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# PROGRESS REPORT.

The committee feel pleasure in announcing that Mr. T. K. Venkataraman, M.A. has got himself elected as a Life-member of our Society.

- 2. The following gentlemen have been elected members of our society:—
- Mr. Srivam Venkatasubba Setti, B.A., D.F.H., A.M.I.C.E.— Superintendent Mechanical Engineering School and Professor, College of Engineering, Bangalore;
- 2. Mr. K. Ananda Rao, M.A., (Smith's Prizeman).-6, Singara-chari Street, Triplicane, Madras;
- 3. Mr.A.V.Subba Rau Esq. M.A., L.T.—Mathematics Assistant, Taluk High School, Tenali;
- Mr. Carcherla Srinivasa Rau.—Clerk, Collector's Office, 1/14, Prospect House, Vepery, Madras;
- 3. As announced along with the last issue of our journal, a conference of our society will be held in Bombay, on or about 9th, 10th and 11th of January 1919, and consequently it is very earnestly requested that members will try to attend the Conference and to make it a success as on the last occasion. Also those of the members who are desirous to read papers at the Conference, should communicate their intention and submit synopsis of their papers for the inspection of the sub-committee appointed for the purpose by the 1st of December 1918.

POONA, 5th Oct. 1918. D. D. KAPADIA,

Honorary Joint Secretary.

# A Note in Combinatory Analysis.

BY R. VYTHYNATHASWAMY.

§ 1. The following lemma will be of use in the course of the present paper.

Lemma:

The number of ways of placing  $\lambda$  distinct things in the first of two rows of r vacant places and  $\mu$  distinct things in the second, where  $\lambda \Rightarrow r$ , and  $\mu \Rightarrow r$ , and each of the  $\lambda$  as well as each of the  $\mu$  things should be used at least once, so as to satisfy the condition that if  $k_1$ ,  $l_1$ , two members of the first row, contain the same  $\lambda$ -thing,  $k_2$ ,  $l_2$ , the corresponding members in the second row should not contain the same  $\mu$ —thin  $l_1$ , is the coefficient of  $l_2$  in

$$\left[ \triangle_1^{\lambda} \triangle_2 \mu (1+x)^{t_1 \cdot t_2} \right]_{t_1 = t_2 = 0}$$

where the operators  $\triangle_1$  and  $\triangle_2$  refer to  $t_1$  and  $t_2$  respectively.

Proof. Let the  $\lambda$  things be  $\alpha_1, \alpha_2, ... \alpha_{\lambda}$  and the  $\mu$  things

 $\beta_1$ ,  $\beta_2$ ,... $\beta_k$ .

Fill the first row with  $k_1$ ,  $\alpha_1$ 's,  $k_2$ ,  $\alpha_2$ 's...k,  $\alpha_k$ 's where  $k_n$  (n=1,2,...k)

is a positive non-zero integer and  $\Sigma k_n = r$ .

The corresponding number of ways of filling the second row will then be

$$\mu^{\mathbf{C}_{k_1}} \cdot \mu^{\mathbf{C}_{k_2}} \cdot \mu^{\mathbf{C}_{k_3}} \cdots \mu^{\mathbf{C}_{k_k}}$$

Hence the total number of ways would be

$$\sum_{k} \mu^{C} k_{1} \mu^{C} k_{2} \cdots \mu^{C} k_{N} \text{ where } \Sigma k_{n} = r \text{ and } k_{n} = 0$$

and this is the coefficient of  $x^r$  in  $[(1+x)^{\mu}-1]^{\lambda}$ .

If now the condition that each of the  $\beta$ s should occur once at least be added, the total number of ways is

east be added, the total number of ways in 
$$[(1+x)^{\mu}-1]^{\lambda} - \mu^{c_1}[(1+x)^{\mu}-1-1]^{\lambda} + \mu^{c_2}[(1+x)^{\mu}-2-1]^{\lambda} - \dots$$

which is easily seen to be a symmetric function of  $\lambda$  and  $\mu$  and equal to

$$\left[ \Delta_1^{\lambda} \Delta_2^{\mu} (1+x)^{t_1 t_2} \right] t_1 = t_2 = 0.$$

This we shall write also as  $\Delta_1^{\lambda} \Delta_2 \mu (1+x)$ .

We may also adopt a different type of argument which is perfectly general.

If there are  $\lambda$  distinct things of one kind and  $\mu$  distinct things of another kind, then the number of ways of forming r distinct combina-

tions of a  $\lambda$  and a  $\mu$  thing is evidently  $\lambda \mu^C r$ , and these correspond to the number of ways of arranging the  $\lambda$  and  $\mu$  things in two rows of r spaces each, with two dissimilar  $\lambda$ —things always opposite to two similar  $\mu$ —things, without, however, the assurance that each  $\mu$ —thing or each  $\lambda$ —thing occurs at least once. Therefore, if we add the condition that each  $\lambda$ —thing is to occur at least once, the number of ways is

$$\lambda \mu^{\mathbf{C}_r} - \lambda^{\mathbf{C}_1} (\lambda - 1) \mu^{\mathbf{C}_r} + \lambda^{\mathbf{C}_2^*} (\lambda - 2) \mu^{\mathbf{C}_r} - \dots = \Delta^{\lambda_0} \mu^{\mathbf{C}_r}$$

(cf. Whitworth: Choice and Chance Adding the further condition that each  $\mu$ -thing should occur also once at least, we see that the total number of ways is

$$\triangle$$
,  $\lambda$   $\triangle$ ,  $\mu_{0.0}$   $C_r$ 

which agrees with the previous result.

Corollary. More generally, we shall also have :

The number of ways of placing  $\lambda_1, \lambda_2, \ldots \lambda_n$  things respectively in n rows of r spaces each, so that each thing occurs at least once, and no two columns are identical, is equal to

 $\triangle_1^{\lambda_1} \triangle_2^{\lambda_2} ... \triangle_n^{\lambda_n} 0 0 0 ... O_n$ 

§ 2. We can now find the number of ways in which a set of r different factors of N can be chosen so as to have the greatest common measure unity.

We will denote this by  $\phi_r$  (N) and seek to express  $\phi_r$  (pq) where p and q are mutually prime in terms of  $\phi_k$  (p) and  $\phi_k$  (q). If p and q are mutually prime, every factor of pq is uniquely resoluble into a factor of p and a factor of q. Thus every factor is specified by a factor of p and a factor of q. To specify r distinct factors of pq, then we have to take  $\lambda$  factors of p and  $\mu$  factors of q where, of course, neither  $\lambda$  nor  $\mu$  is to be greater than r, and fill with them the two rows of r vacant places satisfying the conditions of the Lemma; that is

$$\begin{split} \phi_r(pq) &= \text{coefficient of } x^r \text{ in } \Sigma \Sigma \phi_{\chi} \left( p \right) \phi_{\mu} \left( q \right) \triangle_1^{\chi} \triangle_2^{\mu} \left( 1 + x \right)^{0.0} \\ &= \Sigma \Sigma \phi_{\chi} \left( p \right) \phi_{\mu} \left( q \right) \triangle_1^{\chi} \triangle_2^{\mu} 0.0^{C_r} \end{split}$$

Similarly, if  $a_1, a_2, ...a_n$  are mutually prime, we shall have  $\phi_r(a_1a_2...a_n) = \Sigma \phi_{\lambda_1}(a_1) \phi_{\lambda_2}(a_2)...\phi_{\lambda_n}(a_n) \triangle_1^{\lambda_1} \triangle_2^{\lambda_2}...\triangle_n^{\lambda_n} 000 \_C_r$   $[\lambda_k \not > r \ k = 1, 2...n.]$ 

We have also the result, that if p is a prime

$$\phi_r(p^{\alpha}) = \alpha^{C} r - 1;$$

and therefore, that if  $N = p_1^{\alpha_1} p_2^{\alpha_2} ... p_n^{\alpha_n}$ 

$$\begin{aligned} & \phi_r(\mathbf{N}) = \Sigma \phi_{\lambda_1}(p_1^{\alpha_1}) \dots \phi_{\lambda_n}(p_n^{\alpha_n}) \dots \Delta_1^{\lambda_1} \dots \Delta_n^{\lambda_n} 000 \dots \mathbf{C}_r \\ & = \Sigma \alpha_1^{\mathbf{C}} \sum_{\lambda_1 = 1}^{\mathbf{C}} \dots \alpha_n^{\mathbf{C}} \sum_{\lambda_n = 1}^{\lambda_1} \dots \sum_{n=0}^{\lambda_n} 000^{\mathbf{C}_r} \\ & = (1 + \Delta_1)^{\alpha_1} \dots (1 + \Delta_n)^{\alpha_n} \Delta_1 \Delta_2 \dots \Delta_n 000^{\mathbf{C}_r} \\ & = \Delta_1 \Delta_2 \dots \Delta_n \left[ \alpha_1 \alpha_2 \dots \alpha_n^{\mathbf{C}_r} \right]. \end{aligned}$$

Thus if N = 120 and r = 2,  $\phi_r(N) = 17$ .

 $\S$  3. If  $\varphi_r'(N)$  be defined as the number of sets of r factors of N (not including N itself) which can be found so as to have the greatest common measure unity, we evidently have

 $\phi_r' + \phi'_{r-1} = \phi_r$  (for values of r up to 2)

and we shall also suppose  $\phi_1' = \phi_1 = 1$ . We then have

$$\begin{aligned} & \phi'_r = \varphi_r - \varphi_{r-1} + \varphi_{r-2} - & \dots \\ & = \triangle_1 \triangle_2 \dots \triangle_n [\alpha_1 \alpha_2 \dots \alpha_n C_r - \alpha_1 \alpha_2 \dots \alpha_n C_{r-1} + \dots] \\ & = \triangle_1 \triangle_2 \dots \triangle_n [\alpha_1 \alpha_2 \dots \alpha_n - 1^{C_r}] \end{aligned}$$

We therefore have

from which by writing  $E_r-1$  for  $\triangle_r$  and expanding, we see that the terms in which the index is not zero will vanish. If any one of the as is different from unity, the index of none of the terms in the expanded series vanishes. Hence when N contains repeated prime factors the series identically vanishes. If on the other hand all the as are unity, there is one term for which the index is zero;  $viz (-)^{n+1}(1-1)^o = (-)^{n-1}$ , and therefore, if N contains n prime factors none being repeated the series equals  $(-)^{n-1}$ .

§ 4. We may notice that if  $\psi_r(N)$  denotes the number of sets of refactors of N which can be chosen so as to have the least common multiple N, then  $\psi_r(N) = \psi_r(N)$ . For if we divide N by a set of factors having the least common multiple N, we get a set of factors having the greatest common factor unity, Similarly also we find  $\psi_r(N) = \psi_r(N)$  where  $\psi_r'(N)$  is the same as  $\psi_r(N)$  with unity excluded.

This method of reasoning may be employed to obtain two familiar results.

Let  $t_i, t_i$ ... be the factors of N, N being excluded. If  $\phi(N)$  be the number of factors of N we have,

$$\phi(\mathbf{N}) - 1 = \Sigma \phi(t_1) - \Sigma \phi(t_{12}) + \Sigma \phi(t_{128})...$$

where  $t_{12-r}$  is the G. C. M. of  $t_1, t_2, ...t_r$  and is thus itself one of the ts

that is, 
$$\phi(\mathbf{N}) = 1 + \Sigma \phi(\mathbf{t}_k) \left[ 1 - \phi_s' \left( \frac{\mathbf{N}}{\mathbf{t}_k} \right) + \phi_s' \left( \frac{\mathbf{N}}{\mathbf{t}_k} \right) - \dots \right]$$
  
=  $1 + \Sigma \phi \left( \frac{\mathbf{N}}{n_s} \right) - \Sigma \phi \left( \frac{\mathbf{N}}{n_s n_s} \right) + \dots$ 

which is a familiar result. (Cf. Chrystal: Algebra, Vol. II.)

Similarly, the numbers less than N and not prime to it will be multiples of one or more of the factors  $t_1, t_2...t_n$  and will therefore be  $\sum_{i=1}^{N}$  in all, which enumeration, however, is not simplex. The sim-

plex enumeration is 
$$\Sigma \frac{N}{t_1} - \Sigma \frac{N}{t_{12}} + \Sigma \frac{N}{t_{123}} - \dots$$

where  $t_{12}...$ , is the least common multiple of  $t_1$ ,  $t_2$ ,... $t_r$ .

This last series is by the definition of \u00e4

$$= \sum_{t_h}^{N} [I - \psi_2'(t_h) + \psi_3'(t_h) - \dots]$$

and the expression, in the square brackets, is either 0 or  $(-1)^{n-l}$  according as  $t_h$  contains repeated prime factors, or n non-repeated prime factors. Hence, finally, the number of numbers less than N and not prime to it is

$$\sum_{p_1}^{N} - \sum_{p_1p_2}^{N} + \sum_{p_1p_2p_2}^{N}$$

which is a well known result. (Cf. Smith: Algebra, p. 488).

# Postscript: Extension of Fermat's Theorem.

It is well known that a rational transformation of the nth degree has n+1 fixed points and the mth power of the transformation which is of degree  $n^m$  has  $n^m+1$  fixed points. But if t be any factor of m, all the fixed points of S' (where S is the rational transformation in question) are also fixed points of S''. Hence, if  $t_1 t_2...t_k$  be the factors of m, m itself being excluded, the number of special fixed points of S'' would be equal to

$$(n^m+1)-\Sigma(n^t_{1}+1)+\Sigma(n^t_{12}+1)\Sigma(n^t_{123}+1)$$

where  $t_{12}$ ..., is the G.C.M of  $t_1$   $t_2$ ... $t_r$ .

