THE JOURNAL

OF THE

Indian Mathematical Society.

VI. XII.] JUNE 1920. [No. 3

THE LATE MR. S. RAMANUJAN, B.A., F.R.S.

By P. V. Seshu Aiyar.

As very little is known to the public of the private life of the late Mr. Ramanujan, a few personal details, such as the present writer was privileged to know and learn, may not be out of place in this memorial number of the Journal.

Srinivasa Iyengar Ramanuja Iyengar was born on the 9th day of Margasirsha in the Samvat Sarwajit answering to the English date of 22nd of December 1887, in the house of his maternal grandfather at Erode,—that picturesque spot in South India in whose neighbourhood the Bhavani with overflowing waters meets and combines with the Cauvery. His parentage was humble; his father and his paternal grand-father were both accountants (literally *gumastas*) to cloth merchants at Kumbakonam, while his mother's father had entered Government service as *amin* in the Munsiff's Court at Erode. His mother, who survives him in great sorrow, is a shrewd and cultured lady, and Ramanujan took his features after her. As is usual with Brahmin boys, he was put to school at the age of five, and before seven he was transferred to the Town High School, Kumbakonam. Even at that early age, he used to puzzle his parents and teachers with questions about zero and imaginary quantities, the distances between the stars and the earth, and the like.

From boyhood, Ramanujan was found to be of a contemplative mood and sedentary in habits; he seldom associated with his school-mates at play. A correspondent, who was Ramanujan's class-mate at Kumba-

konam, kindly sends me a description of his early life at school as follows:—
"Throughout his school course he held a freescholarship, and his teachers had already noticed his extraordinary and precocious intellect. He used to borrow Carr's *Synopsis of Pure Mathematics* from the College Library, and took delight in verifying some of the formulæ given there. He would clear in half the time examination papers in Algebra and Geometry and a few seconds' thought always used to suggest to him, the solution to any question however difficult. He used often to entertain his friends with his "theorems and formulæ" even in those early days, which doubtless appeared to his hearers as' mathematical tricks. He had a distinctly religious turn of mind, and was a staunch believer in Hinduism. He never missed, if he could help it, a single *sravanam* ceremony at the Uppiliappan Koil near Kumbakonam. No wonder then that while at Cambridge he could not be persuaded to take the food cooked by others, and always cooked his own food. He had an extraordinary memory and could easily repeat the complete lists of Sanskrit roots (*atmanepada* and *parasmepada*); he could give the values of $\sqrt{2}$, π, e, etc., to any number of decimal places. In manners, he was simplicity itself. On one occasion when I was seriously laid up with fever just before the Matriculation Examination, he used to read for me every night. After his return from England and during his brief stay at Kumbakonam, he insisted on seeing my aunt, who is an old widow on the wrong side of 70, whom he knew while at school some 20 years ago. Perhaps, Ramanujan's peculiar frame of mind and philosophic temperament is forcibly illustrated by the following incident: when about to undergo a serious operation under chloroform at Kumbakonam, he was trying to discover within himself as to which of the five senses lost its power first and which afterwards. * * *."

One of his old school teachers also contributes a similar account, adding that, while yet in the Sixth Form, Ramanujan used to interest himself with problems of plane and spherical trigonometry.

From the Town High School, he passed his Matriculation Examination in 1903, and entered the Government College, Kumbakonam, in 1904 for the F. A. course. As a result of a competitive examination in Mathematics and English Composition, Ramanujan secured the Junior Subramaniam Scholarship. It was there that I, the present writer, came in contact with Mr. Ramanujan for the first time, when I was Lecturer in Mathematics at the Government College. He went to me with a number of results in Finite and Infinite series, which at once struck me as something very ingenious and original. I exhorted him to go on with his results without at the same time neglecting his other studies. Unfortunately however, he was found too weak in English to deserve promotion to the Senior F. A. class in January 1905. He had therefore to forgo also his scholarship at the college, and being too sensitive to ask his parents for help, he next moved to Vizagapatam to see if he could shift for himself to go on with his studies. Not receiving much encouragement there, he came to Madras, joined the Pachaiyappa's College and presented himself for the First Examination in Arts in December 1906 without, as is well-known now, any success. He never repeated his attempt. Till 1909 he was doing nothing very particular, but continued working at Mathematics in his own way and jotting down his results in two good-sized note-books. In the summer of 1909 he married, and his University career abruptly ending, he was in search of a suitable employment to settle down in life.

To this end, he went to Tirukoilur, a small sub-division town in South Arcot District, to see Mr. V. Ramaswami Aiyar, the Founder of the Indian Mathematical Society, but Mr. Aiyar, seeing his wonderful gifts, persuaded him to go to Madras. It was then after some 4 years interval, that Mr. Ramanujan met me at Madras, with his two well-sized note-books referred to above. I sent Ramanujan with a note of recommendation to that true lover of Mathematics, Dewan Bahadur R. Ramachandra Rao, who was then District Collector at Nellore, a small town some 80 miles north of Madras. Mr. Rao sent him back to me saying that it was cruel to make an intellectual giant like Ramanujan rot at a mofussil station like Nellore, and recommended his stay at Madras, generously undertaking to pay Mr. Ramanujan's expenses for a time. This was in December 1910. After a while, other attempts to obtain for him a scholarship having failed, and Ramanujan himself being unwilling to be a burden on anybody for any length of time, he decided to take up a small appointment under the Madras Port Trust in 1911.

But he never slackened his work at Mathematics. His earliest contribution to the *Journal of the Indian Mathematical Society* was in the form of questions communicated by me in Vol. III (1911). His first long article on "Some Properties of Bernoulli's Numbers" was published in the December number of the same volume. Mr. Ramanujan's methods were so terse and novel and his presentation was so lacking in clearness and precision, that the ordinary reader, unaccustomed to such intellectual gymnastics, could hardly follow him. This particular article was returned more than once by the Editor before it took a form suitable for publication. It was during this period that he came to me one day with some theorems on Prime Numbers, and when I referred him to Hardy's Tract on "Orders of Infinity," he observed that Hardy had said on p. 36 of his Tract "the exact order of $\rho(x)$, [defined by the equation $\rho(x) = \phi(x) - \int_2^x dt/\log t$, where $\phi(x)$ denotes the number of primes less than x], has not yet been determined" and that he himself had discovered a result which gave the order of $\rho(x)$. On this I suggested that he might communicate his result to Mr. G. H. Hardy together with some more of his results. That communication excited Mr. Hardy's interest in Ramanujan which never abated, and Mr. Hardy promptly replied in a very kind and appreciative letter, expressing admiration at some of the results and desiring to see more of Ramanujan's work with proofs. This was answered by Ramanujan at some length, and these two letters were enough to rouse interest in Mr. Hardy; he proposed that Mr. Ramanujan should go to Cambridge and wrote to a friend at Madras to induce him to accept the proposal. Mr. Ramanujan however declined, on religious grounds, to go then but set to work more vigorously than ever.

Meanwhile an event occurred which served to push Mr. Ramanujan into public notice and in fact accelerated his voyage to England. Dr. Gilbert T. Walker of the Meteorological Department, when on an official visit to Madras, noticed Mr. Ramanujan's work through the kind offices of Sir Francis Spring, the President of the Port Trust. Dr. Walker immediately realised that a great mathematician was being made to rot in the Port Trust Office. He and Sir F. Spring forthwith induced both the Government and the University of Madras to award Mr. Ramanujan a University Research Scholarship; accordingly a studentship of Rs. 75 per mensem for a period of two years was awarded to him, thus setting him free for mathematical work entirely.

While he was thus engaged as a Research student, Mr. E. H. Neville, then of Trinity College, Cambridge, came to Madras (in January 1914) to deliver a course of University Lectures on Differential Geometry; and Mr. Hardy had requested Mr. Neville to bring Mr. Ramanujan with him to Cambridge. Accordingly, Mr. Neville, assisted by Sir Francis Spring and the Hon'ble Mr. R. Littlehailes, succeeded in influencing the Government of Madras to grant Mr. Ramanujan a liberal scholarship and passage money to enable him to go to Cambridge. Mr. Ramanujan himself was quite willing now to proceed to England and, fortunately or otherwise, Ramanujan sailed for England about the end of March 1914. And just prior to that, the event was marked by a public demonstration held in honour of him by his admirer and well-wisher Mr. S. Sreenivasa Iyengar, Ex-Advocate-General.

His work in England is narrated in appropriate and enthusiastic terms by Mr. Hardy in his official report printed in the issue of this *Journal* for February 1917. And as one reads again the obituary notice of Ramanujan published by the same genial friend in "Nature" of June 1920,* the heart heaves up with melancholy pride.

Reflecting upon this obituary notice and as well as generally upon Mr. Ramanujan's University career, one is forced to admit that Mr. Ramanujan's achievement was a glaring condemnation of the wooden system of our present University education which caters only for the average student and which tries, if possible, to stifle originality and reduce all to a dead level.

It is evident, however, that cruel fate was busy all through his stay at Cambridge, stealthily weakening his health and diminishing his strength. An insidious disease hard to diagnose, and difficult to cure was slowly undermining poor Ramanujan, who however fought against it nobly and courageously. Amidst foreign environments his food was by no means native, and the ailment which caused him pain was strange; and adding to this his peculiar temperament, incessant sedentariness and intense study, one can easily imagine his frail body weakening under the tightening grip of the tuber bacilli. Thanks mainly to the almost parental solicitude of that genial professor, Mr. G. H. Hardy, the best medical aid and nursing attendance was procured for him. And he struggled on with admirable fortitude. There are no papers and researches of his more valued nor more intuitive than those which he thought out during these fateful days. His physical body was failing no doubt, but his intellectual vision grew proportionately keener and brighter—illustrating here again his truly Indian philosophic spirit of self-sacrifice amidst worldliness and materialism. He fully vindicated in the very temple of European learning, where England's sons themselves invoke the muse of number-mystery, the Indian's grit for originality, and for startling discovery. Mr. Ramanujan's disease had assumed serious proportions by the Christmas of 1918 and caused such grave anxiety to his doctors in England, that, hoping to do him good, they advised him to return to his native home in India.

When he arrived in Madras in April 1919, he was very weak and had been reduced to a skeleton. He was lodged in a bungalow in Mylapore and was treated for tuberculosis. At first he seemed to improve slowly and in June 1919 he went for a change to Kodumudi, a cool and bracing place on the banks of the Cauvery. While there, he got slightly better and it was thought he had no specific complaint except weakness, and that with his accustomed food and usual company at Kumbakonam, he would rally. He actually went to Kumbakonam in September and lived peacefully until January 1920, when there was the final and fatal relapse. He was immediately removed to Madras and placed under the treatment of expert doctors. His philanthropic well-wisher, Mr. S. Sreenivasa Iyengar, C. I. E.,

* This is reproduced here on pp. 90, 91.

came generously forward to defray all the expenses for his nursing and medical treatment ; and accordingly he was housed in a neat cottage at Chetput, and was placed under the observation of Dr. P. S. Chandrasekharan, who did his best for him in consultation with Surgeon-General Giffard and other able doctors. But all was in vain Ramanujan himself knew that his end was nearing. He answered the voices and visions which the living do not hear or see, and with courage, peace and good cheer, he passed away at sunrise on the 26th of April 1920, surrounded by a small circle of his friends and relatives, but mourned by an infinite circle of ever-increasing admirers.

Mr. S. Ramanujan's contributions to the *Journal of the Indian Mathematical Society* were the following :—

(A) Articles and Notes.

