JUBILEE COMMEMORATION VOLUME

INDIAN MATHEMATICAL SOCIETY

Silver Jubilee Commemoration Volume

JOURNAL VOL. XX

Published by

The Indian Mathematical Society

THE JOURNAL

OF THE

INDIAN MATHEMATICAL SOCIETY

EDITOR:

R. VAIDYANATHASWAMY, M.A., D.Sc.

JOINT EDITOR:

A. NARASINGA RAO, M.A., L.T.

VOLUME XX

1933

TRICHINOPOLY:

ST. JOSEPH'S INDUSTRIAL SCHOOL PRESS

1934

CONTENTS

List of Papers Published

ERRATA

Page	*line*	*for*	*read*
71	5B*	$p \leq \sqrt{x}$	$p \geq \sqrt{x}$
78	5	$-\dfrac{hx}{\log x}$	$-\dfrac{h_1 x}{\log x}$
120	10	Aitken	R. A. Fisher
140	2 and in the rest of the paper	v	ν
141	3B	$\dfrac{\sigma(2)}{2\nu}$	$\dfrac{\sigma(2\nu)}{2\nu}$
170	6	Clairant	Clairaut
174	6B	$T_n(x)$	$\lvert T_n(x)\rvert$
195	9	x_{k+l}	x_{ks+l}
,,	14	$z_{k,(r-1)}$	$z_{k,(r-1)s}$
,,	21	$x_{x(k-1)s+l}$	$x_{(k-1)s+l}$
198	8	components the collineation	components of the collineation
199	footnote, line 2	e^1	e'
201	8,11,19,20	ps	p_s
,,	11	e_{pq}	$e_{p_s q}$
,,	14	vl	$v-l$
,,	16	$v-p-1$	$v-p_s-1$
,,	18	p_{sq}	$p_s q$
,,	footnote, line 3	v_3	$v-3$
202	7	q_s	p_s
203	10	$n-h$	$n-k$
205	10	$\dfrac{a_1+r_1-t_1^2(a_1+r_1)}{t_1}$	$\dfrac{a_1+r_1-t_1^2(a_1-r_1)}{t_1}$
209	15, 18, 19, 20	$\neq$	$\not\equiv$
210	10	$P^{5}_{24}+P^{3}_{46}$	$P^{3}_{24}+P^{5}_{46}$
211	12	$l^2_{18}+l^4_{35}+l^3_{51}$	$l^2_{13}+l^4_{35}+l^6_{51}$
217	11B, 7B	insert ... (5·1) at the end of line 7B instead of [line 11B	
228	Eq. 15	$l^2 \div m^2 \div n^2$	$l^2+m^2+n^2$
229	2	$l^2 \div m^2 + n^2$	$l^2+m^2+n^2$
238	1B	V_i	(V_i)
239	22	ϕ	φ_j
244	5 and throughout the paper	τ	T
,,	,,	x_1, x_2	$\chi_1\ \chi_2$
245	4	$\exp.\left\{\dfrac{\pm 2\pi i\, E_n\, t/h}{V}\right\}$	$\exp.\left\{\dfrac{\pm 2\pi i\, E_n\, t}{h}\right\}\Big/V$
247	8B	$\bar{\dot{p}}$	$\bar{p}$
247	4B	$\bar{\dot{v}}$	$\bar{v}$

* Read " line 5 from below, (omitting footnote, if any). "

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R. P. Shintre, M.A.	... ...	1932—

REPORT

OF THE

Eighth Conference & Silver Jubilee Celebrations

OF THE

Indian Mathematical Society

(*Bombay, 21st to 24th December 1932*)

General Report

At the invitation of the Bombay University, the Eighth Conference of the Indian Mathematical Society was held at Bombay on the 21st, 22nd, and 23rd December 1932. The Society was fortunate in securing the gracious patronage of His Excellency SIR FREDERICK HUGH SYKES, P.C., G.C.I.E., G.B.E., K.C.B., C.M.G., Governor of Bombay and Chancellor of the Bombay University who delivered the Inaugural Address on the opening day. V. N. CHANDAVARKAR Esq., B.A., Bar-at-law, Mayor of Bombay, was the Vice-Patron of the Conference. Mr. and Mrs. Chandavarkar were "At Home" to the delegates on the 23rd evening, when the latter had an opportunity of meeting the elite of Bombay.

Besides the reading of papers, there were two discussions, one on the teaching of Mathematics in schools and the other on the teaching of Mathematics in Universities. The Society hopes that by thus providing opportunities at its several conferences for teachers of Mathematics in Schools and Colleges to meet together and discuss improvements in courses, syllabuses, and methods of instruction, it is helping to bring about a higher standard of mathematical instruction in the country.

There were three public discourses all of which were well attended; the first by Dr. MEGHNAD SAHA, D.Sc., F.R.S., on "The Present Crisis in the Science of Dynamics", the second by Dr. R. VAIDYANATHASWAMI, D.Sc., F.R.S.E., on "The Nature of the

Continuum", and the last by Rao Bahadur P. V. SESHU AIYAR, B.A., I.E.S. (Retd.), President of the Society, on "The Nature of Mathematics and Religion."

The 24th December was devoted to the celebration of the SILVER JUBILEE of the Foundation of the Society under the Presidentship of the Rev. J. MACKENZIE, M.A., Vice-Chancellor, Bombay University, and included the presentation of an address to Prof. M. T. NARANIENGAR, M.A., Editor of the Society's Journal for nearly two decades from the foundation of the Society. Eloquent tributes were paid to the valuable services rendered by him to the Society. After the Jubilee Celebrations, an excursion was arranged to the Elephanta Caves where light refreshments were served. The Conference terminated on the steps of the Appollo Bunder late in the evening on the 24th December.

The Society wishes to take this opportunity of expressing its gratitude to His Excellency Sir F. H. Sykes, Governor of Bombay, for opening the Conference and for his kind words of appreciation of the Society's work and his good wishes for the success of the Conference; to the Rev. J. Mackenzie, Vice-Chancellor, and the other authorities of the Bombay University for their kind invitation and co-operation; to Mr. and Mrs. V. N. Chandavarkar for their delightful "At Home"; to the Principal of the Royal Institute at which the meetings were permitted to be held; to the Secretary and members of the Reception Committee who were responsible for the excellent arrangements; and to all those, members as well as others, who had contributed their share to the success of the conference.

Detailed Daily Programme of the Conference

Wednesday 21st December 1932.

10-15 A.M.—INAUGURATION OF THE CONFERENCE AT THE COWASJEE JEHANGIR HALL.

His Excellency was received at the steps by the members of the Reception Committee led by The Hon'ble Mr. Justice Mirza Ali Akbar Khan, Ex-Vice-Chancellor of the Bombay University and Mr. V. N. Chandavarkar, the Mayor of Bombay. The President and Members of the Managing Committee of the Society, the Foundation Members and the members of the Local Committee were then presented to His Excellency, who was then taken in procession to the hall.

Mr. Justice Mirza in welcoming the members of the Society, referred to the great value of such Conferences and hoped that the presence of the President Rao Bahadur P. V. Seshu Aiyar, Mr. V. Ramaswamy Aiyar, the Founder of the Society and the other distinguished Mathematicians at the Conference would serve as an inspiration to the younger generation. He then spoke of the efforts of the Bombay University to advance the study of Mathematics and, wishing success to the Conference, requested His Excellency to inaugurate the Conference.

His Excellency then delivered the Inaugural Address and declared the Conference open. (The address is printed on pages 7-8.)

Prof. K. R. Gunjikar, the Secretary of the Local Executive Committee, then read a letter from the Rev. Dr. Mackenzie, Vice-Chancellor of the University and Chairman of the Reception Committee, regretting his unavoidable absence from Bombay on the opening day and wishing all success to the Conference. He also read out a list of distinguished persons and Institutions who had sent Greetings to the Conference on the occasion of its Silver Jubilee. (The list of those from whom Greetings and Congratulations were received is printed on page 9).

The President of the Society then thanked His Excellency on behalf of the Society and referred to the keen interest in the cause of learning which prompted His Excellency to agree to open the Conference in spite of his arduous duties. He also thanked The Hon'ble Mr. Justice Mirza Ali Akbar Khan for his words of welcome and Mr. V. N. Chandavarkar for consenting to become the Vice-Patron of the Conference.

11 A.M.—CONFERENCE SESSION WITH RAO BAHADUR P. V. SESHU AIYAR IN THE CHAIR.

Report of the Society's Activities by Prof. S. B. Belekar, Hon. Joint Secretary.

At the Secretary's request Prof. D. D. Kapadia summarised the activities of the Society during the past 25 years. (The Secretary's report is printed on pages 10-17).

Presidential Address by Rao Bahadur P. V. Seshu Aiyar, President of the Society.

(The Presidential Address is printed on pages 18-30).

12-30 P.M.—BUSINESS MEETING OF THE SOCIETY.

(A report of the Business Meeting and the resolutions passed thereat appears on pages. 31-32).

2 P.M.—LIGHT REFRESHMENTS.

3 P.M.—VISIT TO THE ORIENTAL LIFE ASSURANCE COMPANY'S OFFICES.

At the invitation of Mr. L. S. Vaidyanathan, the delegates paid a visit to the Oriental Life Assurance Company's Offices where they inspected the several calculating machines and the automatic Sorting and Recording Machines used by the Company.

5 P.M.—DR. SAHA'S PUBLIC ADDRESS ON "THE PRESENT CRISIS IN THE SCIENCE OF DYNAMICS."

Thursday the 22nd December 1932.

8-30 A.M. TO 9 A.M.—READING OF PAPERS.

(For the list of Papers read at the Conference *vide* pages 47-52).

9 A.M.—GROUP PHOTO OF MEMBERS OF THE SOCIETY AND THE MEMBERS OF THE RECEPTION COMMITTEE.

9-30 TO 11-30—READING OF PAPERS (*Contd.*)

11-30 TO 2 P.M.—INTERVAL.

The Local Committee had arranged for the meals of those of the delegates whose place of residence was far removed, and for whom it would have been very inconvenient to go and return in time for the reading of papers in the afternoon.

2 P.M.—DISCUSSION ON THE TEACHING OF MATHEMATICS IN SCHOOLS with Rao Bahadur P. V. Seshu Aiyar in the chair.

(For details regarding the discussion *vide* pages. 32-34).

5 P.M.—RECEPTION COMMITTEE "AT HOME" TO THE DELEGATES.

6-30 P.M.—DR. VAIDYANATHASWAMI'S PUBLIC ADDRESS ON "THE NATURE OF THE CONTINUUM".

Friday the 23rd December 1932.

8-30 TO 11-30 A.M.—READING OF PAPERS. (*Contd.*)

11-30 TO 2 P.M.—INTERVAL.

As on the previous day the Local Committee arranged a feast to the delegates who had thus an opportunity of having an informal talk on many topics concerning the affairs of the Society.

2 P.M.—DISCUSSION ON THE TEACHING OF MATHEMATICS IN COLLEGES.

(For details regarding the discussion vide pages 35-36).

5 P.M.—MAYOR'S "AT HOME".

There was a large number of invitees including the Chief Justice of Bombay, the Municipal Commissioner, the Post Master General, and many other prominent citizens of Bombay.

Saturday the 24th December 1932.—Jubilee Day.

11-30 A.M.—JUBILEE CELEBRATIONS COMMENCE WITH THE REV. DR. MACKENZIE IN THE CHAIR.

Prayer by Students of the Royal Institute of Science.

Tributes to the services rendered by Prof. Naraniengar.

The Hon. Secretary gave an account of the work of the Society in its earlier days and paid a warm tribute to the Founder Mr. V. Ramaswami Aiyar, and Prof. M. T. Naraniengar, the first Editor of the Journal. The President, the Founder, Principal Menon, Dr. Vaidyanathaswami, Prof. Arunachala Sastry, and several others spoke eulogising the signal services rendered by Prof. Naraniengar to the Society.

Address to Prof. Naraniengar.

An address in appreciation of his services was then read by the President of the Society and presented to Prof. Naraniengar amid loud cheers. The text of the address will be found on pages 37-38.

Prof. Naraniengar's Reply.

(This will be found on pages 38-40).

Vote of Thanks.

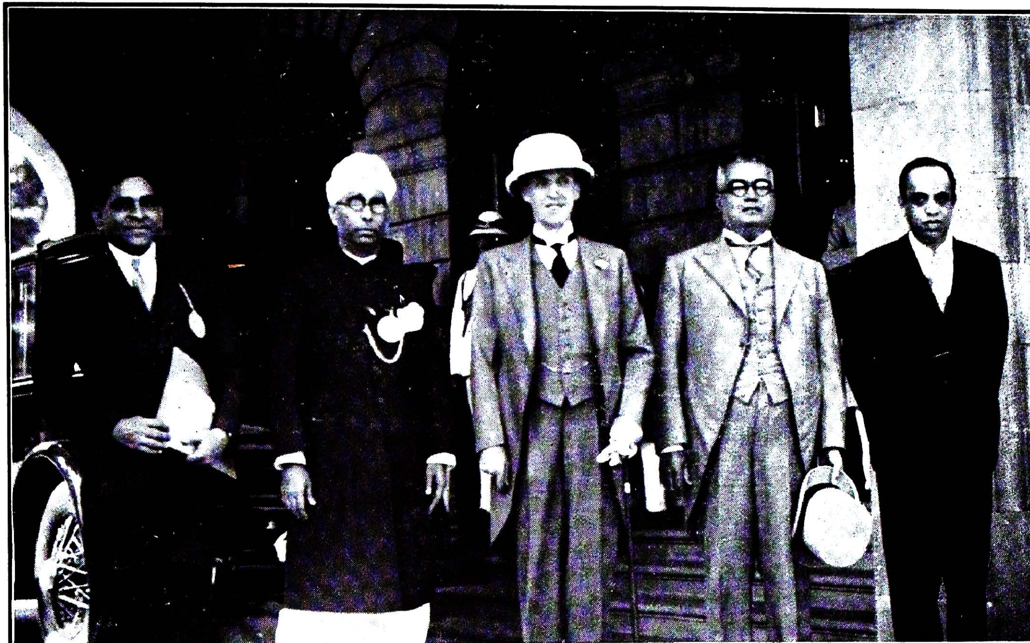
Prof. Gunjikar on behalf of the Local Committee and Prof. Belekar, the Secretary of the Society, on behalf of the Managing Committee of the Society expressed the thanks of the Society to all those whose valuable help had been responsible for the success of the Conference; particularly to His Excellency the Chancellor and the Vice-Chancellor and the other authorities of the Bombay University, to Mr. and Mrs. Chandavarkar, Justice Mirza, Prof. Saha, Dr. Vaidyanathaswamy, Rao Bahadur Seshu Aiyar, Prof. L. S. Vaidyanathan, Mr. K. S. Ramachandra Iyer, President of the South Indian Association, Matunga, who had accommodated the delegates, to the Press and the Broadcasting Service. The thanks of the Society for the excellent arrangements made by Prof. Gunjikar and the Reception Committee and the devoted band of volunteers were proposed by the Secretary of the Society.

The function concluded with a vote of thanks to the Rev. Dr. Mackenzie, the Chairman of the Jubilee Celebrations, and after garlanding the Foundation members of the Society.

12-30 P.M.—LECTURE BY RAO BAHADUR P. V. SESHU AIYAR ON "THE NATURE OF MATHEMATICS AND RELIGION."

1-30 P.M.—EXCURSION TO THE ELEPHANTA CAVES.

The fresh sea breeze was a welcome change from the heated atmosphere of the lecture room and the members and delegates were in a very hilarious mood when the three motor launches began to race towards the island, where tea and light refreshments were awaiting their arrival. The party returned late in the evening and parted after mutual greetings and cheers for the President and the local Secretary.



Left to Right.

Prof K. R. GUNJIKAR, *Hon. Local Secretary.* Rao Bahadur P. V. SESHU AIYAR, *President.* H. E. SIR FREDERICK SYKES, *Governor of Bombay and Patron of the Conference.* Hon. Mr. Justice MIRZA ALI AKBAR KHAN, *Ex-Vice Chancellor, Bombay University.* Mr. V. N. CHANDAVARKAR, *Mayor of Bombay and Vice-Patron of the Conference.*

His Excellency's Speech at the Opening of the Conference.

(*Bombay, 21st December 1932—10-15 A.M.*)

MR. PRESIDENT AND GENTLEMEN,

It gives me very great pleasure warmly to welcome this,—the Indian Mathematical Society's Conference,—to Bombay and to the University. I regret that my engagements have made it impossible for me to do more than pay you this very brief visit; but I am glad indeed that I have found it possible to be here at the opening of the Conference, not, I am afraid, to make a formal speech, but to have the privilege of addressing you a few words of welcome.

I may say that I am impressed by the importance of the work which you as mathematicians are doing. To the general public mathematics is not, and probably cannot be, an absorbingly interesting subject. One meets many people who have painful memories of their early endeavours after mathematical truth. Whether the pains which have so often accompanied the study are due to the subject itself or to the manner in which it has often been taught I am not prepared to say. In addition to trying to develop mathematical research and to encourage young mathematicians, work for which I know your President does so much and for which he deserves all our thanks, I am much interested to learn that you are devoting time at this Conference to the discussion of methods of teaching. If you are able to devise means whereby the study of mathematics may be made more attractive to the average boy and so develop original and sound thinking, and whereby it may be made easier for the average person to apply mathematical methods to the varied material to which they are applicable, you will deserve the gratitude not only of the School-and College-going population but of the whole community.

I should like to take the opportunity also, if I may, to congratulate you on what I hear as to the value of the papers that appear in the *Journal of the Indian Mathematical Society*. No one who knows anything about modern scientific developments can have any doubt whatsoever of the supreme importance of mathematics for the sciences. There are innumerable modern discoveries which would not have been possible if the mathematicians had not prepared the way. We used to think when we heard of mathematicians busying themselves with such subjects as the Geometry of space of n dimensions that they were about as far removed from reality as Alice was in her adventures in Wonderland. But we now learn that a great deal of modern Physics would have been impossible if these adventurers had not prepared the way. Equally fanciful seemed to be the conception of non-Euclidean space. The mathematicians seemed to say "Let's pretend", and the pretence turned out to be truth. Again, I am told, the mathematicians were for long years unconsciously preparing the way for the whole theory of Relativity. Facts like these will encourage the mathematician in the prosecution of his researches. We are all familiar with the old story of the Cambridge don, who said with pride regarding a theorem which he had discovered that the beauty of it was that it could never be put to any practical use. I believe no one will say to-day, in the light of mathematical history, in regard to any new discovery, that the time may not come when it may not be of service in illuminating some great field of enquiry. The mathematician need never apologise for undertaking any line of research, however far removed it may seem to be from any utilitarian end.

There is another practical application of mathematics which in these days has assumed a position of great importance, and to various aspects of which I have no doubt you will be giving some attention in your Conference. I mean Statistics. It is astonishing in what a variety of ways and to what a variety of material statistical methods are now being applied. It is a branch of mathematical science that has often been suspected and not infrequently made a subject for wit. There is no doubt that in the hands of unskilled persons statistics may be abused, but this fact does not in any way detract from the great value which they have, and are increasingly being found to have, in the elucidation of the material with which many sciences deal. It is not merely in the physical sciences, but quite as much in the sciences which relate to human life, and in many of the practical arts, that statistics have their place. We may be far away here from the high disinterestedness of the Cambridge don, but scientists and practical men unite in acknowledging their debt to those who have elaborated the statistical methods which are now at their service.

Let me, in conclusion, express to you my good wishes for the success of your Conference, of which I shall look forward with interest to reading the proceedings. Your work may attract less public attention than that of many Conferences. You will probably have little to say to the plain man, or even to the educated man who is unskilled in mathematics. But you have the satisfaction of knowing that you are rendering a greater service to the community than it is possible for most people to realise. And it may be a source of some encouragement for you to recognise that even among those of us who have little specialised knowledge of your work in its higher reaches there are many who have a high appreciation of its importance and its value both as a great branch of human knowledge and as a fundamental auxiliary to the sciences and the arts.

With your permission, and with no further words other than again wishing that your work may have the greatest possible success, I have now the privilege of declaring this Conference of the Indian Mathematical Society open.

List of Persons

From whom Greetings and Congratulations were received on the occasion of the Eighth Conference and the Silver Jubilee Celebrations.

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1. The President, Calcutta Mathematical Society.
2. The President, Benares Mathematical Society.
3. The Vice-Chancellor, Bombay University.
4. The Hon'ble Sir John Beaumont, Chief Justice.
5. Mr. Justice Kania, Bombay.
6. Mr. Justice B. J. Wadia.
7. Dewan Bahadur R. Ramachandra Rao.
8. Dr. R. P. Paranjpye.
9. Prof. R. N. Apte.
10. Prof. V. Madhava Rao.
11. Mr. R. H. Beckett.
12. Dr. A. Weil, Marseilles.
13. Mr. R. P. Masani.
14. Mr. M. R. Ingle.
15. Sir Dinshaw Wachha.
16. Sir J. B. Petit.
17. Sir C. V. Mehta.
18. Sir H. P. Dastur.
19. Dr. Sir J. J. Modi.
20. Sir Joseph Kay.
21. Sir Reginald Spence.
22. The Rev. Father F. J. Sacasa.
23. Mr. N. V. Mandlik.
24. Mr. G. L. Winterbotham.
25. Mr. L. S. Dabholkar.
26. Mr. S. S. Dabholkar.
27. Mr. V. J. Coltman.
28. Mr. C. B. Nagarkar.
29. Mr. S. N. Pochakhanwalla.
30. Mr. M. J. Antia.
31. Mr. J. G. Ridland.
32. Mr. R. B. Ewbank.
33. Mr. C. W. A. Turner.
34. Mr. S. H. Covernton.
35. Mr. C. B. B. Clee.
36. Mr. C. Y. Freke.
37. Mr. D. C. Pavate.
38. Mr. Lalji Naranji.
39. Mr. A. N. Surve.
40. Mrs. Fyzee.
41. Principal Fyzee.
42. Major S. L. Bhatia.
43. Mr. Marshal.
44. Mr. M. A. Karanjawalla.
45. Mr. J. B. Boman Behram.
46. Mr. Ellis.
47. Dr. D. A. D'Monte.

Report on the Progress of the Society

BY

Prof. S. B. BELEKAR, Hon. Joint Secretary.

LADIES AND GENTLEMEN,

It may not be out of place on this happy occasion to review briefly the early history of our Society:

Prior to its foundation, such Professors and others interested in Mathematics as could afford to do so got a few journals on their own account, but there were no facilities for the interchange of ideas among workers in the same field, and almost everyone was ignorant of what the others were doing. It is a curious fact that a devotee of the subject not himself engaged in teaching should have conceived the formation of a society for bringing together persons engaged in advanced studies and research. It was Mr. V. Ramaswamy Aiyyar, then Deputy Collector at Gooty, who in 1907 addressed a few friends interested in Mathematics for securing facilities for advanced study in the subject by way of Mathematical books and journals. About twenty gentlemen responded and the formation of the "Analytical club" was announced in the Madras Papers on the 4th April 1907. From the very outset the non-parochial and universal character of the Society was in evidence. These first twenty foundation members consisted of two men in revenue service, two Engineers, a Superintendent in the Accountant General's Office, while the rest were teachers in Colleges. Classifying by provinces, there were three Professors from the Bombay Presidency, and the remaining 17 from Madras.

The original idea was only to subscribe for periodicals and to circulate them among the members, but even at that stage higher ideals such as equipment of a Library of standard books of reference and the publication of a journal were kept in view. In spite of his arduous duties as an Executive Officer, the founder spared no pains to promote the interest of the Society and issuing circular after circular had a suitable constitution framed, the affairs of the Society placed on a secure basis by the end of that year.

With the scholar's freedom from provincial bias, the headquarters of the Society was located at Fergusson College, Poona, in the Bombay Presidency. Principal R. P. Paranjpye was made the first Honorary member of the Society and undertook to act as Honorary Librarian which office he held until very recently, when more important duties in the cause of the country forced him to leave the library to his able lieutenant Professor V. B. Naik.

The next two years were taken up by the preparation for the starting of the Journal. With the co-operation of scholars such as Principal Paranjpye, Professor Wilkinson, and others, the first number of our *Journal* appeared in February 1909 under the distinguished Editorship of Prof. M. T. Narainiengar, Professor of Mathematics in the Central College, Bangalore.

Meanwhile the membership was steadily increasing and with it the work of circulating journals and periodicals—in 1908 there were two Assistant Librarians, Profs. V. B. Naik and D. D. Kapadia, both at Poona. A clerk had been appointed, and 30 Mathematical journals and periodicals were being purchased.

The idea of holding conferences was remote in the minds of the organisers of the Society at this stage. They were busy with drafting of constitutions and framing of rules for the efficient working of the circulating Library. By December 1910, the membership had reached 126, the managing Committee had undergone a change in personnel, the Secretaryship had changed hands and the Society had changed its name twice—once from "the Analytical Club" to the "Indian Analytical Club" and then to its present name THE INDIAN MATHEMATICAL SOCIETY.

The Society has been very fortunate in having in the early period of its life, a succession of indefatigable Secretaries to whom the Society owes its rapid progress. We must not however forget that they were strongly supported by learned men like Principal Paranjpye and Prof. Wilkinson and administrators like Dewan Bahadur Ramachandra Rao and Mr. Balak Ram, to mention only a few names.

The Society made steady progress under the paternal care and guidance of its Presidents who were men of experience, the late Mr. Hanumant Rao who was the Professor of mathematics in the Engineering College at Madras, the late Mr. Middlemast, Principal of the Presidency College, Madras, Dewan Bahadur Ramachandra Rao, then a Secretary to Madras Government, Prof. A. C. L. Wilkinson, then Professor of Mathematics and later the Principal of the Deccan College, the late Mr. Balak Ram who though engaged in administrative work could easily give lessons in advanced mathematics to many of us professors at Colleges. Next came Mr. V. Ramaswamy Aiyyar, another administrator whose love and devotion to the subject is embodied in the large number of interesting notes and questions that he has sent to our Journal, Prof. M. T. Narainiengar who ungrudgingly placed at the disposal of the Society, his profound scholarship by editing the Journal for nearly two decades, and finally our present President, Rao Bahadur P. V. Seshu Aiyyar who has served the Society in almost every capacity as Joint Secretary, Joint Editor, and now that he is the President, the Society will make rapid progress under his guidance.

The Society has to-day on its rolls nearly 300 members from all the provinces in India and outside including 9 Honorary members and 23 Life members. Of these about 60 are non-professional, that is, are not actively engaged in the teaching of Mathematics. About 130 ordinary members have been admitted at concessional rates in order to place the services of the Society within the reach of a wider circle of persons. Among the honorary members are eminent Professors of world-wide reputation such as Prof. Whittaker, Prof. G. H. Hardy, Prof. G. A. Miller and Sir C. V. Raman.

Another name whose memory will be fondly cherished, not only by the Society, but by the whole of India, is that of the late S. Ramanujan, F.R.S., and it has been a matter of considerable anxiety to our successive Presidents to institute a befitting memorial in his name.

Donations and other Financial Support.

It is a striking feature of the finances of the Society that its mainstay is the subscription paid by its members. During the last twenty-five years only two noteworthy donations have been received :

(i)	Sir Ratan Tata.	... Rs. 500 in 1910
(ii)	Mr. Balak Ram, I.C.S.	... „ 1000 in 1922

It is, however, a matter of pleasure to record that recently the Universities of Madras, Bombay, and the Annamalai University have sanctioned annual grants ranging from Rs. 100 to Rs. 200. The Society takes this opportunity of expressing its thanks to these bodies for the encouragment they have thus given to it.

The main activities of the Society are connected with—

(1) The central Library at Poona which circulates the Periodicals to the members ;

(2) The publication of the *Journal* ;

and (3) The Biennial Conferences.

I shall take these one by one. As regards the Library, Prof. V. B. Naik has kindly supplied me with the following account of its working since the foundation of the Society for which I am very thankful to him.

The Library of the Society.

Soon after the starting of the Indian Mathematical Club, it was felt desirable that the circulation of mathematical journals and books should proceed from a central place which should also be the Headquarters of the club. The honour of this selection was conferred on Poona, and Dr. R. P. Paranjpye, head of the Fergusson College, was appointed the first Honorary Librarian, with Prof. V. B. Naik also of the same College, as Honorary Assistant Librarian. The whole stock of books and journals belonging to the Club was transferred to Poona and located in a part of the Library Hall of the Fergusson College, which also made the services of one of its clerks and of a peon available at a nominal charge. In 1909, the Library consisted of two large book-cases, containing 247 books, of which 101 had been presented to it, together with 146 volumes of mathematical journals. Since then the Library has been growing both by the addition of standard works on mathematics, and of volumes of journals received in exchange or acquired by purchase. The list attached herewith gives the volumes of Mathematical Journals available in the Library.

Catalogue of Periodical Literature Available in the Society's Library at Poona.

		Years
1.	Abhandlungen aus dem Mathematichen Seminar, Hamburg	1923–1930
2.	Académie des Sciences	... 1907
3.	Acta Mathematica	... 1882–1932
4.	American Journal of Mathematics	... 1907–1932
5.	American Mathematical Monthly	... 1914–1932
6.	Annales de l'école Normale Supérieure	... 1864–1932
7.	Annales de la Faculte des Sciences de Toulouse	... 1887–1923
8.	Annales de l'Observatoire de Paris	... 1885–1913
9.	Annals of Mathematics	... 1899–1932
10.	Astrophysical Journal	... 1888–1932
11.	Bulletin of the American Mathematical Society	... 1906–1932
12.	Bulletin of the Calcutta Mathematical Society	... 1913–1932
13.	Bulletin des Sciences Mathématiques	... 1885–1932
14.	Bulletin des Sciences Mathématiques Astronomiques	... 1870–1884
15.	Crelle's Journal	... 1906–1932
16.	Current Science	... 1932
17.	Jahresbericht der Deut. math. Vereinigung	... 1927
18.	Educational Times	... 1907–1918
19.	Jahrbuch über die Fortschritte der Mathematik	... 1868–1914
20.	Japanese Journal of Mathematics	... 1924–1932
21.	Journal de Mathematiques Elementaires	... 1906–1914
22.	Journal de l'école Imperiale Polytechnique	... 1796–1867
23.	Journal de l'école Polytechnique	... 1854–1894
24.	Journal of the Science Association, Vizianagaram	... 1923–1924
25.	Liouville's Journal	... 1836–1924
26.	L'Intermediaire des Mathematiciens	... 1894–1914
27.	Mathematics from the Educational Times	... 1863–1913
28.	Mathematical Gazette	... 1897–1932
29.	Mathematical Questions and Solutions	... 1916–1917
30.	Mathesis	... 1901–1917
31.	Mysore Half Yearly Journal	... 1927–1932
32.	Mathematische Annalen	... 1869–1932
33.	Messenger of Mathematics	1903, 1907–1930
34.	Monthly Notices of the Royal Astronomical Society	... 1917–1932
35.	Nature	... 1904–1927
36.	Mathematics Teacher	... 1908–1921
37.	Nieuw Archief voor Wiskunde Vols.	... 15–18
38.	Nouvelles Annales de Mathematiques	... 1842–1923

39.	Philosophical Magazine	... 1880–1932
40.	Philosophical Transactions of the Royal Society	... 1906–1932
41.	Popular Astronomy	... 1907–1932
42.	Proceedings of the Edinburgh Mathematical Society	... 1896–1932
43.	Proceedings of the Cambridge Philosophical Society	... 1902–1932
44.	Proceedings of the London Mathematical Society	... 1865–1932
45.	Proceedings of the Physico-Mathematical Society of Japan	1927–1932
46.	Proceedings of the Royal Society of London	... 1905–1932
47.	Publication de la Faculte des Sciences de l'University Masaryk	...
48.	Publications di l'Universidad de la Plata	...
49.	Quarterly Journal of Mathematics	... 1906–1927 1030–31
50.	Rendiconti del Circolo Matematico di Palermo	... 1914–1932
51.	Revue de Mathematiques spéciales	... 1906–1913
52.	Revue Semestrielle des Publications Mathematiques	... 1927–1932
53.	School Science	... 1905–1920
54.	Tohoku Mathematical Journal	... 1906–1932
55.	Transactions of the American Mathematical Society	... 1907–1928
56.	Transactions of the Cambridge Philosophical Society	... 1911–1914
57.	Transactions of the Royal Society of South Africa	1927–1932
58.	Wiskunndige Opgaven Met de Oplessingen Vols.	... 14–18

The small space which the Library Hall of the Fergusson College could afford having proved insufficient for the needs of the Library, the Fergusson College made a large room (28′ x 19′) on the second floor of its amphitheatre available for the purpose. The Library now consists of 450 books, and 1775 bound volumes of journals, stored in 9 large book-cases, together with 125 unbound parts and current journals.

Even now the place is far too crowded with book-cases, leaving little room for those who desire to read in the Library, and a moderate sized building for the Library on a site within the premises of the Fergusson College or in the immediate neighbourhood, is badly needed.

Principal R. P. Paranjpye and Prof. V. B. Naik continued to hold the offices of Honorary Librarian and Assistant Honorary Librarian respectively up to the year 1922, when the former resigned owing to his having accepted the office of Minister of Education of the Bombay Presidency. Prof. V. B. Naik was then appointed Honorary Librarian with Prof. V. A. Apte as Assistant Honorary Librarian. The former has continued to hold the office while Prof. Apte's transfer to the Willingdon College, Sangli, necessitated his being replaced by Prof. S. B. Bondale, and later by Prof. R. P. Shintre, the present Joint Honorary Librarian.

The work of circulation of journals is now being carried on by a clerk working under the direction of the Joint Librarian. From the beginning, this work has presented a number of difficulties which had to be tackled with great care and delicacy. It is needless to go here into a detailed account of how improvement has been steadily effected. In the earlier stages, valuable guidance was received from Mr. V. Ramaswamy Aiyar, the founder of the Mathematical Club and its Secretary. Mention must also be made of a valuable suggestion made by the late Mr. Balak Ram, sometime President of the Society, according to which any journal, instead of moving direct from one member to another in a circulation group, does so through the office of the Library, from which it is re-directed without additional charge to the receiving member. While these, and other improvements, have greatly facilitated the work of circulation it has not yet been possible to eliminate all irregularities. The Honorary Librarian solicits the co-operation of the members in making this part of the work of the Library free from delays, mistakes and losses.

A catalogue of the books in the Library was first published in 1912 and revised and classified both according to authors and according to subjects in 1928. A further revised catalogue is under contemplation.

The following are the names of gentlemen and institutions from whom books and journals have been received in presentation:—

University of Madras
University of Bombay
University of California
Universidad de La Plata
University of Cambridge
University of Sydney
Academie des Sciences, Paris
Cambridge Philosophical Society
Carnegie Institute of Washington
University of Amsterdam
National Research Council of Japan
University of Illinois
Calcutta Mathematical Society
Macmillan & Co., London
Messrs. George Bell & Sons
" G. B. Brown
" B. G. Teubner Leipzig
Prof. E. H. Neville
" G. N. Watson
" C. T. Preece
" Srinivasan
" Rangacharya
" S. B. Bondale

Prof. A. A. Krishnaswami Ayyangar
Sir Thomas Muir
Prof. William Arthur
" R. F. Davis
" Manmohanlal Agarwala
" T. Sunder Rao
Mr. Balak Ram
Prof. S. C. Dhar
" H. P. Petit
" B. P. Reinsch
" G. H. Hardy
Sir Ronald Ross
Dr. N. Kryloff
" Vaidyanathaswami
Mr. G. V. Ramdas
Mr. Venkatasubayya
Prof. Swamynarayan
" D. M. Mehta
" P. V. Seshu Aiyar
" Krishnamachari
" V. B. Naik
" R. D. Karve
Dewan Bahadur Ramchandra Rao

Mr. V. Ramaswami Aiyar

The Society is deeply grateful to all individuals and institutions who have placed their valuable publications at its disposal.

The Journal

The first number of the Journal appeared in February 1909 and the Journal has since appeared every two months thus fulfilling one of the original objects of the Society. The usual features are (i) Original Papers, (ii) Short notes and reviews of books, and (iii) Questions and Solutions.

Many of the questions that have been published have been of considerable interest most of them being original. The whole range of elementary mathematics has been well covered and from the constant stream of questions and the number of those who solve and extend them there can be no doubt that genuine interest is felt by our members.

The Editors of our Journal are giving special attention to the question of making our Journal better known. One of the ways of doing it is by promoting exchange relations with other similar Journals. About twenty-five Journals are, at present, received in exchange.

The following are the names of journals received in exchange :—

1. Abhandlungen aus dem mathematichen Seminar, Hamburg.
2. Académie des Sciences.
3. Acta Mathematica.
4. American Journal of Mathematics.
5. American Mathematical Monthly.
6. Annales de l'école Normale Supérieure.
7. Annals of Mathematics.
8. Bulletin of the American Mathematical Society.
9. Bulletin of the Calcutta Mathematical Society.
10. Bulletin des Sciences Mathématiques.
11. Jahrbuch über die Fortschritte der Mathematik.
12. Jahresbericht der deutsche mathematiker Vereinigung.
13. Japanese Journal of Mathematics.
14. Mathematical Gazette.
15. Mysore Half-Yearly Journal.
16. Proceedings of the Edinburgh Mathematical Society.
17. Proceedings of the Cambridge Philosophical Society.
18. Proceedings of the Physico-Mathematical Society of Japan.
19. Transactions of the Royal Society of South Africa.
20. Current Science.
21. Publications of the Universidad de La Plata.
22. Wiskundige Opgaven met de Oplossungen.
23. Nieu Archief voor Wiskunde.
24. Tohoku Mathematical Journal.
25. Revue de Mathématiques.
26. Rendiconti del circolo Matematico di Palermo.

27. Proc. of the Royal Society of London.
28. Transactions of the Camb. Phil. Society.
29. University of California publications in Mathematics.

Conferences

The success of the first conference at Madras encouraged the Society to arrange periodical conferences and we have been meeting almost every two years. The conferences were held at almost all the University centres. The details regarding the numbers of delegates present and of the papers presented is given below.

			No. of Delegates present	No. of Papers read
First	Conference,	Madras	... 70	13
Second	,,	Bombay	... 50	16
Third	,,	Lahore	... 41	15
Fourth	,,	Poona	... 49	28
Fifth	,,	Bangalore	... 42	30
Sixth	,,	Nagpur	... 39	39
Seventh	,,	Trivandrum	... 75	31
Eighth	,,	Bombay	... 94	35

There is usually besides the reading of papers, a programme of public lectures of a popular kind and discussions on topics of common interest such as methods of teaching, curricula of Elementary Mathematics, and teaching of mathematics in Indian Universities. These conferences provide occasions for the members from all parts of India to get into personal touch with one another.

In conclusion, the Society is deeply indebted to the Bombay University for the liberal grant towards the expenses of the conference and to His Excellency who has been kind enough to grace the occasion with his presence.

The Managing Committee wishes to place on record their thanks to the Local Reception Committee in general, and to the Chairman in particular, for their hospitality and excellent arrangements for the guests. Our special thanks are due to Mr. V. N. Chandavarkar, Bar-at-law, the popular Mayor of Bombay for the keen interest he has evinced throughout the session of the conference and for being a Vice-Patron, and to him and to Mrs. Chandavarkar for the magnificent "At Home," they gave at Mount Pleasant Road.

Also the Managing Committee appreciates the efforts of Prof. K. R. Gunjikar and his band of volunteers which have contributed so much to the success of this conference.

PRESIDENTIAL ADDRESS

BY

RAO BAHADUR P. V. SESHU IYER, *President of the Society.*

Ladies and Gentlemen,

I consider it a high honour and a great privilege to be assigned the function of delivering the Presidential Address before such an educated and enlightened audience. This Conference of Mathematicians and those interested in Mathematics is the eighth conference convened under the auspices of the Indian Mathematical Society, at the kind invitation of the University of Bombay; and with it is also to be associated the celebration of the Silver Jubilee of the Society which, founded in 1907, has completed 25 years of its existence. To have the honour of presiding over this Jubilee Conference is indeed a high privilege. But every privilege is also accompanied by a great responsibility, and I feel I am not competent to discharge that responsibility efficiently and satisfactorily, and I would have very much liked that some one like Dr. Paranjpye, abler than myself, had been chosen for the task. But there is no helping it since it is the *mamool* that the President of the Society delivers the Presidential Address at the Conference. If there are any shortcomings in my address—and many shortcomings are bound to be here—I request you will be pleased to overlook them, remembering that you have put me on to this position.

Before I proceed to the main part of my address I beg leave to express, on behalf of the members of the Indian Mathematical Society our feelings of thankfulness to His Excellency, the Governor of Bombay, for having condescended to open the Conference with an interesting and inspiring speech; and to The Hon'ble Mr. Justice Mirza Ali Akbar Khan for welcoming, on behalf of the University, the delegates and others to the City of Bombay and to this Conference with such warmth and heartiness. In this connection, I may mention that it is in the fitness of things that we meet in Bombay for the celebration of the Jubilee of our Society. For, Bombay is virtually the headquarters of our Society; in fact, in the very first letter announcing the formation of the Society, when Poona was proposed for the headquarters, it was pointed out that, 'Poona is next to Bombay a postal centre for all India, and it is practically Bombay as regards the rest of India.' Further, Lord Sydenham, then the Governor of Bombay, was its Patron. It is gratifying to hold the Jubilee Conference here in Bombay and to have it opened by the official successor of our first Patron, His Excellency Sir Frederick Sykes. Thus our Conference has begun its proceedings under very good auspices, and on behalf of this Society I assure His Excellency that every word of his speech will be treasured up with great devotion in our minds.

On the occasion of this Jubilee Celebration, it is desirable that we briefly survey the life history and work of the Society with a view to know what has been achieved in the past and what remains to be done in the future towards the realisation of the object of our Society, viz., the promotion of mathematical study and research in India. Hence with your permission I proceed to make such a survey.

Mr. V. Ramaswamy Iyer, the founder of our Society, whom we are happy to have in our midst today, and who must be equally happy to take part in these celebrations, having conceived the idea of forming a small mathematical society, sent out on 25th December 1906 a proposal to a few mathematicians in the following terms:—

> "I believe several friends interested in mathematics have felt the present lack of facilities for seeing mathematical periodicals and books. This is a very great disadvantage we are suffering from. I propose, therefore, that a few friends may at once join and form a small mathematical society and subscribe for all the important mathematical periodicals, and as far as possible, for all important books in higher mathematics. We may call the society "The Analytic Club" for the present and have it in view to give it a broader basis with a suitable name by and by. Our work immediately will be to obtain all the important periodicals and new books and circulate them to members.
>
> * * * * *
>
> If half a dozen members could be counted upon to join immediately and each subscribe Rs. 25 per annum, we shall be able to make a good start. Members should be prepared to make a further sacrifice; each member should send the journals and books he receives on to the next. This, in effect, would be to add Rs. 5 to one's subscription. I hope friends interested in Mathematics will not consider this a too heavy sacrifice . . . . . I propose to consider the Club formed as soon as three friends have agreed to the proposal making with me four members."

The founder's enthusiasm for the formation of the Society was such as to make him resolve on forming it even with only four members; and his humble and immediate aim was to subscribe for a few mathematical periodicals and circulate them to members. After a short period of three months, he announced in the Madras Dailies the formation of the Society on 4th April, 1907, with 20 gentlemen enrolled as members. Such was the humble origin of the Society; and it is fortunate that 12 of those enthusiasts prepared to make such sacrifices for the formation and maintenance of the Society, have been spared to us till now, and we must feel happy to have five of them present in our midst to-day to witness this celebration.

After the Society was formed in 1907, new members were enlisted, a constitution was framed and progress reports of the Society were issued every two months. In the progress report of August 1908, a few original questions set by the members were published for solution, and in the report of October 1908, for the first time, an original article "On the Cardioide" by Principal Paranjpye, now Vice-Chancellor of the Lucknow University, and another on "The Nine Points Circle" by Mr. M. T. Naraniengar appeared. The enthusiasm evinced by the members to send in questions and solutions and to contribute articles to the progress report prompted the Managing Committee to resolve to issue regularly a journal of the Club as its organ. The Club having in the meanwhile changed its name into the "Indian Mathematical Club," the first issue of the journal appeared in February 1909 under the title of "*The Journal of The Indian Mathematical Club.*"

The idea of the regular issue of a journal was not in the original letter announcing the formation of the Society. It was the enthusiasm shown by the members that led the Committee to insert in the constitution of 1908 of the Club the following provision:—

> "That the Committee may take such steps as they may deem fit towards the development of a mathematical journal as the organ of the Society."

Soon the Committee resolved to issue a Journal regularly and the first issue as mentioned above came out in February 1909. At that time the strength of the Society was 79 and the number of mathematical periodicals subscribed for by the Club 34.

The development of the Society as evidenced by the increasing strength, the enthusiasm of its members and the quality of the journal was so encouraging that early in 1911 the name of the Society was changed into "The Indian Mathematical Society" and a new constitution was adopted. Practically it is that constitution slightly modified that is in force even now. In 1911, the strength was 132 and the number of periodicals obtained rose to 40.

Thus we had first the *Library* and the circulation of books and periodicals and then a *Journal* as the organ of the Society. The next stage of development of the Society consisted in the holding of periodical *Conferences* under the auspices of the Society. A few years after the formation of the Society and specially after the regular issues of the journal began to appear, the members became naturally anxious to meet in person the several friends who contributed to the journal questions, solutions, short notes and the original papers; and an attempt was made in 1913 to hold a conference; and though the members were anxious to meet one another, there was not a sufficient number of members coming forward to read papers before the conference, and since the reading of papers was considered to be an essential feature of a conference, the idea of a conference had to be dropped. The question was taken up again early in 1916. Then too there was not an encouraging response to the appeal for original papers and the idea was about to be given up. At that time having known, as Assistant Secretary, the intense feeling amongst the members for a conference and fearing that the dropping of the idea of the conference would be a great disappointment to members, I offered to run the conference with success if it should be held in Madras, and I undertook to get at least a dozen papers for the conference. My offer was readily accepted by the then President, Diwan Bahadur R. Ramchandra Rao and the Secretary Prof. D. D. Kapadia, whom I am glad to see now in our midst today. Accordingly the first Conference was held in Madras in December 1916; it was opened by H. E. Lord Pentland, the Governor of Madras. The Bombay Government offered facilities to its mathematical officers to attend the conference by granting them leave and travelling allowance. There was a satisfactory gathering of members and 13 original papers were read by 11 different authors, and there were in addition a number of excursions and socials. The following were the remarks made at the Conference by Principal Wilkinson who delivered the Presidential Address on the occasion:—

> "I consider it a great privilege to realise that I have now become personally acquainted with many known to me previously by name or reputation only . . . . . . I doubt not that, until our next meeting, we shall cherish pleasant recollections of this present gathering and endeavour to maintain by correspondence that personal interest that must necessarily be created during the course of the few days we are able to associate together."

Equally great was the satisfaction that the other members felt after the Conference. Incidentally, the Conference stimulated some of our members to attempt to produce original papers to be read before the Conference; and the Editor of the journal too was glad to find in these papers enough matter for the journal for at least a year. The net result of all this was that the Managing Committee of the Society decided to convene conferences once in two years. Ever since, we have been holding conferences almost regularly once in two years. The conference work of the Society, as I shall call it, thus formed the third stage of its development.

So much for the past. Coming to the present and the immediate future, there are two fields of work demanding the immediate attention of the Society. One is the conduct of an enquiry into the present condition of mathematical teaching in our schools and colleges and the nature of the examination papers set at the University and other public examinations, with a view to bring about some wholesome reforms in connection therewith. The other is a better organization of the research work in mathematics that is now attempted individually and spasmodically by our members and others capable of such work. These two fields of work have been left to me, as President, by my predecessor as a legacy, as he put it. On these I shall dwell in detail later on.

Now, then, to summarise our past work and to survey briefly our present position :—

1. We have gone on enlisting new members and our strength, increasing from the original 20, now stands at 300 in round figures, consisting of 9 honorary members, 23 life members, and 266 ordinary members.

2. Our library commencing from the purchase of a few periodicals for circulation contains now 2,225 volumes of which 450 are books on higher mathematics, and 1775 are back volumes of periodicals, and we are getting nearly 50 journals for circulation amongst our members, of which I am glad to say about 30 are being received in exchange for our journal.

3. Our own journal has completed 24 years of existence appearing once in two months during this period. It is now on a fairly high level attracting exchange copies of high class journals.

4. We have been holding our conferences almost regularly once in two years, thus effecting that personal contact amongst the several workers, which contact is so very useful in stimulating research work, apart from the social amenities arising therefrom. Incidentally, these conferences stimulate our members to attempt to produce original papers to be read before them, and these papers form good matter for our journal.

Thus, Ladies and Gentlemen, we have every reason to be proud and jubilant over our past work and present condition on this Silver Jubilee celebration day of our Society. If we have reason to be proud and jubilant it is but our duty to refer thankfully to the several gentlemen who have mainly contributed to bring about this condition.

First and foremost is the founder, Mr. V. Ramaswamy Aiyar, M.A., whom we are fortunate to have in our midst today and but for whose unabating enthusiasm and indefatigable energy the Society would not have come into existence and developed to such an extent. Rightly we have paid our tribute to him, though very inadequately, when we presented him with an address in token of our appreciation of his services.

Next, and quite on a par with him, comes Mr. M. T. Naraniengar the first Editor of our journal, who took up that work as a labour of love and stuck to that post for full 18 years and spared no pains to bring it up to the high level it occupies today. And it is but proper that we present him with an address on the occasion of this celebration of the Silver Jubilee of our Society. Then I must mention Dr. R. P. Paranjpye the first librarian of our Society who admirably organised the library and the circulation of the periodicals and who in addition was the right hand of Mr. V. R. Aiyar and an earnest collaborator with the editor, specially in the earlier days of our journal. Then the gentleman who deserves to be specially mentioned is Mr. D. D. Kapadia who did yeoman service as the Secretary of the Society for full twelve years from 1910 to 1922, and it was during his regime as Secretary that the Society developed in all the directions mentioned just now. The gentlemen who next deserve special mention are the successive Presidents, Messrs. B. Hanumanta Rao, R. N. Apte, E. W. Middlemast, R. Ramachandra Rao, A. C. L. Wilkinson, R. Balakarm and the successive treasurers Messrs. K. J. Sanjana, C. Pollard and S. Narayana Iyer. I have for obvious reasons omitted to refer to the present office-bearers and other active workers.

Having dwelt at some length upon our past work and present condition, I shall now pass on to the two topics already reserved by me to be dealt with in detail.

The first is the conduct of an enquiry into the present condition of mathematical teaching prevalent in our country and the nature of the papers set at the public examinations. The subject was touched upon at some length by Mr. A. C. L. Wilkinson in his Presidential Address at the Second Conference held in Bombay in January 1919, and at that Conference a Committee was formed to conduct the enquiry, but unfortunately that Committee never worked. Again, at the last Conference at Trivandrum a discussion on mathematical teaching and examination took place and as a result thereof the Managing Committee was asked to take the necessary steps to conduct the enquiry. The Committee has not proceeded far in the matter since the Secretary appointed, viz., Dr. Weil, left our country and since the President appointed, viz., Mr. R. Littlehailes, the late D. P. I. of Madras, has also now retired from service. Thus it will be seen that this work has not been taken up in right earnest by the Society. Unless the members of our Society and others interested in the improvement of mathematical teaching realise the need for such an enquiry and co-operate in the work with earnestness, nothing can be achieved. It is just to make them realise the need for such an enquiry that I propose to dwell at some length on that topic.

This conduct of an enquiry and report on mathematical teaching is one of the methods mentioned in the constitution itself for the furtherance of the aims of the Society, viz., promotion of mathematical study and research in India, and it is satisfactory to note that there is to be a discussion on the teaching of mathematics in Secondary Schools at this Conference. As one who has bestowed some thought and attention to it, I may, as an introduction, touch upon some important aspects of mathematical teaching in our schools. What I may be saying now is based upon my experience in the Madras Presidency and the adjacent States, but I believe that more or less the same conditions prevail elsewhere.

It is a common complaint with all that mental work, specially mental arithmetic, has been greatly neglected, so much so that a student cannot mentally multiply two numbers containing two digits each and that even to do the simplest operations he wants pencil and paper. Our ancients were clever in mental work; they could mentally calculate in no time the interest on capital, the price of any quantity of an article at a given rate, etc. They could mentally do an ordinary problem in square and cubic measure. Even now the old people in the villages do such work mentally, but our present day students require paper and pencil for such work and very often they go wrong.

One consequence of this neglect of mental arithmetic is the non-prevalence of the practice of rough checking of the results arrived at to see if they are at all likely. For want of such rough checking, in the examinations our students are found to give very absurd answers; a result which is to be in tens gets to be given in lakhs, or one which is to be in thousands of a unit comes to be given in thousandths, say through an oversight in the placing of the decimal point.

Another undesirable practice now obtaining in school teaching is the drill in long and complicated or tricky problems, drawn from imagination and having no relation to the real life around and the simultaneous indifference to problems occurring in real life. This is mainly due to the neglect of teachers to observe real life and to find out what problems occur therein. A movement was set on foot in America some twenty-five years ago to find out the applied problems that occur in real life and to publish such problems in journals like the School Science and Mathematics. Some such movement must be started in India and real applied problems published in journals devoted to elementary mathematics or in the elementary portion of our journal.

A third defect in our school teaching is that elementary mathematics is taught in India without any reference to the History of Indian Mathematics. The subject is handled in our classes and text-books as though there was no mathematics in ancient India. The subject consequently grows like an exotic plant in our country and is rendered dull and uninteresting. On the other hand, if the methods and processes in vogue in ancient India be given some prominence in the handling of the subject, giving the names of ancient Indian mathematicians in whose works such methods are to be found, it would rouse considerable interest in the students for the subject and also a feeling of patriotism will be ingrained in our young students. The other day I read in an educational magazine that even in teaching plant and animal life teachers in England do infuse a feeling of patriotism by saying that this plant is *our* plant and this animal is *our* animal meaning thereby the plant or animal found in their motherland.

Of course, such mixing up of ancient Indian methods and processes with those found in English and foreign books requires some effort on our part to study such methods as were in vogue in ancient India and to fit them into the methods now in use. If the principle of setting our mathematical teaching before a historical background with historical references be recognised, there are our friends like Mr. A. A. Krishnaswamy Aiyangar quite willing to do such work for us.

Another defect in the mathematical teaching is the want of attention on the part of the teachers to the fundamental concepts of mathematics. It is a well-known fact that in recent years mathematics has grown both at the top and at the bottom and it is very desirable that our teachers in Secondary Schools should be familiar with the main work done at the foundations of mathematics so that they may be able to have clear ideas for themselves and to avoid imparting wrong notions to their students which they may have to unlearn at a later stage. The pity is our teachers do not realise that they themselves have to learn much as regards the fundamental concepts. A teacher of the lower secondary stage once boasted to me that he knew all the fundamentals required for his classes. Then I asked him what exactly is meant by the formula: Area = L × B. Is there any meaning in multiplying a length by a length? Then he realised his ignorance. Again, another teacher of the high school stage, a graduate in mathematics, once told me that he was satisfied that he knew all the fundamentals required for high school mathematics. I then questioned him regarding the proof of the Remainder Theorem given in the text books which is usually as follows:—

When $f(x)$ is divided by $(x-a)$, let Q be the quotient and R the remainder. We have

$$\frac{f(x)}{x-a} = Q + \frac{R}{x-a}$$

multiplying by $x-a$ we have

$$f(x) = Q(x-a) + R\ .$$

Putting $x=a$ we have

$f(a) = R$. Thus the Remainder is the value of $f(x)$ when $x=a$.

I questioned the teacher thus, "since division by zero is excepted, the first step which involves the division by $(x-a)$ has a meaning only when $x \neq a$, and proceeding from that step at the third step, x is put equal to a. How far is this reasoning logical?" He was non-plussed and confessed his ignorance. The same story we have when we consider some of the college lecturers. Many such instances of self-satisfaction on the part of the teachers of mathematics can be cited, but this is not the place nor the time for such citation. The point is we must make our teachers realise the need for their acquaintance with the fundamental principles of the subject, so that mathematical teaching may be based on clear notions on the part of the teachers. In this unfortunately, the ordinary text books used in the schools and colleges, even books written in England, do not help us much; they even mislead us. So we have to be careful in the selection of our text books.

Again, one chief aim of mathematical study is to lead the students to clear thinking and accurate reasoning, which are so very essential to every civilized man. Hence subjects like Demonstrative Geometry or Algebra must be taught so as to lead to such clear thinking. There are many improvements suggested in the teaching of geometry by the London Mathematical Association and in the Board of Education Circulars, and many text-books have been written on lines suggested by them. But still I am afraid things are not what they ought to be. So far as geometrical teaching is concerned, on account of too many books following different sequences and the difficulty caused by a foreign language being the medium of instruction, the teaching adopted does not secure the end. No doubt students do get by heart and reproduce the geometrical proofs in the examinations but few of them can be said to have clearly grasped the logical reasoning involved. This state of affairs demands an enquiry and methods must be suggested for securing a better teaching of the subject.

Then mathematics is often considered a dull and uninteresting subject and is dreaded by many pupils. We mathematicians must devise methods of handling the subject so as to remove that horror and make it really interesting. My own belief is that, granted some common sense on the part of the pupils—there is that common sense in most of them—we can so handle this subject as to rouse their interest and secure their attention and thus make the subject loved by them. One of the means is to take advantage of their spontaneous activity and make them do things for themselves. They must be encouraged to collect data from real life around and set problems to one another. Also with simple apparatus and measuring instruments they must be made to observe and measure angles, lengths etc. and thus procure data for problems in square and cubic measure and mensuration. As already mentioned by me, applied problems connected with real life must be manufactured and given by the teacher himself. Further, the students must be shown the usefulness and application of mathematics to other subjects, such as Physics, Geography, Domestic Science, etc.

This last point leads me on to dwell a little on the correlation of different branches of mathematics with one another and of mathematics with other subjects. Till recently, the several subjects of mathematics like arithmetic, algebra and geometry were kept in water-tight compartments and the methods of one were not to be applied to another. For instance, algebraic methods were not to be applied in solving problems in arithmetic and algebraic symbols and formulae were not to be used in geometry. But now things are gradually changing for the better. However, even now, in the earlier stages, such as the elementary and lower-secondary, there is not that correlation generally effected. A lot of concrete work as in Kindergarten and Montessori methods must be done closely associated with the teaching of arithmetic in the elementary schools. Unfortunately, these methods are not taught at all, and even if taught, they are done without any relation to what the teacher of arithmetic does in his classes. We must see that this correlation of different branches of mathematics is effected from the earliest stages. In this connection, I may say that in ancient Indian mathematics, these subjects of arithmetic, algebra and geometry are all mixed up and the whole is dealt with as one subject of "*Ganitham*". For instance in the "*Ganita-sara Sangraha*" of Mahaviracharya there are nine chapters dealing with the following subjects: the arithmetical operations of multiplication and division, simplification of fractions, extraction of square and cube roots, summation of arithmetic and geometric series, solution of simple, quadratic and indeterminate equations, mensuration of plain and solid figures, the geometry of the shadows. In addition to this correlation of different branches of mathematics, the teacher must try to correlate his mathematical teaching with the other subjects of physics, geography etc. by ascertaining from the teachers of those subjects what mathematical problems occur in their subjects and giving such problems to his classes.

I am afraid I have dwelt a little too long upon the teaching of mathematics in Secondary Schools; I now proceed to make a few remarks on the teaching of mathematics in Colleges and Universities.

As already mentioned by me, much has been done in recent years 'at the foundations of mathematics. No doubt serious difficulties attach to such topics as irrational numbers and ratios, complex numbers, limits, the notions of infinity, continuity etc. These difficulties are made evident by the fact that, in spite of the attention these topics had received during several centuries, a satisfactory treatment has been found only within the last 50 years. But the difficulties have been overcome, and it is desirable that every student of higher mathematics should be acquainted with the underlying concepts and a satisfactory treatment of those topics. But I regret to have to remark that, judged from the answers of candidates in the Honours examinations and from the recent books written by Indian authors who are all Honours men or M. A's, these topics are not well or clearly grasped by our Honours students. In many cases there is not even an attempt to understand the principles involved because it does not directly pay in the examinations. There is a tendency on the part of the students, as complained of by Prof. Hardy in the preface to the first edition of his book '*A Course of Pure Mathematics*,' to rush through the book work and the underlying principles in order to pass on to exercises bearing on the book-work. This is in a way countenanced by the examiners through the nature of their papers. It is a pity that in our country, examinations are given too much importance; they are made to largely determine our classroom work through such undue importance.

The examination papers are such that a student may be specially prepared or crammed for the examination and get a creditable pass without his understanding the fundamental principles.

The evils of examinations are well known; they are no sure test of a man's knowledge. At best, they are a necessary evil. Nobody has yet devised any other satisfactory method of testing a student's knowledge, and hence they are necessary, but in regard to the evils they must be remedied and it is certainly a hard and difficult task. A number of heads must be put together to suggest improvements in the examination system. At the last Conference, the whole question was well discussed, and as a result thereof, a resolution was passed, viz., that it be a recommendation to the Committee of the Society to take immediate steps to institute an enquiry into the present state of mathematical teaching and examining in Indian Universities. I hereby appeal to you all to render every co-operation to the Committee in the conduct of that enquiry.

My own opinion in regard to the examination papers is that the old Cambridge system of giving a number of book-works with one or two problems under each should be given up. Even in Cambridge that system has been given up. We must have more searching questions set so as to find out whether the students have clearly grasped the fundamentals involved. For this we must have some essay questions requiring a connected discussion of the topics questioned upon. How many such essay questions should be given in a paper will depend upon the topics and the mental capacity and development of the students. If possible, we must also have some papers of such a nature as may be allowed to be answered with the help of books taken into the examination room. Since there is to be an enquiry into the whole system, I do not take up any more of your time by dwelling on this subject, but shall proceed to the next topic reserved by me for consideration, viz., the better organisation of mathematical study and research.

I am sure you will all agree with me when I say that we in India have not studied all the modern subjects that are engaging the attention of European and American mathematicians. There are many branches of mathematics such as groups, several kinds of functions, transformations, differential geometry, tensor calculus, relativity, integral equations, calculus of variations, statistical mechanics, etc. etc., which are not studied in India to any great extent. Of course, there are stray mathematicians here and there acquainted with some of them: all credit to them.

It is very desirable that the Society as a body or the members in groups take the necessary steps to promote the study of the several subjects by our young men. Our founder, Mr. V. R. Iyer, in his Presidential Address at Bangalore in 1926, did throw out some suggestions as to how we may by forming reading circles, promote the study of some of these subjects. I wholly endorse his suggestions. But since little or nothing has yet been done in that direction, I propose to quote his own words once again here and appeal to you to give effect to his scheme. The following are his very words:—

> "We should try to see that not only certain branches of mathematics but the whole field should be cultivated by our members, based on the principle of study in close association. The whole field is recognisable as falling into so many divisions and sub-divisions, which are not water-tight compartments, but have vital connections with one another. I should like to place before members the idea that there should be a few of us studying each of these divisions of fields. Taking any one of these fields, the progress that can be made in its study will be generally as follows: first one must be eager to enter the field. At this stage he is a mere *entrant*, an *embryo*. Then

by study he develops into a *learner*. After this stage with some enlightenment he becomes an *interpreter* of the subject. In this stage, he sees that many of the things learnt are not essential and are mere cobwebs in the mind. He sweeps them out and takes hold of the essential. Finally, one rises to the stage of a *master*. Here he brings a fresh light of his own into the subject, becomes a discoverer and extends the scope of the field. We can rarely rise to be masters, but we can all be *learners* and possibly rise to be *interpreters* by close study. All these stages of progress are symbolised by the letters of the word 'MILE'. M is the stage of the master, I, the stage of the interpreter, L the stage of the learner and E the stage of the entrant or embryo in the subject. Applying this philosophy in order to cultivate any particular branch of mathematics in our Society, I would like to catch some of you, younger members of the Society, as entrants and put you into a compartment, stock around you all the books on the subject, find a senior member, if possible, to serve you as interpreter and find you also a master, if possible. I want to see created in our Society many such groups of interested members, of all grades of advance, to study the different fields of mathematics, so that none is neglected in our Society. Success can only be slowly realised; but we can make an effort from now to secure our progress in all branches, by means of such compact groups consisting of masters, interpreters, learners and entrants. It may be said we have not got masters or interpreters for many subjects. But the Committee have power to appoint honorary members, and we can cast about the wide world for finding such men to assist us."

Thus his scheme is quite complete and ship-shape; only we must have the earnestness to work it up. On the occasion of this Jubilee of our Society, the greatest tribute we can pay to our founder—a tribute which would please him most, is for us to solemnly resolve to give effect to his pet scheme by each one of us offering himself to be enrolled as an entrant, learner or interpreter, as the case may be, for some one branch or other of mathematics, thus forming compact little groups to study the different fields of mathematics. In thus asking you to resolve, I would specially appeal to those who are engaged in the teaching of mathematics, whether as teachers, lecturers or professors. They are the people who ought to regard themselves as wedded to mathematics and to feel bound to do their best for the promotion of its study in all its branches. They have also got greater access to books and periodicals than others not directly engaged in such teaching. Also many mathematical teachers are to be found in any University or College area and hence they can more easily meet and discuss on points occurring in such studies. I hope that at the business meeting of our Society, the necessary steps will be taken to give effect to the scheme.

Such an intensive study in groups of different branches of mathematics must necessarily lead to much research work on the part of our members and the consequent production of original papers on mathematical subjects; and incidentally, our journal too will have ever flowing matter contributed to it so as to enable it to gradually improve in quantity as well as in quality. With a view to encourage such deep study and research, it is the earnest wish of the Society, specially of my predecessor in office Mr. M. T. Naraniengar, that we should raise a fund, to be called the 'Jubilee Fund', in order to enable the Society to found a Research Prize, to be awarded by it once a year or once in two years as funds permit. With this view an appeal was sent to the members of our Society to subscribe liberally for the Jubilee Fund, but till now only about Rs. 500/- have been collected. I take this opportunity of making a special appeal to our members and others interested in mathematics to liberally subscribe for the fund so as to enable the Society to institute a Research Prize and promote mathematical research in our country.

Gentlemen, I have given above a brief history of the Society, what it has done in the past and what has to be done in the immediate future. That is all-right; but before I close I must also refer a little to our present financial position and to the indifference to the Society shown by some of our members. I do this with great regret, but with the hope that with a public appeal from me and with your co-operation things will gradually improve.

I am sorry to say that at present we have only Rs. 6790 to our credit in the bank. Of this, Rs. 3,450 represent the total of the composite sum paid by our life members, which sums were meant to be kept as capital, the interest alone to be spent on current expenses. Though we have today 265 ordinary members, only about 122 are paying their annual subscriptions regularly. The others are in arrears and some of them awfully, so that, though on paper our strength is 265 today, the subscriptions realised last year and this year were much smaller than the collections under subscriptions from members in the year 1919, when the number of ordinary members was only 186. This shows that a larger proportion of our members than before are in arrears now. An earnest attempt will soon be begun by the Committee to collect these arrears and also to enroll new members, and I would take this opportunity of appealing to all the members to persuade the members already enrolled to pay up their arrears and also to persuade others interested in mathematics to join our Society.

In conclusion, I beg to appeal to all those present here and others interested in mathematics to render their hearty co-operation to the Indian Mathematical Society in its efforts to promote mathematical study and research in India, and in the words of its founder, "to hold the banner of mathematics aloft, as the Motherland marches to glory with the rest of the world."

Business Meeting

Minutes of the business meeting held in Bombay at 12-30 P.M. *on 21st December 1932.*

RAO BAHADUR P. V. SESHU AIYAR, I.E.S. (*Retd.*) *in the chair.*

1. A proposal to collect contributions and meet the cost of printing the Society's Jubilee Memorial Volume was moved by Prof. K. S. K. Ayyangar and unanimously accepted. Rs. 50 were immediately collected and promises for Rs. 235 have been given. (*Vide* attached list.)

2. The sense of the meeting was in favour of the publication of the Journal in two separate parts. The question of the amount of the subscription for each part was left to the decision of the Managing Committee with the suggestion that the subscription for each separate part should not exceed Rs. 5 and the subscription for both parts together should not exceed Rs. 9.

3. The meeting recommended, at the suggestion of Prof. Kapadia, Prof. Naik and Dr. G. S. Mahajani, to the managing Committee to consider the feasibility of charging different rates for Part I and Part II. Prof. Arunachala Sastri, Prof. Naik and Prof. Kapadia were in favour of making the elementary part cheaper. Also it was suggested that the second part should be run on lines similar to the *Mathematical Gazette* and the *American Mathematical Monthly*. The meeting could not finally decide these matters in the absence of the Joint Editor and the Treasurer. So it was left to the managing Committee to finally settle the details about the subscription, size and periodicity of each part.

4. It was resolved to publish a special appeal for funds, with the recommendation that the special appeal should occupy a prominent place and be printed on coloured paper so as to attract the attention of every member.

5. The next item before the meeting was Prof. Hansraj Gupta's suggestions regarding the composition fee to be charged for Life membership. The meeting was in favour of the principle involved but as the Treasurer was not present at the meeting, it could not be ascertained how far the proposed scale would adversely affect the revenue of the Society. So it was unanimously resolved to refer the matter to the Managing Committee with the recommendation that an attempt be made to evolve a graduated scale of compounding payments on the following lines :—

	Rs.
At the end of 10 years' membership ...	... 100
" 15 " " ...	... 75
" 20 " " ...	... 50
" 25 " " ...	... Exempted from further payments.

6. It was suggested by some members that the Managing Committee should consider the feasibility of appointing Agents at different centres to collect subscriptions for the Society.

7. The President proposed a vote of thanks to the Principal, Royal Institute of Science for the use of the buildings for the Conference, which was unanimously passed.

Special donations for the Jubilee Volume.

The following donations have been received so far:—

	Rs.		Rs.
Dr. G. S. Mahajani	15	Prof. S. K. Abhyankar	5
„ K. R. Gunjikar	15	D. M. Mehta Esq.	5
Father R. Rafael	10	S. P. Kharas Esq.	5
T. K. Venkataraman Esq.	10	N. D. Doctor Esq.	5
Prof. M. L. Chandratreya	5	Prof. P. K. Kashikar	5
„ J. N. Dharap	5		
„ M. K. Kewalramani	5	Total ...	95
„ S. M. Shah	5		

The following donations have been promised:—

	Rs.		Rs.
Rao Bahadur P. V. Seshu Aiyar	25	S. S. Pillai	5
Prof. K. S. K. Ayyangar	25	A. A. Krishnaswamy Ayyangar	5
Mr. V. Ramaswamy Aiyyar	15	Prof. B. B. Bagi	5
Prof. M. V. Arunachala Sastry	15	A. L. Shaikh Esq.	5
„ D. D. Kapadia	15	Prof. G. V. Bhagwat	5
Prof. M. T. Naraniengar	10	„ D. B. Patravali	5
Dr. R. Vaidyanathaswamy	10	„ K. D. Panday	5
Prof. S. B. Belekar	10		
„ V. B. Naik	10		
„ A. Narasinga Rao	10	Total ...	190
Principal Hemraj	10		

Discussion on School Mathematics

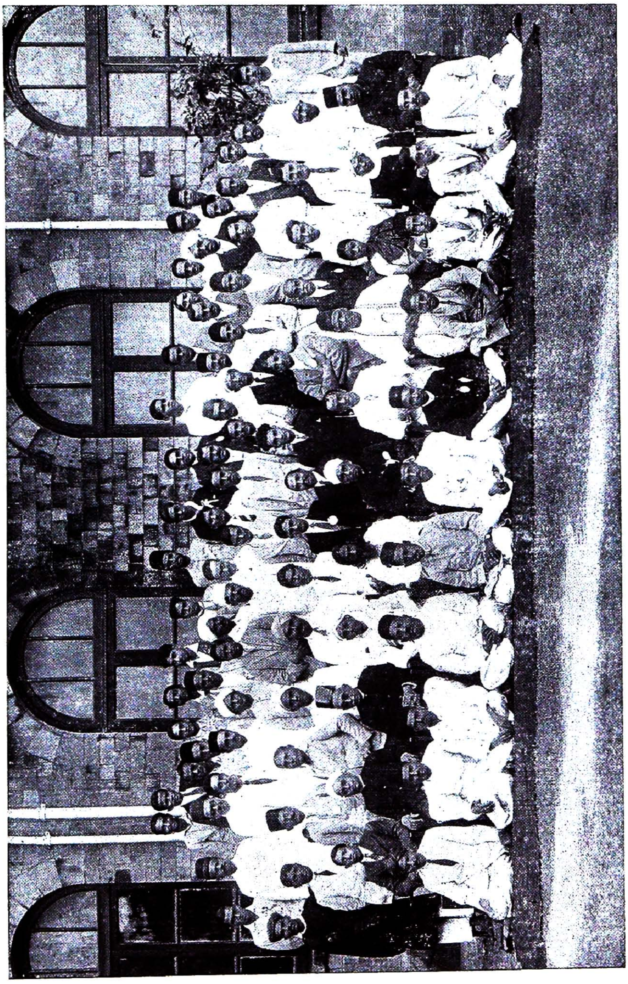
THURSDAY, 22ND DECEMBER 1932, 2 P.M.

The following outline was drawn up and circulated to the members before the meeting.

Outline of discussion suggested.

School Mathematics

1. Stages:
   - (*a*) Middle school stage;
   - (*b*) School Leaving stage:
     - (i) for those continuing their studies in Mathematics further.
     - (ii) for others.



The Indian Mathematical Society

EIGHTH CONFERENCE, SILVER JUBILEE SESSION, BOMBAY
22ND DECEMBER 1932

TOP ROW :—Messrs. M. V. Divatia, A. D. Lawrence, B. V. Patil, S. H. Bhat, P. S. Paralkar, G. M. Fakih, S. S. Kavalekar, S. S. Ghadiali, B. R. Shah, K. R. Gunjikar, G. R. Paranjpe, D. P. Patravali, C. J. Shah, A. R. Rao, J. R. Rana, L. H. Marathe.

