n-4)$ things at a time,

4th term = combinations of $(n+2)$ things taken $(n-6)$ things at a time,

&c. &c. &c.

If we go through the successive steps and make the necessary substitutions, we get

$$\cos\beta=1-\frac{\beta^2}{1\cdot2}+\frac{\beta^4}{1\cdot2\cdot3\cdot4}-\frac{\beta^6}{1\cdot2\cdot3\cdot4\cdot5\;6}+\dots$$

The question of the *convergence* of the series has not been considered here as the object of this note is only to point out a new method of obtaining the well-known series.

Poona, 17th May, 1913. V. B. KEṬAKAR.

Unification of Notations in the Theories of Potential and Elasticity.

First circular.

It is unnecessary to explain at length the great advantages that would follow if uniform notations could be established by international cooperation in all branches of pure and applied science.

In Mathematics and Theoretical Physics, the subjects in respect of which an attempt in this direction may at the present time be made profitably are undoubtedly the theories of Potential and Elasticity, provided that the subjects are not taken in too wide a sense, and that a suitable organization for securing international agreement can be set up

A. Range of subjects.

1. Since it is not to be expected, that the same terminology or notation can be used for the same concept in all languages, the terminology and notation should be so chosen that they may be translated as easily as possible from one language to another.

2. It is proposed in the first instance to establish uniform notations for the quantities which occur in the theories of the equation for the potential and of the differential equations which belong to the theory of elasticity for isotropic media. It would be possible afterwards to extend the conventional notations that may be agreed upon in regard to the theory of potential to more general partial differential equations of elliptic type, and in regard to the elastic differential equations to the corresponding equations for aeolotropic media. The notations should conform as far as possible to existing notations.

B. Plan of organization.

The Committee of organization, herewith, by means of this first circular, makes application to Astronomers, Mathematicians and Physicists who work at the two theories named above, and requests them to cooperate with it to secure the desired uniformity, and in the first place to assist the Committee by answering the following question :—

What are the notions and notations in respect of which it is desirable to establish uniformity ?

The answers which are received during this year will be arranged as soon as possible, and in the course of the year 1914 a second circular will be issued asking for suggestions as to methods by which the desired uniformity may be brought about. Since it is not be expected that the suggestions which may be received will be in complete agreement with each other, it is intended to issue a third circular in the Spring of 1916, setting out the points in dispute, and to arrange a discussion thereon at the next international Congress of Mathematicians in 1916. A fourth circular to be issued in 1917 will contain a report of this discussion and provide an opportunity for those who shall not have been present at the Congress to express their views in writing.

All the proposals and contributions to the discussion will be sifted and arranged, and the Committee will in a fifth circular (1919) state the points in regard to which agreement shall have been obtained and take a vote on those in dispute. The voting will take place at the international Congress of Mathematicians to be held in the year 1920, and an arrangement will be made by which those who do not attend the Congress may record their votes in writing.

The Committee will declare the result of the voting in a sixth circular in 1921, and it is intended that a printed statement of the terminology and notation that may be agreed upon shall be published shortly afterwards.

Please write to the following address (in English, French, German or Italian.)

Herrn **Arthur Korn**, Charlottenburg, Schlüterstrasse 25.

The Committee of organization
for the establishment of uniform notations in the Theories of Potential and Elasticity by international corporation.

Max Abraham (Milano), **Alfred Ackermann-Teubner** (Leipzig) **Robert D'Adhemar** (Lille), **Paul Appell** (Paris), **Serge Bernstein** (Charkow), **Kristian Birkeland** (Kristiania), **Wilhelm Bjerknes** (Leipzig), **Marcel Brillouin** (Paris), **Orest Chwolson** (Petersburg), **Eugéne Cosserat** (Toulouse), **Francois Cosserat** (Paris), **Gaston Darboux** (Paris), **Paul Ehrenfest** (Leiden), **Henri Fehr** (Genève), **Leopold Fejér** (Budapest), **Richard Gans** (La Plata), **Heinrich Graf** (Bern), **Sir George Greenhill** (London), **Jacques Hadamard** (Paris),

Wilhelm Hallwachs (Dresden), **Fritz Hasenöhrl** (Wien), **Tsuruichi Hayashi** (Sendai), **Pierre de Heen** (Liège), **Gavid Hilbert** (Göttingen), **Gustav Jäger** (Wien), **Eugen Jahnke** (Berlin), **Paul Köbe** (Leipzig), **Walter König** (GieBen),**Arthur Korn** (Charlottenburg), **Horace Lamb** (Manchester), **Emil Lampe** (Berlin), **Sir Joseph Larmor** (Cambridge), **Otto Lehmann** (Karlsruhe), **Eugenio Elia Levi** (Genova), **Tullio-Levi-Civitá** (Padova), **Leon Lichtenstein** (Berlin), **Augustus Edward Hough Love** (Oxford), **Roberto Marcolongo** (Napoli), **Max Mason** (Madison, Wis.),**Friedrich Wilhelm Franz Meyer** (Königsberg) **Albert Abraham Michelson** (Chicago), **Gösta Mittag-Leffler** (Stockholm), **Ernst Richard Neumann** (Marburg), **Niels Nielsen** (Kobenhavn), **Wilhelm Oseen** (Upsala), **Michel Petrovitch** (Belgrad), **Emile Picard** (Paris), **Friedrich Pockels** (Heidelberg), **Demetre Pompeiu** (Bukaresti), **Georgios Remundos** ('Αθηναι), **Karl Schwarzschild** (Potsdam), **Carlo Somigliana** (Torino), **Waldimir Stekloff** (Petersburg), **Orazio Tedone** (Genova), **Francisco Gomes Teixeira** (Porto), **Esteban Terradas** (Barcelona), **Vito Volterra** (Roma), **Albert (Wngerin** (Halle), **Otto Wiener** (Leipzig), **Stanislas Zaremba** Kraków).

REVIEWS.

Introduction à la théorie des Nombres Algébriques par Dr. J. Sommer, translated from the German by A. Levy. 15 Francs. Hermann et Fils Paris.

The rich harvest reaped in Analysis after the introduction of the complex quantity could hardly fail to suggest to mathematicians that a similar generalization in the Theory of Numbers would similarly reward the investigator. The great German mathematician Gauss was the first to attempt this generalization in his classical researches into the theory of Cubic and Biquadratic Residues. The extension of the integer concept, thus introduced, was limited however to these two cases; the further attempt to extend the integer concept to quadradic equations, the square of whose discriminant is a positive or negative integer, presented the great difficulty that the fundamental theorem that any integer can be resolved into prime factors in one and only one way no longer held good. Kummer was the first to suggest the way out, but it was R. Dedekind (Diriclhet-Dedekind, Vorlesungen über Zahlentheorie) who finally showed how the difficulty was to be overcome by replacing the integer concept by that of the ideal. Practically the whole of the recent theory of ideals is due to the great German Mathematicians Kronecker, Hilbert, Hensel, Minkowski and Bachmann all of whose works must be regarded as classical. Weber also, in his Lehrbuch der Algebra, gives a masterly description of the theory. It will be seen therefore that this difficult subject has been neglected by all other nations, and as Prof. Hadamard points out in his preface to this work, it is only recently that the University of Paris has inaugurated a series of lectures on this subject. So, in England Vol. II of Mathew's Theory of Numbers, in which he hoped to give an account of ideals, has never appeared. Above all the modern theory is chiefly indebted to D. Hilbert whose lectures at Göttingen and whose superb Theorie der Algebraischen Zahlkörper in the Jahresbericht der deutschen Mathematiker-Vereinigung of 1897 must long remain the standard treatment in this subject. Hilbert's work however is so condensed and difficult that all students desirous of becoming acquainted with the modern theory of numbers will welcome the elementary introduction, under review and more so as it is largely based on Hilbert's own lectures and has received the benefit of his personal criticism.