1.	Some properties of Bernoulli's Numbers,	Vol. 3	pp. 219—235
2.	On Q 330 of Prof. Sanjana ...	Vol. 4	pp. 59— 61
3.	A set of Equations	Vol. 4	pp. 94— 96
4.	Irregular numbers	Vol. 5	pp. 105—107
5.	Squaring the Circle	Vol. 5	pp. 132—133
6.	On the integral $\int_0^x \arctan t . \frac{dt}{t}$ ...	Vol. 7	pp. 93— 96
7.	On the divisors of a Number ...	Vol. 7	pp. 131—134
8.	The sum of the square roots of the first *n* natural numbers... ...	Vol. 7	pp. 173—175
9	On the product $\pi \left[1 + \frac{x^2}{(a+nd)^2} \right]$.	Vol. 7	pp. 209—212
10.	Some definite Integrals	Vol. 11	pp. 81— 88
11.	A proof of Bertrand's Postulate ...	Vol. 11	pp. 181—183
12.	(Communicated by S. Narayana Aiyar).	Vol. 5	p. 60

The following articles relating to Mr. Ramanujan have also appeared in the *Journal* :—

1. Mr. Ramanujan's work in England, by G. H. Hardy, M.A., F.R.S. ... Vol. IX. pp. 30— 46
2. Life Sketch of Ramanujan : Editorial. Vol. XI. p. 122.

(B) Questions proposed and solved.

Nos. 260, 261, 283, 289, 294, 295, 298, 308, 353, 358, 386, 427, 441, 464, 489, 507, 541, 546, 571, 605, 606, 629, 642, 666, 682, 700, 723 724, 739, 740, 753, 768, 769, 783, 785.

(C) Questions proposed but not solved as yet.

Nos. 284, 327, 359, 387, 441, 463, 469, 524, 525, 526, 584, 661, 662, 681, 699, 722, 738, 754, 770, 784, 1049, 1070 and 1076.

IN MEMORIAM

S. RAMANUJAN

By Dewan Bahadur R. RAMACHANDRA RAO.*

AND he is no more—

He whose name shed a lustre on all India,

whose career is understood as the severest condemnation of the present exotic system of education :

whose name was always appealed to, if any one forgetful of India's past, ventured to doubt her intellectual capabilities.

Several years ago, a nephew of mine perfectly innocent of mathematical knowledge, spoke to me, "Uncle, I have a visitor who talks of mathematics ; I do not understand him ; can you see if there is anything in his talk." And in the plenitude of my mathematical wisdom, I condescended to permit Ramanujan to walk into my presence. A short uncouth figure, stout, unshaved, not overclean, with one conspicuous feature—shining eyes—walked in, with a frayed note-book under his arm. He had failed in mathematics in his Arithmetic. He was miserably poor. He had run away from Kumbakonam to get leisure in Madras to pursue his studies. He never craved for any distinction. He wanted leisure, in other words, simple food to be provided for him without exertion on his part, and that he should be allowed to dream on.

He opened his note-book and began to explain some of his discoveries. I saw quite at once that there was something out of the way, but my knowledge did not permit me to judge whether he talked sense or nonsense. Suspending judgment, I asked him to come over again. And he did. And then he had gauged my ignorance and showed me some of his simpler results. These transcended existing books and I had no doubt that he was a remarkable man. Then step by step he led me to elliptic integrals, and hyper-geometric series and at last his theory of divergent series, not yet announced to the world, converted me. I asked him what he wanted. He said he just wanted a pittance to live on so that he may pursue his researches. It is a matter of considerable pride to me that I was in some way useful to this remarkable genius in his earlier days. In a year's time, I introduced him to Sir Francis Spring who gave him a sinecure post in his office.

Ramanujan's later history is well-known. His doubting friends communicated some of his results to Cambridge mathematicians. They first scoffed at them as the methods were uncanonical. Yet the formulæ

* Extracted from *Everyman's Review*, Vol 5, Nos. 7 and 8, 1920, with the kind permission of the Editor.

on application behaved correctly. They were wide generalisations of what were accepted abstruse results in the Western world. And slowly they came to accept that there was something in Ramanujan they are not aware of—results true and unknown and methods irregular. This latter did not suit their training. So efforts were mado to induce Ramanujan to go to Cambridge to study proper methods and to publish his results in the accepted straitcoat moulds.

Ramanujan consulted me and perhaps unfortunately I lent all the weight of my influence to induce him to go. I was tempted to do so because, in the first place, I knew that unless he published in English journals, his results will not be advertised and the only way of benefiting India if not himself was by this course. And in the second place there was no place in India stocked with up-to-date mathematical literature where a higher mathematician can easily find books of reference. Ramanujan demurred a long time to go. And it is well-known to his friends that his ultimate decision he always attributed to divine inspiration.

The last two or three days of his stay in India before he proceeded to England he spent with me in a mofussil station. I took him to the house of a friend living in European style to initiate him in the mysteries of the fork and the knife, under the strict stipulation that nothing but vegetable food should be served. He was not happy. He did not relish food being served by strange servants. He was not very jubilant over his future journey and over his assured distinctions to come. He seemed to have moved as if he was obeying a call. He expounded to me an esoterical interpretation of the Ramayan treated as an allegory which is unrivalled for analysis of detail or comprehensiveness of vision.

Ramanujan's career in England is well-known. All honour to Professor Hardy who took a paternal interest in him. His papers established his reputation which would have been all the more but for the war. But he continued his simple habits. He never deserted the fare which he was accustomed to in India. Nor would he permit strangers to cook for him. His health was never robust. Professor Hardy took the keenest interest in him. He was elected, I believe, the youngest F.R.S. His room was a place of pilgrimage to students. His lustre illuminated India's. But loaded with distinction and honours he returned to India to die.

When I met him alighting from the railway train I foresaw the end. All that his friends could do was to lighten his pains and keep alive the lingering flame of his life for as long awhile as possible. He died young but illustrious.

And he is no more. Is he? So far as he is concerned, if there be any truth in his own beliefs—such beliefs, as he himself acknowledged, led him intuitively to his discoveries—he is still existing and has only quitted a world of shams for a world of realities. His loss is ours. And yet his life, if understood, has valuable lessons. In the field of mathematics itself, he got his results by what I may call intuition for want of a better term. Some of his results for example those on Bernoulli's numbers, have not yet been proved though they are correct. The question arises: What are his ratiocinative methods and have not our educational systems to study fresh processes? It is quite possible that these methods are inapplicable except in the case of the highest intellects, but that is all the more reason to incorporate them in educational methods. Let not the Sadler University be the final word on this intricate subject. One thing is however clear: the more Ramanujan's method was drilled into accepted courses, the less his faculty became inventive.

Secondly Ramanujan had certain definite ideas of the fourth dimension. What those ideas are it is impossible to put on paper as our vocabulary limited to our sensations cannot describe them. Here again, while we deplore his early loss before he announced his rational conclusions from his innate ideas, the question again arises how to adapt educational methods to rise over present stereotyped notions.

Apart from mathematics, his life again is full of lessons even to the most superficial observer. Full of his genius and full of recorded results, he yet never cared for publication or advertisement. All he wanted was to be left alone with his work. His was not the idea to go to England. The really great are never vain. He never prostituted his intellect for gain or selfish advancement. Even when two Continents were publishing his results, he remained the same childish man, with no style in dress or affectation of manner, with the same kind face, with the same simplicity. Pilgrims came to Ramanujan's chambers and wondered if this was he.

You might think that this is an orientally exaggerated panegyric of the deceased man. It is not so The really great are people like him. Such people, especially in India, are not to be found in the Senate, nor are they lime-lighted in the journals. They care not for wealth or adulation. What they conceive their duty to be, they do unostentatiously; and their names, perhaps little known now, will survive a thousand years hence, as Archimedes by his law, when all the temporally great men are forgotten. If I am to sum up Ramanujan in one word, I would say, *Indianality*—to coin a word for the purpose. Highest genius deliberately consecrated to private discharge of duty; considerable possibility of material pelf deliberately resigned ; shrinking even from public notice or worship ; evenness of mind in poverty and riches, in health and disease, amidst strangers and amidst friends ; considerable draft on the hidden resources of the spiritual man shattering all our pre-conceived notions of the humanly possible : with these attributes he beckons forward to the Indian to forget the temporary conflicts of ephemeral things and to brighter realms of thought, not conceivable but for the recorded experiences of his own life. Could it be that the time is approaching for an earth-wave of spiritual life to overpower existing conditions and Ramanujam is just a peak thereof and so visible in advance.

OBITUARY.*

S. RAMANUJAN, F.R.S.

SRINIVASA RAMANUJAN, whose death was announced in "NATURE" of June 3, was born in 1888, in the neighbourhood of Madras, the son of poor parents, and a Brahmin by caste. I know very little of his early history or education, but he became a student in Madras University, and passed certain examinations, though he did not complete the course for a degree. Later, he was employed by the Madras Port Trust as a clerk at a salary equivalent to about 25*l.* a year. By this time, however, reports of his unusual abilities had begun to spread, and, I believe owing to the intervention of Dr. G. T. Walker, he obtained a small scholarship which relieved him from the necessity of office work and set him free for research.

I first heard of Ramanujan in 1913. The first letter which he sent me was certainly the most remarkable that I have ever received. There was a short personal introduction written, as he told me later, by a friend. The body of the letter consisted of the enunciations of a hundred or more mathematical theorems. Some of the formulæ were familiar, and others seemed scarcely possible to believe. A few (concerning the distribution of primes) could be said to be definitely false. There were no proofs, and the explanations were often inadequate. In many cases, too, some curious specialisation of a constant or a parameter made the real meaning of a formula difficult to grasp. It was natural enough that Ramanujan should feel a little hesitation in giving away his secrets to a mathematician of an alien race. Whatever reservations had to be made, one thing was obvious, that the writer was a mathematician of the highest quality, a man of altogether exceptional originality and power.

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* Reprinted from "Nature", June 1920.

It seemed plain, too, that Ramanujan ought to come to England. There was no difficulty in securing the necessary funds, his own University and Trinity College, Cambridge, meeting an unusual situation with admirable generosity and imagination. The difficulties of caste and religion were more serious; but owing to the enterprise of Prof. E. H. Neville, who happened fortunately to be lecturing in Madras in the winter of 1913-14, these difficulties were ultimately overcome, and Ramanujan arrived in England in April, 1914.

The experiment has ended in disaster, for after three years in England Ramanujan contracted the illness from which he never recovered. But for these three years it was a triumphant success. In a really comfortable position for the first time in his life, with complete leisure assured to him, and in contact with mathematicians of the modern school, Ramanujan developed rapidly. He published some twenty papers, which, even in war-time, attracted wide attention. In the spring of 1918 he became the first Indian Fellow of the Royal Society, and in the autumn the first Indian Fellow of Trinity. Madras University endowed him with a research studentship in addition, and early in 1919, still unwell, but apparently considerably better, he returned to India. It was difficult to get news from him, but I heard at intervals. He appeared to be working actively again, and I was quite unprepared for the news of his death.

Ramanujan's activities lay primarily in fields known only to a small minority even among pure mathematicians—the applications of elliptic functions to the theory of numbers, the theory of continued fractions, and perhaps above all, the theory of partitions. His insight into formulæ was quite amazing, and altogether beyond anything I have met with in any European mathematician. It is perhaps useless to speculate as to his history had he been introduced to modern ideas and methods at sixteen instead of at twenty-six. It is not extravagant to suppose that he might have become the greatest mathematician of his time. What he did actually is wonderful enough. Twenty years hence, when the researches which his work has suggested have been completed, it will probably seem a good deal more wonderful than it does to-day.

G. H. Hardy.

JEAN-GASTON DARBOUX
1842—1917.

[The following obituary note was delivered at the Annual Meeting of the *Académie des Sciences de Goettingue* on May 12th, 1917 and was published in the *Nachrichten der K. Gesselschaft der Wirsenschaften Zu Göttingen* for 1917 on pp. 71, *et seq.* A French translation of this by Prof. Hilbert appears on pp. 270—3 of *Acta Mathematica*, Vol. 42 : 3 ; the present note is an attempt at a free English rendering of Prof. Hilbert's translation for the benefit of the readers of the *Journal of the Indian Mathematical Society*. Some additional personal notes of Darboux have also been compiled from the *Annuario Biografico del circolo Mathematico di Palermo* and given here.—K. B. MADHAVA].

If among those mathematicians who have contributed to the development of Mathematics in France during the latter half of the XIXth century the name of Henri Poincaré has been assigned the most eminent place, that of Jean-Gaston Darboux occupies by no means a less prominent position. This is due no doubt to the greatness of his scientific achievements ; but his brilliant career, his engaging personality, his talents for organisation and the sauvity of his temper have also contributed in an equal measure to his eminence.

