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A paper should contain a short and clear summary of the new results obtained and the relations in which they stand to results already known. It should be remembered that, at the present stage of mathematical research, hardly any paper is likely to be so completely original as to be independent of earlier work in the same direction; and that readers are often helped to appreciate the importance of a new investigation by seeing its connection with more familiar results.

The principal results of a paper should, when possible, be enunciated separately and explicitly in the form of definite theorems.

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All contributions should be written legibly on one side only of the paper and all diagrams should be neatly and accurately drawn on separate slips.

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A Statistical Study of Some Examination Marks—II*

BY P. V. SESHU AYYAR AND S. R. RANGANATHAN.

In the first paper † on this subject read before the Madras Session of the Indian Science Congress, it was stated "The average values of the correlation co-efficients of the various subjects with English may be taken to give a quantitative measure of the extent to which the power of expression in good English is necessary in the various subjects. These measures will be more reliable, if the correlation between the marks in the various subjects and the marks in English composition alone are found. It will be of great pedagogic interest to have those measures calculated."

This second paper has been prepared as a supplement on those lines. Marks in the several English papers were separately available only for two years, *viz.*, the 12th and the 13th years, reckoned from the first of the six years considered in the first paper. The correlation co-efficients for these two years have been calculated. It may be pointed out that for the examination considered there are four papers in English for any year—two on text-books prescribed for detailed study, one on composition from books set for non-detailed study, and the last on general composition. For convenience of reference, we have designated all the four papers taken together, by the general name 'English' and the last two papers taken together, by the name 'Composition.' In testing the power of expression, the last two papers are in a sense complementary. For in the third paper, the matter is previously given to the candidates in a crude form in their text-books and they have merely to reproduce the same and marshal details in a suitable manner to produce a desired effect. Whereas, the paper on general composition tests the candidate's general knowledge and his ability to express his ideas in correct English. The following table gives the correlation co-efficients between the various subjects and English and English Composition separately considered :

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* Read before the Eleventh Indian Science Congress, Bangalore, 1924.

† *Vide* J. I. M. S., Vol. XIV. pp. 43—55.

Correlation Co-efficients between

		Maths.	Physics.	Chem.	Mod. Hist.
12th year	English.	·44	·38	·46	·54
	Composition.	·36	·34	·38	·42
13th year	English.	·45	·46	·59	·41
	Composition.	·43	·34	·22	·33

As was expected the correlation co-efficient is slightly different if composition alone is taken into account instead of the whole of English. But no definite conclusion can be drawn from these meagre data. It is desirable to calculate the co-efficient for a longer period of years, say ten years or so, as material becomes available, and take the average value as a suitable measure of the part played by the power of expression in answering the papers in various subjects. Though nothing definite can be said of the correlation between English and the various subjects from these data, yet the bye-products obtained in this analysis, *viz.*, the *mean mark* and the *standard deviation* in the various subjects, seem to call for some interesting remarks.

A comparison of the mean mark and of the standard deviation in the various subjects for the two years with their average values for the first six years considered in the first paper,* discloses certain noteworthy features. The first row contains the average values of these quantities for the six years considered in the first paper.

Mean.

	English.	Maths.	Physics.	Chem.	Modern History.
Average for six years.	39·3	41·5	40·3	39·1	38·0
12th year.	37·13	44·28	39·36	33·52	37·25
13th year	34·82	40·57	35·76	38·54	35·72

* l. c., p. 47.

Standard Deviation.

	English.	Maths.	Physics.	Chem.	Modern History.
Average for six years.	9·26	13·23	11·98	12·50	8·60
12th year.	6·48	19·26	16·50	13·80	8·97
13th year.	5·14	18·48	13·29	10·49	8·24

Recalling to memory Fig. II* of the first paper one may say that there is nothing unusual about the mean in the various subjects. Perhaps one remark may be made regarding the mean in English. Till the end of the 12th year the minimum for a pass in English was 40 and the mean was oscillating about this figure and in the first six years considered the lowest value reached by it was only 36·3. In the 13th year, the minimum was reduced to 35 and the mean had dropped so low as 34·82. It would be of interest to watch the progress of the mean in English for a few more years. It will throw much light on the interaction between the mean in any subject and the minimum for a pass in that subject, as indicated in our first paper. It has to be remembered that the average mean in the other subjects was about 5 marks more than the minimum and that in English alone it was slightly lower than the minimum. Now that the minimum was lowered to 35 in the 13th year, the average also has fallen down. It is not proper to draw any conclusions from the experience of a single year. Observation must be continued over a few more years. If the phenomenon that is observed in the 13th year continues to exist in other years also, reduction of the minimum cannot be said to produce the desired result and under such circumstances the only proper method for fixing the minimum for a pass would be not to define it to be an absolute percentage but as the mark that is at a prescribed distance from the mean, as suggested in our first paper.

While there is nothing markedly abnormal about the behaviour of the mean in the two years considered, it is just the other way with respect to the standard deviation. It must be remembered that the standard deviation would ordinarily be characterised by comparative steadiness. This is also corroborated by Fig. III† of our first paper. But, in English, Mathematics and Physics, the fluctuation in the standard deviation is far from ordinary. It is only just noticeable in Physics; but it is very marked in Mathematics and in English.

* l. c., p. 48. † l. c., p. 50.

In Mathematics, it has gone up considerably, whereas in English it has dwindled down very much; this means that there is a tendency to crowd the candidates about the mean to a greater extent than before in English and to disperse the candidates more evenly than before over the entire range of marks in the case of Mathematics. The question is: what inferences can we draw from this, or on what hypothesis can we explain this?

To take up the case of Mathematics first, much will depend upon whether the frequency curve has undergone any change in shape or type. It is found that it has not undergone any marked change: it continues to be just of the same type as that shown in Fig. I of our first paper and continues to have small skewness. Again, much will also depend upon any variation in the standard of the examination, as determined from other independent considerations. Now, there is a widespread feeling among those competent to judge that the standard of the Mathematics papers has appreciably fallen during the last few of the years considered. Let us take this as an assumption and let us put it alongside the tremendous rise in standard deviation, the constancy of the skewness of the curve and the normality of the distance of the mean from the minimum.

These considerations lead us to infer, in the first place, that the reduction of the standard of the question paper has affected different types of candidates differently. In the first place, the constancy of the skewness and of the type of the frequency curve indicates that there has been no appreciable shift in the frequency from below the minimum to above the minimum. Secondly, the rise in standard deviation shows that the candidates above the average show a tendency to go nearer to the maximum and that those below the average, a tendency to go nearer to zero. The former of these phenomena has, associated with it, the disadvantage that the results of such an examination do not give sufficient help in picking out the best candidates. This is corroborated by the experience of the professors who have to select candidates for the Honours course on the basis of these marks. For, it is found that, not infrequently, getting even 75 or 80 p. c. in these examinations does not ensure a decided aptitude for the subject.

Let us then pass on to candidates who are below the average. It was remarked that they are found to show a tendency to slide down and go nearer to zero. This indicates that the reduction of the standard is not availed of profitably by the very class of students for whom it has probably been intended. On the other hand, reduction in the standard of the paper seems to lead to a tendency to lower the aim itself of the teacher and of the taught and the effect of this tendency is seen most in the case of those who are below the average. These considerations indicate that there is hardly anything gained by the lowering of the standard in Mathematics and that perhaps there is something lost.

The marked lowering of the standard deviation in English is very difficult to explain. Of course, in a subject like English, the standard deviation is bound to be lower than in other subjects. But, in the two years considered, it is much lower than what it was in the first six years. In fact, practically, about 80 % of the candidates came within the range 30 to 40 and nearly 50 % within the range $32\frac{1}{2}$ to $37\frac{1}{2}$. This indicates either that mediocrity is the order of the day in English, or that the examiners concentrate their attention more on whether to pass or fail a candidate than on finding a closer approximation to his actual worth.

This statistical analysis, combined with what is contained in our first paper, clearly proves the desirability of conducting annual statistical audits of the working of large-scale examination systems with a view to find out general tendencies in candidates and examiners and to enable educationists to recognise defects, if there are any, and to devise measures to remove them. The authorities seem to think, however, that, once the papers are set and valued and the results announced, their responsibility ends and that the examination system is functioning properly. But, as a matter of fact, our examination system is very artificial and unreliable, if conducted without proper safeguards and proper vigilance, as it obtains now. It makes us drift, we don't know in what direction, without our even realising that we are so drifting.

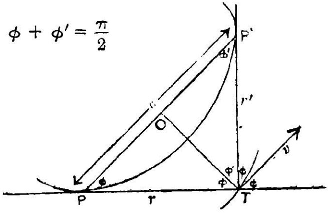
Hence, we once again impress on the public the imperative necessity for the appointment of an Examination Board (having a skilled statistician on it) connected with every University or other examining body—which in the words of the Calcutta University Commission should act as the "conscience" of the authorities in respect of examinations.

The Curvature of the Orthoptic Locus.

By A. Narasinga Rao, M.A.,
College of Engineering, Guindy.

[The orthoptic of a plane curve C is the locus of the intersection of perpendicular tangents to it. Taking such a pair of tangents as a set of moving axes, I obtain expressions (6) and (8) for the curvature of the orthoptic. From these are deduced certain characteristic properties of curves whose orthoptics are straight lines. Besides the parabola, we have such a curve in the unicursal quartic for which the line at infinity is a bitangent in two perpendicular directions.* Hence there arise between these two curves and others for which a straight line is part of the orthoptic locus certain points of similarity which, I think, have not been noticed hitherto. In the last paragraph, I prove that the orthoptic of the quartic referred to above is its directrix.]

1. Let the tangents at P and P′ meet at right angles at T, so that T traces the orthoptic locus. Also let s and s' be the arcual distances of P and P′ from a fixed point on the curve; r, r' the lengths of the tangents PT, P′T and ψ the angle which the tangent PT makes with a fixed line in the plane.



Let the points P and P′ move along the curve dynamically; we have

$$\text{vel. of T} = \text{vel. of P} + \text{vel. of T relative to P}$$
$$= (\dot{s} + \dot{r}) \text{ along PT; and } (r\dot{\psi}) \text{ along TP}'. \quad \dots \quad (1)$$

* Hilton: *Plane Algebraic Curves*—p. 174., Ex. 13. Hilton evidently intends this to be proved by the use of Plucker's numbers for the orthoptic. The demonstration in § 3 is direct and dispenses with the use of these.

Again

vel. of T = vel. of P′ + vel. of T relative to P′

$= r'\dot{\psi}$ along PT; and $(\dot{s}' - \dot{r}')$ along TP′.... (2)

Equating the values derived from (1) and (2), we obtain

$$\left.\begin{aligned} \dot{s} + \dot{r} = r'\dot{\psi} \quad &, \quad r\dot{\psi} = \dot{s}' - \dot{r}'; \\ \text{i.e.} \quad \rho + \frac{dr}{d\psi} = r' \text{ and } &\rho' - \frac{dr'}{d\psi} = r; \end{aligned}\right\} \quad \dots (3)$$

ρ and ρ' being the radii of curvature at P and P′ respectively.

Let v be the velocity of T along the orthoptic and $\overline{\mathrm{K}}$ the curvature of the orthoptic at T, so that the angular velocity of the tangent at T is $\overline{\mathrm{K}}v$.

We have then from the figure*

$$v \sin\phi = \dot{r} + \dot{s}, \quad \dots \quad \dots (4)$$

$$\frac{v\cos\phi}{r} - \dot{\phi} = \overline{\mathrm{K}}v;$$

i.e.

$$\overline{\mathrm{K}} = \frac{\cos\phi}{r} - \frac{\dot{\phi}}{v}. \quad \dots \quad \dots (5)$$

Substitute for v in (5) the value

$$v = \frac{\dot{r} + \dot{s}}{\sin\phi} = \frac{r'\dot{\psi}}{\sin\phi} = c\dot{\psi}$$

derived from (4) and (3). We have thus

$$\overline{\mathrm{K}} = \frac{\cos\phi}{r} - \frac{1}{c}\frac{d\phi}{d\psi}.$$

$$\therefore \quad \overline{\mathrm{K}}c = 1 - \frac{d\phi}{d\psi}. \quad \dots \quad \dots (6)$$

Again since $\qquad \cot\phi = \frac{r}{r'}$,

$$-r'^2 \operatorname{cosec}^2\phi \frac{d\phi}{d\psi} = r'\frac{dr}{d\psi} - r\frac{dr'}{d\psi};$$

i.e.

$$c^2\frac{d\phi}{d\psi} = r'(\rho - r') - r(r - \rho') \quad \text{from (3)}$$

$$= r\rho' + r'\rho - c^2. \quad \dots \quad \dots (7)$$

---

* The normal to the orthoptic locus at T bisects the ortho-chord PP′. (Hilton :—*Plane Algebraic Curves*, p 170) Hence the tangent makes an angle ϕ with PT.

