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ON THE FORMAL STRUCTURE OF THE PROPOSITIONAL CALCULUS I*

BY

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[Received 30 December 1940]

The propositional calculus is the study of the set P of all elementary (or unanalysed propositions) under the three binary operations denoted by *and*, *or*, *implies* and the unary operation of *negation*. The simplest of these are probably the operations '*and*', '*or*' (in symbols $\times$, $+$); the other two operations depend upon each other, and different views of them can be taken. For instance, the negation of a proposition is understood in somewhat different senses in the classical propositional calculus and in the intuitionistic logic, while the implication operation characteristic of the newer logics (e.g. Lewis's logic of strict implication†) has a different meaning from what we attribute to the 'Material Implication' of the *Principia Mathematica*.

The object of this paper is to study the bearing of the possible meanings attributed to negation on the structure of the propositional calculus.

1. *The Proposition as Meaning-Structure.*

A proposition is not merely a statement of meaning but has besides a 'meaning-structure'. If for instance a proposition (say a) is expressed by a sentence, then a

* I am indebted for my interest and work on this subject to the departmental lectures on Symbolic Logic and Intuitionism delivered by Dr. R. Vaidyanathaswamy.

† Lewis (1).

grammatical transformation of the sentence a (e.g. changing the voice of a verb), though it may be held not to affect the meaning, yet may alter the meaning-structure and must therefore be considered to express a new proposition. Two propositions having the same meaning-structure are said to be logically identical (in symbols$=$), that is to say, each may replace the other in any logical context.

It may be admitted at this stage that this concept of meaning-structure is not precise or unambiguous; its connotation may however be fixed by the following plausible postulates relating to the operations 'and', 'or'.

$A(1)$. $a+b$, $a.b$ have the same meaning-structure as $b+a$, $b.a$ respectively.

$a+(b+c)$, $a.(b.c)$ have the same meaning-structure as $(a+b)+c$, $(a.b).c$ respectively.

$A(2)$. $a+a$, $a.a$ have the same meaning-structure as a.

$A(3)$. $a(b+c)$, $(a+b)(a+c)$ have the same meaning-structure as $a.b+a.c$, $a+b.c$ respectively.

$A(4)$. $a+a.b$ has the same meaning-structure as a.

We note that the mutual distributivity of '$\times$', '$+$', is in conformity with the ordinary accepted meanings of 'and', 'or'. Also, from the fourth postulate we can prove that the relations $a+b=b$, $a.b=a$ each imply the other. For, if $b=a+b$, then from $A(3)$, $A(2)$, $A(1)$ and $A(4)$, we have $a.b=a(a+b)=a$; similarly if $a.b=a$, then $b+a.b$ $=b+a$ which by $A(4)$, $A(1)$ reduces to $a+b=b$.

The postulates $A(1)$-$A(4)$ thus make the set of propositions P a distributive lattice* under the lattice operations of $+$, $\times$ standing for 'or', 'and'. In such a distributive lattice, an ordering relation '$<$' may be introduced by saying '$a<b$' whenever either of the

* For the theory from a very general point of view of lattices in general, and distributive lattices in particular, see MacNeille (2).

relations $b = a+b$ or $a = a.b$ (which have been shown to be equivalent) hold in P. It is evident that this relation is transitive and reflexive. Further, if $a < b$ and $b < a$ then $a = b$ since each $= a.b$. It is natural to interpret this ordering-relation as the original form of 'implication' in logic; but obviously it is not possible to identify the two at this stage as we have yet to consider the 'negative' of a proposition and its connection with lattice-order.

The distributive propositional lattice P contains 'unit' elements 0, 1 (necessarily unique), for among the propositions in P are included self-contradictory or 'absurd' propositions and 'logically-necessary' propositions. An absurd proposition x is one which has no meaning-structure whatever; so that whatever the proposition a may be, $a+x$ has substantially the same meaning-structure as a. This means $x < a$ (w. r. t. lattice-order) for every a. Also, it follows that such an absurd proposition is necessarily unique in the sense of lattice-order and is indicated by the element 0 of the lattice. Similarly if y is a logically-necessary proposition, then whatever a may be, $a.y$ has the same meaning-structure as a. In other words $y > a$ for every a and must be identified as the unique element 1 of the lattice P. It is clear that in P, $a+0 = a = a.1$; $a.0 = 0$; $a+1 = 1$ for any a.

The relation of lattice-order in P to implication in the logical sense, may be stated thus: If $a < b$ in the lattice, then a must 'imply' b in any logic of propositions of P. For, by definition $a.b$ has the same meaning-structure as a or $a+b$ as b. Thus while the implication relations which we may impose upon propositions of P may be widely different from each other, they must necessarily satisfy the condition that 'a implies b' whenever $a < b$. This justifies our regarding the order-relation itself as a specific form of implication (which we may call *lattice-implication*), which in consequence must be the strictest of all possible implication-relations. Whether this is the same as Lewis's strict implication or whether it is identical with the logical implication claimed by Emch* to be stricter than Lewis's may be left an open question. Suffice it to say that for any theory of the logical calculus which accepts our postulates $A(1)$-$A(4)$ the above ordering-relation, i.e. our lattice-implication must be basic.

2. *The Negation of a Proposition.*

With each proposition a, we associate a unique proposition a' meaning 'not-a'. The following considerations help to fix the meaning of not-a.

N_1. *Law of Contradiction.* a' is logically inconsistent with a, i.e. $a.a'$ is logically inconceivable: in symbols $a.a' = 0$.

But this property is not characteristic of the negation a', as there may be several propositions inconsistent with a. Accordingly, among the propositions which contradict a, we must give a unique place to the negative of a—hence the next condition which expresses the fact that a' is in a sense the weakest of all propositions contradicting a.

N_2. *Any proposition contradicting a is less than a'.* In symbols, if $ax = 0$, then $x < a'$ or x 'implies' a' in the sense of lattice-implication.

From the above two conditions on negation it follows that a' is identical with what is known as the product-complement† of a in the distributive lattice P. (An element a' of P is defined to be the product-complement of a if $a.a' = 0$ and for all x, $a.x = 0$ implies $x < a'$; if the product-complement exists then obviously it is unique.) It is clear that 0 and 1 possess complements namely

* Emch (3).

† MacNeille (2).

$0' = 1$; $1' = 0$. (This means that our postulates N_1, N_2 entail the negative of a self-contradictory proposition to be logically-necessary and vice-versa.)

N_3. *Law of the Excluded Middle.* The usual meaning of a' leads to the further property that $a+a'$ is a logically necessary proposition, in symbols $a+a' = 1$.

N_4. *Law of Double Negation.* The double negative of a proposition is logically identical with the proposition itself, in symbols $a'' = (a')' = a$.

In order to examine the inter-connection between these laws and their bearing on the structure of the propositional calculus, we study in the next section the properties of the product-complement in a distributive lattice P in which every element has a product-complement.

3. *Theory of Simple and Normal Elements in a Distributive Lattice P admitting Product-complements.**

It is assumed of course that P possesses a, 0 and 1 ; we denote the product-complement by a stroke ' ' '.

Simple and Normal Elements.

(1) An element a of P is said to be a *simple* element if it possesses the property $a+a' = 1$ where a' is the product-complement of a.

(2) An element a of P is said to be a *normal* element if it possesses the property $a'' = a$.

Fundamental Theorems.

THEOREM 1. $(a+b)' = a'.b'$.

Since $a.a' = 0$, $(a+b)a'.b' = 0$. And if $(a+b)x = 0$, then $ax = 0$, $b.x = 0$; hence from the definition of the product-complement, $x < a'$, $x < b'$, i.e., $x < a'.b'$. Thus $a'.b'$ is the product-complement of $a+b$.

* For the case of Boolean rings, the properties of simple and normal elements have been proved by M. H. Stone. See M. H. Stone (4).

THEOREM 2. *$a < b$ implies $a' > b'$.*

For, since $b.b' = 0$ and $a < b$, $a.b' < b.b' = 0$, i.e. $a.b' = 0$; from which we have $b' < a'$ (or $a' > b'$) by definition of the product-complement.

COROLLARY 1. $(a.b)' > (a'+b')$.

For, since $a.b < a$, $a.b < b$, we have from the theorem, $(a.b)' > a'$ and $(a.b)' > b'$, i.e. $(a.b)' > (a'+b')$.

COROLLARY 2. In general, the converse (viz. $a' < b'$ implies $a > b$) of the theorem is not provable but if a, b are normal elements, then $a' < b'$ implies $a'' > b''$ which is the same as $a > b$.

THEOREM 3. (i) $a < a''$; (ii) $a''' = a'$.

(i) Since $a.a' = 0$, $a <$ the product-complement of a' i.e. $< a''$.

(ii) Since the product-complement of a' is a'', we have by (i), $a' < a'''$ or $a''' > a'$. But from (i) and Theorem 2, $a''' < a'$ from which we get $a''' = a'$.

THEOREM 4. *Every simple element in P is a normal element.*

For, let a be a simple element so that $a+a' = 1$. Then, $a'' = a''.1 = a''(a+a') = a.a''$ (since $a'.a'' = 0$) from which $a'' < a$. But from Theorem 3 (i), $a'' > a$. Hence, $(a+a') = 1$ implies $a'' = a$.

THEOREM 5. *If every element is normal, then every element is simple.*

For, let us suppose a is not simple so that $(a+a') \neq 1$. But by Theorem 1, $(a+a')'' = (a'.a'')' = 0' = 1 \neq (a+a')$. Therefore, $(a+a')$ is not normal which contradicts our hypothesis that every element is normal. Hence, a must be simple.

THEOREM 6. *The sub-set S consisting of all simple elements of P constitutes a Boolean Algebra.*

PROOF. We first show that the sum $(a+b)$ and the product $(a.b)$ of two simple elements are simple. For, since $a+a' = b+b' = 1$, we have from Theorem 1, $(a+b)+(a+b)' = (b+a)(b+b')+a'.b' = b'+a.b'+a'.b' = 1$; hence, $(a+b)$ is a simple element. Again, using Theorem 2 Cor. 1, we have $a.b+(a.b)' > a.b+(a'+b')$ which $= (a'+a)(a'+b)+b'$ i. e. $= a'+b+b' = 1$. Therefore, $a.b+(a.b)' = 1$ or $a.b$ is a simple element. [Incidentally, it follows from the above, that $a.b = (a'+b')'$ if a and b are simple. For, we have just now proved that $(a.b)+(a.b)' = 1$ and also that $(a.b)+(a'+b') = 1$, and further we know that whatever a and b might be, $a.b(a'+b') = 0$ and $a.b(a.b)' = 0$; hence the two equations $a.b+x = 1$, $a.b.x = 0$ have two solutions $x = (a.b)'$, $x = a'+b'$. But since in a distributive lattice these equations cannot have more than one solution, the values for x are identical, i.e. $(a.b)' = a'+b'$. As a consequence of this, we may deduce that when a and b are simple, $a.b = (a'+b')'$, for, since we have proved that $(a.b)$ is simple, it must be normal (Theorem 4). Therefore, $a.b = (a.b)'' = (a'+b')$.]

Thus, since the sum and product of two simple elements are also simple, it follows that the totality of simple elements forms a sub-lattice S of P. Next, we observe that the product-complement of a simple element is also simple. For, if a is a simple element, then since by Theorem 4 every simple element is normal, we have $a'+a'' = a'+a = 1$ by hypothesis, so that a' is also a simple element.

But since S is a sub-lattice of P, the product-complement of a in S is also the product-complement a' of a in P. And since $a+a' = 1$ for any a in S, it follows that the sub-lattice S is a Boolean sub-algebra of P, the product-complement in S now becoming the Boolean complement in the Boolean algebra S.

THEOREM 7. *The set of normal elements of P is identical with the set of product-complements of elements of P.*

For, if a is normal, $a = a'' = (a')'$, i.e. a is the product-complement of a'. Conversely if a' be the product-complement of any a, then from Theorem 3, $a' = a'''$ $= (a')''$, i e. any product-complement is normal.

4. *The Propositional Lattice P.*

We shall now apply the results of the previous section to the propositional lattice P which as we have seen, has a 0 and 1. We assume N_1, N_2 to be valid in P, so that every element of P, i.e. every proposition has a product-complement, viz. the negation of the proposition.

