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PROGRESS REPORT.

The following gentlemen have been elected members of our Society :—

1. *Mr. A. M. Koreishi, M.A.*—Professor of Mathematics, M. A. O. College, Aligarh (U. P.)
2. *Mr. G. V. Seshiah, B.A.*—Mathematics Teacher, Athithota, Nellore.

2. The Committee record with great regret the sad and premature death of Mr. P. R. Krishnaswami, M.A., formerly of Pachayappa's College, Madras. Mr. Krishnaswami served as Assistant Secretary of our Society during 1915 and 1916, when his failing health forced him to resign his post. In him the Society has lost an earnest and enthusiastic member.

3. The following books have been received for the Library :—

1. Calcutta University Calendar, for 1917, Parts II & III.
2. Madras University Calendar, for 1917, Part I.

4. The Hon. Treasurer's Statement of Accounts for the year 1917 is given overleaf.

POONA, 31*st January* 1918.

D. D. KAPADIA,
Hon. Joint Secretary.

THE INDIAN MATHEMATICAL SOCIETY.

Statement of Accounts of the Society for the year **1917.**

Receipts.	Rs.	A.	P.	*Expenditure.*	Rs.	A.	P.
To Balance from 1916	1,946	2	1	By ordinary working charges	176	14	8
,, Subscriptions from Members	1,870	0	0	,, Books and Journals	266	8	3
,, Subscriptions for Journal	236	0	0	,, Printing Journal	502	2	0
,, Interest on investments	70	4	8	,, Library	327	6	0
,, Miscellaneous receipts	147	12	6	,, Closing balance*	2,97	4	4
Total ...	4,270	3	3	Total ...	4,270	3	3

	Rs.	A.	P.
* Madras Central Urban Bank	1,200	0	0
The Indian Bank	630	4	8
Treasurer	958	5	8
Assistant Secretary	136	8	6
Secretary	58	6	0
Assistant Librarian	4	0	0
Joint Editor	9	11	6
Total ...	2,997	4	4

I have examined the Treasurers' books and vouchers as also the monthly statements submitted by the Secretary, the Assistant Secretary and the Assistant Librarian and declare the above accounts to be correct.

(Sd.) L. N. SUBRAMANIAN,

3rd February 1918. *Hon. Auditor.*

MADRAS,
30*th January* 1918.

S. NARAYANA AIYAR,
Hon. Treasurer.

Infinite Series and Arithmetical Functions.

By Prof. F. Hallberg.

(*Continued from p.* 186, *Vol. IX.*)

III. Laguerre's series.

12. I now pass on to the interesting series

$$\sum_{n=1}^{\infty} f(n)\frac{x^n}{1-x^n}=\sum_{n=1}^{\infty} x^n\psi(n), \quad |x|<1 \quad \dots \quad \dots \quad (12.1)$$

where $\psi(n)$ is the "numerical integral" of $f(n)$

$$\psi(n)=\sum_{d:\,n} f(d), \quad \dots \quad \dots \quad \dots \quad (12.11)$$

the sum being extended over all divisors d of n. The formula can easily be proved if the left side is developed in a double series.

We know then from the theory of numbers, that

$$f(n)=\sum_{d:\,n} \mu(d)\psi\left(\frac{n}{d}\right), \quad \dots \quad \dots \quad \dots \quad (12\,12)$$

a formula proved by R. Dedekind* and J. Liouville†.
(12.1) may therefore be written in two ways:

$$\sum_{n=1}^{\infty} f(n)\,\frac{x^n}{1-x^n}=\sum_{n=1}^{\infty} x^n \sum_{d:\,n} f(d), \quad \dots \quad \dots \quad (12.13)$$

$$\sum_{n=1}^{\infty} \frac{x^n}{1-x^n}\sum_{d:\,n} \mu(d)\psi\left(\frac{n}{d}\right)=\sum_{n=1}^{\infty} x^n\psi(n). \quad \dots \quad \dots \quad (12.14)$$

The functions $f(n)$, $\psi(n)$ may here be chosen arbitrarily, provided only the series converge.

13. *Some Special Cases.*

For $f(n)=1$ we obtain Lambert's series

$$L(x)=\sum_{n=1}^{\infty} \frac{x^n}{1-x^n}=\sum_{n=1}^{\infty} t(n)x^n, \quad |x|<1 \quad \dots \quad \dots \quad (13.1)$$

a series used by numerous writers in connection with the difficult theory of the asymptotic distribution of the primes.

* J. reine angew. Math. 54 (1857), p. 1.

† J. math. pures appl. (2) 2 (1857), p. 110.

On the other hand, knowing that

$$\sum_{n=1}^{\infty} \frac{x^n}{1-x^n} \sim \frac{1}{1-x} \log\left(\frac{1}{1-x}\right) \text{ [Cesàro]}, \qquad \dots \quad (13.11)$$

we may expect that, as $\sum_{h=1}^{n} t(h)$ is an increasing function of n,

$$\sum_{h=1}^{n} t(h) = n \log n + O(n) \qquad \dots \quad \dots \quad (13.12)$$

In fact, Dirichlet has proved that

$$\sum_{h=1}^{n} t(h) = n \log n + (2C-1)n + O(\sqrt{n}) \qquad \dots \quad (13.13)$$

C being Euler's constant.

For $f(n) = n$, we have Euler's result

$$\sum_{n=1}^{\infty} \frac{nx^n}{1-x^n} = \sum_{n=1}^{\infty} x^n f(n) \qquad , |x| < 1. \qquad \dots (13.2)$$

where $f(n)$ denotes the sum of the divisors of n.

Thus

$$(1-x) \sum_{n=1}^{\infty} \frac{nx^{n-1}}{1+x+x^2+\dots+x^{n-1}} = \frac{xf(1)+x^2f(2)+x^3f(3)+\cdots}{x+2x^2+3x^3+\dots} \quad (13\cdot21)$$

Now Dirichlet has proved, that as $n \to \infty$,

$$\sum_{h=1}^{n} f(h) = \frac{\pi^2 n^2}{12} + O(n \log n), \qquad \dots (13.22)$$

and we see, because of a theorem of Cesàro, that the right side of (13.21) cannot oscillate between wider limits when $x \to 1$ than does

$$\frac{\frac{\pi^2 n^2}{12} + O(n \log n)}{\frac{1}{2} n(n+1)}, \quad \text{when } n \to \infty ,$$

which expression approaches $\frac{\pi^2}{6}$.

Hence we obtain

$$\sum_{n=1}^{\infty} \frac{nx^n}{1-x^n} \sim \frac{\pi^2}{6} \cdot \frac{1}{(1-x)^2} \qquad \dots (13\cdot23)$$

when $\to 1$.

It seems however, that possibly $\pi^2/12$ should be replaced by $\pi^2/6$ in formula (13.22), which is taken from "Encycl des sciences math." Tome I, vol. 3, p. 355. In fact, on p. 359, $n^r\zeta(r+1)$ is given as the "asymptotic mean-value" of the sum of the r^{th} powers of the divisors of a positive integer n, that is

$$\frac{1}{n}\sum_{h=1}^{n}\sum_{d:h} d^r \quad n^r\zeta(r+1),$$

a result attributed to E. Cesàro, L. Gegenbaner and L. Kronecker. Hence the result for $r=1$ should be $n\,\pi^2/6$, while (13.22) gives $n\ \pi^2/12$ ($n\frac{\pi^6}{6}$ is obviously a misprint. Another obvious misprint is "*quadratiques*" instead of "*cubiques*" on the same page (359) of the "Encycl." line 5 *f.b*,). I have neither seen any of the original memoirs in question nor as yet taken the trouble to verify the results, and I am therefore at present not in a position to judge, where the mistake lies.*

It is however easy to verify another formula, also given on p. 359 of the "Encycl.," *viz.*: The asymptotic mean value of the sum of the inverted values of the r^{th} powers of the divisors of n equals $\zeta(r+1)$.

In fact

$$\sum_{n=1}^{\infty}\frac{1}{n^r}\frac{x^n}{1-x^n}=\sum_{n=1}^{\infty}x^n\sum_{d:n}\frac{1}{d^r}, \qquad |x|<1.$$

Thus

$$(1-x)\sum_{n=1}^{\infty}\frac{1}{n^r}\frac{x^{n-1}}{1-x^n}=\frac{x.\frac{1}{1^r}+x^2.\left(\frac{1}{1^r}+\frac{1}{2^r}\right)+x^3\left(\frac{1}{1^r}+\frac{1}{3^r}\right)+\dots}{x+x^2+x^3+\dots},$$

and hence

$$\frac{1}{n}\sum_{h=1}^{n}\sum_{d:h}\frac{1}{d^r}\sim\sum_{n=1}^{\infty}\frac{1}{n^{r+1}}=\zeta(r+1), \qquad \dots (13{\cdot}24)$$

as $n\to\infty$ (since the left side is an increasing function of n, and cannot tend to infinity, thus having a definite limit).

If we put $\psi(n)=\phi(n)$ we get

$$\sum_{n=1}^{\infty}\frac{x^n}{1-x^n}\sum_{d:n}\mu(d)\phi\left(\frac{n}{d}\right)=\sum_{n=1}^{\infty}x^n\phi(n),\ |x|<1 \quad \dots (13.3)$$

(*) Since the above was written, I have proved that Dirichlet's result is the correct one. The proof will be given in the following.

Now Dirichlet has proved, that

$$\sum_{h=1}^{n} \phi(h) = \frac{3n^2}{\pi^2} + O(n^\delta) \quad \ldots \quad \ldots \quad \ldots \quad (13.31)$$

where δ lies between 1 and 2, while according to F. Mertens

$$\sum_{h=1}^{n} \phi(h) = \frac{3n^2}{\pi^2} + O(n \log n). \quad \ldots \quad \ldots \quad (13.32)$$

Hence

$$(1-x) \sum_{n=1}^{\infty} \frac{x^{n-1}}{1+x+x^2+\ldots+x^{n-1}} \sum_{d:n} \mu(d)\phi\left(\frac{n}{d}\right)$$

$$= \frac{\phi(1)x+\phi(2)x^2+\phi(3)x^3+\ldots}{x+2x^2+3x^3+\ldots} \quad \ldots \quad (13.33)$$

whence

$$\sum_{n=1}^{\infty} \frac{x^n}{1-x^n} \sum_{d:n} \mu(d)\phi\left(\frac{n}{d}\right) \sim \frac{6}{\pi^2} \cdot \frac{1}{(1-x)^2} \quad \ldots \quad (13.34)$$

when $x \to 1$.

