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PROGRESS REPORT.

The committee regret to have to report the premature death at Chicago U. S, of Mr. B. K. Srinivasa Iyengar, M.A., who was an Associate of the Society since August 1910. The deceased was a brilliant graduate of the Madras University and held the post of Assistant Professor of Mathematics in the Central College, Bangalore. He went to America as a Mysore Government Scholar to study 'Systems of Education' some two years ago. In him the Society has lost an ardent lover of Mathematics and an enthusiastic member.

POONA,
31*st May* **1918.**

D. D. KAPADIA,
Honorary Secretary.

On the addition Formulae for the Jacobian Functions E and Π.

In the following pages I propose to deduce a few of the expressions for $Eu_1+Eu_2+Eu_3+Eu_4$ and those for $\Pi(u_1,a)+\Pi(u_2,a)+\Pi(u_3,a)+\Pi(u_4,a)$ in terms of the sn, cn, dn functions of u_1, u_2, u_3, u_4, when $u_1+u_2+u_3+u_4$ is congruent to zero.

§ I. Preliminary.

1. To begin with let us establish a few preliminary results which will be helpful to us in what follows.

For this purpose, let us consider the points of intersection of the fixed curve

$$y^2=x(1-x)(1-kx) \quad \dots \quad \dots \quad \dots \quad \text{(i)}$$

with any arbitrary chosen parabola

$$y=l+mx+nx^2 \quad \dots \quad \dots \quad \dots \quad \text{(ii)}$$

Now, if any point (x_1, y_1) be taken on the cubic (i), the equation in u

$$\text{sn}^2(u, k)-x_1=0, \text{ where } k \text{ is constant,}$$

has two solutions $+u_1, -u_1$, and all other solutions are congruent to these two.

Again since $\frac{\partial}{\partial u}(\text{sn}^2u)=2 \text{ sn } u \text{ cn } u \text{ dn } u$, we have

$$\left[\frac{\partial}{\partial u}(\text{sn}^2 u)\right]^2=4y^2.$$

Let us choose u_1 to be the solution for which $\frac{\partial}{\partial u}(\text{sn}^2u)=+2y_1$; and let (x_r, y_r), where $r=1, 2, 3, 4$, be the points of intersection of (i) and (ii); so that x_1, x_2, x_3, x_4 are the roots of the equation

$$F(x)=(l+mx+nx^2)^2-x(1-x)(1-k^2x)=0 \quad \dots \quad \text{(iii)}$$

and

$$F(x)=(x-x_1)(x-x_2)(x-x_3)(x-x_4). \quad \dots \quad \dots \quad \text{(iv)}$$

The variation ∂x_r in one of these abscissae due to the variation in position of the parabola (ii) is given by the equation

$$F'(x_r)\partial x_r-2(l+m\,x_r+n\,x^2_r)(\partial l+x_r\partial m+x^2_r\partial n)=0.$$

Remembering that $l+m\,x_r+n\,x_r{}^2=y_r$, we obtain

$$\frac{\partial x_r}{y_r}=\frac{+2(\partial l+x_r\partial m+x^2_r n)}{F'(x_r)} \quad \dots \quad \dots \quad \text{(v)}$$

$$=\frac{f(x_r)}{F'(x_r)} \text{ suppose.}$$

Then since $xf(x)$ is of lower degree than $F(x)$,

$$\frac{xf(x)}{F(x)}=\sum_{r=1}^{4}\frac{f(x_r)}{F'(x_r)}.\ \frac{x_r}{x-x_r} \qquad \ldots \qquad \ldots \text{ (vi)}$$

provided x_1, x_2, x_3, x_4 are all unequal, and $F'(x_r) \neq 0$.

Putting $x=0$, in (vi)

$$\sum_{r=1}^{4}\frac{f(x_r)}{F'(x_r)}=0.$$

$$\therefore \qquad \sum_{r=1}^{4}\frac{\partial x_r}{y_r}=0.$$

$$\therefore \qquad \sum_{r=1}^{4}\partial u_r=0.$$

Hence the sum of the parameters of the point of intersection is a constant and independent of the position of the parabola and it may be easily shewn to be congruent to zero.

Hence so long as $u_1+u_2+u_3+u_4=0$, we may deduce relations between symmetrical functions of s_r, c_r, d_r, (where s_r, c_r, d_r stand for sn u_r, cn u_r, dn u_r respectively) from the values of the symmetrical functions of the roots of the equation. ... ... (iii)

2. We have from equations (iii) and (iv) above,

$$\sum_{r=1}^{4} s^2_r=\sum_{r=1}^{4} x_r=\frac{k^2}{n^2}-2\frac{m}{n}. \qquad \ldots \qquad \ldots \ (\alpha)$$

$$s_1s_2s_3s_4=\sqrt{x_1x_2x_3x_4}=\frac{l}{n}. \qquad \ldots \qquad \ldots \qquad \ldots \ (\beta)$$

$$c_1c_2c_3c_4=\sqrt{(1-x_1)(1-x_2)(1-x_3)(1-x_4)}=\frac{l+m+n}{n}. \qquad \ldots \ (\gamma)$$

$$d_1d_2d_3d_4=\sqrt{(1-k^2x_1)(1-k^2x_2)(1-k^2x_3)(1-k^2x_4)}$$

$$=1+k^2\frac{m}{n}+k^4\frac{l}{n}. \qquad \ldots \qquad \ldots \qquad \ldots \ (\delta)$$

Hence eliminating l, m, n between (β), (γ) and (δ), we obtain the well known identity of Gudermann

$$k'^2-k^2k'^2s_1s_2s_3s_4+k^2c_1c_2c_3c_4-d_1d_2d_3d_4=0, \qquad \ldots \ (A)$$

where $k'^2=1-k^2$.

Writing in the identity (A) successively, (u_3+K, u_4-K), (u_3-iK', u_4+iK'), $(u_3+K-iK', u_4-K+i'K)$ for (u_3, u_4) respectively, we obtain the following relations quoted in Whittaker and Watson's *Modern Analysis* (Revised Edition) from H. J. S. Smith, *Proceedings*, Lond. Math. Soc. (I) X

$$k^2(s_1s_2c_3c_4-c_1c_2s_3s_4)-d_1d_2+d_3d_4=0 \quad \ldots \quad \ldots \quad (A1)$$

$$k'^2(s_1s_2-s_3s_4)+d_1d_2c_3c_4-c_1c_2d_3d_4=0 \quad \ldots \quad \ldots \quad (A2)$$

$$s_1s_2d_3d_4-d_1d_2s_3s_4+c_3c_4-c_1c_2=0 \quad \ldots \quad \ldots \quad \ldots \quad (A3)$$

3. Again, let us evaluate the functions

$$\sum_{r=1}^{4}\frac{c_rd_r}{s_r},\ \sum_{r=1}^{4}\frac{s_rd_r}{c_r},\ \sum_{r=1}^{4}\frac{s_rc_r}{d_r} \quad \text{in terms of } l, m, n.$$

We have

$$\sum_{r=1}^{4}\frac{c_rd_r}{s_r}=\sum_{r=1}^{4}\frac{y_r}{x_r}=\sum_{r=1}^{4}\frac{l+mx_r+nx^2_r}{x_r}$$

$$=\Sigma\frac{1}{x_r}+n\Sigma x_r+4m,$$

$$=\frac{1}{l}+\frac{k^2}{n}.$$

Again $$\sum\frac{s_rd_r}{c_r}=\sum\frac{y_r}{1-x_r}=\sum\frac{l+mx_r+nx_r^2}{1-x_r}.$$

This function is best evaluated by substituting p_r for $1-x_r$ and forming the equation whose roots are the four values of p_r. We can then by easy algebraical processes deduce that

$$\sum\frac{s_rd_r}{c_r}=\frac{k^2}{n}-\frac{k'^2}{l+m+n}.$$

Similarly from the equation whose roots are $1-kx_r$, we deduce that

$$\sum\frac{s_rc_r}{d_r}=\frac{lk^4+mk^2+n(1+k'^2)}{n(n+mk^2+lk^4)}.$$

From all the foregoing, we are in a position to deduce several relations. We shall select only such as will be useful for our purpose.

4. We have

$$\frac{1}{n}=\frac{1}{k}\sqrt{(\Sigma s_r^2-2s_1s_2s_3s_4+2c_1c_2c_3c_4-2)} \quad \ldots \quad \ldots \quad (B1)$$

$$=\sqrt{(2-\Sigma d_r^2+2d_1d_2d_3d_4-2k^4s_1s_2s_3s_4)} \quad \ldots \quad \ldots \quad (B2)$$

$$=\frac{-1}{k^2+(s_1s_2s_3s_4)^{-1}}\sum\frac{c_rd_r}{s_r} \quad \ldots \quad \ldots \quad \ldots \quad (B3)$$

$$=\frac{1}{k^2-k'^2(c_1c_2c_3c_4)^{-1}}\sum\frac{s_rd_r}{c_r} \quad \ldots \quad \ldots \quad \ldots \quad (B4)$$

$$=\frac{1}{1+k'^2(d_1d_2d_3d_4)^{-1}}\sum\frac{s_rc_r}{d_r} \quad \ldots \quad \ldots \quad \ldots \quad (B5)$$

Further, writing u_r+K for u_r in B3 and in B4, and u_r-iK' for u_r in B5, we obtain

$$\frac{1}{n}=\frac{-1}{(c_1c_2c_3c_4)^{-1}+k^2(d_1d_2d_3d_4)^{-1}}\sum\frac{s_r}{c_rd_r} \quad \ldots \quad \ldots \quad (B6)$$

$$=\frac{1}{k^2k'^2(d_1d_2d_3d_4)^{-1}-(s_1s_2s_3s_4)^{-1}}\sum\frac{c_r}{s_rd_r} \quad \ldots \quad \ldots \quad (B7)$$

$$=\frac{k^2s_1s_2s_3s_4+c_1c_2c_3c_4}{k'^2s_1s_2s_3s_4+c_1c_2c_3c_4}\sum\frac{d_r}{s_rc_r} \quad \ldots \quad \ldots \quad \ldots \quad (B8)$$

the summation extending to the suffixes 1, 2, 3, 4.

By eliminating x between the equations (i) and (ii) of § I, we obtain the equation whose roots are y_1, y_2, y_3, y_4. Evaluating from this equation $\sum_{r=1}^{4}\frac{1}{y^r}$, we can easily prove

$$\frac{1}{n}=-\left\{(s_1s_2s_3s_4)^{-1}+(c_1c_2c_3c_4)^{-1}+k^4(d_1d_2d_3d_4)^{-1}\right\}^{-1}\sum_{r=1}^{4}\frac{1}{s_rc_rd_r} \qquad (B9)$$

§ II. $\Sigma\, E(u_r, k)$ when $u_1+u_2+u_3+u_4=0$.

5. We have $E(u_r, k)=\int(\mathrm{dn}^2u_r)\,\partial u_r$

$$=u_r-k^2\int(\mathrm{sn}^2u_r)\partial u_r$$

$$=u_r-\frac{1}{2}k^2\int\frac{x_r\,\partial x_r}{y_r}.$$

$$\therefore \quad \sum_{r=1}^{4}E(u_r, k)=-\frac{1}{2}k^2\sum_{r=1}^{4}\int\frac{x_r\,\partial x_r}{y_r}. \quad \ldots \quad \ldots \quad (\Sigma)$$

Now going back to our equation (v) of § I

$$x_r\frac{\delta x_r}{y_r}=\frac{+2x_r\,(\partial l+x_r\partial m+x_r^2\,\partial n)}{F'(x_r)}.$$

By the method of partial fractions, we can easily prove that

$$\sum_{r=1}^{4}\frac{x_r}{F'(x_r)}=0, \quad \sum_{r=1}^{4}\frac{x_r^2}{F'(x_r)}=0 \quad \text{and} \quad \sum_{r=1}^{4}\frac{x_r^3}{F'(x_r)}=+\frac{1}{n^2}.$$

Hence
$$\sum_{r=1}^{4}\frac{x_r\,\partial x_r}{y_r}=\frac{2}{n^2}\partial n.$$

$$\therefore \quad \sum_{r=1}^{4}\int\frac{x_r\,\partial x_r}{y_r}=-\frac{2}{n}.$$

$\therefore$ from (Σ), $\sum_{r=1}^{4} E(u_r, k) = k^2\left(\frac{1}{n}\right)$. ... ... (C)

Hence $\sum_{r=1}^{4} E(u_r, k)$ has for its values in terms of s_r, c_r, d_r, each of the expressions in B 1—B 9 in § I multiplied by k^2.

6. $E(u_1,k)+E(u_2, k)+E(u_3, k)$ when $u_1+u_2+u_3=0$.