2ND ROW :—Messrs. P. V. Dandekar, N. D. Doctor, S P. Kharas, A. L. Shaikh, V. A. Pandit, B. B. Bagi, R. Rafael, K. Nagbhushanam, G. V. Bhagvat, A. R. Sayed, K. N. Wani, C. R. Chaturvedi, P. K. Kashikar, S. R. Shaikh, M. V. Pandit.

3RD ROW :—Messrs. D. D. Vania, P. N. Sukeshwala, R. Vaidyanathswamy, L. S. Vaidyanathan, J. N. Dharap, D. M. Patel, B. S. Kalelkar, F. H. Gracias, C. N. Srinivasa Iyengar, B. S. Madhav Rao, S. Sastry, I. Mathai, G. S. Mahajani, K. M. Shah, G. L. Chandratreya.

BEHIND CHAIRS :—Peon, Messrs. B. K. Wagle, K. C. Shah, M. K. Kavalramani, A. K. Krishnaswamy Iyengar, K. D. Panday, S. S. Pillay, D. M. Mehta, S. M. Shah, R. Siddiqui, K. S. K. Iyengar, T. Buell, N. M. Shah, T. S. Wheeler, B. G. Nadkarni.

CHAIRS :—Messrs. Ram Behari, M. V. Arunachala Shastri, D. D. Kappadia, M. T. Narayaniengar, A. V. K. Menon, P. V. SESHU AYYAR (President), V. N. CHANDAVARKAR, (Vice-Patron), V. Ramaswami Aiyar, Mrs. Subramanyam, Messrs. V. B. Naik, S. B. Belekar.

GROUND :—Messrs. D. N. Patankar, S. K. Abhyankar, P. K. Oke, S. V. Pimputkar, M. L. Chandratreya, G. R. Deo, C. A. Sheth, C. K. Koshy, K. V. Iyengar, B. S Gai, N. H. Phadke.

2. Aims of teaching Mathematics:
   - (i) Culture;
   - (ii) Utility;
   - (iii) Discipline;
   - (iv) Preparation for higher studies.

   Their relative importance

3. Curricula:
   - (*a*) Examination of the present syllabuses in Mathematics
     - (i) in secondary schools,
     - (ii) for the Matriculation.

       Their suitability; adequacy; difficulty
   - (*b*) Suggestions for improvement by
     - (i) the inclusion of elements of Trigonometry, Solid Geometry, Calculus.
     - (ii) exclusion of certain topics.

4. Arrangement of subjects:

   Proper stages for beginning different subjects.

5. Examinations:

   Suggestions for their improvement.

6. Need for Experiment:

   Suggestions for experiment.

Discussion

Prof. Kar of the Secondary Teacher's College, opening the discussion, complimented the teachers in Madras on their superior training and equipment. In the Bombay Presidency he considered the Middle School Mathematics course deficient as it excluded Algebra and Practical Geometry. Algebra as a continuation of Arithmetic could be taught much earlier than at present. The subject of *locus* in Geometry was very badly taught and badly learnt. In his opinion the present Matriculation Course in Mathematics was too simple for the clever boy and too heavy for the average one. His suggestion was to make parts of the present curriculum with some additions, optional for the Matriculation. He regretted that mental Arithmetic was neglected in schools.

Mr. D. N. Patankar felt that the F. Y. A. course ought to be grouped with the Matriculation and there should be bifurcation like the Junior and Senior Cambridge Examinations. Junior Matric. Examinations should be held in vernacular and senior Matric. in English. At the Middle school stage, instruction should be in the vernacular. English should come after the stage of the 6th standard.

At present the papers were of two hours duration and the marks allowed were 50 marks for Arithmetic, 50 for Algebra and 50 for Geometry. By this arrangement the boy could get through by studying only one of the subjects. He suggested 75 in Arithmetic, 100 in Algebra and 100 in Geometry, the duration to be three hours.

Mr. C. B. Shaikh pleaded for a more rational teaching of the multiplication tables and their use in problems. This was often very unsatisfactorily done.

Mr. M. R. Paranjpe said that the Matriculation course included Arithmetic, Algebra and Geometry. He felt that Arithmetic ought to stop at an earlier stage.

The Matriculation student must know something of trigonometry, similar figures, approximations, probability, graphs which should be statistical rather than analytical, and logarithms.

Dr. G. S. Mahajani asked if nothing should be taught to the child which is simply to be memorised and which does not appeal to the imagination. In his opinion we should depend more upon the child to memorise rather than to imagine. He pleaded for the teaching of Mathematics being made interesting, by bringing in historical references and by teaching by association of ideas.

Mr. M. R. Ingle agreed that there was a dread of Mathematics because teaching in the middle schools was bad. He was for a special minimum for passing in Arithmetic.

Mr. K. S. K. Iyengar felt that in the school stage it is very difficult to make Mathematics more interesting than what it was at present. In College classes however, it was quite possible to do so. Everything depended on the teacher who must have capacity to inspire ideas in the minds of students.

Mensuration ought to be a compulsory course in the Matric. Examination.

Mr. Mirchandani pointed out that the average student was much more deficient in Arithmetic than in any other subject of study. He felt that it is very essential that more stress should be laid on Mental Arithmetic in Schools than at present.

Mr. S. K. Abhyankar's view was that it was the mechanical nature of teaching that was responsible for the existing low standard in Mathematics.

The cultural value of Mathematics should be brought home to every student.

The Chairman, Rao Bahadur P. V. Seshu Aiyar, in winding up the discussion pointed out the need for frequent meetings of the kind to discuss defects in present day teaching.

Prof. K. R. Gunjikar before proposing the vote of thanks to the Chairman announced that they were contemplating a Mathematical Colloquiem in Bombay one object of which would be to bridge the gulf between different classes of teachers and bring them more closely together.

Discussion on College Mathematics

Friday 23rd December 1932—2 P.M.

PRESIDENT: *Rao Bahadur P. V. Seshu Aiyar.*

The following outline for guiding the discussion was circulated before the meeting:—

1. Stages:

    (*a*) Intermediate stage;

    (*b*) Degree stage (i) as a Principal subject,
    (ii) as a Subsidiary subject,

    (*c*) Post-graduate stage.

2. Aims of study and research in Mathematics.

3. Curricula:

    (*a*) Examination of the present syllabuses; their suitability, adequacy, difficulty;

    (*b*) Suggestions for improvement by rearrangement, inclusion of new topics, exclusion of certain topics.

    (*c*) Question of rigour.

4. Present system of Examinations.
5. Post-graduate studies and Research.

Discussion

Opening the discussion, *Prof. V. B. Naik* raised the question as to whether some of the subjects which are being taught now in the B. A. should not be taught at the Intermediate stage, and whether the course in Mathematical Astronomy should not be replaced by one in Electricity and Magnetism. The B. A. course was not enough for a student to do research in Mathematics for the M. A. Examination.

He invited opinions as to what should be done to initiate research work in the (Bombay) University?

Prof. L. S. Vaidyanathan pleaded for the introduction of a group on Actuarial Mathematics, Statistics, and Economics for the M. A. Examination, as a B. A. in Mathematics was very well equipped for these subjects and there was a great demand for well-trained actuaries and Mathematical Economists.

Dr. Vaidyanathaswamy felt that in teaching Mathematics the aim should be to instill in the minds of the students sound general Mathematical ideas. The object should be to enable students to get a glimpse of certain fundamental mathematical ideas like groups, transformations, potentials and so on. Provided this principle is accepted, there was no objection to teaching Actuarial Science, but he doubted how far it was practicable to introduce it in the general course.

Stress should be laid on Practical Mathematics.

Mr. K. S. K. Iyengar complained that far too little of Analysis was done in the Honours course. It was necessary to teach the elementary Theory of Functions and also something of the foundations of Mathematics. One must not be tied down by the traditional syllabus.

Father Rafael pleaded for the proper teaching of Geometry. The subject must not be taught from the point of view of Examinations only.

Dr. Ram Behari was for rigour. He regretted that the I. C. S. and other examinations controlled the courses a little too much. There should be societies in Colleges and students should be asked to read papers and students should be encouraged to frame questions.

Mr. Krishnaswamy Iyengar: The object of Mathematical teaching is to give ideas about Mathematical discipline.

Prof. Bagi regretted the division of subjects into compartments in the B. A. Hons. papers.

Prof. B. S. Kalelkar wanted a proper framing of the M. A. course which was defective at present.

Prof. S. B. Belekar wanted to know the opinion of the delegates about the stage at which Pure and Applied Mathematics should be separated.

Prof. Gunjikar pointed out that there were certain definite stages in the student's career and different aims that should be kept in mind at each stage. At the School Stage utility should be the sole and principal consideration, as the course was compulsory. At the Intermediate stage it was possible to go further as generally only those students who had any mathematical ability, offered the subject, but there too it was necessary to recognise that a number of these would give up the subject and take up others for the Degree course. For this class of students the courses framed on the lines suggested by Prof. J. Maclean were to be introduced in Bombay. At the degree stage the students should be given a sound training in the subject which should be treated rather as a Science than as an Art. Too much drill and complicated examples should be avoided.

He felt the present tendency to insist too much on rigour in the early stage of Calculus was not sound. He thought that the Intuition of the students needed development first and rigour should come afterwards. In the B. A. course too it is necessary to distinguish clearly between Calculus and Theory of Functions. He doubted the value of the present Essay Paper in the B. A. Examination as the range being limited, there was a tendency among the students to cram a number of likely topics.

Principal N. M. Shah, on the other hand, believed that a rigorous treatment was not necessarily prejudicial to the creation of interest in the subject. Intuition could easily lead into dangers, as seen in text-books which used Differentiation and Integration of Infinite Series indiscriminately. He was for the retention of the Essay Paper at the B. A. Examination.

ADDRESS

PRESENTED TO

M.R.Ry. M. T. Naraniengar Avergal, M.A.

DURING THE

SILVER JUBILEE CELEBRATIONS

To

M.R.Ry. M. T. NARANIENGAR AVL., M.A.,

First Editor, Journal of The Indian Mathematical Society.

DEAR SIR,

On this auspicious occasion when we are met to celebrate the Silver Jubilee of the foundation of the Indian Mathematical Society, it is but fitting that we should recall with affection and gratitude the services of all those devoted workers who laboured unceasingly in the past to bring it to the position it occupies today. Among these pioneer workers, there are few to whom the Society owes as much as to you, its distinguished foundation editor, who for nearly two decades have conducted almost single-handed the Journal which is the embodiment of the creative activity of the Society in the field of Higher Mathematics.

It was in 1907, a quarter of a century ago, that in response to an invitation from M.R.Ry. V. Ramaswami Aiyar a band of enthusiastic devotees of Mathematics formed themselves into the "Indian Mathematical Club" for the advancement of Mathematical study and research in India. The framing of a suitable constitution on an all-India basis and the arrangements for the purchase and circulation of the leading Mathematical periodicals absorbed the early attention of the Society, and the need was soon felt for a medium to enable the members to exchange ideas and to serve as a stimulus to independent study and research. The success of the new venture was largely dependent on the choice of a competent Editor, and the Society was exceedingly fortunate in having secured your services for this work.

Looking back at this distance of time, one appreciates the formidable difficulties which lay in your way. The idea of research in Mathematics was not then as familiar as it is to-day, and the Universities and educational institutions in the country thought mainly in terms of examinations and syllabuses. The number of those undergoing higher courses in the subject was exceedingly small and of these few had any experience in writing papers, and in many cases the task of scrutinising papers for publication involved also the task of rewriting them. Often the necessity of issuing the Journal in time compelled you to step in with contributions of your own within a few days, and the earlier volumes of the Journal bear eloquent testimony to the readiness and the fertility of your response on such occasions. To add to these, there were few printers with experience of mathematical work and the task of getting the matter through the press was by no means a light task.

These services do not however exhaust all that the Society owes you ; for as one of the first Secretaries, as a member of the Managing Committee for several years and lastly as President of the Society your suggestions have been of the greatest help in the conduct of the work of the Society.

By your single-minded devotion to the work you undertook, as a labour of love, your high sense of duty, and the saintliness and simplicity of your life you have set a noble example which will be long cherished by those who have been privileged to know you.

Wishing you many happy years of useful service to the cause of Mathematics and to the Country,

We subscribe ourselves,

Dear Sir,

Your fellow-members of

The Indian Mathematical Society.

P. V. Seshu Aiyar,

President.

Indian Mathematical Society
Bombay, 24th December 1932.

Prof. Naraniengar's Reply

MR. PRESIDENT, LADIES AND GENTLEMEN,

The Committee of the Indian Mathematical Society has placed me under a deep debt of gratitude by voting an Address for me on this important occasion. The honour done to me gains special significance inasmuch as it is associated with the Silver Jubilee Conference of the Society.

I may be pardoned if I refer to my personal feelings in the matter of the propriety of such a function 5 years after my retirement from the Editorship of the Society's Journal. Further, the little services I may have rendered to the Society in my capacity of Editor, do not deserve to be made much of, and it is really flattering to find that the Committee have discovered merits in my services worthy of public recognition.

The Society was founded in 1907, but the Journal was started only in 1909. For a few months previously, the Progress Reports issued by the Secretary used to contain Mathematical Notes and Questions. The Progress Report for October 1908 included the first Mathematical Note by Principal Paranjpye, and the first Question to be published was one by Balakram. The solution of this Question by Professor Wilkinson was published on P. 52 of Vol. I in April 1909.

The First Secretary (Mr. V. Ramaswami Aiyar) while announcing my Joint Secretaryship humorously referred to me as the P. R. S. of South India:— meaning that I was to be the "Progress Report Secretary" of the Society, and generously admitted me to the rank of the Bengal Mathematicians of that high order. I little imagined then that Mr. Ramaswami's P. R. S. would be the recipient of the present honour.

Let me briefly refer to the office-bearers who co-operated in the task of editing the Journal. Mr. S. Narayana Aiyar, M. A., of the Madras Port Trust Office was the First Assistant Secretary appointed by the Committee to supervise the Journal Printing at Madras; and he did his share of the work with commendable zeal. Mr. Narayana Aiyar had to resign the Asst. Secretaryship in 1910, and his place was taken up by Prof. P. V. Seshu Aiyar. On Mr. Seshu Aiyar's election to the Committee in 1915, Mr. Krishnaswami, M. A., of Pachaiyappa's College succeeded the former as Asst. Sec. Owing to the continued illness of Mr. Krishnaswami, Mr. C. N. Ganapati, M.A., had to relieve the former towards the end of 1916. The Journal was thus in charge of the Joint Secretary and the Assistant Secretary for about 8 years. The disinterested services of the several Assistant Secretaries during this period are laudable. In March 1917, a Journal Committee was constituted for the better management of the Journal affairs. Professor Seshu Aiyar was appointed Joint Editor under the new arrangements. When he became the Secretary in 1922, Prof. K. Ananda Rao succeeded him as Joint Editor. Prof. A. Narasinga Rao relieved Mr. Ananda Rao in April 1927; and when I retired from the Editorship in June 1927, Dr. Vaidyanathaswamy was appointed in my place. Ever since, the Journal is being conducted by Dr. Vaidyanathaswamy and Prof. Narasinga Rao with the greatest zeal.

From 1923 onwards, the Journal is being issued on a new plan: 'original articles' appearing with continuous paging; and 'notes and questions' appearing separately with continuous paging. The Progress Report of the Society is now issued as '*extra*' matter. According to the present plan, each volume takes two years to be printed. Thus: volumes I to XIV correspond to the first 14 years (1909 to 1922); and volumes XV to XIX correspond to the next 10 years (1923-32).

I must confess that the editor's work was by no means easy at the commencement. In the first place, there was considerable difficulty in selecting a suitable Press to print our Journal; next, there was the inherent arduousness of editing mathematical matter and passing the proof-sheets.

After trying several Printing Firms, we entrusted the printing to the Kapalee Press owned by Messrs. S. Murthy & Co. of Madras. Though the printing was not everything desirable, we got on fairly well with them till 1919, and the first eleven volumes were issued from that Press.

In 1920, there was a change and the printing was entrusted to Messrs. Srinivasa Varadachari & Co., Mount Road, Madras. It is gratifying to note that this firm is doing its very best to satisfy the requirements of the Society.

As regards the Editorial Work, our main complaint was about the slovenly manner in which Manuscripts intended for publication in the journal were prepared and sent up. There was often difficulty in deciphering 'Contributions.' I had invariably to make press copies of Questions and Solutions and prepare diagrams drawn to scale for making blocks. The work of editing all the solutions to a single question would often involve several hours of close scrutiny and fair-copying. I remember an 'Honours Student' once sent up a solution of a problem by V. Ramaswami Aiyar occupying 20 pages of analytical work, while the geometrical solution of the problem did not occupy even half a page.

The proudest achievement of the Society was perhaps the discovery of the great South Indian Mathematician—the late S. Ramanujan, F.R.S. His contributions began to appear in our Journal in 1911, and his '*first article*' on 'Some Properties of Bernoulli's Numbers' attracted considerable attention. It is however a sad confession to have to say that the Editor's work in connection with Ramanujan's contributions was by no means light. Ramanujan saw intuitively many things and could not bring himself to the level of an ordinary student of mathematics. His 'first article' had consequently to be referred back to him no less than three times.

Among its Contributors, the Journal had the privilege of counting many distinguished mathematicians. To name a few: C. V. Raman, Balakram, Wilkinson, G. A. Miller, V. Ramaswami Aiyar, Philip E. B. Jourdain, K. J. Sanjana, V. Ramesam, Rev. Steichen, R. P. Paranjpye, W. Gallatly, Homersham Cox, T. Hayashi, P. V. Seshu Aiyar, W. J. Greenstreet, E H. Neville... were some of our '*early*' Contributors.

The future of the Journal is linked up with '*Research in India*'; and it is the bounden duty of the Society to create facilities for the research scholars on a permanent basis. Our Silver Jubilee ought, in my opinion, to be availed of for founding a '*Jubilee Prize*' under suitable conditions. My appeal to members in this connection may be fresh in the minds of those present here. The Universities—particularly those of Bombay and Madras—, should help the Society liberally in realizing these laudable objects.

The Journal provides for three Classes of Contributors, viz.—

1st Class; 2nd Class; and 3rd Class.

It should therefore appeal to all grades of mathematicians, from the highest to the lowest.

May we not hope that all Lovers of Mathematics will derive their inspiration from the Journal and endeavour to keep the torch of knowledge ever burning to the lasting Glory of our Society!

In conclusion, let me thank the Committee once again for the honour shown to me today.

PUBLIC LECTURE
ON
MATHEMATICS AND RELIGION

BY

RAO BAHADUR P. V. SESHU AIYAR

There are many points of affinity between Mathematics and Religion. By Mathematics, I mean here the Philosophy of mathematics and mathematical methods, and not the developed Science of Pure or Applied mathematics ; and by Religion I mean the Philosophy of Religion and not its ethical aspect or the daily practices or rituals of religion. Both in contents and in the methods mathematics and religion have much in common between them, so much so, that if religion should be properly evolved it must follow the methods of mathematics.

Both start from the Universe. By Universe I mean all that you can imagine or conceive to be in this world consisting of the Solar system and the Starry region and all the aspects of the World viz , the physical, mental, intellectual and spiritual ; mathematics paying more attention to the physical or the material side of the universe and religion to the spiritual and mental side of it.

While saying so, I am afraid I may be treading on contentious grounds. For, I know that some may start from God and consider the Universe as His creation, and it may be considered a blasphemy to make religion start from the Universe. Whatever that may be, one trained in mathematical methods will be inclined to start only from the known and proceed to the ultimate. That procedure alone will appeal to reason and people who do not like to bring in reason for their aid in understanding religious principles may do just as they please, and mathematics has nothing in common with them. Such people will have to face many difficulties ; one set may start with some idea of God and a particular revelation, and another set may start with some other idea and some other revelation and so on ; and there will be conflicts between these as we have already found in this world by bitter experience, and you have what may be called the clash of religions. But if religion is to be approached with a precision and definiteness for which mathematics is noted and studied by mathematical methods, a common philosophy of religion will evolve out of such a search, and by common consent that philosophy may be made to give us also the ethical aspect of religion. I proceed to show how such a study can be made and how mathematical philosophy and mathematical methods could be used in that study.

I go back to the starting point and say once again that both mathematics and religion start from the Universe. Now I find some mathematicians demurring to that statement of mine. They may say that in mathematics we start from numbers, points, lines, forms etc., and proceed to build up the science of mathematics. What they say is quite true. In the logical development of mathematics we no doubt start from numbers, points, lines etc., but if you look into the history of mathematical philosophy, you will find that we had first the material objects of the Universe and that we have, gradually abstracted these notions of "number', point', 'line' &c.

The truth is that you had the human being with his sense organs, the mind and reasoning faculty and you had the universe around: and then by observing, studying and applying his reason to the physical phenomena and material concrete objects in their various aspects of quantity, size, form etc. man has gradually evolved the science of Mathematics. Similarly by observation, study and application of reason to the mental and spiritual phenomena of the universe the science of Religion, as depicted in our Vedantic Philosophy for instance, has gradually been evolved.

Now coming to the content of mathematics and of religion, we have in them both what is called the finite and what is called the infinite, finite etymologically meaning "that which has an end," अन्तवत् in Sanskrit, and infinite meaning that which has no end (अनन्तं endless). In mathematics we say that space is infinite and time is infinite. Similarly in religion we have आकाशं अनन्तं; कालं आद्यन्तरहितं; etc. By so saying we only mean that the human mind cannot think of an end to space or to time. Here a question may be raised as to whether 'finite' is fundamental and thence 'infinite' is conceived as 'not finite' or whether 'infinite' is fundamental and thence 'finite' as 'not infinite'? This is very difficult to answer. The fact is they are correlative terms like light and darkness. All knowledge is differentiation. Just as one cannot have a clear conception of light without experiencing darkness, similarly we cannot have a clear conception of 'finite' without the notion of 'infinite' or what is not finite. But 'infinite' is etymologically 'not finite'. This is a negative conception. In Sanskrit we have a term denoting a positive conception viz. पूर्णं which when quantitatively viewed may be taken to mean 'infinity'. This notion is fully brought out in the upanishedic saying.

पूर्णमदः पूर्णमिदं
पूर्णात् पूर्णमुदच्यते
पूर्णस्य पूर्णमादाय
पूर्णमेवावशिष्यते.

The last two lines mean that "if infinity is taken away from infinity what remains still is infinity" *i.e.* infinity is the balance left over. In order to have a clear conception of infinity and to understand the meaning of the above lines we may resort to mathematics for help.

Take the integral numbers 1, 2, 3, 4, etc., they are infinite in number *i.e.*, not finite; for after any number n there comes the number $n+1$ *i.e.*, the series of numbers goes on without end. Of these take the odd numbers 1, 3, 5, 7, etc.; they are also similarly infinite in number. Take the even numbers 2, 4, 6, 8, etc. They are also infinite in number. Thus from the infinity of integers if we take away the infinity of odd integers, we have the infinity of even integers still left. Here it may be thought, that the infinity of all the integers is bigger than the infinity of the odd integers or that of the even integers. It is not so; they are the same kind of infinity; that is to say, there are as many even integers as there are integral numbers, no less, no more. To make this clear we resort to the notion of one-to-one correspondence between the elements of two groups or classes and to the notion of cardinal number based on such one-to-one- correspondence.

One-to-one Correspondence

If there are two groups of objects (or *elements* as we shall call them hereafter) to be compared as to their quantity or similarity, a rustic who does not know even counting sets up a correspondence between the elements, setting an element of one group to correspond to an element of the other, and sees whether the two groups exactly tally. If he finds an excess in one group after setting up such a correspondence he considers that group to be bigger or to contain more. If there is an exact tallying between the groups he considers the two groups to be equal. This setting up of one-to-one correspondence between the elements of two groups does not involve the process of counting or the notion of 'number'. For instance at a meeting there is such an one-to-one correspondence between the group of seats and the group of persons present at the meeting, provided every person present is given a seat and a seat accommodates only one person and no seat remains unoccupied. Without any counting or knowing their number, we can say that there are as many seats as there are persons present and there are as many persons as there are seats. Again in a monogamous community, like the Christians, where there is only one wife to every husband and one husband to every wife, there is an one-to-one correspondence between the group of husbands and the group of wives, and we can say without counting or knowing their number that there are as many wives as there are husbands and as many husbands as there are wives.

Two groups which have that one-to-one correspondence between their elements are said to be *similar* or *equivalent*. They are also said to have the same *potency* or *power*. They are also said to have *the same cardinal number*.

Now with this explanation of one-to-one correspondence and the definition of *power* or *cardinal number*, we can easily see that the group of integers and the group of even integers have the same power or cardinal number; for we can set up an one-to-one correspondence between the two groups as follows:—

Write the two groups one below the other thus

1, 2, 3, 4, 5, ...
2, 4, 6, 8, 10, ...

In so writing we have made any number n in the group of integers correspond to $2n$ in the group of even integers and made any number p in the group of even integers correspond to $p/2$ in the group of integers. It will thus be seen that corresponding to every integer in the first group there is one even integer and only one in the second group, and corresponding to every even integer in the second group there is one and only one integer in the first group. Thus there is an one-to-one correspondence between the elements of the group of integers and those of the group of even integers, so that we can say that there are as many even integers as there are integers or the infinity of integers is of the same kind or nature as the infinity of even integers; neither more nor less. Here one may ask " Well, Sir, the integers include all the even integers as well as all the odd integers, and so is not that infinity greater than the infinity of even integers? " No! it is not so; the two infinities are the same. In fact that is one main characteristic of an infinite group, viz., that it can be put into one-to-one correspondence with a part of itself. Some Mathematicians define infinity itself thus :—An aggregate is said to have an *infinite* number of elements if it can be put into one-to-one correspondence with a part of itself. One that cannot be so put is said to have a *finite* number of elements. We can now clearly understand how infinity is such that when an infinity is subtracted from it, the remainder can still be infinity, *i.e.*

पूर्णस्य पूर्णमादाय
पूर्णमेवावशिष्यते ।

Next we have in Purushasuktham which describes the Purusha (the Ultimate or the Absolute), different kinds of infinity, as it were, hinted at. There Purusha is described thus:

सहस्रशीर्षापुरुषः
सहस्राक्षस्सहस्रपात्
स भूमिं विश्वतो वृत्वा
अत्यतिष्ठद्दशांगुलं
पुरुष एवेदं सर्वं
यद्भूतं यच्च भव्यं
उतामृतत्वस्येशानो यदन्नेनातिरोहति
एतावानस्य महिमा
अतोज्यायांश्च पूरुषः
पादोस्य विश्वा भूतानि
त्रिपादस्यामृतं दिवि.

Here Purusha or God is described as an infinite being transcending all space and all time which are themselves infinite. Ordinary people who do not know anything of infinity or different types of infinity are likely to consider this description of Purusha as purely imaginary or mystic. But a mathematician who has studied about transfinite numbers or different types of such numbers can easily give a meaning to such descriptions. What I maintain is that whether our ancients understood such different types of infinities or not, we having the ideas of transfinite numbers can assign a meaning to such expressions as

अतोज्यायांश्च पूरुषः or पादोस्य विश्वा भूतानि त्रिपादस्यामृतं दिवि

I shall explain how it can be done. We have all seen the infinity of integral numbers ; we shall call that infinite cardinal number a. Let us consider the infinity of positive rational numbers [a + ve rational number is of the form p/q where p and q are positive integers]. They are also infinite in number and they appear to be tremendously more infinite than the integral numbers. For between any two rational numbers, however close they may be, there are an infinite number of rational numbers. Though the rational numbers when arranged in order of magnitude are so closely packed, the infinity of rational numbers is the same as the infinity of integers. This can be easily shown thus :—

Every positive rational number is of the form p/q where p and q are positive integers and so we can arrange the rational numbers in the form given below :—

$$\begin{array}{cccccccc}
\frac{1}{1} & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \cdots & \frac{1}{m} & \cdots \\
\frac{2}{1} & \frac{2}{2} & \frac{2}{3} & \frac{2}{4} & \cdots & \frac{2}{m} & \cdots \\
\frac{3}{1} & \frac{3}{2} & \frac{3}{3} & \frac{3}{4} & \cdots & \frac{3}{m} & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\frac{n}{1} & \frac{n}{2} & \frac{n}{3} & \frac{n}{4} & \cdots & \frac{n}{m} & \cdots
\end{array}$$

Every positive rational number is included in this double series consisting of rows and columns. The rational number r/s for instance is found as the sth term in the rth row. Of course there are repetitions. Now if we take the numbers diagonally one after another as shown in the figure we will also be reaching every rational number which any one may mention ; we can thus assign a rank to every rational number excluding all repetitions. In other words the group of rational numbers and the group of integers can be put into an one-to-one correspondence with one another. Therefore the infinity of rational numbers is of the same type as the infinity of integral numbers. Such an infinity is called a *countable* or *enumerable* infinity and the infinity of rational numbers is of that type and belongs to the type we have denoted by a. Incidentally we have also shown that we have $a+a=a$ and $na=a$ and also that $a\times a=a$ or $a^2=a$ and $\therefore a^3=a$ and $a^n=a$. Thus this infinity a has these peculiar properties.

Now take the case of the group of real numbers consisting of the rational numbers above described and irrational numbers such as $\sqrt{2}$ $\sqrt{5}$, π, e, etc. It is shown in mathematics that they form an infinity of a different type altogether. This infinity is denoted by c, the first letter of the word 'continuum.' There are still higher types. In short we have in mathematics transfinite cardinal numbers called 'aleph' numbers, and it is shown that given any 'aleph' number we can get an 'aleph' number higher than that, so that the 'aleph' numbers arranged in an order go on indefinitely i.e., without any limit. Thus if Purusha is described as transcending the universe of space and time, one who knows something of these transfinite numbers can try to understand it, and can say that such descriptions are not altogether imaginary or illusory. Again in religion we have apparently paradoxical statements made about God or the absolute one; such is

सूक्ष्मत्वात्तदविज्ञेयं
दूरस्तं चान्तिके च तत्.

"On account of subtlety, the absolute is unknowable; it is very near and yet far away!" We have illustrations of such subtlety in Mathematics. For instance take any irrational number say $\sqrt{2}$ i.e., the number which when multiplied by itself gives 2 for the product. From the various steps in the ordinary process of the extraction of square root we can see that this $\sqrt{2}$ lies between 1 and 2; again that it lies between 1·4 and 1·5 or again between 1·41 and 1·42 and so on. Thus you can locate it as lying between two decimals which differ from one another by as small a fraction as you please. Yet you can never express it exactly as a terminating decimal. Of course it is not a vulgar fraction. Thus $\sqrt{2}$ which appears to be definitely situated between two series of numbers 1, 1·4, 1·41, 1·414 ...... and 2, 1·5, 1·42, 1·415 ...... which go on approaching one another as closely as one would desire, yet evades an exact grasp because of its subtlety, being a mere point. It appears to be near enough and yet if tried to be approached by means of a decimal, it recedes farther and farther. Thus it is दूरस्तं (far away) though अन्तिके च (near enough).

I have given above some instances of similarity of 'contents', between mathematics and religion. I shall now pass on to give an instance of similarity of methods between them, and close the address You must have often heard it said that mathematics is an abstract science. Yes! it is so. For instance we have the natural numbers or the number concept. This is abstracted from the observation of groups or classes of concrete things and from a comparison of such groups in respect of quantity by means of 'correspondence'. If two groups have an one-to-one correspondence between their elements, we say they are similar or have *the same number*. 'Number' is thus the common characteristic of all 'similar' groups i.e., groups which have an one-to-one correspondence between their elements. If one asks "what is that common characteristic?", we can only say that whatever you can find in common between such similar groups is that common characteristic. Not being able to define that common characteristic, mathematicians have defined 'number' as a class of similar classes, just as the best definition of 'man' is given to be 'the class of men'. Thus 'number' is an abstract concept derived from concrete groups of objects. In religion too we have such abstraction. From the observation of the universe, we find that all things are changing and are being transformed, they are transient, living only for a short time. Yet man cannot regard all these things as having come from 'nothing'; he thinks that there must be a '*reality*' which exists for ever i.e., *eternal* and out of which all the things of the universe have come and into which all the things lapse or are absorbed at Pralayam or deluge. This 'Reality' he calls God. God is thus a 'concept' abstracted from the observation and experience of the universe. If asked to define It, we can only say that It is *That* which exists for ever, which undergoes no change, out of which all the things of the Universe have come and into which they merge at Pralayam. If driven still further to give its exact nature, one only says that it is निर्गुणं (has no attributes) and it is not to be known by the senses or by the 'mind', or one may define It as विश्वं (the universe). In fact the first name of God in Vishnu Sahasranaman is विश्वं. This is quite on a par with the definition of 'number' as the class of similar classes'. Just as 'number' is a concept to be abstracted from 'the class of similar classes,' similarly God is a 'concept' to be abstracted from the 'Universe'. Thus even in method mathematics and religion have much in common between them. From the instances and illustrations given above one could see how a student of mathematics can bring to bear his knowledge of mathematics and mathematical methods to the study and understanding of religious concepts and religious principles.

List of papers communicated to the Conference

1. R. C. ARCHIBALD (Brown University, America): *George Hermann Valentin* 1848-1926.

2. B. B. BAGI (Dharwar): *On the determination of the real foci of a real conic.*

3. S. K. BANERJI (Calcutta): *On the steady rotation of a viscous fluid.*

4. RAM BEHARI (Delhi): *Equilateral Osculating Quadrics of Ruled Surfaces.*

The ruled surfaces whose osculating quadrics are equilateral are determined.

5. ———— : *A Theorem on Normal Rectilinear Congruences.*

"There exist ∞^2 ruled surfaces of an ordinary congruence, the osculating quadrics of which are equilateral, but there are only ∞^1 such ruled surfaces if the congruence is normal."

6. E. T. BELL, (California): *An Algebra of Numerical Compositions.*

7. W. BLASCHKE (Hamburg)—*Hexagonal four-webs of surfaces in 3-space.*

8. S. CHOWLA (Benares): Contributions to the analytic theory of numbers (II).

Let $r_{s,k}(n)$ denote the number of representation of the positive integer n as a sum of s positive k^{th} powers. It has been shewn by S. S. Pillai that $r_{2,3}(n) = \Omega(\log\log n)$.

I shew here that, when k a fixed integer positive or negative,

$$\sum_{\substack{x^3+ky^3=n \\ x>0,\ y>0}} 1 = \Omega(\log\log n)$$

I also shew that $r_{3,4} = \Omega(\log n/\log\log n)$, assuming hypothesis K.

9. M. R. DORESAMIENGAR (Mysore): *Tariff-Policy and Distribution.*

A simple demonstration has been given of Prof. Pigou's formulæ for differential taxation and under certain genuine assumptions, the rise in price to the consumers has been taken as a species of taxation. The work begun in 'Taxation as an instrument for modifying inequalities in Distribution' is thus continued. And the disturbing factors that throw the 'equality-maxim' in 'personal distribution' into a dynamic state, are shown in relief.

10. B. S. GAI (Bombay): *On the roots of real cubic equations.*

The paper constructs a table by the help of which one can get a fairly good idea of the nature and relative positions of the roots of $ax^3+bx^2+cx+d=0$.

11. HANSRAJ GUPTA (Hoshiarpur): *Quotient and Remainder Series.*

In my paper entitled "The Perpetual Calendar Formula" it was shown that under certain limitations a set of n positive integral numbers $a_1, a_2, \ldots a_n$, arranged in ascending order of magnitude and such that $a_r - a_{r-1} = d$ or $d+1$, can be obtained by giving to x in order the values $1, 2, \ldots n$ in an expression of the form $Q_a\ (kx+l)$. In the present paper the limitations are more clearly brought out and the sequences discussed in greater detail.

12. MEGU RAM GUPTA (Hoshiarpur): *Two new perfect numbers.*

13. T. HAYASHI (Sendai, Japan): *A Japanese Problem.*

14. M. V. JAMBUNATHAN (Shimoga): *The Recardian Theory of Rent: A Fallacy*

The Recardian Theory of Rent that the share of land, as a factor of production is obtained only as a residuum, is questioned in this paper. It is shown that the rent of land can be calculated independently without any reference to the shares of the other factors of production; and that it does not stand on a different category from the shares of the other agents of production.

15. H. R. KAPADIA, (Poona): *Some Materials for the Study of Mathematics in Jaina Literature.*

(1) the place assigned to mathematics in Jainism, (2) the mathematical works of the Jainas, (3) different designations for notational places, (4) 27 kinds of numbers and a reference to alef-zero, (5) different types of infinity, (6) 14 kinds of series, (7) the frequent use of permutation and combination and a few typical problems, (8) value of π, (9) measurement of mountains, etc., (10) tables pertaining to different measures, (11) the number of human beings at any time.

16. P. K. KASHIKAR (Bombay): *The Archimedian Solids.*

This contains a proof (believed to be new) of the theorem that there are only 15 different types of these solids. A general method for the construction of card-board models of these figures is explained.

17. ———— : *The Concave Regular Polyhedra.*

This contains a modified form of Cauchy's proof of the theorem that there are only four such polyhedra. Card-board models of these and some other stellated polyhedra were exhibited and the method of construction of some of them explained.

18. D. D. KOSAMBI (Aligarh): *The Differential Invariants of the most general set of curves defined by a set of second order differential equations.*

It has been shown by Professor Cartan, and follows by considering some works of mine from a new point of view, that the essential differential invariants of the system $x_i + \alpha_i(x, x, t) = 0$ $(i = 1, \ldots, n)$ are three in number ε^i, P^i_j, α^i; ; k; l. The first two and their consequences follow from my work quite easily and clearly, but the last cannot be obtained by methods other than those of Prof. Cartan.

19. A. A. KRISHNASWAMI AYYANGAR (Mysore): *A Geometry of Sextuples.*

This paper furnishes a new and simple proof of the theorem that the plane projective geometry of six points to a line is Pascalian and unique, by showing that Pascal's theorem may be regarded as a property of a class of Latin squares. Fano's axiom is proved to be true in this geometry and one characteristic property noted, *viz.*, the existence of sets of four triangles in perspective, which may be so ordered that each triangle is inscribed in its adjacent one in cyclic order.

20. ——— : *On Geodesics.*

When the equation of a surface is referred to non-geodesic orthogonal parametric curves, it is shown that the differential equation of a geodesic can be put in the new form :

$$\frac{d}{ds}\left\{ Ev\left(\frac{du}{ds}\right)^2 + Gu\left(\frac{dv}{ds}\right)^2 \right\} = \frac{du}{ds}\cdot\frac{dv}{ds}\left(P\frac{du}{ds} + Q\frac{dv}{ds}\right)$$

where $P = E + (u - v)\,E_1$, $Q = G + (v - u)\,G_1$ and ds is an element of arc on a geodesic. The above equation immediately puts in evidence the converse, *viz.*, that Liouville's surfaces are the only ones for which $E\iota\left(\frac{du}{ds}\right)^2 + Gu\left(\frac{dv}{ds}\right)^2$ is constant.

21. ——— : *On oriented circles.*

22. G. S. MAHAJANI (Poona): *A definition of steady motion in dynamics.*

The author proposes to define steady motion as one in which the kinetic and potential energies are separately constant.

23. S. L. MALURKAR (Poona): *A note on a particular equation of conduction when radiation is taken into account.*

Some years ago, in 1915, G. I. Taylor found an expression for the heat transported upwards by eddy conduction. He did not take radiation into account and found that $z^2/4t$ was constant, where z is the upward displacement and t is the time taken from the initial stage. Following a few papers that we have published in the *Indian Journal of Physics*, an attempt was made to see if Taylor's expression could not be modified to take account of radiation. By use of contour integrals, a first approximation is derived.

24. K. NAGABUSHANAM, (Madras) *Tensor theory of Jacobi's Multipliers.*

For the equations $\frac{dx^1}{X^1} = \frac{dx^2}{X^2} = \dots = \frac{dx^n}{X^n}$, any function $M(x^1x^2\dots x^n)$ satisfying $\sum_{i=1}^{n} \frac{\partial\,(Mx^i)}{\partial x^i} = 0$ is defined as a *multiplier of Jacobi.* Denoting by (x^i) the contravariant components of the vector defined by the above equations, the condition for M to be a multiplier may be written in the tensor form div. $(MX^i)=0$. In this paper it is shown that

(i) every multiplier is a scalar density;

(ii) the condition div. $(MX^i)=0$ is both necessary and sufficient for the tensor $M\varepsilon_{\alpha_1 \alpha_2 \ldots \alpha_{n-1} i}\, x^i$ to be derivable as a Stokes tensor of an alternating one; and

(iii) the famous theorem of Jacobi on the last integral of the above equations when all integrals but one and a Multiplier are known is a consequence of the more general theorem of Goursat on the integrability of the system of differential equations $Z_{\alpha_1 \alpha_2 \ldots \alpha_{r-1}}\, dx^i = 0$ where $Z_{\alpha_1 \alpha_2 \ldots \alpha_r}$ is derivable as Stokes tensor.

25. S. H. NANAVATI (Bombay): *Mortality curves.*

26. A. NARASINGA RAO (Annamalainagar): *The Metrical Geometry of the cyclic n-point.*

The paper extends to the cyclic n-point, the familiar properties of a triangle associated with the orthocentre, nine-points circle, etc.

27. E. H. NEVILLE (Reading): *Iterative Interpolation.*

28. B. RAMAMURTI (Annamalainagar): *A covariant specification of the unique tetrahedron inscribed in a space cubic curve and circumscribed to a general quadric inpolar to the curve.*

If a space cubic curve c be regarded as the carrier of a binary parameter x, a set of six points on it may be specified by the binary sextic $a^6{}_x$, the roots of which are the parameters of the six points. There is a unique quadric envelope Q, touching the osculating planes at the six points and impolar to the curve. It is well-known that there is, in general, a unique tetrahedron inscribed to c and circumscribed to Q and that if there be more than one such tetrahedron there must be ∞^2 tetrahedra. In the general case if the vertices of the unique tetrahedron correspond to the binary qnartic $b^4{}_x$ in the parameter, it is obvious that $b^4{}_x$ should be a covariant of $a^6{}_x$. The object of this paper is to prove that $b^4{}_x$ is the fourth transvectant of $a^6{}_x$ and itself.

29. V. RAMASAWMY IYER (Chittoor): *Self-Reversible Functions.*

30. C. V. H. RAO (Lahore): *The ϕ-Conic from a projective standpoint.*

31. S. SIVASANKARANARAYANA PILLAI (Annamalainagar): *On the sum-function concerning the number of prime factors of a number.*

Section I of this paper contains a simple proof of the results

$$g(x) = x \log \log x + Ax + O(x/\log x)$$

and

$$G(x) = x \log \log x + Bx + O(x/\log x)$$

where A and B are constants. In Section II by making use of known result about the distributive of primes all the terms are found out till the error term is reduced to the orders of $x/\log^t x$ where t is any fixed positive integer. Section III is devoted to find out a better result by assuming Reimann hypothesis. In the course of the proof the following results are also proved:

$$G(x) = g(x) + Cx + O(\sqrt{x}) \text{ where } C = \Sigma \frac{1}{p(p-1)};$$

$$g(x) = \sum_{r=1}^{\infty} \pi(x/r)\,; \quad \Sigma\left(\frac{x}{p}\right) \sim (1-\gamma)\frac{x}{\log x}$$

where (x) is the fractional part of x, and γ is Euler's constant.

32. R. VAIDYANATHASWAMY (Madras): *An Extension of the Determinant Concept based on Group-characters.*

The Rice-Lecat conception of determinant in which the suffixes are assigned one of two characters, the *signant* or *non-signant*, is here extended by assigning to the suffixes characters which correspond to the Abelian characters of a permutation group.

33. JAGESHWAR DAYAL VAISH (Muzaffarnagar): *A Trigonometrical Formula.*

The formula in question is $\sin\theta = \dfrac{1}{\dfrac{10100}{\theta(180-\theta)} - \dfrac{1}{4}}$ approximately,

where θ is in degrees.

34. K. VENKATACHALIENGAR (Bangalore): *A new method of obtaining the product formula for sin x.*

35. G. N. WATSON (Birmingham): *Proof of certain identities in combinatory analysis.*

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Papers
Presented to the Conference

Proof of Certain Identities in Combinatory Analysis

BY G. N. WATSON, SC.D., F.R.S.

" *Five for the symbol at your door.*"

On this occasion when we celebrate the Silver Jubilee of the foundation of the Indian Mathematical Club, which has now developed into the Indian Mathematical Society, it is fitting that we should have in honour those mathematicians, now no longer with us, who, in the early days of the Society, added to its renown and increased its international reputation, whether by their administrative labours or by their mathematical skill.

It is consequently appropriate to commemorate the name of Srinivasa Ramanujan by making known a few of the most remarkable theorems which he discovered and giving proofs of them.

The majority of the theorems which I propose to discuss are concerned with two functions, $G(x)$ and $H(x)$, defined by the formulæ

$$G(x) = 1 + \frac{x}{1-x} + \frac{x^4}{(1-x)(1-x^2)} + \frac{x^9}{(1-x)(1-x^2)(1-x^3)} + \dots \dots,$$

$$H(x) = 1 + \frac{x^2}{1-x} + \frac{x^6}{(1-x)(1-x^2)} + \frac{x^{12}}{(1-x)(1-x^2)(1-x^3)} + \dots \dots;$$

the indices of the powers of x in the numerators of the $(n+1)$th terms being n^2 and $n(n+1)$ in the respective series.

The well-known Rogers-Ramanujan identities* are expressed by the equations

$$\frac{1}{G(x)} = (1-x)(1-x^4)(1-x^6)(1-x^9)(1-x^{11}) \dots \dots,$$

$$\frac{1}{H(x)} = (1-x^2)(1-x^3)(1-x^7)(1-x^8)(1-x^{12}) \dots \dots,$$

where the indices of the powers of x in the factors differ from multiples of 5 by 1 and 2 in the respective products.

Probably not many mathematicians would dissent from the opinion expressed by Professor Hardy that it would be difficult to find more beautiful formulæ than these identities; but the primary credit for their discovery rests with Rogers whose proofs of them

* For an account of these formulæ with various proofs, see Ramanujan, *Collected Papers* (1927), pp. 214-215, 344-346.

were published some twenty years before they were independently rediscovered by Ramanujan.

Among the formulæ contained in the manuscripts left by Ramanujan is a set of about forty which involve functions of the types $G(x)$ and $H(x)$; the beauty of these formulæ seems to me to be comparable with that of the Rogers-Ramanujan identities. So far as I know, nobody else has discovered any formulæ which approach them even remotely; if my belief is well-founded, the undivided credit for the discovery of these formulæ is due to Ramanujan.

Some of these forty theorems were communicated by Ramanujan to Rogers, and the latter has published his proofs of nine of them *; the method used by Rogers is substantially Schröter's method of constructing modular equations and it is a modification of the method by which Jacobi † proved the fundamental formulæ of theta-functions. Ramanujan's methods of proof are not known.

In this paper I propose to prove ten theorems in all; six of them are members of the set of forty which have hitherto been unpublished, and to which Schröter's method is not obviously applicable; two of the theorems proved by Rogers are added, since they are required in the course of proving the six new theorems, and I give proofs whose main differences from the proofs given by Rogers are in matters of arrangement; as a preliminary I give proofs of the two theorems on which Ramanujan based his work on the modular equation of order 5, since they are continually required in the course of proving the main theorems.

It is convenient to work throughout with the notation used by Ramanujan for partition-functions and theta-functions; he uses a standard notation for the partition-function, but his notation $f(a, b)$ defined below for a theta-function is, I think, his own, and it seems to me to be the notation which is best suited for his type of work on modular equations; he writes

$$\Pi(a, x) = (1+a)(1+ax)(1+ax^2)(1+ax^3)\ldots,$$

$$f(a, b) = 1 + (a+b) + ab(a^2+b^2) + (ab)^3(a^3+b^3) + (ab)^6(a^4+b^4) + \quad \ldots \quad \ldots$$

$$= \sum_{n=-\infty}^{\infty} a^{\frac{1}{2}n(n+1)} b^{\frac{1}{2}n(n-1)},$$

$$\phi(x) = f(x, x) = 1 + 2x + 2x^4 + 2x^9 + \quad \ldots,$$

$$\psi(x) = f(x, x^3) = 1 + x + x^3 + x^6 + \quad \ldots,$$

$$f(-x) = f(-x, -x^2),$$

$$\chi(x) = \Pi(x, x^2) = (1+x)(1+x^3)(1+x^5) \quad \ldots$$

* *Proc. London Math. Soc.* (2) 19 (1920), pp. 392-396.

† *Ges. Math. Werke*, I, pp. 358, 505.

In this notation, the formula for the factorisation of a theta-function is *

$$f(a, b) = \Pi(a, ab)\Pi(b, ab)\Pi(-ab, ab),$$

and, in particular,

$$f(-x) = \Pi(-x, x^3)\Pi(-x^2, x^3)\Pi(-x^3, x^3) = \Pi(-x, x),$$

by rearrangement of factors.

It is also easy to verify that

$$f(1, x) = 2\psi(x), \quad f(-1, x) = 0,$$
$$\phi(x) = \phi(x^4) + 2x\psi(x^8),$$
$$\chi(-x)\chi(x) = \chi(-x^2), \quad \chi(-x)\Pi(x, x) = 1,$$

(the last of these being Euler's formula †) and (from the Rogers-Ramanujan identities) that

$$G(x) = \frac{f(-x^2, -x^3)}{f(-x)}, \quad H(x) = \frac{f(-x, -x^4)}{f(-x)}.$$

The verification of these results and of similar simple formulæ which are used later is left to the reader.

It is convenient to quote here various standard formulae (most, if not all, due to Jacobi‡) which express Ramanujan's functions in terms of moduli and quarter-periods of elliptic functions when the parameter of the elliptic functions, usually denoted by exp. $(-\pi K'/K)$ is equal to x. It seems superfluous to give proofs of any of these standard formulae; they are as follows:

$$\{\phi(x)\}^2 = 2K/\pi, \quad \{\phi(-x)\}^2 = (2K/\pi)k',$$
$$\{\phi(-x^2)\}^2 = (2K/\pi)\sqrt{k'} = \phi(x)\phi(-x),$$
$$\{\psi(x)\}^2 = (K/\pi)\sqrt[4]{(k^2/x)}, \quad \{\psi(x^2)\}^2 = (\tfrac{1}{2}K/\pi)k/\sqrt{x},$$
$$\{\chi(x)\}^6 = \frac{2\sqrt[4]{x}}{\sqrt{(kk')}}, \quad \{\chi(-x)\}^6 = \frac{2k'\sqrt[4]{x}}{\sqrt{k}},$$
$$\{f(-x)\}^6 = \frac{4K^3k'^2\sqrt{k}}{\pi^3\sqrt[4]{x}}, \quad \{f(-x^2)\}^6 = \frac{2K^3kk'}{\pi^3\sqrt{x}},$$
$$\{f(-x^4)\}^6 = \frac{K^3k^2\sqrt{k'}}{2\pi^3 x}.$$

When it is necessary to write down a modular equation in the sequel, the second pair of moduli will, as is customary, be denoted by λ and λ'.

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* Cf. Whittaker and Watson, *Modern Analysis* (1927), pp. 469-473.

† *Ibid.*, p. 472.

‡ See Whittaker and Watson, *Modern Analysis* (1927), pp. 479, 488.

The formulæ which I shall prove are as follows:—

(1) $\{\phi(x)\}^2 = \{\phi(x^5)\}^2 + 4x\{f(-x^{10})\}^2 \dfrac{\chi(x)}{\chi(x^5)}$,

(2) $\{\psi(x)\}^2 = x\{\psi(x^5)\}^2 + \{f(-x^5)\}^2 \dfrac{\chi(-x^5)}{\chi(-x)}$.

(3) $G(x)G(x^4) + xH(x)H(x^4) = \{\chi(x)\}^2 = \dfrac{1 + 2x + 2x^4 + 2x^9 + 2x^{16} + \ldots}{(1-x^2)(1-x^4)(1-x^6)\ldots\ldots}$,

(4) $G(x)G(x^4) - xH(x)H(x^4) = \dfrac{1 + 2x^5 + 2x^{20} + 2x^{45} + 2x^{80} + \ldots}{(1-x^2)(1-x^4)(1-x^6)\ldots\ldots}$,

(5) $G(x)H(-x) + G(-x)H(x) = \dfrac{2}{\{\chi(-x^2)\}^2} = \dfrac{2(1 + x^2 + x^6 + x^{12} + \ldots)}{(1-x^2)(1-x^4)(1-x^6)\ldots}$,

(6) $G(x)H(-x) - G(-x)H(x) = \dfrac{2x(1 + x^{10} + x^{30} + x^{60} + \ldots)}{(1-x^2)(1-x^4)(1-x^6)\ldots}$,

(7) $G(x^{11})H(x) - x^2G(x)H(x^{11}) = 1$,

(8) $G(x^{11})H(-x) + x^2G(-x)H(x^{11})$
$$= \frac{\chi(x^2)\chi(x^{22})}{\chi(-x^2)\chi(-x^{22})} - \frac{2x^3}{\chi(-x^2)\chi(-x^4)\chi(-x^{22})\chi(-x^{44})}.$$

If $G(x)G(x^{44}) + x^9H(x)H(x^{44}) = U$,
and $G(x^4)G(x^{11}) + x^3H(x^4)H(x^{11}) = V$,
then

(9) $U^2 + xV^2 = \{\chi(x)\chi(x^{11})\}^3$,

and

(10) $UV + x = \{\chi(x)\chi(x^{11})\}^2$.

The formulæ proved by Rogers are (3) and (7); an enunciation of (7) had been published previously by Ramanujan*. The essential feature of all the formulæ is that they express theta-functions with various parameters in terms of other theta-functions whose parameters are the fifth powers of the parameters of the original theta-functions. The formulæ, other than (1) and (2), may therefore be regarded as types of modular relations in which transformations of order 5 are combined with transformations of some other order.

Formulæ (1), (2), (3) and (7) are proved by making use of quite simple properties of quadratic forms †. The quadratic forms appear as indices, and, in order to avoid writing complicated expressions in indices, I shall put
$$x^{P/2} = F\{P\},$$
where P is a quadratic form.

* *Proc. London Math. Soc.*, (2) 18 (1920), p. xx.

† Of course formulæ (1) and (2) are very rudimentary examples of the use of quadratic forms. I believe, however, that Ramanujan discovered these two formulæ, not by manipulating quadratic forms, but by transforming series of Lambert's type.

We now proceed to prove (1). Consider

$$\{\phi(x)\}^2 = \sum_{m=-\infty}^{\infty} F\{2m^2\} \sum_{n=-\infty}^{\infty} F\{2n^2\}$$

$$= \sum_{m,\, n=-\infty}^{\infty} F\{2m^2 + 2n^2\}.$$

Take the values of m and n associated with any particular term of this double series and choose the integers M and N such that

$$m + 2n = 5M + \alpha, \quad 2m - n = 5N + \beta,$$

where α and β have values selected from the integers 0, ± 1, ± 2. Since

$$m = M + 2N + (\alpha + 2\beta)/5, \qquad n = 2M - N + (2\alpha - \beta)/5,$$

we see that values of α and β are associated as in the following Table:

α	0	± 1	± 2
β	0	± 2	∓ 1

When α assumes the values -2, -1, 0, 1, 2 in succession, it is easy to see that the corresponding values of $2m^2 + 2n^2$ are respectively

$$10M^2 + 10N^2 - 8M + 4N + 2,$$
$$10M^2 + 10N^2 - 4M - 8N + 2,$$
$$10M^2 + 10N^2,$$
$$10M^2 + 10N^2 + 4M + 8N + 2,$$
$$10M^2 + 10N^2 + 8M - 4N + 2.$$

It is evident from the equations connecting m and n with M and N that there is a one-one correspondence between all pairs of integers (m, n) and all sets of integers (M, N, α). From this correspondence we deduce that

$$\sum_{m,\, n=-\infty}^{\infty} F\{2m^2 + 2n^2\} = x\sum F\{10M^2 + 10N^2 - 8M + 4N\}$$
$$+ x\sum F\{10M^2 + 10N^2 - 4M - 8N\}$$
$$+ \sum F\{10M^2 + 10N^2\}$$
$$+ x\sum F\{10M^2 + 10N^2 + 4M + 8N\}$$
$$+ x\sum F\{10M^2 + 10N^2 + 8M - 4N\}$$

the summations applying to M, N = $-\infty$ to $+\infty$

Now

$$\sum_{M=-\infty}^{\infty} F\{10M^2 \pm 4M\} = f(x^3, x^7),$$

$$\sum_{M=-\infty}^{\infty} F\{10M^2 \pm 8M\} = f(x, x^9),$$

and therefore

$$\{\phi(x)\}^2 = \{\phi(x^5)\}^2 + 4xf(x, x^9)f(x^3, x^7);$$

also

$$f(x, x^9)/(x^3, x^7) = \Pi(x, x^{10})\Pi(x^3, x^{10})\Pi(x^5, x^{10})\Pi(x^7, x^{10})\Pi(x^9, x^{10}) \frac{\{\Pi(-x^{10}, x^{10})\}^2}{\Pi(x^5, x^{10})}$$

$$= \frac{\Pi(x, x^2)}{\Pi(x^5, x^{10})}\{f(-x^{10})\}^2 = \frac{\chi(x)}{\chi(x^5)}\{f(-x^{10})\}^2,$$

whence (1) follows immediately.

The proof of (2) is similar to that of (1). Consider

$$\{f(1, x)\}^2 = \sum_{m=-\infty}^{\infty} F\{m^2 + m\} \sum_{n=-\infty}^{\infty} F\{n^2 + n\}$$

$$= \sum_{m,\, n=-\infty}^{\infty} F\{m^2 + n^2 + m + n\}.$$

Make the same substitutions as before. When α assumes the values -2, -1, 0, 1, 2 in succession, it is easy to see that the corresponding values of

$$m^2 + n^2 + m + n$$

are respectively

$$5M^2 + 5N^2 - M + 3N,$$
$$5M^2 + 5N^2 + M - 3N,$$
$$5M^2 + 5N^2 + 3M + N,$$
$$5M^2 + 5N^2 + 5M + 5N + 2,$$
$$5M^2 + 5N^2 + 7M - N + 2.$$

Hence

$$\{f(1, x)^2\} = 4\, f(x, x^4)\, f(x^2, x^3) + x\{\, f(1, x^5)\}^2;$$

also

$$f(x, x^4)\, f(x^2, x^3) = \Pi(x, x^5)\, \Pi(x^2, x^5)\, \Pi(x^3, x^5)\, \Pi(x^4, x^5)\, \Pi(x^5, x^5) \times \frac{\{\Pi(-x^5, x^5)\}^2}{(\Pi x^5, x^5)}$$

$$= \frac{\Pi(x, x)}{\Pi(x^5, x^5)}\{f(-x^5)\}^2 = \frac{\chi(-x^5)}{\chi(-x^5)}\{f(-x^5)\}^2,$$

whence (2) follows immediately.

To prove (3), consider

$$f(1, x) \quad f(-x^2, -x^2) = \sum_{m,\, n=-\infty}^{\infty} (-)^n F\{m^2 + 4n^2 + m\}.$$

Take the values of m and n associated with any particular term of this double series and choose the integers M and N such that

$$m+n=5\mathrm{M}+\alpha, \quad m-4n=5\mathrm{N}+\beta,$$

where α and β have values selected from the integers $0, \pm 1, \pm 2$. Since

$$m=4\mathrm{M}+\mathrm{N}+(4\alpha+\beta)/5, \quad n=\mathrm{M}-\mathrm{N}+(\alpha-\beta)/5,$$

it is evident that $\beta=\alpha$.

From the one-one correspondence between all pairs of integers (m, n) and all sets of integers $(\mathrm{M}, \mathrm{N}, \alpha)$, combined with the obvious formula

$$m^2+4n^2+m=20\mathrm{M}^2+5\mathrm{N}^2+(4+8\alpha)\mathrm{M}+(1+2\alpha)\mathrm{N}+(\alpha^2+\alpha),$$

we deduce that

$$f(1, x) = \sum_{\alpha=-2}^{2} \sum_{\mathrm{M},\mathrm{N}=-\infty}^{\infty} (-)^{\mathrm{M}+\mathrm{N}} \mathrm{F}\{20\mathrm{M}^2+5\mathrm{N}^2+(4+8\alpha)\mathrm{M}+(1+2\alpha)\mathrm{N}+(\alpha^2+\alpha)\}$$

$$= \sum_{\alpha=-2}^{2} \mathrm{F}(\alpha^2+\alpha) f(-x^{12+4\alpha}, -x^{8-4\alpha}) f(-x^{3+\alpha}, -x^{2-\alpha})$$

$$= 2\ f(-x^8, -x^{12}) f(-x^2, -x^3) + 2x f(-x^4, -x^{16}) f(-x, -x^4) + x^3 f(-1, -x^{20}) f(-1, -x^5),$$

that is to say

$$f(-x^8, -x^{12}) f(-x^2, -x^3) + x f(-x^4, -x^{16}) f(-x, -x^4) = f(x, x^3) f(-x^2, -x^2).$$

Consequently

$$\mathrm{G}(x^4)\mathrm{G}(x) + x\mathrm{H}(x^4)\mathrm{H}(x) = \frac{\psi(x)\phi(-x^2)}{f(-x)f(-x^4)} = \sqrt[12]{\left(\frac{16x}{k^2k'^2}\right)} = \{\chi(x)\}^2 = \frac{\phi(x)}{f(-x^2)},$$

and this proves (3).

Formula (4) is obtained immediately by combining (3) with (1). We have, in fact,

$$[\mathrm{G}(x)\mathrm{G}(x^4) - x\mathrm{H}(x)\mathrm{H}(x^4)]^2$$

$$= [\mathrm{G}(x)\mathrm{G}(x^4) + x\mathrm{H}(x)\mathrm{H}(x^4)]^2 - 4x\mathrm{G}(x)\mathrm{H}(x)\mathrm{G}(x^4)\mathrm{H}(x^4)$$

$$= \frac{\{\phi(x)\}^2}{\{f(-x^2)\}^2} - 4x\frac{f(-x^5)\,f(-x^{20})}{f(-x)\,f(-x^4)}$$

and, since

$$\frac{\{f(-x^2)\}^2}{f(-x)\,f(-x^4)} = \chi(x),$$

we get

$$[\mathrm{G}(x)\mathrm{G}(x^4) - x\mathrm{H}(x)\mathrm{H}(x^4)]^2 = [\{\phi(x)\}^2 - 4x\frac{\chi(x)}{\chi(x^5)}\{f(-x^{10})\}^2] \div \{f(-x^2)\}^2$$

$$= \frac{\{\phi(x^5)\}^2}{\{f(-x^2)\}^2},$$

whence (4) follows by extracting square roots and taking care to select the appropriate sign.

We postpone the proof of (5) for a time and proceed to show how (6) may be obtained by applying (1) to a combination of (3) and (4). From (3) and (4) we obviously have

$$2G(x)=\frac{\phi(x)+\phi(x^5)}{G(x^4)f(-x^2)},\qquad 2xH(x)=\frac{\phi(x)-\phi(x^5)}{H(x^4)f(-x^2)};$$

and hence, by changing the sign of x and combining the results so obtained, we get

$$\begin{aligned}2xG(x)H(-x)-2xG(-x)H(x)&=\frac{\phi(x^5)\phi(-x^5)-\phi(x)\phi(-x)}{G(x^4)H(x^4)\{f(-x^2)\}^2}\\ &=\frac{\{\phi(-x^{10})\}^2-\{\phi(-x^2)\}^2}{G(x^4)H(x^4)\{f(-x^2)\}^2}\\ &=\frac{4x^2\chi(-x^2)\{f(-x^{20})\}^2}{G(x^4)H(x^4)\chi(-x^{10})\{f(-x^2)\}^2}\\ &=\frac{4x^2f(-x^{20})}{\chi(-x^{10})}\cdot\frac{f(-x^4)\chi(-x^2)}{\{f(-x^2)\}^2}\\ &=\frac{4x^2\psi(x^{10})}{f(-x^2)},\end{aligned}$$

since $$f(-x^2)=\psi(x)\chi(-x)=f(-x)/\chi(-x).$$

This gives (6) at once; we now deduce (5) from (6) in the way in which (4) was deduced from (3), using (2) instead of (1). We thus have

$$\begin{aligned}&[G(x)H(-x)+G(-x)H(x)]^2\\ &\quad=[G(x)H(-x)-G(-x)H(x)]^2+4G(x)H(x)G(-x)H(-x)\\ &\quad=\frac{4x^2\{\psi(x^{10})\}^2}{\{f(-x^2)\}^2}+\frac{4f(-x^5)f(x^5)}{f(-x)f(x)}\\ &\quad=\frac{4x^2\{\psi(x^{10})\}^2}{\{f(-x^2)\}^2}+\frac{4\{f(-x^{10})\}^2\ \chi(-x^{10})}{\{f(-x^2)\}^2\ \chi(-x^2)}\\ &\quad=\frac{4\{\psi(x^2)\}^2}{\{f(-x^2)\}^2},\end{aligned}$$

whence (5) follows by extracting square roots and taking care to select the appropriate sign.

In order to prove (7), I transform certain standard combinations of functions of the types $\phi(x)$ and $\psi(x)$. These combinations are not the same as the combinations used by Rogers and, as a consequence, I find it necessary to use the modular equation of order 11 to effect the final simplification. Consider

$$\phi(x)\phi(x^{11})-\phi(-x)\phi(-x^{11})=\sum_{m,\,n=-\infty}^{\infty}\{1-(-1)^{m+n}\}F\{2m^2+22n^2\}$$

Take the values of m and n associated with any particular term of this double series and choose the integers M and N such that

$$3m+11n=20\text{M}+\alpha, \qquad m-3n=20\text{N}+\beta,$$

where α and β have values selected from the integers 0, ±1, ±2, ±3, ±4, ±5, ±6, ±7, ±8, ±9, 10.

Since
$$m=3\text{M}+11\text{N}+(3\alpha+11\beta)/20,$$
$$n=\text{M}-3\text{N}+(\alpha-3\beta)/20,$$

we can construct the following Table of corresponding values of α and β:

α	0	±1	±2	±3	±4	±5	±6	±7	±8	±9	10
β	0	±7	∓6	±1	±8	∓5	±2	±9	∓4	±3	10

Since
$$m+n=4\text{M}-8\text{N}+(\alpha-2\beta)/5,$$
it is evident that $m+n$ is even when, and only when, α is even. Hence, since the terms of the series for which $m+n$ is even cancel, we have to take account of odd values of α only.

When α assumes the values ±1, ±3, ±5, ±7, ±9 in succession, it is easy to see that the corresponding values of $2m^2+22n^2$ are respectively

$$40\text{M}^2+440\text{N}^2\pm\ 4\text{M}\pm308\text{N}+54,$$
$$40\text{M}^2+440\text{N}^2\pm12\text{M}\pm44\text{N}\ +2,$$
$$40\text{M}^2+440\text{N}^2\pm20\text{M}\mp220\text{N}+30,$$
$$40\text{M}^2+440\text{N}^2\pm28\text{M}\pm396\text{N}+94,$$
$$40\text{M}^2+440\text{N}^2\pm36\text{M}\pm132\text{N}+18.$$

Hence

$$\tfrac{1}{4}[\phi(x)\phi(x^{11})-\phi(-x)\phi(-x^{11})]$$
$$=x^{27}f(x^{18},\ x^{22})f(x^{66},\ x^{374})+xf(x^{14},\ x^{26})f(x^{198},\ x^{242})$$
$$+x^{15}f(x^{10},\ x^{30})f(x^{110},\ x^{330})+x^{47}f(x^{6},\ x^{34})f(x^{22},\ x^{418})$$
$$+x^{9}f(x^{2},\ x^{38})f(x^{154},\ x^{286}).$$

Again, consider
$$f(1,\ x^{2})f(1,\ x^{22})+f(-1,-x^{2})f(-1,-x^{22})$$
$$=\sum_{m,\ n=-\infty}^{\infty}\{1+(-1)^{m+n}\}\ \text{F}\{2m^2+22n^2+2m+22n\}$$

We now have to take account of even values of α only.

When α assumes the values $-8, -6, -4, -2, 0, 2, 4, 6, 8, 10$ in succession, it is easy to see that the corresponding values of

$$2m^2+22n^2+2m+22n$$

are respectively

$$\begin{aligned}
&40M^2+440N^2-4M+132N+4,\\
&40M^2+440N^2+4M-132N+4,\\
&40M^2+440N^2+12M-396N+84,\\
&40M^2+440N^2+20M+220N+24,\\
&40M^2+440N^2+28M-44N,\\
&40M^2+440N^2+36M-308N+56,\\
&40M^2+440N^2+44M+308N+60,\\
&40M^2+440N^2+52M+44N+12,\\
&40M^2+440N^2+60M-220N+44,\\
&40M^2+440N^2+68M+396N+112.
\end{aligned}$$

Hence

$$\begin{aligned}
\psi(x^2)\psi(x^{22}) = {} & x^2 f(x^{18}, x^{22}) f(x^{154}, x^{286}) + x^{42} f(x^{14}, x^{26}) f(x^{22}, x^{418})\\
& + x^{12} f(x^{10}, x^{30}) f(x^{110}, x^{330}) + f(x^6, x^{34}) f(x^{198}, x^{242})\\
& + x^{28} f(x^2, x^{38}) f(x^{66}, x^{374}).
\end{aligned}$$

Combining these results, we get

$$\begin{aligned}
&\tfrac{1}{4}[\phi(x)\phi(x^{11})-\phi(-x)\phi(-x^{11})]-x^3\psi(x^2)\psi(x^{22})\\
&\quad = x[f(x^{14}, x^{26}) - x^2 f(x^6, x^{34})][f(x^{198}, x^{242}) - x^{44} f(x^{22}, x^{418})]\\
&\qquad - x^5[f(x^{18}, x^{22}) - x^4 f(x^2, x^{38})][f(x^{154}, x^{286}) - x^{22} f(x^{66}, x^{374})].
\end{aligned}$$

Now it is easy to prove, by rearrangement of series, that

$$f(-a, -b) = f(a^3 b, a b^3) - a f(b/a, a^5 b^3),$$

and therefore, on reduction, we get

$$\begin{aligned}
&\tfrac{1}{4}[\phi(x)\phi(x^{11})-\phi(-x)\phi(-x^{11})]-x^3\psi(x^2)(\psi x^{22})\\
&= x f(-x^2, -x^8) f(-x^{44}, -x^{66}) - x^5 f(-x^4, -x^6) f(-x^{22}, -x^{88})\\
&= x f(-x^2) f(-x^{22})[H(x^2)G(x^{22}) - x^4 G(x^2)H(x^{22})].
\end{aligned}$$

Now the modular equation of order 11, which is usually written in the form

$$\sqrt{(k\lambda)}+\sqrt{(k'\lambda')}+2\sqrt{(4k\lambda k'\lambda')}=1,$$

after multiplication by $\phi(x)\phi(x^{11})$ becomes (in Ramanujan's notation)

$$4x^3\psi(x^2)\psi(x^{22})+\phi(-x)\phi(-x^{11})+4xf(-x^2)f(-x^{22})=\phi(x)\phi(x^{11}).$$

On making use of this form of the modular equation to reduce the preceding result, we immediately find that

$$xf(-x^2)f(-x^{22}) = xf(-x^2)f(-x^{22})[H(x^2)G(x^{22}) - x^4G(x^2)H(x^{22})],$$

whence we evidently obtain (7) by writing $\sqrt{x}$ in place of x.

To prove (8) we use transformations similar to those used in proving (6); by applying (1) and (2) to (3) and (4), we have

$$2G(x) = \frac{\phi(x)+\phi(x^5)}{G(x^4)f(-x^2)}$$
$$= \frac{\phi(x^4)+\phi(x^{20})}{G(x^4)f(-x^2)} + \frac{2x\psi(x^8)+2x^5\psi(x^{20})}{G(x^4)f(-x^2)}$$
$$= \frac{2f(-x^3)}{f(-x^2)}[G(x^{16})+xH(-x^4)],$$

and similarly

$$2xH(x) = \frac{\phi(x)-\phi(x^5)}{H(x^4)f(-x^2)}$$
$$= \frac{\phi(x^4)-\phi(x^{20})}{H(x^4)f(-x^2)} + \frac{2x\psi(x^8)-2x^5\psi(x^{20})}{H(x^4)f(-x^2)}$$
$$= \frac{2f(-x^8)}{f(-x^2)}[x^4H(x^{16})+xG(-x^4)].$$

Now, for brevity, write

$$T(x) = G(x^{11})H(-x) - x^2G(-x)H(x^{11}).$$

Applying these transformations, we have

$$\frac{f(-x^2)}{f(-x^8)}\,\frac{f(-x^{22})}{f(-x^{88})}\,T(x)$$
$$= [G(x^{176})+x^{11}H(-x^{44})][G(-x^4)-x^3H(x^{16})]$$
$$+ x^2[G(x^{16})-xH(-x^4)][G(-x^{44})+x^{33}H(x^{176})]$$
$$= [G(-x^4)G(x^{176})-x^{36}H(-x^4)H(x^{176})]$$
$$+ x^2[G(x^{16})G(-x^{44})-x^{12}H(x^{16})H(-x^{44})]$$
$$- x^3[G(x^{176})H(x^{16})-x^{32}G(x^{16})H(x^{176})]$$
$$- x^3[G(-x^{44})H(-x^4)-x^8G(-x^4)H(-x^{44})].$$

If we write $U(x)$ and $V(x)$ for the functions U and V defined in the enunciations of (9) and (10), this result assumes the form

$$\chi(-x^2)\chi(-x^4)\chi(-x^{22})\chi(-x^{44})T(x) = U(-x^4)+x^2V(-x^4)-2x^3,$$

when (7) is used to reduce the latter half of the expression on the right.

