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CONTENTS.

A paper should contain a short and clear summary of the new results obtained and the relations in which they stand to results already known. Contributors are requested to bear in mind that, at the present stage of mathematical research, hardly any paper is likely to be so completely original as to be independent of earlier work in the same direction; and that readers are often helped to appreciate the importance of a new investigation by seeing its connection with earlier results.

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In (B), put $r = \frac{1}{\sqrt{2}}$ and $\sin\theta = \frac{1}{\sqrt{2}} = \cos\theta$. Then,

$$-\left[\frac{e^{\frac{\pi}{2}}}{1+e^{\pi}} - 3^{2n+1}\,\frac{e^{\frac{3\pi}{2}}}{1+e^{3\pi}} + 5^{2n+1}\cdot\frac{e^{\frac{5\pi}{3}}}{1+e^{5\pi}} - \ldots\ldots\right]$$

$$+\,2^{n+1}.(-1)^{n+1}.\sin(n+1)\frac{\pi}{2}\left[\frac{1}{1-e^{\pi}} + \frac{3^{2n+1}}{1+e^{3\pi}} + \frac{5^{2n+1}}{1-e^{5\pi}} + \ldots\ldots\right.$$

$$= \frac{B_{n+1}}{4(n+1)}.(2^{2n+1}-1)\left\{-\sin(n+1)\frac{\pi}{2}\right\}.2^{n+1}.$$

Here, take n to be *odd*. Then, the 2nd part of the L. H. S. and the R. H. S. are equal to zero, and we get

$$\operatorname{sech}\left(\frac{\pi}{2}\right) - 3^{2n+1}\operatorname{sech}\left(\frac{3\pi}{2}\right) + 5^{2n+1}\operatorname{sech}\left(\frac{5\pi}{2}\right) -.\ \ldots\ldots = 0^{*}, \quad \ldots \quad \text{(IV)}$$

the problem of Ramanujan referred to above.

§ 3. In result IX of Part I, put

$$a = r\,(\cos\theta + i\sin\theta),$$

take the logarithms of both sides, separate into real and imaginary parts and equate the imaginary parts; we have

$$\left[\tan^{-1}\frac{e^{-2\pi r\cos\theta}.\sin(2\pi r\sin\theta)}{1-e^{-2\pi r\cos\theta}.\cos(2\pi r\sin\theta)} + \tan^{-1}\frac{e^{-4\pi r\cos\theta}.\sin(4\pi r\sin\theta)}{1-e^{-4\pi r\cos\theta}.\cos(4\pi r\sin\theta)} + \ldots\ldots\right]$$

$$+\left\{\tan^{-1}\frac{e^{-\frac{2\pi}{r}\cos\theta}\sin\left(\frac{2\pi}{r}\sin\theta\right)}{1-e^{-\frac{2\pi}{r}\cos\theta}\cos\left(\frac{2\pi}{r}\sin\theta\right)} + \tan^{-1}\frac{e^{-\frac{4\pi}{r}\cos\theta}\sin\left(\frac{4\pi}{r}\sin\theta\right)}{1-e^{-\frac{4\pi}{r}\cos\theta}\cos\left(\frac{4\pi}{r}\sin\theta\right)} + \ldots\ldots\right\}$$

$$= \frac{\pi.\sin\theta}{12}\left(r + \frac{1}{r}\right) - \frac{\theta}{2}. \quad \ldots \quad \ldots \text{(C)}$$

* *Vide J. I. M. S.*, Vol. IV., No. 2, Question 358.

It can easily be seen that, in general, such values of r and θ as satisfy the equations

$$\left.\begin{aligned} 2r\sin\theta &= \frac{2n+1}{2} \\ \frac{2}{r}\sin\theta &= \frac{2m+1}{2} \end{aligned}\right\},$$

where m and n are zero or positive integers, will lead to neat results. However, we should have

$$\sin^2\theta = \frac{(2n+1)(2m+1)}{16}.$$

This condition clearly limits the number of cases. The various particular values of n and m, therefore, which would give us distinct results are

$$n = 0;\ m = 0, 1, 2, 3, 4, 5, 6 \text{ or } 7;$$
$$n = 1;\ m = 0, 1, \text{ or } 2;$$
$$n = 2;\ m = 0, \text{ or } 1.$$

Even among these, three cases do not give distinct values for r and θ. Hence, the values of r that may be used are

$$r = 1, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{5}}, \frac{1}{\sqrt{7}}, \frac{1}{3}, \frac{1}{\sqrt{11}}, \frac{1}{\sqrt{13}}, \frac{1}{\sqrt{15}}, 1 \text{ and } \frac{3}{\sqrt{15}}$$

the corresponding values for $\sin\theta$, in order, being

$$\sin\theta = \frac{1}{4}, \frac{\sqrt{3}}{4}, \frac{\sqrt{5}}{4}, \frac{\sqrt{7}}{4}, \frac{3}{4}, \frac{\sqrt{11}}{4}, \frac{\sqrt{13}}{4}, \frac{\sqrt{15}}{4}, \frac{3}{4}, \text{ and } \frac{\sqrt{15}}{4}.$$

Among the above sets of values, two, *viz.*,

$$r = 1, \sin\theta = \tfrac{1}{4}$$
$$r = 1, \sin\theta = \tfrac{3}{4}$$

lead immediately to two single series and the remaining 8 sets to sum or difference of two series.

If, in (C), we put $r = 1$, $\sin\theta = \frac{1}{4}$, and $\cos\theta = \frac{\sqrt{15}}{4}$, then

$$2\left\{\tan^{-1} e^{-\frac{\pi}{2}\sqrt{15}} - \tan^{-1} e^{-\frac{3\pi}{2}\sqrt{15}} + \tan^{-1} e^{-\frac{5\pi}{2}\sqrt{15}} - \ldots\right\}$$
$$= \frac{\pi}{48}\cdot 2 \cdot - \frac{1}{2}\sin^{-1}\frac{1}{4},$$

or $$\tan^{-1} e^{-\frac{\pi}{2}\sqrt{15}} - \tan^{-1} e^{-\frac{3\pi}{2}\sqrt{15}} + \tan^{-1} e^{-\frac{5\pi}{2}\sqrt{15}} - \ldots\ldots$$
$$= \frac{\pi}{48} - \frac{1}{4}\tan^{-1}\frac{1}{\sqrt{15}}. \quad \ldots \text{(V)}$$

Similarly, if we put $r = 1$, $\sin\theta = \frac{3}{4}$, $\cos\theta = \frac{\sqrt{7}}{4}$, we have

$$\tan^{-1} e^{-\frac{\pi}{2}\sqrt{7}} - \tan^{-1} e^{-\frac{3\pi}{2}\sqrt{7}} + \tan^{-1} e^{-\frac{5\pi}{2}\sqrt{7}} - \ldots\ldots$$

$$= \frac{1}{4}\tan^{-1}\frac{3}{\sqrt{7}} - \frac{\pi}{16}. \quad \ldots \quad \text{(VI)}$$

Proceeding in an exactly similar way with the remaining sets of values, we obtain the following :—

$$\left[\tan^{-1} e^{-\frac{\pi}{2}\sqrt{\frac{13}{3}}} - \tan^{-1} e^{-\frac{3\pi}{2}\sqrt{\frac{13}{3}}} + \tan^{-1} e^{-\frac{5\pi}{2}\sqrt{\frac{13}{3}}} - \ldots\right]$$

$$- \left[\tan^{-1} e^{-\frac{\pi}{2}\sqrt{13.3}} - \tan^{-1} e^{-\frac{3\pi}{2}\sqrt{13.3}} + \tan^{-1} e^{-\frac{5\pi}{2}\sqrt{13.3}} - \ldots\right]$$

$$= \frac{\pi}{12} - \frac{1}{2}\tan^{-1}\sqrt{\frac{3}{13}}. \quad \ldots \quad \text{(VII)}$$

$$\left[\tan^{-1} e^{-\frac{\pi}{12}\sqrt{\frac{11}{5}}} - \tan^{-1} e^{-\frac{3\pi}{2}\sqrt{\frac{11}{5}}} + \tan^{-1} e^{-\frac{5\pi}{2}\sqrt{\frac{11}{5}}} - \ldots\right]$$

$$+ \left[\tan^{-1} e^{-\frac{\pi}{2}\sqrt{11.5}} - \tan^{-1} e^{-\frac{3\pi}{2}\sqrt{11.5}} + \tan^{-1} e^{-\frac{5\pi}{2}\sqrt{11.5}} - \ldots\right]$$

$$= \frac{\pi}{8} - \frac{1}{2}\tan^{-1}\sqrt{\frac{5}{11}}. \quad \ldots \quad \ldots \quad \text{(VIII)}$$

$$\left[\tan^{-1} e^{-\frac{\pi}{2}\cdot\frac{3}{\sqrt{7}}} - \tan^{-1} e^{-\frac{3\pi}{2}\cdot\frac{3}{\sqrt{7}}} + \tan^{-1} e^{-\frac{5\pi}{2}\cdot\frac{3}{\sqrt{7}}} - \ldots\right]$$

$$- \left[\tan^{-1} e^{-\frac{\pi}{2}\cdot 3\sqrt{7}} - \tan^{-1} e^{-\frac{3\pi}{2}\cdot 3\sqrt{7}} + \tan^{-1} e^{-\frac{5\pi}{2} 3\cdot\sqrt{7}} - \ldots\right]$$

$$= \frac{\pi}{6} - \tfrac{1}{2}\tan^{-1}\left(\frac{\sqrt{7}}{3}\right). \quad \ldots \quad \ldots \quad \text{(IX)}$$

$$\left[\tan^{-1} e^{-\frac{\pi}{2}\cdot\frac{\sqrt{7}}{3}} - \tan^{-1} e^{-\frac{3\pi}{2}\cdot\frac{\sqrt{7}}{3}} + \tan^{-1} e^{-\frac{5\pi}{2}\frac{\sqrt{7}}{3}} - \ldots\right]$$

$$+ \left[\tan^{-1} e^{-\frac{\pi}{2}\cdot 3\sqrt{7}} - \tan^{-1} e^{-\frac{3\pi}{2}\cdot 3\sqrt{7}} + \tan^{-1} e^{-\frac{5\pi}{2}, 3\sqrt{7}} - \ldots\right]$$

$$= \frac{5\pi}{24} - \frac{1}{2}\tan^{-1}\left(\frac{3}{\sqrt{7}}\right) \quad \ldots \quad \ldots \quad \text{(X)}$$

$$\left[\tan^{-1} e^{-\frac{\pi}{2}\sqrt{\frac{5}{11}}} - \tan^{-1} e^{-\frac{3\pi}{2}\sqrt{\frac{5}{11}}} + \tan^{-1} e^{-\frac{5\pi}{2}\sqrt{\frac{5}{11}}} - \dots\right]$$

$$- \left[\tan^{-1} e^{-\frac{\pi}{2}\sqrt{5.11}} - \tan^{-1} e^{-\frac{3\pi}{2}.\sqrt{5.11}} + \tan^{-1} e^{-\frac{5\pi}{2}\sqrt{5.11}} - \dots\right]$$

$$= \frac{\pi}{4} - \frac{1}{2}\tan^{-1}\left(\frac{\sqrt{11}}{5}\right). \quad \dots \quad \dots \quad \text{(XI)}$$

$$\left[\tan^{-1} e^{-\frac{\pi}{2}\sqrt{\frac{3}{13}}} - \tan^{-1} e^{-\frac{3\pi}{2}\sqrt{\frac{3}{13}}} + \tan^{-1} e^{-\frac{5\pi}{2}\sqrt{\frac{3}{13}}} - \dots\right]$$

$$- \left[\tan^{-1} e^{-\frac{\pi}{2}\sqrt{13.3}} - \tan^{-1} e^{-\frac{3\pi}{2}\sqrt{13.3}} + \tan^{-1} e^{-\frac{5\pi}{2}\sqrt{13.3}} - \dots\right]$$

$$= \frac{7\pi}{24} - \frac{1}{2}\tan^{-1}\left(\frac{\sqrt{13}}{3}\right). \dots \quad \dots \quad \text{(XII)}$$

$$\left[\tan^{-1} e^{-\frac{\pi}{2}\cdot\frac{1}{\sqrt{15}}} - \tan^{-1} e^{-\frac{3\pi}{2}\cdot\frac{1}{\sqrt{15}}} + \tan^{-1} e^{-\frac{5\pi}{2}\cdot\frac{1}{\sqrt{15}}} - \dots\right]$$

$$- \left[\tan^{-1} e^{-\frac{\pi}{2}\sqrt{15}} - \tan^{-1} e^{-\frac{3\pi}{2}\sqrt{15}} + \tan^{-1} e^{-\frac{5\pi}{2}\sqrt{15}} - \dots\right]$$

$$= \frac{\pi}{3} - \frac{1}{2}\tan^{-1}(\sqrt{15}). \quad \dots \quad \dots \quad \text{(XIII)}$$

$$\left[\tan^{-1} e^{-\frac{\pi}{2}\cdot\frac{3}{\sqrt{15}}} - \tan^{-1} e^{-\frac{3\pi}{2}\cdot\frac{3}{\sqrt{15}}} + \tan^{-1} e^{-\frac{5\pi}{2}\cdot\frac{3}{\sqrt{15}}} - \dots\right]$$

$$- \left[\tan^{-1} e^{-\frac{\pi}{2}\cdot\frac{\sqrt{15}}{3}} - \tan^{-1} e^{-\frac{3\pi}{2}\cdot\frac{\sqrt{15}}{3}} + \tan^{-1} e^{-\frac{5\pi}{2}\cdot\frac{\sqrt{15}}{3}} - \dots\right]$$

$$= \frac{1}{2}\tan^{-1}(\sqrt{15}) - \frac{\pi}{6}. \quad \dots \quad \dots \quad \text{(XIV)}$$

