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PROGRESS REPORT.

Mr. Balakram, M.A., I.C.S.—one of the enthusiastic well-wishers of the Society—has been kind enough to present a cheque for Rs. 1,000 to our Society, with a desire that as far as practicable, the money should be utilized in completing our sets of the more important periodicals. The Committee have accepted this kind offer of Mr. Balakram with many thanks, and steps will be taken to utilize the amount in the best possible manner.

(2) The Committee feel great pleasure to announce that *Mr. T. V. Venkatarama Aiyar*, Corner House, Mylapore, Madras—has been elected a Life Member of the Society.

(3) The following gentlemen have been elected members of our Society—

1. *Mr. Indar Singh Puri, M.A.*—Deputy Examiner, A.G.'s Office, Allahabad;
2. *Lala Mehr Chand Suri, M.A.*—Professor of Mathematics, Forman Christian College, Lahore;
3. *Pandit Parama Nand, M.A.*—Professor of Mathematics, Prince of Wales College, Jammu (Kashmere State);
4. *Lala Ram Dass Narda, M.A.*—Professor of Mathematics Sanatan Dharam College, Lahore;
5. *Mr. C. Ranganathan, B.A., L.T.*—Mathematics Assistant, London Mission High School, Gooty (at concessional rate);

6. *Mr. B. S. Madhava Rao, B.Sc.*—M.Sc. Student, Calcutta Univ., 6 Badur Bagan Lane, Calcutta (at concessional rate);

7. *Mr. G. V. Krishnasawmi*—Asst. Prof. of Mathematics, St. Joseph's College, Trichinopoly;

8. *Mr. T. K. Devlalkar, M.A., B.Sc.*—Lecturer in Science, Karnatak College, Dharwar;

9. *Mr. Satyendra Nath Sen B.A., M.Sc.*—Professor of Physics, Canning College, Lucknow.

(4) The following books have been received—

1. Bombay University Calendar, Part I, 1920-21;
2. Madras University Calendar, Vol. I & II for 1920;
3. Theory of Determinants, Vol. III (period 1861 to 1880) by Sir Thomas Muir, Macmillan & Co., London 1920, 35/-;
4. Statics & Dynamics (first part)—by R. C. Fawdry, G. Bell & Sons, London, 1919, 5s/-

D. D. KAPADIA,

Hon. Joint Secretary.

Multiplication of Infinite Integrals

BY K. B. MADHAVA, M. A.,

(*Continued from page* 180, *J.I.M.S.*, *Vol. XI*).

V. The Analogues of Cauchy's and Merten's Theorems

By the analogue of Cauchy's theorem we mean the result that, if

$$\int_0^\infty u_1(x)\,dx$$

is convergent, and if

$$\int_0^\infty v_1(x)\,dx$$

be also convergent, then will

$$\int_0^\infty w_1(x)\,dx$$

be also convergent. ... ... (5·1)

In the case of "Merten's theorem", if

$$\int_0^\infty u_1(x)\,dx \quad \text{and} \quad \int_0^\infty v(x)dx$$

are convergent, we have to prove that

$$\int_0^\infty w(x)\,dx$$

is convergent ; or, if it happens,

$$\int_0^\infty w_1(x)\,dx$$

is convergent. ... ... (5·2

Of these Merten's theorem * is the more general, for if

$$\int_0^\infty w_1(x)\,dx$$

is convergent, it is known that

$$\int_0^\infty w(x)\,dx$$

is convergent.

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* Cf Bromwich : *Infinite Series*, p. 429.

We will therefore prove only Merten's theorem.

Since,

$$\int_0^\infty v\,(x)\,dx$$

is convergent, it is possible to choose a value x_1 of x, such that for all values of $x > x_1$, we shall have

$$\Theta\,(x)=\int_x^\infty v\,(x)\,dx \qquad \dots \qquad \dots \qquad \dots \qquad (5{\cdot}3)$$

an ε function; that is to say $\Theta\,(x)\to 0$ with $1/x$.