(1) *Each of N_3 and N_4 imply the other.*

For, if the law of the excluded middle holds without restriction in P, it would mean that every element of P is simple. But by Theorem 4, every simple element is normal; therefore every element of P is normal or every proposition is logically identical with its double-negation; thus we see that the law of double negation holds universally if the law of the excluded middle holds universally. If on the other hand the law of double negation holds, then every element of P is normal. Therefore by Theorem 5, every element of P is simple, i.e. the law of the excluded middle holds universally in P. In this case, the lattice P becomes a Boolean algebra, namely the Boolean algebra S constituted by the totality of simple elements. (Theorem 6.)

(2) From Theorem 7, it follows that *the elements of P which are normal are identical with the negative propositions.*

Thus if we assume N_1, N_2, N_3 to be valid in P, the structure of P becomes such that every element is both simple and normal so that P reduces to a Boolean algebra S. The product-complement in the distributive lattice P has now become the Boolean complement in S and the 'lattice-relation' in P (i.e. lattice-implication) is now the Boolean ordering-relation. Therefore the calculus of propositions S thus reached is identical with the calculus of propositions in the *Principia Mathematica* except for the notion of implication. For, in the latter, 'implication' or 'material implication' as it is called, requires the ideas of Truth and Truth Values while in our propositional calculus the notion of truth has not yet been introduced. Thus our lattice-implication which has here become 'Boolean implication' is not the same as the material implication of the *Principia*.

5. *Intuitionistic Logic.*

We next assume that N_1, N_2 hold in the distributive lattice P, but N_3 is not universally valid so that N_4 also cannot be universally valid. Then the structure of P is that of a general distributive lattice admitting product-complements, in which every element is not simple and therefore there exist non-normal elements also (Theorem 5). As in the previous section, we apply the results of §3 to the propositional calculus P admitting the negation laws N_1, N_2 but not N_3. Thus :

(1) As already remarked, *the propositions for which the law of double negation holds are identical with negative propositions.*

This follows from Theorem 7, §3.

(2) *The relations of negation to '+' and '×' are not symmetric.*

For, from Theorem 1, §3, $(a+b)'$ is identical with $a'.b'$ but it is not true that in general $(a.b)' = a'+b'$. That is to say, the negative of a proposition 'a or b' is the proposition {'not-a' and 'not-b'}, but the negative of the proposition 'a and b' is not 'not-a or not-b'. Also from Theorem 2, it follows that the negative and the ordering-relation are not symmetrical with respect to each other. Hence there is no question of the law of duality for negation.

(3) *Any proposition for which the law of the excluded middle holds satisfies the law of double negation.*

For, by Theorem 4, every simple element in P is a normal element ; hence the propositions satisfying the law of the excluded middle form a sub-class of the negative propositions. But there is no reason to think that conversely, every normal element would be simple, i.e. that the law of the excluded middle would hold for every negative proposition. This point is discussed further below.

(4) Brouwer's dictum of *the absurdity of the absurdity of the law of the excluded middle holds.*

For, even though $a+a' \neq 1$, we have seen that $(a+a)''=1$ always.

The above are characteristic features of Intuitionistic Logic. We next show that these properties are present in the formal scheme for Intuitionistic Logic developed by Heyting*.

6. *Heyting's Intuitionistic Logic.*

According to the set of axioms of Heyting also, the set of propositions P form a distributive lattice with respect to '$+$' and '$\times$' ('or', 'and'). In the following theorems taken from Heyting's paper, we interpret his implication operation '$\supset$' as our lattice-implication '$<$' and for easy reference, quote Heyting's number for each theorem.

THEOREM 1. (4·5) $b.a \supset a' : \supset : b \supset a'$, *where a and b are any two propositions of P and a' is the negation of a.*

This theorem is tantamount to the statement that the negation a' of a in P is the product-complement of a in P. For, since $b.a \supset a$ and $b.a \supset a'$ (by hypothesis) it follows that $b.a \supset a.a'$. Therefore, $b.a \supset a' : \supset : b.a \supset a.a'$. Conversely since $a.a' \supset a'$, $b.a \supset a.a'$ leads to $b.a \supset a'$. Therefore, $b.a \supset a'$ is logically equivalent to $b.a \supset a.a'$,

* Heyting (5).

that is $b.a = 0$. Hence, Heyting's theorem becomes $b.a = 0 . \supset . b < a'$ which makes Heyting's negation a', the product-complement of a.

THEOREM 2. (4·31, 4·32) $\vdash . a' : \supset : \subset a''' \{ \text{or } a' = a''' \}$.

THEOREM 3. (4·3) $\vdash . \; a \supset a''$.

THEOREM 4. (4·44, 4·53).

(i) $(a+b)' : \supset \subset : a'.b'$ *or* $(a+b)' = a'.b'$.

(ii) $(a'+b') : \supset : (a.b)'$.

THEOREM 5. (4·8, 4·45).

(i) $\vdash . \; (a+a')''$.

(ii) $\vdash . \; (a+a') : \supset : a'' \supset a$.

Theorem 2 above, corresponds to Theorem 3 (ii) of § 3 and is identical with § 6 (1).

Theorem 3 corresponds to Theorem 3 (i) of § 3. This theorem combined with the fact that a theorem of the nature $\vdash a'' \supset a$ is not found among Heyting's theorems, suggests that the law of double negation does not hold for *all* elements a in Intuitionistic Logic.

Theorems 4 (i) and (ii) correspond to Theorem 1 and Corollary (1) of Theorem 2, § 3 respectively, and is identical with § 5 (2).

THEOREM 5 (i) corresponds to § 5 (4).

THEOREM 5 (ii) corresponds to § 5 (3). But the converse of this theorem (as mentioned above), namely, *that every normal element is simple*, is not true in general. In other words, the law of the excluded middle is not valid for every negative proposition. Tarski* has specifically proved this fact by matrix methods in a recent paper, wherein starting from a set of alternative postulates for Intuitionistic Logic, he proves that neither '*a* or

* Tarski (6) pp. 103-8. I am obliged to Dr. Vaidyanathaswamy for drawing my attention to this work.

not-a' nor 'either not-a or not-not-a' are provable formulæ in Intuitionistic Logic. This is also implied in Heyting's paper*. It does not seem easy to state conditions on a distributive lattice P which would imply that every normal element is simple, though it is easy to construct examples of such lattices.

In this connection, Frink† points out the fact that the totality of the negative propositions of an Intuitionistic Logic forms a 'Boolean sub-algebra' of Heyting's* logic, where the Boolean operations of product and complement are identical with the product and product-complement of the lattice P but the Boolean sum is defined differently. As a matter of fact, we can show that the Boolean sum is the 'normalised' sum defined as the double product-complement of lattice sums in P. For, if a', b' are normal elements of P, then the Boolean sum which is known to be $(a''.b'')'$ is the same as $(a'+b')''$ which is precisely the normalised sum of a' and b'. We have then the following theorem:

THEOREM. *The set of all normal elements of P form a Boolean algebra* N, *in which the Boolean product and the complement are the same as lattice-product and product-complement, while addition is defined as 'normalised' sum.*

To prove this, it is only necessary to show that the normalised sum of normal elements distributes and is distributed by the product of normal elements in N. The proof is substantially the same as that for a similar theorem of Huntington‡.

We may remark that since Boolean addition is not the same as a lattice sum, it does not seem proper to call N a Boolean *sub*-algebra.

* See the explanation under (4.45), Heyting (5) p. 50.

† Frink (7).

‡ Huntington (8).

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PATH EQUATIONS ADMITTING THE LORENTZ GROUP—II

BY

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In the first paper* with the same title, it was shown that the fundamental ideas of the theory of relativity could be applied directly to the trajectories of particles, without assuming the existence of a Riemann metric. The results of that paper are valid also for more than four dimensions, with the corresponding extended Lorentz group; in particular, to the Kaluza-Klein theory in five and the Proca-Goudsmit in six dimensions. From the analysis of Cartan and von Neumann dealing with the theory of spinors, it is not to be expected that anything of importance could be obtained from a manifold of more than eight dimensions, but the cases $n = 5$-8 will not be without some value. These path-spaces constitute the most general such extension of Einstein's *special* theory.

The results of my first paper cover somewhat more ground than is apparent therein. Consider that the path-spaces admitting Lorentz ($=$ **L**) and similitude ($=$ **S**) groups are derivable from three-dimensional observations, and have the trajectories of the "fundamental particles", $\dot{x}^i = cx^i$ as solutions:

$$\ddot{x}^i - p^i \frac{\mathbf{Y}}{\mathbf{X}} G(\xi) + 2\dot{x}^i \frac{\mathbf{Z}}{\mathbf{X}} \gamma(\xi) = 0 ; \; i = 0, 1, 2, 3 ;$$
$$2\gamma(1) - G(1) + 1 = 0. \qquad (1)$$

Here, as throughout the rest of this paper, the notation of (**1**) is used, though later on, it will be found more

* Kosambi (**1**).

convenient to make direct use in formulæ of the quantities $g = 1 + G$, $v = 1 + \gamma$. These paths become singular, or indeterminate, for the origin of space-time $x^0, x^1, x^2, x^3 = 0$, for which reason I have called them elsewhere (**4**) cosmogonic instead of cosmological path-spaces. If however, the observer is fixed as the space origin and lets his time coordinate $x^0 = ct$ vary in accordance with the equations, we find that the equations reduce to

$$\ddot{x}^0 + \frac{\dot{x}^{0^2}}{x^0}\left\{2\gamma(1) - G(1)\right\} \equiv \dot{x}^0\left(\frac{\ddot{x}^0}{\dot{x}^0} - \frac{\dot{x}^0}{x^0}\right) = 0. \qquad (2)$$

This has the solution $x^0 = ae^{b\tau}$, so that the relation between the observer's time and the parameter τ of the path-equations is precisely that between Milne's two time-scales. Of course, I do not make the claim that all of Milne's results are covered by the theory of path-equations admitting various special groups. For example, Milne feels bound to express certain views on creation and the deity*, whereas I am unable to venture upon theological applications of the theory of continuous groups.

The rest of this paper, then, will give rather elementary results in the theory of such path-spaces, unifying and illustrating some scattered work published in other papers.†

1. The following formulæ are handy in calculations:

$$-g_{00} = g_{11} = g_{22} = g_{33} = -1\ ;\ g_{ij} = 0,\ i \neq j;\ p^i = x^i,\ \dot{x}^i = \frac{dx^i}{d\tau}$$

$$p_i = g_{ir}p^r\ ;\ \dot{x}_i = \dot{x}^r g_{ir};\ \mathbf{X} = p^r p_r;\ \mathbf{Y} = \dot{x}^r \dot{x}_r;\ \mathbf{Z} = p^r \dot{x}_r\ ;\ \xi = \mathbf{Z}^2/\mathbf{XY}$$

$$\mathbf{X}_{,\,i} = 2p_i;\ \mathbf{X}_{;\,i} = 0;\ \mathbf{Y}_{;\,i} = 2\dot{x}_i;\ \mathbf{Y}_{,\,i} = 0;\ \mathbf{Z}_{,\,i} = \dot{x}_i;\ \mathbf{Z}_{;\,i} = p_i$$

$$\xi_{;\,i} = \frac{2\mathbf{Z}}{\mathbf{XY}^2}(\mathbf{Y}p_i - \mathbf{Z}\dot{x}_i)\ ;\ \xi_{,\,i} = \frac{2\mathbf{Z}}{\mathbf{X}^2\mathbf{Y}}(\mathbf{X}\dot{x}_i - \mathbf{Z}p_i)\ ;\ \xi_{;\,r}\dot{x}^r = \xi_{,\,r}p^r = 0$$

$$\mathbf{X}\xi_{,\,r}\dot{x}^r = \mathbf{Y}\xi_{;\,r}p^r = 2\mathbf{Z}(1 - \xi),\ \text{where}\ F_{;\,r} = \frac{\partial F}{\partial \dot{x}^r};\ F_{,\,r} = \frac{\partial F}{\partial x^r} \qquad (1.1)$$

* Milne, (**2**), 138-40.

† Kosambi, (**3**) and (**4**).

and the tensor summation convention is used.