We could also try a different set of values of r and θ, namely, those that satisfy

$$2r \sin\theta = \frac{2n+1}{2} \text{ and } \frac{2}{r}\sin\theta = m.$$

where m and n are zero or positive integers. Of the various values of r and θ to which these lead, some are useful. When

$$r = \frac{1}{2}, \ \frac{1}{\sqrt{6}}, \text{ or } \frac{1}{\sqrt{8}},$$

and
$$\sin\theta = \frac{1}{2}, \ \sqrt{\frac{3}{8}}, \text{ or } \frac{1}{\sqrt{2}},$$

we obtain

$$\tan^{-1} e^{-\frac{\pi}{2}\sqrt{3}} - \tan^{-1} e^{-\frac{3\pi}{2}\sqrt{3}} + \tan^{-1} e^{-\frac{5\pi}{2}\sqrt{3}} - \ldots\ldots$$

$$= \frac{\pi}{48}. \quad \ldots \quad \ldots \quad \text{(XV)}$$

$$\tan^{-1} e^{-\frac{\pi}{2}\cdot\frac{\sqrt{15}}{3}} - \tan^{-1} e^{-\frac{3\pi}{2}\cdot\frac{\sqrt{15}}{3}} + \tan^{-1} e^{-\frac{5\pi}{2}\cdot\frac{\sqrt{15}}{3}} - \ldots\ldots$$

$$= \frac{\pi}{48} - \frac{1}{2}\tan^{-1}\sqrt{\frac{3}{5}}. \quad \ldots \quad \text{(XVI)}$$

$$\tan^{-1} e^{-\frac{\pi}{2}} - \tan^{-1} e^{-\frac{3\pi}{2}} + \tan^{-1} e^{-\frac{5\pi}{2}} - \ldots\ldots = \frac{\pi}{16}. \quad \text{(XVII)}$$

Certain other values of r and θ, *viz.*,

$$r = \frac{1}{\sqrt{10}}, \frac{1}{\sqrt{12}} \text{ and } \frac{1}{\sqrt{14}},$$

$$\sin\theta = \sqrt{\tfrac{5}{8}},\ \sqrt{\tfrac{3}{4}}, \text{ and } \sqrt{\tfrac{7}{8}};$$

lead to the following :—

$$\tan^{-1} e^{-\frac{\pi}{2}\cdot\frac{3}{\sqrt{15}}} - \tan^{-1} e^{-\frac{3\pi}{2}\cdot\frac{3}{\sqrt{15}}} + \tan^{-1} e^{-\frac{5\pi}{2}\cdot\frac{3}{\sqrt{15}}} - \ldots\ldots$$

$$= \frac{11\pi}{48} - \frac{1}{2}\tan^{-1}\sqrt{\frac{5}{3}}. \quad \ldots \quad \text{(XVIII)}$$

$$\tan^{-1} e^{-\frac{\pi}{2}\cdot\frac{1}{\sqrt{3}}} - \tan^{-1} e^{-\frac{3\pi}{2}\cdot\frac{1}{\sqrt{3}}} + \tan^{-1} e^{-\frac{5\pi}{2}\cdot\frac{1}{\sqrt{3}}} - \ldots\ldots$$

$$= \frac{5\pi}{48} \quad \ldots \quad \ldots \quad \text{(XIX)}$$

and

$$\tan^{-1} e^{-\frac{\pi}{2}\cdot\frac{1}{\sqrt{7}}} - \tan^{-1} e^{-\frac{3\pi}{2}\cdot\frac{1}{\sqrt{7}}} + \tan^{-1} e^{-\frac{5\pi}{2}\cdot\frac{1}{\sqrt{7}}} - \ldots\ldots$$

$$= \frac{5\pi}{16} - \frac{1}{2}\tan^{-1}(\sqrt{7}). \quad \ldots \quad \text{(XX)}$$

(It may be noticed in passing that results XVI and XVIII give separately the values of the two parts in result XIV.)

§ 4. Now, let us take the result 8 of Part I; therein put

$$a = r(\cos\theta + i\sin\theta),$$

take the logarithms of both sides, separate into real and imaginary parts and then equate the imaginary parts. We obtain

$$\left[\tan^{-1}\frac{e^{-\pi r\cos\theta}\sin(\pi r\sin\theta)}{1+e^{-\pi r\cos\theta}\cos(\pi r\sin\theta)}\right.$$

$$\left.+\tan^{-1}\frac{e^{-3\pi r\cos\theta}.\sin(3\pi r\sin\theta)}{1+e^{-3\pi r\cos\theta}\cos(3\pi r\sin\theta)}+\ldots\ldots\right]$$

$$+\left\{\tan^{-1}\frac{e^{-\frac{\pi}{r}\cos\theta}.\sin\left(\frac{\pi}{r}\sin\theta\right)}{1+e^{-\frac{\pi}{r}\cos\theta}\cos\left(\frac{\pi}{r}\sin\theta\right)}\right.$$

$$\left.+\tan^{-1}\frac{e^{-\frac{3\pi}{r}\cos\theta}\sin\left(\frac{3\pi}{r}\sin\theta\right)}{1+e^{-\frac{3\pi}{r}\cos\theta}\left(\cos\frac{3\pi}{r}\sin\theta\right)}+\ldots\ldots\right\}$$

$$=\frac{\pi\sin\theta}{24}.\left(r+\frac{1}{r}\right) \qquad \ldots \quad (D)$$

Here, values of r and θ which make

$$2r\sin\theta = 2n+1,$$

and

$$\frac{2}{r}\sin\theta = 2m+1$$

lead to neat results. Among them are

$$r = 1,\ \frac{1}{\sqrt{3}} \text{ and } \sqrt{3},$$

$$\sin\theta = \tfrac{1}{2},\ \frac{\sqrt{3}}{2} \text{ and } \frac{\sqrt{3}}{2}.$$

Of these, $r = 1,\ \sin\theta = \frac{1}{2}$ give

$$\tan^{-1}e^{-\frac{\pi}{2}\sqrt{3}} - \tan^{-1}e^{-\frac{3\pi}{2}\sqrt{3}} + \tan^{-1}e^{-\frac{5\pi}{2}\sqrt{3}} - \ldots\ldots = \frac{\pi}{48},$$

which is the result XV already got.

Both $r = \frac{1}{\sqrt{3}}$ and $r = \sqrt{3}$ with $\sin\theta = \frac{\sqrt{3}}{2}$ lead to the same series, *viz.*:—

$$\left[\tan^{-1} e^{-\frac{\pi}{2\sqrt{3}}} - \tan^{-4} e^{-\frac{3\pi}{2\sqrt{3}}} + \tan^{-1} e^{-\frac{5\pi}{2\sqrt{3}}} - \ldots\ldots\right]$$

$$-\left[\tan^{-1} e^{-\frac{\pi}{2}.\sqrt{3}} - \tan^{-1} e^{-\frac{3\pi}{2}\sqrt{3}} + \tan^{-1} e^{-\frac{5\pi}{2}.\sqrt{3}} - \ldots\ldots\right]$$

$$= \frac{\pi}{12},$$

not, however, a new result as can be seen on taking the difference between results XIX and XV above.

§ 5. We shall now derive in a direct manner a very important result connected with the above type of series. The result is occasionally useful in evaluating some series among the above.

It is well-known that

$$\int_0^\infty \frac{2\cos 2xt}{\cosh \pi t}\,dt = \operatorname{sech} x.^*$$

$$\therefore \quad \int_0^\infty \frac{2\sin xt \sin at}{\cosh \pi t}\,dt = \tfrac{1}{2}\left[\operatorname{sech}\frac{x-a}{2} - \operatorname{sech}\frac{x+a}{2}\right]$$

Now,
$$\int_0^\infty \frac{2\sin at}{\cosh \pi t}\{2\sin xt - 2\sin 3xt$$

$$+ \ldots\ldots + 2\sin(4n+1)xt\}\,dt$$

$$= \int_0^\infty \frac{2\sin at}{\cosh \pi t}\cdot\frac{\sin(2n+1)2xt}{\cos xt}\,dt. \qquad \ldots \quad \text{(E)}$$

On the L. H. S. of (E), integrate term by term and let n tend to infinity. Then we have

$$\left(\operatorname{sech}\frac{x-a}{2} - \operatorname{sech}\frac{3x-a}{2} + \operatorname{sech}\frac{5x-a}{2} - \ldots\ldots\right)$$

$$-\left(\operatorname{sech}\frac{x+a}{2} - \operatorname{sech}\frac{3x+a}{2} + \operatorname{sech}\frac{5x+a}{2} - \ldots\ldots\right). \qquad \text{(F)}$$

---

* *Vide* Bromwich: *Introduction to Infinite Series*, p. 472, Ex. 26.

Again, taking the R. H. S.,

$$\operatorname*{Lt}_{n\to\infty} \int \frac{\sin(2n+1)\,2xt}{\sin 2xt} \cdot \frac{2\sin xt\, 2\sin at}{\cosh \pi t}\, dt$$

becomes, on changing t into $\frac{t}{2x}$,

$$\operatorname*{Lt}_{n\to\infty} \int_0^\infty \frac{\sin(2n+1)\,t}{\sin t} \cdot \left\{ \frac{\sin\frac{t}{2} \,.\, 2 \,.\, \sin\left(\frac{at}{2x}\right)}{x\cosh\left(\frac{\pi t}{2x}\right)} \right\} dt$$

which on using the result of Ex. 4 on p. 447 of Bromwich's *Introduction to Infinite Series,* gives

$$\frac{\pi}{2} \cdot \left\{ \frac{4}{x} \frac{\sin\left(\frac{a\pi}{2x}\right)}{\cosh\left(\frac{\pi^2}{2x}\right)} - \frac{4}{x} \cdot \frac{\sin\left(\frac{3a\pi}{2x}\right)}{\cosh\left(\frac{3\pi^2}{2x}\right)} + \ldots \right\} \tag{G}$$

Equating (F) and (G),

$$\left[\operatorname{sech}\frac{x-a}{2} - \operatorname{sech}\frac{3x-a}{2} + \operatorname{sech}\frac{5x-a}{2} - \ldots\ldots\right]$$

$$-\left[\operatorname{sech}\frac{x+a}{2} - \operatorname{sech}\frac{3x+a}{2} + \operatorname{sech}\frac{5x+a}{2} - \ldots\ldots\right]$$

$$= \frac{2\pi}{x} \cdot \left\{ \frac{\sin\left(\frac{a\pi}{2x}\right)}{\cosh\left(\frac{\pi^2}{2x}\right)} - \frac{\sin\left(\frac{3a\pi}{2x}\right)}{\cosh\left(\frac{3\pi^2}{2x}\right)} + \ldots\ldots \right\} \tag{XXI}$$