Hence we obtain for the curvature of the orthoptic

$$\left.\begin{aligned}\bar{K} &= \frac{2c^3 - \rho r' - \rho' r}{c^3} \\ &= \frac{2c - \rho \sin\phi - \rho' \sin\phi'}{c^2}.\end{aligned}\right\} \quad \ldots \quad \ldots \quad (8)$$

Corollaries :

i. If P or P′ be a point of inflexion the corresponding point T is a cusp on the orthoptic locus.

ii. If P and P′ are cusps, $\rho = 0$ $\rho' = 0$ and $\bar{K} = \frac{2}{c}$. Hence the circle on PP′ as diameter is the circle of curvature at T.

iii. T will be a point of inflexion if $2c^3 = \rho r' + \rho' r$. Hence inflexions may exist on the orthoptic at points corresponding to ordinary points on C.

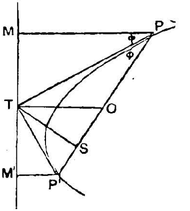
2. If the orthoptic locus be a straight line, $\bar{K}$ vanishes at all points and hence by (6)

$$d\phi = d\psi.$$

$$\therefore \quad \phi = \psi,$$

if the direction $\psi = 0$ be properly chosen.

This result may also be obtained from the figure.



The normal to the orthoptic straight line MTM′ passes through O the middle point of PP′. Hence *TO is fixed in direction* and may be taken as the initial direction from which ψ is measured. If perpendiculars PM and P′M′ are drawn from P and P′ to the orthoptic line, we have

$$\angle MPT = \text{alternate } \angle PTO \ (= \psi)$$
$$= \angle TPO \ (= \phi)$$

since OT = OP = OP′.

The tangent PT bisects the angle between a fixed direction PM and the ortho-chord through P. ... ... (9)

Again from the figure $PM = c\cos^2\phi$, $P'M' = c\sin^2\phi$;

and $PM + P'M' = PP'$.

Hence

There exists a point S on PP′ such that SP = PM and SP′ = P′M. This point is the projection of T on PP′. ... ... (10)

Lastly to each point P corresponds a single tangent PT meeting the orthoptic straight line in the unique point T and through this point passes a unique tangent in a perpendicular direction touching the curve at P′. Also the relation between P and P′ is symmetric. Hence

The points P and P′ are corresponding members of an involution range on the curve. ... ... (11)

These properties follow from a consideration of the curve C in the immediate neighbourhood of P and P′. Hence they will hold even if

(i) The straight line is only a part of the orthoptic locus,

and (ii) P and P′ lie on different algebraic curves provided the tangents at the points meet at right angles on a straight line.

3. There are two cases of the orthoptic locus being a straight line, namely, when C is a parabola and when it is a curve of class 3 for which the line at infinity is a bi-tangent in two perpendicular directions. We have thus a striking resemblance between these two curves in (9), (10) and (11).

In the case of a conic the chord joining point pairs in involution (11) passes through a fixed point. The tangents at the fixed points of the involution being self-perpendicular pass through the circular points. Hence the ortho-chord always passes through a focus and the corresponding directrix is the orthoptic locus.

Consider next a curve of class three for which the line at infinity is a bi-tangent at two points say A and B. Take any fixed tangent t to the curve. If T be any point on t we may draw from T two tangents to the curve other than t and these meet the line at infinity in say π and π'. It is easy to see that π and π' are in one-one correspondence, each being the conjugate of the other. We have thus established an involution on the line at infinity the double points of which are the points at infinity on the two tangents to the curve whose chord of contact is t. If these be the circular points, that is, if t be the *directrix*—chord of contact of tangents from I and J to the curve—π and π' will separate I and J harmonically and hence tangents from T to the curve will be at right angles. Again, in the involution $\pi\ \pi'$, the points A and B form a corresponding pair since the tangents to the curve at these points—both coincident with the line at infinity—fulfil the condition of meeting on t. Hence A and B separate harmonically I and J, that is, the line at infinity must be a bi-tangent, the points of contact being at infinity in perpendicular directions. This condition ensures that I and J shall be the double points of the involution, that is, that t shall be the orthoptic locus.

"On Some Infinite Series and Products."

By M. Bhimasena Rao and M. Venkatarama Ayyar.

Part I.*

This paper deals with the evaluation of certain infinite series and products. A few of them have already been evaluated by Dr. Glaisher and others; but, their mode of evaluation is complex, involving a knowledge of elliptic functions and modular equations. The method adopted here is fairly elementary, consisting of simple transformations of two well-known infinite integrals and is believed to be new.†

§ 1. It is well-known that ‡

$$\int_0^\infty \frac{\sin(bx)\,dx}{e^{2\pi x}-1} = \frac{1}{2}\left(\frac{1}{e^b-1}-\frac{1}{b}+\frac{1}{2}\right), \text{ if } b = p+iq,\ |q| < 2\pi.$$

$$\therefore \int_0^\infty \frac{4\sin(xt)\cos(2ant)}{e^{2\pi t}-1}\,dt = \int_0^\infty \frac{2\sin(2an+x)t}{e^{2\pi t}-1}\,dt - \int_0^\infty \frac{2\sin(2an-x)t}{e^{2\pi t}-1}\,dt$$

$$= \left[\frac{1}{e^{2an+x}-1} - \frac{1}{2an+x}+\frac{1}{2}\right] - \left[\frac{1}{e^{2an-x}-1} - \frac{1}{2an-x}+\frac{1}{2}\right]$$

where the imaginary part of $(2an \pm x) < 2\pi$ in absolute value.

$$\therefore \int_0^\infty \frac{4\sin(xt)\sum_1^n \cos(2ant)\,dt}{e^{2\pi t}-1}$$

$$= \sum_1^n \Big[\{ e^{-(2an+x)} + e^{-2(2an+x)} + e^{-3(2an+x)} + \ldots\ldots \}$$

$$- \{ e^{-(2an-x)} + e^{-2(2an-x)} + e^{-3(2an-x)} + \ldots\ldots \}$$

$$+ \frac{2x}{4a^2n^2-x^2}\Big],$$

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* Read before the Eleventh Session of the Indian Science Congress held at Bangalore in January 1924.

† In this connection, it may be mentioned that the authors have examined the first 45 Volumes of the *Messenger of Mathematics*, the *Fundamenta Nova* of Jacobi, the Treatises on *Elliptic Functions* by Cayley, Greenhill and Hancock, and a few stray issues of the *Quarterly Journal of Mathematics*.

‡ *Vide* Bromwich: *Introduction to Infinite Series*, p. 454, Ex. 1.

which, on expanding each term in the curvilinear brackets, writing in rows one underneath the other and adding column by column, becomes

$$= \left\{ e^{-(2a+x)} \cdot \frac{1-e^{-2na}}{1-e^{-2a}} + e^{-2(2a+x)} \cdot \frac{1-e^{-2n.2a}}{1-e^{-2.2a}} + \ldots\ldots \right\}$$

$$- \left\{ e^{-(2a-x)} \cdot \frac{1-e^{-2na}}{1-e^{-2a}} + e^{-2(2a-x)} \cdot \frac{1-e^{-2n.2a}}{1-e^{-2.2a}} + \ldots\ldots \right\}$$

$$+ \sum_1^n \frac{2x}{4a^2n^2 - x^2}.$$

Now, $$\cot x = \frac{1}{x} + \sum_1^\infty \frac{2x}{x^2 - n^2\pi^2}.$$

$$\therefore \quad \sum_1^\infty \frac{2x}{4a^2n^2 - x^2} = \frac{1}{x} - \frac{\pi}{2a} \cot \frac{\pi x}{2a}.$$

In the result above, use this after letting n tend to infinity; then

$$\lim_{n\to\infty} \int_0^\infty \frac{4 \sin xt \sum_1^n \cos 2ant}{e^{2\pi t} - 1}\, dt$$

$$= \left\{ \frac{e^{-x}}{e^{2a}-1} + \frac{e^{-2x}}{e^{4a}-1} + \frac{e^{-3x}}{e^{6a}-1} + \ldots\ldots \right\}$$

$$- \left\{ \frac{e^{x}}{e^{2a}-1} + \frac{e^{2x}}{e^{4a}-1} + \frac{e^{3x}}{e^{6a}-1} + \ldots\ldots \right\}$$

$$+ \frac{1}{x} - \frac{\pi}{2a} \cot\left(\frac{\pi x}{2a}\right).$$

$$= -2 \left\{ \frac{\sinh x}{e^{2a}-1} + \frac{\sinh 2x}{e^{4a}-1} + \frac{\sinh 3x}{e^{6a}-1} + \ldots\ldots \right\}$$

$$+ \frac{1}{x} - \frac{\pi}{2a} \cot\left(\frac{\pi x}{2a}\right). \quad \ldots \quad \ldots \quad (a)$$

Now, $$\sum_1^n 2 \cos 2ant = \frac{\sin (2n+1)at}{\sin at} - 1.$$

$$\therefore \int_0^\infty \frac{4 \sin xt \sum_1^n \cos 2ant}{e^{2\pi t} - 1}\, dt = \int_0^\infty \frac{\sin (2n+1)at}{\sin at} \cdot \frac{2 \sin xt}{e^{2\pi t} - 1}\, dt$$

$$- \left(\frac{1}{e^x - 1} - \frac{1}{x} + \frac{1}{2} \right).$$

Here let n tend to infinity. In the integral on the right-hand side put $at = z$ and use the well-known result :*

$$\lim_{n \to \infty} \int_0^\infty \frac{\sin(2n+1)z}{\sin z} f(z)\, dz = \frac{\pi}{2}\left[f(0) + 2f(\pi) + 2f(2\pi) + \ldots\ldots\right].$$

We have

$$\lim_{n \to \infty} \int_0^\infty \frac{4 \sin xt \sum_1^n \cos 2ant}{e^{2\pi t} - 1}\, dt$$

$$= \frac{\pi}{2a}\left\{\frac{x}{\pi} + \frac{4\sin\left(\frac{x\pi}{a}\right)}{e^{2\pi^2/a} - 1} + \frac{4\sin\left(\frac{2x\pi}{a}\right)}{e^{4\pi^2/a} - 1} + \ldots\ldots\right\}$$

$$- \left(\frac{1}{e^x - 1} - \frac{1}{x} + \frac{1}{2}\right). \qquad \ldots (b)$$

§ 2. From (*a*) and (*b*), we get

$$\left\{\frac{\sinh x}{e^{2a} - 1} + \frac{\sinh 2x}{e^{4a} - 1} + \frac{\sinh 3x}{e^{6a} - 1} + \ldots\ldots\right\}$$

$$+ \frac{\pi}{a}\left\{\frac{\sin\left(\frac{\pi x}{a}\right)}{e^{2\pi^2/a} - 1} + \frac{\sin\left(\frac{2\pi x}{a}\right)}{e^{4\pi^2/a} - 1} + \ldots\ldots\right\}$$

$$= \frac{1}{2}\left\{\coth\frac{1}{2}\left(\frac{x}{2}\right) - \frac{\pi}{2a}\cot\left(\frac{\pi x}{2a}\right) - \frac{x}{2a}\right\}. \qquad (c)$$

Here, equate the co-efficients of x^{2n+1} on both sides. Then,
(i) when $n \geqslant 1$,

$$\left\{\frac{1^{2n+1}}{e^{2a} - 1} + \frac{2^{2n+1}}{e^{4a} - 1} + \frac{3^{2n+1}}{e^{6a} - 1} + \ldots\ldots\right\}$$

$$+ (-1)^n\left(\frac{\pi}{a}\right)^{2n+2}\left\{\frac{1^{2n+1}}{e^{\frac{2\pi^2}{a}} - 1} + \frac{2^{2n+1}}{e^{\frac{4\pi^2}{a}} - 1} + \frac{3^{2n+1}}{e^{\frac{6\pi^2}{a}} - 1} + \ldots\ldots\right\}$$

$$= \frac{B_{n+1}}{4(n+1)} \cdot \left\{\left(\frac{\pi}{a}\right)^{2n+2} + (-1)^n\right\}. \qquad \ldots (1)$$

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* *Vide* Bromwich : *Introduction to Infinite Series*: p. 447, Ex. 4.

