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PROGRESS REPORT.

The following gentlemen have been elected members of the Society—

1. *Mr. Dattatraya Ganesh Dandekar B.Sc.*—Science Master, Herbert High School, Kotah (Rajputana).

2. *Mr. V. Viraswamaiya B.A., L.T.*,—Assistant, Pachaiyappa's High School, 29, Krishnappa Naick Agraharam, Madras, (at concessional rate).

3. *Mr. Keshav Narasimha Khandekar, B.A* —Assistant Head Master, Mission High School, Beawar (Rajputana), (at concessional rate).

4. *Mr. F. Hallberg.*—Professor of Mathematics, St. Xavier's College, Fort Bombay.

5. *Mr. T. R. Venkatesa Aiyar, B.A.*—Editor, The Indian Engineer, 1/33, Kachaleswara Agraharam, George Town, Madras.

2. The Calcutta University Calendar for 1915, Part I, has been received for the Library.

POONA, 31st Jan., 1916.

D. D. KAPADIA,
Hony. Joint Secretary.

Green's Function.

By P. V. Seshu Aiyar.

Note: In this article Δ stands for the usual symbol ∇^2.

1. *Introduction*:

As a means of solving certain problems in Electrostatics Green introduced a certain function into Analysis which afterwards came to be known after his name. Properties of this function were developed by Green himself from a physical point of view (*vide* Green's *Mathematical Papers*, edited by Ferrers: p. 31 and seq.; or Clarke Maxwell's *Electricity and Magnetism*: p. 133 and seq.]. But continental mathematicans have developed these properties from the point of view of Pure Analysis and the present article gives the exposition from the latter point of view following French mathematicians, chiefly Poincaré.

2. *Definition*:

Let T be a volume bounded by a closed surface S, and M′ a point situated in the interior of T, and M a variable point in the interior of T or upon the surface S. Then the function $G=(1/r+H)$ is called the Function of Green relative to the volume T and to the point M′, where r is the distance of the variable point M from the point M′ and H is a function which is *harmonic* in the volume T (*i.e.*, satisfies the equation of Laplace with the necessary conditions as regards continuity) and which is equal to $-1/r$ upon S.

Thus defined, this function G vanishes upon S, and satisfies the equation of Laplace at every point of volume T except at M′ where it becomes infinite.

The function of Green given by this definition relates to the space enclosed by a closed surface and and a point within it corresponds to the interior problem of Dirichlet [*vide*: the article on the *Problem of Dirichlet* (p. 177) in the issue of October 1915 of this Journal]. As in the case of the problem of Dirichlet, there is also an exterior function of Green, *i.e.*, a function relative to a volume T and to a point M′ outside this volume. In this article we confine ourselves to the interior function.

3. *Physical Interpretation*:

Conceive the surface S to be a perfect conductor put in communication with the earth, and a unit of positive electricity placed at the point M′. Then the value at the variable point M of the total potential function, arising from the unit of electricity at M′ and from the electricity it will induce upon the surface, will be the function of Green as defined above.

For, in consequence of the communication with the earth, the total potential at the surface is zero ; further the potential at M due to the unit of electricity placed at M′ is $1/r$ and that due to the electricity induced on the surface is a harmonic function H ; and hence the total potential is $H+1/r=G$.

This physical interpretation also helps to convince us that such a function exists.

4. *Notation :*

Once the domain T is given, the function G is a function of the co-ordinates (x, y, z) of the variable point M, but it depends also on the co-odinates (x', y', z') of the point M′, the infinity of the function. To put in evidence this fact we may write it as $G(x, y, z, x', y', z')$, or $G(M, M')$ indicating by the latter notation; the value at the point M of the function of Green relative to the domain T and to the point M′.

5. *Properties of the Function :—*

1°. The function is positive at every point M in the domain T.

As r tends to zero (*i.e.*, as the point M tends to the point M′) H remains finite. Therefore $G-1/r$ remains finite.

Hence $(r.G-1)$ tends to zero, and rG tends to unity. Thus when r is very small, the product rG is very near to unity. We can therefore affirm that in the neighbourhood of the point M′, the function G is positive. Enclose this point by a very small sphere Σ. Between Σ and S, we have $\Delta G=0$; upon S, G is zero and upon Σ, $G>0$. Therefore between Σ and S, G is everywhere positive according to the following lemma.

Lemma :

Every harmonic function in a domain lies between the greatest and the least values upon the surface bounding the domain, whether it be simply-connected or multiply-connected. For, otherwise, there must be a point within the domain T, at which the function is a maximum or a minimum, which it cannot be [*vide* : the article on 'Dirichlet's Problem' quoted above.]

2°. In the volume T, G is everywhere less than $1/r$.

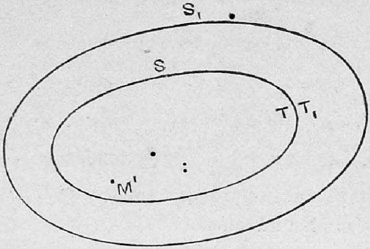
In the volume T, $\Delta H=0$ and $H=-1/r$ upon S. That is, H is negative upon S.

Since at any point within T, H must lie between its greatest and least values upon S, H is negative at every point of T.

$$\therefore \quad G=H+\frac{1}{r}<\frac{1}{r}$$

3°. Let T be a domain bounded by a closed surface S, and T_1 a larger domain bounded by S_1 and containing T within it; and let M′ be a point in T, then if G_1 be the function of Green relative to T_1 and to M′, and G that relative to T and to the same point M′, then, within T,

$$G_1 > G.$$



For, if G_1 be H_1+1/r and G be $H+1/r$, we see that $G_1-G=H_1-H$ is harmonic within T and since $G_1>0$ in T_1, we have $H_1+1/r>0$ upon S but upon S, $H=-1/r$; therefore $H_1-H>0$ upon S and is positive in the volume T enclosed by S according to the Lemma.

Hence $\qquad G_1-G$ is positive ;

i.e. $\qquad G_1>G$ in T.

4.° Consider the surface $G=C$.

To each value of the constant C there corresponds a particular surface.

For very great values of the constant, we have evidently very small surfaces sorrounding the pole M′, since G becomes infinite at M′.

For values very small of C, we have on the contrary surfaces very near to the surface S.

Finally, we pass from one surface to another by a continuous deformation by making the constant C vary in a continuous manner.

Further the function G being uniform in the volume T, two surfaces corresponding to two different values of C cannot cut each other. The surfaces $G=C$, for different values of C, enclose therefore the pole and are mutually enveloping; that is, they are contained within one another, so that the surface corresponding to any one value of C (say C_0) encloses within it all surfaces corresponding to the values of C greater than C_0.

The physical interpretation of all the above mentioned properties is quite plain.

5°. $\int_{S'} \frac{dG}{dn} d\sigma = 4\pi$, where the integration is extended to the whole of any surface S′ contained within S and containing within it the point M′ and the differentiation $\frac{d}{dn}$ is along the normal interior to the surface S′.

Since $G = H + 1/r$ is harmonic everywhere in T except at M′ where it is infinite, $\frac{dG}{dn}$ exists at every point upon S′. Enclose M′ by a small sphere Σ of radius ρ and centre M′. Now in the space between Σ and S′, G is harmonic, and hence $\int \frac{dG}{dn} d\sigma$ extended to the whole of the surfaces bounding that space is zero. [*vide*: Cor. 2 of § 2 in the article on the Problem of Dirichlet].

i.e.

$$\int_{S'} \frac{dG}{dn} d\sigma + \int_{\Sigma} \frac{dG}{dn} d\sigma = 0,$$

the differentiation $\frac{d}{dn}$ being along the normal interior to the space enclosed; in other words

$$\int_{S'} \frac{dG}{dn} d\sigma = \int_{\Sigma} \frac{dG}{dn} d\sigma,$$

the differentiation $\frac{d}{dn}$ being along the normal interior to the geometrical surface in both cases.

But

$$\int_{\Sigma} \frac{dG}{dn} d\sigma = \int_{\Sigma} \frac{d\left(\frac{1}{r}\right)}{dn} d\sigma.$$

$$= \int_{\Sigma} -\frac{1}{r^2} \frac{dr}{dn} d\sigma = \frac{1}{\rho^2} \int_{\Sigma} d\sigma = 4\pi.$$

For, the differentiation being along the normal interior to Σ, $\frac{dr}{dn} = -1$ in the case of the sphere.

$\therefore$

$$\int_{S'} \frac{dG}{dn} d\sigma = 4\pi.$$

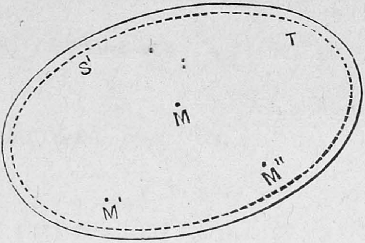
[*Note.*—This could be easily identified as a particular case of Gauss' theorem on the surface integral of normal force.]

6°. Let T be a volume bounded by a surface S, and M′, M″ two points within it. Let G′ denote the function of Green relative to the domain T and to the point M′, and G″ denote the function of Green relative to the same domain T and to the point M″. Then the value of G′ at M″ is equal to the value of G″ at M′.

That is to say, if we denote by G′ (M) and G″ (M) the values of the two functions G′ and G″ respectively at any point M, we shall have

$$G'(M'') = G''(M').$$

Let α' be a very small number and consider the surface S′ corresponding to $G' = \alpha'$. This surface S′ is very near to S and by taking α' sufficiently small, S′ can be made to contain within its interior both the point M′ and M″.



Now consider the integral

$$J = \int_{S'} \left(G' \frac{dG''}{dn} - G'' \frac{dG'}{dn} \right) d\sigma$$

extended to the surface S′, where $\frac{d}{dn}$ is taken along the normal interior to S′.

This integral is well determinate since G′, G″, $\frac{dG'}{dn}$, $\frac{dG''}{dn}$ exist at all points of S′.

Around the two points M′ and M″ as centres, describe small spheres Σ′, Σ″ of radii ρ', ρ'' respectively, and consider the space contained between the surface S′ and the spheres Σ′, Σ″. In this space, G′ and G″ are harmonic, and hence, proceeding as in 5° above, we have

$$\int_{S'} \left(G' \frac{dG''}{dn} - G'' \frac{dG'}{dn} \right) d\sigma = \int_{\Sigma'} \left(G' \frac{dG''}{dn} - G'' \frac{dG'}{dn} \right) d\sigma + \int_{\Sigma''} \left(G' \frac{dG''}{dn} - G'' \frac{dG'}{dn} \right) d\sigma$$

where $\frac{d}{dn}$ is to be taken along the normals interior to the geometric surfaces.

. . . .

Now, let M be the greatest value of G′ upon Σ' ; then

$$\int_{\Sigma'} G'\frac{dG''}{dn}d\sigma < M\left|\int_{\Sigma'}\frac{dG''}{dn}d\sigma\right| = 0 \quad ;$$

because G″ is harmonic in Σ.

Next, consider the integral $\int_{\Sigma'} G''\frac{dG'}{dn}d\sigma$.

G″ is finite and continuous at M′. Therefore, on the sphere Σ', $G'' = G''(M') + \epsilon$ where ϵ vanishes with ρ'.

Thus $\int_{\Sigma'} G''\frac{dG'}{dn}d\sigma$ has for limiting value

$$G''(M')\int_{\Sigma'}\frac{dG'}{dn}d\sigma = 4\pi G''(M'), \text{ [by } 5^\circ].$$

Hence the limiting value of

$$\int_{\Sigma'}\left(G'\frac{dG''}{dn} - G''\frac{dG'}{dn}\right)d\sigma = -4\pi G''(M').$$

Similarly, the limiting value of

$$\int_{\Sigma''}\left(G'\frac{dG''}{dn} - G''\frac{dG'}{dn}\right)d\sigma = 4\pi G'(M'').$$

$$\therefore \qquad J = 4\pi\,\{\,G'(M'') - G''(M')\,\}.$$

Again $J = \int\left(G'\frac{dG''}{dn} - G''\frac{dG'}{dn}\right)d\sigma$ is the difference of two other integrals : *viz*

$$(1)\ \int_{S'} G'\frac{dG''}{dn}d\sigma\ ; \quad (2)\int_{S'} G''\frac{dG'}{dn}d\sigma.$$

But $(1) = \alpha'\int_{S'}\frac{dG''}{dn}d\sigma$, because G′ is constant upon S′ and equals α'.

