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PROGRESS REPORT.

The following gentlemen have been elected members of the Society:

(1) The Hon'ble Mr. Justice J. J. Heaton, I.C.S.—Judge of H. M.'s High Court, Bombay; and Vice-Chancellor, University of Bombay:

(2) Mr. C. S. Anantaram Aiyar, B.A., Acting Under Secretary to Government, Newplace, Mylapore, Madras:

(3) Mr. Vishnu Dattatraya Gokhale, M.A., Teacher, Nutan Marathi Vidyalaya, 11 Kasba Peth, Poona City (at concessional rate):

(4) Mr. Solomon E. Reuben, M.A., Formerly: Fellow, Deccan College,—Law-Student, 8 Y.M.C.A. Quarters, Girgaum, Bombay (at concessional rate):

(5) Mr. Shrimukhrao Laxmilal Mehta, B.A., Fellow, Gujarat College,—1304, Raipoor Haziroo Street, Ahmedabad (at concessional rate):

(6) The Rev. T. Noronha, Ph.D., D.D., S.J., Professor of Mathematics, St. Aloysius' College, Mangalore.

2. According to Art. VIII (d) of our Constitution, the Committee have reappointed Messrs. M. T. Naraniengar, M.A., and D. D. Kapadia, M.A., B. Sc., as Hon. Joint Secretaries for the current year.

3. The following alterations in Art. VIII of our Constitution have been approved of by the General Body, and consequently the amendments will come into operation from the current year:—

Art. VIII—line 1, read "*every two years*" for "*every year*";

" VIII (b)—line 2, read "*two every two years*" for "*two each year*"

" VIII (c)—line 10, read "*in the year in which the nominations are due*" for "*in each year.*"

4. The Audited Balance sheet of accounts for the last year, as well as the Budget for the current year are given below according to Art. X of our Constitution.

5. The following books have presented to the Library—

Elementary Algebra—by Messrs. W. M. Baker and A. A. Bourne, 9th Edition, G. Bell & Sons, London 1912. 4*s*/6*d*.

Madras University Calendar, 1913, Vols. I & II, and Examination Papers, Volume for 1912.

POONA,
31st March 1913.

D. D. KAPADIA,
Hony. Joint Secretary.

Balance Sheet for 1912.

Receipts.	RS.	A.	P.	*Expenditure.*	RS.	A.	P.
Subscription Arrears ...	636	0	0	Debit Balance for 1911 ...	69	1	11
Do. Current ...	1,211	3	8	Books and Journals ...	681	14	2
Do. to Journal ...	156	5	0	Library	265	5	0
Miscellaneous	1	12	0	Journal Printing	506	5	6
				Ordinary Working Expenditure	242	6	6
				Balance to 1913	240	3	7
Total ...	2,005	4	8	Total ...	2,005	4	8

MADRAS,
11th February 1913.

C. POLLARD, M.A.,
Hon. Treasurer.

I have examined the Treasurer's books and vouchers, and the monthly statements and vouchers of the Secretary, Assistant Secretary and Assistant Librarian and declare the above statement as correct.

MADRAS,
14th February 1913.

S. NARAYANA AIYAR, M.A., F.S.S.
Auditor.

Budget for 1913.

Income.	RS.	A.	P.	*Expenditure.*	RS.	A.	P.
Balance in hand	218	9	10	Library Books	450	0	0
Subscription Arrears ...	500	0	0	Periodicals	650	0	0
Do. Current ...	1,500	0	0	Library Expenses... ...	280	0	0
Do. Journal ...	120	0	0	Journal Printing	510	0	0
Miscellaneous	1	6	2	Ordinary Working Expenditure	250	0	0
				Balance to 1914	250	0	0
Total ...	2,340	0	0	Total ...	2,340	0	0

MADRAS,
11th February 1913.

C. POLLARD, M.A.,
Hon. Treasurer.

On Tetrahedral Co-ordinates.

By A. C. L. Wilkinson, M.A., F.R.A.S.

If α,β,γ are the areal coordinates of a point, $\alpha+\beta+\gamma=1$.

Also (α,β,γ) is the centroid of masses α,β,γ placed at A,B,C the vertices of the triangle of reference.

If, therefore, referred to any rectangular axes the coordinates of the vertices of the triangle of reference are (x_1,y_1), (x_2,y_2), (x_3,y_3), and (x,y) are the coordinates of (α,β,γ), we have

$$\left.\begin{aligned} x&=\alpha x_1+\beta x_2+\gamma x_3 \\ y&=\alpha y_1+\beta y_2+\gamma y_3 \end{aligned}\right\}$$

which are the formulae of transition from cartesian to areal coordinates.

These simple relations enable us to solve readily problems in areal coordinates.

§. 1. A similar method applies to tetrahedral coordinates.

Writing—

$$\begin{aligned} x&=\alpha x_1+\beta x_2+\gamma x_3+\partial x_4 \\ y&=\alpha y_1+\beta y_2+\gamma y_3+\partial y_4 \\ z&=\alpha z_1+\beta z_2+\gamma z_3+\partial z_4 \\ 1&=\alpha+\beta+\gamma+\partial \end{aligned}$$

we find for the square of the distance between $(\alpha,\beta,\gamma,\partial)$ $(\alpha',\beta',\gamma',\partial')$

$$\Sigma(x-x')^2=\Sigma[(\alpha-\alpha')x_1+(\beta-\beta')x_2+(\gamma-\gamma')x_3+(\partial-\partial')x_4]^2$$

and reducing by means of $(\alpha-\alpha')^2=-\Sigma(\alpha-\alpha')(\beta-\beta')$ this becomes

$$-\Sigma AB^2(\alpha-\alpha')(\beta-\beta') \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

§. 2. *The angle between two straight lines:*

Any straight line can be written

$$\frac{\alpha-\alpha'}{l}=\frac{\beta-\beta'}{m}=\frac{\gamma-\gamma'}{n}=\frac{\partial-\partial'}{p}=\frac{r}{\sqrt{-\Sigma AB^2 lm}}$$

where r is the distance between $(\alpha,\beta,\gamma,\partial)$ and $(\alpha'\beta'\gamma'\partial')$ and also $l+m+n+p=0$.

The point at infinity on this straight line has coordinates (l,m,n,p).

Thus the direction cosines of the straight line referred to the rectangular axes are $(k\Sigma lx_1,\ k\Sigma ly_1,\ k\Sigma lz_1)$ where the sum of the squares is to be unity.

Thus for the angle between two straight lines $(l,m,n,p)(l',m',n',p')$

$$\cos\Theta=kk'\Sigma(lx_1+mx_2+nx_3+px_4)(l'x_1+m'x_2+n'x_3+p'x_4)\,;$$

replace ll' by $-\frac{1}{2}(l'm+l'n+l'p+lm'+ln'+lp')$, and we find

$$\begin{aligned} \cos\Theta&=kk'\Sigma[(lm'+l'm)\Sigma\{\,x_1x_2+\tfrac{1}{2}(x_1^2+x_2^2\,\}\,)] \\ &=-\tfrac{1}{2}kk'\Sigma(lm'+l'm)AB^2\,; \end{aligned}$$

and by putting l',m',n',p', equal to l,m,n,p, we find

$$-k^2\Sigma lm\ AB^2=1.$$

Thus

$$\cos\theta = -\frac{\frac{1}{2}\Sigma(lm'+l'm)AB^2}{\sqrt{-\Sigma AB^2lm}\ \sqrt{-\Sigma AB^2l'm'}} \quad \dots \quad \dots \quad (1)$$

Example 1.—The straight lines AB, AC of the fundamental tetrahedron are given by

$$\frac{\alpha-1}{1}=\frac{\beta}{-1}=\frac{\gamma}{0}=\frac{\partial}{0} \text{ and } \frac{\alpha-1}{1}=\frac{\beta}{0}=\frac{\gamma}{-1}=\frac{\partial}{0}$$

and $\cos BAC = +\frac{\frac{1}{2}(AB^2+AC^2-BC^2)}{AB.\ AC}$ and it is easy to see that the formula (1) gives that angle corresponding to the directions determined by $l:m:n:p$ and $l':m':n':p'$ where the square roots in the denominator are to be taken positively.

Example 2.—The condition of perpendicularity of AB, CD is seen to be $AC^2+BD^2=AD^2+BC^2$.

Example 3.—The condition that the line joining the middle points of AD, BC should be perpendicular to AD is $BD^2+CD^2-AB^2-AC^2=0$; and if it is also perpendicular to BC, we have $BD^2-CD^2-AC^2+AB^2=0$ whence we obtain $BD=AC$ and $CD=AB$, a result easily seen geometrically.

§. 3. *The shortest distance between two straight lines.*

Consider the straight line

$$\frac{\alpha-\alpha'}{l}=\frac{\beta-\beta'}{m}=\frac{\gamma-\gamma'}{n}=\frac{\partial-\partial'}{p}=\frac{r}{\sqrt{-\Sigma AB^2lm}}.$$

If (x',y',z') is the point $(\alpha',\beta',\gamma',\partial')$ by multiplying these ratios by x_1,x_2,x_3,x_4, etc., and adding we obtain

$$\frac{x-x'}{lx_1+mx_2+nx_3+px_4} = \text{,,} = \text{,,} = \frac{r}{\sqrt{-\Sigma AB^2lm}}.$$

Thus as in § 2 the direction cosines of the straight line are

$$\frac{lx_1+mx_2+nx_3+px_4}{D}, \text{ \&c., where } D=\sqrt{-\Sigma lmAB^2}.$$

Now the shortest distance between two straight lines is given by

$$\begin{vmatrix} x'-x'', & y'-y'', & z'-z'' \\ \lambda, & \mu, & \nu, \\ \lambda', & \mu', & \nu', \end{vmatrix} = p\sin\theta.$$

Consider

$$\begin{vmatrix} \alpha', & \beta', & \gamma', & \delta' \\ \alpha'', & \beta'', & \gamma'', & \delta'', \\ l, & m, & n, & p, \\ l', & m', & n', & p', \end{vmatrix} \begin{vmatrix} x_1, & x_2, & x_3, & x_4 \\ y_1, & y_2, & y_3, & y_4 \\ z_1, & z_2, & z_3, & z_4 \\ 1, & 1, & 1, & 1, \end{vmatrix} = \begin{vmatrix} x' & y' & z' & 1 \\ x'' & y'' & z'' & 1 \\ \lambda & \mu & \nu & 0 \\ \lambda' & \mu' & \nu' & 0 \end{vmatrix} \times D.D'$$

Therefore the shortest distance between two straight lines in tetrahedral coordinates is given by

$$p \sin \theta = 6V \begin{vmatrix} \alpha' & \beta' & \gamma' & \delta' \\ \alpha'' & \beta'' & \gamma'' & \delta'' \\ l & m & n & p \\ l' & m' & n' & p' \end{vmatrix} \div (D.D'),$$

where θ is the angle between the straight lines determined in § 2, and V is the volume of the tetrahedron of reference.

For the shortest distance between AB and CD we get immediately

$$p \sin \theta = \frac{6\ V}{AB \cdot CD},$$

a well known result.

§ 4. *The angle between two planes.*

Take the equations of the planes BCD,...of the tetrahedron of reference, as $l_1x+m_1y+n_1z+p_1=0$, etc.

Thus $$\alpha = \frac{1}{3}\frac{\Delta.BCD}{V}(l_1x+m_1y+n_1z+p_1), \text{ etc.}$$

Any two planes $L\alpha+M\beta+N\gamma+P\partial=0$, $L'\alpha+M'\beta+N'\gamma+P'\partial=0$, when referred to the rectangular axes become

$\Sigma x \{ \triangle BCD.Ll_1+\triangle CAD.Ml_2+\triangle ABD.Nl_3+\triangle ABC.Pl_4 \} =$ constant, and a similar equation in L′, M′, N′, P′.

Hence

$$\cos \theta = \frac{\Sigma\triangle BCD^2.LL' \pm \Sigma\triangle BCD.\triangle ACD.(LM'+L'M) \cos \hat{12}}{\text{Denominator}} \quad \ldots \quad (1)$$

where $\cos \hat{12} = l_1l_2+m_1m_2+n_1n_2 =$ cosine of angle between the faces BCD and CAD of the fundamental tetrahedron, and the denominator is the square root of the product of the two expressions got by writing L′M′N′P′ equal respectively to LMNP and conversely.

To determine $\cos \hat{12}$ we may proceed as follows :

Draw AX, BY perpendiculars on CD. The coordinates of X are easily seen to be

$$\frac{\alpha}{0}=\frac{\beta}{0}=\frac{\gamma}{AD^2+CD^2-AC^2}=\frac{\partial}{AC^2+CD^2-AD^2}$$

and so for Y.

