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PROGRESS REPORT.

1. The following gentlemen have been elected members at the usual concessional rate :—

1. *Mr. S. Mahadevan, B.A. (Hon.)*—47, Teachers College, Saidapet, Madras ;

2. *Mr. Krishnarao Srinivasarao Karpur, B.A., (Hon.)*—Student, Fergusson College, Poona (City) ;

3. *Mr. C. Bhaskaraiya, B.A. (Hon.)*—Acting Science Demonstrator, Nizam's College ; 5566, Emambowry Bazaar, Secunderabad (Deccan).

2. The following books have been received for the Library—

1. *Exercices Numeriques et Graphiques de Mathematique*—by L. Zoretti, Gauthier-Villars, Paris, 1914 : 7 frs. ;

2. *Test Questions in Junior Algebra*—by F. Rosenberg, University Tutorial Press, London 1916 : 1*s.*/—;

3. *Preliminary Geometry*—by F. Rosenberg, University Tutorial Press, London 1916 : 2*s.*/—;

4. *Descriptive Geometry*—by Dr. G. C. Anthony and Mr. G. F. Ashley (Technical Drawing Series), pub. by D. C. Heath & Co., Bosten, Rs. 3-8 ; Purchased ;

5. *Bombay University Calendar, Part I*, 1916—17.

6. *Calcutta University Calendar, Part III* (Examination Papers), 1916.

3. The Committee have pleasure to announce to the General Body that it has been arranged to hold a *Meeting* of the Society at *Madras* on the 26th, 27th and 28th of December 1916—(the dates being liable to alteration). A local working committee consisting of the following gentlemen has been appointed to take the necessary steps in the matter :—Dewan Bahadur R. Ramachandra Rao, Messrs. E. B. Ross, M. T. Naraniengar, S. Narayana Aiyar, P. V. Seshu Aiyar, P. R. Krishnaswami and C. N. Ganapathi; Mr. P. V. Seshu Aiyar will be the *Local Secretary*.

The Committee expect all the members to be present on the occasion and it is requested that the papers to be read at the meeting will be forwarded in time to either of the Joint Hon. Secretaries, or the Local Secretary.

POONA,
8th Sept. 1916.

D. D. KAPADIA,
Hony. Joint Secretary.

The Legendre Expansion of $F\{(1-2r\cos\theta+r^2)^n\}$.

By M. T. Naraniengar.

1. In the March 1916 number of the *Quarterly Journal of Mathematics* * Mr. S. Chapman of Trinity College, Cambridge, investigates a power series for the coefficient of P_m (cos θ) in the expansion of $(1-2r\cos\theta+r^2)^n$ in Legendre's functions, and discusses the validity of the result in accordance with Cesáro's method of summability.†. The formal analysis set out by Mr. Chapman in § 2 leads to the following expression:

$$(1-2rx+r^2)^n=\Sigma[(2m+1).A_m(r).P_m(x)] \quad \ldots \quad (1)$$

where $A_m(r)=(-1)^m r^{-m}\sum_{t=m}\left[{}_nC_t\frac{t_m\cdot(n+\frac{1}{2})_{t-m}}{(t+\frac{1}{2})_t}.\, r^{2t}\right]$, $\quad \ldots \quad (2)$

and x denotes cos θ.

In the course of his analysis, which is somewhat tedious, Mr. Chapman uses the formulae for x^t in terms of Legendre's functions given in Todhunter's *Functions of Laplace, Lame and Bessel* (pp. 18, 19), *viz.*

$$x^{2t}=\frac{1}{2t+1}P_0+\frac{2t}{(2t+1)(2t+3)}.5\,P_2+\frac{2t(2t-2)}{(2t+1)(2t+3)(2t+5)}\,9\,P_4+\ldots$$

$$x^{2t+1}=\frac{1}{2t+3}.3\,P_1+\frac{2t}{(2t+3)(2t+5)}.7\,P_3$$
$$+\frac{2t(2t-2)}{(2t+3)(2t+5)(2t+7)}.11\,P_5+\ldots$$

Byerly's *Harmonic Functions*, (p. 50) contains a simpler expression, *viz.*

$$x^t=\frac{t!}{1\cdot3\cdot5\ldots(2t+1)}\Big[(2t+1)P_t+(2t-3)\frac{2t+1}{2}P_{t-2}$$
$$+(2t-7)\frac{(2t+1)(2t+3)}{2\cdot4}P_{t-4}+\ldots\Big]\ldots \quad (3)$$

with aid of which Mr. Chapman's result can be more readily established.

2. To this end we shall first determine the co-efficient of r^m in the expansion of $(1-2rx+r^2)^n$ as a power series in x.

We have

$$(1-2rx+r^2)^n=[1+r(r-2x)]^n,$$
$$=\sum\left[\frac{n_p}{p!}r^p(r-2x)^p\right],$$
$$=\Sigma(B_m.r^m), \text{ say.}$$

* *On the Expansion of* $(1-2r\cos\theta+r^2)^n$ *in a series of Legendre's Functions* By S. Chapman, (pp. 16—26.)

† *Cf.* Bromwich: *Infinite Series*, p. 310.

Collecting the terms r^m in the double series, we get

$$B_m = \left[(-2x)^m . \frac{n_m}{m!} + (-2x)^{m-2} . \frac{(m-1)_1}{1!} \frac{n_{m-1}}{(m-1)!} + (-2x)^{m-4} \frac{(m-2)_2 . n_{m-2}}{2!\,(m-2)!} + \ldots\ldots \right] \quad \ldots \quad (4)$$

The last term in (4) is $\dfrac{n_{\frac{1}{2}m}}{(\frac{1}{2}m)!}$,

if m is even ; and

$$(-2x)^{\frac{1}{2}(m+1)} . \frac{n_{\frac{1}{2}(m+1)}}{[\frac{1}{2}(m+1)]!}$$

if m is odd.

3. Now, to obtain the required Legendre expansion we have to substitute for the several powers of x in $\Sigma\,(B_m r^m)$ from formula (3) and pick out the co-efficients of P_m. For this purpose, we need not consider B_p for values of $p<m$, as such B's will not involve P_m. The expansions of B_{m+1}, $B_{m+3}, \ldots$ will likewise be free from P_m. The coefficients of P_m will thus be found from the development of

$$B_m r^m + B_{m+2} r^{m+2} + B_{m+4} r^{m+4} + \ldots$$

and may be written

$$(-2)^m . r^m a_m . \frac{n_m}{m!} + r^{m+2} \left[(-2)^{m+2} a_{m+2} \frac{n_{m+2}}{(m+2)!} + (-2)^m a_m \frac{(m+1)_1 . n_{m+1}}{1!\,(m+1)!} \right]$$

$$+ r^{m+4} \left[(-2)^{m+4} . a_{m+4} \frac{n_{m+4}}{(m+4)!} + (-2)^{m+2} . a_{m+2} . \frac{(m+3)_1}{1!} \frac{(n_{m+1})}{(m+3)!} + (-2)^m . a_m \frac{(m+2)_2}{2!} \frac{n_{m+2}}{(m+2)!} \right] + \ldots$$

where a_m, a_{m+2}, $a_{m+4}, \ldots$ denote the coefficients of P_m in the expansions of x^m, x^{m+2}, $x^{m+4}, \ldots$ by (3) ; that is

$$a_m = \frac{m!}{1.3.5\ldots(2m+1)} (2m+1)$$

$$a_{m+2} = \frac{(m+2)!}{1.3.5\ldots(2m+5)} . (2m+1) . \frac{(2m+5)}{2}$$

$$a_{m+4} = \frac{(m+4)!}{1.3.5\ldots(2m+9)} . (2m+1) \frac{(2m+9)(2m+7)}{2.4} .$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

Comparing with (1) we deduce that

$$A_m(r) = \frac{(-2)^m r^m}{1.3.5\ldots(2m+1)} \left[\frac{n_m}{m!} + \left(\frac{2.n_{m+2}}{(2m+3)} + \frac{(m+1)_1}{1!} \frac{n_{m+1}}{m+1} \right) r^2 \right.$$

$$+\left(\frac{2^2.n_{m+4}}{(2m+3)(2m+5).2\,!}+\frac{2.n_{m+3}}{(2m+3)}+\frac{n_{m+2}}{2\,!}\right)r^4+\ldots\Big].$$

$$=\frac{(-1)^m r^m.n_m}{(m+\frac{1}{2})_m}\Big[1+r^2\frac{(n-m)(n+\frac{1}{2})}{(m+\frac{3}{2})}+\frac{r^4(n-m)_2.(n+\frac{1}{2})_2}{2\,!\,(m+\frac{3}{2})(m+\frac{5}{2})}$$

$$+\ldots+\frac{r^{2t}(n-m)_t(n+\frac{1}{2})_t}{t\,!\,(m+t+\frac{1}{2})_t}+\ldots\Big], \quad (5)$$

since by Vandermonde's theorem

$$\frac{(m+\frac{5}{2})_2}{2\,!}+\frac{(m+\frac{5}{2})_1(n-m-2)_1}{1\,!\;1\,!}+\frac{(n-m-2)_2}{2\,!}=\frac{(m+\frac{5}{2}+n-m-2)_2}{2\,!}$$

$$=\frac{(n+\frac{1}{2})_2}{2\,!},$$

and generally

$$\frac{(m+t+\frac{1}{2})_t}{t\,!}+\frac{(m+t+\frac{1}{2})_{t-1}(n-m-t)_1}{(t-1)\,!\;1\,!}+\frac{(m+t+\frac{1}{2})_{t-2}(n-m-t)_2}{(t-2)\,!\;2\,!}$$

$$+\ldots\frac{(n-m-t)_t}{t\,!}=\frac{(n+\frac{1}{2})_t}{t\,!}$$

Finally, therefore we may write (5) in the form

$$A_m(r)=(-1)^m r^m.\sum_{t=0}\Big[\frac{n_{m+t}.(n+\frac{1}{2})_t}{t\,!\;(m+t+\frac{1}{2})_{m+t}}r^{2t}\Big]$$

which is easily seen to agree with Mr. Chapman's result (2).