Put $a = x$ in the above and transpose. We get

$$(\operatorname{sech} x - \operatorname{sech} 2x + \operatorname{sech} 3x - \ldots\ldots)$$

$$+ \frac{\pi}{x}\left(\operatorname{sech}\frac{\pi^2}{2x} - \operatorname{sech}\frac{3\pi^2}{2x} + \operatorname{sech}\frac{5\pi^2}{2x} - \ldots\ldots\right) = \frac{1}{2}. \tag{XXII}$$

§ 6. Now, expand each term on the L. H. S. of XXI. Then

$$2\left[e^{-\frac{x-a}{2}} - e^{-3.\frac{x-a}{2}} + e^{-5.\frac{x-a}{2}} - \ldots\ldots\right]$$

$$-2\left[e^{-\frac{3x-a}{2}} - e^{-3\frac{3x-a}{2}} + e^{-5.\frac{3x-a}{2}} - \ldots\ldots\right]$$

$$+2\left[e^{-\frac{5x-a}{2}} - e^{-3.\frac{5x-a}{2}} + e^{-5.\frac{5x-a}{3}} - \ldots\ldots\right]$$

$$- \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$-2\left[e^{-\frac{x+a}{2}} - e^{-3.\frac{x+a}{2}} + e^{-5.\frac{x+a}{2}} - \ldots\ldots\right]$$

$$+2\left[e^{-\frac{3x+a}{2}} - e^{-3.\frac{3x+a}{2}} + e^{-5.\frac{5x+a}{2}} - \ldots\ldots\right]$$

$$-2\left[e^{-\frac{5x+a}{2}} - e^{-3.\frac{5x+a}{2}} + e^{-5.\frac{5x+3}{2}} - \ldots\ldots\right]$$

$$+ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$= \frac{2\pi}{x}.\left[\sin\left(\frac{a\pi}{2x}\right)\operatorname{sech}\left(\frac{\pi^2}{2x}\right) - \sin\left(\frac{3a\pi}{2x}\right)\operatorname{sech}\left(\frac{3\pi^2}{2x}\right) + \sin\left(\frac{5a\pi}{2x}\right)\operatorname{sech}\left(\frac{5\pi^2}{2x}\right) - \ldots\ldots\right].$$

Adding each set on the L. H. S. column by column,

$$\left[e^{\frac{a}{2}}\operatorname{sech}\left(\frac{x}{2}\right) - e^{\frac{3a}{2}}\operatorname{sech}\left(\frac{3x}{2}\right) - e^{\frac{5a}{2}}.\operatorname{sech}\left(\frac{5x}{2}\right) - \ldots\ldots\right]$$

$$-\left[e^{-\frac{a}{2}}\operatorname{sech}\left(\frac{x}{2}\right) - e^{-\frac{3a}{2}}\operatorname{sech}\left(\frac{3x}{2}\right) + e^{-\frac{5a}{2}}\operatorname{sech}\left(\frac{5x}{2}\right) - \ldots\ldots\right]$$

$$= \frac{2\pi}{x}\left[\sin\left(\frac{a\pi}{2x}\right)\operatorname{sech}\left(\frac{\pi^2}{2x}\right) - \sin\left(\frac{3a\pi}{x}\right)\operatorname{sech}\left(\frac{3\pi^2}{2x}\right) + \sin\left(\frac{5a\pi}{2x}\right)\operatorname{sech}\left(\frac{5\pi^2}{2x}\right) - \ldots\ldots\right].$$

Or,

$$\sinh\left(\frac{a}{2}\right)\operatorname{sech}\left(\frac{x}{2}\right) - \sinh\left(\frac{3a}{2}\right)\operatorname{sech}\left(\frac{3x}{2}\right) + \sinh\left(\frac{5a}{2}\right)\operatorname{sech}\left(\frac{5x}{2}\right) - \ldots\ldots$$

$$= \frac{\pi}{x}.\left[\sin\left(\frac{a\pi}{2x}\right)\operatorname{sech}\left(\frac{\pi^2}{2x}\right) - \sin\left(\frac{3a\pi}{2x}\right)\operatorname{sech}\left(\frac{3\pi^2}{2x}\right) + \sin\left(\frac{5a\pi}{2x}\right)\operatorname{sech}\left(\frac{5\pi^2}{2x}\right) - \ldots\ldots\right] \ldots \quad \text{(XXIII)}$$

§ 7. Here, equate the co-efficients of a^{2n+1} on both sides. We have

$$\frac{1}{2^{2n+1}.(2n+1)!}.\left\{\text{sech}\left[\frac{x}{2}\right]-3^{2n+1}.\text{sech}\left[\frac{3x}{2}\right]+5^{2n+1}.\text{sech}\left[\frac{5x}{2}\right]-\ldots\ldots\right\}$$

$$=\frac{\pi}{x}.\frac{(-1)^n}{(2n+1)!}.\frac{\pi^{2n+1}}{x^{2n+1}.2^{2n+1}}\times .\left\{\text{sech}\left[\frac{\pi^2}{2x}\right]-3^{2n+1}\,\text{sech}\left[\frac{3\pi^2}{2x}\right]+\ldots\ldots\right\}$$

Changing x into πx,

$$\text{sech}\left[\frac{\pi x}{2}\right]-3^{2n+1}.\text{sech}\left[\frac{3\pi x}{2}\right]+\ldots\ldots$$

$$=\frac{(-1)^n}{x^{2n+2}}.\left\{\text{sech}\left[\frac{\pi}{2x}\right]-3^{2n+1}.\text{sech}\left[\frac{3\pi}{2x}\right]+\ldots\ldots\right\}. \quad \text{(XXIV)}$$

If in the above, x is put equal to 1 and n is taken to be odd, we obtain

$$\text{sech}\left[\frac{\pi}{2}\right]-3^{2n+1}\,\text{sech}\left[\frac{3\pi}{2}\right]+\ldots\ldots=0,$$

a result already obtained as result IV though in a different manner.

§ 8. In XXIV, put $n=0$. Then

$$\text{sech}\left[\frac{\pi x}{2}\right]-3\,\text{sech}\left[\frac{3\pi x}{2}\right]+\ldots\ldots$$

$$=\frac{1}{x^2}\left\{\text{sech}\left[\frac{\pi}{2x}\right]-3\,\text{sech}\left[\frac{3\pi}{2x}\right]+\ldots\ldots\right\}.$$

Multiply both sides of the above by $-\frac{\pi}{4}$ and integrate with respect to x. Then,

$$\tan^{-1}\left[e^{-\frac{\pi x}{2}}\right]-\tan^{-1}\left[e^{-\frac{3\pi x}{2}}\right]+\ldots\ldots$$

$$=-\tan^{-1}\left[e^{-\frac{\pi}{2x}}\right]+\tan^{-1}\left[e^{-\frac{3\pi}{2x}}\right]-\ldots\ldots+\text{a constant}.$$

Putting $x = 0$, the constant is found to be equal to $\frac{\pi}{8}$.

Therefore, we get

$$\left\{\tan^{-1}\left[e^{-\frac{\pi x}{2}}\right] - \tan^{-1}\left[e^{-\frac{3\pi x}{2}}\right] + \ldots\ldots\right\}$$

$$+\left\{\tan^{-1}\left[e^{-\frac{\pi}{2x}}\right] - \tan^{-1}\left[e^{-\frac{3\pi}{2x}}\right] + \ldots\ldots\right\}$$

$$= \frac{\pi}{8}.* \qquad \ldots \quad \text{(XXV)}$$

With this result, we could separate into single series results like XIV and evaluate the related series if only one of the two parts above are known in value. If, in this, we put $x = 1$, we get

$$\tan^{-1}\left[e^{-\frac{\pi}{2}}\right] - \tan^{-1}\left[e^{-\frac{3\pi}{2}}\right] + \tan^{-1}\left[e^{-\frac{5\pi}{2}}\right]$$

$$- \ldots\ldots = \frac{\pi}{16},$$

a result otherwise obtained as result XVII.

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* *Cp.* Dr. Glaisher's result in Vol. 5 of the *Messenger of Mathematics*.

Summable Double Series.

By V. THIRUVENKATACHARYA, M.A., L.T.

[*Continued from Vol. XV, No. 10, p. 232.*]

6. Relations between

$$\overset{\infty\,\infty}{\underset{0\,0}{S\,S}}\, u_{mn},\ \overset{\infty\,\infty}{\underset{1\,0}{S\,S}}\, u_{mn},\ \overset{\infty\,\infty}{\underset{0\,1}{S\,S}}\, u_{mn} \text{ and } \overset{\infty\,\infty}{\underset{1\,1}{S\,S}}\, u_{mn}$$

Let
$$S = \overset{\infty\,\infty}{\underset{0\,0}{S\,S}}\, u_{mn} = \int_0^\infty \int_0^\infty e^{-x-y}\, u(x,y)\, dxdy, \ \ldots \qquad 6{\cdot}1$$

$$S_x = \overset{\infty\,\infty}{\underset{1\,0}{S\,S}}\, u^{mn} = \int_0^\infty \int_0^\infty e^{-x-y} \left[\frac{\partial}{\partial x} u(x,y) \right] dxdy, \ \ldots \qquad 6{\cdot}2$$

$$S_y = \overset{\infty\,\infty}{\underset{0\,1}{S\,S}}\, u_{mn} = \int_0^\infty \int_0^\infty e^{-x-y} \left[\frac{\partial}{\partial y} u(x,y) \right] dxdy, \ \ldots \qquad 6{\cdot}3$$

and
$$S_{xy} = \overset{\infty\,\infty}{\underset{1\,1}{S\,S}}\, u_{mn} = \int_0^\infty \int_0^\infty e^{-x-y} \left[\frac{\partial^2}{\partial x \partial y} u(x,y) \right] dxdy. \qquad 6{\cdot}4$$

We have to justify the term by term differentiation of $u(x,y)$ with respect to x and y. It has been already supposed that the u's are such that $u(x, y)$ is convergent in Pringsheim's sense for all values of x and y; and $u(x, y)$ is a double power series in x and y, the associated radii of convergence of which are as large as possible. Therefore the double power series converges uniformly and can be differentiated any number of times with respect to either x or y or both.*

To establish the relations between S, S_x, S_y and S_{xy}, let us take S_x first and integrate it by parts.

$$S_x = \int_0^\infty \left[e^{-x-y}\, u(x,y) \right]_{x=0}^{x=\infty} dy + \int_0^\infty \int_0^\infty e^{-x-y}\, u(x,y)\, dxdy$$

$$= \int_0^\infty \left[e^{-x-y}\, u(x,y) \right]_{x=0}^{x=\infty} dy + S. \qquad \ldots \qquad \ldots \quad 6{\cdot}5$$

Equation 6·5 shows that if S and S_x exist, then

$$\lim_{x \to \infty} e^{-x-y}\, u(x,y)$$

---

* *Vide* Goursat and Hedrick : *Mathematical Analysis*, Vol. I, p. 395.

must exist and be zero since otherwise, the integral $\int_0^\infty e^{-x-y}\, u(x, y)\, dx$ will not converge; but this is necessary for the convergence* and the uniqueness of the double integral S. It also follows from the same equation that $\int_0^\infty e^{-y}\, u(0, y)\, dy$ must converge; but $\int_0^\infty e^{-y}\, u(0, y)\, dy$ is the sum of the first row and the convergence of this integral implies that the row should be summable. Hence provided that the integrals S and S_x are convergent and the first row is summable.

$$S_x = -\int_0^\infty e^{-y}\, u(o, y)\, dy + S$$

i.e.,
$$S - S_x = \int_0^\infty e^{-y}\, u(o, y)\, dy = \text{sum of the first row.} \qquad 6{\cdot}6$$

Integrating S_y by parts, we have

$$S - S_y = \int_0^\infty e^{-x}\, u(x, o)\, dx = \text{sum of the first column,} \qquad 6{\cdot}7$$

provided the integrals S and S_y are convergent and the first column is summable,.

Integrating S_{xy} by parts, both with respect to x and y, we have

$$S_{xy} = u(o, o) - \int_0^\infty e^{-y}\, u(o, y)\, dy - \int_0^\infty e^{-x}\, u(x, o)\, dx + S. \qquad 6{\cdot}8$$

$\therefore\ S - S_{xy}$ = sum of first row + sum of first column $- u(o, o)$.