We know further that $p(n)$ is the numerical integral of $\mu^2(n)$. Thus

$$\sum_{n=1}^{\infty} \frac{x^n}{1-x^n} \mu^2(n) = \sum_{n=1}^{\infty} p(n)x^n \quad , |x|<1. \quad \ldots \quad (13.4)$$

Since now Mertens has proved that

$$\sum_{h=1}^{n} p(h) = \frac{6}{\pi^2} n \log n + O(n) \quad \ldots \quad \ldots (13.41)$$

or more accurately

$$\sum_{h=1}^{n} p(h) = \frac{6n}{\pi^2}\left(\log n + 2C - 1 + \frac{12}{\pi^2} \sum_{h=2}^{\infty} \frac{\log h}{h^2}\right)$$

$$+ O(\sqrt{n} \log n) \quad \ldots (13.411)$$

we have in the usual manner, and by the aid of (13.11) & 13·12),

$$\sum_{n=1}^{\infty} \frac{x^n}{1-x^n} \mu^2(n) \sim \frac{6}{\pi^2} \frac{1}{1-x} \log\left(\frac{1}{1-x}\right), \quad \ldots (13.42)$$

when $x \to 1$.

Combining (13.11) and (13·42), we have also

$$\sum_{n=1}^{\infty} \frac{x^n}{1-x^n}\left(\frac{\pi^2}{6}\mu^2(n)-1\right) = \sum_{n=1}^{\infty} x^n\left(\frac{\pi^2}{6}\cdot p(n)-t(n)\right), \quad \dots \ (13.5)$$

from which we conclude, comparing (13.13) and (13.411), that

$$\sum_{n=1}^{\infty} \frac{x^n}{1-x^n}\left(\frac{\pi^2}{6}\mu^2(n)-1\right) \backsim \frac{12}{\pi^2}\sum_{h=2}^{\infty}\frac{\log h}{h^2}\cdot\frac{1}{1-x} \quad \dots \ (13.51)$$

when $x \to 1$, or

$$\sum_{n=1}^{\infty} \frac{1}{n}\left(\frac{\pi^2}{6}\mu^2(n)-1\right) = \frac{12}{\pi^2}\sum_{h=1}^{\infty}\frac{\log h}{h^2}, \quad \dots \ (13.52)$$

provided the series to the left converges.

Putting $\left.\begin{array}{ll} f(n)=(-1)^{\frac{1}{2}(n-1)} & , n \text{ odd} \\ f(n)=0 & , n \text{ even} \end{array}\right\},$

we obtain

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^{2n-1}}{1-x^{2n-1}} = \sum_{n=1}^{\infty} x^n \sum_{d':n} (-1)^{\frac{1}{2}(d'-1)}, \quad \dots \ (13.6)$$

d' taking the values of all *odd* factors of n.

Since we have

$$\left(\frac{x}{1-x}-\frac{x^3}{1-x^3}+\frac{x^5}{1-x^5}-\dots\right) \sim \frac{\pi}{4}\cdot\frac{1}{1-x}, \quad \dots \quad \dots \ (13·61)$$

[Cesáro]

[Bromwich : p. 172, (21).]

as $x \to 1$, we obtain

$$\operatorname*{Lt}_{n\to\infty} \frac{1}{n}\sum_{h=1}^{n} \mathrm{E}(h) = \frac{\pi}{4}, \quad \dots \quad \dots \ (13·62)$$

where

$$\mathrm{E}(n) = \sum_{d':n} (-1)^{\frac{1}{2}(d'-1)}, \quad \dots \quad \dots \ (13·63)$$

(provided this limit exists).

Now, however, A. Berger * has proved, that

$$\operatorname*{Lt}_{m\to\infty} \frac{\mathrm{E}(1)+\mathrm{E}(3)+\dots+\mathrm{E}(2m-1)}{2m-1} = \frac{\pi}{4}, \quad (13.631)$$

* Acta math. 9 (1886/7), p. 301.

from which we conclude the corresponding

$$\underset{m\to\infty}{\mathrm{Lt}} \frac{E(2)+E(4)+\dots+E(2m)}{2m}=0. \qquad \dots \qquad \dots (13.64)$$

As a consequence of a formula (by J. Liouville†) we have also

$$\sum_{n=1}^{\infty} f(n)\frac{x^n}{1-x^n}=\sum_{n=1}^{\infty} x^n \sum_{d:n}\left(\frac{n}{d}\right)t(d)\ ,\ |x|<1 \qquad \dots (13{\cdot}7)$$

From the relation

$$\sum_{d:n} p(d)=t(n^2) \quad \dots \qquad \dots \qquad \dots (13{\cdot}8)$$

it follows further for $f(n)=p(n)$, that

$$\sum_{n=1}^{\infty}\frac{p(n)x^n}{1-x^n}=\sum_{n=1}^{\infty} t(n^2)x^n,\ |x|<1 \qquad \dots (13{\cdot}81)$$

14. For the values $f(n)=\mu(n)$, $f(n)=\phi(n)$, the right side in (12·1) becomes a rational function. We have in fact in the first case

$$\sum_{n=1}^{\infty}\frac{\mu(n)x^n}{1-x^n}=x,\ |x|<1, \qquad \dots \qquad \dots (14{\cdot}1)$$

because of the well-known formula

$$\sum_{d:n}\mu(d)=0,\ n>1. \qquad (14{\cdot}11{\cdot})$$

It is worth noticing that the same series is obtained by μ—inversion of the geometrical progression

$$\frac{x}{1-x}=\sum_{n=1}^{\infty} x^n.$$

If we sum the inverted series before rearranging the terms we obtain the formula (4·21). This is again a consequence of (14·11).

If we take the limit as $x\to 1$ after multiplication by $1-x$, the result is (11·13).

In the second case we have

$$\sum_{n=1}^{\infty}\frac{x^n}{1-x^n}\sum_{d:n}\mu(d)\left(\frac{n}{d}\right)=\sum_{n=1}^{\infty} nx^n,\ |x|<1. \qquad \dots (14{\cdot}2)$$

† J. math. pures appl.

But as $\sum_{d \mid n} \phi(d) = n$, we have by (12·12)

$$\phi(n) = \sum_{d:n} \mu(d) \left(\frac{n}{d}\right), \quad \dots \quad \dots \quad \dots \quad (14·21)$$

and hence

$$\sum_{n=1}^{\infty} \frac{\phi(n) x^n}{(1-x)^n} = \frac{x}{(1-x)^2}. \quad \dots \quad \dots \quad (14·22)$$

Hence we have also

$$\sum_{n=1}^{\infty} \frac{\phi(n) x^n}{1+x+\dots+x^{n-1}} \backsim \frac{1}{1-x} \quad \dots \quad \dots \quad (14·23)$$

when $x \to 1$, which may be regarded as a complement to the equation

$$\frac{\zeta(s-1)}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\phi(n)}{n^s},$$

in the theory of Dirichlet's series.

15. Let $f(n) = \mu(n) \log n.$

It appears, that the numerical integral in this case is

$$\sum_{d:n} \mu(d) \log d = - \mathrm{V}(n), \quad \dots \quad \dots \quad (15·1)$$

(a function which has been examined by N. V. Bugajev), where $\mathrm{V}(n) = \log p$, if n is a power of a prime p, $\mathrm{V}(n) = 0$, when n contains two or more distinct prime factors, and $\mathrm{V}(1) = 0$.

Consequently

$$\sum_{n=1}^{\infty} \mu(n) \log n \frac{x^n}{1-x^n} = - \sum_{n=1}^{\infty} \mathrm{V}(n) x^n, \ | x | < 1, \quad \dots \quad (15·2)$$

from which follows

$$\sum_{n=1}^{\infty} \mu(n) \log n \frac{x^{n-1}}{1+x+\dots+x^{n-1}} = - \frac{\mathrm{V}(1)x + \mathrm{V}(2)x + \mathrm{V}(3)x^3 + \dots}{x + x^2 + x^3 + \dots} \quad (15·21)$$

By Cesarðs theorem we see that the fraction on the right side cannot oscillate between wider limits, when $x \to 1$, than does $\frac{1}{n}\sum_{h=1}^{n} \nu(h)$ when $n \to \infty$. And as de la Vallee' Poussin*) and J Hadamard**) have proved, that the last fraction tends to 1, we have here a new verification of the theorem

$$\sum_{n=1}^{\infty} \frac{\mu(n) \log n}{n} = -1 \quad \text{***)} \qquad \dots \qquad \dots (15{\cdot}3)$$

16. *The arithmetical function, whose numerical integral is* $\mu(n)$.
We have because of (12.14) for $\psi(n) = \mu(n)$

$$\sum_{n=1}^{\infty} \frac{x^n}{1-x^n} \sum_{d:n} \mu(d).\mu\left(\frac{n}{d}\right) = \sum_{n=1}^{\infty} x^n \mu(n), \; |x| < 1. \qquad \dots (16.1)$$

Put for brevity.

$$\sum_{d:n} \mu(d)\mu\left(\frac{n}{d}\right) = \mu'(n). \qquad \dots \qquad \dots (16.11)$$

By considering all different cases which may arise, we see that $\mu'(n)$ has the following properties:

$$\mu'(1) = 1.$$
$$\mu'(n) = 0,$$

if n is divisible by a cube other than unity.

$$\mu'(n^2) = 1, \text{ if } \mu(n) \neq 0,$$

that is if n is not divisible by a square other than unity.

$$\mu'(n) = (-2)^k, \text{ if } \mu(n) = (-1)^k,$$

that is if n is not divisible by a square other than unity, and k is the number of prime-factors in n.

$$\mu'(n_1 n_2^2) = (-2)^{k_1}$$

if $n_1 > 1, n_2 > 1$ are prime to one another, neither being divisible by a square other than unity, and k_1 is the number of prime-factors in n_1 ... ... (16·12)

Let now

$$\sigma(n) = \sum_{r=1}^{n} \mu(r) \dots \qquad \dots \qquad \dots (16{\cdot}2)$$

* Ann. Soc. Scient. Bruxelles 202 (1895.6), p. 251.