During the years between 1860 and 1870, there was manifest an extreme tendency for specialisation in the domain of sciences in France as also in Germany. Serret, Bouquet, Bonnet, Chasles and Hermite—these were the representatives of this great movement in the mathematical sciences : Chasles for Geometry and Hermite for Analysis. Then followed Darboux and Camille Jordan in the later years, who by opening up new fields for investigation and research, have brought into existence a band of disciples with refreshing enthusiasm for mathematical studies. The result of this great awakening was the new and entire shift in the study of mathematical problems, described by Darboux himself before the International Congress of Mathematicians in Rome (1908), when he instituted a comparison between the mathematical methods of the XIXth and XXth centuries. He said that while in the first half of the nineteenth century scholars were content with merely collating the achievements of the two previous centuries, in the XXth century on the contrary, the research in mathematics opened up altogether new prospects, presenting new and unexplored fields for investigation, and he felt confident that the years following would perfect these tendencies.

During the autumn of the year 1861, Darboux passed the examinations for admission both to the École Polytechnique and to the École Normale Supérieure, obtaining in both cases the first place ; but he decided to enter the latter. The fact that a young man who had attained such high distinction in the entrance examination had decided to abandon the enviable dignity of an engineering officer of the Republic, to exchange it for the humble title of a professor discharging the modest duties that fell to that position, doubtless attracted some general notice; and J. J. Weiess, the celebrated critic of Goethe, devoted an article to this event in the *Journals des débats* (20th Nov. 1861) congratulating the brilliant young man on his great personal sacrifice, and instancing that circumstance as an example for lovers of academic sciences to take to the study of the subjects of their choice for their own sake. At this epoch, Darboux had the valuable privilege of meeting and coming under the personal observation of the most influential scientific personages then living in France. In 1864, the year in which he left the E'cole Normale, as also the year in which he published his first paper "*Sur les sections du tore,*" Pasteur obtained for him a place of some little remuneration which kept him on to work independently, during which time he worked very hard to prepare and publish his Thesis on Orthogonal Surfaces. This fetched him in July 1866, the Doctor's degree in Mathematics at the University of Paris; and two years later, he was successively nominated Professor of Mathematical Physics in the College of France, and later Professor of Advanced Mathematics at the Lyceé—Louis le Grand. Since then, the charges he was entrusted with and the honours that accrued to him, began to increase until towards the latter part of his life he was made permanent Secretary to the Academy of Sciences in Paris. With Gottingen, the University to which Prof. Hilbert belongs, he was specially connected from 1879, first as correspondent to the Society of Göttingen, and later from 1901 as one of its most valued Foreign Members.

Darboux was gifted by nature with a keen geometrical insight, but at the same time he had the capacity to plunge into the most diverse branches of sciences that came within the purview of mathematical enquiry; and carrying with him in each of these his intuitive geometrical insight, he made those subjects the easier. It is due to this varied nature of his talents that we find even among the labours of his youth, three very valuable papers on subjects not purely geometrical, but at which he worked with such completeness and mastery that his name has come to be permanently associated with them. The first is the paper "*Sur les équations aux dérivées partielles* (Ann. E'c. Norm. VII 1870) which played such an important *rolé* subsequently in the labours of Sophius Lie. This memoir established the method of integration of the Linear Partial Differential Equations of the Second Order which bears even to-day the name of Darboux. The method in question constitutes the development of the theory of Monge-Ampere which considers differential equations as a set of ordinary equations of the same type such that the integration of one of them entails those of others.

The two other memoirs had their origin in the studies that Darboux made upon the investigations of Riemann concerning Trigonometric series. The second is the memoir entitled "*Sur les theorie des fonctions discontinues*" (Ann. E'c. Norm. IV. 1875), in which for the first time are introduced the terms "upper limiting integral" (in excess), and "lower limiting integral" (in defect) in the evaluation of a definite integral, the terms which still go by the name of Darboux.

This memoir contains also a large number of results upon the theory of functions of one real variable which were also being given out about the same time by Weierstrass in his lectures but not published then. It is hardly necessary to add that this paper had a decisive influence in the introduction of rigour into modern mathematics in France. The last paper bears the name "*Memoir Sur l'approximation des fonctions de trés grands nombres*" and appeared in *Journal de Liouville*, 3rd Series, tome 4, (1878). It concerned mainly with certain researches of Laplace and related them with the Theory of Fourier's Series. The author has evaluated the coefficients of the Fourier's Series corresponding to an analytic function of a real variable with given real singularities. One striking feature in this paper is the large number of illustrations given applying his results to various important problems selected from the most diverse of mathematical domains. Poincaré has made a very frequent use of this paper of Darboux, particularly for the evaluation of maximum order terms of his "fonction perturbatrice".

The one subject that was dear to Darboux is, as we have already noticed, Geometry, and to this he consecrated his remarkable monograph "*Sur une classe remarquable de courbes et de surfaces algebriques et sur la theorie des imaginaires*" Paris 1873). This monograph introduced for the first time the system of pentaspherical co-ordinates which was essentially due to Darboux himself, and contains the analogue as well as the supplement of the labours of Felix Klein and Sophius Lie about the same epoch.

Darboux undertook to teach Mechanics to the Sorbonne during the years 1873—1878 and this induced him to make a series of new contributions to this part of science. A greater portion of these has been published as notes in the "*Mecanique*" by Despeyrous. His researches epecially those upon the axioms of the parallelogram of forces, have been the starting point of more than one thesis presented to Gottingen—notable among those being the one by M. Schimmack.

Among the problems which Darboux himself attempted will be found his attack of the problem of the motion of a solid body every point of which described ellipses, as the most striking and in some respects the most remarkable. This led him besides, to the discovery of the solution of the problem of constructing those surfaces the geodesic lines upon all of which are closed curves; and indirectly this suggested problems for two theses presented to Gottingen by M. Otto Zoll in 1901 and by M. Paul Furk (1911).

The principal work of Darboux, however, lay in his labours in the domain of the theory of surfaces. They were first collated in his *Lecons Sur les systems orthogonaux*, but his later investigations are included in his copious four volumes of the *Theory of Surfaces*. This last book has not only appropriated a central place in the literature on the subject, but it is also at the same time a rich mine of information on a variety of subjects of importance, such as mechanics, calculus of variations, theory of partial differential equations, the theory of invariants. Each of these subjects the versatile author has found an occasion to apply to the problem he had on hand and has explained them with a penetration and clearness that are his unique qualities. In one word one can safely assert that his labours are as invaluable and indispensable to the library of every true mathematician, as say, the *Cours d' Analyse* of Camille Jordan, or *Traite d' Analyse* of Emile Picard, or *Mecanique Celestiale* of Henri Poincare.

In concluding, we can only recognise that by mere words we cannot chronicle all that was great and valuable in Darboux—not even his high administrative and organising capacities in the official positions he filled from time to time. For six years he was the Doyen of the Faculty of Sciences in France, and to him is due the new constitution of the Sorbonne, the recognition also of the *Counseil Supérieur de l' Instruction Publique*, and of various other offices which had anything to do with the placing of mathematical thought on a higher and more rigorous basis. He worked also with characteristic zeal for the usefulness of the *Association Internationale des Académies Scientifiques*; and it was to the Academy of Sciences in Paris that he gave his best, the most melancholy association with it being that it was at the official residence of the Institute of France, in No. 3, Rue Mazarine, Paris, that he breathed his last in the summer of the year of grace, 1917.

To collect together in one place the official positions he held, or the Societies, French or foreign, with which he was connected will perhaps give an idea of the great influence he wielded over the learned societies of the world.

The following list has accordingly been prepared :—

Doctor of Mathematical Sciences (Paris, 1866).

Honorary Doctor of Science of the Universities of Cambridge, Christiana, Heidelberg.

Perpetual Secretary of the Academy of Sciences of the Institute of France.

Honorary member and for some time (1878) President of the Mathematical Society of France.

Member of the French Association for the Advancement of Science.

President of the Society of Savants and Friends of Science.

President of the First General Congress of the International Association of Academies (Paris 1901)

Honorary Vice-President of the Congress of Art and Science held at Saint-Louis in 1904.

Vice-President of the Council for Advanced Public Instruction in France (1908—1917).

Member of the Bereau des Longitudes.

Delegate of the French Government to the International Geodesy Assn.

President of the Commission of Bolyai's prize (1905)

President of the French Editorial Board for the International Catalogue of Scientific Literature.

President of the Council of the Pasteur Institute.

President of the Council of the Observatory of Paris.

President of the Section of Mathematics at the École Pratique for Advanced Studies.

President of the Editorial Board of "*Annales Scientifiques de l' E'cole Normale Supérieure*".

Editor-in-chief of *Bulletin des Sciences Mathematiques*.

Foreign Member, Honorary Member, Member or Correspondent of the following learned bodies :—

—Alais. Société Scientifique et Littéraire d'—

—Amsterdam, Kon-Akademie Van Wetenschappen te—

Berlin Kgl Preussische Akademie der Wissenschaften

Bern Schweizerische Naturforschende Gesselschaft

—Bologona, Reale Acad, del. Scienze di—

Bordeaux, Société des Sciences Physiques

Brusselles, Académie Royale des Sciences etc. de Belgique)

Budapest, Mathematikai éf Physikai Tàrsulat

Calcutta Mathematical Society

Dublin Royal Irish Academy.

—Edinburgh Royal Society of—

—Gottingen, Kgl. Gesselschaft der Wissenschaften Zu—

Halle Deutsche Akademie der Naturfouscher

Helsingfors Societas scientiarum Fennica
Kasan, Imperial University of—
Kopenhagen, Danske Viden—Selskab
Kristiania Videnskabs—Selskabet
—London, Royal Society of—
London Mathematical Society
Manchester Literary and Philosophical Society
Milano, Reale Institute di Scienze
—Moscow, Société Mathématique de—
—Nimes—Academie de—
Philadelphia—American Phil. Soc. for promoting Useful Knowledge
Rome—Reale Accademia dei Liarcei
—St. Petersburg Académie Impériale des Sciences de—
Santiago, Sociedad Cientifica de Clùle
Venice, Reale Instituto Veneto di Scienze
Washington, National Academy of Sciences

SHORT NOTES

ON THE SET OF POINTS $\frac{\phi(n)}{n}$.*

Let $\phi(n)$ be the number of integers less than n and prime to it and let $F(n) = \frac{\phi(n)}{n}$. It is well known that

$$F(n) = \left(1 - \frac{1}{a}\right)\left(1 - \frac{1}{b}\right)\cdots\left(1 - \frac{1}{k}\right)$$

where $a, b, \ldots\ldots k$ are the different prime factors of n. It follows immediately that for all values of n

$$0 < F(n) < 1.$$

It is the object of this note to prove that every point of the interval $(0, 1)$ is a limiting point of the set of points $F(n)$.

1° Let p_r be the r^{th} prime. Since there are an infinite number of primes and since $$F(p_r) = \left(1 - \frac{1}{p_r}\right) \to 1$$ as r tends to ∞, it is clear that 1 is a limiting point of the set $F(n)$.

2° Consider next the numbers $F(n_r)$, where $n_r = p_1 p_2 \ldots p_r$.

$$\text{Then } F(n_r) = \left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right)\cdots\cdots\left(1 - \frac{1}{p_r}\right).$$

It is known † that this product tends to zero as $r \to \infty$. Hence zero is a limiting point of the set.‡

3° Lastly, let $0 < k < 1$ and $0 < \varepsilon < k$.

It is possible to find a number N so that

$$k - \varepsilon < F(N) < k \tag{1}$$

from which it will follow that k is a limiting point of the set $F(n)$.

We can choose a prime number P_1 so large that

$$P_1 > \frac{1}{1-k}, \; P_1 > \frac{k}{\varepsilon}. \tag{2}$$

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* I am much indebted to Mr. K. Ananda Rau for valuable suggestions in preparing this note.

† See for example Landau, *Handbuch der Lehre von der Verteilung der Primzahlen*, pp. 65, 66.