But $u(o, o) = u_{00}$, the initial term of the double series.

$\therefore\ S - S_{xy}$ = sum of first row + sum of first column $- u_{00}$. 6·9

It will be seen from this that *summable double series possess properties in this respect analogous to convergent series.*

Again, the convergence of the double integral

$$\int_0^\infty \int_0^\infty e^{-x-y}\ u(x, y)\, dx dy$$

does not ensure the existence of a limit for $e^{-x-y}\, u(x, y)$ when either x or y or both tend to infinity. † *Therefore if S exists, it does not follow from this that either S_x or S_y or S_{xy} exists.*

---

* Throughout this article, it is assumed that the infinite double integrals satisfy de la Valee Poussin's conditions, also, whenever they are said to be convergent.

† To see that it is possible, in general, let us use a graphical method. Consider a mountainous tract of country with a number of peaks steadily increasing in heights, the sea-level representing the xy plane. Then it is quite possible to

7. Although the convergence of $\int_0^\infty \int_0^\infty e^{-x-y}\, u(x, y)\, dxdy$ does not imply the existence of a limit for $e^{-x-y}\, u(x, y)$ as x or y or both tend to infinity, yet we can prove that $\lim e^{-x-y}\, u(x, y) = 0$ as x and $y \to \infty$ when $\int_0^\infty \int_0^\infty e^{-x-y} \frac{\partial}{\partial x} u(x, y)\, dxdy$ is convergent*, and from this we can show that

$$\int_0^\infty \int_0^\infty e^{-x-y}\, u(x, y)\, dxdy$$

is convergent.

For, let
$$v = \int_0^y \int_0^x e^{-x-y}\, u(x, y)\, dxdy$$

Then
$$\frac{\partial^2 v}{\partial x\, \partial y} = e^{-x-y}\, u(x, y) \qquad 7{\cdot}1$$

and
$$\frac{\partial^3 v}{\partial^2 x\, \partial y} = e^{-x-y} \frac{\partial}{\partial x} u(x, y) - e^{-x-y}\, u(x, y) \qquad 7{\cdot}2$$

$$\therefore \quad e^{-x-y} \frac{\partial}{\partial x} u(x, y) = \frac{\partial^3 v}{\partial^2 x\, \partial y} + e^{-x-y}\, u(x, y)$$

and consequently integrating both the sides,

$$\int_0^y \int_0^x e^{-x-y} \frac{\partial}{\partial x} u(x, y)\, dxdy = \int_0^y \int_0^x e^{-x-y}\, u(x, y)\, dxdy + \frac{\partial v}{\partial x} - \int_0^y e^{-y}\, u(0, y)\, dy$$

---

suppose that the thickness of the peaks correspondingly decreases in such a way that the cubical contents of each of these peaks form a convergent double series satisfying Pringsheim's conditions and consequently $\int^\infty \int^\infty f(xy)\, dx\, dy$ may converge. For a similar illustration in the case of single infinite integrals, *vide*: Bromwich: *Infinite Series*, page 422.

* It is easy to show that $e^{-x-y}\, u(x, y)$ tends to zero as x tends to infinity, if it is given that

$$\int_0^\infty \int_0^\infty e^{-x-y} \frac{\partial}{\partial x} u(x, y)\, dxdy$$

is convergent. For, when this integral is assumed to be convergent, it is implied that the integral satisfies de la Valee Poussin's conditions. Hence

$$\int_0^\infty e^{-x-y} \frac{\partial}{\partial x} u(x, y)\, dx$$

must be convergent. Therefore it follows that $e^{-x-y}\, u(x, y)$ tends to zero as x tends to infinity (*vide* Bromwich: *Infinite Series*, Art. 101). The case when y tends to infinity can be dealt with similarly.

i. e., $$\int_0^y \int_0^x e^{-x-y} \frac{\partial}{\partial x} u(x, y)\, dxdy = v + \frac{\partial v}{\partial x} - \int_0^y e^{-y} u(0, y)\, dy. \qquad 7{\cdot}3$$

Then provided $\int_0^y e^{-y} u(0, y)\, dy$ is convergent as y tends to infinity, the expression on the right must tend to a definite limit when x and y tend to infinity since we have assumed that

$$\int_0^\infty \int_0^\infty e^{-x-y} \frac{\partial}{\partial x} u(x, y)\, dxdy$$

is convergent and satisfies de la Valee Poussin's conditions.

Let the limit be l and let us write

$$w = v - \int_0^\infty e^{-y} u(0, y)\, dy - l, \text{ and } \frac{\partial w}{\partial x} = \frac{\partial v}{\partial x}.$$

It follows then that $\left| w + \frac{\partial w}{\partial x} \right|$ tends to the limit zero, when x and $y \to \infty$. Thus we can chose X and Y so that

$$\left| w + \frac{\partial w}{\partial x} \right| < \varepsilon, \text{ if } x > X \text{ and } y > Y.$$

If w has any number of extreme values considered as a function of x, $\frac{\partial w}{\partial x}$ vanishes at each of these. Hence at any extreme value beyond X and Y we have $|w| < \varepsilon$, thus $|w| < \varepsilon$, if $x > X$ and $y > Y$.

$$\therefore \quad \lim_{x,y \to \infty} w = 0 \text{ and hence } \lim_{x,y \to \infty} \frac{\partial w}{\partial x} = 0.$$

But if w has not an unlimited number of extreme values as function of x, $\frac{\partial w}{\partial x}$ must be of a constant sign finally, and so w must approach a definite limit k and hence $\frac{\partial w}{\partial x}$ must approach the limit $-k$.

But if $\frac{\partial w}{\partial x}$ approaches the limit $-k$, then $\frac{w}{x}$ approaches the same limit. Thus we have $\lim w = k$; and $\lim \frac{w}{x} = -k$; but w and $\frac{w}{x}$ have the same sign and therefore k must be zero.

Hence $\lim\limits_{x, y \to \infty} w = 0$ and $\lim\limits_{x, y \to \infty} \dfrac{\partial w}{\partial x} = 0$;

or
$$\int_0^\infty \int_0^\infty e^{-x-y}\, u(x, y)\, dx\, dy = \int_0^\infty e^{-y}\, u(0, y)\, dy + \int_0^\infty \int_0^\infty e^{-x-y} \left[\frac{\partial}{\partial x} u(x, y)\right] dx dy.$$

or
$$\mathop{S}_{0}^{\infty}\mathop{S}_{0}^{\infty} u_{m,n} = \mathop{S}_{0}^{\infty} u_{0;n} + \mathop{S}_{0}^{\infty}\mathop{S}_{0}^{\infty} u_{m;n}. \qquad \ldots \quad 7{\cdot}4$$

This shows that if $\mathop{S}_{1}^{\infty}\mathop{S}_{0}^{\infty}$ *is summable and the single series* $\mathop{S}_{0}^{\infty} u_{0;n}$ *is summable, then* $\mathop{S}_{0}^{\infty}\mathop{S}_{0}^{\infty} u_{m,n}$ *is summable; and that the converse is not true.* Similar results can be arrived at by assuming that the integrals for S_y and S_{xy} are convergent and satisfy de la Valee Poussin's conditions.

The results obtained can be summarised as follows:—*To a summable double series any number of summable rows or columns may be added, without destroying the property of summability of the double series; but the removal of either a single row or column may destroy this property.*

8. In article 4 of this paper some examples of summable double series were given and their sums were found. Let us take some more double series the terms of which are circular functions.

(1) Let
$$\begin{aligned} c = \; & 1 + \cos\theta + \cos 2\theta + \cos 3\theta + \ldots \\ & + \cos\theta + \cos 2\theta + \cos 3\theta + \cos 4\theta + \ldots \\ & + \cos 2\theta + \cos 3\theta + \cos 4\theta + \cos 5\theta + \ldots \\ & + \ldots\ldots\ldots\ldots \\ & + \ldots\ldots\ldots\ldots \end{aligned}$$

and
$$\begin{aligned} s = \; & 0 + \sin\theta + \sin 2\theta + \sin 3\theta + \ldots \\ & + \sin\theta + \sin 2\theta + \sin 3\theta + \sin 4\theta + \ldots \\ & + \sin 2\theta + \sin 3\theta + \sin 4\theta + \sin 5\theta + \ldots \\ & + \ldots\ldots\ldots\ldots \\ & + \ldots\ldots\ldots\ldots \end{aligned}$$

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NOTES AND QUESTIONS.

Notes and Questions.

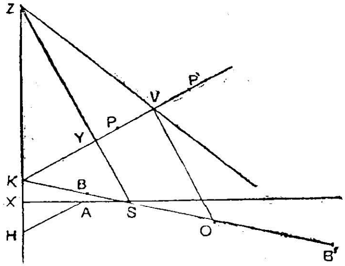
Note on the locus of the middle points of a system of parallel chords of a Conic.

Let S be the focus, A the vertex and X the foot of the corresponding directrix. Draw AH in the direction of the parallel chords to meet the directrix at H, and let KPP′ be any chord of the system cutting the directrix at K.

We know that the points P, P′ on the conic are such that

$$SP : PK = SP' : P'K = SA : AH$$

and therefore if B,B′ divide SK in the ratio SA : AH, the circle on BB′ as diameter passes through P, P′.



Bisect BB′ at O and draw perpendiculars SY, OV on PP′. Produce SY to meet the directrix at Z.

Now $SB : BK = SB' : B'K = SA : AH =$ constant

$\therefore$ $OS : OB = OB : OK =$ constant

$\therefore$ $OS : OK =$ constant.

Hence $YV : KV =$ constant, where V is the middle point of PP′.

Since SZ is drawn in a fixed direction, Z is a fixed point and since K and Y move on fixed lines through Z, V also moves on a fixed line through Z.

In other words, the locus of V is a straight line.

K. SATYAÑARAYAÑA.

S. V. RANADE'S PERPETUAL CALENDAR.

Table I.
Guiding Days of the years.

									← B.D./A.D. →															
									←	–201	–101	–1/0	100	200	300	Julian								
									400	500	600	700	800	900	1000	Julian								
									1100	1200	1300	1400	1500	1600	1700	Julian								
									1500	1600	...	1700	...	1800	...	Gregorian								
									1900	2000	...	2100	—	→	...	Gregorian								
0	...	...	...	...	...	...	...	...	*M*/M	*S*	*St*	*F*/F	*Th*	*W*/W	*Tu*	*								
1	7	*12*	18	...	29	35	*40*	46	Tu	M	S	St	F	Th	W	...	57	63	*68*	74	...	85	91	*96*
2	...	13	19	*24*	30	...	41	47	W	Tu	M	S	St	F	Th	*52*	58	...	69	75	*80*	86	...	97
3	*8*	14	...	25	31	*36*	42	...	Th	W	Tu	M	S	St	F	53	59	*64*	70	...	81	87	*92*	98
...	9	15	*20*	26	...	37	43	*48*	F	Th	W	Tu	M	S	St	54	...	65	71	*76*	82	...	93	99
4	10	...	21	27	*32*	38	...	49	St	F	Th	W	Tu	M	S	55	*60*	66	...	77	83	*88*	94	...
5	11	*16*	22	...	33	39	*44*	50	S	St	F	Th	W	Tu	M	...	61	67	*72*	78	...	89	95	...
6	...	17	23	*28*	34	...	45	51	M	S	St	F	Th	W	Tu	*56*	62	...	73	79	*84*	90	...	...

N.B.—Figure in 'Italics' denotes Leap Year.

The year 4000 A.D. is suggested to be treated as an 'Ordinary year' and not 'Leap'; hence a slight change will be required in the order of Century years thereafter, in Table I.

* Italic letters in half the square denote that only Julian Years were Leap.

Table II.

	JAN.	*→(Leap Year)→*			*FEB.*	
Jan. Oct.	April July	Sept. Dec.	June	Feb. March Nov.	Aug.	May
SLIDING (Week-Day-Tape)						
1	2	3	4	5	6	7
8	9	10	11	12	13	14
15	16	17	18	19	20	21
22	23	24	25	26	27	28
29	30	31	–	–	–	–

[January and February in Leap years are to be noted in the upper line.]

KEY.

