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PROGRESS REPORT.

The committee notes with great pleasure that Mr. S. Ramanujan, a member of the society, who went to Cambridge as a research scholar from Madras, has recently obtained the rare distinction of being elected a Fellow of the Royal Society. This is a great honour of which the whole of India may be proud, and the fact that he was elected to the Fellowship on the first occasion when his name was proposed, and at the comparatively early age of thirty, is eloquent testimony to the excellent work that he has done in the field of Mathematics. It is the earnest hope of the Committee that Mr. Ramanujan may be blessed with long life and health to enable him to engage in further research work, winning fresh laurels for himself and bringing credit and lustre to his motherland, while at the same time serving humanity at large by valuable contributions to the advancement of science.

D. D. Kapadia,

Honorary Secretary.

Complex Roots of Equations.

By M. T. Naraniengar.

Introduction.

It is proposed to discuss in this paper *graphic methods* of visualising the complex roots of an equation. To this end Argand diagrams are freely used, and complex roots obtained as the *real* intersections of two plane curves. The method of the Theory of Functions of Complex Variables is briefly indicated towards the close of the paper.

The paper is conveniently divided into three Sections: (1) The First Section treats of equations up to the fifth degree by means of the elementary methods of curve tracing; (2) In the second Section, Approximations to Complex Roots of a general algebraical equation, are developed by means of Cauchy's expansion and Taylor's theorem, and *incidentally* a "Method of dealing with the Intersections of Plane Curves" is referred to and its connection with the properties of Polar Curves explained; (3) The third and last Section is devoted to a brief discussion of Transcendental equations.

The Post Script deals with simultaneous equations involving two Complex Variables.

The computation of imaginary roots of Numerical Algebraic Equations has not received much attention on the part of mathematicians. There is a reference to a modification of Horner's method so as to apply to imaginary roots on page 12 of Mathematical Monographs, No. 10.—*The Solution of Equations* by M. Merriman*. Mc Clintock in 1894 published a method of development in series of the roots of an equation by means of his Calculus of Enlargement†. His method can be used for approximate computations when the series is convergent. Lambert in 1903 gave out a similar method of expansion by Maclaurin's formula, applicable to Algebraic, as well as, Transcendental Equations‡. This method consists in introducing a second variable into all the terms but two of an equation and putting it equal to unity after the expansion of the first variable is obtained by Maclaurin's theorem.

Dr. F. S. Macaulay in his *Algebraic Theory of Modular Systems*, 1916, No. 19 of Cambridge Tracts, gives the following further references: Transactions of the American Mathematical Society, No. 3, 1902; and No. 5, 1904.

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* Sheffler, Die Auflösung der algebraischen and transzendenten Gleichungen Braunschweig, 1859; and Jelink, Die Auflosung der hoheren numerisehen Gleichungen, Leipzig, 1865.

† Bulletin of American Mathematical Soc., 1894, Vol. I, p. 37; also, American Journal of Mathematics, Vol XVII, pp. 89—110.

‡ American Philosophical Society Proceedings, Vol. 42.

Section (1).

1. Consider the quadratic equation

$$z^2+2pz+q=0. \qquad (1)$$

We may write its roots in the form $(x+iy)$ without loss of generality. Substituting in the equation and separating real and imaginary arts, we have

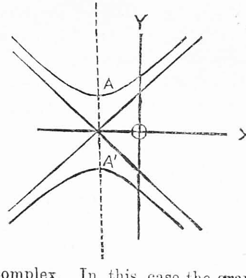
$$(x^2-y^2+2px+q)+2iy(x+p)=0,$$

which is equivalent to the two equations

$$\left.\begin{array}{r} x^2-y^2+2px+q=0, \\ y(x+p)=0. \end{array}\right\} \qquad (2)$$

Drawing the curves (2), we find (x,y) as their intersection. In other words, the Argand representations of the complex roots $(x+iy)$ are the *real* intersections of the Cartesian curves (2). It is quite easy to see that these intersect outside the axis of x only when the roots of

Fig. 1



the quadratic are complex. In this case the graphic points A, A′ will represent the complex roots

$$\{-p \pm i\sqrt{q-p^2}\}$$ [see Fig. 1.]

2. Next, suppose the equation is

$$f(z) \equiv z^3+3pz^2+3qz+r=0. \quad \ldots \quad \ldots \quad (3)$$

Proceeding as before, we have

$$f(x \pm iy)=f(x) \pm iy\, f'(x)-\frac{y^2}{2}f''(x) \mp \frac{iy^3}{6}f'''(x)=0,$$

which breaks up into the two separate equations

$$\left.\begin{array}{r} f(x)-\frac{1}{2}\, y^2 f''(x)=0 \\ y[f'(x)-\frac{1}{6}\, y^2 f'''(x)]=0 \end{array}\right\}$$

The factor $y=0$, in the second equation leads to $f(x)=0$, determining the real roots of the proposed equation. The complex roots (when they exist) are represented graphically as the real intersections of

$$\left.\begin{array}{r} f(x)-\frac{1}{2}\, y^2 f''(x)=0 \\ f'(x)-\frac{1}{6}\, y^2 f'''(x)=0. \end{array}\right\} \quad \ldots \quad \ldots \quad (4)$$

Now the cubic and conic represented by (4) can be readily traced and their intersections obtained; or (4) may be reduced to a cubic equation in x by eliminating y^2 and the real roots of this resulting cubic obtained. The corresponding real values of y will then give the graphic points, required.

[*cf.* MacRobert's *Theory of Functions*, pp. 16—19.]

3. The biquadratic equation $f(z)\equiv 0$, treated similarly leads to the intersections of the curves

$$\left.\begin{array}{l} f(x)-\frac{1}{2}y^2 f''(x)+y^4=0, \\ f'(x)-\frac{1}{6}y^2 f'''(x)=0, \end{array}\right\} \quad \ldots \quad \ldots \quad (5)$$

which are of the 4th and 3rd degrees respectively.

The elimination of y^2 between these gives an equation in x which may also be used for locating the real values of x, but the process is tedious.

4. Instead of the Cartesian $(x+iy)$ for the complex root, we may take the standard form $r(\cos\theta \pm i\sin\theta)$. The auxiliary curves corresponding to the complex roots will then reduce to

$$r^n\cos n\theta+p_1r^{n-1}\cos(n-1)\theta+p_2r^{n-2}\cos(n-2)\theta+\ldots+p_n=0,$$
$$r^n\sin n\theta+p_1r^{n-1}\sin(n-1)\theta+p_2r^{n-2}\sin(n-2)\theta+\ldots+p_n=0.$$

5. In connection with the method of this section the following hints on *curve-tracing* may be of use:

Suppose $F(x,y)=0$ is any plane curve. Then in relation to the curve the set of points in the plane of the curve may be grouped into three classes:

(i) The group of points for which $F(x,y)$ takes a positive value; say, the *positive group*.

(ii) The group of points for which $F(x,y)$ takes a negative value; say, the *negative group*.

(iii) The group of points for which $F(x,y)$ takes a zero value; say the *zero group*.

The last group defines the curve, and the other two groups are separated by it. Thus the whole plane is divided into *compartments* of *positive* and *negative* points, whose boundary is the curve itself.

Now, let P be any positive point and Q any negative point. Then by the principle of continuity we infer that there must be some point between P and Q lying on the curve, which can be approximately found. The method, though laborious, is useful in locating a curve consisting of several branches. Further, its essential importance as a method consists in its negative character; it does not require any knowledge of points on the curve, as in the usual method of curve tracing. By this method we can locate a curve of any degree in relation to two points taken at random.

Ex. 1.—As an illustration, let us consider the equation $z^5-4z-2=0$ [Cajori: *Theory of Equations*, p. 29].

The auxiliary curves being denoted by P and Q, we have

$P\equiv r^5\cos 5\theta-4r\cos\theta-2\equiv x^5-10x^3y^2+5xy^4-4x-2=0,$

$Q\equiv r^5\sin 5\theta-4r\sin\theta\equiv y(5x^4-10x^2y^2+y^4-4)=0.$

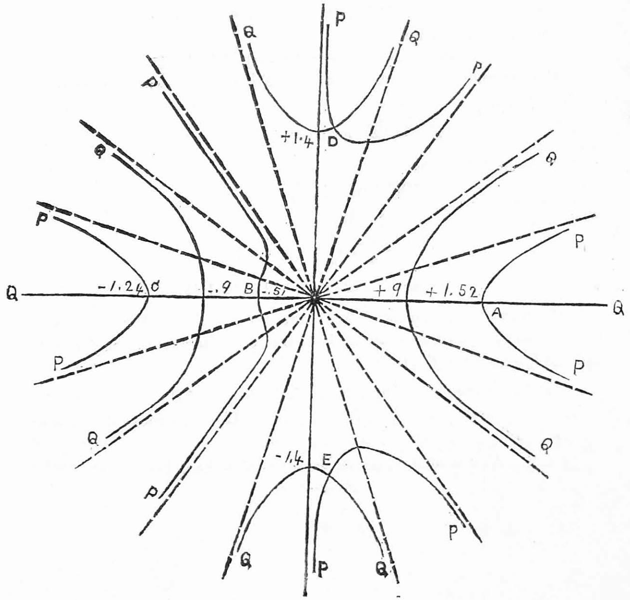
(I) *The P—curve*:—Considering P as a quadratic in y^2, we find that the discriminant

$D=100x^6-20x(x^5-4x-2)=40(2x^6+2x^2+x)$ and is positive for all values of x not lying between 0 and $-\cdot 48$, since $D=0$ for $x=-\cdot 48$ nearly.

Hence y has real values except when x lies between these limits.

Again all the values of y will be real only, if $\phi\equiv(x^5-4x-2)/x$ is positive. The latter changes sign when $x=1{\cdot}52, -{\cdot}51, -1{\cdot}24$ nearly.

Fig. 2.



Thus the following cases arise

(i) $\phi=+$, when $x>1{\cdot}52$;

(ii) $\phi=-$, when $1{\cdot}52>x>0$;

(iii) $\phi=+$, when $0>x>-{\cdot}51$;

(iv) $\phi=-$, when $-{\cdot}51>x>-1{\cdot}24$;

(v) $\phi=+$, when $-1{\cdot}24>x$.

In case (i) y has *four real* values,

(ii) y has *two real* and *two imaginary* values,

(iii) y has *four imaginary* values, if $0>x>-{\cdot}48$, and *four real* values, if $-{\cdot}48>x>-{\cdot}51$,

(iv) y has *two real* and *two imaginary* values,

(v) y has *four real* values.