4. Next, let us consider the more general form $F\,(1-2r\,x+r^2)$, where $F(z)$ is capable of expansion by Maclaurin's theorem. In this case, $A_m(r)$ takes the form

$$(-1)^m\Big[\frac{r^m}{(m+\frac{1}{2})_m}\sum\Big(p_m.\frac{F^p(o)}{p\,!}\Big)+\frac{r^{m+2}}{(m+\frac{3}{2})_{m+1}}\sum(p+\tfrac{1}{2}).\frac{p_{m+1}F^p(o)}{p\,!}$$

$$+\frac{r^{m+4}}{(m+\frac{5}{2})_{m+2}}\sum\frac{(p+\frac{1}{2})_2.p_{m+2}F^p(o)}{2\,!\;p\,!}+\ldots\Big]$$

where the summation includes all integral values of p. Thus we obtain, after some simple reduction,

$$(-1)^m A_m(r)=\frac{r^m}{(m+\frac{1}{2})_m}\,F^m(1)+\frac{r^{m+2}}{(m+\frac{3}{2})_{m+1}}\,[(m+\tfrac{3}{2})F^{m+1}(1)+F^{m+2}(1)]$$

$$+\frac{r^{m+4}}{(m+\frac{5}{2})_{m+2}}\Big[\frac{(m+\frac{5}{2})_2}{2\,!}F^{m+2}(1)+\frac{(m+\frac{5}{2})_1}{1\,!}\,F^{m+3}(1)+F^{m+4}(1)\Big]+\ldots(6)$$

The above result may also be directly derived from the expansion by Taylor's theorem of $F(1-2\,rx+r^2)$, *viz.*

$$\sum\Big[(-2\,rx)^p.\frac{F^p(1+r^2)}{p\,!}\Big]$$

and developing $F^p\,(1+r^2)$ in powers of r^2, and using (3) for the Legendre expansion of x^p.

5. Lastly, we shall discuss the most general case of $F(z)$ where z stands for $(1-2rx+r^2)^n$, n being any number whatever.

This case differs from the preceding in having (pn) instead of p throughout the right-hand member; so that

$$(-1)^m A_m = \frac{r^m}{(m+\frac{1}{2})_m} \sum \frac{(pn)_m F^p(o)}{p\,!} + \frac{r^{m+2}}{(m+\frac{3}{2})_{m+1}} \sum (pn+\tfrac{1}{2}) \frac{(pn)_{m+1} F^p(o)}{p\,!} + \ldots \quad \ldots \quad (7)$$

The summations in the above are effected by means of the following formulae from Boole's *Finite Differences*;

$$\phi(p) = \phi(o) + \Delta\phi(o).p_1 + \Delta^2\phi(o)\frac{p_2}{2\,!} + \Delta^3\phi(o)\frac{p_3}{3\,!} + \ldots \Delta^m\phi(o)\frac{pm}{m\,!}, \text{ [p. 11.]} \quad \ldots \quad (8)$$

$$\Delta^q\phi(x) = \phi(x+q) - q\phi(x+q-1) + \frac{q_2}{2\,!}\phi(x+q-2) - \ldots (-1)^q\phi(x). \text{ [p. 19.]} \quad \ldots \quad (9)$$

Thus, if we write $\phi_m(p) = (pn)_m$, $\phi_{m+2}(p) = (pn)_{m+1}(pn+\frac{1}{2})$, $\phi_{m+4}(p) = (pn)_{m+2}(pn+\frac{1}{2})_2$, &c.; and expand these by (8), then

$$(-1^m A_m = \frac{r^m}{(m+\frac{1}{2})_m}\left\{ \phi(o)\frac{F^p(o)}{p\,!} + \Lambda\phi\frac{F^p}{p-1\,!} + \frac{\Lambda^2\phi F^p}{2\,!\; p-2\,!} + \ldots \right\} + \ldots$$

$$= \frac{r^m}{(m+\frac{1}{2})_m}\left\{ \phi\left(1+\frac{D}{1\,!}+\frac{D^2}{2\,!}\ldots\right)F + \Lambda\phi D\left(1+\frac{D}{1\,!}+\frac{D^2}{2\,!}+\right)F + \frac{\Lambda^2\phi D^2}{2\,!}(1+\ldots)F + \right\} + \ldots$$

$$= \frac{r^m}{(m+\frac{1}{2})_m}.e^{\Lambda D}e^{D}[\phi\, F] + \ldots$$

$$= \frac{r^m}{(m+\frac{1}{2})_m}e^{D(1+\Lambda)}\phi.F + \ldots$$

$$= e^{DD'}\left\{ \frac{r^m\phi_m}{(m+\frac{1}{2})_m} + \frac{r^{m+2}\phi_{m+2}}{(m+\frac{3}{2})_{m+1}} + \ldots \right\} F$$

where D operates on ϕ and D′ on F, and the independent variable is to be put equal to zero after the operations.

Some Deceased Modern Mathematicians.*

1. Weierstrass :—Weierstrass was born in Ostenfelde, Germany, on October 31, 1815, and died at Berlin on February 19, 1897. During the years 1834-38 he studied law and finance at the University of Bonn, and later during 1838-40 he studied mathematics privately under Gudermann at Münster.

Many mathematicians of the first rank gave evidence of their unusual mathematical abilities at an early age. Galois, Gauss, and Abel are instances of this kind. On the other hand there are those who abandoned other lines of work to turn to mathematics, acquiring their special aptitudes along this line at a later period. Weierstrass belongs to the latter class, as he turned to mathematics at an age which exceeded that reached by Galois (1811-1832) and at which Gauss (1777-1855) had already completed his monumental work called *Disquisitiones Arithmeticae.*

Weierstrass constitutes an exception to the supposed rule that elementary teaching is distasteful to those who are specially gifted as regards research ability. He spent a number of years as a teacher of secondary mathematics and began his work as Instructor in the University of Berlin in 1856 in connection with a Technical School.

It was not until 1864 that he received a full Professorship at the University of Berlin and could thus devote all his time to advanced mathematics. He continued to look on his elementary teaching experience with pleasure and had little sympathy with those young men who sought to avoid such experience.

Weirstrass published little compared with the large number of his new results and new methods. He endeared himself to his students by the great liberality with which he allowed them to develop important theories which he himself had started but could not find time to finish. As an instance of this kind we may cite the Doctor's dissertation of Sophie Kowalewski (1850-91) who was one of the most noted women among mathematicians of her time. In her interesting biography of Sophie Kowalewski, Anna Charlotte Leffler remarks that the entire scientific work of Sophie Kowalewski was only the development of the ideas of her great teacher Weierstrass. While this tribute may be too favourable as regards the work of Weirstrass in this connection, it is an evidence of the high regard entertained by his students towards their teacher.

* Extracted from G. A. Miller's *Historical Introduction to Mathematical Literature*, New York, 1916.

While Weierstrass worked in many different fields of mathematics his work in the Theory of Functions is probably the best known. His example of a continuous function without a derivative, or a continuous curve without a tangent, made a profound impression when it first became known and constitutes one of the most striking results in the history of mathematics.

He recognized early the need of greater rigor in mathematics and is regarded as the greatest exponent of the modern tendency termed by F. Klein " Arithmetizing of Mathematics."†

His theory relating to the Irrational Numbers is of fundamental importance in the modern theory of these numbers. He gave an abstract definition of determinants of order n as a function of n^2 independent variables satisfying three characterestic conditions, which is frequently erroneously attributed to Kronecker (1823-1891). This error occurs, for instance, in the Encyclopedie des Sciences Mathematiques, tome 1, volume 1, p. 90. ‡ [pp. 255-57.]

2. Cayley :—The biography of Cayley is especially interesting to English readers in view of the fact that Cayley comes next to Newton among noted English mathematicians. He is, however, first among the English mathematicians who have advanced modern mathematical theories. Like Vieta, Fermat, Sylvester and Weierstrass, Cayley studied law in his early years.

During the fourteen years which he spent in the practice of law, he wrote between two and three hundred mathematical papers, and these included some of his most important discoveries. He regarded his legal occupations as the means of providing a livelihood, and he reserved with particular care a portion of his time for mathematical, investigations, refusing a considerable part of the legal work which came to him.

Arthur Cayley was born at Richmond, England, on August 16 1821, and died at Cambridge on January 26, 1895. He was educated at Trinity College, Cambridge, and in 1863 he was called to the newly established Sadlerian Professorship of Pure Mathamatics in the University of Cambridge, which position he held until his death. The duty of the incumbent of this position was " to explain and teach the principles of pure mathamatics and to apply himself to the advancement of that science". Few men were better qualified than Cayley for such a position, and few men have carried out more completely the duties implied in the acceptance of the position.

† *Bulletin of the American Mathematical Society*, Vol. 2, (1896), p. 241.

‡ *Cf.* G. Frobenius, Crelle, Vol. 129, (1905) p. 179.

He wrote only one book on mathematics, viz, a *Treatise on Elliptic Functions*, which was published in 1876, and contains a considerable amount of new matter. On the other hand he wrote an unusually large number of papers on a great variety of different topics, including the subjects of quantics, geometry, theory of matrices, symmetric functions, theory of invariants, &c. The number of these papers is almost a thousand and they have been collected and published in thirteen large volumes. The subject with which Cayley's name is perhaps most intimately associated is the theory of algebraic invariants, but he contributed his mite to almost every subject in pure mathematics. In particular, he is generally credited with the introduction of the Absolute into geometry.

Cayley endeared himself especially to Americans by the fact that he lectured for a time at John Hopkins University during its early years, when Sylvester (1814-97) was engaged in the fundamental work of establishing research in this country. Cayley and Sylvester were students at Cambridge at the same time and formed then a lifelong friendship which was doubtless increased by their interest in common subjects of research. In the theory of algebraic invariants they were both among the earliest to make important contributions and during the fifties while Cayley was practising law and Sylvester was an actuary in London, they were in the habit of walking around the courts of Lincoln's Inn discussing the theory of invariants and covariants which was then occupying the attention of both; although both were engaged in other work for a livelihood.

While Cayley and Sylvester would doubtless have become great mathematicians under almost any circumstances, the time at which they lived seemed especially ripe for great advances by English mathematicians. The dispute between Newton and Leibnitz had resulted in an isolation of English mathematicians, since these adopted the less convenient notation of Newton as well as his conservative geometrical methods and thus failed to join directly in many of the active advances which were being made on the Continent. The disadvantages of this isolation were fully realized at the time when Cayley and Sylvester began their scientific work and the general appreciation of the analytical methods developed on the Continent acted as a healthy stimulus for the exercise of their special gifts along this line. [pp. 257-59.]

3. **Cremona**:—The word Cremona is met with frequently in the form of an adjective in mathematical literature. We speak of a Cremona congruence, a Cremona curve, a Cremona group, a

Cremona substitution, a Cremona transformation, etc. The student of projective geometry is likely to have received inspiration and pleasure from the reading of *Elements of Projective Geometry* which has been translated into various languages.

Luigi Cremona was born in Pavia on December 7, 1830 and died in Rome on June 10, 1903. He was elected as Professor of Geometry in the University of Bologna in 1860 and many of his brilliant discoveries were made soon after. In 1873 he was appointed Professor of Higher Geometry in the University of Rome and Director of the reorganized Engineering School. Most of his important publications preceded this appointment, as the administrative and political duties connected therewith seemed to have consumed most of his energies during his later years.

Cremona was fully identified with Italian institutions and can be called an Italian Mathematician without reserve, unlike Lagrange (1736-1813) who did a greater part of his scientific work in Germany and France. The wonderful Mathematical advances made by Italy since the middle of the nineteenth century were largely guided by Cremona, Brioschi and Beltrami. [pp. 264—5].