and, (ii) when $n = 0$,

$$\left\{\frac{1}{e^{2a}-1}+\frac{2}{e^{4a}-1}+\frac{3}{e^{6a}-1}+\ldots\ldots\right\}$$

$$+\left(\frac{\pi}{a}\right)^2\left\{\frac{1}{e^{\frac{2\pi^2}{a}}-1}+\frac{2}{e^{\frac{4\pi^2}{a}}-1}+\frac{3}{e^{\frac{6\pi^2}{a}}-1}+\ldots\ldots\right\}$$

$$=\frac{1}{24}\left(\frac{\pi^2}{a^2}+1\right)-\frac{1}{4a}. \qquad \ldots \qquad \ldots (1.1)$$

In (1), let n be an even integer equal to $2m$ (say) and $a = \pi$.* Then,

$$\frac{1^{4m+1}}{e^{2\pi}-1}+\frac{2^{4m+1}}{e^{4\pi}-1}+\frac{3^{4m+1}}{e^{6\pi}-1}+\ldots\ldots=\frac{B_{2m+1}}{4(2m+1)}. \quad \ldots \quad (2)$$

Similarly, in (1.1), put $a = \pi$, we have

$$\frac{1}{e^{2\pi}-1}+\frac{2}{e^{4\pi}-1}+\frac{3}{e^{6\pi}-1}+\ldots\ldots=\frac{1}{24}-\frac{1}{8\pi}. \qquad \ldots (3)$$

a problem proposed by the late S. Ramanujan, F.R.S.†

§ 3. Now, in (1), change a into $\frac{a}{2}$. Then,

$$\left\{\frac{1^{2n+1}}{e^{a}-1}+\frac{2^{2n+1}}{e^{2a}-1}+\frac{3^{2n+1}}{e^{3a}-1}+\ldots\ldots\right\}$$

$$+(-1)^n\left(\frac{\pi}{a}\right)^{2n+2}2^{2n+2}\left\{\frac{1^{2n+1}}{e^{\frac{4\pi^2}{a}}-1}+\frac{2^{2n+1}}{e^{\frac{8\pi^2}{a}}-1}+\frac{3^{2n+1}}{e^{\frac{12\pi^2}{a}}-1}+\ldots\ldots\right\}$$

$$=\frac{B_{n+1}}{4(n+1)}\cdot\left\{\left(\frac{\pi}{a}\right)^{2n+2}.\,2^{2n+2}+(-1)^n\right\}. \quad \ldots \quad (d)$$

Subtract twice (1) from (d). Then

$$\left\{\frac{1^{2n+1}}{e^{a}+1}+\frac{2^{2n+1}}{e^{2a}+1}+\frac{3^{2n+1}}{e^{3a}+1}+\ldots\ldots\right\}$$

$$+(-1)^{n+1}2\left(\frac{\pi}{a}\right)^{2n+2}\left\{\frac{1^{2n+1}}{e^{\frac{2\pi^2}{a}}-1}+\frac{3^{2n+1}}{e^{\frac{6\pi^2}{a}}-1}+\frac{5^{2n+1}}{e^{\frac{10\pi^2}{a}}-1}+\ldots\ldots\right\}$$

$$=\frac{B_{n+1}}{4(n+1)}\left\{\left(\frac{\pi}{a}\right)^{2n+2}.\,2.(2^{2n+1}-1)+(-1)^{n+1}\right\} \ldots \quad (e)$$

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* *Vide* one of the results in Art. 94, p. 81, Vol. 18 of the *Messenger of Mathematics* (1889).

† *Vide* J.I.M.S., Vol. IV, Question No. 387.

Again, subtract 2^{2n+1} times (1) from (d); we obtain

$$\left\{\frac{1^{2n+1}}{e^{a}-1}+\frac{3^{2n+1}}{e^{3a}-1}+\frac{5^{2n+1}}{e^{5a}-1}+\ldots\ldots\right\}$$

$$+(-1)^{n+1}\left(\frac{\pi}{a}\right)^{2n+2}\cdot\left\{\frac{2^{2n+1}}{e^{\frac{2\pi^2}{a}}+1}+\frac{4^{2n+1}}{e^{\frac{4\pi^2}{a}}+1}+\frac{6^{2n+1}}{e^{\frac{6\pi^2}{a}}+1}+\ldots\ldots\right\}$$

$$=\frac{B_{n+1}}{4(n+1)}\left\{\left(\frac{\pi}{a}\right)^{2n+2}.\,2^{2n+1}+(-1)^{n+1}.\,(2^{2n+1}-1)\right\} \quad \ldots \ (f)$$

In (f), change a into $\frac{\pi^2}{a}$. We get

$$\left\{\frac{2^{2n+1}}{e^{2a}+1}+\frac{4^{2n+1}}{e^{4a}+1}+\frac{6^{2n+1}}{e^{6a}+1}+\ldots\ldots\right\}$$

$$=(-1)^{n}\left(\frac{\pi}{a}\right)^{2n+2}\left\{\frac{1^{2n+1}}{e^{\frac{\pi^2}{a}}-1}+\frac{3^{2n+1}}{e^{\frac{3\pi^2}{a}}-1}+\frac{5^{2n+1}}{e^{\frac{5\pi^2}{a}}-1}+\ldots\ldots\right\}$$

$$+(-1)^{n+1}\frac{B_{n+1}}{4(n+1)}\left\{2^{2n+1}+(-1)^{n+1}\left(\frac{\pi}{a}\right)^{2n+2}(2^{2n+1}-1)\right\}. \quad (g)$$

From (g), (f) and (e), we finally arrive at

$$\left\{\frac{1^{2n+1}}{e^{a}+1}+\frac{3^{2n+1}}{e^{3a}+1}+\frac{5^{2n+1}}{e^{5a}+1}+\ldots\ldots\right\}$$

$$+(-1)^{n}\left(\frac{\pi}{a}\right)^{2n+2}\left\{\frac{1^{2n+1}}{e^{\frac{\pi^2}{a}}+1}+\frac{3^{2n+1}}{e^{\frac{3\pi^2}{a}}+1}+\frac{5^{2n+1}}{e^{\frac{5\pi^2}{a}}+1}+\ldots\right\}$$

$$=\frac{B_{n+1}}{4(n+1)}(2^{2n+1}-1)\left\{\left(\frac{\pi}{a}\right)^{2n+2}+(-1)^{n}\right\}, \ldots \ (4)$$

a result true for $n \geq 1$ and also for $n = 0$, as starting with (1.1), we will arrive only at the same result.

Here, put $n = 2m$ and $a = \pi$. Then,

$$\frac{1^{4m+1}}{e^{\pi}+1}+\frac{3^{4m+1}}{e^{3\pi}+1}+\frac{5^{4m+1}}{e^{5\pi}+1}+\ldots\ldots$$

$$=\frac{B_{2m+1}}{4(2m+1)}\cdot(2^{4m+1}-1). \quad \ldots \ (5)$$

§ 4. Now, take result (1), change a into πa therein and differentiate with respect to a. We get

$$-\frac{\pi}{2}\left\{\operatorname{cosech}^2(\pi a)+2^{2n+2}\operatorname{cosech}^2(2\pi a)+\ldots\ldots\right\}$$

$$+(-1)^n\frac{\pi}{2}\frac{1}{a^{2n+4}}\left\{\operatorname{cosech}^2\left(\frac{\pi}{a}\right)+2^{2n+2}\operatorname{cosech}^2\left(\frac{2\pi}{a}\right)+\ldots\ldots\right\}$$

$$+(-1)^{n+1}\cdot\frac{2n+2}{a^{2n+3}}\cdot\left\{\frac{1^{2n+1}}{e^{\frac{2\pi}{a}}-1}+\frac{2^{2n+1}}{e^{\frac{4\pi}{a}}-1}+\frac{3^{2n+1}}{e^{\frac{6\pi}{a}}-1}+\ldots\ldots\right\}$$

$$=\frac{B_{n+1}}{4(n+1)}\cdot\frac{-(2n+2)}{a^{2n+3}}.$$

Here put $n=2m-1$, m being any positive integer, and $a=1$. Then

$$\pi\left\{\operatorname{cosech}^2(\pi)+2^{4m}\operatorname{cosech}^2(2\pi)+3^{4m}\operatorname{cosech}^2(3\pi)+\ldots\ldots\right\}$$

$$=4m\left\{\frac{B^{2m}}{8m}+\frac{1^{4m-1}}{e^{2\pi}-1}+\frac{2^{4m-1}}{e^{4\pi}-1}+\frac{3^{4m-1}}{e^{6\pi}-1}+\ldots\ldots\right\},$$

a result due to Ramanujan* ... ... (6)

In the same way, take result (4), change a into πa and differentiate with respect to a. We obtain

$$-\frac{\pi}{4}\left\{\operatorname{sech}^2\left(\frac{\pi a}{2}\right)+3^{2n+2}\operatorname{sech}^2\left(\frac{3\pi a}{2}\right)+\ldots\ldots\right\}$$

$$+(-1)^n\cdot\frac{\pi}{4}\cdot\frac{1}{a^{2n+4}}\left\{\operatorname{sech}^2\left(\frac{\pi}{2a}\right)+3^{2n+2}\operatorname{sech}^2\left(\frac{3\pi}{2a}\right)+\ldots\ldots\right\}$$

$$+(-1)^{n+1}\cdot\frac{2n+2}{a^{2n+3}}\cdot\left\{\frac{1}{e^{\frac{\pi}{a}}+}+\frac{3^{2n+1}}{e^{\frac{3\pi}{a}}+1}+\frac{5^{2n+1}}{e^{\frac{5\pi}{a}}+1}+\ldots\ldots\right\}$$

$$=\frac{B_{n+1}}{4(n+1)}(2^{2n+1}-1)\cdot\frac{-(2n+2)}{a^{2n+3}}.$$

Here, let $n=2m-1$, m being any positive integer and put $a=1$. Then,

$$\frac{\pi}{2}\left\{\operatorname{sech}^2\left(\frac{\pi}{2}\right)+3^{4m}\operatorname{sech}^2\left(\frac{3\pi}{2}\right)+5^{4m}\operatorname{sech}^2\left(\frac{5\pi}{2}\right)+\ldots\ldots\right\}$$

$$=4m\left\{\frac{B_{2m}(2^{4m-1}-1)}{8m}+\frac{1^{4m-1}}{e^{\pi}+1}+\frac{3^{4m-1}}{e^{3\pi}+1}+\ldots\ldots\right\}$$

a result similar to (6). ... ... (7)

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* *Vide* J. I. M. S., Vol. XIV, No. 3, page 89, result 8.

§ 5. In the result (4), put $n = 0$. Then,

$$\left\{\frac{1}{e^{a}+1}+\frac{3}{e^{3a}+1}+\frac{5}{e^{5a}+1}+\ldots\ldots\right\}$$
$$+\left(\frac{\pi}{a}\right)^{2}.\left\{\frac{1}{e^{\frac{\pi^2}{a}}+1}+\frac{3}{e^{\frac{3\pi^2}{a}}+1}+\frac{5}{e^{\frac{5\pi^2}{a}}+1}+\ldots\ldots\right\}$$
$$=\frac{1}{24}\left(\frac{\pi^2}{a^2}+1\right). \quad \ldots \qquad \ldots \ (h)$$

Integrate both sides with respect to a. Then,

$$[-\log(1+e^{-a})-\log(1+e^{-3a})-\log(1+e^{-5a})-\ldots\ldots]$$
$$+\left[\log\left(1+e^{-\frac{\pi^2}{a}}\right)+\log\left(1+e^{-\frac{3\pi^2}{a}}\right)+\ldots\ldots\right]$$
$$=-\frac{1}{24}\left(\frac{\pi^2}{a}-a\right)+\text{a constant.}$$

On removing the logarithms, changing a into πa and inverting the fraction,

$$\frac{(1+e^{-\pi a})\,(1+e^{-3\pi a})\,(1+e^{-5\pi a})\ \ldots}{\left(1+e^{-\frac{\pi}{a}}\right)\left(1+e^{-\frac{3\pi}{a}}\right)\left(1+e^{-\frac{5\pi}{a}}\right)\ \ldots}$$
$$=Ae^{\frac{\pi}{24}\left(\frac{1}{a}-a\right)}=e^{-\frac{\pi}{24}\left(a-\frac{1}{a}\right)},$$

A being equal to 1 as is easily seen by putting $a = 1$.