(2) *The Q–Curve* :—This consists of the axis of x and two hyperbolic branches as is easily seen.

[*N.B.*—The asymptotes to the P and Q curves are $\cos 5\theta=0$, $\sin 5\theta=0$ respectively].

A tracing of the curves is given in Fig. 2,

Their *real intersections* furnish the roots of the quintic, and they are

A, B, C, corresponding to the real roots;

D, E, corresponding to the imaginary roots.

Their approximations are

$$+1{\cdot}52,\ -{\cdot}51,\ -1{\cdot}24;\ +{\cdot}12\pm1{\cdot}44\sqrt{-1}$$

The approximation to D is found by changing the origin to $(0, \sqrt{2})$, so that Q comes

$$y'=\frac{5\sqrt{2}}{4}x'^2, \text{ nearly};$$

and P becomes (on substitution for y)

$$50x'^3+gx'-1=0,$$

whence $$x'=\tfrac{1}{8}={\cdot}12$$

$$y'=\frac{5\sqrt{2}}{4}\cdot\frac{1}{64}={\cdot}03$$

or $$x=x'=.12!;\ y=\sqrt{2}+y'=1{\cdot}44.$$

It may be remarked here, that the P and Q curves are orthogonal always; and that the P–curve cuts the axis of x in $f(x)=0$, and the Q–curve in $f'(x)=0$.

By Rolle's theorem, therefore, it follows that if P crosses the axis p times, Q crosses it in *at least* $(p-1)$ times. If all the roots of $f(z)$ are real, P and Q will never cut in real points.

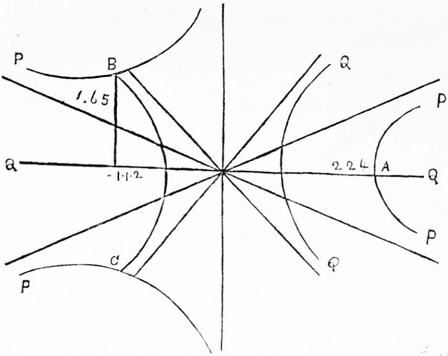
Ex. 2. As a second example, we may take the cubic

$$z^3-z-9=0.$$

Here $$P \equiv x^3 - 3xy^2 - x - 9 = 0,$$
$$Q \equiv y(3x^2 - y^2 - 1) = 0.$$

The graphs are as under.

Fig. 3.



The roots are $+2{\cdot}24$; $-1{\cdot}12 \pm 1{\cdot}65\sqrt{-1}$.

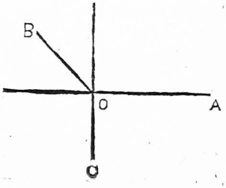
6. The method of '*vectors*' may be employed in a few typical cases, as the following examples will show :—

Ex. (1) : To solve $2z^2+2z+1=0$ ... ... (i)

Let OB represent the vector z, then z^2 is represented by OC, where $A\hat{O}B = B\hat{O}C$, and $OA.OC = OB^2$; so that the equation is written

$$\overline{OA} + 2\overline{OB} + 2\overline{OC} = 0. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \text{(ii)}$$

Fig. 4.



Now, since OB bisects the angle AOC, equation (ii) requires that 2OC should be equal to OA.

Putting OB $= r$, we have

$$2r^2 = 1, \text{ or } r^2 = \tfrac{1}{2}.$$

$$\therefore \qquad r = \frac{1}{\sqrt{2}}.$$

$$\text{Also} \qquad 2r = -2 \cos \text{AOB, from (ii.)}$$

$$\therefore \qquad \cos \text{AOB} = -\frac{1}{\sqrt{2}}.$$

$$\therefore \qquad \text{AOB} = 135^\circ.$$

Hence, the value of z corresponding to OB is

$$r(\cos 135^\circ + i \sin 135^\circ) = -\frac{1}{2}(1-i).$$

Ex. (2) $z^3 - 9z - 12 = 0$.

Here z is easily written in the form $[\sqrt[3]{3} + \sqrt[3]{9}]$ and the several values of z are therefore

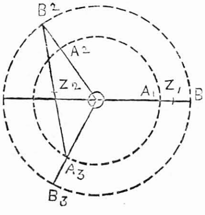
$$a + b, \; a\alpha + b\alpha^2, \; a\alpha^2 + b\alpha,$$

where $a = \sqrt[3]{3}$, $b = \sqrt[3]{9}$, α = cube root of unity.

Hence, we obtain the following construction for the roots of the cubic.

Describe concentric circles of radii a, b; inscribe equilateral triangles $A_1A_2A_3$ $B_1B_2B_3$ in them as in the figure.

Fig. 5.



Then the roots are represented by

$$2\,OZ_1, \; 2\,OZ_2, \; 2\,OZ_3$$

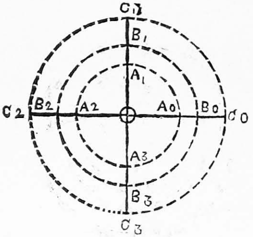
where Z_1 Z_2 Z_3 are the mid-points of A_1B_1, A_2B_3, A_3B_2

Ex. 3. $z^4 - 18z^2 - 48z - 39 = 0$.

In this cas e $z = a\alpha + b\alpha^2 + c\alpha^3$, where $a = 3^{\frac{1}{4}}, b = 3^{\frac{1}{2}}, c = 3^{\frac{3}{4}}$, and $\alpha^4 = 1$. construction for the several values of z corresponding to the four values of α is as follows:

Draw concentric circles of radii a, b, c; inscribe squares in them as shewn in the figure. Then the four roots are represented by $3\ OZ_1$, $3\ OZ_2$, $3\ OZ_3$, $3\ OZ_4$, where $Z_1\ Z_2\ Z_3\ Z_4$ are the mean centres of $A_1B_2C_3$, $A_2B_0C_2$, $A_3B_2C_1$, $A_0B_0C_0$

Fig. 6



7. The *vector* method can be successfully employed in the case of a *circulant* equation of any degree whatever.

For, we know that the roots of such an equation of the n^{th} degree can be written

$$a\alpha + b\alpha^2 + \ldots \qquad l\alpha^{n-1}$$

where $\alpha^n = 1$. [Burnside and Panton: *Theory of Equations*, Vol. II, p. 62.]

To represent the roots graphically draw concentric circles of radii $a, b, c, \ldots l$ and place in them regular n—sided polygons

$$A_1A_2\ldots A_n,\ B_1B_2\ldots B_n, \ldots\ldots L_1L_2\ldots L_n.$$

Then the roots of the circulant equation are given by

$$nOZ_1,\ nOZ_2, \ldots\ldots nOZ_n$$

$Z_1, Z_2\ldots\ldots Z_n$ being the mean centres of $A_1B_2\ldots L_n, A_2B_4C_6\ldots, A_3B_6C_9\ldots$, the suffixes having a period n.

8. The *general* cubic and biquadratic equations can also be solved by the above method.

The circulant equation of the *third* degree is

$$\begin{vmatrix} -x & a & b \\ b & -x & a \\ a & b & -x \end{vmatrix} = 0,$$

which is the same as

$$x^3 - 3abx - a^3 - b^3 = 0;$$

comparing with $x^3 + qx + r = 0$, we have

$$q = -3ab,\ r = -(a^3 + b^3).$$

Thus a^3, b^3 are the roots of

$$t^2+rt-q^3/27=0;$$

and the graphical representation of x is as in § 6, Ex. 2.

Again, the circulant equation of the *fourth* degree is

$$\begin{vmatrix} -x & a & b & c \\ c & -x & a & b \\ b & c & -x & a \\ a & b & c & -x \end{vmatrix}=0,$$

which reduces to

$$x^4-2\,x^2\,(b^2+2\,a\,c)-4\,b\,x\,(a^2+c^2)+b^2\,(b^2-4\,a\,c)-(a^2-c^2)^2=0.$$

Comparing with $x^4+q\,x^2+r\,x+s=0$, we have

$$q=-2\,(b^2+2\,a\,c),\ r=-4\,b\,(a^2+c^2),$$
$$s=b^2\,(b^2-4\,a\,c)-(a^2-c^2)^2.$$

Eliminating a and c, the cubic equation for b^2 is

$$t^3+\tfrac{1}{2}\,q\,t^2+(\tfrac{1}{16}\,q^2-\tfrac{1}{4}\,s)t-\tfrac{1}{64}\,r^2=0,$$

which is identical with the '*auxiliary cubic*' of the biquadratic.

The difference between Euler's method and this method, however, consists in the expression for the roots of the quartic in terms of those of the cubic. According to Euler's method the roots of the quartic are

$$\sqrt{t_1}+\sqrt{t_2}+\sqrt{t_3}$$
$$\sqrt{t_1}-\sqrt{t_2}-\sqrt{t_3}$$
$$-\sqrt{t_1}+\sqrt{t_2}-\sqrt{t_3}$$
$$-\sqrt{t_1}-\sqrt{t_2}+\sqrt{t_3};$$

whereas our method gives them in the form

$$-b\pm(a-c)\,i,\ b\pm(a+c),$$

b, a, c standing for

$$-\sqrt{t_1},\ \tfrac{1}{2}\sqrt{t_3}\,(1-i)-\tfrac{1}{2}\sqrt{t_2}\,(1+i),\ \tfrac{1}{2}\sqrt{t_3}\,(1+i)-\tfrac{1}{2}\sqrt{t_2}\,(1-i)$$

respectively.

Section (II).

9 We shall consider in this Section *approximations* to complex roots of an algebraic equation by means of Cauchy's theorem, *viz.*:

$$f\,(z)=f\,(a)+(z-a)\,f'(a)+\frac{(z-)}{2\,!}\,f''(a)+\ldots$$

This series leads to the following approximations, provided there is a root in the neighbourhood of a point a.

The *first* approximation is $(a+h_1)$, such that

$$f(a)+h_1\,f'(a)=0.$$

$$\therefore\ h_1=-f\,(a)/f'(a). \qquad \text{(i)}$$

The *second* approximation is $(a+h_1+h_2)$, such that

$$f\,(a+h_1)+h_2\,f'\,(a+h_1)=0,$$

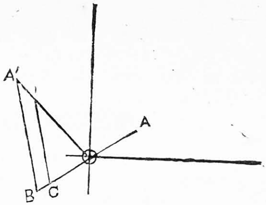
i.e. $$f\,(a)+h_1\,f'(a)+\frac{h_1^2}{2!}f''\,(a)+h_2\,f'\,(a)=0.$$

$$\therefore \qquad h_2=-\tfrac{1}{2}\,[f\,(a)]^2.\ f''\,(a)/[f'\,(a)]^3. \qquad \text{(ii)}$$

And so on.