$= 4\pi\alpha'$ [by 5°.].

$\therefore$ (1) tends to zero with α'.

Nextly let α'' be the superior limit of G″ upon S′, so that we have

$$\left|\int_{S'} G''\frac{dG'}{dn}d\sigma\right| < \alpha''\int_{S'}\left|\frac{dG'}{dn}d\sigma\right|$$

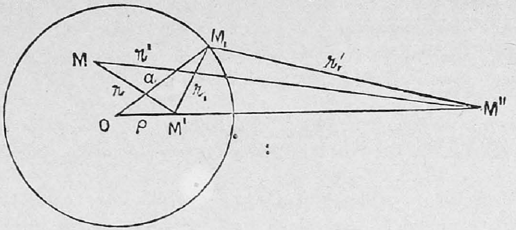
$$< 4\pi\alpha'' \text{ [by } 5^\circ].$$

But α'' tends to zero simultaneously with α'; for, as α' tends to zero, the surface S′ tends to the surface S and hence α'' tends also to zero.

Thus J tends to zero simultaneously with α' and consequently the expression $4\pi\,\{\,G'(M'') - G''(M')\,\}$, which is equal to J, also tends to zero, But this expression has a definite value; it is therefore necessarily zero; and we have

$$G'(M'')=G''(M').$$

6. *Value of Green's function in the case of a Sphere.* Let S be a sphere, O its centre and a its radius. Let M′ be a point within the sphere. Let us calculate the function of Green relative to this point M′.



Let M″ be the point conjugate to M′ with respect to the sphere, so that OM′M″ is a straight line and $OM'.OM''=a^2$.

Let M be any point within the sphere and M_1 a point on the sphere. Join OM_1, MM′, MM″, M_1M', M_1M''; and put $MM'=r$, $MM''=r'$, $M_1M'=r_1$, $M_1M''=r_1'$ and $OM'=\rho$. Then $OM''=\dfrac{a^2}{\rho}$.; and by similar triangles we also have $\dfrac{r_1}{r_1'}=\dfrac{\rho}{a}$, whence

$$\frac{1}{r_1}=\frac{a}{\rho}.\frac{1}{r_1'} \qquad \dots \qquad \dots \qquad \dots \quad (1)$$

Consider now $H=-\dfrac{a}{\rho}.\dfrac{1}{r'}$; H is harmonic within the sphere; for

$$\Delta\left(\frac{1}{r'}\right)=0.$$

Also, on the sphere $H=-\dfrac{a}{\rho}.\dfrac{1}{r_1'}=-\dfrac{1}{r_1}$, in virtue of the relation (1).

Hence $G=H+\dfrac{1}{r}=\dfrac{1}{r}-\dfrac{a}{\rho}.\dfrac{1}{r'}$, is the function of Green required.

Evidently, on the surface $G=\dfrac{1}{r_1}-\dfrac{a}{\rho}.\dfrac{1}{r_1'}=0$, by the relation (1).

Physical Interpretation: The above expression for G in the case of a sphere shows that the function of Green relative to a sphere and to a given point within it is the total potential due to a unit charge of positive electricity placed at the given point and to a charge a/ρ of negative electricity at the conjugate point.

Note.—It could be easily shown that the property G′ (M″) = G″ (M′) in the case of a sphere, leads to Salmon's theorem on poles and polars.

7. *Comparison of the interior problems of Green and of Dirichlet*: The problem of Dirichlet has been given in the article on "The Problem of Dirichlet" quoted above. The following is the corresponding problem of Green:

"Given a volume T bounded by a surface S, calculate the function of Green relative to this volume T and to any one of the points in T."

Let U be the harmonic function in T which it is required to determine, in the problem of Dirichlet, from its values upon the surface S. The value U′ at a point M′ of T is given by the formula

$$4\pi \ U' = \int_S \overline{U}\frac{dG}{dn}d\sigma,$$

where G is the function of Green relative to the volume T and to the point M′, and $\overline{U}$ denotes the given value of U at the element $d\sigma$, and $\frac{d}{dn}$ is to be taken along the interior normal.

[This formula could be proved exactly like the formula $4\pi G'' = \int G''\frac{dG}{dn}d\sigma$, proved in 6°, above.]

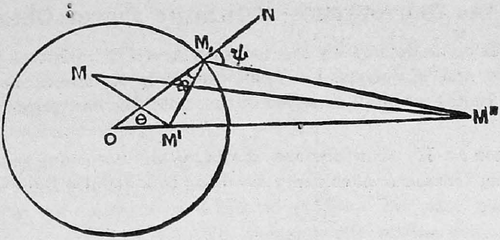
If, therefore, we know how to solve the problem of Green, we can calculate the value U′ of U at each point M′ of T, *i.e.*, we can solve the problem of Dirichlet.

Conversely, if we know how to solve the problem of Dirichlet, we can calculate the function H and consequently the function of Green.

The two problems of Green and of Dirichlet are, therefore, equivalent.

8. *Verification*: For verification take the case of a sphere for which we have solved both the problems independently.

With the notation already employed, we have



$$G=\frac{1}{r}-\frac{a}{\rho}\frac{1}{r'}$$

$$\frac{dG}{dn}=-\frac{1}{r^2}\frac{dr}{dn}+\frac{a}{\rho}\cdot\frac{1}{r'^2}\frac{dr'}{dn}$$

$$\frac{dr}{dn}=-\cos\phi,$$

$$\frac{dr'}{dn}=\cos\psi$$

where $\phi=M'M_1O$, $\psi=M''M_1N$;

$$\therefore \quad \frac{dG}{dn}=\frac{\cos\phi}{r^2}+\frac{a}{\rho}\cdot\frac{\cos\psi}{r'^2}$$

$$=\frac{r\cos\phi}{r^3}+\frac{a}{\rho}\cdot\frac{r'\cos\psi}{r'^3}.$$

But $$r'^3=r^3\cdot\frac{a^3}{\rho^3}$$

$$\therefore \quad \frac{dG}{dn}=\frac{r\cos\phi}{r^3}+\frac{\rho^2}{a^2}\frac{r'\cos\psi}{r^3}.$$

And $r\cos\phi=a-\rho\cos\theta$, where $\theta=M'OM_1$ and $r'\cos\psi=\frac{a^2}{\rho}\cos\theta-a$.

$$\therefore \quad \frac{dG}{dn}=\frac{a-\rho\cos\theta}{r^3}+\frac{\rho^2}{a^2}\cdot\frac{\frac{a^2}{\rho}\cos\theta-a}{r^3}=\frac{a^2-\rho^2}{a\,r^3}.$$

Thus if U is the harmonic function sought in Dirichlet's Problem from its values upon the surface, we have

$$4\pi U=\int\overline{U}\,\frac{dG}{dn}d\sigma$$

$$=\int\frac{a^2-\rho^2}{a\,r^3}\,\overline{U}\,d\sigma$$

$$\therefore \quad U=\frac{1}{4\pi a}\int\frac{a^2-\rho^2}{r^3}\,\overline{U}.\,d\sigma.$$

which we know to be the solution.

SHORT NOTES.

On the Convergence of Infinite Power Chains.*

1. Let a_n be defined by the relation $a_n = a^{a_{n-1}}$, where n is a positive integer and a_0 denotes 1. Then obviously a_n stands for a simple chain of n links; we shall consider in this note the convergence of a_n as $n \to \infty$.

Suppose $a > 1$. It is obvious that a_n is an increasing function of n and hence it increases indefinitely or tends to a definite limit as $n \to \infty$. In the latter case, let $u = f(a)$ be the limit to which the infinite chain tends. Then u satisfies the equation.

$$a^u = u, \text{ or } a = u^{u^{-1}} \qquad \ldots \qquad \ldots \quad (1)$$

It is easily seen the $u^{u^{-1}}$ has a maximum value $k = e^{e^{-1}} = 1{\cdot}444$ nearly. Hence equation (1) has no real root if $a > k$. For such values a_n cannot converge, and so diverges.

If $a = k$, equation (1) gives $u = e$. In this case a_n may converge, in which case it will tend to the value e.

If $a < k$, there are real solutions of (1); and convergence is possible in this case also.

2. We shall now give a formal proof of the convergence of k_n when $n \to \infty$.

We have

$$k_1 = k = e^{e^{-1}};$$

$$k_2 = k^k = e^{e^{-\delta_1}}, \text{ where } \delta_1 = 1 - e^{-1} > 0, < 1;$$

$$k_3 = k^{k_2} = e^{e^{-\delta_2}}, \text{ where } \delta_2 = 1 - e^{-\delta_1} > 0, < 1;$$

and generally

$$k_n = k^{k_{n-1}} = e^{e^{-\delta_{n-1}}}, \text{ where } \delta_{n-1} = 1 - e^{-\delta_{n-2}} > 0 < 1.$$

Now consider the sequence $\delta_1, \delta_2, \delta_3 \ldots\ldots \delta_n$. From the relation

$$\delta_{r+1} = 1 - e^{-\delta_r}$$
$$= \delta_r - \frac{\delta_r^2}{2!} + \frac{\delta_r^3}{3!} - \ldots\ldots,$$

we infer $\delta_{r+1} < \delta_r$, if $\delta_r < 1$; so that $1 > \delta_1 > \delta_2 > \delta_3 \ldots\ldots > 0$. The sequence is therefore a decreasing one, and so either vanishes

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* Read before the Madras Presidency College Mathematical Association, 15-12-1915.

ultimately or tends to a limit $\delta < 1$. In the latter case δ must satisfy the equation

$$\delta = \delta - \frac{\delta^2}{2\,!} + \frac{\delta^3}{3\,!} - \dots\dots$$

the only real root of which is $\delta = 0$.

Hence

$$\underset{n=\infty}{\text{Lt}}\; k_n = e^{e^{-\delta}} = e.$$

Cor. If $1 < a < k$, we have $a_n < k_n$.

$\therefore$ $\text{Lt}\; a_n < \text{Lt}\; k_n < e.$

Hence a_n converges for $a < k$.

4. Suppose $a < 1$,

We have $a < a_n < 1$, or $a_1 < a_2 < 1$.

$\therefore$ $a^{a_1} > a^{a_2} > a$, since $a < 1$.

i.e. $a_2 > a_3 > a_1$.

Similarly $a_3 < a_4 < a_2$;

and $a_4 > a_5 > a_3$.

Thus a_n oscillates, the amplitude gradually decreasing as n increases. Every even a is greater than every odd a. The even a's form a decreasing sequence and the odd a's an increasing one.

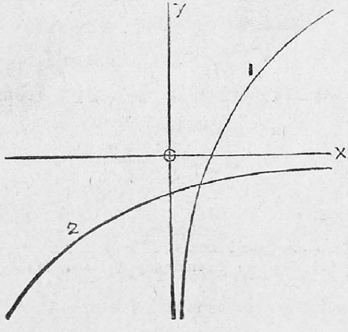
Let the limits of these sequences be λ and μ, respectively; so that λ, μ are real roots of the equation

$$a^{a^x} = x, \text{ or } a^x \log a = \log x,$$

and may be obtained as the points of intersection of the curves:

(1) $y = \log x$, and (2) $y = a^x \log a$, $(a < 1)$.

The graphs of these curves are given below and it is obvious from the figure that there is only one real point of intersection. Hence $\lambda = \mu$, and a_n converges to a definite limit in much the same manner as the convergents of a continued fraction.



5. Let us next consider the general chain $a_1^{a_2^{a_3\cdots}}$ in which the a's are all greater than unity. The chain diverges if $a_r > a > k$, for all values of r beyond a particular number p, say.

For, the part of the infinite chain corresponding to such values is greater than $a^{a^{a\cdots}}$ ...whose divergence has been already proved.

(ii) If $a_r \overline{<} k$, after a certain value of r, we find that the part of the infinite chain beginning with a_r is less than k. Hence the whole chain converges.

A. NARASINGA RAO.

Conical Envelope of a Conicoid.

1. Let $x^2/a^2+y^2/b^2+z^2/c^2=1$, be a conicoid, and (f, g, h) any point T. Also, let (ξ, η, ζ) be a point P on a tangent from T to the conicoid. Then, we have

$$\triangle \text{CPT} : \triangle \text{CDT} = \text{PT} : \text{CD}, \quad \ldots \quad \ldots \quad \ldots \quad \text{(i)}$$

where CD is the semi-diameter parallel to TP.