We have obviously $\quad V=V(x)+\Theta(x). \dots \qquad \dots \qquad \dots \qquad (5{\cdot}31)$

Now,

$$W\,(x)=\int_0^x u(y)\;V(x-y)\;dy \qquad \text{from } (5{\cdot}31)$$

$$=\int_0^x u(y)[V-\Theta(x-y)]dy \quad \text{from } (5{\cdot}31)$$

$$=VU(x)-\int_0^x u\,(y)\Theta(x-y)dy$$

$$=\;VU(x)-H(x),\ \text{say.} \qquad \dots \qquad \dots \qquad \dots \qquad (5{\cdot}4)$$

Consequently,

$$|\,H(x)\,|\leq\int_0^x |\,u(y)\,|\times|\,\Theta(x-y)\,|\,dy.$$

Now since $\Theta\,(x)\to 0$ with $1/x$, the numbers, $\Theta_1\,(x)$ [which means the modulus of $\Theta(x)$] can have at best a finite upper limit, H, say; and in addition, be such that for all values of x greater than some number, say x_2, $\Theta_1(x)<\varepsilon$.

$$\text{Hence}\quad |\,H\,(x)\,|<\Big[\int_0^{x-x_2} dy\;u_1(y)\varepsilon+\int_{x-x_2}^{x} u_1\,(y)\;H\;dy$$

$$\leq \varepsilon.\,U+H[U(x)-U(x-x_2)]$$

$$\therefore \lim_{x\to\infty}|\,H(x)\,|\leq\varepsilon.\overline{U} \qquad \dots \qquad \dots \qquad \dots \qquad \dots \qquad (5{\cdot}5)$$

which tends to zero because $U(x)$ is convergent and when x is indefinitely great $U(x)=U(x-x_2)$.

Consequently, proceeding in (5·4) to the limit as $x \to \infty$,

$$\lim_{x\to\infty} \int w(x)\, dx = \lim_{x\to\infty} W(x)$$

$$= \lim_{x\to\infty} V.U(x) - H(x) = U.V. \qquad \ldots \qquad \ldots \qquad \ldots \quad (5·6)$$

That is to say,

$$\int_0^\infty w(x)\, dx$$

converges, under these circumstances, to the value U·V.

Hence the analogue of Merten's theorem is proved. It may happen that in some cases W is also absolutely convergent.

By proceeding in a similar manner, but replacing the signs of equality in (5·4) etc., by inequalities on either side, we have "the analogue of Hardy-Merten's theorem" for the infinite integrals (Th. 4° of § II), viz.

If $\int_0^\infty u_1(x)\, dx \to U$, and $\int_0^\infty v(x) dx$ oscillates finitely between the limits V_1 and V_2 and $\lim_{x\to\infty} v(x) \to 0$,

then

$$\int_0^\infty dx \int_0^x u(y)\, v(x-y)\, dy$$

also oscillates finitely between the limits U V_1 and U V_2; and may take any value between these limits.

VI. The Analogue of Hardy's first Theorem.

In this theorem, as well as in the theorems following, we do not assume the absolute convergence of the integrals U and V; they should of course be convergent. We shall then write down the integral

$$\int_0^\infty w(x)\, dx,$$

and see what additional conditions could be set upon $u(x)$ and $v(x)$, in order that the integral just mentioned may be convergent.

Consider,

$$W(x) = \int_0^x u(y)\, V(x-y)\, dy \qquad \dots \qquad \dots \qquad \dots \qquad (3{\cdot}5)$$

$$= \int_0^{\xi} u(y)\, V(x-y)\, dy + \int_{\xi}^{x} u(y)\, V(x-y)\, dy, \text{ where } \xi \text{ is}$$

some internal point

$$= I_1 + I_2, \text{ say.} \qquad \dots \qquad \dots \qquad \dots \qquad \dots \qquad (6{\cdot}1)$$

Now $|I_2| = \int_{\xi}^{x} |u(y)| \,.\, |V(x-y)|\, dy$

Since $V(x-y)$ has a definite finite limit, as $y \to \infty$, it has a finite upper limit, say H_2, in the interval $\xi \leq y \leq x$. Also, let $\mathfrak{U}$ be the greatest of the values of $u_1(y)$ in the same interval. Then

$$|I_2| = \int_{\xi}^{x} \mathfrak{U}\, H_2\, dy.$$

$$= H_2\, \mathfrak{U}\, (x-\xi). \qquad \dots \qquad \dots \qquad \dots \qquad (6{\cdot}2)$$

Let us now choose $\xi = \lambda_1 x$, where λ_1 is a proper fraction, so that x, ξ, $x-\xi$ all tend to infinity together.