It is also convenient to know how to calculate the contravariant tensor corresponding to any given covariant tensor of rank two. This, of course, can always be done by taking the normalized cofactors of the elements in the original tensor matrix, but for our work the tensor will be found to have the particular form :

$$T_{ij} = Ag_{ij} + B\dot{x}_i\dot{x}_j + Cp_ip_j + D\dot{x}_ip_j + Ep_i\dot{x}_j. \qquad (1.2)$$

Assuming therefore that the associated contravariant tensor is of type

$$T^{ij} = ag^{ij} + b\dot{x}^i\dot{x}^j + cp^ip^j + d\dot{x}^i p^j + ep^i\dot{x}^j, \qquad (1.3)$$

we solve the equations $T^{ir}T_{rj} = \delta^i_j$, which must be true identically in the quantities concerned. We obtain therefore,

$$\begin{aligned} aA &= 1 \\ aB + b(A + B\mathbf{Y} + E\mathbf{Z}) + d(B\mathbf{Z} + E\mathbf{X}) &= 0 \\ aD + b(C\mathbf{Z} + D\mathbf{Y}) + d(A + C\mathbf{X} + D\mathbf{Z}) &= 0 \\ aC + c(A + C\mathbf{X} + D\mathbf{Z}) + e(C\mathbf{Z} + D\mathbf{Y}) &= 0 \\ aE + c(B\mathbf{Z} + E\mathbf{X}) + e(A + B\mathbf{Y} + E\mathbf{Z}) &= 0. \end{aligned} \qquad (1.4)$$

Eliminating a from the first of these, the remaining fall into two sets, which can be solved if and only if the determinant

$$\Delta = (A + B\mathbf{Y} + E\mathbf{Z})(A + C\mathbf{X} + D\mathbf{Z}) - (B\mathbf{Z} + E\mathbf{X})(C\mathbf{Z} + D\mathbf{Y}) \neq 0.$$

It is clear, of course that $A \neq 0$ is also a necessary restriction; it need not be added that the equations (1.4) can be solved for the coefficients of T_{ij} when T^{ij} is given. The explicit solutions are :

$$\begin{aligned} &a = 1/A; \Delta = A(A + B\mathbf{Y} + C\mathbf{X} + D\mathbf{Z} + E\mathbf{Z}) + (BC - DE)(\mathbf{XY} - \mathbf{Z}^2); \\ &A\Delta b = \mathbf{X}(DE - BC) - AB; \quad A\Delta d = \mathbf{Z}(BC - DE) - AD \\ &A\Delta c = \mathbf{Y}(DE - BC) - AC; \quad A\Delta e = \mathbf{Z}(BC - DE) - AE. \end{aligned} \qquad (1.5)$$

These formulæ become particularly important when we have T_{ij} as the fundamental tensor of a metric path-space, i.e. of the form $f_{;i;j}$, the condition $A\Delta \neq 0$ being then the condition for the metric to be non-degenerate, and the variational problem to be regular. For the special metric $f=\frac{\mathbf{Y}}{\mathbf{X}}\phi(\mathbf{X}, \xi)$ we have:

$$A=\phi-\xi\phi_2;\quad \mathbf{X}C=\phi_2+2\xi\phi_{22};\quad \mathbf{Y}B=2\xi^2\phi_{22};\quad \mathbf{Z}D=\mathbf{Z}E=-2\xi^2\phi_{22}.\quad \phi_2=\partial\phi/\partial\xi, \text{ etc.} \tag{1.6}$$

The condition $\Delta\neq 0$ reduces to

$$\phi^2+(1-2\xi)\phi\phi_2-\xi(1-\xi)\phi_2^2+2\xi(1-\xi)\phi\phi_{22}\neq 0. \tag{1.7}$$

This can be integrated for the cases when the expression vanishes, and the degenerate values of the metric are then given by

$$\sqrt{\phi}=P\sqrt{(\xi-1)}+Q\sqrt{\xi}, \tag{1.8}$$

where P, Q are arbitrary functions of $\mathbf{X}$ alone. This includes the case $A=0$ as well. Any other metric is permissible, if it gives the paths desired as extremals.

2. In the previous paper (1), it was shown that a path-space admitting **L** and **S** in addition to a Riemann metric was, if isotropic, necessarily flat. This result and its possible generalizations really illustrate a theorem in the projective change of connection for the classical path-spaces: that such a change of connection always exists if the space is projectively flat, so that the equations of the paths become those for an ordinary flat space.

Discarding the similitude group, the most general path-spaces with a symmetric affine connection admitting the Lorentz group can, as is obvious, always be put in the form:

$$\ddot{x}^i-p^i\frac{\mathbf{Y}}{\mathbf{X}}(A-1)+\frac{\mathbf{Z}^2}{\mathbf{X}^2}B+2\dot{x}^i\frac{\mathbf{Z}}{\mathbf{X}}(C-1)=0. \tag{2.1}$$

Here A, B, C are as yet arbitrary functions of $\mathbf{X}$ alone. The similitude group applies if and only if all the three functions are constants; the -1 is inserted here, as in later path-equations, to give the simplest ultimate formulæ. A Riemann metric must have the form:

$$f = \alpha \mathbf{Y}/\mathbf{X} + \beta \mathbf{Z}^2/\mathbf{X}^2 = \bar{g}_{ij} x^i x^j; \; \alpha, \beta, \text{ functions of } X \text{ alone.}$$

$$\bar{g}_{ij} = \frac{\alpha}{\mathbf{X}} g_{ij} + \frac{\beta}{\mathbf{X}^2} p_i p_j \; ; \; |\bar{g}_{ij}| = -\alpha^3(\alpha+\beta) \neq 0. \qquad (2.2)$$

Also the covariant derivative with respect to (2.1) of the tensor $\bar{g}_{ij}$ must vanish. These conditions can be calculated by the usual method, to give:

$$\alpha' = \alpha C; \; \beta' = \beta(2C-B) - \alpha B; \; (A-C)\alpha + A\beta = 0. \qquad (2.3)$$

The dash indicates, as usual, differentiation with respect to the independent variable *taken here as* log X. As there are, for any given system of paths, only two unknown functions α, β to be determined, and three equations, we immediately obtain a compatibility condition:

$$A'/A - C'/C = A + B - C; \; AC \neq 0 \text{ (unless } A = C = 0). \qquad (2.4)$$

The latter part of the formulæ is simply the non-degeneracy condition in (2.2).

Direct calculation shows that the projective curvature tensor of the path-space vanishes if and only if we have

$$A^2 + AB - 1 - 2A' = 0. \qquad (2.5)$$

Now, for flatness, we must have in addition to (2.5), the conditions:

$$AC - 1 = 0; \; 2C' + BC - C^2 + 1 = 0. \qquad (2.6)$$

Of these, the condition $AC - 1 = 0$ is crucial, because it makes the other two conditions for flatness compatible. Moreover (if $AC = 1$), the condition for the existence of a non-degenerate metric reduces precisely to the condition for the space to be projectively flat. The relationship between our theory and that of a projective change of connection is furnished by the fact that the function $C(\mathbf{X})$ cannot be determined by three-dimensional (for the extended Lorentz group, $(n-1)$-dimensional) observations. Hence, if we utilize the indeterminacy to obtain the existence of a metric, we can always get projective flatness to coincide with ordinary flatness. This gives us :

THEOREM I. *If A, B are prescribed, with $A \neq 0$, and C may be chosen at will, then the choice $C = 1/A$ in equations (2.1) gives a space which is both isotropic and flat; for the given choice of C, no projectively flat space can exist which is not also flat in the usual sense.*

The actual transformation carrying the metric of (2.2) into a flat space can be found by putting $\bar{x}^i = x^i\phi(\mathbf{X})$, calculating $g_{ij}\dot{\bar{x}}^i\dot{\bar{x}}^j$, and setting it equal to the original metric. The use of (2.3) and the condition $AC-1=0$ leads to the function ϕ at once, $\phi = \exp \int \frac{(C-1)}{2\mathbf{X}} d\mathbf{X}$.

If A, B, C are to be constants and a metric exists, then there is no other choice possible for isotropy except the one which also gives flat spaces. For the general case, it is quite clear that choices of C exist which allow a metric and isotropy, but do not then imply flatness. However, such a statement would mean that the function C has an intrinsic position of its own although it cannot be specified from $(n-1)$-dimensional observations.

3. To extend these results to more general types of spaces, we shall first have to discuss the existence of a metric under more general conditions. Any space whose paths are deducible from $(n-1)$-dimensional observations, and admit the Lorentz group is defined by the path-equations :

$$\ddot{x}^i - b^i\frac{\mathbf{Y}}{\mathbf{X}}\{ g(\mathbf{X}, \xi) - 1 \} + 2\dot{x}^i\frac{\mathbf{Z}}{\mathbf{X}}\{ v(\mathbf{X}, \xi) - 1 \} = 0. \quad (3.1)$$

These admit both $\mathbf{L}$ and $\mathbf{S}$ if g and v are functions of ξ alone. A metric exists for these spaces if and only if there exists a function f satisfying :

$$f_{|i} \equiv f_{,i} - \tfrac{1}{2}\alpha^r_{;i} f_{;r} = 0, \quad |f_{;i;j}| \neq 0, \quad (3.2)$$

where α^i is obtained by regarding the paths as $\ddot{x}^i+\alpha^i=0$. Now the condition of non-degeneracy has been discussed in the first section, so that it only remains to reduce the equations (3.2) to an amenable form. Taking $f=\frac{\mathbf{Y}}{\mathbf{X}}\exp H[\mathbf{X},\ \xi]$, and recalling that with f, any function thereof such as $\log f$ will also be a solution of (3.2), we get the equations:

$$\begin{array}{ll} H_1=PH_2+Q & P=\xi(1-\xi)g_2+\xi v\,; \\ H_2=S/R & Q=-\xi g_2+v+2\xi v_2\,; \\ H_1=\partial H/\partial\log X & R=(1-\xi)(g-\xi g_2)+\xi v\,; \\ H_2=\partial H/\partial\xi \text{ etc.} & S=-(g-\xi g_2)+v-2\xi v_2. \end{array} \tag{3.3}$$

The notation has again been changed from that of my previous work to give the simplest final calculations. Solving the above equations explicitly for H_1, H_2 we have a simple first order partial differential system which is immediately integrable if and only if $H_{12}-H_{21}\equiv(PS/R+Q)_2-(S/R)_1=0$. This is a differential equation, from our present point of view, for the unobservable v in terms of the observed function g, and inasmuch as a solution exists in general, the metric exists unless the only possible solution does not satisfy the condition for non-degeneracy. If the metric be wanted directly, without troubling ourselves as to the choice of v, it can be obtained by regarding (3.3) as linear equations which can be solved for the unknowns v, v_2. We have:

$$\begin{aligned} v(\xi H_2-1)+2\xi v_2&=(g-\xi g_2)\{(\xi-1)H_2-1\}\,; \\ v(\xi H_2+1)+2\xi v_2&=H_1-\xi(\xi-1)g_2H_2+\xi g_2. \end{aligned} \tag{3.4}$$

These give at once

$$2v=H_1+g\{1-(\xi-1)H_2\}. \tag{3.5}$$

Substitution of this value in the solution for v_2 leads to

$$\begin{aligned} 2\xi H_{12}-2\xi(\xi-1)gH_{22}+\xi H_1H_2-\xi(\xi-1)gH_2^2-H_1 \\ +(1-2\xi)gH_2+g=0. \end{aligned} \tag{3.6}$$

The integration of this can be performed by the standard methods of Monge, but it is simplified by the transformation $H = \log \xi + 2 \log \phi$. The new equation in ϕ has then the form

$$\phi_{12} + (1-\xi) g \phi_{22} + (3/2\xi - 2) g \phi_2 = 0. \tag{3.7}$$

This equation has been given* for g, v functions of ξ alone, but our derivation shows it to be valid whenever the path-space admits **L** and possesses a Finsler metric. The integration is obvious, by the standard methods such as that of Charpit, treating (3.7) as a first order linear differential equation in ϕ_2. For $g(=1+G$ in Milne's notation) not zero, the result is equivalent to that of Walker (**2**, 166). For $g=0$, we have at once $\phi = \alpha(\xi)+\beta(X)$, and the second term can be discarded because it amounts to an additive perfect differential, such as is admissible in any variational problem. The metric for $g=0$ (Milne's kinematic case) is then $\mathbf{Y}\phi(\xi)/\mathbf{X}$, the function ϕ being arbitrary, subject only to the condition of non-degeneracy.

