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AN ALGEBRA OF ARITHMETICAL FUNCTIONS.

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Definitions.

§ 1. *Entities.*

An arithmetical function, or A. F., is a function $f(n)$, defined for every positive integral value of the variable n.

The quantity $f(n)$, considered alone, is called the nth element of the A.F.

We shall in the following use square brackets to denote an A.F , thus :

$$[f(1),\ f(2),\ f(3),\ \ldots\ldots], \quad \ldots \quad \ldots \quad \ldots \quad (1\,.\,1)$$

or shortly, when no confusion may arise,

$$[f(n)] \ldots \quad \ldots \quad \ldots \quad \ldots \quad (1\,.\,11)$$

Every A. F. shall further be regarded as an algebraical quantity, or shortly, a number.

In order that every ordinary complex number k may also be considered as an A. F., we define

$$k = [k,\ 0,\ 0,\ 0,\ \ldots\ldots]; \quad \ldots \quad \ldots \quad \ldots \quad (1\,.\,2)$$

thus the quantity k is defined as an A. F., all the elements of which are zero, except the first, which has the value k.

The **A. F.** thus defined must be distinguished from

$$[k] = [k, k, k, \ldots\ldots], \quad \ldots \quad \ldots \quad \ldots \quad (1.3)$$

every element of which has the value k.

Two **A. F.** are called equal if and only if their corresponding elements are equal.

§ 2. *Operations.*

To add (or subtract) two A. F. is to add (or subtract) their corresponding elements; thus

$$[f(n)] \pm [g(n)] = [f(n) \pm g(n)]. \quad \ldots \quad \ldots \quad (2.1)$$

To multiply an A. F. by a quantity k is to multiply every element by k; thus

$$k[f(n)] = [k \,.\, f(n)] \quad \ldots \quad \ldots \quad \ldots \quad (2.2)$$

In particular:

$$k = k[1, 0, 0, 0, \ldots\ldots] = k\,.\,1, \quad \ldots \quad \ldots \quad (2.21)$$

$$[k] = k[1, 1, 1, \ldots\ldots] = k[1]. \quad \ldots \quad \ldots \quad (2.22)$$

To multiply two **A. F.** is to form a new **A. F.**, as indicated by the formula

$$[f(n)] \,.\, [g(n)] = \left[\sum_{d\,:\,n} f(d) \,.\, g\left(\frac{n}{d}\right)\right] \quad \ldots \quad (2.3)$$

the summation Σ being extended over all divisors d of n. It should be noticed, that (2 . 3) includes (2 . 2) as a special case.

To divide an A F. $[g(n)]$ by another $[f(n)]$ is to form a new A. F. $[\partial(n)]$, satisfying the condition

$$[g(n)] = [f(n)] \,.\, [\partial(n)]. \quad \ldots \quad \ldots \quad \ldots \quad (2.4)$$

The A. F. $[\partial(n)]$ will be called the quotient of the two given A. F., and denoted

$$[\partial(n)] = [g(n)] \,.\, [f(n)]^{-1}. \quad \ldots \quad \ldots \quad (2.41)$$

It will be seen later, that the formation of $[\partial(n)]$ is not always possible.

There is now no difficulty to prove, that the associative and commutative laws for addition, as well as the associative, commutative and distributive laws for multiplication, are valid in our extended algebraical domain. We are therefore already in a position to construct an algebra, involving the four fundamental operations, excluding division in certain cases (as when the first element of the divisor is zero, while that of the dividend is different from zero).

§ 3. *Connection with Dirichlet's Series.*

It will be well to note at this stage an important connection between our algebra and the theory of Dirichlet's Series (D. S.). Let for this purpose

$$F(s) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s} \quad \dots \quad \dots \quad (3.1)$$

define an ordinary D. S. associated with the A. F. $[f(n)]$, a corresponding series $G(s)$ being constructed for $[g(n)]$.

It will be seen at once, that the series

$$F(s) \pm G(s),\ F(s) \, . \, G(s),\ G(s)/F(s),$$

where the terms have been rearranged so as to form a new ordinary D. S., are associated in the above manner with the A. F.'s

$$[f(n) \pm g(n)],\ [f(n)] \, . \, [g(n)],\ [g(n)] \, . \, [f(n)]^{-1},$$

respectively.

We are thus led to the following general definition :

A function, algebraical or transcendental, of a given A. F. $[f(n)]$ *is the new A. F., associated with the ordinary D. S., which we obtain by letting the given function operate in the usual manner on* $F(s)$, *and rearranging the terms, provided this process is possible and unique.*

In other words :

The function

$$\gamma[f(n)] = [g(n)] \quad \dots \quad \dots \quad \dots \quad (3.2)$$

is defined by the equation

$$\gamma\{F(s)\} = G(s). \quad \dots \quad \dots \quad \dots \quad (3.3)$$

It is clear, that this general definition includes the former ones as special cases. It is further evident, that our results can never be contradictory, as long as the process indicated in (3.3) is uniquely determined. Even if this were not the case, the definition would remain valid under certain restrictions. In the following, however, multiple-valued functions will not be considered, unless otherwise stated.

In addition, it is to be noted, that questions of convergence of the D. S. involved need not be considered, the developments being purely formal, as will be seen towards the end of the paper, where the units of our algebra are discussed.

By means of the general definition, the entire algebraical analysis is now applicable on arithmetical functions, regarded as algebraical entities. In general, we may state that the first element in each A F. operated upon follows the laws of ordinary algebra, while for every element of an order $n > 1$ a new theorem is obtained, the nature of which depends on the decomposition of n into prime factors, becoming more complicated with n. Hence, for every theorem of ordinary algebraical analysis, we obtain an infinite set of new elemental theorems (identities).

I will now first discuss the process of division from another point of view. Then I will show, how a few known results in the theory of certain elementary A. F., appearing in the theory of numbers, may be found in an extremely simple manner and easily extended. I will then pass over to the discussion of a few elementary analytic functions of an A. F., in order to show the bearings of the theory. Further, the extension into other domains of mathematics, notably the Calculus, will be briefly referred to. Finally, the units of our quasi-linear algebra will be discussed.

It has been thought necessary to give a few individual examples from certain sets of elemental theorems obtained. In general, these examples are carried out for $n = 6$, this number being sufficiently complicated to show some of the most prominent features of the set of theorems studied. To use a more complicated n for this purpose was out of the question, the space available having to be considered.

The process of division in a special case.

§ 4. *The A. F.* $[\mu(n)]$.

We define the important A. F. $[\mu(n)]$ as follows:

$\mu(1) = 1$.

$\mu(n) = 0$, if n is divisible by a square (other than unity).

$\mu(n) = (-1)^k$, if n is not divisible by a square, k being the number of (distinct) prime factors of n.

The A. F. thus defined plays a leading part in the theory of the distribution of the primes.

(See: *Encyclopédie des Sciences Mathématiques*, T. I., 17 —E. Landau, *Handbuch der Lehre von der Verteilung der Primzahlen*, B. II.—F. Halberg, *Infinite Series and Arithmetical Functions—Journ. I.M.S.*, Vol. IX, p. 174, etc.)

§ 5. *A theorem by Kronecker, and its consequences.*

In Kronecker's *Zahlentheorie* I, p. 264, we find the following theorem :

If $[f(n)]$ *is an A. F. such that*

$$f(mm') = f(m) \,.\, f(m'), \quad \dots \quad \dots \quad \dots \quad (5\,.\,1)$$

for all positive integers m, m', $(f(1) = 1)$, *we have, in the notation of* (3 . 1)

$$\frac{1}{F(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)\, f(n)}{n^s}. \quad \dots \quad \dots \quad \dots \quad (5\,.\,2)$$

Translating this theorem into our present notation, we obtain :

If the A. F. $[f(n)]$ *satisfies formula* (5 . 1), *then*

$$[f(n)]^{-1} = [\mu(n)\, f(n)]. \quad \dots \quad \dots \quad \dots \quad (5\,.\,3)$$

In particular, $f(n) = 1$ gives

$$[1]^{-1} = [\mu(n)]. \quad \dots \quad \dots \quad \dots \quad (5\,.\,31)$$

Elementary arithmetical functions in the theory of numbers.

§ 6. *Some general results.*

When $[f(n)]$ is any given A. F., the A. F. $[\psi(n)]$, defined by the equation

$$\psi(n) = \sum_{d\,:\,n} f(d), \quad \dots \quad \dots \quad \dots \quad (6\,.\,1)$$

is called the "numerical integral" of $[f(n)]$. (See the first and last references under § 4).

In our present notation this formula becomes

$$[\psi(n)] = [1] \,.\, [f(n)], \quad \dots \quad \dots \quad \dots \quad (6\,.\,11)$$

from which follows, by (5 . 31)

$$[f(n)] = [\mu(n)] \,.\, [\psi(n)], \quad \dots \quad \dots \quad (6\,.\,12)$$

which is nothing else than the well-known identity

$$f(n) = \sum_{d\,:\,n} \mu(d) \,.\, \psi\left(\frac{n}{d}\right). \quad \dots \quad \dots \quad (6\,.\,2)$$

Assume for the moment

$$\left.\begin{aligned} F(n) &= \sum_{d\,:\,n} f(d)\, h\left(\frac{n}{d}\right) \\ G(n) &= \sum_{d\,:\,n} g(d)\, h\left(\frac{n}{d}\right) \end{aligned}\right\}, \quad \dots \quad \dots \quad (6\,.\,3)$$

$[f(n)]$, $[g(n)]$, $[h(n)]$ being three arbitrarily given A. F.

The relations (6 . 3) may be written

$$\left.\begin{array}{l}[F(n)] = [f(n)]\,.\,[h(n)] \\ [G(n)] = [g(n)]\,.\,[h(n)]\end{array}\right\}. \qquad \dots \qquad \dots (6\,.\,31)$$

Hence, by multiplication

$$[f(n)]\,.\,[G(n)] = [g(n)]\,.\,[F(n)], \qquad \dots \qquad \dots (6\,.\,32)$$

or in ordinary notation

$$\sum_{d\,:\,n} f(d)\, G\left(\frac{n}{d}\right) = \sum_{d\,:\,n} g(d)\, F\left(\frac{n}{d}\right). \qquad \dots \quad (6\,.\,4)$$

The special case $h(n) = 1$ of this theorem has been treated by E. Cesàro (See: *Encycl.*, l. c., p. 231).

Let

$$\sigma(n) = \sum_{h=1}^{n} \mu(h). \qquad \dots \qquad \dots \qquad \dots \quad (6\,.\,5)$$

$$\therefore \quad \sigma(n) - \sigma(n-1) = \mu(n). \qquad \dots \qquad \dots \qquad \dots (6\,.\,51)$$

$$\therefore \quad [\sigma(n)] - [\sigma(n-1)] = [\mu(n)]. \qquad \dots \qquad \dots (6.511)$$

$$\therefore \quad [1]\,.\,[\sigma(n)] - [1]\,.\,[\sigma(n-1)] = 1 = [1, 0, 0, 0, \dots], \quad (6\,.\,52)$$

by (5 . 31).

Hence for $n > 1$, supposing $\sigma(0) = 0$, we have the identity

$$\sum_{d\,:\,n} \sigma(d) = \sum_{d\,:\,n} \sigma(d-1). \qquad \dots \qquad \dots \qquad \dots \quad (6\,.\,6)$$

The last result may be generalised.

Let the A. F. $[f(n)]$ satisfy the relation (5 . 1), and assume

$$\theta(n) = \sum_{h=1}^{n} f(h). \qquad \dots \qquad \dots \qquad \dots \quad (6\,.\,7)$$

Thus

$$\theta(n) - \theta(n-1) = f(n), \qquad \dots \qquad \dots \quad (6\,.\,71)$$

or

$$[\theta(n)] - [\theta(n-1)] = [f(n)]. \qquad \dots \qquad \dots (6\,.\,711)$$

Division by $[f(n)]$ gives

$$\frac{[\theta(n)]}{[f(n)]} = 1 + \frac{[\theta(n-1)]}{[f(n)]}, \qquad \dots \qquad \dots \quad (6\,.\,27)$$

or, by (5 . 3),

$$[\theta(n)]\,[\mu(n)\,f(n)] = 1 + [\theta(n-1)]\,[\mu(n)\,f(n)]. \quad \dots (6\,.\,721)$$

Hence, for $n > 1$, supposing $\theta(0) = 0$, we get

$$\sum_{d:n} \theta(d)\, \mu\left(\frac{n}{d}\right) f\left(\frac{n}{d}\right) = \sum_{d:n} \theta(d-1)\, \mu\left(\frac{n}{d}\right) f\left(\frac{n}{d}\right)$$

$$= \sum_{n} f(v), \quad \ldots \quad \ldots \quad (6.8).$$

where the sum in the last expression is extended over all positive integers $v < n$ and prime to n, this result being obtained by actually carrying out the operation $[\theta(n)]/[f(n)]$.