**) Bull. Soc. math. France 24 (1896), p. 199.

***) C. R. Acad. Sc. Paris 129 (1899), p. 812.

It has been conjectured, that always

$$|\sigma(n)| < |\sqrt{n}|, \quad \ldots \quad \ldots \quad (16.21)$$

the proof of which would have far-reaching consequences, furnishing in particular a proof of the famous proposition of Riemann that the imaginary roots of the equation

$$\zeta(s)=0$$

all have $R(s)=\frac{1}{2}$, while if the latter conjecture were proved correct, it would follow, that for n large enough

$$|\sigma(n)| < n^{\Theta},$$

where Θ is a certain number less than $\frac{6}{7}$*). It would also follow, that the series in formula (11·12) is convergent for $R(s) > \Theta$. *)

At present we know only, that the order of $\sigma(n)$ is not greater than that of $n\, e^{-a\sqrt{\log n}}$ [Landau *], which however is sufficient to find the value, if it exists, of the series

$$\sum_{n=1}^{\infty} \frac{\mu'(n)}{n}.$$

In fact we have by (16·1)

$$\sum_{n=1}^{\infty} \frac{x^{n-1}}{1+x+\ldots+x^{n-1}}\, \mu'(n) = \frac{\mu(1)x+\mu(2)x^2+\mu(3)x^3+\ldots}{x+x^2+x^3+\ldots}. \quad (16·3)$$

As $x \to 1$, the right side of this equation can only oscillate between limits, which are not greater in absolute value than those of

$$\frac{n e^{-a\sqrt{\log n}}}{n}$$

as $n \to \infty$, and as this last expression tends to zero, we may conjecture

$$\sum_{n=1}^{\infty} \frac{\mu'(n)}{n} = 0. \quad \ldots \quad \ldots \quad (16·4)$$

From the general formula in the theory of Dirichlet's series

$$\sum_{n=1}^{\infty} \frac{h(n)}{n^s} = \sum_{n=1}^{\infty} \frac{f(n)}{n^s} \cdot \sum_{n=1}^{\infty} \frac{g(n)}{n^s}, \quad \ldots \quad (16·5)$$

(*) E. Landau, Rend. Circ. Mat. Palermo 26 (1908), p. 264.

where the series are convergent, $f(n)$, $g(n)$ any given arithmetical functions, and

$$h(n) = \sum_{d:n} f(d)\, g\left(\frac{n}{d}\right), \quad \ldots \quad \ldots \quad (16{\cdot}51)$$

it follows for $f \equiv g \equiv \mu$

$$\sum_{n=1}^{\infty} \frac{\mu'(n)}{n^s} = \left\{ \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} \right\}^2 = \frac{1}{\zeta^2(s)} \quad \ldots \quad \ldots \quad (16{\cdot}6)$$

Thus

$$\sum_{n=1}^{\infty} \frac{\mu'(n)}{n^s} . \sum_{n=1}^{\infty} \frac{t(n)}{n^s} = 1, \quad \ldots \quad \ldots \quad (16{\cdot}61)$$

[Bromwich, p. 494 Ex. 45 (2)].

and consequently, if we put

$$\rho(n) = \sum_{n:d} \mu'(d)\, t\left(\frac{n}{d}\right) = \sum_{d:n} \mu'\left(\frac{n}{d}\right) t(d), \quad \ldots \quad (16{\cdot}62)$$

we have by (16·51) $$\sum_{n=1}^{\infty} \frac{\rho(n)}{n^s} = 1, \quad \ldots \quad \ldots \quad \ldots \quad (16{\cdot}7)$$

from which we conclude $\rho(n) \equiv 0$, for $n > 1$ ... ... (16·1)

The formula (16·1) could also have been obtained through μ—inversion of (14·1).

(To be continued.)

Double Lines.

BY M. BHIMASENA RAO.

[This note is in continuation of the Author's Note on "Double Points" on page 19, Vol. IV of the Journal.]

Let ABC and A′B′C′ be any two triangles and P and P′ the mean centres of A, B, C and A′, B′, C′ respectively for any common system of multiples. The line joining any two points P and Q corresponds to the join of P′ and Q′. The self corresponding lines may be called the double lines of the triangles. The line at infinity is evidently a double line.

2. *To find the double lines.*

Let D be the double point of the triangles. Divide AB and A′ B′ in the same ratio at X and X′. DX and DX′ are corresponding lines. X and X′ divide AB and A′ B′ homographically. Hence D X and D X′ are conjugate rays of a homographic pencil whose double lines are the double lines of the triangles.

3. The double lines are evidently the tangents from D to the parabola inscribed in the quadrilateral ABB′A′. Hence the three parabolas inscribed in the quadrilaterals ABB′A′, BCC′B′ and CAA′C′ have two tangents in common which are the double lines of ABC and A′B′C′, and the line at infinity which is also a common tangent of the parabolas is also a double line. The intersection of two double lines is a double point. It follows therefore that any two triangles have *in general** three double lines, and three double points two of which are at infinity. Denote these two points by D_1 and D_2.

4. When ABC and A′ B′ C′ are directly similar triangles, D_1 and D_2 are the circular points. The following properties can be easily inferred by orthogonal projection.

(i) The conics ABC D_1 D_2 and A′ B′ C′ D_1 D_2 are corresponding curves. If X be any point on the former, lines through A′, B′, C′ parallel to AX, BX, CX intersect at a point X′ on the latter. X and X′ are not corresponding points of the triangles.

(ii) If P, P′ and Q, Q′ be pairs of corresponding points, and P P′ and Q Q′ intersect in O, the conics PQD D_1 D_2 and P′ Q′ D D_1 D_2 are corresponding curves passing through O and are the loci of corresponding points whose join passes through O.

5. *To find the condition that the double lines are at right angles.*

In this case D is on the directrices of the three parabolas AB B′A′, BC C′B′, CA A′B′, *i.e.*, on the radical axes of the pairs of circles described on AB′, A′B; BC′, B′C; CA′, C′A; as diameters. If X and X′ be the mid-points of AB′ and A′B, we have

$$DX^2-DX'^2=\frac{1}{4}\,(AB'^2-A'B^2),$$

$$DA^2+DB'^2=2\,DX^2+\frac{AB'^2}{2},$$

$$DA'^2+DB^2=2\,DX'^2+\frac{A'B^2}{2}.$$

* In particular, when A A′, B B′ and C C′ are parallel, any line parallel to these is a double line and any point on the axis of perspective of ABC and A′ B′ C′ is a double point.

$\therefore \quad (DA^2-DB^2)-(DA'^2-DB'^2)=2\ (DX^2-DX'^2)+\frac{1}{2}(AB'^2-A'B^2)$
$\qquad\qquad = AB'^2-A'B^2.$

Adding two similar results, we have

$$AB'^2+BC'^2+CA'^2=A'B^2+B'C^2+C'A^2,$$

showing that the triangles ABC and A′B′C′ are orthologic.

Given that ABC and A′B′C′ are orthologic triangles, it can be shown by a rearrangement of the steps that the radical axes of the three pairs of circles described on AB′ and A′B ; etc. as diameters co-intersect, but it does not follow that the point of intersection is the double point of the triangles.

That the triangles should be orthologic when the double lines are at right angles follows immediately by the application of the first property of § 4. $ABCD_1D_2$ is now a rectangular hyperbola which passes through H the orthocentre of ABC. Therefore lines through A′, B′, C′ parallel to AH, BH, CH, *i.e.*, perpendicular to BC, CA, AB are concurrent on the rectangular hyperbola $A'B'C'D_1D_2$ and for a similar reason perpendiculars from A, B, C, on B′C′, C′A′, A′B′, are concurrent on $ABCD_1D_2$.

6. Given that ABC and A′B′C′ are orthologic triangles, to show the double lines are right angles.

Let the perpendiculars from A, B, C on the corresponding sides of A′B′C′ intersect at S and the perpendiculars from A′B′C′ on the sides of ABC, at S′. Then S and S′ are corresponding point of the triangles.* Let L and L′ be points at infinity on BC and B′C′ and AS, A′S′ intersect the line at infinity in M and M′. Since corresponding lines cut the line at infinity homographically and D_1 and D_2 are the double points of this range we have

$$(D_1D_2,\ LL')=(D_1D_2,\ MM').$$

$$\therefore \quad (D_1D_2,\ LL')=(D_2D_1,\ M'M).$$

$$\therefore \quad (D_1D_2,\ LM',\ L'M) \text{ is in involution.}$$

* This is pointed out by Gallatly in his *Modern Geometry*, *vide* page 56. The following is a geometrical proof :—

Let T be the isogonal conjugate of S and XYZ the pedal triangle of T with respect to ABC. S and T are corresponding points of ABC and XYZ. (Apply Ex. 7 (*a*) and 11 pp. 88 and 89 of M'clelland's Geometry of the circle). The sides of XYZ being at right angles to SA, SB, SC, the triangles A′B′C′ and XYZ ard homothetic, TX, TY, TZ being parallel to S′A′, S′B′, S′C′, T and S′ are correspone-ing points of XYZ and A′B′C′. Therefore S and S′ are corresponding points of ABC and A′B′C′.

But L and M′ are points at infinity in direction at right angles as also L′ and M. Therefore D_1 and D_2 are points at infinity in direction at right angles, *i.e.*, the double lines are at right angles.

Since ABC SD_1D_2 is a rectangular hyperbola, the double lines are parallel to the pedal lines of the extremeties of the circum-diameter of ABC through the isogonal conjugate of S.

7. *Double point of perspective triangles.*

Consider a triangle ABC and its conjugate triangle A′B′C′ with respect to a circle whose centre is D. It is well known that the triangles are in perspective and also orthologic, the points S and S′ determined as in § 6 being coincident at the centre D which shows that the centre is the double point, and the rectangular hyperbolas ABCD and A′B′C′D intersect at the centre of perspective and also on the line at infinity. By orthogonal projection we have.

(1) The double point of two triangles in perspective is the centre of the conic with respect to which they are conjugate (Q. 367).

(2) The double lines are conjugate diameters of the above conic (Q. 374, (2)), and if O is the centre of perspective.

(3) The conics ABCDO and A′B′C′DO intersect the line at infinity in the remaining doule points (Q. 374, (1)) and are the loci of corresponding points whose join passes through O, by the second property of § 4, (Q. 374, (4)). If, in addition, ABC and A′B′C′ be orthologic, S and S′ being the points of concurrence of the perpendiculars from the verticies of either on the corresponding sides of the other, the conics ABCDO and A′B′C′DO are rectangular hyperbolas passing respectively through S and S′. Since S and S′ are corresponding points, they are collinear with the centre of perspective (Q. 643). The first result of Q. 340 is an application of this property.

8. Besides the application and illustrations of the theory of double points and lines given in the above questions set in the Journal, the following are added.

(1) The double lines of the pedal triangles of any two points P and P′ with respect to ABC are the pedal lines of the points in which PP′ intersects the circumcircle of ABC. When PP′ passes through the circumcentre, the pedal lines are at right angles, and therefore the pedal triangles of P and P′ are orthologic; and conversely if the pedal triangles of P and P′ are orthologic, PP′ passes through the circumcentre.

(2) If XYZ and X′Y′Z′ are the pedal triangles of P and P′ on a circum-diameter of a triangle the radical axes of the three pairs of circles described on XY′ and X′Y, etc., as diameters are concurrent on the nine point circle at the orthopole of PP′ (Q. 239 is a particular case of this property).

(3) Let P, P′ be corresponding points and L, L′ corresponding lines of two orthologic triangles F, F′. Then the perpendicular from P on L′ considered as a line of F corresponds to the perpendicular from P′ on L. Inversely similar triangles are orthologic. For inversely similar triangles we have an additional property, viz., the parallel through P to L′ corresponds to the parallel through P′ to L, given by Lachlan: *Modern Geometry*, Ex. 1, § 222.