The equations of AX, BY are

$$\frac{\alpha-1}{-2CD^2}=\frac{\beta}{0}=\frac{\gamma}{AD^2+CD^2-AC^2}=\frac{\partial}{AC^2+CD^2-AD^2},$$

and $$\frac{\alpha}{0}=\frac{\beta}{-2CD^2}=\frac{\gamma}{BD^2+CD^2-BC^2}=\frac{\partial}{BC^2+CD^2-BD^2}.$$

By § 2 we can write down the cosine of the angle between these straight lines, and after considerable reductions, we find

$$4\Delta ACD . 4\Delta BCD . \cos \hat{12} = 2AC^2.BD^2 + 2AD^2.BC^2 - 2AB^2.CD^2 - (AC^2 + AD^2 - CD^2)(BC^2 + BD^2 - CD^2) \quad \ldots \quad (2)$$

We may verify that this and similar expressions give the cosines of the dihedral angles of the tetrahedron. For, by projections of three faces on the fourth, we have equations like

$$\Delta CBD \cos \hat{13} + \Delta ABC \cos \hat{34} + \Delta ACD \cos \hat{23} = \Delta ABD \quad \ldots \quad (3)$$

and these equations will be found, on substitution for the cosines, to be identically satisfied. Otherwise, as in § 2, Ex. 1, the expression for $\cos\hat{12}$ is actually the cosine of the angle between XA and YB; that is, the cosine of the dihedral angle between the planes ACD and BCD of the tetrahedron.

Substituting in (1), we have

$$\cos\Theta = \frac{\Sigma\Delta BCD^2 LL' - \frac{1}{16}\Sigma(LM' + L'M)f(CD)}{[\Sigma\Delta BCD^2L^2 - \frac{1}{8}\Sigma LMf(CD)]^{\frac{1}{2}}[\Sigma\Delta BCD^2L'^2 - \frac{1}{8}\Sigma L'M'f(CD)]^{\frac{1}{2}}} \quad (4)$$

where $f(CD) = 2AC^2.BD^2 + 2AD^2.BC^2 - 2AB^2CD^2 - (AC^2 + AD^2 - CD^2)(BC^2 + BD^2 - CD^2)$.

The adoption of the sign requires justification. Since parallel planes are given by L+K, M+K, N+K, P+K, we have only to verify that the numerator is unaltered by increasing each of (LMNP) by K. If this is done it will be found that the coefficient of M′ is zero in virtue of the relation (3). Or, by taking the origin of the rectangular system within the tetrahedron, we may easily see that in (1) the expression $l_1l_2 + m_1m_2 + n_1n_2$ stands for the cosine of the angle between the perpendiculars from the origin on the planes BCD, ACD and is the supplement of the angle defined as $\cos \hat{12}$ in (2).

§. 5. *The perpendicular distance of a point from a plane.*

With the notation of § 4, the perpendicular from $(\alpha', \beta', \gamma', \delta')$ on $L\alpha + M\beta + N\gamma + P\delta = 0$ is given by

$$\frac{\Sigma\alpha' \{ \Delta BCDLl_1 + + + \}}{\sqrt{\Sigma(\Delta BCDLl_1 + + +)^2}} = \frac{3V(L\alpha' + M\beta' + N\gamma' + P'\delta)}{\{ \Sigma\Delta BCD^2L^2 - \frac{1}{8}\Sigma LMf(CD) \}^{\frac{1}{2}}}.$$

§. 6. These results may also be established from statical considerations. It is at once obvious, by resolving the couples into forces along the sides of the tetrahedron, that four couples in the faces of the tetrahedron represented in magnitude and direction by the areas of the faces BCD, DCA, ACB, ABD are in equilibrium, and the axes of these couples are the inward drawn normals to the faces.

Take $(l_1m_1n_1)$ $(l_2m_2n_2)$ $(l_3m_3n_3)$ $(l_4m_4n_4)$ as the direction cosines of these inward normals.

A system of couples represented in magnitude and direction by $L\Delta BCD$, $M\Delta DCA$, $N\Delta ABD$, $P\Delta ACB$, acting in the four faces of the tetrahedron, have as their resultant a couple whose axis is

$$L.l_1\Delta BCD+M.l_2\Delta DCA+N.l_3\Delta ABD+P.l_4\Delta ACB,$$

Thus the plane of the resultant couple is

$$L\alpha+M\beta+N\gamma+P\partial=0, \text{ or any parallel plane.}$$

Consider two sets of couples (λ,μ,ν,ρ) $(\lambda',\mu',\nu',\rho')$ whose axes are $(l_1m_1n_1)$, etc. If R,R' are the magnitudes of the resultants and (lmn) $(l'm'n')$ their axes, we have

$$Rl=\Sigma\lambda l_1,\quad R'l'=\Sigma\lambda' l_1,$$
$$Rm=\Sigma\lambda m_1,\quad R'm'=\Sigma\lambda' m_1,$$
$$Rn=\Sigma\lambda n,\quad R'n'=\Sigma\lambda' n_1,$$

whence $$R^2=\Sigma\lambda^2-2\Sigma\lambda\mu\cos\hat{12},\; R'^2=\Sigma\lambda^{12}-\Sigma\lambda'\mu'\cos\hat{12},$$

$$RR'\cos\theta=\Sigma\lambda\lambda'-\Sigma(\lambda\mu'+\lambda'\mu)\cos\hat{12},$$

there being no difficulty about the signs.

Replacing λ,λ'...by $L\Delta BCD$, $L'\Delta BCD$...we obtain the result of § 4 :

§. 7. The following statical theorems are well known :

If R is the resultant of a system of forces P_1,P_2......
and R' is the resultant of another system P_1',P_2',......, then

$$R^2=\Sigma P_r^2+2\Sigma P_rP_s\cos\widehat{P_rP_s}$$

$$RR'\cos\widehat{RR'}=\Sigma P_rP_s'\cos\widehat{P_rP_s'}$$

$$RR'.r.\sin RR'=\Sigma P_r.P_s'.p\sin\widehat{P_rP_s'},$$

where r, p are the shortest distances between the forces occurring in the third expression. They may be established simply as follows .

Take (l_r,m_r,n_r) (l_s',m_s',n_s) as direction cosines of P_r, P_s'
(l,m,n) (l',m',n') as direction cosines of R,R'

Then $Rl=\Sigma P_rl_r$, $Rm=\Sigma P_rm_r$, $Rn=\Sigma P_rn_r$
$R'l'=\Sigma P_s'l'$, $R'm'=\Sigma P_s'm_s'$, $R'n'=\Sigma P_s'n_s'$

whence the first two results follow at once.

For the third result, if P_r, P_s' act at (a_r,b_r,c_r) (a_s',b_s',c_s'), then the six components of P_r are

$$P_rl_r,\ P_rm_r,\ P_rn_r,\ P_r(n_rb_r-m_rc_r),\ P_r(l_rc_r-n_ra_r),\ P_r(m_ra_r-l_rb_r)$$

Consider now two forces only P_1,P_2 and denote their six components by X_1,Y_1,Z_1,L_1,M_1,N_1 and $X_2...N_2$, we have

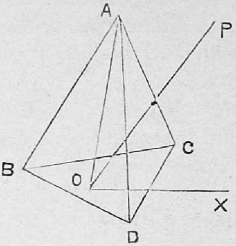
$$\Sigma(X_1+X_2)(L_1+L_2)=\Sigma X_1L_2+X_2L_1)$$
$$=P_1P_2\Sigma[l_1(n_2b_2-m_2c_2)+l_2(n_1b_1-m_1s_1)]$$
$$=P_1P_2\begin{vmatrix} a_1-a_2, & b_1-b_2, & c_1-c_2 \\ l_1, & m_1, & n_1 \\ l_2, & m_2, & n_2 \end{vmatrix}=P_1P_2p_{12}\sin\theta_{12},$$

where p_{12}, θ_{12} are the shortest distance and angle between the forces.

Hence $\Sigma(X_1+X_2+.+X_n)(L_1+L_2+.+L_n)=\Sigma P_rP_sp_{rs}\sin\theta_{rs}$ ... (1)

Taking two sets X...L...X′...L′... and applying (1) to the combined system and the two separate systems, we obtain the third result.

§. 8. Consider any straight line OP as the seat of a force. Let it meet the plane BCD in O and let OX be its projection from A on BCD.



To resolve the force into its six components along the edges of the tetrahedron, we resolve OP along OA and OX and then resolve OX along the sides of the triangle BCD and OA along the three edges of the tetrahedron that meet at A.

Now the equation of OX is $\begin{vmatrix} \beta & \gamma & \partial \\ \beta' & \gamma' & \partial' \\ m & n & p \end{vmatrix}=0$;

also, the tetrahedral coordinates of any point in the plane BCD are the areal coordinates of the same point with respect to BCD.

Hence the components along BD, DC, CB are proportional to

$$BD(\partial'm-\gamma'p),\ DC(\gamma'p-\partial'n),\ CB(\beta'n-\gamma'm).$$

As the resolution of the force into components along the edges of the tetrahedron is unique, by similarly considering the intersections with the other planes we find for the six components along the edges BD, DC, CB, BA, AC, DA, where signs must be regarded, the expressions

$$BD\begin{vmatrix} \partial',\beta' \\ p,m \end{vmatrix},\ DC\begin{vmatrix} \gamma',\partial' \\ n,p \end{vmatrix},\ CB\begin{vmatrix} \beta',\gamma' \\ m,n \end{vmatrix},\ BA\begin{vmatrix} \alpha',\beta' \\ l,m \end{vmatrix},\ AC\begin{vmatrix} \gamma',\alpha' \\ n,l \end{vmatrix},\ DA\begin{vmatrix} \alpha',\partial' \\ l,p \end{vmatrix}$$

§. 9. To apply the theorems of § 7 to two sets of forces whose six components are given by the formulae just obtained, we have

$$R^2=\Sigma BD^2(\partial'm-\beta'p)^2+2\Sigma BD.DC(\partial'm-\beta'p)(\gamma'p-\partial'n)\cos B\hat{D}.DC$$
$$+2\Sigma AC.BD\ (\gamma'l-\alpha'n)(\partial'm-\beta'p)\cos A\hat{C}.BD.$$

By comparing coefficients it will be found that

$$R^2=(\alpha'+\beta'+\gamma'+\partial')^2\Sigma[-AB^2lm]$$

$$RR'\cos\theta=(\alpha'+\beta'+\gamma'+\partial')(\alpha''+\beta''+\gamma''+\partial'')\Sigma[-\tfrac{1}{2}AB^2(lm'+l'm)]$$

$$RR'p\sin\theta=\Sigma AD.BC\left[\begin{vmatrix}\alpha',\partial'\\ l,p\end{vmatrix}\begin{vmatrix}\beta'',\gamma''\\ m',n'\end{vmatrix}+\begin{vmatrix}\alpha'',\partial''\\ l',p'\end{vmatrix}\begin{vmatrix}\beta',\gamma'\\ m,n\end{vmatrix}\right]p_1\sin AD.\hat{BC},$$

where p_1,p_2,p_3 are the shortest distances between opposite edges.

Taking account of the relation $AD.BC.p_1\sin AD.\hat{BC}$ = six times the volume of the tetrahedron, we see that the third expression is identical with that found in § 3 where the determinant has been expanded by Laplace's method.

§. 10. *Isogonal conjugate points.*

We may easily establish that for two points $(\alpha,\beta,\gamma,\partial)$ $(\alpha',\beta',\gamma',\partial')$ which are such that

$$\frac{\alpha\alpha'}{\Delta BCD^2}=\frac{\beta\beta'}{\Delta ACD^2}=\frac{\gamma\gamma'}{\Delta ABD^2}=\frac{\partial\partial'}{\Delta ABC^2}$$

the planes through any edge and the two points are equally inclined to the faces through that edge.

Call the points P,P′ and calculate the angles between the pairs of planes ABP, ABC and ABP′, ABD. These will be found to be equal.

These planes are $\left.\begin{matrix}z\partial-\gamma u=0\\ u=0\end{matrix}\right\}$ $\qquad \left.\begin{matrix}z'\partial'-\gamma'u'=0\\ z=0\end{matrix}\right\}$

where x,y,z,u are written for the current coordinates.

Hence the cosines of the angles between these planes are

$$\frac{\gamma\Delta ABC^2+\frac{1}{16}\partial f(AB)}{\Delta ABC.[\partial^2\Delta ABD^2+\gamma^2\Delta ABC^2+\frac{1}{8}\partial\gamma\, f(AB)]^{\frac{1}{2}}},$$

$$\frac{\partial'\Delta ABD^2+\frac{1}{16}\gamma' f(AB)}{\Delta ABD.[\partial'^2\Delta ABD^2+\gamma'^2\Delta ABC^2+\frac{1}{8}\partial'\gamma' f(AB)]^{\frac{1}{2}}};$$

and these are equal in view of the assumed relation.

The isogonal conjugate of the centroid of the triangle (1,1,1,1) is the point $(\Delta BCD^2,\Delta ACD^2,\Delta ABD^2,\Delta ABC^2)$.

The 8 self isogonal conjugates are

$$(\pm\Delta BCD,\ \pm\Delta ACD,\ \pm\Delta ABD,\ \pm\Delta ABC)$$

and these are centres of the 8 spheres that can be drawn touching the four faces of the tetrahed n.