We reduce the even part of the expression on the right of the last equation in the following manner. It is evident that

$$\chi(-x^2)\chi(-x^4)\chi(-x^{22})\chi(-x^{44})[T(-x)-T(x)] = 4x^3.$$

We can get a second equation connecting $T(x)$ with $T(-x)$ by eliminating the ratios $G(x^{11}) : H(x^{11}) : 1$ from the three equations

$$G(x^{11})H(x) - x^2G(x)H(x^{11}) = 1,$$
$$G(x^{11})H(-x) + x^2G(-x)H(x^{11}) = T(x),$$
$$G(x^{11})H(-x^{11}) + G(-x^{11})H(x^{11}) = \frac{2}{\{\chi(-x^{22})\}^2};$$

for the result of elimination is the determinantal equation

$$\begin{vmatrix} H(x), & -x^2G(x), & 1 \\ H(-x), & x^2G(-x), & T(x) \\ H(-x^{11}), & G(-x^{11}), & \frac{2}{\{\chi(-x^{22})\}^2} \end{vmatrix} = 0$$

which, by expanding in cofactors of the last column and reducing the result with the help of (5) and (7), assumes the simple form

$$1 - T(x)T(-x) + \frac{4x^2}{\{\chi(-x^2)\chi(-x^{22})\}^2} = 0.$$

We now write q for $-x^2$ and we denote the moduli of elliptic functions with parameters q and q^{11} by k, k' and λ, λ' respectively. From the two equations connecting $T(x)$ with $T(-x)$, aided by the modular equation of order 11, it then follows that

$$\begin{aligned}
&[\chi(-x^2)\chi(-x^4)\chi(-x^{22})\chi(-x^{44})]^2[T(-x)+T(x)]^2 \\
&= 4[\chi(q)\chi(-q^2)\chi(q^{11})\chi(-q^{22})]^2 - 16q[\chi(-q^2)\chi(-q^{22})]^2 - 16q^3 \\
&= 4[\chi(q)\chi(-q^2)\chi(q^{11})\chi(-q^{22})]^2[1 - 2\sqrt[6]{(4k\lambda k'\lambda')} - \sqrt{(k\lambda)}] \\
&= 4[\chi(q)\chi(-q^2)\chi(q^{11})\chi(-q^{22})]^2\sqrt{(k'\lambda')} \\
&= 4[\chi(q)\chi(-q^2)\chi(q^{11})\chi(-q^{22})]^2\frac{[\chi(-q^2)\chi(-q^{22})]^2}{[\chi(q)\chi(q^{11})]^4}.
\end{aligned}$$

Hence, taking square roots and selecting the appropriate sign, we get

$$T(-x) + T(x) = 2\frac{\chi(-x^4)\chi(-x^{44})}{[\chi(-x^2)\chi(-x^{22})]^2} = 2\frac{\chi(x^2)\chi(x^{22})}{\chi(-x^2)\chi(-x^{22})},$$

and (8) follows immediately by combining this result with the formula obtained for $T(-x) - T(x)$.

From the equation connecting $U(-x^4)$ and $V(-x^4)$ with $T(x)$ it now follows that

$$U(-x^4) + x^2V(-x^4) = \chi(x^2)\chi(-x^4)\chi(x^{22})\chi(-x^{44}).$$

Multiply this result by the result obtained from it by changing the sign of x^2 throughout; we get

$$\begin{aligned}
\{U(-x^4)\}^2 - x^4\{V(-x^4)\}^2 &= \chi(x^2)\chi(-x^2)\chi(x^{22})\chi(-x^{22})\{\chi(-x^4)\chi(-x^{44})\}^2 \\
&= \{\chi(-x^4)\chi(-x^{44})\}^3.
\end{aligned}$$

In this equation replace $-x^4$ by x and it becomes formula (9).

Finally, to obtain (10), eliminate the ratios $G(x^{44}) : x^8H(x^{44}) : 1$ from the three equations

$$G(x)G(x^{44}) + x^9H(x)H(x^{44}) = U(x),$$
$$H(x^4)G(x^{44}) - x^8G(x^4)H(x^{44}) = 1,$$
$$G(x^{11})G(x^{44}) + x^{11}H(x^{11})H(x^{44}) = \{X(x^{11})\}^2.$$

The result of elimination is the determinantal equation

$$\begin{vmatrix} G(x), & xH(x), & U(x) \\ H(x^4), & -G(x^4), & 1 \\ G(x^{11}), & x^3H(x^{11}), & \{X(x^{11})\}^2 \end{vmatrix} = 0$$

which, by expanding in cofactors of the last column and reducing the result with the help of (3) and (7), gives

$$U(x)V(x) + x - \{X(x)X(x^{11})\}^2 = 0\ ;$$

and this is formula (10).

Of the set of forty theorems, probably about a couple of dozen are more troublesome to prove than those theorems of which the proofs have now been given.

On the sum function of the number of prime factors of N *

BY

S. SIVASANKARANARAYANA PILLAI

Introduction

Let $$n = p_1^{\alpha_1} \, p_2^{\alpha_2} \, \ldots\ldots p_r^{\alpha_r},$$

where the p's are different primes and α's $\geqslant 1$. Also let

$$f(n) = r; \qquad F(n) = \sum_{m=1}^{r} \alpha_m.$$

$$g(x) = \sum_{n=1}^{x} f(n); \qquad G(x) = \sum_{n=1}^{x} F(n).$$

In the paper entitled "*The normal number of prime factors of a number n*"** Hardy and Ramanujan state that by elementary methods, it can be proved that

$$g(x) = x \log \log x + Ax + O(x/\log x),$$

and

$$G(x) = x \log \log x + Bx + O(x/\log x),$$

where A and B are constants. Further, they say, "This problem, however, we shall dismiss for the present, as results still more precise than (1·23) and (1·24) (*i.e.*, the above) can be found by transcendental methods." In the appendix to Ramanujan's collected Papers, a method is indicated for the proof of the above result. In section I, I give a simpler proof of the result. In section II, by making use of known results about the number of primes not exceeding x, all the terms are found out till the error term is reduced to the order of $x/\log^t x$, where t is any positive integer. Section III is devoted to find out a better result by assuming the truth of Riemann's hypothesis.

I

THEOREM I. $g(x) = x \log \log x + Bx + O\,(x/\log x)$.

If p is a prime, the contribution to $g\,(x)$ due to p is the number of multiples of p, which do not exceed x, that is $[x/p]$, where $[x]$ denotes the integral part of x. Hence,

$$g(x) = \sum_{p \leqslant x} [x/p] = \sum_{p \leqslant x} x/p + O\,(x/\log x),$$

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* This paper was read at the Conference of the Indian Mathematical Society held at Bombay in 1932.

** Ramanujan: *Collected Papers,*

since there are $\pi(x)$ terms, and $\pi(x)=O(x/\log x)$, a result which is proved by elementary methods. By elementary methods it can be proved that

$$\sum_{p \leqslant x} 1/p = \log \log x + B + O(1/\log x).$$

Hence, $\quad g(x)=x \log \log x + Bx + O(x/\log x).$

THEOREM II. $\quad G(x)=g(x)+Ax+O(\sqrt{x}),$ *where* $\quad A=\sum \dfrac{1}{p(p-1)}.$

It is easily seen that

$$G(x)=\sum_{p\leqslant x}[x/p] + \sum_{p \leqslant x^{\frac{1}{2}}}[x/p^2] + \ldots\ldots;$$

there being k such sums, where $k=[\log x/\log 2]$.

Therefore, $G(x)-g(x) = \sum \sum_{p \leqslant x^{1/r}} [x/p^r]$

$$= \sum \sum_{p \leqslant \sqrt{x}} [x/p^r]$$

$$= \sum \sum_{p \leqslant \sqrt{x}} x/p^r + O(\sqrt{x})$$

the first summation extending from $r=2$ to $r=[\log x/\log 2]$,

$$= x \sum_{p \leqslant \sqrt{x}} \left\{ \frac{1}{p^2}+\frac{1}{p^3}+ \ldots \text{to } [\log x/\log 2] \text{ terms} \right\} + O(\sqrt{x})$$

$$= x \sum_{p \leqslant \sqrt{x}} \left\{ \sum_{r=2}^{\infty} \frac{1}{p^r} + O\left(p^{-\log x/\log 2} \times \frac{p}{p-1}\right)\right\} + O(\sqrt{x})$$

$$= x \sum_{p \leqslant \sqrt{x}} \left\{ \frac{1}{p(p-1)} + O\left(\frac{1}{x}\right)\right\} + O(\sqrt{x})$$

$$= x \sum \frac{1}{p(p-1)} + O\left(x \sum_{p \leqslant \sqrt{x}} \frac{1}{p(p-1)}\right) + O(\sqrt{x})$$

$$= A x + O\left(x \times \frac{1}{\sqrt{x}}\right) + O(\sqrt{x})$$

$$= A x + O(\sqrt{x}).$$

Thus theorem II has been proved.

Corollary. $G(x)=x \log \log x + (A+B)x + O(x/\log x).$

THEOREM III. $g(x) = \sum_{r=1}^{\infty} \pi\,(x/r)$ *where* $\pi\,(x)$ *denotes the number of primes* $\leqslant x$.

If $(r+1)p > x \geqslant rp$, then the contribution to $g(x)$ due to p is r.

Therefore,

$$g(x) = \sum_{r=1}^{x} r \sum_{\frac{x}{r} \geqslant p \geqslant \frac{x}{r+1}} 1$$

$$= \sum_{r=1}^{x} r \left\{ \pi\left(\frac{x}{r}\right) - \pi\left(\frac{x}{r+1}\right) \right\} = \sum_{r=1}^{\infty} \pi\,(x/r).$$

Now, we shall give an alternative proof for this theorem, which leads to an important transformation formula.

Let (x) denote the fractional part of x. If r is a positive integer, and $(r+1)p > x \geqslant rp$, then $\left(\frac{x}{p}\right) = \frac{x - rp}{p}$.

Therefore,

$$g(x) = \sum_{p \leqslant x} [x/p] = \sum_{p \leqslant x} x/p - \sum_{p \leqslant x} (x/p)$$

$$= \sum_{p \leqslant x} \frac{x}{p} - \sum_{r=1}^{n-1} \sum_{\frac{x}{r} \leqslant p \leqslant \frac{x}{r+1}} \frac{x - rp}{p} - \sum_{p \leqslant x/n} (x/p),$$

where n is a positive integer not exceeding x. Thus,

$$g(x) = \sum_{p \leqslant x} \frac{x}{p} - \sum_{r=1}^{n-1} \sum_{x/r \geqslant p \geqslant x/(r+1)} \frac{x}{p} + \sum_{r=1}^{n-1} \sum_{x/r \geqslant p \geqslant x/(r+1)} r - \sum_{p \leqslant x/n} \left(\frac{x}{p}\right)$$

$$= \sum_{p \leqslant x/n} \frac{x}{p} + \sum_{r=1}^{n-1} \sum_{x/r \geqslant p \geqslant x(r+1)} r - \sum_{p \leqslant x/n} \left(\frac{x}{p}\right)$$

$$= \sum_{p \leqslant x/n} \frac{x}{p} + \sum_{r=1}^{n-1} r \left\{ \pi\left(\frac{x}{r}\right) - \pi\left(\frac{x}{r+1}\right) \right\} - \sum_{p \leqslant x/n} \left(\frac{x}{p}\right)$$

$$= \sum_{p \leqslant x/n} \frac{x}{p} + \sum_{r=1}^{n} \pi\left(\frac{x}{r}\right) - n\,\pi\left(\frac{x}{n}\right) - \sum_{p \leqslant x/n} (x/p), \qquad \text{(A)}$$

Take $n > \frac{x}{2}$; then, since $\pi\,(x) = 0$ when $x < 2$, we have

$$g(x) = \sum_{r=1}^{n} \pi\,(x/r) = \sum_{r=1}^{\infty} \pi\,(x/r)$$

Corollary $\displaystyle\sum_{r=1}^{\infty} \pi\,(x/r) = x \log \log x + Bx + O\left(\frac{x}{\log x}\right)$

The formula A is important. It is with the help of this, that the results in the other two sections are proved.

II

The result proved in this section is this:

THEOREM IV. *If n is any given positive integer, then*

$$g(x) = x \log\log x + Bx + x \sum_{t=1}^{n-1} \frac{h_t}{\log^t x} + O\left(\frac{x}{\log^n x}\right),$$

where $$h_t = (t-1)! \left\{ -1 + A_0 + \frac{A_1}{1!} + \ldots\ldots + \frac{A_{t-1}}{(t-1)!} \right\},$$

and $$A_s = \lim_{m\to\infty} \left\{ \sum_{n=1}^{m} \frac{(\log n)^s}{n} - \frac{(\log m)^{s+1}}{s+1} \right\}$$

To prove our result, we want several lemmas.

Lemma 1. *If* $${}_sH_l = \frac{s(s+1)\ \ .\ \ .\ \ .\ \ (s+l-1)}{l!},$$

then $${}_sH_l \leqslant \frac{(s+l-1)^{s-1}}{(s-1)!}$$

The result is obvious.

Lemma 2. *When* $0 < a \leqslant 1/3$,

$$S = {}_sH_l + {}_sH_{l+1}\cdot a + {}_sH_{l+2}\cdot a^2 + \ldots\ldots\ldots\ldots = O\left(\frac{(s+l-1)^{s-1}}{(s-1)!}\right).$$

By lemma 1,

$$S \leqslant \frac{(s+l-1)^{s-1}}{(s-1)!} \left\{ 1 + \left(1 + \frac{1}{s+l-1}\right)^{s-1} a + \left(1 + \frac{2}{s+l-1}\right)^{s-1} a^2 + \ldots\ldots\ldots \right\}$$

$$\leqslant \frac{(s+l-1)^{s-1}}{(s-1)!} \left\{ 1 + ea + e^2a^2 + \cdot\cdot\cdot\cdot\cdot \right\}$$

$$= O\left(\frac{(s+l-1)^{s-1}}{(s-1)!}\right), \qquad \text{since } ea < 1.$$

Lemma 3. *If* $s \geqslant 2$, *then* $$\frac{{}_sH_{h-1}}{h} = \frac{{}_{s-1}H_h}{s-1}.$$

For $$\frac{{}_sH_{h-1}}{h} = \frac{1}{h} \times \frac{s(s+1)\ldots\ldots(s+h-2)}{(h-1)!}$$

$$= \frac{1}{s-1} \times \frac{(s-1)s\ldots\ldots(s-1+h-1)}{h!}$$

$$= \frac{{}_{s-1}H_h}{s-1}.$$

Lemma 4. *If* $M = \sum_{s=1}^{n} \sum_{h=0}^{n-s} \frac{(s-1)!\ {}_sH_h \cdot A_h}{(\log x)^{h+s}}$,

then $$M = \frac{1}{x} \sum_{h=0}^{n-1} A_h \int_2^x \frac{dt}{(\log t)^{h+1}} + o\left(\frac{1}{\log^n x}\right).$$

Now

$$M = \sum_{s=1}^{n} \sum_{h=0}^{n-s} \left(\frac{(s+h-1)!}{h!}\right) \cdot \frac{A_h}{(\log x)^{h+s}}$$

$$= \sum_{h=0}^{n-1} \frac{A_h}{h!} \sum_{s=1}^{n-h} \frac{(h-1+s)!}{(\log x)^{h+s}}$$

$$= \frac{1}{x} \sum_{h=0}^{n-1} A_h \left\{ \int_2^x \frac{dt}{(\log t)^{h+1}} + O\left(\int_2^x \frac{dt}{\log^{n+1} t} \right) + O(1) \right\}$$

$$= \frac{1}{x} \sum_{h=0}^{n-1} A_h \int_2^x \frac{dt}{(\log t)^{h+1}} + o\left(\frac{1}{\log^n x}\right),$$

since $$\int_2^x \frac{dt}{\log^{n+1} t} = O\left(\frac{x}{\log^{n+1} x}\right).$$

Now we are in a position to prove the theorem. If n is any given positive integer, then

$$\pi(x) = x \left\{ \frac{1}{\log x} + \frac{1}{\log^2 x} + \cdots + \frac{(n-1)!}{\log^n x} + O\left(\frac{1}{\log^{n+1} x}\right) \right\}$$

and

$$\sum_{p \leqslant x} \frac{1}{p} = \log \log x + B + O\left(\frac{1}{\log^{n+1} x}\right).$$

Therefore, from formula A in Section I,

$$g(x) = \sum_{p \leqslant x\, r} \frac{x}{p} + \sum_{t=1}^{r} \pi\left(\frac{x}{t}\right) - r\,\pi\left(\frac{x}{r}\right) - \sum_{p \leqslant x/r} \left(\frac{x}{p}\right)$$

$$= x \left\{ \log \log \frac{x}{r} + B + O\left(\frac{1}{(\log(x/r))^{n+1}}\right) \right\}$$

$$+ \sum_{t=1}^{r} \pi\left(\frac{x}{t}\right) - r\,\pi\left(\frac{x}{r}\right) + O\left(\frac{x}{r \log x/r}\right)$$

Let $r = [\exp.(n \log \log x)]$. Then

$$\log \log \frac{x}{r} = \log \log x + \log\left(1 - \frac{\log r}{\log x}\right).$$

Therefore,

$$g(x)=x\log\log x+Bx+x\log\left(1-\frac{\log r}{\log x}\right)$$

$$+\sum_{t=1}^{r}\pi\left(\frac{x}{t}\right)-r\pi\left(\frac{x}{r}\right)+O\left(\frac{x}{\log^{n+1}x}\right). \qquad (1)$$

Now

$$\sum_{t=1}^{r}\pi\left(\frac{x}{t}\right)=\sum_{t=1}^{r}\left\{\sum_{s=1}^{n}\frac{x(s-1)!}{t\left(\log\frac{x}{t}\right)^{s}}+O\left(\frac{x}{t\left(\log\frac{x}{t}\right)^{n+1}}\right)\right\}.$$

$$=x\sum_{s=1}^{n}(s-1)!\sum_{t=1}^{r}\frac{1}{t(\log x-\log t)^{s}}+O\left(\frac{x}{\log^{n+1}x}\sum_{t=1}^{r}\frac{1}{t}\right)$$

$$=x\sum_{s=1}^{n}(s-1)!\ T_s+o\left(\frac{x}{\log^{n}x}\right), \quad \text{(say)}. \qquad (2)$$

$$T_s=\sum_{t=1}^{r}\frac{1}{t(\log x-\log t)^{s}}$$

$$=\frac{1}{(\log x)^{s}}\sum_{t=1}^{r}\frac{1}{t(1-a)^{s}}, \qquad \text{where } a=\frac{\log t}{\log x},$$

$$=\frac{1}{(\log x)^{s}}\sum_{t=1}^{r}\frac{1}{t}\left\{1+{}_{s}H_1\,a+{}_{s}H_2\,a^{2}+\cdot\ \cdot+{}_{s}H_{n-s}\,a^{n-s}+O\left(\frac{n^{s-1}a^{n-s+1}}{(s-1)!}\right)\right\}$$

by lemma 2

$$=\frac{1}{(\log x)^{s}}\left\{\sum_{t=1}^{r}\frac{1}{t}+\frac{{}_{s}H_1}{\log x}\sum_{t=1}^{r}\frac{\log t}{t}+\cdots\cdots\right.$$

$$\left.+\frac{{}_{s}H_{n-s}}{(\log x)^{n-s}}\sum_{t=1}^{r}\frac{(\log t)^{n-s}}{t}+O\left(\frac{1}{(\log x)^{n-s+1}}\sum_{1}^{t}\frac{(\log t)^{n-s+1}}{t}\right)\right\}$$

$$=\frac{1}{(\log x)^{s}}\left\{\log r+A_0+O\left(\frac{1}{r}\right)+\cdots\cdots\cdots\right.$$

$$\left.+\frac{{}_{s}H_{n-s}}{(\log x)^{n-s}}\left(\frac{(\log r)^{n-s+1}}{n-s+1}+A_{n-s}+O\left(\frac{\log^{n-s}r}{r}\right)\right)+O\left(\frac{\log^{n-s+2}r}{\log^{n-s+1}x}\right)\right\}$$

since $\displaystyle\sum_{s=1}^{r}\frac{(\log t)^{s}}{t}=\frac{(\log r)^{s+1}}{s+1}+A_s+O\left(\frac{\log^{s}r}{r}\right).$

Now, $r=\left[(\log x)^{n}\right]$. Hence

$$T_s = \frac{1}{(\log x)^{s-1}} \sum_{h=1}^{n-s+1} \frac{{}_sH_{h-1}}{h} \left(\frac{\log r}{\log x}\right)^h + \frac{1}{(\log x)^s} \sum_{h=o}^{n-s} \frac{A_h \cdot {}_sH_h}{(\log x)^h}$$

$$+ O\left(\frac{1}{r \log^s x} \sum_{h=o}^{n-s} \left(\frac{\log r}{\log x}\right)^h\right) + o\left(\frac{1}{\log^n x}\right)$$

$$= R_1 + R_2 + O(R_3) + o\left(\frac{1}{\log^n x}\right), \quad \text{say.}$$

$$R_3 = O\left(\frac{1}{(\log x)^n} \times \frac{1}{\log^s x} \times \frac{1}{1 - \frac{\log r}{\log x}}\right) = o\left(\frac{1}{\log^n x}\right).$$

When $s \geqslant 2$, by lemma 3,

$$R_1 = \frac{1}{(s-1)(\log x)^{s-1}} \sum_{h=o}^{n-s+1} {}_{s-1}H_h \left(\frac{\log r}{\log x}\right)^h$$

$$= \frac{1}{(s-1)(\log x)^{s-1}} \left\{ \frac{1}{\left(1 - \frac{\log r}{\log x}\right)^{s-1}} - 1 + O\left(\frac{\log r}{\log x}\right)^{n-s+2} \right\}$$

$$= \frac{1}{(s-1)(\log x - \log r)^{s-1}} - \frac{1}{(s-1)(\log x)^{s-1}} + o\left(\frac{1}{\log^n x}\right).$$

When $s = 1$,

$$R_1 = \sum_{h=1}^{n} \frac{1}{h} \left(\frac{\log r}{\log x}\right)^h = -\log\left(1 - \frac{\log r}{\log x}\right) + o\left(\frac{1}{\log^n x}\right)$$

Therefore, when $s \geqslant 2$,

$$T_s = \frac{1}{(s-1)(\log x - \log r)^{s-1}} - \frac{1}{(s-1)(\log x)^{s-1}}$$

$$+ \frac{1}{(\log x)^s} \sum_{h=o}^{n-s} \frac{{}_sH_h \, A_h}{(\log x)^h} + o\left(\frac{1}{\log^n x}\right), \tag{3}$$

and

$$T_1 = -\log\left(1 - \frac{\log r}{\log x}\right) + \sum_{h=o}^{n-1} \frac{A_h}{(\log x)^{h+1}} + o\left(\frac{1}{\log^n x}\right) \tag{4}$$

Now, from (1) and (2),

$$g(x) = x \log \log x + Bx + x \log\left(1 - \frac{\log r}{\log x}\right)$$

$$+ x \sum_{s=1}^{n} (s-1)!\, T_s - r\pi\left(\frac{x}{r}\right) + o\left(\frac{x}{\log^n x}\right) \tag{5}$$

Therefore, from (5), (4) and (3),

$$g(x)=x\log\log x+Bx+x\log\left(1-\frac{\log r}{\log x}\right)$$

$$-x\log\left(1-\frac{\log r}{\log x}\right)+x\sum_{h=0}^{n-1}\frac{A_h}{(\log x)^{h+1}}$$

$$+x\sum_{s=2}^{n}\left\{\frac{(s-1)!}{(s-1)(\log x-\log r)^{s-1}}-\frac{(s-1)!}{(s-1)(\log x)^{s-1}}\right\}$$

$$+x\sum_{s=2}^{n}(s-1)!\sum_{h=0}^{n-s}\frac{{}_sH_h\cdot A_h}{(\log x)^{h+s}}+o\left(\frac{x}{\log^n x}\right)$$

$$-r\sum_{s=1}^{n}\frac{x}{r}\frac{(s-1)!}{(\log x-\log r)^s}+O\left\{r\cdot\frac{x}{r}\times\frac{1}{(\log x/r)^{n+1}}\right\}$$

$$=x\log\log x+Bx+x\sum_{s=1}^{n}(s-1)!\sum_{h=0}^{n-s}\frac{{}_sH_h\cdot A_h}{(\log x)^{h+s}}$$

$$-x\sum_{s=1}^{n-1}\frac{(s-1)!}{(\log x)^s}+O\left(\frac{x}{\log^n x}\right)$$

$$=x\log\log x+Bx+\sum_{h=0}^{n-1}A_h\int_2^x\frac{dt}{(\log t)^{h+1}}+o\left(\frac{1}{\log^n x}\right)$$

$$-\int_2^x\frac{dt}{\log t}+O\left(\frac{x}{\log^n x}\right)$$

$$=x\log\log x+Bx+\sum_{h=0}^{n-2}A_h\int_2^x\frac{dt}{(\log t)^{h+1}}-\int_2^x\frac{dt}{\log t}+O\left(\frac{x}{\log^n x}\right)$$

$$=x\log\log x+Bx+\sum_{t=1}^{n-1}\frac{x\cdot h_t}{(\log x)^t}+O\left(\frac{x}{\log^n x}\right),\ \text{where}$$

$$h_t=(t-1)!\left\{-1+A_0+\frac{A_1}{1!}+\ \ldots\ +\frac{A_{t-1}}{(t-1)!}\right\}$$

Thus the proof of theorem IV is completed.

Corollary. $\sum_{p\leqq x}\left(\frac{x}{p}\right)\sim(1-\gamma)\frac{x}{\log x}$, *where* γ *is Euler's constant.*

Now $\sum\limits_{p\leqslant x}\left(\frac{x}{p}\right)=\sum\limits_{p\leqslant x}\frac{x}{p}-g(x)$

$$= x\log\log x+\mathrm{B}x+\mathrm{O}\left(\frac{x}{\log^n x}\right)$$

$$-\left\{x\log\log x+\mathrm{B}x+\sum_{t=1}^{n-1}\frac{h_t\,x}{(\log x)^t}+\mathrm{O}\left(\frac{x}{\log^n x}\right)\right\}$$

$$=-\sum_{t=1}^{n-1}\frac{h_t\,x}{(\log x)}+\mathrm{O}\left(\frac{x}{\log^n x}\right)$$

$$\sim -\frac{h_1 x}{\log x}$$

$$=(1-\mathrm{A}_0)\frac{x}{\log x}=(1-\gamma)\frac{x}{\log x}.$$

But, we know that

$$\sum_{n\leqslant x}(x/n)=(1-\gamma)x.$$

Hence

$$\lim_{x\to\infty}\frac{\sum\limits_{p\leqslant x}(x/p)}{\sum\limits_{n\leqslant x}(x/n)}\times\frac{x}{\pi(x)}=1.$$

This result is interesting.

III

Throughout this section, the truth of the Riemann hypothesis is assumed. To prove the theorem in this section, we are in need of some more lemmas.

Lemma 5 *If* $1\leqslant t\leqslant x^{1/3}$, *then*

$$\int_{x^{2/3}}^{x/t}\frac{du}{\log u}=\left[\sum_{m=1}^{n}\frac{(m-1)!\cdot u}{(\log u)^m}\right]_{x^{2/3}}^{x/t}+\mathrm{O}\left(x^{1/3}\right),$$

where $n=\left[\frac{\log x}{3}\right]+1.$

By integration by parts,

$$\int\frac{du}{\log u}=\sum_{1}^{r}\frac{(m-1)!\,u}{(\log u)^m}+r!\int\frac{du}{(\log u)^{r+1}}.$$

Therefore,

$$\int_{x^{2/3}}^{x/t}\frac{du}{\log u}=\left[\sum_{m=1}^{n}\frac{(m-1)!\cdot u}{(\log u)^m}\right]_{x^{2/3}}^{x/t}+\mathrm{R},\tag{6}$$

where $$R = n! \int_{x^{2/3}}^{x/t} \frac{du}{(\log u)^{n+1}}.$$

Now, $$R \leqslant n! \int_{x^{2/3}}^{x} \frac{du}{(\log u)^{n+1}}$$

$$= \left[\sum_{r=n}^{N} \frac{(r-1)!\, u}{(\log u)^{r+1}} \right]_{x^{2/3}}^{x} + N! \int_{x^{2/3}}^{x} \frac{du}{(\log u)^{N+1}}.$$

Now, (1) $$\frac{(r-1)!\, x}{\log^n x} > \frac{r!\, x}{\log^{n+1} x}, \quad \text{if } r < \log x,$$

and (2) $$\frac{x}{\log^r x} > \frac{x^{2/3}}{(\log x^{2/3})^r},$$

if $x^{1/3} > \left(\frac{3}{2}\right)^r$,

or if $r < \frac{1}{3} \log x / \log \frac{3}{2}$,

or if $r < {\cdot}8 \log x$.

Therefore, when $N = [{\cdot}8 \log x - 1]$,

$$R \leqslant 2N \frac{n!\, x}{(\log x)^{n+1}} + N! \int_{x^{2/3}}^{x} \frac{du}{(\log u)^{N+1}}$$

$$= O\left(\frac{n!\, x}{(\log x)^n}\right) + N! \int_{x^{2/3}}^{x} \frac{du}{(\log u)^{N+1}}. \tag{7}$$

$$\int_{x^{2/3}}^{x} \frac{ud}{(\log u)^{N+1}} = \int_{x^{2/3}}^{x^{5/6}} \frac{du}{(\log u)^{N+1}} + \int_{x^{5/6}}^{x} \frac{du}{(\log u)^{N+1}}$$

$$= O\left\{ \left(\frac{3}{2\log x}\right)^{N+1} \cdot x^{5/6} + \left(\frac{6}{5 \log x}\right)^{N+1} \cdot x \right\}.$$

Now,

$${\cdot}8 \log x \left(\log \tfrac{3}{2} - \log\log x\right) + \tfrac{5}{6} \log x - {\cdot}8 \log x \left(\log \tfrac{6}{5} - \log\log x\right) - \log x$$

$$= \log x \left\{ {\cdot}8 \log\left(\tfrac{3}{2} \times \tfrac{6}{5}\right) - \tfrac{1}{6} \right\} > 0.$$

Therefore,

$$\int_{x^{2/3}}^{x} \frac{du}{(\log u)^{N+1}} = O\left(\left(\frac{3}{2 \log x}\right)^{N+1} x^{5/6}\right) \tag{8}$$

Let $$S = N! \left(\frac{3}{2 \log x}\right)^{N+1} \cdot x^{5/6}.$$

Then by Stirling's theorem,

$$\log S \leqslant \cdot 8 \log x \log\log x - \log\log x + \cdot 8 \log x \log(\cdot 8) + \tfrac{1}{2} \log\log x$$
$$- \cdot 8 \log x + \cdot 8 \log x \log\tfrac{3}{2} - \cdot 8 \log x \log\log x + \tfrac{5}{6} \log x + O(1)$$
$$\leqslant \log x\{\cdot 8 \log(\cdot 8) - \cdot 8 + \cdot 8 \log\tfrac{3}{2} + \tfrac{5}{6}\} + O(1)$$
$$\leqslant \log x\{\cdot 8 \log(1 + \tfrac{2}{10}) - \cdot 8 + \tfrac{5}{6}\} + O(1)$$
$$\leqslant \log x(\cdot 8 \times \cdot 2 - \cdot 8 + \cdot 84) + O(1)$$
$$= \cdot 2 \log x + O(1) \leqslant \tfrac{1}{3} \log x + O(1) \tag{9}$$

Therefore, from (9) and (8),

$$N! \int_{x^{2/3}}^{x} \frac{du}{(\log u)^{N+1}} = O(x^{1/3}) \tag{10}$$

Again,

$$\log\left\{\frac{n!\,x}{\log^n x}\right\} = n \log n + \tfrac{1}{2} \log n - n + \log x - n \log\log x + O(1)$$
$$\leqslant \frac{\log x}{3} \log\log x - \frac{\log 3}{3} \log x + \tfrac{1}{2} \log\log x$$
$$- \frac{\log x}{3} + \log x - \frac{\log x}{3} \log\log x + O(1)$$
$$\leqslant \log x \left\{ - \frac{\log 3}{2} - \frac{1}{3} + 1 + \frac{\log\log x}{2 \log x} \right\} + O(1)$$
$$\leqslant \tfrac{1}{3} \log x + O(1)$$

Hence,

$$n! \int_{x^{2/3}}^{x} \frac{du}{(\log^u)^{n+1}} = O(x^{1/3}) . \tag{11}$$

Hence, from (6), (7), (10) and (11), the lemma follows.

Lemma 6. *When* $s \leqslant \log x$ *and* $r = [x^{1/3}]$,

$$L = \left| \frac{\log^s r}{r} - \frac{\log^s(r+1)}{r+1} - \frac{\log^s r}{r(r+1)} \right| \leqslant \frac{s \log^{s-1} r}{r(r+1)} \left(1 + \frac{s}{r}\right).$$

Now,

$$L = \left| \frac{(r+1) \log^s r - r \log^s (r+1) - \log^s r}{r(r+1)} \right|$$

$$= \frac{1}{r+1} \left\{ s \log^{s-1} r \log\left(1 + \frac{1}{r}\right) + \frac{s(s-1)}{2!} \log^{s-2} r \quad \log^2 \left(1 + \frac{1}{r}\right) + \ldots \right\}$$

$$\leqslant \frac{1}{r+1}\left\{\frac{s}{r}(\log r)^{s-1}+\frac{s^2}{2\,!\,r^2}(\log r)^{s-2}+\ldots\ldots\ldots\right\}$$

$$\leqslant \frac{s\log^{s-1}r}{r(r+1)}\left\{1+\frac{1}{2\,!}\cdot\frac{s}{r}+\frac{1}{3\,!}\left(\frac{s}{r}\right)^2+\ \cdots\cdots\right\}$$

$$\leqslant \frac{s\log^{s-1}r}{r(r+1)}\left\{1+\frac{1}{2\,!}\,\frac{s}{r}\left(1+\frac{s}{r}+\left(\frac{s}{r}\right)^2+\ \cdots\cdots\right)\right\}$$

$$\leqslant \frac{s\log^{s-1}r}{r(r+1)}\left(1+\frac{s}{r}\right),\qquad \text{since } \frac{s}{r}<2.$$

Lemma 7. If $s\leqslant \log x$, *and* $r=[x^{1/3}]$, *then*

$$S=\sum_{t=r}^{\infty}\frac{\log^s r}{t(t+1)}=O\left(\frac{s\log^s r}{r}\right).$$

Now, by lemma (6),

$$S\leqslant \frac{\log^s r}{r}+s\left(1+\frac{s}{r}\right)\sum_{t=r}\frac{\log^{s-1}t}{t(t+1)}$$

$$\leqslant \frac{\log^s r}{r}+s\left(1+\frac{s}{r}\right)\frac{\log^{s-1}r}{r}+s(s-1)\left(1+\frac{s}{r}\right)\left(1+\frac{s-1}{r}\right)\sum_{t=1}^{\infty}\frac{\log^{s-2}t}{t(t+1)}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\leqslant \frac{\log^s r}{r}\left\{1+\frac{s}{\log r}\left(1+\frac{s}{r}\right)+\left(\frac{s}{\log r}\left(1+\frac{s}{r}\right)\right)^2+\ \cdots \text{ to } s \text{ terms}\right\}$$

$$\leqslant \frac{\log^s r}{r}\left\{1+\left(1+\frac{s}{r}\right)+\left(1+\frac{s}{r}\right)^2+\ \cdots\cdots\right\}$$

$$\leqslant \frac{\log^s r}{r}s\left(1+\frac{s}{r}\right)^s=O\left(\frac{s\log^s r}{r}\right).$$

Lemma 8. If $s\leqslant \log x$, *and* $r=[x^{1/3}]$, *then*

$$\sum_{t=1}^{r}\frac{\log^s t}{t}=\frac{\log^{s+1}r}{s+1}+A_s+O\left(s\cdot 2^s\,\frac{\log^s r}{r}\right),$$

where A_s *is a constant depending upon* s *alone.*

$$\frac{1}{s+1}\left\{\left(\log(t+1)\right)^{s+1}-(\log t)^{s+1}\right\}$$

$$=\log^s t\log\left(1+\frac{1}{t}\right)+\frac{s}{2\,!}\log^{s-1}t\log^2\left(1+\frac{1}{t}\right)+\cdots\cdots\cdots$$

$$=\frac{\log^s t}{t}+O\left(\frac{\log^s t}{t^2}\right)+O\left(\frac{s}{2!}\cdot\frac{\log^{s-1}t}{t^2}+\frac{s(s-1)}{3\,!}\,\frac{\log^{s-2}t}{t^3}+\ \cdots\cdots\right)$$

$$= \frac{\log^s t}{t} + O\left(\frac{\log^s t}{t^2}\right) + O\left(\frac{\log^{s-1} t}{(s+1)t^2}\left({}_{s+1}C_2 + {}_{s+1}C_3 + \cdots\right)\right)$$

$$= \frac{\log^s t}{t} + O\left(\frac{\log^s t}{t^2}\right) + O\left(\frac{\log^{s-1} t \cdot 2^{s+1}}{s \cdot t^2}\right)$$

$$= \frac{\log^s t}{t} + O\left(\frac{2^s \log^s t}{t^2}\right).$$

Therefore,

$$\sum_{t=1}^{r} \frac{\log^s t}{t} - \frac{(\log(r+1))^{s+1}}{s+1} = O\left(2^s \sum_{t=1}^{r} \frac{\log^s t}{t^2}\right)$$

$$= O\left\{2^s \sum_{t=1}^{r} \frac{\log^s t}{t(t+1)}\right\}$$

$$= A_s + O\left(s \cdot \frac{2^s \log^s r}{r}\right),$$

by lemma (7). Thus, lemma 8 is proved.

Lemma 9. If $r = [x^{1/3}]$ *and* $n = \left[\frac{\log x}{3}\right]$, *then*

$$I = \frac{1}{(\log x)^{s-1}}\left\{ {}_{s-1}H_{n-s+2}\left(\frac{\log r}{\log x}\right)^{n-s+2} + {}_{s-1}H_{n-s+3}\left(\frac{\log r}{\log x}\right)^{n-s+3} + \cdots \right\}$$
$$= O(x^{-1/3}).$$

From lemma 2,

$$I = O\left(\frac{1}{\log^{s-1} x} \times \frac{n^{s-2}}{(s-2)!} \times \left(\frac{\log r}{\log x}\right)^{n-s+2}\right)$$

$$= O\left(\frac{1}{\log^{s-1} x} \times \left(\frac{\log x}{3}\right)^{s-2} \times \frac{1}{(s-2)!} \times \frac{1}{3^{n-s+2}}\right)$$

$$= O\left(\frac{1}{s-2!} \times \frac{1}{3^n}\right) = O(x^{-1/3}).$$

Lemma 10. If $r = \left[x^{1/3}\right]$, $n = \left[\frac{\log x}{3}\right]$, *and*

$$T_s = \sum_{t=1}^{r} \frac{1}{t(\log x - \log t)^s},$$

then $$T_1 = -\log\left(1 - \frac{\log r}{\log x}\right) + \sum_{h=0}^{n-1} \frac{A_h}{(\log x)^{h+1}} + O\left(\frac{1}{x^{1/3}},\right)$$

and $$T_s = \frac{1}{(s-1)(\log x - \log r)^{s-1}} - \frac{1}{(s-1)(\log x)^{s-1}}$$

$$+ \frac{1}{(\log x)^s} \sum_{h=0}^{n-s} \frac{A_h.\ sH_h}{(\log x)^h} + O\left(\frac{1}{x^{1/3}}\right) \quad \textit{when } s \geq 2.$$

Proof.

$$T_s=\frac{1}{(\log x)^s}\sum_{1}^{r}\frac{1}{t(1-a)^s},\qquad \text{where } a=\frac{\log r}{\log x},$$

$$=\frac{1}{(\log x)^s}\sum_{1}^{r}\frac{1}{t}\left\{1+{}_sH_1.a+\ldots+{}_sH_{n-}\ a^{n-s}+O(n^{n-s}a^{n-s+1})\right\}$$

$$=\frac{1}{(\log x)^s}\left\{\sum_{1}^{r}\frac{1}{t}+\frac{{}_sH_1}{\log x}\sum_{1}^{r}\frac{\log t}{t}+\ldots\ldots+\frac{{}_sH_{n-s}}{(\log x)^{n-s}}\sum_{1}^{r}\frac{(\log t)^{n-s}}{t}\right.$$

$$\left.+O\left(\frac{n^{s-1}}{(\log x)^{n-s+1}}\sum_{1}^{r}\frac{(\log t)^{n-s+1}}{t}\right)\right\},\quad \text{(by lemma 2)}$$

$$=\frac{1}{(\log x)^s}\left\{\log r+A_0+O\left(\frac{1}{r}\right)+\ldots\ldots\right.$$

$$+\frac{{}_sH_{n-s}}{(\log x)^{n-s}}\left(\frac{(\log r)^{n-s+1}}{n-s+1}+A_{n-s}+O\left((n-s)2^{n-s}\ \frac{(\log r)^{n-s}}{r}\right)\right.$$

$$\left.\left.+O\left(\frac{n^{s-1}}{(\log x)^{n-s+1}}(\log r)^{n-s+2}\right)\right\}\quad \text{by lemma (8)}$$

$$=\frac{1}{(\log x)^{s-1}}\sum_{h=1}^{n-s+1}\frac{{}_sH_{h-1}}{h}\left(\frac{\log r}{\log x}\right)^h+\frac{1}{(\log x)^s}\sum_{h=0}^{n-s}\frac{A_h\cdot{}_sH_h}{(\log x)^h}$$

$$+O\left\{\frac{n}{r(\log x)^s}\sum_{h=0}^{n-s}\left(\frac{2\log r}{\log x}\right)^h\cdot{}_sH_h\right\}+O\left\{\frac{n^{s-1}\log r}{(\log x)^s}\left(\frac{\log r}{\log x}\right)^{n-s+1}\right\}$$

$$=R_1+R_2+O(R_3)+O(R_4),\ \text{say} \tag{12}$$

$$R_4=O\left(\frac{1}{3^n}\right)=O\left(\frac{1}{x^{1/3}}\right). \tag{13}$$

$$R_3=O\left\{\frac{n}{r(\log x)^s}\times\frac{1}{\left(1-\frac{2\log r}{\log x}\right)^s}\right\}$$

$$=O\left\{\frac{1}{r(\log x)^s}\times\frac{1}{(1-\frac{2}{3})^s}\right\}=O\left\{\frac{1}{r}\left(\frac{3}{\log x}\right)^s\right\}$$

$$=O\left\{\frac{1}{r}\right\}=O\left(\frac{1}{x^{1/3}}\right).$$

When $s\geqslant 2$, by lemma (3), (14)

$$R_1=\frac{1}{(s-1)(\log x)^{s-1}}\sum_{h=1}^{n-s+1}{}_{s-1}H_h\left(\frac{\log r}{\log x}\right)^h$$

$$= \frac{1}{(s-1)(\log x)^{s-1}} \left\{ \frac{1}{(1-\log r/\log x)^{s-1}} - 1 \right.$$

$$\left. + O\left({}_{s-1}H_{n-s+2}(\log r/\log x)^{n-s+2} + {}_{s-1}H_{n-s+3}(\log r/\log x)^{n-s+3} + \cdots\right)\right\}$$

$$= \frac{1}{(s-1)(\log x - \log r)^{s-1}} - \frac{1}{(s-1)(\log x)^{s-1}} + O(1/x^{1/3}), \qquad (15)$$

by lemma (9).

When $s=1$,

$$R_1 = \sum_{h=1}^{n} \frac{1}{h}\left(\frac{\log r}{\log x}\right)^h = -\log\left(1 - \frac{\log r}{\log x}\right) + O(1/x^{1/3}) \qquad (16)$$

By (12), (13), (14), (15) and (16), the lemma follows.

Now we are in a position to prove our main result in this section. We assume the truth of the Riemann hypothesis. On the Riemann hypothesis

$$\pi(x) = \int_2^x \frac{du}{\log u} + O(\sqrt{x} \cdot \log x),$$

and
$$\sum_{p \leqslant x} \frac{1}{p} = \log\log x + B + O\left(\frac{\log x}{\sqrt{x}}\right).$$

Therefore, when $r = [x^{1/3}]$, by (**A**),

$$g(x) = \sum_{p \leqslant \frac{x}{r}} \frac{x}{p} + \sum_{t=1}^{r} \pi\left(\frac{x}{t}\right) - r\pi\left(\frac{x}{r}\right) - \sum_{p \leqslant \frac{x}{r}} \left(\frac{x}{p}\right)$$

$$= x \log\log\frac{x}{r} + Bx + O(\sqrt{xr}\,\log x)$$

$$+ \sum_{t=1}^{r}\left\{\pi\left(\frac{x}{t}\right) - \pi\left(\frac{x}{r}\right)\right\} + O(x^{2/3}), \text{ since } \left(\frac{x}{p}\right) = O(1)$$

$$= x\log\log\frac{x}{r} + Bx + O(x^{2/3}\log x)$$

$$+ \sum_{t=1}^{r}\left\{\int_2^{x/t}\frac{du}{\log u} - \int_2^{x/r}\frac{du}{\log u} + O\left(\sqrt{\frac{x}{t}}\log x\right)\right\}$$

$$= x\log\log\frac{x}{r} + Bx + \sum_{t=1}^{r}\int_{x/r}^{x/t}\frac{du}{\log u} + O(x^{2/3}\log x)$$

$$= x \log\log \frac{x}{r} + Bx + \sum_{t=1}^{r} \left\{ \left[\sum_{m=1}^{n} \frac{(m-1)!\,u}{(\log u)^m} \right]_{x/r}^{x/t} + O(x^{1/3}) \right\}$$

(by lemma 5)

$$= x \log\log \frac{x}{r} + Bx + \sum_{t=1}^{r} x \sum_{m=1}^{n} \frac{(m-1)!}{t(\log x - \log t)^m}$$

$$- \sum_{t=1}^{r} \frac{x}{r} \sum_{m=1}^{n} \frac{(m-1)!}{(\log x - \log r)^m} + O(x^{2/3} \log x)$$

$$= x \log\log \frac{x}{r} + Bx + x \sum_{s=1}^{n} (s-1)! \sum_{t=1}^{r} \frac{1}{t(\log x - \log t)^s}$$

$$- x \sum_{m=1}^{n} \frac{(m-1)!}{(\log x - \log r)^m} + O(x^{2/3} \log x)$$

$$= x \log\log x + x \log\left(1 - \frac{\log r}{\log x}\right) + Bx$$

$$- x \log\left(1 - \frac{\log r}{\log x}\right) + x \sum_{s=2}^{n} \left\{ \frac{(s-1)!}{(s-1)(\log x - \log r)^{s-1}} \right.$$

$$\left. - \frac{(s-1)!}{(s-1)(\log x)^{s-1}} \right\} + x \sum_{s=1}^{n} \sum_{h=0}^{n-s} \frac{(s-1)!\, sH_h A_h}{(\log x)^{h+s}}$$

(by lemma 10)

$$- x \sum_{s=1}^{n} \frac{(s-1)!}{(\log x - \log r)^s} + O(x^{2/3} \log x)$$

$$= x \log\log x + Bx + x \sum_{s=1}^{n-1} \frac{(s-1)!}{(\log x - \log r)^s}$$

$$- x \sum_{s=1}^{n-1} \frac{(s-1)!}{(\log x)^s} + x \sum_{s=1}^{n} \sum_{h=0}^{n-s} \frac{Ah}{h!} \cdot \frac{(s+h-1)!}{(\log x)^{h+s}}$$

$$- x \sum_{s=1}^{n} \frac{(s-1)!}{(\log x - \log r)^s} + O(x^{2/3} \log x)$$

$$= x \log\log x + Bx - x \sum_{s=1}^{n-1} \frac{(s-1)!}{(\log x)^s}$$

$$+ x \sum_{h=0}^{n-1} \frac{A_h}{h!} \sum_{s=1}^{n-h} \frac{(h+s-1)!}{(\log x)^{h+s}} - \frac{x \cdot n!}{(\log x - \log r)^n} + O(x^{2/3} \log x)$$

$$= x \log\log x + Bx - x \sum_{s=1}^{n-1} \frac{(s-1)!}{(\log x)^s}$$

$$+ x \sum_{s=1}^{n} \frac{(s-1)!}{(\log x)^s} \sum_{h=0}^{s-1} \frac{A_h}{h!} + O(x^{2/3} \log x)$$

since
$$\frac{x(n!)}{(\log x - \log r)^n} = O(x^{2/3})$$

$$= x \log \log x + Bx + x \sum_{s=1}^{n} \frac{h_s}{(\log x)^s} + O(x^{2/3} \log x)$$

where
$$h_s = (s-1)! \left\{ -1 + A_0 + \frac{A_1}{1!} + \cdots + \frac{A_{s-1}}{(s-1)!} \right\}.$$

Thus we have proved the main result of this section, namely the following

THEOREM. *If* $n = \left[\frac{\log x}{3}\right]$ *and the Riemann hypothesis is true, then*

$$g(x) = x \log \log x + Bx + x \sum_{s=1}^{n} \frac{h_s}{(\log x)^s} + O(x^{2/3} \log x),$$

where

$$h_s = (s-1)! \left\{ -1 + A_0 + \frac{A_1}{1!} + \cdots + \frac{A_{s-1}}{(s-1)!} \right\},$$

and

$$A_s = \lim_{m \to \infty} \left\{ \sum_{n=1}^{m} \frac{(\log n)^s}{n} - \frac{(\log x)^{s+1}}{s+1} \right\}.$$

Iterative Interpolation

BY*

E. H. NEVILLE.

Contents

The problem with which this paper is concerned is the computation of the numerical value of a tabulated function $u(x)$ for a value of x which is not one of the points of tabulation.

1. Lagrange's Polynomial

The solution is simplest when linear interpolation is adequate, that is to say, when the interval between consecutive entries is so small, in relation to the rate at which $u(x)$ is changing and to the accuracy demanded, that the principle of proportional parts can be used. If only a few significant figures are wanted, this condition may be attainable, but in the vast majority of cases it would imply tables so enormous as to be altogether impracticable, and the condition is not regarded even as an ideal.

The principle of proportional parts may be expressed in the form that for calculation in the interval from a to b we substitute for the function $u(x)$ the known linear function

$$\frac{b-x}{b-a}\,u(a)+\frac{x-a}{b-a}\,u(b)$$

which has the same value as $u(x)$ at a and b. The immediate extension of this principle is to substitute a determinate polynomial which agrees with $u(x)$ at a number of points $a_1, a_2, \cdots a_n$; the polynomial of lowest degree with this property is Lagrange's polynomial, which is of degree $n-1$ and can be written down at once in the form

$$\frac{(x-a_2)(x-a_3)\cdots(x-a_n)}{(a_1-a_2)(a_1-a_3)\cdots(a_1-a_n)}\,u(a_1)$$
$$+\frac{(x-a_1)(x-a_3)\cdots(x-a_n)}{(a_2-a_1)(a_2-a_3)\cdots(a_2-a_n)}\,u(a_2)+\cdots\cdots$$
$$+\frac{(x-a_1)(x-a_2)\cdots(x-a_{n-1})}{(a_n-a_1)(a_n-a_2)\cdots(a_n-a_{n-1})}\,u(a_n).$$

Theoretically, this expression for the polynomial is final, but as a formula for computation it is open to serious objections. For a value of x with ten or a dozen significant figures, the labour of evaluation is immense. The terms are all of the same importance, and all equally heavy to evaluate. None of the intermediate steps can be shortened by means of subsidiary tables, since it is out of the question to tabulate such a function as

$$\frac{(x-a_2)(x-a_3)\cdots\cdots(x-a_n)}{(a_1-a_2)(a_1-a_3)\cdots\cdots(a_1-a_n)}$$

for independent ranges of values of $x, a_1, a_2, \cdots a_n$. Lastly, if an additional point a_{n+1} is introduced, the polynomial is changed not only by the addition of the term with $u(a_{n+1})$, but also by a modification of each of the original terms; in effect, what is calculated is a new value, not the change to be made in an old value, and however little the new value may differ from the old, the work of evaluation is as heavy as if the two values were entirely unrelated.

2. Everett's Formula

In the case of greatest practical importance, in which the function is tabulated at regular intervals, the Lagrangian formula can be modified. Let h be the interval between consecutive entries, and let θ, ϕ be the fractions $(x-a)/h$, $(b-x)/h$ into which x divides the interval ab; let $\delta^2 u_x, \delta^4 u_x, \cdots\cdots$ be the even differences of $u(x)$, defined by

$$\delta^2 u_x = u(x+h)+u(x-h)-2u(x),\quad \delta^4 u_x=\delta^2(\delta^2 u_x),\ \ldots\ \ldots$$

Then Everett has shewn that the function $u_{2n+1}(x)$ defined by

$$u_{2n+1}(x) = E_0(\phi)\,u_a - E_2(\phi)\,\delta^2 u_a + E_4(\phi)\,\delta^4 u_a - \dots \quad \dots + (-)^n\, E_{2n}(\phi)\,\delta^{2r} u_a$$
$$+ E_0(\theta)\,u_b - E_2(\theta)\,\delta^2 u_b + E_4(\theta)\,\delta^4 u_b - \dots \quad \dots + (-)^n\, E_{2n}(\theta)\,\delta^{2n} u_b$$

where $E_{2r}(\phi)$ denotes $\phi(1-\phi^2)(4-\phi^2)\dots \quad \dots (r^2-\phi^2) \div (2r+1)!$ and $E_{2r}(\theta)$ denotes the same function of θ, is a polynomial of degree $2n+1$ in x which agrees with $u(x)$ at the $2n+2$ points of tabulation from $a-nh$ to $b+nh$; in other words, this is the Lagrangian polynomial, differently arranged.

Everett's formula appears completely to meet the objections to Lagrange's. If x is between a and b, the terms in each line diminish in importance, both because each coefficient is less than one quarter of its predecessor and because, for any function in which interpolation can be feasible, the differences also decrease. The coefficient of $\delta^{2r} u_a$ is a function $E_{2r}(\phi)$ of one variable ϕ, and the coefficient of $\delta^{2r} u_b$ is the same function of the complementary argument θ, which is $1-\phi$; the range for which this function is wanted is the definite range from 0 to 1, and therefore tables of this function can be prepared for use, independently of the function $u(x)$ and the interval h. Also the values of x for which the differences $\delta^2 u$, $\delta^4 u$, $\cdots$ are required are the values for which $u(x)$ itself is being tabulated, and if the interests of the user of the tables are the sole consideration, these differences can be printed in line with the function; in practice this does not add to the task of the original computer, since the calculation of differences is invariably part of the work either of building up or of checking a table. Lastly, to include the effect of an additional point on each side of x, we have only to add an additional term in each line, without revising previous calculations: each pair of terms gives the modification due to the introduction of the corresponding pair of points. We may in fact regard the formula as the expression of $u(x)$ by means of an infinite series:

$$u(x) = \{E_0(\phi)u_a + E_0(\theta)u_b\} - \{E_2(\phi)\delta^2 u_a + E_2(\theta)\delta^2 u_b\} + \{E_4(\phi)\delta^4 u_a + E_4(\theta)\delta^4 u_b\}$$
$$- \dots\dots\dots\dots\dots\dots\dots\dots$$

But in spite of the formal perfection of Everett's formula, experience in actual use makes us critical. If twelve places of decimals are wanted in the argument, with a tabular interval of ·01, the values of θ and ϕ must be taken to eight places. The Everett coefficients are not tabulated, and it is not to be supposed that they ever will be tabulated, at an interval of ·0000 0001, and on this account alone the formula is harder* to apply than is at first

* For methods of applying Everett's formula in such cases, see *British Association Mathematical Tables*, vol. I, p. vii.

apparent. The provision of differences adds, and sometimes adds a great deal, to the size and expense of a published table, and yet if they are not given, their computation, simple as it is, takes by no means a negligible fraction of the time of a complete interpolation. The convergence, though certain, may be very slow; if it was not for the diminishing of the differences, five double terms, that is, ten products, would be necessary to advance the accuracy by three places of decimals, but in practice this extreme case would never be allowed.

3. Taylor's Theorem.

There is a classical alternative to the use of Everett's formula. Given the requisite material, the value of $u(x)$ for an untabulated value of x can be calculated by means of Taylor's series, which we may compare with Everett's series by writing it in the two forms

$$u(x) = u_a + \theta h u'_a + \frac{\theta^2}{2!} h^2 u''_a + \cdot \cdot \cdot \cdot \cdot ,$$

$$u(x) = u_b - \phi h u'_b + \frac{\phi^2}{2!} h^2 u''_b - \cdot \cdot \cdot \cdot \cdot$$

The product $h^{2r} u^{(2r)}x$ is of the same order as the difference $\delta^{2r} u_x$, and since the Taylor coefficients tend to zero much more rapidly than the Everett coefficients, the Taylor series has much the better convergence. The Taylor coefficients $\theta^r/r!$ are not better tabulated than the Everett coefficients, and even the powers θ^r are not tabulated at an interval approaching $\cdot 0^71$, but the Taylor coefficients are much the simpler for the computer to determine for himself. Further, if anything at all is to be given in addition to the principal function $u(x)$, there is more satisfaction in providing a number of the derived functions $u'(x)$, $u''(x)$, $\cdot \cdot \cdot \cdot$, which may be expected to have an intrinsic interest, than in providing differences which are aids to computation and nothing else. Also, as we have indicated, we may approach $u(x)$ by means of a Taylor series from each end of the interval and so check the result by two computations which are independent of each other; it is not quite true to say that there is only one Everett series applicable at a given point, but other series which can be used as a check involve considerable formal modifications.

The arguments, however, are not all on one side. It is not to be disputed that derivatives add more than differences to the burden of producing a table. Their computation is not mechanical, and does not serve to check the values of $u(x)$. Also we have to remember that in any case only *even* differences are tabulated. For a given degree of accuracy, more derivatives than even differences are wanted; also the r^{th} even difference is comparable with the $2r^{th}$ derivative; for both reasons the provision of derivatives is a very much more serious undertaking than the provision of differences. Moreover it is an undertaking which must be carried through completely if interpolation by means of Taylor's theorem is the object. Each column of differences is an independent help towards computation by Everett's theorem, and the compiler, if he is not prepared to give all the differences that will be needed, can still save the user about half the labour of differencing by giving $\delta^2 u$ alone. But to evaluate derivatives that are not provided involves numerical work on an altogether different scale, and to give the first two derivatives when six were wanted would still leave the use of Taylor's theorem impracticable.

4. Osculating Polynomials.

The conclusion that for purposes of interpolation we are no better off with two or three derivatives than with none at all is one that we are reluctant to accept. Everett's formula replaces $u(x)$ by a polynomial which agrees with $u(x)$ at any convenient even number of different points. Taylor's theorem replaces $u(x)$ by a polynomial which agrees with $u(x)$ as closely as desired at one particular point. If in order to estimate the value of $u(x)$ at a point x, we replace $u(x)$ by a polynomial determined by the condition of agreeing with $u(x)$ at a given number of points, the closer these points are to x, the better estimate of $u(x)$ we shall expect. Suppose for example that we are given tabulated values of $u(x)$ only, the best quintic is likely to be the one which has the same values as $u(x)$ for the six points from $a-2h$ to $b+2h$. If we are given the values of the first derivative at a and b, we can in effect replace simple agreement at the most distant points $a-2h$ and $b+2h$ by double agreement at the nearest points a and b. If we are given the values also of the second derivative, then agreement of the third order at a and b may replace any agreement at other points. The quintic with treble agreement at two points may be slightly inferior to a quintic obtainable from Taylor's series, but it is certainly much superior to the Everett quintic, and if two derivatives but not five are available, it is well worth while to utilise the doubly osculating quintic if we can.

When we use Everett's formula we recognise only one form of magnitude of order n, namely, the n^{th} difference $\delta^n u(x)$. When we use Taylor's theorem we recognise another form, $h^n\ u^{(n)}(x)$. If we propose to depend on polynomials of an intermediate kind, we are implicitly introducing a whole group of intermediate magnitudes not only of the simple forms $h\delta^{n-1}u^{(1)}(x)$, $h^2\delta^{n-2}u^{(2)}(x)$,... $h^{n-1}\delta u^{(n-1)}(x)$, but also of such complicated forms as differences between two of this kind. The possibilities are so bewildering that we may despair of being able to take advantage of them. If osculating polynomials are desirable, a general formula of which Everett's and Taylor's are the two extreme cases must be complicated; we could hope to deduce a general formula from Lagrange's expression by considering the effects of coalescence, but since, as we have already said, computation from this expression is impracticable anyhow, we can not expect to reach in this way a manageable formula. We must approach the problem differently.

5. The Fundamental Principle of Linear Interpolation.

We are saying that a function $v(x)$ has agreement of order σ with a function $u(x)$ at s if

$$v(s)=u(s), \quad v^{(1)}(f)=u^{(1)}(f), \quad \cdots \quad v^{(\sigma-1)}(s)=u^{(\sigma-1)}(s);$$

agreement of the first order is mere equality, and to say that agreement at s is of order zero means that the functions are not known to have the same value there. Let $p(x)$ be the polynomial of degree $\alpha+\gamma_1+\gamma_2+\ . \ . \ . \ +\gamma_r+\beta-2$ which has agreement of orders $\alpha, \gamma_1, \gamma_2, \ . \ . \ . \ \gamma_r, \beta-1$ at $a, c_1, c_2, \cdots c_r, b$, and let $q(x)$ be the polynomial of the same degree which has the same agreement at $c_1, c_2, \cdots c_r$ but has agreement of orders $\alpha-1, \beta$ at a, b. Consider the polynomial $f(x)$ defined by

$$f(x)=\frac{b-x}{b-a}\,p(x)+\frac{x-a}{b-a}\,q(x).$$

By Leibniz' theorem

$$f^{(m)}(x)=\frac{b-x}{b-a}\,p^{(m)}(x)+\frac{x-a}{b-a}\,q^{(m)}(x)-\frac{m}{b-a}\left\{p^{(m-1)}(x)-q^{(m-1)}(x)\right\}.$$

It follows that for any values of x and m for which both $p^{(m-1)}(x)=q^{(m-1)}(x)$ and $p^{(m)}(x)=q^{(m)}(x)$, $f^{(m)}(x)$ has the same value as $p^{(m)}(x)$ and $q^{(m)}(x)$. Hence $f(x)$ has the same agreement with $u(x)$ as $p(x)$ and $q(x)$ at c_1, c_2 .... c_r; at a, although the common order of agreement is only $\alpha-1$, the value of $q^{(m)}(x)$ does not affect that of $f^{(m)}(x)$ and $f(x)$ has agreement of order α, and similarly at b, $f(x)$ has agreement of order β. That is to say, $f(x)$ is the polynomial of degree $\alpha+\gamma_1+\gamma_2+\ldots\gamma_r+\beta-1$ which has agreements of orders $\alpha, \gamma_1, \gamma_2, \cdots \gamma_r, \beta$ at $a, c_1, c_2 \cdots c_r, b$: with a unit increase in degree, $f(x)$ combines the agreements of $p(x)$ and $q(x)$.

To say that, if X is any value of x, then

$$f(\mathrm{X}) = \frac{b-\mathrm{X}}{b-a}\,p(\mathrm{X}) + \frac{\mathrm{X}-a}{b-a}\,q(\mathrm{X}),$$

seems a mere tautology, but suggests another way of looking at the formula. For $p(\mathrm{X})$ is the estimate of $u(\mathrm{X})$ formed by substitution of $p(x)$ for $u(x)$, and $q(\mathrm{X})$ is the estimate formed by substitution of $q(x)$ for $u(x)$. These are pure numbers, and the formula asserts that an approximation $f(\mathrm{X})$ of a higher order to the same required value $u(\mathrm{X})$ is calculable by linear interpolation between $p(\mathrm{X})$ and $q(\mathrm{X})$, provided that the polynomials $p(x)$, $q(x)$ from which $p(\mathrm{X})$, $q(\mathrm{X})$ are implicitly derived, have the necessary measure of common agreement with $u(x)$.

This is the principle whose application it is the purpose of this paper to describe, and we deal first with problems in which no multiple agreement is postulated, and then with more general problems. In each division of the paper we begin with the simplest cases.

6. Iterative Computation of Lagrangian Approximations.

From two values u_a, u_b of $u(x)$ we have one first approximation L_{ab} to $u(\mathrm{X})$, given by

$$\mathrm{L}_{ab} = \frac{b-\mathrm{X}}{b-a}\,u_a + \frac{\mathrm{X}-a}{b-a}\,u_b\,.$$

This is the value at X of the linear function $l_{ab}(x)$ which agrees with $u(x)$ at a and b. If we are to combine with L_{ab} a different first approximation, the latter must come from a linear function which agrees with $u(x)$ at one of the two points a, b. If this second function is $l_{bc}(x)$, the corresponding first approximation L_{bc} is given by

$$\mathrm{L}_{bc} = \frac{c-\mathrm{X}}{c-b}\,u_b + \frac{\mathrm{X}-b}{c-b}\,u_c,$$

and from L_{ab} and L_{bc} we have an approximation M_{abc} of the second order; the points at which $l_{ab}(x)$ and $l_{bc}(x)$ differ being a and c, we have to interpolate between L_{ab} and L_{bc} by regarding X as belonging to the interval ac:

$$\mathrm{M}_{abc} = \frac{c-\mathrm{X}}{c-a}\,\mathrm{L}_{ab} + \frac{\mathrm{X}-a}{c-a}\,\mathrm{L}_{bc}.$$

Similarly a fourth tabulated value u_d gives another first approximation,

$$\mathrm{L}_{cd} = \frac{d-\mathrm{X}}{d-c}\,u_c + \frac{\mathrm{X}-c}{d-c}\,u_d,$$

another second approximation,

$$M_{bcd} = \frac{d - X}{d - b} L_{bc} + \frac{X - b}{d - b} L_{cd},$$

and an approximation of the third order,

$$N_{abcd} = \frac{d - X}{d - a} M_{abc} + \frac{X - a}{d - a} M_{bcd}.$$

It follows from our fundamental theorem that the number N_{abcd} is precisely the value at X of the cubic which agrees with $u(x)$ at a, b, c, d.

More generally, the value at x of the polynomial of degree n which agrees with $u(x)$ at $n+1$ distinct points is obtained by a set of $\frac{1}{2}n(n+1)$ linear interpolations. We have already remarked that computation from the explicit expression for this polynomial, as given by Lagrange, is laborious if not impracticable. Linear interpolation, however, is one of the most rapid of numerical operations. What is required for computation is a process; whether or not a general formula corresponds to the process is irrelevant.

It is seldom in mathematical tables that the interval of tabulation is irregular. There are nevertheless two important problems which are solved when a practicable process for effecting a Lagrangian interpolation is known, and in the next two sections we deal with them in turn.

7. Inverse Interpolation.

To find the value of x for which $u(x)$ has a given value U, all that is necessary is to regard x as the function of u which acquires the values a, b, c, . . when u has the values u_a, u_b, u_c, ...

An actual example will shew the manner in which a scheme of interpolation is conveniently arranged. The positive root of the equation

$$x^7 + 28x^4 - 480 = 0$$

is easily seen to lie between 1·9 and 2; the trinomial $x^7 + 28x^4 - 480$ can be tabulated as a function $u(x)$ of x, and the evaluation of the root is the determination of the value of x for the value 0 of u. The first step is the calculation of a number of values of $u(x)$:

$$\begin{aligned}
U_0 &= u(1{\cdot}90) = -\ 25{\cdot}71402\ \ 61000 \\
U_1 &= u(1{\cdot}91) = -\ 14{\cdot}62541\ \ 67393 \\
U_2 &= u(1{\cdot}92) = -\ \ 3{\cdot}30746\ \ 39222 \\
U_3 &= u(1{\cdot}93) = +\ \ 8{\cdot}24394\ \ 35400 \\
U_4 &= u(1{\cdot}94) = +\ 20{\cdot}03298\ \ 30120
\end{aligned}$$

We now think of x as the function of u whose values are given by $x(U_0)=1{\cdot}90$, and so on. Considering 0 as a point U in the interval from U_2 to U_3, dividing this interval into the fractions

$$3{\cdot}30746\ 39222/\ 11{\cdot}55140\ 74622, \qquad 8{\cdot}24394\ 35400/\ 11{\cdot}55140\ 74622$$

that is, ·28632 5622, ·71367 4378, we see that the corresponding linear approximation X_{23} divides the interval from X_2 to X_3, that is, from 1·92 to 1·93, in the same ratio; thus this first approximation is given by

$$X_{23}=1{\cdot}92286\ 32562\ 2.$$

The central interval is flanked on one side by the interval from U_1 to U_2; U divides this interval externally, into the fractions

$$14{\cdot}62541\ 67393/\ 11{\cdot}31795\ 28171, \quad -\ 3{\cdot}30746\ 39222/\ 11{\cdot}31795\ 28171,$$

that is, 1·29223 1641, − ·29223 1641, and the corresponding linear approximation X_{12}, which divides the interval from X_1 to X_2 in the same ratio, is given by

$$X_{12}=1{\cdot}92292\ 23164\ 1.$$

The linear functions $x_{23}(u)$, $x_{12}(u)$ to which the approximations X_{23}, X_{12} are implicitly related both agree with the function $x(u)$ for $u=U_2$; the first of them agrees with $x(u)$ also for $u=U_3$, the second also for $u=U_1$. The quadratic function $x_{123}(u)$ defined by

$$x_{123}(u) = \frac{u-U_1}{U_3-U_1}\,x_{23}(u) + \frac{U_3-u}{U_3-U_1}\,x_{12}(u)$$

therefore agrees with $x(u)$ for the three values U_1, U_2, U_3, and the value X_{123} of this quadratic function for the value U of u divides the interval from X_{12} to X_{23} in the ratio in which U divides the interval from U_1 to U_3. Since in fact U divides this last interval into the fractions

$$14{\cdot}62541\ 674\ /\ 22{\cdot}86936\ 028, \qquad 8{\cdot}24394\ 354\ /\ 22{\cdot}86936\ 028,$$

that is, ·63952 02, ·36047 98, we have the quadratic approximation

$$X_{123} = X_{23} + {\cdot}36047\ 98\ (X_{12}-X_{23}) = 1{\cdot}92288\ 45465\ 5.$$

To obtain a cubic approximation we must find a second quadratic approximation, and this in turn requires a third linear approximation. Taking the interval from U_3 to U_4, which flanks the central interval on the other side, we have for the fractions into which this interval is divided at U the values

$$-\,8{\cdot}24394\ 35400/11{\cdot}78903\ 94720, \quad 20{\cdot}03298\ 30120/11{\cdot}78903\ 94720$$

that is, − ·69928 8823, 1·69928 8823, and therefore

$$X_{34}=X_3-\ {\cdot}69928\ 8823\ (X_4-X_3)=1{\cdot}92300\ 71117\ 7.$$

The fractional divisions of the interval from U_2 to U_4 being

$$3{\cdot}30746\ 392/\ 23{\cdot}34044\ 693, \quad 20{\cdot}03298\ 301/\ 23{\cdot}34044\ 693,$$

that is, ·14170 41, ·85829 59, we have

$$X_{234} = X_{23} + \cdot 14170\ 41\ (X_{34} - X_{23}) = 1\cdot 92288\ 36411\ 3.$$

The two quadratic approximations X_{123}, X_{234} agree as far as the fifth place of decimals. To find the cubic approximation X_{1234}, we have to interpolate between X_{123} and X_{234}, or more simply between $X_{123} - 1\cdot 92288$ and $X_{234} - 1\cdot 92288$, in the ratio in which U divides the interval from U_1 to U_4, that is, by means of the fractions

14·6354 / 34·6584, 20·0339 / 34·6584 ; thus

$$X_{1234} = 1\cdot 92288\ 41644\ 7.$$

The cubic approximation now found utilises completely the data on which its value ultimately depends, namely, the values of U_1, U_2, U_3, U_4 ; it is the value for $u=0$ of the cubic $x_{1234}(u)$ whose values for U_1, U_2, U_3, U_4 are 1·91, 1·92, 1·93, 1·94. To improve upon it we take into account the value of U_0, with the corresponding value 1·90 of X_0, and we find in succession

$$X_{01} \quad = \quad X_1 + \frac{14\cdot 62541\ 67393}{11\cdot 08860\ 93607}(X_1 - X_0) = 1\cdot 92318\ 95860\ 5,$$

$$X_{012} \quad = \quad X_{12} - \frac{3\cdot 30746\ 39}{22\cdot 40656\ 22}(X_{01} - X_{12}) = 1\cdot 92288\ 28643\ 7,$$

$$X_{0123} \quad = \quad X_{123} - \frac{8\cdot 2439}{33\cdot 9580}(X_{123} - X_{012}) = 1\cdot 92288\ 41384\ 2.$$

The two cubic approximations agree to the seventh place. Interpolating between the concluding figures 3842 and 6447 with the fractions into which U divides the interval from U_0 to U_4 we have an approximation of the fourth order, 1·92288 41530 6 which is in fact correct to the ninth place. The complete scheme of approximations is best shewn in the following form :

		1·9	2	288	41	
−	25·71402 61000	0				
			318 95860 5			
−	14·62541 67393	1		28643 7		
			292 23164 1		394 2	
−	3·30746 39222	2		45465 5		530 6
			286 32562 2		644 7	
+	8·24394 35400	3		36411 7		
			300 71117 7			
+	20·03298 30120	4				

The positive root of the equation $x^7 + 28x^4 - 480 = 0$ was calculated by W. B. Davis by Horner's method to 42 places (*Educ. Times Reprint*, v. 7 (1867) p. 108). The evaluation was used as an example of inverse interpolation by formula by Whittaker and Robinson (*Calculus of Observations*, p. 61) and adopted by Aitken *Proc. Edinburgh Math. Soc.* Ser. 2, vol. 3. p. 71) to illustrate the process of iterative linear interpolation Aitken however worked from one end of his range, and his scheme consists of the approximations which in the notation of this paragraph are

$$X_{01},\ X_{02},\ X_{03},\ X_{04}\ ;\ X_{012},\ X_{013},\ X_{014}\ ;\ X_{0123},\ X_{0124}\ ;\ X_{01234}.$$

The final approximation is necessarily the same, except perhaps for an accidental figure retained, but the approximations in earlier columns do not cluster round the limit as closely as in the scheme used here, and a scheme which centres on the tabular points nearest to the required value of the independent variable avoids the necessity of considering, however roughly, the relation between the points used and the accuracy to be attained ; the construction proceeds automatically, and since the tip of the triangle is always reached by interpolation in the strict sense, not by extrapolation, the possibility of unsuspected fluctuation is removed.