Dr. Sommer's book comprises three sections; the quadratic field which is treated in great detail, with various applications; an introduction to the more general field illustrated by the cubic field; and an introduction to the relative quadratic field. The discussion of the quadratic field is complete in itself and can be read by any one with only a moderate acquaintance with algebra. It is shown that all ideals can be divided into a certain number of classes whose number can be determined and which are of definite types possessing a certain set of characteristics. A table at the end gives classes, types and characteristics of all ideals of a quadratic field whose discriminants lie between -97 and 101. Applications, such as to prove that the equations $x^2-py^2=\pm 1$, for p prime, admit of integral solutions only when $p\equiv 7$ (mod. 8), or $p\equiv 3$ (mod. 8), according as the upper or lower sign is taken; or if p_1, p_2 are both $\equiv 1$ (mod. 4), the equation $p_1x^2-p_2y^2=\pm 1$ always admits of integral solutions, show only some of the simplest theorems obtained. A discussion of Fermat's "Last" Theorem is given with proofs of Kummer's extension that no quantities of the form $\alpha+i\beta$ where α, β are integers, can satisfy either $x^3+y^3=z^3$, or $x^4+y^4=z^2$. The connection between the ideal theory and Gauss's classical work (Section V of the Disquisitiones Arithmeticae) on the representation of numbers by quadratic forms and the theory of composition of forms is summarised in Chapter III, which concludes with the geometrical representation of ideals including Klein's extension to forms whose discriminant is real. While the quadratic field is only a special case of the more general theory its peculiarities are such that it undoubtedly deserves a special study and the student who has mastered this portion of the book should have no difficulty in entering on the general theory as represented by the discussion of the cubic field in which only the fundamental ideas are developed leaving the study of the laws of reciprocity and the division into types for more advanced study.

It is to be regretted that while the general discussion is admirably clear the proof sheets must have been very indifferently looked over, as the number of minor misprints is excessively large though only in a few instances presenting any difficulty to the reader in following the argument.

A. C. L. W.

The Calculus for Beginners by J. W. Mercer, M.A. 440 pp. (Cambridge University Press). Price 6 *sh.*

This book is written mostly for students of Physics and Engineering, and the whole treatment is inductive. In the first two chapters covering 74 pages the author deals with dy/dx as a *rate-measurer* and as a *gradient*. The examples given for the purpose are mostly numerical.

The functions considered are x^2, x^3 and $1/x$. In the third chapter the differential coefficient of x^n is obtained and the algebraical signs of dy/dx and d^2y/dx^2 are carefully explained and illustrated. Chapter IV is devoted to a discussion of maxima and minima values and many of the results obtained are *graphically* illustrated. In the same chapter the *point of inflexion* is defined as the point at which the *gradient* is a maximum or a minimum and the condition for inflexion is elegantly obtained. In the next chapters, which is on small errors and approximation the bulk modulus $-v\, dp/dv$ of a fluid is obtained and illustrated. In Chapters VI to VIII integration is dealt with ; the determination of areas, volumes of revolution, moment of inertia, centre of gravity, work, and mean value is considered at length. In the next three chapters differential coefficients of more complex functions are discussed. There are also small chapters on approximate solution of equations, integration by parts and by substitution, and polar coordinates.

There is no doubt the work will be helpful to the class of students for whom it is meant.

S. P. S.

SOLUTIONS.

Question 410.

(P. A. SUBRAMANIA AIYAR, B.A., L.T.) :—Solve completely

$$\left\{ \left[1+\left(\frac{dy}{dx}\right)^2\right]\frac{d^3y}{dx^3}-3\frac{dy}{dx}\left(\frac{d^2y}{dx^2}\right)^2 \right\} = a\left[1+\left(\frac{dy}{dx}\right)^2\right]^3$$

Note by A. C. L Wilkinson, M.A., F.R.A.S.

The solution of this equation can be written in the form

$$x=\int\cos(\tfrac{1}{2}au^2+b)du+c$$
$$y=\int\sin(\tfrac{1}{2}au^2+b)du+d$$

where b,c,d are the arbitrary constants of integration.

Question 413.

(M. KANNAN, B. A. L. T.):—Shew that the Apollonian circle through the vertex A of a triangle ABC cuts the nine-points circle and the polar circle of the triangle at $\cos^{-1}$ (cos A sin C−B) and $\cos^{-1}$ ($R\rho^{-1}$ cos A sin $\overline{C-B}$) respectively, ρ being the radius of the polar circle.

Solution (1) by J. C. Swaminarayan, M. A., and (2) by V, B. Naik, M. A.

(1) If O, the middle point of BC be taken as origin and BC and a ⊥ to BC through O as coordinate axes, the equation of the Apollonian. circle through A will be

$$x^2+y^2-ax\,(c^2+b^2)/(c^2-b^2)+a^2/4=0,$$

and its radius $=abc/(c^2-b^2)$.

The equations of the nine−points circle and the polar circle will be

$$x^2+y^2-\mathrm{R}x\sin(\mathrm{C}-\mathrm{B})-\mathrm{R}y\cos(\mathrm{C}-\mathrm{B})=0,$$

and $$x^2+y^2-2\,\mathrm{R}x\sin(\mathrm{C}-\mathrm{B})-4\,\mathrm{R}y\cos\mathrm{B}\cos\mathrm{C}+a^2/4=0,$$

thence if the Apollonian circle cuts the nine-point circle and the polar circle at angles ψ and χ respectively,

$$\cos\psi=\frac{\frac{1}{2}\,\mathrm{R}a\sin(\mathrm{C}-\mathrm{B})\,(c^2+b^2)/(c^2-b^2)-a^2/4}{\mathrm{R}\,abc/(c^2-b^2)}$$

$$=\frac{2\,\mathrm{R}a\,(c^2+b^2)\sin(\mathrm{C}-\mathrm{B})-a^2\,(c^2-b^2)}{4\,\mathrm{R}\,abc}$$

$$=\sin(\mathrm{C}-\mathrm{B})\,\{\,2c^2+2b^2-4\,\mathrm{R}a\sin\mathrm{A}\,\}\,/4bc$$

$$=\sin(\mathrm{C}-\mathrm{B})\,\{\,2b^2+2c^2-2a^2\,\}\,/4bc$$

$$=\sin(\mathrm{C}-\mathrm{B})\cos\mathrm{A}$$

and $$\cos\chi=\frac{\mathrm{R}\sin(\mathrm{C}-\mathrm{B})\,a\,(c^2+b^2)/(c^2-b^2)-a^2/2}{2\rho\,abc/(c^2-b^2)}$$

$$=\mathrm{R}\rho^{-1}\cos\mathrm{A}\sin(\mathrm{C}-\mathrm{B}),$$

whence the result follows.

(2) The equations of the Apollonian circle through A, the nine-points circle and the polar circle, are, respectively,

$$U+\frac{\beta\sin C-\gamma\sin B}{\sin(B-C)}I=0 \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

$$U-\tfrac{1}{2}(\alpha\cos A+\beta\cos B+\gamma\cos C)\,I=0 \quad \ldots \quad \ldots \quad (2)$$

$$U-\ (\alpha\cos A+\beta\cos B+\gamma\cos C)\,I=0 \quad \ldots \quad \ldots \quad (3)$$

where $U=0$, and $I=0$, are the equations of the circumcircle and the line at infinity.

The radii of these circles are 2 R sin B sin C cosec (B−C), R/2 and ρ.