The identity between the two first expressions of (6.8) may be established by subtraction. We get

$$\sum_{d:n} \{\theta(d) - \theta(d-1)\}\, \mu\left(\frac{n}{d}\right) f\left(\frac{n}{d}\right) = \sum_{d:n} f(d)\, \mu\left(\frac{n}{d}\right) f\left(\frac{n}{d}\right)$$

$$= f(n) \sum_{d:n} \mu(d), \text{ by } (5.1)$$

$= 0$, when $n > 1$, in consequence of a well known property of $\mu(n)$

As an example, consider the case $n = 6$ We have

$\mu(1) f(1) \{f(1) + f(2) + f(3) + f(4) + f(5) + f(6)\} +$
$+ \mu(2) f(2) \{f(1) + f(2) + f(3)\} + \mu(3) f(3) \{f(1) + f(2)\} + \mu(6) f(6) f(1)$
$= f(1) = f(5)$;

$\mu(1) f(1) \{f(1) + f(2) + f(3) + f(4) + f(5)\} +$
$+ \mu(2) f(2) \{f(1) + f(2)\} + \mu(3) f(3) f(1) = f(1) + f(5)$;
provided (5.1) is satisfied.

In either of the above ways we may also easily prove the corresponding identity

$$\sum_{d:n} \theta(d)\, \mu(d)\, f\left(\frac{n}{d}\right) = \sum_{d:n} \theta(d-1)\, \mu(d)\, f\left(\frac{n}{d}\right), \quad \ldots \quad (6.9)$$

valid under the same condition as (6.8).

§ 7. *Some special cases.*

We will now deduce in a very simple manner some of the properties of the arithmetical functions :

$t(n)$ = the number of the divisors of n (including 1 and n) ;

$\int(n)$ = the sum of the divisors of n ;

$\phi(n)$ = the number of positive integers $< n$ and prime to n.

(For more details about these functions, see *Encycl.*, l. c., or the above-mentioned paper in this Journal).

We verify at once the following formulæ :—

$$[t(n)] = [1]^2 ; \quad \ldots \quad \ldots \quad \ldots \quad (7.1)$$

$$[\textstyle\int(n)] = [1]\,[n] ; \quad \ldots \quad \ldots \quad \ldots \quad (7.2)$$

$$[\phi(n)] = [1]^{-1}.[n] ; \quad \ldots \quad \ldots \quad \ldots \quad (7.3)$$

(the last being equivalent to the fact, that the numerical integral of $\phi(n)$ is n)

Multiplying (7 . 2) by [1], we get, by the aid of (7 . 1)

$$[1].[\textstyle\int(n)] = [1]^2.[n] = [n]\,[t(n)], \quad \ldots \quad (7.4)$$

or in ordinary notation

$$\sum_{d\,:\,n} \int(d) = \sum_{d\,:\,n} d\,.\,t\left(\frac{n}{d}\right). \quad \ldots \quad \ldots (7.41)$$

Similarly, multiplying (7 . 2) by (7 . 3) and using (7 . 1), we have

$$[\textstyle\int(n)] = [\phi(n)].[1]^2 = [\phi(n)].[t(n)], \quad \ldots \quad (7.5)$$

which implies

$$\int(n) = \sum_{d\,:\,n} \phi(d)\,.\,t\left(\frac{n}{d}\right). \quad \ldots \quad \ldots (7.51)$$

Squaring (7 . 2), we obtain, by means of (7 . 1)

$$[\textstyle\int(n)]^2 = [1]^2\,[n]^2 = [t(n)]\,.\left[\sum_{d\,:\,n} d\,.\,\frac{n}{d}\right]$$

$$= [t(n)].[n\,t(n)], \quad \ldots \quad (7.6)$$

which may be written

$$\sum_{d\,:\,n} \int(d).\int\left(\frac{n}{d}\right) = \sum_{d\,:\,n} d\,.\,t(d)\,.\,t\left(\frac{n}{d}\right). \quad \ldots (7.61)$$

The results (7 . 41), (7 . 51), (7 . 61) are originally due to J. Liouville, (see *Encycl.*, l. c., p. 232). Many more similar formulæ could be deduced, especially by inventing new symbols for the new A. F., as they appear.—The following additional example must suffice.

Multiplying (7 . 2) and (7 . 3) in a different manner, we obtain

$$[\int(n)] \cdot [\phi(n)] = [n]^2 = [n \cdot t(n)], \quad \ldots \quad (7.7)$$

or using the ordinary notation

$$n \cdot t(n) = \sum_{d:n} \int (d) \cdot \phi\left(\frac{n}{d}\right) . \quad \ldots \quad \ldots (7.71)$$

We note further, that formula (5 . 3) becomes, for $[f(n)] = [n]$, (which A. F. satisfies (5 . 1)

$$[n]^{-1} = [n \cdot \mu(n)]. \quad \ldots \quad \ldots \quad \ldots \quad (7.8)$$

Consequently, we have, by (7 . 2), (7 . 3) and (5 . 31)

$$[\int(n)]^{-1} = [\mu(n)] \cdot [n \cdot \mu(n)] = \left[\sum_{d:n} d \cdot \mu(d) \cdot \mu\left(\frac{n}{d}\right)\right] \quad \ldots (7.21)$$

$$[\phi(n)]^{-1} = [1] \cdot [n \cdot \mu(n)] = \left[\sum_{d:n} d \cdot \mu(d)\right] . \quad \ldots (7.13)$$

DOUBLE POINTS AND LINES.

BY M. BHIMASENA RAO.

(*Concluded from page* 135).

PART III.

Correspondence.

§ 1. *One-to-one correspondence.*

Given a triangle ABC and an orthologic triangle A′B′C′ which we have denoted by the notation (P, λ, μ, v), it is evident from the reciprocal relation between P and D, that there is a one-to-one correspondence between P and D. The equations for determining the double point given in page 126, show that when P moves on a line, D moves on a conic and *vice versa*; and generally, any curve of degree n is changed into a curve of degree $2n$ by this transformation. The transformation leaves a triangle invariant for the D locus and another for the P locus. If P is a point such that AA′, BB′ and CC′ are parallel, D becomes indeterminate and may be any point on the axis of perspective of ABC and A′B′C′.[8] This axis of perspective is therefore a side of the triangle invariant for the D locus. Similarly when D is at a vertex of the invariant triangle, P becomes indeterminate and may be any point on the axis of perspective of ABC and A″B″C″ (D, $-\lambda$, $-\mu$, $-v$), which is therefore a side of the invariant triangle of the P locus.

The case when A′B′C′ is the pedal triangle of P (*i.e.* $\lambda = \mu = v = 1$) is interesting geometrically; and we will therefore discuss in the following some of the properties of the invariant triangles, $D_1D_2D_3$ and $P_1P_2P_3$ of the D and P loci.

§ 2. *The orthopoles of the sides of an invariant triangle.*

THEOREM XII.

D_1, D_2, D_3 are respectively the orthopoles of the sides of $P_1P_2P_3$ with respect to ABC.

Let LMN be the pedal triangle of P such that AL, BM, CN are parallel. Now if the pedal triangle of a point with respect to ABC is in

[8] See foot-note on page 265, Vol. X? *J. I. M. S.*

perspective with ABC, it is known that the locus of the centre of perspective is a cubic. Consequently AL is parallel to an asymptote of this cubic. Since there are three asymptotes, there are three positions of the centre of perspective, say Q_1, Q_2, Q_3 and three corresponding positions of P. These are the verticies of the invariant triangle $P_1P_2P_3$. If the double point D is any point on the line D_1D_2, P is at P_3 and if it is any point on D_1D_3, P is at P_2. Hence if D is at D_1, the point of intersection of D_1D_2 and D_1D_3, the position of P becomes indeterminate and may be any point on P_2P_3, showing that the double point of ABC and the pedal triangle of any point on P_2P_3 is D_1. Now since[9] the double point of the pedal triangles of two points is the orthopole of their join, D_1 is the orthopole of P_2P_3. Similarly D_2, D_3 are the orthopoles of P_3P_1 and P_1P_2.

§ 3. *Properties of the triangles $D_1D_2D_3$ and $P_1P_2P_3$.*

We will now establish some properties of the two invariant triangles.

THEOREM XIII.

The triangle $D_1D_2D_3$ is inscribed in the circum-circle of ABC and circumscribed about the Steiner's ellipse of the medial triangle of ABC.

If λ, μ, v be the direction angles of P_2P_3, the orthopole D_1 is the mean centre of the vertices of the pedal triangle of any point on P_2P_3 for the multiples, $a \sec \lambda$, $b \sec \mu$, $c \sec v$. Since D_1 is the double point of ABC and the pedal triangle, the ABC-areal co-ordinates of D_1 are $a \sec \lambda$, *etc.*, showing that D_1 lies on the circum-circle of ABC. Similarly D_2, D_3 lie on the circle ABC.

If LMN be the pedal triangle of P_1, D_2D_3 is the axis of perspective of ABC and LMN. Since AL, BM, CN are parallel, the centre of perspective is at infinity. Also the axis of perspective being the triangular polar of a point at infinity, it touches the Steiner's ellipse of the medial triangle of ABC.

The corresponding theorem for $P_1P_2P_3$ is the following :—

THEOREM XIV.

The triangle $P_1P_2P_3$ is inscribed in a circle concentric with, and radius three times that of the circum-circle of ABC.

If LMN be three points on the sides of a triangle ABC, such that AL, BM, CN are parallel, the area of LMN is twice that of ABC.

[9] *loc. cit.* Vol. IV, p. 21.

Further if the perpendiculars at L, M, N to the sides of ABC concur at P, P lies on a circle concentric with and radius three times that of the circle ABC.

THEOREM XV.

D_1 is equidistant from P_2P_3 and the pedal line of D_1 with respect to ABC.

Let XYZ be the pedal triangle of any point P on P_2P_3. The double point of ABC and XYZ is D_1.

Given D_1, the construction for finding P is as follows:—Draw D_1L, D_1M, D_1N perpendicular to the sides of ABC and produce them to L′, M′, N′ such that D_1 is a point of bisection of LL′, MM′, NN′, then P is the double point of ABC and L′M′N′.

Since D_1 is a vertex of the invariant triangle, AL′, BM′, CN′ are parallel, and P is any point on the axis of perspective of ABC and L′M′N′, *i.e.*, to say P_2P_3 is the axis of perspective. Since L, M, N, are collinear (on the pedal of D_1), L′, M′, N′ are also collinear. Consequently P_2P_3 passes through L′, M′, N′ and the theorem follows.

THEOREM XVI.

The orthocentre of $D_1D_2D_3$ is the median point of ABC.

Let S be the point on the circle ABC diametrically opposite to D_1. The pedal line of D_1 is parallel to P_2P_3 by Theorem XV. Therefore the pedal line of S is perpendicular to P_2P_3, and hence passes through the orthopole of P_2P_3, *viz.*, D_1. Since the pedal line of S passes through its diametrically opposite point D_1, this pedal line passes through the median point of ABC.[10]

Also, the pedal line of a point passes through the isogonal conjugate of its diametrically opposite point. Therefore the pedal line of S which is GD_1 (G is the median point of ABC) passes through the isogonal conjugate of D_1. Therefore D_1 and similarly D_2, D_3 are the foci of parabolas touching ABC whose axes concur at the median point of ABC. But, if the axes of three parabolas escribed to a triangle are concurrent, it is well-known that the point of concurrence is the ortho-centre of the triangle formed by their foci. Hence the result.

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[10] W. F. Beard's theorem, page 25, Gallatly's *Modern Geometry*.

The last theorem suggests the following :—

THEOREM XVII.

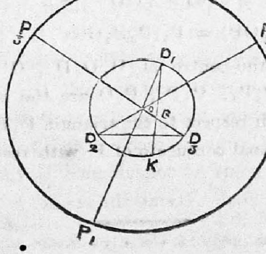
The orthocentre of $P_1 P_2 P_3$ is the ortho-centre of the anti-medial triangle of ABC.