Similarly, by taking result (1.1), integrating both sides with respect to a and proceeding as before, the corresponding result is

$$\frac{(1-e^{-2\pi a})(1-e^{-4\pi a})(1-e)^{-6\pi a})\ \ldots}{\left(1-e^{-\frac{2\pi}{a}}\right)\left(1-e^{-\frac{4\pi}{a}}\right)\left(1-e^{-\frac{6\pi}{a}}\right)\ \ldots}=a^{-\frac{1}{2}}.\,e^{\frac{\pi}{12}\left(a-\frac{1}{a}\right)}. \ \ldots \quad (9)$$

In this, put $a = \frac{1}{\sqrt{2}}$. Then,

$$(1-e^{-\pi\sqrt{2}})\,(1-e^{-3\pi\sqrt{2}})\,(1-e^{-5\sqrt{2}})\ldots\ldots=2^{\frac{1}{4}}\,e^{-\frac{\pi\sqrt{2}}{24}}. \qquad \ldots \ (10)$$

§ 6. In (9), put $a = r\,(\cos\theta + i\,\sin\theta)$ and take the logarithms of both sides. Then,

$$\log\left[\frac{(A+iB)\ \times\ \ldots\ldots}{(A'-iB')\ \times\ \ldots\ldots}\right]$$

$$=\frac{\pi}{12}\left\{r(\cos\theta+i\sin\theta)-\frac{1}{r}(\cos\theta-i\sin\theta)\right\}-\tfrac{1}{2}\log(re^{i\theta})\ldots\quad\text{(i)}$$

where

$$A\equiv 1+e^{-2\pi r\cos\theta}.\cos(2\pi r\sin\theta);\ B\equiv e^{-2\pi r\cos\theta}.\sin(2\pi r\sin\theta);$$

$$A'\equiv 1-e^{-\frac{2\pi}{r}\cos\theta}.\cos\left(\frac{2\pi}{r}\sin\theta\right);\ B'\equiv e^{-\frac{2\pi}{r}\cos\theta}.\sin\left(\frac{2\pi}{r}\sin\theta\right).$$

Separate this into real and imaginary parts and equate the real parts.

$$\tfrac{1}{2}[\log\ \{1-2e^{-2\pi r\cos\theta}\cos(2\pi r\sin\theta)+e^{-4\pi r\cos\theta}\}$$
$$+\log\ \{1-2e^{-4\pi r\cos\theta}\cos(4\pi r\sin\theta)+e^{-8\pi r\cos\theta}\}$$
$$+\ \ldots\ldots\ldots\ldots]$$

$$-\frac{1}{2}\left[\log\left\{1-2e^{-\frac{2\pi}{r}\cos\theta}\cos\left(\frac{2\pi}{r}\sin\theta\right)+e^{-\frac{4\pi}{r}\cos\theta}\right\}\right.$$
$$+\log\left\{1-2e^{-\frac{4\pi}{r}\cos\theta}\cos\left(\frac{4\pi}{r}\sin\theta\right)+e^{-\frac{8\pi}{r}\cos\theta}\right\}$$
$$\left.+\ \ldots\ \ldots\ \right]$$

$$=\frac{\pi\cos\theta}{12}\left(r-\frac{1}{r}\right)-\frac{1}{2}\log r.\ \ldots\ \ldots\ (j)$$

Here put $r=\sqrt{2}$, cancel the terms of the first part of the left-hand side with the even terms of the second part. We get

$$-\tfrac{1}{2}\ [\log\ \{1-2e^{-\pi\sqrt{2}\cos\theta}\cos(\pi\sqrt{2}\sin\theta)+e^{-2\pi\sqrt{2}\cos\theta}\}$$
$$+\log\ \{1-2e^{-3\pi\sqrt{2}\cos\theta}\cos(3\pi\sqrt{2}\sin\theta)+e^{-6\pi\sqrt{2}\cos\theta}\}$$
$$+\ \ldots\ldots\ldots]\ =\frac{\pi\cos\theta}{12\sqrt{2}}-\frac{1}{4}\log 2\ \ldots\ (k)$$

Here put $\sin\theta=\frac{1}{\sqrt{2}}=\cos\theta$. Then, we have

$$-\tfrac{1}{2}[\log(1+2e^{-\pi}+e^{-2\pi})+\log(1+2e^{-3\pi}+e^{-6\pi})+\ \ldots\ldots]$$
$$=\frac{\pi}{24}-\frac{1}{4}\log 2.$$

Or, $(1+e^{-\pi})(1+e^{-3\pi})(1+e^{-5\pi})\ldots\ldots=2^{\frac{1}{4}}e^{-\frac{\pi}{24}}$ * ... (11)

* Compare result 28, p. 178, Vol. 5 of the *Messenger of Mathematics* (1876).

In (k), put $\sin\theta = \dfrac{1}{2\sqrt{2}}$ and $\cos\theta = \dfrac{\sqrt{7}}{2\sqrt{2}}$. Then,

$$(1+e^{-\pi\sqrt{7}})(1+e^{-3\pi\sqrt{7}})(1+e^{-5\pi\sqrt{7}})\ldots\ldots = 2^{\frac{1}{2}}e^{-\frac{\pi\sqrt{7}}{24}}. \qquad (12)$$

Now, in (8), put $a = \sqrt{7}$ and substitute from (12); then

$$(1+e^{-\frac{\pi}{\sqrt{7}}})(1+e^{-\frac{3\pi}{\sqrt{7}}})(1+e^{-\frac{5\pi}{\sqrt{7}}})\ldots\ldots = 2^{\frac{1}{2}}e^{-\frac{\pi}{24\sqrt{7}}}. \quad \ldots \quad (13)$$

Again, in (9) change a into $\frac{a}{2}$; divide (9) by the result so got, we get

$$(1+e^{-\pi a})(1+e^{-2\pi a})(1+e^{-3\pi a})\ldots\ldots$$

$$\times(1+e^{-\frac{2\pi}{a}})(1+e^{-\frac{4\pi}{a}})(1+e^{-\frac{6\pi}{a}})\ldots\ldots = 2^{-\frac{1}{2}}e^{\frac{\pi}{24}\left(a+\frac{2}{a}\right)}. \quad (l)$$

In this, put $a = \sqrt{2}$ and take the square root,

$$(1+e^{-\pi\sqrt{2}})(1+e^{-2\pi\sqrt{2}})(1+e^{-3\pi\sqrt{2}})\ldots\ldots = 2^{-\frac{1}{4}}e^{\frac{\pi\sqrt{2}}{24}}. \qquad (14)$$

Again in (l), put $a = 1$. Then,

$$(1+e^{-\pi})(1+e^{-2\pi})^2(1+e^{-3\pi})(1+e^{-4\pi})^2\ldots\ldots = 2^{-\frac{1}{2}}e^{\frac{\pi}{8}}.$$

But, in result (10), we are given the value of $(1+e^{-\pi})(1+e^{-3\pi})\ldots\ldots$. Hence, using it here,

$$(1+e^{-2\pi})(1+e^{-4\pi})(1+e^{-6\pi})\ldots\ldots = 2^{-\frac{3}{8}}e^{\frac{\pi}{12}}. \quad \ldots \quad (15)$$

It may be noticed, in passing, that, on combining results (11) and (15),

$$(1+e^{-\pi})(1+e^{-2\pi})(1+e^{-3\pi})(1+e^{-4\pi})\ldots\ldots = 2^{-\frac{1}{8}}e^{\frac{\pi}{24}}.$$

Now, in result (l), put $a = r(\cos\theta + i\sin\theta)$, take the logarithms of both sides, separate into real and imaginary parts and equate the real parts. We get

$$\tfrac{1}{2}\left[\log\{1+2e^{-\pi r\cos\theta}.\cos(\pi r\sin\theta)+e^{-2\pi r\cos\theta}\}\right.$$

$$+\log\{1+2e^{-2\pi r\cos\theta}\cos(2\pi r\sin\theta)+e^{-4\pi r\cos\theta}\}$$

$$+\ldots\ldots\ldots\ldots\ldots\ldots$$

$$+\log\{1+2e^{-\frac{2\pi}{r}\cos\theta}\cos\left(\frac{2\pi}{r}\sin\theta\right)+e^{-\frac{4\pi}{r}\cos\theta}\}$$

$$+\log\{1+2e^{-\frac{4\pi}{r}\cos\theta}\cos\left(\frac{4\pi}{r}\sin\theta\right)+e^{-\frac{8\pi}{r}\cos\theta}\}$$

$$\left.+\ldots\ldots\ldots\ldots\ldots\ldots\right]$$

$$= -\frac{1}{2}\log 2+\frac{\pi\cos\theta}{24}\left(r+\frac{2}{r}\right). \quad \ldots \quad (m)$$

In this, put $r = 1$, $\sin\theta = \frac{1}{2}$. Then,

$$\tfrac{1}{2}\log(1+e^{-\pi\sqrt{3}})(1-e^{-\pi\sqrt{3}})^2(1+e^{-3\pi\sqrt{3}})(1+e^{-2\pi\sqrt{3}})^2\ldots$$
$$+\tfrac{1}{2}\log(1-e^{-\pi\sqrt{3}})^2(1+e^{2-\pi\sqrt{3}})^2(1-e^{-3\pi\sqrt{3}})^2\ldots\ldots$$
$$= -\tfrac{1}{2}\log 2 + \frac{\pi\sqrt{3}}{16}.$$

i.e.,
$$(1+e^{-\pi\sqrt{3}})(1-e^{-\pi\sqrt{3}})^4(1+e^{-3\pi\sqrt{3}})(1+e^{-2\pi\sqrt{3}})^4\ldots\ldots = 2^{-1}e^{\frac{\pi\sqrt{3}}{8}}.$$

On putting $e^{-\pi\sqrt{3}} = q$, we have

$$(1+q)(1-q)^4(1+q^3)(1+q^2)^4(1+q^5)(1-q^3)^4\ldots\ldots = 2^{-1}q^{-\frac{1}{8}}.$$

But it is well-known that

$$\prod_1^\infty(1-q^{2n-1})\prod_1^\infty(1+q^{2n})\prod_1^\infty(1+q^{2n-1}) = 1.$$

Using this, we get

$$\frac{1}{(1+q)^3(1+q^3)^3(1+q^5)^3\ldots\ldots} = 2^{-1}q^{-\frac{1}{8}}.$$

$$\therefore\ (1+e^{-\pi\sqrt{3}})(1+e^{-3\pi\sqrt{3}})(1+e^{-5\pi\sqrt{3}})\ldots\ldots = 2^{\frac{1}{3}}e^{-\frac{\pi\sqrt{3}}{24}}.\text{*} \quad (16)$$

Putting $a = \frac{1}{\sqrt{3}}$ in (8), we get

$$(1+e^{-\frac{\pi}{\sqrt{3}}})(1+e^{-\frac{3\pi}{\sqrt{3}}})(1+e^{-\frac{5\pi}{\sqrt{3}}})\ldots\ldots = 2^{\frac{1}{3}}e^{-\frac{\pi}{24\sqrt{3}}}.\dagger \quad \ldots \quad (17)$$

Now, following the notation in Bromwich's *Infinite Series*, pp. 105—107, we may use the equations

$$Q_1Q_2Q_3 = 1 \text{ and } Q_2^8 = Q_3^8 + 16qQ_1^8$$

and derive the values of infinite products related to those got above. A small table of such values is given as an Appendix to this part of the paper.

§ 7. Take result (1) and in it, put $a = \pi r(\cos\theta + i\sin\theta)$. Then, on reducing, splitting into real and imaginary parts, and equating the real parts,

$$\left[\frac{e^{2\pi r\cos\theta}.\cos(2\pi r\sin\theta)-1}{e^{4\pi r\cos\theta}-2e^{2\pi r\cos\theta}\cos(2\pi r\sin\theta)+1}\right.$$
$$\left.+2^{2n+1}\cdot\frac{e^{4\pi r\cos\theta}\cos(4\pi r\sin\theta)-1}{e^{8\pi r\cos\theta}-2e^{4\pi r\cos\theta}\cos(4\pi r\sin\theta)+1}+\ldots\ldots\right]$$

* *Vide*: Result No. 26, p. 178, Vol. 5, *Messenger of Mathematics* (1876).