In the above, putting $u_4=0$, we deduce that when $u_1+u_2+u_3=0$, $E(u_1, k)+E(u_2, k)+E(u_3, k)$

$$=k\sqrt{s_1^2+s_2^2+s_3^2+2c_1c_2c_3-2} \quad \ldots \quad \ldots \quad \ldots \quad (D1)$$

$$=k\sqrt{1-c_1^2-c_2^2-c_3^2+2c_1c_2c_3} \quad \ldots \quad \ldots \quad \ldots \quad (D2)$$

$$=\sqrt{(1-d_1^2-d_2^2-d_3^2+2d_1d_2d_3)} \quad \ldots \quad \ldots \quad \ldots \quad (D3)$$

$$=-k^2s_1s_2s_3 \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (D4)$$

$$=\frac{k^2}{k^2-k'^2(c_1c_2c_3)^{-1}}\left\{\frac{s_1d_1}{c_1}+\frac{s_2d_2}{c_2}+\frac{s_3d_3}{c_3}\right\} \quad \ldots \quad \ldots \quad (D5)$$

$$=\frac{k^2}{1+k'^2(d_1d_2d_3)^{-1}}\left\{\frac{s_1c_1}{d_1}+\frac{s^2c_2}{d_2}+\frac{s_3c_3}{d_3}\right\} \quad \ldots \quad \ldots \quad (D6)$$

§ III. $\Sigma\Pi(u_r, k, a)$ when $u_1+u_2+u_3+u_4=0$.

7. Now, since

$$\Pi(u_r, k, a)=\int\frac{k^2\text{sn}a \text{ cn}a \text{ dn}a \text{ sn}^2 u_r}{1-k^2\text{sn } a \text{ sn}^2 u_r}\partial u_r$$

$$\therefore \quad \Pi(u_r, k, a)+\frac{\text{cn } a \text{ dn } a}{\text{sn } a}.u_r=\frac{\text{cn}a \text{ dn } a}{\text{sn } a}\int\frac{\partial u_r}{1-k^2\text{sn}^2a \text{ sn}^2 u_r}.$$

Then putting $\text{sn}^2 u_r=x_r$; $\frac{1}{k^2\text{sn}^2a}=\lambda$; $\frac{\text{cn } a \text{ dn } a}{k^2\text{sn}^3a}=\mu$; and remembering that $\sum_{r=1}^{4} u_r=0$, we easily deduce that

$$\sum_{r=1}^{4}\Pi(u_r, k, a)=\tfrac{1}{2}\mu\sum_{r=1}^{4}\int\left\{\frac{1}{\lambda-x_r}\frac{\partial x_r}{y_r}\right\}. \quad \ldots \quad (\omega)$$

Again going back to equation ($\dot{\text{v}}$) of § I,

$$\frac{\partial x_r}{y_r}=\frac{+2(\delta l+x_r\delta m+x_r^2\delta n)}{F'(x_r)}=\frac{f(x_r)}{F'(x_r)} \text{ (suppose).}$$

$$\sum_{r=1}^{4}\frac{1}{\lambda-x_r}\frac{\partial x_r}{y_r}=\sum_{r=1}^{4}\frac{f(x_r)}{(\lambda-x_r)F'(x_r)}.$$

Again from (iv) of § 1,

$$F(\lambda)=n^2(\lambda-x_1)(\lambda-x_2)(\lambda-x_3)(\lambda-x_4)$$

$$\therefore \qquad \sum\frac{f(\lambda)}{F(\lambda)}=\sum_{r=1}^{4}\frac{f(x_r)}{(\lambda-x_r)F'(x_r)}$$

since, $f(\lambda)$ is of lower degree than $F(\lambda)$,

$$\text{Hence } \sum_{r=1}^{4}\frac{1}{\lambda-x_r}\frac{\partial x_r}{y_r}=\frac{f(\lambda)}{F(\lambda)}=\frac{+2\partial(l+m\lambda+n\lambda^2)}{(l+m\lambda+n\lambda^2)^2-\mu^2},$$

$$=\frac{1}{\mu}\left[\frac{\partial(l+m\lambda+n\lambda^2)}{l+m\lambda+n\lambda^2-\mu}-\frac{\partial(l+m\lambda+n\lambda^2)}{l+m\lambda+n\lambda^2+\mu}\right]$$

$$=\frac{1}{\mu}\partial\log\frac{l+m\lambda+n\lambda^2-\mu}{l+m\lambda+n\lambda^2+\mu}$$

$$\therefore \qquad \mu\sum_{r=1}^{4}\int\frac{1}{\lambda-x_r}\cdot\frac{\partial x_r}{y_r}=\log\frac{l+m\lambda+n\lambda^2-\mu}{l+m\lambda+n\lambda^2+\mu}.$$

$$\therefore \qquad \sum_{r=1}^{4}\Pi(u_r,k,a)=\frac{1}{2}\log\frac{l+m\lambda+n\lambda^2-\mu}{l+m\lambda+n\lambda^2+\mu}.$$

$$=\tfrac{1}{2}\log\phi,\text{ say.} \qquad \ldots \qquad \ldots \text{(H)}$$

After substituting for λ and μ,

$$\phi=\frac{1+\frac{m}{n}k^2\text{sn}^2a+\frac{l}{n}k^4\text{sn}^4a-\frac{k^2}{n}\text{sn}a\ \text{cn}a\ \text{dn}a}{1+\frac{m}{n}k^2\text{sn}^2a+\frac{l}{n}k^4\text{sn}^4a+\frac{k^2}{n}\text{sn}\,a\ \text{cn}a\ \text{dn}a}$$

Hence after substituting from (β), (γ) of § 1 and from (C) of § II respectively,

$$\phi=\frac{1+\text{P sn}^2a+\text{Q sn}^4a-\text{R sn}a\ \text{cn}a\ \text{dn}a}{1+\text{P sn}^2a+\text{Q sn}^4a+\text{R sn}a\ \text{cn}a\ \text{dn}a},$$

where

$$P=k^2(c_1c_2c_3c_4-s_1s_2s_3s_4-1)=(d_1d_2d_3d_4-k^4s_1s_2s_3s_4-1)$$

$$=\frac{1}{k'^2}(d_1d_2d_3d_4-k^4c_1c_2c_3c_4-1+k^4),$$

$$Q=k^4s_1s_2s_3s_4,$$

$$R=\sum_{r=1}^{4}E(u_r,k),$$

which is already expressed in terms of s_r, c_r, d_r in § II.

8. Now putting $u_4=0$, we have from above, when $u_1+u_2+u_3=0$,
$\Pi(u_1, k, a)+\Pi(u_2, k, a)+\Pi(u_3, k, a)$

$$=\tfrac{1}{2}\log\frac{1+\mathrm{U}\,\mathrm{sn}^2a-\mathrm{W}\ \mathrm{sn}a\ \mathrm{cn}a\ \mathrm{dn}a}{1+\mathrm{U}\,\mathrm{sn}^2a+\mathrm{W}\ \mathrm{sn}a\ \mathrm{cn}a\ \mathrm{dn}a}=\tfrac{1}{2}\log\Omega\ (\text{say}) \qquad \ldots \quad (\mathrm{K})$$

where $\mathrm{U}=k^2(c_1c_2c_3-1)=d_1d_2d_3-1$
$=k^2[s_1s_2c_3d_3-s_3^2]$
$\mathrm{W}=\mathrm{E}(u_1\,k)+\mathrm{E}(u_2,k)+\mathrm{E}(u_3,k)$
$=$ the expressions D1—D6 in § II.
$=-k^2s_1s_2s_3$ in particualr.

With these values

$$\Omega=\frac{\mathrm{dn}^2a+k^2\mathrm{sn}^2a\ c_1c_2c_3+k^2\mathrm{sn}a\ \mathrm{cn}a\ \mathrm{dn}a.s_1s_2s_3}{\mathrm{dn}^2a+k^2\mathrm{sn}^2a\ c_1c_2c_3-k^2\mathrm{sn}a\ \mathrm{cn}a\ \mathrm{dn}a.s_1s_2s_3} \qquad \ldots \quad (\mathrm{K1})$$

$$=\frac{\mathrm{cn}^2a+\mathrm{sn}^2a\ d_1d_2d_3+k^2\mathrm{sn}a\ \mathrm{cn}a\ \mathrm{dn}a\ s_1s_2s_3}{\mathrm{cn}^2a+\mathrm{sn}^2a.d_1d_2d_3-k^2\mathrm{sn}a\ \mathrm{cn}a\ \mathrm{dn}a\ s_1s_2s_3} \qquad \ldots \quad (\mathrm{K2})$$

expressions easily remembered.

9. It will be interesting to deduce from the above the expressions for **Ω** given in Dixon's *Elliptic Functions*.

Taking $\quad \mathrm{U}=k^2(s_1s_2c_3d_3-s_3^2)$

and $\quad \mathrm{W}=-k^2s_1s_2s_3$,

the expression (K) above reduces to

$$\Omega=\frac{(1-k^2\mathrm{sn}^2a\mathrm{sn}^2u_3)+k^2\mathrm{sn}a.s_1s_2(\mathrm{sn}a.c_3d_3+s_3.\mathrm{cn}a\ \mathrm{dn}a)}{(1-k^2\mathrm{sn}^2a\mathrm{sn}^2u_3)+k^2\mathrm{sn}a.s_1s_2(\mathrm{sn}a.c_3d_3-s_3.\mathrm{cn}a\ \mathrm{dn}a)}$$

dividing the numerator and denominator by $1-k^2\mathrm{sn}^2a\mathrm{sn}^2u_3$, we obtain

$$\Omega=\frac{1+k^2\mathrm{sn}a.s_1s_2\mathrm{sn}(a+u_3)}{1+k^2\mathrm{sn}a.s_1s_2\mathrm{sn}(a-u_3)}. \qquad \ldots \qquad \ldots \qquad \ldots \quad (\mathrm{M1})$$

10. Again from the expressions for $\mathrm{sn}\,u_1\ \mathrm{sn}\,u_2$, $\mathrm{sn}\,a\ \mathrm{sn}\,(u_1+u_2-a)$ obtained by substituting in turn $\mathrm{A}=u_1$, $\mathrm{B}=u_2$; $\mathrm{A}=u_1+u_2-a$, $\mathrm{B}=a$ in the formula

$$\mathrm{sn}\mathrm{A}\ \mathrm{sn}\mathrm{B}=\frac{\mathrm{sn}^2\tfrac{1}{2}(\mathrm{A}+\mathrm{B})-\mathrm{sn}^2\tfrac{1}{2}(\mathrm{A}-\mathrm{B})}{1-k^2\mathrm{sn}^2\tfrac{1}{2}(\mathrm{A}+\mathrm{B})\mathrm{sn}^2\tfrac{1}{2}(\mathrm{A}-\mathrm{B})}, \qquad \ldots \qquad \ldots \quad (\mathrm{N})$$

we have $\quad 1-k^2\mathrm{sn}\,u_1\ \mathrm{sn}\,u_2\ \mathrm{sn}(u_1+u_2-a)=$

$$\left\{\frac{1-k^2\mathrm{sn}^4\tfrac{1}{2}(u_1+u_2)}{1-k^2\mathrm{sn}^2\tfrac{1}{2}(u_1+u_2)\mathrm{sn}^2\tfrac{1}{2}(u_1-u_2)}\right\}\left\{\frac{1-k^2\mathrm{sn}^2\tfrac{1}{2}(u_1-u_2)\mathrm{sn}^2\tfrac{1}{2}(u_1+u_2-2a)}{1-k^2\mathrm{sn}^2\tfrac{1}{2}(u_1+u_2)\mathrm{sn}^2\tfrac{1}{2}(u_1+u_2-2a)}\right\}.$$

Writing $-a$ for a in this equation, we have a second equation, which divided by the first gives

$$\Omega=\left\{\frac{1-k^2\mathrm{sn}^2\tfrac{1}{2}(u_1-u_2)\mathrm{sn}^2\tfrac{1}{2}(u_1+u_2-2a)}{1-k^2\mathrm{sn}^2\tfrac{1}{2}(u_1-u_2)\mathrm{sn}^2\tfrac{1}{2}(u_1+u_2+2a)}\right\}\times$$

$$\left\{\frac{1-k^2\mathrm{sn}^2\tfrac{1}{2}(u_1+u_2)\mathrm{sn}^2\tfrac{1}{2}(u_1+u_2+2a)}{1-k^2\mathrm{sn}^2\tfrac{1}{2}(u_1+u_2)\mathrm{sn}^2\tfrac{1}{2}(u_1+u_2-2a)}\right\}. \qquad (\mathrm{M2})$$

Again from the formula (N) above,

$$1 \pm k \text{ sn A sn B} = \frac{\{1 \pm k \text{ sn}^2 \tfrac{1}{2}(\text{A}+\text{B})\}\{1 \mp k \text{ sn}^2 \tfrac{1}{2}(\text{A}-\text{B})\}}{1-k^2 \text{ sn}^2 \tfrac{1}{2}(\text{A}+\text{B}) \text{ sn}^2 \tfrac{1}{2}(\text{A}-\text{B})}.$$

$$\therefore \quad [1-k^2 \text{ sn}^2 \tfrac{1}{2}(\text{A}+\text{B}) \text{ sn}^2 \tfrac{1}{2}(\text{A}-\text{B})]^2 = \frac{\{1-k^2 \text{sn}^4 \tfrac{1}{2}(\text{A}+\text{B})\}\{1-k^2 \text{ sn}^4 \tfrac{1}{2}(\text{A}-\text{B})\}}{(1-k^2 \text{ sn}^2 \text{ A sn}^2 \text{ B})}.$$

In this equation writing in turn

$$\text{A} = u_1 + a, \ \text{B} = u_2 + a\,;$$
$$\text{A} = u_1 - a, \ \text{B} = u_2 - a\,:$$
$$\text{A} = u_1 + u_2 - a, \ \text{B} = a\,;$$
$$\text{A} = u_1 + u_2 + a, \ \text{B} = a\,;$$

we get expressions for the square of each of the factors of the numerator and denominator of the expression M2.