Since G is the ortho-centre of $D_1D_2D_3$, GD_1 is perpendicular to D_2D_3. P_2P_3 is parallel to the pedal line of D_1 with respect to ABC which is perpendicular to GD_1, for GD_1 is the axis of the escribed parabola whose focus is D_1. Therefore P_2P_3 is parallel to D_2D_3, and hence the triangles $P_1P_2P_3$ and $D_1D_2D_3$ are homothetic, the homothetic centre being the circum-centre of ABC.

If LMN is the pedal triangle of P_1, AL, BM, CN are parallel, and D_2D_3 is the axis of perspective of ABC and LMN. But if the pedal triangle of any point with respect to a triangle is in perspective with the triange, it is known that the perpendicular through the point on the axis of perspective passes through the ortho-centre of the antimedial triangle of the fundamental triangle.

Therefore, if H′ is the ortho-centre of the antimedial triangle, $H'P_1$ is perpendicular to D_2D_3. But D_2D_3 is parallel to P_2P_3. Therefore $H'P_1$ is perpendicular to P_2P_3. Hence the result.

In conclusion, it is noticed that the ratio of similitude of the triangles $D_1D_2D_3$ and $P_1P_2P_3$ is easily seen to be $-\frac{1}{3}$, *i.e.*, D_1 and P_1 are on opposite sides of the circum-centre of ABC. Hence when any one of the six points $D_1D_2D_3$ $P_1P_2P_3$, (say D_1) is given, the remaining five are determined in the following manner :



Let O be the circum-centre and G the median point of ABC. Join D_1G and produce it to meet the circum-circle of ABC at K. Bisect GK at right angles by D_2D_3. Produce D_1O, D_2O, D_3O to meet the outer circle (centre O and radius thrice that of circle ABC). These meet in three points which possess the properties of P_1, P_2, P_3 mentioned above.

The correspondence between P and D may now be stated as follows :—

THEOREM XVIII.

If D is the double point of the triangle ABC and of the pedal triangle of P with respect to ABC, then $P_1P_2P_3$—correspondent of D, considered as a point of $D_1D_2D_3$, is the isogonal conjugate of P with respect to $P_1P_2P_3$.

If P moves on a line, D moves in general on a conic through $D_1D_2D_3$. When the line passes through P_1, the conic breaks up into the line D_2D_3 and a line through D_1. Now the double point loci of a pencil of lines through P_1 form a pencil through D_1 and the two pencils correspond anharmonically. Hence P_1P_2 corresponds to D_1D_3 because the double point corresponding to P_2 is any point on D_1D_3. Similarly P_1P_3 corresponds to D_1D_2. The circum-centre O of ABC corresponds to G, the median point of ABC since G is the double point of ABC and the pedal triangle of O.

Therefore we have,

$$P_1(P_2P_3OP) = D_1(D_3D_2GD).$$

If the points D_1, D_2, D_3, G, D be considered as belonging to $D_1D_2D_3$, their correspondents with respect to the homothetic triangle $P_1P_2P_3$ are respectively P_1, P_2, P_3, G′, D′ where G′ and D′ are on GO and DO produced such that $\frac{GO}{OG'} = \frac{DO}{OD'} = 3.$

Therefore $P_1(P_2P_3OP) = P_1(P_3P_2G'D')$.

Since G is the ortho-centre of $D_1D_2D_3$, G′ is the ortho-centre of $P_1P_2P_3$. The lines P_1P_2, P_1P_3, P_1O are the isogonal conjugates of P_1P_3, P_1P_2, P_1G' with respect to the triangle $P_1P_2P_3$. It is now easily seen that D′ is the isogonal conjugate of P with respect to $P_1P_2P_3$, and hence the result.

SHORT NOTES.

Invariants of a Conic.

The Boolians $\frac{ab - h^2}{\sin^2 \omega}$ and $\frac{a + b - 2h \cos \omega}{\sin^2 \omega}$ are quasi-invariants of a conic for any change of cartesian axes, and are largely useful in expressing metrical properties. The equivalents of these for transformations in homogeneous co-ordinates are obtained in Jones' *Algebraical Geometry*, p. 513, in an involved manner. In Salmon's Conic Sections, at p. 351 (6th edition), are found the values of these quantities correct to a constant multiplier. The present note arrives at their exact values, from first principles of linear transformation.

1. The formulæ of transition from oblique cartesians (x, y, z) with $z = 1$ to a system of homogeneous co-ordinates (x', y', z'). when the co-ordinates of the vertices of the triangle of reference PQR are (x_1, y_1), (x_2, y_2), (x_3, y_3), are,

$$\left.\begin{aligned} x &= \alpha x_1 x' + \beta x_2 y' + \gamma x_3 z' \\ y &= \alpha y_1 x' + \beta y_2 y' + \gamma y_3 z' \\ z &= \alpha z_1 x' + \beta z_2 y' + \gamma z_3 z' \end{aligned}\right\} \quad \ldots \quad \ldots \quad (1)$$

where $\alpha x' + \beta y' + \gamma z' = 1$ is the identical relation in the homogeneous system.

Also if (l, m, n) and (l', m', n') are the tangential co-ordinates of a line, in the two systems, then

$$\left.\begin{aligned} l' &= \alpha(x_1 l + y_1 m + z_1 n) \\ m' &= \beta(x_2 l + y_2 m + z_2 n) \\ n' &= \gamma(x_3 l + y_3 m + z_3 n) \end{aligned}\right\} \quad \ldots \quad \ldots \quad (2)$$

In either case, the modulus of transformation ε is equal to $\frac{2\alpha\beta\gamma\delta}{\sin \omega}$, where δ is the area of the triangle of reference.

2. Let $S \equiv ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy$ transform into $S' \equiv a'x'^2 + b'y'^2 + c'z'^2 + 2f'y'z' + 2g'z'x' + 2h'x'y'$ by means of the substitution (1).

Δ denoting the usual determinant with the constants of the conic we have

$$\Delta_{s'} = \Delta_{s} \cdot \varepsilon^2$$

or

$$\frac{\Delta_s}{\sin^2 \omega} = \frac{\Delta_{s'}}{4\alpha^2\beta^2\gamma^2\delta^2} \quad \ldots \quad \ldots \quad \ldots \quad (3)$$

The tangential equations of the conic are, corresponding to

$$S = 0 \text{ and } S' = 0,$$

$$\Sigma_1 \equiv Al^2 + Bm^2 + Cn^2 + 2Fmn + 2Gnl + 2Hlm = 0$$

and $\Sigma_1' \equiv A'l'^2 + B'm'^2 + C'n'^2 + 2F'm'n' + \quad \ldots \quad = 0.$

Therefore the function Σ_1' must, by the substitution (2), transform into $k\Sigma_1$, where k is a constant. Hence the discriminant of $\Sigma_1' . - k . \Sigma_1$ must be equal to that of Σ_1' multiplied by the square of modulus of transformation (2).

$$\therefore \quad k^3 \Delta_{\Sigma'_1} = \varepsilon^2 . \Delta_{\Sigma'_1}.$$

$$\text{But } \Delta_{\Sigma_1} = \Delta_s^2 \text{ and } \Delta_{\Sigma'_1} = \Delta_{s'}^2.$$

while by (3), $\quad \Delta_{s'} = \varepsilon^2 \Delta.$

$$\therefore k^3 = \varepsilon^6,$$

so that $\quad k = \varepsilon^2.$

$$\text{Hence } \sum \equiv \frac{Al^2 + Bm^2 + Cn^2 + 2Fmn + 2Gnl + 2Hlm}{\sin^2 \omega}$$

$$\text{and } \quad \sum{}' \equiv \frac{A'l'^2 + B'm'^2 + C'n'^2 + 2F'm'n' + 2G'n'l' + 2H'l'm'}{4\alpha^2\beta^2\gamma^2\delta^2}$$

are equivalent, in the sense that each transforms into the other, by substitutions (2)

3. Also, the function

$$\Omega' \equiv \frac{p^2}{\alpha^2} l'^2 + \frac{q^2}{\beta^2} m'^2 + \frac{r^2}{\gamma^2} n'^2 - 2\frac{qr}{\beta\gamma} m'n' \cos P \ldots\ldots$$

is equivalent to

$$\Omega \equiv \frac{l^2 + m^2 + 2lm \cos \omega}{\sin^2 \omega}.$$

where p, q, r, P, Q, R denote the sides and angles of the triangle of reference.

For, by substituting from (2) in Ω', the co-efficient of l^2 is found to be

$$\frac{p^2x_1^2 + q^2x_2^2 + r^2x_3^2 - 2qr \cos P\, x_2x_3 \ldots\ldots}{4\delta^2}.$$

But the value of the numerator of this fraction is exactly $\frac{4\,\delta^2}{\sin^2 \omega}$ (*vide* Askwith "*Analytical Geometry of the conic section*" p. 282.]

4. Thus, combining the results of §§ 2 and 3, we see that the function

$$\Sigma' + \lambda\,\Omega' \ldots \quad \ldots \quad \ldots \quad \ldots \quad \text{(A)}$$

will transform into

$$\Sigma + \lambda\,\Omega \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \text{(B)}$$

by means of the substitutions (2).

When λ is such that the discriminant of (A) vanishes, for the same value of λ will the discriminant of (B) vanish. So the roots of the two equations

$$\Delta_{s'}^2 + \lambda\Delta_{s'} (\alpha^2\beta^2\gamma^2)\, I_1 + \lambda^2 (4\delta^2\alpha^2\beta^2\gamma^2)\, I_2 = 0$$

and $\quad \Delta_s^2 + \lambda\Delta_s (a + b - 2h \cos \omega) + \lambda^2 (ab - h^2) \sin^2 \omega = 0,$

must be the same. Identifying the two equations, and using the result (3) we obtain

$$\left.\begin{aligned} \frac{a+b-2h\cos\omega}{\sin^2\omega} &= \frac{I_1}{4\delta^2}, \\ \text{and} \quad \frac{ab-h^2}{\sin^2\omega} &= \frac{I_2}{4\delta^2\alpha^2\beta^2\gamma^2}, \end{aligned}\right\} \quad \ldots \quad \ldots (4)$$

where $I_1 \equiv a' \dfrac{p^2}{\alpha^2} + b' \dfrac{q^2}{\beta^2} + c' \dfrac{r^2}{\gamma^2} - 2f' \dfrac{qr}{\beta\gamma} \cos P - \ldots\ldots$

and $I_2 \equiv A'\alpha^2 + B'\beta^2 + C'\gamma^2 + 2F'\beta\gamma + 2G'\gamma\alpha + 2H'\alpha\beta.$

5. By way of illustration we proceed to obtain the equation giving the squares of the semi-axes of the conic

$$S \equiv ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$$

in general homogeneous co-ordinates.

Transforming to cartesians with the axes of the conic as co-ordinate axes, let S transform into

$$S \equiv C\left(\frac{x^2}{r_1^2} + \frac{y^2}{r_2^2} - z^2\right)$$

where r_1, r_2 are the semi-axes.

Hence, by formulæ (4) of the last section, we should have

$$\left.\begin{aligned} \frac{\Delta}{4\alpha^2\beta^2\gamma^2\delta^2} &= -\frac{C^3}{r_1^2 r_2^2} \\ \frac{I_1}{4\delta^2} &= C\left(\frac{1}{r_1^2} + \frac{1}{r_2^2}\right) \\ \frac{I_2}{4\alpha^2\beta^2\gamma^2\delta^2} &= \frac{C^2}{r_1^2 r_2^2} \end{aligned}\right\} \ldots \quad \ldots \quad \ldots (5)$$

Now, r_1^2, r_2^2 are the roots of the equation

$$\frac{\theta^2}{r_1^2 r_2^2} - \left(\frac{1}{r_1^2} + \frac{1}{r_2^2}\right)\theta + 1 = 0 \ldots \quad (6)$$

So, eliminating c, r_1^2, r_2^2 from (5) and (6), we obtain,

$$I_2^3\,\theta^2 + I_1 I_2 \Delta (\alpha^2\beta^2\gamma^2)\,\theta + 4\Delta^2\alpha^2\beta^2\gamma^2\delta^2 = 0$$

as the equation whose roots are the squares of the semi-axes of the conic S.

It may be noted in passing that this can be readily applied to Mr. A. C. L. Wilkinson's question No. 1038 of this journal.* Taking for S the point equation to the conic whose equation in tangential areals is given, and making $\alpha = \beta = \gamma = 1$, $\delta = 2R\sin A\sin B\sin C$, we get the result in question.

