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PROGRESS REPORT.

1. The following gentlemen have been elected members at the concessional rate :—

1. *Mr. Sonti Purushotam*, M. A., L. T.—Assistant Professor of Mathematics, Presidency College, Madras, 37, Tholasinga Perumal Street, Triplicane ;
2. *Mr. K. B. Madhava*, B. A. (Honours)—Research Scholars, Madras University, 1/11 Singarachari Street, Triplicane, Madras ;
3. *Mr. C. Krishnamachari*, B. A. (Honours)—Lecturer, Collegiate High School, Mysore.

2. The following books have been received for the Library—

1. *Quartric Surfaces with singular points*—by C. M. Jessop. Camb. University Press, 12s., 1915 ;
2. *Ordinary Differential Equations*—by Dr. J. Morris Page. Macmillon & Co., London, 1897. (Presented by Mr. S. Krishnaswami Aiyangar).

POONA, 31st July 1916.

D. D. KAPADIA,
Hony. Joint Secretary.

Stability and Oscillations of Plane Kites.

By J. M. Bose, M.A., B.Sc.

(*Continued from page 49.*)

6. It is obvious now, that the stability of a plane kite depends entirely on the mode of attachment of the string. If the resultant tension always intersects the axis of symmetry, then some of the equations of the previous articles require slight modification.

Let E′ be the point where the resultant tension intersects the axis of symmetry, H the point of bifurcation, and HP the perpendicular on the y-axis, and let

$$GE' = h,$$
$$E'P = x,$$
$$GP = b,$$
$$HP = p.$$

Since $$h + x = b.$$

$$h = b + p\cot\overline{\phi + \chi};$$

so that $$\partial h = -p\,\mathrm{cosec}^2\,\overline{\phi_o + \chi}\;\partial\phi = -p'\epsilon^2 \text{ say},$$

where $$p' = p\,\mathrm{cosec}^2\,\overline{\phi_o + \chi} = p\sec^2\theta,$$

θ being the angle between the y-axis and the perpendicular to the string and ψ is assumed to be constant for the present.

Thus $$h = h_o - p'\epsilon$$

where $h_o = b - p\tan\theta_o$.

Using this value of h the equations (1), (2), (3), of § 2 remain unaltered, but

$$A\dot{\theta}_1 = -T(h_o\sin\theta + p'\cos\theta)\epsilon - \eta_o v R_v - \eta_o w R_w - \eta_o\theta_1 R_1$$
$$-cR_o\phi'(\alpha)\left(\frac{w\cos\alpha}{V} - \frac{v\sin\alpha}{V} + \theta_1\frac{y\cos\alpha}{V}\right) \quad \ldots (4)'$$

$$B\theta_2 = 0 \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots (5)'$$

$$C\dot{\theta}_3 = -h_oT(\theta\cos\chi + \chi\sin x) \quad \ldots \quad \ldots \quad \ldots \quad \ldots (6)'$$

taking (2), (3), (4)′ and proceeding exactly as before, we get the biquadratic

$$a^4\lambda + b\lambda^3 + c\lambda^2 + d\lambda + e = 0,$$

where

$$a = AM^2$$

$$b = M\left[M\left\{R_o\left(\frac{c\,\phi'(\alpha)}{V}y\cos\alpha\right) + \eta_o R_1\right\} + AR_w\right]$$

$$c = \mathrm{M}\Big[\mathrm{MT}\,(p'_1\cos\theta + h_o\sin\theta) + \frac{c\mathrm{R}_o}{\mathrm{V}}\Big\{\phi'(\alpha)\cos\alpha\,(y\,\mathrm{R}_w - \dot{\mathrm{R}}_1)\Big\}\Big]$$

$$d = \mathrm{M}\Big[\mathrm{TR}_w(p'\cos\theta + h_o\sin\theta) + a\mathrm{R}_o\,\mathrm{R}_v - \mathrm{R}_o{}^2\frac{\sin\alpha}{\mathrm{V}}c\phi'(\alpha)\Big]$$

$$e = -\frac{\mathrm{R}_o{}^2}{\mathrm{V}}c\phi'(\alpha)(\mathrm{R}_w\sin\alpha + \mathrm{R}_v\cos\alpha).$$

These coefficients can be put into different forms by using the different equilibrium conditions; for instance, we have

$$\mathrm{R}_o = \frac{\mathrm{W}h_o}{h_o - \eta_o}\cos\alpha = \mathrm{KSV}^2 f(\alpha);$$

hence multiplying the determinental equation in λ by g^3, we may write e in the form

$$\frac{e}{g^4} = -\frac{h_o}{h_o - \eta_o}\cdot\frac{2\mathrm{W}}{g}\,\mathrm{K}^2\mathrm{S}^2\mathrm{V}^2\{f(\alpha)\}^2\,c\phi'(\alpha)\cos\alpha$$

corresponding to Prof. Bryan's (33), *Stability in Aviation*, p. 43.

7. So far we have assumed the length of the string to be infinite. We now proceed to the discussion of the more important case, by taking into account, the variation in the magnitude and direction of the tension, and also assume the length of the string to be finite.

Let H be the point where the string bifurcates into two branches HC, HD; P the foot of the perpendicular from H on the y-axis, and β the angle $\mathrm{H\hat{G}P}$ Also let δ and ϵ be the small increments of χ and ϕ respectively, and $a = \eta_o$

8. *Longitudinal Stability in the case of finite string.*

The roots of the biquadratic given above determine the velocities at the end of any time. We next proceed to show that if we assume the independence of symmetric and asymmetric oscillations, then it is possible to obtain a biquadratic the roots of which determine the co-ordinates of the kite at the end of any time, and hence the nature of the roots of this biquadratic will also determine the conditions for longitudinal *stability of position*.

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* In Prof. Bryan's memoir α is defined as the inclination of a plane to the x-axis, which is the direction of motion, and θ is the inclination of the direction of motion to a fixed time, namely the horizontal. The symbols therefore correspond respectively to our θ and α.

Let χ and ϕ be the inclination of the string and kite to the vertical, and let r† be the length of the string. The accelerations of the point E′, where the string cuts the axis of symmetry are $(\ddot{r}-r\dot{\chi}^2)$ and $\frac{1}{r}\frac{d}{dt}(r^2\dot{\chi})$, and the accelerations of the centre of gravity relative to this point are $(\ddot{h}-h\dot{\phi}^2)$ and $\frac{1}{h}\frac{d}{dt}(h^2\dot{\phi})$. Hence applying these at the centre of gravity and taking moments about E′, and about the point where the string meets the ground, we get the two equations

$$M\,h\cos\Theta\,(\ddot{r}-r\dot{\chi}^2)+M\,h.\,\frac{1}{h}\frac{d}{dt}\,(h^2\dot{\phi})+M\,k^2\,\ddot{\phi}-M\,h\sin\Theta.\,\frac{1}{r}\frac{d}{dt}\,(r^2\dot{\chi})$$
$$=(h-a)\,R-W\,h\sin\phi \qquad \ldots\ (1)$$

$$M\,(h+r\sin\Theta)\,\frac{1}{h}\frac{d}{dt}\,(h^2\dot{\chi})+M\,h\cos\Theta\,(\ddot{r}-r\dot{\chi}^2)$$
$$-M\,(r+h\sin\Theta)\,\frac{1}{r}\frac{d}{dt}\,(r^2\dot{\chi})+M\,k^2\ddot{\phi}-M\,r\cos\Theta\,(\ddot{h}-h\,\dot{\phi}^2)$$
$$=(h-a+r\sin\Theta)\,R-W\,(h\sin\phi+r\sin\chi).\ (2)$$

These are exact equations, but since we are concerned with small motions only, we may reject $\dot{\phi}^2$ and $\dot{\chi}^2$ and regard h, r, $\sin\Theta$, $\cos\Theta$ on the left hand side of the above equations to be constants.

The variable part of r is $HE'=p\sec\Theta$, so that

$$\ddot{r}=\frac{d^2}{dt^2}\,(p\sec\Theta)=p'\sin\Theta_o\,\ddot{\Theta}=p'\sin\Theta_o\,(\ddot{\phi}+\ddot{\chi})$$
$$\dot{r}=p'\sin\Theta_o\ \ (\dot{\phi}+\dot{\chi})$$
$$\ddot{h}=-p'(\ddot{\phi}+\ddot{\chi})\ \text{approximately,}$$

where $\qquad p'=p\sec^2\Theta_o.$

Introducing these and replacing (2) by the difference of (1) and (2), we have the two equations

$$M(h_o p\ \tan\Theta_o+k^2+h^2_o)\ddot{\phi}+M\,h_o(p\tan\Theta_o-r_o\sin\Theta_o)\ddot{\chi}$$
$$=(h-\eta)R-Wh\sin\phi\ \ \ldots \qquad \ldots\ (1)$$

$$M(h_o\sin\Theta_o+p'\cos\Theta_o)\ddot{\phi}+M\ddot{\chi}(p'\cos\Theta_o-r_o)$$
$$=R\sin\Theta-W\sin\chi \quad \ldots \qquad \ldots\ (2)$$

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† The part of the string from the point of bifurcation to the point where it meets the ground is of course assumed to be inextensible and weightless.

We have also for the motion of the centre of gravity

$$M\ddot{\phi}(h+p\tan\theta)+M\ddot{\chi}(p\tan\theta-r\sin\theta)$$
$$=R-W\sin\phi-T\cos\theta. \quad \ldots \quad \ldots (3)$$

If (3) be multiplied by h and subtracted from (1), we shall of course get (4) of § 2.

If we eliminate $\ddot{\phi}$ and $\ddot{\chi}$ from the above equations, we get

$$\begin{vmatrix} hp\tan\theta_o+k^2+h^2, & (p\tan\theta-r\sin\theta)h, & Wh\sin\phi-R(h-\eta), \\ h+p\tan\theta, & p\tan\theta-r\sin\theta, & W\sin\phi-T\cos\theta-R, \\ h\sin\theta+p'\cos\theta, & p'\cos\theta-r, & W\sin\chi-R\sin\theta, \end{vmatrix}=0.$$

This determinant which vanishes in equilibrium owing to the vanishing of the constituents of the third column, vanishes throughout the small motions which are impressed. To simplify it, we replace the first row, by (first row)$-h$. (second row) and since $p'=p\sec^2\theta$ the factor $(p'\cos\theta-r)$ divides out and the determinant reduces to

$$\begin{vmatrix} k^2 & 0 & \eta R-hT\cos\theta \\ h+p\tan\theta & \sin\theta & W\sin\phi+T\cos\theta-R \\ h\sin\theta+p'\cos\theta & 1 & W\sin\chi-R\sin\theta \end{vmatrix}=0,$$

or

$$T(k^2+h^2\cos^2\theta)=R(k^2+\eta h)\cos\theta-Wk^2(\sin\phi\sec\theta-\tan\theta\sin\chi).$$

This equation gives the tension in terms of the air resistance during motion. It will be noticed that it is (as it should be) independent of the length of the string.‡

To discuss small oscillations we take (1) and (2) and put

$$h=h_o-p'(\epsilon+\delta) \text{ from § 6.}$$
$$\phi=\phi_o+\epsilon$$
$$\eta=a+c\phi'(\alpha)\delta\alpha+\frac{c\phi_1(\alpha)}{V}\theta_1$$
$$\chi=\chi_o+\delta$$
$$R=R_o\delta R=R_o+vR_v+wR_w+\theta_1 R_1$$
$$\theta=\theta_o+\delta\theta=\theta_o+\epsilon+\delta.$$

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‡ It was for this relation that I assumed in my previous paper that $T\cos\theta$ or S_g is a function of the same variables as R. Any how this relation shows that the change dT in T owing to a change dR in R is of the first order.

The component velocities of E′ are v and $w+h\Theta_1$, they are also r and $r\dot{\chi}$; hence

$$v=-(r\dot{\chi}\cos\Theta+\dot{r}\sin\Theta)$$

$$w=r\dot{\chi}\sin\Theta-\dot{r}\cos\Theta-h_o\dot{\epsilon}.$$

If these values be substituted in (3) of § 2, it will reduce to (1).