(4) Let A′B′C′ be the pedal triangle of Pwith respect to ABC. To the circumcircle of ABC corresponds a circum-ellipse of A′B′C′. This ellipse is the locus of orthopoles of lines through P with respect to ABC. The double lines being at right angles give the directions of the axes of the ellipse.

(5) If two conics having the same axes are such that triangles inscribed in one are circumscribed to the other, the normals at the points of contact are concurrent. [A well known property.]

For if ABC be an inscribed triangle and A′, B′, C′ the points of contact, ABC and A′B′C′ are conjugate triangles with respect to the second conic, the double point is the common centre of the conics which therefore are corresponding conics, the common axes being conjugate diameters of the conics are the double lines, and these being at right angles, the triangles ABC and A′B′C′ are orthologic, etc. If a concentric conic be inscribed in A′B′C′, the normals at the points of contact are also concurrent.

(6) To project orthogonally orthologic triangles into orthologic triangles, we have only to take the base line parallel to a double line. If inversely similar triangles be projected into orthologic triangles, these latter will also be inversely similar, for the double points and any two corresponding points on the line at infinity, of inversely similar triangles form a harmonic range—a property unaltered by projection.

(To be concluded).

SHORT NOTES.

Note on Question 735.

1. The following *general result* may be of interest :

Let (r_n, θ_n) be the point of the nth pedal corresponding to a point (r, θ) of a curve. Then

$$\theta=\theta_n+n(\tfrac{1}{2}\pi-\phi)\ ;\ r_n=r\sin^n\phi.$$

Now, if C_n is the entire circumference of the nth pedal, we have

$$C_n=\int r_{n+1}d\theta_{n+1}=\int r\sin^{n+1}\phi(d\theta+\overline{n+1}d\phi).$$

[J.I.M.S.. **1916**, p. **137**.)

Also
$$C_{n+2}=\int d\,s_{n+2}=\int r_{n+2}\operatorname{cosec}\phi d\theta_{n+2}$$
$$=\int r\sin^{n+1}\phi(d\theta+\overline{n+2}d\phi)$$

$\therefore$
$$C_{n+2}-C=\int r\sin^{n+1}\phi\ d\phi,$$

and
$$(n+2\ C_n-(n+1)C_{n+2}=\int r\sin^{n}{}_{+}{}^{1}\phi\ d\theta\ ;$$

which may also be written

$$\left.\begin{aligned}(n+1)C_n-nC_{n-2}&=\int r\sin^{n+1}\phi d\psi,\\ (n+2)C_n-(n+1)C_{n+2}&=\int\rho\sin^{n+2}\phi d\psi\end{aligned}\right\}$$

---

2. For the particular case of Q. 735, we write

$$d\psi=\frac{abd\alpha}{r'^2},$$

where α is the eccentric angle of the point (r, θ) and r' is the radius conjugate to r. Thus

$$(n+1)C_n-nC_{n+2}=\int\frac{r.\,p^{n+1}}{r^{n+1}}\cdot\frac{abd\alpha}{r'^2}.$$
$$=\int\frac{(ab)^{n+2}d\alpha}{r^n r'^{n+3}};$$

and
$$(n+2)\ C_n-(n+1)C_{n+2}=\int\frac{r'^2}{ab}\cdot\frac{p^{n+2}}{r^{n+2}}\cdot\frac{abd\alpha}{r'^2}.$$
$$=\int\frac{(ab)^{n+2}d\alpha}{r^{n+2}.\,r'^{n+1}}.$$

Putting $n=0, -1, -2$ in the above, we have

$$C_0 = \int \frac{a^2b^2 d\alpha}{r'^3}$$

$$C_1 = \int \frac{dbr d\alpha}{r'^2}$$

$$2C_0 - C_{-2} = \int \frac{r^3}{r'} d\alpha$$

$$2C_0 - C_2 = \int \frac{a^2b^2}{r^2 r^1} d\alpha$$

$$C_{-1} = \int \frac{ab d\alpha}{r}$$

$$C_0 = \int r' d\alpha$$

whence $$C_1 + C_{-1} = ab(a^2 + b_2 \int \frac{d\alpha}{rr'^2}$$

$$3C_0 - C_{-2} = (a^2 + b^2) \int \frac{d\alpha}{r'}$$

$\therefore$ $$C_{-1}(C_1 + C_{-1}) = (3C_0 - C_{-2})(2C_0 - C_2)$$

since $$\int \frac{d\alpha}{r} \int \frac{d\alpha}{rr'^2} = \int \frac{d\alpha^1}{r'} \int \frac{d\alpha^1}{r^2 r'} = \int \frac{d\alpha}{r'} \int \frac{d\alpha}{r^2 r'}.$$

M. T. Naraniengar.

A Note on Question 837.

1. In the Annual Progress Report for the year 1915—16 of the Superintendent, Hindu and Buddhist monuments, Northern Circle, mention is made of a magic square found on the underside of a fallen lintel at the shrine, locally known as *chota Surang*, at Dudhai, Jhansi Disrict. This temple probably belongs to the first half of the eleventh century A. D. The Square is as follows :—

7	12	1	14
2	13	8	11
16	3	10	5
9	6	15	4

(I)

The Superintendent remarks :—

" Mathematically it is interesting as possessing the following properties:— (i) the sum of each row, each column and each diagonal is 34, (ii) the sum of all the numbers in each sub-square also 34."

Since (i) is true not only of the leading diagonals, but also of broken diagonals, the square is of the type known as pandiagonal (see Ball's Mathematical Recreations,* p. 156). (ii.) however is not an additional property as is evidently supposed by the writer, but is a consequence of (i). In fact if we divide any even Magic square into four groups of cells which we may represent by A,B,C,D as below

A	B
C	D

and if k be the constant sum of rows and columns, we clearly have

$$A+B=\frac{n.k}{2}=A+C \therefore B=C \text{ and } A=D.$$

but in general $A \neq C$ or B

If however the square is of the pandiagonal type, it is easy to show that we have

$$A=B=C=D=\frac{nk}{4}$$

Now in a pandiagonal square we may clearly alter the order of either rows or columns cyclically without losing the properties of the square. Hence it follows that $A=nk/4$ where A is any subsquare of $nk/4$ cells (n^2 = total no. of cells), in this case *e.g.* the four centre cells. It has seemed worth mentioning this property of pandiagonal squares, since it is not mentioned by Ball and is clearly unknown to the writer of the Report, as also to the writer of the Scientific notes in Chambers' Journal where I first came across the square.

2. Squares of the type of (I) are directly connected with those of the type required as a solution to Mr. Malhari Rao's question No. 837, and as a matter of fact the above square leads immediately to a solution of Q. 837 quite distinct from the set of solutions given in J. I. M. S. vol. ix. p. 161.

Making one alteration in rows, one in columns and then writing rows as columns, (I) becomes.

1	15	10	8
12	6	3	13
7	9	16	2
14	4	5	11

(II)

* The references are all to the fifth Edition.

Subtracting unity from each element of (II) and then replacing each element n(say) so obtained by 2^n we obtain the following square which is clearly a solution of Q. 837.

1	2^{14}	2^9	2^7
2^{11}	2^5	2^2	2^{12}
2^6	2^8	2^{15}	2
2^{13}	2^3	2^4	2^{10}

(III)

This solution differs from any of the doubly infinite set obtained in J. I. M. S. vol. ix p. 161 in that in (III) all the 16 elements are different, whereas in the previous solution there were only 8 distinct elements each repeated. Denoting the elements by $a_1, a_2, .. b_1, b_2, ...$ and $a_1.a_2.a_3.a_4 = k$ we have 16 relations connecting 17 quantities but the equations are in reality equivalent to only 14 independent relations ; if therefore, we assign particular values to any two of the quantities we can still assign any arbitrary value to the quantity k and obtain a solution. It does not follow however that the solution so obtained will be in positive integers. For example suppose k is prime, then the only possible solution will have one element in each row $=k$ and all the others unity ; it must be one of the solutions of the problem of placing 4 queens on a chessboard of 16 cells so that no queen can capture any other—there are two solutions of this problem (see Ball, loc. cit. p. 114), but although they make the product of rows and columns $=k$, they break down for the diagonals, hence there is no solution in positive integers if k is prime.

The solution previously given in J. I. M. S. is not the most general solution, the values obtained $\lambda_1 + \lambda_2 = 2$ are only one set of possible values.

3. The question arises whether the solution (III) is the only solution in which the cells are occupied by distinct integers. It is not so as may be seen as follows :—

The square (II) is the sum of the two squares

0	$3m$	$2m$	m
$2m$	m	0	$3m$
m	$2m$	$3m$	0
$3m$	0	m	$2m$

and

1	3	2	4
4	2	3	1
3	1	4	2
2	4	1	3

provided $m=4$; but each of these squares is a pandiagonal magic square for all values of m, and hence so is their sum. But in the sum the cells a_1 and c_4 are occupied by the nos. 1 and 2 for all values of m, hence adding the two squares, subtracting unity from each element and replacing each element so obtained by the corresponding power of 2 we obtain the square

1	2^{3m+2}	2^{2m+1}	2^{m+3}
2^{2m+3}	2^{m+1}	2^2	2^{3m}
2^{m+2}	2^{2m}	2^{3m+3}	2
2^{3m+1}	2^3	2^m	2^{2m+2}

which satisfies the conditions of the problem whatever the value of m and provided $m > 3$ the cells are occupied by 16 distinct integers; in this way we get a singly infinite system of such solutions.

4. Any factor of k must occur as a factor at least once in every line, column and diagonal, but it has been shown above that if it occurs once only no solution is possible; it must therefore occur at least twice and hence for a solution in positive integers k must be a perfect square. But since $c_4=2$, 2 is a factor of k and there 4 is a factor of k, hence it is a necessary condition that k is of the form $4p^2$ (This appears to be proved by the solution previously published where it is found that $k=4b_4^2.d_4^2$, but since this solution was not perfectly general it seemed desirable to give a general proof.)

R. J. Pocock.

Astronomical Notes.

The nearest Star.

For three quarters of a century *alpha centauri* has been the nearest known star. Mr. Innes at Johannesburg has now discovered a star of the eleventh magnitude, the distance of which is almost exactly the same as that of *alpha centauri*. As its proper motion is of approximately the same amount and in the same direction, it seems possible that the two stars are physically connected although the distance separating them is very great, *viz.* 20° 12′.

The particulars of the two stars are given below :—

	Alpha centauri.	New Star.
Parallax ...	0·″756	0·″755
Proper Motion (in arc)	3·″68	3.″76
Direction ...	2814°	2827°

This star is intrinsically the faintest star known.

About a year ago Prof. Barnard discovered a star with a larger proper motion than any previously known. These discoveries show that we still have much to learn about even the nearer stars.