G. A. Srinivasan.

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* *Vide* J. I. M. S., vol. XI, p. 188, and vol. XII, p. 70, for other solutions.

Note on Question 248.

Q. 248. (S. NARAYANAN, B.A., L.T.) :—Shew that the trilinear equation of the Steiner's envelope of a triangle ABC is

$$\Sigma l^4 \cos^2 B \,.\, \cos^2 C \,.\, \sin^2 A$$
$$+\ 2\Sigma l^3 m \sin A \cos B (\sin \overline{A-B} + \sin^3 C)$$
$$+\ \Sigma l^2 m^2 (\cos^2 C \sin^2 C + 2 \cos A \,.\, \cos B \,.\, \sin A \,.\, \sin B \,.\, \overline{\cos^2 C - 4})$$
$$+\ 2\Sigma l^2 mn (3 \sin^2 A + 3 \sin^4 A + 7 \cos A \,.\, \sin^2 A \,.\, \sin B \,.\, \sin C$$
$$-\ \cos B \cos C \,.\, \sin B \,.\, \sin C)$$

where $l = a\alpha$, $m = b\beta$, $n = c\gamma$.

Let θ, ϕ, ψ be the direction angles of any line.

We have

$$a \cos \theta + b \cos \phi + c \cos \psi = 0 \quad \text{.........(1)}$$
$$a \sin \theta + b \sin \phi + c \sin \psi = 0 \quad \text{.........(2)}$$

From (1) and (2) we have $\sin^2 (\phi - \psi) = \sin^2 A$.........(3)
and $\sin 2A \,.\, \sin 2\theta + \sin 2B \,.\, \sin 2\phi + \sin 2C \,.\, \sin 2\psi = 0$.........(4)

The equation of a pedal line is

$$l \tan \theta + m \tan \phi + n \tan \psi = 0.$$

The tangential equation of the envelope is got by eliminating θ, ϕ, ψ from

$$\frac{\tan \theta}{\lambda} = \frac{\tan \phi}{\mu} = \frac{\tan \psi}{v} = k^{\frac{1}{3}}, \text{ say.}$$

Now $\tan \theta (\tan \phi - \tan \psi)^2 = k\lambda (\mu - v)^2$

$$\therefore \quad \frac{\sin \theta \,.\, \cos \theta \,.\, \sin^2 (\phi - \psi)}{\cos^2\theta \,.\, \cos^2 \phi \,.\, \cos^2 \psi} = k\lambda (\mu - v)^2.$$

But $\sin^2 (\phi - \psi) = \sin^2 A.$

$$\therefore \quad \frac{\sin 2\theta \,.\, \sin 2A}{\cos^2 \theta \,.\, \cos^2 \phi \,.\, \cos^2 \psi} = 4k\lambda (\mu - v)^2 \cot A$$

and from (4) we have

$$\lambda(\mu - v)^2 \cot A + \mu (v - \lambda)^2 \cot B + v (\lambda - \mu)^2 \cot C = 0 \quad \text{...(5)}$$

which is the tangential equation of the envelope.

From the equation of the reciprocal of the general cubic (Cayley's Collected Works, Vol. II, pages 328-9) we form the reciprocant of

$$p\lambda (\mu - v)^2 + q\mu (v - \lambda)^2 + rv (\lambda - \mu)^2.$$

The reciprocant is a sextic, but contains the factor $(l+m+n)^2$. On division, we obtain a quartic the co-efficients of whose typical terms are given below :—

l^4		$2m^3n$		$2mn^3$		m^2n^2		$2l^2mn$	
q^2r^2	$+1$	p^2qr	-1	p^2qr	-1	p^2q^2	$+1$	p^2q^2	$+10$
		p^2r^2	$+1$	p^2q^2	$+1$	p^2qr	-4	p^2qr	$+19$
		pqr^2	$+2$	prq^2	$+2$	p^2r^2	$+1$	p^2r^2	$+10$
		pr^3	$+2$	pq^3	$+2$	pq^2r	-8	pq^3	$+6$
		qr^3	$+2$	rq^3	$+2$	pqr^2	-8	pq^2r	$+24$
						q^2r^2	-8	pqr^2	$+24$
								pr^3	$+6$
								q^3r	$+6$
								q^2r^2	$+11$
								qr^3	$+6$

In this put $p = \cot A$, $q = \cot B$, $r = \cot C$, and multiply throughout by $\tan^2 A . \tan^2 B . \tan^2 C$. Observe that the last three terms in the co-efficients of m^3n and m^2n^2 contain the factor $pq + qr + rp$, *i.e.*, $\Sigma \cot A . \cot B = 1$. The equation of the envelope will be found to be

$$\Sigma l^4 \tan^2 A + 2 \Sigma m^3n (\tan^2 B - \tan B \tan C + 2 \tan^2 A . \tan^2 B) + \Sigma m^2n^2 (\tan^2 B + \tan^2 C - 4 \tan B . \tan C - 8 \tan^2 A . \tan B . \tan C) + 2 \Sigma l^2mn \{ 2 \tan^2 A + \tan^2 B + \tan^2 C + \tan B . \tan C + 6 \tan^2 A (\tan^2 B + \tan^2 C) + 9 \tan^2 A . \tan^2 B . \tan^2 C \} = 0 \qquad (6)$$

If the expression given in the question be divided by $\cos^2 A . \cos^2 B . \cos^2 C$, and the result expressed in terms of tangents, we get for the co-efficients of $2m^3n$, m^2n^2, $2l^2mn$,

$$\tan B (1 + \tan^2 A)(\tan B - \tan C) + \tan^2 A \tan B (\tan B + \tan C);$$
$$(\tan B + \tan C)^2 + 2 \tan B \tan C \{1 - 4(1 + \tan^2 A)\};$$

and $3 \tan^2 A (1 + \tan^2 B)(1 + \tan C) + 3 (\tan B + \tan C)^2 \tan^2 A + 7 \tan^2 A \tan B \tan C (\tan B \tan C - 1) - \tan B \tan C (1 + \tan^2 A)$.

The first two co-efficients are easily seen to be identical with those given in (6) and the last after a little simplification by the use of the relation $\tan A + \tan B + \tan C = \tan A \tan B \tan C$.

It is easily seen from Geometry that the Steiner's envelope touches the side BC in a point L which is the isotomic conjugate of the projection of A on BC. It cuts BC in two points D, D′ which are the projections on BC of the extremities of the circum-diameter parallel to BC, The equation of AL is $m \tan B - n \tan C = 0$

... ... AD ... $m(1 + \sin A) + n(1 - \sin A) = 0$

... ... AD′ ... $m(1 - \sin A) + n(1 + \sin A) = 0$

∴ AD, AD′ are represented by

$$m^2 + n^2 + 2mn(1 + 2\tan^2 A) = 0.$$

If M, E, E′ ; N, F, F′ denote similar points with respect to CA, AB, the conic

$$S \equiv \Sigma l^2 \tan^2 A - 2\Sigma mn \tan B \tan C = 0$$

touches the envelope at L, M, N. The six points D, D′, E, E′, F, F′ lie on a conic

$$S' = \Sigma l^2 + 2\Sigma mn(1 + 2\tan^2 A) = 0.$$

The equation of the Steiner's envelope may therefore be put in the form

$$SS' + 4\, lmn\, L = 0 \qquad \ldots \qquad \ldots \qquad \ldots \qquad (7)$$

where L is a linear expression.

The first twelve terms of (6) and (7) will be found to agree. By comparing with (6), L may be shown to be equal to

$$k(l + m + n) + k' \{ l(\tan B - \tan C)^2 + m(\tan C - \tan A)^2 + n(\tan A - \tan B)^2 \},$$

where $k = -\Sigma \tan^4 A + 2\Sigma \tan^2 B \cdot \tan^2 C + 7\tan^2 A \cdot \tan^2 B \cdot \tan^2 C$.

and $k' = \tan A \cdot \tan B \cdot \tan C$.

M. Bhimasena Rao.

Leaves from a Lecturer's Diary.

Many of the elementary properties of points and triangles are but particular cases of theorems relating to circles, and are easily recognised to be such when the point is identified as a circle of zero radius. Thus in parallel columns :—

(1) The perpendicular bisector of the line joining two points is the locus of points equidistant from them.	The locus of points the tangents from which to two circles are equal is a line (the radical axis) perpendicular to the line of centres.
(2) The perpendicular bisectors of the sides of a triangle are concurrent (at the circum-centre).	The three radical axes of three circles taken in pairs co-intersect (at the radical centre).
(3) The circle of Apollonius described on a segment of the line joining two points is the locus of points at distances from the pair of points in a constant ratio.	The locus of points the tangents from which to two given circles are in a given ratio is a circle coaxal with the given circles.

* * *

The asymptotic formula for $(n)!$, when n is large, real and positive, is $\sqrt{2\pi}\left(\frac{n}{e}\right)^n$. To Newton is due the evaluation of $\sum_{r=0}^{n} {}^nC_r$, whatever n may be, in the simple form 2^n. Required the sum when n is large of $\sum_{r=1}^{n} {}^nP_r$. Obviously no simple formula can be given for its exact value; but asymptotically the sum behaves like $e \times (n)!$; for

$$\sum_{r=1}^{n} {}^nP_r = \sum_{r=1}^{n} \left[\frac{n!}{(n-r)!}\right]$$

$$= n!\left[1 + \frac{1}{1!} + \frac{1}{2!} + \ldots\ldots \frac{1}{\overline{n-1}!}\right]$$

$$= n! \times e \text{ when } n \text{ is large.}$$

Even when $n = 9$, $9! \times e = 986410$.

The sum $\Sigma\, {}^9P_r$ actually calculated $= 986409$. The approximation, however, is always in excess by unity neglecting the decimal fraction.

* * *

With respect to a triangle ABC, the isotomic conjugate, Q, of the symmedian point K, will have the trilinear co-ordinates $(1/a^3,\ 1/b^3,\ 1/c^3)$, which are of course the reciprocals of the co-ordinates of the point $(a^3, b^3. c^3)$.

(2) This latter point may be easily seen to be the pole of the line joining the Brocard points Ω, Ω' of the triangle with respect to its Brocard circle.

(3) It can be easily seen that the original triangle and its first Brocard triangle are in perspective, the centre of perspective being Q.

(4) Another interesting property is that the centre of gravity of the triangle $\Omega Q \Omega'$ coincides with that of the original triangle; for the α-co-ordinates of Ω, Q, Ω' are respectively $\frac{2\Delta}{a^2b^2+b^2c^2+c^2a^2}$ times c^2a, $\frac{b^2c^2}{a}$, ab^2 and hence the α-co-ordinate of their centroid is $\frac{2\Delta}{3} \cdot \frac{1}{a}$ which agrees with the corresponding co-ordinate of G.

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SOLUTIONS.

Question 1007.

(Alpha) Prove that, if

$$S_0 = 1! - 2! + 3! - 4! + \ldots\ldots\ldots,$$

then $\quad 1(1!) - 2(2!) + 3(3!) - 4(4!) + \ldots\ldots = 1 - 2S_0$

$$1^2(1!) - 2^2(2!) + 3^2(3!) - 4^2(4!) + \ldots\ldots = 5S_0 - 2$$

and generally $\quad \Sigma\,[(-)^{n-1} n^k (n!)]$ is of the form $\alpha S_0 + \beta$, where α and β are integers (positive or negative).

Solution by Martyn M. Thomas, S. Krishnaswami Iyengar and Tiruvenkatachari, Hemraj and others.