Consequently,

$$\lim_{x\to\infty} |I_2| \leq H_2 (1-\lambda_1) \lim_{x\to\infty} x\,\mathfrak{U}$$

$$\leq K_2 \lim_{x\to\infty} x\,\mathfrak{U}, \text{ say.} \qquad \dots \qquad \dots \qquad (6{\cdot}21)$$

Now consider I_1, where

$$I_1 = \int_0^{\xi} u(y)\, V(x-y)\, dy.$$

We can easily see that we have

$$I_1 = \int_0^{\xi} u(y)\, dy \times \int_0^{x-\xi} v(y)\, dy + \int_0^{\xi} U(y)\, v(x-y)\, dy$$

$$= I + I_3, \text{ say.} \qquad \dots \qquad \dots \qquad \dots \qquad \dots \qquad (6{\cdot}3)$$

[For, differentiating with respect to ξ, we have

$$u(\xi)\, V(x-\xi) = u(\xi) \int_0^{x-\xi} v(y)\, dy - v(x-\xi) \int_0^{\xi} u(y)\, dy$$
$$+ v(x-\xi)\, U(\xi),$$

which is true.]

$$\text{Now} \quad | I_3 | \leq \int_0^{\xi} | U(y) | \; | v(x-y) | \, dy.$$

Since U (y) has a definite limit U as $y \to \infty$, it has a finite upper limit, say H_1, in the interval $0 \leq y \leq \xi$. Also, let $\mho$ be the greatest of the values of v_1 $(x-y)$ in the same interval; we have already chosen ξ, so that it tend to ∞ with x.

Hence

$$\lim_{x \to \infty} | I_3 | \leq H_1 \lim_{x \to \infty} \xi \mho$$
$$\leq K_1 \lim_{x \to \infty} x\, V. \tag{6.41}$$

Moreover since, in the limit that $x \to \infty$, ξ and $x-\xi$ tend to the same limit,

$$\lim_{x \to \infty} I \to \int_0^{\infty} u(y)\, dx \times \int_0^{\infty} v(x)\, dx$$
$$\to U\, V. \tag{6.5}$$

Now proceeding with (6·1.) we find,

$$\lim_{x \to \infty} W(x) = \lim_{x \to \infty} I + I_2 + I_3.$$

$$\left.\begin{aligned} &= UV + H_2 \lim_{x \to \infty} \nu(x-\xi). \\ &\quad + H_1 \lim_{x \to \infty} \mho\, \xi \end{aligned}\right\} \tag{6.6}$$

$$\left.\begin{aligned} &= UV + K_2 \lim_{x \to \infty} x\, \nu \\ &\quad + K_1 \lim_{x \to \infty} \mho\, x \end{aligned}\right\} \tag{6.61}$$

$$= U\, V,$$

where K_1 and K_2 are some constants.

The above will be true if $\lim_{x \to \infty} x\nu = 0$ and $\lim_{x \to \infty} \mho\, x = \infty$,

that is to say, if $\lim_{x\to\infty} x\,u(x)=0$
and $\lim_{x\to\infty} xv(x)=0$ ... ... (6·7)

Hence Hardy's First theorem may be stated as follows :—

If $\int_0^\infty u(x)\,dx$ and $\int_0^\infty v(x)\,dx$ are convergent and have the values U and V; and in addition (6·7) are also satisfied, the integral $\int_0^\infty dx\int_0^x u(y)\,v\,(x-y)\,dy$ converges to the value U. V.

Ex.

$$\int_0^\infty dx\int_0^x \frac{(x-y)^{\alpha-1}}{(1+x^2)\;(1+x-y)}\,dy$$

$$=\frac{\pi^2}{2}\operatorname{cosec}\alpha\pi. \qquad 0<\alpha<1.$$

Here $\lim_{x\to\infty} \dfrac{x.\,x^{\alpha-1}}{1+x}\to 0$

and $\lim_{x\to\infty} x\dfrac{1}{1+x^2}\to 0.$

VII. Analogue of Hardy's second Theorem.

Hardy's second theorem is a generalisation of the first and is easily obtained from (6·6).