8. Bridging.

The conflicting interests of the producer and the user in the matter of the interval of tabulation are often best reconciled by adaptability in the interval. That an interval as small as ·01 is necessary in one part of the table, is no reason for using this interval throughout if ·05 is usually adequate. But the change of interval introduces anomalies in regard to interpolation near the point of change. Suppose that eighth differences are to be allowed, and that the interval is to be ·01 for values of x from 0·00 to 2·00, and ·05 for values from 2·00 upwards. It is of course possible to treat the two parts as independent tables, and to interpolate near $x=2{\cdot}00$ in the same way as near the beginning or end of any other table. If central differences are being used, this implies that the compiler must calculate the function for the values 2·01, 2·02, 2·03, 2·04 in order to be able to provide $u(2{\cdot}00)$ for interpolation between 1·99 and 2·00. The differences at 2·00 for use between 2·00 and 2·05 correspond to the larger interval, and come from the values 1·80, 1·85, 1·90, 1·95, 2·00, 2·05, 2·10, 2·15, 2·20, 2·25 of x ; the first four of these are already available in the lower table, but it is manifestly absurd to accept the slow convergence due to these distant values if the values at no fewer than eight other points between 1·95 and 2·05 are known, explicitly or implicitly. If backward differences only are used between 1·95 and 2·00, the compiler has not to calculate any auxiliary values of $u(x)$, but for a value of x between 1·98 and 1·99 the interpolator is using the values of $u(x)$ at 1·90 and 1·91 instead of the values at 2·00 and 2·05.

To meet these difficulties, Pearson, a powerful advocate of large intervals, gives* "bridging formulae," with numerical coefficients, for a number of the most useful ratios of the two intervals. Not merely are formulae required for the intervals immediately on the two sides of the boundary, but each interval in which any values from across the border are to be utilised requires its own formula. Iterative interpolation replaces each formula by a scheme of coefficients.

* *Tracts for Computers* ; II.

For example, in the case imagined in the last paragraph, if we denote by θ and ϕ the fractions 100 (X − 1·98) and 100 (1·99 − X) into which X divides the interval from 1·95 to 1·99, the scheme is as that given on sheet I (page 99). The structure of this scheme is evident, and it is hardly worth while to record for reference the many variations that may occur.

9. Iterative Linear Interpolation with a Regular Interval.

In inverse interpolation and bridging, we are applying iterative interpolation to cases in which the interval is irregular. But even for a regular interval, the process is not subject to the same criticisms as direct substitution in Everett's formula. The burden of printing differences is avoided altogether, and no subsidiary tables of coefficients are wanted, so that, whatever the number of figures involved, the full value of the argument is used throughout. The convergence takes place, so to speak, under the computer's eye, and the arithmetic is subject to an automatic check. The actual scheme of coefficients is shown on sheet II (page 100).

SHEET I.—SPECIMEN BRIDGING SCHEME

1·94							
	$\left.\begin{matrix}-(3+\theta)\\(4+\theta)\end{matrix}\right\}$						
5		$\left.\begin{matrix}-\frac{1}{2}(2+\theta)\\\frac{1}{2}(4+\theta)\end{matrix}\right\}$					
	$\left.\begin{matrix}-(2+\theta)\\(3+\theta)\end{matrix}\right\}$		$\left.\begin{matrix}-\frac{1}{3}(1+\theta)\\\frac{1}{3}(4+\theta)\end{matrix}\right\}$				
6		$\left.\begin{matrix}-\frac{1}{2}(1+\theta)\\\frac{1}{2}(3+\theta)\end{matrix}\right\}$		$\left.\begin{matrix}-\frac{1}{4}\theta\\\frac{1}{4}(4+\theta)\end{matrix}\right\}$			
	$\left.\begin{matrix}-(1+\theta)\\(2+\theta)\end{matrix}\right\}$		$\left.\begin{matrix}-\frac{1}{3}\theta\\\frac{1}{3}(3+\theta)\end{matrix}\right\}$		$\left.\begin{matrix}\frac{1}{5}\phi\\\frac{1}{5}(4+\theta)\end{matrix}\right\}$		
7		$\left.\begin{matrix}-\frac{1}{2}\theta\\\frac{1}{2}(2+\theta)\end{matrix}\right\}$		$\left.\begin{matrix}\frac{1}{4}\phi\\\frac{1}{4}(3+\theta)\end{matrix}\right\}$		$\left.\begin{matrix}\frac{1}{6}(1+\phi)\\\frac{1}{6}(4+\theta)\end{matrix}\right\}$	
	$\left.\begin{matrix}-\theta\\(1+\theta)\end{matrix}\right\}$		$\left.\begin{matrix}\frac{1}{3}\phi\\\frac{1}{3}(2+\theta)\end{matrix}\right\}$		$\left.\begin{matrix}\frac{1}{5}(1+\phi)\\\frac{1}{5}(3+\theta)\end{matrix}\right\}$		$\left.\begin{matrix}\frac{1}{11}(6+\phi)\\\frac{1}{11}(4+\theta)\end{matrix}\right\}$
8		$\left.\begin{matrix}\frac{1}{2}\phi\\\frac{1}{2}(1+\theta)\end{matrix}\right\}$		$\left.\begin{matrix}\frac{1}{4}(1+\phi)\\\frac{1}{4}(2+\theta)\end{matrix}\right\}$		$\left.\begin{matrix}\frac{1}{10}(6+\phi)\\\frac{1}{10}(3+\theta)\end{matrix}\right\}$	
	$\left.\begin{matrix}\phi\\\theta\end{matrix}\right\}$		$\left.\begin{matrix}\frac{1}{3}(1+\phi)\\\frac{1}{3}(1+\theta)\end{matrix}\right\}$		$\left.\begin{matrix}\frac{1}{9}(6+\phi)\\\frac{1}{9}(2+\theta)\end{matrix}\right\}$		$\left.\begin{matrix}\frac{1}{15}(11+\phi)\\\frac{1}{15}(3+\theta)\end{matrix}\right\}$
9		$\left.\begin{matrix}\frac{1}{2}(1+\phi)\\\frac{1}{2}\theta\end{matrix}\right\}$		$\left.\begin{matrix}\frac{1}{8}(6+\phi)\\\frac{1}{8}(1+\theta)\end{matrix}\right\}$		$\left.\begin{matrix}\frac{1}{14}(11+\phi)\\\frac{1}{14}(2+\theta)\end{matrix}\right\}$	
	$\left.\begin{matrix}(1+\phi)\\-\phi\end{matrix}\right\}$		$\left.\begin{matrix}\frac{1}{7}(6+\phi)\\\frac{1}{7}\theta\end{matrix}\right\}$		$\left.\begin{matrix}\frac{1}{13}(11+\phi)\\\frac{1}{13}(1+\theta)\end{matrix}\right\}$		
2·00		$\left.\begin{matrix}\frac{1}{6}(6+\phi)\\-\frac{1}{6}\phi\end{matrix}\right\}$		$\left.\begin{matrix}\frac{1}{12}(11+\phi)\\\frac{1}{12}\theta\end{matrix}\right\}$			
	$\left.\begin{matrix}\frac{1}{5}(6+\phi)\\-\frac{1}{5}(1+\phi)\end{matrix}\right\}$		$\left.\begin{matrix}\frac{1}{11}(11+\phi)\\-\frac{1}{11}\phi\end{matrix}\right\}$				
05		$\left.\begin{matrix}\frac{1}{10}(11+\phi)\\-\frac{1}{10}(1+\phi)\end{matrix}\right\}$					
	$\left.\begin{matrix}\frac{1}{5}(11+\phi)\\-\frac{1}{5}(6+\phi)\end{matrix}\right\}$						
10							

SHEET II.—LINEAR INTERPOLATION WITH A REGULAR INTERVAL

$u(a-4h)$, $u(a-3h)$, $u(a-2h)$, $u(a-h)$, $u(a)$, $u(b)$, $u(b+h)$, $u(b+2h)$, $u(b+3h)$, $u(b+4h)$

$\begin{Bmatrix}-(3+\theta)\\(4+\theta)\end{Bmatrix}$	$\begin{Bmatrix}-\frac{1}{2}(2+\theta)\\\frac{1}{2}(4+\theta)\end{Bmatrix}$	$\begin{Bmatrix}-\frac{1}{3}(1+\theta)\\\frac{1}{3}(4+\theta)\end{Bmatrix}$	$\begin{Bmatrix}-\frac{1}{4}\theta\\\frac{1}{4}(4+\theta)\end{Bmatrix}$	$\begin{Bmatrix}\frac{1}{5}\phi\\\frac{1}{5}(4+\theta)\end{Bmatrix}$	$\begin{Bmatrix}\frac{1}{6}(1+\phi)\\\frac{1}{6}(4+\theta)\end{Bmatrix}$	$\begin{Bmatrix}\frac{1}{7}(2+\phi)\\\frac{1}{7}(4+\theta)\end{Bmatrix}$	$\begin{Bmatrix}\frac{1}{8}(3+\phi)\\\frac{1}{8}(4+\theta)\end{Bmatrix}$
$\begin{Bmatrix}-(2+\theta)\\(3+\theta)\end{Bmatrix}$	$\begin{Bmatrix}-\frac{1}{2}(1+\theta)\\\frac{1}{2}(3+\theta)\end{Bmatrix}$	$\begin{Bmatrix}-\frac{1}{3}\theta\\\frac{1}{3}(3+\theta)\end{Bmatrix}$	$\begin{Bmatrix}\frac{1}{4}\phi\\\frac{1}{4}(3+\theta)\end{Bmatrix}$	$\begin{Bmatrix}\frac{1}{5}(1+\phi)\\\frac{1}{5}(3+\theta)\end{Bmatrix}$	$\begin{Bmatrix}\frac{1}{6}(2+\phi)\\\frac{1}{6}(3+\theta)\end{Bmatrix}$	$\begin{Bmatrix}\frac{1}{7}(3+\phi)\\\frac{1}{7}(3+\theta)\end{Bmatrix}$	$\begin{Bmatrix}\frac{1}{8}(4+\phi)\\\frac{1}{8}(3+\theta)\end{Bmatrix}$
$\begin{Bmatrix}-(1+\theta)\\(2+\theta)\end{Bmatrix}$	$\begin{Bmatrix}-\frac{1}{2}\theta\\\frac{1}{2}(2+\theta)\end{Bmatrix}$	$\begin{Bmatrix}\frac{1}{3}\phi\\\frac{1}{3}(2+\theta)\end{Bmatrix}$	$\begin{Bmatrix}\frac{1}{4}(1+\phi)\\\frac{1}{4}(2+\theta)\end{Bmatrix}$	$\begin{Bmatrix}\frac{1}{5}(2+\phi)\\\frac{1}{5}(2+\theta)\end{Bmatrix}$	$\begin{Bmatrix}\frac{1}{6}(3+\phi)\\\frac{1}{6}(2+\theta)\end{Bmatrix}$	$\begin{Bmatrix}\frac{1}{7}(4+\phi)\\\frac{1}{7}(2+\theta)\end{Bmatrix}$	
$\begin{Bmatrix}-\theta\\(1+\theta)\end{Bmatrix}$	$\begin{Bmatrix}\frac{1}{2}\phi\\\frac{1}{2}(1+\theta)\end{Bmatrix}$	$\begin{Bmatrix}\frac{1}{3}(1+\phi)\\\frac{1}{3}(1+\theta)\end{Bmatrix}$	$\begin{Bmatrix}\frac{1}{4}(2+\phi)\\\frac{1}{4}(1+\theta)\end{Bmatrix}$	$\begin{Bmatrix}\frac{1}{5}(3+\phi)\\\frac{1}{5}(1+\theta)\end{Bmatrix}$	$\begin{Bmatrix}\frac{1}{6}(4+\phi)\\\frac{1}{6}(1+\theta)\end{Bmatrix}$		
$\begin{Bmatrix}\phi\\\theta\end{Bmatrix}$	$\begin{Bmatrix}\frac{1}{2}(1+\phi)\\\frac{1}{2}\theta\end{Bmatrix}$	$\begin{Bmatrix}\frac{1}{3}(2+\phi)\\\frac{1}{3}\theta\end{Bmatrix}$	$\begin{Bmatrix}\frac{1}{4}(3+\phi)\\\frac{1}{4}\theta\end{Bmatrix}$	$\begin{Bmatrix}\frac{1}{5}(4+\phi)\\\frac{1}{5}\theta\end{Bmatrix}$			
$\begin{Bmatrix}(1+\phi)\\-\phi\end{Bmatrix}$	$\begin{Bmatrix}\frac{1}{2}(2+\phi)\\-\frac{1}{2}\phi\end{Bmatrix}$	$\begin{Bmatrix}\frac{1}{3}(3+\phi)\\-\frac{1}{3}\phi\end{Bmatrix}$	$\begin{Bmatrix}\frac{1}{4}(4+\phi)\\-\frac{1}{4}\phi\end{Bmatrix}$				
$\begin{Bmatrix}(2+\phi)\\-(1+\phi)\end{Bmatrix}$	$\begin{Bmatrix}\frac{1}{2}(3+\phi)\\-\frac{1}{2}(1+\phi)\end{Bmatrix}$	$\begin{Bmatrix}\frac{1}{3}(4+\phi)\\-\frac{1}{3}(1+\phi)\end{Bmatrix}$					
$\begin{Bmatrix}(3+\phi)\\-(2+\phi)\end{Bmatrix}$	$\begin{Bmatrix}\frac{1}{2}(4+\phi)\\-\frac{1}{2}(2+\phi)\end{Bmatrix}$						
$\begin{Bmatrix}(4+\phi)\\-(3+\phi)\end{Bmatrix}$							

Each approximation when computed finds its proper place in a corresponding scheme; the number is to be found at the vertex of a triangle which contains just those lower approximations which are used in the course of its calculation, and the base of the triangle contains the tabular values on which the approximation actually depends. The approximation can be identified by the arguments at the extremities of the base; if these are $a-mh$ and $b+nh$, we denote the approximation by $U_{m,n}$. Either m or n may be negative, since $a-mh$ is identical with $b-(m+1)h$. It is important always to remember that $U_{m,n}$ is a perfectly precise number, namely, the value at X of the polynomial of degree $m+n+1$ which agrees with $u(x)$ at each of the $m+n+2$ tabular points from $a-mh$ to $b+nh$. Since X divides the interval from $a-mh$ to $b+nh$ into the two parts $(m+\theta)h$, $(n+\phi)h$, the approximation $U_{m,n}$ is derived from adjacent approximations of a lower order by the formula

$$U_{m,n} = \frac{(n+\phi)\ U_{m,n-1} + (m+\theta)\ U_{m-1,n}}{m+n+1}$$

a particular case of the general formula of §5, and if to this we add the initial condition

$$U_{m,-(m+1)} = u(a-mh)$$

the whole table is implicit.

To illustrate the uncanny convergence of a regular scheme, we may take the function $\log_{10} \cosh x$, with which Whittaker and Robinson (l. c. p. 41) and Aitken (*Math. Gazette*, v. 16, p. 21) use the values from 0·360 to 0·370 at interval 0·002 to interpolate between 0·364 and 0·366. Aitken's argument is 0·36536 6, implying $\theta = \cdot 683$, $\phi = \cdot 317$. The scheme of approximations, guarded by a digit in the thirteenth place, is:

	0·02			836		4619	6 9
0·362	785	52378	05				
				37425	554		
4	815	73796	65			5 654	
				47847	538		50
6	846	10474	38			7 965	
				43017	164		49
8	876	62389	89			3 504	
				22977	164		
70	907	29521	80				

The required value is therefore 0·02836 46196 95. Whittaker and Robinson evaluate the function for $x = 0·3655$. With the simple values $\theta = \frac{3}{4}$, $\phi = \frac{1}{4}$ the approximations can be computed in a few minutes. Omitting the column of tabulated values, the scheme is:

0·02838			4987	557
39860	600			
		4	404	
31304	948			2
		6	406	
47495	502			0
		3	064	
28475	002			

In each case the value, to the twelfth place, is given without the use of the entry for 0·360, which must be introduced, although it is superfluous, in any process which depends essentially on an even number of tabular values. If the two five-point cubic approximations had not been so close in our schemes, the elements derivable from another entry could have been added without any modification of the work already done.

For a more elaborate example, the reader may use the data given in the next paragraph to evaluate Si 22·12742 983. The calculations can be checked, for the scheme of approximations given there consists of alternate columns of the scheme produced by straightforward application of the scheme of coefficients set out in this section.

10. Quadratic Interpolation.

If in the formula for $U_{m,n}$ we substitute for $U_{m,n-1}$ and $U_{m-1,n}$ their values in terms of $U_{m,n-2}$, $U_{m-1,n-1}$ and $U_{m-1,n-1}$, $U_{m-2,n}$ respectively, we have

$$U_{m,n} = \frac{(n+\phi)(n-1+\phi)U_{m,n-2} + 2(n+\phi)(m+\theta)U_{m-1,n-1} + (m+\theta)(m-1+\theta)U_{m-2,n}}{(m+n)(m+n+1)}$$

a formula for quadratic interpolation, by which we can miss out alternate columns of the scheme, and reach an approximation of assigned order in fewer steps. Similarly by substituting now for $U_{m,n-2}$, $U_{m-1,n-1}$, and $U_{m-2,n}$ we can find a formula for cubic interpolation and advance by steps still longer. This process can be continued, and at length in any specific case we have a formula expressing $U_{m,n}$ in terms of the tabular values from $U_{m,-(m+1)}$ to $U_{-(n+1),n}$ that is, from $u(a-mh)$ to $u(b+nh)$, a formula which can be nothing but a version of Lagrange's polynomial, adapted to the special case of the regular interval. Iterative linear interpolation and the single formula are seen as the extreme members of a chain of methods of reaching what is ultimately the same result. But at each change from one method to the next, the complexity of the coefficients necessarily increases, and as the number of steps is reduced, more time is needed for the construction of the formula and for the computation of the coefficients, and less of the work that is done is of any use in the calculation of any further approximation. The most economical method may well be neither at one end nor at the other of the chain. Experimental evidence is wanted, but since the effect of complexity in the coefficients seems soon to be overwhelming, cubic interpolation is probably not worth consideration. Quadratic interpolation does deserve a few words. The scheme of coefficients is given on Sheet III.

In this scheme the first approximation shewn opposite $u(a-h)$, for example, is calculated as

$$\tfrac{1}{2}(1+\theta)\theta u(a-2h)-(2+\theta)\theta u(a-h)+\tfrac{1}{2}(2+\theta)(1+\theta)u(a).$$

The last two approximations indicated are $U_{1,3}$ and $U_{3,1}$. and these may be combined linearly to give the approximation $U_{1,1}$.

This scheme is much simpler than is at first apparent. In the first column, the set of coefficients

$$\ldots\ldots\tfrac{1}{2}(3+\theta)(2+\theta),\quad \tfrac{1}{2}(2+\theta)(1+\theta),\quad \tfrac{1}{2}(1+\theta)\theta,\quad -\tfrac{1}{2}\theta\phi,\quad \tfrac{1}{2}\phi(1+)\phi,$$
$$\tfrac{1}{2}(1+\phi)(2+\phi),\quad \tfrac{1}{2}(2+\phi)(3+\phi)\ldots\ldots$$

which it will be convenient to call the fundamental set, occurs twice. Now the differences corresponding to this set are simply

$$\ldots\ldots-(2+\theta),\quad -(1+\theta),\quad -\theta,\quad \phi,\quad (1+\phi),\quad (2+\phi),\ldots\ldots$$

and therefore when the product $\tfrac{1}{2}\theta\phi$ has been found, the fundamental set can be written down very quickly. In succeeding columns occur these same numbers, divided in the second column by 6, in the third by 15, in the fourth by 28, in the fifth by 45, and so on, and displaced upwards or downwards in the column. Thus two of the coefficients in each group of three are found without trouble, and since the sum of the three coefficients in each group is necessarily unity, the missing coefficients can be inserted without reference to any formula.

An example which has been used elsewhere* to illustrate other methods is the evaluation of Si X for X = 22·12742 983, from values of the sine integral Si x at interval 0·2. With this interval

$$\theta=\cdot 63714\ 915,\quad \phi=\cdot 36285\ 085,\quad \tfrac{1}{2}\theta\phi=\cdot 11559\ 5055\ ,$$

and the fundamental set of coefficients is

........8·43300 1545, 4·79585 2395, 2·15870 3245, 0·52155 4095,
− 0·11559 5055,

0·24725 5795, 1·61010 6645, 3·97295 7495, 7·33580 8345,.........

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* *B. A. Tables*, I, p. x; Aitken l. c. p. 62.

SHEET III.—QUADRATIC INTERPOLATION WITH A REGULAR INTERVAL.

$u(a-4h)$				
$u(a-3h)$	$\frac{1}{2}(3+\theta)(2+\theta)$ $-(4+\theta)(2+\theta)$ $\frac{1}{2}(4+\theta)(3+\theta)$			
$u(a-2h)$	$\frac{1}{2}(2+\theta)(1+\theta)$ $-(3+\theta)(1+\theta)$ $\frac{1}{2}(3+\theta((2+\theta)$	$\frac{1}{12}(1+\theta)\theta$ $-\frac{1}{6}(4+\theta)\theta$ $\frac{1}{12}(4+\theta)(3+\theta)$		
$u(a-h)$	$\frac{1}{2}(1+\theta)\theta$ $-(2+\theta)\theta$ $\frac{1}{2}(2+\theta)(1+\theta)$	$-\frac{1}{12}\theta\phi$ $\frac{1}{6}(3+\theta)\phi$ $\frac{1}{12}(3+\theta)(2+\theta)$	$\frac{1}{30}\phi(1+\phi)$ $\frac{1}{15}(4+\theta)(1+\phi)$ $\frac{1}{30}(4+\theta)(3+\theta)$	
$u(a)$	$-\frac{1}{2}\theta\phi$ $(1+\theta)\phi$ $\frac{1}{2}(1+\theta)\theta$	$\frac{1}{12}\phi(1+\phi)$ $\frac{1}{6}(2+\theta)(1+\phi)$ $\frac{1}{12}(2+\theta)(1+\theta)$	$\frac{1}{30}(1+\phi)(2+\phi)$ $\frac{1}{15}(3+\theta)(2+\phi)$ $\frac{1}{30}(3+\theta)(2+\theta)$	$\frac{1}{56}(2+\phi)(3+\phi)$ $\frac{1}{28}(4+\theta)(3+\phi)$ $\frac{1}{56}(4+\theta)(3+\theta)$
$u(b)$	$\frac{1}{2}\phi(1+\phi)$ $\theta(1+\phi)$ $-\frac{1}{2}\theta\phi$	$\frac{1}{12}(1+\phi)(2+\phi)$ $\frac{1}{6}(1+\theta)(2+\phi)$ $\frac{1}{12}(1+\theta)\theta$	$\frac{1}{30}(2+\phi)(3+\phi)$ $\frac{1}{15}(2+\theta)(3+\phi)$ $\frac{1}{30}(2+\theta)(1+\theta)$	$\frac{1}{56}(3+\phi)(4+\phi)$ $\frac{1}{28}(3+\theta)(4+\phi)$ $\frac{1}{56}(3+\theta)(2+\theta)$
$u(b+h)$	$\frac{1}{2}(1+\phi)(2+\phi)$ $-\phi(2+\phi)$ $\frac{1}{2}\phi(1+\phi)$	$\frac{1}{12}(2+\phi)(3+\phi)$ $\frac{1}{6}\theta(3+\phi)$ $-\frac{1}{12}\theta\phi$	$\frac{1}{30}(3+\phi)(4+\phi)$ $\frac{1}{15}(1+\theta)(4+\phi)$ $\frac{1}{30}(1+\theta)\theta$	
$u(b+2h)$	$\frac{1}{2}(2+\phi)(3+\phi)$ $-(1+\phi)(3+\phi)$ $\frac{1}{2}(1+\phi)(2+\phi)$	$\frac{1}{12}(3+\phi)(4+\phi)$ $-\frac{1}{6}\phi(4+\phi)$ $\frac{1}{12}\phi(1+\phi)$		
$u(b+3h)$	$\frac{1}{2}(3+\phi)(4+\phi)$ $-(2+\phi)(4+\phi)$ $\frac{1}{2}(2+\phi)(3+\phi)$			
$u(b+4h)$				

The scheme of coefficients is therefore as follows:

21·2

·4	4·79585	2395				
	− 12·22885	3940				
	8·43300	1545				
·6	2·15870	3245	0·08692	568		
	− 5·95455	5640	− 0·49242	594		
	4·79585	2395	1·40550	026		
·8	0·52155	4095	0·01926	584	0·01648	
	− 1·68025	7340	− 0·21995	711	0·42132	
	2·15870	3245	0·79930	873	0·56220	
22·0	− 0·11559	5055	0·04120	930	0·10734	0·142
	0·59404	0960	0·59900	683	0·57294	0·557
	0·52155	4095	0·35978	387	0·31972	0·301
·2	0·24725	5795	0·26835	117	0·26486	0·262
	0·86833	9260	0·64472	315	0·59123	0·567
	− 0·11559	5555	0·08692	568	0·14391	0·171
·4	1·61010	6645	0·66215	958	0·48905	
	− 0·85736	2440	0·35710	626	0·47618	
	0·24725	5795	− 0·01926	584	0·03477	
·6	3·97295	7495	1·22263	472		
	− 4·58306	4140	− 0·26384	402		
	1·61010	6645	0·04120	930		
·8	7·33580	8345				
	− 10·30876	5840				
	3·97295	7495				

23·0

where the middle coefficient in each set operates on the number immediately to its left in the scheme of approximations, and the flanking coefficients operate on the approximations above and below this one. The scheme of approximations so constructed is as follows:

1·		61			566		56	299	
60225	21386								
60822	85319	651	44197	3					
61260	93649	582	54480	6	49625	4			
61525	24777	566	86534	6	56097	5	296	1	
61608	37366	566	86247	7	56640	5	302	9	1
61510	35866	566	03923	3	55766	9	293	6	0
61238	32456	569	46189	9	57819	2	310	8	
60806	11397	601	50598	4	43302	2			
60233	62873	701	12547	0					
59545	94323								

The column on the left contains the tabular values from Si 21·2 to Si 23·0, and the elements of the other columns are evaluated in succession by the operations shewn in the scheme of coefficients; for example,

$$0{\cdot}26486 \times 56640\ 5 + 0{\cdot}59123 \times 55766\ 9 + 0{\cdot}14391 \times 57819\ 2 = 56293\ 6.$$

The figure in the eleventh place serves only as a guard.

As will be understood from the detailed account of the calculations in § 7, the scheme of operations and the scheme of approximations are developed together, each from the centre outwards. The first numbers to be calculated are the three terms nearest the centre of the fundamental set of coefficients. From these are found the two groups of coefficients which are to operate, one on the group of functional values for the arguments 21·8, 22·0, 22·2, the other on the group for the arguments 22·0, 22·2, 22·4. The operations are then performed, giving the central elements in the first column of approximations. These elements agree as far as the fifth place, and since a closer approximation would be obtained by interpolating linearly between them, we can safely assert that the required functional value is between 1·61566 and 1·61567. Assuming that we need greater accuracy than this, since otherwise we should not be working to eleven places of decimals, we must compute two more of the fundamental coefficients, one on each side of those already known. We then write down the group of coefficients which is to operate on the functional values for 21·6, 21·8, 22·0, and the group which is to operate on those for 22·2, 22·4, 22·6, and we perform the operations indicated. Inspection of the four approximations which are now to be combined in groups of three shews the number of decimal places to be retained in the coefficients operating on these groups, and the next step is to write down two more groups of coefficients, the central groups of the second column of the scheme of coefficients; four of the six coefficients in these groups come from the first column by division by 6, and the groups are completed by subtraction from unity. The central elements in the second column of approximations follow, agreeing to the sixth place; these would suffice by linear interpolation for the determination of the seventh place, but nothing is gained by an interruption at this stage.

11. Iterative Computation of Osculatory Approximations.

The only first approximations of which we have as yet made use are those obtained by linear interpolation from the values of $u(x)$ at two distinct points. Developments from this basis are given by Aitken, and but for the remark that quadratic steps are *impossible* except in a certain symmetrical case one would have thought that the substance of the foregoing sections was already implicit in his article. However this may be, when we incorporate into a scheme of approximations values given by polynomials which have multiple agreement with $u(x)$, that is, which are partial sums of the Taylor series for $u(x)$, we are definitely breaking fresh ground. The closest linear approximations of this kind, if X is in the interval a b, are $u_a+(X-a)u'_a$ and $u_b-(b-X)u'_b$; regarding these as determined by contact at a and b respectively, while L_{ab} is determined by intersection at both a and b, we naturally denote these Taylor approximations by L_{aa} and L_{bb} or, turning the suffix into an argument for typographical convenience, by $L(a^2)$ and $L(b^2)$. Since they have no common basis, the approximations L_{aa} and L_{bb} cannot be combined with each other. But each of them can be combined with L_{ab} by the elementary formula, and so we have two approximations of the second order, $M(a^2b)$ and $M(ab^2)$, given by

$$M(a^2b)=\frac{b-X}{b-a}L(a^2)+\frac{X-a}{b-a}L(ab),$$

$$M(ab^2)=\frac{b-X}{b-a}L(ab)+\frac{X-a}{b-a}L(b^2),$$

which are distinct from the three-point quadratic approximations. From these alone we can derive the cubic approximation given by

$$N(a^2b^2)=\frac{b-X}{b-a}M(a^2b)+\frac{X-a}{b-a}M(ab^2),$$

and we can also combine either of them with any other quadratic approximation with which it is already in double agreement. We have, that is to say, if c is distinct from both a and b,

$$N(a^2bc)=\frac{c-X}{c-a}M(a^2b)+\frac{X-a}{c-a}M(abc),$$

$$N(a^2bc)=\frac{c-X}{c-b}M(a^2b)+\frac{X-b}{c-b}M(a^2c),$$

and also

$$N(ab^2c)=\frac{c-X}{c-b}M(ab^2)+\frac{X-b}{c-b}M(abc),$$

$$N(ab^2c)=\frac{c-X}{c-a}M(ab^2)+\frac{X-a}{c-a}M(b^2c).$$

It is to be observed that $N(a^2bc)$ and $N(ab^2c)$ are perfectly definite numbers, the values at X of two special cubics; we have in each case two formulae for computing the same number, not two numbers which we are denoting by the same symbol. In practice there is never any difficulty in seeing which line of approach to any particular approximation uses the smallest fractions and is therefore the most efficient. If b is between a and c, we use $M(abc)$ rather than $M(a^2c)$ to determine $N(a^2bc)$, and $M(b^2c)$ rather than $M(abc)$ to determine $N(ab^2c)$.

In terms of the values of the function and its first derivative at the three points a, b, c only, the approximation of highest order is the value at X of the quintic which has the same value and the same derivative at the three points. The cubic approximations $N(a^2b^2)$, $N(ab^2c)$ give the quartic approximation $P(a^2b^2c)$, the cubic approximations $N(ab^2c)$, $N(b^2c^2)$, give the quartic approximation $P(ab^2c^2)$, and the quintic approximation required follows from these two quartic approximations; the last three interpolations all utilise the same fractions, namely, the parts into which X divides the interval ac.

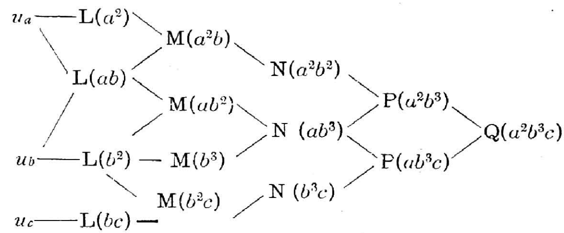
Often we shall know the same number of derivatives at every tabular point, but the use of a derivative at one point is entirely independent of the use of a derivative at any other point. If for example it happens that u_b is not known, or that we have reason to suspect that the value given is not reliable, we can still obtain the quartic approximation $P(a^2bc^2)$, from $N(a^2bc)$ and $N(abc^2)$. Or if $u'(x)$ is tabulated at a wider interval than $u(x)$, or for only part of the range of the table of $u(x)$, we can still utilise such values of $u'(x)$ as we do know.

When the second derivative is tabulated, quadratic Taylor approximations of the form $M(a^3)$, that is, $u_a + (X-a)u'_a + \frac{1}{2}(X-a)^2\ u''_a$, become available. The approximation $M(a^3)$ is independent of any approximation found by interpolation without the use of second derivatives, and it combines with $M(a^2b)$ to give $N(a^3b)$. There is no direct combination of $M(a^3)$ with $M\ (abc)$, but when each of them has been combined separately with $M(a^2b)$, giving $N(a^3b)$ and $N(a^2bc)$, the last two approximations give by linear interpolation the quartic approximation $P(a^3bc)$. The approximation $Q(a^2b^3c)$ uses in addition to the values of $u(x)$ at a, b, c, the values of $u'(x)$ at a and b and the value of $u''(x)$

at b. The Taylor approximation $L(a^2)$, $L(b^2)$, $M(b^3)$ must be computed directly :

$$L(a^2) = u_a + (X - a)u'_a, \qquad L(b^2) = u_b - (b - X)u'_b,$$
$$M(b^3) = L(b^2) + \tfrac{1}{2}(b - X)^2 u''_b ;$$

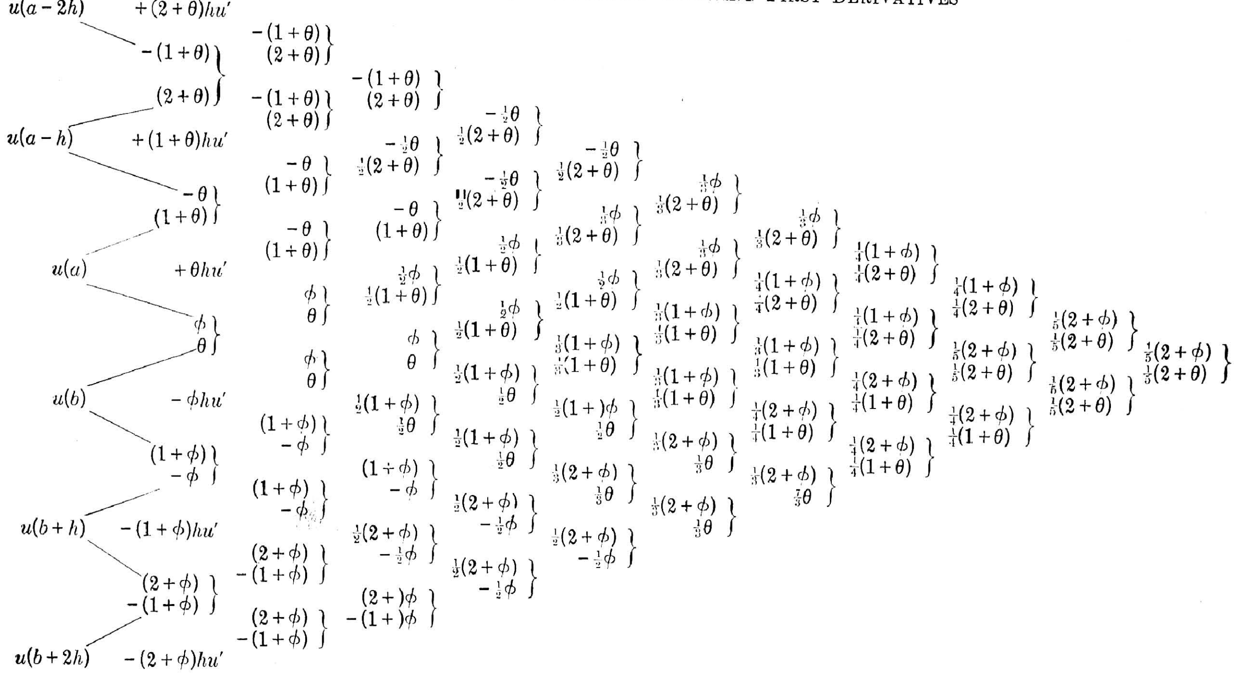
the remaining elements in the scheme of approximations



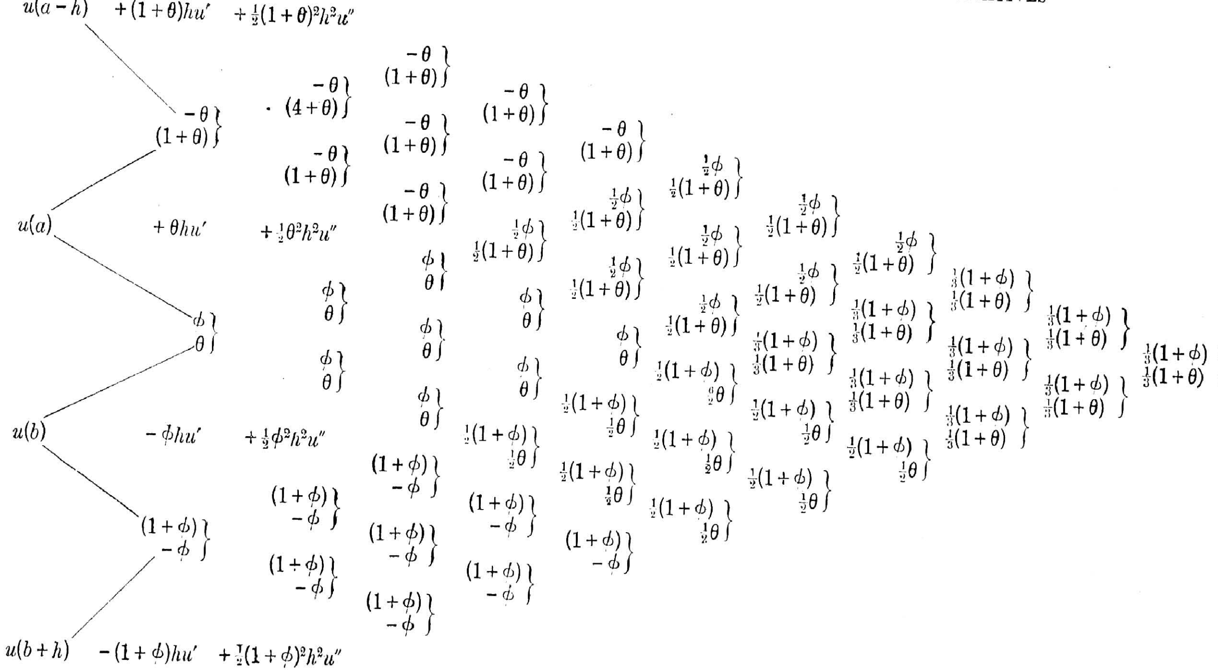
are found by linear interpolation. If the scheme is built up in the most natural way, each interpolation is performed on adjacent members of a column, and the fractions required are the parts into which X divides the interval from the first point involved in the upper symbol to the last point involved in the lower. If in a particular case $P(a^2b^3)$ and $P(ab^3c)$ differ so much that we are doubtful whether $Q(a^2b^3c)$ is sufficiently accurate for our purpose, we must invoke an additional datum, which may take the form of the value of $u(x)$ at another point or of the value of another derivative at one of the points a, b, c. For example, given u''_a, we calculate $M(a^3)$ as $L(a^2) + \frac{1}{2}(x - a)^2 u''_a$ and add to our scheme, without disturbing the results already incorporated, the approximations $M(a^3)$, $N(a^3b)$, $P(a^3b^2)$, $Q(a^3b^3)$; or given u'_c, we calculate $L(c^2)$ and then add $M(bc^2)$, $N(b^2c^2)$, $P(b^3c^2)$, $Q(ab^3c^2)$.

To give schemes of coefficients hardly accords with the emphasis we are placing on the adaptability and simplicity of the iterative process ; it is literally more trouble to verify the position of an approximation in any scheme than to write down the coefficients required for its calculation by mere inspection. If we give on sheets IV and V schemes for the case of a regular interval, firstly when the function and its first derivative are tabulated, and secondly when the function and its first two derivatives are tabulated, the object is rather that the reader by examining them may become confident of the principle involved than that he should refer to them in carrying out a particular interpolation. The notation is that of § 9.

SHEET IV.—REGULAR INTERPOLATION UTILISING FIRST DERIVATIVES



Sheet V.—Regular Interpolation utilising First and Second Derivatives



Since each of these schemes ends in an approximation of the eleventh order, further extension is of formal rather than practical interest. In practice we regard the schemes as ending in two approximations of the tenth order; if these two do not agree to the order of accuracy required, either a mistake has been made in the computation or the interval of tabulation is really excessively large. As to the structure of the schemes, the pairs of fractions are the same as in the earlier scheme, since always they are only the parts into which an interval between tabular points is divided at X, but whereas in the simpler scheme each pair figures only once, these schemes are so to speak tessellated, and each pair occupies the whole of one diamond.

In irregular schemes, in which derivatives are not available to the same order at every tabular point, there is the same kind of tessellation, but the tiles are not equilateral. However irregular the interval of tabulation, and to whatever extent the number of derivatives computed may vary from point to point, it is always natural to build up a scheme in which each element utilises all the data available for values of x between the smallest and the largest on which that element depends, though not necessarily all the data for the extreme values themselves. In other words, the fullest use is made of the data at points of any interval before that interval is extended in either direction. Thus if u'_b is known, and b is between a and c, the scheme does not include M (abc); it includes M (ab^2) and M(b^2c), and it is from these, if u''_b is not given, that N(ab^2c) is found. In a scheme of this kind, the number of available derivatives at any point is shewn at once by the number of Taylor approximations that begin the row corresponding to that point, and each approximation is adequately specified by the multiplicities at its ends, without explicit reference to intermediate multiplicities. For example, in relation to the first scheme of this section, U (m, n) denotes the approximation of order $2(m+n)+1$ which is the value at X of the polynomial of degree $2(m+n)+1$ determined by intersecting $u(x)$ at $a-mh$ and $b+nh$ and touching $u(x)$ at the $m+n$ intermediate tabular points; similarly U(m^2, n), U(m, n^2) are of order $2(m+n)+2$ and come from polynomials which touch $u(x)$ at $a-mh$ and $b+nh$ respectively, while U(m^2, n^2) is of order $2(m+n)+3$ and corresponds to the polynomial which touches $u(x)$ at every tabular point from $a-mh$ to $b+nh$ inclusive. These four approximations are all computed from lower approximations by interpolation with the same fractions $(n+\phi)/(m+n+1)$, $(m+\theta)/(m+n+1)$, since in each case the fractions wanted are the parts into which X divides the interval from $a-mh$

to $b+nh$; the lower approximations are: for $U(m, n)$, $U(\overline{m-1^2}, n)$ and $U(m, \overline{n-1^2})$; for $U(m^2, n)$, $U(m, n)$ and $U(m^2, \overline{n-1^2})$; for $U(m, n^2)$. $U(\overline{m-1^2}, n^2)$ and $U(m, n)$; for $U(m^2, n^2)$, $U(m, n^2)$ and $U(m^2, n)$. The apparent lack of homogeneity is due to the variation in the number of suppressed symbols; in $U(m, n)$ there are $m+n$ of these, and the total order of agreement is $1+2(m+n)+1$; in $U(\overline{m-1^2}, n)$, the $m+n-1$ suppressed symbols imply a total order of agreement of $2+2(m+n-1)+1$, and for $U(m^2, \overline{n-1^2})$ the total order is $2+2(m+n-1)+2$. Similarly if second derivatives are available throughout and the interval is regular, the scheme is constructed of Taylor approximations together with approximations of the form $U(m^\rho, n^\sigma)$, $\rho, \sigma = 1, 2, 3$; there is agreement of order ρ at $a-mh$, of order σ at $b+nh$, and of order 3 at every intermediate tabular point. The fractions for interpolation are the same as before, and the lower approximations from which $U(m\rho, n\sigma)$ is derived are $U(m^{\rho-1}, n^{\sigma})$ and $U(m^{\rho}, n^{\sigma-1})$, if ρ and σ are both greater than 1; if $\rho=1$, $m^{\rho-1}$ must be replaced by $(\overline{m-1})^3$, if $\sigma=1$, $n^{\sigma-1}$ by $\overline{n-1^3}$.

12. Two-Point and Three-Point Interpolation.

One special case of iterated interpolation is particularly simple. Suppose that the values of n derivatives are known at both a and b. Without using values elsewhere, we form the three linear approximations $U(a^2)$, $U(ab)$, $U(b^2)$, the four quadratic approximations $U(a^3)$, $U(a^2b)$, $U(ab^2)$, $U(b^3)$, and so on, until we have $n+2$ approximations $U(a^{n+1})$, $U(a^nb)$ ... ... , $U(b^{n+1})$ of order n. From these, without further reference to tabular values, we can still form sets of higher approximations diminishing in number; the $n+1$ approximations $U(a^{n+1}b)$, $U(a^nb^2)$, ... , $U(a\, b^{n+1})$ are of order $n+1$, the n approximations $U(a^{n+1}b^2)$, $U(a^nb^3)$ ... ... , $U(a^2b^{n+1})$ are of order $n+2$, and at length we have two approximations, $U(a^{n+1}b^n)$, $U(a^nb^{n+1})$, of order $2n$, and a single approximation $U(a^{n+1}\, b^{n+1})$ of order $2n+1$. Since the same pair of fractions ϕ, θ, is used throughout these interpolations, we have in point of fact

$$U(a^{n+1}b^{n+1}) = \phi^{n+1}\; U(a^{n+1}) + \binom{n+1}{1}\phi^n\theta\, U(a^nb) + \binom{n+1}{2}\phi^{n+1}\theta^2\;\; U(a^{n-1}b^2) + \cdot$$

the coefficients being the binomial coefficients. But this identity is not an aid to computation: on the contrary, to compute the value of an expression of the form appearing here, iterative interpolation is certainly the best method. It is interesting to compare the construction of $U(a^{n+1}b^{n+1})$ with the use of Taylor's series alone. To find a Taylor approximation of order $2n+1$ we require derivatives of the first $2n+1$ orders; moreover, to obtain a check on the computation it is necessary to work independently from the two ends of the interval, and only one of the two fractions θ, ϕ can be less than ½. Only the first n derivatives are wanted if the approximation is continued by interpolation; the product $\theta^{n+1}\phi^{n+1}$ takes the place of θ^{2n+2} or ϕ^{2n+2} in governing the accuracy of the approximation, and $\theta\phi$ cannot be greater than $\frac{1}{4}$; also a check on the arithmetic is afforded by a comparison of the two penultimate approximations $U(a^{n+1}b^{n})$, $U(a^{n}b^{n+1})$. As a numerical example, take a function which, with its first two derivatives, is given as follows:

$$u_a = 0{\cdot}86602\ 54 \qquad u'_a = 0{\cdot}13089\ 97 \qquad u''_a = 0{\cdot}05935\ 61$$

$$u_b = 0{\cdot}96592\ 58 \qquad u'_b = 0{\cdot}06775\ 87 \qquad u''_b = 0{\cdot}06620\ 32$$

and let $\theta = {\cdot}75392\ 59$, $\phi = {\cdot}24607\ 41$. The scheme of values is

0.	9	47	2	
86602 54	6471 43	84 52	7 87	
		09 38		5 99
	4134 29		5 38	
		30 60		6 00
			6 21	
96592 58	4925 21	27 78		

whence $u(X) = 0{\cdot}94726\ 00$. The function is in fact the sine, with $a = 60^\circ$, $b = 75^\circ$, $X = 71^\circ\ 18'\ 32''$; second derivatives give seven-figure accuracy with a 15° interval.

The same function furnishes a striking example of three-point interpolation with second derivatives. Among the tables in the British Association volume is one giving sin x and cos x to 15 places of decimals with x in radians at an interval of 0·1, and it is expressly said in the Introduction that this table cannot be interpolated to its own order of accuracy by the ordinary process. Indeed, 12th differences would be necessary, and then there would be no check on the result. Alternatively, Taylor's series could be used, taken to the 9th or 10th derivative. But the required order of approximation is attainable by means of the first two derivatives at three points only. The whole process involves the calculation of three quadratic Taylor approximations, followed by 27 linear interpolations, with 3 distinct pairs of coefficients each used 9 times. The work is comparable with that of interpolation from 8 tabular values, which involves 28 linear stages, no two of which have the same pair of coefficients, and it is well within the range of practical use of a table. The reader who will take the trouble to work out a typical example can hardly fail to be amazed by the complete invisibility of the regularity which must exist in the numbers to account for the convergence of a scheme.

From

sin 38·1 = ·39023 62353 07945, sin 38·2 = ·48020 47804 38257,
sin 38·3 = ·56537 52781 37025

cos 38·1 = ·92071 47661 74999, cos 38·2 = ·87715 64107 06919,
cos 38·3 = ·82483 37983 32416

I find sin 38·24 = ·51489 75686 74725, and from

sin 37·6 = − ·09894 96575 59291, sin 37·7 = + ·00088 81568 05715
sin 37·8 = + ·10071 70969 92503

cos 37·6 = + ·99509 24405 65539, cos 37·7 = + ·99999 96055 88666,
cos 37·8 = + ·99491 51051 08673

I find sin 37·74 = ·04087 67647 89390. The scheme for the latter example is anything but typical, but it is delightfully instructive; throughout the range from 37·6 to 37·8 the even derivatives are small compared with the odd derivatives, and it follows, as will be understood from §14 below, that the convergence is spasmodic; actually the numbers of digits ascertained in the several stages are 2, 0, 4, 1, 3, 0, 5, and the eighth stage is not reached, there being 15-figure agreement between the two approximations of the seventh order.

For another example, let us compute again the value of Si x for $x = 22{\cdot}12742\ 983$, the function Si x being tabulated at an interval of 0·2. By definition the first derivative of this function is $(\sin x)/x$, and successive derivatives can be evaluated rapidly from the formula

$$\mathrm{Si}^{(n+1)}x = \frac{\sin^{(n)}x - n\mathrm{Si}^{(n)}x}{x}$$

where $\sin^{(n)}x$ runs of course through the cycle $\cos x, -\sin x, -\cos x$ $\sin x, \cos x, \ldots$ We are given

Si 22·0 = 1·61608 37366	Si 22·2 = 1·61510 35866
sin 22·0 = − 0·00885 13093	sin 22·2 = − 0·20733 64206
cos 22·0 = − 0·99996 08264	cos 22·2 = − 0·97826 97014

and the scheme of approximations which results, utilising third derivatives, is as follows:

1·61		566	5	6		
608 37366	603 24675	35768	1400			
				283		
			9063		301	
		72329		311		300
	545 92349		4744		299	
		44729		292		299
			7173		299	
				303		
510 35866	578 13541	64261	5807			

whence Si x = 1·61566 56299. It would have been better to retain a figure, itself unreliable, in the eleventh place, to guard the tenth place from accumulation of errors, but the result is correct. It is impossible by considering a page of figures to form a judgment of the work involved in reaching a numerical result. In comparing the scheme just given with accounts of other computations we must bear in mind that if the method of this scheme was the anticipated method of interpolation, the original table would be provided with derivatives rather than with differences.

13. The Utilisation of Derivatives in Bridging and in Inverse Interpolation.

It is evident that in bridging, any available derivatives on either side of the boundary may be utilised; no new principles are involved, and no complications are possible. The problem of inverse interpolation is different. The straightforward process is to calculate derivatives of x as a function of u by the elementary formulae

$$\frac{dx}{du} = 1 \Big/ \frac{du}{dx}, \quad \frac{d^2x}{du^2} = -\frac{d^2u}{dx^2} \Big/ \left(\frac{du}{dx}\right)^3, \quad \frac{d^3x}{du^3} = \left\{ 3\frac{du}{dx}\left(\frac{d^3u}{dx^3}\right) - \left(\frac{d^2u}{dx^2}\right)^2 \right\} \Big/ \left(\frac{du}{dx}\right)^5, \ldots$$

It is true that these formulae soon become heavy and inelegant, but the first two are simple enough, and are worth utilising in practice. In inverse interpolation the evaluation of the fractions into which U divides the various intervals between tabulated values is a substantial part of the labour. For an approximation of the eighth order from tabular values alone, twenty-eight distinct intervals are required, but if second derivatives are used, the number of distinct intervals is reduced to three; also, as we shall see, in § 14, the latter approximation is likely to be accurate to three more places of decimals than the former.

For an example of the utilisation of a derivative in inverse interpolation, we may return to Davis's equation $x^7 + 28x^4 - 480 = 0$. With $u'(x) = 7x^6 + 112x^3$ we have $u'(1{\cdot}92) = 1143{\cdot}39894\ 578$ and therefore, in the notation of § 7, the value of $u'(U_2)$ is the reciprocal of this. Thus the first Taylor approximation corresponding to this value of u, given by $X_{22} = X_2 + (U - U_2)X'_2$, has the value $1{\cdot}92 + 3{\cdot}30\ldots/1143{\cdot}3\ldots$ that is, $1{\cdot}92289\ 26595\ 9$. We can now combine X_{22} and X_{12} to form one quadratic approximation X_{122}, and X_{22} and X_{23} to form a second quadratic approximation X_{223}; the pairs of fractions involved are those already used to form X_{12} and X_{23}, namely, those consisting of the parts into which U divides the intervals from U_1 to U_2 and from U_2 to U_3. Thus we find

$$X_{122} = 1{\cdot}92288\ 39929\ 5, \qquad X_{223} = 1{\cdot}92288\ 42406\ 6.$$

and interpolating between these approximations with the fractions into which U divides the interval from U_1 to U_3 we have the approximation X_{1223}, which although it is a cubic approximation, has the value $1{\cdot}92288\ 41513\ 7$, and is almost as close as the quartic approximation,

X_{01234}. Interpolating between X_{1223} and X_{1234} with the fractions into which U divides the interval from U_2 to U_4 we find that in the quartic approximation X_{12234} the digits from the eighth decimal place to the eleventh are 532 2, and since extrapolation from X_{01234} and X_{12234} in accordance with the relation of U to the interval from U_0 to U_2 replaces the eleventh digit by 4, we can be confident that the root is between 1·92288 41532 and 1·92288 41533. Had we set out from the first with the intention of utilising the first derivative at U_2, the order of evaluation would have been X_{23}, X_{22}, X_{223}, X_{12}, X_{122}, X_{1223}, X_{34}, X_{234}, X_{2234}, X_{12234}, X_{011}, X_{012}, X_{0122}, X_{01223}, X_{012234} and the three approximations of highest order would all have been obtained by interpolation.

14. The Error in an Approximation.

From the practical point of view it is unnecessary to consider in advance the convergence of such schemes as we have been constructing. The convergence is exhibited in the schemes, and affords in itself a very severe check on the computation, for to proceed after a mistake has been made is in effect to introduce a function which is zero at all but one of the tabular points, and an analytic function so defined fluctuates wildly in its numerical values. Nevertheless, we may conclude by examining the two natural assumptions that have been tacit in our work, namely, that an approximation of higher order is likely to be better than one of lower order, and that an approximation is likely to be the more effective the more closely the tabular points on which it depends cluster round the point where the value is actually wanted. These assumptions are justified by the classical expression for the difference between the function $u(x)$ and a polynomial approximation. Given g distinct values $a_1, a_2, \ldots a_g$ of x, let $p(x)$ be the polynomial of degree n, equal to $\sigma_1 + \sigma_2 + \ldots + \sigma_g - 1$, which has agreement with $u(x)$ of order σ_1 at a_1, of order σ_2 at a_2, and so on, and let $\Pi(x)$ denote the product

$$(x-a_1)^{\sigma_1}(x-a_2)^{\sigma_2}\ldots\ldots(x-a_g)^{\sigma_g},$$

a polynomial of degree $n+1$ in which the coefficient of x^{n+1} is unity. Let X be any value of x distinct from $a_1, a_2, \ldots a_g$, determine the constant k by the condition

$$u(X) = p(X) + k\Pi(X),$$

and consider the function $v(x)$ defined by

$$v(x) = u(x) - p(x) - k\Pi(x).$$

By hypothesis, both $u(x) - p(x)$ and $\Pi(x)$ have a_1 for a zero of multiplicity σ_1, a_2 for a zero of multiplicity σ_2, and so on, and therefore $v(x)$ has the same multiple zeroes. Also, by the definition of k,

$v(X)$ vanishes identically. Thus $v(x)$ has zeroes of combined multiplicity $n+2$ for the values $a_1, a_2, \dots a_g$, X of x, and therefore, by Rolle's theorem, the $(n+1)^{\text{th}}$ derivative $v^{(n+1)}(x)$ is zero for at least one value ξ of x between the least and the greatest of $a_1, a_2, \dots a_g$, X. Since $p(x)$ is a polynomial of degree n, $p^{(n+1)}(x)$ is zero identically, and since $\Pi(x)$ differs from x^{n+1} by a polynomial of degree n, $\Pi^{(n+1)}(x)$ has the constant value $(n+1)!$. Hence for all values of x,

$$v^{(n+1)}(x) = u^{(n+1)}(x) - k(n+1)!,$$

and the equation $v^{(n+1)}(\xi) = 0$ is equivalent to

$$k = u^{(n+1)}(\xi)/(n+1)!$$

That is,

There is a value ξ between the least and the greatest of the numbers $a_1, a_2, \dots a_g$, X, such that the difference $u(X) - p(X)$ has the form

$$\frac{u^{(n+1)}(\xi)}{(n+1)!}(X-a_1)^{\sigma_1}(X-a_2)^{\sigma_2}\dots(X-a_g)^{\sigma_g}.$$

Since the precise position of ξ is unknown, the practical value of this result is to furnish limits to the numerical value of the difference $u(X)-p(X)$, in terms of the range of values of $u^{(n+1)}(x)$.

The advantage of values clustering round X is immediately obvious. For example, if the interval is regular and X is between a and b, additional derivatives at a and b introduce a factor $\theta\phi h^2$, whereas additional values at $a-3h$ and $b+3h$ introduce a factor $(3+\theta)(3+\phi)h^2$. Whatever the actual magnitudes of these factors, the second is at least forty-nine times as large as the first. Similarly, with a regular interval, a three-point approximation of the eighth order utilising second derivatives implies a factor $\{\theta\phi(1+\theta)\}^3$, whereas an approximation of this order dependent on tabular values implies a factor

$$\theta\phi(1+\theta)(1+\phi)(2+\theta)\ (2+\phi)\,(3+\theta)\ \ (3+\phi)\,(4+\theta)\,;$$

the maximum value of $\theta\varphi(1+\theta)$, that is, of $\theta(1-\theta^2)$, is $2/3\sqrt{3}$, and the ratio of the first factor to the second is therefore less than that of $4/27$ to $2^2.3^2.4$, that is, than 1 to 972: the difference between the first approximation and the true value is likely to be only about one thousandth of the difference between the second approximation and the true value. This rough estimate has been used in § 13, for although with inverse interpolation the interval is necessarily irregular, inequalities cannot in practice affect the order of the estimate unless the second derivative is so large that the use of the more distant points is out of the question.

To appreciate the effect of an increase in the order of an approximation, we may use the classical theorem that if M is an upper limit to the modulus of an analytic function $u(x)$ on the circumference of a circle in the complex plane, an upper limit to the modulus of $u^{(n+1)}(x)$ throughout a closed region Γ inside the circle is $(n+1)!\,M/R^{n+1}$, when R is the shortest distance between a point of Γ and a point on the circumference. It follows that if $u(x)$ is analytic, throughout a circle which has for diameter an interval ef of the real axis which includes $a_1, a_2, \ldots\ldots a_g$, X as internal points, an upper limit to the numerical value of $u(X) - p(X)$ can be expressed in the form

$$M\left(\frac{X-a_1}{R}\right)^{\sigma_1}\left(\frac{X-a_2}{R}\right)^{\sigma_2}\cdots\cdots\left(\frac{X-a_g}{R}\right)^{\sigma_g},$$

where M is a constant and R is the difference between e and the smallest of the numbers $a_1, a_2, \ldots\ldots\ldots a_g$, X or between f and the greatest of these numbers, which ever of these two differences is the smaller.

For interpolation, in the strictest sense, that is, for a value of X between the least and greatest of $a_1, a_2, \ldots\ldots\ldots a_g$, the value of R, like that of M, is independent of X, but for extrapolation, R may be $f - X$ or $X - e$. Every increase in the order of approximation adds to the number of factors of the form $(X - a_k)/R$, and if the only tabular values of x used are values for which this factor is a proper fraction, every increase in the order means an increase in the reliability of the approximation; this of course is not to say that the closeness of the approximation must improve steadily, since at any stage it may happen that the value of ϕ corresponding to a particular choice of data is so close to a zero of $u^{(n+1)}(x)$ as to imply a much better approximation than could safely be anticipated; such an accidental accuracy is not as a rule transmitted, and an approximation of higher order may be in fact a worse approximation.

The form of the expression for $u(X) - p(X)$ explains at once why we cannot expect to improve an approximation indefinitely by taking account of more and more distant values of x, even if $u(x)$ is bounded for real values of x. With a fixed value of R, there is no reason to suppose that values of a_k for which $X - a_k$ is numerically greater than R are worth using. And for an integral function, although R may be supposed large enough to admit any particular a_k, the value of M necessarily tends to infinity with R, and therefore a variable R defeats its own object.

15. The Discovery of the Iterative Process.

The first attempt to avoid the use of differences by direct operation on the tabular values of a function was made by Ch. Jordan, who published in *Metron* an interpolation formula giving

$u(x)$ in terms of the linear interpolate $\phi u(a)+\theta u(b)$, the linear interpolate $\frac{1}{3}\{(1+\phi)u(a-h)+(1+\theta)u(b+h)\}$, the linear interpolate $\frac{1}{5}\{(2+\phi)u(a-2h)+(2+\theta)u(b+h)\}$, and so on. If these interpolates are denoted by U_0, U_1, U_2, ... Jordan's formula is of the form

$$u(X)=V_0-G_2V_1+G_4V_2+\cdots$$

where G_2, G_4 ,...are coefficients, that is, functions of θ and ϕ already familiar to computers as coefficients in a formula due to Gauss, and V_r is a simple linear combination of U_0, U_1, U_2,......U_r The work of calculating V_0, V_1, V_2, $\cdots$ directly is not quite trivial, and it was presently noticed by Aitken that they can be calculated most readily as differences of increasing order at the midpoint of the series ...U_2, U_1, U_0, U_0, U_1, U_2,... and with this simplification of Jordan's procedure it seemed to be established that for a table unprovided with differences, it is quicker to find Jordan's means and to difference the reflected series formed by them than to difference the original function. Nevertheless, the admission that the best way to handle Jordan's interpolates was to form differences from them seemed a disappointing outcome of the attempt to avoid differencing the original function, and Aitken, appreciating that the practical advantage of Jordan's process was wholly in the first stage of substituting the operation of linear interpolation for one operation of differencing, succeeded in presenting the results of this first stage in such a form that the same substitution could be made again and again; on account of the symmetrical distribution of the data provided by the first stage in Jordan's process, each subsequent step was a quadratic interpolation linear in the square of the variable, but Aitken saw this peculiar form as essentially a process of iterated linear interpolation from tabular values, and discovered the application of the more general process to inverse interpolation and other problems.

Meanwhile, in the hope of reconciling the advocates of Taylor's formula with those of Everett's, I was attempting to utilise derivatives as well as tabular values at more points than one. After looking unsuccessfully for formulae involving differences of derivatives, I approached the problem from the point of view of combining different approximations of the same order. The simple formula $N(a^2b^2)=\phi^2\,L(a^2)+2\phi\theta\,L(ab)+\theta^2\,L(b^2)$ pointed at once to the more general formula of §12, the decomposition into the elementary formulae

$$M(a^2b)=\phi L(a^2)+\theta L(ab),\quad N(a^2b^2)=\phi M(a^2b)+\theta M(ab^2)$$

was the clue to the general theorem of § 5, and it remained only to realise that the practical value of the process did not depend on any possibility of building up elegant general formulae.

Contributions to the analytic theory of numbers (II)*

BY

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Introduction.

Let $r_{s,k}(n)$ denote the number of representations of the positive integer n as a sum of s positive kth. powers,

(1) $r_{s,k}(n) = \sum 1 \qquad (h_1 \ldots h_s \text{ integers} \geqslant 1,\ k \geqslant 2,\ h_1^k + \ldots h_s^k = n.)$

In their researches on Waring's Problem, Hardy and Littlewood have found asymptotic formulae for $r_{s,k}(n)$ for fixed $k \geqslant 3$ and all s exceeding a certain limit depending on k. They have also shown that a good deal of light would be thrown on Waring's Problem if the following unproved hypothesis†, which they call Hypothesis K, is true:

(2) $r_{k,k}(n) = O(n^{\varepsilon})$ *for every positive* ε $(k \geqslant 3)$ HYPOTHESIS K.

A. E. Western has done some computational work which supports (2) for $k=3$. His calculations indicate that**

(3) $$r_{3,3}(n) = O(\log^2 n).$$

It is shown here that

(4) $$r_{3,3}(n) = \Omega(\log n/\log \log n).$$

If Hypothesis K is true for $k=4$, then it follows that

(5) $$r_{3,4}(n) = O(n^{\varepsilon}).$$

I show, however, that

(6) $$r_{3,4}(n) = \Omega(\log n/\log \log n).$$

With regard to the number of representations of n as a sum of two positive cubes, L. J. Mordell†† showed that

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* The present paper formed portion of my dissertation for the Cambridge Ph. D. (1931).

† *Mathematische Zeitschrift*, 23, 1925, 1—37 (4).

** Announced at the meeting of the London Mathematical Society held on 23rd April 1931.

†† In a letter to Prof. G. H. Hardy.

(7) $$r_{2,3}(n) = \Omega(x(n))$$

for some $x(n) \longrightarrow \infty$ (on the other hand $r_{2,3}(n) = O(n^{\varepsilon})$ is trivial).

S. S. Pillai* has improved (7) into

(8) $$r_{2,3}(n) = \Omega(\log\log n).$$

I have generalized Pillai's method to show that if k is any fixed integer (positive or negative) then

(9) $$\sum_{\substack{x^3+ky^3=n \\ x>0,\ y>0}} 1 = \Omega(\log\log n)$$

This result should be contrasted with the following which are due to van der Corput-Jarnik †, Oppenheim ** and Thue-Siegel †† respectively :

(10) $$\sum_{\substack{x^3+ky^3=n \\ x,\ y>0}} 1 = O(n^{2\ 9}), \qquad \sum_{\substack{x^m+ky^m=n \\ x,\ y>0}} 1 = O(n^{2\ 3m}) \qquad (k>0);$$

(11) $$\sum_{\substack{x^3+ky^3=n \\ x,\ y>0}} 1 = O(n^{\varepsilon}), \qquad \sum_{\substack{x^m+ky^m=n \\ x,\ y>0}} 1 = O(n^{\varepsilon}) \qquad (k>0);$$

(12) $$\sum_{\substack{x^3+ky^3=n \\ x,\ y>0}} 1 \text{ is finite } (k<0).$$

§1. Proof of (4)

We have ‡

(13) $$72 = (g_1'(t))^3 + (g_2(t))^3 + (g_3(t))^3$$

where

(14) $$g_1(t) = \frac{12t^3(t^3+1) - (t^3+1)^3}{t(t^3+1)^2}$$

(15) $$g_2(t) = \frac{(t^3+1)^3 - 12t^3(t^3-1)}{t(t^3+1)^2}$$

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* In a letter to Chowla.

† see Jarník : Über die Gitterpunkte auf konvexen Kurven, *Math. Zeitschr.*, 24 (1925), 500-18 (07-08).

** *Crellés Journal* 164, 1931, 133-135.

†† see, for example, Landau's *Vorlesungen über Zahlentheorie* (1927), Bd. 3 (37).

‡ Landau, *Vorlesungen über Zahlentheorie,* 1927 Bd. 3, S. 216. The identity is due to Richmond and was used by him to prove that every positive rational number is expressible as a sum of the cubes of three positive rational numbers.

(16) $$g_3(t) = \frac{12t^3(t^3-1)}{t(t^3+1)^2}$$

Now, $g_1(1)=4$, $g_2(1)=2$, $g_3(1)=0$, $g_3(t)>0$ when $t>1$. It follows that we can find an absolute constant $\theta>0$ such that all the g's are positive in $1<t<1+\theta$, and further that if t_1, t_2 are any two values of t in this range, then

(17) $$g_1(t_1)\neq g_1(t_2) \qquad (t_1\neq t_2)$$

(18) $$g_1(t_1)\neq g_2(t_2)$$

(19) $$g_1(t_1)\neq g_3(t_2)$$

Let m be the least positive integer such that $1/m\leqslant\theta$, it follows from (17), (18), (19) that if we put $t-1=\frac{1}{m+1}, \frac{1}{m+2}, \frac{1}{m+3}, \ldots, \frac{1}{m+n}$ in (13), we obtain n distinct representations of 72 as a sum of the cubes of 3 positive rational numbers. From (14), (15), (16) we have for $r=1, 2, \ldots, n$

(20)
$$g_1\left(1+\frac{1}{m+r}\right)=12\left(\frac{m+r+1}{m+r}\right)^2\frac{(m+r)^3}{(m+r)^3+(m+r+1)^3}$$
$$-\frac{m+r}{m+r+1}\;\frac{(m+r)^3+(m+r+1)^3}{(m+r)^3}$$

(21)
$$g_2\left(1+\frac{1}{m+r}\right)=\frac{m+r}{m+r+1}\cdot\frac{(m+r)^3+(m+r+1)^3}{(m+r)^3}$$
$$-12\left(\frac{m+r+1}{m+r}\right)^2\cdot\frac{(m+r+1)^3-(m+r)^3}{(m+r)^3}\cdot\frac{(m+r)^6}{\{(m+r)^3+(m+r+1)^3\}^2}.$$

(22)
$$g_3\left(1+\frac{1}{m+r}\right)=12\left(\frac{m+r+1}{m+r}\right)^2\cdot\frac{(m+r+1)^3-(m+r)^3}{(m+r)^3}\cdot\frac{(m+r)^6}{\{(m+r)^3+(m+r+1)^3\}^2}.$$

In what follows numerals in **thick print** are to be regarded as references to the equations of this paper. We now see that

(23) $$x=72\prod_{r=1}^{n}\left[(m+r)^2(m+r+1)\left((m+r)^3+(m+r+1)^3\right)\right]^2$$
(20, 21, 22, 13)

is a sum of three positive integral cubes in at least n distinct ways. Let us take $n>m$. Then

$$x\leqslant 72\prod_{r=1}^{n}(4n^2\cdot 2n\cdot 16^2n^6)=72(2048)^n n^{9n}$$ **(23)**

and so,

(24) $\log x \leqslant \log 72 + n \log 2048 + 9n \log n < 10\, n \log n \quad (n > n_0).$

(25) $n = \Omega\left(\dfrac{\log x}{\log\log x}\right)$ (24)

(26) $n \leqslant r_{3,3}(x)$ (23)

(27) $r_{3,3}(x) = \Omega\left(\dfrac{\log x}{\log\log x}\right)$ (26, 25),

which is (4).

§2. Proof of (6)

We start with the identity *

(28) $(k^2 - 2k)^4 + (2k - 1)^4 + (k^2 - 1)^4 = 2(k^2 - k + 1)^4$

and proceed in a manner similar to that of the last section.

§3. Proof of (9)

In this case our starting point is the pair of identities †

(29) $\dfrac{r^3}{p^3} - k\dfrac{s^3}{p^3} = x^3 + ky^3$

where

(30) $x = \dfrac{r}{p} \cdot \dfrac{r^3 - 2ks^3}{r^3 + ks^3}, \qquad y = \dfrac{s}{p} \cdot \dfrac{2r^3 - ks^3}{r^3 + ks^3};$

and

(31) $\dfrac{r^3}{p^3} + k\dfrac{s^3}{p^3} = x^3 - ky^3$

where

(32) $x = \dfrac{r}{p} \cdot \dfrac{r^3 + 2ks^3}{r^3 - ks^3}, \qquad y = \dfrac{s}{p} \cdot \dfrac{2r^3 + ks^3}{r^3 - ks^3}$

We now prove the following

Lemma: If

(33) $r = k \cdot 2^{3(\;+1)}, \; s = 1, \; p = 1, \; k > 0$

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* See Dickson's *History of the theory of numbers*, Vol. 2, page 656, reference No. 222. See also reference No. 205 on page 654.

† The origin of these identities lies in the fact that the tangent at a "rational" point of the curve $x^3 + ky^3 = 1$ cuts it in another "rational" point.

then

$$(34)\quad \frac{r^3+ks^3}{p^3}=\frac{m^3-kn^3}{v^3}=\frac{r_1^3+ks_1^3}{p_1^3}=\frac{m_1^3-kn_1^3}{v_1^3}=\frac{r_2^3+ks_2^3}{p_2^3}=\quad\cdots\quad\cdots$$
$$=\frac{m_t^3-kn_t^3}{v_t^3}=\frac{r_{t+1}^3+ks_{t+1}^3}{p_{t+1}^3}=\quad\cdots\quad\cdots\quad\cdots$$

where the $(m,\ n,\ r,\ s,\ p,\ v=m_0,\ n_0,\ r_0,\ s_0,\ p_0,\ v_0)$

(35) $m_\theta,\ n_\theta,\ r_\theta,\ s_\theta,\ p_\theta,\ v_\theta$ $(0\leqslant\theta\leqslant t+1)$ **are positive integers satisfying the relations.**

(36) $m_\theta=r_\theta(r_\theta^3+2ks^3)_\theta,\quad n_\theta=s_\theta(2r_\theta^3+ks_\theta^3)$ **(31, 32, 34)**

(37) $r_\theta=m_{\theta-1}(m_{\theta-1}^3-2kn_{\theta-1}^3),\quad s_\theta=n_{\theta-1}(2m_{\theta-1}^3-kn_{\theta-1}^3)$
(34, 29, 30).