These approximations are *graphically* equivalent to the following construction :

Corresponding to a in the z—plane, let
A be the point $f\,(a)$ in the w—plane,
A′................ $f'\,(a)$ in the same,



then $$h_1=-f\,(a)/f'\,(a)$$
$$=\overline{OB}/\overline{OA'}.\ [\text{Make } OA^1.\ O1=OB^2=OA^2.$$

In other words, h_1 is the vector equal to OC in the figure whose vectorial angle is equal to A′OB.

The *first* approximation to the root is thus

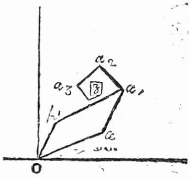
$$a+h_1=a_1 \text{ say}.$$

Similarly, the *second* approximation is

$$a+h_1+h_2=a_2, \text{ say}.$$

And so on, for further approximations.

Fig. 8.



The actual complex root z of the proposed equation is therefore graphically determined as the point of convergence of the polygon $o\ a\ a_1\ a_2 \ldots$

9·1. The assumption that

$$a_1 = a + h_1$$

is a better approximation than a depends upon the following results :

(i) $$\frac{f'(z)}{f(z)} \equiv \sum \left(\frac{1}{z-\alpha}\right)$$

where α is a complex root or real root of $f(z)$. Now, when a is an approximate complex root, the corresponding partial fraction on the right side will have its modulus very great when $z = a$.

Hence $\left|\frac{f'(a)}{f(a)}\right|$ is large when a is an approximate complex root.

that is, $|h_1| = -\left|\frac{f(a)}{f'(a)}\right|$ is small.

(ii) $$f(a_1) = f(a) + h_1 f'(a) + \frac{h_1^2}{2!} f''(a) +$$

$$= \frac{h_1^2}{2} f''(a), \text{ since } h_1 = -f(a)/f'(a).$$

$$\therefore \quad \frac{f(a_1)}{f(a)} = \frac{h_1^2}{2} \frac{f''(a)}{f(a)}.$$

And in general $\left|\frac{f''(a)}{f(a)}\right|$ is small for a similar reason as in (i).

Hence $\left|\frac{f(a_1)}{f(a)}\right|$ is small.

Thus $f(a_1)$ is a better approximation than $f(a)$ under the circumstances.

9·2. The degree of approximation in taking $a_1 = a + h_1$ instead of a may be investigated as in the usual methods for real roots.

10. The approximations found in the last article may also be *directly* obtained by separating the real and imaginary parts at each stage of the approximations.

Thus, putting $$f(a) = \alpha + i\beta,$$
$$f(a) = \alpha' + i\beta', \text{ \&c.,}$$

we have

$$w = u + iv = f(x+iy)$$
$$= \alpha + i\beta + (x+iy-\alpha-i\beta)(\alpha'+i\beta') + \dots$$

Equating real and imaginary parts :

$$u = \alpha + [(x-\alpha)\alpha' - (y-\beta)\beta'] + \dots$$
$$= \alpha + \alpha_1 + \alpha_2 + \dots, \text{ say };$$
$$v = \beta + [\beta'(x-\alpha) + \alpha'(y-\beta)] + \dots$$
$$= \beta + \beta_1 + \beta_2 + \dots, \text{ say}.$$

The *first* approximation is thus the intersection of the straight lines

$$\alpha+\alpha_1=0,\ \beta+\beta_1=0,$$

which are obviously rectangular.

It may be observed that *all* the successive approximations are *orthogonal*, being *conjugate* functions.

11. The approximations here investigated are capable of interpretation by means of 'the theory of polars' of a binary form.

For, corresponding to any point a, the $(n-1)th$ polars with respect to u and v are easily seen to be identical with the *first* approximations in § 10; and therefore, the first approximation is geometrically interpreted as the point of intersection of these polar lines, which are at right angles. Similarly, for further approximations.

12. A general method of dealing with the intersections of two plane algebraic curves suggests itself from the preceding.

Take *any* point in the neighbourhood of a point of intersection and construct the *polar lines* of the point with respect to each curve. Then the point of intersection of these polar lines will be a *first approximation*; similarly we can construct further approximations.

Section (III).

13. In this last Section we shall apply the method of approximations to investigate the complex roots of a *Transcendental Equation*.

Consider the equation

$$z=\cos z,$$

which readily breaks up into the two equations

$$P\equiv x-\cos x\cosh y=0,$$
$$Q\equiv y+\sin x\sinh y=0.$$

The P and Q curves are traced by writing them in the forms

$$\cosh y=x\sec x,$$
$$\sin x=-y\ \text{cosech}\ y;$$

and using the *lambda* table (*vide*: A. Lodge's *Diffl. Calc.*, p. 82), along with the ordinary mathematical tables.

A tracing of portions of P and Q. prepared with reference to the subjoined tabular values, follows:

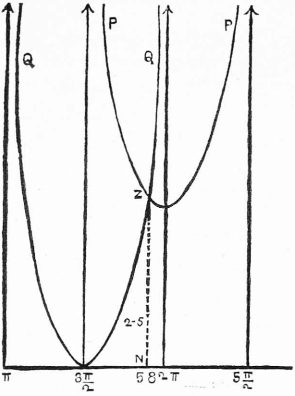
Table (1): $\cosh y=\sec\theta=x.\sec x$.

x	$3\pi/2$	6·11	2π	$5\pi/2$	For values of x between $3\pi/2$ & $5\pi/2$.
$\cos\theta$	0	·16	·156	0	
θ	90°	80°40′	81°	90°	
y	∞	2·5 (min.)	2·54	∞	

Table (2) : $\sin x = -y \operatorname{cosech} y$; $\sinh y = \tan \theta$

y	0	1	2	3	∞	
θ	0°	49°36′	74°40′	84°17′	90°	For values of x between $\frac{3\pi}{2}$ and 2π.
$\sin x$	-1	$-\cdot 851$	$-\cdot 558$	$-\cdot 3003$	0	
x	$3\pi/2 = 270°$	301°40′ 238°20′	326°3′ 213°57′	342°31′ 197°29′	360° 180°	

Fig. 9.



From the diagram it is obvious that the P and Q curves cross in the neighbourhood of the point

$$[x = 5\cdot 8\ ;\ y = 2\cdot 5].$$

To determine the *first* approximation, we take the equations

$$P' + h\frac{dP'}{d\alpha} + k\frac{dP'}{d\beta} = 0,$$

$$Q' + h\frac{dQ'}{d\alpha} + k\frac{dQ'}{d\beta} = 0\ ;$$

where P′, Q′ stand for the values of P and Q, when $x = 5\cdot 8 = \alpha$, say ; $y = 2\cdot 5 = \beta$, say.

Writing $\sin \alpha = \sin 5{\cdot}8 = \sin 332^\circ\ 19'$

$$= -\ {\cdot}4645$$

$$\cos \alpha = \cos 332^\circ\ 19' = {\cdot}8855$$

$$\text{sech}\ \beta = \text{sech}\ 2{\cdot}5$$

$$= \cos 80^\circ\ 40' = {\cdot}1622$$

$$\tanh \beta = \sin 80^\circ\ 40' = {\cdot}9868$$

from the tables and solving the two equations for h and k, we get

$$h = {\cdot}074,\ k = {\cdot}039.$$

Hence, a first approximation to the complex root in the neighbourhood of (α, β) is

$$x = 5{\cdot}874\ ;\ y = 2{\cdot}539\ ;$$

or

$$z = 5{\cdot}874 + 2{\cdot}539\ i.$$

14. We might have proceeded more *directly* thus—

$$f(z) \equiv z - \cos z = (a - \cos a) + (z - a) f'(a), \text{ approximately,}$$

where $a = \alpha + i\beta$; so that

$$z - a = -(a - \cos a)/f'(a)$$

$$= -(a - \cos a)/(1 + \sin a)$$

$$\therefore\ z = a - (a - \cos a)/(1 + \sin a)$$

$$= (a \sin a + \cos a)/(1 + \sin a)$$

$$= [(\alpha + i\beta) \sin (\alpha + i\beta) + \cos (\alpha + i\beta)]/[1 + \sin(\alpha + i\beta)]$$

$$= \alpha_1 + i\beta_1, \text{ say.}$$

Calculating α_1, β_1, we get for the first approximation

$$5{\cdot}874 + 2{\cdot}537i$$

Other complex roots may be obtained similarly by drawing the remaining branches of the P and Q curves and noting their intersections.

Postscript.

15. Two simultaneous equations involving *two variables* are usually represented by plane graphs and their real solutions obtained as the intersections of the graphs. When there are no *real* intersections, however, the corresponding *complex* solutions of the equations may be visualized as follows :—

(i) *First Method.*—Consider z, z' as the unknown complex quantities satisfying the equations

$$f(z, z') = 0,\ \phi(z, z') = 0.$$

Write $z = x + iy$, $z' = x' + iy'$ in the above and separate the real and imaginary parts, so that, we have *four* equations in the *four* unknowns (x, y, x', y').

Now, these four unknowns may be regarded as determining a *real point* in space of four dimensions, and the resulting equations in them as relating to four hyper-surfaces. The common points of these hyper-surfaces would consequently represent the complex solution of the proposed equations.

The difficulty of conceiving four dimensional space detracts from the usefulness of this method of representation.

(ii) *Second Method.*—In this method z, z' may be associated with a line in space in as much as *four co-ordinates* are necessary to determine a line completely. Accordingly the two proposed simultaneous equations on being split up into real and imaginary parts specify four *complexes* in space, and the complex solutions of the equations are geometrically represented by the lines common to the four complexes. The intuitive difficulties of the previous method exist in this case also, in so far as line-geometry and hyper-geometry are intimately connected.

(iii) *Third Method.*—This is only a relic of the line-representation, being a representation of the two variables z and z' by *a point pair* in the same or in different planes. A few simple properties of the two plane representation are discussed in Forsyth's *Theory of Functions of Two Complex Variables*, § § 15—20.

(iv) *Fourth Method.*—Lastly, we may eliminate one of the variables z', between the proposed equations and treat the resulting equation by the methods applicable to functions of a single variable.

16. In simple cases, however, the *vector method* will be found quite suitable, as the following examples will prove :—

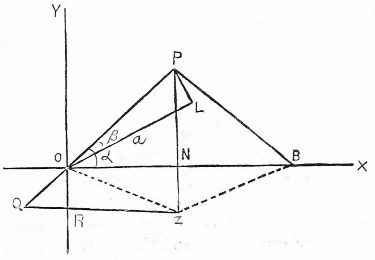
Example 1.—Solve $z^2+z'^2=a^2$, $z+z'=b$; where $b^2>2a^2$.