By Cathcart's Theorem, two circles whose radii are R_1 and R_2 and whose equations are

$$U+(l\alpha+m\beta+n\gamma)\,I=0$$

and

$$U+(l'\alpha+m'\beta+n'\gamma)\,I=0$$

intersect at an angle θ, given by

$$\frac{R_1R_2\cos\theta}{R^2}=1+(l\cos A+m\cos B+n\cos C)+(l'\cos A+m'\cos B+n'\cos C)$$

$$+ll'+mm'+nn'-\Sigma(mn'+m'n)\cos A \quad \ldots \quad (4)$$

For circles (1) and (2), the right hand side of (4) is found to be cos A sin B sin C, and for circles (1) and (3) it is 2 cos A sin B sin C.

$\therefore$ $\cos\theta=\cos A\sin(B-C)$, for (1) and (2).

and $\cos\theta=R\rho^{-1}\cos A\sin(B-C)$, for (1) and (3).

Additional solution by T. P. Trivedi, M.A., L.L.B.

Question 419.

(P. A. Subramania Iyer, B.A., LT.) :—Solve completely

$$\frac{d^2y}{dx^2}+\frac{dy}{dx}\left(\frac{1}{y}\frac{dy}{dx}+\frac{2}{x}\right)-\left(\frac{a^2}{2}+\frac{1}{x^2}\right)y=0.$$

Solution by K. J. Sanjana, T. P. Trivedi and E. R. Seshu Aiyar.

Put $y^2x=v$; then $2yy_1x+y^2=v_1$,

and $2yy_2x+2y_1^2x+4yy_1=v_2$.

Hence

$$v_2=2xy\left\{y_2+\frac{y_1^2}{y}+\frac{2y_1}{x}\right\}$$

$$=2xy^2\left(\frac{a^2}{2}+\frac{1}{x^2}\right)=v\left(a^2+\frac{2}{x^2}\right);$$

or

$$\frac{d^2v}{dx}-a^2v=\frac{1.2}{x^2}v,$$

which is a particular case of $\frac{d^2v}{dx^2}-n^2v=\frac{m(m+1)}{x^2}v.$

The solution of this (see Forsyth, § 112) is

$$v=x^2\left(\frac{1}{x}\frac{d}{dx}\right)^2(\mathrm{A}e^{ax}+\mathrm{B}e^{-ax}).$$

It is found that $y^2x=x^2\left\{\frac{1}{x^2}a^2(\mathrm{A}e^{ax}+\mathrm{B}e^{-ax})-\frac{1}{x^3}a(\mathrm{A}e^{ax}-\mathrm{B}e^{-ax})\right\}$;

$$\therefore \qquad y^2=\frac{a^2}{x}(\mathrm{A}e^{ax}+\mathrm{B}e^{-ax})-\frac{a}{x^2}(\mathrm{A}e^{ax}-\mathrm{B}e^{-ax}).$$

Question 422.

(D. D. Kapadia M.A., B.Sc.) :—Shew that

$$\begin{vmatrix} a, & b, & c, & d, & e, & f \\ f, & a, & b, & c, & d, & e \\ e, & f, & a, & b, & c, & d \\ d, & e, & f, & a, & b, & c \\ c, & d, & e, & f, & a, & b \\ b, & c, & d, & e, & f, & a \end{vmatrix} = \begin{vmatrix} a+d, & b+e, & c+f \\ c+f, & a+d, & b+e \\ b+e, & c+f, & a+d \end{vmatrix} \begin{vmatrix} a-d, & b-e, & c-f \\ f-c, & a-d, & b-e \\ e-b, & f-c, & a-d \end{vmatrix}$$

Additional solution by R. Tata, M.A.

The determinant on the left side is equal to the determinant whose first three columns have for their constituents the sums of the constituents of the 1st and 4th, 2nd and 5th, 3rd and 6th columns, respectively ; thus

$$\triangle=\begin{vmatrix} a+d, & b+e, & c+f, & d, & e, & f \\ c+f, & a+d, & b+e, & c, & d, & e \\ b+e, & c+f, & a+d, & b, & c, & d \\ a+d, & b+e, & c+f, & a, & b, & c \\ c+f, & a+d, & b+e, & f, & a, & b \\ b+e, & c+f, & a+d, & e, & f, & a \end{vmatrix}$$

This is equal to the determinant whose last three rows have for their constituents the differences of the constituents of the 1st and 4th, 2nd and 5th, 3rd and 6th rows :

i.e.

$$\triangle=\begin{vmatrix} a+d, & b+e, & c+f, & d, & e, & f \\ c+f, & a+d, & b+e, & c, & d, & e \\ b+e, & c+f, & a+d, & b, & c, & d \\ o, & o, & o, & a-d, & b-e, & c-f, \\ o, & o, & o, & f-c, & a-d, & b-e, \\ o, & o, & o, & e-b, & f-c, & a-d \end{vmatrix}$$

$$= \begin{vmatrix} a+d, & b+e, & c+f \\ c+f, & a+d, & b+e \\ b+e, & c+f, & a+d \end{vmatrix} \times \begin{vmatrix} a-d, & b-e, & c-f \\ f-c, & a-d, & b-e \\ e-b, & f-c, & a-d \end{vmatrix}$$

The result may also be obtained directly from § 151 in Muir's *Determinants.*

Question 426.

(K. J. SANJANA, M.A.) :—BC is a fixed chord of a given circle and A any point on the arc BAC ; P, P′ are isogonally conjugate with regard to the triangle ABC. If the powers of P and P′ with respect to the circle are proportional to their respective distances from BC, prove that the loci of P and P′ are circles, and that AP : AP′ is constant for all positions of A. (Suggested by Q. 393).

Solution by V. V. Satyanarayana.

Let PD, P′D′ be $\perp^r$ to BC. Let BP,CP′ meet in O and cut the circle in K,K′. Join the necessary lines as in the figure.

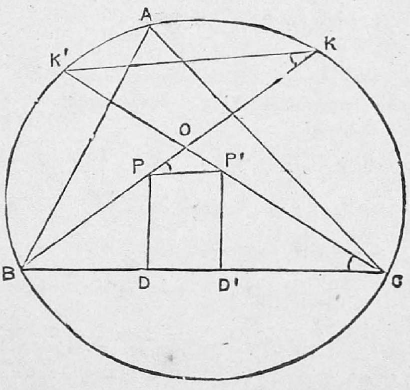
Then BP. PK = power of P with respect to the circumcircle,

and CP′.P′K′ = „ of P′ „ „ „ „

Now, it is given that

$$\frac{PD}{P'D'} = \frac{BP.PK}{CP'.P'K'}$$

But
$$\frac{PD}{P'D'} = \frac{BP\sin\angle OBD}{CP'\sin\angle OCD'} = \frac{BP.OC}{CP'.OB} = \frac{BP}{CP'}\cdot\frac{OK}{OK'}$$



$$\therefore \qquad \frac{BP.PK}{CP'.P'K'} = \frac{BP}{CP'} \cdot \frac{OK}{OK'}.$$

Thus
$$\frac{PK}{P'K'} = \frac{OK}{OK'}.$$

$\therefore$ KK′ and PP′ are parallel.

Now since KK′ is antiparallel to BC (since BK′KC is cyclic), it follows that PP′ also is antiparallel to BC.

Hence BPP′C is cyclic; and as P and P′ are isogonal conjugates, it is easy to see that the circle BPP′C passes through the incentre I (Q. 393, solved in J. I. M. S., Feb. 1913). It is evident from this solution that the ratio AP : AP′ is unity.

Additional solution by N. Sankara Aiyar, M.A.

Question 428.

(S. P. Singaravelu Mudeliar, B.A.) :—Obtain the sextic whose roots are $2\cos\frac{2\pi}{21}$, $2\cos\frac{4\pi}{21}$, $2\cos\frac{8\pi}{21}$, $2\cos\frac{10\pi}{21}$, $2\cos\frac{16\pi}{21}$ and $2\cos\frac{20\pi}{21}$.