(i) Find out in Table I the 'Guiding Day' of the year in question. The day in the *vertical* line of the 'Century' year and *horizontal* line of the 'Odd' year is the Guiding Day of the year.

(ii) Place this 'Guiding Day' below the *required month* in Table II, [by sliding the Week-Day-Tape, or otherwise.]

And we get the Calendar for the Month.

Examples:—To find the day of the week on

1. 1st January 1925—In Table I the day in the vertical line of 1900 and horizontal line of 25 is Thursday, which is the 'Guiding Day' of the year. Placing it below January, we have the 1st on the same day.

2. 18th June 1815 (Battle of Waterloo)—Sunday is the 'Guiding Day' of the year. Placing it below June, we have the 18th on the same day.

3. 3rd August 1492 (when Columbus set sail for America)—Monday is the 'Guiding Day' of the year. Placing it below August, we see that the 3rd is Friday.

4. 1st January 100 B. C.—In the vertical line of (− 101) and the horizontal line of 1 (− 101 + 1 = − 100), Sunday is the 'Guiding Day.' Placing it below January, we have the 1st on the same day.

The General Equation of the Second Degree in Areals.

Let the equation be

$$S \equiv ux^2 + vy^2 + wz^2 + 2u'yz + 2v'zx + 2w'xy = 0, \ldots \quad (1)$$

and let (α, β, γ) be a focus. The distance d of a point (x, y, z) from the focus is given by

$$2d^2 = a'(x-\alpha)^2 + b'(y-\beta)^2 + c'(z-\gamma)^2 \quad \ldots \quad (2)$$

where a', b', c' stand for $b^2 + c^2 - a^2, c^2 + a^2 - b^2, a^2 + b^2 - c^2$ respectively, a, b, c being the sides of the triangle of reference.

Now, the equation to the conic can be written in the form

$$d^2 = k(lx + my + nz)^2 \quad \ldots \quad \ldots \quad (1')$$

where $lx + my + nz = 0$ is the directrix corresponding to the focus (α, β, γ). Hence comparing (1) and (1′),

$$-S + 2\lambda d^2 \quad \ldots \quad \ldots \quad (3)$$

can be made a complete square by properly choosing λ, and then its square root represents the directrix of the conic S.

2. Eliminating z from (1) and (2) by means of the relation

$$x + y + z = 1,$$

we get

$$-S \equiv Ax^2 + 2Hxy + By^2 + 2Gx + 2Fy + C = 0,$$

where, $A = 2v' - w - u, B = 2u' - w - v, H = u' + v' - w - w',$

$$G = w - v', \qquad F = w - u', \qquad C = -w,$$

and $2d^2 = x^2(a' + c') + 2xyc' + y^2(b' + c') - 2x\{a'\alpha + c'(\alpha + \beta)\}$

$$- 2y\{b'\beta + c'(\alpha + \beta)\} + a'\alpha^2 + b'\beta^2 + c'(\alpha + \beta)^2.$$

If the second degree terms in (3), namely,

$$x^2\{A + \lambda(a' + c')\} + 2xy(H + \lambda c') + y^2\{B + \lambda(b' + c')\} \quad (4)$$

form a square, then

$$\{\lambda(a' + c') + A\}\{(\lambda(b' + c') + B\} = (H + \lambda c')^2,$$

and on inserting the values of A, B, H, we have

$$\lambda^2(b'c' + c'a' + a'b') + \lambda[a'(2u' - v - w) + b'(2v' - w - u) + c'(2w' - u - v)]$$

$$+ (vw + wu + uv - 2uu' - 2vv' - 2ww')$$

$$+ (2v'w' + 2w'u' + 2u'v' - u'^2 - v'^2 - w'^2) = 0. \quad \ldots \quad (5)$$

For the value of λ given by (5), the expression (4) takes the form

$$\frac{1}{m}\,[mx + ny]^2,$$

where $m = A + \lambda\,(a' + c')$ and $n = H + \lambda\,c'$. The directrix is therefore of the form $mx + ny + p = 0$,

so that
$$-S + 2\lambda\,d^2 = \frac{1}{m}\,[mx + ny + p]^2. \qquad \dots \quad (6)$$

On equating the co-efficients of x, y and the absolute terms on both sides of (6), we have,

$$\lambda\alpha\,(a' + c') + \lambda\beta c' = G - p, \; \dots \qquad \dots \quad (7)$$

$$\lambda\alpha c' + \lambda\beta\,(b' + c') = F - \frac{np}{m}, \qquad \dots \quad (8)$$

and $\lambda(\alpha + \beta)^2 c' + \lambda a'\alpha^2 + \lambda b'\beta^2 = \frac{p^2}{m} - C$;

which may be reduced by (7) and (8) to

$$\alpha\,(G - p) + \beta\left(F - \frac{np}{m}\right) = \frac{p^2}{m} - C. \; \dots \qquad \dots \quad (9)$$

Eliminating α, β from (7), (8) and (9), we get

$$\begin{vmatrix} \lambda(a' + c') & \lambda c' & G - p \\ \lambda c' & \lambda(b' + c') & F - \frac{np}{m} \\ G - p, & F - \frac{np}{m}. & \frac{p^2}{m} - C \end{vmatrix} = 0 \quad \dots \; (10)$$

which determines p; and α, β are then given by (7) and (8).

If p is eliminated from (7) and (8), we obtain the equation to the axis on which the focus (α, β) lies; it is

$$x\,\{a'n + c'(n - m)\} + y\,\{c'(n - m) - b'm\}$$
$$+ (x + y + z)\left[\frac{(v' - w)n}{\lambda} - \frac{(u' - w)m}{\lambda}\right] = 0. \quad \dots \; (11)$$

The eccentricity of the conic may now be deduced. It turns out to be

$$\frac{1}{2\Delta\sqrt{2\lambda}}\Big[a^2(\lambda a' + v' + w' - u - u') + c^2(\lambda c' + u' + v' - w - w')$$
$$+ \frac{b^2 n}{m}(u + u' - v' - w' - \lambda a')\Big]^{\frac{1}{2}} \quad \dots \; (12)$$

where Δ is the area of the triangle of reference.

3. If the conic S is a circle, then the directrix

$$mx + ny + p = 0$$

should be at infinity. The directrix in homogeneous form is

$$mx + ny + p(x + y + z) = 0$$

i.e., $$x(m + p) + y(n + p) + pz = 0; \qquad \dots \quad (13)$$

and this should be identical with the line at infinity, *viz.*,

$$x + y + z = 0$$

and hence, $$m + p = n + p = p,$$

giving $$m = 0, n = 0;$$

hence we obtain the well-known conditions

$$\lambda = \frac{v + w - 2u'}{2a^2} = \frac{w + u - 2v'}{2b^2} = \frac{u + v - 2w'}{2c^2}.$$

Also from (6), S may be written in the form

$$\frac{1}{m}[mx + ny + p]^2 - 2\lambda d^2 = 0. \qquad \dots \quad (14)$$

When S is a circle, d is equal to the radius, and on making m and n tend to zero in (14), we get

$$\text{lim. } p^2/m = 2\lambda d^2, \quad \dots \qquad \dots \quad (15)$$

and the determinant of (10) becomes

$$\begin{vmatrix} a' + c' & c' & G \\ c' & b' + c' & F \\ G & F & \lambda(2\lambda d^2 - C) \end{vmatrix} = 0;$$

or,

$$2\lambda^2 d^2 (b'c' + c'a' + a'b') + \begin{vmatrix} a' + c' & c' & w - v' \\ c' & b' + c' & w - u' \\ w - v' & w - u' & w\lambda \end{vmatrix} = 0, \quad (16)$$

which determines the radius of the circle, and

$$b'c' + c'a' + a'b' = 16\Delta^2.$$

The centre is given by

$$\alpha(a' + c') + \beta c' = \frac{w - v'}{\lambda}, \quad \alpha c' + \beta(b' + c') = \frac{w - u'}{\lambda}.$$

K. D. PANDAY, M.A., B.Sc.

Solutions.

Question 1274.

(A. A. KRISHNASWAMI IYENGAR):—Prove the following approximate construction for the rectification of a circle:—

Draw AB, CD two perpendicular diameters. Along AO take $AK = \frac{1}{8}AO$. Bisect CO at E. Join CK and from it cut off CG = CO and through G draw GF parallel to KE to meet CE at F. Draw a circle to pass through G and F so as to touch CD at F. Let this circle cut CG again at H.

Then the circumference of the circle ADBC is 6OC+2CH.

Solution by S. A. Mani, Hemraj, P. R. Venkatakrishna Iyer K. Satyanarayana, M. V. Ramakrishnan, etc.

Proof: Since CF touches the circle FGH,

$$CF^2 = CH.CG = CH.r$$

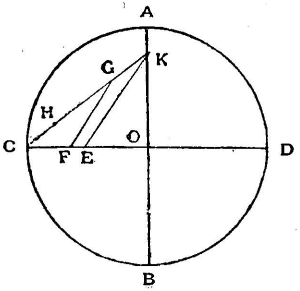
where r = radius.

$$\therefore \quad CH = \frac{CF^2}{r}. \quad \ldots \quad \ldots \quad (1)$$

Also

$$\frac{CF}{CE} = \frac{CG}{CK}.$$

$$\therefore \quad CF = \frac{CG.CE}{CK} = \frac{r^2}{2CK}. \quad \ldots \quad \ldots \quad (2)$$



Again $$CK^2 = r^2\left[1+\left(\frac{7}{8}\right)^2\right] = \frac{113}{64}r^2. \qquad \ldots (3)$$

$$\therefore \quad CH = \frac{CF^2}{r} = \frac{1}{r}\cdot\frac{r^4}{4CK^2} = \frac{r^3}{4}\cdot\frac{64}{113r^2} = \frac{16r}{113}.$$

$$\therefore \quad 6OC+2CH = 6r + \frac{32r}{113}$$

$$= 2r\,(3\cdot1415929\ \ldots\ldots)$$

which agrees with the value $2\pi r$ as far as the 6th decimal place.

Question 1276.

(G. V. TELANG):—If x represents the area of a triangle and x_1, x_2 those of the pedal triangles of the positive (or negative) Brocard point and the 3rd Brocard point, show that

$$xx_2 = 2x_1\,(x - 2x_1).$$

Solution by Hemraj.

Brocard's first triangle $A'B'C'$ is in perspective in three ways with the original triangle ABC:

AC′, BA′, CB′ intersect at Ω (*first* or positive Brocard point)

AB′, BC′, CA′ ,, at Ω' (*second* or negative Brocard point)

AA′, BB′, CC′ ,, at Ω'' (*third* Brocard point).

The trilinear co-ordinates of Ω, Ω', Ω'' are

$$\left(2R\sin^2 w\,\frac{c}{b},\ 2R\sin^2 w\,\frac{a}{c},\ 2R\sin^2 w\,\frac{b}{a}\right)$$

$$\left(2R\sin^2 w\,\frac{b}{c},\ 2R\sin^2 w\,\frac{c}{a};\ 2R\sin^2 w\,\frac{a}{b}\right)$$

$$\left(2R\sin^2 w\,\frac{bc}{a^2},\ 2R\sin^2 w\,\frac{ca}{b^2},\ 2R\sin^2 w\,\frac{ab}{c^2}\right)$$

where w is the Brocard angle of the triangle ABC.

Also $\Omega A = 2R\sin w\,\frac{b}{a}$, $\Omega B = 2R\sin w\,\frac{c}{b}$, $\Omega C = 2R\sin w\,\frac{a}{c}$.

Now $2x = \Sigma\, \Omega B . \Omega C \sin C = \dfrac{2R \sin^3 w}{abc}(a^2b^2 + b^2c^2 + c^2a^2)$;

$$2x_1 = \frac{2R \sin^4 w}{abc}(a^2b^2 + b^2c^2 + c^2a^2).$$

(The pedal triangles of Ω and Ω' are equal in area.)

$$2x_2 = \frac{2R \sin^4 w}{abc}(a^4 + b^4 + c^4).$$

Since $$2 \cos 2w = \frac{a^4 + b^4 + c^4}{a^2b^2 + b^2c^2 + c^2a^2},$$

$\therefore$ $$xx_2 = 2x_1(x - 2x_1).$$

Question 1277.