Hence we have after cancelling like factors in the numerator and denominator,

$$\Omega^2 = \frac{\{1-k^2 \text{ sn}^2 (u_1+a) \text{ sn}^2 (u_2+a)\}}{\{1-k^2 \text{ sn}^2 (u_1-a) \text{ sn}^2 (u_2-a)\}} \times \frac{\{1-k^2 \text{ sn}^2 a \text{ sn}^2 (u_1+u_2-a)\}}{\{1-k^2 \text{ sn}^2 a \text{ sn}^2 (u_1+u_2+a)\}}. \qquad \ldots \quad (\text{M2})$$

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SHORT NOTES.

Pascal Hexagon.

The concurrence of the Pascal lines of a hexagon inscribed in a conic, is proved here by the properties of the Homographic Function. *Reference*: The Note at the end of Salmon's *Conics* and my previous paper on " the Homographic Function as an Operator." (J. I. M. S, vol. VIII, p. 202),

I. *Introductory*: The properties connected with Pascal's Hexagon deal with cases in which three correspondences have their double points in involution, No simple necessary conditions for this can be given, but the following theorems give sufficient conditions and are applicable in many cases.

Theorem I. *If S is a given correspondence and S_1 a variable correspondence with given double points, then the double points of $S_1 S$ belong to a fixed involution.*

Let I represent the involution determined by the double points of S_1 and S. Then S_1 can be decomposed into the product of a variable involution I_1 containing its double points and the involution I (J.I.M.S. vol. VIII, p. 202)

Thus $S_1 S = I_1 I \cdot S$.

Again since I contains the double points of S,IS is a fixed involution I′ containing the double points of S—(*Ibid*)

Hence $S_1 S = I_1 I_1$.

The double points of $S_1 S$ are the double points of $I_1 I'$ *i.e.*, the common pair of I_1 and I′. Thus the double points S_1 S belong to the fixed involution I′.

Cor. In particular if Ω be a correspondence of period three, the double points of S, Ω S, Ω^2 S, belong to an involution.

Theorem II. *If I_1, I_2, I_3, be three involutions having a common pair, the double points of I_1S, I_2S, I_3S, belong to an involution.*

Denote the common pair by *i*, and the double pair of S by *s*' Then by the theorem quoted above I_1 I_2 I_3 can each be split up into the product of the involution I containing *i* and *s* and three correspondences having *p* as their double pair. Hence putting IS=S′, the theorem is reduced to Theorem I.

Theorem III. *If the product of three correspondences is an involution n whichever order they are taken, then their double points belong to an involution.*

If $S_1S_2S_3$ and $S_2S_1S_3$ are involutions, the product of S_1, S_2, S_3 in any order is an involution. (This follows from the theorem that the parameter of the product of a number of correspondences depends not on the actual but only on the cyclical order of the product. (*vide*: the paper on one-one correspondence above referred to)

The condition that the correspondence $\frac{ax+b}{cx+d}$ may be an involution, is $a+d=0$.

Hence forming the products $S_1S_2S_3$, $S_2S_1S_3$ by the ordinary rules of multiplication of matrices, we have, for the condition that these may be involutions, the equations

$$(a_1a_2+b_1c_2)a_3+(a_1b_2+b_1d_2)c_3+(c_1a_2+d_1c_2)b_3+(c_1b_2+d_1d_2)d_3=0.$$

and $(a_1a_2+b_2c_1)a_3+(a_2b_1+b_2d_1)c_3+(c_2a_1+d_2c_1)b_3+(c_2b_1+d_1d_2)d_3=0.$

Subtracting we find

$$\begin{vmatrix} c_1 & d_1-a_1 & b_1 \\ c_2 & d_2-a_2 & b_2 \\ c_3 & d_3-a_3 & b_3 \end{vmatrix}=0,$$

which shews that the double points of S_1, S_2, S_3 belong to an involution.

II. *Properties of three Points*:

Three points pqr determine two periodic correspondences

$$\begin{vmatrix} pqr \\ qrp \end{vmatrix} \text{ and } \begin{vmatrix} prq \\ rqp \end{vmatrix}.$$

If Ω denote one of these, the other is Ω^2. The double points of Ω will be called the *cyclic centres* of pqr. (These must be distinguished from the so-called harmonic centres. The cyclic centres are defined by a purely descriptive process while the harmonic centres, as the name *does not* indicate, are defined by a metrical property.)

(1). From the definition of the cyclic centres as the double points of the correspondence $\begin{vmatrix} pqr \\ qrp \end{vmatrix}$, it follows that each of the involutions (pp,qr) (qq,rp), (rr,pq) contain the cyclic centres.

(2) *Geometrical Interpretation.*

Let PQR be points on the fundamental conic corresponding to pqr. Let the tangents at PQR meet the opposite sides in P′,Q′,R′, Then P′ represents the double pair of the involution (pp,qr). Hence (1) proves that P′Q′R′ are collinear. The intersections of P′Q′R′ with the conic correspond to the cyclic centres of pqr.

(3) Theorem IV. *If a correspondence carries* (pqr) *into* $(p'q'r')$, *it carries the cyclic centres of* (pqr) *into the cyclic centres of* $p'q'r'$.

Let $S=\begin{vmatrix} p\,q\,r \\ p'q'r' \end{vmatrix}$, $\Omega=\begin{vmatrix} q\ r\ p \\ p\ q\ r \end{vmatrix}$, $\Omega^1=\begin{vmatrix} p'q'r' \\ q'r'p' \end{vmatrix}$.

Then we see that $\Omega^1 S\Omega=S$, identically.

Hence if x be a double point of Ω

$$S(x)=\Omega' S\Omega(x)=\Omega' S(x) ;$$

so that S (x) is a double point of Ω'; which proves the theorem, since the double points of Ω and Ω' are the cyclic centres of (pqr) and $(p'q'r')$ respectively.

III. *Six Points*: Six points $pqr\ p'q'r'$ can be divided into two groups of three, in 10 ways. From each division we can get six correspondences by keeping one group in a fixed order and permuting the other. Hence we get sixty correspondences (a correspondence and its inverse not being regarded as distinct).

If the six points are represented as the vertices of a hexagon inscribed in the fundamental conic, the double points of these sixty correspondences are the intersections with the conics of the sixty Pascal lines.

(1) We first shew that the double points of the six correspondences which can be derived from the same division, belong three and three to two involutions.

Consider for example, the division $(pqr)\ (p'q'r')$.

$$\text{Let } S_1=\begin{vmatrix} p\,q\,r \\ p'q'r' \end{vmatrix},\quad S_2=\begin{vmatrix} p\,q\,r \\ q'r'p' \end{vmatrix},\quad S_3=\begin{vmatrix} p\,q\,r \\ r'p'q' \end{vmatrix}.$$

$$S_1'=\begin{vmatrix} p\,q\,r \\ p'r'q' \end{vmatrix},\quad S_2'=\begin{vmatrix} p\,q\,r \\ r'q'p' \end{vmatrix},\quad S_3'=\begin{vmatrix} p\,q\,r \\ q'p'r' \end{vmatrix}.$$

If Ω,Ω^2 be the two periodic correspondences which carry the group $(p'q'r')$ into itself, we have evidently

$$S_1=S_1,\ S_2=\Omega S_1,\ S_3=\Omega^2 S_1,$$
$$S_1'=S_1',\ S_2'=\Omega^2 S_1',\ S_3=\Omega S_1'.$$

Hence, by Th. I, Cor., the double points of $S_1S_2S_3$ belong to an involution I and the double points of S_1',S_2',S_3' belong to another involution I'.

(2) *Each of the involutions I, I' contains the double points of the other*

Let $(t_1t_2),(t_1't_2')$ be the cyclic centres of $(pqr)$and$(p'q'r')$ respectively. Every one of the six correspondences we are considering carries the group (t_1t_2) into the group $(t_1't_2')$ (Th. IV.) Hence if i, i' represent the involutions. $(t_1t_1',\ t_2t_2')$ and $(t_1t_2',t_1't_2)$ respectively, the double points of every one of the six correspondences, must belong either to i or to i'.

Hence the double points of two at least of the three correspondences S_1, S_2, S_3 must belong to one of the involutions i, i' say i and therefore the double points of the third also must belong to i (for we have proved that the double points of S_1,S_2,S_3 belong to an involution).

Hence I must be the same as i.

Similarly it is seen that I' is the same as i'.

Now the product of i and i' is an involution, viz. $(t_1t_2, t_1't_2')$. Hence (Th. II, Cor. in the paper mentioned above) i and i' each contain the double points of the other. This is seen otherwise also, for i, i' are represented in the plane by two sides of the harmonic triangle of the quadrangle $t_1t_2t_1't_2'$.

Thus the involutions determined by the double points of S_1, S_2, S_3 and S_1',S_2',S_3' each contains the double points of the other.

Hence we see that *the sixty Pascal lines meet three by three in* 20 *points (Steiner's points which form ten pairs of conjugate points w.r.t the conic, each conjugate pair corresponding to a division of the six points.* (The latter part is mentioned but not proved in Salmon).

(3) *Kirkman's Points.*

Consider the correspondences

$$S_1 = \begin{vmatrix} p\,q\,r' \\ q'p'r \end{vmatrix},\ S_2 = \begin{vmatrix} q\ r\,p' \\ r'q'p \end{vmatrix},\ S_3 = \begin{vmatrix} r\,p\,q' \\ p'r'q \end{vmatrix}.$$

We have

$$S_1S_2S_3(q') = r$$
$$S_1S_2S_3(r) = q';$$

hence $S_1S_2S_3$ is an involution.

Again $S_1S_3S_2\ (p') = r$

$$S_1S_3S_2\ (r) = p';$$

hence $S_1S_3S_2$ is an involution.

Thus every product of S_1, S_2 and S_3 is an involution and therefore the double points of S_1, S_2 and S_3 belong to an involution. (Theorem III.)

Thus the Pascal lines corresponding to S_1, S_2, S_3 meet in the point (Kirkman's point) which represents the double pair of this involution. In the present case S_2, S_3 are got from S_1 by permuting cyclically two points in the upper group of S_1 and the non-corresponding point in the lower group and also permuting cyclically the corresponding other points in the opposite sense.

There are three ways in which this double permutation can be made.

Thus three types of correspondences S_2S_3 can be obtained from S_1.

Hence every Pascal line contains 3 Kirkman points (and one Steiner's point.)

There are therefore $\frac{60\times 3}{3}$ or 60 Kirkman points.

R. VYTHYNATHASWAMY.

Note on the Gamma Function.

Gauss' Formula.

For n a positive integer $\Gamma(n)=\int_0^\infty e^{-x}x^{n-1}dx=(n-1)!$

$$\Gamma(a)=\int_0^\infty e^{-x}\,x^{a-1}\,dx=\frac{1}{a(a+1)\dots(a+n)}\int_0^\infty x^{a+n}\,e^{-x}\,dx,\text{ where } a>0$$

$$=\frac{1\cdot2\cdot3\dots n}{a(a+1)\dots(a+n)}\cdot\frac{\int_0^\infty x^{a+n}\,e^{-x}\,dx}{\int_0^\infty x^n\,e^{-x}\,dx}=\frac{1\cdot2\cdot3\dots n}{a(a+1)\dots(a+n)}\,n^a\cdot\frac{\int_0^\infty x^{a+n}\,e^{-nx}\,dx}{\int_0^\infty x^n\,e^{-nx}\,dx}$$

It can now be shown that the ratio of the two integrals tends to 1 as n tends to infinity.