Solution by T. P. Trivedi M. A., L. L. B. and E. R. Seshu Aiyar.

The roots of $y^{21}-1=0$, are given by $\cos\left(\frac{2r\pi}{21}+i\sin\frac{2r\pi}{21}\right)$, where r has all the values from 0, 1, 2...to 20.

Again, the roots of

$y^7-1=0$ are given by $\cos\frac{2k\pi}{7}+i\sin\frac{2k\pi}{7}$, where $k=0$, 1, 2...to 6.

Thus the roots of

$\frac{y^{21}-1}{y^7-1}=0$, *i.e.* of $y^{14}+y^7+1=0$ are given by $\cos\frac{2l\pi}{21}+i\sin\frac{2l\pi}{21}$,

where l has all values from 1 up to 20 excluding those which are multiples of three. These may be divided into seven pairs of reciprocal roots.

Put $y+\frac{1}{y}=z$; then $z=2\cos\frac{2l\pi}{21}$, and the equation $y^{14}+y^7+1=0$, *i.e.*, $y^7+1+\frac{1}{y^7}=0$ becomes $z^7-7z^5+14z^3-7z+1=0$.

Rejecting the factor $z+1=0$ corresponding to the root $2\cos\frac{14\pi}{21}$, we have the equation required

$$z^6-z^5-6z^4+6z^3+8z^2-8z+1=0.$$

The roots can also be written in the form

$$2\cos\left(\frac{2\pi}{21}, \frac{4\pi}{21}, \frac{8\pi}{21}, \frac{16\pi}{21}, \frac{32\pi}{21}, \frac{64\pi}{21}\right).$$

Question 430.

(M. BHIMASENA RAO) :—If P is the centre of a conic touching the sides of a triangle ABC at points where the normals are concurrent, shew that

(1) the line joining P to its isogonal conjugate passes through the median point of ABC,

(2) the perpendicular from P on the polar of P with respect to ABC passes through the circumcentre of ABC,

(3) cotPAB + cotPBC + cotPCA = cotPAC + cotPCB + cotPBA.

Hence if a concentric conic be inscribed in the pedal triangle of P, the normals at the points of contact are also concurrent.

Solution by R. Tata, M.A.

Let DEF be the points of contact of the conic with the sides, dividing them into parts $(a_1a_2)(b_1b_2)(c_1c_2)$, respectively, it is readily seen that the equation to the conic is

$$\sqrt{b_2a_2a\alpha}+\sqrt{a_1b_1b\beta}+\sqrt{b_1a_2c\gamma}=0.$$

The coordinates of the centre P of this conic are proportional to

$$bc(b_1a_2+a_1b_1),\ ca(b_2a_2+b_1a_2),\ ab(a_1b_1+b_2a_2)$$

or

$$b_1,\quad a_2,\quad \frac{a_1b_1}{c_2}$$

since $a_1+a_2=a$, $b_1+b_2=b$, $c_1+c_2=c$, and $a_1b_1c_1=a_2b_2c_2$.

The line joining P to its isogonal conjugate will pass through the median point of ABC, if

$$\begin{vmatrix} \frac{1}{a} & b_1 & \frac{1}{b_1} \\ \frac{1}{b} & a_2 & \frac{1}{a_2} \\ \frac{1}{c} & \frac{a_1b_1}{c_2} & \frac{c_2}{a_1b_1} \end{vmatrix}=0.$$

i.e. if $\quad \frac{1}{a}\left(\frac{c_1}{b_2}-\frac{b_2}{c_1}\right)+\frac{1}{b}\left(\frac{a_1}{c_2}-\frac{c_2}{a_1}\right)+\frac{1}{c}\left(\frac{b_1}{a_2}-\frac{a_2}{b_1}\right)=0.$

i.e. if $\quad \frac{c_1}{ab_2}+\frac{a_1}{bc_2}+\frac{b_1}{ca_2}=\frac{b_2}{ac_1}+\frac{c_2}{ba_1}+\frac{a_2}{cb_1}.$

i.e. if $\quad c\,c_1c_2ba_2+a\,a_1a_2cb_2+b\,b_1b_2ac_2=c\,c_1c_2ab_1+a\,a_1a_2bc_1+b\,b_1b_2ca_1.$

i.e. if $\quad c\,c_1c_2\,(ba_2-ab_1)+b\,b_1b_2\,(ac_2-ca_1)+a\,a_1a_2\,(cb_2-bc_1)=0$

i.e. if $\quad c\,c_1c_2\,(b_2a_2-b_1a_1)+b\,b_1b_2\,(a_2c_2-a_1c_1)+a\,a_1a_2\,(c_2b_2-c_1b_1)=0$

i.e. if $\quad c(c_1-c_2)+b(b_1-b_2)+a(a_1-a_2)=0$

i.e. if $\quad c_1^2+b_1^2+a_1^2=c_2^2+b_2^2+a_2^2,$

which is true, since the perpendiculars to the sides of the triangle at D,E,F are concurrent.

(2) Again the trilinear polar of P is

$$\frac{\alpha}{b_1}+\frac{\beta}{a_2}+\frac{\gamma c_2}{a_1 b_1}=0.$$

The join of P and circumcentre is

$$\begin{vmatrix} \alpha & b_1 & \cos A \\ \beta & a_2 & \cos B \\ \gamma & \frac{a_1 b_1}{c_2} & \cos C \end{vmatrix}=0.$$

These will represent two perpendicular lines if

$$\frac{a_2}{b_1}\cos c-\frac{a_1}{c_2}\cos B+\frac{a_1 b_1}{a_2 c_2}\cos A-\frac{b_1}{a_2}\cos C+\frac{c_2}{a_1 b_1}(b_1\cos B-a_2\cos A)$$

$$-\cos A\left[\frac{b_1\cos B-a_2\cos A}{a_2}+\frac{c_2}{a_1 b_1}\left(\frac{a_1 b_1}{c_2}\cos A-b_1\cos C\right)\right]$$

$$-\cos B\left[\frac{c_2}{a_1 b_1}\left(a_2\cos C-\frac{a_1 b_1}{c_2}\cos B\right)+\frac{1}{b_1}(b_1\cos B-a_2\cos A)\right]$$

$$-\cos C\left[\frac{1}{b_1}\left(\frac{a_1 b_1}{c_2}\cos A-b_1\cos C\right)+\frac{1}{a_2}\left(a_2\cos C-\frac{a_1 b_1}{c_2}\cos B\right)=0.\right.$$

This when simplified reduces to

$$\left(\frac{a_2}{b_1}-\frac{b_1}{a_2}\right)(\cos C+\cos A\cos B)+\left(\frac{c_2}{a_1}-\frac{a_1}{c_2}\right)(\cos B+\cos C\cos A)$$

$$+\left(\frac{b_2}{c_1}-\frac{c_1}{b_2}\right)(\cos A+\cos B\cos C)=0.$$

Since $\cos A+\cos B\cos C=\sin B\sin C$, etc., this is equivalent to

$$\frac{1}{c}\left(\frac{a_2}{b_1}-\frac{b_1}{a_2}\right)+\ldots=0,$$

which has been established in (1)

(3) Denoting PBC, PBA by θ_1, θ'_1; PCA, PCB by $\theta_2\theta'_2$; PAB, PAC by $\theta_3\theta'_3$; we have

$$\frac{\sin(B-\theta_1)}{\sin\theta'_1}=-\frac{\gamma}{\alpha},\text{ etc., and }\frac{\sin(B-\theta_1')}{\sin\theta_1'}=\frac{\alpha}{\gamma},$$

$$\therefore\qquad \cos B-\sin B\cot\theta_1=\frac{\gamma}{\alpha}\text{ etc.}$$

$$\therefore\quad \cot\theta_1+\cot\theta_2+\cot\theta_3=\cot A+\cot B+\cot C$$

$$-\frac{a_1}{c_2\sin B}-\frac{b_1}{a_2\sin C}-\frac{c_1}{b_2\sin A},$$

and $\cot\theta_1'+\cot\theta_2'+\cot\theta_3'=\cot A+\cot B+\cot C$

$$-\frac{c_2}{a_1\sin B}-\frac{a_2}{b_1\sin C}-\frac{b_2}{c_1\sin A}.$$

$\therefore \quad \cot\theta_1+\cot\theta_2+\cot\theta_3-(\cot\theta_1'+\cot\theta_2+\cot\theta_3)$

$$=\frac{1}{\sin A}\left(\frac{b_1}{c_2}-\frac{c_2}{b_1}\right)+\frac{1}{\sin B}\left(\frac{c_2}{a_1}-\frac{a_1}{c_2}\right)+\frac{1}{\sin C}\left(\frac{a_2}{b_1}-\frac{b_1}{a_2}\right)=0, \text{ from (I)}.$$

Question 432.