‡ That 1 and 0, are limiting points has been known for a long time, See Landau, *loc. sit.*, pp. 216—218.

Let P_2, P_3,...... be the prime numbers in order following P_1. Consider the numbers N_1, N_2 ...... defined by

$$N_m = P_1 P_2 \ldots\ldots P_m \; (m = 1, 2 \ldots).$$

Then
$$F(N_m) = \left(1 - \frac{1}{P_1}\right)\left(1 - \frac{1}{P_2}\right) \ldots\ldots \left(1 - \frac{1}{P_m}\right).$$

Clearly $F(N_1)$, $F(N_2)$, ... form a decreasing sequence, and from what has been said above

$$F(N_m) \to 0$$

as $m \to \infty$. Also

$$F(N_1) = 1 - \frac{1}{P_1} > 1 - (1 - k) = k$$

from the first of the inequalities (2).

Hence if none of the numbers N_2, N_3......satisfy the inequalities (1), there should exist an integer $p \geqslant 1$, so that

$$F(N_p) \geqslant k \text{ and } F(N_{p+1}) \leq k - \varepsilon.$$

In this case, we shall have

$$\left(1 - \frac{1}{P_1}\right)\left(1 - \frac{1}{P_2}\right) \ldots\ldots \left(1 - \frac{1}{P_p}\right) \geqslant k$$

$$\left(1 - \frac{1}{P_1}\right)\left(1 - \frac{1}{P_2}\right) \ldots\ldots \left(1 - \frac{1}{P_{p+1}}\right) \leq k - \varepsilon$$

and so
$$1 - \frac{1}{P_{p+1}} \leq \frac{k - \varepsilon}{k}$$

or
$$P_{p+1} \leq \frac{k}{\varepsilon}.$$

But
$$P_{p+1} > P_1$$

and therefore
$$P_1 < \frac{k}{\varepsilon},$$

which contradicts the second of the inequalities (2).

Hence a number N exists which satisfies (1), and the theorem is therefore proved.

T. VIJAYARAGHAVAN.

The Generating Planes of a Quadric in Five Dimensions.

It is known that the generating regions of a quadric in four dimensions are straight lines, and in five dimensions, planes. (*Vide:* Whitehead's *Universal Algebra*).

I have shewn elsewhere that the generating lines of a quadric in four dimensions *do not* fall into two distinct systems as in the case of a quadric in three dimensions.

It will be shewn now that *the generating planes of a quadric in five dimensions can be divided into two mutually exclusive systems, such that two planes belonging to the same system intersect in one and only one point and two planes belonging to different systems do not intersect in a single point, i.e., either do not intersect at all or intersect in a straight line.*

The demonstration will depend on line-geometry in three dimensions where the *line* is taken as the fundamental element; the *point* is regarded as the aggregate of lines passing through it, the *plane* is regarded as the aggregate of lines lying on it. In addition to the *point* and the *plane*, there is a third element (called '*coté*' by French writers), which, to some extent, plays the same role in line-geometry as the straight line does in point-geometry. This is the *set of concurrent coplanar straight lines*, and may be conveniently termed a *point-plane*, in as much as it is defined by a point and a plane passing through it.

A point in five dimensions whose homogenous co-ordinates are $(l, m, n, \lambda, \mu, v)$ may be identified with the screw or system of forces in three dimensions whose Cartesian co-ordinates (Resolutes and Moments w, r, t, rectangular axes) are $(l, m, n, \lambda, \mu, v)$. The points on the quadric $S \equiv l\lambda + m\mu + nv = 0$ in five dimensions will then represent all screws which reduce to single forces, in other words, all straight lines.

A generating *line* of S will represent a *linearly closed set* of ∞^1 straight lines, *i.e.*, a set of ∞^1 straight lines which is such that any linear combination of any two lines of the set is a third line of the set. Thus a generating line of S corresponds to a point-plane. Similarly a generating plane of S represents a linearly closed set of ∞^2 straight lines and must therefore correspond either to a *point* or to a *plane*.

Thus there are two kinds of generating planes of S, *viz.*, those which correspond to points in three dimensions and those which correspond to planes.

Two generating planes of the same kind have one and only one point in common ; for, in three dimensions two points or two planes have only one line in common (the joining line and the line of intersection, respectively.)

Two generating planes of different kinds, either have no point in common or have a line in common ; for a point and a plane in three dimensions have either no line in common (when the point does not lie in the plane), or have a point-plane in common (when the point lies in the plane).

It may be shewn from the theory of quadratic forms, that the equation to any non-singular quadric in five dimensions can be reduced to the form $l\lambda + m\mu + n\nu = 0$. The result provcd is therefore general.

R. VYTHYNATHASWAMY.

SOLUTIONS.

Question 754.

(S. RAMANUJAN) :—Show that

$$\frac{e^x\,\Gamma(1+x)}{x^x\sqrt{\pi}} = \sqrt[6]{8x^3+4x^2+x+E}$$

where E lies between $\frac{1}{100}$ and $\frac{1}{30}$ for all positive values of x.

Solution by K. B. Madhava.

It is known (Bromwich § 180), that

$$\log\Gamma(1+x) = (x+\tfrac{1}{2})\log x - x + \tfrac{1}{2}\log 2\pi + \psi(x) \quad \dots \quad (1)$$

where $$\psi(x) = 2\int_0^\infty \frac{\tan^{-1}\left(\frac{y}{x}\right)}{e^{2\pi y}-1}\,dy\,; \quad \dots \quad \dots \quad (2)$$

so that we have

$$\frac{e^x\,\Gamma(1+x)}{x^x\sqrt{\pi}} = \sqrt{2x}\;e^{\psi(x)} = \sqrt[6]{8x^3}\;e^{\psi(x)}$$

$$= \sqrt[6]{8x^3+8ax^2+8bx+8c+\dots}\ \text{(say)} \quad \dots \quad (3)$$

where a, b ... can be determined from

$$e^{\psi(x)} = \sqrt[6]{1+\frac{a}{x}+\frac{b}{x^2}+\frac{c}{x^3}+\dots}$$

i.e. $$\psi(x) = \tfrac{1}{6}\log\left(1+\frac{a}{x}+\frac{b}{x^2}+\frac{c}{x^3}+\dots\right)$$

i.e. $$\int_0^\infty \frac{\tan^{-1}\left(\frac{y}{x}\right)}{e^{2\pi y}-1}\,dy = \tfrac{1}{12}\log\left(1+\frac{a}{x}+\frac{b}{x^2}+\frac{c}{x^3}+\dots\right)$$

$$\text{i.e.}\quad \sum_{r=1}^{\infty}\int_0^{\infty} (-)^{r-1}\frac{y^{2r-1}}{(2r-1)\,x^{2r-1}}\cdot\frac{dy}{e^{2\pi y}-1}$$

$$=\frac{1}{12}\log\left(1+\frac{a}{x}+\frac{b}{x^2}+\frac{c}{x^3}+\dots\right).$$

But it is well-known

$$\int_0^{\infty}\frac{y^{2r-1}\,dy}{e^{2\pi y}-1}=\frac{Br}{4r}\quad\dots\quad\dots\quad(4)$$

where B is the Bernoullian number, so that we have

$$\sum_{r=1}^{\infty}\frac{Br}{r\,(2r-1)}\cdot\frac{1}{x^{2r-1}}=\tfrac{1}{3}\log\left(1+\frac{a}{x}+\frac{b}{x^2}+\frac{c}{x^3}+\dots\right).$$

Hence $a, b, c \dots$ in (3) are determined by algebraic equations.

We shall take Mr. Ramanujan's values for a and b and use E for 8 with a view to calculate E and F, etc.

We have

$$\frac{1}{6x}+\frac{1}{180x^3}+\frac{1}{630x^5}+\frac{1}{840x^7}+\frac{1}{594x^9}$$

$$=\tfrac{1}{3}\log\left(1+\frac{1}{2x}+\frac{1}{8x^2}+\frac{E}{8x^3}+\frac{F}{8x^4}+\dots\right)$$

$$=\tfrac{1}{3}\left[\frac{1}{2x}+\left(\frac{1}{8}-\frac{1}{8}\right)\frac{1}{x^2}+\left(\frac{E}{8}-\frac{1}{16}+\frac{1}{24}\right)\frac{1}{x^3}\right.$$

$$\left.+\left(\frac{F}{8}-\frac{1}{128}-\frac{E}{16}+\frac{1}{64}-\frac{1}{64}\right)\cdot\frac{1}{x^4}+\dots\right];$$

so that we have $\frac{E}{8}=\frac{1}{60}+\frac{1}{48}=\frac{3}{80}$, *i.e.* $E=\frac{3}{10}$;

and $\frac{1}{8}+F=\frac{1}{16}+\frac{3}{20}=\frac{17}{80}$, *i.e.* $F=\frac{7}{80}$.

We finally have then

$$\frac{e^x\,\Gamma(1+x)}{x^x\sqrt{\pi}}=\sqrt[6]{8x^3+4x^2+x+\frac{3}{10}+\frac{7}{80\,.\,x}+\dots}$$

This result gives for all positive values of x (>1) the absolute term to be at most about $\frac{1}{2}$; but Mr. Ramanujan gives the interval $(\frac{1}{30}, \frac{1}{100})$.

Question 791.

(K. Appukuttan Erady, M.A.):—The centres of three circles of radii a, b, c form a triangle of sides l, m, n and area Δ.

If (r_1, r'_1), (r_2, r'_2), (r_3, r'_3) and (r_4, r'_4) be the radii of the four pairs of circles tangential to the three circles (the circles belonging to any pair being inverses of each other with respect to the common orthogonal circle of the three original circles), show that

$$\frac{1}{r_1 r'_1} + \frac{1}{r_2 r'_2} + \frac{1}{r_3 r'_3} + \frac{1}{r_4 r'_4}$$

$$= \frac{16(\Delta^2 - \Sigma a^2 l^2)}{\Sigma l^2(a^2 - b^2)(a^2 - c^2) - \Sigma m^2 n^2(b^2 + c^2) + \Sigma a^2 l^4 + l^2 m^2 n^2}.$$

Solution by Martyn. M. Thomas, M.A.

Lemma. If $\theta + \phi + \psi = 360°$, then

$$\cos^2\theta + \cos^2\phi + \cos^2\psi - 2\cos\theta\cos\phi\cos\psi = 1.$$

Let A, B, C be the centres of the 3 given circles, so that BC $= l$, CA $= m$, AB $= n$. Take a point P in the plane of ABC, and denote the lengths PA, PB, PC by p, q, r.

If the angles at P be θ, ϕ, ψ, then, from the lemma, we can deduce a relation between the six st. lines l, m, n, p, q, r.

Substituting $\dfrac{q^2 + r^2 - l^2}{2qr}$ for $\cos\theta$, $\dfrac{r^2 + p^2 - m^2}{2rp}$ for $\cos\phi$, and $\dfrac{p^2 + q^2 - r^2}{2pq}$ for $\cos\psi$, we have, after simplification

$$\Sigma l^4 p^2 + \Sigma l^2 p^4 + \Sigma q^2 r^2 (l^2 - m^2 - n^2) - \Sigma l^2 m^2 (p^2 + q^2) + l^2 m^2 n^2 = 0.$$

Replacing p, q, r by $x \pm a$, $x \pm b$, $x \pm c$, in all possible combinations of the ambiguities in sign, we get the 8 different values of x to be r_1, r'_1, ..., as the tabular form will show

Length of radius.	p	q	r
r_1	$x + a$	$x + b$	$x + c$
r'_1	$x - a$	$x - b$	$x - c$
r_2	$x - a$	$x + b$	$x + c$
r'_2	$x + a$	$x - b$	$x - c$
r_3	$x + a$	$x - b$	$x + c$
r'_3	$x - a$	$x + b$	$x - c$
r_4	$x + a$	$x + b$	$x - c$
r'_4	$x - a$	$x - b$	$x + c$

Now r_1 is a root of the equation.

$$\Sigma l^4(x+a)^2 + \Sigma l^2(x+a)^4 + \Sigma(x+b)^2(x+c)^2(l^2-m^2-n^2)$$
$$-\Sigma l^2m^2\{(x+a)^2+(x+b)^2\} + l^2m^2n^2 = 0.$$

Arranging in powers of x, we have

$$Ax^3 + Bx^2 + Cx + D = 0, \text{ where}$$

$A = 4\Sigma al^2 + 2\Sigma(b+c)(l^2-m^2-n^2)$.