In § VI, it was made clear that the internal point ξ, which is at our command has to be so chosen that x, ξ, and $x-\xi$ have all to tend to infinity together; and in that article we took the simplest case $\xi=\lambda x$. Now there exists* a whole scheme of functions, say $\psi\,(x)$, *viz.*

$\log x,\ l_2\,(x)\ [i\,e,\ lg_1 lg\,x]$; or $l^{\alpha}\,(x),\ l_2^{\beta}\,(x)\,l_3^{\gamma}\,(x)\ldots_3$;

of which $\log x$ is typical, which are such that they tend to infinity with x; and if we set ξ of the same order of greatness as say,

* Cf. Hardy's Tract *Orders of Infinity.*

$\dfrac{x}{\psi(x)}$ and remember that $\lim\limits_{x\to\infty} \dfrac{x}{\log x} \to\infty$,

we secure the three conditions that we want.

Now resuming (6·6), we have,

$$\lim_{x\to\infty} W(x)-UV < H \lim_{x\to\infty} [\nu.(x-\varepsilon)+\varepsilon.\upsilon] \quad \ldots \quad \ldots \quad (6.6)$$

where H is the greater of the two, H_1 and H_2.

Now ν ψ and υ can obviously be chosen such that the limit of the expression in squared brackets is zero.

$$\begin{array}{l}\text{If now } x\,\psi(x)u(x)\to 0 \\ \text{and } x\ \ v(x)/\psi(x)\to 0\end{array} \quad \text{with } \frac{1}{x} \quad \ldots \quad \ldots \quad (7\,1)$$

We have

$$\lim_{x\to\infty} \left[W(x)-UV\right]$$

$$< H.\,\varepsilon\cdot \lim \left\{\frac{I}{\psi(x)} - \frac{\xi}{x\psi(x)} + \frac{\xi\psi(x)}{x}\right\} \to 0.$$

It is obvious that we can choose an x, say x_4, so large that corresponding to an arbitrarly assigned small positive number ε, we shall have both

$\nu < \dfrac{\varepsilon}{x\psi(x)}$ & $\upsilon < \dfrac{\varepsilon\,\psi(x)}{x}$ for all values of x, greater than x_4

[*Seeing that ν has been obtained § VI, as the greatest of the moduli of $u(y)$ in the interval $(\xi_4\, x)$, it is possible te obtain the inequality

$$\nu < \frac{\varepsilon}{\xi\psi(\xi)}$$

also. In that case the argument needs only slight alteration, thus

$$\left[\lim_{x\to\infty} W(x)-UV\right]$$

$$< H.\,\varepsilon \lim \left\{\frac{x}{\xi\psi(\xi)} - \frac{1}{\psi(\xi)} + \frac{\xi\psi(x)}{x}\right\}$$

$$< H.\,\varepsilon \lim \left\{1 + \frac{\psi(x)}{\psi\left(\dfrac{x}{\psi(x)}\right)}\right\}$$

< 2 H. ε which tends to zero.

Hence "*the analogue of the second* theorem of Hardy,":

If U and V are convergent and in addition (7·1) are also satisfied the integral

$$\int_0^\infty dx \int_0^x v(x)u(x-y)dy,$$

converges to the value UV. ... ... (7·3)

The simplest case of course is to take $\psi(x) = \log x$.

VIII. The Analogue of Hardy's third Theorem.

The theorem is as follows :—

If $\int_0^\infty u(x)\,dx$ and $\int_0^\infty v(x)\,dx$ are convergent respectively to the values U and V, and in addition $|xu(x)| < C_1$ and $|xv(x)| < C_2$ where C_1 and C_2 are constants, then $\int_0^\infty w(x)\,dx$ converges to the value U V.

The proof of this depends upon a lemma analogous to Tauber—Pringsheim's theorem * (usually called the converse of Abel's theorem), *viz.* that a series Σa_n may be inferred to be convergent if $\frac{\Sigma A_n}{n}$ exists and

$$\lim_{n\to\infty} \frac{1}{n}(a_1+2a_2+\ldots na_n)=0.$$

Let us set $F(x) = \int_0^x f(y)\,dy,$

and integrate by parts

$$\frac{1}{x}\int_0^x y f(y)\,dy = F(x) - \frac{1}{x}\int_0^x F(y)\,dy$$

$$= F(x) - \frac{1}{x}\int_0^x dy \int_0^y f(z)dz \qquad (8.11)$$

provided the function behaves all right at the lower limit.

* [Bromwich : *p.* 251., Ex, 28, where other references are given].