This series is equal to 
$$n = \frac{m}{-\sum_n P_1} \frac{m}{P_1 + \sum_n \overline{P_1 P_2}} - \text{etc.}$$

where  $p_1 p_2$ ...are the prime factors of m.

We shall show now that if p is a special fixed point of  $S^m$ , then S(p) is also a special fixed point.

It is evident of course that S(p) is a fixed point of  $S^m$ . To shew that it is a special fixed point, assume it to be a fixed point of  $S^t$  where t is a factor of m. Then we have the following series

$$p, S(p), S(S(p)), \dots S^{t-1}(S(p)), S^{t}(S(p)) \{ = S(p) \}, \text{ etc.}$$

It is evident from this scheme that  $S^{m}(p) = S^{t-1}(S(p)) = S^{t}(p)$ 

Since  $S^m(p) = p$ , this shews that p also is a fixed point of  $S^t$ . This however is contrary to the hypothesis that p is a special fixed point. Hence S(p) is a special fixed point. Hence if  $p_1$  is one special fixed point,  $p_1$ ,  $S(p_1)$ ,  $S^2(p_1)$ .  $S^{m-1}(p_1)$  are all special fixed points and are all distinct. In fact, if  $S^r(p_1) = S^{r_1}(p_1)$ , then  $S^r(p)$  is a fixed point of  $S^{r-r}$ ; which is impossible since  $S^r(p)$  is a special fixed point. Again if  $p_2$  be a special fixed point not included in the above series, then  $p_2$ ,  $S(p_2)$ ,  $S^2(p)$ ... $S^{m-1}(p_2)$ , are all special fixed points and are all distinct both from one another and from the points of the previous series. In fact, the assumption that  $S^r(p_1) = S^{r_1}(p_2)$ , leads to  $S^{m+r-r_1}(p_1) = p_2$ , which contradicts the hypothesis that  $p_2$  is not included in the first series.

Proceeding in this manner we finally come to the conclusion, that the special fixed point can be distributed into sets of cyclic m-groups. Hence.

If n and m be any two integers and  $p_1 p_2 p_3$ ...are the prime factors of m, then the series.

$$n^{m} = \sum_{n} \frac{m}{p_{1} + \sum_{n} p_{1} p_{3}} = \sum_{n} \frac{m}{p_{1} p_{2} p_{3}} +$$

is exactly divisible by m.

If m is a prime, this reduces to Fermat's Theorem.

It will be noticed that in this extension of Fermal's theorem no restrictions are imposed on either of the two numbers n, m.

# Extension of M'cay's Theorem.

BY S. NARAYANAN, B.A., L T., MADURA COLLEGE.

M'Cay's extension of Feuerbach's theorem is 'If the axis major of an inscribed conic passes through the circumcentre, the auxiliary circle touches the nine-points circle.' It is proposed, in this paper, to shew generally the relation between the auxiliary circle of any inconic whose axis major passes through a given point P and the pedal circle of P. The main theorem of this paper may be enunciated as follows:

If the axis major of an inscribed conic of a triangle passes through a given point P, the auxiliary circle of this conic and the pedal circle of P are coaxal with the director circle of the inconic touching the axis major at P, and the common radical axis envelopes the orth polar conic of P.

$$\begin{array}{ll} Lemma\ I.\ \ \mathrm{If}\ x(y^2-z^2)=l,\ y(z^2-x^2)=m,\ z(x^2-y^2)=n,\\ \mathrm{then} & \frac{x(y^2+z^2)}{l(m^2+n^2-l^2)}=\frac{y(z^2+x^2)}{m(n^2+l^2-m^2)}=\frac{z(x^2+y^2)}{n(l^2+m^2-n^2)}=\frac{-xyz}{lmn}. \end{array}$$

This follows immediately on reduction.

Lemma II. If a, b c, denote the circles described on the sides BC, CA AB of a triangle ABC as diameters, then the director circle of any inscribed conic of ABC may be written in the form  $\lambda a + \mu b + \nu c = 0$ .

This is also obvious by rearranging the usual equation of the director circle of an inconic in terms of the varying coefficients of the conic.

Lemma III. The tricyclic equation  $al^2\alpha' + bm^2\beta' + cn^2\gamma' = 0$ ,

represents the director circle of the inscribed conic which touches  $l\alpha + n\beta + n\gamma = 0$  at  $\alpha'$ ,  $\beta'$ ,  $\gamma'$ . [Here a, b, c denote, as before, the circles on the sides as diameters, l, m, n are variables subject to the condition  $l\alpha' + m\beta' + n\gamma' = 0$  and  $\alpha'$ ,  $\beta'$ ,  $\gamma'$  are the normal coordinates of any point].

This follows immediately from Lemma II.

Lemma IV. If P be any point in the plane of a triangle ABC, and if AP, BP, CP meet the opposite sides in D, E, F and if K, L, M be the orthopoles of AP, BP, CP respectively, then the locus of the orthopoles of straight lines passing through P, which is called the orthopolar conic of P, is the conic passing through K, L, M and truching the common chords of the pedal circle of P and the circles on AP, BP, CP as diameters.

Let  $l\alpha + m\beta + n\gamma = 0$  be any straight line passing through  $\alpha', \beta', \gamma'$ , the normal coordinates of the orthopole  $l\alpha + m\beta + n\gamma = 0$  are given by

$$\alpha = (m \cos B + n \cos C - l \cos B \cos C) (m \cos C + n \cos B - l)$$

$$\beta = (n \cos C + l \cos A - m \cos C \cos A) (n \cos A + l \cos C - m)$$

$$y = (l \cos A + m \cos B - n \cos A \cos B) (l \cos B + m \cos A - n)$$
(I)

Now, by means of  $l\alpha'+m\beta'+n\gamma'=0$ , substitute for l in terms of m and n in (I), we have thus three equations in  $m^2$ , mn,  $n^2$ . And again eliminating m and n between these three equations, the result evidently assumes the form  $QR=P^2$  ... ... ... ... (A)

[The details of working are omitted as being too tedious for reproduction here].

Similarly substitute for m in terms of n and l in (I) and eliminate n and l between the three equations. The result will be of the form  $Q_1 B_1 = P_1^2$  ... ... ... ... (B)

Again doing the same thing for 
$$n$$
 we have  $Q_2R_2=P_2^2$  ... (C)

On a simple inspection of the results (A), (B), (C), it will be found that the orthopolar conic in question is *circumscribed* to the triangle formed by

$$\begin{aligned} &\alpha = (\beta' \cos B - \gamma' \cos c) \ (\beta' \cos c - \gamma' \cos B), \\ &\beta = \cos c \ (\beta' + \gamma' \cos A) \ (\beta' \cos A + \gamma'), \\ &\gamma = \cos B \ (\beta' + \gamma' \cos A) \ (\beta' \cos A + \gamma'), \end{aligned}$$

and two other points whose coordinates can easily be written down from cyclic symmetry.

The same is inscribed in the triangle formed by  $\alpha = -\left\{ \begin{array}{l} \alpha'^2 \left(\cos^2 B + \cos^2 C\right) + \alpha' \beta' \cos C \left(1 + \cos^2 B\right) \\ + \alpha' \gamma' \cos B \left(1 + \cos^2 C\right) + 2 \beta' \gamma' \cos B \cos C \right\}, \\ \beta = \alpha'^2 \cos C \left(1 + \cos^2 A\right) + \alpha' \beta' \left(\cos^2 A + \cos^2 C\right) \\ - \alpha' \gamma' \cos A \left(1 + \cos^2 C\right) - 2 \beta' \gamma' \cos C \cos A, \\ \gamma = \alpha'^2 \cos B \left(1 + \cos^2 A\right) - \alpha' \beta' \cos A \left(1 + \cos^3 B\right) \\ + \alpha' \gamma' \left(\cos^2 A + \cos^2 B\right) - 2 \beta' \gamma' \cos A \cos B, \\ \text{and two other points whose coordinates can easily be written down} \end{aligned}$ 

The three former set of points can easily be recognized to be the orthopoles K,L, M of AP, BP, CP respectively. And the joins of the three latter set of points are the chords of intersection of the pedal circle of P with the circles on AP, BP, CP as diameters.

from cyclic symmetry.

Let us now consider the theorem proper.

Let x,y,z be the normal coordinates of the focus of an inconic whose axis major passes through  $P(\alpha',\beta',y')$ .

The equation of the major axis is

$$\alpha x(y^2-z^2) + \beta y(z^2-x^2) + \gamma z(x^2-y^2) = 0.$$

If this passes through  $\alpha', \beta', \gamma'$ , we have

$$\alpha' x(y^2-z^2) + \beta' y(z^2-x^2) + \gamma' z(x^2-y^2) = 0.$$
 ... (D)

The equation of the manxiliary circle, viz, the pedal circle, of (x,y,z) is  $\Sigma \beta y \sin A$ .  $\Sigma yz \sin A$ .  $\Sigma x \sin A$ .

 $=\Sigma \alpha \sin A$ .  $\Sigma \alpha x \sin B \sin C (y+z \cos A) (y \cos A+z)$ 

Arranging this in terms of x,y,z and powers, we have

$$ax(y^2+z^2)+by(z^2+x^2)+cz(x^2+y^2)+k \ xyz=0...$$
 (E)

where a, b, c denote the circles on BC, CA, AB as diameters

$$a \equiv \beta \gamma + \gamma \alpha \cos C + \alpha \beta \cos B - \alpha^2 \cos A$$

$$b \equiv \beta \gamma \cos C + \gamma \alpha + \alpha \beta \cos |A - \beta^2 \cos B|$$

 $c \equiv \beta y \cos B + y \alpha \cos A + \alpha \beta - y^2 \cos C$  $k \equiv 2\Sigma \beta y \cos B \cos C - \Sigma \alpha^2 (1 + \cos^2 A)$ .

and

i.e.

Now let 
$$a(y^2-z^2)=l$$
,  $y(z^2-x^2)=m$ ,  $z(x^2-y^2)=n$ , the relation (D)

assumes the form  $l \alpha' + m\beta' + n\gamma' = 0 \qquad ... \qquad ... \qquad (F)$ 

and with the aid of Lemma I, the relation (E) becomes

al 
$$(m^2+n^2-l^2)+bm (n^2+l^2-m^2)+cn (l^2+m^2-n^2)=k l m n.$$
 (G)

Now (G) is the auxiliary circle of the inconic whose axis major is  $l\alpha + m\beta + n\gamma = 0$ .

Transforming (G) with the aid of (F), we have

$$\Sigma a l^{2} (m\gamma', \overline{\alpha'^{2} - \beta'^{2}} + n\beta', \alpha'^{2} - \gamma'^{2}) + l m n \{ k\alpha'\beta'\gamma' + \Sigma a \alpha' (\beta'^{2} + \gamma'^{2}) \} = 0 \qquad \dots (H)$$

The co-efficient of lmn in(H) can easily be identified with the pedal circle of  $\alpha'$ ,  $\beta'$ ,  $\gamma'$ . Hence, denoting this by  $\theta$ , the auxiliary circle becomes after further reduction with the aid of  $l\alpha + 'm\beta' + n\gamma' = 0$ .

$$l \ m \ n \ \alpha' \ \theta + (m \gamma', \ \overline{\alpha'^2 - \beta'^2} + n \beta', | \overline{\alpha'^2 - \gamma'^2}) (a \ l^2 \ \alpha' + b \ m^2 \beta' + c n^2 \gamma') = 0.$$
 Here let  $a \ l^2 \ \alpha' + b \ m^2 \beta' + c n^2 \gamma' = 0$  be denoted by  $\phi$  and  $m \ \gamma' \ (\alpha'^2 - \beta'^2) + n \ \beta' \ (\alpha'^2 - \gamma'^2) \equiv \lambda$  the auxiliary circle may be written in the form  $l \ m \ \alpha' \ \theta + \lambda \phi = 0$  ... ... ... ... ... ... (J)

Now the relation J shows that the circles  $\phi$  and  $\theta$  are coaxal with the auxiliary circle. And Lemmas II and III tell us that  $\phi$  is the director circle of the inconic touching  $l \alpha + m \beta + n \gamma = 0$  at  $(\alpha', \beta', \gamma')$ 

This proves the first part of the general theorem of this paper, viz that the three circles are coaxal. Now, it remains to prove the second part, viz. the common radical axis envelopes the orthopolar conic of (a' B' y').

Theorem II.—The radical axis of 
$$\Theta$$
 and  $\phi$  is found to be 
$$\frac{\alpha(p_1l^2+p_2m^2+p_8n^8)+\beta(q_1l^2+q_8m^2+q_8n^8)}{+\gamma(r_1l^2+r_2m^2+r_8n^2)=0...} \qquad ... \qquad ... \qquad (M.)$$

where

$$\begin{split} p_1 &= \frac{\alpha'}{\sin A} \left\{ \begin{array}{l} \alpha' \sin B \sin C \left( \beta' + \gamma' \cos | A \right) \left( B' \cos A + \gamma' \right) \\ &- \cos A \ \Sigma \beta' \gamma' \sin A, \quad \Sigma \alpha' \sin A \right\} \\ p_2 &= \alpha' \beta' \sin C \left( \beta' + \gamma' \left| \cos A \right| \right) \left( \beta' \cos A + \gamma' \right) \end{split}$$

 $p_s = \alpha' \gamma' \sin B (\beta' + \gamma' \cos A) (\beta' \cos A + \gamma');$ 

and symmetrical values for q and r.