(38) $p_\theta=v_{\theta-1}(m_{\theta-1}^3+kn_{\theta-1}^3),\quad v_\theta=p_\theta(r_\theta^3-ks_\theta^3)$
(34, 29, 30; 34, 31, 32).

for all $0\leqslant\theta\leqslant t+1$. **Further**

(39) $\dfrac{r_{\theta+1}}{s_{\theta+1}}<\dfrac{r_\theta}{s_\theta}$, and $\dfrac{m_{\theta+1}}{n_{\theta+1}}<\dfrac{m_\theta}{n_\theta}$ $(\theta\leqslant\theta\leqslant t)$.

Proof : Obviously

(40) $m,\ n,\ r,\ s,\ p,\ v$ are integers >0. (36, 33, 38).

Now

(41) $\dfrac{m}{n}=\dfrac{r}{s}\,\dfrac{r^3+2ks^3}{2r^3+ks^3}>\dfrac{1}{2}\,\dfrac{r}{s}$ (36).

(42) $\dfrac{m^3}{n^3}>\dfrac{1}{8}\,\dfrac{r^3}{s^3}=\dfrac{1}{2^3}\,k^3\ 2^{9(t+1)}\geqslant 64k$ (33).

Let

(43) $m_\theta,\ n_\theta,\ r_\theta,\ s_\theta,\ p_\theta,\ v_\theta>0$ $(0\leqslant\theta\leqslant h-1)$

(44) $\dfrac{m_\theta^3}{n_\theta^3}>64k$ $(0\leqslant\theta\leqslant h-1)$;

then, we shall show that

(45) $m_h,\ r_h,\ n_h,\ s_h,\ p_h,\ v_h>0.$

(46) $\dfrac{m_h^3}{n_h^3}>64k.$

When we have done this (43) and (44) will have been proved for all θ in $0 \leqslant \theta \leqslant t+1$ by induction (**40, 42**). We have

(47) $r_h,\ s_h > 0$ **(37, 43, 44).**

(48) $m_h,\ n_h > 0.$ **(36, 47).**

(49) $p_h > 0$ **(38, 43)**

(50) $$\frac{r_\theta}{s_\theta} = \frac{m_{\theta-1}}{n_{\theta-1}} \frac{m_{\theta-1}^3 + 2kn_{\theta-1}^3}{2m_{\theta-1}^3 - kn_{\theta-1}^3}$$ **(37).**

$$> \frac{1}{4} \frac{m_{\theta-1}}{n_{\theta-1}} \quad \textbf{(44)} \quad (1 \leqslant \theta \leqslant h)$$

(51) $$\frac{r_h^3}{s_h^3} > \frac{1}{64} \frac{m_{h-1}^3}{n_{h-1}^3} \quad \textbf{(50)} > k$$ **(44).**

(52) $v_h > 0$ **(49, 51, 38).**

Hence it only remains to prove (46). Now,

(53) $$\frac{m_\theta}{n_\theta} = \frac{r_\theta}{s} \frac{r_\theta^3 + 2ks_\theta^3}{2r_\theta^3 + ks_\theta^3} > \frac{1}{2} \frac{r_\theta}{s_\theta} \textbf{ (36) } > \frac{1}{8} \frac{m_{\theta-1}}{n_{\theta-1}} \textbf{ (50) } (1 \leqslant \theta \leqslant h)$$

Multiplying (41) by the equations (53), we obtain

(54) $$\frac{m_h}{n_h} > \frac{1}{2} \frac{1}{8^h} \frac{r}{s}$$

(55) $$\frac{m_h^3}{n_h^3} > \frac{1}{2^3} \frac{1}{2^{9h}} k^3 2^{9(t+1)} \textbf{ (54, 33) } \geqslant 64k,$$

which proves (46). Further

(56) $$\frac{m_\theta}{n_\theta} < \frac{r_\theta}{s_\theta} \textbf{ (36)}; \quad \frac{r_{\theta+1}}{s_{\theta+1}} < \frac{m_\theta}{n_\theta}$$ **(37).**

and this pair of inequalities proves (39), and the Lemma has been completely proved.

§3. Proof of (9)

Let M be the least common multiple of $p_1, p_2 \ldots, p_t$; it follows from the equations (34) that if

(57) $$N = M^3 \left(k^3 \cdot 2^{9(t+1)} + 1\right)$$

then ($k > 0$)

(58) $$\sum_{\substack{x^3+ky^3=N\\ x,\, y>o}} 1 > t.$$

From (38),

(59) $$v = p(r^3-ks^3)\,;\qquad p_1 = v(m^3+kn^3)\,;\qquad v_1 = p_1(r_1^3-ks_1^3)\,;$$
$$p_2 = v_1(m_1^3+kn_1^3)\,;\quad \ldots\quad \ldots\quad \ldots$$

Thus,

(60) $$p_\theta = (m^3+kn^3)(m_1^3+kn_1^3)\ldots\ldots(m_{\theta-1}^3+kn_{\theta-1}^3)$$
$$\times\,(r_{\theta-1}^3-ks_{\theta-1}^3)\ldots\ldots(r_1^3-ks_1^3)(r^3-ks^3) \qquad (59).$$

and p_t is divisible by the product of $p_1, p_2, \ldots, p_{t-1}$. Hence M $= p_t$. Further

(61) $$p_\theta < 2^\theta \left(mm_1\ldots m_{\theta-1}\right)^3\left(rr_1\ldots r_{\theta-1}\right)^3 \qquad (60,\ 44).$$

(62) $$m = r(r^3+2ks^3) < 3r^4 \qquad (36).$$

(62) $$r_1 = m(m^3-2kn^3) < m^4 < 3^4\ r^{4^2} \qquad (37,\ 62).$$

(62) $$m_1 < 3r_1^4 < 3.\,3^{4^2}.\,r^4$$

(62) $$r_2 < m_1^4 < 3^4.\,3^{4^3}.\,r^{4^4}$$

(62) $$m_2 < 3r_2^4 < 3.3^{4^2}.3^{4^4}.r^{4^5}$$

(62) $$\cdot\ \cdot\ \cdot$$

(62) $$r_{t-1} < m_{t-2}^4 < 3^2.\ 3^{4^3}\ldots 3^{4^{2t-3}}.r^{4^{2t-2}}.$$

(62) $$m_{t-1} < 3r_{t-1}^4 < 3.3^{4^2}.3^{4^4}\ldots\ldots 3^{4^{2t-2}}.r^{4^{2t-1}}$$

Multiplying the equations (62) we obtain on using (61), (33), (57)

(63) $$\log N < ct\cdot 4^{2t}$$

(64) $$\log\log N < ct \qquad (63).$$

From (58) and (64) it follows that for $k>0$,

(65) $$\sum_{\substack{x^3+ky^3=N\\ x,\, y>0}} 1 = \Omega\ (\log\log N).$$

We shall now prove (65) for negative k. Let $k>o$, as before. Then,

(66) $$\sum_{\substack{x^0 - ky^3 = N_1 \\ x,\ y>0}} 1 > t$$ (34)

where

(66) $N_1 = M_1^3(2^{9(t+1)}k^3 + 1)$

and

(67) M_1 is the L. C. M of $v, v_1, \ldots, v_{t-1}$.

But

(68) $M_1 = v_{t-1}$ (59).

(69) $v_{t-1} < p_t = M$ (59)

(70) $M_1 < M$ (68, 69)

(71) $N_1 < N$ (57, 66, 70)

(72) $t = \Omega(\log \log N) = \Omega(\log \log N_1)$ (64, 71)

From (66) and (72) it follows that (65) is also true for negative k, and the proof of (9) is complete.

An Algebra of Numerical Compositions

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The unrestricted partitions of 3 are 3; 1,2; 2,1; 1,1,1. A composition into parts of a prescribed kind is a partition into parts of that kind with attention to the order of the parts. As is well known, the theory of compositions is much simpler than that of partitions. We develop here a complete algebra of compositions, including the new processes of differentiation and integration, from a few definitions and postulates suggested by existing theorems. From this algebra it is a simple matter to convert any algebraic identity into a unique correspondent concerning compositions. The simplest identities yield the known theorems on compositions.

The values of the numerical functions may be elements of any commutative field. For simplicity of statement we have taken the field of complex numbers as the field of values, as the generalization to any commutative field is immediate and can be made by a few slight verbal changes.

1. Definitions and postulates

Let K be a well-defined class of integers $\geqslant c$, where c is a constant integer $\geqslant 0$, and write $K_0 \equiv K$. The vector or one-rowed matrix $(n_1 \ldots, n_p)$ will be called a *composition of* n *of degree* p *over* K_c if $n_1, \ldots, n_p$ are in K_c and $n_1 + \ldots + n_p = n$.

If $f(x)$ is single-valued and finite for finite integer values of x in K_c, we say that $f(x)$, or simply f, is a *numerical function over* K_c. We postulate that the value of $f(x)$ is in the field of complex numbers. A *scalar* is a complex number.

(1) POSTULATE.—If f is a numerical function over K_c, and if n is an integer $\geqslant 0$ not in K_c, the value of $f(n)$ is 0.

This postulate permits us to take $c=0$ in further definitions and postulates. Hence we may omit reference to K_c.

If x is not an integer $\geqslant 0$, $f(x)$ is not defined, and we say that $f(y)$ *exists* when and only when y is in K. (See also § 4 for infinite summations.)

(2) EQUALITY.—The numerical functions f, g are said to be *equal*, $f=g$, if and only if $f(n)=g(n)$ for all integers $n \geq 0$, or, what is equivalent, $f=g$ if and only if $f(x)=g(x)$ whenever $f(x)$, $g(x)$ exist.

(3) ZERO.—The numerical function ζ which is such that $\zeta(x)=0$ when $\zeta(x)$ exists, is the *zero* numerical function. When there can be no confusion, we shall write $\zeta \equiv 0$.

(4) UNIT.—The numerical function υ which is such that $\upsilon(0)=1$, $\upsilon(n)=0$, $n>0$, is the *unit* numerical function. We shall write $\upsilon \equiv 1$ when convenient. For example, the 1 in $1-f$, if f is a numerical function, is υ.

(5) SCALAR PRODUCT.—If α is scalar, the *scalar product* of α and the numerical function f is the numerical function g defined by $\alpha f(x) \equiv g(x)$ whenever $f(x)$ exists. We write $g \equiv \alpha f \equiv f\alpha$.

(6) ADDITION.—The sum h of the numerical functions f, g is defined by $h(x) \equiv f(x)+g(x)$, whenever $f(x)$, $g(x)$ exist, and we write $h \equiv f+g$.

(7) MULTIPLICATION.—The *composite*, or *product*, fg of the numerical functions f, g is defined by

$$h(n) \equiv fg(n) \equiv \sum f(n_1)\, g(n_2), \qquad (n=0,\ 1,\ .\ .\ .\ .\ .\ ),$$

the sum extending over all compositions (n_1, n_2) of degree 2 of n.

Hence, by induction, $f_1 \ldots f_p$ is defined for $p>2$, and is

$$f_1 \ldots f_p(n) \equiv \sum f_1(n_1) \ldots f_p(n_p), \qquad (n=0,\ 1,\ .\ .\ .\ .\ ),$$

summed over all compositions $(n_1, \ldots, n_p)$ of degree p of n.

If $f_1 = \cdots = f_p = f$, we write $f_1 \ldots f_p \equiv f^p$.

If the compositions are taken over K_c, where $c>0$, then $f_1 \ldots f_p(n)=0$ if $p>n$, and hence sums of the form

$$\sum_{p=1}^{\infty} x_p\, f^p\,(n),$$

where the x_p are scalars, are finite (over K_c, $c>0$). This is of importance in questions of convergence.

By definition, $f^0 \equiv \upsilon$ (see (4)).

(8) NILFACTORS.—If $fg=0$, $f \neq 0$, $g \neq 0$, in the notation of (3), each of f, g is said to be a *nilfactor* (of composition, as in (7)).

We dispose of nilfactors here by proving their non-existence. Without loss of generality, for some finite integer $s \geq 0$, let

$$g(j)=0, \qquad (j=0,\ .\ .\ .\ .\ ,\ s), \qquad g(s+1) \neq 0.$$

By (7), if $fg=0$, we have then

$$f(0)\, g(n+s) + f(1)\, g(n+s-1) + \ldots + f(n-1)\, g(s+1) = 0,$$

for $n=1, 2, \ldots$ Now $g(s+1)\neq 0$. Hence, taking $n=1$, we get $f(0)=0$. If it be assumed that $f(j)=0$ for $j=0, \ldots, t$, the induction is completed by showing that $f(t+1)=0$. We have here

$$f(0)\, g(t+s+2) + \ldots + f(t)\, g(s+2) + f(t+1)\, g(s+1) = 0;$$

whence, $f(t+1)=0$.

(9) INDEX.—The least integer n ($\geqslant 0$) for which $f(n)\neq 0$, is called the *index* of f. If $f\neq\zeta$ (ζ as in (3)), the index of f is a finite integer; by definition the index of a scalar is zero, and the index of ζ is ∞. The index of f will be denoted by $\mathrm{I}f$. Hence $\mathrm{I}\,\zeta=\infty$; $\mathrm{I}k=0$, k scalar; $\mathrm{I}v=0$.

(10) DIVISION.—If a unique numerical function h exists such that $fh=g$, where f, g are given numerical functions, we write $h\equiv gf^{-1}\equiv g/f$, and call h the *quotient* of g by f. Necessary and sufficient conditions for the existence of h are given later. We write $v/f\equiv f^{-1}\equiv 1/f$.

(11) DERIVATIVE.—The numerical function f' defined by

$$f'(n)=(n+1)\, f(n+1) \qquad (n=0, 1, \ldots)$$

will be called the *derivative* of the numerical function f. The derivative of a scalar is defined to be zero.

(12) ANTI-DERIVATIVE.—The numerical function $\bar{f}$ defined by

$$\bar{f}\,(0)=0, \qquad \bar{f}\,(n)=\frac{f(n-1)}{n} \qquad (n=1, 2, \ldots),$$

will be called the *antiderivative* of f.

(13) NOTATION.—Referring to (11), (12), we shall write

$$\partial f \equiv f', \quad \partial^{-1}f\equiv \bar{f}.$$

This is consistent, since

$$\partial^{-1}(\partial f)=\partial(\partial^{-1}f)=f.$$

(14) GENERATOR.—If x is a scalar variable, the formal power series

$$\mathrm{F}(x)\equiv f(0)+f(1)x+\ldots+f(n)x^{n}+\ldots.$$

is called the generator of f.

It was shown elsewhere* that the principle of equating coefficients in identities between generators holds independently of considerations of convergence. For the exact statement of the principle in this connection, we refer to the paper cited.

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* E. T. BELL, *Transactions of the American Mathematical Society*, 25, 1923, 135-154.

2. Indices

We first develop the necessary properties of indices as defined in § 1(9). The least of the real numbers $a, b, c, \ldots$ will be denoted by Min $(a, b, c, \ldots)$. The letters f, g, h denote numerical functions.

(15) $\mathrm{I}\,(fg) = \mathrm{I}\,f + \mathrm{I}\,g$.

The proof may be given, as it is typical of more complicated ones which will be omitted. Let $\mathrm{I}f = a$, $\mathrm{I}g = b$. If $a = b = 0$, (15) is obvious. Let $a + b > 0$. It is to be shown that

$$\sum_{j=0}^{n} f(j)g(n-j) = 0, \qquad n = 0, \ldots, a+b-1;$$
$$\neq 0, \qquad n = a+b;$$

and we need discuss only

$$\sum_{j=0}^{a+b-p} f(j)g(a+b-p-j), \qquad p > 0, \qquad a+b > p.$$

By the definitions of a, b, the value of this sum is

$$\sum_{j=0}^{b-p} f(a+j)g(b-p-j),$$

and (15) is proved if $b < p$. If $b = p$, the value of the sum is $f(a)g(0)$, which is different from zero. If $b > p$, the sum has the value

$$\sum_{j=0}^{b-p} g(j)f(a+b-p+j) = 0.$$

The following are obvious from the definition of $\mathrm{I}f$:

(16) $\mathrm{I}\alpha f = \mathrm{I}f$, α scalar $\neq 0$;

(17) $\mathrm{I}\,0\,f = \infty$;

(18) If α, β are scalars such that $\alpha\beta \neq 0$, then:
$\mathrm{I}\,(\alpha f + \beta g) = \mathrm{Min}\,(\mathrm{I}f, \mathrm{I}g)$ if $\mathrm{I}f \neq \mathrm{I}g$: if $\mathrm{I}f = \mathrm{I}g = c$, then
$\mathrm{I}(\alpha f + \beta g) = c$ if $\alpha f(c) + \beta g(c) \neq 0$,
$\mathrm{I}(\alpha f + \beta g) > c$ if $\alpha f(c) + \beta g(c) = 0$.

Referring to (11) or (13) we have

(19) $\mathrm{I}f' = \mathrm{I}f - 1$ if $\mathrm{I}f > 0$;
$\mathrm{I}f' \geqslant 0$ if $\mathrm{I}f = 0$.

3. General Properties

As defined in § 1, the rational operations of composition of numerical functions do not generate a field over the field of scalars, since the quotient g/f (§1, (10)) does not necessarily exist (§1 (1)).

(20) *With respect to addition as in* (6), *multiplication as in* (7), *and scalar multiplication as in* (5), *the set of all numerical functions is a commutative ring over the field of scalars in which the zero, unit elements respectively are* ζ, v *as in* (3), (4).

Division is covered by the following.

(21) *In order that the quotient* h/f *shall exist, it is necessary and sufficient that* $\mathrm{I}h \geqslant \mathrm{I}f$.

If h/f exists, let it be the numerical function g. Then $h = gf$, and hence, by (15), $\mathrm{I}h = \mathrm{I}g + \mathrm{I}f$. Thus, if h/f exists, its index is $\mathrm{I}g$, which is equal to $\mathrm{I}h - \mathrm{I}f$. From this follows the necessity, since $\mathrm{I}g \geqslant 0$. To prove the sufficiency let $\mathrm{I}h = a$, $\mathrm{I}f = b$, assume from what has just been shown that $a - b > 0$, and proceed as in proving (15). This determines $g(a - b + n)$ $(n = 0, 1, \ldots)$ uniquely.

(22) *If* h/f *exists, then* $\mathrm{I}(h/f) = \mathrm{I}h - \mathrm{I}f$.

We show next that derivation, ∂, as defined in (13), has the formal properties of differentiation as in analysis. Thus, if α is scalar, and f, g any numerical functions, then

(23) $\partial(\alpha f) = \alpha \partial f$,

(24) $\partial(fg) = f\partial g + g\partial f$,

(25) $\partial(f/g) = (g\partial f - f\partial g)/g^2$,

the last holding if and only if f/g exists.

To prove (24), write $fg = h$, $\mathrm{I}f = a$, $\mathrm{I}g = b$. Then $\mathrm{I}h = a + b$, by (15), and hence $h(n)$ exists if and only if $n \geqslant a + b$. By (7) we have

$$h(a+b+p) = \sum f(a+n)g(b+m),$$

the sum referring to all integers n, $m \geqslant 0$ such that $n + m = p$. Hence by (11),

$$h'(a+b+p) = (a+b+p+1) \sum f(a+n)g(b+m),$$

summed over n, $m \geqslant 0$, $n + m = p + 1$. Thus

$$h'(a+b+p) = (a+b+p+1) \sum_{j=0}^{p+1} f(a+j)\, g\,(b+p+1-j).$$

Again, by (7), (11),

$$f'g(a+b+p) = \sum (r+1) f(r+1) g(s),$$
$$fg'(a+b+p) = \sum (s+1) f(r)\, g\,(s+1),$$

summed over r, $s \geqslant 0$, $r + s = a + b + p$. Reversing the order of summation in the second, we get for these two the equivalents

$$\sum_{j=0}^{p+1} (a+j) f(a+j)\, g\,(b+p+1-j),$$

$$\sum_{j=0}^{p+1} (b+p+1-j) f(a+j)\, g\,(b+p+1-j),$$

the sum of which is the expression for $h'(a+b+p)$ above.

Since $\partial\alpha=0$, (24) implies (23). If f/g exists in (25), let $f/g=h$. Then, by (20), $f=gh$, and hence, by (24),

$$f'=g'h+gh'=g'f/g+gh'\,;$$

whence, $g^2h'=gf'-g'f$, and (25) will be proved if $\mathrm{I}(g^2)\leqslant \mathrm{I}(gf'-g'f)$. Since f/g exists $\mathrm{I}f\geqslant \mathrm{I}g$. It can be shown from this by (18), (19) that

$$\mathrm{I}(gf'-g'f)\geqslant 2\mathrm{I}g,$$

which completes the proof, by (15).

In the same way, corresponding to partial integration, we have here

(26) $f\partial^{-1}g=fg-g\partial^{-1}f$

From what has been proved we easily see the following comprehensive connection with generators as in (14), which permits us to operate with them, if preferred, rather than directly with the numerical functions.

(27) *If* $F(x)$, $G(x)$, ... , $H(x)$, *are the respective generators of* f, g . . . , h, *and if*

$$\mathrm{R}(\mathrm{F}(x),\ \mathrm{G}(x),\ \ldots,\ \mathrm{H}(x))\equiv 0$$

is a formal identity in x, *then*

$$\mathrm{R}\,(f,\ g,\ \ldots,\ h)=\zeta,$$

where ζ *is as in* (3). *The generator of* ∂f *is the formal derivative* $F'(x)$ *of* $F(x)$ *with respect to* x, *and similarly for* $\partial^{-1}f$ *and integration.*

4. Convergence

Formally we have

$$f(1-f)^{-1}=\sum_{r=1}^{\infty} f^r$$

for any numerical function f. Suppose for a moment that the sum on the right is a numerical function, ϕ, as defined in § 1. Then for all finite integers $n\geqslant 0$, $\phi(n)$ is finite, and $f(1-f)^{-1}=\phi$. Now the sum in question will certainly be a numerical function if the class K_c over which the compositions f are taken has $c>0$, for in that case there exists an integer s such that, if $r>s$, then $f^r(n)=0$. But if the compositions be taken over K_0, the sum does not terminate, since compositions of all degrees $r>0$ exist for all $n\geqslant 0$, and the sum does not then necessarily represent $f(1-f)^{-1}$. However, if $f(0)=0$, the sum does represent $f(1-f)^{-1}$ when the compositions are taken over K_0. Accordingly we make at least one of the following assumptions for any infinite series of the type

(28) $$\alpha_1 f + \alpha_2 f^2 + \ldots + \alpha_r f^r + \ldots,$$

where f is a numerical function and $\alpha_1, \alpha_2, \ldots$ are scalars.

(29) *If the composition in (28) are taken over* K_0, *then* $f(0) = 0$;

or (30) *The compositions in (28) are taken over* K_c, $c > 0$.

If either of (29), (30) holds, we say that (28) is *convergent.* We assume then that all infinite series of compositions occurring are convergent in the sense just defined.

5. Iterations

Let f be a numerical function, α, β scalars. Then (§4) f_α, defined by

$$f_\alpha \equiv \frac{f}{1 - \alpha f} = f + \alpha f^2 + \ldots + \alpha^{r-1} f^r + \ldots$$

is a numerical function. Regarding f_α as having been obtained from $f \equiv f_0$ by the operation R^α,

(31) $$R^\alpha f \equiv f_\alpha = \frac{f}{1 - \alpha f}$$

We call f_α the αth *iterate* of f. Such iterates include many of the functions connected with compositions over K_1 that have appeared in the literature.

By (27), if the generator of f is $F(x)$, the generator of f_α is $F_\alpha(x)$, where

(32) $$F_\alpha(x) \equiv \frac{F(x)}{1 - \alpha F(x)} = \sum_{n=1}^{\infty} f_\alpha(n) x^n;$$

and, by (31),

(33) $$R^\beta(R^\alpha f) = \frac{f}{1 - (\alpha + \beta) f} = R^{\alpha+\beta} f,$$

$$R^\beta(R^\alpha f) = R^\alpha(R^\beta f);$$

whence,

$$f_{\alpha+\beta} = \sum_{r=1}^{\infty} \alpha^{r-1} f_\beta^{\,r} = \sum_{r=1}^{\infty} \beta^{r-1} f_\alpha^{\,r};$$

$$f = \sum_{r=1}^{\infty} (-1)^{r-1} \alpha^{r-1} f_\alpha^{\,r}; \qquad f_\alpha - f_\beta = f_\alpha f_\beta \cdot (\alpha - \beta).$$

By § 4 the infinite sums terminate for every integer $n > 0$ in $f_{\alpha+\beta}(n)$.

As will be seen presently, f_α refers to total compositions, that is, to compositions in which the degree is unrestricted. If the degree is restricted to be $\leqslant r$, $r>0$, the corresponding function is $f^{(r)}$, where

(34) $f_\alpha^{(r)} \equiv f+\alpha f^2+\ldots+\alpha^{r-1} f^r$;

and therefore

(35) $f_\alpha^{(r)} = \dfrac{f(1-\alpha^r f^r)}{1-\alpha f} = f_\alpha(1-\alpha^r f_\alpha^r)$

the limit of which as $r\to\infty$ is f_α;

$$f_\alpha^{(r)}(1-\beta^r f^r)-f_\beta^{(r)}(1-\alpha^r f^r)=(\alpha-\beta)f_\alpha^{(r)} f_\beta^{(r)},$$

the limiting form of which is the last of (33);

(36) $F_\alpha^{(r)}(x) \equiv \dfrac{F(x)[1-\alpha^r(F(x))^r]}{1-\alpha F(x)} \sum_{n=1}^{\infty} f_\alpha^{(r)}(n)x^n.$

From (27), or from the definition of f_α and $\partial f^r = rf^{r-1}$, we get

(37) $f^2 f_\alpha' = f_\alpha^2 f'$; $\quad f_\alpha'/f_\alpha^2 = f'/f^2$

hence, for the second derivatives

$$f_\alpha''/f_\alpha' - 2f_\alpha'/f_\alpha = f''/f' - 2f'/f,$$

and so on. Thus the successive derivatives $\delta^r f_\alpha$ are obtained from $\delta^r(f'/f^2)$ by replacing f by f_α and $\delta^s f$ by $\delta^s f_\alpha$. The next invariant of this kind is

$$\frac{f_\alpha'''}{f_\alpha'} - \left(\frac{f_\alpha''}{f_\alpha'}\right)^2 - 2\left[\frac{f_\alpha''}{f_\alpha} - \left(\frac{f_\alpha'}{f_\alpha}\right)^2\right],$$

which recalls the Schwartzian derivative. Since $f_\alpha = f(1-\alpha f)^{-1}$ expresses f_α as a linear fractional function of f, the derivative in question, $\{f_\alpha, f\}$, vanishes,

$$\frac{f_\alpha'''}{f_\alpha'} - \frac{3}{2}\left(\frac{f_\alpha''}{f_\alpha'}\right)^2 = 0.$$

The above invariant is thus reducible to

$$\frac{1}{2}\left(\frac{f_\alpha''}{f_\alpha'}\right)^2 - 2\left[\frac{f_\alpha''}{f_\alpha} - \left(\frac{f_\alpha'}{f_\alpha}\right)^2\right]$$

The f_α have simple interpretations for positive integer values of α. Let the compositions be taken over K_c, $c>0$ (§ 1). The simplest f_α refer to enumerative functions with respect to these compositions. Let $f(n)=1$ for all n in K_c, and let $f(n)=0$ if n is not in K_c. The generator of $f(n)$ is then $S_c(x) \equiv \Sigma x^{n_c}$, the sum referring to all n_c, where n_c is in K_c. In this case $f_1(n)$ is the total number of compositions of n over K_c, and the generator of f_1 is $S_c(x)[1 - S_c(x)]^{-1}$. For example, if $c = 1$, $f_1(n)$ is the total number of compositions of n into integer parts >0; $S_1(x) = x(1-x)^{-1}$, and the generator of f_1 is here $x(1-2x)^{-1}$, so that for this f_1, we have $f_1(n) = 2^{n-1}$.

Again, for the same K_c, $f^r(n)$ is the number of compositions of n into precisely r parts in K_c. To restrict the magnitude of the parts it is sufficient to define $f(n) = 1$, n in K_c and $n \leqslant m$; $f(n) = 0$ otherwise.

Two further illustrations will suffice. Let K_c be the class of all integers >0 that are congruent to c modulo b. Then

$$S_c(x) = \sum_{n=0}^{\infty} x^{c+nb} = x^c (1 - x^b)^{-1},$$

is the generator of $f(n), = 1$ or 0 according as $n \equiv c$ mod. b or $n \not\equiv c$ mod. b, and the generator of the corresponding f_α is

$$\frac{x^c}{1 - \alpha x^c - x^b}, = \sum_{n=1}^{\infty} f_\alpha(n) x^n.$$

From the expansion of the left the explicit form of $f_\alpha(n)$ can easily be obtained. From the above, for this f_α, we have the following difference equation*,

$$f_\alpha(n) - \alpha f_\alpha(n-c) - f_\alpha(n-b) = 0, \qquad n > \text{Min } (b, c).$$

When $\alpha = 1$, the function enumerates the total number of compositions into parts in arithmetical progression with first term c and common difference b.

Further examples of what are essentially the processes of this section, where the class concerned is that of all square or triangular numbers>0, were given in a previous paper. The results refer to elliptic and theta functions.*

Finally, there is an interesting connection between the functions

$$f, f^{-1}, g \equiv 1 - f, g^{-1}, f_1, f_1^{-1},$$

where f is any numerical function, and the compositions are over any class. For, if $A(\alpha)$ is the unharmonic invariant

$$A(\alpha) \equiv \frac{(\alpha^2 - \alpha + 1)^3}{\alpha(\alpha - 1)^2}$$

of the scalar α, then

$$A(f) = A(f^{-1}) = A(g) = A(g^{-1}) = A(-f_1) = A(-f_1^{-1}).$$

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* For $\alpha = 1$, $b = 1$, the result was stated by Cayley, *Coll. Papers*, Vol. 10, p. 16.

* E. T. Bell, *Annals of Mathematics*, (2), 23, 1921, 56-67.

6. Implicit functions

We give this name to functions $\psi(\phi(f))$ of the numerical function f where $\psi(\phi(x))$ is any function of the scalar x such that $\phi(f)$ exists. The properties of such function for simple choices of ϕ, ψ give rise to an endless variety of theorems on compositions, when combined with the definition of f^p in (7). The simplest theorems thus obtainable refer to enumerative functions, and are given by taking $f(n)=1$ for all n in the class K_c concerned, $f(n)=0$ if n is not in K_c.

We may assume, for example, that ϕ, ψ are inverses of one another, of the following kind,

$$\phi\Big(\psi(F(x))\Big) = \psi\Big(\phi(F(x))\Big) = F(x),$$

provided $F(x)$ exists. The choice $F(x)=1+x$, $\phi = \log$ $\psi = \exp.$ gives

$$g = \sum_{n=1}^{\infty} f^n/n!,$$

where $f(n)=(-1)^{n-1}/n$, $g(1)=1$, $g(m)=0$, $m\neq 1$. By § 4 the sum is convergent. The like for the circular functions and their inverses give composition theorems for the Bernoulli and Euler numbers.

Of the same type is the following, obtained from $\exp.[x(1-x)^{-1}]$ to which, after expansion, is applied the transformation x into $x(1+x)^{-1}$, giving $\exp. x$. Let $f(n)=1$ for all integers $n>0$, so that $f^s(n)$ is the total number of compositions of n into precisely s parts>0. Then

$$\sum_{s=1}^{n}(-1)^s\left[1+\sum_{r=1}^{s}\frac{f^r(s)}{r!}\right]f^s(n) = \frac{(-1)^n}{n!}, \qquad n>1.$$

PASADENA, 16th June, 1932 } CALIFORNIA INSTITUTE OF TECHNOLOGY.

Abundant Numbers

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Let an integer x be called an *abundant integer* if $\sum_{d/x} d \geqslant 2x$, and a *deficient integer* if $\sum_{d/x} d < 2x$.

Let $A(n)$ denote the number of abundant integers $\leqslant n$; and $D(n)$ the number of deficient integers $\leqslant n$. Felix Behrend has proved in his paper "Uber Numeri Abundantes" (*Berliner Sitzungberichte* 1932 XXII) that

$$\overline{\lim_{n\to\infty}} \frac{A(n)}{n} < \cdot 461; \qquad \frac{A(n)}{n} < \cdot 47 \text{ for every } n.$$

I propose to prove in this paper that

$$\overline{\lim_{n\to\infty}} \frac{A(n)}{n} < \cdot 4454; \quad \underline{\lim_{n\to\infty}} \frac{A(n)}{n} > \cdot 2947$$

Let

$A_2(n)$ = number of abundant integers $\leqslant n$ divisible by 2 but not by 3;
$A_3(n)$ = number of abundant integers $\leqslant n$ divisible by 3 but not by 2;
$A_6(n)$ = number of abundant integers $\leqslant n$ divisible by 6;
$A_5(n)$ = number of abundant integers $\leqslant n$ divisible by 5 but prime to 6
$A_1(n)$ = number of abundant integers $\leqslant n$ prime to 30.

We have

$$A(n) = A_1(n) + A_2(n) + A_3(n) + A_5(n) + A_6(n) \tag{1}$$

Let $\sigma(\nu) = \sum_{d/\nu} d$

$\chi_k(\nu)$ the main character modulo k so that $\chi_k(\nu) = 1$ for $(\nu, k) = 1$, and $\chi_k(\nu) = 0$ for $(\nu, k) > 1$

We have also

$$\chi_k(\nu\mu) = \chi_k(\nu)\chi_k(\mu); \qquad \chi_{kl}(\nu) = \chi_k(\nu)\chi_l(\nu)$$

$$\sum_{\nu=1}^{m} \chi_k(\nu) = \frac{m}{k}\phi(k) + K(\phi(k)) \tag{2}$$

where $\mid K(a) \mid < a.$

Consider $\sum_{v \leqslant n} \chi_{30}(\nu) \frac{\sigma(\nu)}{\nu} = \sum_{v \leqslant n} \chi_{30}(\nu) \sum_{d \mid v} \frac{1}{d}$

$$= \sum_{d=1}^{n} \frac{1}{d} \{\chi_{30}(d) + \chi_{30}(2d) + \cdots\cdots + \chi_{30}([n/d]d)\}$$

$$= \sum_{d=1}^{n} \frac{\chi_{30}(d)}{d} \{\chi_{30}(1) + \chi_{30}(2) + \cdots\cdots + \chi_{30}([n/d])\}$$

$$= \sum_{d=1}^{n} \frac{\chi_{30}(d)}{d} \phi(30) \left\{ \frac{n}{30d} + I_d \right\} \text{ where } I_d = K(1)$$

$$= \frac{4n}{15} \sum_{d=1}^{n} \frac{\chi_{30}(d)}{d^2} + K\left(8 \sum_{d=1}^{n} \frac{\chi_{30}(d)}{d}\right)$$

$$= \frac{4n}{15} \left\{ \sum_{d=1}^{\infty} \frac{\chi_{30}(d)}{d^2} - \sum_{n+1}^{\infty} \frac{\chi_{30}(d)}{d^2} \right\} + K\left(8 \sum_{d=1}^{n} \frac{\chi_{30}(d)}{d}\right)$$

Now $\sum_{d=1}^{\infty} \frac{\chi_k(d)}{d^2} = \frac{\pi^2}{6} \prod_{p/k} \left(1 - \frac{1}{p^2}\right)$

$$\therefore \frac{1}{n} \sum_{v \leqslant n} \chi_{30}(\nu) \frac{\sigma(\nu)}{\nu} = \frac{4}{15} \frac{8\pi^2}{75} + O\left(\frac{1}{n}\right) + O\left(\frac{\log n}{n}\right)$$

$$= \frac{32\pi^2}{1125} + O\left(\frac{\log n}{n}\right) \qquad (3)$$

$$\sum_{5v \leqslant n} \chi_6(\nu) \frac{\sigma(5\nu)}{5\nu} = \sum_{5v \leqslant n} \chi_6(\nu) \sum_{d/5v} \frac{1}{d}$$

$$= \sum_{5v \leqslant n} \chi_6(\nu) \left\{ \sum_{d/v} \frac{\chi_5(d)}{d} + \sum_{d/v} \frac{1}{5d} \right\}$$

$$= \sum_{d \leqslant n/5} \left\{ \frac{\chi_5(d)}{d} + \frac{1}{5d} \right\} \left\{ \chi_6(d) + \chi_6(2d) + \cdots + \chi_6\left(\left[\frac{n}{5d}\right]\right)d \right\}$$

$$= \sum_{d \leqslant n/5} \left\{ \frac{\chi_{30}(d)}{d} + \frac{\chi_6(d)}{5d} \right\} \phi(6) \left\{ \frac{n}{30d} + I'_d \right\} \text{ where } I'_d = K\ (1)$$

$$= \frac{n}{15} \sum_{d \leqslant n/5} \frac{\chi_{30}(d)}{d^2} + \frac{n}{75} \sum_{d \leqslant n/5} \frac{\chi_6(d)}{d^2} + O\ (\log n)$$

$$= \frac{29n\pi^2}{3375} + O\ (\log n) \qquad (4)$$

We have also*

$$\frac{1}{n} \sum_{2v \leqslant n} \chi_3(\nu) \frac{\sigma(2\nu)}{2\nu} = \frac{5\pi^2}{81} + O\left(\frac{\log n}{n}\right); \qquad (5)$$

$$\frac{1}{n} \sum_{3v \leqslant n} \chi_2(\nu) \frac{\sigma(3\nu)}{3\nu} = \frac{11\pi^2}{432} + O\left(\frac{\log n}{n}\right). \qquad (6)$$

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* *Loc cit,* Page 324.

Further

$$\sum_{2v\leqslant n} \chi_3(\nu)\frac{\sigma(2\nu)}{2\nu} = \sum_{\substack{2v\leqslant n \\ v \text{ even}}} \chi_3(\nu)\frac{\sigma(2\nu)}{2\nu} + \sum_{\substack{2v\leqslant n \\ v \text{ odd}}} \chi_3(\nu)\frac{\sigma(2\nu)}{2\nu} = \sum{}_1 + \sum{}_2$$

$$\sum{}_1 = \sum_{\substack{2v\leqslant n \\ v \text{ even}}} \chi_3(\nu)\frac{\sigma(2\nu)}{2\nu} = \sum_{4v\leqslant n} \chi_3(2\nu)\frac{\sigma(4\nu)}{4\nu}$$

$$= \sum_{4v\leqslant n} \chi_3(\nu)\frac{\sigma(4\nu)}{4\nu} = \sum_{4v\leqslant n} \chi_3(\nu) \sum_{d/4v} \frac{1}{d}$$

$$= \sum_{4v\leqslant n} \chi_3(\nu) \sum_{d/2v} \left\{ \frac{\chi_2(d)}{d} + \frac{1}{2d} \right\}$$

$$= \sum_{4v\leqslant n} \chi_3(\nu) \left\{ \sum_{d/2v} \frac{\chi_2(d)}{d} + \frac{1}{2}\left(\sum_{d/2v} \frac{\chi_2(d)}{d} + \sum_{d/2v} \frac{1}{2d} \right) \right\}$$

$$= \sum_{4v\leqslant n} \chi_3(\nu) \left\{ \sum_{d/v} \frac{\chi_2(d)}{d} + \frac{1}{2} \sum_{d/v} \frac{\chi_2(d)}{d} + \frac{1}{4} \sum_{d/v} \frac{1}{d} \right\}$$

$$= \sum_{4v\leqslant n} \chi_3(v) \left\{ \frac{3}{2} \sum_{d/v} \frac{\chi_2(d)}{d} + \frac{1}{4} \sum_{d/v} \frac{1}{d} \right\}$$

$$= \sum_{d\leqslant n/4} \left\{ \frac{3}{2}\frac{\chi_2(d)}{d} + \frac{1}{4d} \right\} \left\{ \chi_3(d) + \chi_3(2d) + \ldots \ldots + \chi_3\left(\left[\frac{n}{4d}\right]d\right) \right\}$$

$$= \sum_{d\leqslant n/4} \left\{ \frac{3}{2}\frac{\chi_6(d)}{d} + \frac{\chi_3(d)}{4d} \right\} \left\{ \chi_3(1) + \chi_3(2) + \ldots \ldots + \chi_3\left(\left[\frac{n}{4d}\right]\right) \right\}$$

$$= \sum_{d\leqslant n/4} \left\{ \frac{3}{2}\frac{\chi_6(d)}{d} + \frac{\chi_3(d)}{4d} \right\} \phi(3) \left\{ \frac{n}{12d} + I''_d \right\} \qquad I''_d = K(1)$$

$$= \sum_{d\leqslant n/4} \left\{ \frac{3}{2}\frac{\chi_6(d)}{d} + \frac{\chi_3(d)}{4d} \right\} \left\{ \frac{n}{6d} + 2\, I''_d \right\}$$

$$= \sum_{d\leqslant n/4} \frac{3}{2}\frac{\chi_6(d)}{d^2} \cdot \frac{n}{6} + \sum_{d\leqslant n/4} \frac{\chi_3(d)}{d^2}\frac{n}{24} + O(\log n)$$

$$= \frac{11}{324} n\pi^2 + O(\log n) \tag{7}$$

From (5) and (7)

$$\sum{}_2 = \sum_{2v\leqslant n} \chi_3(\nu)\frac{\sigma(2)}{2\nu}$$

$$= \sum_{4v-2\leqslant n} \chi_3(2\nu-1)\frac{\sigma(4\nu-2)}{4\nu-2} = n\left(\frac{5\pi^2}{81} - \frac{11\pi^2}{324}\right) + O(\log n)$$

$$= \frac{n\pi^2}{36} + O(\log n) \tag{8}$$

We have $$A_5 + D_5 = \frac{n}{15} + O(1)$$

$$\therefore \quad \sum_{5\nu \leqslant n} \chi_6(\nu)\frac{\sigma(5\nu)}{5\nu} \geqslant \frac{2A_5 + \frac{6}{5}D_5}{n}$$

$$\geqslant \frac{2A_5}{n} + \frac{6}{5}\left(\frac{1}{15} - \frac{A_5}{n} + O\left(\frac{1}{n}\right)\right)$$

$$= \frac{4}{5}\frac{A_5}{n} + \frac{2}{25} + O\left(\frac{1}{n}\right)$$

Hence by (4)

$$\frac{4}{5}\frac{A_5}{n} + \frac{2}{25} + O\left(\frac{1}{n}\right) \leqslant \frac{29\pi^2}{3375} + O\left(\frac{\log n}{n}\right)$$

$$\therefore \quad \frac{A_5}{n} \leqslant \frac{29\pi^2}{2700} - \frac{1}{10} + O\left(\frac{\log n}{n}\right) \tag{9}$$

From (3)

$$\frac{32\pi^2}{1125} + O\left(\frac{\log n}{n}\right) = \frac{1}{n}\sum_{\nu \leqslant n} \chi_{30}(\nu)\frac{\sigma(\nu)}{\nu}$$

$$\geqslant \frac{2A_1 + D_1}{n}$$

Also $$\sum_{\nu=1}^{n} \chi_{30}(\nu) = A_1 + D_1$$

$$= \frac{n}{30}\phi(30) + O(1) = \frac{4n}{15} + O(1)$$

$$\therefore \quad \frac{32\pi^2}{1125} + O\left(\frac{\log n}{n}\right) \geqslant \frac{2A_1}{n} + \frac{4}{15} - \frac{A_1}{n} + O\left(\frac{1}{n}\right)$$

$$= \frac{A_1}{n} + \frac{4}{15} + O\left(\frac{1}{n}\right)$$

$$\frac{A_1}{n} \leqslant \left(\frac{32\pi^2}{1125} - \frac{4}{15}\right) + O\left(\frac{\log n}{n}\right) \tag{10}$$

Let $A_4(n)$ = number of abundant integers divisible by 4 but not by 3, and $\leqslant n$; $A'_4(n)$ = number of abundant integers divisible by 2 but not by 4 or 3, and $\leqslant n$; $D_4(n)$ and $D'_4(n)$ deficient integers similarly defined.

Then $$A_4(n) + D_4(n) = \frac{n}{6} + O(1)$$

$$A'_4(n) + D'_4(n) = \frac{n}{6} + O(1)$$

Also $$A_4(n) + A'_4(n) = A_2(n); \qquad D_4(n) + D'_4(n) = D_2(n)$$

Then since $D_4(n) > 0$,

$$A_4(n) < \frac{n}{6} + O(1) \tag{11}$$

Also

$$\frac{\pi^2}{36} + O\left(\frac{\log n}{n}\right) = \frac{1}{n} \sum_{4\nu - 2 \leqslant n} \chi_3(2\nu - 1) \frac{\sigma(4\nu - 2)}{4\nu - 2}$$

$$\geqslant \frac{2A'_4 + \frac{3}{2}D'_4}{n}$$

$$\text{since } \frac{\sigma(4\nu - 2)}{4\nu - 2} > \frac{2\nu - 1 + 4\nu - 2}{4\nu - 2} = \frac{3}{2}$$

$$= \frac{2A'_4}{n} + \frac{3}{2}\left\{\frac{1}{6} - \frac{A'_4}{n} + O\left(\frac{1}{n}\right)\right\}$$

$$= \frac{1}{2}\frac{A'_4}{n} + \frac{1}{4} + O\left(\frac{1}{n}\right)$$

$$\therefore \quad \frac{A'_4}{n} < \left(\frac{\pi^2}{36} - \frac{1}{4}\right) 2 + O\left(\frac{\log n}{n}\right)$$

$$\therefore \quad \frac{A_2}{n} = \frac{A_4 + A'_4}{n} < \frac{\pi^2}{18} - \frac{1}{3} + O\left(\frac{\log n}{n}\right) \tag{12}$$

Now

$$\frac{A_3(n) + D_3(n)}{n} = \frac{1}{6} + O\left(\frac{1}{n}\right)$$

$$\therefore \quad \frac{11\pi^2}{432} + O\left(\frac{\log n}{n}\right) = \frac{1}{n} \sum_{3\nu \leqslant n} \chi_2(\nu) \frac{\sigma(3\nu)}{3\nu}$$

$$\geqslant \frac{2A_3(n) + \frac{4}{3} D_3(n)}{n}$$

$$= \frac{2A_3(n)}{n} + \frac{4}{3}\left\{\frac{1}{6} - \frac{A_3(n)}{n} + O\left(\frac{1}{n}\right)\right\}$$

$$= \frac{2}{3}\frac{A_3(n)}{n} + \frac{2}{9} + O\left(\frac{1}{n}\right)$$

$$\therefore \quad \frac{A_3(n)}{n} < \frac{3}{2}\left\{\frac{11\pi^2}{432} - \frac{2}{9}\right\} + O\left(\frac{\log n}{n}\right) \tag{13}$$

Also $\frac{A_6(n)}{n} \leqslant \frac{1}{6}$

$$\therefore \quad \frac{A(n)}{n} = \frac{A_1(n) + A_2(n) + A_3(n) + A_5(n) + A_6(n)}{n}$$

$$< \pi^2\left\{\frac{33}{864} + \frac{1}{18} + \frac{32}{1125} + \frac{29}{2700}\right\} - \frac{1}{3} + \frac{1}{6} - \frac{1}{3} - \frac{4}{15} - \frac{1}{10} + O\left(\frac{\log n}{n}\right)$$

$$= \cdot 4453 + O\left(\frac{\log n}{n}\right)$$

$$\therefore \quad \overline{\lim_{n\to\infty}} \frac{A(n)}{n} < \cdot 4454$$

and for every $n > n_0$, $\frac{A(n)}{n} < \cdot 4455$

We now prove

$$\lim_{n\to\infty} \frac{A(n)}{n} > \cdot 2947$$

Since the following numbers are primitive abundant integers: (an abundant integer is primitive if it is not a multiple of a smaller abundant number*), 6, 20, 28, 70, 88, 104, 272, 304, 368, 464, 496, 550, 572, 650, 748, 836, 945, ... 8085

$$\therefore \quad A(n) > \left[\frac{n}{6}\right] + \left[\frac{n}{20}\right] + \quad \dots \quad \dots \left[\frac{n}{8085}\right]$$

$$- \left[\frac{n}{6\cdot 20}\right] - \left[\frac{n}{6\cdot 28}\right] \quad \dots \quad \dots$$

$$+ \left[\frac{n}{6\cdot 20\cdot 28}\right] + - + \quad \dots \quad \dots - \left[\frac{n}{6\cdot 20\cdot 28\cdot 70}\right] \quad \dots \quad \dots$$

$$\therefore \quad \frac{A(n)}{n} > \left\{1 - \left(1 - \frac{1}{6}\right)\left(1 - \frac{1}{20}\right) \quad \dots \quad \left(1 - \frac{1}{8085}\right)\right\} + O\left(\frac{1}{n}\right)$$

$$= \cdot 29472 + O\left(\frac{1}{n}\right)$$

$$\therefore \quad \lim_{n\to\infty} \frac{A(n)}{n} > \cdot 29472$$ and for all sufficiently large n.

$$\frac{A_n}{n} > \cdot 2947.$$

Added February 1934:—Since this paper was sent for publication, Felix Behrend has proved (Über Numeri Abundantes II) that for all sufficiently large n

$$\cdot 241 < \frac{A(n)}{n} < \cdot 314$$

Combining this result with our result we have that for all sufficiently large n

$$\cdot 2947 < \frac{A(n)}{n} < \cdot 314$$

* Dickson (1) Odd perfect and primitive abundant numbers. (2) Even Abundant Numbers. *American Journal of Mathematics* Vol. XXXV (1913) *413-426*.

[Received 1-5-33]

Groups generated by an operator of order 2 and an operator of order 3 whose commutator is of order 2

BY

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Let s and t be of orders 2 and 3 respectively and suppose that st^2st is of order 3. Hence it results that $stst^2$ is also of order 3 and that the following equations represent identities

$$st^2stst^2 = t^2stst^2s\,; \qquad stst^2st = tst^2sts.$$

The two operators $stst = C_1$ and $tsts = C_2$ are commutative since $tsts.\ stst = t\ st^2\ st = stst\ tsts.$

The two operators C_1 and C_2 are transformed into each other by s while t transforms C_2 into C_1 and C_1 into $C_1^{-1}\ C_2^{-1}$. Hence t^2 transforms C_1 into C_2 and C_2 into $C_1^{-1}\ C_2^{-1}$.

To construct an infinite system of groups which satisfy these conditions, we may consider the direct product of three cyclic groups whose common order is the order of C_1 and suppose that the generating cycles of these three groups are represented on three distinct sets of letters. If for C_2 we take the product of the first of these cycles into the inverse of the second, and for C_1 the second of these cycles into the inverse of the third, then $C_1\ C_2$ is the product of the first of these cycles into the inverse of the third. The subgroup H generated by C_1 and C_2 involves three and only three subgroups whose common degree is twice the order of C_1 and its order is the square of the order of C_1. This subgroup is transformed into itself by permutations which transform the three given cycles according to the symmetric group of degree 3.

For s we may take the permutation which transforms the second cycle into its inverse and the first into the inverse of the third, while for t we may take the permutation which transforms the three cycles in order cyclically. From the permutations which generate H it results directly that a necessary and sufficient condition that t transforms into itself an operator of H besides the identity is that the order of H is divisible by 3, and when this condition is satisfied exactly three operators of H are transformed into themselves by t.

As these three operators are transformed into their inverses by s, it results that the central of all of these groups is the identity. The quotient group G/H is the non-cyclic group of order 6, and the subgroup generated by H and t involves only operators of order 3, besides those contained in H. When the order of H is prime to 3 all the operators of order 3 appear in a single set of conjugates under the group G generated by s and t. When H is the four-group, G is the octahedral group.

For each of the Sylow subgroups of order p^m in this infinite system of groups, m is even except when $p=2$ or 3. In these two cases m is odd. It is easy to see that when $p=2$, m must always be odd since $C_1 C_2$ must have the same order as C_1. On the other hand, there is an infinite system of groups coming under the heading of the present article in which m is even whenever $p=3$. The smallest group which belongs to this system is the direct product of the symmetric group of order 6 and the group of order 3. For s we may take any of its three operators of order 2 in this direct product, and for t any one of its four operators of order 3 which do not generate an invariant subgroup of this order. To construct the infinite system in question we may proceed as follows:

Let H be any abelian group all of whose Sylow subgroups are the direct products of two Sylow subgroups of the same order, except that the Sylow subgroup whose order is a power of 3 has two invariants such that one is three times the other. The independent generators C_1, C_2 of H may be selected as in the preceding case except that their Sylow subgroups when $p=3$ generate the same subgroup of order 3. We may therefore suppose that t transforms an independent generator of order 3^α of one of these subgroups into itself multiplied by a second independent generator of order $3^{\alpha-1}$ in H, while it transforms the second of these independent generators into itself multiplied by the product of the -3rd powers of both of these generators. The operator s may be supposed to transform the second of these generators into its inverse and the first into itself multiplied by the second. Hence s and t may still be supposed to satisfy the conditions imposed in the first paragraph. Each of the groups of the present infinite category of groups has a central of order 3, while in each of those of the preceding category the central is the identity.

To prove that the two given infinite categories of groups are composed of all the groups which come under the heading of the present article, it is only necessary to note that t could not transform into itself a cyclic subgroup of the group generated by C_1 and C_2 if the order of this subgroup exceeds 3. It is at once evident that this must be the case when t is commutative with a generator of such a subgroup since $C_1^{-1}C_2^{-1}$ would then involve the -2nd power of this generator and this would be equal to it since t transforms C_1 into $C_1^{-1}\ C_2^{-1}$. It could also not transform such a generator into a power of itself which is incongruent to unity with respect to its order since s transforms into its inverse the co-set in which t appears with respect to H. Hence there results the following theorem:

THEOREM: *Every abelian group involving only Sylow subgroups which have two equal invariants, or only such Sylow subgroups except the one in which the larger invariant is three times the smaller, can be extended by two operators of order 3 and 2 respectively whose commutator is of order 3 so as to obtain a group which is generated by these two operators, and whose order is six times the order of this abelian group. Moreover, every group which can be generated by two such operators contains such an invariant abelian subgroup.*

It may be noted that the preceding theorem completes the determination of groups defined by the orders of two generators and the order of their commutator when any one of the following sets of conditions are satisfied: (1) The two given operators are of order 2 and their commutator is of any given arbitrary order, (2) one of the operators is of order 2 and the other is of order 3 while the order of the commutator is 2, (3) one of the operators is of order 2 and the other is of order 4 while the order of the commutator is 2, (4) one operator is of order 2 and the other is of order 3 while the order of the commutator is 3 (the present case). (5) both of the operators are of order 3 and their commutator is of order 2. The following two general theorems are perhaps the most useful among those developed in determining these groups.

If a group is generated by two operators its commutator subgroup is generated by the powers of these operators, and if a group is defined by the orders of two operators and the order of their commutator, it is isomorphic with the abelian group whose independent generators have orders which are equal to the orders of these two generating operators. (Cf. *Proceedings of the National Academy of Sciences*, volume 18, 1932, page 665, and volume 19, 1933, page 199.)

On the behaviour of elliptic theta functions near the line of singularities

BY

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1. Let $\tau = x + iy$ be a complex variable and let $q = e^{i\pi\tau}$. Then the functions [1]

$$\begin{aligned} \vartheta_2(0 \mid \tau) &= 2q^{1/4} + 2q^{9/4} + 2q^{25/4} + \dots\dots\dots \\ \vartheta_3(0 \mid \tau) &= 1 + 2q + 2q^4 + 2q^9 + \dots\dots\dots \\ \vartheta_4(0 \mid \tau) &= 1 - 2q + 2q^4 - 2q^9 + \dots\dots\dots \end{aligned}$$

are, as is well known, analytic for $y > 0$ and have the line $y = 0$ as a natural boundary. In a paper [2] published several years ago Hardy and Littlewood discussed the behaviour of these functions as $y \to +0$ on the straight line $x = \xi$, when ξ is an irrational. In particular they proved that if, when ξ is expressed as a simple continued fraction, the partial quotients form a bounded set, then positive constants K_1, K_2 exist such that

$$\frac{K_2}{\sqrt[4]{y}} < |\vartheta_3(0 \mid \xi + iy)| < \frac{K_1}{\sqrt[4]{y}} \tag{1}$$

for $y > 0$.

2. The object of the present paper is to show that the argument used by Hardy and Littlewood can be used to formulate conditions under which one or other of the inequalities

$$|\vartheta_3(0 \mid \xi + iy)| < \frac{K_1}{\sqrt[4]{y}},$$

$$|\vartheta_3(0 \mid \xi + iy)| > \frac{K_2}{\sqrt[4]{y}}$$

holds for $y > 0$. These conditions are found to be both necessary and sufficient and are given in Theorems A and B of this paper. By combining these two theorems we are enabled to show that the condition of boundedness of the partial quotients of the continued fraction for ξ, which was shown by Hardy and Littlewood to be sufficient for

(1) The notation is that of Tannery and Molk *Elements de la Theorie des Fonctions Elliptiques*, Vol. II. This book will be referred to shortly as T. M.

(2) G. H. Hardy and J. E. Littlewood, Some Problems of Diophantine Approximation (II), *Acta Mathematica*, Vol. 37 (1914) pp. 193-238 (pp. 226-230).

the truth of the inequalities (1), is also necessary. In the present paper I have closely followed the ideas of Hardy and Littlewood, and I have given the arguments in detail only for the sake of completeness; they are not substantially new and different from the arguments of Hardy and Littlewood.

3. Throughout the present paper ξ stands for a positive irrational whose expression as a simple continued fraction is

$$\xi = c_1 + \frac{1}{c_2 +}\ \frac{1}{c_3 +} \ldots + \frac{1}{c_n +} \ldots ;$$

p_n/q_n is the nth convergent of the continued fraction, and f_{n+1} is the complete quotient corresponding to c_{n+1}, that is

$$f_{n+1} = c_{n+1} + \frac{1}{c_{n+2} +} \ldots .$$

It is easy to verify the identity [3]

$$(2) \qquad \frac{1}{|p_n q_n - \xi q_n^2|} = f_{n+1} + \frac{q_{n-1}}{q_n},$$

which we have to apply. We require a few lemmas to begin with.

Lemma 1. *Let* $\tau = x + iy$, $T = X + iY$, *and let* a, b, c, d *be integers such that* $ad - bc = 1$. *Then if* $y > 0$ *and*

$$(3) \qquad \tau = \frac{c + dT}{a + bT}$$

we have

$$\sqrt[4]{y}\,|\vartheta_3(0|\tau)| = \sqrt[4]{Y}\,|\vartheta_m(0|T)|,$$

where the suffix m *stands for* 2, 3 *or* 4 *according to the type of the transformation* (3).

It is known that [4]

$$(4) \qquad |\vartheta_3(0|\tau)| = |\sqrt{a+bT}|\,|\vartheta_m(0|T)|;$$

and it is easy to deduce from (3) and $ad - bc = 1$ that

$$y = \frac{Y}{|a+bT|^2},$$

$$(5) \qquad |\sqrt{a+bT}| = \sqrt[4]{Y/y}$$

The result of the lemma follows from (4) and (5).

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(3) See for example, my paper, On the boundary behaviour of elliptic modular functions, *Acta Mathematica*, Vol. 52 (1928) pp. 143-168 (p. 149).

(4) T. M. p. 41.

Lemma 2. *Let*

$$\phi(y) = \frac{q_n y}{(p_n - \xi q_n)^2 + q_n^2 y^2};$$

then for

$$(6) \qquad \frac{1}{q^2_{n+1}} \leqslant y < \frac{1}{q^2_n}$$

we have [5]

$$(7) \qquad \frac{1}{2} q_n < \phi(y) < q_n \left(\frac{c_{n+1}}{2} + 1\right);$$

and there is a value y_n *of* y *in the interval* (6) *for which*

$$(8) \qquad \phi(y_n) > \tfrac{1}{2} q_n c_{n+1}.$$

It is known that

$$| p_n - \xi q_n | < \frac{1}{q_{n+1}},$$

and so

$$(9) \qquad | p_n - \xi q_n |^2 < \frac{1}{q^2_{n+1}} \leqslant y$$

using the first of the inequalities (6). Also

$$(10) \qquad q_n^2 y^2 < y,$$

since $q_n^2 y < 1$ from the second of the inequalities (6).

By (9) and (10)

$$(p_n - \xi q_n)^2 + q_n^2 y^2 < 2y,$$

and so $$\phi(y) > \frac{q_n y}{2y} = \tfrac{1}{2} q_n,$$
which is the first of the inequalities (7) to be proved.

To prove the second of the inequalities (7) consider

$$F(\lambda) = \frac{\lambda}{\alpha^2 + \lambda^2},$$

where α is a constant greater than zero. By examining the sign of $F'(\lambda)$ it is easy to see that $F(\lambda)$ increases as λ increases from 0 to α, and decreases as λ increases from α onwards. Therefore for $\lambda \geqslant 0$

$$F(\lambda) \leqslant F(\alpha) = \frac{1}{2\alpha}.$$

(5) The inequality $\phi(y) > \frac{1}{2} q_n$ was proved in my paper referred to above (pp. 150-153) by a combination of algebraic and geometric arguments. The proof given here is simpler in detail.

Applying this result with $\lambda = q_n y$, $\alpha = |p_n - \xi q_n|$, and using the identity (2), we see that

$$\phi(y) \leqslant \frac{1}{2|p_n - \xi q_n|}$$

$$= \frac{q_n}{2|p_n q_n - \xi q_n^2|}$$

$$= \tfrac{1}{2} q_n \left(f_{n+1} + \frac{q_{n-1}}{q_n} \right)$$

$$< \tfrac{1}{2} q_n (c_{n+1} + 1 + 1)$$

$$= q_n \left(\frac{c_{n+1}}{2} + 1 \right),$$

which proves the second of the inequalities (7).

To prove (8) we observe that if $c_{n+1} = 1$, (8) follows from what has been already proved. For we have seen that for *all* values of y in the interval (6),

$$\phi(y) > \tfrac{1}{2} q_n = \tfrac{1}{2} q_n c_{n+1}.$$

We next consider the case when $c_{n+1} \geqslant 2$. We shall first show that

$$\text{(11)} \qquad \frac{1}{q_{n+1}^2} < \frac{|p_n - \xi q_n|}{q_n} < \frac{1}{q_n^2}.$$

The latter half is equivalent to a known result in the theory of continued fractions. To prove the first half we observe that

$$\frac{q_{n+1}}{q_n} > c_{n+1},$$

and so

$$\frac{q_{n+1}^2}{q_n^2} > c_{n+1}^2 \geqslant c_{n+1} + 2$$

since $c_{n+1} \geqslant 2$. Therefore

$$\frac{1}{|p_n q_n - \xi q_n^2|} = f_{n+1} + \frac{q_{n-1}}{q_n}$$

$$< (c_{n+1} + 1) + 1$$

$$= c_{n+1} + 2$$

$$< \frac{q_{n+1}^2}{q_n^2},$$

and so

$$\frac{1}{q_{n+1}^2} < \frac{|p_n q_n - \xi q_n^2|}{q_n^2} = \frac{|p_n - \xi q_n|}{q_n},$$

which proves (11). If we take $y_n = \dfrac{|p_n - \xi q_n|}{q_n}$, we have

$$\frac{1}{q_{n+1}^2} < y_n < \frac{1}{q_n^2}$$

and

$$\begin{aligned}\phi(y_n) &= \frac{1}{2|p_n - \xi q_n|} \\ &= \frac{q_n}{2} \cdot \frac{1}{|p_n q_n - \xi q_n^2|} \\ &= \frac{q_n}{2}\left(f_{n+1} + \frac{q_{n-1}}{q_n}\right) \\ &> \tfrac{1}{2} q_n c_{n+1}.\end{aligned}$$

This completes the proof of the Lemma.

Lemma 3. [6] *If* $T = X + iY$ *and* $Y \geqslant \frac{1}{2}$, *then*

$$\tfrac{1}{2} < |\vartheta_3(0|T)| < \tfrac{3}{2},$$

$$\tfrac{1}{2} < |\vartheta_4(0|T)| < \tfrac{3}{2},$$

$$\frac{3}{2} e^{-\pi Y/4} < |\vartheta_2(0|T)| < \frac{5}{2} e^{-\pi Y/4}.$$

4. To facilitate the enunciation of Theorems A and B below, it will be convenient to divide the positive integers 1, 2, 3, ... into two classes according to the following definitions: The integer n is said to belong to the first class if one of p_n, q_n is even [7]. The integer n is said to belong to the second class if both p_n, q_n are odd.

We can now prove the following theorems.

Theorem A. *Let* $\tau = \xi + iy$. *In order that a constant* $K_1 > 0$ *may exist such that*

$$|\vartheta_3(0|\tau)| < \frac{K_1}{\sqrt[4]{y}}$$

for all $y > 0$, *it is necessary and sufficient that the partial quotients* c_{n+1} *for values of* n *belonging to the first class should form a bounded set.*

(6) For proof, see Hardy and Littlewood, *loc. cit.* pp. 227, 228, 230.

(7) The other is necessarily odd.

Let $n = n(y)$ be the integer function of y defined by

$$\frac{1}{q_{n+1}^{\;2}} \leqq y < \frac{1}{q_n^{\;2}},$$

so that as $y \to 0$, $n \to \infty$. If η_n stands for $(-1)^n$, the linear transformation

$$(12) \qquad \tau = \frac{p_{n-1} + \eta_n p_n \mathrm{T}}{q_{n-1} + \eta_n q_n \mathrm{T}} \qquad (\mathrm{T} = \mathrm{X} + i\mathrm{Y})$$

belongs to the modular group since

$$\eta_n p_n q_{n-1} - \eta_n p_{n-1} q_n = \eta_n^{\;2} = 1.$$

Therefore by Lemma 1 we have

$$(13) \qquad \sqrt[4]{y}\,|\vartheta_3(0\,|\,\tau)| = \sqrt[4]{\mathrm{Y}}\,|\vartheta_m(0\,|\,\mathrm{T})|,$$

where the suffix m is 2 if p_n, q_n are both odd, and m is 3 or 4 if one of p_n, q_n is odd and the other even [8].

By writing $\tau = \xi + iy$, $\mathrm{T} = \mathrm{X} + i\mathrm{Y}$ in (12) it is easy to verify that

$$(14) \qquad \mathrm{Y} = \frac{\phi(y)}{q_n},$$

where $\phi(y)$ is the function which occurs in Lemma 2; so that by (7), $\mathrm{Y} > \frac{1}{2}$.

We divide the values of y into two classes: (I) those for which $n(y)$ belongs to the first class and (II) those for which $n(y)$ belongs to the second class.

Let y take values belonging to class II. Then (13) is true with $m = 2$; and since $\mathrm{Y} > \frac{1}{2}$ we have by Lemma 3

$$|\vartheta_2(0\,|\,\mathrm{T})| < \frac{5}{2} e^{-\pi \mathrm{Y}/4},$$

$$(15)^{(9)} \qquad \sqrt[4]{\mathrm{Y}}\,|\vartheta_2(0\,|\,\mathrm{T})| < \frac{5}{2} \sqrt[4]{\mathrm{Y}}\, e^{-\pi \mathrm{Y}/4} < \frac{5}{2}.$$

Combining (13) and (15) we get the result that for values of y belonging to class II

$$(16) \qquad \sqrt[4]{y}\,|\vartheta_3(0\,|\,\tau)| < 5/2.$$

Next let y take values belonging to class I. Then the suffix m in (13) is 3 or 4; and in either case by Lemma 3 (since $\mathrm{Y} > \frac{1}{2}$)

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(8) T. M. pp. 41, 241.

(9) As Y increases from $\frac{1}{2}$ onwards, $\pi\mathrm{Y} - \log \mathrm{Y}$ increases, and so for $\mathrm{Y} > \frac{1}{2}$,

$$\sqrt[4]{\mathrm{Y}} \cdot e^{-\pi\mathrm{Y}/4} = e^{-\frac{1}{4}(\pi\mathrm{Y} - \log \mathrm{Y})} < e^{-\frac{1}{4}\left(\frac{\pi}{2} + \log 2\right)} < 1.$$

(17) $\frac{1}{2} < |\vartheta_m(0|\mathrm{T})| < \frac{3}{2}$;

also by Lemma 2 and the relation (14)

(18) $\sqrt[4]{\mathrm{Y}} < \sqrt[4]{\frac{c_{n+1}}{2}+1}$;

and so from (13), (17), (18) we have

(19) $\sqrt[4]{y}\,|\vartheta_3(0|\tau)| < \frac{3}{2}\sqrt[4]{\frac{c_{n+1}}{2}+1}.$

On the other hand by Lemma 2 there is a value y_n of y in the interval (6) for which

$$\frac{\phi(y_n)}{q_n} > \tfrac{1}{2}\,c_{n+1},$$

and for such a value of y, the corresponding Y will satisfy

(20) $\sqrt[4]{\mathrm{Y}} > \sqrt[4]{c_{n+1}/2}$

and using (13), (17), (20) we will have

(21) $\sqrt[4]{y}\,|\vartheta_3(0|\tau)| > \frac{1}{2}\sqrt[4]{c_{n+1}/2}$

The sufficiency of the condition enunciated in Theorem A follows from (16) and (19), and the necessity of the condition follows from (21).

Theorem B. *Let $\tau = \xi + iy$. In order that a constant $\mathrm{K}_2 > 0$ may exist such that*

$$|\vartheta_3(0|\tau)| > \frac{\mathrm{K}_2}{\sqrt[4]{y}}$$

for all $y > 0$, it is necessary and sufficient that the partial quotients c_{n+1} for values of n belonging to the second class should form a bounded set.

We employ the same transformation (12) as above and use the identity (13). The values of y are divided into two classes I and II as in the last theorem.

First let y take values belonging to class I, so that (13) is true with $m=3$ or 4. In either case we have by Lemma 3 ($\mathrm{Y} > \frac{1}{2}$ as observed previously)

$$|\vartheta_m(0|\mathrm{T})| > \tfrac{1}{2},$$
$$\sqrt[4]{\mathrm{Y}}\,|\vartheta_m(0|\mathrm{T})| > \tfrac{1}{2}\cdot\sqrt[4]{\tfrac{1}{2}},$$

and so by (13) we see that for values of y belonging to class I

(22) $\sqrt[4]{y}\,|\vartheta_3(0|\tau)| > \frac{1}{2}\cdot\sqrt[4]{\frac{1}{2}}.$

Next let y take values belonging to class II, so that

$$(23) \qquad \sqrt[4]{y}\,|\vartheta_3(0\,|\,\tau)| = \sqrt[4]{Y}\,|\vartheta_2(0\,|\,T)|$$

By Lemma 3

$$(24) \qquad |\vartheta_2(0\,|\,T)| > \frac{3}{2} e^{-\pi Y/4}.$$

and by Lemma 2

$$(25) \qquad Y = \frac{\phi(y)}{q_n} < \frac{c_{n+1}}{2} + 1,$$

and therefore by (23), (24), and (25)

$$(26) \qquad \sqrt[4]{y}\,|\vartheta_3(0\,|\,\tau)| > \sqrt[4]{\tfrac{1}{2}} \cdot \frac{3}{2} \cdot e^{-\frac{\pi}{4}(\frac{1}{2}c_{n+1}+1)}.$$

On the other hand by Lemma 2 there is a value y_n of y in the interval (6) for which

$$Y = \frac{\phi(y_n)}{q_n} > \frac{c_{n+1}}{2},$$

and for such a value of y we have on using Lemmas 2, 3

$$|\vartheta_2(0\,|\,T)| < \frac{5}{2} e^{-\pi Y/4} < \frac{5}{2} e^{-\frac{\pi}{8}c_{n+1}},$$

$$\sqrt[4]{Y} < \sqrt[4]{\frac{c_{n+1}}{2} + 1},$$

and therefore using (23)

$$(27) \qquad \sqrt[4]{y}\,|\vartheta_3(0\,|\,\tau)| < \frac{5}{2} \cdot \sqrt[4]{\frac{c_{n+1}}{2} + 1} \cdot e^{-\frac{\pi}{8}c_{n+1}}.$$

The sufficiency of the condition enunciated in Theorem B follows from (22) and (26); and the necessity of the condition follows from (27).

Combining Theorems A and B we get the following Theorem which completes the result of Hardy and Littlewood:

Theorem C. *Let* $\tau = \xi + iy$. *In order that constants* $K_1 > 0$, $K_2 > 0$ *may exist such that*

$$\frac{K_2}{\sqrt[4]{y}} < |\vartheta_3(0\,|\,\tau)| < \frac{K_1}{\sqrt[4]{y}}$$

for all $y > 0$, *it is necessary and sufficient that all the partial quotients* c_n *should form a bounded set.*

5. A few examples may be of interest. Let $c_1 = 1$, $c_2 = 1$ and let all other partial quotients $c_3, c_4 \ldots$ be even. It is easy to verify that $p_1, p_3, p_5, \ldots$ are odd, $p_2, p_4, p_6, \ldots$ are even, and $q_1, q_2, q_3, q_4, \ldots$ are odd. If $c_1, c_3, c_5, \ldots$ is a bounded sequence and $c_2, c_4, c_6, \ldots$ is an unbounded sequence then the condition of Theorem A is satisfied but not that of Theorem B; so that $\sqrt[4]{y}\,|\vartheta_3(0\,|\,\tau)|$ is bounded and has the lower limit zero as $y \to 0$. If, however, $c_2, c_4, c_6, \ldots$ is a bounded sequence and $c_1, c_3, c_5, \ldots$ is an unbounded sequence, the condition of Theorem B is satisfied but not that of Theorem A; so that $\sqrt[4]{y}\,|\vartheta_3(0\,|\,\tau)|$ is unbounded and has a lower limit greater than zero. Generally if the continued fraction is residually periodic [10] to mod. 2, we can decide after a finite number of calculations whether Theorem A or Theorem B applies.