The Planets.

Most of the planets are favourably situated for observation at this time of year.

Below are given the positions on March 15th and the constellations in which the planets will be found.

Planet.	R. A.	Decln.	Constellation.
Mercury ...	23^h 49^m ...	2° 35′ S ...	Pisces.
Venus ...	21 8 ...	11 9 S ...	Aquarius.
Mars ...	11 42 ...	6 1 N ...	Virgo.
Jupiter ...	4 13 ...	20 40 N ...	Taurus.
Saturn ...	8 43 ...	19 7 N ...	Cancer.
Uranus ...	21 52 ...	13 43 S ...	Capricornus.
Neptune ...	8 27 ...	18 59 N ...	Cancer.

The sun's position on the same date is R. A. 23^h 38^m Decl. 2° 20′ S. in the constellation Aquarius. It will be seen that Venus and Uranus are morning stars while all the others are evening stars, though at this date Mercury will be too close to the sun to be visible and Mars will just rise at sunset.

R. J. Pocock.

SOLUTIONS

Question 757

(S. MALHARI RAO) :—Give a solution in positive integers of the equations.

$$(1)\quad x^4-6y^2=1\ ,\quad (2)\ x^{12}-17y^2=x^6,$$

Remarks by S. V. Venkatachala Iyer.

$$(1)\quad x^4-6y^2=1,\quad \text{Let } x^2=z,$$

Then $z^2-6y^2=1$, To solve this in positive integers we express $\sqrt{6}$ as a continued fraction ; we have.

$$\sqrt{6}=2+\frac{1}{2}+\frac{1}{4}+\frac{1}{2}+\frac{1}{4}+\ldots\ldots$$

The successive convergents are 2, $\frac{5}{2}$, $\frac{22}{9}$,...—

$\therefore$ $z=5, y=2$ is obviously a solution of x^2-6y^2, but as this does not give the value of z a perfect square, as it should be, we have to select such a value for z, as will be a perfect square, from the general solution,

$$z=\tfrac{1}{2}\{(5+2\sqrt{6})^n+(5-2\sqrt{6})^n\}.$$

Putting $n=2$, we get $z=49$, a perfect square.

$\therefore$ The value of $x=7$, and the corresponding value of y is 20.

(2) $x^{12}-17y^2=x^6$. Dividing out by x^6, we have

$$x^6-17\frac{y^2}{x^6}=1\quad \text{Put } x^3=\xi,\ y=\xi\eta$$

Then $\xi^2-17\eta^2=1$. To get values for ξ, η, in positive integers, we see $\sqrt{17}=4+\frac{1}{8}+\frac{1}{8}+\cdots$, and the successive convergents are $4, \frac{33}{8}, \frac{26}{65}, \ldots$ Of these $\xi=33, \eta=8$, satisfies $\xi^2-17\eta^2=1$. But since 33 is not a perfect cube, we have to consider the general solution given by

$$\xi=\tfrac{1}{2}\{(33+8\sqrt{17})^n+(33-8\sqrt{17})^n\}$$

$$\eta=\frac{1}{2\sqrt{17}}\left\{(33+8\sqrt{17})^n-(33-8\sqrt{17})^n\right\}$$

and get a value for ξ which is a perfect cube, by properly chosing n and the corresponding value of η also can be found ; The relations $x^3=\xi$, $y=\xi\eta$, give x and y.

Question 761.

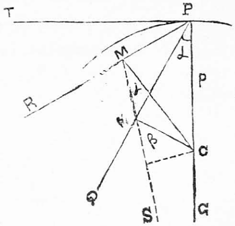
(MARTYN M. THOMAS) :—Apply the method of Instantaneous centre to solve the following·—

" At every point of a plane curve a line is drawn making a given angle with the normal, and let the envelope of these lines be termed the α evolute. Prove that the β. evolute of the α evolute of any curve is the α evolute of the β evolute of the curve"

Solution by S. R. Ranganathan and the Proposer.

Let C be the centre of curvature at P to the given curve. Let CPQ $=\alpha$ and CPR $=\beta$. Draw PT the tangent atP to the given curve.

Consider PT, PR, PQ and PG as four rods rigidly connected, and let PT move tangentially to the given curve.



Then, as P moves on the curve to its consecutive position, PT, PR, PQ, PG will have moved to their consecutive positions, so that

PT moves tangentially	to the given-curve,
PR ,, ,,	to β-evolute,
PQ ,, ,,	to α-evolute,
PG ,, ,,	to evolute proper,

Thus the centre of curvature C is the instantaneous centre, and thus it becomes the centre of curvature for the α-evolute and β-evolute as well.

Hence the perpendiculars from C on PQ and PR become the ρ's for these α and β-evolutes, and M, N the feet of these perpendiculars are, consequently, the points of contact.

Join MN and produce it to S. Then MNCP being a cyclic quadrilateral, NMC $=\alpha$, and SNC $=\beta$.

∴ The curve enveloped by MN becomes the α-evolute of the β-evolute, and is also the β-evolute of the α-evolute.

The foot of the ⊥ from C on MN is the point of contact, and corresponds to P.

Thus as P moves along the given curve, PQ envelopes the α-evolute PR envelopes the β-evolute and MN the α-evolute of the β-evolute and vice versa.

Question 815.

(Martyn M. Thomas, M.A.) :—If

$$\int \frac{dx}{e^{ax^2+bx+c}} = \sum \frac{x^n}{n} k_n$$

prove that $$2a\, k_{n-1} + b\, k_n + n\, k_{n+1} = 0.$$

Solution by K. B. Madhava, R. J. Pocock, H. R. Kapadia, A. A. Krishnaswami Aiyangar, S. R. Ranganathan, C. Krishnamachary, L. S. Vaidyanathan, V. M. Gaitonde, K. A. Erady, S. V. Venkatachalayya, L. N. Bhatt, B.A., K. Santhanam, K. K. R. Aiyar, C. Bhaskaraiya, K. R. Rama Iyer, and S. Gangadharan.

Let $$y = \int exp\,(-ax^2 - bx - c)\, dx = \sum \frac{x^n}{n} k_n,$$

then $$y' = exp\ (-ax^2 - bx - c) = \Sigma\, x^{n-1} k_n,$$

and $$y'' + (2\,ax + b)\, y' = \Sigma\,(n-1)\, k_n\, x^{n-2}.$$

Substituting

$$\Sigma\,(n-1)\, k_n\, x^{n-2} + (2\,ax + b)\, \Sigma\, x^{n-1} k_n = 0$$

and collecting the co-efficient of x^{n-1} we have

$$n\, k_{n+1} + b\, k_n + 2\, a\, k_{n-1} = 0.$$

Question 816.

(Martyn M. Thomas, M.A.) :—If k be the curvature of an ellipse and

$$a_0 = \frac{1}{2\,!} k,\ a_1 = \frac{1}{3\,!}\frac{dk}{ds},\ a_2 = \frac{1}{4\,!}\frac{d^2k}{ds^2},$$

show that

$a_0^{-\frac{8}{5}}\,(4a_0^4 - 5a_1^2 + 4a_0\, a_2)$ has the constant value $\left(\frac{2}{\alpha\beta}\right)^{\frac{4}{3}}$ at all points of the ellipse, α and β being the semi-axes.

Solution (1) *by K. B. Madhava*; (2) *by K. A. Erady, M.A.,*
(3) *K. K. R. Aiyar and several others.*

(1) We know $\frac{d^2}{dx^2}\left(\frac{d^2y}{dx^2}\right)^{-\frac{2}{3}}=\frac{2C}{\Delta^{\frac{1}{3}}}$ for the general conic

$$=\frac{-2}{(\alpha\beta)^{\frac{2}{3}}} \text{ from invariants.}$$

Now $$\frac{dy}{dx}=\tan\psi.$$

$$\therefore \left(\frac{d^2y}{dx^2}\right)^{-\frac{2}{3}}=k^{-\frac{2}{3}}\cos^2\psi,$$

$$\frac{d}{dx}\left(\frac{d^2y}{dx^2}\right)^{-\frac{2}{3}}=-\frac{2}{3}k^{-\frac{5}{3}}\frac{dk}{ds}\cos\psi-2k^{\frac{1}{3}}\sin\psi$$

and $$\frac{d^2}{dx^2}\left(\frac{d^2y}{dx^2}\right)^{-\frac{2}{3}}=-\frac{2}{9}k^{-\frac{8}{3}}\left(-5\left(\frac{dk}{ds}\right)^2-2+3k\frac{d^2k}{ds^2}+9k^4\right)$$

$$=-2^{\frac{1}{3}}(a_o)^{-\frac{8}{3}}(-5a_1^2+4a_o a_2+4a_o^4).$$

Hence

$$a_o^{-\frac{8}{3}}(4a_o^4-5a_1^2+4a_o a_2)=\text{constant and equal to}\left(\frac{2}{\alpha\beta}\right)^{\frac{2}{3}}.$$

(2) Let the equation to the ellipse referred to the tangent and normal at a point be $2y=ax^2+2hxy+by^2$.

Then
$$y_1=(ax+hy)+(hx+by)y_1$$
$$y_2=(a+hy_1)+(hx+by)y_2+(h+by_1)y_1,$$
$$y_3=hy_2+(hx+by)y_3+2(h+by_1)y_2+by_1y_2$$
$$y_4=hy_3+(hx+by)y_4+3(h+by_1)y_3+3by_2^2+by_1y_3.$$

At the origin $y_1=0$, $y_2=a$, $y_3=3ah$, $y_4=3a^2b+12ah^2$.

Also $$\frac{dx}{ds}=(1+y_1^2)^{-\frac{1}{2}}$$

Hence $$k=\frac{1}{\rho}=y_2(1+y_1^2)^{-\frac{3}{2}}$$

$$\frac{dk}{ds}=y_3(1+y_1^2)^{-2}-3(1+y_1^2)^{-3}y_1y_2$$

$$\frac{d^2k}{ds^2}=y_4(1+y_1^2)^{-\frac{5}{2}}-3(1+y_1^2)^{-\frac{7}{2}}y_2^3+y_1(\qquad)$$

$\therefore$ At the origin $k=a$, $\dfrac{dk}{ds}=3ah$ and $\dfrac{d^2k}{ds^2}=3a^2b+12ah^2-3a^3$.