Let $z=a\cos\theta$, $z'=a\sin\theta$, where θ is a complex quantity. Then we may write

$$z=a(\cos\alpha\sec\beta-i\sin\alpha\tan\beta),$$
$$z'=a(\sin\alpha\sec\beta+i\cos\alpha\tan\beta),$$

α, β being real quantities.

Fig. 10.



Hence the following construction for the vectors z, z'.—

In the figure $\overline{OL}=a$, $OB=b$, $OQ=PL$, &c.; and we have

$$ON=OL \sec \cos \alpha = a \sec \beta \cos \alpha,$$
$$NZ=OQ \sin \alpha = PL \sin \alpha = a \tan \beta \sin \alpha.$$
$$\therefore \quad \overline{OZ}=\overline{ON}+\overline{NZ}=a(\cos \alpha \sec \beta - i (\tan \beta \sin \alpha)$$
$$=z.$$

Again $NP=a \sec \beta \sin \alpha.$

$QR=a \tan \beta \cos \alpha$;

$$\therefore \quad iz'=\overline{NP}+\overline{RQ}.$$

But $z'=b-z=\overline{ZB}=\overline{ZN}+\overline{NB}$

$\therefore$ $NB=NP$, and $ZN=RQ$.

In other words, $\alpha=\pi/4$, $ON=\frac{1}{2}b$.

The vector solutions are thus OZ, ZB, corresponding to the complex quantities

$$[\tfrac{1}{2}b \pm i \sqrt{(\tfrac{1}{2}b^2-a^2)}]$$

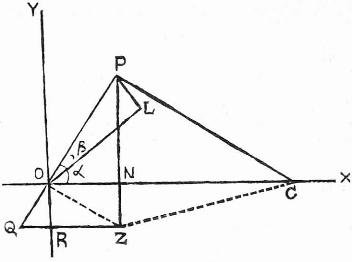
Example 2. Solve $z^2/a^2+z'^2/b^2=1$, $z+z'=c$; where $c^2 > a^2+b^2$.

Here, we can write

$$z=a(\cos \alpha \sec \beta - i \sin \alpha \tan \beta)$$
$$z'=b(\sin \alpha \sec \beta + i \cos \alpha \tan \beta).$$

Proceeding, as before,

Fig. 11.



$$z=\overline{OZ}, \; z'=\overline{ZC}=\overline{ZN}+\overline{NC}.$$

Also $$i z'=\frac{b}{a}\left(\overline{NP}+\overline{RQ}\right).$$

Hence $$NC=\frac{b}{a}NP, \; ZN=\frac{b}{a}RQ.$$

$$\therefore OPC=90^\circ, \tan \alpha=\frac{b}{a}, \; a=c \cos \alpha \cos \beta \text{ ; \&c.}$$

The vector solutions are thus OZ, ZC, corresponding to the complex quantities

$$\frac{a^2c}{(a^2+b^2)} \pm i \frac{ab\sqrt{(c^2-a^2-b^2)}}{(a^2+b^2)}$$

The Function of Mathematics in Scientific Research.

BY PROFESSOR G. A. MILLER.

(Concluded from p. 225 of Vol. IX.)

This common ground of investigators may serve to explain the fact that many of the most influential research organizations, like the National Academy of Sciences in our own country, embrace all the sciences. In recent decades there has been a tendency to organize research separately in the various subjects in the form of national societies named after these subjects. In fact, there are those who think that the latter have assumed such a preponderant sphere of influence as to threaten the very life of the former as serious factors in research. On the other hand, the maintenance of a common scientific life seems to be of the highest importance in view of desirable interactions and special emphasis on what is most fundamental.

The history of mathematics has taught us that some subjects which were apparently far apart and which were long developed separately were later seen to have most important common elements. The discovery of those common elements and their development has led to marked advances in the separate fields themselves. By way of illustration I need only refer to the fields of Algebra and geometry so happily welded through the work of Descartes, Fermat and many others. In modern times the theory of groups and invariants has exhibited many important connections between subjects which had been supposed to be widely separated. The same tendency has, of course, manifested itself in other sciences and may be assumed to become more dominant as knowledge advances.

A pertinent difference between the mathematical investigators and investigators in other sciences is that the former are compelled to stay with their problems until a solution is reached which can be proved to be in accord with deductions from certain definite assumptions, while the latter enjoy much greater freedom in regard to the stage to which they may pursue a problem before announcing results. Hence these may hope for success in dealing with much more difficult questions than the mathematician could reasonably hope to solve at the present time. The limitations thus imposed upon the mathematician are compensated, at least in part, by the finality of his results as regards questions of rigor. Mathematical results *can* never be disproved, other accepted results *have* never been disproved. With respect to simplicity and style, the mathematical developments are seldom final, and in many cases, they appear to admit endless variations.

As instances of final mathematical results may be cited the useful tables which when once computed serve all succeeding generations. Such finality may be said to be a goal of all scientific endeavor, since the results enrich countless ages by increasing their capacity for accomplishments. In fact, such tables may be regarded as typical illustrations of the mathematical contributions to the advancement of knowledge even if they constitute a very minor portion of these contributions. The fact that mathematical results have increased the capacity of the world for doing things may be emphasized by noting, in particular, that in recent years prime numbers have been found which could not have been proved to be prime by the method employed by Eratosthenes, if the entire human race had been working in an organized manner on this single problem since the days of the ancient Greeks.

The present seems to be an especially appropriate time to consider the interrelations of scientific research in view of the rapidly growing public appreciation of the value of such research. Several decades of comparative peace immediately preceding the present great and deplorable conflict were unusually rich in great scientific triumphs. As well-known instances we may cite wireless telegraphy and the construction of the great Panama Canal, which became possible by our advanced knowledge in regard to sanitation. The world-wide health activities under the auspices of the Rockefeller Foundation and the activities of our agricultural colleges in directing attention to advantages resulting from scientific methods of farming are strong forces working towards a popular appreciation of science. Since the great European war began it has become evident through the new elements introduced by the submarines and other scientific devices that the very existence of a great nation may depend upon the scientific attainment of its people, and hence the question of scientific research has taken a prominent place among those of national policy. It is perhaps significant that our National Academy was founded in the midst of the Civil War.

Scientific research is as old as civilization and has often been protected by kings in a patronizing manner, bnt it is a new experience in the history of the world to see kings turn to scientific research for protection. For centuries governments have recognized the value of science and have provided with growing liberality for her development, but now they are calling to her to save them from destruction. They have noticed that in spite of many excellencies in other directions the ignorance of causes may entail their destruction as separate nations. This new attitude towards our field of work may at first tend to gratify us, but a second thought reveals the fact that it is fraught with grave dangers. Kings in government and finance are interested in the dead results of science instead of in the great living and growing organism itself, whose growth seems to have just begun and whose development has always been more keenly inspired by love of truth than by hope of gain.

Is there not a danger that the sudden recognition of the great political importance of certain types of research will have somewhat the same effect on science as the discovery of gold in California and in Australia about the middle of the preceding century had on the development of the regions concerned? People flocked from one mining camp to the other and often neglected duties which are essential for the harmonious development of the resources of a country. Hence there seems to be a special need at present to urge our colleagues to remain at their posts of duty, notwithstanding glowing reports of chances to amass scientific fortunes quickly in certain newly discovered gold fields. The get-rich-quickly schemes in science should be scrutinized as carefully as similar schemes relating to the accumulation of money.

The remaining at one's post of duty in scientific research does not imply a lack of support in the solution of pressing problems or a lack of vacation trips and acquaintance with other fields of work. In fact, such support and acquaintance are highly desirable. It is, however, a question whether the nomadic scientific life, which seems to have become fashionable during the last few decades, at least in mathematics, is the one which will in the long run bring the best results. Science is not primarily a grazing country. Large tracts are suited for agriculture and mining. What is new is not necessarily good and what is good is not necessarily new, and prophesies in regard to the great importance of certain new developments have not always been fulfilled. On the other hand, it should be remembered that reasonable hope and optimism are essential for progress, and that we need prospectors as well as miners in the scientific world.

It should be noted that the miner needs some of the qualifications of the prospector since he is apt to meet with new situations and needs to take advantage of the available by-products. In fact, while he is mining for gold he may strike deposits of copper which are richer than the gold deposits which he was primarily seeking. Some of the richest mathematical discoveries were made while the investigator was looking primarily for other results, and even problems which have not been solved at all up to the present have been the source of very useful developments. I understand that similar conditions hold in other fields of scientific effort and these facts point to the great importance of freedom on the part of the investigator, and, incidentally to the danger of too much organization in scientific research.

As a very recent instance of an unexpected mathematical by-product, I may be pardoned for referring to a somewhat trivial case which has, however, the important property that it can be understood by all. It is well known that the theory of substitution groups was developed for the purpose of clarifying the theory of algebraic equations and not for the purpose of adding to the enjoyment of parties engaged in playing games of cards. In fact, the study of such an advanced mathematical theory as that of substitution groups might appear to involve concepts, which are at the opposite pole from those entering the minds of people seeking recreation at card tournaments.

Notwithstanding this apparent wide separation, I was pleased to be able to say recently to a friend, who desired to have each one of a large party play once and only once with each of the others during a series of successive games, that an arraugement of the players meeting this condition could be determined directly by means of substitutions of certain transitive groups. This should perhaps have been expected, since a transitive substitution group is an ideal republic treating all its etters in exactly the same way. On the contrary, an operation group may have elements enjoying special privileges and hence it has more extensive contact in the actual world of thought.

A little study of the stated problem revealed the interesting fact that when the number of tables is any power of 2 the substitutions of a well-known type of substitution groups and its group of isomorphisms exhibit directly how the players can be arranged so that each one will play once and only once with, and twice and only twice against, each of the others in a certain series of games. To make myself perfectly clear, I may say that if 8 tables, or 32 players, are involved, one can write directly by means of a certain regular substitution group of order 32 a set of possible arrangements so that in 31 successive games each one of these 32 players would play once and only once with each of the others and twice and only twice against each of them. This was, however, not the first solution of the general problem in question. In fact, about twenty years ago Professor E. H. Moore published a different solution of it in Volume 18 of the *American Journal of Mathematics* under the title "Tactical Memoranda."