(A. C. L. Wilkinson, MA., F.R.A.S.,) :—The rectangular hyperbolas $x^2-y^2-2\alpha x+2\beta y=4c(c^2-\alpha\beta)^{\frac{1}{2}}$, and $xy=c^2$ are so related that triangles can be inscribed in the first that are circumscribed to the second. Show that the locus of the centres of circles circumscribing these triangles is the conic

$$(\beta x+\alpha y)^2=4c^2(xy+\alpha\beta-c^2).$$

Solution by Appu Kuttan Erady.

Let $x^2+y^2+2gx+2fy+k^2=0$ be the circumcircle of any triangle inscribed in the first and circumscribed to the second. Then for all values of λ, the conics

$$2xy-2c^2=0$$

and

$$x^2(\lambda+1)+y^2(\lambda-1)+2x(\lambda g-\alpha)-2y(\lambda f+\beta)+(\lambda k^2-4c\sqrt{c^2-\alpha\beta})=0,$$

are so related that triangles can be found which are circumscribed to the first and inscribed in the second.

The condition for this is (in the usual invariant notation) $\Theta'^2=4\Delta\Theta'$; where $\Delta=2c^2$, $\Theta=-(\lambda k^2-4c\sqrt{c^2-\alpha\beta})$ and $\Theta'=2(\lambda g-\alpha)(\lambda f+\beta)-2c^2(\lambda^2-1)$.

Hence $(\lambda k^2-4c\sqrt{c^2-\alpha\beta})^2=16c^2\{(\lambda g-\alpha)(\lambda f+\beta)-c^2(\lambda^2-1)\}$ for all values of λ.

$\therefore \quad k^4=16c^2(fg-c^2)$, and $-8ck^2\sqrt{(c^2-\alpha\beta)}=16c^2(\beta g-\alpha f)$.

Eliminating k between these two equations, we have

$$4(fg-c^2)(c^2-\alpha\beta)=(\beta g-\alpha f)^2$$

or

$$(\beta g+\alpha f)^2=4c^2(fg+\alpha\beta-c^2).$$

Hence the locus of the centre of the circle is

$$(\beta x+\alpha y)^2=4c^2(xy+\alpha\beta-c^2).$$

Question 433.

(J. C. Swaminarayan, M.A.) :—If

$$f(x_1x_2\ldots x_n)=p_1(x_1+a_2x_2+a_3x_3\ldots+a_nx_n)^2$$
$$+p_2(x_2+b_3x_3+\ldots b_nx_n)^2$$
$$+p_3(x_3+c_4x_4+\ldots c_nx_n)^2+\ldots+p_nx_n^2,$$

shew that

$$\begin{vmatrix} \frac{\partial^2 f}{\partial x_1^2}, & \frac{\partial^2 f}{\partial x_1 \partial x_2}, & \frac{\partial^2 f}{\partial x_1 \partial x_3}, & \cdots & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n}, \\ \frac{\partial^2 f}{\partial x_2 \partial x_2}, & \frac{\partial^2 f}{\partial x_2^2}, & \frac{\partial^2 f}{\partial x_2 \partial x_3}, & \cdots & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_n}, \\ \frac{\partial^2 f}{\partial x_n \partial x_1}, & \frac{\partial^2 f}{\partial x_n \partial x_2}, & \frac{\partial^2 f}{\partial x_n \partial x_3}, & \cdots & \cdots & \frac{\partial^2 f}{\partial x_n^2}, \end{vmatrix}$$

is equal to the continued product of $p_1 p_2 p_3 \ldots p_n$.

Solution (1) by E. R. Seshu Aiyar, (2) by T. P. Trivedi M.A., L.L.B., R. Tata M.A., V. D. Gokhale M. A. and N. Sankara Aiyar, M.A.

[The result is incorrect ; the product should be $2^n p_1 p_2 \ldots p_n$]

(1) The given determinant is the Hessian of $f(x_1, x_2, \ldots x_n)$.

$$\text{Let } X_1 = x_1 + a_2 x_2 + \ldots\ldots a_n x_n$$
$$X_2 = x_2 + b_3 x_3 + \ldots\ldots b_n x_n$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$X_n = x_n$$

Since the Hessian is a covariant, we have for the above transformation

$$\frac{\partial\left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots\ldots \frac{\partial f}{\partial x_n}\right)}{\partial(x_1, x_2, \ldots\ldots x_n)} = \mu^2 \frac{\partial\left(\frac{\partial f}{\partial X_1}, \frac{\partial f}{\partial X_2}, \ldots\ldots \frac{\partial f}{\partial X_n}\right)}{\partial(X_1, X_2, \ldots\ldots X_n)}$$

where μ = the modulus of transformation and F denotes $p_1 X_1^2 + p_2 X_2^2 + \ldots p_n X_n^2$.

$$\text{Also } \mu = \begin{vmatrix} 1, & a_2, & \cdots & \cdots & a_n \\ 0 & 1 & b_3 & \cdots & b_n \\ 0 & 0 & 1 & \cdots & c_n \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 1 \end{vmatrix} = 1.$$

Hence the given determinant is equal to

$$\frac{\partial\left(\frac{\partial F}{\partial X_1}, \frac{\partial F}{\partial X_2} \ldots\ldots \frac{\partial F}{\partial X_n}\right)}{\partial(X_1, X_2 \ldots\ldots\ldots X_n)} = \begin{vmatrix} 2p_1 & 0 & 0 & \cdots & 0 \\ 0 & 2p_2 & 0 & \cdots & 0 \\ 0 & 0 & 2p_3 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 2p_n \end{vmatrix}$$

$$=2^n.\ p_1p_2\ .\ \ldots\ldots p_n.$$

(2) We have $\dfrac{\delta^2 f}{\delta x_1^{\,2}}=2p_1$; $\dfrac{\delta^2 f}{\delta x_1\delta x_2}=2p_1a_2$; etc.

$$\frac{\delta^2 f}{\delta x_2^{\,2}}=2p_1a^2_{\,2}+2p_2 : \frac{\delta^2 f}{\delta x_2\delta x_3}=2p_1a_2a_3+2p_2b_3 ;$$

$$\frac{\delta^2 f}{\delta x_3^{\,2}}=2p_1a_3^{\,2}+2p_2b_3^{\,2}+2p_3 ;$$

$$\frac{\delta^2 f}{\delta x_3\delta x_4}=2p_1a_3a_4+2p_2b_3b_4+2p_3c_4 ;\ \text{etc.}$$