4. **Lie** :—The importance of the group-concept in algebraic work was brought to the attention of Mathematicians through the work of Abel and Galois during the first half of the nineteenth century. The fertility of this concept in other lines of work was made clear in the early part of the second half of the century by the researches of Jordan and others. It was not, however, until Lie and Klein had selected this concept as the centralizing and unifying element of their work, that the wide applications of the group concept became generally known and the value of the new method came to be widely appreciated.

Lie permitted his whole soul to be permeated with the group concept and invigourated by its influence he devoted his life with perseverance and great effectiveness to an exposition of various far-reaching Mathematical theories from this new point of view. "The groups do every thing" was the somewhat exaggerated yet inspiring maxim by which he inflamed himself and his pupils.

M. S. Lie was born on December 17, 1842 in Nordfjordeide, Norway and died in Christiania on February 18, 1899. For six and a half years beginning with 1859 he was a student at the University of Christiania, but he did not become specially interested in mathematics. In fact, after leaving the University he was thinking of devoting himself to Astronomy. A few years later he applied himself to the private study of mathematics and read the classic works of Duhamel, Lame', Chasles, Monge, and Poncelet. He became deeply interested and soon began to make original contributions.

In 1861 he went to Berlin having received a stipend from the University of Christiania as a result of his successful investigations. Here he met Klein and both of them went to Paris later coming under the influence of Jordan Darboux and others. The common scientific interests of Klein and Lie led to several papers which they published jointly. While Lie was in Paris, he made in July 1870 that remarkable discovery of a contact transformation by which a sphere can be made to correspond to a straight line.

In 1872 Lie become Professor in the University of Christiania. This special position did not imply that he was expected to give lectures in the University, but permitted him to devote all his time, undisturbed by teaching duties, to his ivestigations. In 1873 he made the beginning of his extensive theory of transformation groups and in 1873-74 he determined all the finite point and contact transformations of the plane.

In 1876 he helped to organize a new journal of mathematics entitled *Archiv for Mathematik og Naturvidenskab*, and ten years later he accepted a call to assume the duties of a Professorship in the University of Leipzig as a successor to F. Klein, who had accepted a call to go to Gottingen. It was at Leipzig that Lie came in contact with a number of students whom he started in his own work. Among these students there were a number of foreigners who were attracted by his great reputation. We referred above to Lie's absorbing interest in group theory, but it should not be inferred that this theory in its abstract form was his chief interest. It served often as a guiding principle where the development of other theories was his main objective. He was especially interested in the theory of differential equations and regarded this theory as the most important among all mathematical subjects.

To make progress in this theory was his chief aim from the beginning to the end of his productive career. Both his geometrical developments and his theory of continuous groups were subsidiary to this end. His great fame led the Norwegian Government to make special efforts to secure his return to his own country, and shortly before his death he did return to accept a very honourable position created for him in the University of Christiania. [pp. 265—68.].

5. **Poincare**:—Notwithstanding the rapid increase in the number of prominent mathematicians during the latter half of the nineteenth and the beginning of the twentieth century, and the tendency to withhold full scientific recognition until after the death of an author, Poincare' stood at the beginning of the twentieth century as the one man whom eminent scholars did not hesitate to speak publicly as the greatest living mathematician. Both in pure and in applied mathematics he worked with remarkable success, and during the latter part of his life he devoted considerable attention to philosophical questions.

By his popular treatment of such fundamental questions as the foundations of geometry and the value of science, he did much to spread scientific knowledge and to popularize our science whose beauties are too apt to escape the attention of the world. These beauties were emphasized in his philosophical work.

Poincaré was born at Nancy, France, on April 29, 1854, and died at Paris in July 1912. He was educated successively at the Lycée de Nancy, l'Ecole Polytechnique , and at l'Ecole Nationale Superieure des Mines, receiving his doctors degree from the University of Paris in 1879. He was a very bright student and received first rank at the entrance examination of l'Ecole Polytechnique. At the early age of 32 he was elected as a member of l'Académie des Sciences, and in view of this occasion he prepared in 1884 a statement entitled "Notice sur les travaux scientifiques de M. Henri Poìncaré."

Although this Notice was written less than five years after he began the publication of his researches, it reviews a large number of his published articles along the following three lines (1) Differential Equations, (2) General Theory of Functions, (3) Arithmetic or Theory of Numbers. He emphasizes the fact that he did not pursue his researches in these three directions independently of each other, but that the results obtained along these various lines threw light on each other, and that his work along each one of them was greatly aided by the work along the other lines.

The breadth of scholarship exhibited by Poincare' in his early writings and his great ability to observe relations between apparently widely different subjects became still more pronounced as he grew older, but we observe even at this early date a mind of very broad sympathies and of extraordinary ability to generalize. His principal writings may be classed under the following four headings :—pure mathamatics, analytic and celestial mechanics, mathematical physics, and the philosophy of science.

In 1909 Emile Borel published in the Journal called *La Revue du Mois*, an article on the method of Poincarè. Parts of this were translated for *the Bulletin of the Calcutta Mathematical Society*. We quote from the translation :—

The method of Poincare′ is essentially active and constructive. He approaches a question, acquaints himself with its present conditions without being much concerned about its history, finds out immediately the new analytical formulas by which the question can be advanced, deduces hastily the essential results, and then passes on to another question. After having finished the writing of a memoir, he is sure to pause for a while, and to think out how the exposition could be improved ; but he would not, for a single instance, indulge in the idea of devoting several days to didactic work. Those days could be better utilized in exploring new regions.

"All this is not specially applicable to mathematics. Let us examine more closely the mechanism made use of for discovery. The essential feature of that mechanism is, as we have already pointed out, the construction of new forumlas. It is not useless that some stress is laid on this point ; for this constructive power is the essential trait of the geinus of Poincare′. The non-mathematical readers can be made to understand all this by means of a comparison. They know what arithmetical calculation is, and are often led to believe that mathematicians are in the habit of making interminable additions, multiplications, etc., and also extractions of cube roots.

In reality arithmetical operations are unique combinations of integral numbers formed of units which are equal to one another. These operations can be compared to the construction of regular walls by means of bricks of uniform size. The work requires only some patience and a little care. On the contrary, analytical operations make use to extremely numerous materials, and their variety is comparable to those of structures where stone, marble, wood, iron, etc, are used. These operations are as different from each other as cuirasse is from a Gothic church. They have also, with the architectural construction, this in common that an impression of beauty is produced by the simplicity and elegance of the essential lines, without exhibiting any of the efforts by means of which the result has been obtained."

Poincaré was a great pioneer boldly entering into unexplored regions and noting some of the most important objective points, and then leaving to others the details of organization. In the words of Borel 'he was more of a conqueror than a colonizer', and he attached little importance to conceptions which cannot be realized in a concrete form. In this respect he may be compared with men of action; his method of work was too active to have much room for such reflections as do not lead to concrete results.

Poincaré won great fame in connection with his prize memoir relating to the problem of three bodies. In 1885 King Oscar II of Sweden offered a prize for the solution of a question in reference to this general problem, and one half of this prize was awarded to Poincare for his article entitled "Sur le proble'me des trois corps et les e'quations de la dynamique" published in the *Acta Mathematica* in 1890. In the *Bibliotheca Mathematica* for 1904 page 198, Eneström calls attention to the interesting fact that the copy of this memoir for which the prize had been actually awarded contained a serious error, and that the given published article was really prepared for the press after the prize had been awarded.

To those who would like to see a connection established between the university athlete and the intellectual giant, between physical powers and intellectual greatness, Poincaré was a decided disappointment. He was only 5 feet 5 inches in height, was somewhat stooped—at least in the latter part of his life, and weighed 154 pounds. Even as a child he was rather weak and did not engage in the rougher sports of the boys of his age. He cared little for politics and achieved his greatness solely through his scholarly services. When he entered the French Academy he was told that he was born a mathematician. He had, however, the good fortune to live in a country where mathematical attainments are held in high esteem even by the general public.

He wrote a number of books especially on mathematical physics, but the three books most commonly known deal with philosophical questions and bear the following titles: *La Science et l'Hypothese*, *La Valeur la Science*, and *Science et Methode.* The first had a circulation of 16000 copies, and had increased his personality tenfold.

The great mainspring of Poincarés activity was seeking the truth. This made his life both simple and beautiful. Seeking the truth implies an open acknowledgment of ignorance. In fact, one of the strongest mathematical methods consists in putting an x for the unknown quantity ; but how could we do this unless we were willing to admit our ignorance of the value of the unknown. Even in his mature years Ponicare' could honestly ask the question "La terre tourne-t-elle ?" Things that are commonly accepted as true, but have not been fully established, frequently offer the most important fields of research, and the great investigator does not always accept the views of the masses as evidence of truth.

At the funeral of Poincaré, the French Minister of Public Instruction remarked that all his work, all his life, was animated, by a prepossession which found expressed in this thought : "The search for truth must be the goal of our activity ; it is the only end that would be worthy of it." [pp. 268-74.]

SHORT NOTES.

A further note on Question 567.

[The solution given in the "Messenger of Mathematics" and on p. 102 of this Journal is unsymmetrical. The following application of the same method preserves symmetry and leads directly to the actual system of equimomental particles required.]

Let each *face* of a tetrahedron be given a surface-density which is uniform, and let the ratios of the densities be such that the faces have all the same mass; since a uniform triangle of mass m is equimomental with three particles each of mass $m/3$ at the mid-points of the sides, the *hollow* tetrahedron so described is equimomental with six particles of equal masses at the mid-points of its edges. It follows at once that if P, Q, R, U, V, W are the mid-points of the edges of a uniform *solid* tetrahedron of mass M, the tetrahedron is equimomental with a distribution along the three lines PU, QV, RW which meet in the centroid G, the total mass of each of the three lines being M/3 and the density in each line bring proportional to the square of distance from G, which is the centroid not only of the tetrahedron but also of each line. With a distribution of this kind in a line, the ratio of the square of the radius of gyration about G to the square of the half-length is

$$\int_0^1 t^4\, dt \Big/ \int_0^1 t^2\, dt,$$

that is, 3/5, and therefore, if the mass of the line is n, the line is equimomental with the system obtained by placing particles of mass $3n/10$ at each end and the residue at the centre. Hence the solid tetrahedron is equimomental with the system obtained by placing particles of mass M/10 at the mid-points of the edges and the residue at the centroid.

The reader is recommended to apply the corresponding process to the triangle; he will obtain at once a system much simpler than that of thirteen particles given inthe June number of the Journal.

ERIC H. NEVILLE.

Infinite Power Chains.

In his Note on the '*Convergence of Infinite Power Chains*', published in the J. I. M. S., Vol. VIII, page 11, Mr. A. Narasinga Rao, B.A. has taken the initial term of the sequence to be unity. The object of the present note is to examine the range within which the initial term a_1 should lie to preserve the convergence of the sequence considered.