(S. Rajanarayanan) :—ABC is a triangle; (D, E), (F, G) are points in AB, AC, such that AD = BE = AF = CG. If DG, FE meet BC in H, K, prove that BH = CK.

Solution by Martyn M. Thomas, M.D. Bhat, B.B. Bagi, M. V. Ramakrishnan, Hemraj, V. V. S. N. Murthy, S. Audinarayanan, K. Satyanarayana, S. A. Mani, P. R. Venkatakrishna Aiyar, N. P. Pandya, C. Ranganathan, etc.

Since DGH and FEK are transversals to the Δ ABC,

$$\frac{BH}{HC} \cdot \frac{CG}{GA} \cdot \frac{AD}{DB} = 1 = \frac{CK}{KB} \cdot \frac{BE}{EA} \cdot \frac{AF}{FC}.$$

Cancelling common factors, $\dfrac{BH}{HC} = \dfrac{CK}{KB}$.

$\therefore$ $$\frac{BH}{BH - HC} = \frac{CK}{CK - KB}.$$

$\therefore$ $$\frac{BH}{BC} = \frac{CK}{BC}.$$

$\therefore$ $$BH = CK.$$

Question 1280.

(M. BHIMASENA RAO and the late C. KRISHNAMACHARI) :—If

$$S_r = 1 + \frac{1}{2^r} + \frac{1}{3^r} \ldots\ldots,$$

and γ is Euler's constant, show that

$$\int_0^\infty e^{-x} (\log x)^3 \, dx = -(\gamma^3 + 3\gamma S_2 + 2S_3);$$

$$\int_0^\infty e^{-x} (\log x)^4 \, dx = \gamma^4 + 6\gamma^2 S_2 + 8\gamma S_3 + 3S_2^2 + 6S_4;$$

and show how to evaluate generally $\int_0^\infty e^{-x} (\log x)^n \, dx$.

Solution by Martyn Thomas.

Now $$\int_0^\infty e^{-x} \,.\, x^{n-1} \, dx = \Gamma(n).$$

Differentiating successively with respect to n,

$$\int_0^\infty e^{-x} x^{n-1} \log x \, dx = \Gamma'(n),$$

$$\int_0^\infty e^{-x} x^{n-1} (\log x)^2 \, dx = \Gamma''(n),$$

$$\int_0^\infty e^{-x} x^{n-1} (\log x)^3 \, dx = \Gamma'''(n),$$

Generally, $$\int_0^\infty e^{-x} x^{n-1} (\log x)^r \, dx = \Gamma^r(n).$$

Putting $n = 1$, we obtain

$$\int_0^\infty e^{-x} (\log x)^3 \, dx = \Gamma'''(1),$$

....

and generally, $$\int_0^\infty e^{-x} (\log x)^r \, dx = \Gamma^r(1).$$

Now

$$\frac{\Gamma'(n)}{\Gamma(n)} + \gamma = 1 + \tfrac{1}{2} + \tfrac{1}{3} + \ldots\ldots + \frac{1}{n-1} = \sum_{k=0}^{\infty} \left(\frac{1}{1+k} - \frac{1}{n+k}\right) \quad \ldots (1)$$

[*Vide*: Wilson's *Calculus*, § 149-(17), or Bromwich's *Infinite Series*.]

Differentiating successively with respect to n,

$$\frac{\Gamma''(n)}{\Gamma(n)} - \left\{\frac{\Gamma'(n)}{\Gamma(n)}\right\}^2 = \sum_{k=0}^{\infty} \frac{1}{(n+k)^2} \quad \ldots \quad (2)$$

$$\frac{\Gamma'''(n)}{\Gamma(n)} - 3\frac{\Gamma''(n)\,\Gamma'(n)}{[\Gamma(n)]^2} + 2\left\{\frac{\Gamma'(n)}{\Gamma(n)}\right\}^3 = \sum_{k=0}^{\infty} \frac{-2}{(n+k)^3} \quad \ldots \quad (3)$$

$$\frac{\Gamma^{iv}(n)}{\Gamma(n)} - 4\frac{\Gamma'''(n)\,\Gamma'(n)}{[\Gamma(a)]^2} + 12\frac{\Gamma''(n)\,[\Gamma'(n)]^2}{[\Gamma(n)]^3}$$

$$-3\left\{\frac{\Gamma''(n)}{\Gamma(n)}\right\}^2 - 6\left\{\frac{\Gamma'(n)}{\Gamma(n)}\right\}^4 = \sum_{k=0}^{\infty} \frac{6}{(n+k)^4}. \quad \ldots \quad (4)$$

Putting $n = 1$ in (1), (2), (3), (4) and remembering $\Gamma(1) = 1$, we get

$$\Gamma'(1) = -\gamma \text{ and } \Gamma''(1) = \gamma^2 + S_2$$

$$\Gamma'''(1) = -(\gamma^3 + 3\gamma S_2 + 2S_3)$$

$$\Gamma^{iv}(1) = \gamma^4 + 6\gamma^2 S_2 + 8\gamma S_3 + 3S_2^{\,2} + 6S_4.$$

Hence the required results.

Question 1281.

(M. BHIMASENA RAO AND THE LATE C. KRISHNAMACHARI):—

If $$S_n = \left(1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}\right)\frac{1}{n},$$

show that

$$\lim_{n\to\infty} \{(S_1 + S_2 + \ldots + S_n) - \gamma \log n - \tfrac{1}{2}(\log n)^2\} = \frac{\gamma^2}{2} + \frac{\pi^2}{12}.$$

Further show that

$$S_1 + S_2 + \ldots\ldots + S_n = \binom{n}{1} - \binom{n}{2}\left(\frac{1}{2^2}\right) + \binom{n}{3}\frac{1}{3^2}$$

$$- \ldots\ldots + (-1)^{n-1}\binom{n}{n}\frac{1}{n^2},$$

where $\binom{n}{r}$ = number of combinations of n things r at a time.

Solution by Martyn Thomas.

First Part.

$$S_1 = \frac{1}{1^2}$$

$$S_2 = \left(1 + \frac{1}{2}\right)\frac{1}{2} = 1 \cdot \frac{1}{2} + \frac{1}{2^2}$$

$$S_3 = 1 \cdot \frac{1}{3} + \frac{1}{2} \cdot \frac{1}{3} + \frac{1}{3^2}$$

$$S_4 = 1 \cdot \frac{1}{4} + \frac{1}{2} \cdot \frac{1}{4} + \frac{1}{3} \cdot \frac{1}{4} + \frac{1}{4^2}$$

....

$$S_n = 1 \cdot \frac{1}{n} + \frac{1}{2} \cdot \frac{1}{n} + \frac{1}{3} \cdot \frac{1}{n} + \ldots\ldots + \frac{1}{(n-1)n} + \frac{1}{n^2}$$

$$\therefore \sum_1^n S_r = \sum_1^n \left(\frac{1}{r^2}\right) + \left\{ \sum_2^n \left(\frac{1}{r}\right) + \frac{1}{2} \sum_3^n \left(\frac{1}{r}\right) + \ldots\ldots + \left(\frac{1}{n-1} \cdot \frac{1}{n}\right) \right\}$$

$$\text{Now } \gamma = \lim_{n \to \infty} \left[\sum_1^n \left(\frac{1}{r}\right) - \log n \right]$$

$$\therefore \gamma^2 = \lim_{x \to \infty} \left[\left(\sum_1^n \left(\frac{1}{r}\right) \right)^2 - 2 \log n \left(\sum_1^n \left(\frac{1}{r}\right) \right) + (\log n)^2 \right]$$

$$= \lim_{n \to \infty} \left[\left(\sum_1^n \frac{1}{r^2} \right) - 2 \log n \, (\gamma + \log n) + (\log n)^2 + 2 \left\{ 1 \cdot \frac{1}{2} + \ldots\ldots + \frac{1}{2} \cdot \frac{1}{3} + \ldots\ldots + \frac{1}{n-1} \cdot \frac{1}{n} \right\} \right]$$

$$= \lim_{n \to \infty} \left[\frac{\pi^2}{6} + 2 \left\{ \sum_1^n S_r - \frac{\pi^2}{6} \right\} - 2\gamma \log n - (\log n)^2 \right]$$

$$\therefore \quad \frac{\gamma^2}{2} + \frac{\pi^2}{12} = \lim_{n \to \infty} \left[\sum_1^n S_r - \gamma \log n - \tfrac{1}{2} (\log n)^2 \right].$$

Second Part.

Consider the fraction

$$\frac{1}{(x+1)(x+2)\ldots\ldots(x+n)},$$

which, when decomposed into its partial fractions

$$\frac{A_1}{x+1}+\frac{A_2}{x+2}+\ldots\ldots+\frac{A_n}{x+n},$$

will have

$$A_1=\frac{1}{1.2\ldots\ldots(n-1)};\ A_2=\frac{1}{(-1))(1.2\ldots\ldots(n-2)},$$

$$A_3=\frac{1}{(-2)(-1)\,1.2\ldots\ldots(n-3)};$$

$$A_{n-1}=\frac{1}{(-n+2)(-n+3)\ldots\ldots(-2)(-1)\,1},$$

$$A_n=\frac{1}{(-n+1)(-n+2)\ldots\ldots(-2)(-1)}.$$

$$\therefore\ \frac{1}{n!\left(1+\frac{x}{1}\right)\left(1+\frac{x}{2}\right)\ldots\ldots\left(1+\frac{x}{n}\right)}=\frac{1}{(n-1)!}\cdot\frac{1}{1+x}$$

$$-\frac{1}{(n-2)!}\cdot\frac{1}{2\left(1+\frac{x}{2}\right)}+\ldots\ldots+\frac{(-1)^{n-1}}{(n-1)!}\,\frac{1}{n\left(1+\frac{x}{n}\right)}.$$

Multiply both sides by $(n-1)!$; then

$$\frac{1}{n\left(1+\frac{x}{1}\right)\left(1+\frac{x}{2}\right)\cdots\left(1+\frac{x}{n}\right)}=(1+x)^{-1}-\frac{n-1}{1!}\cdot\frac{1}{2}\left(1+\frac{x}{2}\right)^{-1}+$$

$$\ldots\ldots+(-1)^{n-1}\cdot\frac{1}{n}\left(1+\frac{x}{n}\right)^{-1}.$$

$$\therefore\ \frac{1}{n}\left\{1+x\left(1+\frac{1}{2}+\ldots+\frac{1}{n}\right)+x^2\left(\frac{1}{1.2}+\frac{1}{1.3}+\ldots\right)+\ldots\right\}^{-1}$$

$$=\text{right side.}$$

Equate the co-efficient of x on both sides and change sign.

$$\therefore\ S_n\equiv\frac{1}{n}\left(1+\frac{1}{2}+\frac{1}{3}+\ldots\ldots+\frac{1}{n}\right)$$

$$=1-\binom{n-1}{1}\cdot\frac{1}{2^2}+\binom{n-1}{2}\cdot\frac{1}{3^2}-\ldots+(-1)^{n-2}\binom{n-1}{n-1}\cdot\frac{1}{n^2}.$$

By the Method of Finite Differences

$$\sum_1^n S_r = \Delta^{-1}[S_{n+1}]$$

$$= \Delta^{-1}\left[1 - \binom{n}{1}\cdot\frac{1}{2^2} + \binom{n}{2}\frac{1}{3^2} + \ldots\ldots(-1)_n\binom{n}{n}\frac{1}{(n+1)^2}\right]$$

But $\quad {}_nC_r + {}_nC_{r-1} = {}_{n+1}C_r$

$\therefore \quad \Delta[{}_nC_r] = {}_nC_{r-1}$

$\therefore \quad \Delta^{-1}[{}_nC_{r-1}] = {}_nC_r.$

$$\therefore \quad \sum_1^n S_r = n - \binom{n}{2}\frac{1}{2^2} + \binom{n}{3}\frac{1}{3^2} - \ldots\ldots + (-1)^{n-1}\binom{n}{n}\cdot\frac{1}{n^2},$$

since $\Delta^{-1}\binom{n}{n} = {}_nC_{n+1} = 0.$

Question 1287.