I have referred to this matter here mainly for the purpose of emphasizing the fact that intellectual penetration is often attended by the most unexpected by-products, but I should also be pleased to have people know that certain kinds of recreation can easily be enriched by making use of results which the mathematician developed for a totally different purpose. Science should and does enrich both work and play. More than a thousand years ago the Hindu astronomer Brahmagupta said:

As the sun obscures the stars, so does the proficient eclipse the glory of other astronomers in an assembly of people by the recital of algebraic problems, and still more by their solutions.*

The playful question, Where do the finger nails find so much dark dirt to put under them ? may serve to arouse a thoughtful attitude on the part of the boy who has been taught to keep his hands clean. In fact, our play and recreation are perhaps as fundamentally affected by questions of science as our serious work and the victrolas and moving pictures should have a marked influence on the popular attitude towards science in view of the fact that they reach so many people. If it is true that the greatest service which science is rendering the human race is the reduction of superstition, it is clear that the efficiency of science depends largely upon its popularity.

The hypothesis that space and the operations of nature are discontinuous clearly excludes the hypothesis that they are continuous, but it is interesting to note that the mathematics relating to the discontinuous does not exclude that relating to the continuous. On the contrary, there are the most helpful interrelations between these two types of mathematics. Such a subject as number theory, relating decidedly to discrete quantities, has been greatly extended by analytic methods relating to continuous quantities, and, on the other hand, processes relating to the study of continuous functions are largely based upon those relating to the discontinuous.

This may perhaps tend to show that even if our hypotheses in regard to the continuity of space and the operations of nature have to be largely modified, as seems now probable, the mathematical methods of attack may require less modification than might at first appear to be necessary. The language which mathematics has provided for science includes not only concepts relating to the continuous and the discontinuous, but fortunately it also shows relations between these concepts and these relations become more pronounced with its development.

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* H. T. Colebrooke, "Algebra with Arithmetic and Mensuration from the Ssnskrit," by Brahmagupta and Bhascara, 1817, p. 379.

In view of the age of this language and its contact with various sciences it may be readily understood why mathematical history occupies a prominent place in the history of science. In fact, the history of science constitutes one of the fields where scientists may find common interests most fully represented, even if the past is too rich in events to be studied completely. It may therefore be appropriate on this occasion to refer to a few recent developments relating to the history of mathematics, especially since the interest in the history of science has increased rapidly during recent decades, as is partly evidenced by the efforts that are now being made to establish an institute of historical scientific research in our land.

One of the most interesting questions relating to the early history of mathematics is the use of positional values of numbers and the closely connected use of a symbol for zero. Until a decade or two ago it was commonly assumed by mathematical historians that the use of zero as a positional number symbol originated in India, and this view has not yet been entirely abandoned, notwithstanding the fact that the Babylonians employed numbers with positional value and a symbol which seems to have fulfilled the main function of our zero several centuries before the Christian era. On the other hand, the first definite evidence of the use of zero among the Hindus falls in the second half of the first millennium of this era.

In view of these facts it is extremely interesting to note the early use of zero, in connection with numbers having positional value, by the Maya, a people inhabiting the Atlantic coast plains of southern Mexico and northern Central America. One of the worthy alumni of your university recently referred to this matter in the columns of SCIENCE in the following words:

Special interest attaches to the occurrence of zero-symbols and the principle of local value among the inhabitants of the flat lands of Central America, at a period as early as the beginning of the Christian era, if not much earlier. It would seem that in this invention, the Maya in Central America possessed priorty over Asiatic people by a margin of five or six centuries.

If further investigation will lead mathematical historians to agree that the zero as a symbol in a numerical notation with positional value was actually first used in America, according to the preserved records, it will effect a very fundamental change as regards interest in the early mathematical attainments of the American aborigines. Unfortunately these early mathematical attainments failed to become the source of extensive further developments on American soil. They exhibit clearly that central concepts may be discovered independently and they direct attention to the danger in trying to establish one source for a particular concept in historical insvestigation. They also show that the small strip of country marked now by Boston has not always been the intellectual hub of America.

The history of some of the mathematical attainments of the Maya people has recently been made more easily accessible through the publication of "An Introduction to the Study of the Maya Hieroglyphs," prepared by S. G. Morley and published as Bullettin 57 of the Bureau of American Ethnology, Smithsonian Institution of Washington. On page 92 of this bulletin a dozen different symbols for zero are noted and on page 131 numbers varying from 21 to 12,489,781, and involving the use of zero, are represented in the Maya notation. It is of interest to note that the value of a unit in a higher position is always 20 times the value of a unit in the next lower position, except in the case of the third place, where its value is only 18 times that of the second place.

In historical research and elsewhere, the mathematician seeks cordial cooperation with other scientists, and he regrets that the confusion of tongues, resembling the experiences at the tower of Babel, is making it more and more difficult to understand each other. In the case of scientists this confusion is mainly due to a rapid growth of language in various directions. May we not hope that as many theories which were supposed to be distinct suddenly exhibited profound connections, so also this extensive language will tend towards unity and simplicity as we see more clearly the fundamental underlying principles. Science knows no bounds in method or in subject-matter and the artificial limitations set by man for his own convenience in making a start must break down before the onward march of truth. All science is a unit and all scientific investigaton should be inspired by their common interests.

[7] F. Cajori. SCIENCE, N. S., Vol. 44 (1916), p. 715.

Astronomical Notes.

Eclipses.

The first eclipse of the year 1918 will occur on June 8, when there will be a total eclipse of the Sun. The eclipse will be visible as a partial eclipse from the north-eastern part of Asia, the north polar regions and the whole of the north American continent. The track of totality runs from a point in the Bahamas Islands nearly to the East coast of China ; as it runs right across the centre of the United States the eclipse is likely to be well observed in spite of the war ; the remainder of the central track lies in the Pacific Ocean.

There will be a partial eclipse of the Moon on the night of June 23—24, the eclipse is a small one, magnitude 0·135 and is invisible in India.

R. A. S. Gold Medal.

The Gold Medal of the Royal Astronomical Society was awarded this year to Mr. J. Evershed, Director of the Solar Physics Observatory at Kodaikanal.

Astronomical Consequence of a Curvature of Space.

In Mon. Not. R. A. S. 1917 Nov. de Litter investigates the effect on astronomical phenomena of a curvature of space taken in conjunction with Einstein's theory.

It is quite clear that two dimensional beings living on the surface of a sphere of very large radius might conceive themselves to be resident on a plane, and a similar notion may be extended to space in three dimensions, but by extending our observations to a distance comparable with the radius of curvature, we may hope to detect the difference ; hence it is to astronomical observations that we must look for any effect of curvature of space.

The new solution of Einstein's equations leads to a form of space with constant positive curvature. Such space may be either

(I) Riemann's spherical space in which all straight lines starting from a point intersect again in the antipodal point, whose distance from the starting point is π. R (R = radius of curvature) along all lines, and this is the greatest possible distance between two points ; or

(II) Newcomb's elliptical space, in which any two lines have at most one point in common, and the largest possible distance between two points is $\frac{1}{2}\pi$. R.

In (I) All lines are of length $2\pi R$; in (II) all lines are of length πR
de Litter finds that (on the assumption of a curved space) no star can possibly have a parallax less than a/R where a is the distance between the Earth and Sun.

Secondly it is found that the lines in the spectra of very distant stars and nebulæ will be displaced towards the red, producing a spurious positive radial velocity; a similar result holds in rectangular space on Einstein's theory, but the displacement is considerably augmented in a curved space. In this way it may be possible to account for the very large velocities of spiral nebulæ, which are certainly very distant, should they turn out to be positive on the whole. de Litter makes a number of estimates of the value of R, the data of course is somewhat scanty, since the smallest parallaxes are naturally the most uncertain, and very few absolute parallaxes are known, while the number of spiral nebulæ of which we have reliable determinations of radial velocity is very small (de Litter uses only three). R comes out about 10^{12} Astronomical Units.

If space is curved we should see an image of the sun at the antipodal point, that is the point of the sky opposite to the sun, but opinions appear to differ as to whether this image would be as large and bright as the sun or whether it would appear as an extremely faint star.

R. J. Pocock.

SOLUTIONS

Question 329.

(M. BHIMASENA RAO) :—If the pedal circle of P with respect to a triangle ABC touches the nine-points-circle of ABC, show that the sum of the angles PAB, PBC, PCA is constant.

Additional Solution by the Proposer.

The following simple result concerning the rectangular hyperbola which may be proved easily by the anharmonic property of conics is here assumed :—

ABCPQ is a rect hyp, perpendiculars QD, QE, QF are dropped on the sides of ABC intersecting PA, PB, PC in A′, B′, C′.

Then $QD. QA' = QE. QB' = QF. QC'$ ... ... (1)

This is the converse of the theorem. 'If two triangles ABC and A′B′C′ are conjugate with respect to a circle, they are in perspective; if P and Q be the centre of perspective and the centre of the circle respectively, the conics ABC PQ and A′B′C′ PQ are rect hyp.

If P and Q be isogonal conjugate points and D′E′F′ the pedal triangle of P

we have $PD'. QD = PE'. QE = PF'. QF$ ... ... (2)

From (1) and (2) we see that when ABC PQ is a rectangular hyp. QA′, QB′ and QC′ are proportional to PD′, PE′ and PF′, and being parallel respectively, the triangles A′B′C′ and D′E′F′ are homothetic. But we know that any inverse triangle of ABC—call it LMN—with respect to P is similar to the pedal triangle of P, the angle of similitude being the complement of $P\hat{A}B + P\hat{B}C + P\hat{C}A$. It follows therefore that A′B′C′ and LMN are similar and being in perspective, one of the two alternatives arises -- either they are homothetic, or ABCP is concyclic as also A′B′C′ P. The second alternative being rejected since P is not on the circum circle of ABC, we see that A′B′C′ and LMN are homothetic. Therefore D′E′F′ which is the pedal triangle of P, A′B′C′ and LMN are homothetic taken two by two, and the angle of similitude vanishes.

Hence PAB+PBC+PCA is equal to one right angle.

Corollary. When a triangle ABC and its inverse with respect to a point P are orthol ogic, the pedal circle of P with respect to either triangle touches the nine-point-circle of that triangle.

The above proof fails when P is on the circumcircle of ABC ; but in this case the result follows more easily. For properties of the points of intersection of McCay's cubic and the circumcircle of ABC, see Mr. S. Narayanan's paper on Three special points' page 85, Vol. I of the Journal.

Question 427.

(S. RAMANUJAN) :—Express

$$(Au^2+Buy+Cy^2)(Ap^2+Bpq+Cq^2)$$

in the form

$$Au^2+Buv+Cv^2,$$

and hence shew that if

$$(2x^2+3xy+5y^2)(2p^2+3pq+5q^2)=2u^2+3uv+5v^2,$$

one set of values of u and v, is

$$u=\tfrac{5}{2}(x+y)(p+q)-2xp,\ v=2qy-(x+y)(p+q).$$

Solution (1) by 'Zero', (2) by S. Narayan.