† " " 27, " " "

$$+(-1)^n.\frac{\cos(2n+2)\theta}{r^{2n+2}}\left\{\frac{e^{\frac{2\pi}{r}\cos\theta}\cos\left(\frac{2\pi}{r}\sin\theta\right)-1}{e^{\frac{4\pi}{r}\cos\theta}-2e^{\frac{2\pi}{r}\cos\theta}\cos\left(\frac{2\pi}{r}\sin\theta\right)+1}\right.$$

$$\left.+2^{2n+1}\frac{e^{\frac{4\pi}{r}\cos\theta}\cos\left(\frac{4\pi}{r}\sin\theta\right)-1}{e^{\frac{8\pi}{r}\cos\theta}-2e^{\frac{4\pi}{r}\cos\theta}\cos\left(\frac{4\pi}{r}\sin\theta\right)+1}+\ldots\ldots\right\}$$

$$+(-1)^n.\frac{\sin(2n+2)\theta}{r^{2n+2}}\left\{\frac{e^{\frac{2\pi}{r}\cos\theta}\sin\left(\frac{2\pi}{r}\sin\theta\right)}{e^{\frac{4\pi}{r}\cos\theta}-2e^{\frac{2\pi}{r}\cos\theta}\cos\left(\frac{2\pi}{r}\sin\theta\right)+1}\right.$$

$$\left.+2^{2n+1}\frac{e^{\frac{4\pi}{r}\cos\theta}\sin\left(\frac{4\pi}{r}\sin\theta\right)}{e^{\frac{8\pi}{r}\cos\theta}-2e^{\frac{4\pi}{r}\cos\theta}\cos\left(\frac{4\pi}{r}\sin\theta\right)+1}+\ldots\ldots\right\}$$

$$=\frac{B_{n+1}}{4(n+1)}.\left\{\frac{\cos(2n+2)\theta}{r^{2n+2}}+(-1)^n\right\}. \qquad \ldots (n)$$

Here, put $r=1$, $\sin\theta=\frac{1}{2}$. We get

$$\left\{\frac{-e^{\pi\sqrt{3}}-1}{(e^{\pi\sqrt{3}}+1)^2}+2^{2n+1}.\frac{e^{2\pi\sqrt{3}}-1}{(e^{2\pi\sqrt{3}}-1)^2}+3^{2n+1}.\frac{-e^{3\pi\sqrt{3}}-1}{(e^{3\pi\sqrt{3}}+1)^2}+\ldots\ldots\right\}$$

$$\times\left\{1+(-1)^n\cos(n+1)\frac{\pi}{3}\right\}$$

$$=\frac{B_{n+1}}{4(n+1)}\left\{\cos(n+1)\frac{\pi}{3}+(-1)^n\right\}. \ldots (o)$$

Here, the common factor $\left[\cos(n+1)\frac{\pi}{3}.+(-1)^n\right]$ can be removed from both sides, whenever n is not of the form $3m-1$.

Hence, when $n=0$ or 1 (modulus 3),

$$\frac{1^{2n+1}}{e^{\pi\sqrt{3}}+1}-\frac{2^{2n+1}}{e^{2\pi\sqrt{3}}-1}+\frac{3^{2n+1}}{e^{3\pi\sqrt{3}}+1}-\frac{4^{2n+1}}{e^{4\pi\sqrt{3}}-1}+\ldots\ldots$$

$$=(-1)^{n+1}\frac{B_{n+1}}{4(n+1)}\ldots \qquad (18)$$

Proceeding in a similar manner with result (1.1) and making the same substitutions for r and θ, we obtain

$$\frac{1}{e^{\pi\sqrt{3}}+1}-\frac{2}{e^{2\pi\sqrt{3}}-1}+\frac{3}{e^{3\pi\sqrt{3}}+1}-\frac{4}{e^{4\pi\sqrt{3}}-1}+\dots\dots\dots$$
$$=\frac{\sqrt{3}}{12\pi}-\frac{1}{24}. \quad \dots (19)$$

Again, in (o), put $r = \sqrt{2}.$, $\sin\theta = \frac{1}{\sqrt{2}}$ and $\cos\theta = \frac{1}{\sqrt{2}}$. Then,

$$\left[\frac{(e^{2\pi}-1)}{(e^{2\pi}-1)^2}+2^{2n+1}.\frac{(e^{4\pi}-1)}{(e^{4\pi}-1)^2}+3^{2n+1}.\frac{(e^{6\pi}-1)}{(e^{6\pi}-1)^2}+\dots\dots\dots\right]$$
$$+\frac{1}{2^{n+1}}\cdot(-1)^n\cos(n+1)\frac{\pi}{2}\cdot\left[\frac{(-e^{\pi}-1)}{(e^{\pi}+1)^2}+\frac{(e^{2\pi}-1)}{(e^{2\pi}-1)^2}\cdot 2^{2n+1}\right.$$
$$\left.+\frac{(-e^{3\pi}-1)}{(e^{3\pi}+1)^2}\cdot 3^{2n+1}+\dots\dots\dots\right]$$
$$=\frac{B_{n+1}}{4\,(n+1)^n}\left\{\frac{\cos(n+1)\frac{\pi}{2}}{2^{n+1}}+(-1)^n\right\}. \quad \dots (p)$$

Here, let $n = 1$. Then,

$$\left(\frac{1^3}{e^{2\pi}-1}+\frac{2^3}{e^{4\pi}-1}+\frac{3^3}{e^{6\pi}-1}+\dots\dots\right)$$
$$+\tfrac{1}{4}\left(\frac{-1^3}{e^{\pi}+1}+\frac{2^3}{e^{2\pi}-1}+\frac{-3^3}{e^{3\pi}+1}+\dots\dots\dots\right)$$
$$=\frac{B_2}{4\,.\,2}\cdot(-\tfrac{1}{4}+1).$$

$$\therefore\ 3\left(\frac{1^3}{e^{2\pi}-1}+\frac{2^3}{e^{4\pi}-1}+\frac{3^3}{2^{6\pi}-1}+\dots\dots\dots\right)$$
$$-\tfrac{1}{4}\left(\frac{1}{e^{\pi}+1}+\frac{3^3}{e^{3\pi}+1}+\frac{5^3}{e^{5\pi}+1}+\dots\dots\right)$$
$$=-\frac{1}{48\times 4}.$$

or,
$$\frac{1}{e^{\pi}+1}+\frac{3^3}{e^{3\pi}+1}+\frac{5^3}{e^{5\pi}+1}+\dots\dots\dots$$
$$=\frac{1}{48}+12\left\{\frac{1}{e^{2\pi}-1}+\frac{2^3}{e^{4\pi}-1}+\frac{3^3}{e^{6\pi}-1}+\dots\dots\right\}\dots \quad (20)$$

§ 8. Equating the imaginary parts in the various equations used above we obtain several neat series; they are dealt with in the next part.

APPENDIX.

Values of q.	Corresponding values of		
	$Q_1 = \prod_1^\infty (1 + q^{2n})$	$Q_2 \equiv \prod_1^\infty (1 + q^{2n-1})$	$Q_3 \equiv \prod_1^\infty (1 - q^{2n-1})$
$e^{-\pi}$	$2^{-\frac{3}{8}} e^{\frac{\pi}{12}}$	$2^{\frac{1}{4}} e^{-\frac{\pi}{24}}$	$2^{\frac{1}{8}} e^{-\frac{\pi}{24}}$
$e^{-\pi\sqrt{2}}$	$(\sqrt{2}+1)^{-\frac{1}{8}} . 2^{-\frac{3}{8}} . e^{\frac{\pi\sqrt{2}}{12}}$	$(\sqrt{2}+1)^{\frac{1}{8}} . 2^{\frac{1}{8}} . e^{-\frac{\pi\sqrt{2}}{24}}$	$2^{\frac{1}{4}} e^{-\frac{\pi\sqrt{2}}{24}}$
$e^{-\pi\sqrt{3}}$	$(\sqrt{3}+1)^{-\frac{1}{4}} . 2^{-\frac{7}{24}} . e^{\frac{\pi\sqrt{3}}{12}}$	$2^{\frac{1}{3}} . e^{-\frac{\pi\sqrt{3}}{24}}$	$(\sqrt{3}+1)^{\frac{1}{4}} . 2^{-\frac{1}{24}} . e^{-\frac{\pi\sqrt{3}}{24}}$
$e^{-\pi\sqrt{7}}$	$(3+\sqrt{7})^{-\frac{1}{4}} . 2^{-\frac{3}{8}} . e^{\frac{\pi\sqrt{7}}{12}}$	$2^{\frac{1}{2}} e^{-\frac{\pi\sqrt{7}}{24}}$	$(3+\sqrt{7})^{\frac{1}{4}} 2^{-\frac{1}{8}} e^{-\frac{\pi\sqrt{7}}{24}}$
$e^{-\frac{\pi}{\sqrt{2}}}$	$(\sqrt{2}+1)^{-\frac{1}{8}} . 2^{-\frac{3}{8}} . e^{\frac{\pi}{12\sqrt{2}}}$	$(\sqrt{2}+1)^{\frac{1}{8}} . 2^{\frac{1}{8}} . e^{-\frac{\pi}{24\sqrt{2}}}$	$2^{\frac{1}{4}} e^{-\frac{\pi}{24\sqrt{2}}}$
$e^{-\frac{\pi}{\sqrt{3}}}$	$(\sqrt{3}+1)^{-\frac{1}{4}} . 2^{-\frac{7}{24}} . e^{\frac{\pi}{12\sqrt{3}}}$	$2^{\frac{1}{3}} . e^{-\frac{\pi}{24\sqrt{3}}}$	$(\sqrt{3}+1)^{\frac{1}{4}} . 2^{-\frac{1}{24}} . e^{-\frac{\pi}{24\sqrt{3}}}$
$e^{-\frac{\pi}{\sqrt{7}}}$	$(3+\sqrt{7})^{-\frac{1}{4}} . 2^{-\frac{3}{8}} . e^{\frac{\pi}{12\sqrt{7}}}$	$2^{\frac{1}{2}} e^{-\frac{\pi}{24\sqrt{7}}}$	$(3+\sqrt{7})^{\frac{1}{4}} . 2^{-\frac{1}{8}} . e^{-\frac{\pi}{24\sqrt{7}}}$

On the Collinearity of three Points on a non-singular Cubic.*

By B. S. Madhavarao, M.Sc. and M. Lakshmanamurthi, M.A.

1. Introductory Remarks.

Every non-singular cubic can be reduced to the canonical form

$$x^3 + y^3 + z^3 + 6mxyz = 0;$$

and if three points (x_i, y_i, z_i) $(i = 1, 2, 3)$ on this cubic be collinear, we have the well-known condition due to Cayley†, *viz.*

$$x_1x_2x_3 + y_1y_2y_3 + z_1z_2z_3 = 0.$$

We have shown here that this condition is necessary but not sufficient.‡ We have further deduced other necessary conditions and we have been able to enunciate the following theorem :—

Theorem:— *When three points on a non-singular cubic in the canonical form are represented in homogeneous co-ordinates* (x, y, z), *there cannot be only one necessary and sufficient condition of collinearity which is a rational integral function of the co-ordinates and symmetrical in* x, y, z.

We have also exhibited sets of conditions necessary as well as sufficient and shown that they must be at least two in number.

2. The General Problem.

We first proceed to show by *a priori* reasoning that there must be a single such condition for collinearity. In fact, we can choose any three points on a cubic in a 3-fold infinity of ways whereas there are only a 2-fold infinity of straight lines in the plane. Thus the imposing of a single condition is necessary and moreover ought to be sufficient for the collinearity of the three points on the cubic. This fact about the existence of a

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* A paper read before the South Indian Science Association, Bangalore, on 30th Sep 1923.

† See Cayley: *Collected Mathematical Papers*, Vol. 2 ; p. 404.

‡ See Ganguli: *Higher Plane Curves* ; Vol 2. where it is stated that this condition is necessary as well as sufficient.

single condition has also been proved from other points of view. Representing the co-ordinates of any point on

$$x^3 + y^3 + z^3 + 6mxyz = 0$$

rationally in terms of the elliptic functions of a parameter u, Clebsch* has shown that, if u_1, u_2, u_3 be the parameters corresponding to the three points the *necessary* and *sufficient* condition for collinearity is

$$u_1 + u_2 + u_3 \equiv 0 \ (\text{mod. } 2\omega,\ 2\omega')$$

or

$$u_1 + u_2 + u_3 \equiv 0 \ (\text{mod. } 2k,\ 2k'i)$$

according as the Elliptic Functions are either Weierstrassian or Jacobian.

Let us consider the same question when the co-ordinates of the three points are simply taken as (x_1, y_1, z_1), (x_2, y_2, z_2) and (x_3, y_3, z_3). The general condition of collinearity for any three points in a plane is given by

$$\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} = 0$$

and the condition we are searching for is the form to which this reduces, when we make use of the facts that

$$x_1^3 + y_1^3 + z_1^3 + 6mx_1y_1z_1 = 0,$$
$$x_2^3 + y_2^3 + z_2^3 + 6mx_2y_2z_2 = 0,$$
$$x_3^3 + y_3^3 + z_3^3 + 6mx_3y_3z_3 = 0.$$

We can therefore solve for z_1, z_2, z_3 from each of the three equations above; substituting their values in the determinant, we obtain a relation among $(x_1, y_1, x_2, y_2, x_3, y_3)$ which is an algebraic function in them since the expression of z in terms of x and y from

$$x^3 + y^3 + z^3 + 6mxyz = 0$$

would involve only square root and cube root. We can thus deduce† a single necessary and sufficient condition even when the points are taken as (x_i, y_i, z_i); but it will not be a rational integral function of the co-ordinates and symmetrical in (x, y, z), and the actual labour involved in obtaining it is very great and, perhaps, not worth the trouble.