Thus we have

$$\delta\alpha=\frac{(w+y\Theta_1)\cos\alpha-v\sin\alpha}{V}$$

$$=\frac{(r\sin\overline{\Theta+\alpha}-p'\sin\Theta\cos\overline{\Theta+\alpha})\dot{\delta}+\dot{\epsilon}(y-h_o\cos\alpha-p'\sin\Theta\cos\overline{\Theta+\alpha}),}{V}$$

$$\delta R=r\dot{\chi}\left\{(R_w\sin\Theta-R_v\cos\Theta)-\frac{p'\sin\Theta}{r}(R_w\cos\Theta+R_v\sin\Theta)\right\}$$

$$+\dot{\epsilon}\left\{(R_1-h\,R_w)-p'\sin\Theta(R_v\sin\Theta+R_w\cos\Theta)\right\}.$$

Hence, substituting, we have the equations of small oscillations

$$\text{(i)}\quad M(h_op\tan\Theta_o+k^2+h^2_o)\ddot{\epsilon}+\frac{Mh}{r}(p\tan\Theta-r\sin\Theta).r\ddot{\delta}$$

$$=r\dot{\delta}\left[(h_o-a)\left\{(R_w\sin\Theta-R_v\cos\Theta)-\frac{p'\sin\Theta}{r}(R_v\sin\Theta+R_w\cos\Theta)\right\}\right.$$

$$\left.-\frac{cR_0\phi'(\alpha)}{V}.\frac{r\sin\overline{\Theta+\alpha}-p'\sin\Theta\cos\overline{\Theta+\alpha}}{r}\right]$$

$$+\dot{\epsilon}\left[(h_o-a)\left\{(R_1-h_oR_w)-p'\sin\Theta(R_v\cos\Theta+R_w\sin\Theta)\right\}\right.$$

$$\left.-\frac{cR_o\phi'(\alpha)}{V}\left\{(y-h)\cos\alpha-p'\sin\Theta\cos\overline{\Theta+\alpha}\right\}-\frac{bR_o\phi_1(\alpha)}{V}\right]$$

$$+\epsilon\left[-p'R_o-W(h\cos\phi-p'\sin\phi)\right]+r\delta\left[\frac{p'}{r}(W\sin\phi-R_o)\right];$$

$$\text{(ii)}\quad M(h\sin\Theta+p'\cos\Theta)\ddot{\epsilon}+M\frac{p'\cos\Theta-r}{r}.r\ddot{\delta}$$

$$\dot{\epsilon}\left[\sin\Theta\left\{(R_w\sin\Theta-R_v\cos\Theta)-\frac{p'\sin\Theta}{r}(R_w\cos\Theta+R_v\sin\Theta)\right]\right\}$$

$$+\dot{\epsilon}\left[\sin\Theta\left\{(R_1-hR_w)-p'\sin\Theta(R_v\sin\Theta+R_w\cos\Theta)\right\}\right]$$

$$+\epsilon.R_o\cos\Theta+r\delta.\frac{R_o\cos\Theta-W\cos\chi}{r}$$

To solve this system we assume as usual ϵ, $r\delta$ proportional to $e^{\lambda t}$ and the determinantal equation in λ will be of the type

$$\begin{vmatrix} A\lambda^2+B\lambda+C, & A'\lambda^2+B'\lambda+C' \\ a\lambda^2+b\lambda+c & a'\lambda^2+b'\lambda+c' \end{vmatrix}=0;$$

expanding, we have a biquadratic

$$a_1\lambda^4+b_1\lambda^3+c_1\lambda^2+d_1\lambda+e_1=0.$$

The actual values of the co-efficients can be written down if desired, but the calculation is somewhat laborious; we have however

$$a_1=\frac{M^2}{r}(r-p'\cos\Theta)(k^2+h^2\cos\Theta),$$

$$d_1=(R_v\cos\Theta-R_w\sin\Theta)[W\sin\Theta(p'\cos\alpha-h\sin\alpha)-R_o\cos\Theta(h_o-a)]$$

$$-\frac{cR_o^2}{V}\phi'(\alpha)\cos\Theta\sin(\Theta+\alpha)+\text{terms in }\frac{1}{r},$$

$$e_1=\frac{W\,h\sin\alpha}{r}(W\cos\chi-R_o\cos\Theta)+\frac{W\,p'\cos\Theta}{r}(R_o-W\sin\phi).$$

The roots of this biquadratic give the position of the kite in space at the end of any time, when small variations are made in the value of the co-ordinates. If the conditions of stability given above are satisfied, then the "positional oscillations" will gradually die out. It follows, therefore, that in both cases the fundamental equation in λ is a biquadratic, one root of the former vanishes, when the shift of the centre of pressure is neglected, and one root of the latter vanishes when the string becomes infinitely long.

10. *Interpretation of the Conditions of Stability.* Since a is essentially positive the conditions of stability require that all the other coefficients as well as the discriminant H, where

$$H=b\,c\,d-a\,d^2-e\,b^2$$

should be positive.

If b be equal to zero, *i.e.* if the string be tied to a fixed point on the axis of symmetry which coincides with the centre of gravity, or if the perpendicular from the point of bifurcation to the axis of symmetry passes through the centre of gravity, then since $a=0$ in § 4, all the coefficients except the first two vanish, showing that in this case the stability is dependent on the shift of the centre of pressure.

If b be not zero then we have

$$c=\frac{T\,W^2}{g}.\,b\cos\Theta\,(\tan\Theta+\tan\beta)$$

Now if Θ is positive or numerically less than β, then $\tan\Theta+\tan\beta>0$ since Θ, β are each $<\frac{\pi}{2}$ and $\Theta+\beta<\pi$.

It follows that c will be positive if b be positive, *i.e.*, if the perpendicular from the point of bifurcation intersects the axis of symmetry at a point above the centre of gravity. This condition can be secured by making $HC < HD$.

We have next

$$d=\frac{W}{g}\left[b\,T\cos^2\theta\,(\tan\theta+\tan\beta)\,R_w+a\,R_oR_v\right].$$

Since $a\,R_o=h_0T\cos\theta=T\cos\theta\,b\,(1-\tan\beta\tan\theta)$,

$$d=\frac{W\,T}{g}\,b\cos\theta\left[R_w\tan\overline{\theta+\beta}+R_v\right]\left(1-\tan\beta\tan\theta\right);$$

again since $\beta<\frac{\pi}{2},\ \theta<\frac{\pi}{2}$

we have $1\tan\beta\tan\theta>0$, since $\beta+\theta<\frac{\pi}{2}$.

Hence $d>0$, if $[R_w\tan\overline{\theta+\beta}+R_v]$ has the same sign as b, *i.e.* positive;

i.e., if $R_w\left[\tan\overline{\theta+\beta}+\frac{R_v}{R_w}\right]>0.$

But by § 3, we have

$$\frac{R_v}{R_w}=\frac{2v_0f(\alpha)-f'(\alpha).w_0}{2w_0f(\alpha)+v_0f'(\alpha)}.$$

Now suppose the angle of attack to be so small that the component velocity of air normal to the plane of the kite is negligible; in this case we have

$$\frac{R_v}{R_w}=\frac{2f(\alpha)}{f'(\alpha)},$$

and if we also put $f(\alpha)=\sin\alpha$,

then $\tan\overline{\theta+\beta}+2\tan\alpha>0.$

The limiting relation at which stability ceases is given by

$$\tan\overline{\theta+\beta}+2\tan\alpha=0.$$

So that in the case of an infinitely long string with forked attachment the critical inclination depends on β.

Proceeding now to the case where the shift of centre of pressure is taken into account, we find that e will be positive if c is positive *i.e.*, if the centre of pressure moves forward as the angle of attack diminishes. This is the case in actual practice.

In the case of a finite string we find the conditions of longitudinal stability far more complicated, and approximations are necessary. If we assume the string to be very long (not necessarily infinite) so that the terms containing $\frac{1}{r}$ are negligible compared with other terms, we find

$$d_1=(R_w \sin\theta-R_v\cos\theta)[\sin\theta\{Wh_o\cos\phi_o-Wp'\sin\phi_o+p'R_0\}+R_o\cos\theta\,(h-a)]$$

and on using the equilibrium conditions

$$R_o-W\sin\phi_o=T\cos\theta,\ (h_o-a)\,R_o=Wh\sin\phi_o,$$

the above is ultimately reduced to

$$d_1=(R_w\sin\theta-R_v\cos\theta)[pT\tan\theta+Wh\cos\overline{\theta-\alpha}]$$

$$=(R_w\sin\theta-R_v\cos\theta)[Wh_o\cos\overline{\theta-\alpha}+pT\cdot\frac{h-a}{a}\tan\alpha].$$

Since

$$\left.\begin{aligned}T\sin\theta&=W\cos\phi\\ T\cos\theta&=\frac{a}{h_o-a}.W\sin\phi\end{aligned}\right\}\quad\ldots\quad\ldots\quad(1)$$

are conditions of equilibrium; remembering that $\theta=\chi-\alpha$, we have

$$\tan\overline{\psi-\alpha}>\frac{R_v}{R_w};$$

or, for stability, the inclination of the string to the vertical must be greater than $\tan^{-1}\frac{R_v}{R_w}$, or $\tan^{-1}(2\tan\alpha)$ if the angle of attack is small.

But in the case of plane kites with forked attachment as used above, equlibrium with a small angle of attack is not always possible; the least value which the angle of attack can have depends on the length HC.

Now in every position of equilibrium, we have

$$b>h>a$$

and $$\tan\theta=\frac{h_0-a}{a}\tan\alpha\ \text{ from (1)},$$

so that the condition $\tan\theta-2\tan\alpha>0$ becomes $h_0>3a$.

Hence the smallest value which the angle of attack can have is that in which the centre of pressure just coincides with the foot of the perpendicular from H to the axis of symmetry.

In any case we may conclude that with a long string *positional stability* may exist so long as the inclination of the string to the vertical is greater than $\left(\alpha+\tan^{-1}\frac{R_v}{R_w}\right)$.

For an interesting graphical discussion of this condition we refer the reader to *Stability in Aviation*, pp. 183, 184.

6. *General Expressions for the Resistance Derivatives.* It follows that the nature of the roots of the above biquadratic depends on the value of certain constants called resistance derivatives. The value of the coefficients depends entirely on the particular law of resistance assumed, but a general expression for these can be obtained without any such assumption.

Let u_o, v_o, w_o,...l_o, m_o, n_o, be the initial values of the velocity components and the direction of motion of air particles referred to any axes. Let x, y, z be the co-ordinates of an element ds of any surface, and let us assume

$$R_o=f(v_o, v_o,\ldots l_o, m_o,\ldots)$$

and when additional velocity components are impressed we have

$$R_o+\delta R=f(u_o+\delta u,\ldots\ldots l_o+\delta l, m_o+\delta m,\ldots)$$
$$=f(u_o\ldots l_o\ldots)+\frac{\partial f}{\partial u_o}\partial u_o+\frac{\partial f}{\partial v_o}\partial v_o+\frac{\partial f}{\partial w_o}\partial w_o+\ldots$$

Let $V^2=u^2{}_o+v_o{}^2+w_o{}^2$

and $\quad U^2=(Vl_o+u+z\Theta_2-y\Theta_3)^2+(Vm_o+v+\ldots)^2+(Vn_o+w+y\Theta_1\ldots)^2$

so that $\quad U=V+l_o(u+z\Theta_2-\ldots)+m_o(v+\ldots)+n_o(w+y\Theta_1-\ldots)$

approximately

also
$$l=l_o+\delta l=\frac{Vl_o+u+z\Theta_2-y\Theta_3}{U}$$
$$m=m_o+\delta m=\frac{Vm_o+v+x\Theta_3-z\Theta_1}{U}$$
$$n=n_o+\delta n=\frac{Vn_o+w+y\Theta_1-x\Theta_2}{U},$$

so that
$$\delta l=l-l_o=\frac{(m_o{}^2+n_o{}^2)(n-y\Theta_3+z\Theta_2)-l_om_o(v+\ldots)-l_on_o(w+\ldots)}{V}$$
$$\delta m=m-m_o=\frac{(l_o{}^2+n_o{}^2)(v+x\Theta_3-z\Theta_1)-\ldots\ldots\ldots-m_on_o(w+\ldots)}{V}$$
$$\delta n=n-n_o=\frac{(l_o{}^2+m_o{}^2)(w+y\Theta_1-\ldots)-m_on_o(v+\ldots)-l_on_o(n+\ldots)}{V}$$

we have also $\delta u=u+z\Theta_2-y\Theta_3$, $\delta v=v+x\Theta_3-z\Theta_1$ $\delta w=w+y\Theta_1-x\Theta_2$.