Also $\text{PT} : \text{CD} = (\xi - f) : al = (\eta - g) : bm = (\zeta - h) : cn$

$$= [(\xi-f)^2/a^2+(\eta-g)^2/b^2+(\zeta-h)^2/c^2]^{\frac{1}{2}}, \quad \ldots \quad \text{(ii)}$$

D being denoted by (al, bm, cn).

From (i) and (ii), we find

$$\triangle \text{CPT} : \triangle \text{CDT} = [(\xi-f)^2/a^2+\ldots]^{\frac{1}{2}} \quad \ldots \quad \ldots \quad \text{(iii)}$$

Now $\quad 2.\triangle \text{CPT} = \text{CP.CT} \sin \text{PCT} = [(\xi g - \eta f)^2 + \ldots]^{\frac{1}{2}}$

$2.\triangle \text{CDT}$ = product of semi-axes of the section CPT

$$= abc/(a^2l'^2+b^2m'^2+c^2n'^2)^{\frac{1}{2}},$$

where $l'x+m'y+n'z=o$ denotes the plane CPT, *viz*:

$$\begin{vmatrix} x & y & z \\ \xi & \eta & z \\ f & g & h \end{vmatrix} = 0.$$

Hence (iii) reduces to

$$[a^2(\xi g-\eta f)^2+\ldots\ldots] = [(\xi-f)^2/a^2+\ldots]a^2b^2c^2$$

$$= [b^2c^2(\xi-f)^2+\ldots\ldots], \quad \ldots \quad \ldots \quad \text{(iv)}$$

which is the equation to the conical envelope.

Cor :—Equation (iv) may be written in the form

$$[(\xi g-\eta f)^2+\ldots\ldots]=a^2[(\xi-f)^2+\ldots\ldots]$$

in the case of a sphere.

2. In the case of a conic $x^2/a^2+y^2/b^2=1$, the equation of the tangents from (f, g) is similarly written

$$(\xi g-\eta f)^2=b^2(\xi-f)^2+a^2(\eta-g)^2,$$

which reduces to

$$(\xi g-\eta f)^2=a^2[(\xi-f)^2+(\eta-g)^2]$$

for a circle.

M. T. NARANIENGAR.

Involution and (1, 1) Correspondence.

1. *Transformation of Point-pairs on a Line* :

For visualising certain relations between points on a line, which we term involution and (1, 1) correspondence, a convenient method may be employed. A line is one-dimensional if we regard it as composed of points, but is two-dimensional if regarded as made up of point-pairs. Thus every point-pair in a line can be placed in (1, 1) correspondence with a point in a plane. This is clearly seen to be possible if we specify the point-pair by two co-ordinates (p, q). It should be remarked however that since we are not supposed to make any distinction between the points composing the pair, (p, q) should be symmetric functions of the co-ordinates of the points. For example, if x_1, x_2, the distances of two points from a fixed point on the line be the roots of the equation

$$x^2+px+q=o,$$

then (p, q) can be taken to be the co-ordinates of the point-pair $(x_1\ x_2)$. (*vide* : Young : *Theory of Sets of Points*, Ch. VIII).

2. *Some Properties* :

Making this transformation of point-pairs in a line to points in a plane, we have

(1) Any continuous one-dimensional series of point-pairs corresponds to a curve.

(2) Any involution corresponds to a straight line. For, since an involution is completely determined by two point-pairs, the corresponding curve must be completely determined by two points.

(3) The locus in the plane of all repeated pairs such as (PP) in the line is a conic which shall be termed the fundamental conic.

For there are two such pairs in every involution. Hence the corresponding curve must cut every straight line in two points.

(4) If two point-pairs have a point in common, the line joining the corresponding points touches the fundamental conic.

For, the double points of the involution determined by two pairs having a common point P, coincide into the single point P. Hence in the plane the line representing the involution touches the fundamental conic.

(5) The point-pair formed by the double points of an involution is represented by the pole of the corresponding line *w. r. t.* the fundamental conic.

For, if the line cuts the conic in (p, q) corresponding to the pairs (PP), (QQ), the pair formed by the double points being (PQ) must, by the last theorem, lie on the tangents at both p and q.

Cor. Two point-pairs separating each other harmonically are represented by conjugate points *w. r. t.* the fundamental conic.

The reciprocal theorem concerning pole and polar is thus seen to follow from the reciprocity of the harmonic relation.

(6) Point-pairs consisting of points in (1, 1) correspondence are represented by a conic having double contact with the fundamental conic.

For, if P is any point, the correspondence will carry P into P_1 and some other point P_2 into P. Thus there are two pairs (PP_1) (PP_2) containing P which shows that any tangent to the fundamental conic cuts the corresponding locus in two points. This locus is therefore a conic.

Since a (1, 1) correspondence has two and only two double pairs, it follows that this conic must have double contact with the fundamental conic in the points corresponding to the double points.

R. Vythynathaswamy.

The Face of the Sky for March and April.

The Sun

enters the first point of Aries on March 21 at 4-30 A.M. and Taurus on April 20 at 3-30 P.M.

The Moon

	March.			*April.*		
	D.	H.	M.	D.	H.	M.
New Moon	4	9	27 A.M.	2	9	51 P.M.
First Quarter ...	12	12	3 A.M.	10	7	65 P.M.
Full Moon ...	19	10	56 P.M.	18	10	37 A.M.
Last Quarter ...	26	9	52 P.M.	25	4	8 A.M.

The Planets.

Mercury attains its greatest elongation (27° 6′ West) on March 2. It is in superior conjunction on April 15 and in conjunction with the Moon on March 2 and April 2, with Jupiter on April 9 and with Uranus on March 5.

Venus continues an evening star. It attains its greatest elongation (45° 33′ East) on April 24. It is in conjunction with the Moon on March 7 at 6-30 P.M. and April 6 at 5-30 P.M. and with Arietis on March 27.

Mars which was in opposition on February 9—one of the most favourable oppositions—becomes stationary on March 21. It is in conjunction with the Moon on March 16 and on April 12 at 7-30 P.M.

Jupiter is in conjunction with the Sun on April 1 at 7-30 P.M., and with the Moon on March 6 and April 3 and April 30.

Saturn is stationary on March 11. It is in quadrature on March 31. It is in conjunction with the Moon on March 13 and on April 9.

Uranus is in conjunction with the Moon on March 2, March 29 and April 26.

Neptune is stationary on April 10. It is in quadrature on April 20. It is in conjunction with the Moon on March 15 and on April 11.

N.B.—The total eclipse of the Sun on February 3 is invisible in India.

V. RAMESAM.

SOLUTIONS.

Question 541.

(S. Ramanujan) :—Prove that

$$1+\frac{1}{1\cdot3}+\frac{1}{1\cdot3\cdot5}+\frac{1}{1\cdot3\cdot5\cdot7}+\ldots+\frac{1}{1+}\,\frac{1}{1+}\,\frac{2}{1+}\,\frac{3}{1+}\,\frac{4}{1+}\,\frac{5}{1+\ldots}=\sqrt{\frac{\pi e}{2}}.$$

Remarks by K. B. Madhava.

We have Prym's identity (Bromwich, p. 9, 3 and 17)

$$\frac{1}{a}+\frac{x}{a(a+1)}+\frac{x^2}{a(a+1)(a+2)}+\frac{x^3}{a(a+1)(a+2)(a+3)}+\ldots\ldots$$

$$=e^x\left[\frac{1}{a}-\frac{x}{1!}\,\frac{1}{a+1}+\frac{x^2}{2!}\,\frac{1}{a+2}-\frac{x^3}{3!}\,\frac{1}{a+3}+\ldots\ldots\right]$$

$$=\frac{e^x}{x^a}\int_0^x e^{-x}x^{a-1}\,dx.$$

Putting $a=x=\frac{1}{2}$, we find

$$2\left\{1+\frac{1}{1\cdot3}+\frac{1}{1\cdot3\cdot5}+\frac{1}{1\cdot3\cdot5\cdot7}+\ldots\infty\right\}=\sqrt{2e}\int_0^{\frac{1}{2}} e^{-x}x^{-\frac{1}{2}}\,dx.$$

$$=2\sqrt{2e}\int_0^{\frac{1}{\sqrt{2}}} e^{-t^2}dt, \text{ putting } x=t^2;$$

so that

$$1+\frac{1}{1\cdot3}+\frac{1}{1\cdot3\cdot5}+\frac{1}{1\cdot3\cdot5\cdot7}+\ldots=\sqrt{2e}\int_0^{\frac{1}{\sqrt{2}}} e^{-t^2}\,dt. \quad \ldots \quad (1)$$

Now consider the continued fraction

$$\frac{1}{1+}\,\frac{1}{1+}\,\frac{2}{1+}\,\frac{3}{1+}\,\frac{4}{1+}\,\frac{5}{1+\ldots\ldots} \qquad (A)$$

which we know to be convergent (Chrystal, p. 525, Ex. 7); this can be transformed in various ways.

The three following methods are given in Oskar Perron "*Die Lehre von den Kettenbruchen.*" (Teubner, 1913).

(i) The first (pp. 294—298) gives at once the desired result of this example.

$$\text{Let } \phi(\alpha,\beta)=\frac{1}{\Gamma(\beta)}\int_0^\infty \frac{e^{-u}\,u^{\beta-1}}{(1+xu)^{\alpha}}du \quad \ldots \quad \ldots \quad (2)$$

where x is real and positive and $\beta>0$; but when $\beta=0$, let $\phi(\alpha,0)=1$. (3)

Integrating (2) by parts

$$\phi(\alpha,\beta)=\phi(\alpha,\beta+1)+\alpha x\phi(\alpha+1,\beta+1),$$

and a set of results follows successively, from which we get (as in Chrystal, Ch. 34, §§ 20—22)

$$\frac{\phi(\alpha, \beta)}{\phi(\alpha, \beta+1)} = 1 + \frac{\alpha x}{1+}\;\frac{(\beta+1)x}{1+}\;\frac{(\alpha+1)x}{1+}\;\frac{(\beta+2)x}{1+\dots} \quad \dots \quad (4)$$

which is true except for $\alpha = 0$ or any negative integer.

Putting $\beta = 0$ and inverting, we have with the aid of (3),

$$\frac{1}{1+}\;\frac{\alpha x}{1+}\;\frac{1\cdot x}{1+}\;\frac{(\alpha+1)x}{1+}\;\frac{2\cdot x}{1+}\dots = \phi(\alpha, 1)$$

$$= \int_0^\infty e^{-u}(1+xu)^{-\alpha}\,du. \qquad (5)$$

Now put $1+xu = xv$ in this last integral and then $x = \frac{1}{z}$ we have;

$$z^{\alpha-1} e^z \int_z^\infty e^{-v} v^{-\alpha}\,dv = \frac{1}{z+}\;\frac{\alpha}{1+}\;\frac{1}{z+}\;\frac{\alpha+1}{1+}\;\frac{2}{z+}\;\frac{\alpha+2}{z+}\;\frac{3}{z+\dots} \qquad (6)$$

wherein the only conditions are $z > o$ and α is real.

This is a useful result and is also given in Legendre: *Fonctions Elliptiques*: Tome ii, Ch. 17.

Now in this first put

$$\alpha = \tfrac{1}{2},\; v = t^2;\; z = \xi^2,$$

and multiply the left hand side of (5) by ξ^2.

We have

$$2\xi\, e^{\xi^2} \int_\xi^\infty e^{-t^2}\,dt = \frac{1}{1+}\;\frac{\frac{1}{2\xi^2}}{1+}\;\frac{\frac{2}{2\xi^2}}{1+}\;\frac{\frac{2}{2\xi^2}}{1+\dots}$$

This result is also given in Laplace, *Celestial Mechanics*, vol. IV, p. 257; and in Jacobi: *Ges Werke*, Bd. VI, pp. 76—78.

If we simply put $\xi^2 = \frac{1}{2}$, we get

$$\frac{1}{1+}\;\frac{1}{1+}\;\frac{2}{1+}\;\frac{3}{1+}\dots\dots = \sqrt{2e}\int_{\sqrt{\frac{1}{2}}}^\infty e^{-t^2}\,dt. \dots \quad \dots \quad (7)$$

Combining (1) and (7) we have

$$1 + \frac{1}{1\cdot 3} + \frac{1}{1\cdot 3\cdot 5} + \dots\dots + \frac{1}{1+}\;\frac{1}{1+}\;\frac{2}{1+}\;\frac{3}{1+}\;\frac{4}{1+}\dots\dots$$

$$= \sqrt{2e}\int_0^\infty e^{-t^2}\,dt = \sqrt{2e}.\,\frac{\sqrt{\pi}}{2} = \sqrt{\frac{\pi e}{2}}.$$

(2) In the second method referred to, M. Perron (pp. 380—392) obtains the same result by taking the corresponding 'modified'

asymptotic series and applies to this the general method for expressing asymptotic series in the form of a continued fraction.