What we are concerned with in this paper is a condition which must involve x, y, z symmetrically and be also algebraic in them and the conclusion arrived at is embodied in the theorem of § 1.

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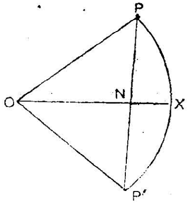
* See Clebsch: *Lecons sur la Géométrie*: t.2 (Fr. Edition).

† This point of view is pointed out to us by Prof. Hilton.

Notes and Questions.

Note on the Hindu Table of Sines.

The early notions of arc and sine are derived from archery where the full arc PXP′ resembles the bow (L. *arcus*, bow, चाप) and the full chord PNP′ the bow-string (L. *sinus*, bosom, शिंजिनी, जीव or ज्या). But in mathematics, it is only the upper halves PX and PN that admittedly have any purpose to serve[1]; and though our old writers first employ the



term ज्यार्ध, sine-half, etc., in order to fix the nature of the line on students' minds[2], they soon drop the qualifying word अर्ध half[3].

The old conventions did not, like the modern ones, treat sine as a fraction and radius as unity, but gave certain values to the

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[1] [अत्र गणिते ऊर्ध्वरेखातोऽर्धज्याया एव प्रयोजनात्—in this mathematical science it is only the upper half that has any purpose to serve (Ranganatha); and अर्ध ज्याभिः कर्म सर्वं ग्रहाणामर्धं ज्यैव ज्याभिधानात्त वेद्या –all calculations relating to the planets are made by taking half-sine, and therefore let it be understood that half sine (ज्यार्ध) is itself treated as sine (ज्या) in this science (Bhaskara-charya)]

[2] (पूर्वं तु स्वरुपो कत्यर्थं पिंडानां ज्यार्धं त्योक्तिः,—previously however their values were spoken of as sine-halves for showing their nature.

[3] इदानीं तेषामेवार्धत्यागेन ज्या पिंडत्वोक्तिः,—but now the very same values are spoken of as ज्या's, sines, without the word अर्ध half.

lines themselves. Further, the Hindus in their zeal to avoid fractions and decimals in their calculations gave sufficiently large values to the lines themselves so as to keep the results within the range of accuracy. They regarded the circumference of a circle as containing 360° or 21600′; and the ratio of the circumference of a circle to its diameter, π, was given by

भनन्दाग्नि मितपरिधौ, ख बाण सूर्यमितो व्यासः[4]

i.e. when the circumference measures 3927, the diameter measures 1250. (Bhāskarâchârya, cited with approval by Ranganâthâ). Thus

$$\frac{\text{circumference of the circle}}{\text{diameter of the circle}} = \pi = \frac{3927}{1250}$$

$$= \frac{3927 \times 8}{1250 \times 8} = \frac{31416}{10000}$$

$$= 3{\cdot}1416.$$

the modern figure itself correct to four places of decimals; the radius (त्रिज्या, so called because it is त्रिराशिज्या, or the sine of 3 × 30° = 90°) was treated as 3438′ (a radian).

To construct a table of 24 sines they divided the arc of a quadrant into 24 parts of 225′ each. Again, this choice of 24 is based upon a very important observation, familiar even from very early days as is borne out by a statement of Sakalya[5],

वृत्तस्य षण्णवत्यंशो दंडवद्दृश्यते तु सः ।

that is, "one-ninety-sixth part of a circle looks (straight) like a rod." Chamber's *Mathematical Tables* give

$$\sin 3^\circ 45' = {\cdot}0654031 = {\cdot}06540;$$

$$\text{arc } 3^\circ 45' = {\cdot}0654498 = {\cdot}06545;$$

and

$$\tan 3^\circ 45' = {\cdot}0655435 = {\cdot}06554,$$

correct to 5 places of decimals. Thus for an angle of 3° 45′ or 225′, the arc is in excess of the sine by 5, and the tangent is in excess of the arc by 9 in 100,000 and these differences may be considered as inappreciable. Certainly a set of Tables calculated at shorter intervals than 225′ would give more exact values and the Hindus who divided 1″ (vikalâ) into 60 paras, 1 para into 60 paratparas,

4 भ = 27, नंद = 9, अग्नि = 3, ख = 0, बाण = 5, सूर्य = 12.

5 Apparently the son of Sakala who first taught Rigveda in its present form,

and 1 paratpara into 60 tatparas, and also marked the division of time into as a unit as त्रटि ($= \frac{1}{33750}$ of a second) were certainly competent to realize this and employ a smaller unit, if they would, for the purpose. But then it would have been consistent to construct a longer table involving more figures too. Apparently this did not appeal to them as they had to trust to their memory as their ready guide, and they chose the present middle course and left the individual to make his own calculations employing as small a unit as he would.

The following rule for calculating the values of sines is given in the Surya-Siddhanta:—

राशिलिप्ताष्टमो भागः प्रथमं ज्यार्धमुच्यते ।
तत्तद्विभक्तलब्धोन मिश्रितं तद्द्वितीयकं ॥
आद्ये नैवं क्रमात्पिंडान् भक्त्वा लब्धोन संयुताः ।
खण्डकाः स्युश्चतुर्विंशज्ज्यार्धं पिंडाः क्रमादमी ॥

(II, 15—16)

This passage has to be carefully followed as it is likely to mislead; its purport, gathered in the light of the commentary, could be expressed thus: (1) one-eighth part of *rasi* (*i. e.*, $\frac{1}{8}$ of $30^\circ = 3^\circ 45' = 225'$) is the first sine; (2) divide it by itself, subtract the *quotient* from itself, add the remainder to itself, and the second sine is obtained; (now, the rule generalizes the process by induction, which is, practically, though not identically, the modern mathematical induction); (3) divide the successive sines already obtained by 225, subtract *the sum of all such quotients* from 225, add the remainder to the last of the sines already obtained, and the result gives the next sine.

In other words, let

S_r denote the r^{th} sine,

and Q_r „ the r^{th} quotient in the successive divisions stated above;

then the rule says,

$$S_1 = 225;$$
$$S_2 = S_1 + (225 - Q_1), \text{ etc.}$$

and generally

$$S_r = S_{r-1} + (225 - \overline{Q_1 + Q_2 + Q_3 + \ldots\ldots + Q_{r-1}})$$
$$= S_{r-1} + S_1 - \Sigma Q_{r-1}$$

$$\therefore \quad \mathbf{S}_r - \mathbf{S}_{r-1} = \mathbf{S}_1 - \Sigma\, \mathbf{Q}_{r-1},$$

which gives the excess of one sine over the one immediately preceding it : *i.e.*, if D_{r-1} denote this excess or the difference to be added to the $(r-1)^{th}$ sine to give the r^{th} sine, then

$$D_{r-1} = S_1 - \Sigma Q_{r-1},$$

and similarly $\quad D_r = S_1 - \Sigma Q_r\,;$

$$\therefore \quad D_{r-1} - D_r = \Sigma Q_r - \Sigma Q_{r-1}$$

i. e., $\quad \mathbf{D}_{r-1} - \mathbf{D}_r = \mathbf{Q}_r.$

Hence each successive difference is in defect of the difference immediately preceding it by Q_r.

Now giving different values to r,

$$Q_1 = \frac{S_1}{225} = \frac{225}{225} = 1\,;\ \therefore S_2 = S_1 + (225 - 1)$$
$$= 225 + 224 = \mathbf{449}.$$

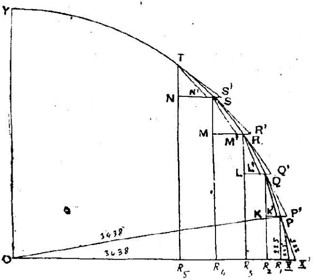
$$Q_2 = \frac{S_2}{225} = \frac{449}{225} = 2\,;\ \therefore S_3 = S_2 + (225 - \overline{1 + 2})$$
$$= 449 + 222 = \mathbf{671}.$$

$$Q_3 = \frac{S_3}{225} = \frac{671}{225} = 3\,;\ \therefore S_4 = S_3 + (225 - \overline{1 + 2 + 3})$$
$$= 671 + 219 = \mathbf{890}.$$

Thus Q_r becomes 1, 2, 3, etc., for different values of r in order and therefore ΣQ_r is the sum of certain natural numbers in a progressive series (which the Hindus called श्रेढी गणित) and it is possibly this that could have suggested to them the formulation of the above rule subject of course to the corrections to be applied.

Now as already considered, the sine, the arc and the tangent corresponding to 225′ present no appreciable differences and therefore if, in the following figure, OXY is a quadrant of a circle, XP, PQ, QR, etc., are divisions ot the circumference into arcs of 225′ each, PX, QP, RQ, etc., can be treated as tangents to the circle at P, Q, R, etc., and if prependiculars PR_1, QR_2, RR_3, etc., are drawn from P_1, Q_1,R_1, etc., to OX, then they are in order the first, second, third, etc., sines.

Now though the first sine PR_1 and the arc PX measure 225 each, they do not coincide and between them there is the



$$\text{versed sine } XR_1 = OX - OR_1 = OX - \sqrt{OP^2 - PR_1^2}$$
$$= 3438 - \sqrt{3438^2 - 225^2}$$
$$= 3438 - 3431 \text{ (clearing of fraction)}$$
$$= 7.$$

Similarly the tangent at P, though practically equal to PX, does not coincide with it, but takes some such position as PX′ in the figure. So also the tangents at Q, R, S, etc., take positions as QP′, RQ′, SR′, etc.

Now if PK, QL, RM, etc., are perpendiculars to QR_2, RR_3, SR_4, etc., from P, Q, R, etc., then QK, RL, SM, etc., are successive differences to be added to the preceding sines to give the following sines.

Draw also QK′, RL′, SM′, etc., parallel to PX′, QP′, RQ′, etc.

Now coming to approximations (which our old writers often do and also deduce values of unknown quantities by comparing similar triangles), R_1PX and $X'PX$ could be treated as triangles equal to one another in all respects.

$$\therefore R\hat{P}_1X = X'\hat{P'}X$$

whence $R\hat{P}X = 2 . 225'$, and $R_1X = XX' = 7$.

Again approximately, QK′, QP′ and QP′ correspond to PR_1, PX, and PX′, and the triangle QKK′ is equal to the triangle PR_1X' in all respects ;

$$\therefore \quad KK' = R_1X' = 14,$$

and $K'P = PP' = 7.$

$\therefore$ $KP = KK' + K'P$

$= 14 + 7$

$= 21,$

whence from the right-angled triangle QKP

$$QK = \sqrt{PQ^2 - KP^2}$$

$$= \sqrt{225^2 - 21^2}$$

$= 224$ (clearing of fraction).

Similarly $LQ = 5 \times 7 = 35;$

$\therefore$ $RL = \sqrt{RQ^2 - LQ^2}$

$$= \sqrt{225^2 - 35^2}$$

$= \mathbf{222}$ (clearing of fraction)

and so on.

Now with the same notation as before,

$$D_1 = 224 = 225 - 1$$

$$= S_1 - Q_1 \quad (\because Q_1 = 1);$$

$$D_2 = 222 = 225 - \overline{1 + 2}$$

$$= S_1 - \overline{Q_1 + Q_2} \quad (\because Q_2 = 2);$$

and generally,

$$D_{r-1} = 225 - [1 + 2 + 3 + \ldots\ldots + (r - 1)]$$

$$= S_1 - \Sigma Q_{r-1}$$

i.e., $S_r - S_{r-1} = S_1 - \Sigma Q_{r-1} \quad (\because S_r - S_{r-1} = D_{r-1})$

and also

$$D_{r-1} - D_r = \Sigma Q_r - \Sigma Q_{r-1} = Q_r,$$

results which prove the rule from either aspect.