(i) Left side $= \sum_{n=1}^{\infty} (-)^{n-1} n \,.\, n!$

$$= \sum_{n=1}^{\infty} (-)^{n-1} \{(n+1) - 1\} n! = \sum_{n=1}^{\infty} (-)^{n-1} \{(n+1)! - n!\}$$

$$= (2! - 3! + 4! - \ldots) - (1! - 2! + 3! - \ldots)$$

$$= (1 - S_0) - S_0 = 1 - 2S_0$$

(ii) Left side $= \sum_{n=1}^{\infty} (-)^{n-1} n^2\, n!$

$$= \sum_{n=1}^{\infty} (-)^{n-1} \{A(n+2)(n+1) + B(n+1) + C\}\, n!, \text{ where } A = 1,$$

$$B = -3, C = 1$$

$$= \sum_{n=1}^{\infty} (-)^{n-1} \{(n+2)! - 3(n+1)! + n!\}$$

$$= (3! - 4! + 5! - \ldots) - 3(2! - 3! + 4! - \ldots) + (1! - 2! + 3! - \ldots)$$

$$= (S_0 - 1! + 2!) - 3(1! - S_0) + S_0 = 5S_0 - 2$$

(iii) Left side $= \sum_{n=1}^{\infty} (-)^{n-1} n^k\, n!$

$$= \sum_{n=1}^{\infty} (-)^{n-1} \left\{ \begin{array}{l} A(n+k)(n+k-1)\ldots(n+2)(n+1) + \\ \qquad B(n+k-1)\ldots(n+2)(n+1) \\ + C(n+k-2)\ldots(n+1) + \ldots + L(n+2)(n+1) + M(n+1) + N \end{array} \right\} n!$$

where A,B,C...L,M,N are numerical constants which can be obtained by comparing co-efficients of different powers of n.

$$= \sum_{n=1}^{\infty} (-)^{n-1} \{ A(n+k)! + B(n+k-1)! + C(n+k-2)! + \dots L(n+2)!$$

$$+ M(n+1)! + N n! \}$$

$$= A(1+k)! - (2+k)! + (3+k)! - \dots] + Bk! - (1+k)! + (2+k)! - \dots)$$
$$+ C(k-1)! - k! + (k+1)! - \dots] + \dots L(3! - 4! + \dots) + M(2! - 3!$$
$$+ \dots) + N(1! - 2! + \dots)$$

$$= A(S_0 - \text{const}) + B(\text{const} - S_0) + \dots + L(S_0 - \text{const})$$
$$+ M(\text{const} - S_0) + N S_e$$

$$= \alpha S_0 + \beta,$$

where α, β are numercial constants, positive or negative.

Question 1008.

(MARTYN M. THOMAS) :—If I_n be written for the integral

$$\int_0^{\frac{\pi}{2}} \cos 2nx \log \left(2 \cos \frac{x}{2}\right) dx,$$

prove that
$$\frac{I_{n+1}}{n} - \frac{I_n}{n+1} = \frac{(-)^{n+1}}{2n(n+1)(2n+1)},$$

and deduce that

$$2n I_n = \left(1 - \frac{1}{3} + \frac{1}{5} - \dots \pm \frac{1}{2_{n-1}}\right) - \frac{\pi}{4}.$$

Solution by V. Tiruvenkatachari and K. B. Madhava and several others.

Integrating by parts

$$I_n = \int_0^{\frac{\pi}{2}} \cos 2nx \log \left(2 \cos \frac{x}{2}\right) dx$$

$$= \frac{1}{2n}\left[\sin 2nx \log\left(2\cos\frac{x}{2}\right)\right]_0^{\frac{\pi}{2}} + \frac{1}{4n}\int_0^{\frac{1}{2}\pi} \sin 2nx \tan\frac{x}{2}\, dx.$$

$$\therefore (n+1) I_{n+1} - n I_n = \tfrac{1}{4} \int_0^{\frac{1}{2}\pi} \tan\frac{x}{2}\left[\sin 2(n+1)x - \sin 2nx\right] dx$$

$$= \frac{(-)^{n+1}}{2(2n+1)},$$

which establishes the first part,

Writing successively $n-1$, $n-2$, ... for n in the above and adding, we have

$$n\,I_n - I_1 = \frac{1}{2}\left[-\frac{1}{3}+\frac{1}{5}-\dots\pm\frac{1}{2n-1}\right].$$

But

$$I_1 = \int_0^{\frac{\pi}{2}} \cos 2x \log\left(2\cos\frac{x}{2}\right)dx$$

$$= \tfrac{1}{4}\int_0^{\frac{\pi}{2}} \sin 2x \tan\frac{x}{2}\,dx = \frac{1}{2}\left(1-\frac{\pi}{4}\right).$$

Hence

$$2n\,I_n = \left[1-\frac{1}{3}+\frac{1}{5}-\dots\pm\frac{1}{2^{n-1}}\right]-\frac{\pi}{4}.$$

Question 1050.

(K. S. Srinivasachari, M.A., L.T.) :—If $(ax^2+bx+c)^{-1} = \sum_0^\infty p_n x^n$,

find in terms of a, b, c, x the value of $\sum_0^\infty p_n^2\, x^{2n}$.

Solution by Hemraj, Nagendranath Dutt and G. R. Narayan Aiyar.

This question is given in Hardy's *Pure Mathematics*. (Ex. 26, p. 160, 2nd Edition).

We suppose that a, b, c are real and that $b^2 < 4ac$; then the roots of $ax^2+bx+c=0$ are conjugate complex numbers. If α and β be the roots, $\alpha = \rho \operatorname{cis}(\phi)$, $\beta = \rho \operatorname{cis}(-\phi)$,

where $\rho^2 = \alpha\beta = \frac{c}{a}$ and $\cos\phi = \frac{b}{2.\sqrt{ac}}$ ....(1)

Now

$$\frac{1}{ax^2+bx+c} = \frac{1}{a}\left[\frac{1}{(x-\alpha)(x-\beta)}\right] = \frac{1}{a\alpha\beta}\left(1-\frac{x}{\alpha}\right)^{-1}\left(1-\frac{x}{\beta}\right)^{-1}$$

$$= \frac{1}{a\alpha\beta}\sum_0^\infty \frac{x^n}{\alpha^n}\times\sum_0^\infty\frac{x^n}{\beta^n} = \sum_0^\infty \frac{\sin(n+1)\phi}{c\sin\phi\rho^n}\,x^n$$

whence $p_n = \dfrac{\sin(n+1)\phi}{\rho^n c\sin\phi}$. ....(2)

$$\therefore \quad \sum_0^\infty p_n^2 x^n = \sum_0^\infty \frac{\sin^2 (n+1)\phi}{c^2 \rho^{2n} \sin^2 \phi} x^n$$

$$= \sum_0^\infty \left[\frac{\sin (n+2)\phi \sin (n+1)\phi}{\sin \phi} + \frac{\sin (n+1)\phi \sin n\phi}{\sin \phi} \right] \frac{x^n}{c^2 \sin 2\phi\, \rho^{2n}}$$

$$= \sum_0^\infty \frac{\sin (n+2)\phi \sin (n+1)\phi}{\sin \phi} \cdot \frac{x^n}{c^2 \sin 2\phi . \rho^{2n}} + \sum_1^\infty \frac{\sin (n+1)\phi \sin n\phi \; x^n}{c^2 \sin \phi\; \rho^{2n} \cdot \sin 2\phi}$$

$$= \sum_0^\infty \frac{\sin (n+2)\phi \sin (n+1)\phi}{c^2 \sin \phi\; \rho^{2n}} \cdot \frac{x^n}{\sin 2\phi}$$

$$+ \frac{x}{\rho^2} \sum_0^\infty \frac{\sin (m+2)\phi \sin (m+1)\phi}{c^2 \sin \phi\; \rho^{2m}} \cdot \frac{x^n}{\sin 2\phi}$$

$$= \left(1 + \frac{x}{\rho^2}\right) \sum_0^\infty \left[\sum_1^{n+1} \sin\ 2r\ \phi \right] \cdot \frac{x^n}{c^2 \rho^{2n} \sin 2\phi}$$

$$= \frac{1}{c^2} \left(1 + \frac{x}{\rho^2}\right) \frac{1}{\sin 2\phi} \cdot \sum_0^\infty \frac{x^n}{\rho^{2n}} \times \sum_0^\infty \frac{\sin 2(n+1)\phi}{\rho^{2n}} x^n$$

$$= \left(1 + \frac{x}{\rho^2}\right) \left(1 - \frac{x}{\rho^2}\right)^{-1} \cdot \frac{1}{a^2 \alpha^2 \beta^2} \cdot \sum_0^\infty \frac{x^n}{\alpha^{2n}} \times \sum_0^\infty \frac{x^n}{\beta^{2n}} \quad \text{from} \quad (2)$$

$$= \left(1 + \frac{x}{\rho^2}\right) \left(1 - \frac{x}{\rho^2}\right)^{-1} \cdot \frac{1}{a^2} \cdot \frac{1}{x - \alpha^2} \cdot \frac{1}{x - \beta^2}$$

$$= \frac{\rho^2 + x}{\rho^2 - x} \cdot \frac{1}{a^2 x^2 - a^2 x (\alpha^2 + \beta^2) + c^2}$$

$$= \frac{c + ax}{c - ax} \cdot \frac{1}{a^2 x^2 - (b^2 - 2ac)x + c^2} \quad \text{from} \quad (1).$$

Changing x into x^2, we get the result.

Question 1051.

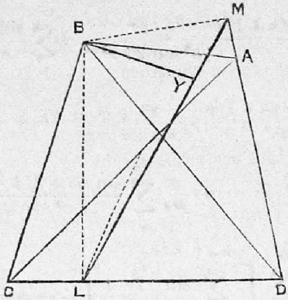
(A. C. L. Wilkinson) :—If ABCD is a cyclic quadrilateral, the points A, B, C, D occurring in this cyclic order, the sum of the reciprocals of the perpendiculars from A, C on their Simpson lines with respect to the triangles BCD, ABD respectively is equal to the sum of the reciprocals of the perpendiculars from B, D on their Simpson lines with respect to the triangles ACD, ABC respectively.

Solution by Hemraj, Kewalramani, V. M. Gaitonde and G. S. Mahajani.

Draw BL and BM perpendicular to CD and AD.

ML is the pedal line of B with respect to ΔACD.

Let BY be perp. to ML, then from the similar triangles BMY and BDL, we have BY . BD = BL . BM.



$$\therefore \quad \frac{1}{BY} = \frac{BD}{BL \cdot BM} = \frac{BD}{BA \cdot BC \sin A \sin C}$$
$$= \frac{BD \sin B}{2\Delta ABC \cdot \sin A \sin C}.$$

If p_1, p_2, p_3, p_4 be the perps. from A, B, C, D on their pedal lines with respect to BCD, etc., then

$$\frac{1}{p_2} + \frac{1}{p_4} = \frac{BD \sin B}{2\Delta ABC \cdot \sin A \sin C} + \frac{BD \sin D}{2\Delta ACD \cdot \sin A \sin C}$$
$$= \frac{2\, BD \sin B \cdot \sigma}{AB \cdot BC \cdot CD \cdot DA \sin A \sin B \sin C \sin D}$$

where σ = quadrilateral ABCD.

Similarly $\dfrac{1}{p_1} + \dfrac{1}{p_3} = \dfrac{2AC \sin A\ \sigma}{AB \cdot BC \cdot CD \cdot DA \sin A \sin B \sin C \sin D}$.

But $\dfrac{AC}{\sin B} = \dfrac{BD}{\sin A}$, $\therefore \dfrac{1}{p_1} + \dfrac{1}{p_3} = \dfrac{1}{p_2} + \dfrac{1}{p_4}$.

Additional solutions by V. V. S. Narayan and S. S. Ramakrishnan.

Question 1052.

(A. C. L. Wilkinson):—If the circle of curvature at any point P of an ellipse be drawn and O be any point on the tangent at P, prove that the straight line joining the points of contact of the other two tangents from O to the ellipse and its circle of curvature, the other common tangent and the tangent at P are concurrent.

Solution (1) *by Hemraj and G. R. Narayana Aiyar.*

Take the tangent and normal at P as axes. Then the equations of the conic and the circle of curvature are

$$2y = ax^2 + 2hxy + by^2 \text{ and } x^2 + y^2 - \frac{2}{a}y = 0.$$

Since the points of contact of the other two tangents from O $(x_1, 0)$ to the conic and the circle are on its chords of contact with respect to them respectively, they are on the lines

$$y = \frac{ax_1x}{1 - hx_1} \text{ and } y = ax_1x.$$

The points of contact are respectively

$$\left\{\frac{2x_1(1 - hx_1)}{1 + (ab - h^2)x_1^2}, \frac{2ax_1^2}{1 + (ab - h^2)x_1^2}\right\}$$

and $$\left\{\frac{2x_1}{1 + a^2x_1^2}, \frac{2ax_1^2}{1 + a^2x_1^2}\right\}.$$

The line joining them meets $y = 0$ at the point given by

$$x = \frac{2h}{a^2 + h^2 - ab}. \qquad \dots \qquad \dots \qquad (1)$$

Again if $x = my + c$ be the common tangent, then

$$h^2c^2 + 1 - 2hc - 2amc = abc^2 \text{ and } a^2c^2 - 1 + 2amc = 0.$$

$$\therefore \quad c = \frac{2h}{a^2 + h^2 - ab}.$$

Hence the result.