(1) Let $f(xy)\equiv Ax^2+Bxy+Cy^2\equiv A(x-\alpha y)(x-\beta y)$,

then $f(pq)\equiv Ap^2+Bpq+Cq^2\equiv A(p-\alpha q)(p-\beta q)$,

where α and β are the roots of $A\lambda^2+B\lambda+C=0$

Now

$$f(xy)\cdot f(p,q)\equiv A^2\{(x-\alpha y)(x-\beta y)(p-\alpha q)(p-\beta q)\}$$
$$\equiv A^2\{px+\alpha^2qy-\alpha(py+qx)\}$$
$$\{px+\beta^2qy-\beta(py+qx)\}$$

But $A\alpha^2+B\alpha+C=0$, and $A\beta^2+B\beta+C=0\cdot$

$\therefore$

$$f(x,y)f(p,q)\equiv\{Apx-A\alpha(py+qx)-qy(B\alpha+C)\}\times$$
$$\{Apx-A\beta(py+qx)-qy(B\beta+C)\}$$
$$\equiv A(u-\alpha v)(u-\beta v)$$
$$\equiv Au^2+Buv+Cv^2,.$$

where $Apx-Cqy=u\sqrt{A},\ A(py+qx)+Bqy\equiv v\sqrt{A}.$

A *general* solution can be obtained by the following artifice. Take (a,b) such that

$$f(a,b)=a\text{ square}$$
$$=c^2\text{ (say)}.$$

Then $f(xy).f(pq).f(ab)=A^3(x-\alpha y)(x-\beta y)(p-\alpha q)(p-\beta q)$
$(a-\alpha b)(a-\beta b)$.

By virtue of the quadratic $A\alpha^2+B\alpha+C=0$, the product $(x-\alpha y)(p-\alpha q)(a-\alpha b)$ can be reduced to the form $(u-\alpha v)$, where u and v are rational functions of $(x\ y\ p\ q\ a\ b)$

Similarly $(x-\beta y)(p-\beta q)(a-\beta b)=u-\beta v.$

$\therefore$ $f(x\,y).\,f(p\,q).\,f(a\,b)=A^3(u-\alpha v)(u-\beta v)$
$=A^2(A\,u^2+Buv+Cv^2)$

i.e., $c^2.\,f(xy).\,f(p\,q)=A^2(A\,u^2+Buv+Cv^2)$

Hence $f(x\,y).\,(f\,p\ q)=A\,u'^2+Bu'v'+Cv'^2,$

where $u'=Au/c,\ v'=Av/c.$

The *general* solution of the indeterminate equation

$$f(a,b)=c^2$$

leads to a corresponding *general* solution of the proposed question.

In the particular case of A=2, B=3, C=5, one solution of

$$2a^2+3ab+5b^2=c^2$$

is $a=1$, $b=-1$, $c=2$, and we easily write a set of values of u and v different from those given by the proposer.

(2) $Ax^2+Bxy+Cy^2$ may be easily thrown into one of the three forms

$$(lx+ly)^2+(mx+ny^2,$$
$$(lx+my)^2+(nx+y)^2,\ (lx+my)^2+(x+ny)^2,$$

provided $B^2+4AC<0$.

Similarly $\qquad Ap^2+Bpq+Cq^2$ may be expressed.

And since $\qquad (a^2+b^2)(c^2+d^2)$

can be expressed as the sum of two squares in two ways, it follows that

$$(Ax^2+B\,xy+Cy^2)\ (Ap^2+Bp\,q+Cq^2)$$

can be expressed as the sum of two perfect squares. Hence expressing

$$Au^2+Buv+Cv^2$$

as the sum of two squares and comparing with the above form, the values of u and v are easily obtained. It is clear that there are 108 ways of doing this. The values of u, v given by the proposer for the particular question constitute one of these ways.

And
$$u=\frac{3}{2}px-\frac{7}{2}y-\frac{1}{8}(x+y)\,(p+q),$$
$$v=\frac{3}{2}qy+\frac{1}{2}px-\frac{11}{8}(x+y)\,(p+q).$$

constitute another solution.

Question 508.

(S. P. Singaravelu Mudaliar):—If s_n stand for the sum of the reciprocals of the first n natural numbers, find the sum of the infinite series.

$$s_1-\frac{1}{2}\left(\tfrac{1}{2}\right)^2 s_2+\frac{1}{3}\left(\frac{1\cdot 3}{2\cdot 4}\right)^2 s_3-\dots$$

Remarks by H. Br.

We are required to evaluate

$$S=\sum_{n=0}^{\infty}(-1)^n\,a_n^2\,\frac{s_{n+1}}{n+1}$$

where $\qquad a_n=\dfrac{1\cdot 3\ldots\ldots(2n-1)}{2\cdot 4\ldots\ldots(2n)}$, and $s_{n+1}=1+\dfrac{1}{2}+\ldots+\dfrac{1}{n+1}$

It is easy to obtain a variety of definite integrals, (simple, double, and triple) for the value of S, but I have not been able to find a formula giving its numerical value--

(1) As
$$-\int_0^1 \log(1-x).\, x^n dx = \frac{s_{n+1}}{n+1},$$

$$S = -\int_0^1 dx \log(1-x).\left(\sum_0^\infty a_n^2 (-x)^n\right).$$

Now
$$(1+x\sin^2\theta)^{-\frac{1}{2}} = \sum_0^\infty a_n (-x)^n \sin^{2n}\theta\ ;$$

$\therefore$
$$\int_0^{2\pi} \frac{d\theta}{(1+x\sin^2\theta)^{\frac{1}{2}}} = \frac{\pi}{2}\sum_0^\infty a_n^2 (-x)^n,$$

and
$$S = -\frac{2}{\pi}\int_0^{\frac{\pi}{2}} d\theta \int_0^1 \frac{\log(1-x).}{(1+x\sin^2\theta)^{\frac{1}{2}}} dx,$$

The integration with respect to x can be easily performed, but the result does not throw any fresh light on the subject.

(2) Again we may take $a_n^2 = \frac{4}{\pi^2}\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} \sin^{2n}\theta \sec^{2n}\phi\ d\theta\ d\phi$, and we get

$$S = -\frac{4}{\pi^2}\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\int_0^1 \frac{\log(1-x).\, dx\, d\theta\, d\phi.}{1+x\sin^2\theta\cos^2\phi}$$

(3) We have $a_{n+1} = a_n.\frac{2n+1}{2n+2} = a_n\left[1 - \frac{1}{2(n+1)}\right]$

$$\therefore \frac{a_n^2}{n+1} = 2(a_n^2 - a_n\, a_{n+1})$$

$$= \frac{8}{\pi^2}\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} \sin^{2n}\theta.\ \sin^{2n}\phi.\ \cos^2\phi.\, d\phi$$

and $(-1)^n s_{n+1}$ is the coefficient of

$$+\, t \text{ in } \frac{(t-1)(t-2)(t-3)\ldots(t-\overline{n+1})}{(n+1)!}$$

$$\therefore \quad S = \text{coefficient of } t \text{ in } \frac{8}{\pi^2}\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} \frac{d\theta\, d\phi.\ \cos^2\phi.}{\sin^2\theta\ \sin^2\phi} \sum_{n=0} \frac{(t-1)(t-2)\ldots(t-n-1)}{(n+1)!} \sin^{2n+2}\theta\ \sin^{2n+2}\phi$$

$$= \text{coefficient of } t \text{ in } \frac{8}{\pi^2}\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} \frac{d\theta\, d\phi.\ \cos^2\phi}{\sin^2\theta\ \sin^2\phi}\left[-1+(1+\sin^2\theta\ \cos^2\phi)^{t-1}\right]$$

$$= \frac{8}{\pi^2}\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} \frac{d\theta\ d\phi.\ \cos^2\phi}{\sin^2\theta.\ \sin^2\phi} \frac{\log(1+\sin^2\theta\ \sin^2\phi)}{1+\sin^2\theta\ \sin^2\phi}.$$

(4) Another from of the integral may be obtained by using the fact that

$$\int_0^{2\pi} \frac{d\theta}{\sqrt{1-2r\cos\ \theta+r2}} = 2\pi\Sigma\, a_n^2\, r^{2n}.$$

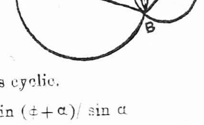
Question 591.

(A. H. Krishnaswami Aiyangar):—Two circles intersect at A and B; the tangents at the extremities of a double chord through A meet in X. Shew that XY perpendicular to BX envelopes a circle.

Solution by 'Zero'.

Let PAQ be the double chord. then

$$XPA + XQA = PBA + QBA = PBQ.$$



$\therefore$ PBQX is cyclic.

Now $BP = AB \sin(\phi+\alpha)/\sin\alpha$

$BQ = AB \sin(\phi+\alpha)/\sin\beta$

Also BP. BQ = BX. BA, since in the cyclic quadrilateral PBQX

PBA $= \phi =$ XBQ.

Hence $BX = AB \sin^2 (\phi+\alpha)/(\sin \alpha \sin \beta)$

$$= AB \{ (1-\cos (2\phi+2\alpha) \} /(2 \sin \alpha \sin \beta)$$

$$\propto \{ 1-\cos (\pi+\theta-\beta+\alpha) \},$$

where $\theta = ABX = (2\phi-\pi+\alpha+\beta)$.

In other words, the locus of X is the cardioide

$$r = a \{ (1+\cos (\theta+\alpha-\beta) \},$$

shewing that the envelope of XY is a circle.

Question 650.

(N. B. PANDYA) :—Circumscribe an ellipse about a given triangle so that incentre of ellipse may coincide with the incentre of the triangle.

Solution by C. Bhaskaraiya.

The areal co-ordinates of the in centre are

$$a : b : c.$$

If $Fyz + Gzx + Hxy = 0$ be the required ellipse, the polar *w. r. t.* if of $(a : b : c)$ ought to be identical with the line at infinity, $x+y+z=0$.

$$\therefore \quad cG+bH = Ha+cF = Ga+bF.$$

$$\therefore \quad F : G : H = a\ (a-b-c) : b\ (b-c-a) : c\ (c-a-b)$$

Hence the ellipse is

$$a\ (b+c-a).\ yz + b\ (c+a-b)\ zx + c\ (a+b-c)\ xy = 0,$$

To trace the ellipse put $\frac{y}{b} = \frac{z}{c}$, and see where it cuts the bisector of the angle A; and so on.