Hence in the notation of the question

$$a_0=\frac{a}{2},\ a_1=\frac{ah}{2},\ a_2=\frac{a^2b+4ah^2-a^3}{8}$$

$$\therefore \quad 4a_0^4-5a_1^2+4a_0a_2=\frac{a^2(ab-h^2)}{4}.$$

But if α, β are the semi-axes of $ax^2+2hxy+by^2=2y$,

$$\frac{1}{\alpha^2\beta^2}=\frac{(ab-h^2)^3}{a^2}.$$

Hence $a_0^{-\frac{8}{3}}(4a_0^4-5a_1^2+4a_0a_2)=\left(\dfrac{a}{2}\right)^{-\frac{2}{3}}(ab-h^2)$

$$=\left(\frac{2}{\alpha\beta}\right)^{-\frac{2}{3}}.$$

(3) We have $2a_0=k=\dfrac{d\psi}{ds}=\dfrac{1}{\rho}=\dfrac{p^3}{\alpha^2\beta^2}$ ... ... (i)

$$\frac{1}{k}\frac{dk}{ds}=\frac{3}{p}\cdot\frac{dp}{d\psi}\cdot k$$

so that $\dfrac{dp}{d\psi}=\dfrac{p}{2}\left(\dfrac{a_1}{a_0^2}\right)$ ... ... (ii)

Also $\left(\dfrac{d^2k}{ds^2}\div\dfrac{dk}{ds}\right)=\dfrac{4a_2}{a_1}=\dfrac{d}{ds}\log\left[\dfrac{3k^2}{\rho}\dfrac{dp}{d\psi}\right]$

$$=\left\{\left(\frac{d^2p}{d\psi^2}\Big/\frac{dp}{d\psi}\right)+\frac{6}{p}\frac{dp}{d\psi}-\frac{1}{p}\frac{dp}{d\psi}\right\}k$$

$$=\left[\frac{\rho-p}{\dfrac{p}{2}\dfrac{a_1}{a_0^2}}+\frac{5}{2}\frac{a_1}{a_0^2}\right]2a_0,\ \text{from (ii) since } \rho=p+\frac{d^2p}{d\psi^2}.$$

That is $\dfrac{4a_2}{a_1}\cdot\dfrac{a_1}{a_0^2}=4a_0\left(\dfrac{\rho-p}{p}\right)+\dfrac{5a_1^2}{a_0^3}$,

or $[4a_0a_2-5a_1^2+4a_0^4]=4a_0^4\cdot\dfrac{1}{2a_0}\cdot(2\alpha^2\beta^2a_0)^{-\frac{1}{3}}$

$$=2^{\frac{2}{3}}a_0^{\frac{8}{3}}(\alpha\beta)^{-\frac{2}{3}}.$$

Hence $a_0^{-\frac{8}{3}}[4a_0a_2-5a_1^2+4a_0^4]=\left(\dfrac{2}{\alpha\beta}\right)^{\frac{2}{3}}$

is the differential equation to a conic in the intrinsic form.

Question 817.

(S. Malhari Rao) :—If xC_3 is equal to the continued product of three primes, whose sum is 327, find x.

Solution by Martyn M. Thomas, M.A., and K. B. Madhava.

$$xC_3=\frac{x(x-1)(x-2)}{1\times2\times3}$$

∴ We have the following cases:

$$x+\frac{x-1}{2}+\frac{x-2}{3}=327\ ;\ x+\frac{x-1}{3}+\frac{x-2}{2}=327$$

$$\frac{x}{2}+\frac{x-1}{3}+(x-2)=327\ ;\ \frac{x}{2}+(x-1)+\frac{x-2}{3}=327$$

$$\frac{x}{3}+\frac{x-1}{2}+(x-2)=327\ ;\ \frac{x}{3}+(x-1)+\frac{x-2}{2}=327.$$

From the first case, we have $x=179$; all the others do not give integral values of x.

∴ The primes, as obtained from the first, are 179, 89 and 59.

Question 818.

(S. Malhari Rao) :—Complete the following magic squares by inserting prime numbers in vacant cells :—

(*a*)

	1	
13		
		7

(*b*)

			47
		19	
	61		
53	83	1	43

(*c*)

		1		
		3		
103	97	449	43	37
		137		
		139		

Solution by R. J. Pococh.

I assume that the proposer requires the cells to be occupied by distinct integers.

(a)

Consider the square

$\alpha+6$	$1+0$	$\beta+12$
$1+12$	$\beta+6$	$\alpha+0$
$\beta+0$	$\alpha+12$	$1+6$

It will clearly be magic provided $\alpha+\beta+1+18=3\beta+18$ *i.e.* $\alpha+1=2\beta$; we have then to find two sets of 3 primes each in A. P. (common difference $=6$) such that the first of one set *plus* unity is double the first of the other set. If the numbers are to be distinct we easily find that $p=31$ satisfies all the conditions.

(b)

Consider the square

a_1	a_2	a_3	47
b_1	b_2	19	b_4
c_1	6_1	c_3	c_4
53	83	1	43

we have $b_2+c_3=a_2+a_3+4$ assume $b_2=a_2+2, c_3=a_3+2$ then we have $a_2=17$, $b_2=19$, $a_3=79$, $c_3=81$.

$$\therefore \quad a_1=180-43-b_2-c_3=37.$$

$\therefore$ $\qquad b_1+c_1=90=b_4+c_4$

and $\qquad b_1+b_4=142$; $c_1+c_4=38$

The only solution of $c_1+c_4=31$ in primes, other than those already assigned to cells is 7, 38 therefore b_1 and b_4 must be 83, 59.

Hence the square is completely determined.

(c) We notice that the pairs of cells already filled which are symmetrically situated *w-r-t* the centre add up to 140. This suggests filling the remaining cells in the leading diagonals in a similar manner. One arrangement is shown below

79	a_2	1	a_4	7
b_1	67	3	53	$b5$
103	97	449	43	37
d_1	87	137	73	$5d$
93	e_2	139	e_4	61

we then have $\left.\begin{aligned} a_2+a_4&=602 \\ e_2+e_4&=436 \\ a_2+e_2&=478 \\ a_4+e_4&=560 \end{aligned}\right\}$ and $\left.\begin{aligned} b_1+b_5&=606 \\ d_1+d_5&=432 \\ b_1+d_1&=454 \\ b_5+d_5&=584 \end{aligned}\right\}$

Hence $a_4-e_2=124$ and $b_5-d_1=752$.

There two equations are easily solved in primes, so that we have

$e_2=17,\ a_4=141 \quad \therefore\ a_2=461,\ e_4=419$

and $d_1=11,\ b_5=163 \quad \therefore\ b_1=443,\ d_5=421.$

Hence the square is completely determined.

Question 840.

(A. C. L. Wilkinson) :—If A′, B′, C′ are the points of contact of the inscribed circle of a triangle ABC with the sides and P the Feuerbach point corresponding to the inscribed circle, show that

$$PA' : PB' : PC' = \left| \sin \frac{B-C}{2} \right| : \left| \sin \frac{C-A}{9} \right| : \left| \sin \frac{A-B}{2} \right|$$

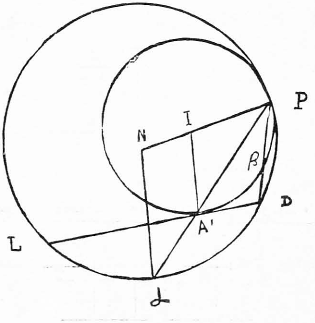
Question 849.

(K. Appukuttan Erady) :—If D, E, F be the feet of the altitudes of the triangle in the above question, shew that

$$DP : EP : FP = \operatorname{cosec} \frac{A}{2} \sin \frac{B-C}{2} : \operatorname{cosec} \frac{B}{2} \sin \frac{C-A}{2} : \operatorname{cosec} \frac{C}{2} \sin \frac{A-B}{2}.$$

Solution (1) *by S. Muthukrishnan and Zero*; (2) *by F. H. V. Gulasekharan and K. A. Erady*; (3) *by K. B. Madhava.*

(1) (i) Produce PA′ to meet the nine point circle at α, and let L, M, N be the mid-points of the sides. Then obviously



$$PA' \propto P\alpha$$

$\therefore$ $$PA'^2 \propto \alpha A' . \alpha P$$
$$\propto \alpha D^2.$$

$\therefore$ $$PA' \propto R \sin \tfrac{1}{2} (B-C).$$

Hence the result.

(ii) Let PD cut the incircle at β. Then

$$PD \propto P\beta$$

$\therefore$ $$PD^2 \propto D\beta . DP$$
$$\propto DA'^2.$$

$\therefore$ $$PD \propto DA'.$$
$$\propto AI \sin \tfrac{1}{2} (B-C)$$
$$\propto r \operatorname{cosec} \frac{A}{2} \sin \tfrac{1}{2} (B-C).$$

Hence the result.

(2) (i) We have

$$\frac{PA'}{A'\alpha} = \frac{r}{\rho - r}$$

where r is the in-radius and ρ the radius of the nine point circle.

Again since the chords $P\alpha$ and DL of the nine point circle intersect at A',

$$PA' . A'\alpha = A'D . A'L.$$

$\therefore$ $$PA'^2 = \frac{r}{\rho - r} . A'D . A'L.$$
$$= \frac{r}{\rho - r} . \frac{b-c}{2} . \frac{(b-c)(b+c-a)}{2a}$$
$$= \frac{32\rho^2 r}{\rho r} \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} \sin^2 \frac{B-C}{2}$$

Hence $PA' : PB' : PC' = \left| \sin \frac{B-C}{2} \right| : \sin \left| \frac{C-A}{2} \right| : \left| \sin \frac{A-B}{2} \right|$

Here $$\frac{DP}{D\beta} = \frac{\rho}{\rho - r}$$

and $$DP . D\beta = DA'^2$$

$\therefore$ $$DP^2 = \frac{\rho}{\rho - r} \times DA'^2.$$

$\therefore$ $$DP : EP : FP$$
$$= DA' : EB' : FC'$$
$$= \frac{(b-c)(b+c-a)}{a} : \frac{(c-a)(c+a-b)}{b} : \frac{(a-b)(a+b-c)}{c}$$
$$= \operatorname{cosec} \frac{A}{2} \sin \frac{B-C}{2} : \operatorname{cosec} \frac{B}{2} \sin \frac{C-A}{2} : \operatorname{cosec} \frac{C}{2} \sin \frac{A-B}{2}.$$

(3)

(i) The areal co-ordinates of the Feuerbach point P with respect to the triangle A′B′C′ are (cf : J. I. M. S., Vol ix, No. 2 § 6),

$$\frac{a}{b-c};\ \frac{b}{c-a};\ \frac{c}{a-b}$$

i.e. $\cos\frac{A}{2}\operatorname{cosec}\frac{B-C}{2};\ \cos\frac{B}{2}\operatorname{cosec}\frac{C-A}{2};\ \cos\frac{C}{2}\operatorname{cosec}\frac{A-B}{2}$.