Bromwich (p. 267 and §§ 136—139) has also the same discussion; he shows that the series

$$x-1!\,x^2+2!\,x^3-3!\,x^4+\dots \qquad \dots \qquad \dots \qquad \dots \quad (8)$$

which formally satisfies the differential equation

$$x^2\frac{dy}{dx}+y=x \qquad \dots \qquad \dots \qquad \dots \quad (9)$$

has the the definite integral solution

$$y=\int_0^\infty \frac{xe^{-tx}}{1+xt}\,dt, \qquad \dots \qquad \dots \qquad \dots \quad (10)$$

and the solution in the form of a continued fraction

$$y=\frac{x}{1+}\,\frac{x}{1+}\,\frac{x}{1+}\,\frac{2x}{1+}\,\frac{2x}{1+}\,\frac{3x}{1+}\,\frac{3x}{1+}\dots \qquad \dots \qquad \dots \quad (11)$$

which is the same as (6), with $\alpha=1$ and $z=1$.

(3) The third method of M. Perron (pp. 469—472) is to be deduced from the general result of expressing a function in continued fraction if it satisfies a differential equation: *viz.*,

If $$y=Q_0y'+P_1y'' \qquad \dots \qquad \dots \qquad \dots \quad (12)$$

since the n^{th} differential of this is

$$y^{(n)}=Q_n\,y^{(n+1)}+P_{n+1}\,y^{(n+2)} \qquad \dots \qquad \dots \quad (13)$$

where $$Q_n=\frac{Q_{n-1}+P_n'}{1-Q'_{n-1}},\ \text{and}\ P_{n+1}=\frac{P_n}{1-Q'_{n-1}}, \qquad \dots \qquad \dots \quad (14)$$

we easily see that

$$\frac{y}{y'}=Q_0+\frac{P_1}{Q_1+}\,\frac{P_2}{Q_2+}\,\frac{P_3}{Q_3+}\dots \qquad \dots \qquad \dots \qquad \dots \quad (15)$$

where the P's and Q's are determined as in (14).

If we adopt this method for the function

$$y=\int_0^\infty e^{ux-\frac{1}{2}u^2}\,u^{\alpha-1}\,du \qquad \dots \qquad \dots \quad (16)$$

by partial integration we have

$$y=\frac{1}{\alpha}\int_0^\infty e^{ux-\frac{1}{2}u^2}\,d(u^\alpha)$$

$$=-\frac{1}{\alpha}\int_0^\infty e^{ux-\frac{1}{2}u^2}(x-u)\,u^\alpha\,du$$

$$=-\frac{x}{\alpha}y'+\frac{1}{\alpha}y''; \qquad \dots \qquad \dots \qquad \dots \quad (17)$$

and hence differentiating (17)

$$y' = -\frac{1}{\alpha}y' - \frac{x}{\alpha}y'' + \frac{1}{\alpha}y'',$$

i.e.

$$y' = \frac{-x}{\alpha+1}y'' + \frac{1}{\alpha+1}y''' :$$

or generally

$$y^{(n)} = \frac{-x\,y^{(n+1)}}{\alpha+n} + \frac{1}{\alpha+n}y^{(n+2)}. \quad \cdots \quad (18)$$

Therefore from the general formula (15)

$$\frac{y}{y'} = \frac{-x}{\alpha} + \cfrac{\frac{1}{\alpha}}{\cfrac{-x}{\alpha+1} + \cfrac{\frac{1}{\alpha+1}}{\cfrac{-x}{\alpha+2} + \cdots\cdots}}$$

which can be transformed into

$$\frac{-x}{\alpha -}\;\frac{1+1/\alpha}{-\alpha +}\;\frac{\alpha+2}{-x+}\;\frac{\alpha+3}{-x+} \quad \cdots \quad \cdots \quad \cdots \quad (19)$$

by multiplying by $\alpha+1$ after the second term.

If in this we put $x=-\xi$ and multiply by α, we have

$$\xi + \frac{\alpha+1}{\xi+}\;\frac{\alpha+2}{\xi+}\;\frac{\alpha+3}{\xi+}\cdots = \alpha\,\frac{\int_0^\infty e^{-u\xi-\frac{1}{2}u^2}u^{\alpha-1}\,du}{\int_0^\infty e^{-u\xi-\frac{1}{2}u^2}u^{\alpha}\,du} \quad \cdots \quad (20)$$

If in this again, we put $\alpha=\xi=1$, we fall back on (A)

$$1 + \frac{2}{1+}\;\frac{3}{1+}\;\frac{4}{1+\ldots} = \frac{\int_0^\infty e^{-u-\frac{1}{2}u^2}\,du}{\int_0^\infty e^{-u-\frac{1}{2}u^2}u\,du} \quad \cdots \quad \cdots \quad (21)$$

which are familiar integrals.

Question 593.

(S. Narayana Aiyangar, M.A.) :—Shew that

$$\frac{\frac{1}{a} + \frac{x}{a+1} + \frac{x^2}{1\cdot 2}\cdot\frac{1}{a+2} + \ldots\ldots \text{to } \infty}{\frac{1}{a+1} + \frac{x}{a+2} + \frac{x^2}{1\cdot 2}\,\frac{1}{a+3} + \ldots\ldots \text{to } \infty}$$

is equal to

$$\frac{\frac{1}{a} - \frac{x}{a(a+1)} + \frac{x^2}{a(a+1)(a+2)} - \ldots \text{to } \infty}{\frac{1}{1+a} - \frac{x}{(a+1)(a+2)} + \frac{x^2}{(a+1)(a+2)(a+3)} - \ldots \text{to } \infty}.$$

Additional Solution by C. Krishnamachary.

This can be easily solved by the application of the method of finite differences. We have the equality of operators

$$E=1+\Delta.$$

Hence

$$1+xE+\frac{x^2E^2}{2!}+\ldots=1+x(1+\Delta)+\frac{x^2}{2!}(1^2+\Delta)^2+\ldots$$

$$=(1+x+\frac{x^2}{2!}+\ldots)\ (1+x\Delta+x^2\frac{\Delta^2}{2!}+\ldots);$$

since if $E\ (x)=1+x+\frac{x^2}{2\,!}+\cdots$,

$$E(x)\times E(y)=E(x+y)]\ ;$$

$$=e^x(1+x\Delta+\frac{x^2}{2!}.\Delta^2+\ldots).$$

Operate upon the function $\frac{1}{a}$. Then

$$(1+xE+\frac{x^2}{2!}E^2+\ldots)\frac{1}{a}=\frac{1}{a}+xE\ \left(\frac{1}{a}\right)+\frac{x^2}{2!}E^2\left(\frac{1}{a}\right)+\ldots$$

$$=\frac{1}{a}+\frac{x}{a+1}+\frac{x^2}{2\,!}\cdot\frac{1}{a+2}+\frac{x^3}{3\,!}\cdot\frac{1}{a+3}+\ldots$$

and

$$e^x(1+x\Delta+\frac{x^2}{2\,!}.\Delta^2+\ldots)\frac{1}{a}=e^x.\left(\frac{1}{a}+x.\Delta\left(\frac{1}{a}\right)+\frac{x^2}{2!}.\Delta^2\left(\frac{1}{a}\right)+\ldots\right)$$

$$=e^x\left(\frac{1}{a}-\frac{x}{a(a+1)}+\frac{x^2}{a(a+1)(a+2)}-\ldots\right);$$

since $\Delta\ \left(\frac{1}{a}\right)=(E-1)\frac{1}{a}=\frac{1}{a+1}-\frac{1}{a}=-\frac{1}{a(a+1)}$.

$$\Delta^2\left(\frac{1}{a}\right)=\Delta\left(\Delta\ \left(\frac{1}{a}\right)\right)=\frac{\lfloor 2}{a(a+1)(a+2)};\ \text{and so on}].$$

Hence we have the equality of the two series

$$\frac{1}{a}+\frac{x}{a+1}+\frac{x^2}{2!}\cdot\frac{1}{a+2}+\frac{x^3}{3!}\cdot\frac{1}{a+3}+\ldots$$

$$=e^x\left\{\frac{1}{a}-\frac{x}{a(a+1)}+\frac{x^2}{a(a(+1)(a+2)}+\ldots\right\}\qquad\ldots\quad(1)$$

Put $a+1$ for a, and we similarly have

$$\frac{1}{a+1}+\frac{x}{a+2}+\frac{x^2}{2!}\cdot\frac{1}{a+3}+\ldots$$

$$=e^x\left\{\frac{a+1}{1}-\frac{x}{(a+1)(a+2)}+\frac{x^2}{(a+1)(a+2)(a+3)}\right\}\qquad\ldots\quad(2)$$

Hence from (1) and (2) we have

$$\frac{\frac{1}{a}+\frac{x}{a+1}+\frac{x^2}{2!}\cdot\frac{1}{a+2}+\ldots}{\frac{1}{a+1}+\frac{x}{a+2}+\frac{x^2}{2!}\cdot\frac{1}{a+3}+\ldots}=\frac{\frac{1}{a}-\frac{x}{a(a+1)}+\ldots}{\frac{1}{a+1}-\frac{x}{(a+1)(a+2)}+\ldots}.$$

Question 617

(S. Narayana Aiyar, M.A.) :—If $F(\alpha, \beta, \gamma, x)$ denote the hypergeometric series $1+\frac{\alpha.\beta}{\gamma.1}x+\frac{\alpha(\alpha+1).\beta(\beta+1)}{\gamma(\gamma+1).\quad 1.2}x^2+\ldots$, show that

$$\sum_{r=\infty}^{r=0}\left\{\frac{P(b)P(\gamma)P(a-b+1)P(\alpha+r)P(\beta+r)}{P(\alpha)P(\beta)P(b+r)P(\gamma+r)P(a-b-r+1)}\cdot\frac{x^r}{P(r+1)}.F(\alpha',\beta',\gamma',x)\right\}$$

$$=1+\frac{a.\alpha.\beta}{b.\gamma.1}x+\frac{a(a+1).\alpha(\alpha+1).\beta(\beta+1)}{b(b+1).\gamma(\gamma+1).\quad 1.2}x^2+\ldots$$

where P stands for Γ, and

$$\alpha'=\alpha+r\,;\ \beta'=\beta+r\,;\ \gamma'=\gamma+r.$$

Examine the case when $\alpha=\gamma$.

Solution by K. B. Madhava.

By comparing the co-efficients of x^n, we have to show that

$$\frac{P(\alpha+n)P(\beta+n)P(a+n)P(b)P(\gamma)}{P(\alpha)P(\beta)P(a)P(b+n)P(\gamma+n)P(n+1)}$$

is equal to the series

$$\sum_{r=0}^{n}\frac{P(\alpha+n)P(\beta+n)P(\gamma+r)}{P(\alpha+r)P(\beta+r)P(\gamma+n)P(n-r+1)}\times$$

$$\frac{P(b)P(\gamma)P(a-b+1)P(\alpha+r)P(\beta+r)}{P(\alpha)P(\beta)P(b+r)P(\gamma+r)P(a-b-r+1)P(r+1)}$$

i.e. $$\sum_{r=0}^{n}\frac{P(\alpha+n)P(\beta+n)P(b)P(\gamma)P(a-b+1)}{P(\gamma+n)P(\alpha)P(\beta)P(b+r)P(a-b-r+1)P(r+1)P(n-r+1)};$$

hence it is required to show that

$$\frac{P(a+n)}{P(a)P(b+n)P(n+1)}=\sum_{0}^{n}\frac{P(a-b+1)}{P(b+r)P(a-b-r+1).P(r+1)P(n-r+1)}.$$

Multiplying both sides by $P(b)P(n+1)$, we have to show

$$1+\frac{(a-b)}{b}\cdot\frac{n}{1}+\frac{(a-b)(a-b-1)}{b(b+1)}\cdot\frac{n(n-1)}{1.2}+\ldots\text{up to }(n+1)\text{ terms}$$

$$=\frac{(a+n-1)(a+n-2)\ldots(a+1)a}{(b+n-1)(b+n-2)\ldots(b+1)b};$$

and this has been done in the solution to Q. 616 (J. I. M. S., Vol. VII, p. 195); putting $b-a=k$ and $b=l$, we get, as required to be shown,

$$1-\frac{k}{l}\cdot\frac{n}{1}+\frac{k(k+1)}{l(l+1)}\cdot\frac{n(n-1)}{1.2}-\ldots \text{ up to } (n+1) \text{ terms}$$

$$=\frac{(l-k)(l-k+1)\ldots(l-k+n-1)}{l(l+1)\ldots(l+n-1)}.$$

When $\alpha=\gamma$, the right-hand side of the given series reduces to $F(a,\beta,b,x)$, while in the left-hand side $F(\alpha', \beta', \gamma', x)$ reduces to

$$(1-x)^{-\beta-r};$$

and therefere we get

$$F(a,\beta, b, x) = \sum_0^\infty \left\{ \frac{P(b)P(a-b+1)P(\beta+r)}{P(\beta)P(b+r)P(a-b-r+1)} \frac{x^r}{P(r+1)}(1-x)^{-\beta-r} \right\}$$

which, after some manipulations, is seen to be

$$=(1-x)^{-\beta}.F\left(\beta, b-a, b, \frac{x}{x-1}\right)$$

being one of the 23 other forms in which F can be expressed.