$B = -16\Delta^2 + \Sigma l^2\{4a^2+6(bc-ab-ac)\}$.

$C = 2\Sigma al^4 + 4\Sigma a^3l^2 + 2\Sigma bc(b+c)(l^2-m^2-n^2) - 2\Sigma(a+b)l^2m^2$.

$D = \Sigma l^4a^2 + \Sigma l^2(a^2-b^2)(a^2-c^2) - \Sigma m^2n^2(b^2+c^2) + l^2m^2n^2$.

Changing a, b, c to $-a, -b, -c$, in the above equation, we find that r'_1 is a root of the equation $-Ax^3 + Bx^2 - Cx + D = 0$.

$\therefore$ r_1 and r'_1 both satisfy the equation $Bx^2 + D = 0$.

$$\therefore \quad r_1r'_1 = -\frac{D}{B}.$$

$$\therefore \frac{1}{r_1r_1'} = -\frac{B}{D} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

$$= \left[\begin{array}{l}16\Delta^2 - l^2\{4a^2+6(bc-al-ac)\} \\ -m^2\{4c^2+6(ac-bc-ab)\} - n^2\{4c^2+6(ab-ac-bc)\}\end{array}\right] \div D$$

changing a alone to $-a$ in (i), we have

$$\frac{1}{r_2r_2'} = \left[\begin{array}{l}16\Delta^2 - l^2\{4a^2+6(bc+al+ac)\} \\ -m^2\{4b^2+6(-ac-bc+ab)\} - n^2\{4c^2+6(-ab+ab-bc)\}\end{array}\right] \div D$$

changing b alone to $-b$ in (i),

$$\frac{1}{r_3r_3'} = \left[\begin{array}{l}16\Delta^2 - l^2\{4a^2+6(-bc+ab-ac)\} \\ -m^2\{4b^2+6(ac+bc+ab)\} - n^2\{4c^2+6(-ab-ac+bc)\}\end{array}\right] \div D$$

changing c alone to $-c$ in (i),

$$\frac{1}{r_4r_4'} = \left[\begin{array}{l}16\Delta^2 - l^2\{4a^2+6(-bc-ab+ac)\} \\ -m^2\{4b^2+6(-ac+bc-ab)\} - n^2\{4c^2+6(ab+ac+bc)\}\end{array}\right] \div D$$

$$\therefore \frac{1}{r_1r_1'} + \frac{1}{r_2r_2'} + \frac{1}{r_3r'_3} + \frac{1}{r_4r_4'} = \frac{64\Delta^2 - 16\Sigma a^2l^2}{D}$$

$$= \frac{16(4\Delta^2 - \Sigma a^2l^2)}{\Sigma l^4a^2 + \Sigma l^2(a^2-b^2)(a^2-c^2) - \Sigma m^2n^2(b^2+c^2) + l^2m^2n^2}$$

Question 876.

(M. Bhimasena Rao) :—Shew that the radical axis of the N. P. circle and the Pedal circle of isogonal conjugates, P and Q, cuts the N. P. circle at the angle POQ, where O is the circumcentre of ABC.

Solution by the Proposer.

Let B′ C′ be the side of the medial triangle of ABC and L, L′ the feet of the perpendiculars from A on OP and OQ. If w and w' be the reflections of L and L′ in B′C′, the radical axis of the Pedal circle and the N. P. circle is ww'. The Nine point circle is the reflection of the circle AB′C′ LL′O in B′C′. Therefore angle POQ which is the same as the angle LOL′ is the angle subtended by LL′ at any point on the circle AB′C′ LL′O and is equal to that subtended by ww' at any point on the N. P. circle. Hence the result.

The length of the radical axis is R sin POQ, where R is the circumradius of ABC.

Question 883.

(K. J. Sanjana, M. A.) ;—Prove that the determinant of $2n$ lines and columns

$$\begin{vmatrix} \frac{1}{1!} & 1 & 0 & 0 & 0 & 0 & \cdots \\ \frac{1}{3!} & \frac{1}{2!} & \frac{1}{1!} & 1 & 0 & 0 & \cdots \\ \frac{1}{5!} & \frac{1}{4!} & \frac{1}{3!} & \frac{1}{2!} & \frac{1}{1!} & 1 & \cdots \\ \frac{1}{7!} & \frac{1}{6!} & \frac{1}{5!} & \frac{1}{4!} & \frac{1}{3!} & \frac{1}{2!} & \cdots \\ \frac{1}{4n-3!} & \frac{1}{4n-4!} & \frac{1}{4n-5!} & \frac{1}{4n-6!} & \frac{1}{4n-7!} & \frac{1}{4n-8!} & \cdots \\ \frac{1}{4n-1!} & \frac{1}{4n-2!} & \frac{1}{4n-3!} & \frac{1}{4n-4!} & \frac{1}{4n-5!} & \frac{1}{4n-6!} & \cdots \end{vmatrix}$$

is equivalent to $\{ 1^{2n}. 3^{2n-1}. 5^{2n-2}...(4n-3)^2, (4n-1)^1 \}^{-1}$

Solution by the late R. J. Pocock and S. V. Venkatachala Iyer.

Multiply the rows by 1!, 3!, 5!, 7!,...(4n—3)!, (4n—1)! respectively, so that the whole determinant is multiplied by the continued product of these factorials. Then subtract the top row from each of the others and the determinant reduces to the $(2n-1)$ th order determinant

$$\begin{vmatrix} 2 & 3\cdot2 & \dots \\ 4 & 5\cdot4 & \dots \\ 6 & 7\cdot6 & \dots \\ \dots & & \\ 4n-2 & (4n-1)(4n-2) & \dots \end{vmatrix}$$

Dividing the rows by $2, 4, 6, \dots 4n-2$ respectively, and subtracting the top row, the determinant again reduces to one of lower order. Again, we divide by $2, 4, 6, \dots (4n-4)$ and continuing this process; the determinant finally reduces to unity multiplied by $(4n-2)\,(4n-4)^2\,(4n-6)^3 \dots 2^{2n-1}$ and therefore the original determinant is equal to

$$\{1^{2n}.\ 3^{2n-1}.\ 5^{2n-2} \dots (4n-3)^2,\ (4n-1)^1.\}^{-1}.$$

Question 884.

(M. K. KEWALRAMANI) :—A B C is a triangle and ω is its positive Brocard point, PQR are the points of contact of the inscribed Brocard ellipse with the sides BC, CA, AB respectively. If $A\omega$, $B\omega$, $C\omega$, meet the sides in D, E, F respectively, prove that (1) the triangles PQR and DEF are equivalent and (2)

$$\frac{\Delta\ PQR}{\Delta\ ABC} = \frac{2\,a^2 b^2 c^2}{(a^2+b^2)(b^2+c^2)(c^2+a^2)}.$$

Solution by K. B. Madhava, and S. Muthukrishnan.

In areal co-ordinates

$$\omega \text{ is } \left\{\frac{1}{b^2}, \frac{1}{c^2}, \frac{1}{a^2}\right\}$$

and the Brocard ellipse is $\dfrac{\sqrt{X}}{a} + \dfrac{\sqrt{Y}}{b} + \dfrac{\sqrt{Z}}{c} = 0.$

Hence the points P, Q, R are

$(0, b^2, c^2)$; $(a^2, 0, c^2)$; $(a^2, b^2, 0)$ respectively.

Hence

$$\frac{\Delta\ PQR}{\Delta\ ABC} = \frac{1}{(a^2+b^2)(b^2+c^2)(c^2+a^2)} \times \begin{vmatrix} o & b^2 & c^2 \\ a^2 & o & c^2 \\ a^2 & b^2 & o \end{vmatrix}$$

$$= \frac{2\,a^2 b^2 c^2}{(a^2+b^2)(b^2+c^2)(c^2+a^2)}.$$

Again D, E, F are $(0, a^2, c^2)$; $(a^2, 0, b^2)$ $(c^2, b^2, 0)$ respectively.

and
$$\frac{\Delta\ DEF}{\Delta\ ABC} = \frac{2\,a^2 b^2 c^2}{(a^2+b^2)(b^2+c^2)(c^2+a^2)}.$$

Hence the triangles DEF and PQR are equivalent.

Question 880.

(S. MAHADEVAN) :—If V be the potential at an external point of a solid homogeneous ellipsoid semiaxes a, b, c and density ρ, and if $a^2 + \lambda$, $b^2 + \lambda$, $c^2 + \lambda$, be the squares on the semiaxes of the confocal ellipsoid passing through the external point, s the arc measured from any fixed point of the curve of intersection of the two confocal hyperboloids passing through the external point, prove that

$$\frac{d^2 V}{d\lambda^2} = \frac{4\pi\,\rho\,abc}{(a^2+\lambda)^{\frac{1}{2}}(b^2+\lambda)^{\frac{1}{2}}(c^2+\lambda)^{\frac{1}{2}}}\left(\frac{ds}{d\lambda}\right)^2.$$

Solution by the late Appukuttan Erady.

The potential V at the external point (x, y, z) is given by

$$V = \int_\lambda^\infty \frac{\pi\rho\,abc\,.\,du}{\{(a^2+u)(b^2+u)(c^2+u)\}^{\frac{1}{2}}}\left\{1 - \frac{x^2}{a^2+u} - \frac{y^2}{b^2+u} - \frac{z^2}{c^2+u}\right\};$$

where λ is the parameter of the confocal ellipsoid passing through the point.

Differentiating V with respect to λ, *i.e.* along the curve of intersection of the two confocal hyperboloids that pass through the point and remembering that x, y, z depend upon λ we have

$$\frac{dV}{d\lambda} = -\frac{\pi\rho\,abc}{\{(a^2+\lambda)(b^2+\lambda)(c^2+\lambda)\}^{\frac{1}{2}}}\left[1 - \frac{x^2}{a^2+\lambda} - \frac{y^2}{b^2+\lambda} - \frac{z^2}{c^2+\lambda}\right]$$

$$-\int_\lambda^\infty \frac{\pi\rho\,abc\,du}{\{(a^2+u)(b^2+u)(c^2+u)\}^{\frac{1}{2}}}\left[\frac{1}{a^2+u}\cdot\frac{2x\,dx}{d\lambda} + \frac{1}{b^2+u}\cdot\frac{2y\,dy}{d\lambda} + \frac{1}{c^2+u}\cdot\frac{2z\,dz}{d\lambda}\right].$$

Also $$\frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} + \frac{z^2}{c^2+\lambda} = 1.$$

$$\therefore \frac{2x}{a^2+\lambda}\cdot\frac{dx}{d\lambda} + \frac{2y}{b^2+\lambda}\cdot\frac{dy}{d\lambda} + \frac{2z}{c^2+\lambda}\cdot\frac{dz}{d\lambda}$$

$$= \frac{x^2}{(a^2+\lambda)^2} + \frac{y^2}{(b^2+\lambda)^2} + \frac{z^2}{(c^2+\lambda)^2} \quad \ldots\ldots\ldots\ldots\ldots \text{(A)}$$