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(10) That is to say, if λ_n is defined to be 1 or 2 according as c_n is odd or even, and if the sequence $\lambda_1, \lambda_2, \ldots$ is periodic. Cf. my paper, Additional note on the boundary behaviour of elliptic modular functions, *Acta Mathematica*, Vol. 53, (1929), p. 78.

Singular Solutions of Ordinary Differential Equations of the Second Order

BY

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PART I

1. We shall start with the following result, which will be considered in detail in a paper (to be published later) on the singular solutions of simultaneous equations.

THEOREM: *Let $u(x, y, z) = a$; $v(x, y, z) = b$ be the general solutions of the differential equations $y' = \psi_1(x, y, z)$; $z' = \psi_2(x, y, z)$. Let $w(x, y, z) = 0$ represent a relation in virtue of which a first partial derivative of ψ_1 or ψ_2 (or $1/\psi_1$ or $1/\psi_2$) as well as a first partial derivative of u or v becomes infinite. The singular solutions (if any exist) of the given equations will be included in the equations*

$$\left.\begin{array}{l} w(x, y, z) = 0 \\ y' = \psi_1, \text{ or } z' = \psi_2 \end{array}\right\} \qquad \left(y' = \frac{dy}{dx}\,;\ z' = \frac{dz}{dx}\right)$$

Replacing now the equation

$$F(x, y, y', y'') = 0 \qquad \ldots \qquad (1)$$

by the simultaneous system

$$F(x, y, z, z') = 0\,; \quad z = y', \qquad \ldots \qquad (2)$$

we can now write down as corollaries of the above theorem, the following:

THEOREM: (1) *Let $w_1(x, y, y') = 0$ be a relation in virtue of which and of the equation $F = 0$, we have either $\partial F/\partial y'' = 0$, or a first partial derivative of $F(x, y, y', y'')$ with respect to any of x, y, y', or y'' infinite. Then the complete primitive* of $w_1(x, y, y') = 0$ might give singular solutions of $F = 0$.*

(2) *If $\phi_1(x, y, y') = a$ and $\phi_2(x, y, y') = b$** are the two first integrals of $F = 0$, and if $w_2(x, y, y') = 0$ be a function in virtue of*

* As regards the singular solution of $w_1(x, y, y') = 0$, *vide* §§ 9—10.

** If instead of $\phi = c$, we have $\psi(x, y, z, a) = 0$, then $w_2 = 0$ will be a function in virtue of which $\partial\psi/\partial a = 0$, or $\partial\psi/\partial x$ or $\partial\psi/\partial y$ or $\partial\psi/\partial z$ is infinite.

which any of the first partial derivatives of ϕ_1 or ϕ_2 becomes infinite, then the complete primitive of $w_2(x, y, y')=0$ might give singular solutions of $F=0$.

The conditions are of course necessary conditions only, but either of the processes (1) or (2) gives the singular solutions of $F=0$ exhaustively.

It may happen that w_1 or w_2 is independent of y', but provides a singular solution—the only singular solution of equation (1) (*vide* § 8)

Some Remarks

2. (1) It may happen that either of the equations $\phi_1=a$, $\phi_2=b$ considered as a differential equation of the first order admits of singular solutions for one or more values of the constants a or b. Such a singular solution may or may not satisfy (1). [*vide*: Johnson: *Differential Equations.* §§ 81 and 83]. Conversely, singular solutions of (1) are not necessarily singular solutions of some first integral.

(2) If we consider the congruence of curves $\phi_1(x, y, z)=a$; $\phi_2(x, y, z)=b$ where $\phi_1(x, y, y')=a$, $\phi_2(x, y, y')=b$ are a pair of independent first integrals of (1), the envelope of any singly infinite system of curves selected out of the congruence leads to singular solutions of $F=0$. Otherwise expressed, if $E(x, y, z)=0$ is the focal surface of the congruence (assuming that E involves z), $E(x, y, y')=0$ will necessarily give solutions of (1), and these solutions are usually singular.

[We know that the equation (1) *in general* does not admit of any singular solutions. This means that the corresponding congruence $\phi_1(x, y, z)=a$; $\phi_2(x, y, z)=b$ in the general case does not possess a focal surface.]

Examples:

3. In one or two other papers, a distinction has been made between singular solutions and "infinite solutions"* (or limiting forms of the general primitive). This distinction will assume greater prominence and importance in the case of differential equations of the second order.

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* *Vide* a paper in the *Half-Yearly Journal of the Mysore University.* Vol. V. No. 2. The nomenclature is from E. B. Wilson's *Advanced Calculus* § 101.

EXAMPLE 1. $y(1-\log y)y''+(1+\log y)y'^2=0.$

$\partial F/\partial y''=0$ leads to the solution $y=e$ which is however a particular case of the solution $y=$ const. The complete primitive is $\log y=(x+a)/(x+b)$ The solution $y=$ constant may be considered to be included in the complete primitive by making a and b infinite in such a way that $a/b=k$. $y=k$ provides an infinite solution. There is no singular solution.

EXAMPLE 2. $$\phi\left(x-\frac{y'(1+y'^2)}{y''},\ y+\frac{1+y'^2}{y''},\ \frac{(1+y'^2)^{\frac{3}{2}}}{y''}\right)=0$$

The complete primitive is given by the congruence of circles $(x-\alpha)^2+(y-\beta)^2=r^2$ where α, β, r are connected by the relation $\phi\ (\alpha, \beta, r)=0$. When ϕ is written as a polynomial in y'', the equation $1+y'^2=0$ satisfies the equation. The isotropic lines $y+ix=\mathrm{A}$ and $y-ix=\mathrm{B}$ therefore always furnish solutions of the given differential equation, but *these solutions will in no case be singular.* Singular solutions however exist which may be obtained by the rules in § 1.

To prove that $1+y'^2=0$ does not furnish singular solutions, we shall compare the equations

$$x^2+y^2-2\alpha x-2\beta y+(\alpha^2+\beta^2-r^2)=0$$

and $(y+ix-\mathrm{A})(y-ix-\mathrm{B})=0$. If these are identical, we must have

$$-2\alpha=i(\mathrm{A}-\mathrm{B}),\qquad 2\beta=\mathrm{A}+\mathrm{B},\qquad r^2=\mathrm{AB}-\alpha^2-\beta^2=0.$$

The equation $\phi(\alpha, \beta, 0)=0$ gives β in terms of α. Keeping α arbitrary, the equation $x^2+y^2-2\alpha x-2\beta y+\alpha^2+\beta^2=0$ gives the isotropic lines $y=\pm ix+$ constant.

PARTICULAR CASES OF EXAMPLE 2:

(I) $$a(a-2x)y''^2+2ay'y''(1+y'^2)+(1+y'^2)^3=0.\ *$$

Writing the equation in the form

$$a^2+\frac{(1+y'^2)^3}{y''^2}-2a\left(x-\frac{y'(1+y'^2)}{y''}\right)=0,$$

the complete primitive is $(x-\alpha)^2+(y-\beta)^2=r^2$, where $a^2+r^2-2a\alpha=0$. Putting $\alpha=\frac{a}{2}$, $r=0$, we obtain $y=\pm ix+$ const. Prof. Forsyth's statement that $1+y'^2=0$ gives singular integrals is an error.

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* Forsyth: *Theory of Differential Equations.* Vol. III. § 239

The two first integrals may be written

$$\phi_1 \equiv \frac{y(1+y'^2) - ay' \pm \sqrt{2ax(1+y'^2) - a^2}}{1+y'^2} = \text{const.}$$

$$\phi_2 \equiv \frac{x(1+y'^2) - a \pm y'\sqrt{2ax(1+y'^2) - a^2}}{1+y'^2} = \text{const.}$$

From § 1, we should expect singular solutions, if any exist, to be given by $2ax(1+y'^2) - a^2 = 0$. This is verified to be true, and we obtain on integration

$$y + c = \frac{a}{2}\sin^{-1}\sqrt{\frac{2x}{a}} + \frac{\sqrt{2x(a-2x)}}{2}$$

(II)** (i) $x^2 + y^2 - \dfrac{2xy'(1+y'^2)}{y''} + \dfrac{2y(1+y'^2)}{y''} = k^2$

(ii) $x^2 + y^2 - \dfrac{2xy'(1+y'^2)}{y''} + \dfrac{2y(1+y'^2)}{y''} + \dfrac{(1+y'^2)^3}{y''^2} = k^2$

(iii) $y - mx + \dfrac{1+y'^2}{y''} + \dfrac{my'(1+y'^2)}{y''} = 0.$

All the three examples belong to the general type considered and their primitives can be written straight off. They are

(i) $x^2 + y^2 - 2ax - 2by + k^2 = 0$

(ii) $(x-a)^2 + (y-b)^2 = c^2$ where $a^2 + b^2 = k^2$

(iii) $x^2 + y^2 + 2a(x + my) + b = 0.$

For (i) what is the nature of the solution y/x = const.? This is not a singular solution, but an infinite solution. For, writing the primitive in the form $y + \dfrac{a}{b}x = \dfrac{x^2+y^2+k^2}{b}$, we obtain the solution $y/x =$ const. by making a and b infinite in such a way that $a/b =$ const. This fact can also be seen by writing a first integral in the form $\dfrac{xy' - y}{y^2 - x^2 - 2xyy' + k^2}$ = constant. Taking the constant as zero, we obtain y/x = constant.

For (ii), we have the singular solutions

$$\tan^{-1}\frac{y}{x} = A \pm \left(\frac{\sqrt{x^2+y^2-k^2}}{k} - \cos^{-1}\frac{k}{\sqrt{x^2+y^2}}\right)$$

For (iii), the solutions $x + my =$ const. provide infinite solutions.

** K. J. Sanjana : *Journal of the Indian Mathematical Society*, Qn. 1162.

PART II

4. The geometrical interpretation of the singular solutions of the differential equation of the second order introduces us to the subject of "osculants" or "osculating envelopes". *The osculant of a singly infinite system of curves may be defined as a curve having contact of the second order with every curve of the system.*

Consider now a doubly infinite system of curves $f(x, y, a, b)=0$. If the singly infinite system given by $b=\theta(a)$ possess an osculant, we must have at the point where the osculant touches a curve

$$f\ (x,\ y,\ a,\ b)=0 \tag{3}$$

$$\frac{\partial f}{\partial a}+\frac{\partial f}{\partial b}\frac{db}{da}=0 \tag{4}$$

$$\frac{\partial f}{\partial x}\Big/\frac{\partial f}{\partial y}=\left(\frac{\partial^2 f}{\partial x\partial a}+\frac{\partial^2 f}{\partial x\partial b}\frac{db}{da}\right)\Big/\left(\frac{\partial^2 f}{\partial y\partial a}+\frac{\partial^2 f}{\partial y\partial b}\frac{db}{da}\right)^{*} \tag{5}$$

These equations of the osculant have been obtained otherwise in Forsyth: *Theory*. Vol III. § 245.

Eliminating x and y from (3), (4), (5) we obtain a relation $\lambda(a, b, db/da)=0$. Let the general primitive of this be $G(a, b, c)=0$. Eliminating a and b between

$$f(x,\ y,\ a,\ b)=0\ ;\qquad \frac{\partial f}{\partial a}+\frac{\partial f}{\partial b}\left(-\frac{\partial G}{\partial a}\Big/\frac{\partial G}{\partial b}\right)=0\ ;\qquad G(a,\ b,\ c)=0,$$

we obtain $H\ (x, y, c)=0$ which *includes* the osculating envelopes of $f(x, y, a, b)=0$.

The singly infinite system of osculants so obtained evidently satisfy the differential equation of the second order derived from the primitive $f(x, y, a, b)=0$, since contact of the second order is equivalent to equality of curvature.

We shall obtain an alternate set of equations (I believe this method to be new) giving the osculants of $f(x, y, a, b)=0$.

* The conditions that the envelope of $f(x, y, a)=0$ might have everywhere contact of the second order with the curves of the system are usually given in the form $f=0$; $\partial f/\partial x=0$; $\partial^2 f/\partial x^2=0$. But in the present case, $\partial^2 f/\partial x^2=0$ is unsuitable as it leads to an equation of the second order involving d^2b/da^2. Consequently, its equivalent viz $\frac{\partial f}{\partial x}\frac{\partial^2 f}{\partial x\partial y}-\frac{\partial f}{\partial y}\frac{\partial^2 f}{\partial x\partial x}=0$ is employed. *Vide* Fowler: *Cambridge Mathematical Tracts.* No. 20. § 5· 420

In the general case, the osculants represent singular solutions of the differential equation of the second order derived from $f(x, y, a, b)=0$. It follows that the differential equation of the system of osculants can be derived from one of the first integrals by the process mentioned in the last theorem of § 1. Now the first integrals are obtained by the elimination of one of the constants a or b between

$$f(x, y, a, b)=0 \tag{6}$$

$$\psi(x, y, a, b)\equiv\frac{\partial f}{\partial x}+y'\frac{\partial f}{\partial y}=0 \tag{7}$$

The solutions of (6) and (7) for a and b therefore yield functions of x, y, y' such that a first partial derivative of at least one of the solutions becomes infinite in virtue of the singular first integral $w(x, y, y')=0$. By the theory of implicit functions, the condition that the solutions for a and b may possess this property is that

$$\frac{\partial(f, \psi)}{\partial(a, b)}=0 \tag{8}$$

assuming that f involves the variables rationally.

The differential equation of the system of osculants is therefore obtained by eliminating a and b from (6), (7) and (8).

This fact is also deduced from § 2 (2) by using the fact that the two congruences $\phi_1(x, y, z)=a$, $\phi_2(x, y, z)=b$ and $f(x, y, a, b)=0$, $\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}z=0$ must possess the same focal surface.

5. It should be noted that in the above discussion multiple points are left entirely out of consideration. The theory of contact is usually discussed with reference to ordinary points only. It is desirable to define contact of a given order at a point which may be a multiple point for one or both of two given curves, and to establish the conditions for contact. I shall not digress into this subject here, and in the rest of this paper the osculants are considered to touch the curves at ordinary points only.

6. If the osculants of $f(x, y, a, b)=0$ happen to possess an osculant of their own, the latter will in general be itself an osculant of $f=0$. The proof is easily supplied. In exceptional cases, this new curve may be a locus of double points of a ∞^1 curves of the primitive. The new osculant in the general case is not included in the original system $H(x, y, c)=0$.

7. *For a differential equation of the second order, can there exist a singular solution which does not represent an osculant of a system of curves selected out of the general primitive?*

In order to answer this important question, it is first necessary to have a clear idea of the nature of the general primitive of a differential equation. The general equation $F(x, y, y', y'')=0$ admits of a general primitive involving two arbitrary constants. This is a consequence of the fundamental existence-theorem of Cauchy. Now it may happen that the general primitive $\psi(x, y, a, b)$ breaks up into two factors $g(x, y, c) \times f(x, y, a, b)$ where g involves only one arbitrary constant. This solution $g(x, y, c)=0$ may not be derivable by the rules of § 1 for singular solutions, so that $g(x, y, c)=0$ must be regarded as forming part of the general primitive.

Let us next *start* with a *given* equation $f(x, y, a, b)=0$ and form the corresponding differential equation $F(x, y, y', y'')=0$. *Then it is incorrect to presume that $f(x, y, a, b)=0$ always represents the complete general primitive of $F=0$.* There may often exist an additional solution $g(x, y, c)=0$ such that for values of x, y, y' satisfying $g(x, y, c)=0$; $y'=-\frac{\partial g}{\partial x} \frac{\partial g}{\partial y}$, the property of *uniqueness* in Cauchy's existence theorem holds. The solution $g(x, y, c)=0$ should be preferably called non-singular on account of this property coupled with the fact that it may not be obtainable by the rules of § 1.

These facts will be illustrated by some of the following examples. That the answer to the question raised at the beginning of this section may be said to be in the affirmative at least in a sense, is seen in Example 9.

EXAMPLE 3. $\quad F \equiv 3x^2y'' + 2y'^3 - 6xy' = 0.$

This equation is derived from the equation $(y+b)^2 = x^3 + a$. The differential equation is obviously satisfied by $y=$ constant. What is the nature of this solution? It is not obtained by means of the equation $\partial F/\partial y''=0$, so that it cannot be a singular solution. This is confirmed by writing down a first integral in the form $b=\frac{3x^2-2yy'}{2y'}$. The value $b \rightarrow \infty$ gives $y'=0$ which is therefore included in the general first integral. But if in the primitive $(y+b)^2 = x^3 + a$, we change b to $1/c$ and put $c=0$ on simplification, we do not obtain $y=$ constant. Cauchy's existence theorem holds when the initial values are chosen so that $y'=0$. For these reasons, I call $(y+b)^2 = x^3 + a$ as the *incomplete general primitive* and $y=$ constant as the *residual primitive*.

EXAMPLE 4. $9\,y'^4 + 8y'' = 0.$

Incomplete general primitive : $(y+a)^3 = (x+b)^2$

Residual primitive : y constant.

A general first integral is given by $a = \dfrac{4 - 9y'^2 y}{9y'^2}$. As $a \to \infty$, $y' = 0$.

It may be added that when we solve the given equation by writing $y' = p$, $y'' = p\dfrac{dp}{dy}$, the equation $p = 0$ is at once thrown out, giving $y =$ constant.

EXAMPLE 5. $F \equiv (y' - xy'')^2 - 6xy'^5 = 0.$

Incomplete primitive : $(y+a)^3 = x^2 + b.$

Residual primitive : $y =$ constant.

In this case, $F = 0$, $\dfrac{\partial F}{\partial y''} = 0$ lead to $y' = 0$ which is however not a singular first integral, for it is included in the general first integral $3y'(y+a)^2 = 2x$, when $a \to \infty$. The curves $y =$ constant have no special geometrical significance in relation to $(y+a)^3 = x^2 + b$.

EXAMPLE 6. $F \equiv (y'^2 - 2yy'')^2 - 9yy'^6 = 0.$

Incomplete primitive : $(x+a)^2 = y(y-b)^2$.

Residual primitive : $y =$ constant.

(The particular case $y = 0$ is however included in the former, as may be seen by writing c^{-1} for b and putting $c = 0$ after simplification.) $y' = 0$ is included in the first integral $3y - b = 2y^{\frac{1}{2}}\,y'^{-1}$ when $b \to \infty$. $y' = 0$ is also given by $F = 0$, $\partial F / \partial y'' = 0$. The curves $y =$ constant happen to represent nodal loci of the primitive when a is varied.

EXAMPLE 7. $2\,(1+y')\,y'' + y'^3 = 0.$

Incomplete primitive : $(x + y + a + b - \frac{1}{3})^2 = \frac{4}{9}(y+a)^3$.

Residual primitive : $y =$ constant.

EXAMPLE 8. $F \equiv 4y''^2 + 24y'^2 y'' + 8y'' - (9y'^4 + 27y'^6) = 0.$

Incomplete primitive : $(x-b)^2 = (y+a)^2\,(1 - y - a).$

Residual primitive : $y =$ constant, which is the nodal loci for variation of the parameter b. $y' = 0$ is not given by the equations $F = 0$, $\partial F / \partial y'' = 0$.

EXAMPLE 9. The primitive $(x-c)^3 + y^3 = 3(x-c)y + a^3$ leads to the equation $(y^2 y'' + 2yy'^2 - 2y')\;(y^2 y'' + 2yy'^2 - y'y'') + (y'' - 2)^2\;(y^2 y' - y) = 0.$

A first integral is given by

$$(x-c)^2 - y'(x-c) + y^2y' - y = 0$$

or

$$2(x-c) = y' \pm \sqrt{y'^2 - 4y^2y' + 4y}.$$

There is no residual primitive. Forming the y'' - discriminant of the differential equation, it may be verified that $y=0$ satisfies it. $y=0$ also satisfies the differential equation. It furnishes a singular solution. Does this represent an osculant of the primitive curves?

$y=0$ meets any curve of the primitive at the point $x-c=a$ or $a\omega$ or $a\omega^2$ where ω is an imaginary cube root of unity. The values of y' and y'' at this point for the primitive are found from the equations

$$(x-c)^2 - y'(x-c) + y^2y' - y = 0$$

$$2(x-c) - y''(x-c) - 2y' + y^2y'' + 2yy'^2 = 0.$$

We obtain at $(c+a, 0)$, $y'=a$ and $y''=0$; similarly at $(c+a\omega, 0)$, $y'=a\omega$ and $y''=0$. Contact of the second order between the primitive and the line $y=0$ is thus impossible if $a \neq 0$. But when $a=0$, the point $(c, 0)$ is a node, and the value of y'' for the branch of the primitive touching $y=0$ is evidently not zero. $y=0$ is thus not an osculant in the sense employed.

The differential equation thus possesses a solution not included in the general primitive and not constituting a proper osculant of the primitive.

The following will be an explanation for the occurrence of the solution $y=0$. For the initial values $y=0$, $y'=0$, the differential equation will be satisfied *by every value of* y'' i.e. any curve touching the x - axis will satisfy the differential equation at the point of contact. The x - axis itself therefore furnishes a solution throughout its length.

EXAMPLE 10. $(y+c)^2 = (x+a)^3$ is the primitive of $8y'y''=9$. When c is kept constant and a is varied, $y+c=0$ gives the cuspidal locus which is an envelope as well. This however does not satisfy the differential equation. A nodal or cuspidal locus which is also an envelope is thus not necessarily a singular solution.

8. We shall next consider the case where the differential equation admits of *only one singular solution*. This happens, as has been mentioned in § 1, when a first integral $\phi(x, y, y') = a$ is such that the surfaces $\phi(x, y, z) = a$ admit cf an envelope whose equation involves only x and y. *This case is found to occur whenever all the* *curves of a doubly infinite system of plane curves can be enveloped by the same curve.* If E be the envelope and P any point on it, every one of the singly infinite number of curves of the system passing through P touches E there. In the general case, we may expect one curve out of these, to have contact of the second order with E at P. E thus appears as an osculant of a particular ∞^1 curves of the system. The equation of the envelope therefore provides a solution of the differential equation of the second order satisfied by the system, and the solution will in general be singular. In exceptional cases, however, there may be no curve of the system which has contact of the second order with E at any point along it, and yet E may furnish a solution. (I have found that this solution is usually particular or is an infinite solution). These facts will be illustrated in the course of the following examples:

We shall first consider a general type:

$$[f(x, y, a, b)]^2 = \phi(x, y).$$

Every curve of the system touches the curve $\phi(x, y) = 0$. We shall show that if (x_1, y_1) is any point on $\phi = 0$, it is possible in general to find values of a and b such that the corresponding curve has contact of the second order with $\phi = 0$ at (x_1, y_1).

Since the curve $f^2 = \phi$ has to pass through (x_1, y_1) we must have $f(x_1, y_1, a, b) = 0$. The values of y' and y'' at (x_1, y_1) for the curve $f^2 = \phi$ are given by the equations

$$2f.\left(\frac{\partial f}{\partial x_1} + \frac{\partial f}{\partial y_1} y'\right) = \frac{\partial \phi}{\partial x_1} + \frac{\partial \phi}{\partial y_1} y'$$

$$2f.(\text{some expression}) + 2\left(\frac{\partial f}{\partial x_1} + \frac{\partial f}{\partial y_1} y'\right)^2 = \frac{\partial^2 \phi}{\partial x_1^2} + 2\frac{\partial^2 \phi}{\partial x_1 \partial y_1^2} y' + \frac{\partial^2 \phi}{\partial y_1} y'^2 + \frac{\partial \phi}{\partial y_1} y''.$$

The first equation gives $y' = -\frac{\partial \phi}{\partial x_1} \Big/ \frac{\partial \phi}{\partial y_1}$. Writing down the value of y'' for the curve $\phi = 0$, we easily see that it is equal to the value of y'' given by the second of the above equations if $\frac{\partial f}{\partial x_1} + \frac{\partial f}{\partial y_1} y' = 0$. If therefore we determine a and b so as to satisfy the equations

$$f(x_1, y_1, a, b) = 0 \tag{9}$$

$$\frac{\partial f}{\partial x_1}\frac{\partial \phi}{\partial y_1} - \frac{\partial f}{\partial y_1}\frac{\partial \phi}{\partial x_1} = 0, \tag{10}$$

the corresponding curve $[f(x, y, a, b)]^2 = \phi(x, y)$ has contact of the second order with $\phi = 0$ at (x_1, y_1). In general, therefore, $\phi = 0$ is an osculant of a ∞^1 of curves selected out of the system $f^2 = \phi$, and the system possesses no other osculant.

In exceptional cases, however, it may be impossible to solve the above equations for a and b at any point whatever along the curve $\phi = 0$. *The equation $\phi = 0$ will nevertheless be a solution of the differential equation derived from $f^2 = \phi$.* If (x_1, y_1) be any point on the curve $\phi = 0$, then for the set of initial values $x = x_1$, $y = y_1$, $y' = -\frac{\partial \phi}{\partial x_1} \Big/ \frac{\partial \phi}{\partial y_1}$, the differential equation will be satisfied by the value of y'' corresponding to any curve of the system $f^2 = \phi$ that passes through (x_1, y_1). Assuming in the general case, that the singly infinite number of curves passing through (x_1, y_1) do not all have the same curvature at (x_1, y_1) which is different from that of $\phi(x, y) = 0$ at the point, it follows that for the above initial values, the differential equation will be satisfied by a certain singly infinite number of values of y'', and *hence by any value of y''*. This being so at any point along $\phi(x, y) = 0$, it follows that $\phi = 0$ itself must be a solution of the differential equation. It is also clear that whether this solution be singular or particular, it satisfies the equation $\partial F/\partial y'' = 0$ where $F = 0$ is the rationalised differential equation of the congruence.

EXAMPLE 11. $(x + ay + b)^2 = x^2 - y$.

The corresponding differential equation is

$$(x^2 - y)y''\{2x - y' - 2\sqrt{x^2 - y}\} = (x^2 - y)(2 - y'')y' - \tfrac{1}{2}y'(2x - y')^2.$$

$x^2 - y = 0$ is the only singular solution and the only osculant of the primitive curves. [The point (0, 0) presents an exception. The curves of the system passing through this point are $(x + ay)^2 = x^2 - y$. They break up into $y = 0$ and $2ax + a^2y + 1 = 0$. The values of y'' for $x^2 - y = 0$ and for either of the degenerate curves are different].

The above is a typical example of the general case. It is unnecessary to give more examples. The following examples illustrate the various possibilities of exception to the above general discussion.

EXAMPLE 12. $(x + ay + b)^2 = y$.

In this case, the equation (10) gives $1 = 0$. The envelope $y = 0$ is therefore not an osculant. But $y =$ constant constitute infinite solutions, as is easily verified, of the corresponding differential equations.

EXAMPLE 13. $b(x + y + a)^2 = \phi(x, y)$.

This illustrates a trivial case. The envelope $\phi = 0$, though not an osculant, is a particular solution corresponding to $b = 0$.

EXAMPLE 14. $\{ay^3 - x^2(x-b)\}^2 = y.$

$y=0$ is an envelope of the doubly infinite system and is a cusp-locus for the system $ay^3 = x^2(x-b)$. The equations (9) and (10) determining a and b such that the corresponding curve may have its y' and y'' equal to zero at any point $(x_1, 0)$ are

$$x_1^2(x_1-b)^2 = 0 ; \qquad x_1^2 + 2x_1(x_1-b) = 0$$

which are inconsistent when $x_1 \neq 0$. $y=0$ is therefore not an osculant. $y=0$ may be regarded as the single infinite solution obtained by making $a \to \infty$; for the given equation can be written

$$y^3 - \frac{1}{a}(x-b)x^2 = \frac{1}{a}y^{\frac{1}{2}}.$$

EXAMPLE 15. $\{f(x, y, a, b)\}^3 = \phi(x, y).$

$\phi = 0$ is an envelope having contact of the second order with every curve of the system. All the singly infinite number of curves that pass through a given point on the curve $\phi=0$ have the same values of y' and y'' as the curve $\phi=0$ has at the point.

9. I do not quite agree with the existing theory about *singular solutions of the second order.* If $w(x, y, y')=0$ be the singular first integral leading to the singular solutions of the first order, then according to the present theory, the existence of a singular solution of the second order requires that the curves defined by $w(x, y, y')=0$ should possess an envelope which has contact of the second order with every one of the curves. This condition is however not *in general* necessary. *Any singular solution of $w(x, y, y')=0$ will satisfy the differential equation of the second order, in a large number of types of equations.* In other words, for the existence of a singular solution of the second order, it is sufficient in many cases if the curves $w(x, y, y')=0$ possess an *ordinary* envelope (not included in the curves themselves) and not necessarily an osculant.

To explain this, let us suppose for the sake of simplicity that $w(x, y, y')=0$ is rational in x, y and y'. If $S(x, y)=0$ be its singular solution, the three equations $w=0$; $\frac{\partial w}{\partial y'}=0$; $\frac{\partial w}{\partial x} + y'\frac{\partial w}{\partial y} = 0$ are all satisfied by the value $y' = -\frac{\partial S}{\partial x} \Big/ \frac{\partial S}{\partial y}$ in virtue of the equation $S(x, y)=0$. Now from the last theorem of § 1, in order that $w(x, y, y')=0$ may be a singular first integral, one of the general first integrals $\phi_1 = a$ or $\phi_2 = b$ should involve $u=0$ irrationally or transcendentally.

A wide class of examples is obtained by taking the first integral in the form

$$\phi_1(x, y, y') \equiv f(x, y, y') + \{w(x, y, y')\}^{1/n} = a \qquad \text{where } n > 1.$$

The differential equation of the second order is

$$w^{1-1/n}\left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\right) + \frac{1}{n}\left(\frac{\partial w}{\partial x} + \frac{\partial w}{\partial y}y' + \frac{\partial w}{\partial y'}y''\right) = 0.$$

It may be rationalised if required, or we might eliminate a from the equation $(a-f)^n = w$, and obtain the differential equation in the form

$$\left(\frac{\partial w}{\partial x} + \frac{\partial w}{\partial y}y' + \frac{\partial w}{\partial y'}y''\right)^n = (-1)^n n^n w^{n-1}\left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\right)^n$$

The singular solution of $w=0$ evidently satisfies the differential equation of the second order, since we have simultaneously $w=0$, $\frac{\partial w}{\partial y'}=0$, $\frac{\partial w}{\partial x} + \frac{\partial w}{\partial y}y' = 0$. It must be noticed that for the initial values of x, y, y' satisfying the singular solution of $w=0$, the differential equation of the second order is satisfied *irrespective of the value of* y''.

EXAMPLE 16. Let $\phi_1(x, y, y') \equiv x^2 + y' - \sqrt{y - xy' - y'^2} = a$.

The differential equation of the second order is

$$2(2x + y'')(y - xy' - y'^2)^{\frac{1}{2}} + (x + 2y')y'' = 0.$$

The equation $y = xy' + y'^2$ leads to singular solutions of the first order viz. $y = cx + c^2$. Its singular solution $x^2 + 4y = 0$ satisfies the differential equation of the second order and constitutes the singular solution of the second order.

EXAMPLE 17. A more general case than that discussed is obtained by taking

$$\phi_1 \equiv f(x, y, y', \sqrt[k]{w}) = a, \qquad (k > 1)$$

w being rational in x, y, y' as before.

We obtain the differential equation of the second order in the form

$$\left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\right)w^{1-1/k} + \frac{\partial f}{\partial (w^{1/k})}\left(\frac{\partial w}{\partial x} + \frac{\partial w}{\partial y}y' + \frac{\partial w}{\partial y'}y''\right) = 0.$$

$w=0$ and the singular solution of $w=0$ obviously both satisfy the differential equation and yield the singular solutions of the first and second orders respectively.

10. It becomes necessary to explain the exceptional examples wherein the singular solution of the singular first integral does not satisfy the given differential equation.

EXAMPLE 18. $(1+x^2)y''^2 - y''(2xy' + \frac{1}{2}x^2) + (y'^2 + xy' - y) = 0.$

This example has been discussed by various writers including Goursat and belongs to the extended Clairant-type*

$$\phi(y'',\ xy''-y',\ x^2y''-2xy'+2y)=0.$$

The general primitive is given by

$$y = ax^2 + bx + 4a^2 + b^2.$$

The singular first integral is

$$y'^2 + \left(x + \frac{x^3}{2}\right)y' - (1+x^2)y - \frac{x^4}{16} = 0, \text{ whose solution}$$

viz. $\sqrt{16y+4x^2+x^4} - x\sqrt{1+x^2} - \log(x+\sqrt{1+x^2}) = \text{constant}$

represents the singular solutions of the first order. The singular first integral admits of the singular solution $16y+4x^2+x^4=0$ which does not satisfy the given equation.

We shall now explain why the general remarks of § 9 fail in this case. By putting a for y'' in the given equation, we obtain a general first integral

$$a^2(1+x^2) - a(2xy' + \tfrac{1}{2}x^2) + y'^2 + xy' - y = 0.$$

If we eliminate a from this by differentiation, we obtain in addition to the given differential equation of the second order, *another differential equation of the first order.* The result of differentiation gives

$$(a - y'')\ (2ax - x - 2y') = 0.$$

Eliminating a by using the equation $2ax - x - 2y' = 0$, we obtain the extraneous equation of the first order

$$(x+2y')^2 = 2x^2(2y - xy')$$

and this equation is satisfied by $16y + 4x^2 + x^4 = 0$.

In other words, if, as in § 9, we write

$$\phi_1 \equiv \frac{2xy' + \frac{1}{2}x^2}{2(1+x^2)} \pm \sqrt{\frac{x^4}{16} + y(1+x^2) - (x + \frac{x^3}{2})y' - y'^2} \Big/ \sqrt{1+x^2} = a,$$

the singular solution of the singular first integral *does constitute* a solution of the differential equation obtained by differentiating $\phi_1(x,\ y,\ y') = a$; but this differential equation happens to be degenerate,

* *Vide* Forsyth : Vol III. § 237.

breaking up into two equations, one of the second order, and the other of the first order; the solution in question is for the latter. Similar remarks are true if we start with any other general first integral of the given equation.

EXAMPLE 19. $y = -\frac{x^2}{2}y'' + xy' + y''(y' - xy'')$*

The complete primitive is

$$y = ax^2 + bx + 2ab.$$

The singular solutions of the first order are given by

$$y = -\frac{x^2}{18} + \frac{2}{3}cx^{\frac{3}{2}} + \frac{c^2}{4}.$$

These curves envelop the curve $y = -x^3/2$ which does not satisfy the given equation. The differential equation belongs to the same type as the equation of example 18. A first integral is

$$y = -\frac{ax^2}{2} + xy' + a(y' - ax)$$

It will be found that when a is eliminated from this by differentiation, we obtain the differential equation of the second order *together with the extraneous equation* $y + \frac{x^3}{2} = 0$.

It thus appears that the remarks of § 9 are true provided that the differential equation admits of two first integrals $\phi_1(x, y, y') = a$, $\phi_2(x, y, y') = b$ which are such that when the constant is eliminated by differentiation, we do not obtain any extraneous equation involving x, y, y', or only x and y. Examples 18 and 19 are examples wherein this proviso does not hold. Such types of examples may be multiplied, but there exist a large number of examples for which the statements of § 9 are true, and *§ 9 appears to cover the more general case.*

When, as in § 9, the curves representing singular solutions of the first order possess an *ordinary* envelope satisfying the differential equation, this envelope also constitutes in general an osculant of the primitive curves, though not in a way quite similar to the case of § 6. Let H be a curve of the system $H(x, y, c) = 0$ constituting singular solutions of the first order, and let E be the envelope of the curves H, touching H at P. We have seen in § 9 that, for the initial values of x, y, y' corresponding to any point P on E, the differential equation is satisfied by all values of y''. In the general case, there will therefore be a singly infinite number of curves of the primitive that pass

* P. Burgatti: *Rendiconti del cir. Matematico di Palermo*: Vol 20. (1905) pp. 256—264.

through P and have a common tangent. Out of these curves, one curve C will have its curvature at P equal to that of H, and a different curve D will have its curvature at P equal to that of E. H constitutes the osculant of a system of which C is a typical curve, while E constitutes the osculant of a system of which D is a typical curve.

It is extremely difficult to obtain examples wherein a singular solution of a differential equation of the second order does not constitute an osculant of a ∞^1 of primitive curves, chosen in some way or other, although such examples are not altogether impossible.

Bibliography

FORSYTH : *Theory of Differential Equations*, Vol. III, pp. 247-275.

GOURSAT: Part III of his memoir in the *American Journal of Mathematics*, Vol. XI.

A. C. DIXON : *Phil. Trans. Roy. Soc. London*, 1895, Part I-A, Arts. 53-56.

MAYER : § 3 of his paper in *Math. Annalen.* Vol. XXII.

BURGATTI : *Rendiconti del Circolo Math. di Palermo.* Vol. 20 (1905). pp. 256-264.

COON AND GORDON : A paper in *Annals of Mathematics*, Vol. 21, (1919-20.)

[This paper deals with the extraneous loci given by the y'' - discriminant of F $(x, y, y', y'')=0$].

HAMBURGER : *Crelle's Journal.* Vol. 121.

(Received 22-5-1933).

On series whose terms as well as the sum-function are continuous in an interval, and which converges non-uniformly in every sub-interval

BY

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1. In the following paper, a simple method for the construction of such series is given. Lebesgue [1] has given a Trigonometric series behaving in this way, and Osgood [2] has constructed series which converge non-uniformly in any sub-interval, but here the sum function is not continuous.

2. We start by partitioning all positive integers into an enumerable infinity of groups, each containing an infinity of elements. Thus the first group may consist of all those integers which are not powers of any number, arranged in the order of magnitude; the second, of square integers which are not higher powers; the third of those which are only cubes and not higher powers; and so on. Let these groups be represented by

$$[\alpha_1],\ [\alpha_2],\ [\alpha_3],\ \cdots\cdots\ [\alpha_r],\ \cdots\cdots$$

Now for simplicity take the interval to be (0, 1). We next take an enumerable set of numbers x_1, x_2, x_3 $\cdots\cdots$ everywhere dense in (0, 1), for instance the rational fractions. Lastly we define an auxiliary sequence $S_n(x)$ thus:—

If n is the k^{th} element in $[\alpha_r]$, then

$$S_n(x) = 0, \text{ for all values of } x, \quad \text{if } k \leqslant 4.$$

For $k \geqslant 4$,

(i) $S_n(x) = 0$ for all x such that, $|x - x_r| > 2/k$,

(ii) $S_n(x) = k\left\{x - (x_r - \frac{2}{k})\right\}$, if $x_r - \frac{2}{k} \leqslant x \leqslant x_r - \frac{1}{k}$

(1) See GANESH PRASAD *Fourier Series.*

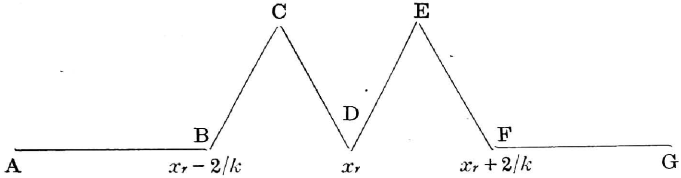
(2) OSGOOD: *Lehrbuch der Funktion theorie*; Bd I.

(iii) $S_n(x) = k(x_r - x)$ if $x_r - \frac{1}{k} \leqslant x \leqslant x_r$

(iv) $S_n(x) = k(x - x_r)$ if $x_r \leqslant x \leqslant x_r + \frac{1}{k}$

(v) $S_n(x) = k\left\{ (x_r + \frac{2}{k}) - x \right\}$ if $x_r + \frac{1}{k} \leqslant x \leqslant x_r + \frac{2}{k}$

The graph ABCDEFG of the function $S_n(x)$ is indicated in the following figure.



Suitable modification is to be made if x_r coincides with either 0 or 1.

Next let a_r be any sequence which tends to zero when r becomes infinite. Then the required sum function $T_n(x)$ is defined by

$$T_n(x) = a_r \; S_n(x)$$

if n belongs to the group $[\alpha_r]$.

3. Proof of the convergence of $T_n(x)$ at any given point (x).

Given any $\varepsilon > 0$, it is possible to find a number t such that for all $r > t$, $| a_r S_n(x) | < \varepsilon$, since $\lim_{r \to \infty} a_r = 0$. Hence for all values n belonging to groups whose indices are greater than t, we have $| T_n(x) | < \varepsilon$.

Now consider the finite number of groups, $[\alpha_1]$, $[\alpha_2]$ . . . . $[\alpha_t]$. Now any infinite sequence of values of n for which $T_n(x)$ does not tend to zero, and which belong to the above groups, should contain at least one infinite subsequence, belonging to one of these groups. This is however not possible. Hence we can find a value n_0 such that $T_n(x) < \varepsilon$, for all $n \geqslant n_0$ (x being fixed). Hence $T_n(x)$ for all x tends to zero as $n \to \infty$. Hence the sum-function is certainly a continuous function of x.

4. Proof of non-uniform convergence in any sub-interval.

If we take any sub-interval, there is at least one point x_r in it. Now consider only those $T_n(x)$'s for which n takes the values included in the group $[\alpha_r]$, we see that for any n however large, there are points in the interval for which $T_n(x) = a_r$. Hence the series is non-uniformly convergent in any sub-interval.

5. It is possible to modify the argument suitably to construct examples of the following.

$f(x,y)$ is a continuous function of x for $0 \leqslant x \leqslant 1$, when y takes a fixed value y_0 belonging to an interval say $[0,1]$;

also
$$\lim_{y \to 0} f(x,y) = \phi(x)$$
exists at every point x, and $\phi(x)$ is continuous in $[0,1]$, and yet the convergence is not uniform with respect to y, i.e. $f(x,y)$ is not a continuous function in (x,y) in the region considered.

Such an example can be constructed as follows. $f(x,y) = 0$ for all irrational values of y, whatever x may be. We can arrange the rational values of y in a sequence $y_1, y_2, y_3, \cdots y_r, \cdots$ where $y_r \to 0$. And we may divide this sequence into groups in exactly the same way as before, and define the sequences also in the same way.

The Φ-Conic from a projective standpoint

BY

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There are several ways of obtaining the equation of the conic under reference. One method turns on interpreting the vanishing of the middle coefficient of a quadratic equation. Secondly there is the method of obtaining two circles by projection and then utilising the elementary properties of the circle. Thirdly there is the straightforward analytic method, where it may be noticed that a certain factor is thrown out with or without any reasons given. Finally one may locate a certain number of points on the locus, and prove as best as one can that the locus is a conic. We shall have to do mainly with this last proof.

There is no proof available on purely projective lines, that the locus is a conic. Reye must have sought for such a proof; and a recent work on conics published at Warsaw draws attention to the need.

The following proof is offered as meeting the need; it recognises the fact that a proposition on conics may, sometimes, best be established by an appeal to higher elements, the plane cubic in this case.

We begin with a

LEMMA: Given two pairs of points A_1 B_1 and A_2 B_2 on a line (or a conic), the fourth harmonics C_1 and C_2 of any point P on the line will be a pair of the involution fixed by the two given pairs if and only if the two given pairs are harmonic.

There is one exception, namely when C_1 and C_2 coincide at C, and in this case P and C are the united points of the involution fixed by the given pairs. Such a case thus arises when P is one of the united points of this involution, and in this exceptional case it does not follow that A_1 B_1 and A_2 B_2 are harmonic, but only that A_1B_1, A_2B_2, PP, CC are all pairs of one involution.

Now consider two conics S_1 and S_2. Take an arbitrary point P and let p_1 and p_2 be its polars in regard to the conics. We may then consider S_1, S_2 and the conic consisting of p_1 and p_2 as three conics. It is known that the envelope of a line which meets three conics in pairs of points of an involution is a curve of class three; of such lines therefore there are thus three which pass through the point P.

One of these three lines is the join of P to C the intersection of p_1 and p_2; on this line CP we have pairs of points A_1B_1 and A_2B_2 cut out by S_1 and S_2, and the fourth harmonics of P with respect to these pairs coincide at C. The points C, P are thus the united points of the involution fixed by the pairs A_1B_1 and A_2B_2, and these pairs are not harmonic. On each of the other two lines possible through P the pairs A_1B_1, A_2B_2 cut out by the conics S_1, S_2 and C_1C_2 (the intersections of the line in question with p_1, p_2) are pairs of an involution, whence C_1, C_2 being the harmonics of P in regard to A_1B_1 and A_2B_2 it follows from the Lemma that A_1B_1 and A_2B^2 are harmonic. The class of the envelope required is thus two only, and the envelope is a conic.

A Japanese Problem

BY

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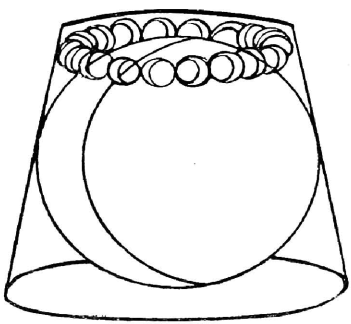
The enunciation of the problem is as follows:—

"There is a right circular conical frustum circumscribed about a large sphere and many small equal spheres are arranged as in the figure in a rosary form within the space over the surface of the sphere and under the top circle of the frustum. Given the diameter of the top circle, find the diameter of the bottom circle which is minimum."



This was dedicated to a Budha in a temple in Kazusa (near Tokyo) prefecture in 1814 by SEITO BABA, an able mathematician of that time, and thereafter to a hero-god in a Shinto (the indigenous religion of Japan only) shrine in Osaka prefecture in 1835 by RIKEN FUKUDA, also an able mathematician. Also it was proposed and solved by KOZEN FUKUCHI, in the book "*Juntendo Sampu*" second volume, 1847, compiled by SEIVO IWATA, a disciple of FUKUDA, together with the case of a right regular polygonal frustum. Especially the tablet dedicated by R. FUKUDA was famous, because his teacher SHINGEN TAKEDA questioned the result of R. FUKUDA, although SHUKI KOIDE, a contemporary mathematician, supported it as true, and on that account FUKUDA became a disciple of KOIDE thereafter.

The answer was:

The minimum diameter of the bottom circle is

$$\sqrt{\frac{\sqrt{\sqrt{8}+4}+\sqrt{8}+4}{8}} \times \text{ the given diameter of the top circle.}^{*}$$

* Even in the case of a polygonal frustum, the ratio of the corresponding sides of its upper and lower bases is the same according to the proposer FUKUCHI.

It is difficult to explain the methods for solving the problem actually used by these mathematicians, because they are in peculiar Japanese style. The following one is mine, but not theirs, and is not so easy.

Take the plane section of the figure (an isosceles trapezoid), passing through the centre of the large sphere, the two centres of the top and bottom circles, and the centre of one small sphere, and let the given radius of the top circle (centre A), the radius of the bottom circle (centre B) (to be minimized), the radius of the large sphere, and the radius of the small sphere be denoted by a, y, R, r, respectively. Let the distance between the points of contact of the two spheres with the side of the trapezoid, the distance of the point of contact of the two spheres from the height AB, and the angle between the height AB and the radius of the large sphere drawn from its centre to the point of contact with the side of the trapezoid be denoted by a', d, θ, respectively. Then

$$\tan\theta = \frac{2R}{y-a}, \qquad y \geqslant a; \qquad \ldots \quad (1)$$

$$d = R\sin\theta/2.$$

Again $\quad d = a' - r\sin\theta/2; \qquad a' = a - r\tan\theta/2.$

Therefore $\quad R\sin\frac{\theta}{2} = a - r\tan\frac{\theta}{2} - r\sin\frac{\theta}{2},$

i.e., $$a - r\tan\theta/2 = (R+r)\sin\theta/2 \qquad \ldots \quad (2)$$

But $\quad a'^2 = (R+r)^2 - (R-r)^2 = 4Rr.$

Therefore $$a - r\tan\theta/2 = 2\sqrt{Rr}. \qquad \ldots \quad (3)$$

Next, the distance between the feet of perpendiculars dropped from the centres of two neighbouring small spheres upon the plane of the top circle is $2r$, and the side and base of the isosceles triangle formed by joining the feet of perpendiculars to the centre A of the top circle are a' and $2r$, respectively. Let the vertical angle of the isosceles triangle be ϕ. Then $\sin\phi/2 = r/a'$.

Put $$s = \sin\phi/2 \qquad \ldots \quad (4)$$

Then $$r = \left(a - r\tan\frac{\theta}{2}\right)s. \qquad \ldots \quad (5)$$

From the four equations (1), (2), (3), (5), eliminate R, r, θ, and express y in terms of a and s.

First, from (2) and (3)

$$(R+r)\sin\theta/2 = 2\sqrt{Rr}.$$

Therefore $$\sin\frac{\theta}{2} = \frac{2\sqrt{Rr}}{R+r}$$

Hence $$\cos\frac{\theta}{2} = \frac{R-r}{R+r},$$

But $$\tan\theta = \frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{\cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2}}.$$

Hence by (1) $$\frac{R}{y-a} = \frac{2(R-r)\sqrt{Rr}}{R^2+r^2-6Rr}. \qquad \ldots \quad (6)$$

But from (3) and (5)

$$\sqrt{r/R} = 2s.$$

Hence (6) changes into

$$\frac{R}{y-a} = \frac{2(1-4s^2)2s}{1+16s^4-6(4s^2)}. \qquad \ldots \quad (7)$$

Now from (3)

$$a - r\,\frac{2\sqrt{Rr}}{R-r} = 2\sqrt{Rr},$$

whence $$\frac{a}{R} = \frac{4s}{1-4s^2}.$$

Multiplying this and (7)

$$\frac{a}{y-a} = \frac{4t}{t^2-6t+1},$$

where $t = 4s^2$. Hence

$$4\left(\frac{y}{a}-1\right) = \frac{t^2-6t+1}{t}, \qquad \ldots \quad (A)$$

whence $$\frac{y}{a} = \frac{(t-1)^2}{4t}. \qquad \ldots \quad (B)$$

The conditions of the problem require $t>0$, $y \geqslant a$.

Therefore from (A) $t^2 - 6t + 1 \geqslant 0$ and therefore if we put $t_1 = 3 - 2\sqrt{2}$, $t_2 = 3 + 2\sqrt{2}$, then $t \leqslant t_1$, or $t \geqslant t_2$.

But $$t = 4s^2 = 4\sin^2\phi/2$$

So $$4\sin^2\phi/2 \leqslant 3 - 2\sqrt{2}, \quad \text{or} \quad 4\sin^2\phi/2 \geqslant 3 + 2\sqrt{2}.$$

But since $3+2\sqrt{2} > 4$, the latter is not necessary for us. Therefore

$$4\sin^2 .\phi/2 \leqslant 3-2\sqrt{2}$$

Now all the small spheres are in external contact side by side without any vacant space between them. Let their number be n, so that $\phi = 2\pi/n$.

Then

$$\sin\frac{\pi}{n} \leqslant \frac{\sqrt{3-2\sqrt{2}}}{2} = \frac{\sqrt{2}-1}{2} < 0{\cdot}208 < \sin 12^\circ$$

Therefore

$$180^\circ < n{\cdot}12^\circ, \quad \text{i.e.} \quad n > 15.$$

Therefore the number of the small spheres is arbitrary if it exceeds 15, the minimum being 16. If it is 16, then t^2-6t+1 is nearest zero, and y is greater than a and nearest to a. This gives the answer to the problem.

When the number of the small spheres is 16,

$$t = 4s^2 = 4\sin^2\frac{180^\circ}{16} = 4\sin^2 11^\circ\ 15',$$

and

$$\sin 11^\circ\ 15' = \tfrac{1}{2}\sqrt{2-\sqrt{2+\sqrt{2}}}$$

Substituting this in the expression of y, we arrive at the result:

The minimum y is

$$\frac{2(2+\sqrt{2})+\sqrt{2(2+\sqrt{2})}}{8}\cdot a$$

February 5th, 1932
SENDAI, JAPAN.

Hexagonal 4-webs of Surfaces in 3-Space

BY

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In a 3-Space with the arbitrary point co-ordinates x_i; $(i = 1, 2, 3)$ we consider a "*4-web of Surfaces*"

$$u_\alpha (x_1, x_2, x_3) = \text{const.}; \quad \alpha = 0, 1, 2, 3;$$

the *matrix* $(\partial u_\alpha / \partial x_i)$ having the rank 3.

If we consider the intersections of one of the surfaces, for instance, of the sheaf $u_0 = \text{const.}$ by the surfaces of the three other sheaves, we get on our surface a "*3-web of Curves*". We consider the special case, when all this 3-webs of curves are "*hexagonal*". This means that they are topologically equivalent to three sheaves of parallel straight lines in the same plane. In this case we call our 4-web of Surfaces *hexagonal*.

A special case of such a hexagonal 4-web is given by four pencils of planes.

In this paper, I shall prove a theorem of uniqueness.

A hexagonal 4-web of surfaces is except for topological transformations uniquely determined by three functions each of one variable.

The corresponding *existence theorem* has still to be found.[1]

Two years ago I found the following theorem[1]:

The four sheaves of surfaces

$$(1) \qquad u_\alpha(x_1, x_2, x_3) = \text{const.}; \qquad \alpha = 0, 1, 2, 3$$

in the 3-Space x_1, x_2, x_3 with

$$(2) \qquad \text{rank } (\partial u_\alpha / \partial x_i) = 3$$

form a hexagonal web of surfaces, if it is possible to find operators

---

1 The following is related to the paper:
W. BLASCHKE, Topologische Fragen der Differentialgeometrie 19 ("T_{19}") *Hamburg Abhandlungen* 1930.

(3) $$\Delta_i = \sum_{k=1}^{3} a_{ik}(x) \frac{\partial}{\partial x_k}, \qquad |a_{ik}| \neq 0$$

such that the u_α's satisfy the conditions

(4) $$\begin{aligned} + \Delta_1 u_0 &= + \Delta_2 u_0 = + \Delta_3 u_0, \\ + \Delta_1 u_1 &= - \Delta_2 u_1 = - \Delta_3 u_1, \\ - \Delta_1 u_2 &= + \Delta_2 u_2 = - \Delta_3 u_2, \\ - \Delta_1 u_3 &= - \Delta_2 u_3 = + \Delta_3 u_3 \end{aligned}$$

and the operators the following relations:—

(5) $$\begin{aligned} \Delta_2\Delta_3 - \Delta_3\Delta_2 &= g_1\Delta_1, \\ \Delta_3\Delta_1 - \Delta_1\Delta_3 &= g_2\Delta_2, \\ \Delta_1\Delta_2 - \Delta_2\Delta_1 &= g_3\Delta_3, \end{aligned}$$

the functions g_i *satisfying the equation*

(6) $$g_1 + g_2 + g_3 = 0$$

and the differential equations

(7) $$\Delta_1 g_1 = \Delta_2 g_2 = \Delta_3 g_3 = 0.$$

Given the u_α's, the Δ_i are determined except for a common constant factor λ:

(8) $$\Delta_i^* = \lambda \Delta_i, \qquad g_i^* = \lambda g_i.$$

Writing for shortness

$$\Delta_i \ldots \ldots \Delta_k \; g_l = g_{lk} \ldots \ldots$$

we have (7) or

(7) $$g_{11} = g_{22} = g_{33} = 0.$$

Using (7) it follows from (6) by derivation

(9) $$g_{21} + g_{31} = 0, \quad g_{12} + g_{32} = 0, \quad g_{13} + g_{23} = 0.$$

Therefore knowing the derivatives

(10) $$g_{23}, \; g_{31}, \; g_{12}$$

by (7), (9) we know all the first derivatives g_{ik}.

Let us consider now the derivatives of order $n \geqslant 2$ of g_2

(11) $$\underbrace{g_{2ik\ldots\ldots l}}_{n}$$

If one of the indices $i, k \ldots\ldots l$ is 2, we can using (5) and their $(n-1)^{th}$ derivatives change the order of these indices, so that the first becomes equal to 2. Then we have

(12) $$g_{22k\ldots l} = 0$$

using (7) therefore of one of the indices $i, k, \ldots l$ is equal 2, the n^{th} derivative of g_2 is reducible to $(n-1)^{th}$ derivatives.

If one of the indices $i, k, \ldots l$ is equal 1, we can assume again that the first index $i=1$, and we have

(13) $$g_{21k\ldots l} = -g_{31k\ldots l}$$

from (9) and we can reduce again, if one of the indices $k\ldots\ldots l$ is equal 3.

Therefore the only cases, where no reduction is possible are the cases

(14) $$g_{2\underbrace{333\ldots3}_{n}}, \quad g_{3\underbrace{111\ldots1}_{n}}, \quad g_{1\underbrace{22\ldots2}_{n}},$$

We thus see that:

If at one point O of our 3-space the derivatives of the

0^{th} order	g_1, g_2, g_3;	relation $g_1+g_2+g_3=0$	;
1^{st} ,,	g_{12}, g_{23}, g_{31};	no relation	;
2^{nd} ,,	$g_{122}, g_{233}, g_{311}$;	,, ,,	;

. . . .

are known we are able to calculate all derivatives $g_{ik\ldots l}$ of the functions g_i in this point.

There exists a "*completnes theorem*" found by *G. Bol* and *G. Howe*(2), that by the derivatives of the "composition functions" g_i in one point, the operators Δ_i and therefore the web is uniquely determined except for topological transformations.

If we introduce along the curve Δ_i, i.e.

$$dx_1 : dx_2 : dx_3 = a_{i1} : a_{i2} : a_{i3}$$

through O the paramater t_i determined by the conditions $\Delta_i(t_i)=1$, and $t_i=0$ at O, we have

$$g_{2\underbrace{33\ldots3}_{n}} = \frac{d^n}{dt_3^n} g_2(0).$$

Therefore:

Given each function g_i along the curve Δ_{i+1} (indices modulo 3) through the same point O, this function satisfying at O the initial condition

$$(g_1+g_2+g_3)_O = 0,$$

the hexagonal web is uniquely determined.

I do not know if it is possible to prescribe these three functions (each of one variable) arbitrarily.

Singapore, March 1932.

(2) *Hamburg Abhandlungen.*

The Problem of Differential Invariants

BY

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The classical problem of characterizing a surface regardless of the co-ordinate system used, was shown by Gauss to be that of determining its invariant curvature. For the general n dimensional Riemannian space, the solution depends on the Riemann-Christoffel curvature tensor. In fact, we say after Lie and Ricci that all essential differential invariants of such a space are given by the curvature tensor and its successive covariant derivatives.

Further generalizations of the concept of space, such as those with an affine connection have an additional number of invariants, as the *torsion*, and other well-known tensors depending on the special type of connection or parallelism used. I have shown elsewhere * that a geometry can be associated with second order differential equations, the *paths* being integral curves of

$$\ddot{x}^i + \alpha^i(x, \dot{x}, t) = 0 \tag{1}$$

There are two procedures for obtaining differential invariants; the first (my own) is motivated, but incomplete; the second is more powerful, but has a slight disadvantage in needing a good many *a priori* assumptions.

With the tensor-invariance of (1) and their equations of variation

$$\ddot{u}^i + \alpha^i_{;r}\,\dot{u}^r + \alpha^i_{,r}\,u^r = 0 \tag{2}$$

we deduce the existence of a vector differential operator, the *bi-derivate*;

$$D\,(u)^i \equiv \dot{u}^i + \tfrac{1}{2}\,\alpha^i_{;r}\,u^r$$

(1) can now be written as

$$D\,(\dot{x})^i + (\alpha^i - \tfrac{1}{2}\,\alpha^i_{;r}\,\dot{x}^r) = 0 \tag{3}$$

* *Cf.* D. D. Kosambi: *Math. Zeitschrift* vol. 37 (1933) pp. 608-618.
E. Cartan: *Ibid.* pp. 619-622.

It is seen that

(4) $$\varepsilon^i = \alpha^i - \tfrac{1}{2}\,\alpha^i_{\;;r}\,\dot{x}^r$$

is a vector, a first differential invariant of the system. The second can be obtained by reducing the equations of variation to the normal form :

(5) $$D^2(u)^i = \overset{i}{\underset{r}{P}}\,u^r$$

$$P^i_{\;j} = -\alpha^i_{\;,j} + \tfrac{1}{2}\dot{x}^k\alpha^i_{\;,k;j} + \tfrac{1}{2}\frac{\partial}{\partial t}\alpha^i_{\;;j} - \tfrac{1}{2}\alpha^k\alpha^i_{\;;j;k} + \tfrac{1}{4}\alpha^i_{\;;k}\,\alpha^k_{\;;j}$$

The mixed tensor P^i corresponds to the Riemann-Christoffel tensor in this scheme. Our invariants are thus ε^i and $P^i_{\;j}$ with the two differential processes

(6) $$\partial/\partial\dot{x}^r \qquad \text{and} \qquad D^i$$

For more general connections, we may use

(7) $$D(u)^i \equiv \dot{u}^i + \gamma^i_{\;r}\,u^r$$

where the $\gamma^i_{\;j}$ are only restricted to having the same law of transformation as $\frac{1}{2}\alpha^i_{\;;j}$. The consequent scheme of invariants can be deduced from the preceding *intrinsic invariants*, with the one *invariant of the connection*:

(8) $$\sigma^i_{\;j} = \tfrac{1}{2}\,\alpha^i_{\;;j} - \gamma^i$$

It is, however, not at all clear that all invariants have been so obtained. To settle this important point, we follow the procedure of Prof. Elie Cartan of Paris, the second procedure mentioned above. The space considered is now of $2n+1$ dimensions in x, $\dot{x}$, t, the last being an absolute time-like parameter. We take

(9) $$\omega^i_{\;d} \equiv dx^i - \dot{x}^i\,dt \neq 0$$

and ascribe to this Pfaffian, a vector character under all admissible transformations. A tensorial operator is then defined *a priori*,

(10) $$\mathcal{D}\,(u)^i = du^i + \gamma^i_{\;r}\,u^r + \gamma^i_{\;kr}\,u^r\,\omega^k_{\;d}$$

The difference $\mathcal{D}\omega^i_{\;\delta} - \Delta\,\omega^i_{\;d}$ can be expressed as

(11) $$\mathcal{D}\,\omega^i_{\;\delta} - \Delta\,\omega^i_{\;d} = \theta^i_{\;\delta}\,dt - \theta^i_{\;d}\,\delta t + (\gamma^i_{\;kr} - \gamma^i_{\;rk})\omega^k_{\;d}\,\omega^r_{\;\delta}$$

where

(12) $$\overset{i}{\underset{d}{\theta}} = d\dot{x}^i + \alpha^i\,dt + \gamma^i_{\;r}\,\omega^r_{\;d}$$

This is also tensor-invariant, $\omega^i_{\;d} = \theta^i_{\;d} = 0$ being equivalent to the original system (1). In all succeeding formulæ, we eliminate dx, $d\dot{x}$, by the use of the Pfaffian differential vectors in (9) and (12).

A first set of differential invariants now appears as the coefficients of the various linear and bilinear terms in dt, ω, θ, in the expression $D\,\theta^i_\delta - \Delta\,\theta^i_d$

An intrinsic choice of coefficients would be one that minimises the number of these invariants, and for that we must choose

$$\gamma^i_k = \tfrac{1}{2}\,\alpha^i_{;k} \qquad\qquad \gamma^i_{kr} = \tfrac{1}{2}\,\alpha^i_{;k;r}$$

Another differential invariant appears in computing $(\mathcal{D}\Delta - \Delta\mathcal{D})\,u^i$ and the full intrinsic set is then seen to be:

$$(13) \qquad \dot{x}^i \qquad \varepsilon^i \qquad P^i_j \qquad \alpha^i_{;j;k;l}$$

The invariantive vector differential processes are now *three* in number to be obtained by writing

$$\mathcal{D}(u)^i = D(u)^i\,dt + u^i_{/k}\,\omega^k_d + u^i_{;k}\,\theta^k_d$$

where

$$(14) \qquad D(u)^i = \frac{\partial u^i}{\partial t} + \dot{x}^r\,\frac{\partial u^i}{\partial x^r} - \alpha^h\,\frac{\partial u^i}{\partial x^k} + \tfrac{1}{2}\,\alpha^i_{;r}\,u^r$$

$$u^i_{/k} = u^i_{,k} - \tfrac{1}{2}\,\alpha^r_{;k}\,u^i_{;r} + \tfrac{1}{2}\,\alpha^i_{;k;r}\,u^r$$

For the most general connections of this type, we shall have to add the following invariants of the connection:

$$\sigma^i_j = \tfrac{1}{2}\,\alpha^i_{;j} - \gamma^i_j \qquad\qquad \sigma^i_{jk} = \tfrac{1}{2}\,\alpha^i_{;j;k} - \gamma^i_{jk}$$

It is seen that with proper restrictions on our absolute parameter and the transformation group, no further differential invariants are to be obtained, except by using the differential operators on these. There are differential relations between the invariants, but none that prevent the set from consisting of independent members. The various special types of spaces hitherto considered can be described by the vanishing of one or more of these invariants.

NOTE: I find that not even the procedure of Prof. Cartan includes all the differential invariants for the transformation group [A].:

$$\bar{x}^i = F^i\,[\,x^k \cdots\,] \qquad\qquad \bar{t} = t$$

These can be derived from the rather arbitrary but classical procedure of alternating all fundamental operations which are tensor-invariant. This, in essence, is the method of Christoffel for the derivation of the invariants from compatibility conditions, and is equivalent to calculating the Poisson brackets for a system of linear partial differential equations.

The fundamental differential operations which carry a vector u^i into another are:

(15) $$Du^i \equiv \dot{u}^i + \tfrac{1}{2}\alpha^i_{;r}\, u^r \qquad \frac{\partial u^i}{\partial \dot{x}^j} = u^i_{;j} \qquad \frac{\partial u^i}{\partial t}$$

In the preceding, I only considered the first two. We find on alternating upon a vector u^i:

(16) $$u^i_{;j;k} - u^i_{;k;j} = 0 \qquad \frac{\partial}{\partial t} u^i_{;j} - \left(\frac{\partial u^i}{\partial t}\right)_{;j} = 0$$

The alternant of D and $\partial/\partial \dot{x}$ is another operator, precisely the covariant differentiator suggested by Cartan:

(17) $$(Du^i)_{;k} - D(u^i_{;k}) = u^i_{/k} = u^i_{,k} - \tfrac{1}{2} u^i_{;r}\, \alpha^r_{;k} + \tfrac{1}{2} u^r \alpha^i_{;r;k}$$

The rest give us immediately:

(18) $$u^i_{/j;k} - u^i_{;k/j} = \tfrac{1}{2} u^r \alpha^i_{;r;j;k}$$

(19) $$u^i_{/j/k} - u^i_{/k/j} = u^r R^i_{jk;r} - u^i_{;r} R^r_{jk}$$

where $$R^i_{jk} = \tfrac{1}{3}(P^i_{j;k} - P^i_{k;j})$$

(20) $$(Du^i)_{/j} - Du^i_{/j} = u^r (R^i_{jr} - P^i_{j;r}) + u^i_{;r} P^r_j$$

(21) $$\frac{\partial}{\partial t} D u^i - D\frac{\partial u^i}{\partial t} = u^r \frac{\partial}{\partial t}\tfrac{1}{2}\alpha^i_{;r} - u^i_{;r}\frac{\partial \alpha^r}{\partial t}$$

(22) $$\frac{\partial}{\partial t} u^i_{/j} - \left(\frac{\partial u^i}{\partial t}\right)_{/j} = u^r \frac{\partial}{\partial t}\tfrac{1}{2}\alpha^i_{;r;j} - \tfrac{1}{2} u^i_{;r} \frac{\partial \alpha^r_{;j}}{\partial t}$$

Thus, for intrinsic invariants, the fundamental list is:

(23) $$\dot{x}^i \qquad P^i_j \qquad \alpha^i_{;j;k;l}$$

The rest are derivable from these by means of the operations $\frac{\partial}{\partial t}$, D and $\partial/\partial x$. The following relations are seen to hold for the invariants:

(24)
$$\dot{x}^i_{;k} = \delta^i_k$$
$$D\dot{x}^i = -\varepsilon^i$$
$$\dot{x}^i_{/k} = -\varepsilon^i_{;k}$$
$$-\varepsilon^i_{;k;l} = \tfrac{1}{2}\alpha^i_{;k;l;m}\dot{x}^m$$
$$D\varepsilon^i = -P^i_j \dot{x}^j + \frac{\partial \alpha^i}{\partial t}$$
$$\varepsilon^i_{/k} = -P^i_k - \dot{x}^r R^i_{rk} + \frac{\partial}{\partial t}\alpha^i_{;k}$$

Thus, the term $\frac{\partial \alpha^i_{;j}}{\partial t}$ can be omitted from P^i_j without destroying its tensor invariance.

A covariant specification of the simplex inscribed in a rational norm curve in a space of odd dimensions and circumscribed to a quadric inpolar to the curve.*

BY

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1. Regarding a twisted cubic R_3 as the carrier of a binary variable t, a set of six points on it given parametrically by the binary sextic a_t^6, determines a unique quadric envelope Q, touching the osculating planes at these points, and inpolar to the curve. In general, there is a unique tetrahedron inscribed in R_3 and circumscribed to Q. If the vertices of this tetrahedron correspond to the binary quartic b_t^4, it is known that b_t^4 is the fourth transvectant of a_t^6 with itself. The object of this paper is to extend the above result to a space of odd dimensions.

Taking a norm curve R_{2n-1} in a space S_{2n-1} of $2n-1$ dimensions, a set of $4n-2$ points on it, given parametrically by the binary $(4n-2)$-ic a_t^{4n-2}, determines uniquely a quadric envelope Q, touching the osculating primes at these points and inpolar to R_{2n-1}. There is, in general, a unique simplex T inscribed in R_{2n-1} and circumscribed to Q. If the vertices of the simplex are given parametrically by the binary $2n$-ic b_t^{2n}, it is evident that b_t^{2n} should be a covariant of a_t^{4n-2}. In this paper it is proved that

$$b_t^{2n} \equiv (a_1 a_2)^4 \;.\;.\;.\; (a_1 a_n)^4 \;.\;.\;.\; (a_{n-1} a_n)^4 \; a_{1,t}^2 \;.\;.\;.\; a_{n,t}^2$$

where $a_t^{4n-2} \equiv a_{1,t}^{4n-2} \equiv \cdots\cdot \equiv a_{n,t}^{4n-2}$

2. The following correspondence between quadric envelopes Q in S_{2n-1} and linear line complexes L in S_{2n}, considered by Dr. Vaidyanathaswamy, is required for our investigation†.

An inpolar quadric Q′ determines a pencil of simplexes inscribed in R_{2n-1} and self-polar with respect to Q′. Any simplex inscribed in R_{2n-1}, and given parametrically by a binary $2n$-ic α_t^{2n}, can be made to correspond to the common point of intersection of the osculating

* I am indebted to Dr. R. Vaidyanathaswamy for suggestion and criticism in the preparation of this paper.

† 'On the rational norm curve II' R. Vaidyanathaswamy *Jour. London. Math. Soc.* (1932.)

primes to a rational norm curve R_{2n} in S_{2n} at the points on it, given parametrically by a_t^{2n}. Then the pencil of simplexes determined by an inpolar quadric Q' corresponds to a line in S_{2n}. Dr. Vaidyanathaswamy has proved that

the lines in S_{2n} corresponding to inpolar quadrics Q' outpolar to a quadric envelope Q belong to a linear complex L.‡ ...(2·1)

In general a linear complex L in S_{2n} has one singular point, and it has been proved that

the singular point of L in S_{2n} corresponds to the simplex T inscribed in R_{2n-1} and circumscribed to Q. ...(2·2)

We shall now obtain the equation of the linear complex L and its singular point.

3. If ξ_t^{2n} and η_t^{2n} give parametrically two of the simplexes inscribed in R_{2n-1} and self-polar with respect to Q', Q' intersects R_{2n-1} at the points given by $(\xi\eta)\ \xi_t^{2n-1}\eta_t^{2n-1}$. Let the quadric envelope Q touch the osculating primes of R_{2n-1} at the points given by a_t^{4n-2}. Since Q is inpolar to R_{2n-1} and Q' is outpolar to Q, a_t^{4n-2} is apolar to $(\xi\eta)\ \xi_t^{2n-1}\eta_t^{2n-1}$‡. Hence

$$(\xi\eta)\ (a\xi)^{2n-1}(a\eta)^{2n-1}=0 \qquad ...(3·1)$$

If the simplex ξ_t^{2n} be represented by the point in S_{2n} whose coordinates are

$x_r=\xi_1^r\ \xi_2^{2n-r}$ $(r=0,\ 1,\ 2,\ldots\ldots 2n)$, the equation of the linear complex L corresponding to the quadric Q is

$$\sum A_{rs}p_{rs}=0 \qquad (r,\ s=0,\ 1,\ldots 2n)$$

where $\qquad p_{rs}=x_ry_s-x_sy_r,\quad A_{rs}=\alpha_r\alpha_{s-1}-\alpha_s\alpha_{r-1}$

and α_r is symbolically $(-1)^r\ {}_{2n-1}C_r\ a_1^{2n-r-1}a_2^r$ ...(3·2)

4. When the equation of the linear complex L in S_{2n} in Dr. Weitzenböck's complex symbolic notation § is $(Ap)^2=0$, the prime equation of its singular point is $(uA_1^2\ .\ .\ A_n^2)=0$ where $A,\ A_1\ .\ .\ .\ A_n$ are equivalent symbols. ...(4·1)

The points x and y are conjugate points with respect to L if $(Ax)(Ay)=0$. Hence the points y conjugate to the point given by $(uA_1^2\ldots\ A_n^2)=0$ lie on the prime $(AA_1^2\ldots\ A_n^2)\ (Ay)=0$. We shall now prove that $(AA_1^2\ldots A_n^2)\ (Ay)$ vanishes identically, so that the point given by $(uA_1^2\ .\ .\ A_n^2)$ is the singular point.