Question 627.

(K. V. Anantanarayana Sastri, B.A.) :—Four spheres of radii a,b,c,d intersect at right angles. Show that the volume of the tetrahedron formed by their centres is

$$\frac{1}{6}\, a\, b\, c\, d\, (a^{-2}+b^{-2}+c^{-2}+d^{-2})^{\frac{1}{2}}.$$

Additional Solutions (1) *by K. B. Madhava and* (2) *by R. Vythynathaswam y*

(1) Take for the equations of the four spheres

$$x^2+y^2+z^2-a^2=0,$$

and $\quad x^2+y^2+z^2+2x_rx+2y_ry+2z_rz-k_r=0 \quad (r=1, 2, 3).$

By the conditions of the problem

$$x_r^2+y_r^2+z_r^2-k_r=b^2, c^2 \text{ or } d^2 \text{ according as } r \text{ is } 1, 2, 3.$$

and $$k_r=a^2$$

Let $\quad (\lambda\mu)\equiv x_\lambda x_\mu+y_\lambda y_\mu+z_\lambda z_\mu=\tfrac{1}{2}\left(k_\lambda+k_\mu\right)=a^2,$

where λ is any of the quantities 1, 2, 3 and μ either of the other. The volume of the tetrahedron formed by the centres

$$=\frac{1}{6}\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} = \frac{1}{6}\begin{vmatrix} \Sigma x_1^2 & (12) & (13) \\ (12) & \Sigma x_2^2 & (23) \\ (13) & (23) & \Sigma x_3^2 \end{vmatrix}^{\frac{1}{2}}$$

$$=\frac{1}{6}\begin{vmatrix} a^2+b^2 & a^2 & a^2 \\ a^2 & a+c^2 & a^2 \\ a^2 & a^2 & a^2+d^2 \end{vmatrix}^{\frac{1}{2}} = \tfrac{1}{6}a^3 \begin{vmatrix} 1+b^2/a^2 & 1 & 1 \\ 1 & 1+c^2/a^2 & 1 \\ 1 & 1 & 1+d^2/a^2 \end{vmatrix}^{\frac{1}{2}}$$

$$=\frac{1}{6}b\,c\,d\left(1+\frac{a^2}{b^2}+\frac{a^2}{c^2}+\frac{a^2}{d^2}\right)^{\frac{1}{2}}$$

$$=\frac{1}{6}\,a\,b\,c\,d\left(a^{-2}+b^{-2}+c^{-2}+d^{-2}\right)^{\frac{1}{2}}.$$ (Burnside and Panton, p. 299, Ex. 20)

This is set as an exercise in Aldis's *Solid Geometry*, p. 137.

(2) The volume of a tetrahedron in terms of the edges is given by

$$144V^2=\begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & (12)^2 & (13)^2 & (14)^2 \\ 1 & (21)^2 & 0 & (23)^2 & (24)^2 \\ 1 & (31)^2 & (32)^2 & 0 & (34)^8 \\ 1 & (41)^2 & (42)^2 & (43)^2 & 0 \end{vmatrix}$$

This is got at once by multiplying the determinants

$$\begin{vmatrix} 1 & 0 & 0 & 0 & 0 \\ x_1^2+y_1^2+z_1^2 & x_1 & y_1 & z_1 & 1 \\ x_2^2+y_2^2+z_2^2 & x_2 & y_2 & z_2 & 1 \\ x_3^2+y_3^2+x_3^2 & x_3 & y_3 & z_3 & 1 \\ x_4^2+y_4^2+z_4^2 & x_4 & y^4 & z_4 & 1 \end{vmatrix} \text{ and } \begin{vmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & -2x_1 & -2y_1 & -2z_1 & x_1^2+y_1^2+z_1^2 \\ 1 & -2x_2 & -2y_2 & -2z_2 & x_2^2+y_2^2+z_2^2 \\ 1 & -2x_3 & -2y_3 & -2z_3 & x_3^2+y_3^2+z_3^2 \\ 1 & -2x_4 & -2y_4 & -2x_4 & u_4^2+y_4^2+z_4^2 \end{vmatrix}$$

Putting in the above expression $(12)^2=a^2+b^2, (13)^2=a^2+c^2$, etc., the resulting determinant $=8\Sigma(b^2c^2d^2)$. (See Question 657.)

Hence $V=\frac{1}{6}abcd({}^{-1}a^{-2}+b^{-2}+c^{-2}+d^{-2})^{\frac{1}{2}}$.

Question 629.

(S. Ramanujan) :—Prove that :

$$\frac{\frac{1}{2}+\sum_{n=1}^{\infty} e^{-\pi n^2 x}\cos(\pi n^2\sqrt{1-x^2})}{\sum_{n=1}^{\infty} e^{-\pi n^2 x}\sin(\pi n^2\sqrt{1-x^2})}=\frac{\sqrt{2}+\sqrt{1+x}}{\sqrt{1-x}}$$

and deduce the following :

$$(a)\quad \frac{\frac{1}{2}+\sum_{n=1}^{\infty} e^{-\pi n^2}}{\frac{1}{2}+\sum_{n=1}^{\infty} e^{-5\pi n^2}}=\sqrt{(5\sqrt{5}-10)}.$$

$$(b)\quad \sum_{1}^{\infty} e^{-\pi n^2}\left(\pi n^2-\frac{1}{4}\right)=\frac{1}{8}.$$

Solution (1) *by K. B. Madhava,* (2) *by N. Durai Rajan,* (3) *by M. Bhimasena Rao.*

(1) The denominator and the second term of the numerator in the first problem are respectively given by the imaginary and the real parts of the integral

$$\int_0^{\infty} e^{-\pi(x-i\sqrt{1-x^2})t^2}\,dt$$

the value of which $=\dfrac{\sqrt{\pi}}{2}\cdot\dfrac{1}{\sqrt{\pi(x-i\sqrt{1-x^2})}}=\dfrac{1}{2}\cdot\dfrac{1}{\sqrt{(x-i\sqrt{1-x^2})}}.$

To separate the real and imaginary parts of this, put $x=\cos\psi$; and it is easy to see that the expression is equal to

$$\frac{\frac{1}{2}+\frac{1}{2}\sqrt{\dfrac{1+x}{2}}}{\frac{1}{2}\sqrt{\dfrac{1-x}{2}}}=\frac{\sqrt{2}+\sqrt{1+x}}{\sqrt{1-x}}.$$

Now, for (*b*) $\displaystyle\sum_{1}^{\infty} \pi n^2 e^{-\pi n^2}=\pi\int_0^{\infty} t^2 e^{-\pi t}\,dt$

$$=\frac{1}{2\sqrt{\pi}}\int_0^{\infty} e^{-t}.t^{\frac{3}{2}}dt=\frac{1}{4}$$

$$-\frac{1}{4}\sum_{8}^{\infty} e^{-\pi n^2}=-\tfrac{1}{4}\int_0^{\infty} e^{-\pi t^2}\,dt=-\frac{1}{4}\cdot\frac{1}{\sqrt{\pi}}\cdot\frac{\sqrt{\pi}}{2}=-\frac{1}{8};$$

whence result (*b*) follows.

(2) Let

$$C=1+2\sum_{n=1}^{n=\infty} e^{-\pi n^2 x}\cos\left(\pi n^2\sqrt{1-x^2}\right)$$

$$S=\quad 2\sum e^{-\pi n^2 x}\sin\left(\pi n^2\sqrt{1-x^2}\right).$$

Substitute $x=\cos\Theta$;

$$C+iS=1+2\sum e^{-\pi n^2\cos\Theta}e^{\pi n^2 . i\sin\Theta}$$

$$=1+2\sum e^{-\pi n^2 e^{-i\Theta}} \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

$$C-iS=1+2\sum e^{-\pi n^2 e^{-\Theta}} \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

Let $$q=e^{-\pi\frac{K'}{K}}=e^{-\pi e^{i\Theta}}.$$

$$\therefore \quad \frac{K'}{K}=e^{i\Theta}=\frac{\psi(\sqrt{1-k})}{\psi(k)} \text{ say.}$$

$$\therefore \quad e^{-i\Theta}=\frac{\psi(k)}{\psi(\sqrt{1-k^2})}=\frac{\psi(\sqrt{1-k'^2})}{\psi(k')}, \text{ where } k^2+k'^2=1.$$

Let $$q_1=e^{-\pi\frac{\Delta'}{\Delta}}=e^{-\pi e^{-i\Theta}}.$$

$$\therefore \quad e^{-i\Theta}=\frac{\Delta'}{\Delta}.$$

We see that $\Delta'=K$ and $\Delta=K'$.

Now

$$C-iS=1+2q+2q^4+2q^9+\cdots$$

$$=\sqrt{\frac{K}{\frac{1}{2}\pi}}. \quad \text{(See, Greenhill's \textit{Elliptic Fns})}$$

and $$C+iS=\sqrt{\frac{\Delta}{\frac{1}{2}\pi}}.$$

$$\therefore \quad \left(\frac{C+iS}{C+iS'}\right)^2=\frac{\Delta}{K}=\frac{K'}{K}=e^{i\Theta}.$$

$$\therefore \quad \frac{C+iS}{C+iS}=e^{\frac{1}{2}\Theta i}=\cos\tfrac{1}{2}\Theta+i\sin\tfrac{1}{2}\Theta.$$

Multiplying and equating real and imaginary parts, we get

$$(1+\cos\tfrac{1}{2}\Theta)S=C\sin\tfrac{1}{2}\Theta;\ C(1-\cos\tfrac{1}{2}\Theta)=S.\sin\tfrac{1}{2}\Theta.$$

$$\therefore C.\sin\frac{\Theta}{4}=S\cos.\frac{\Theta}{4}$$

$$\therefore \quad \frac{C}{S}=\frac{\cos\frac{\Theta}{4}}{\sin\frac{\Theta}{4}}=\cot\frac{\Theta}{4}.$$

Replacing $x=\cos\Theta$, we get the expression given in the question.

$$(b) \quad \sum_{n=1}^{n=\infty} e^{-\pi n^2}\left(\pi n^2-\frac{1}{4}\right)=\tfrac{1}{8}.$$

Let S be equal to the left hand side and, $e^{-\pi}=q$,

$$\text{so that } \frac{K'}{K}=1 \text{ and } k=k'=\frac{1}{\sqrt{2}}.$$

$$\therefore\ S=\Sigma q^{n^2}(\pi n^2-\tfrac{1}{4})$$

$$=\pi(q+4q^4+9q^9+16\,q^{16}+\ldots)-\tfrac{1}{4}(q+q^4+q^9+\ldots)$$

Now $\qquad 1+2q+2q^4+\ldots=\sqrt{\frac{K}{\frac{1}{2}\pi}}.$

Differentiating with respect to q and remembering that

$$\log q=-\pi K'/K$$

$$2+8\,q^3+18\,q^8+\ldots=\tfrac{1}{2}\left(\frac{1}{\pi K}\right)^{\frac{1}{2}}\frac{dK}{dq}$$

$$\therefore\ 2(q+4q^4+9q^9+\ldots)=\tfrac{1}{2}\left(\frac{1}{\frac{1}{2}\pi K}\right)^{\frac{1}{2}}\frac{q\,dK}{dq}.)$$