We have now to find the envelope of M subject to the condition  $l\alpha' + m\beta' + n\gamma' = 0$ . The envelope is easily seen to be of the form  $BC\alpha' \sin A + CA\beta' \sin B + AB\gamma' \sin C = 0$ ,

where 
$$\begin{aligned} \frac{\text{A}\,\alpha'}{\sin\,\text{A}} &\equiv \alpha p_1 + \beta q_1 + \gamma r_1 \\ \frac{\text{B}\beta'}{\sin\,\text{B}} &\equiv \alpha p_2 + \beta q_3 + \gamma r_3 \\ \frac{\text{C}\gamma'}{\sin\,\text{C}} &\equiv \alpha p_3 + \beta q_3 + \gamma r_3. \end{aligned}$$

$$\frac{C\gamma'}{\sin C} \equiv \alpha p_{8} + \beta q_{8} + \gamma r_{8}.$$

The relation (N) shews that the envelope which is evidently a conic is circumscribed to the triangle formed by the three straight lines A, B. And on finding the normal coordinates of the vertices of the triangle formed by A, B, C they are easily found to be the former set of points in Lemma IV.

Hence the whole theorem is proved.

# Euclid's Book on Divisions of Figures. \*

Euclid, it is now well known, was the author of no less than nine works. Four of these-the Elements, the Data, the Optics, and the Phenomena, have been carefully preserved and are well known, even to the extent that the first, perhaps, is in the hands of every schoolgirl. On the fifth, the Porisms, there is an elaborate commentary of Pappus while in the case of the Pseudaria, the Surface loci and the Conics how. ever, our fragmentary knowledge, derived wholly from Greek sources, enables us to make a conjecture of the vaguest nature, as to their possible contents. Of the ninth book, On Division (of figures), nothing however was known till about the middle of the last century; nor, curiously enough, is there any trace of it in the extant Greek sources of mathematics, except perhaps a casual mention by Proclus. But it is however certain, that the Problems of this book, thirty-six in number, concerned mainly with the division of rectilinear figures and the circle into a certain number of 'like' and 'unlike' parts by straight lines, was very popular among Arabians, as well as in Europe. This popularity was perhaps largely due to the possible practical applications of the problems in the division of parcels of lands of various shapes, the areas of which, according to the Rhind papyrus, were discussed in empirical fashion about 1800 B. C. Indeed, in the first century before Christ, we find that Heron of Alexandria, dealt with the division of surfaces and solids in the third book of his Metriks, (Surveying) though not in the actual form in which Euclid dealt with the same. But nothing in a definite form was known until Woepcke, whose name is so familiar to every oriental researcher, discovered at the Bibliotheque Nationale in Paris, a treatise in Arabic on the division of plane figures, which he translated (into French) and published in the Journal Asiatique, September 1851, (XVIII), under the title " Notice sur des traductions Arabes de deux ouvrages perdus d'Euclide". In the original Arabic not only is the full. account given (though unfortunately the proofs, save of four theorems. are passed over as being too easy !), but the scribe expressly attributes the authorship to Euclid, and contents himself with the unassuming office of a 'translator'. Such a treatise, significant alike for its historical interest as for its technical import, had to be restored and reedited, if for no other reason, at least as a mark of our regard fro the author. Prof. Heiberg gives an account of the book only in a Latin publication, Opera Omnia, while Sir Thomas Heath, the foremost living authority in England on Greek Mathematics, dismisses the work in the brief space of two pages in his Thirtzen Books of Euclid's Elements; there is no reference to it even in the article on Euclid in the last

<sup>\*</sup> By R, C, Archibald, Ph. D. Cambridge University Press, 1915.

edition of the Encyclopælia Brittanica. -Under these circumstances, Dr. R. H. Archibald of Providence, Rhode Island, has done a lasting service to the English reader in restoring in the English language this long buried treasure; and the scholarly way in which he has done his task is proved by the readiness with which the publication of the book was undertaken by the Cambridge University Press.

To follow in historical sequence how this treatise came to be unearthed or to dismiss the myth of Dee's making a copy of any Cottonian MSS earlier than 1563, is not our purpose in this article; the interested reader must be referred to the learned discussion in Archibald's book itself. But it must be noticed to what extent Woepcke's translation referred to above, is in agreement with and is supplemented by the writings of Leonardo Pisano (Fibonaci), who occupies such an important place in the history of Mathematics of the 13th century. It is well known that in 1220 he wrote his Practica Geometriae, a book largely used by the Mathematicians of the middle ages, published nearly 650 years after its compilation by Prince Boncompagni, and still preserved in MS in the Vatican library. Section IVof this book is wholly devoted to the enunciation and proof and numerical illustration of propositions concerning the division of figures. No less than twenty-two of Woepcke's propositions are practically identical in statement with propositions in Leonardo ; the solutions of eight more of Woepcke are either given or are clearly indicated by Leonardo, and all the six of the remaining Woepcke's are assumed as known in the book of Leonardo. Indeed the two works have a remarkable similarity. Not only are practically all of Woepcke's theorems in Leonardo, but proofs called for by the order of the theorems and by the auxiliary propositions are invariably the kind of proofs which Euclid might have given - no other propositions than those which had gone before or which were to be found in the Elements being regarded in the successive constructions.

In this treatise of Euclid, the figures divided are the triangle, the parallelogram, the trapezium, the quadrilateral, the circle and a figure formed by the arc of a circle and two lines; and the division is by transversals through assigned points or parallel to given lines into two or several parts in a given ratio, equality included. It is difficult to choose from this interesting collection but the three following are specially instructive, and suggest generalisations. Thus Steiner solved in 1827 how through a given point on a sphere to draw arc of a great circle cutting two given great circles such that the intercepted area is equal to

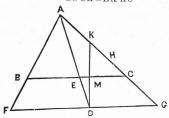
given area; and Bobillier solved: to draw by means of places and spheres only, a given line in a plane which shall cut off from a given cone of revolution a required volume. The three theorems referred to above are given as propositions 27, 28 and 29 by Woopeke and are extracted below:

# Proposition 27.

To cut off a certain fraction of a triingle by a straight line drawn from a given point situated outside of the triangle:

Let ABC be the given triangle and D, the given point outside. It is required to cut off from the triangle a certain fraction, say a third, by a line through D. Join AD cutting BC in E. If either BE or EC be a third of BC, then the line AD cuts off the required fraction from the given triangle. If it is not, produce AB and AC to meet in F and G respectively a line drawn through D parallel to BC. Make area DG-CH = \frac{1}{2} area AC-CB and apply to the line CH, a rectangle equal to the rectangle GC-CH, but exceeded by a square, Then

 $GC \cdot CH = HK \cdot KC$ 



and draw the line KD cutting BC in M.

Then KMC shall be a third of the triangle ABC.

Proof. For since area GC.CH = area CK.KH, and therefore GC:CK = KH:HC.

it follows that GK: CK = CK: CH,
But GK: KC = DG: CM.

Hence DG CM = CK : CH

i.e.  $\begin{array}{c} \operatorname{CK} \cdot \operatorname{CM} = \operatorname{CH} \operatorname{DG}, \\ \operatorname{DG} \cdot \operatorname{CH} = \frac{1}{3} \cdot \operatorname{area} \operatorname{AC} \cdot \operatorname{CB}, \end{array}$ 

since  $\triangle$  KCM:  $\triangle$ ACB=CK·CM: CA·CB, it follows that  $\triangle$ KCM= $\frac{1}{3}$ 

In a similar manner any part of a triangle may be cut off by a straight line drawn from a given point on a side of the triangle produced, &c.

Q.E.F.

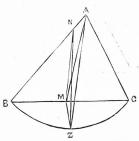
# Proposition 28.

To divide into two equal parts a given figure bounded by an arc of a circle and by two straight lines which form a given angle.

Let ABC be the given figure bounded by the arc BC and two lines AB and AC which include the angle BAC. It is required to divide by means of a straight line the figure ABC into two equal parts.

Construction. Bisect the line BC at M, and erect MZ perpendicular to BC join MA. If AMZ are in a line, obviously the figure is divided into two equal parts by AZ. If not; join AZ, and through M draw a line parallel to AZ cutting AB at N; and join NZ.

Then NZ shall be the required line, and the figure BNZ shall be equal in area to the figure NACZ.



Proof. Obviously,  $\Delta \Delta BM = \Delta ACM$ .

and

figure BMZ = figure CMZ.

figure ABZM = figure ACZM.  $\Delta$ MAZ =  $\Delta$ NAZ.

Hence But Therefore,

figure ACZN = figure ACZ+ $\Delta$ NAZ.

=figure  $ACZ + \Delta MAZ$ .

=figure ACZM,

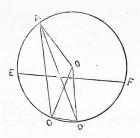
= igure ACZM, = i the figure ABZC.

QE,F.

#### Proposition 29.

"To draw in a given circle two parallel lines cutting off a certain fraction, say, a third, from the circle.

Take the circle with centre O, and inscribe in it the equilateral triangle ABC. Join AO and CO., and draw OD parallel to AC. Join CD. Bisect the arc AC at E and draw EF parallel to CD. Then shall the figure EC DF be a third of the given circle. Join AD.



Proof.

Since AC and OD are parallel,  $\triangle ACO = \triangle ACD$ ;

and, adding to leach side the figure AEC, the figure, AECD=sector  $ACO = \frac{1}{3}$  of the circle.

Again, since EF and CD are parallel,

arc DF = arc EC

= the equal arc AE.

Add are ED to these.

 $arc\ AD = arc\ EF$ 

and

chord AD = chord EF;
segment AED = segment ECDF

Take away the common segment CD,

and the figure ECDF = figure AECD =  $\frac{1}{3}$  of the circle, as was required to be shown.

Q. E. F.

In the Arabic Manuscript it seems to have been stated that the method is also available, when the fraction is my fraction. But Dr-Archibald Pastens to add in a footnote that this problem is clearly not susceptible of solution with ruler and compasses, in such a case as when a

'certain fraction' 1/n, is one-seventh. In fact the only cases in which the problem is possible, for a fraction of this kind, is when n is of the

form  $2^{p}(2^{z^{g_1}}+1)(2^{z^{g_2}}+1)...(2^{z^{g_m}}+1)$  where p, and s's (all different), are positive integers or zero,

and  $2^{2^{S_m}} + 1(m = 1, 2, ...m)$  is a prime number.

It is notwonder that such interesting theorems were considered independently or by suggestion by mathematicians, great and small-Thus the problems "To bisect, by a straight line, an equilateral, but not equiangular heptagon", "Through a point without a triangle to draw a line which will cut off a third" went as a challenge from L. F. (L. Ferari) to N. T (N. Tartaglia). Kepler, in his investigations of the path of a planet was led in 1609 to the problem "To divide the area of a semicircle in a certain ratio by a straight line drawn through a given point on the diameter or on the diameter produced" and this was attacked among others by Wallis, Cassini, Lagrange and Laplace. Newton and Bernoulli, D'Alembert and Euler, Huyghens and L'Hospital attacked several of the problems that have been given in this treatise of Euclid. These theorems, or their immediate generalisations have also fascinated the readers of popular magazines: thus in 1814, in the Gentlemen's Diary was solved " within a given triangle to find thus, that if lines be drawn from it to cut each side at right angles, the three parts into which the triangle thus becomes divided, shall obtain a given ratio"; while a century later in 1914we find in the two most popular mathematical journals, the problem how "to divide a circle into two equal parts by means of an arc with its centre on the circumference of the given circle" (Q. 4327, L'Intermediar, des Mathematiciens) and "to divide a square into five right angled triangles the areas of which shall be in Arithmetic progression" (Q. 17197 Educational Times). Among Indians who interested themselves in such problems, Dr. Archibald mentions the name of the late Mr. S. Radhakrishna Aiyar of Pudukotah.

#### SHORT NOTES.

# On Legendre's relation in Elliptic Functions.

# § 1. The relation $EK' + E'K - KK' = \frac{\pi}{2}$

connecting the two kinds of complete Elliptic integrals was first given by Legendre in his Exercises de Calcul, and several procés of it are well known. [Cf. Dixon: Elliptic Functions § 65; Greenhill: Elliptic Functions § 121; Whittaker and Watson: Modern Analysis, § 22.735, Appellet Lacour: Fonctions Elliptiques, § 80; or Jacobi: Fundamenta Nova, p. 161]. Following however, the proof (cf. Forsyth: Theory of Functions, § 129 or Henry: Fonctions Elliptiques, p. 53) of the analogous relation in the Weierstrassian theory, viz,  $\eta_1\omega_2 - \eta_1\omega_1 = \frac{\pi i}{2}$ , we may proceed directly as follows.