The sequence (a_n), where $a_{n+1}=a^{a_n}$ converges, if at all, to a root of the equation $x=a^x$, *i.e.* a point of intersection of the straight line $y=x$ and the exponential curve $y=a^x$.

The following results were arrived at by Mr. A. Narasinga Rao:

(i) $e^{e^{-1}}<a$, no real root, convergence impossible;

(ii) $e^{e^{-1}}=a$, one real root, convergence possible;

(iii) $1<a<e^{e^{-1}}$, two real roots, convergence possible;

(iv) $0<a<1$, one real root, convergent, though oscillating.

(i) The first case requires no further consideration.

(ii) When $a=e^{e^{-1}}$, the limit to which the sequence may converge is e, and there are three special cases to be considered.

First. If $e<a_1$, the sequence diverges to ∞.

For, in this case, the sequence is an increasing monotone; and given any number N, however great, we can find a term in the sequence which is greater than N.

The graphical discussion of this and the succeeding cases is interesting. Let us represent the sequence (see Fig. I.) by points on the line $y=x$, the point D_n, whose abscissa is a_n, representing a_n. Then, the construction for D_{n+1} is as follows. Let the ordinate through D_n cut the exponential curve in D'_{n+1}. Let the line through D'_{n+1} parallel to the x-axis cut the line $y=x$ at D_{m+1}; then D_{n+1} shall represent a_{n+1}. By means of this construction, we get the range of points D_1, D_2...which is readily seen to diverge to ∞.

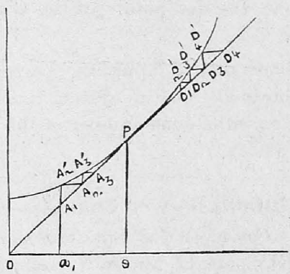


Fig. I.

Secondly. If $a_1=e$, the sequence converges to e.

For, each term of the sequence is e.

Thirdly. If $a_1 < e$ (a_1 should be finite), the sequence converges to e

For, starting with any point A_1, which corresponds to any value of a_1 which is less than e, we get the range of points $A_1, A_2 \ldots$ which ascend up the straight line $y = x$, but cannot cross the point P $(x = e)$ and hence converge to e.

(iii) When $1 < a < e^{e^{-1}}$, the two possible values to which the sequence may converge are x_1 and x_2, $(x_1 < x_2)$ the two roots of the equation $x = a^x$.

Here, we have five special cases to consider.

(*a*) If $x_2 < a_1$, the sequence diverges to ∞.

(*b*) If $x_2 = a_1$, the sequence converges to x_2.

These results follows readily for reasons similar to those given in (ii).

(*c*) If $x_1 < a_1 < x_2$, the sequence converges to x_1.

For, in this case, the sequence is a decreasing monotone, none of whose terms is less than x_1. Graphically, (see Fig. II) we have the range of points $A_1, A_2 \ldots$ which descend down the straight line $y = x$ and converge to the point P_1.

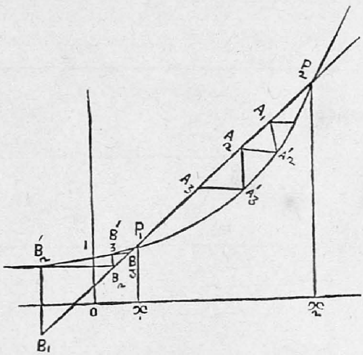


Fig. II.

(*d*) If $x_1 = a_1$, the sequence converges to x_1.

(*e*) If $a_1 < x_1$ (a_1 should be finite), the sequence converges to x_1.

For, in this case, the sequence is an increasing monotone none of whose terms can be greater than x_1. Graphically, (see Fig. II.) we have the range of points $B_1, B_2 \ldots$ which ascend up the straight line $y = x$ and converge to P_1.

(iv) If $0 < a < 1$, the sequence oscillates and can be split up into two parts, consisting of the odd terms and the even terms respectively, each part being monotonic. If x_1 is the root of $x = a^x$, in this case, and if $a_1 < x_1$, the sequence of odd terms is an increasing monotone and the sequence of even terms is a decreasing one. And if $x_1 < a_1$, the former is decreasing and the latter is increasing. In either case, both the sequences converge to the same limit x_1. Graphically, we have the range of points A_1, A_3...on one side of P_1 and the range A_2, A_4...on the other ide, each converging to P_1. (See Fig. III.)

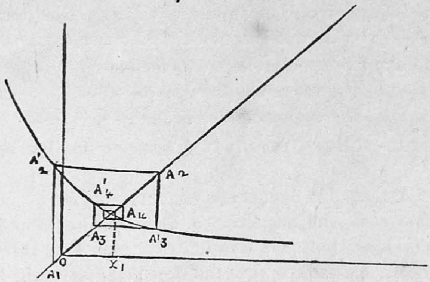


Fig. III.

The different results may be tabulated as follows :—

a	a_1	Limit.
$e^{e^{-1}} < a$	a_1 (finite).	∞
$e^{e^{-1}} = a$	$e < a_1$	∞
	$a_1 \leq e$	e
$1 < a < e^{e^{-1}}$	$x_2 < a_1$	∞
	$x_2 = a_1$	x_2 (the greater root).
	$a_1 < x_2$	x_1 (the smaller root).
$0 < a < 1$	a_1 (finite).	x_1 (the only root).

S. R. RANGANATHAN.

Note on Question 742.

[Q. 742. (MARTYN M. THOMAS) :—Shew that a curve and all its pedals, positive and negative, have the same potential at the pedal origin.]

The following general theorem may be stated :

Let $x_o = x_o\ (t_o)$, $y_o = y_o\ (t_o)$ be the equations of a curve; and let $x_1 = x_1\ (t_1)$, $y_1 = y_1(t_1)$ be those of an associated curve, such that

$$\begin{aligned} t_o - t_1 + \lambda = z &= f(x_o,\ y_o,\ dy_o/dx_o, \ldots) \\ &= f(x_1,\ y_{12}\ dy_1/dx_1, \ldots)\ ; \\ &= \text{a quantity related to the curves } (x_o\, y_o). \end{aligned}$$

$(x_1\, y_1)$ geometrically in the same manner.

Then

$$\int_0^{t_o} \phi'(z)dt_o = \int_0^{t_1} \phi'(z)dt_1 + \int_0^{z} \phi'(z)dz$$

$$= \int_0^{t_1} \phi'(z)dt_1 + [\phi(z)]$$

provided $\phi(z)$ is a single valued function.

Cor. In the case of a closed curve C_o

$$\int_{C_o} \phi'(z)dt_o = \int_{C_1} \phi'(z)d_1$$

since $\int \phi'(z)dz$ round a closed curve is zero.

Applications :

1. The potential of a closed curve is equal to that of its n^{th} positive or negative pedal.

$$\text{For } V_o = \int_{C_o} \frac{ds}{r} = \int_{C_o} \frac{d\theta_o}{\sin\phi_o} = \int \frac{d\theta_1}{\sin\phi_1}, \quad \left[\begin{aligned} \theta_o &= \theta_1 + \frac{\pi}{2}\phi_o \\ \theta_o &= \phi_1 \end{aligned} \right]$$

since $\int \frac{d\phi}{\sin\phi}$ round a closed curve is zero.

Hence $V_o = V_1 = V_2 = \ldots V_n$, where n is positive or negative.

2. When n is any fraction, the *nth.* pedal is geometrically defined as in the article on *Root Pedals*, J. I. M. S., Vol. II, p. 99.

In this case also, proceeding as before we have $V_o = V_n$.

3. For a similar reason the potentials of the inverse and the polar reciprocal of a closed curve are the same.

4. In the case of an *arc* of a curve (not closed)

$$V_o = V_n - \int \frac{d\phi}{\sin\phi} = V_n - [\log\tan\tfrac{1}{2}\phi]$$

$$= V_n - \log(\tan\tfrac{1}{2}\beta.\cot\tfrac{1}{2}\alpha)$$

β, α being the limiting values of ϕ at the extremities of the arc.

5. Let $I_o = \int_{C_o} f(\phi)\, d\Theta_o$ round a closed curve,

and $I_n = \int_{C_n} f(\phi)\, d\Theta_n$, the corresponding integral of the n^{th} pedal; then $I_o = I_n$.

For, $\int f(\phi)\, d\phi = 0$, round a closed curve, provided $f(\phi)$ is a single-valued function of ϕ. Hence the result stated.

M. T. NARANIENGAR.

A small theorem.

By giving particular values to $f(x)$ in the following theorem, neat indentities can be obtained; the theorem is:

$$\frac{f(o)}{a} + \frac{f'(o)}{1!}\,\frac{x}{a+1} + \frac{f''(o)}{2!}\,\frac{x^2}{a+2} + \frac{f'''(o)}{3!}\,\frac{x^3}{a+3} + \ldots$$

$$= \frac{f(x)}{a} - \frac{xf'(x)}{a(a+1)} + \frac{x^2 f''(x)}{a(a+1)(a+2)} - \frac{x^3 f'''(x)}{a(a+1)(a+2)(a+3)} + \ldots$$

This may be demonstrated as follows :—

$$\int_0^1 z^{a-1} f(zx)\,dz = \int_0^1 z^{a-1} f\left\{x - x(1-z)\right\} dz.$$

Expanding each side by Taylor's theorem, we have

$$\int_0^1 z^{a-1}\left\{f(o) + zxf'(o) + \frac{z^2x^2}{2!}f''(o) + \ldots\right\} dz$$

$$= \int_0^1 z^{a-1}\left\{f(x) - x(1-z)\,f'(x) + \frac{x^2(1-z)^2}{2!}f''(x) - \ldots\right\} dz.$$

Integrating each term separately we get the theorem.

S. NARAYANA AIYAR.

The Face of the Sky for November & December 1916.

The Sun

enters Sagittarius on November 22 at 8 P. M. and Capricorns on December 22 at 9 A. M.

Phases of the Moon.

	November.			December.		
	D.	H.	M.	D.	H.	M.
First Quarter ...	4	16	26	3	23	0
Full Moon ...	11	8	31	10	19	1
Last Quarter ...	18	17	35	18	13	8
New Moon ...	26	19	34	26	8	37

Eclipses.

There is a partial eclipse of the Sun on December 25 invisible except in the Antartic regions.

The Planets.

Mercury is in superior conjunction with the Sun on November 23 It is in conjunction with the Moon on November 24 and on December 25, with Mars on December 22, with *m* Virginis on November 1 and with θ Ophinehus on December 5.

Venus is in conjunction with the Moon on November 22 and on December 6. It is in conjunction with θ Virginis on November 16.

Mars is in conjunction with the Moon on November 27 and on December 26.

Jupiter is stationary on December 20. It is in conjunction with the Moon on November 8 at 8-30 P. M. and on December 5 at 10-30 P. M.

Saturn is stationary on November 11. It is in conjunction with the Moon on November 16 and on December 13.

Uranus is in quadrature to the Sun on November 8. It is in conjunction with the Moon on November 3, November 30 and December 28.