‡ Meyer: *Apolarität and rationale kurven.* Page 370.

§ Weitzenböck: *Invariantentheorie.* Pages 73—90.

To prove this, consider the following identity in ordinary symbols.

$$(a_1 a_2 a_3 \ldots\ldots a_{2n} a_{2n+1})(ay) =$$

$$(aa_2a_3\ldots a_{2n+1})(a_1y) - (aa_1a_3\ldots a_{2n+1})(a_2y) \;\ldots\; + \;(-1)^{2n}\;(aa_1\ldots a_{2n})(a_{2n+1}y)$$

To get the corresponding identity in complex symbols, we arrange the symbols in each term in the same order $a, a_1, \ldots, a_{2n}, a_{2n+1}$ and then substitute the complex symbols A for a and a_1, A_1 for a_2 and a_3, and so on thus we have

$$2\,(AA_1^2\ldots A_n^2)(Ay) + 2\,(A^2A_1\;A_2^2\ldots A_n^2)(A_1y)\ldots + 2\,(A^2A_1^2\ldots A^2_{n-1}\;A_n)\;(A_ny)$$

Let now $A, A_1 \ldots A_n$ be equivalent symbols. Then interchanging A and A_1 in the second term, A and A_2 in the third term and so on, we have

$$2(n+1)\;(AA_1^2\ldots A_n^2)\;(Ay) \equiv 0$$

Hence the point given by $(uA_1^2\ldots A_n^2) = 0$ is the singular point of the linear complex L.

5. From (2·2) the simplex T inscribed in R_{2n-1} and circumscribed to Q is given by the $2n$-ic b_t^{2n}, which corresponds in S_{2n} to the singular point of the linear complex L.

Hence

$$b_t^{2n} \equiv \begin{vmatrix} u_0 & u_1 & \ldots & u_{2n} \\ A_{1,0} & A_{1,1} & \ldots & A_{1,2n} \\ A_{1,0} & A_{1,1} & \ldots & A_{1,2n} \\ \ldots & \ldots & \ldots & \ldots \\ A_{n,0} & A_{n,1} & \ldots & A_{n,2n} \\ A_{n,0} & A_{n,1} & \ldots & A_{n,2n} \end{vmatrix} \quad \text{where } u_r = {}_{2n}C_r\; t_1^r\; t_2^{2n-r} \qquad \ldots(5\cdot1)$$

It remains to express b_t^{2n} in ordinary symbols.

Now $A_{rs} = A_rA_s = A_{1r}A_{1s} = \;\ldots\ldots = A_{nr}A_{ns}$

$$= \alpha_r\alpha_{s-1} - \alpha_s\alpha_{r-1} = \alpha_{1,r}\,\alpha_{1,s-1} - \alpha_{1,s}\alpha_{1,r-1} = \cdots = \alpha_{n,r}\,\alpha_{n,s-1} - \alpha_{n,s-1}\;\alpha_{n,r-1}$$

in equivalent symbols.

Hence substituting ordinary symbols for the complex symbols A_rA_s, we have the coefficient of the highest term namely t_1^{2n}, to be

$$\begin{vmatrix} \alpha_{10} & \alpha_{11} & \ldots & \ldots & \alpha_{1,2n-1} \\ 0 & \alpha_{10} & \ldots & \ldots & \alpha_{1,2n-2} \\ \alpha_{20} & \alpha_{21} & \ldots & \ldots & \alpha_{2,2n-1} \\ 0 & \alpha_{20} & \ldots & \ldots & \alpha_{2,2n-2} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \alpha_{n0} & \alpha_{n1} & \ldots & \ldots & \alpha_{n,2n-1} \\ 0 & \alpha_{n0} & \ldots & \ldots & \alpha_{n,2n-2} \end{vmatrix} \qquad \ldots(5\cdot2)$$

where $\quad \alpha_{s,r} = (-1)^r\;{}_{2n-1}C_n\;a_{s,1}^{2n-r-1} \times a_{s,2}^r = (-1)^r\;{}_{2n-1}C_r\;a_{s1}^{2n-r-1}$

if we suppose $\quad a_{12} = a_{22} = a_{s2}\;\ldots\; = a_{n,2} = 1.$

Lemma 1. The determinants of the matrix formed by the first two rows of (5·2) have each the factor a_{11}^2.

Taking the (r_1+1)th and the (r_2+1)th columns, the determinant is $(-1)^{r_1+r_2}\ {}_{2n}C_{r_1}\ {}_{2n}C_{r_2}\ (r_1-r_2)\ a_{11}^{4n-1-r_1-r_2}$. Since $r_1 \leqslant 2n-1$ and $r_2 \leqslant 2n-1$ and $r_2 \neq r_1$, the degree in $a_{11} \geqslant 2$.

Lemma 2. The determinants of the matrix formed by the first four rows have each the factor $(a_{11}-a_{21})^4$. Considering the determinant formed by the (r_1+1), (r_2+1), (r_3+1) and (r_4+1)th columns, we have on expanding, and omitting a numerical factor

$$\Sigma(r_1-r_2)\ (r_3-r_4)\ (a_{11}^{4n-1-r_1-r_2}\ a_{21}^{4n-1-r_3-r_4}+a_{11}^{4n-1-r_3-r_4}\ a_{21}^{4n-1-r_1-r_2})$$

the summation corresponding to the permutations (12)(34), (23)(14) and (31)(24). The above expression being homogeneous in a_{11} and a_{21} can be regarded as a polynomial in a_{11}/a_{21}. This as well as its first three derivatives with respect to a_{11}/a_{21}, can be shown to vanish when $a_{11}/a_{21}=1$, by making use of the identity

$$\Sigma\{(r_1+r_2)^k+(r_3+r_4)^k\}\{r_1-r_2\}\{r_3-r_4\}=0 \qquad [k=0,\ 1,\ 2,\ \text{or}\ 3.]$$

Hence $(a_{11}-a_{21})^4$ is a factor. Or if we introduce the variables a_{12}, a_{22} etc., the factor is $(a_{11}a_{22}-a_{12}a_{21})^4=(a_1a_2)^4$.

From Lemmas 1 and 2, and the symmetrical nature of (5·2) in the equivalent symbols, it follows that (5·2) has the factors

$$(a_1a_2)^4\ \ldots\ldots\ (a_1a_n)^4\ \ldots\ (a_{n-1}\ a_n)^4\ a_{11}^2\ .\ .\ a_{n1}^2.$$

Noting that in the above product, we have the requisite degree namely $4n-2$, in each of the equivalent symbols, (5·2) is but for a numerical factor, equal to

$$(a_1a_2)^4\ (a_1a_n)^4\ (a_{n-1}\ a_n)^4\ a_{11}^2\ .\ .\ a_{n1}^2.$$

This is the coefficient of the leading term in b_t^{2n}. It is evident from its form that it is a semi-invariant, the corresponding covariant b_t^{2n}, which is sought for in this paper being

$$(a_1a_2)^4\ (a_1a_3)^4\ .\ .\ .\ .\ (a_1a_n)^4\ .\ .\ .\ (a_{n-1}\ a_n)^4\ a_{1,t}^2\ a_{2,t}^2\ .\ .\ .\ .\ a_{n,t}^2.$$

Collineations in n-Space *

BY

S. KRISHNAMURTHY RAO, B.A. (HONS.)

The object of this paper is to study how quadrics and sub-regions of order k ($<n$) in a space $[n]$ are transformed by a given point-collineation. In other words, given the latent orders or the invariant-sequence of a point-collineation, we shall find the latent orders or the invariant-sequences of the collineations induced in

i. the ${}_nC_2 + n$ terms $x_1^2, x_1x_2, \ldots x_rx_s, \ldots$

and ii. the line-co-ordinates.

$(x_1, x_2, \ldots x_n)$ being the homogeneous co-ordinates of any point in the space $[n]$. We shall also find whether a given sub-region is latent or semi-latent with respect to the collineation induced in its regional co-ordinates by the given point-collineation and establish a formula for the species of that sub-region.

SECTION A

Transformation of Quadrics :

We shall, first, obtain the invariant-sequences of two types of collineations, for immediate application to the problem of quadrics.

1. TYPE I. Let

$$S_1: \quad x_1' = \alpha x_1 + x_2\,;\; x_2' = \alpha x_2 + x_3\,;\; \cdots\,;\; x_r' = \alpha x_r.$$
$$(s > r)$$
$$S_2: \; y_1' = \beta y_1 + y_2\,;\;\; y_2' = \beta y_2 + y_3\,;\; \cdots\,;\; y_s' = \beta y_s.$$

be two collineations in canonical form, in the two systems of variables, the x's and the y's, each having a single invariant-factor. We know that S_1 and S_2 can, for the sake of convenience, be thrown in the forms :—

$$S_1: x'_1 = \alpha x_1 + \alpha x_2\,;\; x'_2 = \alpha x_2 + \alpha x_3\,;\; \ldots\; x'_r = \alpha x_r$$
$$(s > r)$$
$$S_2: y_1' = \beta y_1 + \beta y_2\,;\; y_2' = \beta y_2 + \beta y_3\,;\; \ldots\; y_s' = \beta y_s$$

Let S be the collineation induced by S_1 and S_2 in the rs quantities $x_k y_l$ $(k = 1, 2, \cdots r\,;\; l = 1, 2, \cdots s.)$

* I am greatly indebted to Dr. R. Vaidyanathaswamy for suggesting this problem to me and kindly discussing it with me.

It is clear that S has a single characteristic-root $\alpha\beta$, and that our problem will not, in any way, be affected by putting $\alpha\beta=1$. Now the matrix S can be written.

$$\begin{vmatrix} C & C & O & O & . & . & O \\ O & C & C & O & . & . & O \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ O & . & . & . & O & C & C \\ O & . & . & . & O & O & C \end{vmatrix} \qquad (1)$$

which is a matrix of order r in the matrix elements O and C, O denotes a null-matrix of order s and C stands for the matrix

$$\begin{vmatrix} 1 & 1 & 0 & . & . & 0 \\ 0 & 1 & 1 & . & . & 0 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 0 & 0 & . & 0 & 1 & 1 \\ 0 & 0 & . & . & 0 & 1 \end{vmatrix}$$

of order s. Again it is clear that S is equivalent to the matrix

$$\begin{vmatrix} C & I & O & . & . & . & O \\ O & C & I & O & . & . & O \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ O & . & . & . & O & C & I \\ O & . & . & . & O & O & C \end{vmatrix} \qquad (2)$$

of order r in the matrix-elements O, C and I, where I denotes the unit-matrix of order s.

2. Consider the following matrix M of order rs. Let D_1^k represent the kth element in the first over-diagonal of M. Let each element in the principal diagonal be unity, and let

$$D_1^s=0\,; \quad D_1^{2s-1}=D_1^{2s}=0\,; \quad D_1^{3s-2}=D_1^{3s-1}=D_1^{3s}=0, \quad \cdots\cdots\cdot;$$
$$D_1^{rs-r+1}=D_1^{rs-r+2}= \quad \cdots \quad = D_1^{rs}=0\,;$$

every other element of D_1 being unity. The D_s of M is obtained by replacing the unit elements of D_1 by zeros and the zero elements by unity, and then moving D_1 parallel to itself into the position of D_s. All the other elements of M are zero. If in M, we bring together the first s^{th}, $(2s)^{th}$, $(3s)^{th}$, ... , $(rs)^{th}$ rows and columns together without affecting the other rows and columns, then M will be in the form of a direct product of a canonical matrix with the single invariant-factor $(\lambda-1)^{s+r-1}$, and another matrix M_1 which is of the same form as M but of order $(r-1)(s-1)$. Applying this process successively, we obtain the invariant-sequence of M to be

$$(s+r-1),\ (s+r-3),\ (s+r-5),\ \cdots,\ (s-r+1) \qquad (r \text{ terms}).$$

3. We shall now establish by the method of induction that S can be reduced to the form M. Let R be the minor-matrix of S obtained by cancelling the last s rows and the last s columns. R is of the same form as S but of order $(r-1)s$.

Assume that the invariant-sequence of R is

$$s+r-2,\ s+r-4,\ \cdots,\ s-r+2. \qquad (r-1)\text{terms}$$

Then there exists a matrix Z such that $Z^{-1}RZ=M'$, where M' is a matrix of the same form as M but of order $(r-1)s$. It is also clear that Z is in its normal form. From (2) of §1, we see that the collineation S can be written in the form

$$x'_{(k-1)s+l}=x_{(k-1)s+l}+x_{(k-1)s+l+1}+x_{k+l} \qquad [l=1,\ 2,\ \cdots\ (s-1)]$$
$$x'_{ks}=x_{ks}+x_{p(k+1)s} \qquad (k=1,\ 2,\ \cdots,\ r).$$

p being zero for $k=r$ and unity for $k=1,\ 2,\ \cdots\ (r-1)$.

First apply to the collineation S the scheme of transformation.

$$\xi_k=[z_{kk}x_k+\ \cdots\ +z_{k,(r-1)s}\,x_{(r-1)s}]$$
$$-[z_{k,(r-2)s+1}x_{(r-1)s+1}+z_{k,(r-2)s+2}x_{(r-1)s+2}+\ \cdots\ +z_{k,(r-1)}\,x_{rs}]$$
$$k=1,\ 2,\ \cdots,\ (r-1)s\,;$$
$$\xi_{(r-1)s+l}=x_{(r-1)s+l} \qquad [l=1,\ 2,\ \ldots,\ s.]$$

Then the matrix S will be transformed into T, in which the elements outside of the boundary of the minor-matrix R remain unaltered, while R itself is transformed into M'. That is, the collineation T, can be written as follows:

$$(a)\quad \begin{cases} x'_{(k-1)s+l}=x_{(k-1)s+l}+x_{(k-1)s+l+1} & [l=1,\ 2,\ \cdot\ (s+k)] \\ x'_{(k-1)s+m}=x_{(k-1)s+m}+x_{ks+m} & \\ \quad (m=s-k+1,\ s-k+2,\ \ldots,\ s) & (k=1,\ 2,\ \ldots\ r-2) \end{cases}$$

$$(b)\quad \begin{cases} x'_{(r-2)s+l}=x_{(r-2)s+l}+x_{(r-2)s+l+1}+x_{(r-1)s+l} & \\ & [l=1,\ 2,\ \ldots,\ (s-1)]. \\ x'_{(r-1)}=x_{(r-1)s}+x_{rs} & \end{cases}$$

$$(c)\quad \begin{cases} x'_{(r-1)s+l}=x_{(r-1)s+l}+x_{(r-1)s+l+1} & [l=1,\ 2,\ \ldots,\ (s-1)] \\ x'_{rs}=x_{rs}. & \end{cases}$$

Secondly, applying to T the transformation

$$(s-r+2)\,\xi_{(r-2)s+k}=x_{(r-2)s+k}+(k-1)x_{(r-1)s+k-1}$$
$$[k=1,\ 2,\ \ldots,\ (s-r+2)]$$

the other variables being unchanged, we obtain the collineation T_1, which is the same as T except for the equations (b), which are now of the form

$$x'_{(r-2)s+k}=x_{(r-2)s+k}+x_{(r-2)s+k+1} \qquad [k=1,\ 2,\ \ldots,\ (s-r+1)].$$
$$x'_{(r-2)s+l}=x_{(r-2)s+l}+x_{(r-1)s+l} \qquad [l=s-r+2,\ s-r+3,\ \ldots,\ s]$$

Finally by means of the transformation

$$\xi_{(r-l)s+s-r+k} = x_{(r-l)s+s-r+k} - x_{(r-l-1)s+s-r+k+1} \qquad (l=1,\ 2,\ \ldots\ k-1)$$
$$\xi_{(r-k)s+m} = x_{(r-k)s+m} - x_{(r-k-1)s+m+1}$$
$$[m=s-r+k-1,\ s-r+k-2,\ \ldots\ldots,\ 1];\ [k=1,\ 2,\ \ldots\ (r-1)]$$

the other variables being unchanged, T_1 will be reduced to the form M. Thus we have shown that the matrix S, of order rs, can be reduced to the form M, on the assumption that the minor-matrix R of order $(r-1)s$, which is of the same form as S, can be reduced to the form M′, which is of the same form as M but of order $(r-1)s$. That is, we have shown that the invariant-sequence of $S(rs)$ is

$$s+r-1,\quad s+r-3,\ \ldots\ldots,\quad s-r+1 \qquad (r \text{ terms})$$

on the assumption that this law of sequence holds for all values of r up to $r-1$. But we know that for the case $r=2$, $s=s$, the sequence is $s+1$, $s-1$. Hence, by induction, we obtain the invariant-sequence of the type I collineation to be

I. $(s+r-1),\ (s+r-3),\ (s+r-5),\ \ldots,\ (s-r+1)$ $[r \text{ terms } s>r]$

4. Collineations of Type II

Let Σ be the collineation of Type II, induced by

$$x'_1=\alpha x_1+\alpha x_2;\quad x'_2=\alpha x_2+\alpha x_3;\quad \ldots;\quad x'_r=\alpha x_r.$$

in the ${}_{r+1}C_2$ terms x_k^2, $x_k x_{k+1}$, $x_k x_{k+2}$, $\ldots$, $x_k x_r$ arranged in the order $(k=1,\ 2,\ \ldots\ r)$.

Also let S be the collineation of type I, induced by

$$x'_1=\alpha x_1+\alpha x_2;\quad x'_2=\alpha x_2+\alpha x_3;\quad \ldots;\quad x'_r=\alpha x_r$$
$$y'_1=\alpha y_1+\alpha y_2;\quad y'_2=\alpha y_2+\alpha y_3;\quad \ldots;\quad y'_r=\alpha y_r$$

in the r^2 quantities arranged in the order,

$$x_k y_1,\ x_k y_2,\ \ldots\ldots,\ x_k y_r \qquad (k=1,\ 2,\ \ldots,\ r).$$

It is clear that the matrix Σ can be derived from the matrix S by the following process:

Add the

$$(r+1)^{th};\quad (2r+1)^{th},\ (2r+2)^{th};\quad (3r+1)^{th},\ (3r+2)^{th},\ (3r+3)^{th};\ \ldots$$
$$\ldots;\quad \{(r-1)r+1\}^{th},\ \{(r-1)r+2\}^{th},\ \ldots\ldots,\ (r^2-1)^{th} \qquad (3)$$

columns to the

$$2^{nd};\ 3^{rd},\ (r+3)^{th};\ 4^{th},\ (r+4)^{th},\ (2r+4)^{th};\ \ldots\ r^{th},\ (2r)^{th},\ (3r)^{th},\ \ldots,\ (r-1)r^{th} \qquad (4)$$

columns respectively, and then cancelling the rows and columns indicated in (3). Therefore Σ is equivalent to the minor-matrix of S obtained by cancelling the rows and columns indicated in (3). Also, since the rows and columns of S and M correspond as they are, $\sum$ is equivalent to the minor-matrix $\sum_1$ of M obtained by cancelling in M the rows and columns indicated in (3). Now, if in $\sum_1$, of order $(r+1)r/2$, we bring together the r^{th}, $(2r-1)^{th}$, $(3r-3)^{th}$, ... $\frac{1}{2}(r+1)r^{th}$ rows and columns, without affecting the other rows and columns, then $\sum_1$ will be reduced to a direct product of a Canonical matrix with the single invariant-factor of index $(2r-1)$ and a matrix $\sum_2$ of order $(r-1)(r-2)/2$, which is of the same form as $\sum_1$, but only $r=r-2$. Hence by successive application of this process, we will finally obtain the invariant-sequence of $\sum_1$ i.e. of $\sum$ (type II) to be

$$\text{II.} \quad \begin{cases} 2r-1,\ 2r-5,\ 2r-9,\ \ldots,\ 1 & [(r+1)/2 \text{ terms if } r \text{ is odd}], \text{ and} \\ 2r-1,\ 2r-5,\ 2r-9,\ \ldots,\ 3 & [r/2 \text{ terms if } r \text{ is even}] \end{cases}$$

Here it may be observed that the sequences I and II are but an extension of the results of Art. 2, Chapter VII in Hilton's '*Linear Substitutions*'.

5. Problem of Quadrics

Let the given collineation, possessing any number of characteristic-roots and any number of invariant-factors, be taken in its canonical form C,

$$x_1' = \mu x_1 + x_2 ; \quad x'_2 = \mu x_2 + x_3 ; \quad . . . ; \quad x'_m = \mu x_m$$

$$\mu = \alpha : \quad m = a_1 ; \quad m = a_2 ; \ . . . . .$$

$$\mu = \beta : \quad m = b_1 ; \quad m = b_2 ; \ . . . . .$$

$$. . . . . . . \text{ etc } . . . . . .$$

where $a_1 \geqslant a_2 \geqslant a_3 \geqslant . . . ; \quad b_1 \geqslant b_2 \geqslant b_3 \geqslant . . . ;$ etc.

Then it is clear that the matrix of the collineation Q induced by C in the quantities $(x_r x_s)$ is a compartite matrix, some of whose components are of type I and the rest are of type II. Hence to obtain the invariant-sequence of Q, we have only to apply the following

THEOREM * The invariant-sequence of a compartite matrix is the compound of the invariant-sequences of its component parts.

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* c. f. § 10. (A). of 'The Invariant-factors and Integer-sequences' by R. Vaidyanathaswamy, *Jour. Ind. Math. Soc.*—June, 1924.

SECTION B

6. Transformation of Lines

If $(x_1, x_2, \ldots x_n)$ and $(y_1, y_2, \ldots y_n)$ be any two points on a line, then the co-ordinates of the line are defined to be the ${}_nC_2$ quantities

$$(x_k y_l - x_l y_k) \qquad (k, l = 1, 2, \ldots n)$$

Let C be the Canonical form of a given collineation and let A (of order r) and B (of order s) be two components the collineation in the x's and let A′ (of order r) and B′ (of order s) be the corresponding components of the collineation in the y's (co-gredient with the x's).

$$A \equiv x'_1 = \alpha x_1 + \alpha x_2 ; \quad x'_2 = \alpha x_2 + \alpha x_3 ; \; \ldots ; \quad x'_r = \alpha x_r$$

$$B \equiv x'_{r+1} = \beta x_{r+1} + \beta x_{r+2} ; \quad \ldots . ; \quad x'_{r+s} = \beta x_{r+s}$$

$$A' \equiv y'_1 = \alpha y_1 + \alpha y_2 ; \quad y'_2 = \alpha y_2 + \alpha y_3 ; \; \ldots ; \quad y'_r = \alpha y_r$$

$$B' \equiv y'_{r+1} = \beta y_{r+1} + \beta y_{r+2} ; \quad \ldots . ; \quad y'_{r+s} = \beta y_{r+s} .$$

Let L′ be the collineation induced in the ${}_{r+s}C_2$ quantities

$$(x_k y_l - x_l y_k) \qquad (k, l = 1, 2, \ldots , r+s)$$

by these two components. Then it is evident that L′ is a direct product of three collineations L_1', L_2', L_3', where

(1) L_1' is the collineation induced by A and A′ in the quantities

$$(x_k y_l - x_l y_k) \qquad [k, l = 1, 2, \cdots r]$$

(2) L_2' is the collineation induced by B and B′ in the quantities

$$(x_k y_l - x_l y_k) \qquad [k, l = r+1, r+2, \cdots r+s]$$

3. L_3' is induced by A, B ; A′, B′ in the quantities

$$(x_k y_l - x_l y_k) \qquad [k = 1, 2, \cdots r ; l = r+1, r+2, \cdots r+s]$$

On actually writing down the matrices L_1', L_2', L_3', it will be found that L_1' and L_2' are each in their canonical forms having the single invariant-factors of indices ${}_rC_2$ and ${}_sC_2$ respectively; while L_3' is of type I, so that its invariant-sequence is

$$s+r-1, \; s+r-3, \; \cdots , \; s-r+1 \qquad (r \text{ terms} ; \; s > r).$$

Now, since the collineation L, induced by C in the ${}_nC_2$ line-co-ordinates, can be expressed as a direct product of components of the types L_1', L_2', L_3', the invariant-sequence of L is easily found.

SECTION C

7. Transformation of sub-regions of a given space $[n]$.

Let $(e_1, e_2, \ldots e_n)$ be the n reference points of the given space $[n]$. Let Δ be the given point-collineation with the single latent root α and let

$$r_1, r_2, \ldots, r_m$$

be its latency-sequence. Also let the first r_1 reference points be latent, the second r_2 be semi-latent points of the first species, and so on. Now arrange the first r_1 latent points in the first row, the second r_2 semi-latent points of the first species in the second row, and so on, so that points in the same column are corresponding points.† With this array of reference points in view, we can denote any reference point by the double-suffix notation e_{pq}, where $(p+1)$ is the number of the row, and q the number of the column in which the point appears. The first suffix p denotes the species of the point and the second suffix q gives all its corresponding points. (Points in the same column have the same second suffix q and are corresponding points). So we can write

$$\Delta \; e_{pq} = \alpha e_{pq} + e_{p-1,\, q}$$

8. Now consider a sub-region of order k defined by the k points

$$e_{p_1 q_1}, e_{p_2 q_2}, \ldots, e_{p_k q_k}$$

We shall study how this sub-region is transformed by Δ *i.e.* we shall find a formula for the species of this region. Let

$$R = e_{p_1 q_1} e_{p_2 q_2} \ldots e_{p_k q_k},$$

where the right member is a progressive product of the e's. We know that, if k is the least positive integer such that

$$(\Delta - \alpha)^k e = 0,$$

then e is a semi-latent point of the $(k-1)^{th}$ species. Also, if k is the least positive integer such that

$$(\Delta - \alpha^p)^k \; R = 0,$$

where R is a sub-region of order p, then R is a semi-latent sub-region of the $(k-1)^{th}$ species. With these fundamental principles, we shall prove the following

THEOREM I: If $E_\mu = e_{p_1 q} e_{p_2 q} \ldots e_{p_s q}$

is a semi-latent sub-region of species μ and order s, then its species μ is given by the formula

$$p_1 + p_2 + \ldots + p_s - {}_sC_2.$$

† If e and e' are two corresponding points with respect to Δ, then there exists an integer $k > 0$, such that, e being supposed to be of higher species than e', $(\Delta - \alpha)^k e = e^1$.

There is no loss of generality if we assume that

$$p_1 < p_2 < p_3 \ldots . < p_s.$$

for such an arrangement of the e's only affects the sign of the progressive product, and still represents the same sub-region of order s.

Let $\qquad E_v = e_{p_1 q}\, e_{p_2 q} \ldots . e_{p_{s-1} q}$

we shall now find the species of $E_\mu = E_v\, e_{p_s q}$, given v and p_s. Assume that the species of E_μ is given by the formula

$$v + p_s - l \ldots \ldots \tag{5}$$

for all values of v from 0 up to $v-1$, and for all values of p_s from $(s-1)$ up to $(p_s - 1)$. If this hypothetical formula is to be true, it must also be true for the case $v = 0$ and $p_s = (s-1)$. But the species of $E_0\, e_{s-1,\,q}$ is zero, for the only form for E_0 is $(e_{0q}\, e_{1q} \ldots e_{s-1,\,q})$

Hence we have $\quad 0 + (s-1) - l = 0. \qquad$ i.e. $\quad l = s - 1.$

Now $\Delta\, E_v\, e_{p_s q} = (\beta\, E_v + E_{v-1})\,(\alpha e_{p_s q} + e_{p_s - 1, q})$

where $\quad \beta = \alpha^{s-1}$. Hence

$$(\Delta - \beta\alpha)\, E_v\, e_{p_s q} = \beta E_v\, e_{p_s - 1,\,q} + \alpha\, E_{v-1}\, e_{p_s q} + E_{v-1}\, e_{p_s - 1,\,q}$$

Multiplying both sides by $(\Delta - \beta\alpha)^{v + p_s - l}$, we have

$$(\Delta - \beta\alpha)^{v+p_s-l+1}\, E_v\, e_{p_s q} = (\Delta - \beta\alpha)^{v+p_s-l}\,(\beta E_v\, e_{p_s-1,\,q} + \alpha E_{v-1}\, e_{p_s q}) \tag{6}$$

since, by hypothesis,

$$(\Delta - \beta\alpha)^{v+p_s-l}\, E_{v-1}\, e_{p_s-1,\,q} = 0.$$

Again the right member of (6) is equal to

$$(\Delta - \beta\alpha)^{v+p_s-l-1} \Big\{ \beta(\beta E_v\, e_{p_s-2,\,q} + \alpha E_{v-1}\, e_{p_s-1,\,q}) + \alpha(\beta E_{v-1}\, e_{p_s-1,\,q} \;+\; \alpha E_{v-2}\, e_{p_s,\,q}) \Big\}$$

$$= (\Delta - \beta\,\alpha)^{v+p_s-l-1} \Big\{ \beta^2\, E_v\, e_{p_s-2,\,q} + \alpha^2\, E_{v-2}\, e_{p_s q} \Big\}$$

since, by hypothesis,

$$(\Delta - \beta\alpha)^{v+p_s-l-1}\, E_{v-1}\, e_{p_s-1,\,q} = 0.$$

By proceeding in this manner, we obtain the general equation

$$(\Delta - \beta\alpha)^{v+p_s-l+1-k}(\beta^k E_v\, e_{p_s-k,\,q} + \alpha^k E_{v-k}\, e_{p_s q}) \tag{7}$$

where E_{v-k} is a sub-semi-latent* sub-region of the $(v-k)^{th}$ species corresponding to E_v and is a linear combination of regions of the type

$$e_{p_1-l_1,\,q}\; e_{p_2-l_2,\,q} \dots e_{p_{s-1}-l_{s-1},\,q}$$

where the l's are suitable integers such that the species of each such region does not exceed $(v-k)$

When $k=p_s$, the right member of (7) is

$$(\Delta-\beta\alpha)^{v-l+1}\;(\beta^{p_s} E_v\, e_{oq} \;+\; \alpha^{p_s}\, E_{v-p_s}\, e_{p_s q})$$

But $(\Delta-\beta\alpha)^{v-l+1} E_v\, e_{oq} = (\Delta-\beta\alpha)^{v-l}\, E_{v-1}\, e_{oq}$

$$= 0 \quad \text{(by hypothesis)}$$

$$\therefore \qquad (\Delta-\beta\alpha)^{v+p_s-l+1}\, E_v\, e_{p\,q} = \alpha^{p_s}(\Delta-\beta\alpha)^{v-l+1} E_{v-p_s}\, e_{p_s q} \qquad \dots(8)$$

$$(\Delta-\beta\alpha)^{v-l+1} E_{v-p_s}\, e_{p_s q} = (\Delta-\beta\alpha)^{v-l}\{\beta E_{v-p_s}\, e_{p_s-1,\,q} + \alpha E_{v-p_s-1},\, e_{p_s q}$$

$$+ E_{v-p_s-1}\, e_{p_s-1,\,q}\}$$

$$= (\Delta-\beta\alpha)^{vl}\; \alpha\; E_{v-p_s-1}\, e_{p_s q}$$

Since, by hypothesis $(\Delta-\beta\alpha)^{v-l}\, E_{v-p_s}\, e_{p_s-1,\,q} = 0$

and $\qquad (\Delta-\beta\alpha)^{v-l}\, E_{v-p-1}\, e_{p_{s-1}\,q} = 0$

By proceeding in this manner, we obtain the general equation

$$(\Delta-\beta\alpha)^{v-l+1}\; E_{v-p_s}\; e_{p_s q} = (\Delta-\beta\alpha)^{v-k-l+1}\;\alpha^k\; E_{v-p_s-k}\; e_{p_s q} \qquad \dots(9)$$

When $k=v-p_s$, the right member of (9) $= \alpha^{v-p_s}\,(\Delta-\beta\alpha)^{p_s-l+1}\; E_o\, e_{p_s q}$

$$= \alpha^{v-p_s+1}\;(\Delta-\beta\alpha)^{p_s-l}\; E_o\; e_{p_s-1,\,q}$$

$$= 0 \qquad\qquad \text{(by hypothesis)}$$

Thus we have proved that the formula (5), where $l=(s-1)$, holds good for the case $v=p_s$ also. But we know that it is true for the case $v=0$ and $p_s=(s-1)$. Hence it is true universally, the inequality

$$p_1 < p_2 < p_3 \;\dots\;\dots < p_s$$

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* If E_v is a semi-latent sub-region of order $(s-1)$ and species v, then $(\Delta-\alpha^{s-1})\,E_v = E_{v-1}$, $(\Delta-\alpha^{s-1})^2 = E_{v-2}$, ... $(\Delta-\alpha^{s-1})^k\, E_v = E_{v-k}$
The regions E_{v-1}, E_{v-2}, E_{v3}, ... E_{v-k} ... are called the sub-semi-latent sub-regions corresponding to E_v

being always maintained. Hence it follows that the species of

(i) $e_{p_1q}\ e_{p_2q}\ \cdots$ is p_1+p_2-1

(ii) $e_{p_1q}\ e_{p_2q}\ e_{p_3q}\ \cdots$ is $p_1+(p_2-1)+(p_3-2)$

$\cdot\ \cdot\ \cdot\ \cdot$ etc.

$(s-1)$ $e_{p_1q}\ e_{p_2q}\ \cdots\ e_{p_sq}$ is $p_1+(p_2-1)+(p_3-2)+\ \cdots+(p_s-\overline{s-1})$

Thus we have proved that the species of

$$e_{p_1q}\ e_{p_2q}\ \cdots\ e_{q_sq}$$

is given by

$$p_1+p_2+\ \cdots\ +\ p_s-{}_sC_2$$

9. Next we shall prove the following

THEOREM II: If $E_m=e_{p_1q_1}\ e_{p_2q_2}\ \cdots\ e_{p_kq_k}$ be a sub-region of order k, in which s of the q's are equal, the other q's being all different, then the species of E_m is given by

$$p_1+p_2+\ \cdots\ +p_k-{}_sC_2$$

There is no loss of generality in assuming the first s q's to be equal, so that

$$E_m=e_{p_1q}\ e_{p_2q}\ \cdots\ e_{p_sq}\ e_{p_{s+1}q_{s+1}}\ \cdots\ e_{p_kq_k}$$

$$=E_v\ e_{p_{s+1}q_{s+1}}\ \cdots\ e_{p_kq_k}$$

First it can be easily proved, by the method of induction similar to that in § 8, that the species of $E_ve_{p_{s+1}q_{s+1}}$ is given by $v+p_{s+1}$

Then we obtain the species of $E_v\ e_{p_{s+1}q_{s+1}}\ e_{p_{s+2}q_{s+2}}$ to be $v+p_{s+1}+p_{s+2}$

and so on, so that, finally, we derive that the species of E_m is given by

$$v+p_{s+1}+p_{s+2}+\ \cdots\ +p_k$$

$$=p_1+p_2+\ \cdots\ +p_s-{}_sC_2+p_{s+1}+p_{s+2}+\cdots\ +p_k$$

$$=\quad p_1+p_2+\ \cdots\ +\ p_k-{}_sC_2$$

Now we immediately deduce the most general

THEOREM III: If $E_m=e_{p_1q_1}\ e_{p_2q_2}\ \cdots\ e_{p_kq_k}$ be a sub-region of order k, in which s_1 of the q's are equal to a_1; s_2 of the q's equal to a_2; $\cdots\cdots$; s_l of the q's equal to a_l, where

$$s_1+s_2+\ \cdots\ +s_l\leqslant k,$$

then the species of E_m is given by the formula

$$p_1+p_2+\ \cdots\ +p_k-({}_{s_1}C_2+{}_{s_2}C_2+\ \cdots\ +{}_{s_l}C_2)$$

10. Since this geometrical method of investigating the transformation of sub-regions of a given space [n] by a given point-collineation fails to obtain easily the latent orders of the collineation induced in regional-co-ordinates by the point-collineation, it will be interesting to know whether the algebraical method, indicated in the problem of the line-co-ordinates, can also be pursued in the case of the regional co-ordinates. However, we shall conclude this paper by explaining the duality between the two collineations induced in the co-ordinates of a given sub-region S_k of order k and in the co-ordinates of the complementary sub-region S_{n-k} of order $(n-k)$. Let S_k and S_{n-k} be defined by the first k and the last $(n-k)$ of the points

$$(x_{11}, x_{12}, \ldots x_{1n})\,;\ (x_{21}, x_{22} \ldots x_{2n})\ (x_{n1}, x_{n2} \ldots x_{nn})$$

respectively. Then the co-ordinates $(x_1, x_2, \ldots x_m)(m = {}_nC_k)$ of S_k are the ${}_nC_k$ square determinants of order k formed from the first k rows of the above array. If A be the matrix of the collineation in point-co-ordinates and a its determinant, and if the co-ordinates $y_1, y_2, \ldots y_m$ of S_{n-k} are taken to be the minor determinants complementary to $x_1, x_2, \ldots x_m$, each divided by a, then it is easily seen that

$$x_1y_1 + x_2y_2 + \cdots + x_my_m$$

is an absolute invariant with respect to the collineation A. Hence the two sets of variables, the x's and the y's, are contragredient with respect to the transformation A. Therefore the two collineations, induced in the x's and the y's by the collineation A, have the same invariant-sequence, so that we have the following important

THEOREM IV: If P and Q be the collineations induced by a given point-collineation in the co-ordinates of a sub-region S_k of order k and in the co-ordinates of the complementary sub-region S_{n-k} of order $(n-k)$, then P and Q have the same invariant-sequence.

Oriented Circles

BY

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§1. In order to secure precision in the treatment of angles and common tangents between circles, we assign to the radius of a circle a positive or negative sign and introduce the concept of orientation. A circle is said to be oriented positively or negatively and has a positive or negative radius according as it is described by a point moving along the circumference in the anti-clockwise or clock-wise direction. Dr. J. L. Coolidge † has given a treatment of the oriented circle by both geometrical and analytical methods, the analytic domain being the complex cartesian plane including the line at infinity. We investigate here certain results using the concept of the signed radius r, and a parameter t in the cartesian plane. We write the equation of an oriented circle in the parametric form :

$$x = a + 2tr/(1+t^2), \qquad y = a_1 + r(1-t^2)/(1+t^2)$$

where the radius r may be a positive or negative number and t takes in order, all real values from $+\infty$ to $-\infty$. This circle is briefly referred to as (a, a_1, r).

The appropriateness of the parametric form lies in the fact that 'r' occurs therein explicitly *in the first degree.*

§2. Consider two oriented circles of radii r_1, r_2 given by

$$x = a + 2tr_1/(1+t^2), \quad y = a_1 + r_1(1-t^2)/(1+t^2) \qquad ...(1)$$

$$x = a + 2tr_2/(1+t^2), \quad y = a_2 + r_2(1-t^2)/(1+t^2) \qquad ...(2)$$

The X-axis is taken as the radical axis of the two circles, so that

$$a_1^2 - r_1^2 = a_2^2 - r_2^2 \qquad ...(3)$$

† *Vide* J. L. COOLIDGE, *A Treatise on the Circle and the Sphere,* Chap. X. This work will hereafter be referred to as 'COOLIDGE'.

The equations of the tangents at 't_1' on (1) and 't_2' on (2), ($t_1 \neq t_2$), are respectively:

$$\begin{vmatrix} x-a & y & 1 \\ r_1 & t_1(a_1-r_1) & t_1 \\ t_1 r_1 & a_1+r_1 & 1 \end{vmatrix} = 0 \qquad \text{...(4)}$$

and

$$\begin{vmatrix} x-a & y & 1 \\ r_2 & t_2(a_2-r_2) & t_2 \\ t_2 r_2 & a_2+r_2 & 1 \end{vmatrix} = 0 \qquad \text{...(5)}$$

If (4) and (5) meet on the radical-axis, we have

$$\frac{a_1+r_1-t_1^2(a_1+r_1)}{t_1} = \frac{a_2+r_2-t_2^2(a_2-r_2)}{t_2}$$

Hence
$$t_1 t_2 = \frac{t_1(a_2+r_2)-t_2(r_1+a_1)}{t_1(r_1-a_1)-t_2(r_2-a_2)}$$

$$= \frac{r_2+a_2}{r_1-a_1} = \frac{r_1+a_1}{r_2-a_2} \quad \text{by virtue of (3)}$$

Thus, when the tangents at 't_1', 't_2', meet on the radical-axis,

$$t_1 t_2 = \frac{r_1+a_1}{r_2-a_2} \quad \text{(a constant)}\dagger \qquad \text{...(6)}$$

and the relation between the parameters is symmetrical.

Conversely, when $t_1 t_2 = k$ (a constant) we can associate with each proper tangent to the oriented circle (a, a_1, r_1) a unique proper tangent to a definite oriented circle (a, a_2, r_2) such that the two tangents meet on a given oriented line, which is taken as the X-axis.

For, we have only to put

$$r_2+a_2 = k(r_1-a_1)\,; \qquad r_2-a_2 = \frac{1}{k}(r_1+a_1)$$

and get

$$r_2 = \frac{1}{2}\,r_1\left(k+\frac{1}{k}\right) - \frac{1}{2}a_1\left(k-\frac{1}{k}\right); \qquad a_2 = \frac{1}{2}r_1\left(k-\frac{1}{k}\right) - \frac{1}{2}a_1\left(k+\frac{1}{k}\right) \qquad \text{...(7)}$$

§**3.** The process by which the oriented circle (a, a_1, r_1) is transformed into the other oriented circle (a, a_2, r_2) as in §2 is the

† *Vide* COOLIDGE p. 354.

well-known Laguerre Transformation or Laguerre Inversion†. We wish to point out that this transformation is algebraically equivalent to the linear involutory transformation of the type‡

$$x = -\alpha x' + \beta y' \; ; \quad y = -\beta x' + \alpha y' \qquad \ldots(8)$$

where $\alpha = \frac{1}{2}\left(k + \frac{1}{k}\right)$; $\beta = \frac{1}{2}\left(k - \frac{1}{k}\right)$; and $\alpha^2 - \beta^2 = 1$.

Hence, we may deduce the properties, such as invariants, of Laguerre Inversion easily from those of the linear transformation (8)

$$\left.\begin{aligned} \text{Obviously } x^2 - y^2 &= x'^2 - y'^2 && \ldots \text{ (i)} \\ xx_1 - yy_1 &= x'x'_1 - y'y'_1 && \ldots \text{ (ii)} \\ (x - x_1)^2 - (y - y_1)^2 &= (x' - x'_1)^2 - (y' - y'_1)^2 && \ldots \text{ (iii)} \\ (y - y')/(x - x') &= \beta/(\alpha + 1) && \ldots \text{ (iv)} \end{aligned}\right\} \quad (9)$$

From (9), we infer easily the following:

(i) The linear transformation (8) changes the rectangular hyperbola $x^2 - y^2 = a^2$ into itself and the tangent at any point on it into the tangent at the corresponding point on the same hyperbola; the locus of the intersection of the tangents at the corresponding points is a diameter of the hyperbola;

(ii) The line joining corresponding points is parallel to a given direction.

The result (9), (iii) may be interpreted as the well-known theorem:

The common proper tangential segment of two oriented circles remains invariant for Laguerre Inversion.

Hence, *if two oriented circles have proper contact, their transforms also have proper contact.*

In particular, *an oriented circle and a point on it transform into circles having proper contact.*

Again, $x = x'$ and $y = y'$ when $x/y = \beta/(1 + \alpha) = (k - 1)/(k + 1)$.

Hence, *an oriented circle* (a, a_1, r_1) *can be transformed into itself*, by choosing k such that $r_1 + a_1 = k(r_1 - a_1)$

Also, by choosing k such that $a_1 + r_1 = k^2(a_1 - r_1)$, we can transform *the oriented circle* (a, a_1, r_1) *into a non-linear null-circle*; for if

$$x/y = \alpha/\beta = (k^2 + 1)/(k^2 - 1), \text{ then } y' = 0.$$

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† COOLIDGE P. 355.

‡ We may conveniently put $\alpha = \cosh\theta$, $\beta = \sinh\theta$.

Corresponding to the two values of k, there are two point-circles, which are the images of each other in the fundamental line and are evidently the limiting points of the co-axal system to which the given circle and the fundamental line belong.

§**4.** THEOREM : *Three oriented circles simultaneously transform into three other oriented circles of the same radius, only with respect to any oriented line parallel to the line joining the proper centres of similitude of the given circles taken in pairs.*

In particular, the transforms degenerate into point-circles, when the fundamental line coincides with the line containing the proper centres of similitude.

Suppose the given oriented circles (a, a_1, r_1), (b, a_2, r_2), and (c, a_3, r_3) transform into three oriented circles of radius ρ, the constant of transformation being k.

Then
$$\begin{aligned} 2\rho &= r_1\ (k+1/k) - a_1(k-1/k) \\ &= r_2\ (k+1/k) - a_2(k-1/k) \\ &= r_3\ (k+1/k) - a_3(k-1/k) \end{aligned}$$

whence we easily derive

$$k^2 = (a_1 - a_2 + r_1 - r_2)/(a_1 - a_2 - r_1 + r_2) = \text{two similar expressions} \quad \ldots(10)$$

and
$$a_1(r_2 - r_3) + a_2(r_3 - r_1) + a_3(r_1 - r_2) = 0 \quad \ldots(10)'$$

If the centres of the three oriented circles be taken as the vertices of a triangle of reference for areal co-ordinates, we may write the equations of the fundamental line and the line of proper centres of similitude, respectively as

$$a_1x + a_2y + a_3z = 0 \ldots \quad \ldots(11)$$

and
$$r_1x + r_2y + r_3z = 0 \ldots \quad \ldots(12)$$

The condition (10)′ shows that the lines given by (11) and (12) are parallel; *i.e.* the fundamental line is parallel to the line of proper centres of similitude.

Conversely, we may take for our fundamental line any line, say $p_1x + p_2y + p_3z = 0$, parallel to (12) and choose k such that

$$k^2 = \frac{p_1 - p_2 + r_1 - r_2}{p_1 - p_2 - r_1 + r_2} = \frac{p_2 - p_3 + r_2 - r_3}{p_2 - p_3 - r_2 + r_3} = \frac{p_3 - p_1 + r_3 - r_1}{p_3 - p_1 - r_3 + r_1} \quad \ldots(13)$$

where p_1, p_2, p_3 are the algebraic distances of the centres of the given circles (or the vertices of the triangle of reference) from

$$p_1x + p_2y + p_3z = 0.$$

Now, each of the given oriented circles transforms into a circle of radius ρ, given by

$$\rho = \frac{1}{k} \frac{p_1 r_2 - p_2 r_1}{p_1 - p_2 - r_1 + r_2} = \text{two similar expressions.} \qquad \ldots(14)$$

When $\rho = 0$, $\frac{p_1}{r_1} = \frac{p_2}{r_2} = \frac{p_3}{r_3}$, *i.e.* the line of proper centres of similitude coincides with the fundamental line; and vice-versa.

As an immediate application of the above theorem, we point out a solution of the Gergonne problem, viz. to draw a circle to touch three given circles.

Regarding the three given circles as oriented circles we first transform them into three non-linear null-circles or points, and then re-transform the circle through these three points. The final circle thus obtained has proper contact with the given oriented circles.

(*N. B.*)—(1) As observed already towards the end of § 3, there are two triads of points into which the given oriented circles can be transformed, and these are images of each other in the fundamental line. Hence the circles through them are also images of each other in the same line, which is therefore their radical-axis. Further these circles are coaxal with their transforms. Thus, an axis of similitude of three circles is the radical axis of two of the circles touching them.

(2) The eight possible circles are obtained by taking all the possible combinations in the orientation of the given circles.

§5. We now proceed to give an important group of dual theorems in circle-geometry, the duality being exhibited between the angle of intersection of two circles and their common proper tangential segment. It is also interesting to notice that the group of theorems involving angles and the duals thereof can be proved by nearly analogous reasoning making use of the corresponding dual principles—ordinary inversion and Laguerre-inversion.

Definition: If A is a point of intersection of two oriented circles C_p, C_q, we define A_{pq} as the angle through which the proper tangent to C_p at A has to be turned in the anti-clockwise direction so as to coincide with the proper tangent to C_q at the same point.

Let B be the other point of intersection of the two circles; then it is easily seen that

$$A_{pq} + B_{pq} = A_{pq} + A_{qp} = B_{pq} + B_{qp} = 0 \pmod{2\pi} \qquad \ldots(15)$$

Dually, $$l_{pq}+m_{pq}=l_{pq}+l_{qp}=m_{pq}+m_{qp}=0 \qquad \ldots(16)$$
where l, m are the proper common tangents to the oriented circles C_p, C_q, and l_{pq} represents the common proper tangential segment measured from the point of contact of C_p to that of C_q in the direction of the proper tangent l; l_{pq} will be considered positive or negative according as the direction of measurement coincides with or is opposite to that of the proper tangent.

Dual Theorems

THEOREM I (*a*): *If three oriented circles C_1, C_2, C_3 meet in P, then* $P_{12}+P_{23}+P_{31}\equiv 0 \pmod{2\pi}$.

THEOREM I (*b*): *If three oriented circles C_1, C_2, C_3 have a common proper tangent p, then* $p_{12}+p_{23}+p_{31}=0$.

THEOREM II (*a*): *If three oriented circles C_1, C_2, C_3, be such that C_2, C_3 intersect in (A, A'), C_3, C_1 in (B, B') and C_1, C_2 in (C, C'), then* $A_{23}+B_{31}+C_{12}\neq 0 \pmod{2\pi}$.

This can be proved easily by elementary angle considerations, after inverting two of the circles into straight lines.

Similarly, we can prove that $A_{23}+B'_{31}+C_{12}\neq 0 \pmod{2\pi}$.

Since $B'_{31}\equiv -B_{31} \pmod{2\pi}$, by (15) above, $A_{23}-B_{31}+C_{12}\neq 0 \pmod{2\pi}$. Thus, we can prove that $A_{23}\pm B_{31}\pm C_{12}\neq 0 \pmod{2\pi}$, if the oriented circles C_1, C_2, C_3 do not have a common point.

THEOREM II (*b*): *If three oriented circles C_1, C_2, C_3 be such that C_2, C_3 have (a, a'), C_3, C_1 have (b, b'), and C_1, C_2 have (c, c') as their common proper tangents, then* $a_{23}\pm b_{31}\pm c_{12}\neq 0$.

This theorem is readily proved by transforming the three given circles into point-circles by Laguerre inversion, as in §4, and using the invariant property of the common proper tangential segments.

THEOREM III (*a*): *If P^1, P^2, P^3 be three non-collinear points such that the oriented circles C_1, C_2, C_3 pass respectively through the point-pairs (P^2, P^3), (P^3, P^1) and (P^1, P^2), and $P^1_{23}+P^2_{31}+P^3_{12}\equiv 0 \pmod{2\pi}$, then C_1, C_2, C_3 have a common point.*

This follows readily from Theorems I (*a*) and II (*a*).

M 27

THEOREM III (b): *If l^1, l^2, l^3 be three non-concurrent oriented lines, in the same plane such that the oriented circles C_1, C_2, C_3 have as their respective proper tangents the line-pairs (l^2, l^3), (l^3, l^1), (l^1, l^2) and $l^1_{23} + l^2_{31} + l^3_{12} = 0$, then C_1, C_2, C_3 have a common proper tangent.*

THEOREM IV (a): *If P^1, P^2, ... P^6 be any six coplanar points and a series of oriented circles C_1, C_2, ... C_6 drawn to pass through the triads (P^6, P^1, P^2), (P^1, P^2, P^3),......(P^5, P^6, P^1) respectively, then*

$$P^2_{13} + P^4_{35} + P^6_{51} + P^5_{24} + P^3_{46} + P^1_{62} \equiv 0 \quad (mod.\ 2\pi).$$

Since every three consecutive circles of the series C_1, C_2, ... C_6 taken in cyclic order, have a common point, we have the following relations:

$$P^2_{12} + P^2_{23} + P^2_{31} \equiv 0 \ (\text{mod. } 2\pi).$$

$$P^3_{23} + P^3_{34} + P^3_{42} \equiv 0 \quad ,,$$

$$P^4_{34} + P^4_{45} + P^4_{53} \equiv 0 \quad ,,$$

$$P^5_{45} + P^5_{56} + P^5_{64} \equiv 0 \quad ,,$$

$$P^6_{56} + P^6_{61} + P^6_{15} \equiv 0 \quad ,,$$

$$P^1_{61} + P^1_{12} + P^1_{26} \equiv 0 \quad ,,$$

Adding these equations and using the relations

$$P^2_{12} + P^1_{12} \equiv 0; \quad P^2_{31} + P^2_{13} \equiv 0; \quad (\text{mod. } 2\pi) \quad \text{etc.}$$

from (15) we get the required result.

The above theorem can be obviously extended to any number of coplanar points. If any three consecutive points in the series be collinear, the corresponding oriented circle becomes an oriented line.

Cor:

If $P^2_{13} + P^4_{35} + P^6_{51} \equiv 0$ (mod. 2π), then $P^3_{24} + P^5_{46} + P^1_{62} \equiv 0$ (mod. 2π).

Hence, if C_1, C_3, C_5 are concurrent, so also are C_2, C_4, C_6.

If C_1, C_3, C_5 degenerate into oriented lines, then evidently P^2_{13} is the angle between the oriented lines P^6P^2 and P^2P^4 and so on, so that

$$P^2_{13} + P^4_{35} + P^6_{51} \equiv 0 \pmod{2\pi}.$$

Hence, the circles C_2, C_4, C_6 meet in a point.

This result can be immediately recognised as the Miquel Theorem. †

THEOREM IV (b): *If* l^1, l^2, ... l^6 *be any six coplanar oriented lines and a series of oriented circles* C_1, C_2, ...C_6, be *drawn properly tangent to the triads* (l^6, l^1, l^2), (l^1, l^2, l^3), ...(l^5, l^6, l^1) *respectively then*

$$l^2_{18} + l^4_{35} + l^3_{51} + l^3_{24} + l^5_{46} + l^1_{62} = 0.$$

The proof of this theorem follows on dualising the steps employed in Th. IV (a).

This theorem also admits of an obvious extension to any number of coplanar lines.

Cor: If $l^2_{13} + l^4_{35} + l^6_{51} = 0$, then $l^3_{24} + l^5_{46} + l^1_{62} = 0$ †† i.e. if the oriented circles C_1, C_3, C_5 have a common proper tangent, then C_2, C_4, C_6 also have a common proper tangent.

We conclude with the remark that it is possible to draw a very large number of corollaries from our theorems IV (a) and (b) †††

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† COOLIDGE, p. 85. †† *Ibid.* p. 363. Theorem 8.
††† *Ibid.* pp. 364-366.

Equilateral Osculating Quadrics of Ruled Surfaces.

BY

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1. The object of this paper is to obtain the condition that the osculating quadric of a skew ruled surface be equilateral and to find a new expression and a new geometrical meaning for Laguerre's function.*

Several other new theorems on equilateral osculating quadrics are also obtained.

The results of this paper formed part of my dissertation for the degree of Ph. D. of the University of Dublin, and were obtained under the able guidance of Prof. C. H. Rowe of Trinity College, Dublin, to whom I am indebted for much assistance and advice.

2. Differential Equation of the Curved Asymptotic Lines.

Let the equations of the ruled surface be

$$x = p + lu,\ \ y = q + mu,\ \ z = r + nu$$

where p, q, r ; l, m, n are functions of v, the arc of the base curve. The fundamental magnitudes of the second order L, M, N are given by

$$\mathrm{LV} = [x_1, x_2, x_{11}] = [l, p' + ul', 0] = 0$$

where the notation $[x_1, x_2, x_{11}]$ denotes $\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_{11} & y_{11} & z_{11} \end{vmatrix}$

$$\mathrm{MV} = [x_1, x_2, x_{12}] = [l, p' + ul', l'] = \sum l(q'n' - r'm') \equiv \delta, \text{ say.}$$

$$\begin{aligned} \mathrm{NV} &= [x_1, x_2, x_{22}] = [l, p' + ul', p'' + ul''] \\ &= \sum l(q'r'' - r'q'') + u \sum l\{(m'r'' - m''r') + (q'n'' - n'q'')\} \\ &\quad + u^2 \sum l(m'n'' - n'm'') \\ &= \lambda + \mu u + \nu u^2, \text{ say,} \end{aligned}$$

where

$$\begin{aligned} \lambda &= \sum l(q'r'' - r'q'') \\ \mu &= \sum l\{(m'r'' - m''r') + (q'n'' - n'q'')\} \\ \nu &= \sum l(m'n'' - n'm''). \end{aligned}$$

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* See WEATHERBURN, *Differential Geometry*, Vol. II, (1930) p. 139.
DARBOUX, *Théorie générale des Surfaces*, t. II, (1915), p. 411.
A. J. MCCONNELL, *Applications of the Absolute Differential Calculus*, p. 217.
FORSYTH, *Differential Geometry*, (1912) p. 194.
BLASCHKE, *Vorlesungen über Differentialgeometrie*, Vol. I, p. 87.
GOURSAT, *Cours d'analyse*, t. I, 5 ed. p. 650.

Hence, the general equation of the asymptotic lines on the surface $Ldu^2 + 2Mdudv + Ndv^2 = 0$, reduces to $dv \,.\, (2Mdu + Ndv) = 0$. Hence the differential equation of the curved asymptotic lines of the ruled surface is

$$\frac{du}{dv} = -N/2M = -(\lambda + \mu u + \nu u^2)/2\delta$$

it being assumed that $\delta \neq 0$. (As we shall concern ourselves here with non-developable ruled surfaces, we shall make this assumption throughout, because $\delta = 0$ is the condition that the surface be developable). Thus

$$\frac{du}{dv} = \alpha + \beta u + \gamma u^2$$

where $$\alpha = -\frac{\lambda}{2\delta}, \qquad \beta = -\frac{\mu}{2\delta}, \qquad \gamma = -\frac{\nu}{2\delta}.$$

3. Osculating Quadrics of a Ruled Surface.

The tangent to the asymptotic line at the point (u,v) is given by

$$\frac{x-p-lu}{p'+l'u+(\alpha+\beta u+\gamma u^2)l} = \frac{y-q-mu}{q'+m'u+(\alpha+\beta u+\gamma u^2)m} = \frac{z-r-nu}{r'+n'u+(\alpha+\beta u+\gamma u^2)n}$$

When u varies, the point (u, v) describes a generator and the tangents form a quadric. Also since the generator and the tangent to the asymptotic line are two distinct lines that touch both the quadric and the ruled surface it follows that :—

> The quadric generated by the tangents to the curved asymptotic lines at their points of intersection with a generator touches the ruled surface all along that generator.*

4. Condition that the Osculating Quadrics of a Ruled Surface be equilateral.

We shall first obtain the required condition by taking for our base curve an orthogonal trajectory of the generators, so that $\Sigma lp' = 0$, and shall then apply the result to find the required condition when the base curve is any arbitrary curve.

The quadric osculating a ruled surface along a generator is equilateral if the two tangents to the curved asymptotics which are perpendicular to the generator lie in perpendicular planes through the generator, because then the generator and these two tangents are three mutually perpendicular generators of the osculating hyperboloid. Now the points on the generator at which the curved asymptotics

* BIANCHI, *Lezioni di Geometria Differenziale*. Vol. I. P. 394.

are perpendicular to the generator are given by the two values of u that satisfy $\alpha+\beta u+\gamma u^2=0$. These two values of u must belong to the involution formed by pairs of points along the generator the tangent planes at which are perpendicular and hence must be harmonic with respect to the roots of $1+2\text{B}u+\text{A}u^2=0$, the condition for which is $\gamma-\beta\text{B}+\alpha\text{A}=0$.

Hence

$\gamma-\text{B}\beta+\alpha\text{A}=0$ *is the condition that the osculating quadric of the ruled surface be equilateral.*

Substituting the values of α, β, γ in $\gamma-\beta\text{B}+\alpha\text{A}=0$,

we get $\sum l'^2[p', p'', l]-\Sigma l'p'[l', p'', l]=\Sigma l'p'[p', l'', l]-[l', l'', l]$,

i.e. $[p'\Sigma l'^2-l'\Sigma l'p', p'', l]=[p'\Sigma l'p'-l', l'', l]$.

$$\text{Since } p'\Sigma l'^2-l'\Sigma l'p'=p'(l'^2+m'^2+n'^2)-l'(l'p'+m'q'+n'r')$$
$$=m'(p'm'-l'q')-n'(l'r'-p'n'),$$

$$\text{the L.H.S.}=[m'(p'm'-l'q')-n'(l'r'-p'n'), p'', l]$$
$$=\Sigma\left\{\begin{vmatrix} m' & n' \\ l'r'-p'n' & p'm'-l'q'\end{vmatrix}\times\begin{vmatrix} q'' & r'' \\ m & n\end{vmatrix}\right\}$$
$$=\begin{vmatrix} l' & m' & n' \\ n'q'-r'm' & l'r'-p'n' & p'm'-l'q'\end{vmatrix}\times\begin{vmatrix} p'' & q'' & r'' \\ l & m & n\end{vmatrix}$$

by the usual rule for expressing the product of two rectangular arrays when the number of columns exceeds the number of rows, as a sum of determinants.*

$$=\begin{vmatrix} \Sigma p''l' & 0 \\ \Sigma p''(n'q'-r'm') & \Sigma l(n'q'-r'm')\end{vmatrix}^{**}=\Sigma p''l'.\Sigma l(q'n'-r'm').$$

$$\text{R.H.S.}=[p'\Sigma l'p'-l', l'', l]=[q'(p'm'-l'q')-r'(l'r'-p'n'), l'', l]$$
$$=\begin{vmatrix} p' & q' & r' \\ q'n'-m'r' & l'r'-p'n' & p'm'-l'q'\end{vmatrix}\times\begin{vmatrix} l'' & m'' & n'' \\ l & m & n\end{vmatrix}$$
$$=\begin{vmatrix} \Sigma p'l'' & 0 \\ \Sigma l''(q'n'-m'r') & \Sigma l(q'n'-m'r')\end{vmatrix}=\Sigma p'l''.\Sigma l(q'n'-m'r')$$

$\therefore\ \Sigma p''l'-\Sigma p'l''=0$, *which is the required condition*† *that the osculating quadric be equilateral, when the base curve is an orthogonal trajectory of the generators.*

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* See BURNSIDE and PANTON, *Theory of Equations,* Vol. II, p. 34.

** See BÔCHER, *Introduction to Higher Algebra.* p. 63.

† This condition retains this form also when v is not the arc, but any arbitrary parameter.

{ REMARKS :

(i) This condition is identically satisfied for a right-helicoid whose equations are $x=u$, $y=uf(v)$, $z=kv$. Hence,

A right-helicoid is a ruled surface such that all its osculating quadrics are equilateral.

(ii) Consider the ruled surface whose generators are parallel to a fixed plane. Then

$$\nu \equiv \sum l(m'n''-n'm'')=0$$

$\therefore$ The equation of the curved asymptotic lines reduces to the linear equation

$$\frac{du}{dv}+\frac{\lambda}{2\delta}+\frac{\mu}{2\delta}u=0.$$

Taking the fixed plane as the plane of xy, the equations of the surface can be written in the form

$$x=u, \qquad y=v+f(v)\,u, \qquad z=\mathrm{F}(v).$$

Here $\delta \equiv \sum l(q'n'-r'm') = -f'\,\mathrm{F}'$,

$$\mu \equiv \sum l\{(m'r''-m''r')+(q'n''-n'q'')\} = f'\mathrm{F}''-f''\,\mathrm{F}',$$

$$\lambda \equiv \sum l(q'r''-r'q'') = \mathrm{F}''.$$

$\therefore$ the equation of the curved asymptotic lines becomes

$$\frac{du}{dv}-\frac{\mathrm{F}''}{2f'\,\mathrm{F}'}-\frac{1}{2f'\,\mathrm{F}'}(f'\,\mathrm{F}''-f''\,\mathrm{F}')u=0.$$

The equations of the tangents to the curved asymptotic lines at the points (u, v) where they cut a generator v, are given by

$$\frac{x-u}{(du/dv)}=\frac{y-v-fx}{1+f'u}=\frac{\mathrm{Z}-\mathrm{F}}{\mathrm{F}'},$$

where v is constant, but u varies; and

$$\frac{du}{dv}=\frac{\mathrm{F}''}{2f'\,\mathrm{F}'}+\frac{1}{2f'\,\mathrm{F}'}(f'\,\mathrm{F}''-f''\,\mathrm{F}')u.$$

Eliminating u between these equations, the equation of the surface generated by the tangents to the curved asymptotic lines at the points where they meet a generator is

$$\frac{(y-v-fx)\mathrm{F}'-(z-\mathrm{F})}{(z-\mathrm{F})f'}=\frac{x\mathrm{F}'-(z-\mathrm{F})\,\mathrm{F}''/2f'\,\mathrm{F}'}{\mathrm{F}'+\frac{1}{2}(z-\mathrm{F})\left(\frac{\mathrm{F}''}{\mathrm{F}'}-\frac{f''}{f'}\right)} \quad \dots \quad (4\cdot1)$$

which being an equation of the second degree, represents a quadric.

This quadric is the quadric containing three consecutive generators of the ruled surface and is therefore an osculating hyperboloid, but as these generators are parallel to a plane, it is an osculating paraboloid.

Equating to zero the second degree terms in the equation of the Hyperbolic Paraboloid, the equations of the two director planes are

$$z=0, \text{ and } \{(y-fx)F'-z\}\cdot\frac{1}{2}\left(\frac{F''}{F'}-\frac{f''}{f'}\right)-f'\,F'x+\frac{zF''}{2F'}=0.$$

These two planes are at right angles if

$$-1\cdot\frac{1}{2}\left(\frac{F''}{F'}-\frac{f''}{f'}\right)+\frac{F''}{2F'}=0, \text{ i.e., if } f'=\text{const.}=c_1 \text{ (say)},$$

that is $f=c_1v+c_2$. *

In this case all the generators meet a line perpendicular to the z-plane. Hence any three generators determine an equilateral paraboloid.

The equations $x=u$, $y=v+(c_1v+c_2)u$, $z=F(v)$ show that for $u=-1/c_1$, we get $x=-1/c_1$, $y=-c_2/c_1$ which are the equations of the axis of the right-conoid.

Incidentally we get the

THEOREM: *If the osculating quadric of a ruled surface whose generators are parallel to a fixed plane is always an equilateral paraboloid, the ruled surface is a right conoid.*}

Let the base curve now be any arbitrary curve and let x, y, z be the co-ordinates of the point where the generator meets it, p, q, r being the co-ordinates of the point where the generator meets an orthogonal trajectory of the generators. Also let the parameter be arbitrary for the curve (x, y, z).

Then we have

$$p=x+Rl, \;\ldots\;\ldots\;\ldots\;\ldots$$
$$\therefore\; p'=x'+Rl'+lR', \;\ldots\;\ldots\;\ldots$$
$$p''=x''+Rl''+2l'R'+lR'', \;\ldots\;\ldots\;\ldots$$

The condition $\sum p''l'-\sum p'l''=0$, that the osculating quadric be equilateral becomes

$$\sum l'(x''+Rl''+2l'R'+lR'')-\sum l''(x'+Rl'+lR')=0$$

$$\text{ie. } \sum l'x''-\sum l''x'+2R'(\sum l'^2-\sum ll'')=0. \qquad \ldots(4\cdot2)$$

* Cf. *Nouvelles Annales de Mathematiques* 1924. P. 144.

But $\sum lp' = 0 \quad \therefore \quad \sum l(x' + Rl' + lR') = 0$, i.e. $R' = -\sum lx'$.
Putting this value of R′ in (4·2) we get

$$\sum x''l' - \sum x'l'' + 2\sum lx'(\sum ll'' - \sum l'^2) = 0,$$

which is the required condition that the osculating quadric be equilateral, when the base curve is any arbitrary curve.

When the arbitrary base curve is an orthogonal trajectory of the generators, $\sum lx' = 0$, and this reduces, as it should, to $\sum x''l' - \sum x'l'' = 0$.

5. A new geometrical meaning of Laguerre's function.

Take a curve on the surface as the base curve and let x, y, z be the co-ordinates of a point P on the base curve and X, Y, Z the direction cosines of the normal to the surface at P. Also let l_1, m_1, n_1; l_2, m_2, n_2; l_3, m_3, n_3 be the direction cosines of the tangent, principal normal and binormal to the base curve at P. Then if ϕ is the angle between the principal normal to the curve and normal to surface at P we have

$$X = l_2 \cos\phi + l_3 \sin\phi, \quad Y = m_2 \cos\phi + m_3 \sin\phi, \quad Z = n_2 \cos\phi + n_3 \sin\phi.$$

We know that the condition that the ruled surface formed by the normals along the curve has equilateral osculating quadrics is

$$\sum X''x' - \sum X'x'' = 0$$

Denoting differentiations with respect to s by dashes and making use of Frenet's formulae, we have

$$X' = -\cos\phi\left(\frac{l_3}{\sigma} + \frac{l_1}{\rho}\right) - l_2 \sin\phi\cdot\phi' + \sin\phi\cdot\frac{l_2}{\sigma} + l_3 \cos\phi\cdot\phi'$$

$$= -l_3 \cos\phi\left(\frac{1}{\sigma} - \phi'\right) + l_2 \sin\phi\left(\frac{1}{\sigma} - \phi'\right) - \frac{l_1}{\rho}\cos\phi. \qquad \text{...(5·1)}$$

$$\therefore \qquad \sum X'x'' \equiv \frac{1}{\rho}\sum X'l_2 = \frac{\sin\phi}{\rho}\cdot\left(\frac{1}{\sigma} - \phi'\right) \qquad \text{...(5·2)}$$

Also $\qquad \sum X'x' = \sum X'l_1 = -\dfrac{\cos\phi}{\rho}$,

$$\therefore \sum X'x'' + \sum X''x' = -\frac{d}{ds}\left(\frac{\cos\phi}{\rho}\right), \text{ i.e. } \sum X''x' = -\frac{d}{ds}\left(\frac{\cos\phi}{\rho}\right) - \sum X'x''.$$

Hence, (5·1) gives $\quad 2\sum X'x'' + \dfrac{d}{ds}\left(\dfrac{\cos\phi}{\rho}\right) = 0,$

i.e. $\quad \dfrac{d}{ds}\left(\dfrac{\cos\phi}{\rho}\right) + \dfrac{2\sin\phi}{\rho}\cdot\left(\dfrac{1}{\sigma} - \phi'\right) = 0$, from (5·2)

i.e. $$\frac{d}{ds}\left(\frac{1}{R}\right) + \frac{2}{\tau}\cdot\frac{1}{\gamma} = 0, \qquad \text{...(5·3)}$$

where R, τ, γ are the radii of normal curvature, geodesic torsion and geodesic curvature of the base curve. Conversely, if

$$\frac{d}{ds}\left(\frac{\cos\phi}{\rho}\right)+\frac{2\sin\phi}{\rho}\left(\frac{1}{\sigma}-\phi'\right)=0,$$

we have $\sum X''x'-\sum X'x''=0$. For

$$\frac{d}{ds}\left(\frac{\cos\phi}{\rho}\right)\equiv\frac{d}{ds}(\sum Xx'')=\sum Xx'''+\sum X'x'' \quad \ldots \quad (5{\cdot}4)$$

Also differentiating $\sum Xx'=0$ twice by Leibnitz Theorem, we get

$$\sum X''x'+2\sum X'x''+\sum Xx'''=0 \quad \ldots \quad (5{\cdot}5)$$

Subtracting (5·4) from (5·5) we get

$$\sum X''x'+\sum X'x''=-\frac{d}{ds}\left(\frac{\cos\phi}{\rho}\right).$$

$$\therefore\ \sum X''x'-\sum X'x''=-\frac{d}{ds}\left(\frac{\cos\phi}{\rho}\right)-2\sum X'x''$$

$$=-\frac{d}{ds}\left(\frac{\cos\phi}{\rho}\right)-\frac{2\sin\phi}{\rho}\left(\frac{1}{\sigma}-\phi'\right)$$

since $\sum X'x''=\frac{1}{\rho}\sum X'l_2=\frac{\sin\phi}{\rho}\left(\frac{1}{\sigma}-\phi'\right)$ from (5·2).

$\therefore\ \sum X''x'-\sum X'x''=0$ by hypothesis.

Thus we see that $\sum X''x'-\sum X'x''=0$ is equivalent to Laguerre's function * $\quad \mathcal{L}', =\frac{d}{ds}\left(\frac{1}{R}\right)+\frac{2}{\tau}\cdot\frac{1}{\gamma}=0.$

Hence

we get a new expression for Laguerre's function $\mathcal{L}$ viz. $\sum X''x'-\Sigma X'x''$ and a new geometrical meaning of Laguerre's function, viz. that its vanishing along a curve on the surface is the condition that the osculating quadric of the ruled surface formed by drawing normals to the surface along the curve be equilateral.

COROLLARY.—If the curve on the surface is a geodesic, $1/\gamma=0$, and also $1/R=(\cos\phi)/\rho=1/\rho$, since $\phi=0$ for a geodesic, therefore if $\mathcal{L}=0$ along the curve, we have ρ stationary. Hence

If the ruled surface formed by drawing normals to a surface along a curve on it has equilateral osculating quadrics and if the curve is a geodesic, the circular curvature of the curve will be stationary

* loc. cit.