Thus the determinant is equal to

$$2^n\begin{vmatrix} p_1, & p_1a_2, & p_1a_3 & \ldots & \ldots \\ p_1a_2, & p_1a_2^{\,2}+p_2, & p_1a_2a_3+p_2b_3,\ldots & p_1a_2a_n+p_2b_n \\ p_1a_3, & p_1a_2a_3+p_2b_3, & p_1a_3^{\,2}+p_2b_3^{\,2}+p_3, & & \ldots \\ \ldots & \ldots & \ldots\ \ldots\ \ldots & \ldots & \ldots \\ p_1a_n, & p_1a_na_2+p_2b_n, & p_1a_na_3+p_2b_nc_3+p_3c_n,\ldots & & p_n, \end{vmatrix}$$

Multiply the first column by a_2 and subtract it from the 2nd, multiply it by a_3 and subtract from the 3rd, etc. The determinant now reduces to

$$2^np_1\begin{vmatrix} p_2, & p_2b_3 & \ldots & \ldots & p_2b_n \\ p_2b_3, & p_3b_3^{\,2}+p_3, & \ldots & \ldots & p_2b_3b_n+p_3c_n \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ p_2b_n & \ldots & \ldots & \ldots & p_n \end{vmatrix}$$

Repeating the process with the b's, c's, &c., the determinant is easily seen to be equal to $2^np_1p_2\ldots\ldots\ldots p_n$.

Question 439.

(S. P. Singaravelu Mudaliar):—The circle of curvature at P to the Folium of Descartes passes through the node and cuts the curve again at Q. Shew that the envelope of PQ is a rectangular hyperbola. Also, find the locus of the intersection of the tangents at P and Q to the Folium.

Solution by G. Ramachandran and H. V. Venkataramiengar, B.A.

Let the equation of the Folium be $x^3+y^3=axy$.

The coordinates of any point on the curve are

$$\frac{at}{(1+t^3)}, \frac{at^2}{(1+t^3)}$$

The equation of any circle through the node is

$$x^2+y^2+2gx+2fy=0.$$

The points of intersection of this circle with the cubic are given by the equation

$$2ft^4+t^3(a+2g)+t(2f+a)+2g=0.$$

For the circle of curvature at P, these points are P and Q and

$$3t^2+3tt_1=\text{coefft. of } t^2=0.$$

$$\therefore \qquad t+t_1=0.$$

Hence, Q is the point $-t$, and the equation of the chord PQ

$$t^4x-t^2a+y=0,$$

which touches the rectangular hyperbola $4xy=a^2$.

Again, the equation of the tangent at P is

$$t\,x(2-t^3)-y(1-2t^3)=at^2,\ldots \qquad \ldots \quad (1)$$

and that of the tangent at Q is

$$-t\,x(2+t^3)-y(1+2t^3)=at^2 \quad \ldots \qquad \ldots \quad (2)$$

The required locus is obtained by eliminating t between these equations.

Adding (1) and (2)

$$x\,t^4+y+at^2=0.$$

Substituting in equation (1)

$$2t\,x+2yt^3=0. \quad \therefore \quad t^2=-x/y.$$

Hence the locus of the intersection of the tangents is

$$x^3+y^3=axy,$$

which is the Folium itself.

Question 440.

(P. V. SESHU AIYAR) :—Show that

(1) $$\int_0^\infty \frac{\sinh px}{\sinh qx}\cos rx\,dx=\frac{\pi}{2q}\cdot\frac{\sin\frac{p\pi}{q}}{\cos\frac{p\pi}{q}+\cosh\frac{r\pi}{q}}$$

(2) $$\int_0^\infty \frac{\cosh px}{\sinh qx}\sin rx\,dx=\frac{\pi}{2q}\frac{\sin\frac{r\pi}{q}}{\cos\frac{p\pi}{q}+\cosh\frac{r\pi}{q}}$$

Solutions by K. Appukuttan Erady, M.A., and V. K. Aravamudan, B.A.

(1) $$\int_0^\infty \frac{\sinh px}{\sinh qx}.\cos rx\,dx = \int_0^\infty \frac{e^{px}-e^{-px}}{e^{qx}-e^{-qx}}\cdot\frac{e^{irx}+e^{-irx}}{2}.\,dx$$

$$=\tfrac{1}{2}\int_0^\infty \frac{e^{(p+ir)x}-e^{-(p+ir)x}}{e^{qx}-e^{-qx}}.dx$$

$$+\tfrac{1}{2}\int_0^\infty \frac{e^{(p-ir)x}-e^{-(p-ir)x}}{e^{qx}-e^{-qx}}.dx$$

$$=\frac{\pi}{2q}\int_0^\infty \frac{e^{\frac{\pi(p+ir)}{q}z}-e^{\frac{-\pi(p+ir)}{q}z}}{e^{\pi z}-e^{-\pi z}}\,dz$$

$$+\frac{\pi}{2q}\int_0^\infty \frac{e^{\frac{\pi(p-ir)}{q}z}-e^{\frac{-\pi(p-ir)}{q}z}}{e^{\pi z}-e^{-\pi z}}\,dz$$

$$=\frac{\pi}{4q}\tan\frac{\pi(p+ir)}{2q}+\frac{\pi}{4q}\tan\frac{\pi(p-ir)}{2q}.$$

[Williamson, *Integral Calculus* p. **142**.]

$$=\frac{\pi}{4q}.\frac{\sin\frac{p\pi}{q}}{\cos\frac{\pi(p+ir)}{2q}\cos\frac{\pi(p-ir)}{2q}}=\frac{\pi}{2q}\frac{\sin\frac{p\pi}{q}}{\cos\frac{p\pi}{q}+\cosh\frac{r\pi}{q}}.$$

(2) $$\int_0^\infty \frac{\cosh px}{\sinh qx}\sin rx.dx = \int_0^\infty \frac{e^{px}+e^{-px}}{e^{qx}-e^{-qx}}.\frac{e^{irx}-e^{-irx}}{2i}dx.$$

$$=\frac{1}{2i}\int_0^\infty \frac{e^{(p+ir)x}-e^{-(p+ir)x}}{e^{qx}-e^{-qx}}.dx$$

$$-\frac{1}{2i}\int_0^\infty \frac{e^{(p-ir)x}-e^{-(p-ir)x}}{e^{qx}-e^{-qx}}.dx$$

$$=\frac{\pi}{4iq}\tan\frac{\pi(p+ir)}{2q}-\frac{\pi}{4iq}\tan\frac{\pi(p-ir)}{2q}, \text{ as in (1)}$$

$$=\frac{\pi}{4iq}\frac{\sin\frac{ir\pi}{q}}{\cos\frac{\pi(p+ir)}{2q}\cos\frac{\pi(p-ir)}{2q}}=\frac{\pi}{2q}\frac{\sinh\frac{r\pi}{q}}{\cos\frac{p\pi}{q}+\cosh\frac{r\pi}{q}}$$

Question 443.

(K. Appukuttan Erady, M.A.) :—Points are taken on the principal normals to a curve in space at distances from the curve equal to c times the circular curvature. Prove that

$$\left(\frac{ds'}{ds}\right)^2=\left(1-\frac{c}{\rho^2}\right)^2+\frac{c^2}{\rho^2}\left\{\frac{1}{\sigma^2}+\frac{1}{\rho^2}\left(\frac{d\rho}{ds}\right)^2\right\}$$

where s' refers to the locus of the points in question.

Solution by E. R. Seshu Aiyar.