(B. B. BAGI):—If regular polygons $A_1A_2 \ldots\ldots A_m$; $B_1B_2 \ldots\ldots B_n$ are inscribed in circles with centres a_1, a_2 and radii r_1, r_2 respectively, then show that

$$\sum_{x=1}^{m}\sum_{y=1}^{n}(A_xB_y)^2 = mn\left[(a_1a_2)^2 + r_1^2 + r_2^2\right].$$

[The circles are not necessarily in one plane.]

Solution (1) *by Martyn Thomas* (2) *by K. J. Sanjana.*

(1) Lemma. If D, E, F ...... are fixed points in space, G their centre of mean position, and P any variable point, then

$$\Sigma PD^2 = \Sigma GD^2 + n.PG^2.$$

Choosing P to coincide with $A_1, A_2 \ldots A_m$ successively, and D, E, F ... being respectively placed at the angular points $B_1, B_2, B_3 \ldots\ldots B_n$, and remembering that G must coincide with the centre a_2, of the circle of radius r_2, we have

$$\sum_{y=1}^{n}(A_1B_y)^2 = \sum_{y=1}^{n}(a_2B_y)^2 + n(A_1a_2)^2, \text{ by Lemma}$$

$$= nr_2^2 + n(A_1a_2)^2$$

Similarly, $\sum_{y=1}^{n} (A_2B_y)^2 = n \cdot r_2^2 + n(A_2a_2)^2$

$$\sum_{y=1}^{n} (A_3B_y)^2 = nr_2^2 + n(A_3a_2)^2$$

....

Finally, $\sum_{y=1}^{n} (A_mB_y)^2 = n \cdot r_2^2 + n(A_ma_2)^2$

Summing up these 'm' equalities,

$$\sum_{x=1}^{m} \sum_{y=1}^{n} (A_xB_y)^2 = m(n\, r_2^2) + n \sum_{x=1}^{m} (a_2A_x)^2$$

$$= mn\, r_2^2 + n\,[\sum_{x=1}^{m} (a_1A_x)^2 + m(a_1a_2)^2],$$

by Lemma

$$= mn(r_2^2 + n\,[m\,r_1^2 + m\,(a_1a_2)^2]$$
$$= mn\,[r_2^2 + r_1^2 + (a_1a_2)^2].$$

(2) Let A_xM be the $\perp$r from A_x on the plane of the circle $B_1B_2 \ldots\ldots B_n$; let the line B_1a_2 make with Ma_2 an angle θ. Join A_xB_y, MB_y.

Then $A_xB_y^2 = A_xM^2 + MB_y^2 = A_xM^2 + Ma_2^2 + a_2B_y^2$

$$- 2\,Ma_2 \cdot a_2B_y \cdot \cos\left(\theta + \frac{2y\pi}{n}\right)$$

$$= A_xa_2^2 + r_2^2 - 2r_2 \cdot Ma_2 \cos\left(\theta + \frac{2y\pi}{n}\right).$$

$$\therefore \sum_{y=1}^{y=n} A_xB_y^2 = n\,A_xa_2^2 + nr_2^2 - 2r_2 \cdot Ma_2 \cdot \sum_{1}^{n} \cos\left(\theta + \frac{2y\pi}{n}\right)$$
$$= n\,A_xa_2^2 + nr_2^2.$$

$$\therefore \sum_{x=1}^{x=m} \sum_{y=1}^{y=n} A_xB_y^2 = n \sum_{1}^{m} A_xa_2^2 + nmr_2^2$$
$$= n(m \cdot a_1a_2^2 + mr_1^2) + nmr_2^2,$$

by similar reasoning,

$$= nm(a_1a_2^2 + r_1^2 + r_2^2).$$

[*Additional Analytical Solutions by Nandlal M. Mehta and I. Totadri Iyengar.*]

Question 1291.

(MARTYN M. THOMAS, M.A.):—If the equation of a curve in multiple angular co-ordinates be

$$\theta_1 + \theta_2 + \theta_3 + \ldots\ldots + \theta_n = \text{a constant},$$

show that the equation of its orthogonal trajectory in multiple polar co-ordinates is

$$r_1 \, r_2 \, r_3 \ldots\ldots r_n = \text{a constant}.$$

[*Particular case*—The orthogonal trajectories of rectangular hyperbolas are Cassini's Ovals.]

Solution (1) *by A. Narasinga Rao,* (2) *by Miss Y. Bhate, I. B. Mukherji, M. V. Seshadri, S. Audinarayanan, S. M. Shah, C. Ranganatha Iyengar, M. K. Kewalramani and K. N. Srikanta Sastry.*

(1) It is a well-known result that, if $u(x, y) + iv(x, y)$ where u and v are real, is a function of the complex variable $x + iy$, then the curves $u =$ const. and $v =$ const. are orthogonal.

Consider the function

$$\phi(z) \equiv \log(z - \alpha_1) + \log(z - \alpha_2) + \ldots\ldots + \log(z - \alpha_n)$$

where $\alpha_1 \alpha_2 \ldots\ldots \alpha_n$ are complex numbers corresponding to the n poles $O_1 \, O_2 \ldots\ldots O_n$.

The real part of $\phi(z)$ is $\Sigma \log r$, while the unreal part is $\Sigma \, i\theta$; on equating these to arbitrary constants, we obtain the two series of curves mentioned in the question.

(2) The differential equation of the curve is

$$\sum_{t=1}^{n} \frac{d\theta_t}{ds} = 0$$

where ds is an element of the arc of the curve. If ϕ_t denote the angle made by the tangent at any point with the radius vector r_t, the equation may be written

$$\sum_{t=1}^{n} \frac{\sin \phi_t}{r_t} = 0.$$

The orthogonal trajectory is obtained by changing every ϕ_t into

$$\left(\phi_t + \frac{\pi}{2}\right).$$

Hence its differential equation is

$$\sum_{t=1}^{n} \frac{\cos\phi_t}{r_t} = 0, \ i.e., \sum_{t=1}^{n} \frac{1}{r_t}\frac{dr_t}{ds} = 0, \ i.e., \sum_{t=1}^{n} \frac{dr_t}{r_t} = 0.$$

Integrating $\sum_{t=1}^{n} \log r_t \equiv \log r_1 r_2 \ldots\ldots r_n =$ a constant.

$$\therefore \qquad r_1 r_2 \ldots\ldots r_n = \text{a constant.}$$

[When $n = 2$, $\theta_1 + \theta_2 = k$. This represents a rectangular hyperbola, the two poles being the extremities of the major axis. The orthogonal trajectories are $r_1 r_2 =$ a constant and are Cassini's Ovals.]

Question 1292.

(V. RAMASWAMY AIYAR) :—If a and b be positive and unequal, show that

(1) $$\frac{\frac{1}{2}(a^2 - b^2)}{a - b} < \frac{\frac{1}{3}(a^3 - b^3)}{\frac{1}{2}(a^2 - b^2)} < \frac{\frac{1}{4}(a^4 - b^4)}{\frac{1}{3}(a^3 - b^3)} < \ldots\ldots\ldots$$

without limit.

(2) Show further that

$$\left\{\frac{a^m - b^m}{m(a - b)}\right\}^{\frac{1}{m-1}}$$

always increases as m increases.

(3) Hence or otherwise, show that

$$\frac{a^m - b^m}{a - b} \gtrless m\,(ab)^{\frac{m-1}{2}}, \text{ according as } m\,(m^2 - 1) \gtrless 0,$$

and that

$$\frac{a^m - b^m}{a - b} \gtrless m\left(\frac{a + b}{2}\right)^{m-1}, \text{ according as } m\,(m - 1)\,(m - 2) \gtrless 0.$$

(4) Show also that

$$\sqrt{(ab)} < \frac{a - b}{\log a - \log b} < \frac{a^{\frac{a}{a-b}} \cdot b^{\frac{b}{b-a}}}{e} < \frac{a + b}{2}.$$

Solution and Remarks by J. B. Freeman, M. A., L. T.

§ 1. Let $u_{m-1} \equiv \dfrac{\frac{1}{m}(a^m - b^m)}{\frac{1}{m-1}(a^{m-1} - b^{m-1})}$.

Then $u_m > u_{m-1}$

if $$\frac{1}{m^2-1}(a^{m-1} - b^{m-1})(a^{m+1} - b^{m+1}) > \frac{1}{m^2}(a^m - b^m)^2,$$

i.e., $$\frac{1}{m^2-1}[a^{2m} - a^m b^m \left(\frac{a}{b} + \frac{b}{a}\right) + b^{2m}]$$

$$> \frac{1}{m^2}[a^{2m} - 2a^m b^m + b^{2m}]$$

or $$m^2 . [a^{2m} - a^m b^m \left(\frac{a^2+b^2}{ab}\right) + b^{2m}]$$

$$> (m^2 - 1)[a^{2m} - 2a^m b^m + b^{2m}]$$

or $$-m^2 . a^m b^m \left(\frac{a^2+b^2}{ab}\right) > -a^{2m} + 2a^m b^m - b^{2m} - 2m^2 a^m b^m,$$

or $$m^2 a^m b^m \left(\frac{a^2+b^2}{ab} - 2\right) < (a^m - b^m)^2,$$

i.e., if $$m^2 . a^{m-1} b^{m-1} (a-b)^2 < (a^m - b^m)^2$$

or $$\frac{1}{m} . \frac{a^m - b^m}{a-b} > a^{\frac{m-1}{2}} . b^{\frac{m-1}{2}}.$$

$$\text{L. H. S.} = \frac{a^{m-1} + a^{m-2} b + a^{m-3} b^2 + \ldots\ldots + ab^{m-2} + b^{m-1}}{m}$$

$$> \sqrt[m]{\{a^{(m-1)+(m-2)+\ldots\ldots+2+1} \times b^{(m-1)+(m-2)+\ldots\ldots+2+1}\}}$$

i.e., $$> \sqrt[m]{\left\{a^{\frac{m(m-1)}{2}} . b^{\frac{m(m-1)}{2}}\right\}}$$

i.e., $$> a^{\frac{m-1}{2}} b^{\frac{m-1}{2}},$$

§ 1.1. In the classic equality

$$\frac{f(a)-f(b)}{F(a)-F(b)}=\frac{f'(\xi)}{F'(\xi)},\quad (b<\xi<a),$$

if we put $f(x)=x^m$ and $F(x)=x^{m-1}$,

we get $$\frac{a^m-b^m}{a^{m-1}-b^{m-1}}=\frac{m\,\xi^{m-1}}{(m-1)\,\xi^{m-2}}$$

or $$u_{m-1}\equiv\frac{(m-1)(a^m-b^m)}{m(a^{m-1}-b^{m-1})}=\xi.$$

Hence ξ is an increasing function of m.

§ 2. If u_m has the value assigned above, we find that

$$\left[\prod_{r=1}^{m-1}u_r\right]^{\frac{1}{m-1}}=\left\{\frac{1}{m}\,\frac{a^m-b^m}{a-b}\right\}^{\frac{1}{m-1}}$$

and $$\left[\prod_{r=1}^{m}u_r\right]^{\frac{1}{m}}=\left\{\frac{1}{m+1}\,\frac{a^{m+1}-b^{m+1}}{a-b}\right\}^{\frac{1}{m}}.$$

Thus $$\left[\prod_{r=1}^{m}u_r\right]^{\frac{1}{m}}>\left[\prod_{r=1}^{m-1}u_r\right]^{\frac{1}{m-1}},$$

if $$u_1^{m-1}.u_2^{m-1}\ldots\ldots u_m^{m-1}>u_1^m.u_2^m\ldots\ldots u_{m-1}^m$$

that is, if $$u_m^{m-1}>u_1u_2\ldots\ldots u_{m-1}$$

which is true as $u_m>u_{m-1}>u_{m-2}$ etc.;

Hence the required result follows.

§ 3. Applying the result obtained in § 1.1, we have

$$\left\{\frac{a^m-b^m}{m(a-b)}\right\}^{\frac{1}{m-1}}>\left\{\frac{a^2-b^2}{2(a-b)}\right\}^{\frac{1}{2-1}},\ \text{if } m\geqslant 2;$$

i.e., $$>\frac{a+b}{2}.$$

$$>\sqrt{ab}.$$

$$\therefore\quad \frac{a^m-b^m}{a-b}>m\,(ab)^{\frac{m-1}{2}}.$$ See also § 1.

If $m<-1$, put $m=-n$ and apply the result where $n>1$, and a and b are unequal as before.