Hence S = Limit of

$$\pi.\tfrac{1}{4}\left(\frac{1}{\frac{1}{2}\pi K}\right)^{\frac{1}{2}}.\ q.\ \frac{dK}{dq}-\frac{1}{4}\left(\tfrac{1}{2}\sqrt{\frac{K}{\frac{1}{2}\pi}}-\tfrac{1}{2}\right)$$

$$=\frac{1}{8}+\text{Lt}\frac{1}{4}\sqrt{\frac{1}{\frac{1}{2}\pi}}\left[\pi.K^{-\frac{1}{2}}.\ q\frac{dK}{dq}-\tfrac{1}{2}K^{\frac{1}{2}}\right]$$

∴ The expression within the square brackets is seen to be zero,

(3) We know that $\int_0^\infty e^{-t^2}2\cos 2\,xt\,dt=\sqrt{\pi}e^{-x^2}.$

$$\therefore\ \int_0^\infty e^{-t^2}\left(1+2\cos 2\,xt+2\cos 4\,xt+\ldots\ldots+2\cos 2nxt\right)dt$$

$$=\sqrt{\pi}\left\{\frac{1}{2}+\sum_1^n e^{-n^2x^2}\right\}.$$

$$\text{The left hand side}=\int_0^\infty e^{-t^2}.\frac{\sin(2n+1)xt}{\sin xt}dt$$

$$=\int e^{-t^2/x^2}.\frac{\sin(2n+1)t}{\sin t}\frac{dt}{x}, \text{ by putting } t \text{ for } xt.$$

Let n become infinite, the value of the integral is

$$\frac{\pi}{x}\left\{\tfrac{1}{2}+e^{-\pi^2/x^2}+e^{-4\pi^2/x^2}+\ldots\ldots\right\},$$ *Bromwhich, Infinite Series.* Ex 4, p. 447,

$$\therefore \sqrt{\pi}\left\{\tfrac{1}{2}+\sum_{1}^{\infty} e^{-n^2x^2}\right\}=\frac{\pi}{x}\left\{\tfrac{1}{2}+\sum_{1}^{\infty} e^{n^2\pi^2/x^2}\right\};$$

writing πx for x^2, we get

$$\tfrac{1}{2}+\sum_{1}^{\infty} e^{-n^2\pi x}=\frac{1}{\sqrt{x}}\left\{+\sum_{1}^{\infty} e^{-n^2\pi/x}\right\}. \quad \dots \quad \dots \quad (1)$$

Both sides are equal when $x=1$.

Differentiating with respect to x, we have,

$$\sum_{1}^{\infty}- e^{-n^2\pi x}.\, n^2\pi=-\frac{1}{2x^{\frac{3}{2}}}\left\{\tfrac{1}{2}+\sum_{1}^{\infty} e^{-n^2\pi/x}\right\}$$

$$+\frac{1}{\sqrt{x}}\left\{\sum_{1}^{\infty} e^{-n^2\pi/x}.\, \frac{n^2\pi}{x^2}\right\}.$$

Putting $x=1$

$$\sum_{1}^{\infty} e^{-n^2\pi}(2n^2\pi-\tfrac{1}{2})=\tfrac{1}{4},$$

$$\sum_{1}^{\infty} e^{-n^2\pi}\ (n^2\pi-\tfrac{1}{4})=\tfrac{1}{8}.$$

In (1) write $r\ (\cos\theta+i\sin\theta)$ for x and we see that

$$\tfrac{1}{2}+\sum_{1}^{\infty} e^{-n^2\pi r\cos\theta}.\cos\,(n^2\pi r\sin\theta)$$

$$=\frac{\cos\frac{\theta}{2}}{2\sqrt{r}}+\frac{1}{\sqrt{r}}\mathrm{R}\left\{e^{-i\frac{\theta}{2}}\sum_{1}^{\infty} e^{-n^2\frac{\pi}{r}(\cos\theta-i\sin\theta)}\right\}$$

where R (x) denotes the real part of x;

$$=\frac{\cos\frac{\theta}{2}}{2\sqrt{r}}+\frac{1}{\sqrt{r}}\mathrm{R}\sum_{1}^{\infty} e^{-n^2\frac{\pi}{r}\cos\theta+i\left(n^2\frac{\pi}{r}\sin\theta-\frac{\theta}{2}\right)}$$

$$=\frac{\cos\frac{\theta}{2}}{2\sqrt{r}}+\frac{1}{\sqrt{r}}\sum_{1}^{\infty} e^{-n^2\frac{\pi\cos\theta}{r}}.\cos\left(n^2\frac{\pi}{r}\sin\theta-\frac{\theta}{2}\right) \quad \dots \quad (2)$$

Put $r=1$, we obtain

$$\tfrac{1}{2}+\sum_{1}^{\infty} e^{-n\pi\cos\theta}\cos(n^2\pi\sin\theta)$$

$$=\cos\frac{\theta}{2}\left\{\tfrac{1}{2}+\sum_{1}^{\infty} e^{-n^2\pi\cos\theta}.\cos(n^2\pi\sin\theta)\right\}.$$

$$+\sin\frac{\theta}{2}\sum_{1}^{\infty} e^{-n^2\pi\cos\theta}\sin(n^2\pi\sin\theta).$$

Writing x for $\cos\theta$, $\sqrt{1-x^2}$ for $\sin\theta$, and simplifying we get

$$\frac{\tfrac{1}{2}+\sum_{1}^{\infty} e^{-n^2\pi x}\cos(n^2\pi x\sqrt{1-x^2})}{\sum_{1}^{\infty} e^{-n\pi^3 x}\sin(n^2\pi\sqrt{1-x^2})}=\frac{\sqrt{2}+\sqrt{1+x}}{\sqrt{1-x}}.$$

(a) Let $\frac{\cos\theta}{r}=1$, $\frac{\sin\theta}{r}=2$; then

$$r=\frac{1}{\sqrt{5}},\ r\cos\theta=\frac{1}{5},\ r\sin\theta=\frac{2}{5};$$

$$\tan\theta=2,\ \cos\frac{\theta}{2}=\frac{1}{5^{\frac{1}{4}}}\sqrt{\frac{\sqrt{5}+1}{2}};$$

making these substitutions in (2), we have

$$\tfrac{1}{2}+\sum_{1}^{\infty} e^{-n^2\frac{\pi}{5}}\cos\left(2n^2\frac{\pi}{5}\right)=5^{\frac{1}{4}}\cos\frac{\theta}{2}\left\{\tfrac{1}{2}+\sum_{1}^{\infty} e^{-n^2\pi}\right\};$$

when n is not a multiple of 5, $\cos\left(2n^2\frac{\pi}{5}\right)=\cos\frac{n\pi}{5}$; when n is a multiple of 5, $\cos\left(2n^2\frac{\pi}{5}\right)=1$.

$$\therefore\ \tfrac{1}{2}+\sum_{1}^{\infty} e^{-\frac{n^2\pi}{5}}\cos\left(2n^2\frac{\pi}{5}\right)$$

$$=\tfrac{1}{2}+\sum_{1}^{\infty} e^{-n^2\frac{\pi}{5}}\cos\frac{2\pi}{5}+\sum_{1}^{\infty} e^{5n^2\pi}\left(1-\cos\frac{2\pi}{5}\right).$$

By (1) $\frac{1}{2}+\sum_{1}^{\infty} e^{n^2\pi/5} = \sqrt{5}\left\{\frac{1}{2}+\sum_{1}^{\infty} e-\frac{n^2\pi}{5}\right\}$

$$\therefore \frac{1}{2}+\sum_{1}^{\infty} e^{-\frac{n^2\pi}{5}} \cos\left(2n^2\frac{\pi}{5}\right)$$

$$=\frac{1}{2}\left(1-\cos\frac{2\pi}{5}\right)+\sqrt{5}\cos\frac{2\pi}{5}\left\{\frac{1}{2}+\sum_{1}^{\infty} e^{-5n^2\pi}\right\}$$

$$+\sum_{1}^{\infty} e^{-5n^2\pi}\left(1-\cos\frac{2\pi}{5}\right)$$

$$=\left(1-\cos\frac{2\pi}{5}+\sqrt{5}.\cos\frac{2\pi}{5}\right)\left\{\frac{1}{2}+\sum_{1}^{\infty} e^{-5n^2\pi}\right\}$$

$$\therefore \frac{\frac{1}{2}+\sum_{1}^{\infty} e^{-\pi n^2}}{\frac{1}{2}+\sum_{1}^{\infty} e^{-5\pi n^2}} = \frac{1+\cos\frac{2\pi}{5}\left(\sqrt{5}-1\right)}{5^{\frac{1}{4}}\cos\frac{\theta}{2}} = \frac{1+\frac{(\sqrt{5}-1)^2}{4}}{\sqrt{\frac{\sqrt{5}+1}{2}}}$$

$$=\frac{(10-2\sqrt{5})}{4}\sqrt{\frac{\sqrt{5}-1}{2}}=\sqrt{5}.\left(\frac{\sqrt{5}-1}{2}\right)^{\frac{3}{2}}=\sqrt{5}\sqrt{\sqrt{5}-2}$$

$$=\sqrt{5\sqrt{5}-10}.$$

(b) $\frac{1}{2}+\sum_{1}^{\infty} e^{-\pi n^2 x}\cos\left(\pi n^2\sqrt{1-x^2}\right)$

$$=\frac{\sqrt{2}+\sqrt{1+x}}{\sqrt{1-x}}\sum_{1}^{\infty} e^{-\pi n^2 x}\sin\left(\pi n^2\sqrt{1-x^2}\right).$$

Since $\underset{x=1}{\text{Lt}}\ \frac{\sin\left(\pi n^2\sqrt{1-x^2}\right)}{\sqrt{1-x}} = \underset{x=1}{\text{Lt}}\sqrt{1+x}\ \frac{\sin\left(\pi n^2\sqrt{1-x^2}\right)}{\sqrt{1-x^2}} = \sqrt{2}.\ \pi n^2$;

we get, by putting $x=1$,

$$\frac{1}{2}+\sum_{1}^{\infty} e^{-\pi n^2} = 4\pi\sum_{1}^{\infty} e^{-\pi n^2}.n^2$$

Hence $$\sum_{1}^{\infty} e^{-\pi n^2}\left(\pi n^2-\frac{1}{4}\right)=\frac{1}{8},$$

a result already obtained directly by differentiating **(1)**.

Question 665.

(R. N. Apte, M. A., F. R. A. S.) :—Find the value of

$$\iint xyz\left\{1+\overline{\frac{\partial z}{\partial x}}\Big|^2+\overline{\frac{\partial z}{\partial y}}\Big|^2\right\}^{\frac{1}{2}}dxdy \text{ where } z=c\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)^{\frac{1}{2}} \text{ and}$$

the integration is over the positive quadrant of $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

Solution by K. B. Madhava and H. V. Venkataramiengar.

Since $\frac{dz}{dx}=-\frac{c^2x}{a^2z}+\frac{dz}{dy}=-\frac{c^2y}{b^2z}$ the given integral

$$=\iint cxy\sqrt{1+\frac{c^2-a^2}{a^4}x^2+\frac{c^2-b^2}{b^4}y^2}\,dxdy,$$

where the integration extends to all positive values of $x+y$ subject to the condition $\frac{x^2}{a^2}+\frac{y^2}{b^2}<1$. Put $\xi=x^2/a^2$, $\eta=y^2/b^2$;

$$\mathrm{I}=c.\frac{a^2b^2}{4}\iint(1+\frac{c^2-a^2}{a^2}\xi+\frac{c^2-b^2}{b^2}\eta)^{\frac{1}{2}}d\xi d\eta \text{ where } \xi+\eta<1$$

$$=\frac{2}{3}\cdot\frac{c.a^2b^2}{4}\cdot\frac{a^2}{c^2-a^2}\int_0^1\left\{\frac{c^2}{a^2}-c^2\left(\frac{1}{a^2}-\frac{1}{b^2}\right)\eta\right\}^{\frac{3}{2}}-\left\{1+\frac{c^2-b^2}{b^2}\eta\right\}^{\frac{1}{2}}d\eta$$

$$=\frac{c}{15}\cdot\frac{1}{(a^2-b^2)(b^2-c^2)(c^2-a^2)}[a^2(c^5-b^5)(a^2-b^2)-c^2(c^2-b^2)(a^5-b^5)].$$

Question 666.

(S. Ramanujan) :—Solve in positive rational numbers $x^y=y^x$.