Substituting these values and arranging, we have

$$\delta R=uR_u+vR_v+wR_w+\Theta_1R_1+\Theta_2R_2+\Theta_3R_3$$
$$=u\left[fu_o+fl_o\frac{m_o{}^2+n_o{}^2}{V}-fm_o\frac{l_om_o}{V}-fn_o\frac{l_on_o}{V}\right]$$

$$+v\left[fv_o-fl_o\frac{l_om_o}{V}+fm_o\frac{l_o^2+n_o^2}{V}-fn_o\frac{m_on_o}{V}\right]$$

$$+w\left[fw_o+fn_o\frac{l_o^2+m_o^2}{V}-fl_o\frac{l_on_o}{V}-fm_o\frac{m_on_o}{V}\right]$$

$$+\Theta_1\left[yfw_o-zfv_o+fl_o\left(\frac{l_om_o}{V}z-\frac{l_on_o}{V}z\right)\right.$$
$$\left.-fm_o\left(\frac{l_o^2+n_o}{V}z^2+\frac{m_on_o}{V}y\right)+fn_o\left(\frac{l_o^2+m_o^2}{V}y+\frac{m_on_o}{V}z\right)\right]$$

$$+\Theta_2\left[zfu_o-xfw_o+fl_o\left(\frac{m_o^2+n_o^2}{V}z+\frac{l_on_o}{V}x\right)\right.$$
$$\left.+fm_o\left(\frac{m_on_o}{V}-\frac{l_om_o}{V}z\right)-fn_o\left(\frac{l_o^2+m_o^2}{V}x+\frac{l_on_o}{V}z\right)\right]$$

$$+\Theta_3\left[xfv_o-yfu_o-fl_o\left(\frac{m_o^2+n_o^2}{V}y+\frac{l_om_o}{V}x\right)\right.$$
$$\left.+fm_o\left(\frac{l_o^2+n_o^2}{V}x+\frac{l_om_o}{V}y\right)-fn_o\left(\frac{m_on_o}{V}x-\frac{l_on_o}{V}y\right)\right].$$

where $fu_o, \ldots fl_o, \ldots$, stand for $\frac{\partial f}{\partial u_o}, \ldots \frac{\partial f}{\partial l_o}, \ldots$

The values of R_u, R_v, R_w can now be obtained by multiplying the coefficients of u, v, ... by an element ds of a plane and integrating over the whole plane. If however the breadth of a plane be small compared with the distance from the origin, then the resultant velocity of every element ds may be assumed to be the same, in which case the above expressions will slightly simplify.

For plane kites with axes chosen as above, the terms containing z will vanish; the terms containing x *will also disappear if the y-axis is an axis of symmetry.* Further we have

$$l_o,\ l,\ m,=0,\ m_o=\sin\alpha,\ n_o=\cos\alpha,\ n=1;$$

when these values are substituted we find that the coefficients of u, Θ_2, Θ_3 vanish.

In any other case the value of these coefficients can be simplified by a proper choice of axes and the law of resistance. For instance, if we assume that the thrust on an element of a plane is given by

$$dR=KV^2.\sin\alpha.\ ds,$$

we have

$$f(u_o,\ v_o \ldots l_o,\ m_o, \ldots)=ds\ (u^2_o+v^2_o+w^2_o)(ll_o+min_o+nn_o);$$

if we also assume the air to blow along the axis of x with a velocity U then we may put after differentiation

$$u_o=U;\ v_o,\ w_o,\ m,_o\ n_o=0;\ l_o=1;$$

we thus find $\int u_o = 2\text{K}\text{U}l ds, \frac{fl_o}{\text{V}} = \text{K}\text{U}l ds \frac{fm_o}{\text{V}} = \text{K}\text{U}m ds$, etc.,

and $\text{R}_u = 2\int \text{K}l\text{U}ds$, $\text{R}_v = \int \text{K}m\text{U}ds$ $\text{R}_w = \int \text{K}n\text{U}ds$

$$\text{R}_1 = \int \text{KU}(ny - mzds), \text{ etc.}$$

i.e., they reduce to the forms used by Prof. Bryan in *Stability in Aviation*, p. 124.

In the above case we have assumed that the thrust on an element depends on the velocity of that element only, and is independent of the motion of other elements surrounding it. Again a current of air impinging on a rotating lamina cannot be expected to behave in the same way as if the motion of the lamina were of uniform translation. Owing to the difference of velocities between the different parts of the lamina, the total thrust cannot be expected to be the same, and a consideration of these facts led Prof. Byran to assume that the thrust on the lamina is a function, not only of the linear velocities and the directions of motion, but also of the angular velocities. If we introduce Θ_1/V, Θ_2/V, Θ_3/V in the above functional form for R, the effect will be to introduce additional terms $\frac{f_1}{\text{V}}, \frac{f_2}{\text{V}}, \frac{f_3}{\text{V}}$ in the coefficients of $\Theta_1, \Theta_2, \Theta_3$. They are termed "*rotary derivatives*" by Prof. Byran.||

For plane kites let us assume

$$\text{R} = \text{KSV}^2 f(\alpha);$$

we assume that for motion in the $y - z$ plane, $f(\alpha)$ is a function not only of α but also of $\frac{\Theta_1}{\text{V}}$.

With this assumption we may immediately deduce the expressions for R_v, R_w, R_1 from the above on putting

$$f(u_o \ldots l_o \ldots n_o) = \text{K}ds(u_o^2 + v_o^2 + w_o^2) f\left(\frac{\pi}{2} - \cos^{-1}(ll_o + mm_o + nn_o)\right)$$

where, after differentiation, we may put

$$n_o, l_o = 0;\ v_o = \text{V} \sin \alpha,\ w_o = \text{V} \cos \alpha$$

$$m_o = \sin \alpha,\ n_o = \cos \alpha;$$

also $z = 0$, and if the y-axis is an axis of symmetry then we may put $x = 0$.

[In a recent issue of the *Bulletin* of the Calcutta Mathematical Society (vol. v) there is a note by Mr. B. N. Rau, questioning the

|| See *Stability in Aviation* §§ 24-29.

validity of certain assumptions made in my first paper on this subject For instance one of the equations was

$$M\dot{u} = W\,\theta + S_x$$

where $$S_x = S_x\,(u,\ \theta_2,\ \theta_3).$$

We need not discuss any further why those particular variables, u, θ_2 θ were taken in the x—component of the tension. But Mr. Rau gives (what he considers to be) an example, to prove that if after the elimination of other unknown quantities, the tension is expressed as a function of the velocities, such a function is not necessarily expansible in a series whose terms diminish. His example is as follows:

For small oscillation

$$u = a \sin \omega\, t, \text{ where } a \text{ is small};$$

and if the equation of motion is

$$M\dot{u} = S_x,$$

we get $$S_x = Ma\omega\left(1 - \frac{u^2}{a^2}\right)^{\frac{1}{2}};$$

and since u is of the same order as a the successive terms in the expansion of $a\left(1-\frac{u^2}{a^2}\right)^{\frac{1}{2}}$ are not negligible.

This is of course true. But the fallacy lies in the fact, that the expression $Ma\,\omega\left(1-\frac{u^2}{a^2}\right)^{\frac{1}{2}}$ is deduced from $a \sin \omega t$, which cannot again represent u unless the successive terms in the expansion of the "force function" are negligible. This is a necessary condition for small oscillation. As an example we may take the approximate equation of the pendulum

$$\ddot{\theta} + a^2\theta = 0, \text{ where } a^2 = \frac{g}{l},$$

which leads to

$$\dot{\theta} = \ a\left(1 - \frac{\theta^2}{a^2}\right)^{\frac{1}{2}};$$

and the right hand side cannot be expanded as stated above. But it in easy to show that it is the first term of the expansion of the integral of the exact equation

$$\ddot{\theta} + a^2 \sin\theta = 0.$$

In this paper as well as in my previous papers, I have assumed

$$R = R(v,\ w,\ \theta_1);$$

but Mr. Rau maintains that the "angle of attack" should have been included among the variables on the right hand side. But he has probably overlooked the relation

$$\delta\alpha = \frac{(w + y\Theta_1)\cos\alpha - v\sin\alpha}{V}$$

which shows that the angle of attack is not an independent variable and therefore cannot occur explicitly in the function. (See § 8 of this paper ; and §§ 16, 18, 23 and also the last four lines of p. 172 of Prof. Bryan's *Memoir*)].

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SHORT NOTES.

The Roots of a Derivative of a Rational Function.

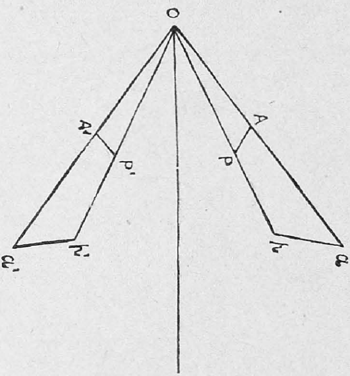
1. MR. L. R. FORD, M.A., contributes a *Note* on the above subject to the Edinburgh Mathematical Society [Vol. XXXIII, 1915, Part 2, p. 103]. The theorem 'that in the complex plane of the variable the smallest convex rectilinear polygon surrounding the roots of a polynomial surrounds also the roots of its first derivative'*, with allied theorems and extensions follows from the properties of the *Harmonic Centre* given by me in this Journal [Vol. IV, 1912, p. 96]. The extension of the above theorem to the more general case of a rational function with a *pole*, discussed by Mr. Ford in the above *Note*, is—

'"*If $f(z)$ is a rational function of z whose only pole is at the point a, the smallest circular polygon surrounding the roots of $f(z)$—the sides of the polygon passing through a, and the polygon lying entirely without or entirely within each of its bounding circles—surrounds also the roots of $f'(z)$, with the possible exceptions of two roots at infinity.*"

The method of transformation employed by Mr. Ford virtually amounts to saying that, if $zz'=1=\alpha\alpha'$, then

$$\frac{1}{(z-\alpha)}\equiv\frac{1}{\left(\frac{1}{z'}-\frac{1}{\alpha'}\right)}\equiv\frac{\alpha'z'}{\alpha'-z'}\equiv z'^2\left(\frac{1}{z'}-\frac{1}{z'-\alpha'}\right),$$

which according to *Vector Algebra* signifies that



$$\frac{1}{\overline{\mathrm{AP}}}\equiv\overline{\mathrm{O}p'}^2\left\{\frac{1}{\overline{\mathrm{O}p'}}-\frac{1}{\overline{a'p'}}\right\} \qquad \ldots \qquad \ldots \ (1)$$

* Osgood: *Lehrbuch der* Funktionen theorie, Vol. I, 1912, p 211;
Hayashi, in the *Annals of Mathematics*, Vol. 15, 1914, p 112;
Irwin, in the Annals of Mathematics, Vol. 16, 1915, p. 138.

where the points p, a are inverses of P, A with respect to the unit circle and the accented letters denote the reflections of the corresponding unaccented letters in the initial line.

Hence, we deduce the general result

$$\sum\left(\frac{1}{z-\alpha_r}\right)=z'^2\left\{\sum\left(\frac{1}{z'}\right)-\sum\left(\frac{1}{z'-\alpha'_r}\right)\right\} \quad \dots (2)$$

for values of r form 1 to n.