Also dx, dy, dz are proportional to the direction cosines of the normal to the ellipsoid through (x, y, z), we have therefore

$$\frac{\frac{dx}{d\lambda}}{\frac{x}{a^2+\lambda}} = \frac{\frac{dy}{d\lambda}}{\frac{y}{b^2+\lambda}} = \frac{\frac{dz}{d\lambda}}{\frac{z}{c^2+\lambda}} = \frac{\Sigma\frac{ax}{a^2+\lambda}\cdot\frac{dx}{d\lambda}}{2\Sigma\frac{x^2}{(a^2+\lambda)^2}} = \tfrac{1}{2} \quad \ldots\ldots\ldots\ldots\ldots \text{from (A)}$$

$$\text{Hence } \frac{dv}{d\lambda} = -\int_\lambda^\infty \frac{\pi\rho\, abc\, du}{\{(a^2+u)(b^2+u)(c^2+u)\}^{\frac{1}{2}}}\left[\frac{x^2}{(a^2+u)(a^2+\lambda)} + \frac{y^2}{(b^2+u)(b^2+\lambda)} + \frac{z^2}{(c^2+u)(c^2+\lambda)}\right]$$

$$\therefore \frac{d^2v}{d\lambda^2} = \frac{\pi\rho\, abc}{\{(a^2+\lambda)(b^2+\lambda)(c^2+\lambda)\}^{\frac{1}{2}}}\left[\frac{x^2}{(a^2+\lambda)^2} + \frac{y^2}{(b^2+\lambda)^2} + \frac{z^2}{(c^2+\lambda)^2}\right]$$

$$-\int_\lambda^\infty \frac{\pi\rho\, abc\, du}{\{(a^2+u)(b^2+u)(c^2+u)\}^{\frac{1}{2}}}\left[\frac{1}{a^2+u}\cdot\frac{d}{d\lambda}\left(\frac{x^2}{a^2+\lambda}\right) + \frac{1}{b^2+u}\cdot\frac{d}{d\lambda}\left(\frac{y^2}{b^2+\lambda}\right) + \frac{1}{c^2+u}\,\frac{d}{d\lambda}\left(\frac{z^2}{c^2+\lambda}\right)\right]$$

From the values of $\frac{dx}{d\lambda}$, $\frac{dy}{d\lambda}$, $\frac{dz}{d\lambda}$ obtained above, we have

$$\left(\frac{ds}{d\lambda}\right)^2 = \left(\frac{dx}{d\lambda}\right)^2 + \left(\frac{dy}{d\lambda}\right)^2 + \left(\frac{dz}{d\lambda}\right)^2$$

$$= \tfrac{1}{4}\left[\frac{x^2}{(a^2+\lambda)^2} + \frac{y^2}{(b^2+\lambda)^2} + \frac{z^2}{(c^2+\lambda)^2}\right];$$

and $$\frac{d}{d\lambda}\left(\frac{x^2}{a^2+\lambda}\right) = \frac{2x}{a^2+\lambda}\cdot\frac{dx}{d\lambda} - \frac{x^2}{(a^2+\lambda)^2} = 0\,;$$

similarly for $$\frac{d}{d\lambda}\left(\frac{y^2}{b^2+\lambda}\right) \text{ and } \frac{d}{dy}\left(\frac{z^2}{c^2+\lambda}\right).$$

$$\text{Hence } \frac{d^2v}{d\lambda^2} = \frac{4\pi\rho\, abc}{\{(a^2+u)(b^2+u)(c^2+u)\}^{\frac{1}{2}}}\cdot\left(\frac{ds}{d\lambda}\right)^2.$$

Question 885.

(M. K. KEWALRAMANI) :—Two particles are describing an equiangular spiral such that at any instant the angular velocity of the one with respect to the other is zero. If the speeds are proportional to the radii vectores, show that the two points subtend at the pole an angle whose cosine is $\frac{s^2 + s_1^2}{2ss_1}$, where s and s_1 are the arcs measured from the pole to the two points respectively.

Solution by Sadanand.

Let P and Q be (r_1, θ_1) and (r_2, θ_2) and let PQ make an angle ψ with the initial line, also let the tangents at P and Q meeting in O′ make angles ϕ_1 and ϕ_2 with PQ, (α being the angle of the spiral).

Then according to the condition ψ is constant

$$\left.\begin{aligned} v_1 \sin\phi_1 &= v_2 \sin\phi_2 \\ O^1P \sin\phi_1 &= O'Q \sin\phi_2 \end{aligned}\right\}$$

also

$$\therefore \quad \frac{O'P}{v_1} = \frac{O'Q}{v_2}, \quad i.e. \quad \frac{O^1P}{r_1} = \frac{O'Q}{r_2} = k, \text{ suppose ;}$$

also $\quad \phi_1 = \psi - \theta_1 - \alpha$ and $\phi_2 = \alpha - \psi + \theta_2$.

$\therefore \quad \phi_1 + \phi_2 = \theta_2 - \theta_1$; hence OPO′Q is cyclic.

$$\therefore \quad PQ^2 = k^2(r_1^2 + r_2^2 + 2r_1r_2 \cos(\theta_2 - \theta_1) \quad \ldots \quad \ldots (1)$$

$$= r_1^2 + r_2^2 - 2r_1r_2 \cos(\theta_2 - \theta_1) \quad \ldots \quad \ldots (2)$$

$$\therefore \quad \cos(\theta_2 - \theta_1) = \frac{k^2 - 1}{k^2 + 1} \times \frac{r_1^2 + r_2^2}{2r_1r_2} \qquad \text{from (1) \& (2)}$$

$$= \frac{k^2 - 1}{k^2 + 1} \times \frac{s_1^2 + s_2^2}{2s_1s_2} \quad \text{since } s = r \sec\alpha.$$

Hence $\cos(\theta_2 - \theta_1)$ is proportional to

$\dfrac{s_1^2 + s_2^2}{2s_1s_2}$ and not *equal* to $\dfrac{s_1^2 + s_2^2}{2s^1s_2}$ as stated.

Question 886.

(K. Appukuttan Erady):—

$$\text{If } \phi(\alpha,\beta) \equiv 1 + \frac{\alpha.\beta}{1.(\alpha+\beta+1)} + \frac{\alpha(\alpha+1).\beta(\beta+1)}{1.2.(\alpha+\beta+1)(\alpha+\beta+3)}$$

$$+ \frac{\alpha(\alpha+1)(\alpha+2).\beta(\beta+1(\beta+2)}{1.2.3.(\alpha+\beta+1)(\alpha+\beta+3)(\alpha+\beta+5)} + \ldots\ldots$$

show that

$$\phi(\alpha + 2n.\ \beta + 2n)$$

$$\equiv \frac{(\alpha+\beta+1)(\alpha+\beta+3)\ldots(\alpha+\beta+4n-1)}{(\alpha+1)(\alpha+3)\ldots(\alpha+2n-1)(\beta+1)(\beta+3)\ldots(\beta+2n-1)}\,\phi(\alpha,\beta).$$

Solution by K. B. Madhava.

We have $\phi(\alpha,\beta) = F\left(\alpha, \beta, \dfrac{\alpha+\beta+1}{2}, \tfrac{1}{2}\right)$

where F is the hypergeometric function.

This, by Gauss's transformation,

$$= 2^{\alpha} F\left(\alpha, \frac{\alpha-\beta+1}{2}, \frac{\alpha+\beta+1}{2}, -1\right)$$

$$= \frac{2^{\alpha}.\Gamma\left(\dfrac{\alpha+\beta+1}{2}\right)}{\Gamma(\alpha).\Gamma\left(\dfrac{\beta-\alpha+1}{2}\right)} \times$$

$$\cdot \int_0^1 t^{\alpha-1} (1-t)^{\frac{\beta-\alpha-1}{2}} (1+t)^{\frac{\beta-\alpha-1}{2}} dt$$

$$= \frac{2^{\alpha-1}\Gamma\left(\frac{\alpha+\beta+1}{2}\right)}{\Gamma(\alpha).\Gamma\left(\frac{\beta-\alpha+1}{2}\right)}\int_0^1 t^{\frac{\alpha}{2}-1}(1-t)^{\frac{\beta-\alpha-1}{2}}\,dt$$

$$= \frac{2^{\alpha-1}\Gamma\left(\frac{\alpha+\beta+1}{2}\right)\Gamma\left(\frac{\alpha}{2}\right)}{\Gamma(\alpha)\Gamma\left(\frac{\beta+1}{2}\right)}.$$

Hence $\dfrac{\phi(\alpha+2n,\ \beta+2n)}{\phi(\alpha,\ \beta)}$

$$= \frac{2^{2n}\Gamma\left(\frac{\alpha+\beta+4n+1}{2}\right)\Gamma\left(\frac{\alpha}{2}+n\right)\Gamma(\alpha)\Gamma\left(\frac{\beta+1}{2}\right)}{\Gamma(\alpha+2n)\Gamma\left(\frac{\beta+2n+1}{2}\right)\Gamma\left(\frac{\alpha+\beta+1}{2}\right)\Gamma\left(\frac{\alpha}{2}\right)}$$

$$= \frac{(\alpha+\beta+1)(\alpha+\beta+3)\ \dots\ (\alpha+\beta+4n-1)}{(\alpha+1)(\alpha+3)\ \dots\ (\alpha+2n-1).(\beta+1)(\beta+3)\ \dots\ (\beta+2n-1)}.$$

Question 891.

(A. C. L. Wilkinson :—Show that the radii of the 16 spheres which can be drawn to touch 4 given spheres of radii are r_1, r_2, r_3, r_4 are given by the question.

$$\begin{vmatrix} 0 & r_{12}^2 & r_{13}^2 & r_{14}^2 & (R\pm r_1)^2 & 1 \\ r_{21}^2 & 0 & r_{23}^2 & r_{24}^2 & (R\pm r^2)^2 & 1 \\ r_{31}^2 & r_{32}^2 & 0 & r_{34}^2 & (R\pm r_3)^2 & 1 \\ r_{41}^2 & r_{42}^2 & r_{43}^2 & 0 & (R\pm r_4)^2 & 1 \\ (R\pm r_1)^2 & (R\pm r_1)^2 & (R\pm r_3)^2 & (R\pm r_4)^2 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 \end{vmatrix} = 0.$$

where r_{pq} is the distance between the centres of two spheres whose radii are r_p and r_q.

Remarks and solution by the late K. Appukuttan Erady.

Multiplying together the two matrices

$$\begin{vmatrix} x_1^2+y_1^2+z_1^2 & 2x_1 & 2y_1 & 2z_1 & 1 \\ x_2^2+y_2^2+z_2^2 & 2x_2 & 2y_2 & 2z_2 & 1 \\ x_3^2+y_3^2+z_3^2 & 2x_3 & 2y_3 & 2z_3 & 1 \\ x_4^2+y_4^2+z_4^2 & 2x_4 & 2y_4 & 2z_4 & 1 \\ x_5^2+y_5^2+z_5^2 & 2x_5 & 2y_5 & 2z_5 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{vmatrix}\begin{vmatrix} 1 & x_1 & y_1 & z_1 & x_1^2+y_1^2+z_1^2 \\ 1 & x_2 & y_2 & z_2 & x_2^2+y_2^2+z_2^2 \\ 1 & x_3 & y_3 & z_3 & x_3^2+y_3^2+z_3^2 \\ 1 & x_4 & y_4 & z_4 & x_4^2+y_4^2+z_4^2 \\ 1 & x_5 & y_5 & z_5 & x_5^2+y_5^2+z_5^2 \\ 0 & 0 & 0 & 0 & 1 \end{vmatrix}$$

it is easily seen that if A, B, C, D, P be five points in space

$$\begin{vmatrix} 0 & AB^2 & AC^2 & AD^2 & AP^2 & 1 \\ AB^2 & 0 & BC^2 & BD^2 & BP^2 & 1 \\ AC^2 & BC^2 & 0 & CD^2 & CP^2 & 1 \\ AD^2 & BD^2 & CD^2 & 0 & DP^2 & 1 \\ AP^2 & BP^2 & CP^2 & DP^2 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 \end{vmatrix} = 0.$$

Now let A, B, C, D be the centres of the four spheres in question and P the centre of a sphere of radius R touching each of them. Then $PA = R \pm r_1$, $PB = R \pm r_2$, $PC = R \pm r_3$, $PD = R \pm r_4$, also writing r_{12} for AB, r_{13} for AC, etc., we get

$$\begin{vmatrix} 0 & r^2_{12} & r^2_{13} & r^2_{14} & (R\pm r_1)^2 & 1 \\ r^2_{12} & 0 & r^2_{23} & r^2_{24} & (R\pm r_2)^2 & 1 \\ r^2_{13} & r^2_{23} & 0 & r^2_{34} & (R\pm r_3)^2 & 1 \\ r^2_{14} & r^2_{24} & r^2_{34} & 0 & (R\pm r_4)^2 & 1 \\ (R\pm r_1)^2 & (R\pm r_2)^2 & (R\pm r_3)^2 & (R\pm r_4)^2 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 \end{vmatrix} = 0.$$

The above determinantal equation represents 2^4 or 16 equations in R as there are four ambiguous signs.