It could be seen that $Q_1 = 1$, and for all other values of r, Q involves a fraction, and is reduced to an integer by the rule for clearing of fractions, *viz.*—

(i) *A general convention* to drop the fractional part of a number and retain only the integral part of it when the fractional part is less than $\frac{1}{2}$; and in all other cases, the fractional part is dropped and the integral part is increased by unity (अर्धाधिकावयवस्यैकाधिकत्वेन ग्रहस्य सांप्रदायिकत्वात्—Ranganatha.)

n rigidly following the above rule certain excesses might continually accumulate or diminish and in either case the accuracy of the results is interfered with. Therefore to keep them within the range of accuracy the following two rules are adopted :—

(ii) A *special convention* which has to be observed only in calculating this table of 24 sines is this:—

ऐकविंशाच्च विंशाच्च षष्ठात्पंचदशादपि ।
सप्तमाद्द्वादशात्सप्तदशान्नाधौंत्तरं मतम् ॥

(*Brahma-Siddhanta* cited by Ranganatha)

which in effect means—the fractional part though not less than $\frac{1}{2}$ should be dropped and the integral part only retained in Q_6, Q_7, Q_{12}, Q_{15}, Q_{17}, Q_{20}, and Q_{21}.

(iii) Certain other *corrections* there are which could in effect be stated thus :

Diminish ΣQ_r by 1, 2, 3, 4, and 5 in order, before subtracting from S_1 when r = 7, 8, 9, 10 and 11; 7, 9 and 11, when r = 12, 13, and 14; 14, 17, and 20; when r = 15, 16, and 17; 25 and 30. when r = 18 and 19; 36, when r = 20; 43, when r = 21; 50, when r = 22; and 58, when r = 23.

Thus a complete general rule for calculating the 24 sines is laid down and it also satisfies the rule of proportional parts within those limits.

Example. To calculate $\sin 62° \, 28'$.

Since $62°28' = 62 \times 60 + 28 = 3478'$,

$$\therefore \quad \frac{3478}{225} = 16\frac{148}{225},$$

and the required sine lies between the 16th (= 2978) and 17th (=3084) sines, the difference between which is 3084 — 2978 = 106, which corresponds to 225′;

∴ diff. corresponding to 148 is

$$\frac{148}{225} \times 106 = 69\frac{163}{225}$$

i.e., $= 70$ (clearing of fraction)

$$\therefore \quad \text{the required sine} = 2978 + 70 = 3048$$

It should be observed that by dividing any one of the sines thus obtained by 3438 (or by 3437·74677 for greater accuracy), the values of Natural Sines as given in any modern book of Tables is obtained, *e.g.*, dividing 3048 above obtained by 3437·74677, we get

$$\frac{3048}{3437{\cdot}74677} = {\cdot}8866270 \qquad \ldots \quad (1)$$

and from Chamber's *Tables* sin 62° 28′ = ·8867420, which is practically the same as (1).

The above facts are based on the texts and the conventions of the Sûrva-Siddhânta. If any of the values are differently given by other writers it is not only because the value of π is differently taken, [*e.g.*, $\sqrt{10}$, 600/191 etc.], but also the part of the circumference which presents no curvature is differently assumed.

For example, it is only[6] the 110th part of the circumference that looks flat according to Aryabhatta, and so on. But it is hard to account for the values Brahmagupta gives, for though he apparently takes $\pi = \sqrt{10}$ and therefore his value of the radius ought to have been $\frac{21600}{2 \times \sqrt{10}} = 3430$, yet he gives it as 3270 and the first sine as 214. The proportions however hold good throughout: *e.g.*, if instead of 3438, 3270 is taken, then instead of 225, we have to take

$$\frac{225 \times 3270}{3438} = \frac{25 \times 3270}{382} = \frac{40875}{191} = 214\frac{1}{191} \; i.e., \; 214,$$

which is Brahmagupta's figure. It might be that, since he says राश्यष्टांशे ष्वङ्कान्. (XXI . 17) and not राशिलिप्ताष्टमोभागः etc., he might have chosen arbitrary figures to fix the relation between the sines and the radius; or, it is suggested with much diffidence, that he might have chosen the quadrant of an ellipse (the form of the orbit of any planet) whose minor axis is 3270, though its major axis might be 3430 or whatever it is[7].

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[6] पपनोभागः परिधेः (प = 1, प = 1, न = 0).

[7] The late Pandit Sudhakara Dvivedi in the course of his commentary on Brahmagupta (II. 6—9) says, आचार्येण किमर्थं त्रिज्यास्व मुनिरद (3270) मिता गृहीतेत्यस्य मीमांसा भूमिकायां भविष्यति । Certainly it would have been interesting to have the view of so high an exponent of Indian Astronomy as the late Pandit, and it is disappointing that his introduction contains nothing on the subject.

Lastly, the rule laid down by Aryabhatta thus—

प्रथमाच्चापज्यार्धाद्यैरूनं खंडितं द्वितीयार्धं ।
तत्प्रथमज्यार्धांशै स्त्यै स्त्यै रूनानि शेषाणि ॥

means exactly the same as the rule of the Surya Siddhanta quoted above, if it is interpreted as under : प्रथमात् चापज्यार्धात् खण्डितं भक्तं—प्रथममिति शेषः यैः भागलब्धांशैः ऊनं हीनं प्रथम मित्येव शेषः—(तैः सह) तत् प्रथमज्यार्धं द्वितीयार्धं द्वितीयज्यार्धं भवति—प्रथमज्यार्धांशैः तैः तथैव प्रथमज्यार्धविभक्तफलांशैः ऊनानि यानि फलानि तैः फलैः सह इति यावत्, शेषाणि इतराणि ज्यार्धानि स्युः ॥— that is, the first sine being divided by itself and the quotient obtained, the same (*i. e.*, the first sine) together with the same (*i.e.*, the first sine) *minus* the quotient is the second sine ; and the other sines are obtained by successively subtracting the sum of all the quotients from the first sine and adding the results successively to the last of the already obtained sines. Symbolically,

$$S_r = S_{r-1} + S_1 - \Sigma Q_{r-1}$$

which has already been proved.

S. N. NARAHARAYYA.

Solutions.

Question 359.

(S. RAMANUJAN): If $\sin(x+y) = 2\sin\frac{1}{2}(x-y)$ and $\sin(y+z) = 2\sin\frac{1}{2}(y-z)$, prove that

$$\sqrt[4]{(\tfrac{1}{2}\sin x\cos z)} + \sqrt[4]{(\tfrac{1}{2}\cos x\sin z)} = \sqrt[12]{(\sin 2y)}$$

and verify the result when

$$\sin 2x = (\surd 5 - 2)^3 (4 + \surd 15)^2 ; \sin 2y = \surd 5 - 2 ;$$
$$\sin 2z = (\surd 5 - 2)^3 (4 - \surd 15)^2 .$$

Solution by T. R. Srinivasa Iyer, M.A.

Now

$$\cos^4\frac{x+y}{2} - \sin^4\frac{x+y}{2} = \left(\cos^2\frac{x+y}{2} + \sin^2\frac{x+y}{2}\right)$$
$$\left(\cos^2\frac{x+y}{2} - \sin^2\frac{x+y}{2}\right) = \cos(x+y), \text{ identically} \quad \ldots \ (1)$$

and
$$\cos^4\frac{x+y}{2} + \sin^4\frac{x+y}{2} = \left(\cos^2\frac{x+y}{2} + \sin^2\frac{x+y}{2}\right)^2$$
$$- 2\sin^2\frac{x+y}{2}\cos^2\frac{x+y}{2}$$
$$= 1 - \frac{\sin^2(x+y)}{2}$$
$$= 1 - 2\sin^2\frac{x-y}{2} \text{ (by hyp.)}$$
$$= \cos(x-y). \quad \ldots \quad \ldots \ (2)$$

From (1) and (2), we get

$$\cos^4\frac{x+y}{2} = \cos x\cos y, \quad \ldots \quad \ldots \ (3)$$

$$\sin^4\frac{x+y}{2} = \sin x\sin y. \quad \ldots \quad \ldots \ (4)$$

Similarly, since $\sin(y+z) = 2\sin\frac{y-z}{2}$, we get

$$\cos^4\frac{y+z}{2} = \cos y\cos z, \quad \ldots \quad \ldots \ (5)$$

$$\sin^4\frac{y+z}{2} = \sin y\sin z, \quad \ldots \quad \ldots \ (6)$$

From (4) and (5),

$$\sqrt[4]{(\sin y \cos y \sin z \cos z)} = \sin \frac{x+y}{2} \sin \frac{y+z}{2},$$

and from (3) and (6)

$$\sqrt[4]{(\sin y \cos y \sin x \cos x)} = \cos \frac{x+y}{2} \sin \frac{y+z}{2}.$$

$$\therefore \quad \sqrt[4]{\sin 2y}\ [\sqrt[4]{(\tfrac{1}{2} \sin x \cos z)} + \sqrt[4]{(\tfrac{1}{2} \sin z \cos x)}]$$

$$= \sin \left(\frac{x+y}{2} + \frac{y+z}{2}\right). \quad \ldots (7)$$

It will now be shown that

$$\sin \left(\frac{x+y}{2} + \frac{y+z}{2}\right) = (\sin 2y)^{\frac{1}{3}}.$$

The given conditions may be written

$$2 \sin \frac{x+y}{2} \cos \frac{x+y}{2} = 2 \sin \left(\frac{x+y}{2} - y\right)$$

and

$$2 \sin \frac{y+z}{2} \cos \frac{y+z}{2} = 2 \sin \left(y - \frac{y+z}{2}\right).$$

Putting $\frac{x+y}{2} = \alpha$ and $\frac{y+z}{2} = \beta$, the given conditions are

$$\sin (\alpha - y) = \sin \alpha \cos \alpha \quad \ldots \quad \ldots (8)$$

and

$$\sin (y - \beta) = \sin \beta \cos \beta. \quad \ldots \quad \ldots (9)$$

$$\therefore \quad \sin \alpha \cos y - \cos \alpha \sin y = \sin \alpha \cos \alpha,$$

$$- \sin \beta \cos y + \cos \beta \sin y = \sin \beta \cos \beta.$$

Solving for $\sin y$ and $\cos y$,

$$\sin y (\cot \beta - \cot \alpha) = \cos \beta + \cos \alpha,$$

and

$$\cos y (\tan \alpha - \tan \beta) = \sin \alpha + \sin \beta.$$

$$\therefore \quad \sin y = \frac{\cos \beta + \cos \alpha}{\cot \beta - \cot \alpha} = \frac{2 \cos \frac{1}{2}(\alpha+\beta) \cos \frac{1}{2}(\alpha-\beta) . \sin \alpha \sin \beta}{\sin (\alpha - \beta)},$$

or

$$\sin y = \frac{\cos \frac{1}{2}(\alpha+\beta) . \sin \alpha \sin \beta}{\sin \frac{1}{2}(\alpha-\beta)}. \quad \ldots (10)$$

$$\text{Also } \cos y = \frac{\sin \alpha + \sin \beta}{\tan \alpha - \tan \beta} = \frac{2 \sin \frac{1}{2}(\alpha+\beta) \cos \frac{1}{2}(\alpha-\beta) \cos \alpha \cos \beta}{\sin(\alpha - \beta)}$$

or

$$\cos y = \frac{\sin \frac{1}{2}(\alpha+\beta) \cos \alpha \cos \beta}{\sin \frac{1}{2}(\alpha-\beta)}. \quad \ldots \quad \ldots (11)$$

From (10) and (11), we get

$$\sin^2\tfrac{1}{2}(\alpha+\beta)\cos^2\alpha\cos^2\beta+\cos^2\tfrac{1}{2}(\alpha+\beta)\sin^2\alpha\sin^2\beta=\sin^2\tfrac{1}{2}(\alpha-\beta)$$

i.e. $$\sin^2\tfrac{1}{2}(\alpha+\beta)\{\cos(\alpha+\beta)+\cos(\alpha-\beta)\}^2+\cos^2\tfrac{1}{2}(\alpha+\beta)\{\cos(\alpha-\beta)-\cos(\alpha+\beta)\}^2=4\sin^2\tfrac{1}{2}(\alpha-\beta),$$

i.e. $$\{\cos^2(\alpha+\beta)+\cos^2(\alpha-\beta)\}\left(\cos^2\frac{\alpha+\beta}{2}+\sin^2\frac{\alpha+\beta}{2}\right)$$

$$-2\cos(\alpha+\beta)\cos(\alpha-\beta)\left(\cos^2\frac{\alpha+\beta}{2}-\sin^2\frac{\alpha+\beta}{2}\right)=4\sin^2\frac{\alpha-\beta}{2}.$$

i.e. $$\cos^2(\alpha+\beta)+\cos^2(\alpha-\beta)-2\cos^2(\alpha+\beta)\cos(\alpha-\beta)=4\sin^2\frac{\alpha-\beta}{2}.$$