Now, let $\phi(z)\equiv f(z)/z^n=(z-\alpha_1)(z-\alpha_2)\dots(z-\alpha_n)/z^n=F(z')$; so that $f(z)=F(z').z^n$. Then since $\phi(z)=F(z')$,

$$\phi'(z)=F'(z').\frac{dz'}{dz}=F'(z')\cdot(-z'^2) \quad \dots \quad \dots (3)$$

$$\therefore \quad \frac{\phi'(z)}{\phi(z)}=-z'^2\frac{F'(z')}{F(z')}=\frac{f'(z)}{f(z)}-\frac{n}{z}, \text{ from (2)}$$

$$=\sum\left(\frac{1}{\overline{A_rP}}\right)-\frac{n}{\overline{OP}}. \quad \dots \quad \dots (4)$$

In other words, the roots of $\phi'(z)$ must satisfy the condition that *the resultant of forces inversely proportional to A_rP is n times the inverse of OP.*

Further, the roots of $\phi'(z)$ correspond to those of $F'(z')$ by (3), and the latter lie within the rectilinear polygon determined by the polynomial $F(z')$. Also, the inverse relation of z and z' shows that the equivalent polygonal boundry for z must be formed by arcs of circles passing through the origin. Hence, the result stated by Mr. Ford.

2. More generally, putting

$$\phi(z)=f(z)/(z-\alpha)^m$$

so that the pole of $\phi(z)$ is α and its roots are the same as those of $f(z)$, we find

$$\frac{\phi'(z)}{\phi(z)}=\frac{f'(z)}{f(z)}-\frac{m}{z-\alpha}$$

$$=\sum\left(\frac{1}{\overline{A_rP}}\right)-\frac{m}{\overline{AP}},$$

where A is the pole.

Hence, in this case, the roots of the derivative $\phi'(z)$ must satisfy the relation

$$\sum\left(\frac{1}{\overline{A_rP}}\right)=\frac{m}{\overline{AP}};$$

that is *the resultant of forces inversely proportional to (A_rP) must be m times the inverse of AP.*

M. T. NARANIENGAR.

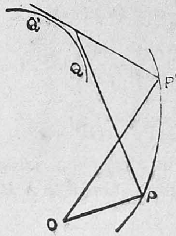
Geometrical representation of a Definite Integral.

A definite integral is usually represented as an *area* in Text-books on the Calculus. It may, however, with equal facility, be expressed as an *arc* as the following method shows :

Consider the integral

$$I=\int_{\alpha}^{\beta} f(\Theta)\, d\Theta \quad \dots \quad \dots \quad \dots \quad (1)$$

in relation to the curve $r=f(\Theta)$, which is the locus of P.



Suppose the *first negative* pedal is the locus of Q. Then approximately

$$r\delta\Theta=PQ+QQ'-P'Q'=\delta s'-\delta t', \quad \dots \quad \dots \quad (2)$$

denoting PQ by t' and QQ' by $\delta s'$.

Integrating (2), we have

$$\int r d\Theta=[s'-t']$$

between proper limits.

In other words, the integral I is represented by an arc of the first negative pedal of $r=f(\Theta)$, and its bounding tangents.

Cor. If the curve $r=f(\Theta)$ is a closed curve C, the contour integral

$$\int_C r d\Theta=C'$$

where C′ is the whole arc of the first negative pedal of C.

M. T. NARANIENGAR.

Note on Question 737.

The importance of the result depends on the extension made which itself presents no difficulty; the definition of the set B_k is in no way dependent on the characters of C, and the quotient $A_k/2k$ can be found for any function that is not negative, for any set of points C, and for any positive value of k. If then $A_k/2k$ tends to a definite limit, this limit may be used to define the linear interval of the function f for the set of points C, and the linear integral of a function g which is sometimes positive and sometimes negative may be defined as Q−R, where Q is the linear integral of the function which is equal to g when g is positive and is elsewhere zero, and R is the linear integral of the function which is equal to $-g$ when g is negative and is elsewhere zero. The convergence of $A_k/2k$ in general deserves investigation, and since the method may obviously be extended to the definition of line integrals and of surface integrals in space, a research is suggested which may be difficult. It is to be noticed that the regions and numbers obtained depend only on the form of the set found by completing C (adding to C all of its limiting points which it does not include), and that therefore the definitions may be valuable only in the case of complete sets.

The theorem given affords a simple exercise in the kinematical treatment of differential geometry. As a current point O describes a curve C, the circle with centre O and radius kf describes a band of variable width, and the edges of this band are traced by two points Q, R in which the circle touches the boundaries. In the circumstances described, if k is sufficiently small the region B_k consists of the regions swept by the two lines OQ, OR as O describes the curve, together with sectors corresponding to the end-points if the curve is not closed.

At the current point O, let OT be the tangent, OE the normal making a positive right angle with OT, and κ the curvature; let the current circle meet its envelope in a point P such that the angle EOP is Θ; then the co-ordinates of P with respect to OT and OE are $-kf\sin\Theta$, $kf\cos\Theta$, and therefore the velocity of P with respect to the arc of C has components

$$1-kf'\sin\Theta-kf(\kappa+\Theta')\cos\Theta,\ kf'\cos\Theta-kf(\kappa+\Theta')\sin\Theta.$$

But because the circle touches its envelope, the velocity of P is at right angles to OP, and therefore

$$\sin\Theta=kf',$$

and the velocity has components

$$\{\cos\Theta-kf(\kappa+\Theta')\}\cos\Theta,\ \{\cos\Theta-kf(\kappa+\Theta)\}\sin\Theta;$$

thus if α is the angle between $-\frac{1}{2}\pi$ and $\frac{1}{2}\pi$ whose sine is kf', we may take Θ as α for Q and as $\pi-\alpha$ for R.

We note in passing that if kf' is numerically greater than unity, the circles have no envelope; indeed, the circle with centre O then contains the neighbouring circles with centres on one side of O and is contained in the neighbouring circles with centres on the other side of O. The case in which kf' is numerically equal to unity is interesting as being the case of circles of curvature of a plane curve; as de la Vallée Poussin pointed out although these circles have the curve from which they are derived for a genuine envelope it is entirely false to say that consecutive circles of curvature cut on this envelope. This is a digression, for in the problem with which we are dealing these difficulties are left behind when k has become sufficiently small.

If two points F, G are moving with respect to a parameter t, and if for every value of t their velocities u, v normal to the line through them are positive in the same direction, then if the chord FG is of length l, the rate with respect to t at which this chord sweeps out area is $\frac{1}{2}\ l(u+v)$; this result may be applied in two ways to the present problem. Since Q is moving at right angles to OQ at the rate

$$\cos \alpha - kf\ (\kappa+\alpha'),$$

which is positive if k is sufficiently small (α depends on k, but both α and α' tend to zero with k), and the velocity of O inthe same direction is $\cos \alpha$, the area swept by OQ is

$$\tfrac{1}{2}\int kf\ \{2 \cos \alpha - kf(\kappa+\alpha')\}\, ds;$$

similarly the area swept by OR is

$$\tfrac{1}{2}\int kf\ \{2 \cos \alpha + kf\ (\kappa-\alpha')\}\, ds,$$

and if the curve is closed the total area A_k is

$$\int kf\ (2 \cos \alpha - kf\alpha')\, ds.$$

If the curve has a beginning and an end, the total area is the sum of this integral and of the area $k^2f^2\ (\frac{1}{2}\pi-\alpha)$ of a sector at the beginning and of the area $k^2f^2\ (\frac{1}{2}\pi+\alpha)$ of a sector at the end—the values of f and α being of course those which correspond to these points. Alternatively with respect to the arc of C the velocities of Q and R at right angles to QR are

$$\{\cos \alpha - kf(\kappa+\alpha')\} \cos \alpha,\ \{\cos \alpha + kf\ (\kappa-\alpha')\} \cos \alpha,$$

which are positive if k is sufficiently small, and since the length of QR is $2kf \cos \alpha$ the value of A_k is

$$\int 2k\, f\ (\cos \alpha - k\, f\alpha')\, \cos^2\alpha\ ds;$$

if C is closed and is the sum of this integral and the areas of segments corresponding to the end-points if C is a curve from one point to a distinct point. The two integrals obtained differ by

$$\int k f (\sin \alpha \sin 2\alpha + k f \alpha' \cos 2\alpha)\, ds,$$

but from the relation of α to f this integral is

$$\tfrac{1}{2}\int \{ d(k^2 f^2 \sin 2\alpha)/ds \}\, ds,$$

which is immediately seen to vanish if C is closed and to represent the difference between the sectors and the segments at the end-points in the more general case. It is interesting to remark that for all sufficiently small values of k the value of A_k is independent of the curvature of the curve the form of the curve affects only the limit below which k must lie for the result to be true.

ERIC H. NEVILLE.

The Face of the Sky for September and October 1916.

The Sun

enters Lebra on September 23 at 3 P. M. and Scorpio on October 23 at 11 P. M..

Phases of the Moon.

	September.			October.		
	D.	H.	M.	D.	H.	M.
First Quarter ...	5	9	57 A. M.	4	4	31 P. M.
Full Moon ...	12	2	1 A. M.	11	12	31 A. M.
Last Quarter ...	19	11	5 A. M.	19	6	39 A. M.
New Moon ...	27	1	4 P. M.	27	2	7 A. M.

The Planets.

Mercury attains its greatest elongation (26° 54′ E) on September 9, is stationary on September 23, is in inferior conjunction on October 5, is stationary on October 14 and attains its greatest elongation (18° 17′ W) on October 21. It is in conjunction with the Moon September 28 and on October 25.

Venus attains its greatest elongation (46° 1′ W) on September 12. It is in conjunction with the Moon on September 23 and on October 23, with Saturn on September 5, with Neptune on September 13 and with ρ Leairs on October 12.

Mars is in conjunction with the Moon on September 2 and September 30 and on October 29.

Jupiter is in opposition to the Sun on October 24. It is in conjunction with the Moon on September 15 and on October 12.

Saturn is in conjunction with the Moon on September 22 at 4-45 A.M. and on October 19.

Uranus is stationary on October 26. It is in conjunction with the Moon on September 9 at 10-30 P. M. and on October 7.

Neptune is in quadrature to the Sun on October 28 and is in conjunction with the Moon on September 22 and on October 20 at 2-16 A.M.

V. RAMESAM.

SOLUTIONS.

Question 495.

(A. C. L. Wilkinson) :—Prove that

$$\begin{vmatrix} 1, & \cos c, & \cos b, & \cos (b-c) \\ \cos c, & 1, & \cos a, & \cos (c-a) \\ \cos b, & \cos a, & 1, & \cos (a-b) \\ \cos (b-c), & \cos (c-a), & \cos (a-b), & 1 \end{vmatrix} = -16\sigma^2$$

where $\sigma = \sin (s-a) \sin (s-b) \sin (s-c)$.

Solution by N. Sankara Aiyar, M.A.

Let l, m, n denote $\cos (s-a)$, $\cos (s-b)$, $\cos (s-c)$ and p, q, r $\sin (s-a)$, $\sin (s-b)$, $\sin (s-c)$, so that $l^2=1-p^2$, $m^2=1-q^2$, $n^2=1-r^2$;

and $\cos a = \cos (s-b+s-c) = mn-qr$,

$\cos (b-c) = \cos (s-c-s+b) = mn+qr$.