In (A) subtracting R^2 times the last column from the fifth column and then R^2 times the last row from the fifth row, we get the equivalent determinantal equation

$$\begin{vmatrix} 0 & r^2_{12} & r^2_{13} & r^2_{14} & \pm 2Rr_1 + r_1^2 & 1 \\ r^2_{12} & 0 & r^2_{23} & r^2_{24} & \pm 2Rr_2 + r_2^2 & 1 \\ r^2_{13} & r^2_{23} & 0 & r^2_{14} & \pm 2Rr_3 + r_3^2 & 1 \\ r^2_{14} & r^2_{24} & r^2_{34} & 0 & \pm 2Rr_4 + r_4^2 & 1 \\ \pm 2Rr_1 + r_1^2 & \pm 2Rr_2 + r_3^2 & \pm 2Rr_3 + r_3^2 & \pm 2Rr_4 + r_4^2 & -2R^2 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 \end{vmatrix} = 0.$$

This shows that each of the 16 equations in R is of the second degree. It will also be seen that eight of the 16 equations may be obtained from the remaining eight by changing the signs of the roots, *e.g.*, equation obtained when the ambiguous signs are replaced by +, —, —, +, has its roots equal in magnitude but opposite in sign to those of the equation obtained when the ambiguous signs are replaced by —, +, +, —.

Hence we draw the following conclusions :—

(i) The radii of the 16 spheres which touch four given spheres can be obtained from 8 quadratic equations ;

(ii) Each quadratic equation (when its roots are real) gives a positive root and a negative root. The positive root and the negative root with its sign changed give the radii of two of the tangential spheres ;

(iii) The two tangential spheres whose radii are given by a quadratic equation are such that the number of external contacts of the one with the original spheres is equal to the number of internal contacts of the other and *vice-versa* ;

(iv) Two such tangential spheres are also inverses of each other with respect to the common orthogonal sphere of the four original spheres.

Question 899.

(HEMRAJ) :—There is a pair of numbers, each consisting of three different digits other than zero, such that if the sum of the digits of each is subtracted from it, each resulting number consists of three different digits other than zero and can be obtained from the other by reversing the digits. Find the pair when it is similar to the resulting pair.

Solution by P. R. Venkatakrishna Iyer and R. D. Karve.

Let $100x + 10y + z$ and $100z + 10y + x$ be the pair of numbers. Since $10y + z \geqslant x + y + z$ and $10y + x \geqslant x + y + z$, the numbers obtained by subtracting $x + y + z$ from each will have x and z respectively for their hundreds place.

Let y' be the digit in the tens place in the resulting numbers. Then we have $\quad 100x + 10y + z - x - y - z = 100x + 10y' + z.$

$$\therefore \quad \frac{x + y' + z}{9} + y' = y \text{ and } \therefore \frac{x + y + z}{10} = \frac{x + y' + z}{9}.$$

Since $x + y + z$ and $x + y' + z$ are each < 24, $x + y + z$ can only have the value 10 or 20 and $x + y' + z$ the value 9 or 18.

Taking into consideration that the digits must all be different we get the following pairs of numbers :

136, 631 ; 145, 541 ; 163, 361 ; 172, 271 ; 253, 352 ; 325, 523 ; 389, 983 ; 398, 893 ; 479, 974 ; 569, 965 ; 596, 695 ; 587, 785 ; 659, 956 ; 749, 947 ; 758, 857 ; 839, 938.

Question 900.

(S. Malhari Rao) :—Two sides of a triangle are 40 inches and 39 inches, and its area contains an integral number of square inches. Find the length of the third side which contains an integral number of inches.

Solution by K. J. Sanjana, R. D. Karve, R. J. Pocock and N. B. Mitra.

If a, b are the given sides, c is the side required and $\triangle$ the area, we have at once $(2ab)^2 - (a^2 + b^2 - c^2)^2 = (4\triangle)^2$.

Thus $2ab$ is the hypotenuse of a right-angled triangle of which $a^2 + b^2 - c^2$ and $4\triangle$ are sides. We may therefore put $2ab = k(l^2 + m^2)$, and $a^2 + b^2 + c^2 = k(l^2 - m^2)$ or $2klm$.

In the present case, $2ab = 78 \cdot 40 = 78(6^2 + 2^2)$; so that one set of values of k, l, m is 78, 6, 2; and $3121 - c^2 = 78 \cdot 32$ or $78 \cdot 24$,

The second is inapplicable as it does not make c rational; the first gives $c^2 = 3121 - 2496 = 625$, and the third side is 25.

Of course $2ab$ can be put into the form $k(l^2 + m^2)$ in several ways. The number of solutions is limited on account of $(a^2 + b^2) - k(l^2 - m^2)$ or $2klm$ being required to be a square.

In the easy case here given a better method would be to try all the admissible values of c, *viz.*, the odd numbers from 3 up to 77, and write down s, $s - a$, $s - b$, $s - c$; it can then be seen at a glance which values make the product a square.

Question 901.

(S. Mahadevan, B.A.) :—Prove that the potential of a homogeneous spheroid of revolution of very small ellipticity at a point anywhere without it is

$$\frac{4\pi}{3} \cdot \frac{k^3}{r} - \frac{8\pi}{15} \frac{k^5}{r^3} e \left\{ \tfrac{3}{2}\lambda^2 - \tfrac{1}{2} \right\}$$

where k is the radius of the sphere whose volume is the same as that of the spheroid, e the ellipticity, r the radius vector of the attracted point, λ the cosine of the angle the radius vector makes with the axis of revolution, the density of the spheroid being taken as unity.

Hence prove that the potential of the same spheroid at an internal point is $$2\pi\left(k^2 - \frac{r^2}{3}\right) - \frac{8\pi}{15} r^2 e \left\{ \tfrac{3}{2}\lambda^2 - \tfrac{1}{2} \right\}.$$

Solution by R. J. Pocock.

If a, c are the axes, $c = a\,(1 - e)$; $k = a\,(1 - \frac{1}{3}e)$; $a = k\,(1 + \frac{1}{3}e)$.

If μ is the parameter of the confocal through the point (r, λ)

$$\frac{r^2}{a^2+\mu}(1-\lambda^2) + \frac{r^2\lambda^2}{c^2+\mu} = 1 \quad \therefore \frac{r^2}{a^2+\mu} + \frac{2r^2\lambda^2a^2e}{(a^2+\mu)^2} = 1.$$

Hence $\dfrac{1}{a^2+\mu} = \dfrac{1}{r^2}\cdot\left[1 - \dfrac{2\lambda^2a^2e}{r^2}\right]$; and

$$\frac{1}{(a^2+\mu)^{\frac{1}{2}}} = \frac{1}{r}\left[1 - \frac{\lambda^2a^2e}{r^2}\right].$$

If V is the potential, we have

$$\frac{V}{\pi a^2c} = \int_0^\alpha \frac{du}{\{(a^2+\mu+u)^2(c^2+\mu+u)\}^{\frac{1}{2}}}\left[1 - \frac{r^2(-\lambda^2)}{a^2+\mu+u} - \frac{r^2\lambda^2}{c^2+\mu+u}\right].$$

$$= \int_\mu^\alpha \frac{du}{(a^2+u)\,(c^2+u)^{\frac{1}{2}}}\left[1 - \frac{r^2(1-\lambda^2)}{a^2+u} - \frac{r^2\lambda^2}{c^2+u}\right].$$

$$= \int_\mu^\alpha du\left\{(a^2+u)^{-\frac{3}{2}}\left(1 + \frac{a^2e}{a^2+u}\right)\left[1 - \frac{r^2(1-\lambda^2)}{a^2+u} - \frac{r^2\lambda^2}{a^2+u}\left(1 + \frac{2a^2e}{a^2+u}\right)\right]\right\}.$$

$$= \frac{2}{(a+\mu)^{\frac{1}{2}}} - \frac{2r^2}{3(a^2+\mu)^{\frac{3}{2}}} + \frac{2}{3}\cdot\frac{a^2e}{(a^2+\mu)^{\frac{3}{2}}} - \frac{2}{5}\cdot\frac{a^2e\,.\,r^2}{(a^2+\mu)^{\frac{5}{2}}} - \frac{4}{5}\cdot\frac{r^2\lambda^2a^2e}{(a^2+\mu)^{\frac{5}{2}}}$$

$$= \frac{2}{r} - \frac{2\lambda^2a^2e}{r^3} - \frac{2}{3r} + \frac{2\lambda^2a^2e}{r^3} + \frac{2}{3}\,\frac{a^2e}{r^3} - \frac{2}{5}\,\frac{a^2e}{r^3} - \frac{4}{5}\,\frac{\lambda^2a^2e}{r^3}$$

$$\therefore\ V = \frac{4\pi}{3}\cdot\frac{k^3}{r} - \frac{8}{15}\,\frac{\pi k^5e}{r^3}\,\{\tfrac{3}{2}\lambda^2 - \tfrac{1}{2}\}.$$

The potential at an internal point is found by putting $\mu = 0$ and we have

$$\frac{V'}{\pi a^2c} = \frac{2}{a} - \frac{2r^2}{3a^3} + \frac{2}{3}\,\frac{e}{a} - \frac{2}{5}\,\frac{er^2}{a^3} - \frac{4}{5}\,\frac{r^2\lambda^2e}{a^3}.$$

which reduces to $V' = 2\pi\left(k^2 - \dfrac{r^2}{3}\right) - \dfrac{8\pi}{15}\,r^2.e\,\{\tfrac{3}{2}\lambda^2 - \tfrac{1}{2}\}$.

Question 906.

(K. APPUKUTTAN ERADY, M.A.):—Shew that

$(a_0, a_1, a_2, \ldots\ldots\ldots a_{2n})(x, y)^{2n}$ is reducible to the form

$p_1(x + b_1 y)^{2n} + p_2(x + b_2 y)^{2n} + \ldots\ldots + p_n(x + b_n y)^{2n}$

if
$$\begin{vmatrix} a_0, & a_1, & a_2 \ldots\ldots & a_n \\ a_1, & a_2, & a_3 \ldots\ldots & a_{n+1} \\ \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ a_n, & a_{n+1}, & a_{n+2}, \ldots\ldots & a_{2n} \end{vmatrix} = 0$$

Solution by F. H. V. Gulasekharam, K. R. Rama Iyer and G. V. Seshiah.

On equating the co-efficients of like powers of x of the two expressions, we have the following $2n+1$ equations:—

$$p_1 + p_2 + p_3 + \ldots\ldots\ldots + p_n = a_0$$
$$p_1 b_1 + p_2 b_2 + p_3 b_3 + \ldots\ldots + p_n b_n = a_1$$
$$p_1 b_1^2 + p_2 b_2^2 + p_3 b_3^2 + \ldots\ldots + p_n b_n^2 = a_2$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$p_1 b_1^{2n} + p_2 b_2^{2n} + p_3 b_3^{2n} + \ldots\ldots + p_n b_n^{2n} = a_{2n}.$$

Let $b_1, b_2, \ldots\ldots b_n$ be the n roots of the equation,

$$\lambda_0 + \lambda_1 u + \lambda_2 u^2 + \ldots\ldots + \lambda_n u^n = 0.$$

Then we easily obtain the following $n+1$ relations:—

$$a_0\lambda_0 + a_1\lambda_1 + a_2\lambda_2 + \ldots\ldots + a_n\lambda_n = 0$$
$$a_1\lambda_0 + a_2\lambda_1 + a_3\lambda_2 + \ldots\ldots + a_{n+1}\lambda_n = 0$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$a_n\lambda_0 + a_{n+1}\lambda_1 + a_{n+2}\lambda_2 + \ldots\ldots + a_{2n}\lambda_n = 0.$$

Hence, eliminating $\lambda_0, \lambda_1, \lambda_2, \ldots\ldots \lambda_n$, the required relation is obtained.