For example : $x=4$, $y=2$; $x=3\frac{3}{8}$, $y=2\frac{1}{4}$.

Solution by J. C. Swaminarayan, M.A., and R. Vythynathaswamy.

Put $x=ky$. Then $y^k=ky$.

$$\therefore \qquad y^{k-1}=k.$$

The solution will be a rational number only if k is of the form $1+\frac{1}{n}$.

$$\therefore \qquad y=\left(1+\frac{1}{n}\right)^n \text{ and } x=\left(1+\frac{1}{n}\right)^{n+1}.$$

When $n=1$, $x=4$, $y=2$; when $n=2$, $x=\frac{27}{8}$, $y=\frac{9}{4}$ when $n=3$,

$$x=\frac{256}{81}, y=\frac{64}{27}.$$

Thus infinite solutions in positive rational integers will be obtained corresponding to all the integral values of n.

Question 667.

(S. NARAYANA AIYAR, M.A.) :—Establish the truth of the following theorems:

1. If $F(y)=\phi(y)-\frac{x}{1!}\,\phi(y+1)+\frac{x^2}{2!}\phi(y+2)-\frac{x^3}{3!}\phi(y+3)+\ldots$

then $\phi(y)=F(y)+\frac{x}{1!}F(y+1)+\frac{x^2}{2!}F(y+2)+\frac{x^3}{3!}F(y+3)+\ldots$

2. If $F(y)=\phi(y)-\frac{n}{1}\phi(y+1)+\frac{n(n+1)}{1\cdot 2}\phi(y+2)-\ldots$

then $\phi(y)=F(y)+\frac{n}{1}F(y+1)+\frac{n(n-1)}{1\cdot 2}F(y+2)+\ldots$

Solution (1) *by J. C. Swaminarayan, M.A., and R. Srinivasan M.A.*; (2) *by K.B. Madhava.*

(1) Let E be an operator which when applied to a function of y changes y into $y+1$.

$$\text{Then } F(y)=\left\{1-\frac{xE}{1!}+\frac{x^2E^2}{2!}-\frac{x^3E^3}{3!}+-\right\}\phi(y)$$

$$=e^{-xE}\phi(y)$$

$$\therefore \quad \phi(y)=e^{xE}F(y)$$

$$=\left\{1+\frac{xE}{1!}+\frac{x^2E^2}{2!}+\frac{x^3E^3}{3!}+\ldots\ldots\right\}F(y)$$

$$=F(y)+\frac{x}{1!}F(y+1)+\frac{x^2}{2!}F(y+2)+\frac{x^3}{3!}F(y+3)+\ldots$$

$$\text{Also } F(y)=\left\{1-\frac{nE}{1}+\frac{n(n+1)E^2}{1\cdot 2}-\ldots\ldots\right\}\phi(y)$$

$$=(1+E)^{-n}\phi(y).$$

$$\therefore \quad \phi(y)=(1+E)^nF(y)$$

$$=\left\{1+\frac{n}{1}E+\frac{n(n+1)}{1\cdot 2}E^2+\ldots\right\}F(y)$$

$$=F(y)+\frac{n}{1}F(y+1)+\frac{n(n-1)}{1\cdot 2}F(y+2)+\ldots\ldots$$

(2) Substituting in the right hand side expressions for F (y) F $(y+1)$ etc., in terms of ϕ (y), ϕ $(y+1)$...in the first part, we have for the co-efficient of ϕ $(y+r)$

$$\frac{x^r}{r!}\left\{1-{}_rC_1+{}_rC_2-\ldots 1\right\}=0$$

except in the case of the first term, which is unity, whence the identity.

Substituting similarly, in the second part the co-efficient of $\phi\ (y+r)$

$$\frac{n(n+1)\ldots(n+r-1)}{r\,!}-\frac{n}{1}\cdot\frac{n(n+1)\ldots(n+r-2)}{(r-1)\,!}$$
$$+\frac{n(n-1)}{1\cdot 2}\frac{n(n+1)\ldots(n+r-3)}{(r-2)\,!}-\ldots$$

which is the co-efficient of x^n in the product of $(1-x)^{-n}$ and $(1-x)^n$ and is zero.

The co-efficient of $\phi\ (y)$ however is unity, ; hence the result.

Question 677.

(D. Krishnamurti) :—A chain of lenth l and mass M is coiled at the edge of a table. A particle of mass m is fastened to one end of the chain and the other end is gently let slip over the edge of the table. Show that the velocity of the particle immediately after it leaves the table is $k\sqrt{[2\,gl\,(1+k^2)/3]}$, where $k=\mathrm{M}/(m+\mathrm{M})$.

Solution by K. B. Madhava and Martyn M. Thomas.

A particular case of this where $\mathrm{M}=m$ is given in Loney, p. 133 Ex. 11. [See also Routh, § 150.]

From Newton's Second Law as applied to motion where the moving mass varies, we have

$$\frac{d}{dt}\left(\frac{\mathrm{M}}{l}x\dot{x}\right)=g.\frac{\mathrm{M}}{l}.x. \qquad \ldots \qquad \ldots\ (1)$$

To integrate this multiply by $\frac{\mathrm{M}}{l}x\dot{x}$, and then we have

$$\frac{1}{2}\cdot\left(\frac{\mathrm{M}}{l}x\dot{x}\right)^2=g\cdot\frac{\mathrm{M}^2}{l^2}\cdot\frac{x^3}{3} \qquad \ldots \qquad \ldots\ (2)$$

$$\therefore \qquad \dot{x}^2=\frac{2g}{3}l. \qquad \ldots \qquad \ldots\ (3)$$

Calling this velocity u, we have by the principle of conservation of momentum, for the horizontal and vertical velocities

$$(\mathrm{M}+m)v=\mathrm{M}u \text{ and } (\mathrm{M}+m)w=\mathrm{M}v \qquad \ldots \qquad \ldots\ (4)$$

i.e. $\qquad v=ku$ and $w=k^2u \qquad \ldots \qquad \ldots \qquad \ldots\ (5)$

The sq. of the vel. when the particle has just left the table is

$$v^2+w^2=k^2(1+k^2)u^2=\tfrac{2}{3}gl(1+k^2) \text{ from (2).}$$

Question 683.

(GANPATRAM R. JANI) :—If S_r is put for $1^r+2^r+3^r+\ldots n^r$, prove that

$$\begin{vmatrix} S_0 & S_1 & \ldots & S_{n-1} \\ S_1 & S_2 & & S_n \\ \ldots & \ldots & & \\ S_{n-1} & S_n & & S_{2n-2} \end{vmatrix} = \left\{ \frac{(n\,!)^n}{1^1 2^2 3^3 \ldots n^n} \right\}^2$$

Solution by K. B. Madhava, A. Narasinga Rao, K. R. Rama Aiyar and Martyn M. Thomas.

This is just a particular case of the example due to Sylvester solved on p. 229 of Boole's *Finite Differences*, (1880) and the given determinant is, as shown there, seen to be equivalent to

$$\begin{vmatrix} 1 & 1 & 1 & \ldots & 1 \\ 1 & 2 & 3 & & n \\ 1^2 & 2^2 & 3^2 & & n^2 \\ 1^n & 2^n & 3^n & & n^n \end{vmatrix}^2$$

and this is equal to the product of the squares of the differences $1, 2, \ldots n$ taken with the proper sign.

Hence $$D = -\left\{ \frac{(n\,!)^n}{1^1 2^2 3^3 \ldots n^n} \right\}^2$$

It is also shown in the reference (*loc. cit.*) that

$$\begin{vmatrix} S_x & S_{x+1} & \ldots & S_{x+n-1} \\ S_{x+1} & S_{x+2} & \ldots & S_{x+n-2} \\ S_{y+n-1} & S_{x+n} & \ldots & S_{x+n} \end{vmatrix} = (n\,!)^x D$$

Question 686.

(S. NARAYANA AIYAR, M.A.) :—Establish the following identities :—

(i) $$\frac{a(a+1)(a+2)\ldots(a+n-1)}{b(b+1)(b+2)\ldots(b+n-1)} = 1 + {}_nC_1.\frac{a-b}{b} + {}_nC_2.\frac{(a-b)(a-b-1)}{b(b+1)} + \ldots + \frac{(a-b)(a-b-1)\ldots(a-b-n+1)}{b(b+1)\quad\ldots\quad(b+n-1)};$$

(ii) $$\frac{(a-b)(a-b-1)\ldots(a-b-n+1)}{b(b+1)\quad\ldots\quad(b+n+1)} = (-1)^n \left\{ 1 - {}_nC_1.\frac{a}{b} + {}_nC_2\frac{a(a+1)}{b(b+1)} - \ldots (-1)^n \frac{a(a+1)\ldots(a+n-1)}{b(b+1)\ldots(b+n-1)} \right\}$$

(iii) $F(a,b,c,x)=(1-x)^{-a} F\left(a,\ c-b,\ c, \frac{x}{x-1}\right)$, where F stands for the hypergeometric series,

(iv) (v) and (vi).

Remarks by K. B. Madhava and R. Srinivasan, M.A.

(i) is proved in the solution of Q. (616) ;

(ii) change a into $b-a$ in (i) and the result follows ;

(iii) is proved in the solution to Q. (616) ;

(iv) if in Q. (616) ϕ (Q) = log $(1+x)$, the result follows ;

(v) if in Q. (616) $\phi(x)=\sin x$ and $x=\pi$, the result follows ;

(vi) if in Q. (616) $\phi(n)=$ los x and $x-\pi$, the result follows ;

In connection with (iii) the following algebraic proof extracted from the *Messenger of Mathematics*, Vol. XLIV, No. 8, December 1914, may be of interest. The proof is by Prof. M. J. M. Hill, University College, London :

The righthand side is $(1-x)^{-a} F\left(a, c-b, c, \frac{x}{x-1}\right)$

$$=(1-x)^{-a}+\frac{a(c-b)}{1.c}(-x)(1-x)^{-a-1}$$

$$+\frac{a_2.(c-b)_2}{2.c_2}(-x)^2(1-x)^{-a-2}+\ldots$$

$$+\frac{a_n(c-b)_n}{n!\,c_n}(-x)^n(2-x)^{-a-n}.$$

Hence the coeff. of x^n is

$$\frac{a_n}{n!}+\frac{a.c-b}{1.c}.(-)\frac{(a+1)_{n-1}}{(n-1)!}+\frac{a_2.(c-b)_2}{2!\,c_2}.(-)^2.\quad\frac{(a+2)_{n-2}}{n-2!}$$

$$\ldots(-)^n\frac{a_n(c-b)_n}{n!\,c_n}$$

$$=\frac{a_n}{n!\,c_n}\{\,c_n-nc_1.\,(c-b)(c+1)_{n-1}+nc_2(c-b)_2\,(c+2)_{n-2}$$

$$+\ldots(-)^n(c-b)_n\,\}$$

The $(r+1)^{th}$ term in brackets is

$(-)^r nc_r(c-b)_r(c+r)_{n-r}$

$$=(-1)^r nc_r(c-b)(c-b-1)\ldots(c-b+r-1)\times$$

$$(c+r)(c+r+1)\ldots(c+n-1)$$

$$=nc_r(b-c)(b-c-1)\ldots(b-c-r+1)\times(c+n-1)(c+n-2)\ldots(c+r).$$

If we now adopt, as in the statement of Vandermonde's theorem, an abbreviation for $a(a-1)\ldots(a-r+1)$, say $\overline{a}_r$, so that V's theorem can be written :

$$\overline{(a+b)}_n = \sum_{r=0}^{n} nc_r \overline{a}_r \overline{b}_{n-r}$$

we have for the $(r+1)^{th}$ term in brackets the expression

$$nc_r \overline{(b-c)}_r \overline{(c+n-1)}_{n-r}$$

and therefore by Vandermonde's theorem, the whole expression in brackets

$$=\overline{(b-c+c+n-1)}_n$$

$$=\overline{(b+n-1)}_n$$

$$=(b+n-1)(b+n-2)\ldots(b+1)b.$$

$=b_n$, in the notation of this paper, where throughout we have adopted for abbreviation c_n for $c(c+1)(c+2)\ldots(c+n-1)$,

and therefore the coeft. of x^n is $\dfrac{a_n b_n}{n!\, c_n}$ which is the coeft. of x^n in $F(a,b,c,x)$.

Thus the identity is demonstrated.

Question 696.