When, however, the similitude group applies, the situation is better treated in another way, though the above methods are quite valid. Here, the metric, to admit both **L** and **S**, with relative invariance under the latter, must have the form $\frac{\mathbf{Y}}{\mathbf{X}}\mathbf{X}^a\phi(\xi)$, and ϕ is given as $\exp \int (S/R)\,d\xi$. The condition for this to be possible, i.e. the condition of integrability of (3.3) when g, v do not contain **X**, is $(PS/R+Q)_2 = 0$, which is $PS/R+Q = a-1$, the constant a being the same as that which enters into the metric. In this case, we can determine the proper choice of v, for any given g and the existence of a metric, as a solution of the Riccatian equation

* See (**1**), formula (12).

$$v' = \frac{a(g-\xi g')}{2\xi g} + \frac{v}{\xi g}\left(\xi g' - g - \frac{g+a\xi}{2(\xi-1)}\right) + \frac{v^2}{(\xi-1)g} \cdot \quad (3.8)$$

The above equation always has the solution $v = \frac{1}{2}a + g/2\xi$, but this always leads to a degenerate metric, and can only be used to deduce the general solution :

$$v = a/2 + g/2\xi + 1/u,$$
$$u' + u\{g'/g + 1/2(\xi-1) - 3/2\xi + a/2(\xi-1)g\} + 1/(\xi-1)g = 0. \quad (3.9)$$

As an application, it will be found that the metric spaces which have identically the relationship $2v = g$ (which need hold only for $\xi = 1$ to give us the solutions $x^i = cx^i$ for the path-equations) have a g given by

$$g = \frac{a\xi}{\xi - 1 + b\sqrt{\xi - 1}}, \quad (3.10)$$

a, as in the metric ; b, arbitrary constant.

So, the only sub-case which is also Riemannian is $v = g = 0$, the "kinematic" case again, metric **Y**/**X**.

4. Coming now to the question of isotropy, we note that the theorem of Schur admits of a partial extension to our general path-spaces, if the concept of isotropy is redefined (3). This amounts to the restriction that the first curvature tensor P^i_j of the space should reduce to the form $\lambda\delta^i_j - x^i q_j$. This curvature tensor is, for the spaces (3.1) admitting the Lorentz group and deducible from three-dimensional observations, of the form

$$\mathbf{X}^2 P^i_j = A\mathbf{X}\mathbf{Y}\delta^i_j + \mathbf{X}B\dot{x}^i\dot{x}_j - C\mathbf{Z}\dot{x}^i p_j + Ep^i(\mathbf{Z}\dot{x}_j - \mathbf{Y}p_j) ;$$
$$A = g\{v + 2\xi(1-\xi)v_2\} + \xi(2v_1 - v^2) - 1 ; \quad B = C\xi - A ;$$
$$C = 1 + vg_2 - v^2 + 2v_1 - 4\xi v_{12} + v_2\{2\xi(1-\xi)g_2 - 6g + 8\xi g - 2\xi v\}$$
$$- 4\xi(1-\xi)gv_{22} ;$$
$$E = \xi(1-\xi)(2gg'' - g'^2) + (1-2\xi)gg' + g^2 - 1 + 2(g_1 - \xi g_{12}) ;$$
$$\text{where} \quad v_1 = \frac{\partial v}{\partial \log \mathbf{X}}, \quad v_2 = \frac{\partial v}{\partial \xi}, \text{ etc.} \quad (4.1)$$

For simplicity, I consider the case where both **L** and **S** groups are admitted, and g, v are functions of ξ alone.

The condition of quasi-isotropy is, for these restricted path-spaces

$$\Gamma \equiv \xi(1-\xi)(2gg''-g'^2)+(1-2\xi)gg'+g^2-1=0. \qquad (4.2)$$

This differential equation has the integrating factor g'/g^2; the first integration gives

$$\xi(1-\xi)g'^2/g+g+1/g=\text{const.} \qquad (4.3)$$

The complete solution is best presented in the form

$$g=p\sqrt{\xi(\xi-1)}+q(\xi-1/2)\pm\tfrac{1}{2}\sqrt{(p^2-q^2+4)}. \qquad (4.4)$$

If we wish to make $P^i_j=0$, we get three more equations, of which only two are independent, and admit the common solution:

$$\left.\begin{aligned} v &= b(\xi-1)^{\frac{1}{2}}\xi^{-\frac{1}{2}}+c \\ g &= 2b\sqrt{\xi(\xi-1)}+\frac{b^2-1}{c}(\xi+1)+c\xi \end{aligned}\right\}, \; b, c, \text{ any constants.}$$

$$(4.4)$$

In spite of the apparent difference in form, it will be seen that the form of the solution for g is precisely that given in (4.4), with proper adjustment of the two sets of arbitrary constants. One further adjustment can be made by substitution of the above values of v, g in (3.8): the arbitrary constant c in (4.4) must be the same as the exponent in the metric; $c=a$. It is clear, then, that *a choice always exists for v which gives a metric, and whenever the g is such that the space is quasi-isotropic (the condition of quasi-isotropy being independent of v), we automatically have $P^i_j=0$.*

It does not follow, however, that the space is flat even then. To this end, it would be necessary and sufficient to have an additional condition $\alpha^i_{;j;k;l}=0$, which would bring us back to the symmetric affine connection discussed before. The facts of the matter here are as follows. For any system of paths, and for a sufficiently restricted piece of a given path thereof, it is possible to choose a coordinate system making $\alpha^i_{;j}=0$

along the path. When, as here, we have $\alpha^i - \frac{1}{2}\alpha^i_{;r}\dot{x}^r = 0$, the path has the equation of a straight line $\ddot{x}^i = 0$. If, in addition, P^i_j is zero, the equations of variation admit along the chosen path as base, solutions for which the components of the vector variation are linear in the parameter τ. This means that the whole infinite sheaf of paths which can be obtained from the given path by giving it successive "small variations" all have the form of straight lines. Beyond this it is not possible to go unless $\alpha^i_{;j;k;l} = 0$, in which case alone is it possible to assert that *all* paths become straight lines in the chosen system of coordinates.

For non-homogeneous α^i, it is not possible to go even as far as the conclusions of the last paragraph. But the discussion of that case is beyond our scope here.

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THE LINEAR LINE - CONGRUENCE

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1. A linear congruence of lines is formed by the lines common to two linear line-complexes. The lines of such a congruence possess two transversals, and the congruence is called hyperbolic, elliptic or parabolic according as these transversals are real and distinct, imaginary and distinct, or coincident. This paper is devoted to the consideration of a number of properties connected with hyperbolic and parabolic linear congruences. The properties of a hyperbolic congruence will of course be true for an elliptic congruence by the introduction of imaginary elements when necessary.

2. Taking the directrix lines of a hyperbolic congruence as the lines $x_1 = x_2 = 0$ and $x_3 = x_4 = 0$, the line coordinates p_{ik} of the ray of the congruence joining $(0, 0, x_3, x_4)$ and $(x_1, x_2, 0, 0)$ are given by

$$p_{14} : p_{24} : p_{34} : p_{23} : p_{31} : p_{12} = x_1x_4 : x_2x_4 : 0 : x_2x_3 : -x_1x_3 : 0. \quad (1)$$

Hence the congruence consists of the lines common to the two special linear complexes $p_{12} = 0$, $p_{34} = 0$. Plucker's fundamental identity reduces, for these lines, to

$$p_{14}p_{23} = p_{24}p_{13}.$$

If a set of homogeneous coordinates are considered, wherein

$$X_1 : X_2 : X_3 : X_4 = p_{14} : p_{24} : p_{23} : p_{13} \quad (2)$$

we obtain from the above, the quadric surface

$$X_1X_3 = X_2X_4. \quad (3)$$

A (1, 1) correspondence is thus set up between the rays of a linear hyperbolic congruence, and the points on a quadric surface. This correspondence was set up by Hænzel*, who further transformed the points of the quadric by means of a stereographic projection on a plane from a fixed point on the quadric. Part of Hænzel's work is devoted to the study of the correspondence between curves on this plane and the ruled surfaces of the congruence. Some properties of these ruled surfaces can however be deduced from considerations of the quadric itself *directly* in relation to the congruence. §§ 3-4 are devoted to these properties.

3. The two systems of generators of the quadric (3) are

$$X_1 = \lambda X_2, \quad X_4 = \lambda X_3$$

and

$$X_1 = \mu X_4, \quad X_2 = \mu X_3.$$

Any λ-generator corresponds, by (2), to

$$p_{14} = \lambda p_{24}, \quad p_{13} = \lambda p_{23}$$

both of which are by (1) equivalent to $x_1 = \lambda x_2$, which is a plane through one of the directrices. The points on the λ-generator thus correspond to the pencil of lines joining the point $(\lambda, 1, 0, 0)$ to all points on the line $x_1 = x_2 = 0$. Similarly a μ-generator corresponds to the pencil of lines joining the point $(0, 0, 1, \mu)$ to all points on $x_3 = x_4 = 0$.

The points on the conic given by the section of the quadric (3) and the plane

$$aX_1 + bX_2 + cX_3 + dX_4 = 0$$

correspond to those lines of the congruence which also belong to the linear complex

$$ap_{14} + bp_{24} + cp_{23} + dp_{13} = 0. \tag{4}$$

But the lines common to this linear complex and the

* *Jour. fur reine und ang. Math.* 173 (1935), 91-113. This paper is followed by further papers: Hænzel, *Ibid.* 175 (1936), 169-81, Hænzel and Reutter, 178 (1938), 229-52.

complexes $p_{12}=p_{34}=0$ form a regulus, the quadric containing this regulus being

$$ax_1x_4+bx_2x_4+cx_2x_3+dx_1x_3=0.$$

This is the general equation of any quadric belonging to the congruence, and shows that there are ∞^3 such quadrics. Since three points on the quadric (3) determine a conic section on it, it follows that *the quadric through any three rays of the congruence is a quadric belonging to the congruence.* When the conic breaks up into a pair of lines, i.e. when the plane of the conic is a tangent plane of (3), the corresponding quadric of the congruence breaks up into a pair of planes.

The familiar property that four fixed generators of a quadric of the same system cut a variable generator of the opposite system in a constant cross-ratio, follows at once by this method; for this configuration corresponds to the lines joining four fixed points on one directrix to a variable point on the other.

4. Let us consider a curve of type (p, q) on the quadric (3), i.e. a curve which cuts any generator of one system in p points, and any generator of the other system in q points. This curve corresponds to a ruled surface belonging to the congruence, which is such that through any point on one directrix there pass p generators while through any point on the other directrix there pass q generators. In other words, we get a ruled surface of order $p+q$ on which the two directrices are multiple lines of orders p and q.

Conversely, the generators on any ruled surface with two directrix lines may be made to correspond to points on a curve on a quadric.

The osculating plane at a point on a curve on (3) corresponds to the osculating quadric through the corresponding ray of the ruled surface of the congruence. It follows from § 3 that *any osculating quadric of any scroll of the congruence itself belongs to the congruence.*

The statement that a skew curve on a quadric is of order n and class m is equivalent to the following: A scroll of order n of the congruence has n rays in common with any quadric of the congruence (the directrices forming the residual intersection), and a definite number m of osculating quadrics pass through any generator of the scroll.

We now write down the following properties of certain scrolls by considering known properties of corresponding curves on a quadric, or *vice versa*.

A curve of type (p, q) on a quadric is determined by $pq+p+q$ points*.	A scroll of order $p+q$ with two multiple lines of orders p and q respectively is determined by $pq+p+q$ transversals of these lines.
Through any point there pass three osculating planes of a twisted cubic, and the points of contact are coplanar with the given point.	Through any ray of the linear congruence determined by the two directrices of a cubic scroll of the first species, there pass three osculating quadrics. The quadric through the generators along which the osculation takes place passes through the ray considered.
A skew quartic of type (2, 2) has 16 stationary planes whose points of contact lie on four planes.	A quartic scroll having two double lines has 16 osculating quadrics each of which has contact of the third order with the scroll along a generator. These generators, which may be called *flecnodal generators,* lie four by four on four quadrics.
A skew quartic of type (2, 2) is touched by four generators, of each system, of any quadric containing the curve.	A quartic scroll with two double lines has four pinch-points on each line.†

* Sommerville, *Analytical Geometry of Three Dimensions*, p. 307.

† Use § 3.