6. Some other Theorems on Equilateral Osculating Quadrics

THEOREM I. *If a ruled surface R having equilateral osculating quadrics be taken, and if the ruled surface formed by the normals to R along an orthogonal trajectory of its generators also has equilateral osculating quadrics, then the orthogonal trajectory is of constant curvature.*

For if l, m, n are the direction cosines of the generator of R which meets the orthogonal trajectory at P (x, y, z), then since R has equilateral osculating quadrics, we have $\sum l''x' = \sum l'x''$

But $l = -l_2 \sin\phi + l_3 \cos\phi$ where l_1, m_1, n_1; l_2, m_2, n_2; l_3, m_3, n_3 are the direction cosines of the tangent, principal normal and binormal to the orthogonal trajectory at P, and ϕ is the angle between the principal normal (l_2, m_2, n_2) and normal to the surface.

$$\therefore\; l' = \sin\phi\left(\frac{l_3}{\sigma} + \frac{l_1}{\rho}\right) - l_2\cos\phi\cdot\phi' + \cos\phi\left(\frac{l_2}{\sigma}\right) - l_3\sin\phi\cdot\phi'.$$

$$\therefore\; \sum l'x'' \equiv \sum l'\cdot\frac{l_2}{\rho} = \frac{\cos\phi}{\rho}\left(\frac{1}{\sigma} - \phi'\right);\quad \sum l'x' = \sum l'l_1 = \frac{\sin\phi}{\rho}.$$

so that
$$\sum l'x'' + \sum l''x' = 2\sum l'x'' = \frac{d}{ds}\left(\frac{\sin\phi}{\rho}\right) \text{ since } \sum l'x'' = \sum l''x'.$$

$$\therefore\quad \frac{2\cos\phi}{\rho}\left(\frac{1}{\sigma} - \phi'\right) = \frac{d}{ds}\left(\frac{\sin\phi}{\rho}\right). \qquad \ldots (6\cdot1)$$

Also since the ruled surface formed by the normals to R along an orthogonal trajectory of the generators has equilateral osculating quadrics, we have

$$\frac{d}{ds}\left(\frac{\cos\phi}{\rho}\right) + \frac{2\sin\phi}{\rho}\cdot\left(\frac{1}{\sigma} - \phi'\right) = 0 \qquad \ldots (6\cdot2)$$

From (6·1) and (6·2), equating the values of $\left(\frac{1}{\sigma} - \phi'\right)$ we get

$$\frac{\rho}{2\cos\phi}\cdot\frac{d}{ds}\left(\frac{\sin\phi}{\rho}\right) = -\frac{\rho}{2\sin\phi}\frac{d}{ds}\left(\frac{\cos\phi}{\rho}\right).$$

that is
$$\sin\phi\cdot\frac{d}{ds}\left(\frac{\sin\phi}{\rho}\right) + \cos\phi\cdot\frac{d}{ds}\left(\frac{\cos\phi}{\rho}\right) = 0$$

that is
$$\sin\phi\cdot\left(\frac{\cos\phi}{\rho}\cdot\phi' - \frac{\sin\phi}{\rho^2}\cdot\rho'\right) + \cos\phi\cdot\left(-\frac{\sin\phi}{\rho}\cdot\phi' - \frac{\cos\phi}{\rho^2}\cdot\rho'\right) = 0$$

which gives $-\frac{1}{\rho^2}\cdot\rho' = 0$, that is $\rho = \text{constant}$. Hence the orthogonal trajectory is of constant curvature, which proves the theorem.

We proceed to examine whether the converse is true, i.e. if a ruled surface R possessing equilateral osculating quadrics be taken and if an orthogonal trajectory of the generators has constant curvature, then is it true that the ruled surface formed by the normals to R along the orthogonal trajectory also has equilateral osculating quadrics ?

Since R has equilateral osculating quadrics we have as before

$$\frac{d}{ds}\left(\frac{\sin \phi}{\rho}\right) = \frac{2 \cos \phi}{\rho} \cdot \left(\frac{1}{\sigma} - \phi'\right) \text{ from } \quad (6\text{·}1)$$

Since $\rho =$ constant, this gives $\quad \frac{1}{\rho} \cos \phi \left(3\phi' - \frac{2}{\sigma}\right) = 0.$

Hence either $1/\rho = 0$, or $\phi = \pi/2$, or $3\ \phi' - 2/\sigma = 0$.

Now if the ruled surface formed by the normals to R along the orthogonal trajectory also has equilateral osculating quadrics, we must have

$$\frac{d}{ds}\left(\frac{\cos \phi}{\rho}\right) + \frac{2 \sin \phi}{\rho} \cdot \left(\frac{1}{\sigma} - \phi'\right) = 0,$$

i.e.
$$\frac{\sin \phi}{\rho} \cdot \left(3\phi' - \frac{2}{\sigma}\right) = 0.$$

which is satisfied if $3\phi' - 2/\sigma = 0$. Hence,

THEOREM II. *If a ruled surface R with equilateral osculating quadrics has an orthogonal trajectory C of its generators which has constant curvature, the normals along C generate a second ruled surface with equilateral osculating quadrics except (possibly) when C is an asymptotic line on R.*

Again if the normals to a curve of constant curvature generate a ruled surface which possesses equilateral osculating quadrics, we have as before $\frac{\sin \phi}{\rho}\left(3\phi' - \frac{2}{\sigma}\right) = 0$ so that, either $\phi = 0$, or $3\ \phi' - 2/\sigma = 0$.

Hence either the curve of constant curvature is a geodesic, or for it $3\phi' - 2/\sigma = 0$. Disregarding the exceptional cases when $\phi = 0$, or $\pi/2$, we also get the following results :—

THEOREM III. *If two ruled surfaces intersect along a common orthogonal trajectory of their generators at a constant angle, and if this trajectory is of constant curvature, then neither or both have equilateral osculating quadrics.*

Conversely, if two ruled surfaces, both having equilateral osculating quadrics intersect at a constant angle along a common orthogonal trajectory, then this trajectory has constant curvature.

Otherwise

THEOREM IV. *If a ruled surface with equilateral osculating quadrics is formed by normals to a curve of constant curvature, we get a second ruled surface of the same kind if we rotate each generator through a constant angle in the normal plane of the curve.*

On a Method of Computing Gravity Anomalies.

BY

G. P. RAO, M.A., F.R.A.S.

In the *Transactions of the Cambridge Philosophical Society* (Vol. VIII, Part V, p. 694: 1849) Professor Stokes has given an elegant formula for computing the elevations of geoid above the spheroid of reference from known gravity anomalies. Following his method of transformation of a series of spherical harmonics into a definite integral, a method is herein developed of solving the converse problem, namely of computing gravity anomalies from the known elevations of geoid over the spheroid.

It is well known that if a distribution of matter is such as to make the surface $r = R\left(1 + \sum_{n=1}^{\infty} u_n\right)$ a level surface under the potentials of its rotation about the polar axis and the gravitational attraction, the value of gravity at the level surface is

$$g = G\left[1 - \frac{5}{2}m\left(\frac{1}{3} - \cos^2\theta\right) + \sum_{n=2}^{\infty}(n-1)u_n\right]$$

where R is mean radius of the surface, m is mass of internal matter, u_n is a Laplace function of order n, small compared with unity, and mG is centrifugal force at the equator. Let g_c be the gravity calculated according to Clairaut's Theorem which applies to a generalised oblate spheroid. Let r_c be the radius vector of the spheroid measured from the common centre of gravity of mass and volume. If $r = r_c + \Delta r$, and $g = g_c + \Delta g$, then

$$\Delta r = R(u_2 + u_3 + u_4 + \cdots\cdots) \qquad \ldots(1)$$

$$\Delta g = G(u_2 + 2u_3 + 3u_4 + \cdots\cdots) \qquad \ldots(2)$$

Now, as is well known, the geoid or the earth's sea-level surface is an equipotential surface nearly spheroidal in form. The elevations of the geoid above the spheroid of reference are known from the observations of plumb-line deflections at a great many stations scattered over the surface of the earth, and are usually exhibited in the form of contours for all the geodetically surveyed areas of the world.

It is required to determine the anomalies of gravity due to the known undulations of the geoid from the spheroid. Let $\Delta r = RF(\theta, \phi)$ and let α be the angle between the directions determined by the co-ordinates (r, θ, ϕ) and (r', θ', ϕ'). Also let x denote the ratio r'/r.

Then

$$(1 - 2x \cos\alpha + x^2)^{-\frac{1}{2}} = P_0 + P_1 x + P_2 x^2 + P_3 x^3 + \cdots \quad \text{...(3)}$$

where P_0, P_1, $P_2 \cdots$ are Legendre functions.

Expanding $F(\theta, \phi)$ in a series of Laplace's functions and equating terms of the same order

$$u_n = \frac{2n+1}{4\pi} \int_0^{\pi} \int_0^{2\pi} F(\theta', \phi') \; P_n \sin\theta' d\theta' d\phi \quad \text{...(4)}$$

Substituting in (2) we get

$$\Delta g = \frac{G}{4\pi} \int_0^{\pi} \int_0^{2\pi} F(\theta', \phi')[1{\cdot}5 \; P_2 + 2{\cdot}7 \; P_3 + 3{\cdot}9 \; P_4 + \cdots] \sin\theta' \, d\theta' \, d\phi' \quad \text{...(5)}$$

Denoting the series within the square brackets by S, it is easily seen that

$$S = 2 \sum_{n=2}^{\infty} n^2 P_n - \sum_{n=2}^{\infty} n P_n - \sum_{n=2}^{\infty} P_n \quad \text{...(6)}$$

Putting $x = 1$ in equation (3)

$$(2 - 2\cos\alpha)^{-\frac{1}{2}} = P_0 + P_1 + P_2 + P_3 + \cdots$$

since $P_0 = 1$ and $P_1 = \cos\alpha$, it follows

$$\sum_{n=2}^{\infty} P_n = \tfrac{1}{2} \operatorname{cosec} \alpha/2 - 1 - \cos\alpha \quad \text{...(7)}$$

Let $\displaystyle\sum_{n=2}^{\infty} n P_n \, x^{n-1} = \gamma$

$$\int_0^x \gamma dx = \sum_{n=2}^{\infty} P_n \, x^n = (1 - 2x \cos\alpha + x^2)^{-\frac{1}{2}} - P_0 - P_1 x$$

$$\gamma = \frac{d}{dx}\left[(1 - 2x\cos\alpha + x^2)^{-\frac{1}{2}} - 1 - x\cos\alpha\right]$$

$$= \frac{\cos\alpha - x}{(1 - 2x\cos\alpha + x^2)^{3/2}} - \cos\alpha$$

Putting $x = 1$ in the above, we get

$$\sum_{n=2}^{\infty} n P_n = -\tfrac{1}{4} \operatorname{cosec} \alpha/2 - \cos\alpha \quad \text{...(8)}$$

Again let $\sum_{n=2}^{\infty} n^2 P_n x^{n-1} = \sigma$

$$\int_0^x \sigma dx = \sum_{n=2}^{\infty} nP_n x^n = x \sum_{n=2}^{\infty} n P_n x^{n-1} = x\gamma$$

$$= \frac{x(\cos \alpha - x)}{(1-2x\cos\alpha + x^2)^{3/2}} - x \cos \alpha$$

$$\sigma = \frac{3x(\cos\alpha - x)^2}{(1-2x\cos\alpha+x^2)^{5/2}} + \frac{\cos\alpha - 2x}{(1-2x\cos\alpha+x^2)^{3/2}} - \cos\alpha$$

Putting $x=1$ in the above, we get

$$\sum_{n=2}^{\infty} n^2 P_n = \tfrac{1}{8} \operatorname{cosec} \alpha/2 - \tfrac{1}{8} \operatorname{cosec}^3 \alpha/2 - \cos\alpha \qquad ...(9)$$

Substituting (7), (8) and (9) in equation (6)

$$S = 1 - \frac{1}{4} \operatorname{cosec}^3 \frac{\alpha}{2}$$

Using this in equation (5) and replacing F (θ',ϕ') by $\Delta r/R$ we get

$$\Delta g = \frac{G}{R}\frac{1}{4\pi}\int_0^{\pi}\int_0^{2\pi} \Delta r \left(1 - \frac{1}{4}\operatorname{cosec}^3\frac{\alpha}{2}\right)\sin\theta' d\theta' d\phi' \qquad ...(10)$$

If the radius vector to the station at which the gravity anomaly is sought is taken as the axis of spherical polar co-ordinates, and if the angles α, β correspond with the θ, ϕ that refer to the north pole, then the expression for Δg becomes

$$\Delta g = \frac{G}{R}\frac{1}{2\pi}\int_0^{\pi}\int_0^{2\pi} \Delta r\, f(\alpha)\, d\alpha\, d\beta \qquad ...(11)$$

where $f(\alpha) = \tfrac{1}{2}\sin\alpha - \tfrac{1}{8}\sin\alpha \operatorname{cosec}^3 \alpha/2$.

In practice, the integration is effected by numerical summation. A series of concentric circles are drawn around the station at which the gravity anomaly is required, and within each of the zones bounded by these circles, an average value of Δr is determined. Let Δr_m be the mean value of Δr over an entire zone bounded by the radii α_1 and α_2, and let $f_m(\alpha)$ be the mean value of $f(\alpha)$ between the limits α_1 and α_2:

Then $$\Delta g = \frac{G}{R} \sum \Delta r_m . f(\alpha)(\alpha_2 - \alpha_1) \qquad ...(12)$$

If $\phi(\alpha)$ is a function such that $\int f(\alpha)\, d\alpha = \phi(\alpha)$

then
$$f_m(\alpha) = \frac{1}{(\alpha_2 - \alpha_1)} \int_{\alpha_1}^{\alpha_2} f(\alpha)\, d\alpha = \frac{1}{(\alpha_2 - \alpha_1)} [\, \phi(\alpha_2) - \phi(\alpha_1) \,]$$

where
$$\phi(\alpha) = \tfrac{1}{2} \text{ cosec } \alpha/2 - \tfrac{1}{2} \cos \alpha$$

Therefore, a working form of the formula is

$$\Delta g = \frac{G}{R} \sum \Delta r_m \; [\phi(\alpha_2) - \phi(\alpha_1)\,] \qquad \text{...(13)}$$

This formula enables us to compute the gravitational effects due to the deviations of the geoid from the spheroid of reference. When these effects are removed from the gravity anomalies reckoned from spheroidal formulæ, the residual anomalies are then more localised and are likely to be of industrial interest.

On the Equation of Heat Conduction in Wave-Mechanics

BY

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Introduction

It has been shown[1] that in a crystal, the equation of heat conduction according to Wave-Mechanics takes the non-linear form:

$$\text{I.} \qquad \sum_{r=1}^{3} \frac{\partial^2 u}{\partial x_r^2} - \beta\frac{\partial u}{\partial t} = \mathrm{P}(u, x_r, t),$$

where β is a constant and P is non-linear in u.

In this paper we consider a special boundary value problem for this type of equation, and prove the existence and uniqueness of the solution for the particular equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} - \frac{1}{q^2}\frac{\partial u}{\partial t} = u^2$$

for the boundary values:

$$u = 0 \text{ for } \begin{cases} x = 0, \ \pi \text{ and all } y, z; \\ y = 0, \ \pi \text{ and all } z, x; \\ z = 0, \ \pi \text{ and all } x, y; \end{cases}$$

$$u = f(x, y, z) \text{ for } t = 0 \text{ and all } x, y, z.$$

Finally, we remark that the method holds equally well for the more general equation I.

We consider the partial differential equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} - \frac{1}{q^2}\frac{\partial u}{\partial t} = u^2 \tag{1}$$

where q is a real finite constant, and determine its solution which is unique and regular[2] in the domain[3]

$$0 \leqslant x \leqslant \pi, \quad 0 \leqslant y \leqslant \pi, \quad 0 \leqslant z \leqslant \pi, \quad 0 \leqslant t, \tag{2}$$

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[1] R. Peierls: *Zeitschrift für Physik*, (1930).

[2] We say that a solution is regular when it is continuous in the whole domain, along with all its derivatives that enter into the differential equation.

[3] For simplicity, we have taken the length of the interval for x, y, z as π. It is obvious that any three constants a, b, c can be taken without any material change in the process.

and which satisfies the boundary conditions:

$$\left.\begin{array}{l} u=0 \text{ for } x=0 \text{ and } x=\pi \text{ for all } y,\ z \text{ and } t \text{ in (2)}, \\ u=0 \text{ for } y=0 \text{ and } y=\pi \text{ for all } x,\ z \text{ and } t \text{ in (2)}, \\ u=0 \text{ for } z=0 \text{ and } z=\pi \text{ for all } x,\ y \text{ and } t \text{ in (2)}, \end{array}\right\} \quad (3)$$

$$u=f(x, y, z) \text{ for } t=0 \text{ and all } x,\ y,\ z \text{ in (2)} \quad (4)$$

We assume that $f(x, y, z)$ can be expanded in a multiple Fourier Series [4]

$$f(x, y, z) = \sum_{l, m, n} c_{l, m, n} \sin lx \sin my \sin nz, \quad (5)$$

and that the series $\sum_{l, m, n} (l^2+m^2+n^2) \mid c_{l, m, n} \mid$ is convergent.

For the solution we write

$$u(x, y, z ; t) = \sum_{l, m, n} v_{l, m, n} (t) \sin lx \sin my \sin nz, \quad (6)$$

and assume, for the present, that the series in (3) is absolutely and uniformly convergent in the domain (2).

Evidently, (6) satisfies the conditions (3). If we determine $v_{l, m, n}(t)$ so that for all $l,m,n \geqslant 1$

$$v_{l, m, n} (0) = c_{l, m, n} \quad (7)$$

then the condition (4) will also be satisfied. Further, we have:

$$u^2(x, y, z ; t) = \sum_{l, m, n} F_{l, m, n} (t) \sin lx \sin my \sin nz \quad (8)$$

where

$$F_{l,m,n,}(t) = \frac{8}{\pi^3} \int_0^\pi \int_0^\pi \int_0^\pi u^2(\alpha, \beta, \gamma; t) \sin l\alpha \sin m\beta \sin n\gamma \, d\alpha d\beta d\gamma$$

$$= \sum_{\substack{\kappa\mu\lambda \\ \nu\sigma\rho}} a^{(l)}{}_{\kappa\nu} \; a^{(m)}{}_{\lambda\rho} \; a^{(n)}{}_{\mu\sigma} \; v(t)_{\kappa\lambda\mu} \; v(t)_{\nu\rho\sigma} \quad (9)$$

with

$$a^{(l)}{}_{k\nu} = \frac{2}{\pi} \int_0^\pi \sin k\alpha \sin \nu\alpha \sin l\alpha \, d\alpha. \quad (10)$$

We assume [5] now that the series

$$\sum_{l, m, n,} (l^2+m^2+n^2) v_{l, m, n}(t) \text{ and } \sum_{l, m, n,} dv_{l, m, n}/dt$$

---

[4] The summation extends from 1 to ∞, unless otherwise stated.

[5] Of course, all the above assumptions about the uniform convergence of the series shall be proved later.

are absolutely and uniformly convergent for all t. Then we get on substituting the series for u and u^2 in the differential equation (1)

$$-\sum_{l,m,n}(l^2+m^2+n^2)\,v_{l,m,n}(t)\sin lx\sin my\sin nz$$

$$-\frac{1}{q^2}\sum_{l,m,n}\frac{dv_{l,m,n}}{dt}\sin lx\sin my\sin nz$$

$$=\sum_{l,m,n}F_{l,m,n}(t)\sin lx\sin my\sin nz. \qquad (11)$$

Therefore

$$\frac{dv_{l,m,n}}{dt}+q^2(l^2+m^2+n^2)v_{l,m,n}(t)=-q^2F_{l,m,n}(t) \quad (l,m,n=1,2,\ldots) \qquad (12)$$

The solution of this differential equation which satisfies the boundary condition (7) is given by

$$v_{l,m,n}(t)=c_{l,m,n}e^{-q^2(l^2+m^2+n^2)t}-q^2\int_0^t e^{-q^2(l^2+m^2+n^2)(t-\delta)}F_{l,m,n}(\delta)d\delta,$$

$$=c_{l,m,n}\,e^{-q^2(l^2+m^2+n^2)t}-q^2\int_0^t e^{-q^2(l^2+m^2+n^2)(t-\delta)}d\delta \qquad (13)$$

$$\times\sum_{\substack{\kappa\lambda\mu\\ \nu\rho\sigma}}a^{(l)}_{\kappa\nu}\,a^{(m)}_{\lambda\rho}\,a^{(n)}_{\mu\sigma}\,v(\delta)_{\kappa\lambda\mu}\,v(\delta)_{\nu\rho\sigma}.$$

If we write

$$\phi_{l,m,n}(t)=(l^2+m^2+n^2)\,v_{l,m,n}(t)\;;\qquad \gamma_{l,m,n}=(l^2+m^2+n^2)\,c_{l,m,n} \qquad (14)$$

then we get

$$\phi_{l,m,n}(t)=\gamma_{l,m,n}\,e^{-q^2(l^2+m^2+n^2)t}-q^2(l^2+m^2+n^2)\int_0^t e^{-q^2(l^2+m^2+n^2)(t-\delta)}d\delta$$

$$\times\sum_{\substack{\kappa\lambda\mu\\ \nu\rho\sigma}}\frac{a^{(l)}_{\kappa\rho}\,a^{(m)}_{\lambda\rho}\,a^{(n)}_{\mu\sigma}}{(\kappa^2+\lambda^2+\mu^2)(\nu^2+\rho^2+\sigma^2)}\,\phi(\delta)_{\kappa\lambda\mu}\,\phi(\delta)_{\nu\rho\sigma}, \qquad (15)$$

$$(l,\ m,\ n=1,\ 2,\ \ldots).$$

This is a triply infinite system of non-linear integral equations for the determination of the $\phi_{l,m,n}(t)$. We solve this system by successive approximations, and for this purpose write

$$\phi^{(0)}_{l,m,n}(t)=\gamma_{l,m,n}\,e^{-q^2(l^2+m^2+n^2)t}. \qquad (16)$$

and for $r \geqslant 1$

$$\phi^{(r)}_{l,m,n}(t) = \gamma_{l,m,n}\, e^{-q^2(l^2+m^2+n^2)t} - q^2(l^2+m^2+n^2)\int_0^t e^{-q^2(l^2+m^2+n^2)(t-\delta)}\,d\delta$$

$$\times \sum_{\substack{\kappa\lambda\mu\\ \nu\rho\sigma}} \frac{a^{(l)}_{\kappa\nu}\, a^{(m)}_{\lambda\rho}\, a^{(n)}_{\mu\sigma}\, \phi^{(r-1)}_{\kappa\lambda\mu}(\delta)\, \phi^{(r-1)}_{\nu\rho\sigma}(\delta)}{(\kappa^2+\lambda^2+\mu^2)(\nu^2+\rho^2+\sigma^2)} \tag{17}$$

Lemma I.

For all k, ν, $l \geqslant 1$, $\quad |a^{(l)}_{\kappa,\nu}| / k^2\nu^2 \leqslant 8/l^2.$ (18)

We have $\quad a^{(l)}_{\kappa\nu} = \dfrac{2}{\pi}\displaystyle\int_0^\pi \sin k\alpha \sin \nu\alpha \sin l\alpha \, d\alpha.$

On integrating twice by parts, we get

$$a^{(l)}_{\kappa\nu} = -\frac{2}{\pi l^2}\int_0^\pi \sin l\alpha \frac{d^2}{d\alpha^2}(\sin k\alpha \sin \nu\alpha)\, d\alpha,$$

$$= -\frac{2}{\pi l^2}\int_0^\pi \sin l\alpha \{-(k^2+\nu^2)\sin k\alpha \sin \nu\alpha + 2k\nu \cos k\alpha \cos \nu\alpha\} d\alpha.$$

Therefore

$$\frac{|a^{(l)}_{\kappa\nu}|}{k^2\nu^2} \leqslant \frac{2}{\pi l^2}\frac{k^2+\nu^2}{k^2\nu^2}\int_0^\pi |\sin k\alpha \sin \nu\alpha \sin l\alpha| d\alpha$$

$$+ \frac{4}{\pi l^2}\frac{k\nu}{k^2\nu^2}\int_0^\pi |\cos k\alpha \cos \nu\alpha \sin l\alpha| d\alpha.$$

Now since k, ν are $\geqslant 1$, we have for all l, $\dfrac{|a^{(l)}_{\kappa\nu}|}{k^2\nu^2} \leqslant \dfrac{8}{l^2}$,

which proves our Lemma.

Lemma II. The series

$$\sum_{l,m,n} q^2(l^2+m^2+n^2)\int_0^t e^{-q^2(l^2+m^2+n^2)(t-\delta)} d\delta \left| a^{(l)}_{\kappa\nu} a^{(m)}_{\lambda\rho} a^{(n)}_{\mu\sigma} \right| \Big/ (k^2+\lambda^2+\mu^2)(\nu^2+\rho^2+\sigma^2) \tag{19}$$

is uniformly convergent for all t and all $\kappa, \lambda, \mu, \nu, \rho, \sigma \geqslant 1$.

We have, for all $t > 0$

$$\int_0^t e^{-q^2(l^2+m^2+n^2)(t-\delta)}\, d\delta = e^{-q^2(l^2+m^2+n^2)t} \int_0^t e^{q^2(l^2+m^2+n^2)\delta}\, d\delta$$

$$= \left\{ e^{-q^2(l^2+m^2+n^2)t} - 1 \right\} \Big/ q^2(l^2+m^2+n^2)\, e^{q^2(l^2+m^2+n^2)t}$$

$$\leqslant \frac{1}{q^2(l^2+m^2+n^2)} \tag{20}$$

Further we have, for a properly chosen constant A,

$$\frac{|a^{(l)}_{k\nu}\, a^{(m)}_{\lambda\rho}\, a^{(n)}_{\mu\sigma}|}{(\kappa^2+\lambda^2+\mu^2)(\nu^2+\rho^2+\sigma^2)} < A \frac{|a^{(l)}_{\kappa\nu}|}{\kappa^2\nu^2} \cdot \frac{|a^{(m)}_{\lambda\rho}|}{\lambda^2\rho^2} \cdot \frac{|a^{(n)}_{\mu\sigma}|}{\mu^2\sigma^2} < \frac{8^3\, A}{l^2m^2n^2} \tag{21}$$

from (18). Thus we see that the series (19) is less than

$$\sum_{l,m,n} q^2(l^2+m^2+n^2) \cdot \frac{1}{q^2(l^2+m^2+n^2)} \cdot \frac{8^3\, A}{l^2m^2n^2}$$

$$= 8^3\, A \,.\left(\sum_l \frac{1}{l^2}\right)^3 = \frac{8^3\, \pi^6}{6^3} = \frac{64}{27}\pi^6\, A. \tag{22}$$

Therefore the series (19) is uniformly convergent.

According to the hypothesis, the series $\sum\limits_{l,\, m,\, n} |\gamma_{l,m,n}|$ is convergent. We write

$$\sum_{l,\, m,\, n} |\gamma_{l,m,n}| = c. \tag{23}$$

From (17) we get, if $\sum\limits_{l,\, m,\, n} |\phi^{(r-1)}_{l,m,n}(t)|$ converges for all t;

$$\sum_{l,\, m,\, n} |\phi^{(r)}_{l,m,n}(t)| < c + \frac{64\pi^6 A}{27} \left\{ \max \sum_{l,\, m,\, n} |\phi^{(r-1)}_{l,m,n}(t)| \right\}^2, \tag{24}$$

which shows that $\sum\limits_{l,\, m,\, n} |\phi^{(r)}_{l,m,n}(t)|$ is also convergent for all t.

Now from (16) we have

$$\text{Max} \sum_{l,\, m,\, n} |\phi^{(0)}_{l,m,n}(t)| = c\,; \tag{25}$$

substituting this in (24) for $r = 1$ we get:

$$\sum_{l,\, m,\, n} |\phi^{(1)}_{l,m,n}(t)| < c + \frac{64\pi^6 A}{27} c^2. \tag{26}$$

We assume that $\qquad c < 27/256\pi^6 A < 1. \qquad (27)$

Then we get

$$\sum_{l,\, m,\, n} |\phi^{(1)}_{l,m,n}(t)| \leqslant 2c. \tag{28}$$

Substituting (28) in (24) for $r=2$, we get

$$\sum_{l,\,m,\,n} \left| \phi^{(2)}_{l,m,n}(t) \right| < c + \frac{64\pi^6 A}{27}\,(2c)^2 < 2c. \tag{29}$$

on account of (27).

In general we get for all $r \geqslant 1$,

$$\sum_{l,\,m,\,n} \left| \phi^{(r)}_{l,m,n} \right| < 2c < 1. \tag{30}$$

We shall prove now that the series

$$\sum_{r=0}^{\infty} \sum_{l,m,n}^{1 \text{ to } \infty} \left| \phi^{(r+1)}_{lmn}(t) - \phi^{(r)}_{lmn}(t) \right| \tag{31}$$

is uniformly convergent for all t.

We have from (17).

$$\phi^{(r+1)}_{lmn}(t) - \phi^{(r)}_{lmn}(t) = -q^2(l^2+m^2+n^2) \int_0^t e^{-q^2(l^2+m^2+n^2)(t-\delta)}\, d\delta$$

$$\times \sum_{\substack{\kappa\lambda\mu \\ \nu\rho\sigma}} \frac{a^{(l)}_{\kappa\nu}\, a^{(m)}_{\lambda\rho}\, a^{(n)}_{\mu\sigma}}{(\kappa^2+\lambda^2+\mu^2)\ (\nu^2+p^2+\sigma^2)} \left\{ \phi^{(r)}_{\kappa\lambda\mu}(\delta)\, \phi^{(r)}_{\nu\rho\sigma}(\delta) - \phi^{(r-1)}_{\kappa\lambda\mu}(\delta)\, \phi^{(r-1)}_{\nu\rho\sigma}(\delta) \right\} \tag{32}$$

$$= -q^2(l^2+m^2+n^2) \int_0^t e^{-q^2(l^2+m^2+n^2)(t-\delta)}\, d\delta \times \sum_{\substack{\kappa\lambda\mu \\ \nu\rho\sigma}} \frac{a^{(l)}_{\kappa\nu}\, a^{(m)}_{\lambda\rho}\, a^{(n)}_{\mu\sigma}}{(n^2-\lambda^2+\mu^2)\ (\nu^2+p^2+\sigma^2)}$$

$$\times \left\{ \phi^{(r)}_{\kappa\lambda\mu} \left(\phi^{(r)}_{\nu\rho\sigma} - \phi^{(r-1)}_{\nu\rho\sigma} \right) + \phi^{(r-1)}_{\nu\rho\sigma}\, \phi\left(^{(r)}_{\kappa\lambda\mu} - \phi^{(r-1)}_{\kappa\lambda\mu} \right) \right\}.$$

Therefore on account of (22) and (30), we get

$$\sum_{lmn} \left| \phi^{(r+1)}_{lmn}(t) - \phi^{(r)}_{lmn}(t) \right| < \frac{64\pi A^6}{27} \cdot 2c,\ \max. \sum_{lmn} \left| \phi^{(r)}_{lmn}(t) - \phi^{(r-1)}_{lmn}(t) \right|. \tag{33}$$

Repeating the same process r times, we get :

$$\sum_{lmn} \left| \phi^{(r+1)}_{lmn}(t) - \phi^{(r)}_{lmn}(t) \right| < \left(\frac{64\pi^6 A}{27} \times 2c \right)^r \max \sum_{lmn} \left| \phi^{(1)}_{lmn}(t) - \phi^{(0)}_{lmn}(t) \right| \tag{34}$$

and, therefore, summing over r from 0 to ∞, we have

$$\sum_{r=0}^{\infty} \sum_{lmn} \left| \phi^{(r+1)}_{lmn}(t) - \phi^{(r)}_{lmn}(t) \right| < \sum_{r=0}^{\infty} \left(\frac{128\pi^6 Ac}{27} \right)^r \max \sum_{lmn} \left| \phi^{(1)}_{lmn}(t) - \phi^{(0)}_{lmn}(t) \right|. \tag{35}$$

But on account of (27), $\frac{128\pi^6 Ac}{27} < 1$, and therefore

$$\sum_{r=0}^{\infty}\left(\frac{128\pi^6 Ac}{27}\right)^r = \frac{27}{27-128\pi^6 Ac}. \tag{36}$$

Moreover,

$$\phi^{(1)}_{lmn}(t) - \phi^{(0)}_{lmn}(t) = -q^2(l^2+m^2+n^2)\int_0^t e^{-q^2(l^2+m^2+n^2)(t-\delta)}d\delta$$

$$\times \sum_{\substack{\kappa\lambda\mu\\ \nu\rho\sigma}} \frac{a^{(l)}_{\kappa\nu}\; a^{(m)}_{\lambda\rho}\; a^{(n)}_{\mu\sigma}}{(\kappa^2+\lambda^2+\mu^2)(\nu^2+\rho^2+\sigma^2)} \cdot \phi^{(0)}_{\kappa\lambda\mu}(\delta)\; \phi^{(0)}_{\nu\rho\sigma}(\delta)$$

Therefore

$$\sum_{lmn} \left| \phi^{(1)}_{lmn}(t) - \phi^{(0)}_{lmn}(t) \right| \leqslant \frac{64}{27}\pi^6 Ac^2. \tag{37}$$

On substituting (36) and (37) in (35), we get:

$$\sum_{r=0}^{\infty} \sum_{l,m,n} \left| \phi^{(r+1)}_{l,m,n}(t) - \phi^{(r)}_{l,m,n}(t) \right| < \frac{27}{27-128\pi^6 Ac} \cdot \frac{64}{27}\pi^6 Ac^2 \tag{38}$$

which shows that the series (31) is uniformly convergent for all t.

From the convergence of (31) it follows that all the limits

$$\lim_{r\to\infty} \phi^{(r)}_{lmn}(t) \qquad (l, m, n = 1, 2, \ldots\ldots)$$

exist, and that on writing

$$\phi_{l,m,n}(t) = \lim_{r\to\infty} \phi^{(r)}_{l,m,n}(t) \tag{39}$$

the functions $\phi_{l,m,n}(t)$ are continuous for all $t \geqslant 0$.

We see also that

$$\sum_{l,m,n} \left| \phi_{l,m,n}(t) \right| < 2c < 1 \tag{40}$$

and

$$\phi(t) = \sum_{l,m,n} \phi_{l,m,n}(t) = \lim_{r\to\infty} \sum_{l,m,n} \phi^{(r)}_{l,m,n}(t) \tag{41}$$

From the equation (17) we get for $r \to \infty$,

$$\phi_{lmn}(t) = \gamma_{lmn} e^{-q^2(l^2+m^2+n^2)t} - q^2(l^2+m^2+n^2)\int_0^t e^{-q^2(l^2+m^2+n^2)(t-\delta)} d\delta$$

$$\times \sum_{\substack{\kappa\lambda\mu\\ \nu\rho\sigma}} \frac{a^{(m)}_{\kappa\nu}\; a^{(l)}_{\lambda\rho}\; a^{(n)}_{\mu\sigma} \quad \phi_{\kappa\lambda\mu}(\delta)\; \phi_{\nu\rho\sigma}(\delta)}{(\kappa^2+\lambda^2+\mu^2)(\nu^2+\rho^2+\sigma^2)} \tag{42}$$

We put now, for all $l,m,n \geqslant 1$.

$$v_{l,m,n}(t) = \frac{1}{l^2+m^2+n^2}\,\phi_{l,m,n}(t)\,; \qquad c_{l,m,n} = \frac{1}{l^2+m^2+n^2}\,\gamma_{l,m,n} \tag{43}$$

The function $v_{l,m,n}(t)$ satisfies the integral equation (13), and therefore the differential equation (12)

Moreover, on account of (18) and (30) we see from (9) that the series $\sum\limits_{l,m,n} | F_{l,m,n}(t) |$ is uniformly convergent for all t. Also the series

$$\sum_{l,m,n} (l^2+m^2+n^2) \mid v_{l,m,n}(t) \mid = \sum_{l,m,n} \mid \phi_{l,m,n}(t) \mid$$

is convergent from (40). This proves the uniform convergence of all the series mentioned in the beginning.

We conclude therefore that

$$u\,(x,\,y,\,z\,;\,t) = \sum v_{l,\,m,\,n}(t) \sin lx \sin my \sin nz \tag{44}$$

is the required solution of equation (1).

We proceed now to show that this solution (44) is the only one of its kind which can be expressed as a Fourier series whose coefficients are such that the series

$$\sum_{l,m,n} (l^2+m^2+n^2) \mid v_{l,m,n}(t) \mid \text{ and } \sum_{l,m,n} \mid d\,v_{l,m,n}/dt \mid$$

are uniformly convergent.

If possible, let the Integral equation (15) have solutions $\bar{\phi}_{lmn}(t)$ $(l,\,m,\,n=1,\,2,\,.\,.\,.\,.\,.\,)$ other than the $\phi_{lmn}(t)$ found already in (39), and suppose that the series $\sum\,|\bar{\phi}_{lmn}(t)|$ uniformly converges, and

$$\sum_{lmn} \mid \bar{\phi}_{lmn}(t) \mid = \bar{c} < 2c. \tag{45}$$

Let us consider the series $\sum\limits_{lmn} |\bar{\phi}_{lmn}(t) - \phi^{(r+1)}_{lmn}(t)|$. We have for all $l,\,m,\,n \geqslant 1$:

$$\phi^{(r+1)}_{lmn}(t) = \gamma_{lmn}\, e^{-q^2(l^2+m^2+n^2)t} - q^2(l^2+m^2+n^2) \int_0^t e^{-q^2(l^2+m^2+n^2)(t-\delta)}\, d\delta$$

$$\times \sum_{\substack{u\lambda\mu\\ \nu\rho\sigma}} \frac{a^{(l)}_{u\nu}\, a^{(m)}_{\lambda\rho}\, a^{(n)}_{\mu\sigma}}{(k^2+\lambda^2+\mu^2)(\nu^2+\rho^2+\sigma^2)}\, \phi^{(r)}_{\kappa\lambda\mu}(\delta)\, \phi^{(r)}_{\nu\rho\sigma}(\delta)\,,$$

$$\bar{\phi}_{lmn}(t) = \gamma_{lmn}\, e^{-q^2(l^2+m^2+n^2)t} - q^2(l^2+m^2+n^2)\int_0^t e^{-q^2(l^2+m^2+n^2)(t-\delta)}$$

$$\times \sum_{\substack{\kappa\lambda\mu \\ \nu\rho\sigma}} \frac{a^{(l)}_{\kappa\nu}\, a^{(m)}_{\lambda\rho}\, a^{(n)}_{\mu\sigma}}{(\kappa^2+\lambda^2+\mu^2)(\nu^2+\rho^2+\sigma^2)}\, \bar{\phi}_{\kappa\lambda\mu}(\delta)\, \bar{\phi}_{\nu\rho\sigma}\, d\delta$$

Therefore

$$\bar{\phi}_{lmn}(t) - \phi^{(r+1)}_{lmn}(t) = -q^2(l^2+m^2+n^2)\int_0^t e^{-q^2(l^2+m^2+n^2)(t-\delta)}\, d\delta$$

$$\times \sum_{\substack{\kappa\lambda\mu \\ \nu\rho\sigma}} \frac{a^{(l)}_{\kappa\nu}\, a^{(m)}_{\lambda\rho}\, a^{(n)}_{\mu\sigma}}{(\kappa^2+\lambda^2+\mu^2)(\nu^2+\rho^2+\sigma^2)} \cdot \left\{ \bar{\phi}_{\kappa\lambda\mu}(\delta)\, \bar{\phi}_{\nu\rho\sigma}(\delta) - \phi^{(r)}_{\kappa\lambda\mu}(\delta)\, \phi^{(r)}_{\nu\rho\sigma}(\delta) \right\}$$

$$= -q^2(l^2+m^2+n^2)\int_0^t e^{-q^2(l^2+m^2+n^2)(t-\delta)}\, d\delta \times \sum_{\substack{\kappa\lambda\mu \\ \nu\rho\sigma}} \frac{a^{(l)}_{\kappa\nu}\, a^{(m)}_{\lambda\rho}\, a^{(n)}_{\mu\sigma}}{(\kappa^2+\lambda^2+\mu^2)(\nu^2+\rho^2+\sigma^2)} \cdot$$

$$\times \left\{ \bar{\phi}_{\kappa\lambda\mu}(\delta) \left(\bar{\phi}_{\nu\rho\sigma}\, \delta - \phi^{(r)}_{\nu\rho\sigma}(\delta) \right) + \phi^{(r)}_{\nu\rho\sigma}(\delta) \left(\bar{\phi}_{\kappa\lambda\mu}(\delta) - \phi^{(r)}_{\kappa\lambda\mu}(\delta) \right) \right\}$$

Therefore from (22), (30) and (45) we have

$$\sum_{l,m,n} \left| \bar{\phi}_{l,m,n}(t) - \phi^{(r+1)}_{l,m,n}(t) \right| < \frac{64}{27}\pi^6 \mathrm{A}2\cdot 2c\cdot \max \sum_{l,m,n} \left| \bar{\phi}_{l,m,n}(t) - \phi^{(r)}_{l,m,n}(t) \right|$$

Repeating the process r-times, we get

$$\sum_{l,m,n} \bar{\phi}_{l,m,n}(t) - \phi^{(r+1)}_{l,m,n}(r) \Big| < \left(\frac{64 \times 4}{27}\pi^6 \mathrm{A}\, c \right)^r \max \sum_{l,m,n} \left| \bar{\phi}_{l,m,n}(t) - \phi^{(o)}_{l,m,n}(t) \right| \quad (46)$$

Now on account of (27) $\dfrac{256\pi^6 \mathrm{A}c}{27} < 1$, and since evidently $\mathrm{Max} \sum_{l,m,n} \left| \phi_{l,m,n}(t) - \phi^{(o)}_{l,m,n}(t) \right|$ is finite, therefore

$$\lim_{r\to\infty} \sum_{l,m,n} \left| \bar{\phi}_{l,m,n}(t) + \phi^{(r+1)}_{l,m,n}(t) \right| = 0 \quad (47)$$

so that

$$\bar{\phi}_{l,m,n}(t) = \lim_{r\to\infty} \phi^{(r)}_{l,m,n}(t) = \phi_{l,m,n}(t) \quad (48)$$

Thus the two solutions $\bar{v}_{l,m,n}(t)$ and $v_{l,m,n}(t)$ are identical, and the solution

$$u(x, y, z; t) = \sum_{l,m,n} v_{l,m,n}(t) \sin lx \sin my \sin nz \quad (49)$$

is unique.

Concluding Remarks.

Now let us consider the more general equation (50)

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}-\frac{1}{q^3}\frac{\partial u}{\partial t}=P(u,\ x,\ y,\ z,\ t),$$

for the same boundary values (3) and (4). We assume that the function P can be expanded in a power series of the form

$$\sum_{\nu=2}^{\infty} p_\nu(x,\ y,\ z\ ,t)\ u^\nu, \tag{51}$$

where the functions p_ν are continuous and uniformly bounded along with their first and second derivatives with respect to x, y, z.

It is easy to see that nothing essential is changed by taking $\sum_{\nu=2}^{\infty} p_\nu\ (x,\ y,\ z,\ t)u^\nu$ instead of u^2 in the right-hand side of equation (1), as all the leading steps in the demonstration remain the same. This will be still further apparent on referring to a previous paper by the author: Zur theorie der nicht-linearen partiellen differential gleichungen vom parabolischen Typus; *Math Zeitschrift* 35 (1931).

On the Transformation Theory of Dynamics in the Manifold of States and Time.*

BY

K. NAGABHUSHANAM.

§**1.** E. T. Whittaker in his *Analytical Dynamics* has constructed the transformation theory of Dynamics in the Manifold of States. Referring to the Manifold of Configurations, J. L. Synge suggests † the study of the extremals of $\int ds$ where $ds = 2\,\mathrm{L}dt$ is not the square root of a homogeneous quadratic form. In this paper, I wish to study some aspects of the transformation theory in the $(2n+1)$-*Manifold of States and Time*, in which the form

$$\mathrm{L}dt = \sum_{r=1}^{n} p_r\, dq^r - \mathrm{H}dt$$

is a Pfaffian of rank § $2n+1$. Thus the study of the variational problem of a Pfaffian * of odd rank ($2n+1$, if the dynamical system under consideration have n degrees of freedom) becomes essential.

Notation.

§**2.** The usual conventions of Tensor Calculus are followed. Any repeated index stands for summation over the range of its variation. The range of variation for r is from 1 to n, for j from 1 to $2n$, and for any other index from 1 to $2n+1$.

The Trajectories.

§**3.** Let us denote by $x^1\ x^2 \ldots x^{2n+1}$ the variables of the coordinate system used. All of them are supposed to be on equal footing; ordinarily, any variable x^i does not separately stand for position in the n-space, or momentum, or time. We consider the Hamilton's

* Submitted in April 1933, revised in March 1934.

† *Phil. Trans. Roy. Soc.* A. vol. 226 (1927) 35-36.

§ The rank of the Pfaffian $\sum_{r=1}^{n} \mathrm{X}_r\, dx^r$ is the rank of the matrix

$$\begin{vmatrix} a_{11} & a_{12} & \cdot & \cdot & a_{1n} \\ a_{21} & a_{22} & \cdot & \cdot & a_{2n} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{n1} & a_{n2} & \cdot & \cdot & a_{nn} \\ \mathrm{X}_1, & \mathrm{X}_2, & \cdot & \cdot & \mathrm{X}_n \end{vmatrix} \quad \text{where } a_{ik} = \frac{\partial \mathrm{X}_i}{\partial x^k} - \frac{\partial \mathrm{X}_k}{\partial x^i}.$$

* Cf. G. D, Birkhoff: *Dynamical Systems*, 55.

principle in the general form as $\delta \int X_i \, dx^i = 0$ along the natural trajectories, the symbol δ standing for the difference in the corresponding quantities on the natural and variant paths. We obtain * the equations of the natural trajectories as the Pfaff's First System of equations, *viz*

$$a_{ik} \, dx^k = 0 \qquad (i = 1, 2 \ldots 2n+1) \qquad \ldots(1)$$

where $a_{ik} = \frac{\partial X_i}{\partial x^k} - \frac{\partial X_k}{\partial x^i}$. For convenience the word 'natural' before trajectories will be hereafter dropped, for we shall throughout refer to them only.

A Subgroup of Infinitesimal Contact Transformations :

§**4.** The most general type of the infinitesimal transformations, $\delta x^i = \varepsilon \xi^i$ $(i=1, 2 \ldots 2n+1)$ ε being a small constant, which changes $X_i \, dx^i$ into $X_i \, dx^i + \varepsilon d\varphi$, transforms the a_{ik}'s into themselves, and so the totality of trajectories into itself. We here consider the subgroup which transforms each trajectory into itself, by transforming every point of a trajectory into a neighbouring point of the same. On performing the transformation, we obtain†

$$X_i \, dx^i + \varepsilon \left(a_{ki} \, \xi^i \, dx^k + dA \right)$$

where $A = X_i \, \xi^i$, it being assumed that d and δ are commutative.

Therefore the vector ξ^i is a solution of the system of equations

$$a_{ki} \, \xi^i \, dx^k + dA = d\varphi$$

where φ is an arbitrary function ; that is

$$a_{ki} \, \xi^i \, dx^k = d \, (\varphi - A) = d\psi .$$

The ξ's which are solutions of

$$a_{ki} \, \xi^i = 0, \qquad [k=1, 2, \cdot \cdot \; 2n+1] \qquad \ldots (2)$$

obviously give rise to a subgroup of the transformations. It is also evident from a comparison with the equations of the trajectories (1), that the vector ξ^i is codirectional with the trajectories. We shall call this subgroup of transformations the Infinitesimal

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* E. Goursat: *Leçons Sur le problème de Pfaff*, 21. The Hamiltonian equations constitute the First System, when $X_i \, dx^i$ has the form $p_r dq^r - H dt$.

† *Encyk. Math. Wiss.* II A. 5 Partielle Diffi. Glei. 318

Tangential (or shortly I. T) transformations, as ξ^i is tangential to the trajectories. These play an important rôle in the present paper; and hereafter the ξ's will stand for solutions of the equations (2)

The equations of the trajectories may be written alternatively as

$$\frac{dx^1}{\xi^1} = \frac{dx^2}{\xi^2} = \dots \dots = \frac{dx^{2n+1}}{\xi^{2n+1}}.$$

The Poisson's Bracket Weights:

§5. The minors (with the proper sign) A^{ik} of a_{ik} in the determinant $|a_{ik}|$ constitute the contravariant components of a symmetric tensor of order two and weight two.

If u_i and v_i are any two covariant vectors, we write

$$\{u_i \; v_k\} = A^{ik} \; u_i \; v_k.$$

It is evident that $\{v_i \; u_k\} = \{u_i \; v_k\}$.

If $f\,(x^1 x^2 \;.\;.\; x^{2n+1})$ is a scalar, and u_i any covariant vector we write

$$\{f \; u_k\} = A^{ik} \frac{\partial f}{\partial x^i} u_k.$$

If $f(x^1 \, x^2 \;.\;.\; x^{2n+1})$ and $\varphi\,(x^1 \, x^2 \;.\;.\; x^{2n+1})$ are two scalars, we write

$$\{f \; \varphi\} = A^{ik} \frac{\partial f}{\partial x^i} \frac{\partial \varphi}{\partial x^k} = \{\varphi \; f\}.$$

These expressions are scalars of weight two and will be called the Poisson's Bracket weights*

The Poisson's bracket weights as symbols of the I. T. transformations:

§6. Consider the system of equations $a_{ik} \, \xi^k = 0$, $(i = 1, 2 \cdot \cdot 2n+1)$ If we substitute $A^{ki} \, V_i$ for ξ^k, where (V_i) is any covariant vector, the equations are obviously satisfied, for the coefficient of every V_i is zero, or $|a_{ik}| = 0$. In order to exclude the trivial case $\xi^1 = \xi^2 = \cdot \cdot = \xi^{2n+1} = 0$, (V_i) should be such that $A^{ki} V_i \neq 0$. This condition can be physically interpreted as the non-incidence of the contravariant direction of the trajectories with V_i; for, if $A^{ki} V_i \neq 0$ at least for one

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* Clebsch's expression for the Poisson's Bracket $(f\phi) = A^{ik} \frac{\partial f}{\partial x^i} \frac{\partial \varphi}{\partial x^k} \Big/ \Delta$, where $\Delta = |a_{ik}|$ is here modified by dropping the division by Δ which vanishes identically in the present case.

See E. T. Whittaker, *loc. cit* (3rd edition), 311 example 2.

value of k, then $A^{ki}\ V_iV_k \neq 0$.* This condition may be rewritten $\xi^k\ V_k \neq 0$. Again, if $\xi^k\ V_k \neq 0$, we have $A^{ki}\ V_i\ V_k \neq 0$. Hence $A^{ki}V_i \neq 0$ at least for one value of k, for otherwise every coefficient of V_k in $A^{ki}\ V_i\ V_k$ is zero, giving $A^{ki}\ V_iV_k \equiv 0$. Thus the condition, $A^{ki}\ V_i \neq 0$ for at least one value of k, is identical with the condition $\xi^k V_k \neq 0$, i.e. identical with the non-incidence of (ξ^k), and (V_k)

With every covariant vector (V_i), non-incident with the contra-variant direction of the trajectories, we can associate a system of ξ's and hence an I. T. transformation. The increment in a function $f(x^1 \cdots x^{2n+1})$ due to this transformation is $\delta f = \varepsilon\ \xi^i \dfrac{\partial f}{\partial x^i}$, which is proportional to

$$A^{ik}\ V_k \frac{\partial f}{\partial x^i} = \{ f V_k \}.$$

We have thus

THEOREM 1 †: *The increment in a function f due to the I. T transformation associated with (V_i) non-incident with the direction of the trajectories is proportional to $\{fV_k\}$.*

Integrals of Motion

§**7.** Any function $\varphi(x^1, x^2 \ldots x^{2n+1})$ which has a constant value at all points of any trajectory, *i.e.* an integral of the Pfaff's first system, is called an *Integral of Motion.* Naturally we get sets of $2n$ independent integrals of motion, for the First System contains $2n$ independent equations. We shall denote them by $\phi\ (j = 1, 2 \ldots 2n)$.

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* If we adopt the coordinate system in which $X_i\ dx^i$ has the cannonical form $p_r dq_r - dT$ the matrix $\| a_{ik} \|$ becomes, on taking the variables in the order $q^1 q^2 \ldots q^n, p_1 \ldots p_n, T$

$$\left| \begin{array}{l} 0 \ldots .\ 0\ 1\ 0 \ldots .\ 0\ 0 \\ 0 \ldots .\ 0\ 0\ 1\ 0 \ldots 0\ 0 \\ 0 \ldots .\ 0\ 0\ 0\ 1\ 0 \ldots 0\ 0 \\ \ldots\ldots\ldots\ldots\ldots \\ 0 \ldots .\ 0\ 0 \ldots .\ 0\ 1\ 0 \\ -1\ 0 \ldots 0\ 0 \ldots\ldots 0\ 0 \\ 0\ {-1}\ 0 \ldots 0\ 0 \ldots\ldots 0\ 0 \\ 0\ 0\ {-1}\ 0 .\ 0\ 0 \ldots\ldots 0\ 0 \\ \ldots\ldots\ldots\ldots\ldots \\ 0 \ldots .\ 0\ {-1}\ 0 \ldots .\ 0\ 0 \\ 0 \ldots .\ 0\ 0 \ldots\ldots 0\ 0 \end{array} \right|$$

All the A^{ik}'s except $A^{(2n+1)\,(2n+1)}$ are zero. The nonvanishing of $A^{ki}\ V_i$ for at least one value of k here reduces to the nonvanishing of $A^{(2n+1)\,(2n+1)}\ V_{(2n+1)}$, and hence to the nonvanishing of $A^{(2n+1)\,(2n+1)}\ V_{2n+1}\ V_{2n+1}$. If the transformation of coordinates considered be nonsingular $A^{ki}\ V_i\ V_k$ cannot vanish in any coordinate system.

† The corresponding result in the manifold of states is: The Poisson's Bracket of f and an arbitrary function is a symbol of the most general type of infinitesimal contact transformation.

See E. T. Whittaker, *loc. cit.*, 303.

Since φ has a constant value along the trajectories, $\delta\varphi$ for the I. T. transformations is zero, so that $\{\varphi\ V_k\}=0$, when (V_k) is non-incident with the direction of the trajectories. In the trivial case $\{\varphi V_k\}\equiv 0$.

The converse that, if $\{\varphi V_k\}=0$ for (V_k) which is non-incident with the trajectories, φ is an integral of motion, is seen to be true by the consideration that all I. T. transformations are obtained by putting $\chi\,\xi^i$ for ξ^i where χ is arbitrary. If for the I. T. transformation with the symbol $X(f)=\xi^i\dfrac{\partial f}{\partial x^i}$, $\delta\varphi$ which is proportional to $\{\varphi V_k\}$ is zero, then for any I. T. transformation with the symbol $X(f)=\chi\xi^i\dfrac{\partial f}{\partial x^i}$ $\delta\phi$ which is proportional to $\chi\,\{\varphi V_k\}$ is also zero. Thus φ is constant for all I. T. transformations at the point under consideration; and this property holds at all points of the trajectory. Therefore $\delta\varphi=0$ along the entire trajectory, or φ is an integral of motion.

The trivial case is excluded now. We state

THEOREM 2*.—*If φ is an integral of motion, and (V_k) is non-incident with the contravariant direction of the trajectories, $\{\varphi V_k\}=0$ and conversely.*

§8. We shall next prove

THEOREM 3. *If f and φ are two scalars, and if at least one of them is an integral of motion, $\{f\ \varphi\}=0$, and conversely.*

Firstly, if φ is an integral of motion, then by theorem 2

$$\{f\varphi\} = \{\varphi f\} = \{\varphi,\ \partial f/\partial x^i\}=0.$$

Similarly if f is an integral of motion.

Let us next suppose that $\{f\varphi\}=0$. Either $\dfrac{\partial\varphi}{\partial x^i}\,\xi^i$ is zero or not. If $\dfrac{\partial\varphi}{\partial x^i}\,\xi^i\neq 0$, then by theorem 2, f is an integral of motion, for $\left\{f\ \dfrac{\partial\varphi}{\partial x^i}\right\}=0$. If $\dfrac{\partial\varphi}{\partial x^i}\,\xi^i=0$, we get $\dfrac{\partial\varphi}{\partial x^i}\,dx^i\ (=d\varphi)$ vanishes, where

$$\frac{dx^1}{\xi^1}=\frac{dx^2}{\xi^2}=\ldots=\frac{dx^{2n+1}}{\xi^{2n+1}}.$$

Hence $d\varphi=0$, along the trajectories, or φ is an integral of motion.

* The corresponding result in the manifold of States is: The Poisson's Bracket of two integrals of motion is constant.

See E. T. Whittaker, *loc cit*, 320.

Functions which are integrals of rank two for $X_i\, dx^i$

§**9.** For the Pfaffian $X_i dx^i$ (of rank $2n+1$) any function φ, which is such that the substitution φ = constant, $d\varphi = 0$ in $X_i dx$ depresses the rank by exactly two units, is called an integral of rank two. Such functions are Integrals of the Pfaff's First System.

THEOREM 4. *Any integral of rank two for $X_i dx^i$ is an integral of motion, and conversely.*

Time and the Lagrangian Function:

§**10.** Any function $t(x^1 \cdot\cdot x^{2n+1})$ which is such that the substitution t = constant, and $dt=0$, in $X_i dx^i$ depresses the rank by one and only one unit, and the *differential consequence* $dt=0$ *must necessarily be made use of to depress the rank* may be taken to denote *a measure of Time**.

Let us write the equations of the trajectories in the form

$$\frac{dx^1}{\xi^1} = \frac{dx^2}{\xi^2} = \quad \cdot \quad \cdot \quad \cdot \quad = \frac{dx^{2n+1}}{\xi^{2n+1}} = \varepsilon.$$

If t is a measure of time $dt = \frac{\partial t}{\partial x^i}\, dx^i = \frac{\partial t}{\partial x^i}\, \xi^i\, \varepsilon$, so that $\varepsilon = dt \Big/ \frac{\partial t}{\partial x^i}\, \xi^i$. Hence $X_i\, dx^i = X_i\, \xi^i\, \varepsilon = X_i\, \xi^i\, dt \Big/ \frac{\partial t}{\partial x^i}\, \xi^i = L\, dt$.

where
$$L = X_i\, \xi^i \Big/ \frac{\partial t}{\partial x^i}\, \xi^i.$$

Now $\delta \int X_i\, dx^i = 0$ reduces to $\delta \int L\, dt = 0$ along the trajectories. The function L may be called the *Lagrangian Function* for the measure of time t.

§**11.** We shall consider in this and the succeeding paragraphs special coordinate systems of the type $(\varphi_{(1)}, \varphi_{(2)}, \ldots \varphi_{(2n)}, t)$, where the φ's are integrals of motion, and t is a measure of time. In such a coordinate system, the First System of Pfaff becomes.

$$d\varphi_{(1)} = d\varphi_{(2)} \quad \cdot \quad \cdot \quad \cdot \quad = d\varphi_{(2n)} = 0, \qquad a_{i,(2n+1)}dt = 0.$$

Since t is not an integral of motion, $a_{i(2n+1)} = 0$. We may state this as

THEOREM 5. *In the coordinate system $(\varphi_{(1)}, \ldots \varphi_{(2n)}, t)$ any component of the bilinear covariant a_{ik}, involving the suffix $2n+1$ vanishes.*

Since the Lagrange Bracket expressions in the Manifold of States

* Not being an integral of motion t changes along the trajectories. The restriction that $dt=0$ must necessarily be used to depress the rank is necessary to treat the theory of the Hamiltonian from the view point of the rank of $X_i dx^i$. Results appear elsewhere.

correspond to the components* of a_{ik} in the coordinate system of independent variables, we may display the result of theorem 5 as

$$a_{j(2n+1)} = [\varphi_{(j)}, t] = 0, \qquad (j = 1, 2 \cdot \cdot \cdot 2n+1,)$$

a form which puts one in mind of the similar result in the $2n-$ Manifold, viz. the Lagrange Brackets of two integrals of motion is constant.

Next we shall prove the converse.

THEOREM 6. *If in a coordinate system* $(x^1 ... x^{2n+1})$, $[x^j x^{2n+1}] = 0$, $j = 1, 2 ... 2n$, *the coordinates* $x^1 ... x^{2n}$ *are integrals of motion, and* x^{2n+1} *is a measure of time, where the* x*'s are all functionally independent.*

The equations of motion now become

$$a_{ij'}\, dx^j = 0, \quad (j', j = 1, 2 ... 2n.)$$

The determinant $|a_{j'j}| \neq 0$†. Hence the variables $dx^1, dx^2 ... dx^{2n}$ in the linear Homogeneous equations must be separately zero, *i.e.* $x_1 = c_1, ... x_{2n} = c_{2n}$ must be a system of $2n$ independent integrals of motion, the c's being arbitrary constants, x^{2n+1}, being independent of $x^1 ... x^{2n}$, is not an integral of motion. Hence the substitution $x^{2n+1} = \text{constant}$, $dx^{2n+1} = 0$ cannot depress the rank of $X_i\, dx^i$ by two units. The substitution reduces the number of independent variables to $2n$. Hence the Pfaffian cannot have its rank after the substitution greater than $2n$.

Thus the substitution depresses the rank by one and only one unit. Also the Pfaffian cannot appear with more than n differentials in $x^1 ... x^{2n}$, the x's being independent integrals of the First system. Writing it as $X_r\, dx^r + X_{2n+1}\, dx^{2n+1}$ it also becomes evident that the

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* E. T. Whittaker, *loc. cit.*, 298.

† $|a_{j'j}| = A^{(2n+1)(2n+1)} = A^{ik} \dfrac{\delta x^{2n+1}}{\delta x^i} \dfrac{\delta x^{2n+1}}{\delta \bar{x}^k} \left| \dfrac{\delta \bar{x}}{\delta x} \right|^2$

with the usual notation. In the coordinate system $(\bar{x}) = (q^1 ... q^n, p_1 .. p_n, T)$ in which $X_i dx_i$ has the cannonical form $p_r\, dq^r - dT$,

$$\bar{A}^{(2n+1)(2n+1)} = 1; \; \bar{A}^{j',j} = 0; \qquad (j', j = 1, 2 ... 2n.)$$

Hence $|a_{j'j}| = \left(\dfrac{\delta x^{2n+1}}{\delta \bar{x}^{2n+1}}\right)^2 \left| \dfrac{\delta \bar{x}}{\delta x} \right|^2$ But $\left| \dfrac{\delta \bar{x}}{\delta x} \right| \neq 0$ since all the transformations considered are non-singular. Also $\dfrac{\delta x^{2n+1}}{\delta \bar{x}^{2n+1}}$ cannot be zero, for if, in the transformation from (x) to $(\bar{x})$, x^{2n+1} which is not an integral of motion does not explicitly depend on $\bar{x}^{2n+1}$, it becomes a function of $\bar{x}^1 ... \bar{x}^{2n} (= q^1 ... q^n p_1 ... p_n)$ which are all integrals of motion.

differential consequence $dx^{2n+1}=0$ must be necessarily used to depress the rank.

§**12.** If we consider the cannonical form $p_r dq^r$—dT, all the q's and p's are integrals of motion; and in the cannonical coordinate system the trajectories can be looked upon as the flow of time. The problem of finding the integrals of motion is therefore identical with the reduction $X_i dx^i$ to the form $p_r\ dq^r - d\mathrm{T}$.

I thank Dr. R. Vaidyanathaswamy for his kind help in preparing this paper.

The Normalisation in Wave Statistics

BY

K. K. MUKHERJEE.

Section I

In some of my papers [1] already published an attempt has been made to establish a relation between the classical Mechanics and the Wave Statistics and also to find an expression for the current. But the condition of normalisation *viz.*, $\int \frac{1}{4} D\bar{D}\, d\tau = 1$, where $\bar{D}$ is the conjugate of D, the phase-density, which has been adopted there by analogy of Wave Mechanics does not clearly bring out the physical meaning of the problem which is the striking feature of Wave Statistics. Again, the particular constant *viz.* ¼ taken above is not essential, in as much as it may be replaced by any constant whatever.

The object of the present paper is to obtain the results of the papers referred to above by adopting a normalisation condition which is better from the physical point of view.

It has been shewn by Kar [2] that the number of electrons per unit phase volume having energy E_n is given by

$$D_n = \frac{N}{\Phi} x_{1_n} x_{2_n} \exp. \left\{ \frac{\psi_n + \psi - E_n}{K\tau} \right\} \exp. \{\pm 4\pi i E_n t/h\} \exp. \{\pm 2\pi i E_n t/h\} \quad \ldots(1)$$

where x_{1_n} and x_{2_n} are functions of q's and p's respectively.

Again, if we integrate throughout the volume V at a particular instant, we have from Equation. (1)

$$\int_0^V D_n \, d\tau_{q_n} d\tau_{p_n} = \frac{N}{\Phi} \exp. \left\{ \frac{\psi_n + \psi - E_n}{K\tau} \right\} \exp. \{\pm 2\pi i E_n t/h\} \int_0^V x_{1_n} x_{2_n} d\tau_{q_n} d\tau_{p_n}$$

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[1] K. K. Mukherjee—*Physikalische Zeitschrift.* 32, 485, 1931.

K. K. Mukherjee—*Journal of the Indian Mathematical Society*, Vol. XIX (1931).

K. K. Mukherjee—*Indian Physico-Mathematical Journal*, Vol. III, No. 1, Jan. 1932.

[2] K. C. Kar. *Zeit. f. Phys.* 61, 675, 1930.

Now, for the n^{th} elementary volume $\Delta\tau_n = d\tau_{q_n}\, d\tau_{p_n}$, $d\tau_{p_n}$ may be taken as a small constant. Hence we have

$$\int_0^V D_n\, d\tau_{q_n} = \frac{N}{\Phi} \exp.\left\{\frac{\psi_n + \psi - E_n}{K\tau}\right\} \exp.\{\pm 2\pi i E_n t/h\} \int_0^V x_{1_n}\, x_{2_n}\, d\tau_{q_n} \qquad \ldots(2)$$

$$\text{or,}\quad D_n = \frac{N}{\Phi} \exp.\left\{\frac{\psi_n + \psi - E_n}{K\tau}\right\} \exp.\left\{\frac{\pm 2\pi i E_n t/h}{V}\right\} \int_0^V x_{1_n}\, x_{2_n}\, d\tau_{q_n}$$

$$= \frac{N}{\Phi} \exp.\left\{\frac{\psi_n + \psi - E_n}{K\tau}\right\} C \int_0^\infty x_{1_n}\, x_{2_n}\, d\tau_{q_n}. \qquad \ldots(3)$$

as V is large.

It may be remarked here that $x_2\,(p/p_0) = x_2\,(q/q_0)$ because of the relation $p = \alpha q$, proved before; and so x_{2_n} in (3) may be regarded as a function of q's.

Now, Equation (3) gives the classical value of D_n if we take

$$C \int_0^\infty x_{1_n}\, x_{2_n}\, d\tau_{q_n} = 1. \qquad \ldots(4)$$

which we adopt as the normalisation condition in Wave Statistics. And as the left hand expression of (4) represents the average of $x_{1_n}\, x_{2_n}$, it is clear that normalisation, as we conceive it, is nothing but the process of averaging. It may also be pointed out that in (4) we have taken the product of x_1 and x_2 waves of the same elementary phase volume, the n^{th} one in this case. The x_1 and x_2 waves belonging to two different elementary volumes cannot influence one another. This would mean mathematically

$$C \int x_{1_p}\, x_{2_s}\, d\tau_q = 0. \qquad (p \neq s).$$

Section II

With the above normalisation condition we proceed to establish the well-known law $\dot{P} = M$ of classical mechanics.

We have shewn in a previous paper [2] that the velocity of the x_1 – waves is given by

$$v = \sqrt{2(E - V)/m} \qquad \ldots(5)$$

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(2) K. C. Kar and K. K. Mukherjee—*Zeit. f. Phys.* 59, 102, 1929.

Hence we have

$$p = mv = \sqrt{2m(E - V)} \qquad \ldots(6)$$

Again, we have from classical mechanics

$$P = [r.\ p] = [r,\ \sqrt{2m(E - V)}] \qquad \ldots(7)$$

Hence we have (*l. c.*)

$$\dot{P} = \left[r.\ \frac{d}{dt}\ \sqrt{2m(E - V)}\right] \qquad \ldots(8)$$

$$= \left[r.\ \frac{m}{\sqrt{2m(E - V)}}\ \frac{\sqrt{2(E - V)}}{m}\ \frac{d}{dx}\ (E - V)\right] \qquad \ldots(9)$$

Now, for an electron moving in a field, E is constant and thus we have from (9)

$$\dot{P} = -[r,\ \text{grad}\ V] \qquad \ldots(10)$$

And denoting the statistical average of $\dot{P}$ by $\overline{\dot{P}}$, we have

$$\overline{\dot{P}} = C \int \dot{P}\ x_1 x_2\ d\tau = -C \int [r\ \text{grad}\ V] x_1\, x_2\, d\tau = \overline{M}$$

We shall next find an expression for the current in wave statistics.

If we differentiate (6) with respect to the time, we have (cf, Eq. (10))

$$\dot{p} = \frac{d}{dt}\sqrt{2m(E - V)}$$

$$= -\text{grad}\ V. \qquad \ldots(11)$$

Now, the differential equation for x_1-waves is

$$\Delta\ x_1 + \frac{8\pi^2 m}{h^2}\ (E - V) x_1 = 0 \qquad \ldots(12)$$

Introducing the new variable $D_1 = x_1$ exp. $\{2\pi i E t/h\}$, it becomes

$$\dot{D}_1 = \frac{h}{4\pi i m}\left(\Delta - \frac{8\pi^2 m}{h^2}\ V\right) D_1 \qquad \ldots(13)$$

Again, the wave equation for x_2-waves namely

$$\Delta_p\, x_2 + \frac{8\pi^2 m}{\alpha^2 h^2}\ (E - V)\ x_2 = 0 \qquad \ldots(14)$$

where, $$\Delta_p = \frac{\partial^2}{\partial p_1{}^2} + \frac{\partial^2}{\partial p_2{}^2} + \frac{\partial^2}{\partial p_3{}^2} \quad \text{and} \quad p = \alpha q \qquad \ldots(15)$$

may, on introducing the new variable $D_2 = x_2 \exp. \{-2\pi i Et/h\}$ be written

$$\dot{D}_2 = -\frac{h}{4\pi im}\left(\Delta - \frac{8\pi^2 m}{h^2}V\right)D_2 \quad ...(16)$$

where

$$\Delta = \frac{\partial^2}{\partial q_1^2} + \frac{\partial^2}{\partial q_2^2} + \frac{\partial^2}{\partial q_3^2}.$$

Now, from (11) we have

$$\bar{p} \left(\text{or, } \bar{\dot{p}}\right) = -C\int x_1 x_2 \text{ grad } V\, d\tau \quad ...(17)$$

$$= C\left[-\int x_2 \text{ grad } (Vx_1)d\tau + \int Vx_2 \text{ grad } x_1\, d\tau\right] \quad ...(18)$$

Again, because

$$\int(\Delta x_1 \text{ grad } x_2 + \Delta x_2 \text{ grad } x_1)dT = 0,^{(3)}$$

equation (18) may be written in the form

$$\bar{\dot{p}} = \frac{Ch}{2\pi i}\left[\int \frac{h}{4\pi im}\left(\Delta - \frac{8\pi^2 mV}{h^2}\right)x_1 \text{ grad } x_2\, d\tau + \int \frac{h}{4\pi im}\left(\Delta - \frac{8\pi^2 mV}{h^2}\right)x_2 \text{ grad } x_1\, d\tau\right]$$

$$= \frac{Ch}{2\pi i}\left[\int \frac{h}{4\pi im}\left(\Delta - \frac{8\pi^2 mV}{h^2}\right)D_1 \text{ grad } D_2\, d\tau + \int \frac{h}{4\pi im}\left(\Delta - \frac{8\pi^2 mV}{h^2}\right) D_2 \text{ grad } D_1\, d\tau\right]$$

and with the help of (13) and (16)

$$= \frac{Ch}{2\pi i}\left[\int\left(\dot{D}_1 \text{ grad } D_2 + D_1 \text{ grad } \dot{D}_2\right)d\tau \quad ...(19)$$

Hence we have

$$\bar{\dot{p}} = \frac{Ch}{2\pi i}\int D_1 \text{ grad } D_2\, d\tau$$

$$= \frac{Ch}{2\pi i}\int x_1 \text{ grad } x_2\, d\tau \quad ...(20)$$

which is our expression for the wave statistical average of impulse.

Again, from (20) we have

$$\bar{\dot{v}} = \frac{Ch}{2\pi im}\int x_1 \text{ grad } x_2\, d\tau \quad ...(21)$$

Now, if we denote the current by S, we have $S = +e\,\bar{v}$.
Hence using (21), we have

$$S = +e.\frac{Ch}{2\pi im}\int x_1 \text{ grad } x_2\, d\tau \quad ...(22)$$

(3) Sommerfeld—*Wave Mechanics* (Engl), p. 247.

We next proceed to show that for a circular orbit the above wave statistical expression for the current leads to the well-known Bohr value.

Now, because the current is a vector having no gradient along r or θ, we have from (22), since x_1 or $x_2 = \text{C.R.P}_l^n(\cos\theta)e^{in\phi}$,

$$\begin{aligned} S &= + \frac{enh}{2\pi mr \sin\theta}.\ \text{C.} \int x_1\, x_2\, d\tau \\ &= + \frac{enh}{2\pi ma} \end{aligned} \qquad \text{...(23)}$$

where a $(= r \sin\theta)$ is the radius of the n^{th} Bohr orbit.

If we substitute in (23) the value of a, viz. $\frac{n^2h^2}{4\pi^2me^2}$, we get

$$S = +e.\frac{2\pi e^2}{nh} = +e\ v. \qquad \text{...(24)}$$

where v is the well-known Bohr value of the velocity in the n^{th} orbit.

We may remark also that Fermi's [4] expression for the current can be easily reduced to the form given in (23) as has been shewn in a previous paper.[5]

In conclusion my best thanks are due to Dr. K. C. Kar, for his valuable advice in preparing this paper.

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(4) Fermi, *Nature*, December 18, 1926.
(5) K. K. Mukherjee—*Journal of the Indian Mathematical Society*, Vol. XIX, (1931).