(S. Krishnasawmi Aiyangar) :—If λ, μ be the latera-recta of the parabola and rectangular hyperbola of closest contact with a curve at any point, prove that $2\lambda\rho=\mu^2$.

Solution by R. Srinivasan, M.A., Martyn M. Thomas and K. B. Madhava.

With the usual notation, in a parabola

$$\frac{SP}{SY}=\frac{SY}{SA}=\sec\phi.$$

$$\therefore \quad SP=SA\sec^2\phi.$$

$$\therefore \quad \rho=2a\sec^3\phi. \quad (SA=a).$$

Also in a rectangular hyperbola,

$$\rho=\frac{CP^2}{CY},\ \frac{CP}{CY}=\sec\phi,\ CD\,.\,CY=\alpha^2,$$

where α is the semi-axis (or semi latus-rectum.)

$$\therefore \quad \rho=\alpha\sec^{\frac{3}{2}}\phi.$$

$$\therefore \quad \lambda=2\rho\cos^3\phi \text{ and } \mu=2\rho\cos^{\frac{3}{2}}\phi.$$

$$\therefore \quad 2\lambda\rho=\mu^2.$$

Question 703.

(N. SANKARA AIYAR, M.A.) :—If AP is the symmedian through A, prove that

$$\Sigma \{ (b^2+c^2)(\text{AK. KP}) \} = 3a^2b^2c^2/(a^2+b^2+c^2).$$

Solution by R. Srinivasan, M.A., K. B. Madhava, R. D. Karve and L. N. Subramanian.

Let the distance of K from BC be x, and the attitude AD be p.

Now $\qquad \text{BP} : \text{PC} :: c^2 : b^2.$

$\therefore \qquad b^2\,\text{AB}^2 + c^2.\text{AC}^2 = b^2.\text{BP}^2 + c^2\text{CP}^2 + (b^2+c^2)\text{AP}^2.$

i.e. $\qquad \text{AP}^2 = \dfrac{b^2c^2(2b^2+2c^2-a^2)}{(b^2+c^2)^2}. \qquad \dots \qquad \dots \text{(i)}$

Again $\qquad \dfrac{\text{KP}}{\text{AP}} = \dfrac{x}{p} = \dfrac{2a\triangle/(a^2+b^2+c^2)}{2\triangle/a}$

$$= \frac{a^2}{a^2+b^2+c^2}.$$

and $\qquad \dfrac{\text{AK}}{\text{AP}} = \dfrac{b^2+c^2}{a^2+b^2+c^2}.$

$\therefore \qquad \text{AK}\cdot\text{KP} = \dfrac{a^2(b^2+c^2)}{(a^2+b^2+c^2)^2}\,\text{AP}^2.$

$\therefore \qquad (b^2+c^2)\ \text{AK.KP} = \dfrac{a^2b^2c^2}{(a^2+b^2+c^2)^2}(2b^2+2c^2-a^2).$

$\therefore \qquad \Sigma \{ (b^2+c^2)\ \text{AK}\cdot\text{KP} \} = \dfrac{3\,a^2b^2c^2}{a^2+b^2+c^2}.$

Question 704.

(S. MALHARI RAO, B.A.) :—If the sum of a number of 3 digits and the number formed by reversing the digits be divisible by 37, the sum of all such pairs of numbers is 480×37.

Solution by S. V. Venkatarayasastri, M.A., L.T. and R. D. Karve.

If a, b, c be the 3 digits, the sum of the number and the number formed by reversing the digits $= 101\ (a+c) + 20b$.

If this is divisible by 37, the values of b and $a+c$ are the 9 pairs 1, 2 ; 2, 4 ; 3, 6 ;......9, 18.

Taking the first set of values 1, 2 for b, $a+c$, we get 210 and 111. As these do not give us a pair of 2 numbers satisfying the given conditions, we reject them.

Taking the set 2, 4, we get 2 numbers, omitting numbers 420 and 222 for the above reason. Their sum $= 222 \times 2$.

Taking the set 3, 6, we get 4 numbers or 2 pairs. Their sum $= 333 \times 4$.

And so on.

The sum of all such pairs $= 222 \times 2 + 333 \times 4 + 444 \times 4 + \dots + 888 \times 2$
$= 480 \times 37.$

QUESTIONS FOR SOLUTION.

725. (K. B. MADHAVA) :—Show that

$$\int^{\infty} \frac{x^8 dx}{1+x^9 \sin^2 x} \text{ converges,}$$

but that

$$\int^{\infty} \frac{x^8 dx}{1+x^8 \sin^2 x} \text{ diverges.}$$

726. (K. B. MADHAVA) :—Shew that

$$\sum_{n=1}^{\infty} \left\{ \frac{\Gamma(n)}{\Gamma(n+4)} \right\}^2 = \frac{1}{6^3} (20\pi^2 - 197) ;$$

and that

$$\sum_{1}^{\infty} \left\{ \frac{\Gamma(n)}{\Gamma(n+5)} \right\}^2 = \frac{1}{12^4 . 2^2} (1680\pi^2 - 16575).$$

727. (Communicated by MR. P. V. SESHU AIYAR) :—Three persons A, B, C had a, b, c mangoes respectively. A gave B and C $\frac{b(b+1)}{2}$, and $\frac{c(c+1)}{2}$ mangoes respectively out of what he had ; then B gave C and A similarly ; and finally C gave A and B similarly. It was found that A, B, and C had the same number of mangoes ultimately. How many had each at first ?

728. (K. APPUKUTTAN ERADY, M.A.) :—If $u \equiv (abcfgh)(xyz)^2$, show that

$$\iiint u^n \, dx \, dy \, dz$$

taken throughout the space bounded by $u=1$, is

$$\frac{4\pi}{2n+3} \Delta^{-\frac{1}{2}}$$

where Δ is the discriminant of u.

729. (K. PADMANABHULU, B.A.) :—If the earth were to break up into an indefinite number of fragments at any point in its course round the sun by any sudden explosion, prove that all the fragments meet again at the same point ; and that at the middle of the interval between the explosion and junction all the pieces will be moving with equal velocities in parallel directions.

730. (S. KRISHNASWAMI AIYANGAR) :—Shew that the locus of the orthopoles of tangents to the maximum inscribed ellipse of a triangle is a straight line through the ortho-centre.

731. (S. KRISHNASWAMI AIYANGAR) :—In a spherical triangle prove that

$$\begin{vmatrix} \sin a & \sin b & \sin c \\ \operatorname{cosec} a & \operatorname{cosec} b & \operatorname{cosec} c \\ \operatorname{cosec}^2 a \operatorname{cosec} A & \operatorname{cosec}^2 b \cos B & \operatorname{cosec}^2 c \operatorname{cosec} C \end{vmatrix}$$

is equal to

$$\operatorname{cosec}^2 a \operatorname{cosec}^2 b \operatorname{cosec}^2 c (\cos a - \cos b)(\cos b - \cos c)(\cos c - \cos a) \times (1 - \cos a - \cos b - \cos c).$$

What is the corresponding formula in a plane.

732. (R. VYTHYNATHASWAMY) :—There are two kinds of elements α, β, so related that n elements of either kind determine uniquely an element of the other kind. Prove that the aggregate of each must be n-dimensional. Further, supposing

β_r to be the α-element determined by $(\alpha_{r1}, \alpha_{r2}, \ldots \alpha_{rn})$ $(r=1,2\ldots n+1)$

$\beta_{p,r}$ the element determined by $(\alpha_{p1}, \alpha_{p2} \ldots \alpha_{pr-1}, \alpha_{pr+1} \ldots; \alpha_{pn+1})$ $(p=1,2\ldots n)$

α_r the element determined by $(\beta_{1,r}, \beta_{2,r} \ldots \beta_{n,r})$ $(r=1,2\ldots n+1)$; shew that if the $(n+1)$ elements β_r determine the same α-element, then the $(n+1)$ elements α_r determine the same β-element and conversely.

733. (R. VYTHYNATHASWAMY) :—Required a definition of continuity which does not involve the idea of number ; also a definition or explanation and not pre-supposing parameters of 'dimension' as applied to a dense aggregate of similar elements.

Does the idea of dimension involve the idea of continuity or even that of being dense?

Is the aggregate of all numbers, real and complex, to be regarded as one-dimensional or two-dimensional ?

734. (J. C. SWAMINARAYAN, M.A.) :—Solve the differential equation

$$\left(y^2 - b^2\right)\frac{dy}{dx} + xy = \sqrt{b^2x^2 + a^2y^2 - a^2b^2}.$$

735. (SELECTED) :—A, B, C, D, E represent the entire circumferences of a curve and its successive pedals. If C pertains to an ellipse having its centre at the pedal origin, show that

$$B(B+D) = (2C-E)(3C-A).$$ (M.A. 1903, Madras.)

736. (R. SRINIVASAN, M.A.) :—Shew that the common tangent to the nine-point and inscribed circles of a triangle ABC cuts the sides a, b, c in the ratios

$$\frac{a-b}{a-c}, \frac{b-c}{b-a}, \frac{c-a}{c-b}.$$

737. (E. H. NEVILLE) :—A continuous function of position f which is nowhere negative is defined for every point of a closed plane curve C which does not extend to infinity ; m is a positive number, and B_m is the region of the plane containing every point whose distance from some point of C is less than m times the value of f for that point of C (in other words, a circle is described round each point of C with radius mf, and a point belongs to B_m if it is in the interior of one of the circles) ; shew that if C has everywhere a definite tangent and a finite curvature and if m is sufficiently small, the area A_m of the region B_m is

$$2m\int \{ \cos \alpha - \tfrac{1}{2}\, m^2 f(d^2f/ds^2) \sec \alpha \} f ds,$$

where α is the angle between $-\frac{1}{2}\pi$ and $\frac{1}{2}\pi$ whose sine is $m\, df/ds$, and the integral is taken along C. Find a corresponding expression valid if C has end-points, and further shew that if C is composed of any finite number of distinct arcs and has cusps, multiple points, and vertices, then provided that these singular points and the end-points are finite in number the quotient $A_m/2m$ tends to $\int f ds$ as m tends to zero.

738. (S. RAMANUJAN) :—If

$$\phi(x) = e^{-x} + \frac{x}{1!} e^{-2x} + \frac{3x^2}{2!} e^{-3x} + \frac{4^2 x^3}{3!} e^{-4x} + \frac{5^3 x^4}{4!} e^{-5x} + \ldots,$$

show that $\phi(x) = 1$ when x lies between 0 and 1 ; and $\phi(x) \neq 1$ when $x > 1$. Find the limit of

$$\frac{\phi(1+E) - \phi(1)}{E}$$

as $E \to 0$ by positive values.

739. (S. RAMANUJAN) :—Show that

$$\int_0^\infty e^{-nx} (\cot x + \coth x) \sin nx\, dx = \frac{\pi}{2} \left(\frac{1 + e^{-\pi n}}{1 - e^{-\pi n}} \right)^{(-1)^n}$$

for all positive integral values of n.

740. (S. RAMANUJAN) :—If

$$\phi(x) = \left\{ \frac{e^x \lfloor [x] ^2}{x^{[x]}} \right\} - 2\pi x,$$

where $[x]$ denotes the greatest integer in x, show that $\phi(x)$ is a continuous function of x for all positive values of x and oscillates from $\frac{\pi}{3}$ to $-\frac{\pi}{6}$ when x becomes infinite. Also differentiate $\phi(x)$.

Statement of Accounts of the Indian Mathematical Society for the year 1915.

Receipts.	Rs.	A.	P.	*Expenditure.*	Rs.	A.	P.
To opening balance	508	10	2	By books and Journals	332	10	2
,, subscriptions from members	2,294	7	0	,, Library	293	0	0
,, ,, for Journal	245	4	0	,, printing Journal	477	3	6
,, miscellaneous receipts	41	1	0	,, ordinary working expenses	202	12	8
				,, closing balance*	1,783	11	10
	3,089	6	2		3,089	6	2

* Fixed deposit with the

	Rs.	A.	P.
Madras Central Urban Bank	800	0	0
Treasurer	839	11	7
Secretary	52	9	0
Asst. Secretary	55	3	3
Asst. Librarian	36	4	0
	1,783	11	10

I have examined the Treasurer's books and vouchers and the monthly statements submitted by the Secretary, Asst. Secretary and Asst. Librarian and declare the above accounts to be correct.

C. N. GANAPATHI AIYAR.

MADRAS,
5th February 1916.

S. NARAYANA AIYAR,
Hony. Treasurer.