Neptune is stationary on November 7. It is in conjunction with the Moon on November 16 and December 13.

V. RAMESAM.

SOLUTIONS.

Question 613.

(P. V. SESHU IYER) :—Show that

$$\lim_{z\to o} \frac{\frac{P(s)}{P(s+z)}\left(\frac{x}{2}\right)^{z}-\frac{P(n+s)}{P(n+s-z)}\left(\frac{x}{2}\right)^{-z}}{z}$$

$$=2\log\frac{x}{2}-\psi(s)-\psi(n+s),$$

where $\psi(x)$ stands for $\frac{d \log Px}{dx}$, and $P(x)$ denotes $\Gamma(x+1)$.

Remarks and Solution by K. B. Madhava.

Let us write $f(z)$ for the expression in the numerator.

Then $$\lim_{z\to o}\frac{f(z)}{z}=f'(o).$$

$$\text{Now } f'(z)=\frac{P(s)\left(\frac{x}{2}\right)^{z}\log\frac{x}{2}}{P(s+z)}-\frac{P(s)\left(\frac{x}{2}\right)^{z}P'(s+z)}{[P(s+z)]^2}$$
$$+\frac{P(n+s)\left(\frac{x}{2}\right)^{-z}\log\frac{x}{2}}{P(n+s-z)}-\frac{P(n+s)\left(\frac{x}{2}\right)^{-z}P'(n+s-z)}{[P(n+s-z)]^2}.$$

$\therefore$ $\quad f'(o)=2\log\frac{x}{2}-\psi(s)-\psi(n+s)$, where ψ is as defined

Here z is made to approach zero through positive values ; if otherwise it is easy to see that the second and third terms enter with changed signs. The limit that we have just calculated has a very interesting application in Bessel Functions, see for instance, Gray and Mathews.

For let it be required to express

$$x^n\int_o^{\frac{\pi}{2}}\sin(x\sin\theta)\cos^{2n}\theta\,d\theta-x^n\int_o^{\infty}e^{-x\sinh u}\cosh^{2n}u\,du \qquad \ldots \quad (A)$$

in terms of Bessel and Neumann functions.

This can of course be done by the usual definite integral definition of these functions, but if we take the integral

$$I=x^n\int_{+\infty i}^{+\infty i}e^{ixz}\,(1-z^2)^{n-\frac{1}{2}}dz$$

along a contour through the imaginary axis beginning at $+\infty i$ and ending again there, but only enclosing the points -1 and $+1$, we can successively, by Cauchy's theorem, deduce (with some restrictions on the magnitudes of n and z) results to establish the equation

$$G_n\sin n\pi=J_n\cos n\pi-J_{-n} \qquad \ldots \qquad \ldots \quad (B)$$

where $\sqrt{\pi}2^{n-1}\Gamma(n+\frac{1}{2})\,G_n$ is the expression (A)

When n is an integer however, (B) gives an apparently indeterminate form for G_n; in such a case, we need only understand (B) to mean

$$\lim_{z\to 0} \sin (n-z)\pi.\ G_{n-z}=J_{n-z} \cos (n-z)\pi - J_{-n+z}$$

i.e.
$$\pi G_n=\lim_{z\to 0} \frac{(-)^n J_{-(n-z)}-J_{(n-z)}}{z}$$

$$=\lim_{z\to 0}\left\{\left(\frac{x}{2}\right)^{-n+z}\sum_{n=0}^{n-1}\frac{(-)^{n+r}}{\Gamma(r+1)}\cdot\frac{1}{z.\Gamma(-n+r+1+z)}\left(\frac{x}{2}\right)^{z}\right.$$

$$\left.+\left(\frac{x}{2}\right)^{n}\sum_{r=0}^{\infty}(-)^r\frac{F(z)}{z}\left(\frac{x}{z}\right)^{2r}\right\}$$

where $F(z)=\Gamma(r+1)\Gamma(n+r+1)f(z)$. Putting in the limit we have determined

$$\frac{\pi}{2}G^n=Y_n-J_n\ (\log 2+\psi(0))$$

where J, Y are Bessel and Neumann functions, respectively.

Here ψ (0) of Mr. Seshu Iyer's notation

$=\psi$ (1) of Dr. Bromwich's $=-C$. (Euler's constant).

[See Bromwich, P. 475, Ex. 42].

For other properties of Hankel's function G_n reference may be made to §§ 17.6 and 17.61 of Whittaker and Watson "*Modern Analysis*" (Revised Edition, 1915).

Question 652.

(S. P. SINGARAVELU MUDALIAR) :—From a point (eccentric angle ϕ) of an ellipse of semi-axes a, b, three normals are drawn to the ellipse; shew that the square on the radius of the circle passing through the feet of the normals is

$$(a+b^2/2a)\cos^2\phi+(b+a^2/2b)\sin^2\phi$$

Additional solution by E. H. Neville.

We prove first that

If a circle cuts an ellipse in the points whose eccentric angles are θ_1, θ_2, θ_3, θ_4, then the centre of the circle has co-ordinates

$$\{(a^2-b^2)/4a\}\,\Sigma\cos\theta_r,\ \{(b^2-a^2)/4b\}\,\Sigma\sin\theta_r,$$

a familiar theorem, virtually given in Wolstenholme's *Mathematical Problems* (No. 1026 of the 3rd Edition). If the radius of the circle is r and the centre is p, q, the four eccentric angles are given by the equation

$$(a\cos\theta-p)^2+(b\sin\theta-q)^2=r^2;$$

arranged as an equation in cos Θ, this is

$$\{(a\cos\Theta-p)^2+b^2(1-\cos^2\Theta)+q^2-r^2\}^2=4\,b^2q^2\,(1-\cos^2\Theta)$$

and an inspection of the coefficients of $\cos^4\Theta$ and $\cos^2\Theta$ gives

$$(a^2-b^2)\,\Sigma\cos\Theta_r=4\,ap,$$

and the corresponding equation

$$(b^2-a^2)\,\Sigma\sin\Theta_r=4\,bq,$$

follows from symmetry.

We can now solve Q. 652; if ψ_1, ψ_2, ψ_3 are the eccentric angles of the feet of the normals from the point of eccentric angle ϕ, then ψ_1, ψ_2, ψ_3, ϕ are the four angles given by

$$a^2\cos\phi\sec\psi-b^2\sin\phi\,\mathrm{cosec}\,\psi=a^2-b^2\,;$$

arranged as an equation in cos ψ this is

$$\{(a^2-b^2)\cos\psi-a^2\cos\phi\}^2\,(1-\cos^2\psi)=b^4\sin^2\phi\cos^2\psi,$$

and therefore

$$(a^2-b^2)\{\cos\phi+\Sigma\cos\psi_r\}=2\,a^2\cos\phi, \quad \ldots \quad \ldots \quad (1)$$

from which follows by symmetry

$$(b^2-a^2)\{\sin\phi+\Sigma\sin\psi_r\}=2b^2\sin\phi. \quad \ldots \quad \ldots \quad (2)$$

Since the circle through the feet of the normals passes also through the point diametrically opposite to the given point, the centre of the circle has co-ordinates

$$\{(a^2-b^2)/4a\}\,\{(\Sigma\cos\psi_r)-\cos\phi\},\ \{(b^2-a^2)4b\}\,\{(\Sigma\sin\psi_r)-\sin\phi\}$$

and by (1), (2) these are equal to $(b^2/2a)\cos\phi$, $(a^2/2b)\sin\phi$. The circle with this centre which passes through the point $(-a\cos\phi, -b\sin\phi)$ has the square of its radius equal to

$$\left(a+\frac{b^2}{2a}\right)^2\cos^2\phi+\left(b+\frac{a^2}{2b}\right)^2\sin^2\phi.$$

The result which we have to prove, and Q. 652 can be expressed differently:

If α, β, γ are the executive angles of the vertices of a triangle inscribed in an ellipse, and if δ denotes $-(\alpha+\beta+\gamma)$, then the circum-centre has co-ordinates:

$$\{(a^2-b^2)/4a\}\,(\cos\alpha+\cos\beta+\cos\gamma+\cos\delta),$$
$$\{(b^2-a^2)/4b\}\,(\sin\alpha+\sin\beta+\sin\gamma+\sin\delta).$$

Since the centroid has co-ordinates

$$(a/3)(\cos\alpha+\cos\beta+\cos\gamma),\ (b/3)(\sin\alpha+\sin\beta+\sin\gamma)$$

and is the point of trisection nearer to the circumcentre of the line joining the circumcentre, the first co-ordinate of the ortho-centre is

$$3(a/3)(\cos\alpha+\cos\beta+\cos\gamma)-2\{(a^2-b^2)/4a\}(\cos\alpha+\cos\beta+\cos\gamma+\cos\delta),$$

that is,

$$\{(a^2+b^2)/2a\}(\cos\alpha+\cos\beta+\cos\gamma)-\{(a^2-b^2)/2a\}\cos\delta,$$

and the second co-ordinate of the ortho-centre is

$$\{(b^2+a^2)/2b\}(\sin\alpha+\sin\beta+\sin\gamma)-\{b^2-a^2)/2b\}\sin\delta;$$

these expressions also are given by Wolstenholme in the problem quoted.

The above solution of Q. 652 depends on the same equation as that given by Mr. K. B. Madhava on p. 110; the principal difference is that instead of referring to the complete equations we observe only certain coefficients which are sufficient for our purpose; there is a slight difference also in the fact that the form required for the square of the radius is exhibited as natural and not as an artificial transformation.

Question 655.

(A. Narasinga Rao) :—Solve for f from the relation

$$f'''(x)=\alpha.f(x+\beta),$$

α and β being any constants

A curve and its *nth.* evolute are similar. Find the intrinsic equation of the curve.

Solution by N. Durairajan.

In similar polygons the ratios of corresponding sides are equal, and angles at corresponding points are equal. Considering the curves to be limits of polygons of an infinite number of sides, we see that at corresponding points,

$$\text{(i) } ds : ds'=\text{const., (ii) } 180^\circ-\delta\psi=180^\circ-\delta\psi'.$$

$$\therefore\ \rho/\rho'=\text{const.; and } \psi=\psi'+\beta \ (\beta \text{ being a constant}).$$

For inversely similar figures,

$$\rho/\rho'=\text{const.; and } \psi+\psi'=\beta.$$

Let $\rho=f(\psi)$, $\rho=g(\psi)$ be the equations to the two curves. If the point ψ_1 on the first corresponds to the point ψ_2 on the second, then

$$\rho_1/\rho_2=k,\ \psi_1=\psi_2+\beta.$$

$$\therefore\qquad f(\psi_2+\beta)=k.g(\psi_2).$$

The condition of similarity is that the functions f and g should be related in such a manner as to permit of the constants k and β being so chosen as to have $f(\psi_2+\beta)\equiv k.g(\psi_2)$ for all values of ψ_2.