$\therefore$ $$\cos^2(\alpha+\beta)+\cos^2(\alpha-\beta)-2\cos^2(\alpha+\beta)\left\{1-2\sin^2\frac{\alpha-\beta}{2}\right\}=4\sin^2\frac{\alpha-\beta}{2}$$

i.e. $$\cos(\alpha-\beta)+\cos^2(\alpha+\beta)=4\sin^2\frac{\alpha-\beta}{2}.\sin^2(\alpha+\beta)$$

i.e. $$\{\cos(\alpha-\beta)+\cos(\alpha+\beta)\}\{\cos(\alpha-\beta)-\cos(\alpha+\beta)\}=4\sin^2\frac{\alpha-\beta}{2}\sin^2(\alpha+\beta)$$

i.e. $$\cos\alpha\cos\beta\,.\,\sin\alpha\sin\beta=\sin^2\frac{\alpha-\beta}{2}\sin^2(\alpha+\beta).$$

$\therefore$ $$\sin^2(\alpha+\beta)=\frac{\cos\alpha\cos\beta}{\sin\frac{1}{2}(\alpha-\beta)}\cdot\frac{\sin\alpha\sin\beta}{\sin\frac{1}{2}(\alpha-\beta)}$$

$$=\frac{\cos y}{\sin\frac{1}{2}(\alpha+\beta)}\cdot\frac{\sin y}{\cos\frac{1}{2}(\alpha+\beta)}\quad\text{from (10) and (11)}$$

$\therefore$ $$\sin^2(\alpha+\beta)=\frac{\sin 2y}{\sin(\alpha+\beta)}$$

i.e. $$\sin^3(\alpha+\beta)=\sin 2y$$

i.e. $$\sin(\alpha+\beta)=(\sin 2y)^{\frac{1}{3}}.$$

Replacing α and β, by $\frac{x+y}{2}$ and $\frac{y+z}{2}$, we have

$$\sin\left(\frac{x+y}{2}+\frac{y+z}{2}\right)=\sin(2y)^{\frac{1}{3}}. \quad \ldots \quad \ldots (12)$$

But from (7), we have

$$(\sin 2y)^{\frac{1}{4}} \{ (\tfrac{1}{2} \sin x \cos z)^{\frac{1}{4}} + (\tfrac{1}{2} \sin z \cos x)^{\frac{1}{4}} \}$$
$$= \sin \left[\frac{x+y}{2} + \frac{y+z}{2} \right].$$

$\therefore$ From (7) and (12), we get

$$(\sin 2y)^{\frac{1}{4}} \{ (\tfrac{1}{2} \sin x \cos z)^{\frac{1}{4}} + (\tfrac{1}{2} \sin z \cos x)^{\frac{1}{4}} \}$$
$$= (\sin 2y)^{\frac{1}{3}}$$

$$\therefore \quad (\tfrac{1}{2} \sin x \cos z)^{\frac{1}{4}} + (\tfrac{1}{2} \sin z \cos x)^{\frac{1}{4}} = (\sin 2y)^{\frac{1}{12}},$$

i.e. $$\sqrt[4]{(\tfrac{1}{2} \sin x \cos z)} + \sqrt[4]{(\tfrac{1}{2} \sin z \cos x)} = \sqrt[12]{\sin 2y},$$

which is the result to be proved.

The numerical values given have been verified and found correct.

N.B.—Results (3), (4) and (12) were communicated to me by Mr. M. Bhimasena Rao, who obtained them by the use of Elliptic Functions.

Question 1226.

(B. B. Bagi):—ABC, DEF are six points on a conic, DE, DF meet BC in L_1, L_2; EF, ED meet CA in M_1, M_2; and FD, FE meet AB in N_1, N_2; then if AD, BE, CF meet in a point O, then L_1N_2, N_1M_2, M_1L_2 meet in the same point O.

Solution by P. R. Venkatakrishna Iyer.

Let O be taken as the origin and OCF and OBE be the axes of co-ordinates. If OC $= c$, OB $= b$, OF $= f$, OE $= e$, then the equation of any conic through B, C, E, F is

$$\left(\frac{x}{c} + \frac{y}{b} - 1 \right) \left(\frac{x}{f} + \frac{y}{e} - 1 \right) = \lambda xy. \qquad \text{... (i)}$$

Let A be (x_1, mx_1) and D be (x_2, mx_2), where x_1 and x_2 are the roots of the quadratic

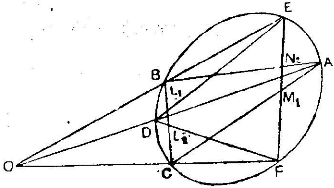
$$\left(\frac{x}{c} + \frac{mx}{b} - 1 \right) \left(\frac{x}{f} + \frac{mx}{e} - 1 \right) = \lambda mx^2,$$

i.e., of the equation

$$x^2 \left\{ \left(\frac{1}{c} + \frac{m}{b} \right) \left(\frac{1}{f} + \frac{m}{e} \right) - \lambda m \right\} - x \left(\frac{1}{c} + \frac{m}{b} + \frac{1}{f} + \frac{m}{e} \right) + 1 = 0. \text{(ii)}$$

The straight lines DE and BC meet in L_1, whose co-ordinates (x, y) are given by the equations

$$y - e = \frac{x}{x_2}(mx_2 - e) \text{ and } \frac{x}{c} + \frac{y}{b} = 1.$$



From these two equations, we get

$$\frac{y}{x} = \frac{bx_2(e + mc) - bce}{(b - e)\,cx_2}.$$

Similarly, if N_2 is (x', y'), we get

$$\frac{y'}{x'} = \frac{ex_1(b + mf) - bef}{(e - b)\,fx_1}.$$

If L_1 and N_2 are in the same straight line with O, then

$$\frac{y}{x} = \frac{y'}{x'}\text{; that is, } \frac{bx_2(e + mc) - bce}{(b - e)\,cx_2} = \frac{ex_1(b + mf) - bef}{(e - b)\,fx_1}.$$

i.e., $$bcef\left(\frac{1}{x_1} + \frac{1}{x_2}\right) = bf(e + mc) + ce(b + mf),$$

i.e., $$\frac{1}{x_1} + \frac{1}{x_2} = \frac{1}{c} + \frac{m}{e} + \frac{1}{f} + \frac{m}{b},$$

which is true as x_1 and x_2 are the roots of the equation (ii).

Therefore, L_1N_2 passes through O.

Similarly N_1M_2 and M_1L_2 pass through O.

Question 1228.

(Prof. K. J. SANJANA) :—Given the intrinsic equation of a curve show how to find the intrinsic equations of its first positive and negative pedals.

Solution by S. Audinarayanan and K. Satyanarayana.

Suppose that the intrinsic equation of the curve is

$$s = f(\psi).$$

This can be transformed into tangential-polar equation thus:

$$\frac{d^2p}{d\psi^2} + p = f'(\psi).$$

Solving this, we have

$$p = A\sin\psi + B\cos\psi + \int_0^\psi f'(\omega)\sin(\psi - \omega)\,d\omega', \quad \ldots \quad (1)$$

where A and B are constants depending on the origin.—(Edward's *Integral Calculus*, Vol. I, p. 557.)

It is evident that the tangential-polar equation of a curve is the polar equation of its first positive pedal. Hence the polar equation of the first positive pedal is given by (1).

Let us suppose that (1) is represented by $r = \phi(\theta)$.

When this equation is transformed into an intrinsic equation, we have

$$s = \int \sqrt{\{\phi(\theta)\}^2 + \{\psi'(\theta)\}^2}\,d\theta.$$

(*Ibid*, p. 552). ... (2)

Thus we can get the intrinsic equation of the first positive pedal.

When we come to the negative pedal, we have to reverse the process.

From the intrinsic equation of the curve

$$s = f(\psi)$$

get the polar equation of the curve by eliminating x, y and ψ from the equations

$$x = \int \cos\psi f'(\psi)d\psi$$

$$y = \int \sin\psi f'(\psi)d\psi$$

$$r = \sqrt{(x^2 + y^2)}$$

$$\theta = \tan^{-1}\frac{y}{x}.$$

Having obtained the polar equation which is the tangential polar equation of the first negative pedal, we can transform it into an intrinsic equation.

If the first be $p = \phi(\psi)$, then $s = \phi'(\psi) + \int \phi(\psi)d\psi$.

Questions for Solution.

1320. (V. V. SATYANARAYANA MURTI) :—P, Q are points on the sides AB, AC of a triangle ABC such that the circle APQ touches BC at its middle point M. N is the middle point of PQ. R is a point on the circle so that RM = MN. RN produced cuts the circle at K. The tangent at K cuts CA produced at L. Show that AB divides CL and MK in the same ratio.

1321. (R. GOPALASWAMI) :—If two points be the isogonal conjugates of a pair of polar conjugate points of a triangle, the pole of their joining line with respect to the circum-conic passing through them is the symmedian point of the triangle.

Deduce that the polar of the symmedian point with respect to the rectangular hyperbola of the circum-centroidal pencil is the Euler line.

1322. (R. GOPALASWAMI) :—O is the circum-centre of a triangle ABC and AO, BO, CO meet the opposite sides at L, M, N. P, Q, R are points on the sides, so that DP = DL, EQ = EM, FR = FN, where DEF is the middle-point-triangle of ABC. Show that the ellipse which touches the sides of ABC at P, Q and R also touches the nine-points circle.

1323. (B. B. BAGI) :—Two circles intersect in A, B. A straight line through A meets them again in P, Q. Show that the circum-circle of the triangle PBQ envelopes a cardioid.

1324. (K. J. SANJANA, M.A.) :—Find the values of λ for which the conics $x^2+y^2-2xy\cos\omega=\lambda^2$ and $xy=kx+hy$ have single contact,—ω being the angle between the axes of co-ordinates and (h, k) being a fixed point. If in this case P be the point of contact, PM and PN its ordinate and abscissa, and D the foot of the perpendicular from the origin on MN, prove that the distance of D from M is equal to the distance of (h, k) from N.

1325. (S. M. SHAH, M.A.) :—Sum the series

$$\sum_{n=0}^{\infty} b^n . e^{-\frac{(x+2na)^2}{c}}$$

where a, b, c are constants.

1326. (N. DURAI RAJAN) :—If OA, OB, OC are three rods in a plane jointed at O, show how to determine the triangle ABC such that the radius of the circle ABC is a minimum.

LIST OF JOURNALS RECEIVED

(From 5th of January 1924 to 5th of February 19

1 Rendiconti del Circolo Mathematico di Palermo. To.
Fascicolo II.
Supplements to the above XII. Anno. 1921.
2 Annals of Mathematics, Series Second, Vol. 24, No. 3.
March 192
3 Abhandlungen aus dem Mathematischen Seminar, Band III,
1 heft.
4 Journal für die Reine und angewandte Mathematik,
Band, 152, complete
" 153, 1/2 heft.
5 Proceedings of the Royal Society, Series A, Vol. 104, Nos. A.728, 729, 730.
6 The Astrophysical Journal, Vol. LVIII, No. 4, Nov. 1923.
7 Philosophical Transactions of the Royal Society of London, Vol. 224, No. A. 616.
8 Nature, Vol. 112. No. 2811, Sept. 1923, No. 2820, Nov. 1923, Nos 2823, 2824, 2825, 2826, Dec. 1923, Nos. 2827, 2829, 2830, Jan. 1924, No. 2831.
9 Popular Astronomy, Vol. 31, No. 10. (Two copies).
10 Mathematical Gazette, Vol. 11, No. 167, and Vol. 12, No. 168. (Two copies each).
11 Bulletin of the American Mathematical Society. Vol. 29, No. 10. (Two copies).
12 The Messenger of Mathematics, Vol. 53, Nos. 3, 4.
13 Proceedings of the London Mathematical Society: Series 2, Vol. 22, Part 4, Dec. 1923.
14 L'Intermediaire des Mathematiciens, 2nd Series, Tome 11, Nos. 9 & 10, Sept. & Oct. 1923.
15 Philosophical Magazine, Vol. 47, Nos. 277, 278.
16 Transactions of the American Mathematical Society, Vol. 25, No. 3, July 1923.
17 Proceedings of the Edinburgh Mathematical Society, Vol. 41. (Session 1922-1923).
18 The American Mathematical Monthly, Vol. 30, No. 8.
19 Acta Mathematica, Vol. 44, No. 4.
20 Quarterly Journal of Mathematics, Vol. 50, No. 1.
21 Monthly Notices of the Royal Astronomical Society, Vol. 84, No. 1.
22 Mathematische Annalen. Band 81. (Complete).

Other Publications Received.

1 Publications de la Faculte Des Sciences, de L'Universite Masaryk, Nos. 25, 26. 28, 29, 30, 32.
2 Revista de Mathematicas y Fisicas Elementales, Ano. V, Nos. 1, 2, 3, 4, 5, 6, 7 & 8.

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