Further, isogonally conjugate points are foci of conicoids of revolution touching the four faces of the tetrahedron. For consider the conicoid whose tangential equation is

$$9V^2\Sigma(L\alpha).\Sigma(L\alpha')=K^2[\Sigma\Delta BCD^2L^2-\tfrac{1}{8}\Sigma LMf(CD)]$$

where $$\frac{K^2}{9V^2}=\frac{\alpha\alpha'}{\Delta BCD^2}=\dots=\dots=\dots$$

This represents the envelope of a plane such that the product of the perpendiculars from two fixed points is constant and is therefore a conicoid of revolution having the two fixed points for foci ; further it is satisfied by the planes (1,0,0,0), etc.

§. 11. *Relations connecting* $f(AB)$, $f(AC)$, etc ,

Since $V^2 = \frac{1}{36} AB^2.AC^2.AD^2 \begin{vmatrix} 1 & \cos\nu & \cos\mu \\ \cos\nu & 1 & \cos\lambda \\ \cos\mu & \cos\lambda & 1 \end{vmatrix}$

$$= \frac{1}{288} \begin{vmatrix} 2AB^2 & , AB^2+AC^2-BC^2, & AB^2+AD^2-BD^2 \\ AB^2+AC^2-BC^2, & 2AC^2 & , AC^2+AD^2-CD^2 \\ AB^2+AD^2-BD^2, & AC^2+AD^2-CD^2, & 2AD^2 \end{vmatrix}$$

Border the determinant with a column AB^2, (AC^2, AD^2, 1) and a row (0,0,0,1) and subtract the bordered column from each of the other columns. Next border by a row (AB^2, AC^2, AD^2, 0, 1) and a column (0,0,0,0,1) and subtract the bordered row from the first three rows ; we obtain the well known expression

$$288V^2 = \begin{vmatrix} 0, & BC^2, & BD^2, & AB^2, & 1 \\ BC^2, & 0, & CD^2, & AC^2, & 1 \\ BD^2, & CD^2, & 0, & AD^2, & 1 \\ AB^2 & AC^2, & AD^2, & 0, & 1 \\ 1 & 1 & 1 & 1 & 0 \end{vmatrix} = \Delta \text{ say.,}$$

It will be found by expansion that

(1) the minor of AB^2 is $f(CD)$

(2) the minor corresponding to the zero in the first row and column is $-16\,\Delta ACD^2$

(3) the minors of the last row or column give the coordinates of the centre of the circumscribing sphere and a function depending on the radius of the circumsphere.

Thus for the reciprocal determinant we have

$$\Delta^4 = \begin{vmatrix} -16\Delta ACD^2, & f(AD), & f(AC), & f(CD), & K_2 \\ f(AD), & -16\Delta ABD^2, & f(AB), & f(BD), & K_3 \\ f(AC), & f(AB), & -16\Delta ABC^2, & f(BC), & K_4 \\ f(CD), & f(BD), & f(BC), & -16\Delta BCD^2, & K_1 \\ K_2, & K_3, & K_4, & K_1, & K_5 \end{vmatrix}.$$

Considering the second minors we obtain

$$f(AC)\,f(BD)-f(AB)\,f(CD)=(AB^2+CD^2-BD^2-AC^2)\Delta \quad \ldots \quad (1)$$

$$16.\Delta\, ACD^2.16\Delta\, ABD^2-f(AD)^2=2AD^2.\Delta \quad \ldots \quad \ldots \quad (2)$$

From § 4 (2) we have

$$\cos \hat{24}=\frac{f(AC)}{16\Delta ACD.\Delta ABC}$$

whence

$$\sin \hat{24}=\frac{3V.AC}{2\Delta ACD.\Delta ABC} \quad \ldots \quad \ldots \quad (3)$$

Again $BC^2.f(AD)+BD^2f(AC)+AB^2f(CD)+K_2=\Delta$

$$-16\Delta ACD^2.BC^2+f(AC).CD^2+f(CD).AC^2+K_2=0$$

$$-16\Delta ACD^2+f(AD)+f(AC)+f(CD)=0$$

These are obtained by multiplying the first row of the reciprocal determinant by the first, second and last rows of the original determinant.

We thus obtain

$$16\ \Delta ACD^2=f(AD)+f(AC)+f(CD) \quad \ldots \quad \ldots \quad \ldots \quad (4)$$

$$\Delta=2BC^2f(AD)+(BD^2+BC^2-CD^2)f(AC)+(AB^2+BC^2-AC^2)f(CD)$$

$$=2BD^2f(AC)+(BC^2+BD^2-CD^2)f(AD)+(AB^2+BD^2-AD^2)f(CD)$$

$$=2BA^2f(CD)+(BD^2+AB^2-AD^2)f(AC)+(BC^2+AB^2-AC^2)f(AD)$$

the last two results being written down from symmetry.

Hence we have

$$f(CD)\{BD^2+AC^2-BC^2-AD^2\}=f(AC)\{BC^2-CD^2-BD^2\}-f(AD)\{BD^2-CD^2-BC^2\}$$

$$f(CA)\{BA^2+DC^2-BC^2-AD^2\}=f(DC)\{BC^2-AC^2-AB^2\}-f(AD)\{AB^2-AC^2-BC^2\}$$

$$f(AD)\{AB^2+CD^2-BD^2-AC^2\}=f(CD)\{BD^2-AD^2-AB^2\}-f(AC)\{AB^2-AD^2-BD^2\}$$

and similar relations between any other set of three ... ... (5)

Consider

$$\begin{vmatrix} -16\Delta ACD^2, & f(AD), & f(AC) \\ f(AD), & -16\Delta ABD^2, & f(AB) \\ f(AC), & f(AB), & -16\Delta ABC^2 \end{vmatrix}=-\Delta^2.$$

Replace $-16\Delta ACD^2$ etc., by the expressions (4), and we get on expanding the determinant

$$\Sigma f(BC)f(CD)f(BD)+\Sigma f(AC)f(BD).X=\Delta^2 \quad \ldots \quad \ldots \quad (6)$$

where X denotes $f(AB)+f(AD)+f(BC)+f(CD)$.

Again $K_1+K_2+K_3+K_4=\Delta$, and from expressions as

$$BC^2f(AD)+BD^2f(AC)+AB^2f(CD)+K_2=\Delta,$$

and the similar expressions for K_1, K_3, K_4 we get

$$\Sigma AB^2f(CD)=432V^2 \quad \ldots \quad \ldots \quad \ldots \quad (7)$$

§. 12. *The direction cosines of a straight line perpendicular to a plane:*

If $\frac{\alpha-\alpha'}{\lambda}=\frac{\beta-\beta'}{\mu}=\frac{\gamma-\gamma'}{\nu}=\frac{\partial-\partial'}{\rho}$ is perpendicular to the plane $L\alpha+M\beta+N\gamma+P\delta=0$, then any plane $L'\alpha+M'\beta+N'\gamma+P'\delta=0$, which contains the given straight line is perpendicular to the given plane; whence

$$\Sigma L'\lambda=0 \text{ and } \Sigma[a\,LL'-(LM'+L'M)f(CD)]=0$$

are identical, where a is written for ΔBCD^2, etc.

Therefore
$$\begin{aligned}\lambda&=aL-Mf(CD)-Nf(BD)-Pf(BC)\\ \mu&=bM-Lf(CD)-Nf(AD)-Pf(AC)\\ \nu&=cN-Lf(BD)-Mf(AD)-Pf(AB)\\ \rho&=dP-Lf(BC)-Mf(AC)-Nf(AB).\end{aligned}$$

In particular, the straight line through $(\alpha',\beta',\gamma',\delta')$ perpendicular to $\alpha=0$ is

$$\frac{\alpha-\alpha'}{-a}=\frac{\beta-\beta'}{f(CD)}=\frac{\gamma-\gamma'}{f(BD)}=\frac{\delta-\delta'}{f(BC)} \quad \ldots \quad \ldots \quad (1)$$

where $a=16\Delta BCD^2=f(BC)+f(BD)+f(CD)$.

The condition the points where this straight line meets $\alpha=0$ and the three other points in the planes $\beta=0$, $\gamma=0$, $\delta=0$ should be coplanar is (dropping accounts)

$$\begin{vmatrix} 0, & \beta+\frac{\alpha f(CD)}{a}, & \gamma+\frac{\alpha f(BD)}{a} & \delta+\frac{\alpha f(BC)}{a} \\ \alpha+\frac{\beta f(CD)}{b}, & 0, & \gamma+\frac{\beta f(AD)}{b}, & \delta+\frac{\beta f(AC)}{b} \\ \alpha+\frac{\gamma f(BD)}{c}, & \beta+\frac{\gamma f(AD)}{c}, & 0, & \delta+\frac{\gamma f(AB)}{c} \\ \alpha+\frac{\delta f(BC)}{d}, & \beta+\frac{\delta f(AC)}{d}, & \gamma+\frac{\delta f(AB)}{d}, & 0 \end{vmatrix}=0;$$

adding the rows we notice that it is divisible by $\alpha+\beta+\gamma+\delta$.

Border the determinant by a horizontal line 0,0,0,0,−1 and a vertical column 1,1,1,1,−1. Multiply the bordering column by $\alpha,\beta,\gamma,\delta$ respectively and subtract from the first four rows; we get

$$\begin{vmatrix} -a, & f(CD), & f(BD), & f(BC), & \frac{a}{\alpha} \\ f(CD), & -b, & f(AD), & f(AC), & \frac{b}{\beta} \\ f(BD), & f(AD), & -c, & f(AB), & \frac{c}{\gamma} \\ f(BC), & f(AC), & f(AB), & -d, & \frac{d}{\delta} \\ \alpha & \beta & \gamma, & \delta, & -1 \end{vmatrix}=0.$$

Expand in terms of products of the elements of the last row and column and by use of the theorems of § 11, it becomes

$$-\Delta^2(a+b+c+d)-\Delta^2\Sigma\left(\frac{a\beta}{\alpha}+\frac{b\alpha}{\beta}\right)=0,$$

or
$$\Sigma(a\beta^2+b\alpha^2)\gamma\delta+(a+b+c+d)\alpha\beta\gamma\delta=0.$$

Dividing this by the factor $\alpha+\beta+\gamma+\delta$, we have, for the surface which is the locus of points such that the feet of the perpendiculars on the four faces of a tetrahedron are coplanar, the equation

$$\frac{a}{\alpha}+\frac{b}{\beta}+\frac{c}{\gamma}+\frac{d}{\delta}=0.$$ [Salmon. Edited by Rogers § 121 Ex. 17.]

Otherwise, it is the locus of the foci of all paraboloids of revolution touching the four faces of the tetrahedron.

For if $(\alpha'\beta'\gamma'\delta')$ is a focus its isogonal conjugate must lie in the plane at infinity $\alpha+\beta+\gamma+\delta=0$. Thus the locus of $(\alpha', \beta', \gamma' \delta')$ is $\frac{a}{\alpha'}+\frac{b}{\beta'}+\frac{c}{\gamma'}+\frac{d}{\delta'}=0$, where a, b, c, d are written for the squares of the areas of the faces of the tetrahedron.

§ 13. The surface $\frac{a}{\alpha}+\frac{b}{\beta}+\frac{c}{\gamma}+\frac{d}{\delta}=0$ has been investigated for the special case of $a=b=c=d$. Compare: Cesaro, *Calcolo Differenziale ed Integrale*, p. 458, from which the following section is taken:

For a tetrahedron whose opposite edges are equal the faces are all similar triangles and the three straight lines joining the middle points of opposite edges meet in a point and are mutually orthogonal. Take these lines as axes, or what is the same thing, considering the rectangular parallelopiped that circumscribes such a tetrahedron the axes are at the centre of the parallelopiped and parallel to the edges of the parallelopiped.

The faces of the tetrahedron are

$$\frac{x}{\lambda}+\frac{y}{\mu}+\frac{z}{\nu}+1=0,\ \frac{x}{\lambda}-\frac{y}{\mu}-\frac{v}{\nu}+1=0, \text{ etc},$$

and the surface is $\Sigma(1/p)=0$, where p is the perpendicular on any face.

Substituting, we get for the surface,

$$\frac{xyz}{\lambda\mu\nu}+\tfrac{1}{2}\left(\frac{x^2}{\lambda^2}+\frac{y^2}{\mu^2}+\frac{z^2}{\nu^2}-1\right)=0.$$

Consider the section of this surface by the series of planes $z=0$, to $z=\nu$; they consist of a series of ellipses inscribed in the series of rectangles in which the parallelopiped is cut by the planes, starting from $\frac{x^2}{\lambda^2}+\frac{y^2}{\mu^2}=1$, and ending up with the diagonal $\frac{x}{\lambda}+\frac{y}{\mu}=0$, counted as a double line and then passing into a series of hyperbolas. Similarly the series of planes $z=0$, to $z=-\nu$, may be considered.