In the case of a curve and its n^{th} evolute: let $\rho=f(\psi)$ be the curve. The first normal being at point 'α,' the radius of curvature of the n^{th} evolute is

$$\left[\left(\frac{d}{d\psi}\right)^n f(\psi)\right]$$

when $\psi=\alpha$.

But the inclination of the tangent there is $\alpha+n\frac{\pi}{2}$.

The equation of the n^{th} evolute is $\rho=f^{(n)}\left(\psi-n\frac{\pi}{2}\right)$

The condition of similarity is

$$k.f^{(n)}\left(\psi-n\frac{\pi}{2}\right)=f\left(\psi-n\frac{\pi}{2}+\beta\right)$$

and for inversely similar curves

$$k f^{(n)}\left(\psi-n\frac{\pi}{2}\right)=f\left(\psi-n\frac{\pi}{2}+\beta\right).$$

Hence we have to solve equations of the type

$$\left(\frac{d}{dx}\right)^n . f(x)=\alpha f(\pm x+\beta) \qquad \dots \quad (1)$$

Writing $y=f(x)$

$$y^{(n)}=\alpha.e^{\beta\frac{d}{dx}}.y$$

i.e. $$D^n=\alpha e^{\beta D} \text{ where } D=\frac{d}{dx}.$$

This is a transcendental equation and is not in general solvable. One solvable case is got by assuming $\beta=0$, so that

$$D^n=\alpha=a^n \text{ say.}$$

If $n=2m$, the primitive is (Forsyth: *Differential Equations.*)

$$y=Ce^{-ax}+De^{ax}$$

$$+\sum_{r=1}^{m-1} e^{ax\cos\frac{r\pi}{m}}\left[A_r \cos ax \sin\frac{r\pi}{m}+B_r \sin ax \sin\frac{r\pi}{m}\right].$$

Question 680.

(R. Vythynathaswamy):—If $l, m, n\dots\lambda$ are all positive integers, find the greatest value of

$$lx+my+nz+\dots$$

subject to the condition

$$x+y+z+\dots=\lambda,$$

$x, y, z\dots$ being zero or positive integers.

Solution by A. Narasinga Rao, D. R. Karve and H. K. Chakrabuty, B. Sc.

Let the coefficients arranged in the order of descending magnitude be $l, m, n...$ Then the maxium value is obviously $l\lambda$. For

$$l\lambda-(lx+my+...)$$
$$=l(x+y+z...)-(lx+my...)$$
$$=y(l-m)+z(l-n)+...$$

which is an essentially positive quantity, since $l>m>n,...$

In the same way, we may show that the minimum value of this function is $r\lambda$ where r is the least of the quantities $l, m, n...$

Question 742.

(MARTYN M. THOMAS, M.A.) :—Show that a curve and all its pedals, positive and negative, have the same potential, at the pedal origin.

Solution by K. B. Madhava.

Evidently the proposer has in view the law of the inverse square; and with that assumption, consider the potential V at any point O for a thin uniform bar of small cross section k and mean density ρ.

$$\text{Then } V=\gamma k\rho\int\frac{ds}{OP}$$
$$=\gamma k\rho\int_A^{\pi-B}\frac{d\theta}{\sin\theta}$$
$$=\gamma k\rho \log \cot\frac{OAB}{2}\cot\frac{OBA}{2}.$$

Now let ABCD...NA be any closed polygon of any number of sides with no reentrant angles, and let A′ B′...N′ A′ be its 'pedal polygon' with respect to the origin O.

Then the total potential V of the polygon ABC...NA

$$=\gamma k\rho \log\cot\frac{OAB}{2}\cot\frac{OBA}{2}$$
$$=\gamma k\rho \log\cot\frac{OBC}{2}\cot\frac{OCB}{2}+...$$
$$=\gamma k\rho \log\left(\cot\frac{OAB}{2}\cot\frac{OBA}{2}\cot\frac{OBC}{2}\cot\frac{OCB}{2}...\right)$$
$$=\gamma k\rho \log\left(\cot\frac{OA'B'}{2}\cot\frac{OB'A'}{2}\cot\frac{OB'C'}{2}\cot\frac{OC'B'}{2}...\right)$$

=V of the 'pedal polygon.' (Cf. Minchin p. 310.)

Indefinitely increasing the number of sides of the polygon which in the limit becomes a curve, we have the theorem stated in the problem for the positive pedals. Considering any of these positive pedals as the original curve, we see how to extend the theorem for all pedals, positive or negative.

Question 691.

(A. NARASINGA RAO) :—A marble slab of n pounds breaks into k pieces with which a tradesman finds himself able to weigh goods from 1 to n pounds (fractions excluded). Show that the least value of k is the smallest integer satisfying the relation $3^k \geq 2n+1$.

What are the weights of the several pieces?

Solution by N. Durairajan.

Let $a_1\ a_2 \ldots a_k$ be the k pieces, so that $a_1+a_2+\ldots a_s=n$. Also the weights may be put simultaneously in both scales if necessary. Now consider the product $P\equiv(1+x^{a_1}+x^{-a_1})(1+x_a{}^2+x^{-a_2})\ldots(1+x^{ak}+x^{-ak})$

(1) This product has coefficient of x^r = coefficient of x^{-r}, as is seen by changing x into x^{-1};

(2) The indices represent all possible combinations of $a_1 \ldots a_k$, either additively or subtractively;

(3) The coefficients of all terms are positive integers $\geq$ unity.

Hence $P\equiv1+\Sigma p_r(x^r+x^{-r})$ ... ... (1)

Now, if $a_1 \ldots a_k$ be the k pieces with which we can weigh out 1 to n lbs., r in (1) has all values from 1 to n.

Hence $$P\equiv1+\sum_{1}^{n} p_r(x^r+x^{-r}).$$

When $x=1$, left side $=3^k$; and since $p_r \geq$ an right side is $\geq 1+2n$.

Thus $$3^k \geq 2n+1.$$

This is a necessary condition. In the critical case $3^k=2n+1$, $p_1=p_2=\ldots p_n$; and

$$P=\frac{(1+x^{a_1}+x^{2a_1})\ldots}{x^{a_1+\ldots a_k}}=\frac{1+x+\ldots x^{2n}}{x^n}.$$

$\therefore\ P\,(1+x^{a_1}+x^{2a_1})=1+x+\ldots x^{3^k-1}=(1+x+x^2)(1+x^3+x^6)\ldots$

Thus, if a, be the least of the k quantities $a_1 \ldots a_k$, $a_1=1$, $a_2=3$, $a_3=3^2$, etc.

In other words, the k pieces are 1, 3, 9, 27...etc.

When $2n+1$ is not an exact power of 3, the problem of finding the values of $a_1 \ldots a_k$ is indeterminate, but the least value of k is such that $3^k > 2n+1$.

For example, take the case $n=44$ The value of k here is $k=5$.

In this case the five weights may either be

$$1, 3, 9, 27, 4$$
$$1, 3, 9, 26, 5.$$

The reason is that it is possible to choose $p_1 \ldots p_k$ so that $(p_k \geq 1)$

$1+\sum_{1}^{44} p_k \ (x^k+x^{-k})$ may be the product of five factors of the form $(1+x^a+x^{-a})$.

Question 702 (ii).

(Martyn M. Thomas) :—Prove that

$$\text{(ii)} \int_0^\infty \sin x \log|\cos x| \frac{dx}{x} = \pi \log \frac{1}{\sqrt{2}}$$

Additional Solution by K. B. Madhava.

More generally, let $f(\xi)$ be an odd function of ξ and consider the integral

$$\int_0^\infty f(\sin x)\frac{dx}{x} \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

Splitting up the interval into parts,

$$\int_0^\infty = \int_0^{\frac{1}{2}\pi} + \int_{\frac{1}{2}\pi}^{\pi} + \int_{\pi}^{\frac{3}{2}\pi} + \ldots \quad \ldots \quad \ldots \quad (2)$$

Now in the second interval put $y=\pi-x$.

third $\quad z=x-\pi \ldots \quad \ldots \quad \ldots \quad$ (A)

fourth $\quad u=2\pi-x$ and so on.

$$\text{Thus} \int_0^\infty f(\sin x)\frac{dx}{x} = \int_0^{\frac{\pi}{2}} f(\sin x)\frac{dx}{x} + \int_0^{\frac{\pi}{2}} f(\sin x)\frac{dx}{\pi-x} + \ldots$$

$$= \int_0^{\frac{\pi}{2}} f(\sin x)\left[\frac{1}{x} - \left(\frac{1}{x-\pi}+\frac{1}{x+\pi}\right)\right] + \ldots dx$$

$$= \int_0^{\frac{\pi}{2}} f(\sin x) \operatorname{cosec} x \, dx.$$

Thus if both integrals are convergent, and f is an odd function we shall have

$$\int_0^\infty f(\sin x)\frac{dx}{x} = \int_0^{\frac{\pi}{2}} f(\sin x)\frac{dx}{\sin x}$$

This converts the infinite range into a finite range, and all the expressions involved are circular functions, and often it will be found easier to evatuate the right hand side.

In this particular example, the transformation (A) leaves the function unaltered, except that the cos x changing sign, its logarithm introduces some infinities. To avoid this we put the modulus sign before cos x and read the question as written at the top of this paper.

The question therefore reduces to

$$\int_0^\infty \sin x \log |\cos x| \frac{dx}{x}$$

$$=\int_0^{\frac{\pi}{2}} \sin x \log |\cos x| \operatorname{cosec} x \, dx.$$

$$=\int_0^{\frac{\pi}{2}} \log |\cos x| \, dx.$$

$$=\frac{\pi}{2} \log \tfrac{1}{2},$$

$$=\pi \log \frac{1}{\sqrt{2}}.$$ [*Vide*: Bromwich: App. III, Exs. 11, 12].

Question 724.

(S. Ramanujan) :—Show that

(i) $$\tan^{-1}\frac{1}{2n+1}+\tan^{-1}\frac{1}{2n+3}+\tan^{-1}\frac{1}{2n+5}\ldots\text{to } n \text{ terms}$$

$$=\tan^{-1}\frac{1}{1+2\cdot 1^3}+\tan^{-1}\frac{1}{1+2\cdot 2^3}+\tan^{-1}\frac{1}{1+2\cdot 3^3}+\ldots\text{to } n \text{ terms};$$

(ii) $$\tan^{-1}\frac{1}{(2n+1)\sqrt{3}}+\tan^{-1}\frac{1}{(2n+3)\sqrt{3}}+\ldots\text{to } n \text{ terms}$$

$$=\tan^{-1}\frac{1}{(\sqrt{3})^3}+\tan^{-1}\frac{1}{(3\sqrt{3})^3}+\tan^{-1}\frac{1}{(5\sqrt{3})^3}+\ldots\text{to } n \text{ terms}.$$

Solution by K. B. Madhava.

The question is wrongly printed; the first part ought to read as follows.