With the usual notation the equations of the principal normal are

$$\frac{X-x}{l_2}=\frac{Y-y}{m_2}=\frac{Z-z}{n_2}=\frac{c}{\rho},$$

where the distance of (XYZ) from the foot of the normal is c times the circular curvature. Thus, we have

$$X=x+cl_2/\rho, \text{ \&c.}$$

$$\therefore \quad \frac{dX}{ds}=\frac{dx}{ds}+\frac{ds}{ds}\left(\frac{cl_2}{\rho}\right)=l_1+\frac{c}{\rho}\cdot\frac{dl_2}{ds}-\frac{cl_2}{\rho^2}\cdot\frac{d\rho}{ds}$$

$$=l_1+\frac{c}{\rho}\left(-\frac{l_1}{\rho}-\frac{l_3}{\sigma}\right)-\frac{cl_2}{\rho^2}\frac{d\rho}{ds}, \text{ Frenet's Formulae.}$$

Similarly for $\frac{dY}{ds}$ and $\frac{dZ}{ds}$.

Squaring and adding we get, since $(dX)^2+(dY)^2+(dZ)^2=(ds')^2$,

$$\left(\frac{ds'}{ds}\right)^2=\sum\left\{l_1\left(1-\frac{c}{\rho^2}\right)-\frac{cl_3}{\rho\sigma}-\frac{cl_2}{\rho^2}\frac{d\rho}{ds}\right\}^2$$

$$=\left(1-\frac{c}{\rho^2}\right)^2+\frac{c^2}{\rho^2\sigma^2}+\frac{c^2}{\rho^4}\left(\frac{d\rho}{ds}\right)^2,$$

if we remember that $l^2_1+m^2_1+n^2_1=1$, &c., and $l_1l_2+m_1m_2+n_1n_2=0$, &c.,

Question 444.

(J. C. Swaminarayan, M.A.) :—On the base BC of a triangle ABC equilateral triangles BPC and BQC are described. Shew that the bisectors of the angle PAQ are parallel to the axes of the maximum incribed ellipse of the triangle ABC.

Solution by V. K. Aravamudhan B. A., and G. Ramachandran.

The maximum inscribed ellipse is the momental ellipse of the $\triangle$ at the *c.g.* and its axes are the principles axes. If these be taken as coordinate axes and if A,B,C be (x_1y_1), (x_2y_2), (x_3y_3), the coordinates of the mid-points of the sides where the ellipse touches them are

$$\left(\frac{x_2+x_3}{2}, \frac{y_2+y_3}{2}\right), \text{ etc.}$$

Now by the principles of dynamics,

$$\Sigma \frac{x_2+x_3}{2}0, \text{ or } \Sigma x_1=0, \Sigma y_1=0 ;$$

and
$$\Sigma\frac{x_2+x_3}{2}\frac{y_2+y_3}{2}=0, \text{ or } \Sigma x_1y_1=0.$$

Now, again, the coordinates of P,Q are given by

$$x=\frac{x_2+x_3}{2} \pm \sqrt{3}\ \frac{(y_2-y_3)}{2},\ y=\frac{y_2+y_3}{2} \pm\sqrt{3}\frac{(x_2-x_3)}{2}.$$

To prove the question, it is enough to show that AP and AQ are equally inclined to the x axis.

The condition for this is

$$\frac{y_2+y_3-2y_1-\sqrt{3}(x_2-x_3)}{x_2+x_3-2x_1+\sqrt{3}(y_2-y_3)}-\frac{y_2+y_3-2y_1+\sqrt{3}(x_2-x_3)}{x_2+x_3-2x_1-\sqrt{3}(y_2-y_3)}=0.$$

By using $\Sigma x_1=0$ and $\Sigma y_1=0$, this reduces to

$$3x_1y_1+(x_2-x_3)\ (y_2-y_3)=0,$$

or
$$2x_1y_1+(x_2+x_3)\ (y_2+y_3)+(x_2-x_3)\ (y_2-y_3)=0,$$

or
$$\Sigma x_1y_1=0, \text{ which is true.}$$

Additional solution by N. P. Pandya.

Question 445.

(D. D. Kapadia, M.A., B.Sc.) :—The equation of a family of curves is $f(x,y,a)=0$, where a is the variable parameter. If the envelope of this family of curves has a contact of the second order with the curve, prove that at the point of contact

$$\frac{\partial f}{\partial x}\cdot\frac{\partial^2 f}{\partial y\partial a}-\frac{\partial f}{\partial y}\cdot\frac{\partial^2 f}{\partial x\partial a}=0.$$

Solution by K. J. Sanjana, M.A. and K. Appu Kuttan Erady, M.A.

If the curves A and B, B and C, of the family cut at P, Q respectively, P and Q lie ultimately on the envelope. If the envelope has contact of the second order with B, it has ultimately a third point common with

B; but this must be a point of a contiguous curve, as it is on the envelope. Hence A and B, or B, and C must touch. But points of contiguous intersection lie on the curves

$$f=0, \frac{\partial f}{\partial a}=0.$$

Hence, applying the condition of tangency, we get

$$\frac{\dfrac{\partial f}{\partial x}}{\dfrac{\partial f}{\partial y}}=\frac{\dfrac{\partial}{\partial x}\left(\dfrac{\partial f}{\partial a}\right)}{\dfrac{\partial}{\partial y}\left(\dfrac{\partial f}{\partial a}\right)},$$

which is the given condition.

As ultimately ∂a is infinitesimal and the curves A and B, or B and touch, we shall further have $\dfrac{\partial^2 f}{\partial a^2}=0$. (Wolstenhol me, Q. 1813 2nd Ed.)

Additional solution by H. V. Venkataramiengar.

Question 446.

(A. C. L. Wilkinson, M.A., F.R.A.S.):—In Mr. Swaminarayan's Note on "A generalised form of Claraiut's Equation" J. I. M. S. Vol. IV, No. 6, discuss the cases where $b=1$, and $a+b=1$. Illustrate by solving the equation $(y-xp)^2=4\,py$.

Solution by N. Sankara Aiyar, M.A. and E. R. Seshu Aiyar.

Case (i): $b=1$. Here

$$y=m\,xp+k\,y^a\,p.$$

Interchanging the dependent and independent variables, we get

$$qy-mx=ky^a, \text{ where } q=\frac{dx}{dy}.$$

i.e. $$\frac{q}{y^m}-\frac{mx}{y^{m+1}}=ky^{a-m-1}$$

i.e. $$\frac{d}{dy}\left(\frac{x}{y^m}\right)=ky^{a-m-1}$$

i.e. $$\frac{x}{y^m}=\frac{k}{a-m}y^{a-m}+c$$

i.e. $$x=cy^m+\frac{k}{a-m}y^a.$$

Case (ii): $a+b=1$. The equation is

$$y=mxp+ky^a\,p^{1-a}.$$

Let $\log y=z/m$, so that $p/y=q/m$. Then, dividing out by y, we have

$$1=mxp/y+k(p/y)^{1-a}.$$

$$\therefore \qquad 1 = xq + k(q/m)^{1-a},$$

which can be solved by the known methods.

Illustration: $(y-px)^2 = 4py$. Here

$$y - px = \pm 2\, p^{\frac{1}{2}} y^{\frac{1}{2}}.$$

$$\therefore \qquad y = px \pm 2p^{\frac{1}{2}} y^{\frac{1}{2}}.$$

This comes under Case (ii), where $m=1$, $a=b=\frac{1}{2}$; the transformed equation is

$$1 = xq \pm 2\, q^{\frac{1}{2}}.$$

Solving for q

$$q = x - 2 \pm 2\sqrt{1-x}.$$

Hence $$z = \frac{x^2}{2} - 2x \pm \frac{4}{3}(1-x)^{\frac{3}{2}} + \log c.$$

i.e. $$\log y = \frac{x^2}{2} - 2x \pm \frac{4}{3}(1-x)^{\frac{3}{2}}.$$

Question 447.

(G. Ramachandran) :—Construct a triangle ABC having given the rectangle contained by AB and AC, the median from A to BC, and the sum or difference of the angles ABC, ACB.

Solution by N. P. Pandya.