The diagonal plane $\frac{y}{\mu}+\frac{z}{\nu}=0$ cuts the surface in a parabola; for, writing $kY=\frac{y}{\mu}+\frac{z}{\nu}$, $kZ=\frac{y}{\nu}-\frac{z}{\mu}$, where $k^2=\frac{1}{\mu^2}+\frac{1}{\nu^2}$, the surface becomes

$$\frac{X}{\lambda}\cdot\frac{1}{k^2\mu\nu}\left(\frac{Y}{\mu}+\frac{Z}{\nu}\right)\left(\frac{Y}{\nu}-\frac{Z}{\mu}\right)$$
$$+\tfrac{1}{2}\left\{\frac{X^2}{\lambda^2}+k^2Y^2-\frac{2}{k^2\mu\nu}\left(\frac{Y}{\mu}+\frac{Z}{\nu}\right)\left(\frac{Y}{\nu}-\frac{Z}{\mu}\right)-1\right\}=0;$$

which, for $Y=0$, reduces to the section $Y=0$, $X=\lambda$, and the parabola

$$\frac{X}{\lambda^2}+1+\frac{2Z^2}{k^2\mu\nu}=0.$$

From these indications the general form of the surface can be realized.

§. 14. *On the hyperboloid containing the four perpendiculars from the vertices on the opposite faces.*

Consider the plane through AB perpendicular to the plane BCD; by § 4, it is

$$\gamma f(BC)-\delta f(BD)=0.$$

Similarly the planes through BD perpendicular to BAC, and BC perpendicular to BAD are given by

$$\alpha f(AB)-\gamma f(BC)=0$$
$$\alpha f(AB)-\delta f(BD)=0.$$

These three planes intersect in the straight line

$$\alpha f(AB)=\gamma f(BC)=\delta f(BD)$$

passing through the vertex B. There are three similar lines through the vertices A,C,D.

The four lines thus obtained are generators of a hyperboloid. For consider the hyperboloid

$$\{f(AB)f(CD)-f(AD)f(BC)\}\ \{\alpha\gamma f(AC)+\beta\delta f(BD)\}$$
$$+\{f(AD)f(BC)-f(AC)f(BD)\}\ \{\alpha\beta f(AB)+\gamma\delta f(CD)\}$$
$$+\{f(AC)f(BD)-f(AB)f(CD)\}\ \{\beta\gamma f(BC)+\alpha\delta f(AD)\}=0. \quad (1)$$

It is easily seen that

$$\alpha:\gamma:\delta=\frac{1}{f(AB)}=\frac{1}{f(BC)}=\frac{1}{f(BD)},$$

satisfies it identically, and is therefore a generator; from symmetry the other three lines through the vertices A,C,D are also generators.

Again, consider the four perpendiculars from A,B,C,D on the opposite faces ; from § 12 (1) they are :

$$\left.\begin{array}{l} \dfrac{\alpha-1}{-a}=\dfrac{\beta}{f(CD)}=\dfrac{\gamma}{f(BD)}=\dfrac{\delta}{f(BC)} \\ \dfrac{\alpha}{f(CD)}=\dfrac{\beta-1}{-b}=\dfrac{\gamma}{f(AD)}=\dfrac{\delta}{f(AC)} \\ \dfrac{\alpha}{f(BD)}=\dfrac{\beta}{f(AD)}=\dfrac{\gamma-1}{-c}=\dfrac{\delta}{f(AB)} \\ \dfrac{\alpha}{f(BC)}=\dfrac{\beta}{f(AC)}=\dfrac{\gamma}{f(AB)}=\dfrac{\delta-1}{-d} \end{array}\right\} \quad \ldots \quad \ldots \quad (2)$$

It is at once seen that these straight lines are generators of the hyperboloid (1). Cp. Bell : *Coordinate Geometry*, § 111 Ex. 3.

Suppose the first two perpendiculars intersect, we see that

$$f(BD)\,f(AC)=f(BC)\;\;f(AD)\,;$$

whence also the last two perpendiculars must intersect and the hyperboloid containing the four generators reduces to the pair of planes

$$\alpha\,f(AC)-\beta\,f(BC)=0, \gamma\,f(AC)-\partial\,f(AD)=0.$$

(*To be continued.*)

Mersenne's Numbers.

By Balak Ram M.A., I.C.S.

The following method of obtaining the factors, if any, of the smaller of Mersenne's Numbers has been in my possession for four or five years; but I did not publish it partly because I thought it was too simple to be new, and partly because I wished to obtain some new results before publishing the details of the method. Pressure of other duties having prevented me from devoting attention to the problem, I was glad to communicate last year the method to Mr. V. Ramesam who had obtained by another method the smallest factor of $2^{71}-1$. My method enabled him to factorise the number completely. The result on being published, attracted the attention of Lt. Col. Cunningham, who wrote to Mr. Ramesam asking for details of the method. Mr. Ramesam has very kindly suggested that I should publish a note on the subject, and I gladly accept the suggestion, as there seems to be a possibility (though a slight one) of the method not having been previously applied to the problem in hand.

The reader is referred to Ball's *Mathematical Recreations* and *Essays* for a full account of Mersenne's Numbers, and to the concluding portion of the present paper for results obtained since Ball's book was published.

1. The problem is to determine the integral values of n for which 2^n-1 is a prime. If n is composite (equal to xy say), then 2^x-1 and 2^y-1 are obviously factors of 2^n-1; it is therefore not necessary to consider any but prime values of n. Primarily the problem is not the determination of the factors of 2^n-1; but, so far as I am aware, none of the numbers hitherto considered have been shown to be composite without one or more factors being obtained, though theoretically it is possible to prove the composite character of a number without factorising it.

Let p be the number, c be a number prime to p, and x the lowest number giving $c^x \equiv 1$ (mod, p); then if x is not a factor of $p-1$, p is not a prime. If however x is a factor of $p-1$, we cannot say whether p is a prime or not. These statements follow from the generalized form of Fermat's theorem.

2. I will illustrate my method by considering the number $2^{31}-1$ a number known to be prime. All factors of 2^n-1 (n an odd prime) are of the form $an+1$, and also of the form $8b\pm1$. Further, there must be an odd number of factors of the form $8b-1$. The factors

of $2^{31}-1$ are thus of the form $k\alpha+1$, and $\lambda\alpha+63$, where α is written for 8×31. We may therefore put

$$2^{31}-1=(k\alpha+1)\ (\lambda\alpha+63)$$

where the factors are either prime, or (if composite) have factors of these forms and no others.

3. It is easily shown that

$$2^{31}-1=140\alpha^3+196\alpha^2+40\alpha+63$$

$$i.e.,=(k\alpha+1)\ (\lambda\alpha+63)$$

$$\therefore \quad k\lambda.\alpha+63\ k+\lambda=140\alpha^2+196\alpha+40 \quad \ldots \quad (1)$$

$$\therefore \quad k+\lambda\equiv40\ (\text{mod},\ 62)$$

$$=40+62\ \mu,\ \text{say}\ ; \quad \ldots \quad \ldots \quad (2)$$

the more nearly equal k and λ are, the smaller μ is.

Substituting for λ in (1), and dividing out by 62,

$$4k\ (9-k)+\kappa+\mu\equiv9\ (\text{mod},\ 31)$$

or

$$(2k+14)^2\equiv\mu+1\ (\text{mod},\ 31)$$

$\therefore$ $\mu+1$ is a quadratic residue of 31 ... (3)

Again,

$$(k\alpha+1)\ (\lambda\alpha+63)=2^{31}-1$$

gives us

$$4(k+2)\ (\lambda+1)\equiv2\ (\text{mod},\ 5),$$

or

$$(k+2+\lambda+1)^2-(\overline{k+2-\lambda+1})^2\equiv2,$$

or

$$(k+\lambda+3)^2\equiv2+X^2,\ \text{say}.$$

The possible values of X^2 are 0, 1, 4 (mod, 5) ; also $2+X^2$ must be a quadratic residue of 5 ;

$$\therefore \quad X^2\equiv4,\ \text{and}\ (k+\lambda+3)^2\equiv1\ (\text{mod},\ 5),$$

or

$$(62\mu+43)^2\equiv1.$$

Solving this congruence we get

$$\mu\equiv3\ \text{or}\ 4\ (\text{mod},\ 5).$$

Similarly $\mu\equiv1$, 2, 10 or 20 (mod, 27).

Combining the two, we get

$$\mu\equiv28,\ 29,\ 64,\ 74,\ 83,\ 109,\ 118,\ 128,\ (\text{mod},\ 135.)$$

Introducing the condition that $\mu+1$ is a quadratic residue of 31, we see that the lowest possible value of μ is 128.

4. Returning to the original equation, we see that if either k or λ is 5 or more, the other number is $<\frac{141\alpha}{5}$ and therefore $\mu<112$. Thus the only possible factors are those given by $k<5$, and $\lambda<5$. Of these eight numbers the only prime is that corresponding to $\lambda=1$. Hence if 311 is not a factor, $2^{31}-1$ is a prime.

Taking 311 to be the modulus,

$$1\equiv312\equiv2^3.39\equiv2^3\ (39-311)\equiv2^7\ (-17)\equiv[2^7\ (-17)]^2\equiv2^{14}.\ (289)$$
$$\equiv2^{14}\ (-22)\equiv2^{15}\ (-11)\equiv[2^{15}\ (-11)]^2=2^{30}.\ (121).$$

$\therefore$ 1 is not $\equiv 2^{31}$ (mod, 311)

$\therefore$ $2^{31}-1$ is a prime.

5. I applied this method to $2^{41}-1$, and obtained the following results :—

α being $8\times 41=328$,

$$(k\alpha+1)(\lambda\alpha+247)=2^{41}-1=189\alpha^4+325\alpha^3+82\alpha^2+169\alpha+247$$

$$k+\lambda=5+82\mu$$

$$\mu+8\equiv(2k-16)^2(\text{mod}, 41)$$

$$\left.\begin{array}{l}\mu+3k\equiv 2\ (\text{mod}, 8)\\ \lambda-k\equiv 1\ (\text{mod}, 8)\end{array}\right\}$$

$$\mu\equiv 2, 7, 9, 10, 12, 17, 22, (\text{mod}, 25)$$

or $\mu\equiv 1, 3$ (mod, 5) are excluded ;

and if $\mu\equiv 0$ or (4 mod, 5)

it must be $\equiv 9, 10$ (mod, 25)

$$\mu\equiv 6, 22, 33, 42, 49, 60, 67, 76\ (\text{mod}. 81)$$

Also $\mu\equiv 1\cdot 3\cdot 6\cdot$ (mod, 7)

$\equiv 1\cdot 2\cdot 6\cdot 7\cdot 8\cdot$ (mod, 11)

$\equiv 1\cdot 5\cdot 7\cdot 9\cdot 10\cdot 12$ (mod, 13)

$\equiv 2\cdot 4\cdot 5\cdot 7\cdot 8\cdot 9\cdot 11\cdot 12\cdot 14$ (mod, 17)

$\equiv 0, 4\cdot 7\cdot 8\cdot 9\cdot 12\cdot 16\cdot 17\cdot 18\cdot$ (mod, 19).

These did not require long calculations. I next drew up a table, the typical row of which was

	6	22	33	42	49	60	67	76
$81m+$								

By using the various moduli (5, 7,...19,and 41), one by one and putting crosses in the cells belonging to inadmissible numbers, I found (again without long calculations) that

$$\mu>82\cdot 64+22,$$

i.e., that the smaller factor was less than 48α. Omitting such values of λ as made $(\alpha\lambda+247)$ a multiple of primes ≤ 41, I got $\lambda=43, 40, 34...$

$\lambda=43$ gave $\mu=19\alpha+92$, which was inadmissible ;

$\lambda=40$ gave a factor, the other factor being $4\alpha^3+217\alpha^2+47\alpha+1$, which can be proved to be a prime.

6. The peculiarity of this method consists in the fact that the nearer a factor is to the square root of the number the more easily it is found. The range of values for μ extends to $4\,(2^n-1)/\omega^2$, but may be

reduced considerably by trying all factors given by k and λ equal to $1,2,\ldots l$, where l is some convenient number. Within the range, the possible values of μ are (as seen above) reduced by using various test-moduli, 81 reducing the number to 1/8, and every other modulus reducing it (generally speaking) by half. In some cases, 25 is a good modulus to employ instead of 5.

7. It should be noted that if the residue of either k or λ to the base α is known, the residue of the other number and of μ is also known. For larger numbers it may be necessary to take the factors to be $K\alpha^2+k_0\alpha+1$ and $L\alpha^2+\lambda\ \lambda_0\alpha+c$, where k_0, λ_0 are known and both less than α, and K, L are to be determined.

8. The following information supplements that given in Ball's *Recreations* :—

(i) $2^{71}-1=228479\times48544121\times212885883$.

The smallest factor was found by Cunningham ; Ramesam factorised the number completely.

(ii) $2^{89}-1$ is declared by two independent computers (Powers and Terry) to be a prime, Mersenn asserted that it was composite.

(iii) 150287 is a factor of $2^{163}-1$. [Cunningham.]

(iv) 730753 is a factor of $2^{173}-1$. [Cunningham.]

(v) 43441 is a factor of $2^{181}-1$. [Woodall.]

(vi) In announcing a factor of $2^{163}-1$, Cunningham asserted that all factors below 200,000 of all Mersenne's numbers had been discovered. Woodall's subsequent discovery of 'a factor of $2^{181}-1$ showed that the assertion was not quite correct.