We assume of course that

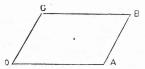
$$\mathbf{E}(u) = \int_{0}^{u} \mathrm{d}\mathbf{n}^{2} \ u \ du$$

and that the function  $\operatorname{dn} u$  is a doubly periodic function of u with periods  $2\mathbf{K}$  and  $4i\mathbf{K}'$ , that it is analytic except at the points congruent to  $i\mathbf{K}'$  and to  $3i\mathbf{K}'$  which are simple poles, the residues thereat being -i and +i respectively.

It follows that 
$$\operatorname{dn} u = -i/u + O(u)$$
 at  $u = iK'$   
 $\operatorname{dn}^2 u = -1/u^2 + O(1)$ 

whence we notice that E(u) has got a simple pole with residue 1 at u = iK' which is the only pole in the parallelogram [-K, K, K + 2iK',

-K+2iK'



Integrating now E(u) along the parallelogram OABC, we have  $E(u)du = 2\pi i \times \text{the sum of the residues} = 2\pi i,$  ABCO

$$\begin{bmatrix} \mathbf{K} & \mathbf{E}(\mathbf{u})du + i \int_{0}^{2\mathbf{K}} & \mathbf{E}(\mathbf{K} + iv)dv + \int_{\mathbf{K}}^{-\mathbf{K}} & \mathbf{E}(u + 2i\mathbf{K}')du \\ + i \int_{2\mathbf{K}'}^{0} & \mathbf{E}(-\mathbf{K} + iv)dv \end{bmatrix} = 2\pi i,$$

$$i.e., \int_{-K}^{-K} [E(u+2iK')-E(u)]du$$

$$-i \int_{0}^{2K'} [E(-K+iv)-E(K+iv)]dv = 2\pi i$$

$$i.e., \int_{-K}^{-K} 2i(K'-E')du+i \int_{0}^{2K'} 2 E du = 2\pi i$$

by the periodicity formulæ, cf. Dixon Elliptic Functions § 64.

i e,  $4 i (E K' + E'K - K K') = 2\pi i$ 

which establishes Legendre's relation connecting; the two complete

§ 2. In passing, it may also be noticed that the above relation is only a special case of a more general result in the theory of Abelian functions (Cf. Baker; Abelian Functions p.p. 197-200): that between the modulii of periodicity A', B',...of a function P(z) of the first kind (Normal function on a Riemann Surface) and A, B,...of a function Q(z) of the second kind, all the infinities of which are simple,

$$\sum_{r=1}^{p} (\mathbf{A}, \mathbf{B}_{r}' - \mathbf{B}_{r} \mathbf{A}_{r}') + 2\pi i \Sigma \mathbf{H}_{s} \left( \frac{d\mathbf{P}}{d\mathbf{z}} \right) a_{s}$$

where  $a_j$  is a simple pole of Q(z) and  $H_s$  its residue thereat. Legendre's relation is of course included in this; for, we know  $P(z) \equiv F(z)$  in the notation of the Elliptic Functions

$$= \int_0^z \left[ (1-z^2)(1-k^3z^2) \right]^{-\frac{1}{2}} dz$$

$$\therefore \frac{dP}{dz} = \frac{1}{K} \text{ in the lower sheet, and } -\frac{1}{K} \text{ in the upper for } \mathbf{Z} = \infty$$

and  $Q(z) \equiv E(z) = \int_0^z \left[ \frac{1 - K^2 Z^2}{1 - Z^2} \right]^{\frac{1}{2}} dz$ .

This is infinite at  $z=\infty$  in each sheet, and the residue is +K in the upper and -K in the lower sheet, so that  $\Sigma H_s! \left(\frac{dP}{dz}\right)_{\sigma_s} = -2$ .

Moreover  $A_1'$  and  $B_1'$  are 4K and 2iK' $A_1$  and  $B_1$  are 4E and 2i(K'-E'), respectively.

Hence substituting in the above we have immediately

4 E. 2iK' - 4K.  $2i(K' - E') - 4\pi i = 0$ ,

which leads to  $EK' + KE' - KK' = \pi/2$ .

K. B. MADHAVA.

# Astronomical Notes.

# Eclipse of the Sun.

The third and last eclipse of the year will occur on December 3. This will be an annular eclipse of the sun. The central line runs from a point just inside the west coast of Africa to a point in the Pacific Ocean, crossing the Argentine Republic. The eclipse is partially visible over nearly the whole of South America and a considerable portion of Africa. The southern limit does not quite reach the South Pole.

# Nova Aquilae.

The new star in Aquila was discovered by about a dozen different observers in England within a few hours of one another, and also by numerous other observers in different parts of the world.

Both as regards magnitude and changes in its spectrum, it seems to have followed pretty close the changes observed in other novae. Perhaps the most interesting point about it is that it appears to be identical with a star which has been photographed several times during the last quarter of a century, the magnitude of which appears to have varied from the eighth to the tenth magnitude. With the exception of the nova known as T Coronæ, (which was also exceptional in that it did not appear in the Milky Way), there is no other case in which a star is known to have existed previously in the place where a nova has appeared.

#### New Comet.

The first discovery of a comet this year was made by Mr Reid at the Cape of Good Hope in June, when it appeared as a faint round nebulosity. As yet only very meagre details are to hand, so that it is impossible to give any further information at present.

# New Trojan Planet.

A new planet having the same mean distance as Jupiter and thus illustrating Lagranges particular Solution of problem of the three bodies has been discovered. This is the fifth such planet known, three of the planets librate in the neighbourhood of the triangular point on one side of the line joining the Sun and Jupiter, and the other two, of which the new discovery is one, librate about the apex of the equilateral triangle on the opposite side of the Sun—Jupiter line.

R. J. Pocock.

## SOLUTIONS.

#### Question 685.

(S. Malhari Rao):—If n and p are any two positive integers, show that the fraction—may be expressed as the sum of a series of p different fractions, the numerator of each of which is 1 and the denominator a positive integer. If the least of the fractions be (n-1)!/(n+p-1)! find the rth fraction.

Solution by Prof. N. B. Mitra and N. Sankara Aiyar M. A.

The question is vague; for,  $\frac{1}{n}$  can be resolved into the sum of p unit-numerator fractions in several ways. Thus, one way would be as follows:—

We easily see that 
$$\frac{1}{n} = \frac{1}{n(n+1)} + \frac{1}{n+1}$$
. ... ... (1)

By applying the same formula to  $\frac{1}{n+1}$ , the second fraction on the right, we have,

$$\frac{1}{n} = \frac{1}{n(n+1)} + \frac{1}{(n+1)(n+2)} + \frac{1}{n+2}.$$

By repeating the same process we set,

$$\frac{1}{n} = \frac{1}{n(n+1)} + \frac{1}{(n+1)(n+2)} + \dots + \frac{1}{(n+p-2)(n+p-1)} + \frac{1}{n+p-1}.$$

Another way is to substitute in the first fraction on the right of (1), the value of 1/n given by (1). If this is repeated p-1 times we get ultimately

$$\frac{1}{n} = \frac{1}{n(n+1)^{p-1}} + \frac{1}{(n+1)^{p-1}} + \frac{1}{(n+1)^{p-2}} + \dots + \frac{1}{n+1}.$$

A third way is as follows:—In (1) apply the same formula to  $\frac{1}{n+1}$  in the first fraction on the right. We get

$$\frac{1}{n} = \frac{1}{n(n+1)(n+2)} + \frac{1}{n(n+2)} + \frac{1}{n+1}.$$

By applying the process altogether p-1 times, we get

By applying the process altogether 
$$p-1$$
 times, we get 
$$\frac{1}{n} = \frac{1}{n(n+1)...(n+p-1)} + \frac{1}{n(n+1)...(n+p-3)(n+p-1)} + \frac{1}{n(n+1)...(n+p-4)(n+p-2)} + ... + \frac{1}{n(n+2)} + \frac{1}{n+1} = [n-1] \left[ \frac{1}{n+p-1} + \frac{1}{n+p-3(n+p-1)} + \frac{1}{n+p-4(n+p-2)} + ... + \frac{1}{n-1} \frac{1}{(n-1)} \right]$$
The  $t^{th}$  fraction on the right is evidently 
$$\frac{1}{n+p-r-1} \frac{1}{(n+p-r+1)} = \frac{1}{n+p-r-1} \frac$$

the least being the first.

This is perhaps what the proposer seeks by the second part of his question.

#### Question 954.

(K. B. MADHAVA) :- When asked his age De Morgan once humorously observed: "I am one of those whose age shall be in a certain year belonging to the century of their birth the square root of that year. "For being born in 1806 A. D., he was 43 years old in 1849 and 1849=43. Show that the same observation will be true in the case of those born in 9 particular years in all subsequent centuries.

S lution by N. B. Mitra, K. J. Sanjana and R. D. Karve.

It would appear that the figure 9 in the last line of the question is wrong. The problem may be solved thus:-Let x be the year of birth, y the age at which the year is square of the age. Then we have to solve the indeterminate equation  $y^2=x+y$  in non-zero positive integers subject to the conditions (1) that x>1900 and (2) that y+Least positive residue of x, modulus 100 100 this residue itself being >0. This gives 15 sets of values of x and y which are exhibited below:-

x = 2450; 3306; 3422; 3540; 3906; 4032; 4422; 4830; 5112; 5402; 6006; 6320; 6806; 7310; 8010. y = 50; 58; 59; 60; 63; 64; 67; 70: 72; 74; 78; 80; 83.86; 90.

It may be interesting to know the years in the centuries previous to the 20th in which De Morgan's observation was true. They are given below together with the corresponding ages at which the year of the Christian era was square of the age :-

x=2; 6; 12; 20; 30; 42; 56; 72; 90; 110; 132; 156; 182; 210; 240; 272; 306; 342; 380; 420; 462; 506; 552; y=2; 3; 4; 5; 6; 7; 8; 9; 10; 11; 12; 13; 14; 15; 16; 17; 18; 19; 20; 21; 22; 23; 24; x = 690; 702; 756 : 812; 870; 930; 1056; 1122; 12600; 1332; 1406; 1560; 1640; 1722; 1806 (De Morgan's case.) y = 26; 27; 28; 29; 30; 31; 33; 34; 36; 37; 38; 49; 41; 42; 43;

#### Question 831.

(Enquirer) Find, by elementary methods, the invariant relation expressing the condition that hexagons may be inscribed in the conic S=0 which are also circumscribed to the conic S'=0.

If a hexagon ABCDEF circumscribes a conic S and is inscribed in S', then the Brianchon point of the Pascal line of the hexagon are pole and polar.

Projecting S' into a circle with the Brianchon point as centre we have hexagons inscribed in  $S'\equiv x^2+y^2-c^2=0$  and circumscribed to

$$S \equiv \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0.$$

To find the relation between a, b, c consider that hexagon ABCDEF of which A and D lie on the minor axis of the ellipse and BC, EF touch the ellipse at the extremities of the major axis.

Let AB touch the ellipse at  $(a \cos \alpha, b \sin \beta)$  then A is  $(0, \frac{b}{\sin \alpha})$ 

and B is 
$$\left(a, b \tan \frac{\alpha}{2^i}\right)$$
.

Hence 
$$c^2 = a^2 + b^2 \tan \frac{2a}{3} = b^2 \csc^2 \alpha$$

giving 
$$(e^2-a^2+b^3)^2=4e^2(e^2-a^2)$$
  
or  $3e^4-2e^2(a^2+b^3)-(a^2+b^2)^2+4a^2b^2=0$  ... (1

Also 
$$\Delta = -\frac{1}{a^2b^2}$$
,  $\Theta = -\frac{1}{a^2b^2}(a^2 + b^2 + c^2)$ ,  $\Theta' = -\frac{a^2b^2 + a^2c^2 + b^2c^3}{a^2b^2}$ ,  $\Delta' = -c^3$ 

write 
$$p = \frac{\Theta}{\Delta} = a^2 + b^2 + c^2$$
,  $q = \frac{\Theta}{\Delta} a^2 b^2 + a^2 c^2 + b^2 c^3$ ,  $v = \frac{\Delta'}{\Delta} = a^2 b^2 c^3$ .

Then  $a^2$ ,  $b^2$ ,  $c^2$  are the roots of

$$x^3 - px^2 + qx - r = 0$$

and from (1) 
$$3x^2 - 2x(p-x) - (p-x)^2 + \frac{4r}{x} = 0$$
or 
$$4x^3 - p^2x + 4r = 0.$$

Write 2p for p, so that  $x^s-2px^2+qx-r=0$ ,  $x^s-p^2x+r=0$ ; whence

$$2x^2-2px+q-p^2=0$$
 and  $2px^2-(p^2+q)x+2r=0$ ;

which gives

or 
$$\{(p^4-q^2)+4pr\}(p^2-q)+2\{p^5-pq+2r\}^2=0,$$
  
or  $(3p^2+q)(p^2-q)^2+12pr(p^2-q)+8r^2=0,$ 

Hence the invariant relation is  $(3\theta^2 + 4\Delta\theta')(\theta^2 - 4\Delta\theta')^2 + 96\theta\Delta^2\Delta'(\theta^2 - 4\Delta\theta') + 512\Delta'^2\Delta^4 = 0$ 

This will be found to agree with  $\begin{vmatrix} D,E\\E,F \end{vmatrix}=0$  of Salmon after removing the factor  $\theta^a-4\Delta\theta'$ , and is easily identified with Halphen's for n which is

 $4(\theta^3-4\Delta\theta\theta'+8\Delta^2\Delta')^3+4(\theta^3-4\Delta\theta\theta'+8\Delta^2\Delta').8\Delta^2\Delta'-(\theta^2-4\Delta\theta')^3=0.$ 

#### Question 844.

(A. A. Krishnaswamiengar, m.a.):—Shew that the locus of the centre of the conic of minimum eccentricity having contact of the third order with the conic  $Ax^3+By^3=1$ , has for its equation  $(x^2+y^3)\sqrt{(Ax^2+By^2)}=(Ax^2-By^3)\left(\frac{1}{A}-\frac{1}{B}\right)$ 

Solution (1) by N. B. Mitra, Hemraj, and S. V. Venkatachala Aiyar;
(2) by F. H. V. Gulasekharam.