Suppose a to be between r and $r+1$, where r is a positive integer, then

$$\int_1^\infty x^{r+n}\,e^{-nx}\,dx<\int_1^\infty x^{a+n}\,e^{-nx}\,dx<\int_1^\infty x^{r+1+n}\,e^{-nx}dx;\quad\dots\quad(1)$$

also $\int_0^1 x^{a+n}e^{-nx}\,dx$ lies between 0 and $\int_0^1 x^{a+n}\,dx$, or $\frac{1}{a+n+1}$,

whence we have

$$\int_0^1 x^{a+n}\,e^{-nx}\,dx=\frac{\theta}{a+n+1},\text{ where } 0<\theta<1.$$

Hence from the inequalities (1) we can write

$$\frac{\frac{\theta}{a+n+1}-\frac{\theta'}{r+n+1}+\int_0^\infty x^{r+n}\,e^{-nx}\,dx}{\int_0^\infty x^n\,e^{-nx}\,dx}<\frac{\int_0^\infty x^{a+n}\,e^{-nx}\,dx}{\int_0^\infty x^n\,e^{-nx}\,dx}$$

$$<\frac{\frac{\theta}{a+n+1}-\frac{\theta''}{r+n+2}+\int_0^\infty x^{r+1+n}\,e^{-nx}\,dx}{\int_0^\infty x^n\,e^{-nx}\,dx}.$$

Consider
$$\frac{\int_0^\infty x^{r+n}\,e^{-nx}\,dx}{\int_0^\infty x^n\,e^{-nx}\,dx}=\frac{1}{n^r}\cdot\frac{1\cdot2\cdot3\dots(r+n)}{1\cdot2\cdot3\dots n}$$

$$=\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)\dots\left(1+\frac{r}{n}\right);$$

since there are only a finite number of factors the limit as n tends to infinity of the ratio of the integrals tends to 1, and hence

$$\Gamma(a) = \lim_{n\to\infty} \frac{1\cdot 2\cdot 3 \ldots n}{a(a+1)\ldots(a+n)} n^a.$$

2. *The relation between the Beta and Gamma Functions.*

$$\frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} = \lim_{\mu\to\infty} \frac{1\cdot 2\cdot 3 \ldots \mu}{n(n+1)\ldots(n+\mu)} \cdot \frac{(m+n)\ldots(m+n+\mu)}{m(m+1)\ldots(m+\mu)}.$$

$$B(m,n) = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}} dx = \frac{m+n}{n} \int_0^\infty \frac{x^n\, dx}{(1+x)^{m+n+1}}$$

$$= \frac{(m+n)(m+n+1)\ldots(m+n+\mu)}{n(n+1)\ \ldots\ (n+\mu)} \int_0^\infty \frac{x^{n+\mu}}{(1+x)^{m+n+\mu}} d$$

$$= \frac{(m+n)(m+n+1)\ldots(m+n+\mu)}{n(n+1)\ldots(n+\mu)} \int_0^1 x^{n+\mu} (1-x)^{m-1} dx$$

$$= \frac{(m+n)(m+n+1)\ldots(m+n+\mu)}{n(n+1)\ldots(n+\mu)} \cdot \frac{1.2.3\ldots\mu}{m(m+1)\ldots(m+\mu)} \times \frac{\int_0^1 x^{n+\mu}(1-x)^{m-1} dx}{\int_0^1 x^\mu (1-x)^{m-1} dx}.$$

If n lies between r and $r+1$

$$\frac{\int_0^1 x^{r+\mu}(1-x)^{m-1} d}{\int_0^1 x^\mu (1-x)^{m-1} dx} > \frac{\int_0^1 x^{\mu+n}(1-x)^{m+1} dx}{\int_0^1 x^\mu (1-x)^{m-1} dx}$$

$$> \frac{\int_0^1 x^{\mu+r+1}(1-x)^{m-1} dx}{\int_0^1 x^\mu (1-x)^{m-1} dx};$$

and

$$\operatorname*{Lt}_{\mu\to\infty} \frac{\int_0^1 x^{r+\mu}(1-x)^{m-1} dx}{\int_0^1 x^\mu (1-x)^{m-1} dx} = \operatorname*{Lt}_{\mu\to\infty} \frac{1.2.3\ldots(\mu+r)}{m(m+1)\ldots(m+\mu+r)}$$

$$\times \frac{m\ m+1)\ldots(m+\mu)}{1.2.3\ldots\mu} = \operatorname*{Lt}_{\mu\to\infty} \frac{(\mu+1)\ldots(\mu+r)}{(m+\mu+1)\ldots(m+\mu+r)} = 1.$$

A. C. L. WILKINSON.

A Diophantine Problem.

1. If $ax^2-y^2=1$ is a conic passing through the *rational* points (p, q), (p', q'), then we can write

$$\left.\begin{array}{l} pa^{\frac{1}{2}}=\cosh u,\ q=\sinh u\ ; \\ p'a^{\frac{1}{2}}=\cosh v,\ q'=\sinh v. \end{array}\right\}$$

$$\therefore \quad \left.\begin{array}{l} \sinh (u\pm v)=qp'a^{\frac{1}{2}}\pm pq'a^{\frac{1}{2}}, \\ \cosh (u\pm v)=pp'a\pm qq'. \end{array}\right\}$$

Hence, the *rational* points $(qp'\pm pq')$, $(app'\pm qq')$ lie on the conjugate hyperbola $ax^2-y^2=-1$.

The *rational* points may, therefore, be readily constructed as the ends of chords conjugate to the radius to (p, q) or (p', q'); and the process can be continued indefinitely.

Ex. (1). $2x^2-y^2=1$.

Here $(1, 1)$ is an obvious *rational* point. Therefore, if (p, q) is any other such, the points $(q\pm p)$,$(2p\pm q)$ lie on the conjugate $y^2-2x^2=1$; and so on.

Ex. (2). $2x^2-3y^2=1$.

Here, if (p, q) is a point, $\{ (5p\pm 6q), (4p\pm 5q) \}$ is also a point on it.

2. If $ax^2+y^2=1$, we may put $\left.\begin{array}{l} pa^{\frac{1}{2}}=\cos\theta,\ q=\sin\theta, \\ p'a^{\frac{1}{2}}=\cos\phi,\ q'=\sin\phi, \end{array}\right\}0;$

and deduce $\left.\begin{array}{l}\sin(\theta+\phi)=(qp'+pq')a^{\frac{1}{2}}, \\ \cos(\theta+\phi)=(app'-qq'.)\end{array}\right\}$

Thus $\{ (qp'+pq'),(app'-qq') \}$ is also a point on the ellipse $ax^2+y^2=1$

3. When the equation is $ax^2+by^2=1$, put

$$pa^{\frac{1}{2}}=\cos\theta,\ qb^{\frac{1}{2}}=\sin\theta\ ;$$

and suppose

$$p'=\cos\phi,\ q'\ (ab)^{\frac{1}{2}}=\sin\phi,$$

so that (p', q')lies on

$$x^2+aby^2=1.$$

Then
$$\cos(\theta+\phi)=(pp'-bqq')a^{\frac{1}{2}},$$
$$\sin(\theta+\phi)=(qp'+pq'a)b^{\frac{1}{2}};$$

and the points $\{ \pm(qp'+apq'), \pm(pp'-bqq') \}$ will also lie on $ax^2+by^2=1$, being the ends of a chord conjugate to the radius to (p, q).

A similar method applies to $ax^2-by^2=1$.

Example : $2x^2-3y^2=1$.

Take the auxiliary conic $x^2-6y^2=1$, which is satisfied by $(5, 2)$.

Hence if (p, q) is a point on $2x^2-3y^2=1$,

$$\{ (5p+6q), (5q-4p) \},$$

also lies on the same, and so on.

M. T. Naraniengar.

Note on Q. 831.

(ENQUIRER) :—Find by elementary methods the invariant relation expressing the condition that hexagons may be inscribed in the conic $S=0$ which are also circumscribed to the conic $S'=0$.

[*N.B.*—The result quoted in Salmon's *Conic Sections*, 6th Edition p. 343, *foot-note*, contains an extraneous factor corresponding to degenerate hexagons. The result is given in Halphen, *Functions Elliptiques*, Tome II.]

Solution by A. C. L. Wilkinson.

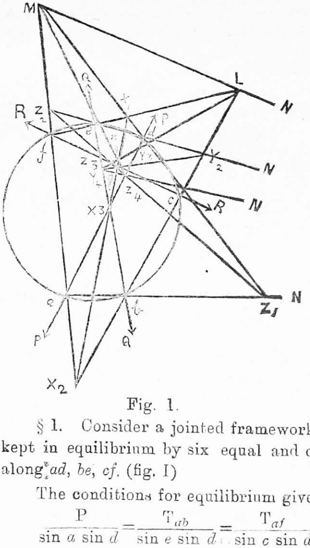


Fig. 1.

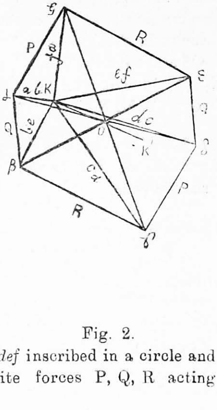


Fig. 2.

§ 1. Consider a jointed framework *abcdef* inscribed in a circle and kept in equilibrium by six equal and opposite forces P, Q, R acting along *ad*, *be*, *cf*. (fig. I)

The conditions for equilibrium give

$$\frac{P}{\sin a \sin d}=\frac{T_{ab}}{\sin e \sin d}=\frac{T_{af}}{\sin c \sin d}=\frac{Q}{\sin b \sin e}=\frac{T_{bc}}{\sin e \sin f}=\text{etc.}$$

which are symmetrical and thus the frame-work is in equilibrium. Consider the force diagram for any point K (fig. 2) where $K\alpha$, $K\beta$ ......are parallel to *ab*, *bc*......

Then *abcdef* is a funicular corresponding to K.

Now the hexagon $\alpha\beta\gamma\delta\epsilon\zeta$ is symmetrical about O; thus considering K′, a point such that O is the middle point of KK′, then $K'\alpha$ is parallel to $K\delta$, etc.

Thus *defabc* is another funicular and the corresponding sides of these two funiculars intersect on a straight line parallel to KK.'

Hence *ab*, *de* ; *bc*, *ef* ; *cd*, *fa* intersect on a straight line NLM, the Pascal line of the hexagon *abcdef*, and this line is parallel to KOK'.

§ 2. Consider any funicular corresponding to O, the centre of the force diagram ; in general it consists of a hexagon inscribed in the triangle $X_3Y_3Z_3$, formed by the lines *ad*, *be*, *cf*, whose sides meet the corresponding sides of the funiculars *abcdef*, *defabc* in points lying on two straight lines parallel to LMN or KOK'.

Consider a straight line NY_4Z_4 parallel to $\alpha O\delta$ and complete the funicular $Z_4Y_4X_4Z_4'Y_4'X_4'$; since corresponding sides of this funicular and *abcdef* intersect on a straight line parallel to OK, that is, parallel to LMN, and L is the point of intersection of Z_4Y_4 and *ab*, therefore LMN is the line of intersection of the corresponding sides. Hence Y_4X_4 passes through L, X_4Z_4' through M and $Z_4'Y_4'$ through N. But $Z_4'Y_4'$ is parallel to $O\delta$ and hence to Y_4Z_4. Thus the funicular degenerates to the triangle $X_4Y_4Z_4$ inscribed in the triangle $X_3Y_3Z_3$, whose sides pass respectively through L, M, N.

§ 3. The pencil at K consisting of $K\beta$, $K\varepsilon$, KO and the line through K parallel to $\beta\varepsilon$ is harmonic ; therefore L*cb*, L*ef*, LMN, LX_4Y_4 form a harmonic pencil.

Produce *ab*, *cd*, *ef* to form the triangle $X_1Y_1Z_1$ and *bc*, *de*, *fa* to form the triangle $X_2Y_2Z_2$.

It follows from the harmonic pencils at L and M that $X_1X_4X_2$ are collinear.

§4. *The points* X_1, X_2, X_3, X_4 *are collinear.*

Consider the forces, P, Q acting at *a*, *b* ; their resultant passes through X_2 for they can be replaced by T_{ba}, T_{fa} and T_{ab}, T_{db} and thus X_3X_2 is the resultant.

Similarly by considering P, Q acting at *d*, *e*, X_3X_1 is the line of action of the resultant.

Again P at Z_4 can be replaced by forces along Z_4Y_4, X_4Z_4 represented by $O\alpha$, εO ; and Q at Y_4 can be replaced by forces along Y_4Z_4 and X_4Y_4 represented by αO, $O\beta$; whence X_4 is also a point on the resultant.

Similarly $Y_1Y_2Y_3Y_4$ and $Z_1Z_2Z_3Z_4$ are collinear.

Let X_3X_4, Y_3Y_4 meet in *g* ; then the moments about *g* for P, Q at *a*, *b* is zero and that about *g* for P, R at *d*, *c* is zero and the P forces are equal and opposite, hence the sum of the moments about g for Q, R at b, c is zero.

Hence g also lies in Z_3Z_4.

Now $X_1X_2X_3$ is the Pascal line of *afe bcd*, $Y_1Y_2Y_3$ the Pascal line of *abc fed* and $Z_1Z_2Z_3$ of *afcdeb*. Thus g is the Steiner point of the Pascal lines X_1X_2, Y_1Y_2, Z_1Z_2.

The point g is $\begin{pmatrix} af, cd, be \\ bc, fe, ad \\ de, ab, cf \end{pmatrix}$ in the usual notation.

§5. The four triangles $X_1Y_1Z_1$, $X_2Y_2Z_2$, $X_3Y_3Z_3$, $X_4Y_4Z_4$ are copolar with a common pole at g.

The three axes of homology of any three triangles which are copolar with a common pole are known to be concurrent (a very simple proof by projecting the line joining the common pole to the intersection of two of the axes of homology to infinity). The axes of homology are seen to be

(12) the Pascal line LMN, (13)the Pascal line $\begin{pmatrix} ab, cd, ef \\ cf, be, ad \end{pmatrix}$

(14) LMN, (23) the Pascal line $\begin{pmatrix} ad, cf, be \\ bc, de, af \end{pmatrix}$, (24) LMN

and the axis of mology (34).

Hence the 3 Pascal lines

$$\begin{pmatrix} af, bc, de \\ cd, fe, ab \end{pmatrix}, \begin{pmatrix} af, bc, de \\ be, ad, cf \end{pmatrix}, \begin{pmatrix} cd, fe, ab \\ be, ad, cf \end{pmatrix}$$

are concurrent and the axis of homology of the triangles $X_3Y_3Z_3$, $X_4Y_4Z_4$ also passes through this point of concurrence of the 3 Pascal lines.

Thus the axis of homology of the triangles $X_3Y_3Z_3$, $X_4Y_4Z_4$ meets LMN in a second g point, and we shall show that these two g points are conjugate with respect to the conic. (Staudt's Theorem).