Show that

(i) $$\tan^{-1}\frac{1}{2n+1}+\tan^{-1}\frac{1}{2n+3}+\tan^{-1}\frac{1}{2n+5}+\ldots\text{to } n \text{ terms}$$

$$=\tan^{-1}\frac{1}{1+2\cdot 1^2}+\tan^{-1}\frac{1}{3(1+2\cdot 3^2)}+\tan^{-1}\frac{1}{5(1+2\cdot 5^2)}+\ldots\text{to } n \text{ terms};$$

The results can be established easily by Mathematical Induction; for, assuming it to be true for n, we shall prove it for $n+1$; when we have

$$S_{n+1}\equiv\tan^{-1}\frac{1}{2n+3}+\tan^{-1}\frac{1}{2n+5}+\ldots$$

$$\tan^{-1}\frac{1}{4n+1}+\tan^{-1}\frac{1}{4n+3}\quad [n+1 \text{ terms in number}]$$

$$=S_n+\tan^{-1}\frac{1}{4n+1}+\tan^{-1}\frac{1}{4n+3}-\tan^{-1}\frac{1}{2n+1}.$$

But

$$\tan^{-1}\frac{1}{4n+1}+\tan^{-1}\frac{1}{4n+3}=\tan^{-1}\frac{4n+2}{8n^2+8n+1},$$

and $$\tan^{-1}\frac{4n+2}{8n^2+8n+1}-\tan^{-1}\frac{1}{2n+1}=\tan^{-1}\frac{1}{(2n+1)(8n^2+8n+3)}$$

$$=\tan^{-1}\frac{1}{(2n+1)[1+2\cdot(2n+1)^2]},$$

which is exactly the term to be newly added on the right side.

And to complete the proof, we can prove the result true for the first few cases.

When $n=1$; $\tan^{-1}\frac{1}{3}=\tan^{-1}\frac{1}{3}$.

When $n=2$; $\tan^{-1}\frac{1}{5}+\tan^{-1}\frac{1}{7}=\tan^{-1}\frac{6}{17}$

$$\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{57}=\tan^{-1}\frac{6}{17},$$

(ii) Similarly with the second result.

With analogous notation and method, S_{n+1} differs from S_n by

$$\tan^{-1}\frac{1}{(4n+1)\sqrt{3}}+\tan^{-1}\frac{1}{(4n+3)\sqrt{3}}-\tan^{-1}\frac{1}{(2n+1)\sqrt{3}}$$

$$=\tan^{-1}\frac{(2n+1)\sqrt{3}}{12n^2+12n+2}-\tan^{-1}\frac{1}{(2n+1)\sqrt{3}}$$

$$=\tan^{-1}\frac{1}{\sqrt{3}}\cdot\frac{1}{(2n+1)(12n^2+12n+3)}$$

$$=\tan^{-1}\frac{1}{[(2n+1)\sqrt{3}]^3},$$

which is precisely the term to be added when n is increased by unity.

The first few cases when $n=1$, 2 etc, being easily verified as before the proof by Induction is complete.

Question 729.

(K. Padmanabhulu, B.A.) :—If the earth were to break up into an indefinite number of fragments at any point in its course round the sun by any sudden explosion, prove that all the fragments meet again at the same point ; and that at the middle of the interval between the explosion and junction all the pieces will be moving with equal velocities in parallel directions.

Solution by Martyn M. Thomas.

When the explosion takes place, the fragments are all scattered in different directions with an equal velocity, say v.

They describe elliptic paths with the sun at one focus.

The relation $v^2=\mu\left(\frac{2}{r}-\frac{1}{a}\right)$ shows that v, μ, r being the same for all these paths, a also must be the same.

Thus the periodic time which is $\frac{2\pi}{\sqrt{\mu}}a^{\frac{3}{2}}$ is also the same for all the paths.

Hence the fragments will again pass through the point at which the explosion took place, after the lapse of time $\frac{2\pi}{\sqrt{\mu}}\;{}^{3}$

At the middle of the interval between explosion and junction, the several fragments will be situated at the diametrically opposite points of their respective orbits, and must be moving with the same velocity v, but in directions opposite and parallel to their initial directions Hence the theorem.

Question 736.

(R. Srinivasan, M.A.) :—Show that the common tangent to the nine-point and inscribed circle of the triangle ABC cuts the sides a, b, c in the ratios

$$\frac{a-b}{a-c},\ \frac{b-c}{b-a},\ \frac{c-a}{c-b}.$$

Solution by K. V. A. Sastri, B.A., M. M. Thomas, K. B. Madhava, V. Anantaraman and K. Appukuttan Erady, M.A.

The equation of the radical axis of two circles is

$$(t_1^2-t'^2_1)x+(t_2^2-t'^2_2)y+(t_3^2-t'^2_3)z=0,$$

where the t's are the lengths of the tangents from the vertices to the two circles. (See Milne : *Homogeneous Co-ordinates*, p. **111**.)

But, for the nine-point circle and the incircle, the radical axis is, by Feuerbach's theorem, the common tangent. Hence its equation is

$$\left[\frac{bc\cos A}{2}-(s-a)^2\right]x+\left[\frac{ca\cos B}{2}-(s-b)^2\right]y$$
$$+\left[\frac{ab\cos C}{2}-(s-c)^2\right]z=0,$$

which reduces to

$$\frac{x}{b-c}+\frac{y}{c-a}+\frac{z}{a-b}=0;$$

hence it cuts the sides of the triangle in the ratios

$$\frac{a-b}{a-c},\ \frac{b-c}{b-a},\ \frac{c-a}{c-b},$$

as given in the problem.

Additional Solutions by R. D. Karve, S. V. Venkatarayasastri, M. K. Kewalramani, M.A., and Chas. Saldanha.

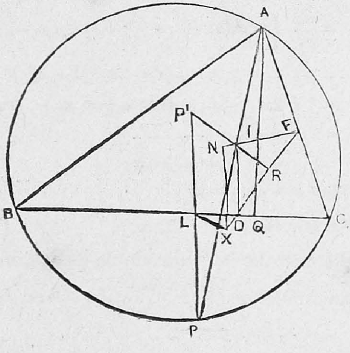
Question 743.

(N. Salva) :—Prove the following construction for the Feuerbach-point :

If AI meet the circumcircle in P and P′ be the reflection of P in BC, then the Feuerbach-point F is the reflection of D in IP′, where D is the point of contact of the incircle and BC.

Solution by K. J. Sanjana.

Draw NX the radius of the nine-point circle perpendicular to BC ; then X corresponds to D in the incircle, and XD (like NI) will go through F, the external centre of similitude. Draw IR perpendicular to DF to meet in P′ the line PL perpendicular to BC.



Then $\angle PP'I = \angle DIR = \angle RDQ = \angle LDX$; also $\angle P'PI = \angle IAQ = \frac{1}{2}(C-B)$, and $\angle DLX = \angle DFQ = \frac{1}{2} \angle LFQ = \frac{1}{2}(C-B)$ from the nine-point circle. Thus the $\triangle s$ P′IP and XLD are similar, and $PP' : PI = LD : LX$.

Now $PI = 2R \sin \frac{A}{2}$; $LD = \frac{1}{2}(c-b) = 2R \sin\frac{A}{2} \sin\frac{C-B}{2}$; and $LX = R \sin \frac{C-B}{2}$, from the nine-point circle. Thus, we get

$$PP' = 4R \sin^2 \frac{A}{2} = 2PL,$$

so that P′ is the reflection of P in BC. Hence the theorem.

It will similarly be found that the point of contact of the nine-point with the first scribed circle is the reflection in $P'I_1$ of D_1, the point of contact of the ex-circle with BC.

Question 744.

(N. Salva) :—If ABCD, AB′ CD′ be two harmonic ranges, such that D′ is the midpoint of CD, prove that $BC^2 = BB'.BD$.

Solution by Lakshmishankar Bhatt and K. J. Sanjana.

Referring to A as origin, let B, C, D be denoted by x_1, x_2, x_3 respectively, so that $2x_1 x_3 = x_2(x_1 + x_3)$...(i)

Now D′ will be $\frac{1}{2}(x_2 + x_3)$, so that if $AB' = y$

$$2y.\tfrac{1}{2}(x_2 + x_3) = x_2 \{ y + \tfrac{1}{2}(x_2 + x_3) \},$$

or
$$y = (x_2^2 + x_2 x_3)/2x_3.$$

Hence
$$y(x_3 - x_1) = \frac{1}{2x_3} \left\{ x_2^2 x_3 - x_2^2 x_1 + x_2 x_3^2 - x_1 x_2 x_3 \right\}$$

$$= \frac{1}{2x_3} \left\{ x_2 x_3 (x_2 + x_3 - x_1) - x_2(2x_1 x_3 - x_2 x_3) \right\}, \text{ from (i)}$$

$$= \tfrac{1}{2} \{ x_2^2 + x_2 x_3 - x_2 x_1 - 2x_2 x_1 1 + x_2^2 \}$$

$$= \tfrac{1}{2} \{ 2x_2^2 - 3x_2 x_1 + 2x_1 x_3 - x_2 x_1 \}, \text{ from (i)}$$

$$= x_2^2 - 2x_2 x_1 + x_1 x_3$$

$$= (x_2 - x_1)^2 + x_1(x_3 - x_1);$$

thus finally $(x_2 - x_1)^2 = (x_3 - x_1)(y - x_1)$

or
$$BC^2 = BD.BB'.$$

There is a slight mistake in the example as originally given.

Additonal Solution by S. V. Venkatachala Iyer.

Question 745.

(V. Anantaraman) :—How can 49 diamonds whose values are in A. P., be divided among 7 persons so that each may get the same number of diamonds, the total value of the diamonds got by each being the same.

Solution by R. D. Karve, S. Venkataraya Sastri, S. V. Vekatachala Iyer and others.

Let the values of the diamonds be denoted respectively by

$$a+d,\ a+2d, \ldots\ldots\ldots\ldots a+49d.$$

Then the total value of the diamonds that each of the 7 persons should get is denoted by $7a+175d$.

The question thus reduces to selecting 7 groups of 7 numbers each, from the numbers 1 to 49 so that the sum of each group may be 175. This may be done by taking the groups of numbers, either along the rows or along the columns in a magic square filled with numbers 1 to 49 as represented below.

30	39	48	1	10	19	28
38	47	7	9	18	27	29
46	6	8	17	26	35	37
5	14	16	25	34	36	45
13	15	24	33	42	44	4
21	23	32	41	43	3	12
22	31	40	49	2	11	20

Question 758.

(S. Malhari Rao, B. A.) :—If the integers x, y, z represent the sides of a right-angled triangle ; and x, z, are primes greater than 5, shew that y is a multiple of 60 ; and that $x+z=1800$, when $y=1740$.

Solution (1) *by K. Appukuttan Erady, M.A. and 'Q'* ;

(2) *by K. J. Sanjana, M.A. and R. D. Karve, M.A.*

(1) We have $x^2+y^2=z^2$.