This relation enables us to infer that

$$\lim_{x \to \infty} F(x) = \int_0^{\infty} f(y)\, dy,$$

exists and is equal to a number F if

$$\lim_{x \to \infty} \frac{1}{x} \int_0^{x} y f(y) dy \longrightarrow 0 \qquad \ldots \qquad \ldots \quad (5 \cdot 19)$$

and

$$\lim_{x \to \infty} \frac{1}{x} \int_0^{x} dy \int_0^{y} f(z) dz \longrightarrow F. \quad \ldots \qquad \ldots \quad (8 \cdot 13)$$

In fact, the last two are necessary and sufficient conditions for the convergence of the integral $\int_0^{\infty} f(x) dx$ to F.

We shall now proceed with our main theorem: we have to show that the integral $\int_0^{\infty} w(x) dx$ is convergent. Hence we have got to show that

$$\frac{1}{x} \int_0^{x} y\, w(y) dy \to 0 \text{ with } \frac{1}{x}.$$

It is convenient to write

$$\overline{W}(x) = \int_0^{x} y\, w(y) dy \quad \ldots \qquad \ldots \qquad \ldots \qquad \ldots \quad (8 \cdot 2)$$

and analogous notation for $\overline{V}$ and $\overline{U}$.

Also we can easily verify that

$$\overline{W}(x) = \int_0^{x} y\, u(y) V(x-y) dy + \int_0^{x} y v(y) U(x-y) dy$$

$$= I_2 + I_1, \text{ say.}$$

Now $| I_1 | \leq \int_0^{x} | y v(y) | \cdot | U(x-y) | dy.$

and since $U(x)$ is convergent, it follows that we can choose a number x_5 of x such that for all values of x greater than x_5, we can have $U(x-y)=U+\varepsilon$, when y lies in the interval considered.

Also, since V is a convergent integral

$\int_0^x y\, v(y)$ is of the (small o) order of x, by (8·11).

Hence

$$|\, I_1 \,| \leq \int_0^x [y\, v(y)]\, [U+\varepsilon_1]\, dy$$

$$\leq U\, o(x) + \varepsilon_1 \int_0^x |\, y\, v(y)\, |\, dy$$

so that

$$\lim_{x \to 0} \left| \frac{1}{x} I_1 \right| \to 0, \text{ if } \varepsilon_1 \int_0^x |\, yv(y)\, |\, dy \text{ can be made to tend to zero.}$$

This latter will certainly happen if

$|\, yv(y)\, | <$ a constant C_2, when $y \to \infty$. ... ... (8·3)

Similarly we can show that

$$\lim_{x \to \infty} \left| \frac{1}{x} I_3 \right| \to 0.$$

if $|\, yu(y)\, | <$ a constant C_1, when $y \to \infty$. ... ... (8·31)

Hence under these two conditions

$$\lim_{x \to \infty} \frac{1}{x} \overline{W}(x) = 0 \qquad \text{... ... ... (8·11)}$$

Similarly it may be shown that (8·12) is satisfied.

Hence the integral $\int_0^\infty w(x)\, dx$ is convergent and has the value U V.

Finally, we may permit ourselves to make the following observation. Hardy has replaced (in the paper cited already) our (8·13) with the condition

$$\lim_{x\to\infty} \frac{1}{x}\int_0^x dy\int_0^y f(z)\,dz \to 0$$

for the convergence of (8·1). This no doubt is sufficient, but as we have shown in our proof this is not necessary. It seems to us therefore that we can replace his (*small o*) inequality with a (capital O) inequality; and unless his result is a casual slip, it seems to us that the latter case is the one of greater interest. We need only take a simple illustration with $f(x) = e^{-x}$, to bear out the force of this remark; for, with this example,

$$\lim_{x\to\infty} \frac{1}{x}\int_0^x dy\int_0^y e^{-z}\,dz$$

$$= \lim_{x\to\infty} \frac{1}{x}\int_0^x (1-e^{-y})\,dy$$

$$= \lim_{x\to\infty} \frac{1}{x}\left[x+e^{-x}-1\right]$$

which is 1 and not zero.

CERTAIN DEFINITE INTEGRALS AND SERIES

Connected with Bernoulli's Numbers.

[By C. Krishnamachari, M.A., Assistant Professor,

College of Engineering, Bangalore.]