(1) The equation giving the eccentricity of the general conic in rectangular coordinates is  $\frac{e^4}{1-e^2} = \frac{(a+b)^2}{ab-v^2} - 4$ 

If e < 1, e will be least when  $\frac{(a+b)^2}{ab-n^2}$  is least since  $\frac{e^4}{1-e^2}$  in this case diminishes continually with e.

Now the conic having four pointic contact with  $Ax^2 + By^3 = 1...(1)$  at the point  $\left(\frac{\cos \theta}{\sqrt{A}}, \frac{\sin \theta}{\sqrt{B}}\right)$  has for its equation

$$Ax^2 + By^2 - 1 - \lambda(\sqrt{Ax}\cos\theta + \sqrt{By}\sin\theta - 1)^2 = 0 \dots$$
 (2)

The eccentricity of this is least when

$$\frac{\{\ A(1-\lambda\ \cos^2\!\theta) + B(1-\lambda\ \sin^2\!\theta)\ \}^{\ 2}}{AB(1-\lambda\ \cos^2\!\theta)(1-\lambda\ \sin^2\!\theta) - AB\ \sin^2\!\theta\ \cos^2\!\theta}$$

or 
$$\frac{\left\{\begin{array}{l} A+B-\lambda(A\,\cos^{2}\!\theta+B\,\sin^{2}\!\theta)\,\right\}^{2}}{AB(1-\lambda)}\,\text{or}\,\frac{(A\,\cos^{2}\!\theta+B\,\sin^{2}\!\theta)^{2}}{AB(1-\lambda)}\\ \left\{\begin{array}{l} \frac{A\,\sin^{2}\!\theta+B\,\cos^{2}\!\theta}{A\,\cos^{2}\!\theta+B\,\sin^{2}\!\theta}\,+1-\lambda \end{array}\right\}$$

is least or putting  $p = A \sin^2\theta + B \cos^2\theta$  &  $q = A \cos^2\theta + B \sin^2\theta$ , when

$$\frac{q^{2}}{AB} \begin{cases} \frac{p^{2}}{q^{2}} \\ (1-\lambda) \end{cases} + 2\frac{p}{q} + (1-\lambda) \begin{cases} \text{is least } i.e. \text{ when } 1-\lambda = \frac{p}{q} \\ \text{or } \lambda = \frac{(A-B)\cos 2\theta}{q} & \dots \end{cases}$$
(3)

The centre of (2) is given by A  $(1-\lambda \cos^2\theta)x - \lambda \sqrt{AB} \cos\theta \sin\theta y$  $+\lambda\sqrt{A}\cos\theta=0$ .

$$\lambda(\sqrt{Ax}\cos\theta + \sqrt{By}\sin\theta - 1) = \sqrt{Ax}\cos\theta + \sqrt{By}\sin\theta$$
or  $\sqrt{Ax}\cos\theta + \sqrt{By}\sin\theta = \frac{\lambda}{\lambda - 1} = \frac{(B - A)\cos^3\theta}{\theta}$  from (3) ... (6)

also from (4) and (5) 
$$\frac{\sqrt{Ax}}{2} = \frac{\sqrt{By}}{2} = \sqrt{Ax^2 + By^2} = K$$
 suppose.

also from (4) and (5) 
$$\frac{\sqrt{Ax}}{\cos \theta} = \frac{\sqrt{By}}{\sin \theta} = \sqrt{\frac{Ax^2 + By^2}{1}} = K$$
 suppose.

$$\therefore \cos \theta = \frac{\sqrt{Ax}}{k}, \sin \theta = \frac{\sqrt{By}}{k} \therefore \cos \theta = \frac{Ax^2 - By^2}{k^2}$$
and  $\theta = \frac{AB(x^2 + y^2)}{k^2}$ 

$$\therefore \text{ from } (6), \frac{\mathbf{A}x^2 + \mathbf{B}y^2}{k} = \frac{(\mathbf{B} - \mathbf{A})(\mathbf{A}x^2 - \mathbf{B}y^2)}{\mathbf{A}\mathbf{B} (x^2 + y^2)} \text{ or } (x^2 + y^2) \sqrt{(\mathbf{A}x^2 + \mathbf{B}y^3)} \\
= \left(\frac{1}{\mathbf{A}} - \frac{1}{\mathbf{B}}\right)(\mathbf{A}x^2 - \mathbf{B}y^3)$$

[See: Wolstenholme, Ex. 1209, p. 202.]

(2) The equation to the conic having contact of the third order with  $Ax^2 + By^2 = 1$  at any point P  $(x_1, y_1)$  on it is

 $Ax^2 + By^2 - 1 - k(Axx_1 + Byy_1 - 1)^2 = 0$ (1)

The coordinates (x, y) of the centre Q of this conic are given by the equations,

$$\frac{x}{x_1} = \frac{y}{y_1} = k(Axx_1 + Byy_1 - 1)$$
  
= \(\lambda \say \tau \cdots \c

Then since 
$$(x_1, y_1)$$
 is a point on the given conic,
$$\lambda^2 = Ax^2 + By^2$$
and
$$k = \frac{\lambda}{\lambda - 1}$$
... ... (3)

Also from (2), it is clear that the centre of the conic (1) lies on the diameter through P of the given conic.

Again from Ex. 6 Chap XI, C. Smith's Conic Sections (Revised Edition), "Of all conics which have a third order contact with a given conic at a given point on it, the one of least eccentricity will be that in which one of the equi-conjugate diametres passes through that point."

Hence if  $r_1$ ,  $r_2$  be the semi-axes of the conic (1),

$$2PQ^{2}=r_{1}^{2}+r_{2}^{3};$$

$$\therefore 2[(x-x_{1})^{3}+(y-y_{1})^{2}] = \frac{A+B-k(A^{2}x_{1}^{2}+B^{2}y_{1}^{2})}{{}_{5}AB(1-k)}$$

(Use the equation to determine the length of the axes of a central conic given by the general equation of the second degree.)

Now substituting for  $x_1$ ,  $y_1$  and k from (2) and (3), the above equation becomes.

$$2\left[1-\frac{1}{\lambda}\right]^2(x^2+y^3) = \frac{A+B-\frac{A^2x^2+B^2y^3}{\lambda(\lambda-1)}}{\frac{AB}{(\lambda-1)^3}},$$

which, when simplified and  $Ax^2 + By^2$  is written for  $\lambda^2$ , reduces to

$$(x^3+y^2)\sqrt{Ax^2+By^2} = \left(\frac{1}{A} - \frac{1}{B}\right)(Ax^2-By^3)$$

giving the equation of the locus required.

## Question 845.

(M. K. Kewaleamani):—ABC is a triangle such that cos 2 B, cos 2 A, cos 2 C are in A. P. A rectangular hyperbola circumscribes this triangle and passes through its centroid. AD is the common chord of this hyperbola and the osculating circle at A meeting the opposite side in D. show that AD divides BC in the ratio  $\sin \overline{A-B}$  [ $b \sin \overline{B-C}-a \sin \overline{C-A}$ ]:  $\sin \overline{C-A}$  [ $c \sin \overline{B-C}-a \sin \overline{A-B}$ ].

Solution by Hemraj, S. V. Venkatachıla Aiyar.

If  $\cos 2 B + \cos 2 C = 2 \cos 2 A$ , it is easy to show that

$$b^2 + c^3 = 2 a^2, \frac{\sin (B - C)}{\sin (C - A)} = -2 \frac{b}{a}, \frac{\sin (B - C)}{\sin (A - B)} = -2 \frac{c}{a} \dots$$
 (1)

From the last two relations of (1), it follows at once that the ratio sin A - B [b sin B - C - a sin C - A]:

$$\sin \overline{C-A} \left[C \sin \overline{B-C} - a \sin \overline{A-B}\right] = 2b^2 + a^2 : 2c^2 + a^2$$

So we have to show that AD divides CB in the ratio

$$2b^2 + a^2$$
:  $2c^2 + a^2$ .

Now take the triangle ABC as the triangle of reference, and the aquation of the rectangular hyperbola is

$$uyz + \dot{v}zx + wxy = 0$$

with the conditions  $bc \cos A u + ca \cos Bv + ab \cos C w = 0$ and 4+ from which we have

 $\frac{u}{b^2-c^2} = \frac{v}{c^2-a^2} = \frac{w}{a^2-b^2}.$ 

Hence the equation of the rectangular hyperbola takes the form  $(b^2-c^2)$   $yz+(c^2-a^2)$   $zx+(a^2-b^2)$  xy=0.

A is (1, 0, 0) and the tangent thereat is  $(a^2-b^2)y+(c^2-a^2)z=0$ ; i.e. y+z=0 from the first relation of (1).

And any line through A is  $\mu y + vz = 0$ .

Hence the equation of a conic having a contact of the 2nd order at A with the hyperbola is

 $2(b^2-c^2)yz + 2(c^2-a^2)zx + 2(a^2-b^2)xy + (y+z)(\mu y + vz) = 0$ 

It represents a circle if

$$\frac{\mu + v - 2 \left(b^{\frac{3}{2}} - c^{\frac{3}{2}} + \frac{1}{2} \mu + \frac{1}{2} v\right)}{a^{2}} = \frac{v - 2 \left(c^{2} - a^{2}\right)}{b^{4}} = \frac{\mu - 2 \left(a^{2} - b^{2}\right)}{c^{3}},$$

which gives

$$\mu = 2 (u^3 - b^2) - 2 \frac{c^2}{a^2} (b^9 - c^2) = \frac{2}{a^2} [(c^2 - a^2) a^2 - c^2 (b^9 - c^2)]$$
$$= \frac{2}{a^2} (c^9 - a^2) (2 c^2 + a^2)$$

and 
$$v=2(c^2-a^2)-2\frac{b^2}{a^2}(b^2-c^2)=\frac{2}{a^2}[(a^2-b^2)a^2-b^3(b^2-c^2)]$$
  
=\frac{2}{a^2}(a^2-b^2)(2b^2+a^2) from (1).

 $\therefore v/\mu = (2b^2 + a^2)/(2c^2 + a^2).$ 

Now the equation of the chord of curvature is evidently  $\mu y + vz = 0$ and it divides CB in the ratio  $\frac{-y}{z}$  i.e.  $\frac{v}{u}$  i.e.  $(2b^2+a^2)/(2c^2+a^2)$ . Hence the result.

# Question 862.

(Communicated by Mr. Hem Raj) :- Angles A, B, C of a triangle are trisected by AQ, AR; BR, BP; CP, CQ. Show that the triangle PQR is equilateral.

Solution (1) by F. H. V. Gulasekaram, H. R. Kapadia, Sadanand, K R. Rama Iyer, (2) by Hemraj and S. V. Venkatachala Aiyan.

(1) Let P, Q, R be such that 
$$\stackrel{\wedge}{CBP} = \frac{B}{3}$$
,  $\stackrel{\wedge}{PCB} = \frac{C}{3}$ ,  $\stackrel{\wedge}{QCA} = \frac{C}{3P}$ .

 $\stackrel{\wedge}{CAQ} = \frac{A}{3}$ ,  $\stackrel{\wedge}{RAB} = \frac{A}{3}$ ,  $\stackrel{\wedge}{RBA} = \frac{B}{3}$ .

Let BR, CQ meet at D; CP, AR meet at E, and AQ, BP meet at F. Hence P is the incentre of the △ DBC

$$\begin{aligned} & \vdots & \text{BP} = a \sin \frac{C}{3} \operatorname{cosec} \left( \frac{B+C}{3} \right) \\ & = a \sin \frac{C}{3} \operatorname{cosec} \left( 60 - \frac{A}{3} \right) \\ & \text{Also} & a = 2R \sin A = \frac{R}{2} \sin A/3 \sin \left( 60 + \frac{A}{3} \right) \sin \left( 60 - \frac{A}{3} \right) \\ & \vdots & \text{BP} = \frac{R}{2} \sin \frac{A}{3} \sin \frac{C}{3} \sin \left( 60 + A/3 \right). \\ & \text{Similarly BR} = \frac{R}{2} \sin \frac{A}{3} \sin \frac{C}{3} \sin \left( 60 + C/3 \right) \end{aligned}$$

Similarly BR =  $\frac{R}{2} \sin \frac{A}{2} \sin \frac{C}{2} \sin (60 + C/3)$ 

$$PR^{2} = \frac{R^{3}}{4} \left\{ \sin^{2}\left(\frac{A}{3}\right) \sin^{2}\left(\frac{C}{3}\right) \right\} \left[ \sin^{2}\left(60 + \frac{A}{3}\right) + \sin^{2}\left(60 + \frac{C}{3}\right) - 2\cos\frac{B}{3}\sin\left(60 + \frac{A}{3}\right)\sin\left(60 + \frac{C}{3}\right) \right]$$

$$= \frac{R^{3}}{4}\sin^{2}\left(A/3\right)\sin^{3}\left(B/3\right)\sin^{2}\left(C/3\right).$$

$$PR = \frac{1}{2}R\sin\frac{A}{3}\sin\frac{B}{3}\sin\frac{C}{3}.$$

Similarly  $RQ = QP = \frac{1}{2}R \sin A/3 \sin B/3 \sin C/3$ .