$\therefore$ $x^2=z^2-y^2=(z+y)(z-y)$.

Hence since x is prime

$$z-y=1 \text{ and } z+y=x^2.$$

Thus when $y=1740$, $z=1741$, $x^2=3481$, $x=59$ and $x+z=1800$.

Again since $z-y=1$ and $z+y=x^2$, $2y=x^2-1$.

Now x being prime, $x-1$ & $x+1$ are two consecutive even numbers

$\therefore$ x^2-1 contains 8 as a factor.

Since $(x-1)\,x\,(x+1)$ contains 3 as a factor, and x is prime, x^2-1 contains 3 as a factor.

Again $(x-2)(x-1)x(x+1)(x+2)$ is divisible by 5; but $x^2-4=2y-3=2z-5$, and z is a prime greater than 5;

$\therefore$ x^2-1 contains 5 as a factor.

Hence x^2-1 contains $8\times3\times5$ as a factor. That is y is multiple of 60.

(2) Evidently x, y, z must be of the form

$$m^2-n^2,\ 2mn,\ m^2+n^2$$

Since x and z are primes, y is of the form $2mn$ and x, z of the forms m^2-n^2 and m^2+n^2, and

$$(2mn)^2=(m^2+n^2)^2-(m^2-n^2)^2.$$

Now $(m^2+n^2)^2-1$ and $(m^2-n^2)^2-1$ are each divisible by 3 by Fermat's Theorem. Hence their difference (the right hand side above) is divisible by 3,

$\therefore$ $2mn$ is divisible by 3.

Again all primes (>2) being of the form $4k\pm1$, their squares have the form $8p+1$ and the difference of squares of primes is therefore divisible by 8,

$\therefore$ $(2mn)^2$ is divisible by 8 and hence by 16 being a square,

$\therefore$ $2mn$ is divisible by 4.

Also all squares are of the forms $5k$, or $5k\pm1$. Hence either m^2 or n^2 must be of the form $5k$, otherwise either m^2+n^2 or m^2-n^2 would be composite. Hence $2mn$ is divisible by 5.

$\therefore$ Thus $2mn$ is a multiple of $3\times4\times5$ or 60.

Again, if $y=1740$, $2mn=1740$, and $mn=870$. Obviously, since m^2-n^2 *i.e.* $(m+n)(m-n)$ is to be a prime, $m-n$ must be 1.

Hence $m=30$ and $n=29$, and $x+z=2m^2=1800$.

QUESTIONS FOR SOLUTION.

789. (K. J. Sanjana, M. A.) :—P, Q, R, S are four conormal points on an ellipse whose centre is C and axes $2a$ and $2b$, O being the point of concurrence of the normals. If x_r, y_r ($r=1, 2, 3, 4$) denote the centres of the circles Q R S, P R S, P Q S, P Q R respectively, prove that $\Sigma x=$ the abscissa of O, $\Sigma y=$ the ordinate of O, and that each centre lies on an ellipse whose centre is at the mid point of CO and whose axes are

$$\frac{a^2-b^2}{2a}, \frac{a^2-b^2}{2b}.$$

[Suggested by Mr. Neville's Question No. 788.]

790. (K. J. Sanjana, M. A.) :—Integrate the equation

$$(a^2+x^2)\frac{d^2y}{dx^2}+2(m+1)x\frac{dy}{dx}+m(m+1)y=f(x).$$

[Two particular integrals when $f(x)$ is absent are

$$\int_0^\infty e^{-az}\cos zx.z^{m-1}\,dz, \int_0^\infty e^{-az}\sin.xz.z^{m-1}\,dz.]$$

791. (K. Appukuttan Erady, M.A.) :—The centres of three circles of radii a, b, c form a triangle of sides l, m, n and area Δ. If $(r_1\, r_1')$, (r_2, r'_2) (r_3, r_3') and (r_4, r_4'), be the radii of the four pairs of circles tangential to the three circles (the circles belonging to any pair being inverses of each other with respect to the common orthogonal circle of the three original circles), show that

$$\frac{1}{r_1 r_1'}+\frac{1}{r_2 r_2'}+\frac{1}{r_3 r_3'}+\frac{1}{r_4 r_4'}$$

$$=\frac{16(\Delta^2-\Sigma a^2l^{21})}{\Sigma l^2(a^2-b^2)(a^2-c^2)-\Sigma m^2n^2(b^2+c^2)+\Sigma a^2l^4+l^2m^2n^2}.$$

792. (K. Appukuttan Erady, M.A.) :—With the usual notation in elliptic functions, show that if $\alpha+\beta+\gamma+\delta=0$,

(1) k'^2 (sn α sn $\beta-$sn γ sn δ)

$\quad=$cn α cn β dn γ dn$\delta-$cn γ cn δ dn α dn β ;

(2) cn α cn $\beta-$cn γ cn $\delta=$

$\quad$sn α sn β dn γ dn $\delta-$sn γ sn δ dn α dn β ;

(3) $\frac{1}{k^2}$ (dn α dn $\beta-$dn γ dn δ)

$\quad=$sn α sn β cn γ cn $\delta-$sn γ sn δ cn α cn β.

793. (Martyn M. Thomas, M. A.) :—Two stars rise together, and are observed to come simultaneously over a vertical λ degrees west of the meridian. Show that they must have risen $\frac{1}{15}\cos^{-1}(-\sin^2\lambda)$ hours before; and that the latitude of the place is $\tan^{-1}(\sin\lambda)$.

794. (Martyn M. Thomas, M.A.) :—The plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ moves in such a manner that the centroid of the triangle ABC always lies on the surface $x^{-2}+y^{-2}+z^{-2}=9$, A, B, C being the points where the plane meets the rectangular axes of co-ordinates. Prove that the Nine-points centre of ABC will lie on the surface

$$x\sqrt{4x^2-1}+y\sqrt{4y^2-1}+z\sqrt{4z^2-1}+2(x^2+y^2+z^2)=1.$$

795. (V. M. Gaitonde) :—Prove that

$$\tan\frac{3\pi}{11}+4\sin\frac{2\pi}{11}=\sqrt{11}$$

796. (M. K. Kewalramani, M.A.) :—Prove that

$$\underset{x=0}{\mathrm{Lt}}\left[\frac{d^{2n}}{dx^{2n}}\left(x\cot x\right)^{2n+1}\right]=(-1)^n(2n)!$$

797. (K. S. Karpur) :—If two sides of a given polygon touch each a fixed circle, prove that the remaining sides also touch each a fixed circle.

798. (S. Sengodaiyan) :—Establish the identity

$$S(n, r)=n_r \cdot c_r-n_{r-1} \cdot c_{r-1}S(n-r, 1)+n_{r-2}c_{r-2}S(n-r+1, 2)$$
$$-+\ldots(-1)^rS(n-1, r),$$

where n_p denotes $n(n-1)\ldots(n-p+1)$, c_p denotes the number of combinations of n things p together, and $S(x, y)$ denotes the sum of the products of the first x natural numbers y at a time.

799. (S. Malhari Rao.) :—Find three primes in A. P. such that the sum of their squares is 35427.

800. (S. Malhari Rao.) :—Shew that the sum of all fractions which may be represented by a recurring decimal of the form $(.\dot{a}\,b\,c\,\dot{d})$ is 50, provided $a+c=b+d=9$.

801. (S. Krishnaswami Aiyangar) :—

$$\text{If } a_n=1-\frac{3^n}{3!}+\frac{5^n}{5!}-\frac{7^n}{7}+\ldots\ldots$$

and

$$-b_n=\frac{2^n}{2!}-\frac{4^n}{4!}+\frac{6^n}{6!}-\ldots\ldots$$

prove that

$$\text{(i) } a_n+a_{n+1}\log p+a_{n+2}\frac{(\log p)^2}{2!}+a_{n+3}\frac{(\log p)^3}{3!}+\ldots\ldots$$
$$=p-3^n.\frac{p^3}{3!}+5^n.\frac{p^5}{5!}-\ldots\ldots$$

(ii) $\frac{1}{2}\sin(e^x+e^{-x})$

$$=\sum_{0}^{\infty}\left\{a_{2n}b_0+{}^{2n}C_2a_{2n-2}b_2+\ldots\ldots,+a_0b_{2n}\right\}\frac{x^{2n}}{2n\,!}$$

802. (S. Krishnaswami Aiyangar) :—Prove that

(i) $$\sum_{1}^{\infty}\left\{\frac{\Gamma(n+1)}{\Gamma(n+\frac{3}{2})}\right\}^2\frac{1}{n}=4\left(\frac{4}{\sqrt{\pi}}-\sqrt{\pi}\right)$$

(ii) $$\sum_{n=0}^{\infty}\frac{\Gamma(n+\frac{1}{2})}{\Gamma(n+1)}\cdot\frac{1}{(2m+2n+1)(m+n+1)}$$

$$=\pi\left\{\frac{\Gamma(m+\frac{1}{2})}{\Gamma(m+1)}-2.\frac{\Gamma(m+1)}{\Gamma(m+\frac{1}{2})}\right\}$$

803. (Selected) :—Prove that the Rectangular Hyperbola $x^2-y^2-4ax\cos^3\alpha+4ay\sin^3\alpha+3a^2\cos 2\alpha=0$ osculates its envelope.

804. (Selected) :—In the curve whose intrinsic equation is $\frac{ds}{d\psi}=a$ sec. 3ψ, show that the rectangle under the distances of any point from the foci of the osculating conic is constant.

805 (E. H. Neville).—With the notation of Questions 412 and 761, but allowing the angles α, β to be variable, shew that the necessary and sufficient condition for the current point of the α—evolute of the β—evolute to be the current point of the β—evolute of the α—evolute is that either α or β is a right angle or that the difference between α and β is constant.

806. (S. Narayana Aiyar, M. A.) :—Demonstrate the following :—

(1) $$\frac{d}{dx}\frac{\Gamma(x+a)}{\Gamma(x+b)}=\frac{\Gamma(x+a)}{\Gamma(x+b)}\cdot\left\{\frac{a-b}{x+b}-\frac{1}{2}\frac{(a-b)(a-b-1)}{(x+b)(x+b+1)}+\frac{1}{3}\frac{(a-b)(a-b-1)(a-b-2)}{(x+b)(x+b+1)(x+b+2)}-\ldots\right\}$$

(2) $$\frac{d}{dx}\frac{\Gamma(a-x)}{\Gamma(b-x)}=\frac{\Gamma(a-x)}{\Gamma(b-x)}\cdot\left\{\frac{b-a}{a-x-1}-\frac{1}{2}\frac{(b-a)(b-a+1)}{(a-x-1)(a-x-2)}+\frac{1}{3}\frac{(b-a)(b-a+1)(b-a+2)}{(a-x-1)(a-x-2)(a-x-3)}-\ldots\right\}$$

807. (F. H. V. Gulasekaran) :—Construct a trapezium having given the lengths of diagonals and the oblique sides.