If on a skew quartic with two inflexions, we take four points A, B, C, D such that their osculating planes pass through a given point O, then A, B, C, D, O lie on a plane; also if the osculating planes meet the curve again in A', B', C', D' then O, A', B', C', D' are also coplanar.*

If a quartic scroll with a triple line and a simple directrix possesses two pinch-points through each of which three coincident generators pass, then there are four generators of the scroll whose osculating quadrics pass through a given generator. The five lines lie on a quadric surface. Also these osculating quadrics meet the scroll again in four generators which lie on another quadric through the given generator.†

5. We next proceed to consider some metrical properties of the hyperbolic linear congruence. For this purpose we have to use Cartesian rectangular coordinates. Taking the two directrices in the standard form

$$y = mx,\ z = c \text{ and } y = -mx,\ z = -c$$

the coordinates of the middle point of the line joining $(\alpha, m\alpha, c)$ and $(\beta, -m\beta, -c)$ are

$$x_0 = \frac{\alpha+\beta}{2},\ y_0 = \frac{m(\alpha-\beta)}{2},\ z_0 = 0.$$

Since the foci of any ray of the congruence lie on the directrices, it follows that

The middle surface of a hyperbolic linear congruence is a plane parallel to the directrices.

The middle envelope is the envelope of the plane

$$(x-x_0)y_0 + (y-y_0)m^2x_0 + mcz = 0,$$

as x_0 and y_0 vary. Hence,

The middle envelope is the rectangular paraboloid,

$$cz(1+m^2) + mxy = 0.$$

* C. N. Srinivasiengar, *J. I. M. S.* (2) 2 (1936-37), 306.

† Use §3 to work out the correspondence between these properties.

The equations of the ray having its middle point at (x_0, y_0) are

$$\frac{x-x_0}{y_0/m} = \frac{y-v_0}{mx_0} = \frac{z}{c}.$$

Solving for x_0, y_0 in terms of x, y, z, we get

$$x_0 = \frac{c(cmx-yz)}{m(c^2-z^2)}, \; y_0 = \frac{c(cy-mxz)}{c^2-z^2}. \tag{5}$$

If (x_0, y_0) describes a curve on the middle plane $z=0$, (x, y, z) which is any point on the ray through (x_0, y_0) traces out a ruled surface.

6. The condition that the foot of the common perpendicular between the ray through (x_0, y_0) and the ray through (x_0+dx_0, y_0+dy_0)—points on the middle plane—, should fall at (x_0, y_0) is worked out to be

$$(y_0dx_0+m^2x_0dy_0)(y_0dy_0+m^4x_0dx_0)$$
$$= (1+m^2)(y_0^2+m^4x_0^2+c^2m^2)\,dx_0dy_0.$$

Writing p for dy_0/dx_0, we obtain the differential equation

$$x_0y_0p^2 - p(y_0^2+m^2x_0^2+c^2\overline{1+m^2}) + m^2x_0y_0 = 0. \tag{6}$$

Putting $x_0^2 = \xi$, $y_0^2 = \eta$, we obtain the general solution in the form

$$y_0^2 = \lambda x_0^2 - \frac{\lambda c^2(1+m^2)}{\lambda - m^2}, \tag{7}$$

where λ is an arbitrary constant.

From the properties of mean ruled surfaces,* it follows that

The lines of striction of the mean ruled surfaces of a hyperbolic linear congruence are a family of concentric conics on the middle plane.

From (5) and (7), we obtain the equations of the mean ruled surfaces as

$$(cy-mxz)^2 - \frac{\lambda}{m^2}(cmx-yz)^2 + \frac{\lambda(1+m^2)}{\lambda-m^2}(c^2-z^2)^2 = 0.$$

* Weatherburn, *Differential Geometry* Vol. I. § 99.

The singular solution of (6) represents four straight lines, which are however imaginary. A straight line $Ax_0+By_0+C=0$ on the middle plane generates a paraboloid in virtue of (5). Hence,

The mean ruled surface of any ray of a hyperbolic congruence is a quartic scroll having the directrices as double lines. The mean ruled surfaces of the different rays of the congruence form a singly infinite system enveloping four imaginary paraboloids.

If β_1 and β_2 are the parameters of distribution of the mean surfaces of the ray through $(x_0, y_0, 0)$, and if the ray meets one of the directrices in $(\alpha, m\alpha, c)$, then the square of the distance* between $(x_0, y_0, 0)$ and $(\alpha, m\alpha, c)$ is equal to $-\beta_1\beta_2$. Hence

$$-\beta_1\beta_2 = (y_0^2+m^4x_0^2+c^2m^2)/m^2.$$

The two focal planes of a ray are the planes through it and each of the directrices. If ϕ is the angle between a focal plane and one of the central planes, 2ϕ is the angle between the focal planes. Also

$$\tan^2\phi = -\beta_1/\beta_2,$$

so that

$$(\beta_1+\beta_2)^2 = -4\beta_1\beta_2 \cot^2 2\phi.$$

Writing down the directions of the normals to the two focal planes through the ray, and calculating $\cot 2\phi$, we obtain after some simplification

$$\beta_1+\beta_2 = \pm\frac{m^2x_0^2 - y_0^2+c^2(1-m^2)}{cm}.$$

Using equations (5), we obtain

Rays of a hyperbolic linear congruence for which the mean parameter of distribution is equal to a given constant lie on one or other of two quartic scrolls having the directrices as double lines.

7. We shall investigate a few properties concerning asymptotic curves on scrolls belonging to a hyperbolic

* C. N. Srinivasiengar, *Proc. Indian Acad. Sc.* 12 (1940), 352.

congruence, and we start with the following theorem due to Snyder.*

Any asymptotic curve of a scroll belonging to a linear congruence belongs to a linear complex. The linear complexes corresponding to different asymptotic curves are of the form $\Omega + k\Omega' = 0$, *where* k *is any constant, and* $\Omega = 0$, $\Omega' = 0$ *are the complexes defining the congruence.*

The theorem is true for hyperbolic and elliptic as well as for parabolic congruences. For the sake of completeness, the proof will be outlined here.

Consider a generator of any scroll belonging to the congruence. The polar plane of any point P on the generator with regard to the complex $\Omega + k\Omega' = 0$ contains the generator and hence touches† the scroll at some point Q on the generator. Let L, L' be the double points of the homography (P, Q). The locus of L, L' as the generator is varied is a curve cutting each generator in two points. The tangent to this curve obviously belongs to the complex, and since the polar plane of any point on a curve belonging to a linear complex is the osculating plane thereat, it follows that the curve is an asymptotic curve. To complete the proof, we must ascertain whether the double points L, L' are distinct from the points on the two directrices. For a hyperbolic congruence, this is easily verified since the polar plane of a point on a directrix passes through the other directrix, and hence cannot be the tangent plane at the point. For a parabolic congruence, however, one of the double points is on the directrix, and the locus of the other traces out the asymptotic curve. Any asymptotic curve in this case cuts a generator in one point only.

* V. Snyder, Asymptotic lines on ruled surfaces having two rectilinear directrices, *Bull. Amer. Math. Soc.* 5 (1899), 343.

† This statement is completely true only if we include among the "tangent planes", the nodal planes at the multiple points of the surface lying on the generator.

8. Now consider the hyperbolic congruence whose directrices are $y = mx$, $z = c$ and $y = -mx$, $z = -c$. Any transversal of these satisfies the equations

$$\Omega \equiv p_{23} + mcp_{14} = 0 \text{ and } \Omega' \equiv mp_{13} + cp_{24} = 0, \tag{8}$$

and the asymptotic curves of any scroll of the congruence form a system belonging to the pencil of complexes

$$\Omega + \lambda\Omega' = 0, \tag{9}$$

where Ω, Ω' are the above expressions, and λ is an arbitrary constant. The axis of the linear complex given by (9) i.e. the unique line such that the polar plane of any point on the line is perpendicular to the line is given by*

$$\left.\begin{aligned} y + \lambda mx &= 0 \\ z(1 + \lambda^2 m^2) + \lambda c(1 + m^2) &= 0. \end{aligned}\right\} \tag{10}$$

The elimination of λ from (10) gives

$$mz(x^2 + y^2) = c(1 + m^2)xy.$$

Hence,

The axes of the linear complexes of the asymptotic curves of any scroll of a hyperbolic congruence are all parallel to the middle plane, and their locus is a cylindroid.

9. If (x', y', z') is any point in space, the ray through (x', y', z') is the transversal of the two directrices, and is hence given by

$$\frac{y - mx}{y' - mx'} = \frac{z - c}{z' - c}; \quad \frac{y + mx}{y' + mx'} = \frac{z + c}{z' + c}$$

which reduce to

$$\frac{x - x'}{cy' - mx'z'} = \frac{y - y'}{m^2cx' - my'z'} = \frac{z - z'}{m(c^2 - z'^2)}. \tag{11}$$

In any linear complex, if r is the shortest distance between a ray and the axis, and if θ is the angle between them, we have the relation*

$$r \tan \theta = \text{a constant } h, \text{ say.}$$

* Salmon, *Analytical Geometry of Three Dimensions*, Vol. II, p. 41.

From the fact that the line (11) belongs to the linear complex (9), we obtain

$$h = \frac{mc(1-\lambda^2)}{1+m^2\lambda^2}.$$

All curves of a linear complex passing through a given point have the same torsion*, which is equal to $(\sin^2 \omega)/h$, where ω is the angle that the axis makes with the polar plane of the point. For the complex (9), this expression works out at (x', y', z') to

$$\frac{1}{\sigma} = \frac{mc(1-\lambda^2)}{\lambda^2(m^2x'^2+m^2z'^2+c^2)+2\lambda(mx'y'+cz'.\overline{1+m^2})+(y'^2+z'^2+c^2m^2)}. \tag{12}$$

Now the torsion of an asymptotic curve is equal to $(-K)^{\frac{1}{2}}$, where K is the Gaussian curvature of the surface at the point considered. From the form of the expression (12), it follows that for a given point (x', y', z'), when we consider variable scrolls of the congruence passing through (x', y', z'), there are two values of λ for which K is a maximum or a minimum. The asymptotic curves to these surfaces correspond to values of λ given by the quadratic

$$(\lambda^2+1)(mx'y'+cz'.\overline{1+m^2})+\lambda\{m^2x'^2+y'^2+(z'^2+c^2)(1+m^2)\} = 0. \tag{13}$$

We observe that *the product of the two values of* λ *is equal to unity.*

10. The argument in § 7 gives another result of Snyder, viz. every asymptotic curve cuts the ray in two points which harmonically separate the feet of the directrices (i.e. the intersections of the ray and the directrices). Hence, given one of the two points, the other point is the same for all scrolls of the congruence that contain the given ray. Let (x', y', z') and (x'', y'', z'') be such a pair of corresponding points.

* Salmon, *loc. cit.*

The feet of the directrices, A and B, can be found as the intersections of either directrix with the plane through (x', y', z') and the other directrix. We thus get their coordinates:

$$A \equiv \frac{c(y'+mx')}{m(z'+c)},\ \frac{c(y'+mx')}{z'+c},\ c$$

$$B \equiv \frac{c(y'-mx')}{m(z'-c)},\ \frac{-c(y'-mx')}{z'-c},\ -c.$$

The planes joining these to any line, $x=0$, $y=0$, must harmonically separate those joining the same line to (x', y', z') and (x'', y'', z''). Hence

$$(yx'-xy')(yx''-xy'')=0$$

harmonically separate $y^2-m^2x^2=0$. This gives

$$y'y''-m^2x'x''=0. \qquad (14)$$

Also from (11),

$$\frac{x''-x'}{cy'-mx'z'} = \frac{y''-y'}{m^2cx'-my'z'} = \frac{z''-z'}{m(c^2-z'^2)}. \qquad (15)$$

Solving (14) and (15), we obtain

$$(x'', y'', z'') = \left(\frac{cy'}{mz'},\ \frac{mcx'}{z'},\ \frac{c^2}{z'}\right). \qquad (16)$$

Since $z'z''=c^2$, the points form an involution range whose centre is the middle point of the ray, as is otherwise obvious.

The polar plane of (x', y', z') w. r. t. the complex $\Omega+\lambda_1\Omega'=0$ is given by

$$mx(c+\lambda_1 z')+y(z'+\lambda_1 c)-z(y'+\lambda_1 mx')-c(\lambda_1 y'+mx')=0. \qquad (17)$$

The polar plane of (x'', y'', z'') w. r. t. the complex $\Omega+\lambda_2\Omega'=0$ is given by

$$mx(z'+\lambda_2 c)+y(\lambda_2 z'+c)-z(\lambda_2 y'+mx')-c(y'+\lambda_2 mx')=0. \qquad (18)$$

The planes (17) and (18) coincide if $\lambda_1\lambda_2=1$, i.e. if λ_1, λ_2 are the roots of (13).