§ 6. *The axis of homology of the triangles $X_3Y_3Z_3$, $X_4Y_4Z_4$ and the Pascal line LMN are harmonic conjugates with respect to the Pascal lines*

$$\begin{pmatrix} ab, cd, ef \\ cf, be, ad \end{pmatrix} \textit{and} \begin{pmatrix} ab, cf, be \\ bc, de, af \end{pmatrix}.$$

Project LMN to infinity so that we obtain fig. 3, where ab, de ; bc ef ; cd, af are respectively parallel. Then X_4, Y_4, Z_4 are the middle points of X_1X_2, Y_1Y_2, Z_1Z_2. Also X_2Z_3, X_4Z_4, X_1Z_1 are three parallel straight lines and hence they meet X_3Z_3 in three points U, V, W where V is the middle point of UW, and this establishes the above theorem. Also since X_4 is the middle point of cf, we see that in fig. 1 the point where ad meets LMN is fourth harmonic with respect to Z_4 of a and d.

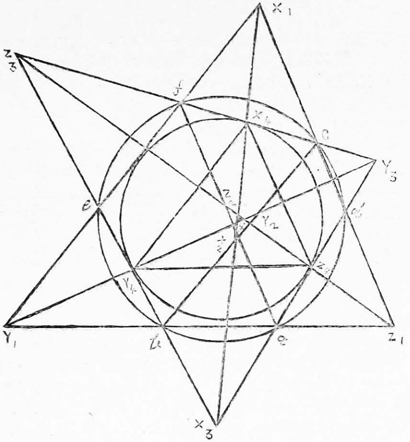


Fig. 3.

§ 7. Taking the conic of fig. 3 as a circle, since $ad = be = cf$ and Z_4, Y_4, X_4 are their middle points, we see that $X_4Y_4Z_4$ lie on a circle inscribed in the triangle $X_3Y_3Z_3$ and concentric with the circle $abcdef$.

Hence in fig. 1. the conics $abcdef$ and $X_4Y_4Z_4$ touching the sides of $X_3Y_3Z_3$ have double contact, the Pascal line LMN being the chord of contact; and more generally if we take any other hexagon $a'b'c'd'e'f'$ whose verties lie on the lines of action of the forces P, Q, R and whose corresponding sides $a'b'$, $d'e'$ pass through N, etc., the conics $a'b'c'd'e'f'$ all have double contact at the same points with the Pascal line LMN.

In fig. 3, g is the symmedian point of the triangle $X_4Y_4Z_4$ and its polar with respect to the circle $X_4Y_4Z_4$ is the axis of homology of $X_4Y_4Z_4$ and $X_3Y_3Z_3$; hence the polar of g with respect to $adcdef$ is parallel to the axis of homology of $X_4Y_4Z_4$, $X_3Y_3Z_3$ and thus passes through the point of intersection of LMN (the line at infinity) and the axis of homology of $X_4Y_4Z_4$, $X_3Y_3Z_3$; this was shown in § 5 to be the intersection of the three Pascal lines, *viz.* the reciprocal g point. We thus have a proof of Staudt's, Theorem that the two reciprocal g points are conjugate with respect to the fundamental conic.

A. C. L. Wilkinson.

Astronomical Notes.

A new minor planet.

Among the recent discoveries of minor planets is one of some importance. This planet (as yet unnamed) has a perihelion distance of 1·182 astronomical units, very little greater than that of Eros, so that on certain favourable occasions it may approach to within about seventeen million miles of the Earth, and will therefore be useful for determining the solar parallax. The eccentricity of the planet's orbit is very great so that its aphelion distance is 3·879. Its period is almost exactly four years. The planet is of the tenth magnitude and its diameter about four miles. The planet was discovered by Wolf who recently announced the discovery of a minor planet attended by a satellite which moved through eight degrees in a little over half an hour—a discovery of so startling a nature that one awaits its confirmation with some curiosity.

In connection with minor planets another point of interest occurs to me. It has been known for many years that at distances from the Sun at which the motion would be commensurate with that of Jupiter, there are no minor planets, any planets which may have originally existed in these positions having been diverted therefrom by the perturbing action of Jupiter. A recent investigation of the orbits of the many hundreds of planets now known fully confirms these facts, gaps being found at the ratios 2/1, 7/3, 5/2, 8/3, 3/1. It is however found that for values of the daily motion less than 500″ this avoidance of commensurability does not hold and planets are found with ratios 1/1, 4/3, 3/2.

The planets with ratio 1/1 are a special case, forming with Jupiter and the Sun an equilateral triangle and illustrating Lagrange's particular solution of the problem of three bodies; there are four such planets known, usually referred to as the Trojan group since they are named after the heroes of the Trojan war: Hector, Patroclus, &c.

R. J. Pocock.

SOLUTIONS.

Question 64.

(A. C. L. WILKINSON, M.A.) :—Prove that no solution in positive integers exists of the equations :

$$x^2-y^2=y^2-z^2=z^2-u^2.$$

Solution by H. Br.

1. *Lemma.* The equations

$$\alpha^2=\beta^2+\delta^2=\gamma^2+9\delta^2 \qquad \dots \quad (1)$$

have no solution in integers, if $\delta \neq 0$.

Proof : (i) A commun factor f of any two of the quantities is a factor of the other two also. [This is easily seen except possibly in the case where $f=2$ and is a factor of β, γ. In this case

$$\left(\frac{\beta}{2}\right)^2-\left(\frac{\gamma}{2}\right)^2=2\delta^2.$$

$\therefore$ $\frac{\beta}{2}, \frac{\gamma}{2}$ are both odd or both even ; hence the difference of their squares is a multiple of 4 ; *i. e.*, δ is even].

We may therefore without loss of generality take α, β, γ, δ prime to each other. As the sum of two odd squares is not a square δ is even.

(ii) By the ordinary rules, we must have

$$m^2+n^2=\alpha=M^2+N^2 \qquad \dots \quad (2)$$

$$2mn=\delta \; ; \; 2\,M\,N=3\,\delta$$

or

$$3\,m\,n=MN \qquad \dots \quad (3)$$

where m n are co-primes and n is even ; and M, N are co-primes and N is even.

From (3) we get

either
$$\left.\begin{matrix} m=a\,b, & M=3\,a\,c \\ n=c\,d & N=b\,d \end{matrix}\right\} \qquad \dots \quad (4)$$

or
$$\left.\begin{matrix} m=a\,b & M=a\,c \\ n=c\,d & N=3\,b\,d \end{matrix}\right\} \qquad \dots \quad (5)$$

a, b, c, d being prime to each other, and d being even. We take the two cases separately.

(iii) From (2) and (4),

$$a^2\,b^2+c^2\,d^2=9\,a^2\,c^2+b^2\,d^2$$

or

$$b^2\,(d^2-a^2)=c^2\,(d^2-9a^2)$$

giving

$$\frac{d^2-a^2}{c^2}=\frac{d^2-9a^2}{b^2}=\text{an integer}=v.$$

$\therefore$ $8\,a^2$ is divisible by c^2-b^2 which must therefore be the common factor of a^2 and d^2.

As a, b, c, d are prime to each other and b and c are odd, v can only be $+1$ or -1.

If $v=1$, $d^2=a^2+c^2$, which is impossible as d is even and, a c are odd.

If $v=-1$, $9a^2=b^2+d^2$ which is also impossible as b, d are not both multiples of 3.

Hence the relations (4) do not hold.

(iv) From (2) and (5) we get

$$a^2 b^2+c^2 d^2=a^2 c^2+9b^2 d^2 .$$

or

$$\frac{a^2-d^2}{b^2}=\frac{a^2-9d^2}{c^2}=\pm 1.$$

The lower sign gives $9 d^2=a^2+c^2$ which is impossible as a, c are not both multiples of 3.

The upper sign gives

$$a^2=b^2+d^2=c^2+9 d^2 \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (6)$$

an equation of the same form as (1). Also $a=\frac{m}{b}< m^2+n^2<\alpha$.

(v) Proceeding in this manner we get a series of equations with diminishing values of α, and ultimately arrive at the equations

$$1=b_o{}^2+d_o{}^2=c_o{}^2+9 d_o{}^2$$

which have no solution (if $d_o \neq 0$).

2. To prove that the system of equations

$$x^2-y^2=y^2-z^2=z^2-u^2$$

has no integral solution except the trivial one

$$x^2=y^2=z^2=u^2.$$

(i) Clearly x, y, z, u may be taken to be prime to each other without loss of generality $\qquad x^2=2 y^2-z^2$

We may solve this by putting

$$x+z=X,\ x-z=Y,$$

or we may apply known results in the theory of quadratic forms, and noting that

$$x^2=2 y^2-z^2=(2 y+z)^2-2 (y+z)^2$$

we may at once put down

$$\left.\begin{aligned} \pm x&=A^2-2 B^2 \\ 2 y+z&=A^2+2 B^2 \\ y+z&=2 A B \end{aligned}\right\} \quad \ldots \quad \ldots \quad \ldots \quad (7)$$

where A is odd, and A and B are prime to each other.

Similarly

$$\left.\begin{aligned} \pm u&=C^2-2 D^2 \\ y+2 z&=C^2+2 D^2 \\ y+z&=2 C D \end{aligned}\right\} \quad \ldots \quad \ldots \quad \ldots \quad (8)$$

C being odd, and C and D prime to each other.

Eliminating y and z

$$A^2+C^2+2B^2+2D^2=6AB=6CD \quad \ldots \quad \ldots \quad \ldots \quad (9)$$

As AB = CD, we put

$$\left.\begin{aligned} A &= pr, \ B = qs \\ C' &= ps, \ D = qr \end{aligned}\right\} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (10)$$

Then p, q, r, s are prime to each other and p, r, s are odd

(ii) Substituting in (9)

$$p^2r^2 + p^2s^2 + 2q^2s^2 + 2q^2r^2 = 6pqrs$$

or

$$(p^2 + 2q^2)(r^2 + s^2) = 6pqrs$$

Now $r^2 + s^2$ has no factor common with r, s and $p^2 + 2q^2$ has no factors common with p, q. Also $r^2 + s^2$ is even, but is not a multiple of 3

$\therefore$

$$\left.\begin{aligned} p^2 + 2q^2 &= 3rs \\ 2pq &= r^2 + s^2 \end{aligned}\right\}$$

or

$$\left.\begin{aligned} 2(p+q)(2q+p) &= 3(r+s)^2 \\ 2(p-q)(2q-p) &= 3(r-s)^2 \end{aligned}\right\} \quad \ldots \quad \ldots \quad (11)$$

It follows that $r+s$ may be put in the form $2\alpha\beta$, where α, β are prime to each other, and $r-s$ may be put in the form $2\gamma\delta$ where γ, δ are prime to each other. As $p^2 - q^2 = -3r^2 + 3rs =$ a multiple of 3, and as p and q are not both multiples of 3, one of these quantities $p+q$, $p-q$ is a multiple of 3 but not the other. Two cases arise

(1)

$$\left.\begin{aligned} p+q &= 6\alpha^2, \\ 2p+q &= \beta^2 \\ 2q-p &= 3\gamma^2 \\ p-q &= 2\delta^2. \end{aligned}\right\} \quad \ldots \quad \ldots \quad \ldots \quad (12)$$

Eliminating p and q, we get

$$9\alpha^2 = \beta^2 + \delta^2,$$

or β, δ are multiples of 3, implying r, s multiples of 3, which is not the case as r, s are co-prime's. $\therefore$ (12) is inadmissible.

(2) In the second case

$$\left.\begin{aligned} p+q &= 2\alpha^2 \\ 2q+p &= 3\beta^2 \\ 2q-p &= \gamma^2 \\ p-q &= 6\delta^2. \end{aligned}\right\} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (13)$$

Eliminating pq, we get

$$\left.\begin{aligned} \alpha^2 &= \beta^2 + \delta^2 \\ &= \gamma^2 + 9\delta^2 \end{aligned}\right\}$$

This has no integral solutions unless $\delta = 0$

If $\delta = 0$, $r = s$

$$A = C, \ B = D$$

$$y = z,$$

and finally

$$x^2 = y^2 = z^2 = u^2.$$

Question 378.

(J. C. SWAMINARAYAN) :—Having given that

$$\{ \Sigma(b+c)x^2-2\Sigma(fyz) \}^2=4(x^2+y^2+z^2)[\Sigma(bc-f^2)x^2+2\Sigma(gh-af)yz],$$

prove that $\dfrac{bz^2+cy^2-2fyz}{y^2+z^2}=\dfrac{cx^2+az^2-2gzx}{z^2+x^2}=\dfrac{ay^2+bx^2-2hxy}{x^2+y^2}$

Remarks by H. Br.

The given expression may be written

$$[(a+b+c)(x^2+y^2+z^2)-(ax^2+by^2+cz^2+2fyz+2gzx+2hxy)]^2$$
$$=4(x^2+y^2+z^2)(Ax^2+By^2+Cz^2+2Fyz+2Gzx+2Hxy).$$

Take (x, y, z) to be co-ordinates of a point, the axes being supposed to be rectangular. As $a+b+c$ is an invariant of the quadric $ax^2+\ldots$ it is clear that the above expression retains its form if the axes are rotated in any way. Consequently, the conclusions (whatever they be) to be drawn from the data will also retain their form unaltered if the axes are rotated in any way. But the conclusion that is asserted in the question does not satisfy the test.