[In solving Q. 913 of Prof. Sanjana, I arrived at certain definite integrals which I set as Q. 951. A study of Whittaker, page 126, suggested transformations of these integrals. Mr. Bhimasena Rao pointed out that the solution of Q. 609 by Mr. Madhava assumes that

$$\sum_{1}^{\infty} \frac{r^{4n+1}}{e^{2\pi r}-1}=\int_{0}^{\infty} \frac{x^{4n+1}}{e^{2\pi x}-1}dx$$

and suggested that I may solve Questions 387 and 609 by the theory of residues. A successful attempt at this problem led to some further results. These form the subject of the present paper.]

Part I.

§ 1. (1) Lemma— $\lim\limits_{y\to 0} y^r (\log y)^s=0$

where r and s are any positive integers. This can be proved by the substitution

$$y=e^{-x}.$$

(2) We easily obtain by integration by parts and using the above lemma, or by means of the transformation suggested in the above lemma, that

$$\int_0^1 (\log y)^{2n}\, y^m\, dy=\frac{(2n)!}{(m+1)^{2n+1}}$$

$$\int_0^1 (\log y)^{2n-1}\, y^m\, dy=-\frac{(2n-1)!}{(m+1)^{2n}}.$$

§ 2. Expanding $\dfrac{1}{1+y^2}$ and integrating term by term, we get

$$\int_0^1 \frac{(\log y)^{2n}}{1+y^2}dy=(2n)!\left\{\frac{1}{1^{2n+1}}-\frac{1}{3^{2n+1}}+\frac{1}{5^{2n+1}}-\dots\right\}$$

$$=\frac{E_n}{2}\left(\frac{\pi}{2}\right)^{2n+1}$$

whence by some obvious transformations, we obtain the following integrals for E_n:

$$E_n = \left(\frac{2}{\pi}\right)^{2n+1} \int_0^{\frac{1}{2}\pi} \left(\log \tan \frac{x}{2}\right)^{2n} dx \qquad [y=\tan \frac{x}{2}$$

$$= \left(\frac{2}{\pi}\right)^{2n+1} \int_0^{\infty} \frac{z^{2n}}{\cosh z}\, dz. \qquad [y=e^{-z}.$$

$$= 4^{n+1} \int_0^{\infty} \frac{z^{2n}}{e^{\pi z}+e^{-\pi z}} dz.$$

Since we have

$$\sec z = \Sigma\, E_n \frac{z^{2n}}{(2n)\,!}$$

we obtain by substituting for E_n the above integral,

$$\sec z = 2\int_0^{\infty} \frac{e^{2zt}+e^{-2zt}}{e^{\pi t}+e^{-\pi t}}\, dt.$$

§ 3. We similarly obtain

$$\int_0^1 \frac{(\log y)^{2n-1}}{1+y} dy = -\frac{(2^{2n-1}-1)}{2n} B_n\ \pi^{2n},$$

whence as before

$$B_n = -\frac{2n}{\pi^{2n}(2^{2n-1}-1)} \int_0^{\frac{1}{4}\pi} \frac{(\log \tan x)^{2n-1}}{\cos x\,(\cos x+\sin x)} dx \qquad [y=\tan x.$$

$$= +\frac{2n}{2^{2n-1}-1} \int_0^{\infty} \frac{z^{2n-1}}{e^{\pi z}+1} dz. \qquad [y=e^{-\pi z}$$

Since

$$\operatorname{cosec} z = \frac{1}{z} + 2 \sum_1^{\infty} \frac{2^{2n-1}-1}{(2n)\,!} B_n\, z^{2n-1}, \text{ (Hobson, page 363)};$$

we obtain

$$\operatorname{cosec} x = \frac{1}{x} + \int_0^{\infty} \frac{e^{xz}-e^{-xz}}{e^{\pi z}+1} dz.$$

§ 4. We also have

$$\int_0^1 \frac{(\log y)^{2n-1}}{1-y^2} dy = -\frac{\pi^{2n}(2^{2n}-1)}{4n} B_n.$$

$$\therefore \quad B_n = -\frac{4n}{(2^{2n}-1)\,\pi^{2n}} \int_0^{\frac{\pi}{4}} \frac{(\log \tan x)^{2n-1}}{\cos 2x} dx \ [y = \tan x.$$

$$= \frac{2n}{\pi^{2n}(2^{2n}-1)} \int_0^{\infty} \frac{z^{2n-1}}{\sinh z} dz. \quad [\text{Whittaker page 126, } y = e^{-z}.$$

$$= \frac{2n}{2^{2n}-1} \int_0^{\infty} \frac{z^{2n-1}}{\sinh \pi z} dz.$$

From the expansion

$$\frac{1}{e^z+1} = \frac{1}{2} - \sum_1^{\infty} (-1)^{n-1} \frac{Bn}{\underline{|2n}} (2^{2n-1})\ x^{2n-1} \quad \text{(Bromwich, page 235)},$$

we obtain

$$\frac{1}{e^x+1} = \frac{1}{2} - \int_0^{\infty} \frac{\sin xz}{\sinh \pi z}\ dz.$$