The given determinant

$$\begin{aligned}
&=1-\Sigma\cos^2 a-\Sigma\cos^2(b-c)+2\Sigma\cos a \cos (a-b) \cos (c-a) \\
&\quad +\Sigma\cos^2 a\cos^2(b-c)+2 \cos a \cos b \cos c \\
&\qquad -2\Sigma\cos a \cos b \cos(c-a) \cos (b-c) \\
&=1-\Sigma(lm-pq)^2-\Sigma(lm+pq)^2+2\Sigma(lm-pq)(mn+qr)(ln+pr) \\
&\quad +\Sigma(l^2m^2-p^2q^2)^2+2\Pi(lm-pq)-2\Sigma(l^2m^2-p^2q^2)(m^2n^2-q^2r^2) \\
&=1-2\Sigma(l^2m^2+p^2q^2)+\Sigma(l^2m^2-p^2q^2)^2 \\
&\qquad -2\Sigma(l^2m^2-p^2q^2)(m^2n^2-q^2r^2)+8(l^2m^2n^2-p^2q^2r^2) \\
&=1-2\Sigma(1-p^2-q^2+2p^2q^2)+\Sigma(1-p^2-q^2)^2- \\
&\qquad 2\Sigma(1-p^2-q^2)(1-q^2-r^2)+8(1-\Sigma p^2+\Sigma p^2q^2-2p^2q^2r^2) \\
&=1-6+4\Sigma p^2-4\Sigma p^2q^2+3-4\Sigma p^2+\Sigma(p^2+q^2)^2-6 \\
&\qquad +8\Sigma p^2-2\Sigma(p^2+q^2)(q^2+r^2)+8-8\Sigma p^2+8\Sigma p^2q^2-16p^2q^2r^2 \\
&=\Sigma(p^2+q^2)^2-2\Sigma(p^2+q^2)(q^2+r^2)+4\Sigma p^2q^2-16p^2q^2r^2 \\
&=2\Sigma p^4+2\Sigma p^2q^2-2\Sigma p^4-6\Sigma p^2q^2+4\Sigma p^2q^2-16p^2q^2r^2 \\
&=-16p^2q^2r^2 \\
&=-16\sigma^2.
\end{aligned}$$

Question 668.

(J. C. Swaminarayan, M.A.) :—If r and n are integers, prove that the expression

$$1-\frac{2n+1}{1}.{}_rC_1+\frac{(2n+1)(2n+3)}{1.3}{}_rC_2-\frac{(2n+1)\,(2n+3)\,(2n+5)}{1.3.5}.{}_rC_3$$

$$\frac{\ldots\ldots(-1)^r(2n+1)(2n+3)\ldots(2u+2n-1)}{1\cdot3\cdot5\ldots(2r-1)}{}_rC_r$$

is equal to

$$\frac{(-2)^r.{}_nP_r}{1\cdot3\cdot5\ldots(2r-1)}$$

as long as r is not greater than n, but vanishes if $r>n$.

Solution by K. R. Rama Aiyar.

We have

$$(1+x)^{-(n+\frac{1}{2})}=1-\frac{(2n+1)x}{2}+\frac{(2n+1)(2n+3)x^2}{2\cdot 4}-\ldots$$

$$+(-)^r\frac{(2n+1)(2n+3\ldots\ldots(2n+2r-1)}{2\cdot 4\cdot 6\ldots\ldots 2r}x^r+\ldots,$$

and

$$(1+x)^{r-\frac{1}{2}}=1+\frac{(2r-1)}{2}x+\frac{(2r-1)(2r-3)x^2}{2\cdot 4}+\ldots$$

$$+\frac{(2r-1)(2r-3)\ldots\ldots 3\cdot 1}{2\cdot 4\cdot 6\ldots\ldots 2r}x^r+\ldots$$

Multiplying the two series we find the coefficient of x^r in the expansion of $(1+x)^{r-n-1}$ is equal to

$$\frac{(2r-1)(2r-3)\ldots 3\cdot 1}{2\cdot 4\cdot 6\ldots\ldots 2r}-\frac{2n+1}{2}\cdot\frac{(2r-1)(2r-3)\ldots 3}{2\cdot 4\cdot 6\ldots 2r-2}+\ldots\ldots$$

$$+\ldots\ldots+(-)^r\frac{(2n+1)(2n+3)\ldots(2n+2r-1)}{2\cdot 4\cdot 6\ldots\ldots 2r}.$$

$$=\frac{(2r-1)(2r-3)\ldots 3\cdot 1}{2\cdot 4\cdot 6\ldots 2r}\left\{1-\frac{2n+1}{2\cdot 1}2r+\frac{(2n+1)(2n+3)}{1\cdot 3\cdot 2\cdot 4}.2r(2r-2)+\ldots\right.$$

$$\left.+\ldots+(-)^r\frac{(2n+1)(2n+3)\ldots\ldots(2n+2r-1)}{1\cdot 3\ldots\ldots(2r-1)\,2\cdot 4\ldots 2r}.2r(2r-2)\ldots 4\cdot 2\right\}$$

$$=\frac{(2r-1)(2r-3)\ldots\ldots 1}{2^r.r\,!}\left\{1-\frac{2n+1}{1}.{}^rC_1+\frac{(2n+1)(2n+3)}{1\cdot 3}{}^rC_2-\ldots+\right.$$

$$\left.+\ldots+(-)^r\frac{(2n+1)(2n+3)\ldots\ldots(2n+2r-1)}{1\cdot 3\cdot 5\ldots\ldots(2r-1)}{}^rC_r\right\}$$

$$=\frac{1\cdot 3\cdot 5\ldots\ldots(2r-1)}{2^r.r\,!}.\quad \text{S, where S denotes the series given.}$$

But we know that the coefficient of x^r in the expansion of $(1+x)^{r-n-1}$ is

$$(-)^r.\frac{(n-r+1)(n-r+2)\ldots n}{r\,!}$$

if $r\leq n$, and zero if $r>n$.

Hence the result stated.

Question 669.

(J. C. SWAMINARAYAN, M. A.) :—Prove that when n is a positive integer,

$$(b^2-a^2)^n+(2n+1)\frac{2n}{2\,!}(b^2-a^2)^{n-1}a^2$$
$$+\frac{(2n+1)(2n+3)(2n)(2n-2)}{4\,!}(b^2-a^2)^{n-2}a^4+\dots$$
$$=b^{2n}+\frac{(2n)^2}{2\,!}b^{2n-2}a^2+\frac{(2n)^2(2n-2)^2}{4\,!}b^{2n-4}a^4+\dots$$

Solution by R. D. Karve and K. R. Rama Aiyar.

The coeff. of $b^{2n-2r}a^{2r}$ on the left side

$$=(-1)^r[{}_nC_r-(2n+1)\frac{2n}{2\,!}\cdot{}_{n-1}C_{r-1}$$
$$+(2n+1)(2n+3)\frac{2n(2n-2)}{4\,!}\cdot{}_{n-2}C_{r-2}-\dots]$$

$$=(-1)^r nC_r[1-(2n+1)\frac{2n}{2\,!}\cdot\frac{n-1\,!}{n\,!}-rC_1$$
$$+(2n+1)(2n+3)\frac{2n(2n-2)}{4\,!}.rC_2.\frac{n-2\,!}{n\,!}-\dots]$$

$$=(-1)^r{}_nC_r[1-\frac{2n+1}{1}{}_rC_1+\frac{(2n+1)(2n+3)}{1\cdot3}{}_rC_2-\dots]$$

$$=\frac{2^r\cdot{}_nC_r\cdot{}_nP_r}{1\cdot3\cdot5\dots(2r-1)}=\frac{2^{2r}.({}_nP_r)^2}{2^r.r\,!\,.\,1.3.5\dots(2r-1)}$$

$$=\frac{(2n)^2(2n-2)^2\dots(2n-2r+2)^2}{(2r)\,!}.$$

Hence the result.

Question 670.

(K. J. SANJANA, M. A.) :— Prove that

$$\frac{\frac{1}{1^2}-\frac{1}{3^2}+\frac{1}{5^2}-\frac{1}{7^2}+\dots}{1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\dots}=1+\frac{1}{3\,!}\pi^2\frac{B_1}{2}+\frac{7}{5\,!}\pi^4\frac{B_3}{2^3}+\frac{3_1}{7\,!}\pi^6\frac{B_5}{2^5}+\dots$$

where B_1, B_3...are the numbers of Bernoulli.

Solution by (1) *K. B. Madhava and* (2) *K. R. Rama Aiyar.*

(1) We have

$$\text{cosec } z=\frac{1}{z}+\frac{2(2-1)}{2\,!}B_1z+\frac{2(2^3-1)}{4\,!}B_3z^3+\dots$$
$$+\frac{2(2^{2n-1}-1)}{(2n)\,!}B_{2n-1}z^{2n-1}+\dots\text{(Hobson, p. 340)}$$

$$\therefore \int z \operatorname{cosec} z\, dz = z\left\{1+\frac{2(2-1)}{3!}B_1 z^2+\frac{2(2^3-1)}{5!}B_3 z^4+\ldots\right.$$
$$\left.+\frac{2(2^{2n-1}-1)}{(2n+1)!}B_{2n-1}z^{2n}+\ldots\right\}$$

$$\therefore \int_0^{\frac{1}{2}\pi} z \operatorname{cosec} z\, dz = \frac{\pi}{2}\left\{1+\frac{2(2-1)}{3!}B_1\left(\frac{\pi}{2}\right)^2+\frac{2(2^3-1)}{5!}B_3\left(\frac{\pi}{2}\right)^4\right.$$
$$\left.+\ldots+\frac{2(2^{2n-1}-1)}{(2n+1)!}B_{2n-1}\left(\frac{\pi}{2}\right)^{2n}+\ldots\right\}$$

$=\frac{\pi}{2}$ times the expression on the right hand side,

and since we know that the denominator on the left hand side is equal to $\frac{\pi}{4}$, the problem reduces to showing

$$\int_0^{\frac{1}{2}\pi} z \operatorname{cosec} z\, dz = 2\sum_0^{\infty}\frac{(-)^n}{(2n+1)^2}.$$

This is a theorem established in Bromwich, p. 289, by applying Borel's rule for uniform summability of non-convergent and asymptotic series.

In fact it is easily with the definition of the term 'uniform summability' that

$$\sin z+\sin 3z+\sin 5z+\ldots=\tfrac{1}{2}\operatorname{cosec} z$$

is uniformly summable in an interval $\left(\delta, \frac{\pi}{2}\right)$ where $0<\delta<\frac{\pi}{2}$.

and that $\int_\delta^{\frac{1}{2}\pi} z \operatorname{cosec} z\ dz = 2\int_\delta^{\frac{1}{2}\pi} \overset{\infty}{\underset{0}{S}}\, z \sin(2n+1)z\, dz.$

$$= 2\ \overset{\infty}{\underset{0}{S}} \int_\delta^{\frac{\pi}{2}} z\sin(2n+1)z\, dz.$$

But $\int_\delta^{\frac{1}{2}\pi} z\sin(2n+1)\,z\,dz = \frac{\delta\cos(2n+1)\delta}{2n+1}-\frac{\sin(2n+1)\delta}{(2n+1)^2}+\frac{(-)^n}{(2n+1)^2}.$

Again $\sum\frac{\cos(2n+1)\delta}{2n+1}=\frac{1}{2}\log(\cot\frac{1}{2}\delta.)\ \ 0<\delta<\pi.$

Also since $\left|\frac{\sin(2n+1)\delta}{(2n+1)^2}\right| \leq \frac{1}{(2n+1)^2}$ (a quantity independent of δ)

$$\sum_0^{\infty}\left|\frac{\sin(2n+1)\delta}{(2n+1)^2}\right| \leq \sum_0^{\infty}\frac{1}{(2n+1)^2}\leq\frac{\pi^2}{8}\leq 1{\cdot}23,$$

the series $\sum_{0}^{\infty} \frac{\sin(2n+1)\delta}{(2n+1)^2}$ is uniformly convergent by Weierstrars's M—Test.

Again

$$\lim_{\delta \to 0} \sum_{0}^{\infty} \left[\frac{\delta \cos(2n+1)\delta}{2n+1} - \frac{\sin(2n+1)\delta}{(2n+1)^2} \right] = 0.$$

Hence $\int_0^{\frac{\pi}{2}} z \operatorname{cosec} z dz = 2 \sum_{0}^{\infty} \frac{(-)^n}{(2n+1)^2}$.

Hence the result as stated.

(2) Left hand member $= \frac{4}{\pi} \int_0^1 \left\{ x - \frac{x^3}{3} + \frac{x^5}{5} - \dots \right\} dx$

$$= \frac{4}{\pi} \int_0^1 \frac{\tan^{-1} x}{x} dx = \frac{2}{\pi} \int_0^1 \frac{\sin^{-1} x}{x\sqrt{1-x^2}} dx$$

[J.I.M.S , Vol. VII, p. 140]

$$= \frac{2}{\pi} \int_0^{\frac{1}{2}\pi} z \operatorname{cosec} z \, dz. \text{ [put } z = \sin^{-1} x].$$

The rest as before.