To verify: Suppose the quadric referred to its principal axes, so that $f=g=h=0$.

$$[(b+c)x^2+(c+a)y^2+(a+b)z^2]^2=4(x^2+y^2+z^2)(bcx^2+cay^2+abz^2).$$

Expanding and simplyfying we get

$$\Sigma(b-c)^2x^4-2\Sigma(a-b)(c-a)y^2z^2=0.$$

Which implies

$$\sqrt{(b-c)}.x\pm\sqrt{(c-a)}.y\pm\sqrt{(a-b)}.z=0. \quad \ldots \quad \ldots \quad (1)$$

We are required to prove that

$$\frac{bz^2+cy^2}{y^2+z^2}=\frac{cx^2+az^2}{x^2+z^2}=\frac{ay^2+bx^2}{x^2+y^2}$$

i.e.

$$\left.\begin{aligned} z^2[(b-c)x^2+(c-a)y^2-(a-b)z^2]=0 \\ y^2[(b-c)x^2-(c-a)y^2+(a-b)z^2]=0 \\ x^2[-(b-c)x^2+(c-a)y^2+a-b)z^2]=0 \end{aligned}\right\} \quad \ldots \quad \ldots \quad (2)$$

The set of relations (2) is not a necessary consequence of (1).

Question 723.

(S. RAMANUJAN) :—If $[x]$ denote the greatest integer in x and n is a positive integer, show that

(i) $\left[\dfrac{n}{3}\right]+\left[\dfrac{n+2}{6}\right]+\left[\dfrac{n+4}{6}\right]=\left[\dfrac{n}{2}\right]+\left[\dfrac{n+3}{6}\right]$,

(ii) $[\tfrac{1}{2}+\sqrt{(n+\tfrac{1}{2})}]=[\tfrac{1}{2}+\sqrt{(n+\tfrac{1}{4})}]$,

(iii) $[\sqrt{n}+\sqrt{(n+1)}]=[\sqrt{(4n+2)}]$,

Solution by H. Br.

(i) Put $\phi(n)=\left[\dfrac{n}{3}\right]+\left[\dfrac{n+2}{6}\right]+\left[\dfrac{n+4}{6}\right]-\left[\dfrac{n}{2}\right]-\left[\dfrac{n+3}{6}\right]$ ()

Let $\left[\frac{n}{6}\right]=k$, so that $n=6k+m$, where $m<6$.

$$\left[\frac{n}{3}\right]=\left[2k+\frac{m}{3}\right]=2k+\left[\frac{m}{3}\right], \text{ \&c.}$$

$\therefore$ $\phi(n)=\phi(m)$, where $m=0$, 1, 2, 3, 4, or 5.

It is easily verified that $\phi(m)=0$ and therefore also $\phi(n)=0$.

(ii) Put $n=k^2+a$, where $a<2k+1$. Then $[\sqrt{n}]=k$.

Let
$$n+\tfrac{1}{2}=k^2+a+\tfrac{1}{2}=(k+\xi)^2,$$
$$n+\tfrac{1}{4}=k^2+a+\tfrac{1}{4}=(k+\eta)^2.$$

We are required to prove $[\tfrac{1}{2}+\sqrt{n+\tfrac{1}{2}}]=[\tfrac{1}{2}+\sqrt{n+\tfrac{1}{4}}]$

or, in effect, $[\tfrac{1}{2}+\xi]=[\tfrac{1}{2}+\eta]$

If $a<k$, $0<\eta<\xi<\tfrac{1}{2}$, and $[\tfrac{1}{2}+\xi]=[\tfrac{1}{2}+\eta]=0$,

If $a\geq k$, $\tfrac{1}{2}\leq\eta<\xi<1$, and $[\tfrac{1}{2}+\xi]=[\tfrac{1}{2}+\eta]=1$.

(iii) As before let $n=k^2+a$, where $a<2k+1$.

Put

$$\left.\begin{aligned} n&=k^2+a=(k+\xi)^2,\\ n+\tfrac{1}{2}&=k^2+a+\tfrac{1}{2}=(k+\eta)^2,\\ n+1&=k^2+a+1=(k+\zeta)^2.\end{aligned}\right\} \quad \ldots \quad \ldots \quad (1)$$

We are required to prove

$$[\sqrt{n}+\sqrt{n+1}]=[\sqrt{4n+2}]$$

or in effect $[\xi+\zeta]=[2\eta]$.

If $a<k$, we have $0\leq\xi<\eta<\zeta<\tfrac{1}{2}$, and $[\xi+\zeta]=[2\eta]=0$

If $a>k$, we have $\tfrac{1}{2}<\xi<\eta<\zeta\leq 1$, and $[\xi+\zeta]=[2\eta]=1$.

If $a=k$, $\xi<\tfrac{1}{2}$, and η, ζ are $>\tfrac{1}{2}$, but <1. Thus $[2\eta]=1$. We have to show that $[\xi+\zeta]=1$. If $\xi=\tfrac{1}{2}-x$, and $\zeta=\tfrac{1}{2}+z$, we have to show that $z>x$.

Substituting for ξ and ζ in (1), we get

$$\tfrac{3}{4}=z\,(2k+1)+z^2,$$

and
$$-\tfrac{1}{4}=-x\,(2k+1)+x^2$$

$$\therefore \quad (z-x)(2k+1)=\tfrac{1}{2}-(x^2+z^2).$$

As x and z are both less than $\tfrac{1}{2}$, x^2+z^2 is $<2(\tfrac{1}{2})^2<\tfrac{1}{2}$.

$\therefore$ $(z-x)(2k+1)$ is positive, or $z-x$ is positive.

Hence in every case $[\xi+\zeta]=[2\eta]$.

Question 836.

(Martyn M. Thomas, M.A.) :—At a given instant, the same star is observed to be at the horizon of one place (lat. ϕ_1, long. L_1) ; on the prime vertical of another place (lat. ϕ_2, long. L_2) and at the zenith of a third place whose latitude is unknown, and long. L_3. Prove that

$$\cos(L_3-L_2)\tan\phi_1+\cos(L_3-L_1)\cot\phi_2=0,$$

and find the latitude of the third place.

Solution by R. J. Pocock, S. Muthukrishnan, K. R. Rama Iyer, A. K. Anantanarayanan, R. D. Karve and C. Krishnamachary.

Let the hour angles of the star be $h_1\ h_2$ in the first two cases; then

$$h_1 = L_3 - L_1;\quad h_2 = L_3 - L_2;$$

and if δ is the star's declination $\delta = \phi_3$.

Also
$$\tan\delta . \tan\phi_1 = -\cos h_1,$$
$$\tan\delta . \cot\phi_2 = \cos h_2.$$

Eliminating δ, h_1, h_2, we have

$$\cos(L_3 - L_1)\cot\phi_1 + \cos(L_3 - L_2)\tan\phi_2 = 0,$$

or
$$\cos(L_3 - L_2)\tan\phi_1 + \cos(L_3 - L_1)\cot\phi_2 = 0;$$

and
$$\tan\phi_3 = \tan\delta = \cos(L_3 - L_2)\tan\phi_2$$
$$= -\cos(L_3 - L_1)\cot\phi_2.$$

Question 843.

(Lakshmishanker N. Bhatt):—From any point C tangents CD and CE are drawn to a circle, and from any fixed point O on the circle, chords OD, OE are drawn. If from any other variable point O′ on the circle, chords O′D, O′E are drawn cutting the former chords in P, Q and the tangents in R, S; then prove that (i) the line PQ always passes through C and (ii) the line RS always touches a conic.

Solution (1) by Hemraj; (2) by V. M. Gaitonde, S. R. Ranganathan, and C. Bhaskaraiya, B.A. (Hon.)

(1) Reciprocating with respect to O, we have :—

TP, TQ are tangents to a parabola at P, Q; and a variable tangent cuts them in R, S. Show that the lines through R and S parallel to TQ and TP respectively meet on PQ, and (ii) the locus of the point of intersection of the diagonals PS, RQ of the quadrilateral PQSR is a conic.

Take TP, TQ as axes. The parabola is

$$\sqrt{\frac{x}{a}} + \sqrt{\frac{y}{b}} = 1,$$

and a variable tangent is $\frac{x}{f} + \frac{y}{g} = 1$ with the condition $\frac{f}{a} + \frac{g}{b} = 1$, ... (1)

(i) $$PQ \equiv \frac{x}{a} + \frac{y}{b} - 1 = 0$$

and the point of intersection of lines through R, S parallel to TQ, TP is evidently (f, g). It lies on PQ, if

$$\frac{f}{a}+\frac{g}{b}-1=0, \text{ which is (1)}$$

[For a geometrical proof, see Lachlan's *Modern Pure Geometry* Art 272].

(ii) $$PS\equiv\frac{x}{a}+\frac{y}{g}-1=0 \quad \ldots \quad \ldots \quad (2)$$

$$QR\equiv\frac{x}{f}+\frac{y}{b}-1=0. \quad \ldots \quad \ldots \quad (3)$$

Eliminating f, g between (1), (2), (3) we have the locus required, *viz.*

$$x^2/a^2+xy/ab+y^2/b^2-2x/a-2y/b+1=0.$$

It has double contact with the parabola at P and Q. Hence the theorem.

(2) (i) Let OO′ and DE meet in K. Then by the quadrangle construction for a polar, PQ is the polar of K. Also since K lies on DE the polar of C, the polar of K always passes through C.

Thus PQ always passes through C.

(ii) It can easily be shown that R and S generate homographic ranges on CE, CD respectively and that the ranges are not in perspective. Therefore RS touches a conic touching the fixed lines CD, CE.

Additional Solutions by S. V. Venkatachala Iyer and H. R. Kapadia.

Question 846.

(S. MALHARI RAO, B.A.) :—Solve in positive integers $10^x+95^x=101^y$.

Solution (1) *by* H. Br. (2) *by S. V. Venkatachala Aiyar and N. B. Mitra.*

(1) $$10^x+95^x\equiv 0 \text{ (mod, 101)}$$

Multiply both sides by $(-17)^x$, and simplify. We get

$$32^x+1\equiv 0$$

or $$2^{5x}+1\equiv 0 \quad \ldots \quad \ldots \quad \ldots (1)$$

As 2 is not a quadratic residue of 101, $2^{50}+1\equiv 0$.

Henceif x_0 is the lowest positive value of x satisfying (1), 50 must be an odd multiple of $5\,x_0$.

$$\therefore \quad x_0=2 \text{ or } 10.$$

$x_0=2$, gives $2^{5x}\equiv 2^{10}\equiv 1024 \not\equiv -1$; $x_0=10$ is obviously admissible.

The general value of x is any odd multiple of x_0 ;

or $$x=20\,k+10\,;$$

the value of y can be deduced when x is known.

(2) Forming the residues to the modulus 101 of the successive powers of 10^x, we have

$$10, -1, -10, 1, \ldots\ldots$$

(the same set recurring after this),

Similarly the residues of 95^x are

$$-6, 36, -14, -17, +1, \ldots\ldots$$

(the same set recurring).

In order that $10^x + 95^x$ may be divisible by 101, the sum of their residues must be zero. This is obviously the case when x has any of the values 10, 30, 50,...... ; for, when x has a value of the form $2+4n$, the residue in the first case is -1, and when x has a value of the form $5m$, the residue in the second case is $+1$;

Thus the values of x having been found, the corresponding values of y can be easily calculated.

Question 861.

(Communicated by Mr. J. H. Grace, F. R. S.) :—A heterogeneous medium is such that the path of every ray through it is a plane curve. Find the index of refraction at any point.

Solution by Balak Ram.

If μ be the index of refraction at any point, the differential equations of a ray are

$$\frac{d}{ds}\left(\mu \frac{dx}{ds}\right) = \frac{\partial \mu}{\partial x},$$

and two similar equations. Denoting $\frac{d\theta}{ds}$ by θ', we have

$$\mu x'' = \frac{\partial \mu}{\partial x} - x'\mu',$$

and

$$\mu x''' = -2\mu' x'' - \frac{\partial \mu'}{\partial x} - x'\mu'', \text{ \&c.}$$

If the path is a plane curve, the torsion is zero. Simplifying the determinant factor of the value of the torsion, we get

$$\begin{vmatrix} x' & y' & z' \\ \frac{\partial \mu}{\partial x} & \frac{\partial \mu}{\partial y} & \frac{\partial \mu}{\partial z} \\ \frac{\partial \mu'}{\partial x} & \frac{\partial \mu'}{\partial y} & \frac{\partial \mu'}{\partial z} \end{vmatrix} = 0.$$

We must have relations of the form

$$\left.\begin{aligned} \frac{\partial \mu'}{\partial x} + u\frac{\partial \mu}{\partial x} &= vx', \\ \frac{\partial \mu'}{\partial y} + u\frac{\partial \mu}{\partial y} &= vy', \\ \frac{\partial \mu'}{\partial z} + u\frac{\partial \mu}{\partial z} &= vz'. \end{aligned}\right\}$$

The first equation is equivalent to

$$d\left(e^{\int u ds}\frac{\mu}{\partial\mu}\right)=ve^{\int u ds}dx.$$

This shows that the right hand side is a perfect differential.

Therefore $ve^{\int u ds}$ is independent of y and z. Similarly by considering the other equations, we find that it is independent of x, and is consequently a constant. Integrating, we get

$$\frac{\partial\mu}{\partial x}=v(x-x_0),$$

$$\frac{\partial\mu}{\partial y}=v(y-y_0),$$

$$\frac{\partial\mu}{\partial z}=v(z-z_0);$$

or $$d\mu=v\,[(x-x_0)dx+(y-y_0)dy+(z-z_0)dz].$$

Changing to polar co-ordinates, the new origin being taken to be at (x_0, y_0, z_0),

$$d\mu=\frac{\partial\mu}{\partial r}dr+\frac{\partial\mu}{\partial\theta}d\theta+\frac{\partial\mu}{\partial\phi}d\phi=vrdr.$$

$$\therefore \qquad \frac{\partial\mu}{\partial\theta}=\frac{\partial\mu}{\partial\phi}=0,$$

or $$\mu=f(r)=F[(x-x_0)^2+(y-y_0)^2+(z-z_0)^2],$$

where (x_0, y_0, z_0) is an arbitrary point and F is any function.