§ 3.1. We have shown that

$$\left\{\frac{a^m - b^m}{m(a-b)}\right\}^{\frac{1}{m-1}}$$

is an increasing function of m.

Hence if $m > 2$

$$\left\{\frac{a^m - b^m}{m(a-b)}\right\}^{\frac{1}{m-1}} > \left\{\frac{a^2 - b^2}{2(a-b)}\right\}^{\frac{1}{2-1}}$$

i. e.,

$$> \frac{a+b}{2}.$$

$$\therefore \qquad \left\{\frac{a^m - b^m}{m(a-b)}\right\} > \left\{\frac{a+b}{2}\right\}^{m-1}$$

provided that $m > 2$.

If $m = 2$ or 1, the inequality becomes an equality.

N.B.—From this result by putting $b = ak$ it follows that

$$\frac{1 + k + k^2 + \ldots\ldots + k^{m-1}}{m} > \left\{\frac{1+k}{2}\right\}^{m-1},$$

if $m > 2$ and $k > 0$.

§ 3.2. All the above inequalities have been proved for the case where m is a positive integer.

In the case where m is fractional of the form p/q, we have by writing

$$A^q = a, \ B^q = b,$$

$$\left\{\frac{a^m - b^m}{m(a-b)}\right\}^{\frac{1}{m-1}} = \left\{\frac{a^{\frac{p}{q}} - b^{\frac{p}{q}}}{\frac{p}{q}(a-b)}\right\}^{\frac{1}{\frac{p}{q}-1}} = \left\{\frac{\frac{1}{p}(A^p - B^q)}{\frac{1}{q}(A^q - B^p)}\right\}^{\frac{1}{\frac{p}{q}-1}}$$

$$= u_{p+1} \cdot u_p \cdot u_{p-1} \ldots\ldots u_{p-q}, \text{ if } p > q$$

$$= \prod_{s=0}^{s=p}(u_{s+1}) / \prod_{s=0}^{s=q}(u_{s+1})$$

where u has the same meaning as in § 1, and p and q have integral values.

This is an increasing function of p when q is fixed, or a decreasing function of q when p is fixed.

Hence the result (2) is true when m is fractional.

When m is negative, let $m = -n$ where n is positive; then write

$$v_m \equiv \left\{\frac{a^m - b^m}{m(a-b)}\right\}^{\frac{1}{m-1}} = \left[\frac{a^{-n} - b^{-n}}{-n(a-b)}\right]^{\frac{1}{-n-1}}$$

$$= \left\{\frac{A^n - B^n}{\frac{n}{AB}(A-B)}\right\}^{\frac{1}{-n-1}}, \text{ where } a^{-1} = A,\ b^{-1} = B$$

$$= \left\{\frac{n(A-B)}{AB(A^n - B^n)}\right\}^{\frac{1}{n+1}}$$

$$= \left[\frac{1}{AB}\right]^{\frac{1}{n+1}} \left[\frac{1}{v_n}\right]^{\frac{n-1}{n+1}}$$

which is evidently a decreasing function of n since v_n is an increasing function of n.

Further it is seen that v_m has the limits

$$\frac{a^{\frac{a}{a-b}} \cdot b^{\frac{b}{b-a}}}{e} \cdot \frac{a-b}{\log a - \log b}$$

as m tends to the values 1 and 0, respectively.

§ 3.3. We have shown that

$$\left\{\frac{a^m - b^m}{m(a-b)}\right\}^{\frac{1}{m-1}}$$

is a continuous function of m and increases as m increases. ... (2)

§ 3.4. Consider now the function

$$\left\{\frac{a^m - b^m}{(a-b)}\right\} - m\,(ab)^{\frac{m-1}{2}} \equiv f(m)$$

which is a continuous function of m.

This is positive when $m = 2$, as can be seen by applying (2). It is equal 0 when $m = 1$, and when $m = 0$ and also when $m = -1$. Hence between the values -1 and 0, 0 and $+1$, the expression has the same sign. That is, $f(m)$ is positive for $m > 1$, negative for $0 < m < 1$, positive for $-1 < m < 0$ and negative for $m < -1$.

Thus $\dfrac{a^m - b^m}{a - b} \lessgtr m\,(ab)^{\frac{m-1}{2}}$, according as $m\,(m^2 - 1) \gtrless 0$.

§ 3.5. Consider the expression

$$\frac{a^m - b^m}{a - b} - m\left(\frac{a + b}{2}\right)^{m-1} \equiv F(m)$$

which is a continuous function of m.

This is positive when $m > 2$ (see § 3), zero when $m = 2$, 1 and 0.

Hence the expression has the same sign for $m > 2$, for $1 < m < 2$, for $0 < m < 1$ and $m < 0$; that is, $F(m)$ is positive for $m < 2$, negative for $1 < m < 2$, positive for $0 < m < 1$ and negative for $m < 0$.

Thus
$$\frac{a^m - b^m}{a - b} \gtrless m\left(\frac{a + b}{2}\right)^{m-1},$$

according as
$$m\,(m - 1)(m - 2) \lessgtr 0.$$

§ 4. Let
$$v_m \equiv \left\{\frac{a^m - b^m}{m(a - b)}\right\}^{\frac{1}{m-1}}.$$

Then
$$\sqrt{ab} < \lim_{m \to 0} v_m < \lim_{m \to 1} v'_m < \lim_{m \to 2} v_2$$

or
$$\sqrt{ab} < \frac{a - b}{\log a - \log b} < \frac{a^{\frac{a}{a-b}} \times b^{\frac{b}{b-a}}}{e} < \frac{a + b}{2}.$$

Partial solutions by Hans R. Gupta, A. Mahadevan, and K. N. Srikanta Sastry.

Questions for Solution.

1359. (M. BHIMASENA RAO):—Show that the sixteen points of contact of the in- and ex-circles with the nine-point circle and the sides of a triangle lie on a bi-circular quartic whose focal conics are concentric with the circum-circle of the triangle.

1360. (M. BHIMASENA RAO):—Show that the director circles of the in-conics of a triangle passing through the in-centre (or an ex-centre) and the corresponding Gergonne point touch the in-circle (or the ex-circle).

1361. (M. BHIMASENA RAO):—Evaluate the determinants

$$A \equiv \begin{vmatrix} -\left(a^2+\frac{bcd}{a}\right), & ab+cd, & ac+bd, & ad+bc \\ ba+cd, & -\left(b^2+\frac{acd}{b}\right), & bc+ad, & bd+ac \\ ca+bd, & cb+ad, & -\left(c^2+\frac{abd}{c}\right), & cd+ab \\ da+bc, & db+ac, & dc+ab, & -\left(d^2+\frac{abc}{d}\right). \end{vmatrix}$$

$$B \equiv \begin{vmatrix} -2, & \frac{c}{d}+\frac{d}{c}, & \frac{d}{b}+\frac{b}{d}, & \frac{b}{c}+\frac{c}{b} \\ \frac{c}{d}+\frac{d}{c}, & -2, & \frac{d}{a}+\frac{a}{d}, & \frac{a}{c}+\frac{c}{a} \\ \frac{d}{b}+\frac{b}{d}, & \frac{d}{a}+\frac{a}{d}, & -2, & \frac{a}{b}+\frac{b}{a} \\ \frac{b}{c}+\frac{c}{b}, & \frac{a}{c}+\frac{c}{a}, & \frac{a}{b}+\frac{b}{a}, & -2 \end{vmatrix}$$

and show that $A = B\,(abcd)^2$.

1362. (S. NARAYANA AIYAR):—Show that

$$\frac{1-dn\,u\;dn\,v\;dn\,(u+v)}{1-cn\,u\;cn\,v\;cn\,(u+v)} = k^2.$$

1363. (S. NARAYANA AIYAR) :—In Dr. Glaisher's notation show that

$$(1)\quad \mathrm{cs}\frac{k}{n}\,.\,\mathrm{cs}\frac{2k}{n}\,.\,\mathrm{cs}\frac{3k}{n}\ldots\ldots\mathrm{cs}\frac{(n-1)k}{n}=(1-k^2)^{\frac{n-1}{4}}$$

$$(2)\quad \mathrm{ds}\frac{k+ik'}{n}\,.\,\mathrm{ds}\frac{2}{n}(k+ik')\,.\,\mathrm{ds}\frac{3}{n}(k+ik')\ldots\ldots\mathrm{ds}\frac{n-1}{n}(k+ik')$$

$$=k^{n-1}\,.\,\{\mathrm{cn}\,(k+ik')\}^{\frac{n-1}{2}}$$

1364. (R. GOPALASWAMI) :—Prove that a conic touching the sides of a triangle ABC at L, M and N has double contact with the polar conic of the centre of perspective of the triangles ABC, LMN; and that the chord of contact is their axis of perspective.

Deduce that the Brocard ellipse has double contact with the circum-circle.

1365. (P. L. SRIVASTAVA):—Solve by the principle of *virtual work* the following :—

" A smooth rod passes through a smooth ring at the focus of an ellipse, whose major axis is horizontal and rests with its lower end on the quadrant of the curve furthest removed from the focus. Show that its length must at least be $\frac{3a}{4}+\frac{a}{4}\sqrt{1+8e^2}$, where a is the semi-major axis and e the eccentricity of the ellipse." [*Math. Tripos*, 1883.]

1366. (S. D. CHOWLA) :—With the usual notation for the Elliptic Functions prove that

$$\text{(i)}\quad \int_0^{\frac{\pi}{2}}\left\{\frac{1-2q\cos 2x+q^2}{1+2q\cos 2x+q^2}\right\}\left\{\frac{1-2q^3\cos 2x+q^6}{1+2q^3\cos 2x+q^6}\right\}$$

$$\times\left\{\frac{1-2q^5\cos 2x+q^{10}}{1+2q^5\cos 2x+q^{10}}\right\}\ldots\ldots dx=\frac{\pi^2}{4\,k\,k'^{\frac{1}{2}}}.$$

$$\text{(ii)}\quad \int_0^{\pi}\sin^3 x\sin px\prod_{n=0}^{\infty}\left\{\frac{1-2q^{2n}\cos 2x+q^{4n}}{1-2q^{2n-1}\cos 2x+q^{4n-2}}\right\}^3 dx=0,$$

where p is an even positive integer. Evaluate the definite integral when p is an odd positive integer. $(q=e^{k-\pi'/k})$.

LIST OF JOURNALS RECEIVED

(From 25th September to 22nd November 1924)

1 Bulletin of the Calcutta Mathematical Society, Vol. 15, No. 1.
2 Philosophical Magazine, Vol. 48, Nos. 285, 286.
3 The Tohoku Mathematical Journal, Vol. 23, Nos. 3 and 4.
4 The Messenger of Mathematics, Vol. 54, No 2.
5 The American Mathematical Monthly, Vol. 31, No. 7.
6 Transactions of the American Mathematical Society, Vol. 26, No. 2.
7 Rendiconti del Circolo Mathematico Di Palermo, Tome 43, Fascicolo 2 and Tomes 44, 45, 46 and 47 (complete).
8 Popular Astronomy, Vol. 2, No. 8.
9 Nature, Vol. 114, Nos. 2860, 2861, 2862, 2863 and 2864.
10 American Journal of Mathematics, Vol. 46, Nos. 1, 2, 3 and 4.
11 Mathematical Gazette, Vol. 12, No. 172 (three copies).
12 Proceedings of the London Mathematical Society, Vol. 23, Part 4.
13 Crelle's Journal, Band 154, Heft 1.
14 The Astrophysical Journal, Vol. 60, No. 2.
15 Proceedings of the Royal Society, Series A, Vol. 106, No. 738 A.
16 Universidad Nacionale de La Plata.
   Anuario para El Ano 1923, No. 13.
   Creacian de la Estacian Experimental de Hydraulica Y de Electroticnica.
17 Annals de L Ecolé Normale Superieure :—1864 to 1870, 1872 to 1903, 1905 to 1924 (July) and three tables of contents.
18 Annals de Toulouse :—1887 to 1922 (all).
19 Bulletin des Sciences Mathematiques :—
   1st Series, Vols. 1 to 11.
   2nd Series, Vols. 1 to 30, 33, 39 to 47.
   Also all issues of 1924 from January to August and one table of contents.

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