[*N. B.*—It is noteworthy that for every number factorised so far either k or λ is less than α.]

SHORT NOTES.

The Distribution of Primes.

The following results in the theory of numbers have been obtained by Mr. S. Ramanujan of Madras, and are published for the information of mathematicians. Proofs will be supplied later.

1. The number of prime numbers less than e^{a} is

$$\int_{0}^{\infty} \frac{a^{x}dx}{x.S_{x+1}\,\Gamma(x+1)}, \text{ where } S_{x+1}=\frac{1}{1^{x+1}}+\frac{1}{2^{x+1}}+\frac{1}{3^{x}}+\frac{1}{4^{x+1}}+\cdots$$

2. The number of prime numbers less than n is

$$\frac{2}{\pi}\left\{\frac{2}{B_2}\cdot\left(\frac{\log n}{2\pi}\right)+\frac{4}{3.B_4}\left(\frac{\log n}{2\pi}\right)^{3}+\frac{6}{5.B_6}\left(\frac{\log n}{2\pi}\right)^{5}+\frac{8}{7.B_8}\left(\frac{\log n}{2\pi}\right)^{7}+\cdots\right\},$$

where $B_2=\frac{1}{6}$; $B_4=\frac{1}{30}$; $B_6=\frac{1}{42}$, &c., are the Bernoullian numbers.

3. The number of prime numbers less than n is

$$\int_{\mu}^{n}\frac{dx}{\log x}-\frac{1}{2}\int_{\mu}^{\sqrt{n}}\frac{dx}{\log x}-\frac{1}{3}\int_{\mu}^{\sqrt[3]{n}}\frac{dx}{\log x}-\frac{1}{5}\int_{\mu}^{\sqrt[5]{n}}\frac{dx}{\log x}$$

$$+\frac{1}{6}\int_{\mu}^{\sqrt[6]{n}}\frac{dx}{\log x}-\frac{1}{7}\int_{\mu}^{\sqrt[7]{n}}\frac{dx}{\log x}+\frac{1}{10}\int_{\mu}^{\sqrt[10]{n}}\frac{dx}{\log x}-\frac{1}{11}\int_{\mu}^{\sqrt[11]{n}}\frac{dx}{\log x}$$

$$-\frac{1}{13}\int_{\mu}^{\sqrt[13]{n}}\frac{dx}{\log x}+\frac{1}{14}\int_{\mu}^{\sqrt[14]{n}}\frac{dx}{\log x}+\frac{1}{15}\int_{\mu}^{\sqrt[15]{n}}\frac{dx}{\log x}-\frac{1}{17}\int_{\mu}^{\sqrt[17]{n}}\frac{dx}{\log x}$$

where $\mu=1{\cdot}45136380$ nearly.

The numbers, 1, 2, 3, 5, 6, 7, 10, 11, 13, 14, 15, 17, 19 etc., above are the natural numbers containing dissimilar prime divisors. Thus the numbers 4, 8, 9, 12, 16, 18, 20 etc., are excluded. The sign is positive for an even number of prime divisors and negative for an odd number of prime divisors.

In practice, as soon as a term becomes less than unity, we should stop at the next term where the asterisk is marked and not anywhere; asterisks should be marked over the terms corresponding to the numbers 5, 7, 11, 14. 17,... ; hence the first four terms are absolutely essential even when n is very small.

4. In the above theorems unity is not considered as a prime number.

For practical purposes

$$\int_{\mu}^{n}\frac{dx}{\log x}=n\left\{\frac{1}{\log n}+\frac{\lfloor 1}{(\log n)^2}+\frac{\lfloor 2}{(\log n)^3}+\ldots\frac{\lfloor k-1}{(\log n)^k},\Theta\right\}7$$

where Θ is equal to

$$\left(\frac{2}{3}-\delta\right)+\frac{1}{\log n}\left\{\frac{4}{135}-\frac{\delta^2(1-\delta)}{3}\right\}$$

$$+\frac{1}{(\log n)^2}\left\{\frac{8}{2835}+\frac{2\delta(1-\delta)}{135}-\frac{\delta(1-\delta^2)(2-3\delta^2)}{45}\right\}+\&c., \&c., \ldots$$

δ denoting $(k-\log n)$. It would be advantageous to choose k to be the integer just greater than $\log n$.

In accordance with the above formula the number of primes less than 50 is 14·9, while the actual number is 15,
" 300 is 61·9, " " 62,
" 1000 is 168·2, " " 168,
and so on.

S. NARAYANA AIYAR.

27th February 1913.

A set of Simultaneous Equations.

On page 94 of the Journal Mr. S. Ramanujan has discussed the solution by 'partial fractions,' of the following set of equations :—

$$x_1+x_2+x_3+\quad\ldots\quad\ldots\quad x_n=a_1$$
$$x_1y_1+x_2y_2+x_3y_3+\quad\ldots\quad x_ny_n=a_2$$
$$x_1y_1^2+x_2y_2^2+x_3y_3^2+\quad\ldots\quad x_ny_n^2=a_3$$
$$* \quad * \quad * \quad *$$
$$\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots$$
$$x_1y_1^{2n-1}+x_1y_2^{2n-1}+\quad\ldots\quad x_ny_n^{2n-1}=a_{2n}.$$

where $x_1, x_2,\ldots x_n$; $y_1, y_2,\ldots y_n$ are $2n$ unknown quantities.

Following Burnside and Panton (Vol. II, p. 106), we may find the values of $x_1,x_2\ldots x_n$ in terms of the y's and the a's, by solving the first n linear equations simultaneously. Thus, we have

$$\begin{vmatrix} 1, & y_1 & y_1^2 & \ldots & y_1^{n-1} & x_1 \\ s_0 & s_1 & s_2 & \ldots & s_{n-1} & a_1 \\ s_1 & s_2 & s_3 & \ldots & s_2 & a_2 \\ & \ldots & & \ldots & \ldots & \\ s_{n-1} & s_n & s_{n+1} & \ldots & s_{2n-2} & a_n \end{vmatrix}=0, \text{ etc.}$$

where $s_k=y_1^k+y_2^k+\ldots\ldots y_n^k$.

Similarly by solving the 2nd, 3rd,...$(n+1)^{th}$ equations for (x_1y_1), $(x_2y_2)\ldots(x_ny_n)$, we have determinantal results differing from the above only in having (x_1y_1)......., in place of x_1,......

And so on for $x_1y_1^2, x_2y_2^2 \ldots\ldots x_ny_n^2$; etc.

Hence, these determinantal equations may be rewritten

$$\begin{array}{lllll}
x_1X_1+a_1A_1+ & \ldots & \ldots & a_n\,A_n=0 \\
x_1y_1X_1+a_2A_1+ & \ldots & \ldots & a_{n+1}\,A_n=0 \\
x_1y_1^2X_1+a_3A_1+ & \ldots & \ldots & a_{n+2}\,A_n=0 \\
\ldots & \ldots & \ldots & \ldots & \ldots \\
x_1y_1^nX_1+a_{n+1}A_1+ & \ldots & \ldots & a_{2n}\,A_n=0.
\end{array}$$

Eliminating x_1 X_1, A_1, A_2, ...A_n from these, we finally obtain

$$\begin{vmatrix}
1 & a_1 & a_2 & \ldots & a_n \\
y_1 & a_2 & a_3 & \ldots & a_{n+1} \\
y_1^2 & a_3 & a_4 & \ldots & a_{n+2} \\
\ldots & \ldots & \ldots & & \ldots \\
y_1^n & a_{n+1} & a_{n+2} & \ldots & a_{2n}
\end{vmatrix} = 0$$

an equation of the n^{th} degree in y whose roots are $y_1, y_2, \ldots y_n$. When the y's are known, the x's are known too.

Practically, the labour involved in the reduction of the above determinant is as great as the work of finding the B's in Mr. Ramanujan's method ; and both methods are equally tiresome.

25th April 1912. M. T. NARANIENGAR.

The Face of the Sky for May and June 1913.

Sidereal time at 8 p.m.

			May.				*June.*		
			H.	M.	S.		H.	M.	S.
1	...	...	10	35	38	...	12	37	51
8	...	...	11	3	14	...	13	5	27
15	...	...	11	30	50	...	13	33	3
22	...	...	11	58	26	...	14	0	39
29	...	...	12	26	2	...	14	28	15

From this table the constellations visible during the evenings of May and June can be ascertained by a reference to the positions as given in a star atlas.

Phases of the Moon.

		May.					*June.*			
		D.	H.	M.			D.	H.	M.	
New Moon	...	6	1	54	P.M.	...	5	1	27	A.M.
First Quarter	...	13	5	15	"	...	11	10	7	P.M.
Full Moon	...	20	0	48	"	...	18	11	24	"
Last Quarter	...	28	5	34	A.M.	...	26	11	11	"

Planets.

Mercury is in superior conjunction with the Sun on June 2, when it becomes an evening star. It is in conjunction with the moon on May 4 and with Neptune on June 24.

Venus attains maximum brilliancy as a morning star on May 31 and is in conjunction with the moon on June 1.

Mars is in conjunction with the moon on May 2, on May 31 and on June 29.

Jupiter is stationary on May 6. It is in conjunction with the moon on May 24 at 3-53 A.M. and on June 20.

Saturn is in conjunction with the moon on May 8 and June 4.

Uranus is stationary on May 13 and is in conjunction with the moon on June 22 at 1-38 A.M.

Neptune is in conjunction with the moon on May 11 and June 7.

[**Erratum**.—The phases as given in the February number of the Journal were for May and June instead of March and April].

V. Ramesam.

Occultation of β Tauri (1681 B.A.C.)

9-20 p.m. Saturday, 15th March.

As I was sitting on the beach last evening trying to recognize the brighter stars as they began to appear in the twilight I noticed a star of about the second magnitude in the line joining the centers of the sun and the moon. I guessed that the distance between the dark limb of the moon and the star was about 1° and that as the time was then 6-45 P.M. an occultation would occur at about 8-45 P.M. I therefore procured a pair of binoculars to observe it better. As the star neared the moon's limb the limb began to show in contrast with the light of the star. I was able to follow the star until 9.20 P.M., when as if by a jerk the moon swallowed up the star. As I was not acquainted with the star I made a map of that portion of the heavens relative to the bigger stars. I was this morning able to identify the star as β Tauri; and it is of magnitude 1·8.

I calculated that for the path that the star seemed to take the occultation would last for 1 hr. 5 min., and at 10·15 P.M. I turned to the sky to watch the reappearance of the star. The most remarkable thing was that the darker limb of the moon was traceable now only with great difficulty. It was not until 10-35 P.M. that I was able to see the star again. But it was then quite clear of the moon's disc and the glare of this portion of the lunar disc made it impossible for the star to be recognized even as a pin head until it had separated from the disc by about 2″. I calculated that the star must have emerged about 5 min. before and that the occultation must have ended at about 10-30 P.M.

MADRAS,
16th March 1913.

N. SANKARA AIYAR.

SOLUTIONS.

Question 295.

(S. RAMANUJAN) :—If $\alpha\beta=\pi$, shew that

$$\sqrt{\alpha}\int_0^\infty \frac{e^{-x^2}dx}{\cosh\alpha x}=\sqrt{\beta}\int_0^\infty \frac{e^{-x^2}dx}{\cosh\beta x}$$

Solution by the Proposer.

Since $$\int_0^\infty \frac{\cos 2nz}{\cosh \pi z}dz=\frac{1}{2\cosh n}$$

$$\sqrt{\alpha}\int_0^\infty \frac{e^{-x^2}dx}{\cosh\alpha x}=2\sqrt{\alpha}\int_0^\infty\int_0^\infty \frac{e^{-x^2}\cos 2\alpha xz}{\cosh \pi z}dx\,dz$$

$$=\sqrt{\alpha\pi}\int_0^\infty \frac{e^{-\alpha^2z^2}}{\cosh\pi z}dz=\sqrt{\frac{\pi}{\alpha}}\int_0^\infty \frac{e^{-z^2}}{\cosh\frac{\pi}{\alpha}z}dz$$

$$=\sqrt{\beta}\int_0^\infty \frac{e^{-x^2}dx}{\cosh\beta x}, \text{ since } \alpha\beta=\pi.$$

Question 364.

(ZERO) :—Explain a simple method of rationalizing the equation

$$x^{\frac{4}{5}}+ax^{\frac{3}{5}}+bx^{\frac{2}{5}}+cx^{\frac{1}{5}}+d=0,$$

and discuss the rationalized form of the general equation $f(x^{1/p})=0$.

Solution by M. R. Sadasiva Iyer.