Next, the torsion at (x'', y'', z'') to the asymptotic curve which is defined by the complex $\Omega+\lambda\Omega'=0$ is obtained by changing (x', y', z') in (12) to (x'', y'', z''). Using (16), this becomes

$$\frac{m(1-\lambda^2)z'^2}{c[\lambda^2(y'^2+z'^2+c^2m^2)+2\lambda(mx'y'+cz'.\overline{1+m^2})+(m^2x'^2+m^2z'^2+c^2)]}. \qquad (19)$$

Putting $\lambda=\lambda_1$ in (12) and $\lambda=\lambda_2$ in (19) where $\lambda_1\lambda_2=1$, the ratio of the two expressions is $-c^2/z'^2$ or $-z''/z'$.

We also observe that (13) remains unaltered if x', y', z' are replaced by x'', y'', z'' given by (16). We have thus the following results*:

Of all scrolls belonging to a hyperbolic congruence, there are two for which the Gaussian curvature at a given point P is a maximum or a minimum. The asymptotic curve through P of either scroll meets the ray through P again at a point Q such that the tangent plane at P to one of the scrolls is also the tangent plane at Q to the other. The two scrolls having maximum or minimum Gaussian curvature at Q are identical with the two scrolls at P.

The ratio of the radius of torsion at P for the asymptotic curve of one of the scrolls to the radius of torsion at Q for the asymptotic curve of the other scroll is equal to the negative of the ratio of the distances of P and Q from the middle plane. Hence the product of the torsions of the two asymptotic curves at P is equal to the product of the torsions of the two curves at Q.

Also, for either of the scrolls, the tangent planes at P and Q are at right angles.

The last property is verified by using equations (17) and (18), after replacing λ_2 by λ_1. The condition for perpendicularity will be a consequence of equation (13).

* See post-script, p. 91.

It is also verified that *the asymptotic curves of the two scrolls through P or Q belong to conjugate* linear complexes,* for the condition that the complexes λ_1 and λ_2 should be conjugate is $\lambda_1\lambda_2 = 1$.

11. Let p_1 and p_2 be the parameters of distribution for the two scrolls having maximum and minimum Gaussian curvatures at the point P. Since the tangent planes, for either scroll, at P and Q are at right angles, we have by a known result

$$\frac{1}{p_r} = (-K_P^r)^{\frac{1}{2}} + (-K_Q^r)^{\frac{1}{2}}, \quad r = 1, 2$$

where the two values of r refer to the two scrolls, and K_P^r, K_Q^r are the Gaussian curvatures at P and Q. Hence, using (12) and (19) and Enneper's formula for the torsion of an asymptotic curve, we obtain

$$\frac{1}{p_1} = \frac{mc(1-\lambda_1^2)}{A\lambda_1^2+2B\lambda_1+C} + \frac{m(1-\lambda_1^2)z'^2}{c(A+2B\lambda_1+C\lambda_1^2)}$$

$$\frac{1}{p_2} = \text{a similar expression in } \lambda_2$$

$$= \frac{-mc(1-\lambda_1^2)}{A+2B\lambda_1+C\lambda_1^2} - \frac{m(1-\lambda_1^2)z'^2}{c(A\lambda_1^2+2B\lambda_1+C)},$$

where A, B, C stand for the expressions $m^2(x'^2+z'^2)+c^2$, $mx'y'+cz'(1+m^2)$, $y'^2+z'^2+c^2m^2$, respectively. Using (13), we obtain,

$$\frac{1}{p_1} + \frac{1}{p_2} = \frac{m(c^2-z'^2)(A-C)}{c(AC-B^2)}.$$

If now $(x_0, y_0, 0)$ be the middle point of the ray, we have by equations (5),

$$x_0 = \frac{c(cmx'-y'z')}{m(c^2-z'^2)}, \quad y_0 = \frac{c(cy'-mx'z')}{c^2-z'^2}.$$

Simplifying $A-C$ and $AC-B^2$, we obtain

$$\frac{1}{p_1} + \frac{1}{p_2} = \frac{m}{c} \cdot \frac{(m^2x_0^2-y_0^2+c^2-c^2m^2)}{y_0^2+m^4x_0^2+c^2m^2}.$$

* For the definition of conjugate linear complexes, see Sommerville *loc. cit.* p. 351.

Using the results of § 6, we obtain

$$\frac{1}{p_1}+\frac{1}{p_2}=\frac{1}{\beta_1}+\frac{1}{\beta_2}.$$

Hence,

If P be a variable point on a given ray, the sum of the reciprocals of the parameters of distribution of the surfaces having maximum and minimum Gaussian curvatures at P is constant, being equal to the sum of the reciprocals of the parameters of distribution for the mean ruled surfaces of the ray.

The result further suggests that when P is taken at the middle point of the ray, p_1 and p_2 become β_1 and β_2— a fact which can be verified. Hence,

The mean ruled surfaces of a ray of a hyperbolic congruence are also the surfaces having maximum and minimum Gaussian curvatures at the middle point of the ray.

THE PARABOLIC CONGRUENCE.

12. The rest of this paper will be devoted to the parabolic linear congruence. The two directrices now coincide. Taking $x=y=0$ as the directrix, one of the linear complexes can be taken as $p_{12}=0$. If

$$\Omega \equiv ap_{12}+bp_{13}+cp_{14}+dp_{23}+ep_{24}+fp_{34}=0$$

be any linear complex, the condition that this together with $p_{12}=0$ should define a parabolic congruence having the z-axis as directrix is that the quadratic in λ which represents the condition that $\Omega+\lambda p_{12}=0$ defines a special linear complex, must have both roots infinite. Hence we obtain $f=0$; $be-cd\neq 0$. The polar plane of any point on the directrix with respect to Ω will then be a plane through the directrix. The directrix of a parabolic linear congruence is therefore itself a ray of the congruence.

This leads to the following mode of generation of a parabolic linear congruence.

Consider any plane meeting a given line l at a point O. Set up a (1, 1) correspondence between lines through O on the *plane, and points on the line l. If OM be the line corresponding to the point P on l, then the rays joining P to points of OM generate a parabolic linear congruence.*

PROOF. Choose $x = y = 0$ as the line l, and $z = 0$ as the equation of the plane, O being taken as the origin. Let $y = mx$, $z = 0$ correspond to the point $(0, 0, \gamma)$ where

$$Am\gamma + Bm + C\gamma + D = 0,$$

A, B, C, D being constants. The line coordinates of the join of $(0, 0, \gamma)$ and $(x, mx, 0)$ satisfy the relations

$$p_{12} = 0, \; Ap_{23} + Bp_{24} + Cp_{13} + Dp_{14} = 0. \tag{20}$$

Since there is no term in p_{34}, these determine a parabolic congruence whose directrix is the line l. This proves the result.

The polar plane of $(0, 0, \gamma)$ with respect to the second complex in (20) is given by

$$\gamma(Ay + Cx) + (By + Dx) = 0.$$

If we choose $x = 0$ as the polar plane of O, and $y = 0$ as that of the point corresponding to $\gamma = \infty$, we must have $B = C = 0$. By an interchange of these planes, we could also take $A = D = 0$. Hence,

The equations of a parabolic linear congruence can by a suitable choice of coordinate axes be reduced to either of the forms

$$p_{12} = 0, \; p_{23} = kp_{14} \tag{21}$$

or

$$p_{12} = 0, \; p_{24} = kp_{13}, \tag{22}$$

where k is a constant.

13. Our method of procedure furnishes a general method of constructing ruled surfaces whose generators belong to a parabolic linear congruence. If H is any point on the z-plane, and if K is the pole of the plane formed by H and the line l with respect to the second complex in (20), or its simplified forms in (21) or (22), then HK is a ray of the congruence. If H traces out a curve on the z-plane, HK will generate a scroll of the required type.

Let the congruence be taken in the form given by (21). Let $f(x, y) = 0$ be the curve on the z-plane, and $(0, 0, \gamma)$ the pole of the plane joining the z-axis and the point $(x, y, 0)$, with respect to the second complex in (21). Then, from the coordinates of the line joining $(x, y, 0)$ and $(0, 0, \gamma)$, we must have $y\gamma = kx$. The ray of the congruence through $(x, y, 0)$ is given by

$$\frac{X}{x} = \frac{Y}{y} = \frac{Z-\gamma}{-\gamma},$$

where X, Y, Z denote current coordinates. Hence

$$x = \frac{-\gamma X}{Z-\gamma} = \frac{kX^2}{kX-YZ}$$

$$y = \frac{-\gamma Y}{Z-\gamma} = \frac{kXY}{kX-YZ}.$$

The coordinates may be made homogeneous in the usual way. Hence the equation of the scroll generated is

$$f\left(\frac{kx^2}{kxw-vz}, \frac{kxy}{kxw-yz}\right) = 0.$$

If the curve on the z-plane is of order n and does not pass through O, the scroll is of order $2n$ having the directrix as a multiple line of order n. If the curve passes through O, the order of the scroll is reduced according to the nature of the point O in relation to the curve.

The following particular cases are of interest :

(1) A straight line on the z-plane not passing through O generates a quadric, the rays of the congruence meeting the line being in fact the lines joining corresponding points of homographic ranges on two skew lines.

It follows that there are ∞^2 quadrics each of which touches a given plane and has one regulus included in a given parabolic linear congruence.

(2) The conic $x^2+y=0$ generates the familiar Cayley's cubic scroll

$$kx^3+kxyw=y^2z,$$

excluding the factor x.

(3) The conic $y^2+x=0$ gives only a quadric $ky^2+kxw-yz=0$, together with $x=0$ twice.

(4) The semi-cubical parabola $y^3=x^2$ also generates Cayley's cubic scroll with the equation

$$ky^3=kx^2w-xyz.$$

The folium of Descartes $x^3+y^3=3axy$ also generates a Cayley's cubic scroll.

(5) On the other hand, $y^2=x^3$ generates

$$kx^4=kxy^2w-zy^3,$$

a quartic scroll with a triple line. This surface is the Cayley scroll of order four. More generally, *a Cayley scroll of order m* has its generators belonging to a parabolic linear congruence, and is generated by the curve* $y^{m-2}=x^{m-1}$.

The curve $y^{m-1}=x^{m-2}$ also gives a Cayley scroll, but of order $m-1$.

(6) In my paper referred to, Cayley's scroll of order m has been generalized to the form

$$(xz+yw)^n=x^{n-m}y^{n+m},$$

where m and n are integers, positive or negative. This surface also possesses the property that the generators belong to a parabolic linear congruence In fact the curve $y^{m+n}=x^m$ in the z-plane generates the scroll

$$k^nx^{n-m}y^{n+m}=(kxw-yz)^n.$$

In particular, the conic $xy=1$ which corresponds to $n=2$, $m=-1$ gives

$$k^2x^3y=(kxw-yz)^2.$$

This surface has been explained in the paper cited, but we may add here that the surface is the limiting form

* Vide my paper, *J.I.M.S. loc. cit.* p. 304.

of a quartic with two non-intersecting double lines and a third double line meeting both of them, when the first two lines tend to coincide. The surface could be somewhat generalized into the form

$$ax^4+bx^3y+cx^2y^2+(dx^2+exy)(kxw-yz)=(kxw-yz)^2,$$

which corresponds to the general equation of the conic on the z-plane. A similar generalization can be carried out for the scroll of order $2n$ given above.

14. We conclude by working out for the parabolic congruence the counterpart of the result of § 8.

We shall take $x=y=0$ as the directrix, and use rectangular coordinates. The equations (§ 12) of the congruence are then given by

$$p_{12}=0,\ ap_{23}+bp_{24}+cp_{13}+dp_{14}=0,$$

where $bc-ad\neq 0$. The equations of the axis of the linear complex

$$ap_{23}+bp_{24}+cp_{13}+dp_{14}+\lambda p_{12}=0$$

are

$$\frac{cz+\lambda y+d}{a}=\frac{az-\lambda x+b}{-c}=\frac{-ay-cx}{\lambda}.$$

Elimination of λ leads to the cylindroid

$$ay+cx=\frac{\alpha(ax-cy)(\beta z+\gamma)}{(\beta z+\gamma)^2+\beta(ax-cy)^2},$$

where α, β, γ stand for $bc-ad$, a^2+c^2 and $ab+cd$ respectively.