Solution (2) *by V. M. Gaitonde.*

On page 261 in Russel's *Pure Geometry* (2nd Edition), it has been proved that :—If a system of conics have three point contact at A and pass through D, a fixed line through A cuts them in P, P′ ...... and another fixed line in Q, Q′ ......, then all the lines PQ, P′Q′, ... are concurrent in a point on AD.

Now if AP and AQ coincide, the lines PQ, P′Q′, ... become the tangents at P, P′, ...

Hence the tangents at the intersections of any chord through A with the conic and its circle of curvature at A, meet on AD where D is their common point.

Reciprocating this theorem, we get the desired result at once.

Question 1054.

[MARTYN M. THOMAS]. If S_r denote the sum of the squares of the fractions $\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \dots \frac{1}{r}$; and if p_r denote the sum of their products taken two at a time, prove that

$$^nC_1 S_1 - {}^nC_2 (S_2 + p_2) + {}^nC_3 (S_3 + p_3) - \dots \&c. = \frac{1}{n^2}.$$

Solution by G. S. Mahajani and others.

Obviously, $S_r + p_r =$ coeff. of x^2 in

$$(1 + x/1)^{-1}.(1 + x/2)^{-1}(1 + x/3)^{-1}\dots\left(1 + \frac{x}{r}\right)^{-1}.$$

The left-hand side = co-efficient of x^2 in

$$\phi(x) = \frac{^nC_1}{(1 + x/1)} - \frac{^nC_2}{(1 + x/1)(1 + x/2)} + \dots \&c.$$

Now, the sum can be easily obtained thus:

$$\frac{U_r}{U_{r-1}} = \frac{r^{\text{th}} \text{ term}}{\overline{r-1}^{\text{th}} \text{ term}} = -\frac{n - r + 1}{r + x}.$$

$$\therefore U_r (x + r) = U_{r-1}(r - n - 1) = \text{say } V_r$$

$$\therefore U_r (r - n) = V_{r+1}. \quad \therefore U_r (x + n) = V_r - V_{r+1}.$$

$$\therefore (x + n)\sum_1^n U_r = V_1 - V_{n+1} = V_1 = U_1 (x + 1) = {}^nC_1 = n,$$

since $V_{n+1} = U_n (n - n) = 0$.

$$\therefore \phi(x) = \frac{n}{x + n} = \frac{1}{\frac{x}{n} + 1},$$

and the co-efficient of x^2 in $\phi(x) = \frac{1}{n^2}$.

Numerous identities can be deduced by equating co-efficients of other powers of x. In particular, equate the coefficients of x and we get,

$$^nC_1 \sigma_1 - {}^nC_2 \sigma_2 + {}^nC_3 \sigma_3 - , \& = \frac{1}{n},$$

where $\sigma_r = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \dots \frac{1}{r}$.

Question 1055.

(ENQUIRER) :—Show that

$$\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{6}} = \frac{1}{2^2} + \frac{5}{2^7} + \frac{7 . 9}{2\,!} \cdot \frac{1}{2^{12}} + \frac{9 . 11 . 13}{3\,!} \cdot \frac{1}{2^{17}} + \ldots\ldots$$

Solution by M. M. Thomas and N. G. Leather.

$$\text{Now } \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{6}} = \tfrac{1}{2}\left[\sqrt{2} - \sqrt{\tfrac{2}{3}}\right] = \frac{1}{2}\left[\left(1 - \frac{1}{2}\right)^{-\frac{1}{2}} - \left(1 + \frac{1}{2}\right)^{-\frac{1}{2}}\right]$$

$$= \tfrac{1}{2} \cdot \tfrac{1}{2} + \frac{\frac{1}{2} \cdot \frac{3}{2} \cdot \frac{5}{2}}{1 . 2 . 3} \cdot \frac{1}{2^3} + \frac{1 . 3 . 5 . 7 . 9}{1 . 2 . 3 . 4 . 5} \cdot \frac{1}{2^{10}}$$

$$+ \frac{1 . 3 . 5 . 7 . 9 . 11 . 13}{1 . 2 . 3 . 4 . 5 . 6 . 7} \cdot \frac{1}{2^{14}} + \ldots\ldots$$

$$= \frac{1}{2^2} + \frac{5}{2} \cdot \frac{1}{2^6} + \frac{7 . 9}{2 . 4} \cdot \frac{1}{2^{10}} + \frac{9 . 11 . 13}{2 . 4 . 6} \cdot \frac{1}{2^{14}} + \ldots\ldots$$

$$= \frac{1}{2^2} + \frac{5}{2^7} + \frac{7 . 9}{2\,!} \cdot \frac{1}{2^{12}} + \frac{9 . 11 . 13}{3\,!} \cdot \frac{1}{2^{17}} + \ldots\ldots$$

Question 1056.

[M. K. KEWALRAMANI]. If $\Sigma \cos \alpha = \Sigma \sin \alpha = 0$, prove,

$$\sum \frac{\cos 2\,(p\alpha + q\beta + r\gamma)}{(2p)\,!\,(2q)\,!\,(2r)\,!} = \frac{\cos 2\,n\alpha + \cos 2\,n\beta + \cos 2\,n\gamma}{(2\,n)\,!} \cdot 2^{2n-2}$$

and a similar result for sines, p, q, r being any positive integers subject to the condition that $p + q + r = n$.

Solution by G. S. Mahajani and N. G. Leather.

Put $a = \cos \alpha + i \sin \alpha$; $b = \cos \beta + i \sin \beta$
$c = \cos \gamma + i \sin \gamma$.

Then consider the coefficient of x^{2n} in the expansion of, $\cosh ax \cosh bx . \cosh cx$.

$$\text{i.e., in } \left[1 + \frac{a^2x^2}{2\,!} + \frac{a^4x^4}{4\,!} + - . \frac{a^{2p}\,x^{2p}}{(2p)\,!} + - .\right]$$

$$\left[1 + \frac{b^2\,x^2}{2\,!} + - . \frac{b^{2q}\,x^{2q}}{(2q)\,!} + -\right]\left[1 + \frac{c^2\,x^2}{2\,!} + \ldots + \frac{c^{2r}\,x^{2r}}{(2r)\,!} + - .\right]$$

$$= \sum \frac{a^{2p} . b^{2q} . c^{2r}}{(2p)\,!\,(2q)\,!\,(2r)\,!} = \frac{\cos 2\,(p\alpha + q\beta + r\gamma) + i \sin 2\,(p\alpha + q\beta + r\gamma)}{(2p)\,!\,(2q)\,!\,(2r)\,!}$$

Now, $\cosh ax \cdot \cosh bx \cdot \cosh cx =$

$$\frac{1}{2^2}\left[\cosh(a+b+c)x + \Sigma\cosh(a+b-c)x\right]$$

$$= 2^{-2}\,(\Sigma\cosh 2ax), \text{ [since } \Sigma a = 0 \text{ by hypothesis].}$$

and the coefficient of x^{2n} in this is

$$\frac{a^{2n}+b^{2n}+c^{2n}}{(2n)!}\cdot 2^{2n-2} = \frac{\Sigma\cos 2n\alpha + i\,\Sigma\sin 2n\alpha}{(2n)!}2^{2n-2}.$$

Hence, equating the real and imaginary parts, the result follows.

In fact, more generally we have

$$\sum\frac{(b+c)^{2p}(c+a)^{2q}(a+b)^{2r}}{(2p)!\,(2q)!\,(2r)!} = \frac{\Sigma a^{2n}+(\Sigma a)^{2n}}{(2n)!}\,2^{2n-2},$$

where $p+q+r=n$, and p, q, r are $+$ve integers including 0.

Question 1057.

(M. K. Kewalramani.): —To prove that

$$\sum\frac{(a+b+c)^{2p}(b+c+d)^{2q}(c+d+a)^{2r}(d+a+b)^{2s}}{(2p)!\,(2q)!\,(2r)!\,(2s)!}$$
$$= \frac{2^{2n-3}}{(2n)!}\left[3^{2n}\sigma^{2n} + \Sigma(\sigma+a)^{2n} + \Sigma(\sigma-a-b)^{2n}\right]$$

where $2\sigma = a+b+c+d$; and $p+q+r+s=n$; &c.

Solution by G. S. Mahajani and N. G. Leather.

As in the preceding question, it is easy to see that, the left side = coefficient of x^{2n} in the product,—

$\cosh(a+b+c)x.\cosh(b+c+d)x\cosh(c+d+a)x\cosh(d+a+b)x.$

$$= 2^{-3}\Big[\cosh 3(a+b+c+d)x + \Sigma\cosh(a+b+c+3d)x + \cosh(a+b-c-d)x\Big]$$

$$= 2^{-3}\Big[\cosh 6\sigma x + \Sigma\cosh 2(\sigma+a)x + \Sigma\cosh 2x(\sigma-a-b)\Big];$$

and the coefficient of x^{2n} in this is obviously

$$\frac{2^{-3}}{(2n)!}\left\{6^{2n}\cdot\sigma^{2n} + 2^{2n}\,\Sigma(\sigma+a)^{2n} + 2^{2n}\,\Sigma(\sigma-a-b)^{2n}\right\}$$
$$= \frac{2^{2n-3}}{(2n)!}\left[3^{2n}\cdot\sigma^{2n} + \Sigma(\sigma+a)^{2n} + \Sigma(\sigma-a-b)^{2n}\right]$$

The method can be extended to five or more quantities a, b, c, d, e ...... where $p + q + r + s + t + ... = n$.

Question 1058.

(N. P. Pandya) :—A fixed ellipse is intersected by a variable parabola in four points, A, B, C, D. If the axis of the parabola be always perpendicular to the axis major of the ellipse, find the locus of the centre of gravity of the quadrilateral ABCD.

Solution by N. G. Leather.

Let the equation of the ellipse be $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

and that of the parabola $x^2 + 2gx + 2fy + c = 0$, where $(\pm 2f)$ is the latus rectum.

Any point on the ellipse will be $(a \cos \theta, b \sin \theta)$ and this point will also lie on the parabola if $a^2 \cos^2\theta + 2\,ag \cos \theta + 2\,bf \sin \theta + c = 0$, whence $[a^2 + c + 2bf \sin \theta - a^2 \sin^2 \theta]^2 - 4\,a^2g^2\,(1 - \sin^2 \theta) = 0$

$$\therefore \qquad a^4 \sin^4 \theta - 4a^2\,bf \sin^3 \theta + \text{etc.} = 0$$

$\therefore$ if $\theta_1, \theta_2, \theta_3, \theta_4$ be the eccentric $\angle$s of A,B,C,D,

$$\sum \sin \theta_r = \frac{4\,bf}{a^2}.$$

But if $\bar{x}, \bar{y}$ be the co-ordinates of the centroid of A,B,C,D,

$$\bar{y} = \frac{b\,\Sigma \sin \theta_r}{4} = \frac{b^2}{a^2} f = \pm \frac{b^2}{a^2}\, l$$

where l is the semi-latus rectum of the parabola.

$\therefore$ the locus of the centroid is two straight lines parallel to the major axis of the ellipse and distant $\frac{b^2}{a^2}\, l$ from it on either side.

Question 1065.

(S. Krishnaswamiengar) :—If ρ be the radius of curvature of the curve $r^m = a^m \sin m\theta$ at the point whose distance measured along the curve from a fixed point is s, prove that

$$(m - 1)\,(m + 1)^2\, \rho \frac{d^2\rho}{ds^2} - m(m + 1)^2 \left(\frac{d\rho}{ds} \right)^2 - m(m - 1)^2 = 0.$$

Show how to solve this equation, and hence find the equation of given curve in terms of ρ and s.

Solution by Martyn M. Thomas, N. G. Leather and several others.