Question 671.

(K. J. SANJANA, M.A.) :—The numbers from 1 to 2^n-1 being arranged on n cards on which the least numbers are respectively 2^0, 2^1 ... ... 2^{n-1} as in Question 640, prove that the sum of the numbers on the r^{th} card is $2^{2n-2}+2^{n+r-3}-2^{n-2}$

Solution by N. Sankara Aiyar and V. Anantaraman.

The first number on the r^{th} card is 2^{r-1} and all numbers will be found on it which when divided by 2^r leave remainders greater than or equal to 2^{r-1}.

Hence the numbers on that card are

$$\{ 2^{r-1}+(2^{r-1}+1)+..+(2^{r-1}+2^{r-1}-1) \} + \{ (2^r+2^{r-1})+(2^r+2^{r-1}+1)+. \}$$

$$+ \{ (2 \cdot 2^r+2^{r-1})+(2 \cdot 2^r+2^{r-1}+1)+ \quad \dots \quad \dots \quad \} \quad \dots \quad \dots$$

$$+ \{ (2^n-2^{r-1})+(2^n-2^{r-1}+1)+ \quad \dots \quad +(2^n-1) \} \quad \dots \quad \dots$$

$$\therefore \quad = 2^{r-1} \times 2^{r-1}(1+3+5+ \quad \dots \quad +2^{n-r+1}-1) \} \dots \quad \dots$$

$$+ \quad \frac{2^{n-r+1}}{2}(0+1+ \dots \quad \dots +2^{r-1}-1).$$

$$= 2^{2r-2}. \left(\frac{2^{n-r+1}}{2}\right)^2 + 2^{n-r}. \frac{2^{r-1}}{2}(2^{r-1}-1)$$
$$= 2^{2r-2}. 2^{2n-2r} + 2^{n-r}. 2^{r-2}(2^{r-1}-1)$$
$$= 2^{2n-2} + 2^{n+r-3} - 2^{n-2}.$$

Question 682.

(S. RAMANUJAN) :—Show how to find the cube root of surds of the form $A+\sqrt[3]{B}$; and deduce that

$$\sqrt[3]{(\sqrt[3]{2}-1)} = \sqrt[3]{\frac{1}{9}} - \sqrt[3]{\frac{2}{9}} + \sqrt[3]{\frac{4}{9}}.$$

Solution by 'Zero.'

If α denote a cubic surd any expression in α can be written in the form $(a+b\,\alpha+c\alpha^2)$ and further

$$(a+b\alpha+c\alpha^2)^3 \equiv P+Q\alpha+R\alpha^2 \text{ (say)}.$$

Thus, when either Q or R is zero, we have a binomial cubic surd on the right side.

Hence we can write

$$\sqrt[3]{(A+\sqrt[3]{B})} \equiv r^{\frac{1}{3}}(a+b\alpha+c\alpha^2),$$

and proceed to find the values of a, b, c, α satisfying the following:

$$\left.\begin{aligned} rP &= A \\ Q &= 0 \\ rR\alpha^2 &= B^{\frac{1}{3}} \end{aligned}\right\} \text{ or } \qquad \begin{aligned} rP &= A \\ R &= 0 \\ rQ\alpha &= B^{\frac{1}{3}} \end{aligned}$$

In the particular case stated, $A=-1$, $B=2$;

$$\therefore \qquad P \equiv a^3+b^3\alpha^3+c^3\alpha^6+6abc\alpha^3,$$
$$Q \equiv 3a^2b+3ac^2\alpha^3+3b^2c\alpha^3,$$
$$R \equiv 3ab^2+3a^2c+3bc^2\alpha^3,$$

and suitable values for a, b, c, r are seen to be

$$a=-b=c=1,\ r=\frac{1}{9}.$$

More generally if $AQ=BP$, we may put $a=c$ and $\alpha=\sqrt[3]{B}$, so that $R=0$ reduces to

$$a^2+b^2+abB=0$$

$$\therefore \qquad b=-m\,a, \text{ if } B=\left(m+\frac{1}{m}\right);$$

and we may, without loss of generality, write $a=c=1$, $b=-m$.

The problem is thus solved, when B is of the form $\left(m+\frac{1}{m}\right)$, and $3A\{B(1+m)-m\} = B\{B^2-B(m^2+6m)+1\}$

Question 725.

(K. B. MADHAVA) :—Shew that

$$\int^{\infty} \frac{x^3 dx}{1+x^9 \sin^2 x} \text{ converges,}$$

but that

$$\int^{\infty} \frac{x^3 dx}{1+x^8 \sin^2 x} \text{ diverges.}$$

Solution by S. R. Ranganathan.

The condition for the convergence of

$$\int^{\infty} \frac{x\beta}{1+x^{\alpha} \sin^2 x}$$

are fully discussed in Bromwich: *Infinite Series*, App. III. Art. 166. It is proved there that the integral converges or diverges with the series

$$\sum \left(n^{\beta - \frac{\alpha}{2}}\right),$$

i.e., according as $\alpha > 2(\beta+1)$ or $\alpha \leq 2(\beta+1)$.

Now taking $\beta = 3$, we get that

$$\int^{\infty} \frac{x^3}{1+x^{\alpha} \sin^2 x} dx$$

converges if $\alpha > 8$, and diverges if $\alpha \leq 8$.

Hence the results given in the question.

Question 800.

(S. MALHARI RAO) :—Shew that the sum of all fractions which may be represented by a recurring decimal of the form $.a\,b\,\dot{c}\,\dot{d}$ is 50, provided

$$a+c=b+d=9.$$

Solution (1) by G. L. Gupta, M.A. and S. V. Venkatachalayya.

(i) For all values of a, b, c and d the recurring decimal $\cdot a\,b\,\dot{c}\,\dot{d}$

$$= \frac{abcd - ab}{9900} = \frac{(1000a+100b+10c+d)-(10a+b)}{9900}$$

$$= \frac{990a+99b+10c+d}{9900}. \qquad \ldots \qquad \ldots \qquad \ldots \qquad (1)$$

But here $\qquad a+c=b+d=9.$

$\therefore$ (1) which may be written as

$$\frac{10(a+c)+(b+d)+98(10a+b)}{9900}$$

reduces to

$$\frac{99+98(10a+b)}{9900} = \frac{99+98(ab)}{9900} \qquad \ldots \qquad \ldots \qquad (2)$$

But since each of a and b may have any of the values 0,1,2,3,...9 ab includes all integral numbers from 1 to 99, besides 0.

Hence the required sum

$$=\frac{1}{9900}\{(99\times100)+98(1+2+3+\ldots+99)\}$$

$$=\frac{1}{9000}\left\{9900+\frac{98\times90\times100}{2}\right\}=50.$$

Note:—There is a slight mistake in the printed question in which the decimal reads ($\cdot\dot{a}bcd$).

Solution by H. Br. and (2) K. B. Madhava.

(2) The recurring fraction ($.\dot{a}bc\dot{d}$)

$$=\frac{1000a+100b+10c+d}{9999}$$

$$=\frac{99.(10a+b)+10(a+c)+(b+d)}{9999}$$

$$=\frac{10a+b+1}{101},\text{ as } a+c=b+d=9,$$

As a and b assume independently all values from 0 to 9 (inclusive). the numerator assumes (once only) every integral values from 1 to 100, The sum of the fractions is therefore

$$\frac{1+2+3+\ldots\ldots+100}{101}=50.$$

Question 802.

(S. Krishnaswami Iyengar) :—Prove that

(i) $$\sum_{1}\left\{\frac{\Gamma(n+1)}{\Gamma(n+\frac{3}{2})}\right\}^2\cdot\frac{1}{n}=\frac{4}{\sqrt{\pi}}\left(\frac{4}{\sqrt{\pi}}-\sqrt{\pi}\right),$$

(ii) $$\sum_{0}^{\infty}\frac{\Gamma(n+\frac{1}{2})}{\Gamma(n+1)}\cdot\frac{1}{(2m+2n+1)}\cdot\frac{1}{(m+n+1)}$$

$$=\pi\left\{\frac{\Gamma(m+\frac{1}{2})}{\Gamma(m+1)}-\frac{2}{2m+1}\cdot\frac{\Gamma(m+1)}{\Gamma(m+\frac{1}{2})}\right\}.$$

Solution by S. R. Ranganathan and S. V. Venkatachallayya.

(i) This is the same as Q 664 solved on p. 69 of Vol. VIII, J.I.M.S.

(ii) We have, for $0\leq x\leq 1$,

$$(1-x)^{-\frac{1}{2}}=1+\tfrac{1}{2}x+\frac{1\cdot3}{2\cdot4}x^2+\ldots$$

so that, $$\Gamma(\tfrac{1}{2})(1-x)^{-\frac{1}{2}}=\sum_{0}^{\infty}\frac{\Gamma(n+\frac{1}{2})}{\Gamma(n+1)}x^n,$$

so that, $$\Gamma(\tfrac{1}{2})\int_0^1 x^m (1-x)^{-\frac{1}{2}}dx = \int_0^1 \sum_0^\infty \frac{\Gamma(n+\frac{1}{2})}{\Gamma(n+1)} x^{m+n} dx$$

$$= \sum_0^\infty \frac{\Gamma(n+\frac{1}{2})}{\Gamma(n+1)} \int_0^1 x^{m+n} dx$$

$$= \sum \frac{\Gamma(n+\frac{1}{2})}{\Gamma(n+1)} \frac{1}{m+n+1};$$

i.e., $$\{\Gamma(\tfrac{1}{2})\}^2 \frac{\Gamma(m+1)}{\Gamma(m+\frac{3}{2})} = \sum \frac{\Gamma(n+\frac{1}{2})}{\Gamma(n+1)} \frac{1}{m+n+1}.$$

In the above the change of order of integration and summation is permissible since the integral on the left is convergent and the terms in the series on the right are all positive (cf. Bromwich *Infinite Series* App. III, Art. 175. Theorem B.)