Combining this result with that of § 8, we may state

The axes of a pencil of linear complexes generate a right cylindroid.

This result is in substance equivalent to that in statics regarding the locus of the central axis of wrenches acting on two given screws. But the method of work here is geometrical*, and considers the two cases wherein

* The result for the hyperbolic case was first given by Plucker. See Ball, *Theory of Screws*, p. 20.

the linear congruence determined by the two screws is non-parabolic and parabolic respectively. The parabolic case perhaps admits of a statical interpretation and discussion worth investigating.

[*Postscript.* The results of §§ 9-11 require a little modification. What we actually obtain in § 9 is that at any point P on a given ray, there are two extremum values of K, but it does not follow that there is only one scroll corresponding to each of these values. In fact, if we consider a scroll through the ray and having a given value of K at P, any scroll which can touch this at all points along the ray has the same value of K at P. There may exist an infinite number of such scrolls, the osculating quadric being the simplest example. For a value of K intermediate between the maximum and minimum values, two such sets exist, since equation (12) gives two different values of λ. But when K is maximum or minimum, the two sets coincide. The theorems of §§ 9-11 are valid if by the two scrolls having maximum and minimum values of K at P, we understand *any* two scrolls taken from the $K_{max.}$ and $K_{min.}$ sets respectively. The words "two" in line 12, and "two scrolls" in lines 16-17, p. 84 should be read "two families" and "two families of scrolls" respectively.]

A GENERALIZATION OF LEGENDRE FUNCTIONS

BY

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1. *Introduction.* Legendre functions are defined as solutions of the differential equation

$$(1-z^2)\frac{d^2y}{dz^2}-2z\frac{dy}{dz}+n(n+1)y=0. \qquad (1.1)$$

This equation may be identified with the hyper-geometric equation

$$\left[z^2\frac{d}{dz^2}\left(z^2\frac{d}{dz^2}-\frac{1}{2}\right)-z^2\left(z^2\frac{d}{dz^2}-\frac{n}{2}\right)\left(z^2\frac{d}{dz^2}+\frac{n+1}{2}\right)\right]y=0, \qquad (1.2)$$

and hence the solutions may be obtained as hyper-geometric functions. When n is a positive integer, the differential equation admits of a polynomial solution and a non-polynomial solution, both of which have well-known properties.

I consider here the generalized hyper-geometric equation

$$\left[\vartheta'\left(\vartheta'-\frac{1}{s}\right)\left(\vartheta'-\frac{2}{s}\right)\cdots\left(\vartheta'-\frac{s-1}{s}\right)-z^s\left(\vartheta'-\frac{n(s-1)}{s}\right)\right.$$

$$\left.\times\left(\vartheta'+\frac{n+1}{s}\right)\left(\vartheta'+\frac{n+2}{s}\right)\cdots\left(\vartheta'+\frac{n+s-1}{s}\right)\right]y=0, \quad (1.3)$$

where $\vartheta'=z^s d/dz^s$. When n is a positive integer, this has one polynomial solution and $s-1$ other solutions. Several properties of these solutions analogous to those of the Legendre functions will be investigated here.

2. The solutions of (1.3) are the s functions

$$z^r {}_sF_{s-1}\left[\begin{array}{c} \frac{-(s-1)n+r}{s}, \frac{n+r+1}{s}, \frac{n+r+2}{s}, \ldots, \frac{n+r+s-1}{s}; z^s \\ \frac{s+r-1}{s}, \frac{s+r-2}{s}, \ldots, \frac{s+1}{s}, \frac{s+r}{s}, \frac{s-1}{s}, \ldots, \frac{r+1}{s} \end{array}\right]$$

$$r = 0, 1, \ldots, s-1, \qquad (2.1)$$

where

$${}_sF_{s-1}\left[\begin{array}{c} a_1, a_2, \ldots, a_s; z \\ b_1, \ldots, b_{s-1} \end{array}\right]$$

denotes the series

$$1 + \frac{a_1 a_2 \ldots a_s}{1!\, b_1 b_2 \ldots b_{s-1}} z + \frac{a_1(a_1+1)a_2(a_2+1)\ldots a_s(a_s+1)}{2!\, b_1(b_1+1)\ldots b_{s-1}(b_{s-1}+1)} z^2 + \ldots .$$

The solutions are distinct and valid for $|z| < 1$. When n is a positive integer, one of the solutions reduces to a polynomial. If n is a negative integer $-ps-t$, $0 \leqslant t \leqslant s-1$, then t or $s-1$ of the functions reduce to polynomials according as t is not or is equal to zero. When $|z| > 1$, the solutions of (1.3) are easily obtained by putting $z = 1/z'$ and are given by

$$z^{n(s-1)} {}_sF_{s-1}\left[\begin{array}{c} \frac{-n(s-1)}{s}, \frac{1-n(s-1)}{s}, \ldots, \frac{(s-1)-n(s-1)}{s}; z^{-s} \\ \frac{s-ns-1}{s}, \ldots, \frac{-ns+1}{s} \end{array}\right] \qquad (2.2)$$

and

$$z^{-(n+r)} {}_sF_{s-1}\left[\begin{array}{c} \frac{n+r}{s}, \frac{n+r+1}{s}, \ldots, \frac{n+r+s-1}{s}; z^{-s} \\ \frac{r+s+ns}{s}, \frac{r+s-1}{s}, \ldots, \frac{s+1}{s}, \frac{s-1}{s}, \ldots, \frac{r+1}{s} \end{array}\right],$$

$$r = 1, 2, \ldots, s-1. \qquad (2.3)$$

When n is a positive integer, (2.2) reduces to a polynomial. We shall have occasion, later, to study this polynomial in detail.

3. We shall now obtain certain other forms of the differential equation (1.3).

Now $\vartheta' = z^s \dfrac{d}{dz^s} = \dfrac{z}{s}\dfrac{d}{dz} = \dfrac{\vartheta}{s}$, say.

Equation (3) now becomes

$$[\vartheta(\vartheta-1)\ldots(\vartheta-s+1)-z^s(\vartheta-\overline{s-1}.n)\,(\vartheta+n+1)\ldots \times(\vartheta+n+s-1)]\,y=0, \quad (3.1)$$

which may also be written

$$\frac{d}{dz}\left[z^{-ns}\,\frac{d^{s-1}}{dz^{s-1}}\,(z^{n+s-1}\,y)\right]=z^{-(s-1)n-1}\,\frac{d^s y}{dz^s}. \quad (3.2)$$

In particular, Legendre's equation is

$$\frac{d}{dz}\left[z^{-2n}\,\frac{d}{dz}\,(z^{n+1}\,y)\right]=z^{-(n+1)}\,\frac{d^2y}{dz^2}, \quad (3.3)$$

an equivalent form being

$$\frac{d}{dz}\left[z^{2(n+1)}\,\frac{d}{dz}\,(z^{-n}\,y)\right]=z^n\frac{d^2y}{dz^2}. \quad (3.4)$$

Working out the differentiations in (3.2), we get

$$(1-z^s)\frac{d^sy}{dz^s}+\sum_{r=1}^{s}\binom{s}{r}(n+s-1,\,r-1)(rn-n-s+r)\frac{z^{s-r}d^{s-r}y}{dz^{s-r}}=0, \quad (3.5)$$

where $\binom{s}{r}$ denotes the binomial coefficient, and $(a,\,p)$ denotes $a(a-1)\ldots(a-p+1)$.

4. *Solution in other forms.*

Let n be a positive integer. Then it can be shown that

$$\frac{d^n}{dz^n}\,(z^s-1)^n$$

is a solution of (3.5). For, denoting $(z^s-1)^n$ by u, we have

$$\frac{du}{dz}\,(z^s-1)=nsz^{s-1}u. \quad (4.1)$$

Differentiating this $(n+s-1)$ times and replacing $\dfrac{d^nu}{dz^n}$ by y, we get (3.5).

If we denote (2.2) by $P(z)$ and (2.3) by $Q(r, z)$, it is seen that when n is a positive integer,

$$P(z) = C\frac{d^n(z^s-1)^n}{dz^n}, \tag{4.2}$$

C being a determinate constant. When n is a negative integer, it can be shown that

$$P(z) = C\int_z^\infty\int_z^\infty\cdots\int_z^\infty\frac{(dz)^{-n}}{(z^s-1)^{-n}}, \tag{4.3}$$

C being a definite constant. The Q's may also be expressed in similar forms. For, if instead of equation (4.1) we take the equation

$$\frac{du}{dz}(z^s-1) = nsz^{s-1}u + \sum_{r=0}^{n+s-2} c_r z^r, \tag{4.4}$$

the c's being arbitrary constants, differentiating it $(n+s-1)$ times and replacing $\frac{d^n u}{dz^{n-1}}$ by y, we still get (3.5). Hence it follows that the nth derivative of any solution of (4.4) satisfies (3.5). Now from (4.4) we get

$$u = A(z^s-1)^n + (z^s-1)^n\int_z^\infty\frac{c_0+c_1z+\ldots+c_{n+s-2}z^{n+s-2}}{(z^s-1)^{n+1}}dz,$$

from which we easily deduce that

$$Q(r, z) = C\frac{d^n}{dz^n}\left[(z^s-1)^n\int_z^\infty\frac{z^{s-r-1}dz}{(z^s-1)^{n+1}}\right], \tag{4.5}$$

where C is a determinate constant.

5. *The generalized Legendre polynomial.*

Let

$$P_{n,s}(z) = \frac{1}{n!\,s^n}\frac{d^n}{dz^n}(z^s-1)^n. \tag{5.1}$$

It is readily seen that $P_{n,s}(z) = \frac{(ns)!}{\{n(s-1)\}!\,n!\,s^n}\times$

$$\left[z^{n(s-1)} - \frac{n(s-1)\{n(s-1)-1\}\ldots\{n(s-1)-s+1\}}{1!\,ns(ns-1)\ldots(ns-s+1)}z^{n(s-1)-s} + \ldots\right]$$

$$= \frac{ns!}{\{n(s-1)\}!\; n!\; s^n} z^{n(s-1)}$$

$$\times {}_sF_{s-1}\left[\begin{matrix} \frac{-n(s-1)}{s}, \frac{1-n(s-1)}{s}, \ldots, \frac{(s-1)-n(s-1)}{s}; z^{-s} \\ \frac{s-ns-1}{s}, \ldots, \frac{-ns+1}{s} \end{matrix}\right],$$

$$= (-1)^{n+\frac{n+r}{s}} \frac{(n+r)!}{s^n . r! \left(\frac{n+r}{s}\right)! \left(n-\frac{n+r}{s}\right)!}$$

$$\times z^r F\left[\begin{matrix} \frac{-(s-1)n+r}{s}, \frac{n+1+r}{s}, \ldots, \frac{n+s-1+r}{s}; z^s \\ \frac{s+r-1}{s}, \ldots, \frac{s+1}{s}, \frac{s+r}{s}, \frac{s-1}{s}, \ldots, \frac{r+1}{s} \end{matrix}\right],$$

$$(n = ps-r,\ 0 \leqslant r \leqslant s-1). \tag{5.2}$$

The following properties of $P_{n,s}(z)$ are easily proved.

(i) $P_{n,s}(z) = \rho^n P_{n,s}(\rho z)$, *where* ρ *is any sth root of unity.* (5.3)

(ii) *When* n *is not a multiple of* s

$$P_{n,s}(0) = 0, \tag{5.4}$$

and when n *is a multiple of* s

$$P_{n,s}(0) = (-1)^{\frac{n(s+1)}{s}} \frac{n!}{s^n \left(\frac{n}{s}\right)! \left(n-\frac{n}{s}\right)!}. \tag{5.5}$$

More generally, when n *is of the form* $ps-r$ $(0 \leqslant r \leqslant s-1)$,

$$\lim_{z\to 0} \frac{P_{n,s}(z)}{z^r} = \frac{(-1)^{n+\frac{n+r}{s}} (n+r)!}{s^n r! \left(\frac{n+r}{s}\right)! \left(n-\frac{n+r}{s}\right)!}. \tag{5.6}$$

(iii) $P_{n,s}(1) = 1$. (5.7)

PROOF. It is clear that $P_{1,s}(1) = 1$. Let us assume that $P_{n,s}(1) = 1$ is true for a particular value $(m-1)$ of n so that $P_{m-1,s}(1) = 1$.