Also since

$$\tan x = \sum_1^{\infty} \frac{2^{2n}\ (2^{2n-1})}{\underline{|2n}}\ B_n\ x^{2n-1}, \text{ (Hobson, page 363)}$$

we at once obtain

$$\tan x = 2 \int_0^{\infty} \frac{e^{2xz} - e^{-2xz}}{e^{\pi z} - e^{-\pi z}}\ dz.$$

§5. From

$$\int_0^1 \frac{(\log y)^{2n-1}}{1-y}\ dy = -\frac{2^{2n-1}\ \pi^{2n}}{2n}\ B_n,$$

we derive

$$B_n = -\frac{2n}{\pi^{2n}\ 2^{2n-1}} \int_0^{\frac{\pi}{4}} \frac{(\log \tan x)^{2n-1}}{\cos x\ (\cos x - \sin x)}\ dx.$$

$$= 4n\int_0^\infty \frac{t^{2n-1}}{e^{2\pi t}-1}\,dt. \qquad [y=e^{-2\pi t}$$

$$=\frac{2n\,(2n-1)}{\pi}\int_0^\infty t^{2n-2}\log\left(\frac{1}{1-e^{-2\pi t}}\right)dt,$$

by an easy integration by parts and noting that

$$\frac{1}{e^{2\pi t}-1} = \frac{1}{2\pi}\frac{d}{dt}\log\frac{1}{1-e^{-2\pi t}}.$$

From the expansion

$$\frac{1}{e^z-1} = -\frac{1}{2}+\frac{1}{z}+\sum_1^\infty (-1)^{n-1}\,B_n\,\frac{z^{2n}}{(2n)!}$$

we obtain the well-known result

$$\frac{1}{e^x-1} = -\frac{1}{2}+\frac{1}{x}+2\int_0^\infty \frac{\sin xt}{e^{2\pi t}-1}\,dt.$$

And since it is known that, (see Bromwich page 233,)

$$\frac{1}{e^x-1} = \frac{1}{x}-\frac{1}{2}+\sum\frac{2y}{y^2+4n^2\pi^2},$$

we get

$$\sum_1^\infty \frac{y}{y^2+4n^2\pi^2} = \int_0^\infty \frac{\sin xt}{e^{2\pi t}-1}\,dt.$$

§ 6. A simple consideration shows that the formulae can be transformed into one another. Thus

$$\frac{B_n}{4n} = \int_0^\infty \frac{t^{2n-1}}{e^{2\pi t}-1}\,dt$$

$$=\frac{1}{2}\int_0^\infty t^{2n-1}\left(\frac{1}{e^{\pi t}-1}-\frac{1}{e^{\pi t}+1}\right)dt.$$

$$=2^{2n-1}\left\{\int_0^\infty t^{2n-1}\left(\frac{1}{e^{2\pi t}-1}-\frac{1}{e^{2\pi t}+1}\right)dt.\right.$$

$$=2^{2n-1}\frac{B_n}{4n}-2^{2n-1}\int_0^\infty \frac{t^{2n-1}}{e^{2\pi t}+1}dt.$$

$$\therefore \int_0^\infty \frac{t^{2n-1}}{e^{2\pi t}+1}dt = \left(1-\frac{1}{2^{2n-1}}\right)\frac{B_n}{4n}.$$

But $\coth \pi x = \dfrac{1+e^{-2\pi x}}{1-e^{-2\pi x}}$.

$$= 1+2\,e^{-2\pi x}+2\,e^{-4\pi x}+2\,e^{-6\pi x}+\ldots\ldots$$

And $\displaystyle\int_0^\infty x^{2n-1}\left(\frac{1}{\tanh \pi x}-1\right)dx = 2\int_0^\infty x^{2n-1}\left(e^{-2\pi x}+e^{-4\pi x}\ldots\right)dx$

$$=\frac{B_n}{2n}.$$

$$\therefore \quad B_n = 2n\int_0^\infty \frac{x^{2n-1}}{\sinh \pi x\,(\cosh \pi x+\sinh \pi x)}dx.$$