(1) Let m be the given median. Then

$$4m^2 = b^2 + c^2 + 2bc \cos A.$$

This determines the $\triangle$, if $B+C$ is given.

(2) Again $$4m^2 = (b+c)^2 - 2bc(1-\cos A).$$

$$\therefore \qquad 4m^2 + 2bc(1-\cos A) = (b+c)^2$$

Also $$4m^2 = (b-c)^2 + 2bc(1+\cos A).$$

$$\therefore \qquad 4m^2 - 2bc(1+\cos A) = (b-c).$$

$$\therefore \qquad \frac{4m^2 + 2bc(1-\cos A)}{4m^2 - 2bc(1+\cos A)} = \frac{(b+c)^2}{(b-c)^2}$$

$$= \frac{\tan^2 \frac{1}{2}(B+C)}{\tan^2 \frac{1}{2}(B-C)}$$

$$= \frac{(1+\cos A)(1+\cos \overline{B-C})}{(1-\cos A)(1-\cos \overline{B-C})}.$$

This gives $(B+C)$ when $(B-C)$ is known, and the Δ is determined.

Question 448

(V. V. SATYANARAYANA) :—In any triangle ABC, circles BQRC' CRPA, APQB are described shew that QC.RA.PB = QA.RB.PC.

Also, if (x,x'), (y,y'), (z,z') are the distances of (Q,R), (R,P), (P,Q) from BC, CA, AB respectively, prove that $xyz = x'y'z'$.

Solution by (1) *K. J. Sanjana, T. P. Trivedi and N. P. Pandya* ;
(2) *by G. Ramachandran.*

(1) Because AP, BQ, CR are the radical axes of the ⊙s taken two and two, they meet at a point O ; draw OL, OM, ON. perp. to BC, CA, AB.

Also draw QB′, RC′ perp. to BC ; RC″, PA′ to CA, PA″, QB″ to AB ; and let d_1, d_2, d_3 be the diameters of the circles in order.

By Euclid, VI-C, $QC.BQ = QB'.d_1 = xd_1$; so also
[$RC.RB = x'd_1$. $RC.RA = yd_2$, $PC.PA = y'd_2$, $PA.PB = zd_3$, $QA.QB = z'd_3$.

Multiplying alternate equations and cancelling, we get

$$\frac{QC \cdot RA \cdot PB}{RB \cdot PC \cdot QA} = \frac{xyz}{x'y'z'}.$$

Again $\frac{x}{z'} = \frac{QB'}{QB''} = \frac{OL}{ON}$; so also $\frac{y}{x'} = \frac{OM}{OL}, \frac{z}{y'} = \frac{ON}{OM}$.

Hence $\frac{xyz}{x'y'z'} = 1$; and $QC \cdot RA.PB = RB.PC.QA$.

(2) The triangles BPO and AQO are similar, since $P\hat{B}O = Q\hat{A}P$ being angles in the same segment, and $A\hat{O}B$ is common to both the triangles.

$$\therefore \quad \frac{PB}{BO} = \frac{AQ}{AO}, \ ie. \ \frac{PB}{QA} = \frac{BO}{AO}.$$

Similarly

$$\frac{QC}{BR} = \frac{CO}{BO}, \text{ and } \frac{AR}{CP} = \frac{AO}{CO}.$$

$$\therefore \quad PB \cdot QC \cdot RA = QA \cdot RB \cdot PC.$$

For the second part, *vide* : Casey's *Sequel to Euclid*, Book, VI., Prop. 12, page 76.

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QUESTIONS FOR SOLUTION.

468. (N. SANKARA AIYAR, M A.) :—Solve the equation

$$\int_0^{v_0} \left\{ \frac{x^2\left(\frac{dy}{dx}\right)^2+1}{v_0^2-x^2} \right\}^{\frac{1}{2}} dx = k.$$

469. (S. RAMANUJAN) :—The number $\{1+\underline{|n}\}$ is a perfect square for the values 4, 5, 7 of n. Find other values.

470. (S. KRISHNASWAMI AIYANGAR) :—If ρ, ρ' be the radii of curvature at corresponding points of a curve and its evolute, prove that

$$\rho/'\rho = \{3y_1y_2^2 - y_3(1+y_1^2)\}/y_2^2.$$

471. (K. J. SANJANA, M.A.) :—Prove that

$$\frac{1}{2}\frac{x^2}{2}+\frac{1\cdot3}{2\cdot4}\frac{x^4}{4}+\frac{1\cdot3\cdot5}{2\cdot4\cdot6}\frac{x^6}{6}+\ldots\ldots=\log\frac{2+2\sqrt{(1-x^2)}}{x^2}$$

and find the sum of

$$\frac{2}{3}\frac{x^3}{3}+\frac{2\cdot4}{3\cdot5}\cdot\frac{x^5}{5}+\frac{2\cdot4\cdot6}{3\cdot5\cdot7}\cdot\frac{x^7}{7}+\ldots\ldots\ldots\ldots$$

472. (SELECTED) :—Evaluate $\int_0^\infty \frac{\cos x}{(1+x^2)^2}dx$; $\int_0^\infty \frac{\cos 2x}{(1+x^2)^2}dx.$

473. (A. C. L. WILKINSON, M.A., F.R.A.S.) :—PQRS is a spherical quadrilateral such that PR and QS are quadrants. If A, B, C are the intersections of (PR,QS), (PS,QR) and (PQ,RS) respectively ; prove that

(i) cosPA. cosPB = cosQA. cosQB,

(ii) cosPA. cosPC + cosQA. cosQC = cosAC.

474. (N. P. PANDYA) :—P is a point on a parabolic mirror (vertex A, focus S). A ray proceeds from a point L on the axis of the parabola, is reflected at P, and meets the axis again at M. If N is the geometrica, focus of a small pencil from L after reflection at P, prove that

$$SL.\ MN = AS^2(1+\sec^2\alpha)\tan^2\alpha,$$

where 2 α is the angle between SP and the axis.

475. (K. Appukuttan Erady, M.A.) :—The space bounded by the coordinate planes and the surface $(x/a)^n+(y/b)^n+(z/c)^n=1$ is filled with an elastic fluid without weight. Prove that the pressures on the curved surface reduce to a single resultant whose line of action is

$$a(x-\lambda a)=b(y-\lambda b)=c(z-\lambda c)$$

where $\lambda=2[\Gamma(2/n)]^2/3\Gamma(1/n).\Gamma(3/n)$.

476. (Zero) :—Solve the difference equations

$$\frac{d^2x_k}{dt^2}+b\frac{dx_k}{dt}+c(x_{k-1}+x_{k+1}-2x_k)=0,$$

where $[k=1, 2, \ldots\ldots n-1]$, on the supposition that $x_o=o=x_n$.

477. (V. V. Satyanarayana) :—ABC is a triangle of given perimeter. If the vertex A is fixed and BC is of constant length, find the locus of B when C describes (1) a straight line, (2) a conic.

478. (M. T. Naraniengar) :—If S, H be the foci and O the centre of the maximum inscribed ellipse of a triangle ABC, prove that

$$AS{\cdot}BS{\cdot}CS+AH{\cdot}BH{\cdot}CH=2\ AO{\cdot}BO{\cdot}CO.$$

479. (S. P. Singaravelu Modeliar) :—Shew that the equation

$$(x^5-10x^3y^2+5xy^4)\cos 5\alpha+(y^5-10y^3x^2+5yx^4)\sin 5\alpha$$
$$-5ar^4+20a^3r^2-16a^5=0,$$

represents the sides of a regular pentagon, and find a similar equation for the sides of a regular heptagon.

480. (V. Ramaswami Aiyar) :—If a straight line cut a three-cusped hypocycloid at P, Q, R, S, show that the extremities of the tangent-chords touching the tricusp at P, Q, R and S lie on a rectangular hyperbola.