They are also

$$\frac{\sin B'\hat{A}'C'}{PA'};\ \frac{\sin A'\hat{B}'C'}{PB'};\ \frac{\sin A'\hat{C}'B'}{PC'}$$

that is
$$\frac{\cos\frac{A}{2}}{PA'};\ \frac{\cos\frac{B}{2}}{PB'};\ \frac{\cos\frac{C}{2}}{PC'}.$$

Hence we have at once

$$PA' : PB' : PC' = \sin\frac{B-C}{2} : \sin\frac{C-A}{2} : \sin\frac{A-B}{2}.$$

(ii) The areal co-ordinates of P *w. r. t.* the △DEF (cf. J.I.M.S. vol. ix p. 66).

$$\frac{a(b-c)\cos A}{\sin^2\frac{B-C}{2}};\ \frac{b(c-a)\cos B}{\sin^2\frac{C-A}{2}};\ \frac{c(a-b)\cos C}{\sin^2\frac{A-B}{2}}$$

i.e.
$$\frac{\sin 2A\sin\frac{A}{2}}{\sin\frac{B-C}{2}};\ \frac{\sin 2B\sin\frac{B}{2}}{\sin\frac{C-A}{2}};\ \frac{\sin 2C\sin\frac{C}{2}}{\sin\frac{A-B}{2}}$$

They are also

$$\frac{\sin EDF}{DP},\ \frac{\sin DEF}{EP},\ \frac{\sin DFE}{FP}$$

i.e.
$$\frac{\sin 2A}{DP},\ \frac{\sin 2B}{EP},\ \frac{\sin 2C}{FP}$$

Hence we have at once

$$DP : EP : FP :$$
$$=\operatorname{cosec}\frac{A}{2}\sin\frac{B-C}{2} : \operatorname{cosec}\frac{B}{2}\sin\frac{C-A}{2} : \operatorname{cosec}\frac{C}{2}\sin\frac{A-B}{2}.$$

Solution of Q. 840 *by Sadanand, R. D. Karve and K. R. Rama Iyer.*

If θ be the angle A′P subtends at I

$$\left(\frac{R}{2}-r\right)\cos\theta=\frac{1}{2}(R\cos A+2R\cos B\cos C)-r$$

$$=\left(\frac{R}{2}-r\right)-R\sin^2\left(\frac{B-C}{2}\right).$$

$$\therefore\quad \left(\frac{R}{2}-r\right)\sin\frac{\theta}{2}=R\sin^2\frac{B-C}{2}$$

or $$\left|\sin\frac{\theta}{2}\right| = \left(\frac{R}{R-2r}\right)^{\frac{1}{2}} \left|\sin\frac{B-C}{2}\right|$$

$\therefore$ $$A'P = 2r\sin\frac{\theta}{2} = 2r\left(\frac{R}{R-2r}\right)^{\frac{1}{2}} \left|\sin\left(\frac{B-C}{2}\right)\right|$$

$\therefore$ A'P : B'P : C'P

$$= \left|\sin\frac{B-C}{2}\right| : \left|\sin\frac{C-A}{2}\right| : \left|\sin\frac{A-B}{2}\right|$$

Additional Solutions by M. K. Kewalramani, Hem Raj, N. P. Pandya and S. V. Venkatachalayyar.

Question 855.

(MARTYN M. THOMAS, M A.) :—An ellipse is inverted with respect to any point on its circumference. Show that the inverse curve has three and only three points of inflexion, and that these are collinear.

Solution (1) by K. B. Madhava, (2) by Sadanand and K. R. Rama Aiyer.

(1) The equation of an ellipse referred to a point on the circumference can be put in the form

$$ax^2 + 2hxy + by^2 + 2gx + 2fy = 0;$$

and its inverse *w.r.t* a circle of radius $=k$ can be easily seen to be the cubic,
$$2(gx+fy)(x^2+y^2) + k^2(ax^2+2hxy+by^2) = 0,$$
(*Cf*: Edwards: Diff. Cal. p. 173 Ex. 2).

This form shows at once that the origin is a node with the nodal tangents $ax^2 + 2hxy + by^2 = 0.$

Hence by Plücker's second equation (Edwards p. 255), the number of inflexional points $= 9 - 6\delta = 3$; and only three; and it is well-known that the three points of inflexion of a cubic are collinear.

(2) Let the ellipse be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Let P be any point on the ellipse and let α be its eccentic angle. Then we know that there are three and only three points on the ellipse whose circles of curvature pass through P and that their eccentric angles are $\frac{1}{3}(2\pi - \alpha)$, $\frac{1}{3}(4\pi - \alpha)$ and $\frac{1}{3}(6\pi - \alpha)$. If Q be one of these points then its circle of curvature cuts the ellipse in three adjacent points. If we invert the ellipse *w. r. t.* P the circle of curvature inverts into a straight line which cuts the inverse curve in three adjacent points. Therefore Q inverts into a point of inflexion. Similarly the other two points invert into two points of inflexion. Hence the inverse curve has three and only three points of inflexion. Also P and the three points on the ellipse form a cyclic quadrilateral because the sum of their eccentric angles $=\alpha+\frac{1}{3}\{2\pi-\alpha+4\pi-\alpha+6\pi-\alpha\}=4\pi$.

The circumcircle of this cyclic quadrilateral inverts into a straight line passing through the three points of inflexion in the inverse curve. Hence the three points of inflexion are collinear.

Question 881.

(S. Krishnaswami Aiyangar):—Prove that

$$\text{(i)} \int_0^{\frac{\pi}{2}} \frac{(\log \sin x)^2}{\sqrt{\sin x}} dx = \frac{\{\Gamma(\frac{1}{4})\}^2 (\pi^2+16)}{8\sqrt{2\pi}}$$

$$\text{(ii)} \int_0^{\frac{\pi}{2}} \frac{(\log \sin x)^3}{\sqrt{\sin x}} dx = \frac{\{\Gamma(\frac{1}{4})\}^2 (\pi^2-32\pi-112)}{16\sqrt{2\pi}}$$

Remarks by K. B. Madhava and Sadanand.

The first part of the question is completely solved in Bromwich, Ex. 45 p. 476, taking $\alpha=\frac{1}{4}$; second part follows from the same, by differentiation. It appears the right hand sides of these integrals are wrongly given.

Question 892.

(S. Krishnaswami Iyengar):—Show that

$$\int_0^{\frac{\pi}{2}} \sqrt[3]{\sin x} \log \sin x dx = \frac{\pi\sqrt[3]{2}.\sqrt{3}}{\Gamma(\frac{1}{3})}\left\{\frac{\pi}{3}\sqrt{3}+\log 2-3\right\}.$$

Question 893.

(S. Krishnaswami Iyengar):—Find the value of

$$\int_0^{\frac{\pi}{2}} \frac{\log \sin x. \log \cos x dx.}{\sqrt{\sin x \cos x}}$$

Solution by R. J. Pocock, S. R. Ranganathan and K. B. Madhava.

We have

$$\int_0^{\frac{\pi}{2}} \sin^{2\alpha-1} x.\ \log\ \sin x dx = \frac{\sqrt{\pi}}{4}\cdot \frac{\Gamma(\alpha)}{\Gamma(\alpha+\frac{1}{2}\delta)} \left[\psi(\alpha)-\delta(\alpha+\tfrac{1}{2}\psi)\right]$$

(Bromwich p. 476 Ex. **45.**)

Write $\alpha=\frac{2}{3}$. Now $\psi\left(\frac{2}{3}\right)=-c-\frac{3}{2}\log 3+\frac{2}{6}\pi\sqrt{3}$

(loc. cit. Ex. **43.**)

Also $$\psi\left(\frac{2}{3}\right)+\psi\left(\frac{7}{6}\right)=2\psi\left(\frac{4}{3}\right)-2\log 2$$

and $$2\psi\left(\frac{4}{3}\right)-2\psi\left(\frac{1}{3}\right)=6.$$

(loc. cit. Ex. **42.**)

Hence $$\psi\left(\frac{7}{6}\right)=-c+6-2\log 2-\frac{3}{2}\log 3-\pi\frac{\sqrt{3}}{2}.$$

$$\therefore \psi\left(\frac{2}{3}\right)-\psi\left(\frac{7}{6}\right)=\frac{2}{3}\pi\sqrt{3}+2\log 2-6.$$

Also $$\frac{2^{2x}.\Gamma(x+\frac{1}{2}).\Gamma(x+1)}{\Gamma(2x+1)}=\sqrt{\pi}$$

(Bromwich p. **461.**)

$$\therefore \quad \Gamma\left(\frac{7}{6}\right).\Gamma\left(\frac{2}{3}\right)=\frac{3}{2}\cdot\frac{\sqrt{\pi}}{2^{\frac{4}{3}}}.\Gamma\left(\frac{1}{3}\right). \quad \text{Also } \Gamma\left(\frac{2}{3}\right)=\frac{2\pi}{\sqrt{3}\Gamma(\frac{1}{3})}.$$

$$\therefore \quad \int_0^{\frac{\pi}{2}} \sqrt[3]{\sin x}.\ \log\ \sin x dx=\frac{\sqrt{\pi}}{2}\cdot\frac{\Gamma(\frac{2}{3})}{\Gamma(\frac{7}{6})}\cdot\left(\frac{\pi\sqrt{3}}{3}+\log 2-3\right)$$

$$=\frac{2\sqrt[3]{2}.\ \pi^2}{\{\Gamma(\frac{1}{3})\}^3}=\left\{\frac{\pi\sqrt{3}}{3}+\log 2-3\right\}.$$

[*N.B.*—The result as printed appears to be inaccurate.].

Again,

$$\int_0^{\frac{\pi}{2}} \frac{\log\ \sin x\ \log\ \cos x}{\sqrt{\sin x\ \cos x}},$$

$$=\frac{1}{4}\left[\frac{d^2}{d\alpha d\beta}\int_0^{\frac{\pi}{2}} \sin^{2\alpha+1} x\ \cos^{2\beta+1} x\ dx\right]\alpha=\beta=\tfrac{1}{4}$$

$$=\frac{1}{8}\left[\frac{d^2}{d\alpha d\beta}\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}\right]\alpha=\beta=\tfrac{1}{4}$$

$$=\frac{1}{8}\frac{\{\Gamma(\frac{1}{4})\}^2}{\Gamma(\frac{1}{2})}\left[\{\psi(\tfrac{1}{4})-\psi(\tfrac{1}{2})\}^2-\psi'(\tfrac{1}{2})\right]$$

$$=\frac{1}{8}\frac{\{\Gamma(\frac{1}{4})\}^2}{\sqrt{\pi}}\left[\left(\log 2+\frac{\pi}{2}\right)^2-\frac{\pi^2}{2}\right]$$

[For values of $\psi(\frac{1}{4})$, $\psi(\frac{1}{2})$, $\psi'(\frac{1}{2})$, see Bromwich: *Infinite Series*, Appendix III, Exs. 42 and 43.]

Question 911.