Question 874.

(S. MALHARI RAO) :—Solve in primes

$$x+y=206,\ x+z=132,\ z+w=82.$$

Solution by H. Br. N. B. Mitra, C. Krishnamachary, K. B. Madhava, R. D. Karve, R. J. Pocock, G. Joshi and S. V. Venkatachala Aiyer.

As x, y, z, w are primes, and the value 2 is clearly inadmissible for any of them, each of them is of the form $6k\pm1$. From the given relations if follows that x and y are of the form $6k+1$, and z and w of the form $6k-1$. The values of w that make w and $82-w$ prime are

$$w=11,\ 23,\ 29,\ 41,\ 53,\ 59,\ 71,$$

$$z=82-w=71,\ 59,\ 53,\ 41,\ 29,\ 23,\ 11\ ;$$

of these $w=29, 53, 59$ give prime values of x and y, as follows :—

x	y	z	w
79,	127,	53,	29
103,	103,	29,	53
109,	97,	23,	59

Question 875.

(**M. T. Naraniengar**) :—P is any point on the circle of similitude of two circles A, B; and Q, R are the inverses of P with respect to A, B. Show that QR is bisected by the radical axis of A and B.

Solution (1) *by Hemraj, S. Muthukrishnan, and K. R. Rama Iyer,*

(2) *by Hemraj, M. K. Kewalramani, and S. V. Venkatachala Iyer.*

(1) Denote the centres of A and B by A and B, and the circle of similitude by C.

Since P is on C, $\frac{AP}{BP}=\frac{e}{e_1}$, where e, e_1 are the radii of A, B;

and since Q,R are the inverses of P *w.r.t.* A, B,

$$\therefore \qquad \frac{AP}{e}=\frac{e}{AQ}, \text{ and } \frac{BP}{e_1}=\frac{e_1}{BR}$$

$$\therefore \qquad \frac{e}{AQ}=\frac{e_1}{BR} \text{ i.e. } \frac{AP}{BP}=\frac{AQ}{BR}$$

QR ∥ AB *i.e.* the radical axis of A, B is ⊥ QR. ... (1)

Evidently the circle PQR is orthogonal to A, B and C which is coaxal with A, B,

Thus the centre of the circle PQR is on the radical axis and on the right bisector of QR as well. Hence by (1) the radical axis bisects at right angles QR unless it is parallel to QR, *i.e.*, unless the centre of the circle PQR is at infinity, *i.e.*, unless P coincides with either centre of similitude; even in this case QR is bisected by the radical axis.

(2) Taking the line of centres and the radical axis of A, B as axes we have the equation of A, B in the form

$$x^2+y^2-2kx+\delta=0 \qquad x^2+y^2-2k_1x+\delta=0.$$

Hence the equation of C is

$$x^2+y^2-2\frac{kk_1+\delta}{k+k_1}x+\delta=0. \qquad \ldots \qquad \ldots \qquad \ldots (1)$$

Now P (x_1, y_1) is any point on (1),

$$\therefore \qquad x_1^2+y_1^2-2\frac{kk_1+\delta}{k+k_1}x_1+\delta=0; \quad \ldots \qquad \ldots \qquad \ldots (2)$$

and Q is the point of intersection of the polar of P *w.r.t.* A and the line through the centre of A perpendicular to the polar; *i e.* the intersection of the lines

$$(x_1-k)x+y_1y-kx_1+\delta=0, \; y_1x-(x_1-k)\; y-y_1k=0.$$

Hence Q is $\left(\frac{k_1-k}{k+k_1-2x_1}x_1, \frac{k+k_1}{k+k_1-2x_1}y_1\right)$. ... ...

Similarly R is $\left(\frac{k-k_1}{k+k_1-2x_1}x_1, \frac{k+k_1}{k+k_1-2x_1}y_1\right)$.

Hence the esult follows at once.

Question 878.

(S. NARAYANA AIYAR, M.A.):—Show that

$$(1-k^2x^2)^m(1-x^2)^n=\sum_{r=0}^{\infty}\frac{(-x^2)^r}{\Gamma(r+1)}\cdot\frac{\Gamma(n+1)}{\Gamma(n-r+1)}\,F(-m,-r,n-r+1,k^2)$$

when F denotes the hypergeometric function.

Hence show that

$$\int(1-k^2x^2)^{-\frac{1}{2}}(1-x^2)^{-\frac{1}{2}}dx=\sum_{r=0}^{\infty}\frac{(-)^r x^{2r+1}}{\Gamma(r+1)(2r+1)}\cdot\frac{\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{2}-r)}\times$$

$$F(\tfrac{1}{2},-r,\tfrac{1}{2}-r,k^2).$$

Solution by K. B. Madhava, S. R. Ranganathan, C. Krishnamachary and Sadanand.

If neither m nor n is a positive integer,

$$(1-k^2x^2)^m=\sum_{r=0}^{\infty}\frac{\Gamma(m+1)}{\Gamma(m-r+1)\,\Gamma(r+1)}k^{2r}(-x^2)^r$$

and

$$(1-x^2)^n=\sum_{r=0}^{\infty}\frac{\Gamma(n+1)}{\Gamma(n-r+1)\,\Gamma(r+1)}(-x^2)^r.$$

Hence the coefficient of $(-x^2)^r$ in the product $(1-x^2)^n(1-k^2x^2)^m$ is

$$\sum_{s=0}^{s=r}\frac{\Gamma(n+1)\,\Gamma(m+1)}{\Gamma(n-r+1)\,\Gamma(r+1)}\cdot k^{2s}\frac{\Gamma(n-r+1)\,\Gamma(r+1)}{\Gamma(m-s+1)\Gamma(r-s+1)\Gamma(s+1)\Gamma(n-r+s+1)}$$

$$=\frac{\Gamma(n+1)}{\Gamma(n-r+1)\Gamma(r+1)}\sum_{s=0}^{s=r}\frac{\Gamma(m+1)}{\Gamma(m-s+1)}\cdot\frac{\Gamma(r+1)}{\Gamma(r-s+1)}\cdot$$

$$\frac{\Gamma(n-r+1)}{\Gamma(n-r+s+1)}\cdot\frac{k^{2s}}{\Gamma(s+1)}$$

$$=\frac{\Gamma(n+1)}{\Gamma(n-r+1)\,\Gamma(r+1)}\,F(-m,-r,n-r+1,k^2),$$

where the hypergeometric series contains only a finite, *viz.* $r+1$ terms.

Since neither m nor n is a positive integer, the product can be summed for r fron 0 to ∞.

Hence $$(1-k^2x^2)^m(1-x^2)^n=\sum_{r=0}^{\infty}\frac{(-x^2)^r}{\Gamma(r+1)}\cdot\frac{\Gamma(n+1)}{\Gamma(n-r+1)}$$

$$F(-m,-r,n-r+1,k^2).$$

The second result follows by a simple integration, and putting $m=-\frac{1}{2}$ and $n=-\frac{1}{2}$.

Question 889.

(HEMRAJ) :—Two parabolas have each double contact with a given ellipse of semi-axes a, b, and their axes parallel to its axes respectively PP′, QQ′, are the chords of contact, and ρ_1, ρ_2 the radii of curvature of the parabolas; and ρ, ρ', those of the ellipse at P, Q. If the distances of the chords of contact from the centre of the ellipse are in the ratio of its axes, prove that $\rho\, \rho_2 = \rho' \rho_1$. In particular if the distances are equal, then

$$a^2\, \rho\, \rho_2 = b^2\, \rho'\, \rho_1.$$

But if Q coincides with P, then

$$\frac{1}{\rho} = \frac{1}{\rho_1} + \frac{1}{\rho_2}.$$

Solution (1) *by K. Appukuttan Erady and K. B. Madhava;*
(2) *by C. Krishnamachary, S. V. Venkatachala Aiyer and Sadanund.*

(1) Let the equations to the chords P P′ and Q Q′ be $x - f = 0$ and $y - g = 0$ respectively.

Then the equations to the two parabolas are

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 - \frac{1}{a^2}(x-f)^2 = 0, \qquad \ldots \qquad \ldots \quad \text{(i)}$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 - \frac{1}{b^2}(y-g)^2 = 0. \qquad \ldots \qquad \ldots \quad \text{(ii)}$$

For the ellipse at P,

$$\frac{y^2}{b^2} = 1 - \frac{x^2}{a^2},$$

$$\frac{y}{b^2} \cdot \frac{dy}{dx} = -\frac{x}{a^2},$$

$$\frac{y}{b^2} \cdot \frac{d^2y}{dx^2} = -\frac{1}{a^2} - \frac{1}{b^2}\left(\frac{dy}{dx}\right)^2.$$

For the parabola (i) at P,

$$\frac{y^2}{b^2} = 1 - \frac{2fx}{a^2} + \frac{b^2}{a^2},$$

$$\frac{y}{b^2} \cdot \frac{dy}{dx} = -\frac{f}{a^2},$$

$$\frac{y}{b^2} \cdot \frac{d^2y}{dx^2} = -\frac{1}{a^2} - \frac{1}{b^2}\left(\frac{dy}{dx}\right)^2.$$

Since the parabola (i) touches the ellipse at P, their curvatures are in the ratio of their $\dfrac{d^2y}{dx^2}$ at P.

Hence in the notation of the Question

$$\frac{\rho}{\rho_1} = \frac{\dfrac{1}{b^2}\left(\dfrac{dy}{dx}\right)^2}{\dfrac{1}{a^2} + \dfrac{1}{b^2}\left(\dfrac{dy}{dx}\right)^2} = \frac{\dfrac{f^2 b^2}{a^4 y^2}}{\dfrac{1}{a^2} + \dfrac{f^2 b^2}{a^4 y^2}} = \frac{\dfrac{b^2}{a^2}}{\dfrac{y^2}{b^2} + \dfrac{f^2}{a^2}} = \frac{f^2}{a^2}.$$

And it follows from symmetry that at R,

$$\frac{\rho'}{\rho_2}=\frac{g^2}{b^2}.$$

Hence if $\frac{f}{a}=\frac{g}{b}$,

$$\frac{\rho}{\rho_1}=\frac{\rho'}{\rho_2},$$

or

$$\rho\rho_2=\rho'\rho_1;$$

and if $f=g$,

$$\frac{a^2\rho}{\rho_1}=\frac{b^2\rho'}{\rho_2},$$

or

$$a^2\rho\rho_2=b^2\rho'\rho_1.$$

Also if P coincides with Q,

$$\rho'=\rho \text{ and } \frac{f^2}{a^2}+\frac{g^2}{b^2}=1,$$

and consequently

$$\frac{\rho}{\rho_1}+\frac{\rho}{\rho_2}=1,$$

i.e.,

$$\frac{1}{\rho}=\frac{1}{\rho_1}+\frac{1}{\rho_2}.$$

(2) Let the eqnation of the ellipse be

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

and P, Q, be the points whose excentric angles are α, β.

If P′ be α', the eqn. to the parabola with its axis parallel to the x—axis is

$$\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)-k\left\{\frac{x}{a}\cos\frac{\alpha+\alpha'}{2}+\frac{y}{b}\sin\frac{\alpha+\alpha'}{2}-\cos\frac{\alpha-\alpha'}{2}\right\}=0.$$

Since the axis of this parabola is parallel to the x--axis,

$$k=1, \text{ and } \alpha+\alpha'=0;$$

and the equation to the parabola is

$$\frac{y^2}{b^2}+\frac{2x}{a}\cos\alpha-(1+\cos^2\alpha)=0. \quad \dots \quad \dots \quad \dots \quad (1)$$

Similarly the equation to the parabola with Q Q′ as chord of contact and axis parallel to the y—axis is

$$\frac{x^2}{a^2}+\frac{2y}{b}\sin\alpha-(1+\sin^2\alpha)=0. \quad \dots \quad \dots \quad \dots \quad (2)$$

Now $\rho=\frac{(a^2\sin^2\alpha+b^2\cos^2\alpha)^{\frac{3}{2}}}{ab}$, $\rho'=\frac{(a^2\sin^2\beta+b^2\cos^2\beta)^{\frac{3}{2}}}{ab}$.

Also we can easily find ρ_1, ρ_2, the radii of curvature of the parabolas (1) and (2) at P, Q, and these are given by

$$\rho_1=\frac{(a^2\sin^2\alpha+b^2\cos^2\alpha)^{\frac{3}{2}}}{ab\cos^2\alpha},\ \rho_2=\frac{(a^2\sin^2\beta+b^2\cos^2\beta)^{\frac{3}{2}}}{ab\sin^2\beta}.$$

The perpendiculars from the origin on the chords of contact PP′ QQ′ are $a\cos\alpha$, $b\sin\beta$.

If these are in the ratio of the axes

$$\cos\alpha=\sin\beta\ ;\ \rho\rho_2=\rho'\rho_1.$$

If these are equal, $a\cos\alpha=b\sin\beta$,

$$a^2\rho\rho_2=b^2\rho'\rho_1.$$

But if P, Q coincide, $\alpha=\beta$, and the relation

$$\frac{1}{\rho}=\frac{1}{\rho_1}+\frac{1}{\rho_2}$$

easily follows.