Question 673.

(S. Krishnaswami Aiyengar) :—If the sides of a polygon of n sides subtend equal angles at a point S, then with S as focus two conics can be described, one circumscribing the polygon and the other inscribed in it. Show that the envelope of polars with respect to the inscribed conic of points on the circumconic is a conic having one focus and one directrix coincident with those of the original conics.

Solution by K. R. Rama Aiyar.

We know that a circle can be circumscribed about and another inscribed in a regular polygon of n sides ; these two circles are concentric and the locus of the poles with respect to the circum-circle of tangents to the incircle is another concentic circle. Now reciprocate with respect to a circle with S as centre. Then the polygon reciprocates into another polygon of n sides each of which subtends the same angle at S. Also the circum-circle and the incircle reciprocate into the inscribed conic and the circumconic, both having S for one of their foci and having a common directrix, since the circles are concentric; and the envelope of polars with respect to the inscribed conic of points on the circum-conic is a conic having S for one of its foci and one directrix the same as that of the two other conics.

Question 678.

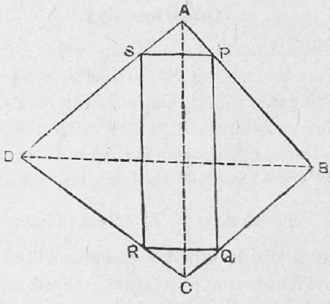
(D. KRISHNA MURTI) :—Show that the centroids of parallelograms inscribed in a skew quadrilateral, so as to have their sides parallel to the diagonals of the quadrilateral, all lie in a straight line which is also the locus of the centres of the conicoids having the sides of the quadrilateral for generators.

Solution (1) *by K. B. Madhava*, (2) *by A. Narasinga Rao.*

(1) By taking the lines joining the mid-points of the sides and of the diagonals as axes, we can obtain the equation to any conicoid as the one passing through the planes ABC and ADC and through ADB and BCD and thus obtain the general equation in the form

$$\left(\frac{x}{a}+\frac{y}{b}\right)^2-\left(\frac{z}{c}-1\right)^2=\lambda\left\{\left(\frac{x}{a}-\frac{y}{b}\right)^2-\left(\frac{z}{c}+1\right)^2\right\}$$

where λ is a variable parameter and by the usual methods available for the purpose the equation to the locus of the centres can be got to be the z-axis. The above form is easily written down by obtaining the equations of the planes from the intercepts they make on the axes.



(2) Let ABCD be the skew quadrilateral and PQRS an inscribed parallelogram.

Consider the centroids of 4 unit masses placed at P, Q, R and S. It is obviously the centroid of PQRS.

The mass at P can be replaced by two masses $\frac{BP}{AB}, \frac{AP}{AB}$. respectively at A and B. Replacing Q, R, S, in the same manner, we have four masses, at A,B, C,D of magnitudes

$$\left(\frac{BP}{AB}+\frac{DS}{AD}\right), \left(\frac{AP}{AB}+\frac{CQ}{BC}\right), \left(\frac{BQ}{BC}+\frac{DC}{CD}\right), \left(\frac{CR}{CD}+\frac{AS}{AD}\right)$$

i.e. $$2.\frac{BP}{AB}, \quad 2.\frac{AP}{AB}, \quad 2.\frac{BQ}{BC}, \quad 2\frac{CR}{CD}$$

i.e. λ, μ, λ, μ, since the sides of the parallelogram are parallel to the diagonals.

Since the masses at A and C are equal as also those at B and D, we may replace the whole system by masses $(2\lambda, 2\mu)$ at the mid-points, O_1, O_2 of AC and BD.

Hence the centroid of parallelogram lies on the line $O_1 O_2$.

Now let O be the centre of a conicoid passing through the quadrilateral. The tangent planes at A and C meet in BD of which the mid-point is O_2. Therefore OO_2 meets AC.

Similarly OO_1 meets BD. Hence O lies on the line O_1O_2.

N.B.—The condition that the sides of the parallelogram should be parallel to the diagonals is unnecessary.

For, it is readily seen that if a parallelogram is inscribed in a quadrilateral the sides must be parallel to the diagonals, unless the quadrilateral is plane.

Question 688.

(Ganpatram R. Jáni) :—S_r being the sum of the r^{th} powers of the first n natural numbers, prove that

$$S_{r+1}=(r+1)\int^n S_r \, dn+qn.$$

When r is even, $q=0$; when r is odd, q may be found by giving numerical values to n.

Solution (1) *by K. B. Madhava and K. R. Rama Iyer,*

(2) *by R. Srinivasan, M.A. and S. V. Venkatarayasastri, M.A., L.T.*

(1) Denoting by $\phi_n(x)$, the Bernoullian polynomial of degree n, *viz*: the coefficient of $\frac{t^n}{n!}$ in the expansion of $t\frac{e^{xt}-1}{e^t-1}$, we easily obtain,

$$\phi_n(x+1)-\phi_n(x)=\text{coefficient of } \frac{t^n}{n!} \text{ in } \frac{t}{e^t-1}\left[e^{(1+x)t}-e^{xt}\right]$$

$$=\text{coefficient in } te^{xt}$$

$$=n\,x^{n-1}.$$

Putting $x=1, 2, \ldots$ in succession

$$1+2^{n-1}+3^{n-1}+\ldots x^{n-1}=\frac{1}{n}\,\phi_n{}^{(x+1)}$$

a well-known result.

Differentiating w. r. t. x,

$$\phi'_n(x)=\text{coefficient of } \frac{t^n}{n!} \text{ in } \frac{t^2 e^{xt}}{e^t-1}$$

$$= \quad ,, \qquad \text{in } t\left[\frac{t(e^{xt}-1)}{e^t-1}+\frac{t}{e^t-1}\right];$$

from which we have

(i) $\qquad \phi'_{2m}(x)=2m\phi_{2m-1}(x)$ when $(m>1)$

and (ii) $\phi'_{2m+1}(x)=(2m+1)\{\phi_{2m}(x)+(-)^{m-1}B_m\}$ $(m\geq 1)$.

Thus $\phi'_2(x)=2\phi_1(x)-1$ and all the even $\phi'_{2m}(x)=2m\phi_{(2m-1)}(x)$.

Hence combining the two cases and changing the notation and integrating

$$S_{r+1}=(r+1)\int^n S_r dn+qn.$$

where q is zero when r is even, and when r is odd q is given by (ii).

(2) We have

$$S_r=\frac{n^{r+1}}{r+1}+\tfrac{1}{2}n^r+B_1\frac{r}{2!}n^{r-1}-B_3\frac{r(r-1)(r-2)}{4!}n^{r-3}$$
$$+B_5\frac{r(r-1)(r-2)(r-3)(r-4)}{6!}n^{r-5}-\ldots,$$

the last term containing n or n^2 according as r is even or odd.

[*Vide : Higher Algebra* by Hall and Knight, § 406]

$$\therefore\ (r+1)\int^n S_r dn=\frac{n^{r+2}}{r+2}+\tfrac{1}{2}n^{r+1}+B_1\frac{r+1}{2!}n^r-B_3\frac{(r+1)r(r-1)}{4!}n^{r-2}$$
$$+B_5\frac{(r+1)r(r-1)(r-2)(r-3)}{6!}n^{r-4}-\ldots$$

the last term containing n^2 or n^3 according as r is even or odd.

Hence the integral is equal to S_{r+1}, if r is even; and equal to $S_{r+1}\pm$ a term containing n, if n is odd.

Hence the result.

Question 693.

(R. VYTHYNATHASWAMY) :—If ABC, A′B′C′ be triangles inscribed in the same circle, L_1, L_2, L_3 the latera recta of parabolas having A, B, C for foci and touching the sides of A′BC′ ; L'_1, L_2', L_3' the latera recta of parabolas having A′, B′, C′ for foci and touching the sides of ABC ; show that $L_1.L_3.L_3 = L_1'.L_2'.L_3'$.

Solution by R. Srinivasan, M. A. and K. R. Rama Aiyar.

The pedal line of any point P is the tangent at the vertex of the parabola inscribed in the triangle and having P for its focus. Hence the distance of P from its pedal line is $\frac{1}{4}$ of the latus rectum. If p be this distance and PD the perpendicular to BC, then

$$\frac{p}{\text{PD}} = \text{sine of the angle between the pedal line and BC}$$

$$= \text{sine of angle subtended at the O}^{ce} \text{ by arc AD.}$$

$$\therefore \qquad \frac{p}{\text{PD}} \propto \text{chord PA.}$$

$$\therefore \qquad p \propto \text{PA. PD.}$$

Now PB. PC = PD × (diameter of circle ABC)

$$\therefore \qquad p \propto \text{PA. PB. PC.}$$

$$\therefore \qquad L_1 L_2 L_3 = k.\ \text{A'A. A'B. A'C. B'A. B'B. B.C. C'A. C'B. C'C}$$

The symmetry shows that

$$L_1L_2L_3 = L_1'L_2'L_3'.$$

Question 696.

(S. KRISHNASWAMI AIYANGAR) :—If λ, μ be the latera recta of the parabola and the rectangular hyperbola of closest contact with a curve at any point, prove that

$$2\lambda\rho = \mu^2$$

Additional solution by K. Appukuttan Erady, M.A.

Since, the parabola and the retangular hyperbola have closest contact with the curve at the point in question, the values of $\rho, \frac{d\rho}{ds}$ for these conics must be the same as those for the given curve.

The relation between ρ and $\frac{d\rho}{ds}$ for the parabola may be obtained as follows :—

Referred to the axis and tangent at the vertex as axes of co-ordinates the equation to the parabola is $y^2 = \lambda x$.

$$\therefore \quad y\frac{dy}{dx}=\frac{\lambda}{2} \therefore \frac{dy}{dx}=\frac{\lambda}{2y}$$

$$\therefore \quad \frac{d^2y}{dx^2}=\frac{-\lambda}{2y^2}\frac{dy}{dx}=\frac{-\lambda}{4y^3}$$

$$\therefore \quad \rho=\frac{\left(1+\frac{\lambda^2}{4y^2}\right)^{3/2}}{\frac{\lambda^2}{4y^3}}=\frac{(\lambda^2+4y^2)^{3/2}}{2\lambda^2} \qquad \ldots \quad (1)$$

$$\frac{d\rho}{ds}=\frac{1}{2\lambda^2}\times\frac{3}{2}(\lambda^2+4y^2)^{\frac{1}{2}}+8y\frac{dy}{ds}$$

$$=\frac{6y}{\lambda^2}(\lambda+4y^2)^{\frac{1}{2}}\left\{1+\left(\frac{dx}{dy}\right)^2\right\}^{-\frac{1}{2}}$$

$$=\frac{6y}{\lambda^2}(\lambda^2+4y^2)^{\frac{1}{2}}\left\{1+\frac{4y^2}{\lambda^2}\right\}^{-\frac{1}{2}}$$

$$=\frac{6y}{\lambda}. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

From (1) and (2) by eliminating y we get

$$2\lambda^2\rho=\left\{\lambda^2+\frac{\lambda^2}{9}\left(\frac{d\rho}{ds}\right)^2\right\}^{3/2} \quad \ldots \quad \ldots \quad (A)$$

The equation to the rectangular hyperbola referred to its as symptotes is $xy=\mu^2/8$, where μ is the latus rectum.

$$\therefore \quad \frac{dy}{dx}=-\frac{\mu^2}{8x^2},\ \frac{d^2y}{dx^2}=\frac{\mu^2}{4x^3}.$$

$$\therefore \quad \rho=\frac{\left\{1+\frac{\mu^4}{64x^4}\right\}^{3/2}}{\frac{\mu^2}{4x^3}}=\frac{(\mu^4+64x^4)^{3/2}}{128\mu^2x^3}. \qquad \ldots \quad (\alpha)$$

$$\mu^2\frac{d\rho}{ds}=\left\{-3\frac{(\mu^4+64x^4)^{3/2}}{128x^4}+\frac{3}{2}\frac{(\mu^4+64x^4)^{\frac{1}{2}}}{128x^3}\times 64x4x^3\right\}\frac{dx}{ds}$$

$$i.e_1 \quad \frac{\mu^2 d\rho}{ds}=3\left\{(\mu^4+64x^4)^{\frac{1}{2}}-\frac{(\mu^4+64x^4)^{3/2}}{128x^4}\right\}\left\{1+\frac{\mu^4}{64x^4}\right\}^{-\frac{1}{2}}$$

$$=3\times 8x^2\left\{1-\frac{\mu^4+64x^4}{128x^t}\right\}=3.\frac{64x^4-\mu^4}{16x^2}.$$

$$\therefore \quad \frac{16}{3}\mu^2\frac{d\rho}{ds}=64x^2-\frac{\mu^4}{x2};\ \text{and from } (\alpha),\ (128\,\mu^2\rho)^{2/3}=64x^2+\frac{\mu^4}{x^2}.$$

$$\therefore \quad (128\mu^2\rho)^{\frac{4}{3}}-\frac{256}{9}\left(\frac{d\rho}{ds}\right)^2=256\mu^4 \quad \ldots \quad \ldots \quad \ldots \quad (B)$$

Eliminating $\frac{d\rho}{ds}$ from (A) & (B), we have

$$2\lambda\rho=\mu^2.$$

Question 700.