Put $x=z^5$ we get

$$z^4+az^3+bz^2+cz+d=0 \quad \ldots \quad \ldots \quad (1)$$

Multiply (1) by (z) and substitute for z^5 and we get

$$az^4+bz^3+cz^2+dz+x=0 \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

similarly, we get

$$bz^4+cz^3+dz^2+xz+ax=0 \quad \ldots \quad \ldots \quad (3)$$

$$cz^4+dz^3+xz^2+axz+bx=0 \quad \ldots \quad \ldots \quad (4)$$

$$dz^4+xz^3+axz^2+bxz+cx=0 \quad \ldots \quad \ldots \quad (5)$$

Eliminating z, z^2, z^3, z^4 we get the result of reationalization, which is the persymmetric determinant

$$\begin{vmatrix} 1 & a & b & c & d \\ a & b & c & d & x \\ b & c & d & x & ax \\ c & d & x & ax & bx \\ d & x & ax & bx & cx \end{vmatrix} = 0 \ldots \qquad \ldots \quad (6)$$

Similarly, $f(x^{1/p})=0$ may be rationalized. Writing $x=z^p$, we get a persymmetric determinant of the p^{th} order.

The other factors of the determinant (6) are easily seen to be derived from the given expression by multiplying the terms by w^4, w^3, w^2, w, 1, where w is a primitive root of $x^5-1=0$.

Question 371.

(K. J. SANJANA, M.A.) :—I, I_1, I_2, I_3 are the centres of the four contact circles of a triangles, and N, N_1, N_2, N_3 the nine-point centres of the triangles formed by their respective points of contact; prove that NI, N_1I_1, N_2I_2, N_3I_3 are concurrent.

Solution by M. Bhimasena Rao, R. Tata, M.A. & G. Ramachandran, B.A.

Let ABC be the given triangle and L. M, N. The points of contact corresponding to I. It is well-known that the inverse of the nine point circle of LMN with respect to the circumcircle of LMN is the circumcircle of ABC (Casey's *Sequel*, Bk. VI, Prop. 12). Since a circle and its inverse are co-axal with the circle of inversion, it is evident that NI passes through the circumcentre of ABC, through which point N_1I_2, N_2I_2, N_3I_3 pass for a similar reason.

Analytical solution by J. C. Swaminarayan, M.A.

Question 392.

(A. C. L. WILKINSON) :—If in Question 313, the circle TQQ′ be replaced by any conic through TQQ′ touching the ellipse at P, the radii of curvature of the two conics at P are in the ratio 1 : 2.

Solution by J. C. Swaminarayan, M.A.

Take the tangent and normal at P as the axes of x and y respectively Then the equation of the given conic can be put in the form

$$ax^2+2hxy+by^2-2y=0 \qquad \ldots \qquad \ldots \qquad \ldots \quad (1)$$

Now, QQ′ is a chord of (1) and T is the pole of QQ′ with respect to (1). The conic through QQ′ touching (1) at P, can be put in the form

$$(1)\quad +\lambda y\,\{\,axx'+by\,y'+h\,(x'y+xy')-(y+y')\,\}=0 \quad \dots \quad (2)$$

where (x',y') are the coordinates of P. (2) will pass through T if $1+\lambda\,y'=0$.

Hence the equation of the conic TQQ′ assumes the form

$$ax^2y'+xy\,(y'h-ax')-y^2(hx'+1)-yy'^2=0 \quad \dots \quad \dots \quad (3)$$

The radius of curvature of (1) is $(1/a)$ and the radius of curvature of (3) is $(1/2\,a)$

Hence the result.

Question 398.

(A. C. L. Wilkinson, M.A., F.R.A.S.) :—A hyperboloid is described having for generators the edges AB, BC, CD, DA of a tetrahedron ABCD and with its centre at the centroid of the tetrahedron, shew that the algebraical sum of the squares of its axes is equal to

$$\tfrac{1}{4}(3AC^2+3BD^2-AB^2-BC^2-CD^2-DA^2)$$

and that the absolute values of the product of the axes is three times the volume of the tetrahedron ABCD.

Solution by R. N. Apte, M.A., F.R.A.S., and J. C. Swaminarayan, M.A.

If θ, θ' are the angles between the generating lines at the points A, C respectively.

$$\cot\theta=\frac{p(r^2-a^2-b^2+c^2)}{2\,abc},\quad \cot\theta'=\frac{p'(r'^2-a^2-b^2+c^2)}{2\,abc},$$

where p, p' are the perpendiculars on the planes ABD, BCD from the centre of the hyperboloid $x^2/a^2+y^2/b^2-z^2/c^2=1$. [See : Frost, Art 212.]

Generally, putting $4s^2$ = alg. sum of the squares of the axes, we have

$$\frac{\cot\theta}{\cot\theta'}=\frac{p\ \ (r^2-s^2)}{p'\ \ (r'^2-s^2)}- \quad \dots \quad \dots \quad (1)$$

where r, r' are the distances of A, C from the centre.

The centre is at the centroid G.

$\therefore$ $r=GA,\ r'=GC,$

$4\,p=$ the perpendicular CN from C on the plane ABD

$4\,p'=$ „ AN „ A „ BCD

But vol. V of tetrahedron $=\frac{1}{6}$ CN.AB.AD $\sin\theta$,

and $$\cos\theta=\frac{AB^2+AD^2-BD^2}{2\,AB.AD}.$$

Similarly, for sin θ', cos θ'.

$$\therefore \quad \frac{\cot\theta}{\cot\theta'}=\frac{p\,(AB^2+AD^2-BD^2)}{p'\,(BC^2+CD^2-BD^2)} \quad \ldots \quad (2)$$

From (1) and (2) $$\frac{r^2-s^2}{r'^2-s^2}=\frac{AB^2+AD^2-BD^2}{BC^2+CD^2-BD^2},$$

and $$\frac{r^2-s^2}{r^2-r^1}=\frac{AB^2+AD^2-BD^2}{AB^2+AD^2-BC^2-CD^2}. \quad (3)$$

Now if g be the centroid of the triangle BCD and E the middle point of BC,

$$2\,AE^2+AD^2=3\,Ag^2+\tfrac{2}{3}\,DE^2$$

$$\therefore \quad 9\,Ag^2=6\,AE^2+3\,AD^2-2\,DE^2;$$

and $$2\,(AE^2+BE^2)=AB^2+AC^2,\ 2\,DE^2+2\,DE^2=BD^2+CD^2$$

$$\therefore \quad 16\,GA^2=9\,Ag^2=(3\,AB^2+3\,AC^2+3\,AD^2-BC^2-BD^2-CD^2)$$

so $$16\,GC^2=(3BC^2+3\,CD^2+3\,CA^2-AB^2-BD^2-DA^2).$$

Hence from (3)

$$\frac{3\,AB^2+3\,AC^2+3\,AD^2-BC^2-BD^2-CD^2-16\,s^2}{4\,AB^2+4\,AD^2-4\,BC^2-4\,CD^2}=\frac{AB^2+AD^2-BD^2}{AB^2+AD^2-BC^2-CD^2}.$$

$$\therefore \quad \tfrac{1}{4}\,\{3\,AC^2+3BD^2-AB^2-BC^2-CD^2-AD\}=4s^2.$$

Question 401.

(S. P. Singaravelu Mudeliar, B.A) :—A straight line is drawn in the plane of a parabola, through the foot of the directrix. If the plane of the parabola be vertical and its axis horizontal, shew that the time of quickess descent from the straight line to the parabola is $\sqrt{2\,l\cos 2\theta/g\,(2\cos\theta+\sin 2\theta)}$, where θ is the inclination of the straight line to the vertical, and l the latus rectum of the parabola.

Solution by V. B. Naik, M.A., T. P. Trivedi, M.A., L.L.B., and others.

If PQ is the line of quickest descent from the straight line to the parabola, then PQ is the diagonal of the rhombus formed by the normals at P and Q to the straight line and parabola, and the verticals through P and Q. The tangent at Q to the parabola is, therefore, parallel to the given line and hence makes an angle θ with the axis. The coordinates of Q are $(a\tan^2\theta,\ 2a\tan\theta)$, the vertex of the parabola being the origin and its axis the axis of x. The equation of the line through the foot of the directrix is $x\cos\theta-y\sin\theta+a\cos\theta=0$, and the perpendicular from Q on it is

$$a \tan^2\theta \cos\theta - 2\,a \tan\theta \sin\theta + a\cos\theta = a\sec\theta - 2\,a\sin^2\theta\sec\theta$$
$$= a\cos 2\theta \sec\theta.$$

This perpendicular is evidently the projection of PQ on the normal at Q and makes with PQ an angle $= \frac{1}{2}(\pi/2-\theta)$.

$\therefore \qquad PQ = a\cos 2\theta \sec\theta \sec(\pi/4-\theta/2.)$.

Again, PQ makes with the vertical the same angle and the acceleration down PQ is $g\cos\left(\frac{\pi}{4}-\frac{\theta}{2}\right)$.

Hence the time down PQ is equal to

$$\left[2\,PQ/g\cos\left(\frac{\pi}{2}-\frac{\theta}{2}\right)\right]^{\frac{1}{2}} = \left[2\,a\cos 2\theta\sec\theta/g\cos^2\left(\frac{\pi}{4}-\frac{\theta}{2}\right)\right]^{\frac{1}{2}}$$
$$= [4\,a\cos 2\theta\sec\theta/g\,(1+\sin\theta)]^{\frac{1}{2}}$$
$$= \sqrt{2\,l\cos 2\theta/8\,(2\cos\theta+\sin 2\theta)}.$$

Question 402.

(J. C. Swaminarayan, M.A.) :—Prove that

$$\int_0^1 \left(\frac{\sqrt{(1-z^2)}}{z}\right)(\tanh^{-1}z)dz = \frac{\pi^2}{4}-\frac{\pi}{2}.$$

Solution (1) *by V. B. Naik, N. Sankara Aiyar, and T. P. Trivedi,* (2) *by R. Srinivasan, M.A.,* (3) *by M. R. Sadasiva Aiyar, and the Proposer.*

(1) Since, when, $0<|z|<1$, $\tanh^{-1}z = z+\frac{z^3}{3}+\frac{z^5}{5}+\dots\ \dots$, the given integral

$$= \sum_{n=0}^{\infty}\int_0^1 \frac{z^{2n}\sqrt{(1-z^2)}}{2n+1}dz$$

$$= \sum_{n=0}^{\infty}\int_0^{\frac{\pi}{2}} \frac{\sin^{2n}\theta\cos^2\theta}{2n+1}d\theta, \text{ putting } z=\sin\theta.$$

$$= \frac{\pi}{2}\sum_{n=0}^{\infty}\left(\frac{1}{2}\cdot\frac{3}{4}\cdots\cdots\frac{2n-1}{2n}\right)\left(\frac{1}{(2n+1)(2n+2)}\right).$$

Now $$\sin^{-1}z = z + \frac{1}{2}\frac{z^3}{3} + \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{z^5}{5} + \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot\frac{z^6}{7} + \cdots \quad \cdots \quad \cdots$$

$$\therefore \quad \int_0^1 \sin^{-1}z\,dz = \frac{1}{1\cdot 2} + \frac{1}{2}\cdot\frac{1}{3\cdot 4} + \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{1}{5\cdot 6} + \cdots \quad \cdots \quad \cdots$$

$$= \sum_{n=0}^{\infty} \left(\frac{1}{2}\cdot\frac{3}{4}\cdots\cdots\frac{2n-1}{2n}\right)\left(\frac{1}{(2n+1)(2n+2)}\right).$$

Also $$\int_0^1 \sin^{-1}z\,dz = \int_0^{\frac{\pi}{2}} \phi\cos\phi\,d\phi = \Big[\phi\sin\phi + \cos\phi\Big]_0^{\frac{\pi}{2}}$$

$$= \frac{\pi}{2} - 1.$$

$\therefore$ Given integral $= \frac{\pi}{2}\left(\frac{\pi}{2} - 1\right) = \frac{\pi^2}{4} - \frac{\pi}{2}$.

(2) Let $z = \tanh y$; the integral is equal to

$$\int_0^\infty \frac{y}{\sinh y}\operatorname{sech}^2 y\; dy = \left[\frac{y}{\sinh y}\tanh y\right]_0^\infty - \int_0^8 \tanh y.\frac{\sinh y - y\cosh y}{\sinh^2 y}dy$$

$$= \int_0^\infty \frac{y\;dy}{\sinh y} - \int_0^\infty \operatorname{sech} y\;dy.$$

Now $$\frac{y}{\sinh y} = \frac{2ye^{-y}}{1-e^{-2y}} = 2y\left\{e^{-y} + e^{-3y} + e^{-5y} + \cdots\cdots\right\}$$

$$\therefore \quad \int_0^\infty \frac{y\;dy}{\sinh y} = 2\Sigma \int_0^\infty ye^{-(2n+1)}y\;dy$$

$$= 2\Sigma\frac{1}{(2n+1)}$$

$$= 2\left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2}\cdots\cdots\cdots\right) = \frac{\pi^2}{4}.$$

And $$\operatorname{sech} y = \frac{2e^{-y}}{1+e^{-2y}} = 2(e^{-y} - e^{-3y} + e^{-5y} - \cdots\cdots)$$

$$\therefore \quad \int_0^\infty \operatorname{sech} y = 2\Sigma\int_0^\infty (-1)^n e^{-(2n+1)y}dy = 2\Sigma\frac{1}{2n+1}(-1)^n$$

$$= 2\left(1 - \frac{1}{3} + \frac{1}{5}\cdots\cdots\right) = 2.\,\frac{\pi}{4} = \frac{\pi}{2}.$$

Hence, the given integral $= \frac{\pi^2}{4} - \frac{\pi}{2}$.