Question 907.

(H. R. KAPADIA):—If $\phi_n(x) = \sum_{r=0}^{\infty} \frac{(-)^r x^r}{(n+r)!\, r!}$

and u denote the continued fraction $\frac{1}{n+1-}\;\frac{x}{n+2-}\;\frac{x}{n+3-}\ldots$

prove that $\phi_n(x) = \frac{1}{n!} e^{-\int u\, dx}.$

Solution by K. B. Madhava.

It is easy to verify that $\phi_n(x)$ satisfies the differential equation

$$x\frac{d^2y}{dx^2}+(n+1)\frac{dy}{dx}+y=0.$$

Now there is a general result for expressing a function in continued fraction if it satisfies a differential equation (See, Oskan Perron: *Kettenbruchen*, pp. 469 to 472, or, my note on p. 19 of J. I. M. S., vol. viii).

Since $y = Q_0 y' + P_1 y''$, the n^{th} differential of this is

$$y^{(n)} = Q_n\, y^{(n+1)} + P_{n+1}\, y^{(n+2)}$$

where $Q_n = \dfrac{Q_{n-1} + P_n'}{1-Q'_{n-1}}$ and $P_{n+1} = \dfrac{P_n}{1-Q_n'{}_{-1}}$.

We accordingly see that $\dfrac{y}{y'} = Q_0 + \dfrac{P_1}{Q_1} + \dfrac{P_2}{Q_2} + \dfrac{P_3}{Q_3}$ ....

If we adopt that method here, we see that $Q_r = (n+r+1)$ and $P_n = -x$, and therefore inverting

$$\frac{\phi'_n(x)}{\phi^n(x)} = -u, \text{ where } u \text{ is the C. F. given.}$$

Integrating this we have at once the desired result; the constant can of course be determined by considering a particular case.

It is perhaps instructive to identify the problem with a result in Bessel Functions.

Thus, $$\phi_n(x) = x^{-\frac{n}{2}} J_n(2\sqrt{x}).$$

But $$J_{n+1}(z) + J'_n(z) = \frac{n}{z} J_n(z). \quad \ldots \quad \text{(Whittaker, p. 354)}$$

$$\therefore\ \frac{J_{n+1}(z)}{J_n(z)} = \left[-\frac{J'_n(z)}{J_n(z)} + \frac{n}{z}\right] = -\frac{d}{dz}\log\left[z^{-n} J_n(z)\right]$$

$$= -\frac{z}{2}\frac{d}{dx}\log\left[2^{-n} x^{-\frac{n}{2}} J_n(2\sqrt{x})\right]$$

$$= -\frac{z}{2}\frac{d}{dx}\log\phi_n(x) \text{ on putting } z = 2\sqrt{x}.$$

But $$\frac{J_{n+1}(z)}{J_n(z)} = \frac{\frac{1}{2}z}{n+1-}\ \frac{(\frac{1}{2}z)^2}{n+2-}\ \frac{(\frac{1}{2}z)^2}{n+3-}\ \ldots\ldots$$ (Whittaker, p. 373).

$$\therefore\ -\frac{d}{dx}\log\phi_n(x) = \frac{1}{n+1-}\ \frac{x}{n+2-}\ \frac{x}{n+3-}\ \ldots\ldots$$

Hence $$\frac{\phi'_n(x)}{\phi_n(x)} = -u, \text{ as before.}$$

Question 935.

(K. J. SANJANA, M A.):—Let $a, b, c,$ be numbers representing the sides of a real triangle and

$$S = a^2(b + c) + b^2(c + a) + c^2(a + b)$$
$$\Sigma = a^3 + b^3 + c^3,\ P = abc$$

prove that $6S >$ or $< l\Sigma + (36 - 3l)$ P according as $l = 0, 1, 2, 3$ or $= 6, 7, 8, 9, 10, 11, 12$.

Also examine the inequality when $l=4$ and $l=5$. [Suggested by Q. 917].

Solution by N. Sankara Aiyar.

Let $a < b < c$, and $a = b - p = c - q$.

Then $6S > =$ or $< l\Sigma + (36 - 3l)P$ according as

$$6\{6a^3 + 6a^2(p + q) + 2a(p + q)^2 + pq(p + q)\} > = \text{ or } < l(p^2 - pq + q^2)(3a + p + q) + 36(a^3 + a^2\overline{p + q} + apq),$$

i.e., $12a(p^2 - pq + q^2) + 6pq(p + q) > =$ or $< l(p^2 - pq + q^2)(3a + p + q)$,

i.e., $6pq(p+q)+(p^2-pq+pq^2)[12a-l(3a+p+q)] > =$ or < 0.

When $l = 3$, we get

$$6pq(p + q) + (p^2 - pq + q^2)(3a - 3p - 3q) > 0.$$

The worst case arises when $p = q$. The inequality then becomes $12p^3 + p^2(3a - 6p) > 0$, which is clearly true.

This proves the theorem for $l < 3$.

When $l = 4$, we get

$$6pq(p + q) + (p^2 - pq + q^2)(-4p - 4q) > = \text{ or } < 0,$$

according as $(p + q)(-4p^2 + 10pq - 4q^2) > =$ or < 0,

i.e., $-(2p - q)(2q - p) > =$ or < 0,

i.e., $2p - q < =$ or > 0,

i.e., $2(b - a) - (c - a) < =$ or > 0,

i.e., $2b - c - a < =$ or > 0.

When $l = 5$, we get

$$6pq(p + q) + (p^2 - pq + q^2)(-30 - 5p - 5q) > = \text{ or } < 0,$$

according as $-(p+q)(5p^2-11pq+5q^2)-3a(p^2-pq+q^2) > =$ or < 0.

The first term is negative except when $p > \dfrac{11 - \sqrt{21}}{10}q$. The value of a for which the whole left-hand side is negative can then be found for the particular value of p. The limiting case is when $p = q$.

Then $+ 2p^3 - 3ap^2 > =$ or < 0, according as $2p > =$ or $< 3a$,

i.e., $b + c > =$ or $< 5a$.

When $l = 6$, we get

$$6pq(p + q) + (p^2 - pq + q^2)(-6a - 6p - 6q) > = \text{ or } < 0$$

according as $-(p + q)(p^2 - 2pq + q^2) - a(p^2 - pq + q^2) > =$ or < 0,

i.e., $-(p + q)(p - q^2) - a(p^2 - pq + q^2) > =$ or < 0.

The L. H. S. being negative, we get that for values of

$$l > 6,\ 6S < l\Sigma + (36 - 3l)P.$$

QUESTIONS FOR SOLUTION.

1101. (M. Bhimasena Rao):—Show that the centre of a three-cusped hypo-cycloid escribed to a given triangle is equidistant from the orthocentre and the circum-centre of the triangle.

1102. (M. Bhimasena Rao) :—P is the inverse of the incentre of a triangle ABC with respect to the circum-circle of ABC. Show that the isögonal conjugate of P with respect to ABC lies on the common diameter of the in-circle and the nine-point circle of ABC.

1103. (M. Bhimasena Rao):—A parabola is drawn touching two sides of a triangle and the line joining their middle points, and passing through the Nagel point of the triangle. Show that it touches the in-circle of the triangle, and find the condition that it may touch one of the ex-circles also.

1104. (M. Bhimasena Rao):—A circle cuts the sides of the triangle of reference at angles α, β, γ. Show that it cuts the nine-point circle of the triangle at the angle θ given by the equation

$$\cos\theta.\ (a\cos\alpha + b\cos\beta + c\cos\gamma) =$$
$$a\cos A\sin^2\alpha + b\cos B\sin^2\beta + c\cos C\sin^2\gamma + a\cos\beta\cos\gamma + \ldots$$

If $\theta = \alpha + \beta + \gamma$ show that either

(i) $\sin A\sin\alpha + \sin B.\ \sin\beta + \sin C\sin\gamma = 0$,

or (ii) $\cos A\sin\alpha + \cos B\sin\beta + \cos C\sin\gamma + \sin(\alpha+\beta+\gamma) = 0$.

Interpret these results geometrically.

1105. (N. P. Pandya) :—S is a fixed point in the plane of a given circle. A variable parabola is described with S as focus to cut the circle in P and Q so that PSQ is always a straight line. Find the locus of the vertex of the parabola.

1106. (N. P. Pandya) :—Find four integers in A. P. each of which is equal to the sum of the divisors of some one of the other three.

1107. (Charles Saldhana) :—If x be any odd integer, not divisible by 3, prove that the integral part of

$$4x - (2 + \sqrt{2})x \text{ is a multiple of } 112.$$

1108. (Communicated) :—Prove that

$$\int_0^{\frac{\pi}{2}} \text{arc tan } (a \sqrt{\tan x})\, dx = \pi \text{ arc tan } (a \sqrt{2} + 1) - \frac{\pi^2}{4}.$$

Hence or otherwise, prove that

$$\int_0^{\frac{\pi}{2}} \text{arc tan } \left[\frac{\sqrt{2 \tan x}}{1 + \tan x}\right] dx = \frac{\pi^2}{12}$$

and $$\int_0^{\frac{\pi}{2}} \text{arc tan } \left\{\frac{2 \sqrt{\tan x}}{1 + \tan x}\right\} dx = \pi \text{ arc sin } \tfrac{1}{3}.$$

1109. (Saldhana and Madhava) :—Prove that

$$e^{\lambda_1} \; [f(x)] = \left(1 + \frac{1}{x}\right) f(x+1),$$

where $\lambda_1 = \dfrac{d}{dx} + \dfrac{1}{x}$;

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and extend the result as follows :—

$$e^{\lambda_2} \; [\tfrac{1}{2} f(x)] = \left(\frac{1}{2} + \frac{1}{x} + \frac{1}{2x^2}\right) f(x+1),$$

$$e^{\lambda_3} \; [\tfrac{1}{3} f(x)] = \left(\frac{1}{3} + \frac{1}{x} + \frac{1}{x^2} + \frac{1}{3x^3}\right) f(x+1),$$

where $\lambda_2 = \dfrac{d}{dx} + \dfrac{2}{x}$ and $\lambda_3 = \dfrac{d}{dx} + \dfrac{3}{x}$.

1110. (A. Narasinga Rao) :—If for a curve there exists a functional relation connecting the area of every escribed triangle, and that of the triangle formed by the points of contact, such a curve must be a straight line or a parabola.

1111. (A. Narasinga Rao) :—Determine generally the form of a vessel whose contents are just spilling over in the position of equilibrium, whatever the amount of liquid it contains (1) when it rests on a horizontal plane, (2) when it is suspended about a horizontal axis.

1112. (C. Krishnamachari, M.A.) :—If V is the normal velocity drawn outwards to a closed surface S in a liquid, show that

$$\iint \rho V \, dS + \iiint \frac{d\rho}{dt} \, dx \, dy \, dz = 0,$$

where the volume integral extends throughout the volume enclosed by S and the surface integral over the surface S.

Hence establish the equation of continuity.

THE INDIAN MATHEMATICAL SOCIETY.

Statement of Accounts for the year 1919.

Receipts :—	Rs.	A.	P.	Expenditure :—	Rs.	A.	P.
Balance from 1918	4210	3	1	Ordinary working expenses ...	154	0	0
Subscriptions from Members ...	2092	6	0	Second Mathematical Conference ...	160	0	9
Life Members Subscription ...	300	0	0	Books and Journals	33	14	0
Subscriptions for Journal... ...	192	13	0	Printing Journal	821	1	6
Interest on investments	143	9	3	Library expenses	240	0	0
Donation from Mr. Balak Ram ...	1000	0	0	Closing Balance...	6529	15	1
	7938	15	4		7938	15	4

The Closing Balance is distributed as follows :

Central Urban Bank ...	2000	0	0
Indian Bank ...	1330	14	2
Savings Bank ...	1000	0	0
Asst. Secretary ...	1315	13	6
Treasurer ...	782	15	8
Secretary ...	66	3	6
Joint Editor ...	22	0	3
Librarian ...	12	0	0
TOTAL ...	6529	15	1

Madras,
10th June 1920.

S. NARAYANA AIYAR,
Hon. Treasurer.