Taking the derivate logarithmic differential, $\frac{1}{r}\frac{dr}{d\theta} = \cot m\theta$.

$$\therefore \quad \phi = m\theta \text{ and } \psi = (m+1)\theta.$$

$$\therefore \quad \rho = r\frac{dr}{dp} = \frac{r}{\frac{d}{dr}(r \sin m\theta)} = \frac{r}{\sin m\theta \,.\, (1+m)},$$

$$\frac{d\rho}{ds} = \frac{1}{1+m}\left\{\frac{dr}{ds}\operatorname{cosec} m\theta - r \operatorname{cosec} m\theta \cot m\theta \,.\, m \,.\, \frac{d\theta}{ds}\right\}$$

$$= \frac{1-m}{1+m} \cot m\theta \quad \ldots \quad \ldots \quad \ldots \quad \ldots \ (1)$$

$$\frac{d^2\rho}{ds^2} = \frac{1-m}{1+m}\left(-\operatorname{cosec}^2 m\theta \,.\, m\frac{d\theta}{d\psi} \,.\, \frac{d\psi}{ds}\right) = \frac{m(m-1)\operatorname{cosec}^2 m\theta}{(1+m)^2\rho}.$$

$$\therefore \quad \rho\frac{d^2\rho}{ds^2} = \frac{m(m-1)}{(1+m)^2} \,.\, \operatorname{cosec}^2 m\theta. \quad \ldots \quad \ldots \quad \ldots \ (2)$$

Eliminating θ from (1) and (2),

$$\rho\frac{d^2\rho}{ds^2} - \frac{m}{m-1}\left(\frac{d\rho}{ds}\right)^2 = \frac{m(m-1)}{(1+m)^2}.$$

To solve this, put $\frac{d\rho}{ds} = p$ and hence $\frac{d^2\rho}{ds^2} = p\frac{dp}{d\rho}$.

$$\therefore \quad p\frac{dp}{d\rho} - \frac{m}{(m-1)\rho} \cdot p^2 = \frac{m(m-1)}{(1+m)^2\rho}.$$

Putting $p^2 = u$,

$$\frac{du}{d\rho} - \frac{2m}{(m-1)\rho} \cdot u = \frac{2m(m-1)}{(1+m)^2\rho}, \text{ which is linear in } u.$$

Since the integrating factor is $\rho^{-\frac{2m}{m-1}}$, we get

$$\therefore \quad u\rho^{-\frac{2m}{m-1}} = C + \frac{2m(m-1)}{(1+m)^2}\int \rho^{-\frac{3m-1}{m-1}} d\rho$$

$$= C - \left(\frac{m-1}{m+1}\right)^2 \rho^{-\frac{2m}{m-1}}$$

$$\therefore \quad u = C\rho^{\frac{2m}{m-1}} - \left(\frac{m-1}{m+1}\right)^2$$

$$\therefore \quad \frac{d\rho}{ds} = \left\{C\rho^{\frac{2m}{m-1}} - \left(\frac{m-1}{m+1}\right)^2\right\}^{\frac{1}{2}}.$$

$$\therefore \quad s + C' = \int \frac{d\rho}{\sqrt{\left\{C\rho^{\frac{2m}{m-1}} - \left(\frac{m-1}{m+1}\right)^2\right\}}}$$

Let $$C\rho^{\frac{2m}{m-1}} = \left(\frac{m-1}{m+1}\right)^2 \sec^2\phi;$$

then logarithmic differentiation gives

$$\frac{2m}{m-1}\frac{d\rho}{\rho} = 2\tan\phi\,.\,d\phi.$$

$$\therefore\ s+C' = \frac{m+1}{m}\int\rho\,d\phi = \frac{m+1}{m}\left(\frac{m-1}{m+1}\right)^{\frac{m-1}{m}}\cdot\frac{1}{C^{\frac{m-1}{2m}}}\int\sec^{\frac{m-1}{m}}\phi\,.\,d\phi.$$

For a parabola we have $m = -\frac{1}{2}$, and for a rect. hyp. $m = -2$, while for a lemniscate $m = 2$, and for a cardioide $m = \frac{1}{2}$.

Question 1066.

(P. A. Subramani Aiyar) :—Solve completely

$$axy\left(\frac{dy}{dx}\right)^2 + (x^2 - ay^2 - b)\frac{dy}{dx} - xy = 0.$$

Solution by G. R. Narayana Ayyar.

Change the variables to s and t such that $y^2 = s$ and $x^2 = t$.

The differential equation now becomes

$$at\left(\frac{ds}{dt}\right)^2 + (t - as - b)\frac{ds}{dt} - s = 0. \quad \dots \quad \dots \quad (1)$$

Differentiating with respect to t

$$\frac{d^2s}{dt^2}\left\{2at\frac{ds}{dt} + t - as - b\right\} = 0.$$

$$\therefore \qquad \frac{d^2s}{dt^2} = 0, \quad \dots \quad \dots \quad \dots \quad (2)$$

or $$2at\frac{ds}{dt} + t - as - b = 0. \quad \dots \quad \dots \quad \dots \quad (3)$$

From (2) $\frac{ds}{dt} = c$, any constant.

Hence one solution is $atc^2 + (t - as - b)c - s = 0$.

Changing to the original variables, the solution is

$$cx^2(ac+1) - y^2(ac+1) - bc = 0,$$

or $$y^2 = cx^2 - \frac{bc}{ac+1}. \quad \dots \quad \dots \quad (4)$$

Another solution is obtained by eliminating $\frac{ds}{dt}$ between (1) and (3).

The solution is

$$at\ \frac{(as+b-t)^2}{4a^2t^2}+(t-as-b)\ \frac{(as+b-t)}{2at}-s=0,$$

i.e. $$(t-as-b)^2+4\,at\,s=0.$$

Changing to the original variables the solution is

$$(x^2-ay^2-b)^2+4ax^2y^2=0.$$

This is the singular solution as it does not contain any arbitrary constants.

Question 1067.

(V. THIRUVENKATACHARI) :—Show that

$$\int_0^{\frac{\pi}{2}} \frac{\log \sin x \log \cos x}{\sqrt{\sin x \cos x}}\ dx = \tfrac{1}{8}\ \frac{(\Gamma\frac{1}{4})^2}{\sqrt{\pi}}\ \left\{(\log 2)^2+\pi\log 2-\frac{\pi^2}{4}\right\}$$

Solution by Martyn M. Thomas and several others.

$$\text{Now} \int_0^{\frac{\pi}{2}} \sin^{2\alpha-1} x\ \cos^{2\beta-1} x\ dx = \tfrac{1}{2}\ \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}.$$

Differentiating under the integral sign, successively with respect to α, β, we have

$$4\int_0^{\frac{\pi}{2}} (\sin x)^{2\alpha-1}\log\sin x.\,(\cos x)^{2\beta-1}\log\cos x\,dx = \tfrac{1}{2}\,\frac{d}{d\beta}\left[\frac{d}{d\alpha}\left\{\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}\right\}\right]$$

$$=\tfrac{1}{2}\,\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}\left[\{\psi(\alpha+\beta)-\psi(\alpha)\}\ \{\psi(\alpha+\beta)-\psi(\beta)\} - \psi'(\alpha+\beta)\right],$$

where $\psi(\alpha)=\dfrac{\Gamma'(\alpha)}{\Gamma(\alpha)}$. Putting $\alpha=\frac{1}{4}$ and $\beta=\frac{1}{4}$

$$\int_0^{\frac{\pi}{2}} \frac{\log\sin x\log\cos x}{\sqrt{\sin x\cos x}} = \tfrac{1}{8}\,\frac{(\Gamma\frac{1}{4})^2}{\Gamma(\frac{1}{2})}\left[\{\psi(\tfrac{1}{2})-\psi(\tfrac{1}{4})\}^2-\psi'(\tfrac{1}{2})\right]$$

$$=\tfrac{1}{8}\,\frac{(\Gamma\frac{1}{4})^2}{\sqrt{\pi}}\left[\left\{(-c-2\log 2)-\left(-c-3\log 2-\frac{\pi}{2}\right)\right\}^2-\frac{\pi^2}{2}\right]$$

$$=\tfrac{1}{8}\,\frac{\{\Gamma(\frac{1}{4})\}^2}{\sqrt{\pi}}\left\{\left(\log 2+\frac{\pi}{2}\right)^2-\frac{\pi^2}{2}\right\}$$

$$=\tfrac{1}{8}\,\frac{(\Gamma\frac{1}{4})^2}{\sqrt{\pi}}\left\{(\log 2)^2+\pi\log 2-\frac{\pi^2}{4}\right\}.$$

Question 1069.

(A. C. L. Wilkinson) :—Normals PA, PB, PC, PD are drawn to a hyperbola ; PA meets the hyperbola again in A′, PB in B′, PC in C′, PD in D′. If A′B′C′D′ are concyclic, prove that the locus of P is an ellipse of which the equi-conjugate diameters coincide, with the asymptotes of the hyperbola.

Solution by N. G. Leather.

Let the hyperbola be $xy = c^2$, w being the angle between the asymptotes, taken as axes of co-ordinates.

The co-ordinates of any point can be written $\left(cp, \frac{c}{p} \right)$.

Let k_1, k_2, k_3, k_4 be the parameters of A, B, C, D and p_1, p_2, p_3, p_4 those of A′, B′, C′, D′.

The normal at A can easily be shewn to be

$$k_1(k_1^2 - \cos w)\, x + k_1(k_1^2 \cos w - 1)\, y - c(k_1^4 - 1) = 0. \quad \text{(i)}$$

Writing $x = cp$, $y = c/p$, we find the parameters of the points in which this normal meets the hyperbola in the form

$$(p - k_1)\,[k_1(k_1^2 - \cos w)\, p - (k_1^2 \cos w - 1)] = 0,$$

$$\therefore \qquad p_1 = \frac{k_1^2 \cos w - 1}{k_1(k_1^2 - \cos w)},$$

and similar expressions for p_2, p_3 and p_4.

Now if A′, B′, C′, D′ are concyclic A′B′ and C′D′ will be equally inclined to either axis of the hyperbola, *i.e.*, the angle between A′B′ and one asymptotes will equal the angle between C′D′ and the other asymptote, the angle between A′B′ and C′D′ being $\pm w$.

Also the line through the origin parallel to A′B′ will be $\frac{y}{x} = -\frac{1}{p_1 p_2}$

and the line through the origin parallel to C′D′ will be $\frac{y}{x} = -\frac{1}{p_3 p_4}$.

Hence the required condition obviously is $p_1 p_2 p_3 p_4 = 1$.

$$\therefore \quad \Pi\,(k_1^2 \cos w) = k_1 k_2 k_3 k_4\, \pi\, (k_1^2 - \cos w).$$

If (x, y) be the co-ordinates of P, we see from (i) that k_1, k_2, k_3, k_4 are the roots of $ck^4 - (x + y \cos w)\, k^3 + (x \cos w + y)\, k - c = 0$.

$$\therefore \qquad k_1 k_2 k_3 k_4 = -1,$$

and $k_1^2, k_2^2, k_3^2, k_4^2$ will be the roots of

$$c^2[\lambda^2 - 1]^2 - \lambda[(x + y \cos w)\, \lambda - (x \cos w + y)]^2 = 0.$$

$$\therefore\ \Pi\ (k_1^2 \cos w - 1) = c^2(1 - \cos^2 w)^2 - \cos w\ [(x + y \cos w) - \cos w(x \cos w + y)];$$

and $k_1 k_2 k_3 k_4\ \Pi\ (k_1^2 - \cos w)$

$$= -\lfloor c^2(1 - \cos^2 w)^2 - \cos w \{ (x + y \cos w) \cos w - (x \cos w + y) \}^2].$$

Hence the required locus will be

$$2c^2 \sin^4 w - x^2 \cos w \sin^4 w - y^2 \cos w \sin^4 w = 0,$$

that is

$$x^2 + y^2 = 2c^2 \sec w,$$

which is an ellipse whose equi-conjugate semi-axes coincide with the asymptotes of the hyperbola.

Question 1071.

(T. P. Trivedi, M.A., LL.B.) :—

If $S_r = 1^r . n^{C_r} + 2^{r-1} . (n-1)^{C_{r-1}} + 3^{r-2} . (n-2)^{C_{r-2}} + \ldots 1 . (n-r)^{C_0}$,

prove that

$$\sum_{r=1}^{r=n} S_r = 1^{n+1} + 2^n + 3^{n-1} + \ldots\ldots + (n+1).$$

Solution by M. V. Ramakrishnan and G. R. Narayana Aiyar.

On expansion

$$S_r = 1^r . n^{C_r} + 2^{r-1} . (n-1)^{C_{r-1}} . + 3^{r-2}(n-2)^{C_{r-2}} + \ldots + 1 . (n-r)^{C_0}.$$

$$\therefore \sum_{r=1}^{r=n} S_r = S_1 + S_2 + \ldots\ldots + S_n$$

$$= \sum_{r=1}^{r=n} \left[1^r . n^{C_r} + 2^{r-1} . n - 1^{C_{r-1}} + 3^{r-2} . n - 2^{C_{r-2}} + \ldots\ldots \right]$$

$$= (n^{C_1} + n^{C_2} + \ldots\ldots + n^{C_n})$$

$$+ \{ n - 1^{C_0} + 2 . n - 1^{C_1} + 2^2 . n - 1^{C_2} + \ldots\ldots + 2^{n-1} n - 1^{C_{n-1}} \}$$

$$+ \{ n - 2^{C_0} + 3 . n - 2^{C_1} + 3^2\ n - 2^{C_2} + \ldots\ldots + 3^{n-2} n - 2^{C_{n-2}} \}$$

$$+ \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

$$= \{ (1+1)^n - 1 \} + (1+2)^{n-1} + (1+3)^{n-2} + \ldots\ldots + (1+n)^1$$

$$= 2^n + 3^{n-1} + 4^{n-2} + \ldots\ldots + (1 + n) - 1$$

$$= 1^{n+1} + 2^n + 3^{n-1} + 4^{n-2} + \ldots\ldots + (n + 1).$$

Question 1072.