Hence the triangle PQR is equilateral.

We can similarly shew that the  $\triangle$  DEF is also equilateral.

(2) Put  $A = 3\lambda$ ,  $B = 3\mu$ ,  $C = 3\nu$ . In  $\triangle AQR$ ,

$$\begin{split} \frac{\sin \alpha}{\sin \beta} &= \frac{\dot{A}R}{\dot{A}Q} = \frac{c\sin \mu}{\sin(\lambda + \mu)} \times \frac{\sin(\lambda + \nu)}{b \sin \nu} \\ &= \frac{\sin 3\nu \sin \mu \sin(\lambda + \nu)}{\sin 3\mu \sin \nu \sin(\lambda + \mu)} \\ &= \frac{(3 - 4 \sin^2\nu)\sin(60^\circ - \mu)}{(3 - 4 \sin^2\nu)\sin(60^\circ - \nu)} = \frac{\sin(60^\circ + \nu)}{\sin(60^\circ + \mu)} \end{split}$$

There arise three cases . (a)  $\alpha = 60^{\circ} + \nu$ ,  $\beta = 60^{\circ} + \mu$ .

(b)  $\alpha = 60^{\circ} + \nu$ ,  $\beta = 120^{\circ} - \mu$ ; or  $\alpha = 120^{\circ} - \nu$ ,  $\beta = 60 + \mu$ ; (inadmissible).

(c)  $\sin \alpha = k_1 \sin (60^\circ + \nu)$ ,  $\sin \beta = k_1 \sin (60^\circ + p_1)$ . It is easy to show that (c) is also inadmissible, as

$$\frac{\sin \alpha}{\sin (60^{\circ} + \nu)} = \frac{\sin \beta}{\sin (60^{\circ} + \mu)}$$

and 
$$\frac{AR}{\sin \alpha} = \frac{AQ}{\sin \beta} = \frac{QR}{\sin \lambda}$$
,  $\therefore \frac{AR}{k_1 \sin(60^o + \nu)} = \frac{AQ}{k_1 \sin(60^o + p_1)} = \frac{QR}{\sin \lambda}$ 

$$i.e. \frac{AR}{\sin \alpha'} = \frac{AQ}{\sin \beta'} = \frac{QR}{\sin \lambda}$$

Now  $\alpha+\beta+\lambda=180^\circ$  and  $60^\circ+\nu+60+\mu+\lambda=180^\circ$ , it follows that  $\alpha'+\beta'+\lambda=180^\circ$  unless  $k_1=1$ 

Hence (a) is the only admissible solution.

$$\therefore \quad \alpha = 60^{\circ} + \nu, \ \beta = 60^{\circ} + \mu, \ r = 60^{\circ} + \lambda$$

$$\angle RPQ = 360^{\circ} - \angle BPC - \alpha - \beta = 60^{\circ} = \angle PRQ = \angle PQR$$

:. PQR is equilateral.

# Question 913.

(PROF. K. J. SANJANA):—If the expansion of (see  $x + \tan x$ ) is written in the form

$$1 + S_1 - \frac{x}{1!} + S_3 + \frac{x^2}{2!} + S_3 + \frac{x^3}{3!} + \dots$$

prove that

$$S_n = 2^{n+2} I/\pi^{n+1}$$

where 
$$I \equiv \int_0^a \frac{(\log \sec x)^n}{\sin x} dx$$
 or  $\log (\sqrt{2}+1) \int_0^b (\log \operatorname{cosech} x)^n / \cosh x dx$ ,

according as n is odd or even and  $a = \frac{1}{2}\pi$ ,  $b = \log(\sqrt{2} + 1)$ .

Solution by Hemraj, R. Srinivasan, S. R. Ranganathan, C. Krishnamachary, H. R. Kapadia, A. K. Krishnawami Iyer, and M. K. Kewalramani.

From Chap. XVII. Hobson's Trignometry, or Arts. 573, 574: Diff. Cal. by Edwards, we have

$$\sec x + \tan x = 1 + S_1 \frac{x}{11} + S_2 \frac{x^2}{21} + S_3 \frac{x^3}{31} + \dots$$

whone

$$\mathbf{S}_n = \frac{2^{n+s_n}!}{\pi^{n+1}} \left\{ 1 + \left( -\frac{1}{3} \right)^{n+1} + \left( \frac{1}{5} \right)^{n+1} + \left( \frac{1}{7} \right)^{n+1} + \left( \frac{1}{9} \right)^{n+1} + \dots \right\}$$

Hence we have to show that

(i) 
$$\int_{0}^{a} \frac{(\log \sec x)^{2n-1}}{\sin x} dx = (2n-1)! \left\{ 1 + \frac{1}{3^{2n}} + \frac{1}{5^{2n}} + \dots \right\}$$
(ii) 
$$\int_{0}^{a} \frac{(\log \operatorname{cosech} x)^{n}_{1}}{\cosh x} dx = 2n! \left\{ 1 - \frac{1}{3^{2n+1}} + \frac{1}{5^{2n+1}} - \frac{1}{7^{2n+1}} + \dots \right\}.$$

#### Question 919.

(C. Krishnamachart):—Prove that  $1 - \frac{1}{2} - \frac{1}{5} + \frac{1}{7} + \frac{1}{0} - \frac{1}{11} - \frac{1}{12} + \dots = \frac{1}{\sqrt{2}} \log (1 + \sqrt{2})$ 

Solution (1) by K. J. Sanjana and H. R. Kapadia; (2) by S. R. Rans ganathan, R. Srinivasan and G. A. Kamtekar, (3) Hemraj,

Sadanand and N. B. Mitra.

Let  $S = x - \frac{x^3}{3} - \frac{x^4}{5} + \frac{x^7}{7} + \frac{x^9}{9} - \dots;$ then  $\frac{dS}{dx}$ 

is the sum of two G.P.'s. Finding their sums and integrating we shall be able to evaluate S. The following is a trigonometrical method.

In Gregory's series change x to  $x\sqrt{\iota}$  and multiply by  $\sqrt{\iota}$ ; we get  $\sqrt{\iota}$  tan<sup>-1</sup> $(x\sqrt{\iota}) = \iota \left(x - \frac{x^5}{5} + \frac{x^9}{9} - \dots\right) + \left(\frac{x^9}{3} - \frac{n^7}{7} + \frac{x^{11}}{11} = \dots\right)$ =  $\iota S_1 + S_2$ , suppose.

Then  $S=S_1-S_2$ , and we have thus to separate  $\sqrt{t}$  tan-1  $z\sqrt{t}$  into real and imaginary parts. Taking  $+\sqrt{t}\equiv (1+t)/\sqrt{2}$ , we have

$$\begin{split} &\frac{1+i}{\sqrt{2}}\tan^{-1}\left(\frac{x}{\sqrt{2}}+\frac{x_{2}}{\sqrt{2}}\right)\\ =&\frac{1+i}{\sqrt{2}}\left\{\frac{1}{2}\tan^{-1}\frac{x\sqrt{2}}{1-\frac{1}{2}x^{2}-\frac{1}{2}x^{2}}+\frac{1}{2}i\tanh^{-1}\frac{x\sqrt{2}}{1+\frac{1}{2}x^{2}+\frac{1}{2}x}\right\}\\ =&\frac{1+i}{2\sqrt{2}}\left\{\tan^{-1}\frac{x\sqrt{2}}{1-x^{2}}+i\tanh^{-1}\frac{x\sqrt{2}}{1+x^{2}}\right\}, \end{split}$$

by the usual procedure. This gives at once

$$\begin{aligned} \mathbf{S}_{1} &= \frac{1}{2\sqrt{2}} \left( \tan^{-1} \frac{x\sqrt{2}}{1-x^{2}} + \ \tanh^{-1} \frac{x\sqrt{2}}{1+x^{2}} \right), \ \mathbf{S}^{1} &= \frac{1}{2\sqrt{2}} \\ & \left( \tan^{-1} \frac{x\sqrt{2}}{1+x^{3}} - \ \tanh^{-1} \frac{x\sqrt{2}}{1+x^{3}} \right), \end{aligned}$$

whence

$$S = \frac{1}{2\sqrt{2}} \left\{ 2 \tanh^{-1} \frac{x\sqrt{2}}{1+x^2} \right\} = \frac{1}{2\sqrt{2}} \log \frac{1+x\sqrt{2}+x^3}{1-x\sqrt{2}+x^3}$$

The result in the question follows by making z=1, which is allowable as the signs are - and + in pairs.

We also get

$$S_1 + S_2 = \frac{1}{\sqrt{2}} \tan^{-1} \frac{x\sqrt{2}}{1-x^2}$$

so that

$$1 + \frac{1}{3} - \frac{1}{5} - \frac{1}{7} + \frac{1}{9} + \frac{1}{11} - \dots = \frac{\pi}{2\sqrt{2}}.$$

(2)

We have for  $-\frac{\pi}{2} < x < \frac{\pi}{2}$ 

(Bromwich: Art. 90 Ex. 2).

$$e^{xi} - \frac{e^{xxi}}{3} + \frac{e^{xxi}}{5} \dots = \frac{\pi}{4} + \frac{1}{2}i \log (\sec x + \tan x)$$

Equating the imaginary parts, we have,

$$\sin x - \frac{\sin 3x}{3} + \frac{\sin 5x}{5} - \dots = \frac{1}{2} \log (\sec x + \tan x)$$

Putting  $x = \frac{\pi}{4}$ ,

we get,

$$1 - \frac{1}{3} - \frac{1}{5} + \frac{1}{7} + \frac{1}{9} = -\sqrt{2\log(\sqrt{2} + 1)}$$
$$= \frac{1}{\sqrt{2}} \log(\sqrt{2} + 1)$$

(3) It follows from  $\tan^{-1}(\cos\theta + i\sin\theta)$  or  $\frac{1}{2}\log\left(\frac{1+Z}{1-Z}\right)$ 

that if  $0 < x < \frac{\pi}{2} \sin x - \frac{1}{3} \sin 3x + \frac{1}{5} \sin 5x - \frac{1}{7} \sin 7x + \dots$ =  $\frac{1}{2} \log (\sec x + \tan x)$ 

Putting  $z = \frac{\pi}{4}$  we have the result.

# Questions 919, & 968.

919. (C. Krishnamachary):-Prove that

$$1 - \frac{1}{3} - \frac{1}{5} + \frac{1}{7} + \frac{1}{9} - \frac{1}{11} - \frac{1}{13} + \frac{1}{\sqrt{2}} \log (1 + \sqrt{2})$$

968. (V. ANANTARAMAN) :- Sum the series

$$\frac{1}{3} - \frac{1}{7} + \frac{1}{11} - \frac{1}{15} + \frac{1}{19} \dots$$

Solution (i) by T. H. V. Gulasekharam, N. G. Leather and S. Muthukrishnan; (ii) L. S. Vaidyanathan and M. K. Kewalramani.

(i) We have on page 303, Hobson's Plane Trigonometry, the formula  $\log (2 \cos \frac{1}{2} \theta) = \cos \theta - \frac{1}{2} \cos 2 \theta + \frac{1}{3} \cos 3 \theta - \frac{1}{4} \cos 4 \theta + \dots;$ 

write  $\theta = \frac{\pi}{4}$ .

Then remembering that

$$\log 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$$

we deduce

$$1 - \frac{1}{3} - \frac{1}{5} + \frac{1}{7} + \frac{1}{9} - \frac{1}{11} - \frac{1}{13} + \dots = \frac{1}{\sqrt{2}} \log (1 + \sqrt{2}) \cdot \dots$$
 (1)

Again, we have the formula

$$\frac{1}{2}\theta = \sin\theta - \frac{1}{2}\sin 2\theta + \frac{1}{8}\sin 3\theta - \frac{1}{4}\sin 4\theta + \dots$$
;

write  $\theta = \frac{\pi}{4}$ .

Then remembering

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$$

we deduce that

$$1 + \frac{1}{3} - \frac{1}{5} - \frac{1}{7} + \frac{1}{9} + \frac{1}{11} - \frac{1}{13} - \dots = \frac{\pi}{2\sqrt{2}} \qquad \dots \qquad \dots \qquad (2)$$

Hence subtracting (1) from (2) and dividing the result by 2,

$$\frac{1}{3} - \frac{1}{7} + \frac{1}{11} - \frac{1}{15} + \frac{1}{19} - \dots = \frac{\pi}{4\sqrt{2}} - \frac{1}{2\sqrt{2}} \log(1 + \sqrt{2}).$$

(ii) Consider the equality.

$$\frac{1}{a} - \frac{1}{a+b} + \frac{1}{a+2b} - \frac{1}{a+3b} + \dots = \int_{-\infty}^{\infty} \frac{x^{a-1}}{1+x^b} dx \quad (a, b, >0)$$

Here by giving particular values to a and b we can get the sum of not only the above series, but also of many others of its type.

The given series corresponds to the values a=3 and b=4.