(S. RAMANUJAN) :—Sum the series

$$(a+b+1)\left(\frac{a}{b}\right)^2+(a+b+3)\left\{\frac{a(a+1)}{b(b+1)}\right\}^2$$
$$+(a+b+5)\left\{\frac{a(a+1)(a+2)}{b(b+1)(b+2)}\right\}^2+\dots \text{ to } n \text{ terms.}$$

Solution by **(1)** *K. R. Rama Aiyar,* **(2)** *by K. Appukuttan Erady.*

(1) Euler's Identity gives

$$(1-a_1)+a_1(1-a_2)+a_1a_2(1-a_3)+\dots$$
$$+a_1a_2\dots a_n(1-a_{n+1})=1-a_1a_2\dots a_{n+1}.$$

In this put

$$a_1=a^2,\ a_2=\left(\frac{a+1}{b}\right)^2,\ a_3=\left(\frac{a+2}{b+1}\right)^2,\quad a_{n+1}=\left\{\frac{a+n}{b+n-1}\right\}^2.$$

Then

$$1-a^2+a^2\left\{1-\left(\frac{a+1}{b}\right)^2\right\}+\frac{a^2}{b^2}(a+1)^2\left\{1-\left(\frac{a+2}{b+1}\right)^2\right\}$$
$$+\dots+\left\{\frac{a(a+1)(a+2)\dots(a+n-2)}{b(b+1)(b+2)\dots(b+n-2)}\right\}^2(a+n-1)^2\left\{1-\left(\frac{a+n}{b+n-1}\right)^2\right\}$$
$$=1-\frac{a^2}{b^2}\left(\frac{a+1}{b+2}\right)^2\left(\frac{a+1}{b+2}\right)^2\dots\dots\left(\frac{(a+n-1}{b+n-1}\right)^2(a+n)^2.$$

i.e., $$(1-a^2)+\frac{a^2}{b^2}(a+b+1)(b-a-1)+\left\{\frac{a(a+1)}{b(b+1)}\right\}^2(a+b+3)(b-a-1)$$
$$+\dots+\left\{\frac{a(a+1)\dots(a+n-1)}{b(b+1)\dots(b+n-1)}\right\}^2(a+b+2n-1)(b-a-1)$$
$$=1-\left\{\frac{a}{b}\cdot\frac{a+1}{b+1}\cdot\frac{a+2}{b+2}\dots\dots\frac{a+n-1}{b+n-1}\cdot(a+n)\right\}^2.$$

Hence the given series S is equal to

$$\frac{1}{b-a-1}\left\{a^2-\left[\frac{a(a+1)\dots\dots(a+n-1)(a+n)}{b(b+1)\dots\dots(b+n-1)}\right]^2\right\}.$$

(2) Let $u_r=(a+b+2r-1)\left\{\frac{a(a+1)\dots(a+r-1)}{b(b+1)\dots(b+r+1)}\right\}^2.$

Take the auxiliary series whose $v_r=\left\{\frac{a(a+1)\dots(a+r-1)}{b(b+1)\dots(b+r-2)}\right\}^2.$

Then $v_r-v_{r+1}=\left\{\frac{a(a+1)\dots(a+r-1)}{b(b+1)\dots(b+r-1)}\right\}^2\left\{(b+r-1)^2-(a+r)^2\right\}$

$=(b-a-1)u_r.$

$\therefore \quad v_1-v_{n+1}=(b-a-1)\mathrm{S}.$

Hence $\quad \mathrm{S}=\frac{1}{b-a-}\left\{a^2-\left(\frac{a(a+1)\dots(a+n)}{b(b+1)\dots(b+n-)}\right)^2\right\}.$

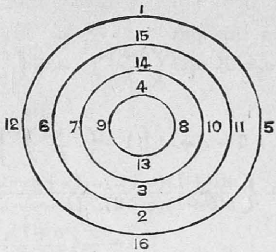
Question 705.

(S. Malhari Rao, B.A.) :—The circumferences of four concentric circles are cut by two diameters. Arrange the numbers 1, 2, 3,...16 at the points of intersection in such a way that the sum of the numbers on each radius and on each circumference may be the same, and also the sum of every pair of diametrically opposite numbers may be the same.

Solution by R. D. Karve and others.

The usual magic square

1	15	14	4
12	6	7	9
8	10	11	5
13	3	2	16



suggests the accompanying arrangement. Of course the arrangement may be varied.

Question 712.

(J. C. Swaminarayan, M.A.) :—If $a > b$ and

$$f(a, b) = \int_0^{\pi} \log (a + b \cos \theta) \, d\theta,$$

prove that

$$f(a, b) = \frac{1}{2} f\left(a^2 - \frac{b^2}{2}, \frac{b^2}{2} \right) = \pi \log \left(\frac{a + \sqrt{a^2 - b^2}}{2} \right).$$

Solution by (1) *D. Krishnamurti, L. N. Subramanyam and K. B. Madhava,* (2) *by R. Srinivasan, M.A. and* (3) *by G. A. Kamekar.*

(1) Now $f(a, b) = \int_0^{\pi} \log (a + b \cos \theta) \, d\theta$

$$= \int_0^{\pi} \log (a - b \cos \theta) \, d\theta \text{ (changing } \theta \text{ to } \pi - \theta).$$

$$\therefore \quad 2 f(a\, b) = \int_0^{\pi} \log (a^2 - b^2 \cos^2 \theta)\, d\theta$$

$$= \int_0^{\pi} \log \left(a^2 - \frac{b^2}{2} - \frac{b^2}{2} \cos 2\theta \right) d\theta;$$

changing 2θ to $\pi - \theta$, we get

$$= \int_0^{\pi} \log \left(a^2 - \frac{b^2}{2} + \frac{b^2}{2} \cos \theta \right) d\theta$$

$$= f\left(a^2 - \frac{b^2}{2}, \frac{b^2}{2} \right).$$

Again $f(a, b) = \pi \log a + \int_0^{\pi} \log \left(1 + \frac{b}{a} \cos \theta \right) d\theta.$

$$= \pi \log a \cos^2 \frac{1}{2} \alpha + \int_0^{\pi} \log (1 + te^{i\theta})(1 + te^{-i\theta})\, d\theta$$

putting $\frac{b}{a} = \sin \alpha = \frac{2t}{1+t^2}$, where $t = \tan \frac{1}{2} \alpha$

$$= \pi \log a \cos^2 \tfrac{1}{2} \alpha + \int_0^{\pi} \left(te^{i\theta} - \frac{t^2}{2} e^{2i\theta} + - \ldots \right)$$

$$\times \left(te^{-i\theta} - \frac{t^2}{2} e^{-2i\theta} + \ldots \right) d\theta$$

$$= \pi \log a \cos^2 \tfrac{1}{2} \alpha + \int_0^{\pi} 2 \left(t \cos \theta - \frac{t^2}{2} \cos 2\theta + - \ldots \right) d\theta$$

$$= \pi \log a \cos^2 \tfrac{1}{2} \alpha + 2 \left[-t \sin \theta + \frac{t^2}{2^2} \sin 2\theta^2 + \ldots\ldots \right]_0^{\pi}$$

$$= \pi \log \frac{a}{2} \left(1 + \frac{\sqrt{a^2 - b^2}}{a} \right)$$

$$= \pi \log \left(\frac{a + \sqrt{a^2 - b^2}}{2} \right)$$

$$\therefore f(a, b) = \tfrac{1}{2} f\left(a^2 - \frac{b^2}{2}, \frac{b^2}{2} \right) = \pi \log \left(\frac{a + \sqrt{a^2 - b^2}}{2} \right)$$

(2) $\int_0^{\pi} \log (1 + n \cos \theta)\, d\theta = \pi \log \frac{1 + \sqrt{1 - n^2}}{2}$

(page 177, Roberts' *Integ. Cal.*).

In this, putting $\frac{b}{a}$ for n, we get the required result.

Also

$$f(a, b) = \pi \log \frac{a+\sqrt{a^2-b^2}}{2}.$$

$$\therefore f\left(a^2-\frac{b^2}{2}, \frac{b^2}{2}\right) = \pi \log \left\{\frac{a^2-\frac{b^2}{2}+\sqrt{a^4-a^2b^2}}{2}\right\}$$

$$= \pi \log \frac{2a^2-b^2+2a\sqrt{a^2-b^2}}{4}$$

$$= \pi \log \left(\frac{a+\sqrt{a^2-b^2}}{2}\right)^2$$

$$= 2f(a, b).$$

(3) Let $I = f(a,b) = \int_0^\pi \log (a+b \cos \Theta) d\Theta,\ a>b$;

$$J = f\left(a^2-\frac{b^2}{2}, \frac{b^2}{2}\right) = \int_0^\pi \log\left(a^2-\frac{b^2}{2}+\frac{b^2}{2} \cos \Theta\right) d\Theta.$$

Differentiating I and J with respect to a, we get

$$\frac{\partial I}{\partial a} = \int_0^\pi \frac{1}{a+b \cos\Theta} d\Theta = \left[\frac{2}{\sqrt{a^2-b^2}} \tan^{1}\sqrt{\frac{a-b}{a+b}} \tan\frac{\Theta}{2}\right]_0^\pi$$

$$\frac{\partial J}{\partial a} = \left[\frac{4}{1\sqrt{a^2-b^2}} \tan^{1}\sqrt{\frac{a^2-b^2}{a^2}} \tan \Theta/2\right]_0^\pi$$

$$\therefore \quad \frac{\partial I}{\partial a} = \frac{\pi}{\sqrt{a^2-b^2}};\ \text{and}\ \frac{\partial J}{\partial a} = \frac{2\pi}{\sqrt{a^2-b^2}}$$

$\therefore$ $I = \pi (\log a+\sqrt{a^2-b^2})+c$; and $J = 2\pi \log (a+\sqrt{a^2-b^2}+c')$.

Now both I and J vanish if $a=1$ and $b=0$.

$$\therefore \quad c = -\pi \log 2 \text{ and } c' = -2\pi \log 2.$$

Hence

$$I = \pi \log \frac{a+\sqrt{a^2-b^2}}{2}, \text{ and } J = 2\pi \log \frac{a+\sqrt{a^2-b^2}}{2}$$

$$\therefore \quad I = \tfrac{1}{2} J = \pi \log \frac{a+\sqrt{a^2-b^2}}{2}.$$

Question 716.

(S. Krishnaswami Aiyangar, B.A.) :—Show how to find the sum of

$$1+\frac{a}{1!}+\frac{2^r a^2}{2!}+\frac{3^r a^3}{3!}+\ldots\frac{p^r a^p}{p!}+\ldots$$

Remarks by K. B. Madhava, R. Srinivasan, M.A., and S. R. Ranganathan.