(R. J. POCOCK) :—Show that the locus of a point such that the eccentric angles of the feet of the four normals which can be drawn from it to the ellipse $x^2/a^2+y^2/b^2=1$ satisfy the relations

$$\Sigma \cos\phi_1 \cos\phi_2=\Sigma \sin\phi_1 \sin\phi_2=0$$

is the ellipse

$$x^2+(1-e^2)y^2=a^2e^4.$$

Solution by K. J. Sanjana, H. R. Kapadia and K. B. Madhava.

The normal at the point of eccentric angle ϕ is

$$ax \sec\phi-by \operatorname{cosec}\phi=a^2-b^2=c^2;$$

if this goes through (h, k) we have $ah \sec\phi-bk \operatorname{cosec}\phi=c^2$.

Replacing by $\cos\phi$ and $\sin\phi$ and finding each in turn by rationalising, this condition can be written in the form—

$$c^4\cos^4\phi-2c^2ah\cos^3\phi+(a^2h^2+b^2k^2-c^4)\cos^2\phi$$
$$+2c^2ah\cos\phi-a^2h^2=0,$$

$$c^4\sin^4\phi+2c^2bk\sin^3\phi+(a^2h^2+b^2k^2-c^4)\sin^2\phi$$
$$-2c^2bk\sin\phi-b^2k^2=0.$$

Both the given relations give rise to the same result, *viz.*

$$a^2h^2+b^2k^2-c^4=0.$$

Thus the locus of the intersection of the normals is

$$x^2+\frac{b^2}{a^2}a^2=y^2\left(\frac{c^2}{a^2}\right)^2,$$

which is the result in the question. [See J. I. M. S., Vol. VIII, p. 233.]

Additional Solution by R. Srinivasan.

Question 914.

(Prof. K. J. SANJANA) :—The sum of the $2n+1$ terms of the type $\frac{\alpha}{\alpha^2}$ where α is a $(2n+1)^{th}$ root of unity is $(-)^n(n+\frac{1}{2})$.

Solution by K. B. Madhava.

Since α satisfies $x^{2n+1}-1=0$,

$\frac{1}{1\pm i\alpha}$ satisfy $\pm i^{2n+1}(z-1)z^{2n+1}+z^{2n+1}=0$ respectively.

Hence,

$$\sum\frac{\alpha}{1+\alpha^2}=\frac{1}{2i}\sum\left[\frac{1}{1-i\alpha}-\frac{1}{1+i\alpha}\right]$$

$$=\frac{1}{2i}\left[\frac{(2n+1)i^{2n+1}}{1+i^{2n+1}}+\frac{(2n+1)i^{2n+1}}{1-i^{2n+1}}\right]$$

$$=\frac{1}{2i}\times(2n+1)i_2^{\,n+1}=(-)^n(n+\tfrac{1}{2}).$$

Slightly different solutions by S. R. Ranganathan and R. Srinivasan.

Question 917.

(HEM RAJ) :—In any triangle shew that

$$2\Sigma a^2(b+c)\geq\Sigma(a^3)+9abc.$$

Solution by K. J. Sanjana.

It may be shown that if G is the centroid and I the incentre,

$$GI^2+4Rr=\tfrac{1}{3}(bc+ca+ab)-\tfrac{1}{9}(a^2+b^2+c^2).$$

[see Hobson, *Plane Trig*, p. 200.]

Hence, unless the triangle is equilateral in which case the inequality becomes an equality, we get

$$\tfrac{1}{3}(bc+ca+ab)>\tfrac{1}{9}(a^2+b^2+c^2)+4Rr$$

i.e. $$>\tfrac{1}{9}(a^2+b^2+c^2)+\frac{2abc}{a+b+c};$$

so that $$3\,(a+b+c)(bc+ca+ab)>(a+b+c)(a^2+b^2+c^2)+18abc,$$

or $$3\Sigma a^2(b+c)+9abc>\Sigma a^2(b+c)+\Sigma(a^3)+18abc,$$

i.e. $$2\Sigma a^2(b+c)>\Sigma(a^3)+9abc.$$

QUESTIONS FOR SOLUTION.

927. (M. K. KEWALRAMANI) :—ABCDE .. is a polygon whose sides taken in order are a, b, c, d, e.... If the angles which the sides b, c, d, e...make with the positive direction of the side a, be respectively called ab, ac, ad, ae..., prove that

$$a^n = \sum \left[\frac{n!}{r!\, s!\, t! \ldots} b^r c^s d^t \ldots\ldots (\cos \{ r.ab + s.ac + t.ad + \ldots \}) \right]$$

were n is a positive integer and r, s, t, ... are also positive integers including zero, but such that $r+s+t+\ldots=n$ and Σ denoting summation of like expressions when, r, s, t, ... are taken in their entirety subject to the above restriction ; of course on the right hand side, a not occurring.

928. (M. K. KEWALRAMANI) :—Show that the infinite product

$$\left(1-\frac{1}{2^{12}}\right)\left(1-\frac{1}{3^{12}}\right)\left(1-\frac{1}{4^{12}}\right)\cdots$$

$$= \frac{\sinh \pi \cosh^2 \frac{\sqrt{3}\pi}{2}}{24\pi^6} [\cosh \pi - \cos \sqrt{3}\pi]$$

929. (S. MALHARI RAO) :—Express 2595600 as the product of two factors such that the sum of the factors of either factor may be equal to the other factor.

930. (S. MALHARI RAO) :—Fill up the vacant cells in the following figure with the remaining numbers of the series 1, 2, 3, ..., 63, 64, so that the whole figure and each of the four corner sub-squares of 16 cells, may all be pan-diagonal magic squares.

39	28	38	25				
24	29	35	32				
27	40	26	37				
30	33	31	36				

931. (C. KRISHNAMACHARY):—Sum to n terms and discuss the convergence to infinity of the series.

$$\frac{y}{x+y}+x\frac{y+z}{(x+y)(x+y+z)}+x^2\frac{y+2z}{(x+y)(x+y+z)(x+y+2z)}+\ldots$$

932. (T. KRISHNA RAO) :—In Q. 772 prove the following construction :—Take MH equal to the given sum of the sides and bisect it at F. Draw the angle FMK equal to the vertical angle. Cut off MK such that the rect. MF. MK = the given rectangle. Join FK and draw the bisector of the ∠MFK. On the bisector cut off FG such that $FG^2 = FM.FK$. Join GM and GH. From GM cut off GL = GH. Let the bisector of ∠MGH meet MH in N. Join LN. Then △ LMN is the required triangle.

933. (K. B. MADHAVA):—With the usual notation of the Elliptic Functions, show that

$$\int_0^{\frac{1}{\sqrt{2}}} k\,KK'\,dk = 2^6\int_0^{\frac{1}{\sqrt{2}}} k\,EE'\,dk = \frac{\pi^3}{32}.$$

934. (HEMRAJ):—If n be a positive integer, show that

$$\frac{1\cdot3\cdot5\ldots\ldots(2n-1)}{2^{n-1}}\left\{n+\frac{n(n-1)}{1\cdot2}\left(1-\tfrac{1}{3}\right)+\frac{n(n-1)(n-2)}{1\cdot2\cdot3}\left(1-\tfrac{1}{3}+\tfrac{1}{5}\right)+\ldots\right.$$

$$\ldots+\frac{n(n-1)}{1\cdot2}\left(1-\tfrac{1}{3}+\tfrac{1}{5}-\ldots\ldots+(-1)^{n-2}\frac{1}{2n-5}\right)$$

$$+n\left(1-\tfrac{1}{3}+\tfrac{1}{5}-\ldots\ldots+(-1)^{n-2}\frac{1}{2n-3}\right)$$

$$\left.+\left(1-\tfrac{1}{3}+\tfrac{1}{5}-\ldots\ldots+(-1)^{n-1}\frac{1}{2n-1}\right)\right\}$$

is an integer.

935. (K. J. SANJANA, M.A.) :—Let a, b, c, be numbers representing the sides of a real triangle, and

$$S\equiv a^2(b+c)+b^2(c+a)+c^2(a+b),$$
$$\Sigma\equiv a^3+b^3+c^3,\ P\equiv abc:$$

prove that $6S > \text{or} < l\Sigma+(36-3l)\,P$,

according as $l=0, 1, 2, 3$ or $=6, 7, 8, 9, 10, 11, 12$.

Also examine the inequality for $l=4$ and $l=5$.

[Suggested by Q. 917.]

936. (K. J. SANJANA, M. A.) :—If

$$f(x)=x^6+55x^4+439x^2+225,$$

shew that

$$\sum_1^\infty\frac{(-)^{m-1}}{2mf(2m-1).f(2m+1)}=\frac{\log 4-1\cdot2\dot{3}}{2073600},$$

(Suggested by Q. 903.)

937. (H. R. Kapadia) :—If n have any positive integral value except unity, and r be any positive integer which is not a perfect power, show that

$$\Sigma(n-1)/(r^n-1)=\frac{\pi^2}{6};$$

and if $d(n)$ denote the number of divisors of n, that $\Sigma(d(n)-1)/r^n=1$; also that $\Sigma(n-1)/r=\Sigma 1/(r-1)^2$.

938 (H. R. Kapadia) :—Prove that, if $m_1, m_2, \ldots \ldots m_s$ are the integers less than and prime to m, and if $p_1, p_2, \ldots\ldots$ are the different prime factors of m,

$$\prod_1^s \sin\left(\theta+\frac{m_r\pi}{m}\right)=\frac{\sin m\theta, \prod \sin\frac{m\theta}{p_1p_2}. \prod \sin\frac{m\theta}{p_1p_2p_3p_4}\ldots\ldots}{2^s\prod \sin\frac{m\theta}{p_1}. \prod \sin\frac{m\theta}{p_1p_2p_3}}.$$

939. (S. R. Ranganathan) :—If ρ, ρ_0 and ρ_α are the radii of curvature, at the corresponding points of a curve, its evolute, and its α—evolute respectively, show that as α varies from $-\frac{\pi}{2}$ to $+\frac{\pi}{2}$, ρ_α has for its maximum value the expression $\sqrt{\rho^2+\rho_0^2}$ and that it attains it when $\tan\alpha=\rho/\rho_0$.

940 (R. Srinivasan, M.A., M.R.A.S.) :—Show that

$$\int_0^1 \frac{x^n \log x}{1+x^2}dx=\frac{\pi^2}{4}\sec\frac{n\pi}{2}\tan\frac{n\pi}{2};$$

941. Balak Ram) :—[Suggested by Q. 817].

Solve $$x!=(x-y)!\ (y!)\ PQR$$

where x and y may be any integers satisfying

$$3\angle y\frac{x}{2},$$

and P, Q, R are primes.

922. (Hemraj) :—Show that if p be prime,

$$\left[\left(\sum_1^{p-1}\frac{1}{n}\right)\div\frac{p3}{p!}\right]$$

is an integer.