By treating similarly the expansion of $(1-x^2)^{-\frac{1}{2}}$ we get

$$\tfrac{1}{2}\{\Gamma(\tfrac{1}{2})\}^2 \frac{\Gamma(m+\frac{1}{2})}{\Gamma(m+1)} = \sum_0^\infty \frac{\Gamma(n+\frac{1}{2})}{\Gamma(n+1)} \frac{1}{(2m+2n+1)}$$

Now, $$\sum_0^\infty \frac{\Gamma(n+\frac{1}{2})}{\Gamma(n+1)} \frac{1}{(2m+2n+1)(m+n+1)}$$

$$= \sum_0^\infty \frac{\Gamma(n+\frac{1}{2})}{\Gamma(n+1)} \left\{ \frac{2}{2m+2n+1} - \frac{1}{m+n+1} \right\}$$

$$= \{\Gamma(\tfrac{1}{2})\}^2 \left\{ \frac{\Gamma(m+\frac{1}{2})}{\Gamma(m+1)} - \frac{\Gamma(m+1)}{\Gamma(m+\frac{3}{2})} \right\}$$

$$= \pi \left\{ \frac{\Gamma(m+\frac{1}{2})}{\Gamma(m+1)} - \frac{2}{(2m+1)} \frac{\Gamma(m+1)}{\Gamma(m+\frac{3}{2})} \right\}.$$

Question 810.

(T. P. TRIVEDI, M.A., L.L.B.) :—Find the values other than zero which satisfy the equations :—

$$x^2 = y - z\,;\quad y^2 = z - x\,;\quad z^2 = x - y.$$

Solution by H. Br. S. Gangadharan, K. Santanam, Kasturi Reddi, B. Sc., and F. H. V. Gulasekkaran, B. A.

We have

$$\left.\begin{aligned} x^2&=y-z\\ y^2&=z-x\\ z^2&=x-y.\end{aligned}\right\}\quad\ldots\quad (A)$$

$$\left.\begin{aligned} \therefore\quad x^2+y^2+z^2&=0;\\ x^3+y^3+z^3&=0;\\ \text{and } x^4+y^4+z^4=\Sigma x^2(y-z)&=-(y-z)(z-x)(x-y)=-x^2y^2z^2 \end{aligned}\right\}\quad B$$

[The system of equations B is composed of the system A together with the system derived from A by changing the signs of x, y, z.]

Let (x, y, z) be the roots of $t^3-pt^2+qt-r=0$. Then from (B),

$$\left.\begin{aligned} p^2&=2q;\\ pq&=3r\ldots\\ \text{and}\qquad r(p+r)&=0.\end{aligned}\right\}$$

If $r=0$, we get $p=q=0$, or $x=y=z=0$.
If $r\neq 0$, we get $q=-3$, $r=-p_1$; $p^2=-6$.

Thus x, y, z are the roots of

$$t^3-pt^2-3t+p=0, \text{ where } p^2=-6.$$

Put $\mu=pt+2$, and we get

$$\mu^3+6\mu+16=0,$$

The roots of this equation are

$$\mu_1=\sqrt{2}\ \{(3-2\sqrt{2})^{\frac{1}{3}}-(3+2\sqrt{2})^{\frac{1}{3}}\}$$

$$\mu_2=\sqrt{2}\ \{(3-2\sqrt{2})^{\frac{1}{3}}\omega-(3+2\sqrt{2})^{\frac{1}{3}}\omega^2\}$$

$$\mu_3=\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

From these we deduce the following types of values of x, y, z.

$$\left.\begin{aligned} 100x&=-153\dot{2}\\ 100y&=-11\dot{8}-46\dot{2}\\ 100z&=\ \ 11\dot{8}-46\dot{2}\end{aligned}\right\}\text{approximately.}$$

QUESTIONS FOR SOLUTION.

943. (HEMRAJ) :—If n be prime and

$$(x+1)(x+2)\dots(x+n-1)$$
$$=x^{n-1}+A_1\,x^{n-2}+A_2\,x^{n-3}+\dots A_{n-1}$$

shew that all the odd A's except A_1 are divisible by n^2—

944. (SADANAND) :—Prove the identity

$$\phi(x)+S_1\phi'(x).\frac{x}{1!}-S_3\phi''(x).\frac{x^3}{3!}+\dots\dots$$

$$\equiv\phi(2x)-S_2\phi''(2x)\frac{x^2}{2!}+S_4\phi''^1(2x)\frac{x^4}{4!}-\dots\dots$$

where S_{2n} is the nth Eulerian number and S_{2n} is the n the'' prepared Bernoullean number.

Deduce that $\log 2 = \frac{S_1}{1}-\frac{S_2}{2^3.1}-\frac{S_3}{3}+\frac{S_4}{2^5.2}+\frac{S_5}{5}\dots\dots$

945. (SADANAND) :—Fill up the vacant cells of the following magic square :—

46				16
	54		39	
		27		
	50		35	
73				43

946. (M. K. KEVALRAMANI) :—Show that there are forty points on the curve $x^2/a^2+y^2/b^2=8x^2y^2$. Such that if tangents be drawn from them to the ellipse $x^2/a^2+y^2/b^2=1$, the points of contact have got their eccentric angles in the ratio 1 : 2.

947. (M. K. KEVALRAMANI):—A triangle ABC inscribed in any lipse, touches a confocal ellipse at the points D, E, F respectively. Show that the ratio of the triangle DEF to ABC$=r/(2R)$, where r, R refer to the triangle ABC as usual.

948. (M. K. KEVALRAMANI):—A perfectly elastic particle, acted on by no forces is projected from the centre of a rectangle whose sides are $2a$ and $2b$ $(a>b)$ to strike the bigger side first and then goes on rebounding from its sides. If it ever pass through an angular point show that the direction of projection makes with the smaller side an angle $\tan^{-1}\left(\frac{a}{b}\frac{2n+1}{2m+1}\right)$ where m & n are integers, and if it ever strike any point of the smaller side other than the extremities, the angle must be of the form $\tan^{-1}\left(\frac{a}{b}\cdot\frac{2n+1}{2m}\right)$

949. (C. KRISHNAMACHARY):—With the usual notation for the numbers of Bernoulli and Euler, show that

$$2n = B_1\, 2^2\,(2^2-1)\binom{2n}{2} - B_2\, 2^6\,(2^4-1)\binom{2n}{4} + B_3\, 2^6\,(2^6-1)\binom{2n}{6}$$
$$- \qquad + (-1)^{n-1} Bn\, 2^{2n}\,(2^{2n}-1).$$
$$(-1)^n E_n\,(2n+1) = (-1)^n B_n\, 2^{2n}\,(2^{2n}-1)\binom{2n+1}{1} +$$
$$(-1)^{n-1} B_{n-1}\, 2^{2n-2}\,(2^{2n-2}-1)\binom{2n+1}{3} + \quad -B_1\, 2^2\,(2^2-1)\binom{2n+1}{2}$$
$$+ \quad (2n+1)$$

950. (C. KRISHNAMACHARY) :—Collect the co-efficients of x^n in the series—

$$\frac{1}{1-x} - \frac{x}{(1-x)^3} + \frac{1\cdot3}{2\,!}\,\frac{x^2}{(1-x)^5} - \frac{1\cdot3\cdot5}{3\,!}\,\frac{x^3}{(1-x)^4} + \quad \ldots \quad \ldots$$

and show that its sum is zero when n is odd, and $(-1)^m$

$$\frac{1\cdot3\cdot5\cdot(2m-1)}{2\cdot4\cdot6\cdot\ \ 2m} \quad \text{when } n \text{ is } 2m.$$

951. (C KRISHNAMACHARY) :—Prove that

$$(1)\ \int_0^{\frac{\pi}{4}} (\log \tan x)^{2n} dx = \frac{E_n}{2^{2n+2}}\, \pi^{2n+1},$$

$$(2)\ \int_0^{\frac{\pi}{4}} \frac{(\log \tan x)^{2n-1}}{\cos x\,(\cos x + \sin x)}\, dx = -\frac{(2^2n^{-1}-1)}{2n} B_n \pi^{2n},$$

$$(3)\ \int_0^{\frac{\pi}{4}} \frac{(\log \tan x)^{2n-1}}{\cos 2x}\, dx = -\frac{(2^{2n}-1)B_n}{4n}\pi^{2n},$$

$$(4)\ \int_0^{\frac{\pi}{4}} \frac{(\log \tan x)^{2n-1}}{\cos x\,(\cos x - \sin x)}\, dx = -\frac{2^{2n-1}B_n}{2n}\pi^{2n}.$$

952. (Communicated by Mr. Hemraj) :—The polar reciprocal of an equiangular spiral with respect to a rectangular hyperbola having its centre at the pole and touching the spiral is the curve itself.

953. (Communicated by Mr. Hemraj) :—Two of the common tangents to a circle S and a conic T meet in P and the other two in Q. Show that P and Q lie on the same confocal to T.

954. (K. B. Madhava) :—When asked his age De Morgan once humorously observed : "I am one of those whose age shall be in a certain year belonging to the century of their birth the square root of that year." For, being born in 1806 A. D., he was 43 years old in 1849, and $\sqrt{(1849)}=43$. Show that the same observation will be true in the case of those born in 9 particular years in all subsequent centuries.

955. (K. B. Madhava) :—The 5th, 13th, 17th, 29th and 37th roots (if rational) of a number end in the same digit as the number itself, the numbers when fractional being expressed in decimals. Prove this property and illustrate how it enables one to give out at sight the above rational roots of big numbers.

956. (K. B. Madhava) :—Find the years of the present century in which the month of February will have (i) no full moon, (ii) no new moon.

957. (Martyn M. Thomas, M. A.) :—From a flexible envelope in the form of a surface of revolution formed by the curve $s=f(y)$ revolving about the axis of x, the part between two meridians the planes of which are inclined to each other at an angle $\frac{2\pi}{m}$ is cut away, and the edges are then sewed together. Prove that the meridian curve of the new envelope will be $s=f\left(\frac{m\,y}{m-1}\right)$.

Hence show that, if a lune of angle $\frac{4\pi}{5}$ be cut off from an oblate spheroid, the minor axis of whose generating ellipse is c, and eccentricity $\frac{3}{5}$, the meridian curve of the new surface of revolution will be the curve of sines $y=\frac{3}{4}c \sin \frac{x}{c}$.

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958. (Martyn M. Thomas, M. A.,) :—Establish the formula

$$\frac{d^n u}{dx^n}=x^{r-n}\frac{d^r}{dx^r}\left[x^q\frac{d^{q-r}}{dx^{q-r}}\left\{x^{p-r}\frac{d^{p-q}}{dx^{p-q}}\left(x^{n-q}\frac{d^{n-p}}{dx^{n-p}}\cdot\frac{u}{x^p}\right)\right\}\right],$$

where $n>p>q>r$.

959. (M. T. Naraniengar) :—The sides AB, AC of a triangle ABC are produced to D, E such that BD = CE = BC.

If the circle ADE cuts the ~~straight line~~ AI_1 in J, prove that

$$I_1J=\tfrac{1}{2}\,II_1.$$