Hence S = 
$$\int_{0}^{\infty} \frac{x^{2}dx}{1+n^{4}}$$
  
i.e. S =  $\frac{1}{4\sqrt{2}} \left[ \log \frac{x^{2}-x\sqrt{2}+1}{x^{4}+x\sqrt{2}+1} \right]_{0}^{\prime} + \frac{1}{2\sqrt{2}}$   
 $\left[ \tan^{-1}(x\sqrt{2}+1) + \tan^{-1}(x\sqrt{2}-1) \right]$   
=  $\frac{1}{4\sqrt{2}} \log \frac{2-\sqrt{2}}{2+\sqrt{2}} + \frac{1}{2\sqrt{2}} \left[ \tan^{-1}(\sqrt{2}+1) + \tan^{-1}(\sqrt{2}-1) \right]$   
=  $\frac{1}{4\sqrt{2}} \log \frac{\sqrt{2}-1}{\sqrt{2}+1} + \frac{\pi}{4\sqrt{2}}$   
=  $\frac{\pi}{4\sqrt{2}} - \frac{1}{2\sqrt{2}} \log (\sqrt{2}+1)$ 

#### Question 921.

(H. R. KAPADIA) :- Prove that the infinite product

$$(1+x^{2H})\left(1+\frac{x^{2H}}{3^{2H}}\right)\left(1+\frac{x^{2H}}{5^{2H}}\right) \qquad \cdots \qquad \cdots$$

$$=\frac{1}{n}\prod_{1}\left(\cosh \pi x\alpha_{r}+\cos \pi x\beta_{r}\right)$$

$$2^{\frac{1}{2}}$$

$$or = \frac{1}{2^{\frac{1}{2}(n-1)}} \cosh \frac{\pi x}{2} \prod_{1} (\cosh \pi x \alpha_{r} + \cos \pi x \beta_{r}).$$

according as n is even or odd, if  $\alpha_n$   $\beta_r$  denote  $\sin \frac{r\pi}{2n}$  and  $\cos \frac{r\pi}{2n}$  respectively, where r is an odd number.

Solution by Hemraj and M. K. Kewalramani.

Consider the equation 
$$\frac{x^{2n}}{k^{2n}}+1=0$$
; clearly  $\frac{x}{k}=\cos\frac{2\rho+1}{2n}\pi+i$   

$$\sin\frac{2\rho+1}{2n}\pi \text{ where } \rho=0,\ 1,2 \quad \dots \quad (2n-1)$$
i.e. if  $r$  is odd  $=\cos\frac{r\pi}{2n}+i\sin\frac{r\pi}{2n}=\beta_r+i\alpha$ ,

If one root is w, the other must be—w, the third  $w^{-1}$ , and the fourth  $(-w^{-1})$ ,—provided n is even. But if n is odd, there are two roots  $\pm i$  corresponding to  $\rho = \frac{n-1}{2}$  and  $\rho = n + \frac{n-1}{2}$ ; and the other (2n-2) roots can be divided into fours. Also it is evident that if any other factor be equated to zero, the roots will depend on k and w's must be the same.

Hence 
$$\frac{x^{1n}}{k^{2n}} + 1 = \prod_{\rho = 0}^{n-1} \left( \frac{x^2}{k^2} - \omega_{\rho}^2 \right) = (-1)^n \prod_{\rho = 0}^{n-1} (\omega_{\rho})^n$$

$$\times \prod_{\rho = 0}^{n-1} \left( 1 - \frac{x^2}{k^2 \omega_{\rho}^2} \right) = \prod_{\rho = 0}^{n-1} \left( 1 - \frac{x^2}{k^2 \omega_{\rho}^2} \right)$$

 $\therefore \prod (\boldsymbol{\psi_{\rho}}^2) = \pm 1 \text{ according as } n \text{ is even or rodd.}$ 

Hence 
$$\frac{\infty}{1} \left(1 + \frac{x^{2n}}{k^{2n}}\right) = \frac{\infty}{k-1} \left[\frac{n-1}{p-0} \left(-\frac{x^2}{k^2 \omega^2 \rho}\right)\right]$$

$$= \frac{n-1}{0} \left[\frac{\infty}{1} \left(1 - \frac{x^2}{k^2 \omega^2 \rho}\right)\right]$$

$$= \frac{n-1}{0} \left(\cos \frac{x}{2\omega \rho}\right) \quad \therefore k \text{ is odd} \quad \dots \quad (A)$$

$$\frac{2n-1}{0} \left\{\cos \frac{\pi x}{2} (\beta_r - i\alpha_r)\right\} \quad \therefore r = 2 \rho + 1$$

$$= \frac{2n-1}{1} \left(\cos \frac{1}{2} \pi x \beta_r \cosh \frac{1}{2} \pi x \alpha_r + i \sin \frac{1}{2} \pi x \beta_r \sinh \frac{1}{2} \pi x \alpha_r\right)$$

$$= \frac{n-1}{1} \left(\cos^2 \frac{1}{2} \pi x \beta_r \cosh^2 \frac{1}{2} \pi x \alpha_r + \sin^2 \frac{1}{2} \pi x \beta_r \sinh^2 \frac{1}{2} \pi x \alpha_r\right)$$

$$= \frac{n-2}{1} \left(\cos^2 \frac{1}{2} \pi x \beta_r \cosh^2 \frac{1}{2} \pi x \alpha_r + \sin^2 \pi x \beta_r \sinh^2 \frac{1}{2} \pi x \alpha_r\right)$$

$$= \frac{n-2}{1} \left(\cos^2 \frac{1}{2} \pi x \beta_r \cosh^2 \frac{1}{2} \pi x \alpha_r + \sin^2 \pi x \beta_r \sinh^2 \frac{1}{2} \pi x \alpha_r\right)$$

$$= \frac{n-2}{1} \left(\cos^2 \frac{1}{2} \pi x \beta_r \cosh^2 \frac{1}{2} \pi x \alpha_r + \sin^2 \pi x \beta_r \sinh^2 \frac{1}{2} \pi x \alpha_r\right)$$

$$= \frac{n-2}{1} \left(\cos^2 \frac{1}{2} \pi x \beta_r \cosh^2 \frac{1}{2} \pi x \alpha_r + \sin^2 \pi x \beta_r \sinh^2 \frac{1}{2} \pi x \alpha_r\right)$$

$$= \frac{n-2}{1} \left(\cos^2 \frac{1}{2} \pi x \beta_r \cosh^2 \frac{1}{2} \pi x \alpha_r + \sin^2 \pi x \beta_r \sinh^2 \frac{1}{2} \pi x \alpha_r\right)$$

$$= \frac{n-2}{1} \left(\cos^2 \frac{1}{2} \pi x \beta_r \cosh^2 \frac{1}{2} \pi x \alpha_r + \sin^2 \pi x \beta_r \sinh^2 \frac{1}{2} \pi x \alpha_r\right)$$

$$= \frac{n-2}{1} \left(\cos^2 \frac{1}{2} \pi x \beta_r \cosh^2 \frac{1}{2} \pi x \alpha_r + \sin^2 \pi x \beta_r \sinh^2 \frac{1}{2} \pi x \alpha_r\right)$$

$$= \frac{n-2}{1} \left(\cos^2 \frac{1}{2} \pi x \beta_r \cosh^2 \frac{1}{2} \pi x \alpha_r + \sin^2 \frac{1}{2} \pi x \beta_r \sinh^2 \frac{1}{2} \pi x \alpha_r\right)$$

$$= \frac{n-2}{1} \left(\cos^2 \frac{1}{2} \pi x \beta_r \cosh^2 \frac{1}{2} \pi x \alpha_r + \sin^2 \frac{1}{2} \pi x \beta_r \sinh^2 \frac{1}{2} \pi x \alpha_r\right)$$

$$= \frac{n-2}{1} \left(\cos^2 \frac{1}{2} \pi x \beta_r \cosh^2 \frac{1}{2} \pi x \alpha_r + \sin^2 \frac{1}{2} \pi x \beta_r \sinh^2 \frac{1}{2} \pi x \alpha_r\right)$$

$$= \frac{n-2}{1} \left(\cos^2 \frac{1}{2} \pi x \beta_r \cosh^2 \frac{1}{2} \pi x \alpha_r + \sin^2 \frac{1}{2} \pi x \beta_r \sinh^2 \frac{1}{2} \pi x \alpha_r\right)$$

$$= \frac{n-2}{1} \left(\cos^2 \frac{1}{2} \pi x \beta_r \cosh^2 \frac{1}{2} \pi x \alpha_r + \sin^2 \frac{1}{2} \pi x \beta_r \sinh^2 \frac{1}{2} \pi x \alpha_r\right)$$

$$= \frac{n-2}{1} \left(\cos^2 \frac{1}{2} \pi x \beta_r \cosh^2 \frac{1}{2} \pi x \alpha_r + \sin^2 \frac{1}{2} \pi x \alpha_r\right)$$

$$= \frac{n-2}{1} \left(\cos^2 \frac{1}{2} \pi x \beta_r \cosh^2 \frac{1}{2} \pi x \alpha_r + \sin^2 \frac{1}{2} \pi x \alpha_r\right)$$

$$= \frac{n-2}{1} \left(\cos^2 \frac{1}{2} \pi x \beta_r \cosh^2 \frac{1}{2} \pi x \alpha_r\right)$$

$$= \frac{n-2}{1} \left(\cos^2 \frac{1}{2} \pi x \beta_r \cosh^2 \frac{1}{2} \pi x \alpha_r\right)$$

$$= \frac{n-2}{1} \left(\cos^2 \frac{1}{2} \pi x \beta_r \cosh^2 \frac{1}{2} \pi x \alpha_r\right)$$

$$= \frac{n-2}{1} \left(\cos^2 \frac{1}{2} \pi x \beta_r \cosh^2 \frac{1}{2} \pi x \alpha_r\right)$$

$$= \frac$$

[Notice that  $\cosh \frac{1}{2} \pi x$  corresponds to  $\omega_{\rho} = i$  and that if one factor in (A) is  $\cos \frac{\pi x}{2\omega}$ , the other must be  $\cos \frac{\pi x}{2}\omega_{\rho}$ 

$$= \frac{1}{2^{\frac{1}{2}n}} \prod_{i=1}^{n-1} (\cosh \pi x_i \alpha_r + \cos \pi x_i \beta_r) \text{ or } =$$

$$\frac{\cosh \frac{1}{2} \pi x}{2^{\frac{1}{2}(n-1)}} \frac{n-1}{ } \left( \cosh \pi x \, \alpha_r + \cos \pi x \, \beta_r \right)$$

Note.—This question was originally constructed by Glaisher. See. Hobson's Trigonometry. Q. 32 Ch. XVII.

# Question 924.

(S. MALHARI RAU.) :- Complete the following pandiagonqual Bears.

24				
	30		-	-
		1		
		1		
			7	
				13

Solution by R. J. Pocock and Sadanand The numbers given may be written 20+4, 25+5, 0+1, 5+2, 10+3.

Consider the following squares A, B and C. If each number in A is multiplied by 5 and added to the corresponding number in as we get the required pandiagonal magic square C.

	Α					
4	0	2	5	1		
2	5	1	4	0		
1	4	0	2	5		
0	2	5	1	4		
5	1 .	4	0	2		

		В		
4	3	2	1	5
1	5	4	3	2
3	2	1	5	4
5	4	3	2	1
2	1	5	4	3

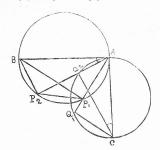
	-		_	-
24	3	. 12	26	10
11	30	9	23	2
8	22	1	15	29
5	14	28	7	21
27	6	25	4	13

# Question 926.

(S. R. RANGANATHAN):—If AB, AC be two straight lines intersecting at right-angles  $P_1$ ,  $P_2$ |any two points on the circle on A B as diameter and  $Q_1$ ,  $Q_2$  the points of intersection of  $P_1$  A and  $P_2$  A respectively with the circle on AC as diameter, show that  $P_1P_2$  and  $Q_1Q_2$  are at right angles to each other.

Solution by R.J. Pocock, P.M.H. Baker, C. Krishnamachary, R. Srinivasan K.B. Madhava, P.R. Venkatakrishnaiya, Hemraj and many others.

In the circle  $AP_1F_2 B \angle P_2AB = \angle P_2P_1B$ .



Also  $P_2AB = AQ_1Q_2$  since BA touches the circle  $ACQ_1Q_2$ .  $\therefore P_3P_1Q_1 + P_1Q_1Q_2 = BP_1Q_1 = 90^\circ.$ 

Hence P1P2 and Q1Q2 are at right angles.

[An obvious extension of the Question is the following :-

"Any two circles intersect at A and A  $P_1Q_1$ , A  $P_2Q_2$  are drawn to cut them again in  $(P_3,Q_1)$ ,  $(P_2,Q_2)$ , Show that the angle between  $P_1P_2$  and  $Q_1Q_2$  is equal to the angle between the circles."]

# Questions 942 and 943.

942. (HEMRAJ): - Show that if p be prime

$$\left[\sum_{1}^{p-1} \left(\frac{1}{n}\right) \div \frac{p^{n}}{p!}\right] \text{ is an integer}$$

943. (HEMRAJ) :- If w be prime and

$$(x+1) (x+2).....(x+n-1)$$

$$\equiv x^{n-1} + A_1 x^{n-2} + A_2 x^{n-3} + ... A_{n-1}$$

shew that all the odd A's except A, are divisible n2.