It is quite easy to see that the sum of this is

$$1+\left(a\frac{d}{da}\right)^r e^a.$$

The proposer is referred to p. 184 of the Journal for October 1913 for more general results of this kind.

Question 717.

(C. KRISHNAMACHARY) :—Show that:

$$\phi(x+h)-\phi(x+3h]+\phi(x+5h)-\ldots$$
$$=\phi(x-h)-\phi(x-3h)+\phi(x-5h)-\ldots$$

Solution (1) by K. B. Madhava and D. G. Dandehar, B.Sc; (2) by R. Srinivasan, M.A.

(1) Adopting the method of Edwards' *Diffl. Calc.* § 553, we take the result $\sin\theta-\sin 3\theta+\sin 5\theta\ldots=0$

i.e. $$e^{i\theta}-e^{3i\theta}+e^{5i\theta}-\ldots\equiv e^{-i\theta}-e^{-3i\theta}+e^{-5i\theta}-\ldots$$

and write for $e^{i\theta}$, the operator $e^{h\frac{d}{dx}}=\mathrm{E}h$, when we get

$$\mathrm{E}^h-\mathrm{E}^{3h}+\mathrm{E}^{5h}-\ldots\equiv\mathrm{E}^{-h}-\mathrm{E}^{-3h}+\mathrm{E}^{=5h}-\ldots$$

applying this to operate upon $\phi(x)$, we at once have the desired result.

(2) We know that $\dfrac{x}{1+x^2}=\dfrac{x^{-1}}{1+x^{-2}}$.

In this put $x=e^{h\frac{d}{dx}}=\mathrm{E}^h$ and let both sides operate on $\phi(x)$, then

$$(1+\mathrm{E}^{2h})^{-1}\phi(x)=\mathrm{E}^{-h}(1+\mathrm{E}^{-2h})\mathrm{E}^{-1}\phi(x)$$

i.e. $$\phi(x+h)-\phi(x+3h)+\ldots=\phi(x-h)-\phi(x-3h)+\ldots$$

Remarks by K. B. Madhava.

Expanding in powers of h, by Taylor's theorem we see that the even powers of h agree and the question amounts to showing that the coefficients of the odd powers of h, *viz*:

$$1^{2s+1}-3^{2s+1}+5^{2s-1}-\ldots\text{vanish.}$$

From the fact that the integral $\int_0^\infty e^{-t}u_p'(xt)\,dt$ converges uniformly if $\cos\theta\leq 1-a$ (in the notation of Bromwich, §§ 109, 110), we infer that the results for $\mathop{\mathrm{S}}\limits_1^\infty \begin{matrix}\cos\\ \sin\end{matrix}\, n\theta$ can be differentiated any number of times, whence we obtain the results:

$$\mathop{\mathrm{S}}\limits_1^\infty n^{2s+1}\sin n\theta=0,\text{ if }\theta\text{ is not an even multiple of }\pi.$$

$$\mathop{S}_{1}^{\infty}(-)^{n-1}n^{2s}\sin n\theta=(-)^{s}\frac{d^{2s}}{d\theta^{2s}}\left(\tfrac{1}{2}\tan\frac{\theta}{2}\right),\text{ if }\theta\text{ is an odd }\pi;$$

from which we have

$$1^{2s+1}-3^{2s+1}+5^{2s+1}-\ldots=0.$$

and $$1^{2s}-3^{2s}+5^{2s}-\ldots=\tfrac{1}{2}(-)^{s}E_{s};$$

where E is the Euler number.

Question 728.

(K. Appukuttan Erady, M. A.) :—If $u\equiv(a\ b\ c\ f\ g\ h)(x\ y\ z)^2$, show that $\iiint u^n\,dx\,dy\,dz$ taken throughout the space bounded by $u=1$, is $\frac{4\pi}{2n+3}\Delta^{-\frac{1}{2}}$, where Δ is the discriminant of u.

Solution by Martyn M. Thomas, K. V. A. Sastri and C. Bhaskaraiya.

Since $u=1$ represents a central conicoid, transforming the equation to the standard form $\lambda_1 x^2+\lambda_2 y^2+\lambda_3 z^2=1$, by turning the co-ordinate axes so as to coincide with the axes of the conicoide, we find $\lambda_1\lambda_2\lambda_3=\Delta$.

Now consider a concentric, similar ellipsoidal shell

$$\lambda_1 x^2+\lambda_2 y^2+\lambda_3 z^2=h$$

over the surface of which u has the constant value h.

Extended over the *volume* of $\lambda_1 x^2+\lambda_2 y^2+\lambda_3 z^2=h$, included in the first octant,

$$\iiint dx\,dy\,dz=\frac{\left(\frac{h}{\lambda_1}\right)^{\frac{1}{2}}\cdot\left(\frac{h}{\lambda_2}\right)^{\frac{1}{2}}\left(\frac{h}{\lambda_3}\right)^{\frac{1}{2}}}{2\times2\times2}\frac{\left[\Gamma\left(\frac{1}{2}\right)\right]^3}{\Gamma\left(1+1\frac{1}{2}\right)}=\frac{h^{\frac{3}{2}}}{6\Delta^{\frac{1}{2}}}.$$

The differential of this gives the value of $\iiint dx\,dy\,dz$ extended all over the *surface* of the shell, *viz.* $\frac{\pi}{4\Delta^{\frac{1}{2}}}h^{\frac{1}{2}}dh$.

$$\therefore\ \iiint u^n\,dx\,dy\,dz=\frac{\pi}{4\Delta^{\frac{1}{2}}}\int h^n.\,h^{\frac{1}{2}}\,dh,$$ since $u=h$ all over the surface of the shell.

Hence the value of $\iiint u^n\,dx\,dy\,dz$ throughout the solid

$$=8\frac{\pi}{4\Delta^{\frac{1}{2}}}\int_0^1 h^{n+\frac{1}{2}}\,dh$$

$$=\frac{4\pi}{\Delta^{\frac{1}{2}}}\cdot\frac{1}{2n+3}.$$

QUESTIONS FOR SOLUTION.

772. (T. RAJARAMA RAO, B.A., B.L.) :—To construct a triangle having given the sum of the sides, the vertical angle, and the area of the rectangle formed by the two segments into which the base is divided by the external bisector of the vertical angle.

Wanted a *purely geometrical* solution.

773. (K. APPUKUTTAN ERADY, M.A.):—The conic $l\beta\gamma+m\gamma\alpha+n\alpha\beta=0$ circumscribes ABC, the triangle of reference. If ρ_1, ρ_2, ρ_3 be the radii of curvature of the conic at A, B, C respectively, show that

$$a^3\rho_1 : b^3\rho_2 : c^3\rho_3 :: (m^2+n^2-2mn\cos A)^{\frac{3}{2}} : (l^2+n^2-2ln\cos B)^{\frac{3}{2}} : (l^2+m^2-lm\cos C)^{\frac{3}{2}}$$

774. (N. P. PANDYA) :—Find five sets of two numbers each such that their sum is a perfect square, and the sum of their cubes is a perfect fifth power.

775. (K. SRINIVASAN) :—Prove geometrically

$$1+\text{dn}\, 2z=\frac{2\text{dn}^2 z}{1-k^2\,\text{sn}^4 z}.$$

776. (K. SRINIVASAN) :—Expand in a Fourier Series $\text{cn}^3 z$.

777. (S. KRISHNASWAMI IYENGAR) :—Shew that the envelope of the axes of the parabolas having double contact with the ellipse $x^2/a^2+y^2/b^2=1$ at the ends of a focal chord is

$$4a^2(x^2+y^2)^2+27b^4e^6x^2-48ab^2e^3x^2y=0.$$

778. (S. MALHARI RAO, B. A.) :—Solve in positive integers $x^2-y^2=4z^2$, $z^2-w^2=15w$, $x+y+z+w=192$.

779. (S. MALHARI RAO, B. A.) :—ABC and ADC are two rational triangles. If their areas are rational but unequal, and if BC=CD=13 inches, and AC=37 inches, shew that ABCD is a cyclic quadrilateral and that the angle BAD is bisected by AC.

780. (MARTYN M. THOMAS) :—Solve the difference equation

$$(1+x)u_{x+2}-2e^{2x+1}.x.u_{x+1}+e^{4x}(x-1)u_x=e^{1+x+x^2}$$

781. (Selected) :—Two vectors OP, OQ of the curve

$$r=2a\cos^3\left(\frac{\pi}{4}+\frac{\theta}{3}\right)$$

are drawn equally inclined to the linitial line. Prove that, if s be the length of the arc intercepted, the area included betweeen the curve and the radii is

$$\frac{5as}{8}-\frac{9a^2}{16}\sin\left(\frac{2s}{3a}\right).$$

782. (Selected) :—If S_r is the sum of the squares of the reciprocals of the first r odd numbers, prove that

$$\frac{S_1}{3^2}+\frac{S_2}{5^2}+\frac{S_3}{7^2}+\ldots=\frac{\pi^4}{384}.$$

783. (S. RAMANUJAN) :—If $x=y^n-y^{n-1}$,

and
$$J_n=\int_0^1\frac{\log y}{x}dx,$$

show that

(i) $$J_o=\frac{\pi^2}{6};\ J_{\frac{1}{2}}=\frac{\pi^2}{10};\ J_1=\frac{\pi^2}{12}\ J_2=\frac{\pi^2}{15}.$$

(ii) $$J_n+J_{\frac{1}{n}}=\frac{\pi^2}{6}.$$

784. (S. RAMANUJAN) :—If $F(x)$ denotes the fractional part of x (*e.g.* $F(\pi)=.14159\ldots$) and N is a positive integer, show that

(i) $$\lim_{N\to\infty} NF(N\sqrt{2})=\frac{1}{2\sqrt{2}};\ \lim_{N\to\infty} NF(N\sqrt{3})=\frac{1}{\sqrt{3}};$$

$$\lim_{N\to\infty} NF(N\sqrt{5})=\frac{1}{2\sqrt{5}};$$

$$\lim_{N\to\infty} NF(N\sqrt{6})=\frac{1}{\sqrt{6}};\ \lim_{N\to\infty} NF(N\sqrt{7})=\frac{3}{2\sqrt{7}}.$$

(ii) $$\lim_{N\to\infty} N(\log N)^{1-p}F\left(Ne^{\frac{2}{n}}\right)=0,$$

where n is any integer and p is any positive number.

(iii) In (ii) show that p cannot be zero.

785. (S. RAMANUJAN) :—Show that

$$\sqrt[3]{3(\sqrt[3]{a^3+b^3}-a)(\sqrt[3]{a^3+b^3}-b)}=\sqrt[3]{(a+b)^2}-\sqrt[3]{a^2-ab+b^2}.$$

This is analogous to

$$\sqrt{2(\sqrt{a^2+b^2}-a)(\sqrt{a^2+b^2}-b)}=a+b-\sqrt{a^2+b^2}.$$

786. (MARTYN M. THOMAS) :—Pairs of tangents are drawn to a closed oval curve, without singularities, at a constant angle 2α ; and lines are drawn from their point of intersection, inclined to them, externally, at β and γ. If p and q be the entire lengths of the curves enveloped by these lines, and l that of the given curve, show that

$$p \operatorname{cosec}(\alpha+\beta)=l \operatorname{cosec}\alpha=q \operatorname{cosec}(\alpha+\gamma).$$

787. (M. K. KEWALRAMANI, M. A.) :—Prove that if a be not an integer

$$\frac{\pi}{2}\,\frac{f(x+ah)-f(x-ah)}{\sin a\pi}=\frac{f(x+h)-f(x-h)}{1^2-a^2}-2\,\frac{f(x+2h)-f(x-2h)}{2^2-a^2}$$
$$+3\,\frac{f(x+3h)-f(x-3h)}{3^2-a^2}-\ldots$$

788. (E. H. NEVILLE) :—From the point (u, v) can be drawn four normals to the ellipse $(x^2/a^2)+(y^2/b^2)=1$; four circles are drawn, each through the feet of three of these normals ; shew that the sum of the squares of the radii of these circles is given by

$$2a^2b^2\Sigma r^2=(a^2+b^2)^3-(a^2-b^2)(a^2u^2-b^2v^2).$$