Then

$$P_{m,s}(1) = \left[\frac{1}{m!\, s^m}\frac{d^m}{dz^m}(z^s - 1)^m\right]_{z=1}$$

$$= \left[\frac{1}{(m-1)!\, s^{m-1}}\frac{d^{m-1}}{dz^{m-1}}\left\{z^{s-1}(z^s-1)^{m-1}\right\}\right]_{z=1}$$

$$= \left[\frac{1}{(m-1)!\, s^{m-1}}z^{s-1}\frac{d^{m-1}}{dz^{m-1}}(z^s-1)^{m-1} + (\ldots)\frac{d^{m-2}}{dz^{m-2}}(z^s-1)^{m-1} + \ldots\right]_{z=1}$$

$$= P_{m-1,s}(1),$$

since

$$\left[\frac{d^{m-r}}{dz^{m-r}}(z^s-1)^{m-1}\right] = 0, \text{ if } r > 1.$$

The required result follows by induction.

(iv) $P_{m,s}(\rho) = \rho^{-n}$, *where ρ is any sth root of unity.* (5.8)

This is an immediate consequence of (i) and (iii).

6. We now proceed to prove a result concerning the zeros of $P_{n,s}(z)$.

THEOREM. *All the roots of $P_{n,s}(z) = 0$ lie within the unit circle $|z| = 1$, symmetrically situated on the radial lines to the points representing the sth roots of unity.*

We first observe that if

$$P_{n,s}(a) = 0,$$

then

$$P_{n,s}(\rho a) = 0. \tag{6.1}$$

We next investigate the nature of distribution of the zeros of $P_{n,s}(z)$ between 0 and 1 on the real axis. When n is of the form $ps-r$, the origin is easily seen to be a zero of order r. We shall show further that there are exactly $\left\{\frac{n(s-1)-r}{s}\right\}$ non-zero real roots between 0 and 1, so that, since $P_{n,s}(z)$ is of degree $n(s-1)$ in z, our theorem follows by means of (6.1).

Let $y = (z^s - 1)^n$, and let $y^{(n)}$ denote the nth derivative of y. Then at $z = 1$,

$$\left.\begin{aligned} &y^{(t)} = 0, \quad (t = 0, 1, \ldots, n-1) \\ &y^{(n)} \neq 0. \end{aligned}\right\} \qquad (6.2)$$

Also if $n = ps - r$ $(0 \leqslant r \leqslant s-1)$, we have at $z = 0$

$$\left.\begin{aligned} &y^{(qs-t)} = 0 \left(\begin{matrix} t = 1, \ldots, s-1 \\ 1 \leqslant q < p \end{matrix}\right), \\ &y^{(qs)} \neq 0, \\ &y^{(ps-t)} = 0 \quad (t = r+1, \; r+2, \ldots, s-1). \end{aligned}\right\} \qquad (6.3)$$

By means of (6.2) and (6.3) we see that $y^{(t)} = 0$, $(t = 1, 2, \ldots, s)$ has at least t distinct non-zero roots in the interval $(0, 1)$. Hence it follows that the equation $y^{(s+t)} = 0$, $(t = 1, 2, \ldots, s)$ has at least $s+t-1$ distinct non-zero roots in $(0, 1)$. Proceeding in this manner, we see that if $q < p$, the equation $y^{(qs-t)} = 0$, $(t = 1, 2, \ldots, s)$ has at least $(qs-t-q+1)$ distinct non-zero roots in $(0, 1)$. This in turn shows that $y^{(ps-t)} = 0$ $(t = r+1, \ldots, s)$ has at least $(ps-t-p+1)$ distinct non-zero roots in $(0, 1)$ and that the equation $y^{(n)} = 0$ has at least $(n-p)$, i.e. $\left(n - \frac{n+r}{s}\right)$ distinct non-zero roots in $(0, 1)$. The required result at once follows from this.

COROLLARY. *When $n = ps - r$, the roots of*

$${}_sF_{s-1}\left[\begin{matrix} \frac{-(s-1)n+r}{s}, \frac{n+1+r}{s}, \ldots, \frac{n+s-1+r}{s}; \; z \\ \frac{s+r-1}{s}, \ldots \frac{s+1}{s}, \frac{s+r}{s}, \frac{s-1}{s}, \ldots, \frac{r+1}{s} \end{matrix}\right] = 0$$

are all real and distinct and lie in the interval $(0, 1)$.

7. *Integral properties of $P_{n,s}(z)$.*

(*a*) *If ρ, ρ' be any two sth roots of unity then, for any path of integration*

$$\int_\rho^{\rho'} z^k P_{n,s}(z)\,dz = 0 \quad (k = 0, 1, \ldots, n-1). \qquad (7.1)$$

This is a particular case of the more general result :

For any path of integration

$$\int_{a_i}^{a_j} z^k \frac{d^n}{dz^n}\left\{\prod_{t=1}^{s}(z-a_t)^n\right\}dz = 0 \quad \begin{pmatrix} k = 0, 1, \ldots, n-1 \\ i, j = 1, 2, \ldots, s \end{pmatrix}. \tag{7.2}$$

The proof is easily supplied by repeated integration by parts. Conversely it can be shown that the only rational integral function $f(z)$ of degree $n(s-1)$ in z which is such that

$$\int_{a_i}^{a_j} f(z)z^k dz = 0, \quad \begin{pmatrix} i, j = 1, 2, \ldots, s \\ k = 0, \ldots, n-1 \end{pmatrix}$$

is of the form

$$f(z) = A\frac{d^n}{dz^n}\left\{\prod_{t=1}^{s}(z-a_t)^n\right\},$$

A being an arbitrary constant. If $\phi(z)$ be any polynomial of degree less than n it follows from (7.2) that

$$\int_{a_i}^{a_j} \phi(z)f(z) = 0, \quad (i, j = 1, 2, \ldots, s). \tag{7.3}$$

In particular, if ρ, ρ' be any two sth roots of unity,

$$\int_{\rho}^{\rho'} \phi(z)P_{n,s}(z)dz = 0.$$

(*b*) *If r be a positive integer or zero, and ρ, ρ' any two sth roots of unity, then*

$$\int_{\rho}^{\rho'} P_{n,s}(z)z^{n+r}dz = \frac{(n+r)!\,(\rho'^{r+1}-\rho^{r+1})}{r!\,(r+1)(r+1+s)\ldots(r+1+ns)}. \tag{7.4}$$

The proof readily follows by integrating the left side n times by parts.

In particular,

$$\int_{\rho}^{\rho'} P_{n,s}(z)z^{n+ks-1}dz = 0, \quad (k = 1, 2, \ldots). \tag{7.5}$$

(*c*) *For all values of k for which the integral*

$$\int_0^1 z^k P_{n,s}(z)dz$$

is convergent, it is equal to

$$\frac{k(k-1)\ldots(k-n+2)}{(k-n+1+s)(k-n+1+2s)\ldots(k-n+1+ns)}. \tag{7.6}$$

PROOF. $P_{n,s}(z)$ is of the form $\alpha z^{n(s-1)}+\beta z^{n(s-1)-s}+\dots$. Therefore

$$\int_0^1 z^k P_{n,s}(z)\,dz=\frac{\alpha}{k+n(s-1)+1}+\frac{\beta}{k+n(s-1)+1-s}+\dots .$$

The right side of this is reducible to the form

$$\frac{f(k)}{[k+n(s-1)+1][k+n(s-1)+1-s]\dots[k-n+1+s](k-n+1)},$$

where $f(k)$ is a polynomial in k. Suppose for the moment that $k>n$; then, on integrating by parts n times,

$$\int_0^1 z^k P_{n,s}(z)dz=\frac{1}{n!\,s^n}\int_0^1 z^k\frac{d^n}{dz^n}(z^s-1)^n dz$$

$$=\frac{k(k-1)\dots(k-n+1)}{n!\,s^n}\int_0^1 z^{k-n}(1-z^s)^n dz$$

$$=\frac{k(k-1)\dots(k-n+1)}{(ns+k-n+1)\dots(k-n+1)}.$$

Hence we find that

$$f(k)=k(k-1)\dots(k-n+1),\ k>1.$$

Since the form of $f(k)$ is invariable, it follows that for all k for which the integral is convergent

$$f(k)=k(k-1)\dots(k-n+1).$$

Thus we get the required result.

8. *Recurrence relations.*

(*a*) We have by application of Cauchy's theorem

$$P_{n,s}(z)=\frac{1}{2\pi i\,s^n}\int_C\frac{(t^s-1)^n dt}{(t-z)^{n+1}}, \qquad (8.1)$$

where C is any contour for which z is an interior point.

Now
$$\frac{d}{dt}\left[\frac{t(t^s-1)^n}{(t-z)^{n+1}}\right]$$

$$=n(s-1)\frac{(t^s-1)^n}{(t-z)^{n+1}}+sn\frac{(t^s-1)^{n-1}}{(t-z)^{n+1}}-(n+1)z\frac{(t^s-1)^n}{(t-z)^{n+2}}. \qquad (8.2)$$

Integrating both sides of (8.2) along C, we have

$$n(s-1)P_{n,s}(z)dz+\frac{d}{dz}P_{n-1,s}(z)-z\frac{d}{dz}P_{n,s}(z)=0. \qquad (8.3)$$

Integrating (8.3) we see that

$$\frac{n(s-1)}{n!\, s^n}\frac{d^{n-1}}{dz^{n-1}}(z^s-1)^n = zP_{n,s}(z)-P_{n-1,s}(z). \tag{8.4}$$

(*b*) Applying Lagrange's expansion theorem to the equation

$$z-a=\alpha\frac{z^s-1}{s}, \tag{8.5}$$

where α is sufficiently small so that

$$|z-a| > \left|\alpha\frac{z^s-1}{s}\right| \tag{8.6}$$

we get for the root ζ which is such that

$$|\zeta| \leqslant |a|,$$

the expansion

$$\zeta = a+\alpha\frac{a^s-1}{s}+\frac{\alpha^2}{2!}\frac{d}{da}\left(\frac{a^s-1}{s}\right)^2+\ldots+\frac{\alpha^n}{n!}\frac{d^{n-1}}{da^{n-1}}\left(\frac{a^s-1}{s}\right)^n+\ldots \tag{8.7}$$

Differentiating (8.7) with respect to a, we get

$$\frac{\partial\zeta}{\partial a} = 1+\alpha P_{1,s}(a)+\alpha^2 P_{2,s}(a)+\ldots+\alpha^n P_{n,s}(a)+\ldots \tag{8.8}$$

But since ζ satisfies (8.5), we get

$$\frac{\partial\zeta}{\partial a} = \frac{1}{1-\alpha\zeta^{s-1}}. \tag{8.9}$$

Substituting this in (8.8) we get

$$\frac{1}{1-\zeta^{s-1}\alpha} = 1+\alpha P_{1,s}(a)+\alpha^2 P_{2,s}(a)+\ldots. \tag{8.10}$$

If we take $a=\rho$, an sth root of unity, so that $\zeta=\rho$, we see by equating the coefficients of α^n on both sides of (8.10) that $P_{n,s}(\rho)=\rho^{-n}$, a result which we obtained earlier by induction.

Again, it can be seen that

$$\alpha\frac{\partial\zeta}{\partial\alpha} = (\zeta-a)\frac{\partial\zeta}{\partial a}. \tag{8.11}$$

Expanding both sides of (8.11) in powers of α and comparing coefficients, we find that

$$\frac{1}{(n-1)!}\frac{d^{n-1}}{da^{n-1}}\left(\frac{a^s-1}{s}\right)^n = \sum_{r=0}^{n-1} P_{n-r-1,\,s}(a)\frac{1}{(r+1)!}\frac{d^r}{da^r}\left(\frac{a^s-1}{s}\right)^{r+1} \qquad (8.12)$$

Hence, making use of (8.4), we get

$$\frac{n}{n(s-1)+1}\Big[P_{n-1,\,s}(a)-aP_{n,\,s}(a)\Big] = \sum_{r=1}^{n}\frac{P_{n-r,\,s}(a)}{r(s-1)+1}\Big[P_{r-1,\,s}(a)-aP_{r,\,s}(a)\Big]. \qquad (8.13)$$

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