Solution and Remarks by H. Br.

Writing f(x) for the given expression, we have  $f(-x) = (-x+1) (-x+2) \dots (-x+n-1)$   $= (x-1) (x-2) \dots (x-n+1)$   $= (x-n+n-1) (x-n+n-2) \dots (x-n+1)$  = f(x-n)  $= f(x) - n! f'(x) + \frac{n^2}{2} f''(x) - \dots$   $= f(x) - n \{ (n-1) x^{n-2} + (n-2) | \Lambda_1 x^{n-2} + \dots \} + \dots$   $= f(x) - n (n-1) x^{n-2} + n^2 F(x),$ 

where F (x) is a polynomial in x with integral co-efficients. [This follows from the fact that the A's, upto  $A_{n-2}$  are multiples of n.]

Hence

$$f(x)-f(-x)\equiv n \ (n-1) \ x^{n-2}+0 \ (\text{mod}, \ n^2).$$

This being an identity, we can equate co-efficients of powers of x. We get

$$A_1 = \frac{1}{2} n \ (n-1); \ A_8 \equiv A_5 \equiv \dots \equiv A_{n-2} \equiv 0 \ (\text{mod}, \ n^2).$$

Note: By considering terms involving  $n^2$  and  $n^8$ , we get 2  $A_k \equiv n \ (n-k) \ A_{k-1}$ 

where k is odd and >3. This is easily proved.

Remarks on Q. 943 by Mr. N. G. Leathar St. Stephen's College, Delhi.

This problem was set by me in the Punjab University M. A. Exam. of this year in the following form:—

By considering the equation 
$$(x-1)(x-2)...(x-p+1)$$
  

$$\equiv x^{b-1} - C_1x^{b-2} + C_2x^{b-2} + ...(-)^{b-1}(p-1) != 0$$

show from the results of Qn. IV that, if p be a prime all the coefficients  $C_0$ ,  $C_2$ ,...are divisible by p, and that the odd coefficients  $C_3$ ,  $C_5$ ,...are divisible by  $p^2$ .

This was set as a rider to Newton's relations between the sums of the powers of the roots and the coefficients of an algebraical equation. The reference to Qn. IV is to the theorem that:-

If  $S_r$  denote the sum of the  $r^{th}$  powers of the first (p-1) natural numbers,  $S_{2r+1}=S_1^g$ .  $F(S_1)$  and  $S_{2r}=S_2$   $G(S_1)$ , where F and G are polynomials of degree (r-1).

Newton's relations give

$$r C_r - S_1 C_{r-1} + S_2 C_{r-2} - S_3 C_{r-3} + \dots + (-)^r S_r = 0 \dots (r = 1, 2, \dots, p-1)$$
 (A)

I. S<sub>1</sub>, S<sub>2</sub>, ...S<sub>r</sub> are all divisible by p (assumed); therefore since r is prime to p, C, is divisible by p, for all values of r from 1 to (p-1).

II. If r is odd and  $\not \subset 3$ ,  $S_r$  is divisible by  $p^s$ ,  $S_1, S_2 \dots S_{r-1}$  are all divisible by p, and  $C_{r-1}, C_{r-2} \dots C_1$  are also all divisible by p,

Thus every term of (A), except the first, is divirible by  $p^{s}$ . Also r is prime to p.

Hence  $C_r$  is divisible by  $p^2$ , if r is odd and  $\angle 3$ 

#### Question 944.

(SADANAND) :- Prove the identity

$$\phi(x) + S_1 \phi'(x) \cdot \frac{x}{1!} - S_2 \phi'''(x) \cdot \frac{x^8}{3!} + \dots$$
  
 $\equiv \phi(2x) - S_2 \phi''(2x) \cdot \frac{x^9}{2!} + S_4 \phi''(2x) \cdot \frac{x^4}{4!} \dots$ 

where  $S_{2n}$  is the  $n^{th}$  Eulerian number and  $S_{2n-1}$  is the  $n^{th}$  "prepared Bernoullian" number.

Deduce that 
$$\log 2 = \frac{S_1}{1} - \frac{S}{2^s.1} - \frac{S_s}{3} + \frac{S_4}{2^s.2} + \frac{S_s}{5} - \dots$$

Solution by A. C. L. Wilkinson.

In the identity 
$$\frac{2e^x}{e^x + e^{-x}} = 1 + S_1 x - S_8 \frac{x_8}{3!} + S_8 \frac{x^5}{5!} - \dots$$

where  $S_1$ ,  $S_2$ ,  $S_3$ , ... are the "prepared Bernoullians," put  $\lambda \frac{d}{dx}$  for x and operate with both sides on  $\phi$  (2), we get

$$\phi(x) + \lambda S_1 \phi'(x) - \frac{\lambda^2}{3!} S_3 \phi'''(x) + \dots = \frac{2}{e^{\lambda} \frac{d}{dx}} - \frac{e^{\lambda} \frac{d}{dx}}{e^{-\lambda} \frac{d}{dx}} \phi(x) - \frac{2}{e^{\lambda} \frac{d}{dx} + e^{-\lambda} \frac{d}{dx}} \phi(x)$$

$$= \phi(x+\lambda) - \frac{\lambda^2}{2!} S_2 \phi''(x+\lambda) + \frac{\lambda^4}{4!} S_4 \phi''''(x+\lambda) - \dots$$

Put  $\lambda = x$  and we have the result given.

If we put  $\phi(x) = \log x$  and then put  $\lambda = 1$ , we get  $\log 2$  in the form given which is however a divergent series.

#### Ouestion 949.

(C. Krishnamachary):—With the usual notation for the numbers of Bernoulli and Euler, show that

$$\begin{split} \mathbf{2n} = & \mathbf{B_r} \ 2^2 \left( 2^3 - 1 \right) \left( \frac{2^n}{1} \right) - \mathbf{B_s} \ 2^4 \left( 2^4 - 1 \right) \left( \frac{2^n}{4} \right) + \mathbf{B_s} \ 2^5 \left( 2^6 - 1 \right) \left( \frac{2^n}{6} \right) - \dots \\ & + (-1)^{n-1} \ \mathbf{B_n} \ 2^{2^n} \left( 2^{2^n} - 1 \right) \\ & \left( 2^n + 1 \right) - \mathbf{B_{n-1}} \ 2^{2^{n-2}} \left( 2^{2^{n-2}} - 1 \right) \\ & \left( \frac{2^n + 1}{1} \right) - \mathbf{B_{n-1}} \ 2^{2^{n-2}} \left( 2^{2^{n-2}} - 1 \right) \\ & + (-1)^{n-1} \ \mathbf{B_1} \ 2^2 \left( 2^5 - 1 \right) \left( \frac{2^n + 1}{2} \right) + (-1)^n \left( 2^n + 1 \right). \end{split}$$

Solution by A. C. L. Wilkinson.

The first result is proved in Edwards: Differential Calculus, second edition, p.502. It can also be proved as follows:

$$\begin{split} &\frac{2x}{e^{2x}-1} = 1 - n + \Sigma(-)^{n-1} \frac{2^{n}}{(2n)} B_n x^{2n} \\ &- \frac{4x}{e^{4x}-1} = 1 - 2x + \Sigma(-)^{n-1} \frac{2^{4n}}{(2n)} B_n x^{2n} \;; \end{split}$$

whence subtracting

$$\frac{2x}{e^{2}x+1} = x + \Sigma(-)^{n} \frac{2^{2n} (2^{2n}-1) - B_{n}}{(2n)!} x^{2n}$$
or
$$e^{-x} = \left\{ 1 + \Sigma(-)^{n} \frac{2^{2n} (2^{2n}-1) B_{n}}{2n!} x^{2n-1} \right\} \times \left\{ 1 + \sum_{n=1}^{\infty} \frac{x^{2n}}{2n!} \right\}.$$

Equating the coefficients of  $x^{gii-1}$ , we obtain the first result. Again, in

$$\begin{aligned} &\frac{2c^{x}}{e^{2x}+1} = \mathrm{sech} \ x = 1 + \Sigma(-)^{n} \ \mathrm{E}_{n} \frac{x^{2^{n}}}{(2n)} \\ &= \left\{ 1 + \Sigma(-)^{n} \ \mathrm{E}_{n} \frac{2^{2^{n}}(2n-1)}{(2n)!} x^{2^{n}-1} \right\} \times \left\{ 1 + \frac{\nabla x^{n}}{n!} \right\}, \end{aligned}$$

equating the coefficient of xon on both sides, we obtain the second result.

## QUESTIONS FOR SOLUTION.

**984.** (S. R. RANGANATHAN):—A prism has for its cross section a quadrilateral ABCD with  $B=75^{\circ}$ ,  $C=105^{\circ}$  and  $D=90^{\circ}$  and  $\sqrt{3}$  AB  $=\sqrt{2}$  BC. Show that a ray of light incident on the face AB at the angle sin -1 M and emerging out after once being totally reflected would undergo a deviation of 90°, M being the coefficient of refraction of the substance of the prism.

985. (M. K. Kewalramani) :- If A', B', C' are points in the sides

of the triangle ABC, such that

BA': A'C = CB': B'A = AC': C'B = M:1,

and an expression for the radius of the circumcircle of the triangle XYZ formed by the intersections of AA', BB', CC'.

986. (M. K. Kewalramani): -At any point P of a curve equal arcs PQ, PR are measured in the same direction along the curve and along the parabola of closest contact. Prove that ultimately

$$QR = \left\{ 3 + \frac{1}{3} \left( \frac{d\rho}{ds} \right)^2 - \rho \frac{d^2 \rho}{ds^2} \right\} \frac{(ds)^4}{24\rho^4}$$

987. (V. TIRUVENKATACHARI) :- Show th

$$\int_{0}^{\frac{1}{2}\pi} \sin \Theta(\log \cos \Theta) d\Theta = -\frac{1}{3} \left\{ 1 + \frac{2}{5} + \frac{2.4}{57} + \frac{2.4.6}{5.7.9} + \dots \right\} = -1.$$

988. (S MALHARI RAG) :- Complete the following magic squares of 16 cells, by placing 8 other prime numbers in the vacant cells:

7	43	71	113
		-	
73	97	23	41

989. (S. Malhari Rao) :- Give three solutions in positive integers of the equation

 $(x+y+z)^2=xyz$ . 990. (S. Malhari Rao):—Give a solution in positive integers of

$$(x_1 + x_2 + ... x_n)^{\frac{1}{2} {n-1 \choose 2}} = x_1.x_2.x_3...x_n$$

 $(x_1+x_2+\dots x_n)^{\frac{1}{2}\binom{n-1}{2}}=x_1.x_2.x_3\dots x_n$  when n is odd; and derive a method of solution of

$$(x_1+x_2+...x_n)^{\frac{1}{2}^n}=x_1x_2...x_n$$

when n is even and greater than 2.

Hence deduce an expression for any fraction  $\binom{1}{2}$  as the sum of a series of r fractions

$$\frac{1}{x_1} + \frac{1}{x_2} + \dots + \frac{1}{x_r}$$
.

- 991. (S. NARAYANAN):—ABC is a triangle whose incentre is f and orthocentre H. If the incentre to the ex-medial triangle of AB ! be K and KH bisected at L, prove that the nine-points centre of ABC is the middle point of LI.
- 992. (S. NARAYANAN):—The internal bisectors of the angles A, B,C of a triangle meet the sides of the medial triangle DEF at P, Q, R. Prove that the conic which touches these sides at P, Q, R touches also the principal Fenerbach tangent of DEF.
- 993 (COMMUNICATED BY PROF. HEMRAJ):—From point P perpendiculars PL, PM, PN are drawn to the sides of a triangle ABC. If AL, BM, CN meet at a point Q, find the locus of P and prove that PQ passes through a fixed point.
- 934. (K. B. Madhava):—Conduct the following game of Bridge with Hearts as trumps in which A has to lead and A and B together have to make 7 tricks.

Hearts: Q. Clubs: 7; 6. Spades: K; 8.

Hearts: 7; 6. Spades: K; 8.

Hearts: 9; 8.

Clubs: Q; J: 9.

Spades: None.

Hearts: A; K; 5. Diamonds: K.

Clubs: A; 10; 8; 4. Spades: None.

995. (T. P. TRIVEDI, M.A., LL.B.):-Prove that

$$\int_{0}^{1} x^{x} dx = \sum_{n=0}^{n=\infty} (-1)^{n} \left\{ \frac{1}{1^{n+1}} - n \cdot \frac{n+1}{2^{n+2}} + \frac{n^{2}}{2!} \frac{(n+1)(n+2)}{3^{n+1}} - \frac{n^{2}}{3!} \frac{(n+1)(n+2)(n+3)}{4^{n+4}} + \cdots \right\}$$

996. (T. P. TRIVEDI, M.A., LL. B.) :-Find the conditions that the conics

 $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ 

and

 $a'x^2 + 2h'xy + b'y^2 + 2g'x + 2f'y + c' = 0$ 

have one asymptote in common ..

997. (C. KRISHNAMACHARY):—A uniform chain in partly coiled on a table with one end just carried over a smooth pulley at a height 2h vertically above the coil, and to this end is attached a mass M. Discuss the motion, shewing that the system will come to rest when M reaches the table provided its weight is nearly equal to that of a length h of the chain.