Question 898.

(HEMRAJ):—A parabola has double contact with a given ellipse of semi-axes a, b and its axis parallel to either axis of the latter. Find the equation of the locus of the centre of curvature of the parabola corresponding to the point of contact; and show that

$$a^2b^2\rho_1^2\rho_2^8=\{(\rho_2-\rho_1)a^2+\rho_1b^2\}^3,\ \text{or}\ \{(\rho_2-\rho_1)b^2+\rho_1a^2\}^3$$

where ρ_1, ρ_2 are the radii of curvature of the curves at a point of contact.

Solution by K. B. Madhava and C. Krishnamachary.

We notice that

$$y^2=4\,A\,(x+C)\qquad\ldots\qquad\ldots\qquad\ldots\qquad(1)$$

is a parabola with its axis parallel to the major axis of

$$1=\frac{x^2}{a^2}+\frac{y^2}{b^2}\qquad\ldots\qquad\ldots\qquad\ldots\qquad(2)$$

and having double contact with it, if

$$4\,AC\,b^2=b^4+4\,A^2\,a^2.\qquad\ldots\qquad\ldots\qquad\ldots\qquad(3)$$

The points of contact of (1) and (2) are

$$-\frac{2\,A\,a^2}{b^2},\ \pm\frac{1}{b}-\sqrt{b^4-4\,A^2\,a^2}$$

where $\rho_1^2\ (\text{Ellipse})=\frac{1}{a^2b^2}(a^2-x^2e^2)=\frac{a^4}{b^{14}}(b^4-4\,A^2/a^2e^2)^3\quad\ldots\quad(4)$

and $\rho_2^2\ (\text{Parabola})=\frac{4}{A}(A+x+C)^3=\frac{1}{16\,A^4b^6}(b^4-4A^2a^2e^2)^3\ ;\quad(5)$

whence the required relation is got by eliminating A between (4) and (5) *i.e.* between (4), and $\rho_1 b^4 = 4 A^2 a^2 \rho_2$.

$$\therefore \qquad \rho_1^2 = \frac{a^4 b^{12}}{b^{14} \rho_2^3} (\rho_2 - \rho_1 e^2)^3,$$

i.e.
$$a^2 b^2 \rho_1^2 \rho_2^3 = \{ a^2 (\rho_2 - \rho_1) + b^2 \rho_1 \}^3.$$

Similarly the other result (when the axis of the parabola is parallel to the minor axis of the ellipse) can be got by interchanging a and b.

To obtain the locus of the centre of curvature, we eliminate x, y, A and C from (1), (3) and

$$\xi = 2A + 2C + 3x \text{ and } \eta = -\frac{y^3}{4A^2}.$$

Question 904.

(R. J. Pocock, B.A., B. Sc.) :—Show that

$$\int_0^{\frac{\pi}{2}} \frac{\sin^{2n}\phi}{\left(1 - \frac{1}{2}\sin^2\phi\right)^{n+\frac{1}{2}}} d\phi = 2^{2n-3} \frac{\Gamma^2\left(\frac{2n+1}{4}\right)}{\Gamma\left(\frac{2n+1}{2}\right)}.$$

Solution (1) *by F. H. V. Gulasekharam and by the Proposer*; (2) *by A. C. L. Wilkinson and K. R. Rama Iyer.*

(1) Consider
$$I = \int_0^\infty x^{2n} e^{-x^4} dx = \frac{1}{4}\Gamma\left(\frac{2n+1}{4}\right);$$

by substituting $x^4 = y$

whence
$$2^{2n} I^2 = 2^{2n} \int_0^\infty \int_0^\infty x^{2n} y^{2n} e^{-x^4 - y^4} dx dy$$

$$= \int_0^\infty \int_0^{\frac{\pi}{2}} r^{4n} \sin^{2n} 2\theta \, e^{-r^4 (1 - \frac{1}{2}\sin^2 2\theta)} r \, dr \, d\theta$$

$$= \frac{1}{4} \int_0^\infty \int_0^\pi z^{2n} \sin^{2n}\phi \, e^{-z^2 (1 - \frac{1}{2}\sin^2\phi)} dz \, d\phi$$

$$= \frac{1}{2} \int_0^\infty \int_0^{\frac{\pi}{2}} \frac{\sin^{2n}\phi}{\left(1 - \frac{1}{2}\sin^2\phi\right)^{\frac{2n+1}{2}}} \cdot t^{2n} e^{-t^2} dt \, d\phi$$

$$= \frac{1}{4}\Gamma\left(n + \frac{1}{2}\right) \int_0^{\frac{\pi}{2}} \frac{\sin^{2n}\phi}{\left(1 - \frac{1}{2}\sin^2\phi\right)^{\frac{2n+1}{2}}} d\phi$$

This solution is suggested by a similar analysis, due to Forsyth given in G. H. Darwin's paper "On the Mechanical Conditions of a Swarm of Metorites" (Phil. Trans. Series A, Vol. 180 (1880) pp. 1—69; also Collected Papers, Vol. IV, pp. 362—431).

Putting $n=0$, we get $\Gamma\left(\frac{1}{4}\right)=2\pi^{\frac{1}{4}}F^{\frac{1}{2}}$,

where F is the complete elliptic integral of the first kind, modulus $\frac{1}{\sqrt{2}}$.

From $\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{4}\right)=\pi\sqrt{2}$, and Legendres formula connecting the complete elliptic integrals of the first and second kind, which for modulus $\frac{1}{\sqrt{2}}$ becomes $2\,FE-F^2=\frac{1}{2}\pi$, we obtain

$$\Gamma\left(\frac{3}{4}\right)=\pi^{\frac{1}{4}}(2E-F)^{\frac{1}{2}};$$

and we thus obtain the expressions for

$$\int_0^\infty e^{-x^4}dx,\quad \int_0^\infty x^2 e^{-x^4}dx$$

given in the paper quoted above; and more generally

$$\int_0^\infty x^{2n}e^{-x^4}dx=\frac{1}{4}\Gamma\left(\frac{2n+1}{4}\right)$$

can be reduced to one or other of these two expressions.

(2) Write $\sin x=\dfrac{\frac{1}{2}\sin^2\phi}{1-\frac{1}{2}\sin^2\phi}$, $\cos x=\dfrac{\cos\phi}{1-\frac{1}{2}\sin^2\phi}$; so that

$$\cos x\,dx=\frac{\sin\phi\cos\phi\,d\phi}{\left(1-\frac{1}{2}\sin^2\phi\right)^2},\quad dx=\sqrt{2}\sin x\,\frac{d\phi}{\left(1-\frac{1}{2}\sin^2\phi\right)^{\frac{1}{2}}}$$

$$\therefore\qquad I=2^{n-\frac{1}{2}}\int_0^{\frac{1}{2}\pi}\sin x^{n-\frac{1}{2}}\,dx=2^{3n-3}\,\frac{\Gamma^2\left(\frac{2n+1}{4}\right)}{\Gamma\left(\frac{2n+1}{2}\right)}.$$

Question 905.

(T. P. Bhaskara Sastri) :—Investigate the effect of precession and nutation on the position angle of a double star.

[See : Ball's *Spherical Astro*: p. 185, Ex. 16.]

Solution by C. Krishnamachary and the Proposer.

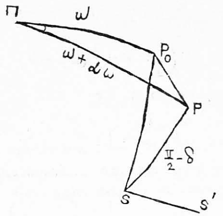
Let π be the pole of the ecliptic.

P_o, the celestial pole at epoch T_o.

P, the same at a subsequent epoch T.

S, the principal star and S′ his companion

Fig. 1.



also let the position angle at the first epoch be p_o, and p that at the second epoch.

$\angle P \pi P_o$ = the arc between the noles of the circles πP_o and πP $= k.$

From the triangle $P_o S P$,

$$\cos\delta \sin(p-p_o) = -\sin P P_o \sin \angle P P_o S.$$

But $\angle P P_o S = \angle P P_o \pi - \angle S P_o \pi = \angle P P_o \pi - (90^\circ + \alpha_o)$
$= -(90^\circ - \angle P P_o \pi) - \alpha_o.$

$$\therefore \quad \cos\delta \sin(p-p_o) = \sin P P_o \sin P P_o \pi \sin \alpha_o + \sin P P_o \cos P P_o \pi \cos \alpha_o$$

From the triangle $P P_o \pi$,

$\sin P P_o \sin P P_o \pi = \sin(\omega + d\omega) \sin k$ and $\sin P P_o \cos P P_o \pi = \cos(\omega + d\omega) \sin\omega - \sin(\omega + d\omega)\cos\omega \cos k.$

Making these substitutions, we get

$$(p-p_o)\cos\delta = [k \sin\omega \sin\alpha - d\omega \cos\alpha],$$

neglecting small quantities of order higher than the first.

i.e. $$p - p_o = [n \sin\alpha - d\omega \cos\alpha] \sec\delta.$$

QUESTIONS FOR SOLUTION.

903. ***Correction :*** Insert at the end of the question the following:—" where the summation is extended only to odd values of n ".

960. (V. RAMASWAMI AIYAR, M.A.):—Given a triangle ABC, if a point P and the feet of the perpendiculars from it on the sides and altitudes of ABC all lie on a Cartesian Oval, show the negative pedal of the oval with respect to P touches the nine points circle. Show that the pedals of the inscribed and escribed circles of a triangle, taken with respect to the orthocentre, are cases of such ovals, illustrating the property.

961. (H. BR.):—Show that

$$\left.\begin{array}{l} x^2=y^2+z^2 \\ t^2=y^2+4z^2 \end{array}\right\}$$

have no solutions in non-zero integers.

962. (A. C. L. WILKINSON):—If two circles are such that hexagons can be inscribed in the one which are circumscribed to the other, prove that the sum of the products of the diagonals taken two at a time is constant and equal to

$$\frac{4\,R^2 r^2 \{ (R^2-c^2)^2+4\,R^2 r^2 \}}{(R^2-c^2)^3},$$

where R, r, are the radii of the two circles and c the distance between their centres.

963. (A. C. L. WILKINSON):—If two circles are such that quadrilaterals can be inscribed in the one which are circumscribed to the other, the ratio of the areas of these quadrilaterals is constant.

964. (H. BR.):—If $f(x, y)$ is a rational algebraic function of x and y, such that $f(-x, y)=(-)^{k+1} f(x, y)$,

(i) Show that

$$\int_0^\infty f(\sin 2\theta, \cos 2\theta)\frac{d\theta}{\theta^{k+1}}$$

$$=\frac{(-1)^k}{k!}\int_0^\infty f(\sin 2\theta, \cos 2\theta)\tan\theta.\frac{d^k}{d\theta^k}\cot\theta.\frac{d\theta}{\theta}$$

$$= \frac{(-1)^k}{k!} \int_0^{\frac{\pi}{2}} f(\sin 2\theta, \cos 2\theta). \frac{d^k}{d\theta^k} \cot \theta. \, d\theta$$

and

$$\text{(ii)} \int_0^{\infty} f(\sin \theta, \cos \theta) \frac{d\theta}{\theta^{k+1}} = \frac{(-1)^k}{k!} \int_0^{\frac{\pi}{2}} f(\sin \theta, \cos \theta) \frac{d^k \lambda}{d\theta^k} \, d\theta$$

where $\lambda = \cot \theta$ or $\operatorname{cosec} \theta$, according as

$$f(-x, -y) = f(x, y) \text{ or } -f(x, y).$$

[The integrals are assumed to be finite.]

965. (HEMRAJ) :—ABC is a triangle. A point J is taken on AI_1 produced, such that I_1J subtends at B or C an angle α where $\cot \alpha = 3 \cot \frac{1}{2} C$ or $3 \cot \frac{1}{2} B$. Show that there exists a circle through A and J which cuts AB and AC in D and E such that BD = CE = BC.

[Suggested by Q. 959.]

965. (A. C. L. WILKINSON) :—Prove that

$$\frac{1-2(2+k^2)s_1^2+5k^2s_1^4}{2s_1c_1d_1} + \frac{1-2(1+k^2)s_2^2+3k^2s_2^4}{2s_2c_2d_2} - \frac{1-2(1+2k^2)s_3^2+5k^2s_3^4}{2s_3c_3d_3}$$
$$= -3k^2 s_1 s_2 s_3$$

where $$s_1 = \operatorname{sn}\left(\frac{K}{3}\right) s_2 = \operatorname{sn}\frac{iK'}{3}, \; s_3 = \operatorname{sn}\frac{K+iK'}{3} \text{ etc.}$$

966 (A. C. L. WILKINSON) :—Prove that

$$'E - 3E\left(\frac{K}{3}\right) = -\frac{k^2}{k'} \operatorname{cn} \frac{^3K}{3}.$$

967. (M. K. KEWALRAMANI, M.A) :—Prove that

$$1^{-5} + 2^{-5} - 4^{-5} - 5^{-5} + 7^{-5} + 8^{-5} - 10^{-5} - 11^{-5} + \ldots\ldots = \frac{35\pi^5}{8748\sqrt{3}}.$$

968 (V. ANANTARAMAN) :—Sum the series

$$\frac{1}{3} - \frac{1}{7} + \frac{1}{11} - \frac{1}{15} + \frac{1}{19} - + \ldots\ldots$$