(3) Let I denote the value of the given integral. Put $z=\sin\theta$, so that $dz=\cos\theta d\theta$.

$$I=\int_0^{\frac{\pi}{2}}\frac{\cos^2\theta}{\sin\theta}.\tanh^{-1}(\sin\theta)d\theta$$

$$=\Big[(\log\tan\frac{\theta}{2}+\cos\theta)(\tanh^{-1}\sin\theta)\Big]_0^{\frac{\pi}{2}}$$

$$-\int_0^{\frac{\pi}{2}}\Big\{\log\tan\frac{\theta}{2}+\cos\theta\Big\}\sec\theta d\theta.$$

Now, since $\text{Lt}\left(\log\tan\frac{\theta}{2}\right)\left(\tanh^{-1}\sin\theta\right)=\text{Lt}\left(\sin\theta\log\tan\frac{\theta}{2}\right)$

$$\text{Lt}\frac{\log\tan\frac{\theta}{2}}{\operatorname{cosec}\theta}=\text{Lt}\frac{\operatorname{cosec}\theta}{\operatorname{cosec}\theta\cot\theta}=\text{Lt}\,(-\tan\theta)=0.$$

$$\therefore\qquad I=-\int_0^{\frac{\pi}{2}}d\theta-\int_0^{\frac{\pi}{2}}\frac{\log\tan\frac{\theta}{2}}{\cos\theta}d\theta$$

$$=-\frac{\pi}{2}-\int_0^1\left(\frac{\log z}{1-z^2}\right)2.dz,\text{ where } z=\tan\frac{\theta}{2}.$$

$$=-\frac{\pi}{2}-2\int_0^1\log z(1+z^2+z^4+\ldots)dz$$

$$=-\frac{\pi}{2}+2\Big\{\frac{1}{1^2}+\frac{1}{3^3}+\frac{1}{5^2}+\cdots\Big\}$$

$$=\frac{\pi^2}{4}-\frac{\pi}{2}$$

Question 406.

(K. J. Sanjana, M.A.) :—If m_1, m_2 m_3 be the medians and s_1, s_2, s_3 the symmedians of a triangle ABC, prove that

(i) $\Sigma[a(b^2+c^2).\,s_1\,m_1]=\frac{3}{2}\,abc\,(a^2+b^2+c^2)$

(ii) $\Sigma\,[a(b^2+c^2)\,s_1/m_1]=24\,\Delta R.$

Solution by V. B. Naik and several others.

If θ and θ' be the angles which the median and symmedian through A, make with BC, $\theta'=\pi-(C-B+\theta)$, then

$$\frac{m_1}{s_1}=\frac{\sin\theta'}{\sin\theta'}=\frac{\sin(C-B+\theta}{\sin\theta}=\sin(C-B)\cot\theta+\cos(C-B)$$

$$=\tfrac{1}{2}\sin(C-B)[\cot B-\cot C]+\cos(C-B)$$

$$=\tfrac{1}{2}\left[\frac{\sin C}{\sin B}+\frac{\sin B}{\sin C}\right],\text{ by reduction,}$$

$$=\frac{b^2+c^2}{2bc}.$$

Also $m_1 s_1=\frac{n_1}{m_1}\times m_1^2=\frac{2bc}{b^2+c^2}\times\tfrac{1}{4}(2b^2+2c^2-a^2)$

$$=bc-\frac{a^2bc}{2(b^2+c^2)}$$

Hence (i) $\Sigma[a(b^2+c^2)\,s_1\,m_1]=abc\Sigma(b^2+c^5)-\tfrac{1}{2}\Sigma a^3bc=\tfrac{3}{2}\,abc\Sigma a^2$,

(ii) $\Sigma[a(b^2+c^2)s_1/m_1]=6\,abc=24\triangle R$.

Question 409.

(S. P. Singaravelu Moodeliar, B.A.) :—Q is any point from which four normals are drawn to the ellipse $ax^2+by^2=1$, whose centre is C. If N_1, N_2, N_3, N_4 are the projections upon CQ of P_1, P_2, P_3, P_4, the feet of the normals, shew that

$$\Sigma CP_1^2-CQ\Sigma CN_1=2(1/a+1/b).$$

Solution by T. P. Trivedi, M. A., L. L. B. and others.

Let Q be the point (h,k) and P_1, P_2, P_3, P_4 be the points x_1, y_1, x_2, y_2 etc.

Equation of CQ is $hy=kx$ and that of $P_1 N_1$ perpendicular to CQ is $k(y-y_1)+h\,(x-x_1)=0$.

$\therefore$ $\quad N_1$ is $\dfrac{(hx_1+ky_1)\,h}{h^2+k^2},\ \dfrac{(hx_1+ky_1)k}{h^2+k^2}.$

Similar values exist for N_2, N_3, N_4

$\therefore$ $\quad CQ\cdot CN_1=hx_1+ky_1$ and $CP_1^2=x_1^2+y_1^2$.

$\therefore$ $\quad \Sigma CP_1^2-CQ\cdot\Sigma CN_1=\Sigma x_1^2+\Sigma y_1^2-h\Sigma x_1-k\Sigma y_1$.

But $P_1P_2P_3$ and P_4 lie upon the ellipse $ax^2+by^2=1$, as well as on the Apollonian hyperbola

$$xy\,(a-b)+bhy-akx=0.$$

Hence x_1, x_2, x_3, x_4 are the roots of

$$ax^4\,(a-b)^2+2\ abh\,(a-b)\ x^3+\{\,ab(bh^2+ak^2-(a-b)^2\,\}$$
$$x^2-2\,bhx\,(a-b)-b^2h^2=0,$$

Similarly y_1, y_2, y_3, y_4 are the roots of a similar equation in y with and b interchanged, as well as h and k interchanged.

Hence $$\Sigma x_1 = -\frac{2\,bh}{a-b}\,;\ \Sigma x_1^2 = \frac{2\,b^2h^2 - 2\,abk^2}{(a-b)^2} + \frac{2}{a}$$

$$\Sigma y_1 = -\frac{2\,ak}{b-a}\,;\ \Sigma y_1^2 = \frac{2\,a^2k^2 - 2\,abh^2}{(a-b)^2} + \frac{2}{b}.$$

Thus

$$\Sigma CP_1^2 - CQ\ \Sigma CN_1 = \frac{2}{a} + \frac{2}{b}, \text{ on simplifying.}$$

Question 410

(P. A. Subramania Iyer, B.A., L.T.) :—Solve completely

$$\left[1+\left(\frac{dy}{dx}\right)^2\right]\frac{d^3y}{dx^3} - 3\frac{dy}{dx}\left(\frac{d^2y}{dx^2}\right)^2 = a\left[1+\left(\frac{dy}{dx}\right)^2\right]^3.$$

Solution by K. J. Sanjana, V. B. Naik, M.A., J. C. Swaminarayan, and others.

Dividing by y_2^2, we have

$$\frac{(1+y_1^2)y_3}{y_2^2} - 3\,y_1 = a\,\frac{(1+y_1^2)^2}{y_2^2}.$$

But $\rho = \dfrac{(1+y_1^2)^{3/2}}{y_2}$, and $\dfrac{d\rho}{ds} \equiv \dfrac{d\rho}{dx} \div \dfrac{ds}{dx} = 3y_1 - \dfrac{(1+y_1^2)y_3}{y_2^2}.$

Hence the equation reduces to

$$-\frac{d\rho}{ds} = a\rho^2, \text{ which gives } \frac{1}{\rho} = a(s-c_1).$$

Thus $\dfrac{d\psi}{ds} = a(s-c_1)$, so that $\psi = c_2 + \frac{1}{2}as^2 - ac_1s$, which may be written

$$2\psi = as^2 + bs + c.$$

Question 412.

(Martyn M. Thomas) :—Apply the theory of moving axes to solve the following. "At every point of a plane curve a line is drawn making a given angle α with the normal; let the envelope of the line be termed the α-evolute of the curve. Prove that the α-evolute of the β-evolute of any curve is the β-evolute of the α-evolute of the same"

[M. A. Degree examination, 1911, Madras.]

Solution by A. C. L. Wilkinson M.A., F.R.A.S., V.B. Naik, M.A., and others.

Take moving axes so that the origin moves along the curve, the axis of x being a tangent to the curve and the positive direction of the normal the axis of y, the positive direction of rotation being from x to y.

The point of contact of a straight line $ax+by+c=0$, and its envelope is given by its intersection with

$$x\frac{da}{ds}+y\frac{db}{ds}+\frac{dc}{ds}+\frac{a}{\rho}y-\frac{b}{\rho}x-a=0.$$

For if ∂s is the element of arc, $\partial\psi$ the angle of contingence, (x,y) (X,Y) the coordinates of a point P referred to the axes at any point of the curve and the axes at a consecutive point, we have, by projection, the relations

$$X=x-\partial s+y\partial\psi,\ Y=y-x\partial\psi.$$

Thus the consecutive line to $ax+by+c=0$, is, referred to the consecutive axes, $(a+\partial a)\ X+(b+\partial b)\ Y+c+\partial c=0$, and on substituting for X, Y their values in terms of x, y, we get the result stated.

The α-evolute of a curve is given therefore by

$$x\cos\alpha-y\sin\alpha=0, \quad \ldots \quad \ldots \quad (1)$$

and its point of contact with its envelope is given by

$$y\cos\alpha+x\sin\alpha-\rho\cos\alpha=0. \quad \ldots \quad (2)$$

Solving (1) and (2) the point of contact is given by

$$\frac{x}{\cos\alpha}=\frac{y}{\sin\alpha}=\rho\cos\alpha.$$

The β-evolute of this is given by a straight line through this point making an angle $\alpha+\beta-\frac{\pi}{2}$ with the normal to the original curve; this evolute is therefore the envelope of

$$(x-\rho\cos^2\alpha)\cos(\alpha+\beta)+(y-\rho\sin\alpha\cos\alpha)\sin(\alpha+\beta)=0,$$

that is of $\quad x\cos(\alpha+\beta)+y\sin(\alpha+\beta)-\rho\cos\alpha\cos\beta=0.$

The symmetry of this result proves the theorem stated.

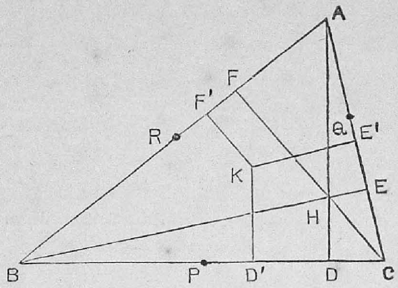
Question 414.

(R. Tata, M.A.) :—If DEF is the pedal triangle of ABC, and D′E′F′ that of its symmedian point, shew that

$$a.\ DD'+b.\ EE'+c.\ FF'=0.$$

Solution by J. C. Swaminarayan, M.A.

Let K be the symmedian point and let H be the ortho-centre. Let P, Q, R be the middle points of BC, CA, AB respectively.



Now $BK^2-CK^2+CK^2-AK^2+AK^2-BK^2=0.$

$\therefore$ $2a\ PD+2b\ QE+2C\ RF=0.$ ... ... (1)

Similarly, because

$$BH^2-CH^2+CH^2-AH^2+AH^2-BH^2=0$$

$\therefore$ $2a.\ PD+2b.\ QE+2c.\ RF=0$ ... ... (2)

Subtracting (1) from (2) and dividing by 2, we get

$$a.\ DD'+b.\ EE'+c.FF'=0.$$

N.B.—This proof is quite general and consequently the given result holds good in the case of the pedal triangles DEF and D′E′F′ of *any two* points X and X′.

Additional solutions by V. B. Naik, M.A., and M. R. Sadasiva Aiyar, M.A., B.E.

Question 415.

(A.A. Krishnaswami Aiyangar):—If $s_r=1^r+2^r+\ldots+n^r$, and $c_p={}_rc_p$ shew that $1+c_0s_0+c_1s_1+c_2s_2+\ldots+c_{r-1}\ s_{r-1}=(1+n)^r.$

Solution by N. Ganapati Subba Iyer and several others.

$$(1+n)^r-n^r=c_0+c_1n+c_2n^2+\ldots+c_{r-1}n^{r-1}$$

$$n^r-(n-1)^r=c_0+c_1(n-1)+c_2(n-1)^2+\ldots+c_{r-1}(n-1)^{r-1}$$

... ...

$$3^r-2^r=c_0+c_1\cdot2+c_2\cdot2^2+\ldots+c_{r-1}\cdot2^{r-1}$$

and $2^r-1^r=c_0+c_1\cdot1+c_2\cdot1^2+\ldots+c_{r-1}\cdot1^{r-1}$

Adding all these results, we get

$$(1+n)^r=1+c_0s_0+c_1s_1+c_2s_2+\ldots+c_{r-1}s_{r-1}.$$

Question 416.