(T. P. TRIVEDI, M.A., LL.B.):—Prove that

$$\int_0^{\pi/2} \sqrt{\tan x} \,.\, \log (\tan x)\, dx = \frac{\pi^2}{2\sqrt{2}},$$

$$\int_0^{\pi/2} \sqrt[5]{\tan x} \,.\, \log (\tan x)\, dx = \frac{\pi^2}{4}\left(\frac{3}{\sqrt{5}} - 1\right).$$

Solution (1) *by K. J. Sanjana, M.A.* (2) *by Martyn M. Thomas, M.A.*

(1) We have $\int_0^{\pi/2} (\tan x)^{2a-1}\, dx = \int_0^{\pi/2} (\sin x)^{2a-1} (\cos x)^{1-2a}\, dx$.

$$= \frac{\Gamma(a)\,\Gamma(1-a)}{2\Gamma(1)} = \frac{\pi}{2 \sin a\pi}.$$

Differentiating with respect to a,

$$2\int_0^{\pi/2} (\tan x)^{2a-1} . \log \tan x\, dx = -\frac{\pi^2}{2}\,\frac{\cos a\pi}{\sin^2 a\pi},$$

a being taken to be positive and less than 1.

Put $a = \frac{3}{4}$; $2\int_0^{\pi/2} \sqrt{(\tan x)} . \log \tan x\, dx = \frac{\pi^2}{2}$; hence the first result should be $\pi^2/2\sqrt{2}$.

Putting $a = \frac{3}{5}$, $2\int_0^{\pi/2} \sqrt[5]{(\tan x)} . \log \tan x\, dx = \frac{\pi^2}{2}\,\frac{\cos 72°}{\sin^2 72°}$

$$= \frac{2\pi^2(\sqrt{5}-1)}{10+2\sqrt{5}} = \frac{\pi^2}{\sqrt{5}}\,\frac{(\sqrt{5}-1)^2}{4} = \frac{\pi^2}{2}\left(\frac{3}{\sqrt{5}} - 1\right).$$

(2) Now $\int_0^{\pi/2} \tan^n x\, dx = \int_0^{\pi/2} \sin^n x \cos^{-n} x\, dx$

$$= \frac{\Gamma\left(\frac{n+1}{2}\right)\Gamma\left(\frac{1-n}{2}\right)}{2\Gamma\left(\frac{n+1}{2} + \frac{1-n}{2}\right)}$$

$$= \frac{\Gamma\left(\frac{1+n}{2}\right)\Gamma\left(\frac{1-n}{2}\right)}{2\Gamma(1)} = \frac{\pi \operatorname{cosec}\left(\frac{1+n}{2}\right)\pi}{2}.$$

Differentiating both sides with respect to n,

$$\int_0^{\pi/2} \tan^n x \log \tan x\, dx = \frac{\pi}{2}\left\{-\operatorname{cosec}\left(\frac{1+n}{2}\right)\pi . \cot\left(\frac{1+n}{2}\right)\pi\right\}.\frac{\pi}{2}$$

$$\therefore \int_0^{\pi/2} \tan^{\frac{1}{2}} x \log \tan x\, dx = \frac{\pi}{2}\left\{-\operatorname{cosec}\frac{3\pi}{4} \cot\frac{3\pi}{4}\right\}\frac{\pi}{2}$$

$$= \frac{\pi}{2}\left\{(-\sqrt{2})(-1)\right\}\frac{\pi}{2} = \frac{\pi^2}{2\sqrt{2}},$$

$$\text{Also} \int_0^{\pi/2} \tan^{\frac{1}{5}} x \log \tan x \, dx = \frac{\pi}{2}\left\{ -\operatorname{cosec}\frac{3\pi}{5} \cot\frac{3\pi}{5} \right\} \frac{\pi}{2}$$

$$= \frac{\pi}{2}\left\{ (-\operatorname{cosec} 72) (-\cot 72) \right\} \frac{\pi}{2}$$

$$= \frac{\pi^2}{4} \cdot \frac{\sin 18^\circ}{\cos^2 18^\circ} = \frac{\pi^2}{4} \cdot \frac{\sqrt{5}-1}{4} \cdot \frac{16}{10+2\sqrt{5}}$$

$$= \frac{\pi^2(\sqrt{5}-1)(10-2\sqrt{5})}{80} = \frac{\pi^2}{4}\left(\frac{3}{\sqrt{5}}-1\right).$$

Question 1090.

(S. MAHADEVAN) :—The focus S is joined to any point P_1 on the ellipse, P_1S cuts the ellipse again in Q_1, Q_1S' cuts the ellipse again in P_2; and so on. Show that SP_n ultimately tends to coincide with the major axis.

Solution by K. J. Sanjana and S. Ranganathan.

Let $\alpha_1, \alpha_2, \ldots \alpha_n$ be the eccentric angles of $P_1, P_2, \ldots P_n$, and $\beta_1 . \beta_2 \ldots$ those of $Q_1, Q_2, \ldots$. The equation of P_1SQ_1 is

$$\frac{x}{a}\cos\frac{\alpha_1+\beta_1}{2} + \frac{y}{b}\sin\frac{\alpha_1+\beta_1}{2} = \cos\frac{\alpha_1-\beta_1}{2};$$

as this line goes through S, we have

$$e\cos\frac{\alpha_1+\beta_2}{2} = \cos\frac{\alpha_1-\beta_1}{2},$$

which gives

$$\cot\frac{\beta_1}{2} = -\frac{1+e}{1-e}\tan\frac{\alpha_1}{2}.$$

The equation of $Q_1S'P_2$ is

$$\frac{x}{a}\cos\frac{\beta_1+\alpha_2}{2} + \frac{y}{b}\sin\frac{\beta_1+\alpha_2}{2} = \cos\frac{\beta_1-\alpha_2}{2};$$

as this line goes through S′, we have

$$-e\cos\frac{\beta_1+\alpha_2}{2} = \cos\frac{\beta_1-\alpha_2}{2},$$

which gives

$$\cot\frac{\beta_1}{2} = -\frac{1-e}{1+e}\tan\frac{\alpha_2}{2}.$$

Equating, we get

$$\cot\frac{\alpha_2}{2} = \left(\frac{1-e}{1+e}\right)^2 \cot\frac{\alpha_1}{2}.$$

Continuing the process, we shall have $\cot\frac{\alpha_n}{2} = \left(\frac{1-e}{1+e}\right)^{2n-2} \cot\frac{\alpha_1}{2}$;

hence, proceeding to the limit, as $n \to \infty$, $\cot\frac{\alpha^n}{2} = 0$.

Therefore $\alpha_n = 2(n+\frac{1}{2})\pi$, and SP coincides with the major axis ultimately.

QUESTIONS FOR SOLUTION.

1126. (V. Ramaswami Aiyar, M.A.):—Given a triangle ABC; prove that the orthopolar ellipse of every point P is *inscribed* in the Steiner's tricusp, that is, has triple contact with it; and each of its auxiliary circles touches the nine points circle of ABC.

1127. (K. J. Sanjana, M.A.):—Prove that there are two and only two Tucker circles of a triangle which touch a given straight line in the plane of the triangle. These circles coalesce when the given line is one of the sides of the triangle.

If O and K be the circum-centre and symmedian point of a triangle ABC, T_1 the centre and R_1 the length of the radius of the Tucker circle touching BC, prove that

$$KT_1 : T_1O = b^2 + c^2 - a^2 : b^2 + c^2 + a^2, \text{ and } R_1 : R = bc : b^2 + c^2.$$

1128. (K. J. Sanjana and M. K. Kewalramani):—If ${}_rS_p$ = the sum of the products p at a time of the first r natural numbers (${}_3S_0 \equiv 1$), show that

$$(1)\quad \frac{{}_3S_0}{3!} - \frac{{}_5S_2}{5!} + \frac{{}_7S_4}{7!} - \dots = \frac{\pi}{96}\left[\frac{\pi^2}{4} - 3\pi(\log 2)^2\right].$$

$$(2)\quad \frac{{}_3S_1}{4!} - \frac{{}_5S_3}{6!} + \frac{{}_7S_5}{7!} - \dots = \frac{\pi^2}{64}\log 2 - \frac{1}{48}(\log 2)^3.$$

1129. (S. Malhari Rao):—Find a number which when multiplied by 2 or 3 or 4 or 5 or 6 or 7 or 8 or 9 gives in each case a product which contains the same figures as the number itself.

1130. (A. A. Krishnaswami Aiyangar):—Prove that a triangle ABC in which $\tan A = \tan^{2n+1} B$ and $\tan 2B = m \tan C$ where m, n are positive integers, is right-angled and isosceles.

1131. (A. A. Krishnaswami Aiyangar):—Solve in positive integers the equation $9x^4 - 516x^2y^2 + 7168y^4 = z^2$.

1132. (N. Dorai Rajan):—If O, I, H, be the circum-, in- and ortho-centres of a triangle ABC, and if the circle OIH passes through one angular point, prove that it also passes through another angular point. Find the necessary condition. (Suggested by Q. 1037 of Prof. Wilkinson).

1133. (N. Dorai Rajan):—Show that, in an epicycloid, where the radius of the fixed circle is n (integer) times that of the rolling circle, the feet of the normals to the curve from any point on the fixed circle form the angular points of a regular polygon. Show that a nearly similar theorem holds for the tangents.

1134. (N. B. Mitra):—If p is a positive integer $\not\equiv 0 \pmod 3$ prove that $\sum_{r=1}^{r=p} C_{2r}\, 2^{p-r}(2^r - 1) \equiv 0 \pmod 7$ where C_r denotes $2p+1\, C^r$.

1135. (N. B MITRA) :—If n is a prime number of the form $4m + 1$ $(m > 1)$ prove that the sum of the products of the first n natural numbers taken $n - 2$ at a time is divisible by the sum of the cubes of these natural numbers.

1136. (Selected by T. KRISHNA RAO) :—Parallel lines through the foci of an ellipse meet the tangent at the vertex A in P, Q and the lines joining P, Q, to the other vertex A′ meet the circle on AA′ as diameter again in R and S. Prove geometrically that RS is a tangent to the ellipse.

1137. (MARTYN M. THOMAS) :—Two particles moving with velocities $1 : k$ are at corresponding points of a curve and its nth negative pedal. Show that the differential equation of the original curve is

$$\frac{dr}{d\theta} + r^{\frac{n-1}{n}} \cdot \sqrt{(kr + a)^{\frac{2}{n}} - r^{\frac{2}{n}}}.$$

where 'a' is an arbitrary constant.

Each radius vector of the curve $r = b \cos(\theta \sin \alpha)$ is diminished by $b \cos \alpha$. Show that this new curve and its negative pedal can be described by two particles simultaneously with velocities $\sec \alpha : 1$.

1138. (R. VYTHYNATHASWAMY) :—If the product of three quaternions in every order is a vector, the axes of the three quaternions are || to a plane.

Prove this and relate it to the following theorem :

'If the product of three homographies in every order be an involution, the double points of the homographies belong to an involution.'

1139. (R. VYTHYNATHASWAMY) :—If $x_0 x_1 \ldots\ldots x_n$ are homogenous co-ordinates in n dimensions, find the complete curve of intersection of the following surfaces :—

$$x_0 x_2 - x_1^2 = 0$$
$$x_1 x_3 - x_2^2 = 0$$
$$\cdots\cdots\cdots\cdots\cdots$$
$$\cdots\cdots\cdots\cdots\cdots$$
$$x_{n-2}\, x_n - x^2_{n-1} = 0.$$

1140. (R. VYTHYNATHASWAMY) :—If the super-osculation points of a rational space-quartic are coplanar, prove that the curve can be conically projected into a plane tricusp. Shew also that the locus of the vertices of projection is a conic through the super-osculation points.