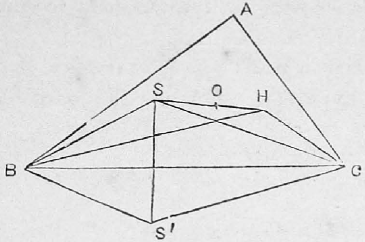
(M. T. Naraniengar) :—If l, l' ; m, m' ; n, n' are the distances of the foci of an inconic from the angular points, shew that the areal co-ordinates of the centre of the conic are

$$\frac{1}{4\Delta}(mm' \sin B + nn' \sin C)\&c.$$

Solution by V. B. Naik and N. Bhimasena Rao.

Let S, H, be the foci, O the centre and S′ the reflection of S in the side BC

Since O is the middle point of SH



$$\triangle BOC = \tfrac{1}{2}(\triangle BSC + \triangle BHC)$$
$$= \tfrac{1}{2}(\triangle BS'C + \triangle BHC)$$
$$= \tfrac{1}{2}(\triangle BHS' + \triangle CHS')$$
$$= \tfrac{1}{4}(BH \cdot BS' \sin HBS' + CH \cdot CS' \sin HCS')$$

Now $BS' = BS$, $CS' = CS$,

$\angle HBS' = B$, $\angle HCS' = C$.

$\therefore \triangle BOC = \tfrac{1}{4}(mm' \sin B + nn' \sin C)$.

Other Solutions by J. C. Swaminarayan, T. P. Trivedi, N. B. Pendse and N. P. Pandya.

Question 422.

(D. D. Kapadia, M.A., B.S.C.) :—Shew that—

$$\begin{vmatrix} a, & b, & c, & d, & e, & f \\ f, & a, & b, & c, & d, & e \\ e, & f, & a, & b, & c, & d \\ d, & e, & f, & a, & b, & c, \\ c, & d, & e, & f, & a, & b \\ b, & c, & d, & e, & f, & a \end{vmatrix} = \begin{vmatrix} a+d, & b+e, & c+f \\ c+f, & a+d, & b+e \\ b+e, & c+f, & a+d \end{vmatrix} \begin{vmatrix} a-d, & b-e, & c-f \\ f-c, & a-d, & b-e \\ e-b, & f-c, & a-d \end{vmatrix}$$

Solution by K. J. Sanjana, T. P. Trivedi, and Appu Kuttan Erady, M.A.

The determinant on the left is divisible by $a+b+c+d+e+f$; so also $a-b+c-d+e-f$, $a+wb+w^2c+d+we+w^2f$,
$a-wb+w^2c-d+we-w^2f$, $a+w^2b+wc+d+w^2e+wf$,
$a-w^2b+wc-d+w^2e+wf$,
are seen to be divisors, where w is a complex cube root of unity. The numerical facter is clearly unity.

The factors of the first determinant on the right are similarly found to be

$$a+b+c+d+e+f,$$
$$a+wb+w^2c+d+we+w^2f, \quad a+w^2b+wc+d+w^2e+wf;$$

the factors of the second determinant are the three remaining divisors of the left side. No numerical factors are required in addition. Hence the result follows.

Other solutions by V. D. Gokhale and V. K. Aravamudan.

Question 425.

(R. SRINIVASAN, M. A.) :—Shew that the envelope of the polar of the focus of the parabola $y^2=4ax$, with respect to any rectangular hyperbola which has four point contact with it, is a parabola having the same axis as the given parabola.

Solution by G. Ramachandran, T. P. Trivedi and V. K. Aravamudan, B.A.

The equation of the rectangular hyperbola osculating the parabola at the point "t" can be easily shown to be

$$x^2-2xyt-y^2+2ax(3t^2+2)-2ayt^3+a^2t^4=0.$$

The polar of the focus of the parabola with respect to the rectangular hyperbola is

$$3ax(1+t^2)-aty(1+t^2)+a^2(t^4+3t^2+2)=0,$$

(*i.e*) $$a^3t^2-aty+(3ax+2a^2)=0.$$

The envelope of the above line is

$$y^2=4a(3x+2a),$$

which is a parabola having the same axis as the given parabola.

Question 435.

(R. TATA, M.A.) :—If the lines joining the symmedian point of ABC to the middle points of the sides, make angles $\theta_1, \theta_2, \theta_3$ with them, shew that,

$$\cot\theta_1+\cot\theta_2+\cot\theta_3=0.$$

Solution by V. V. Satyanarayanan and S. Krishnaswamiengar.

Let K be the symmedian point, AP the perpendicular, and AD, the median corresponding to BC.

We know "the lines joining the middle points of the sides of a $\triangle$ to the middle points of the corresponding altitudes pass through the symmedian point."

Hence, DK passes through M, the middle point of AP. That is

$$\cot\theta_1=2\cot\angle ADP=\cot B-\cot C.$$

Similarly $$\cot\theta_2=\cot C-\cot A, \quad \cot\theta_3=\cot A-\cot B.$$

Hence $$\cot\theta_1+\cot\theta_2+\cot\theta_3=\Sigma(\cot B-\cot C)=0.$$

Question 437.

(A. A. Krishnasami Aiyangar B.A.) :—Obtain the following property of Bernoulli's numbers—

$$\frac{B_{4n+1}}{|\underline{4n+2}}.\frac{B_1}{|\underline{2}}+2\frac{B_{4n-1}}{|\underline{4n}}.\frac{B_3}{|\underline{4}}+\ldots+2\frac{B_{2n+3}}{|\underline{2n+4}}\frac{B_{2n-1}}{|\underline{2n}}+\frac{B_{2n+1}}{|\underline{2n+2}}\frac{B_{2n+1}}{|\underline{2n+2}}=(4n+5)\frac{B_{4n+3}}{|\underline{4n+4}}.$$

Correction and Solution by G. Ramachandran and V. K. Aravamudan B A.

The first term on the left hand side should be $2\frac{B_{4n+1}}{|\underline{4n+2}}.\frac{B_1}{|\underline{2}}$.

Now

$$\frac{x}{2}\cot\frac{x}{2}=1-B_1\frac{x^2}{|\underline{2}}-B_3\frac{x^4}{|\underline{4}}-\frac{B_5}{|\underline{6}}x^6\ldots\ldots B_{2n+1}\frac{x^{2n+2}}{|\underline{2n+2}}\ldots\ldots\frac{B_{4n+3}}{|\underline{4_n+4}}x^{4n+4}\ldots\ldots$$

[*vide* : page 106, Ex. 149, Edward's Diff. Calc.]

Squaring both sides and collecting the coefficient of x^{4n+4} on the right hand side, we get

$$2\left\{\frac{B_{4n+1}}{|\underline{4n+2}}.\frac{B_1}{|\underline{2}}+\frac{B_{4n-1}}{|\underline{4n}}.\frac{B_3}{|\underline{4}}+\ldots\ldots\frac{B_{2n+3}}{|\underline{2n+4}}.\frac{B_{2n-1}}{|\underline{2n}}\right\}+\frac{B_{2n+1}}{|\underline{2n+2}}\frac{B_{2n+1}}{|\underline{2n+2}}-2\frac{B_{4n+3}}{|\underline{4n+4}}.$$

But $\frac{x^2}{4}\cot\frac{{}^2x}{2}=-\frac{x^2}{4}\left\{1+2\frac{d}{dx}\cot\frac{x}{2}\right\}$

$$=-\frac{x^2}{4}\left\{1+2.\ \frac{d}{dx}\left(\frac{2}{x}-B_1\frac{2x}{|\underline{2}}-B_3\frac{2x^3}{|\underline{4}}-\ldots-B_{4n+3}\frac{2x^{4n+3}\ldots}{|\underline{4n+4}}\right).\right\}$$

$$=-\frac{x^2}{4}\left\{1-4\left(\frac{1}{x^2}+\frac{B_1}{|\underline{2}}+3B_3\frac{x^2}{|\underline{4}}+\ldots\ldots+(4n+3)\frac{B_{4n+3}}{|\underline{4n+4}}x^{4n+2}\right)\right\}$$

Equating the co-efficient of x^{4n+4}

$$2\left\{\frac{B_{4n+4}}{|\underline{4n+2}}\frac{B_1}{|\underline{2}}+\frac{B_{4n-1}}{|\underline{4n}}\frac{B_3}{|\underline{4}}+\ldots\ldots+\frac{B_{2n+3}}{|\underline{2n+4}}\frac{B_{2n-1}}{|\underline{2n}}\right\}+\frac{B_{2n+1}}{|\underline{2n+2}}\frac{B_{2n+1}}{|\underline{2n+2}}-2\frac{B_{4n+3}}{|\underline{4n+4}}=(4n+3)\frac{B_{4n+3}}{|\underline{4n+4}}.$$

$$\therefore\ 2\ \frac{B_{4n+1}}{|\underline{4n+2}}\frac{B_1}{|\underline{2}}+2,\frac{B_{4n-1}}{|\underline{4n}}\frac{B_3}{|\underline{4}}+\ldots\ldots+2\frac{B_{2n+3}}{|\underline{2n+4}}\frac{B_{2n-1}}{|\underline{2n}}+\frac{B_{2n+1}}{|\underline{2n+2}}\frac{B_{2n+1}}{|\underline{2n+2}}=(4n+5)\frac{B_{4n+3}}{|\underline{4n+4}}.$$

QUESTIONS FOR SOLUTION.

447. (G. RAMACHANDRAN) :—Construct a triangle ABC having given the rectangle contained by AB and AC, the median from A to BC, and the sum or difference of the angles ABC, ACB.

448. (V. V. SATYANARAYANA) :—In any triangle ABC circles BQRC, CRPA, APQB are described. Shew that QC.RA.PB= QA.RB.PC.

Also, if (x, x'), (y, y'), (z, z') are the distances of (Q,R), (R,P), (P,Q), from, BC, CA, AB respectively, prove that $xyz = x'y'z'$.

449. (S. KRISHNASWAMI AIYANGAR) :—Establish the results

$$\frac{1}{1^4}-\frac{1}{3^4}-\frac{1}{5^4}+\frac{1}{7^4}+\frac{1}{9^4}-\dots=\frac{11\pi^4}{768\sqrt{2}}.$$

450. (K. APPUKUTTAN ERADY, M.A.) :—If A,B,C...are the minors of a,b,c ... in the determinant $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$, and if

$$\frac{al+hm+gn}{l}=\frac{hl+bm+fn}{m}=\frac{gl+fm+cn}{n}$$

prove that

$$\frac{\mathrm{A}l+\mathrm{H}m+\mathrm{G}n}{l}=\frac{\mathrm{H}l+\mathrm{B}m+\mathrm{F}n}{m}=\frac{\mathrm{G}l+\mathrm{F}m+\mathrm{C}n}{n}.$$

Hence, prove that the cone $(abcfgh)(xyz)=0$, and its reciprocal are coaxial.

451. (K. J. SANJANA, M.A.) :—Shew how to find pairs of isosceles triangles with rational areas such that the perimeters are in one given ratio and the areas in another given ratio. *Example* : the perimeters are as 25 : 9 and the areas are equal.

452. (A. C. L. WILKINSON, M.A., F.R.A.S.) :—In a tetrahedron in which the opposite edges are perpendicular the perpendiculars from the vertices on the opposite faces meet in a point O. If G is the centroid and A,B,C,D the vertices, prove the following construction for the points of contact of the ellipsoid of revolution inscribed in the tetrahedron with one focus at O :—produce AG to A′ so that GA′=GA, then A′O meets meets BCD in the point of contact of the ellipsoid with the face BCD.

453. (S. P. SINGARAVELU MODELIAR, B.A., L.T.) :—A hollow right cone is filled with a homogeneous liquid and held with axis vertical and vertex down. Find the parabolic section of the cone on which the thrust is a maximum.

454. (A. A. KRISHNASWAMI AIYANGAR) :—If $s_r = 1^r + 2^r + 3^r + \ldots n^r$, prove that

$$ {}_rC_1 . s_{r-1} + ({}_{r-1}C_1 - {}_rC_2) s_{r-2} + ({}_{r-2}C_1 - {}_{r-1}C_2 + {}_rC_3) s_{r-3} + \ldots \text{to } r \text{ terms} = (n^{r+1} - n)/(n-1). $$

455. (J. C. SWAMINARAYAN, M.A.) :—From an external point O, a tangent OT and a secant OPQ are drawn to a circle ; D is the middle point of PQ and TD cuts the circle again in A. If any other secant OXY is drawn and AX, AY cut PQ in R, S, prove that RD = DS.

456. (SELECTED) :—If $f(x)$ is a continuous function of x, not necessarily satisfying the conditions for expansion as a Fourier series, prove that—

$$ \frac{1}{2\pi} \int_0^{2\pi} [f(x)]^2 dx = a_0^2 + \tfrac{1}{2} \sum_{n=1}^{\infty} (a_n^2 + b_n^2), $$

where $a_0, a_1 a_2, \ldots \; b_0, b_1, b_2 \ldots$ are the Fourier constants of $f(x)$.

Shew also that

$$ a_0 x + \Sigma \frac{1}{n} \{ a_n \sin nx + b_n (1 - \cos nx) \} = \int_0^x f(\xi) d\xi $$