[*Note.*—In passing, the following integrals which can be easily transformed into integrals having 0 and ∞ for limits may be noted:—

$$\frac{1}{1^{2n+1}}+\frac{1}{2^{2n+1}}+\frac{1}{3^{2n+1}}+\ldots=\frac{1}{(2n)!}\int_0^1 \frac{(\log y)^{2n}}{1-y}$$

$$=\frac{1}{(2n)!}\int_0^\infty \frac{x2n}{e^x-1}dx.$$

$$\frac{1}{1^{2n+1}}-\frac{1}{2^{2n+1}}+\frac{1}{3^{2n+1}}-\ldots=\frac{1}{(2n)!}\int_0^\infty \frac{x^{2n}}{e^x+1}dx.$$

$$\frac{1}{1^{2n+1}}+\frac{1}{2^{2n+1}}+\ldots+\frac{1}{k^{2n+1}}=\frac{1}{(2n)!}\int_0^1 (\log y)^{2n}\frac{1-y^k}{1-y}dy.$$

$$\frac{1}{1^{2n}}+\frac{1}{2^{2n}}+\ldots\ldots+\frac{1}{k^{2n}}=-\frac{1}{(2n-1)!}\int_0^1 (\log y)^{2n-1}\frac{1-y^k}{1-y}dy.$$]

Part II.

§ 8. We can make a very interesting application of the theory of residues to establish some well-known results. Thus

$$\frac{z}{e^z-1}=1-\frac{z}{2}+\sum_{1}^{\infty}(-1)^{n-1}\frac{B_n}{(2n)!}z^{2n}.$$

$\therefore\quad (-1)^{n-1}\frac{B_n}{(2n)!}$ is the residue of the function

$$f(z)=\frac{1}{z^{2n}\,(e^z-1)}\text{ at the origin.}$$

The other poles of $f(z)$ are $z=\pm 2\pi ri \quad (r=1, 2, \ldots\ldots)$.

The residue at $\pm\, 2\pi ri$ is

$$\frac{1}{(2\pi ri)^{2n}}\cdot\frac{1}{e^{2\pi ri}}=(-1)^n\frac{1}{(2\pi r)^{2n}}.$$

And $z\,f(z)\rightarrow 0$ as $z\rightarrow\infty$ uniformly. Hence the sum of the residues of the function at all its poles is zero. Hence we obtain

$$(-1)^{n-1}\frac{Bn}{\underline{|2n}}+(-1)^n\;2\frac{1}{(2\pi r)^{2n}}=0$$

i.e. $$B_n=\frac{2\,(2n)!}{(2\pi)^{2n}}\sum_{1}^{\infty}\frac{1}{r^{2n}}.$$ (a well-known result).

Similarly consider the function

$$\sec z=\sum_{0}^{\infty}E_n\frac{z^{2n}}{(2n)!}.$$

The poles are $z=\pm(2n+1)\frac{\pi}{2}$, and observing that $\frac{E_n}{(2n)!}$ is the residue of $f(z)=\frac{1}{z^{2n+1}}\cdot\frac{1}{\cos z}$ at the origin, we obtain as before

$$E_n=\frac{2\,(2n)!}{\left(\frac{\pi}{2}\right)^{2n+1}}\sum_{1}^{\infty}\frac{(-1)^{r-1}}{(2r-1)^{2n+1}}.$$ (a familiar result).

Note.—§ 8 is suggested in MacRobert's *Theory of Functions.*

Similarly, we can find an expression for $\phi_n(z)$, the Bernoullian polynomial function of degree n, *viz.*

$$\phi_n(x) = x^n - \frac{n}{2} x^{n-1} + B_1\binom{n}{2} x^{n-2} - B_4\binom{n}{4} xn - 4 + \ldots\ldots$$

We have

$$z\,\frac{e^{xz}-1}{e^z-1} = \sum_1^\infty \phi_n(x)\,\frac{z^n}{n!}.$$

We see at once that $\phi_n(x)/(n)!$ is the residue of the function

$$f(z) = \frac{1}{z^n} \cdot \frac{e^{xz}-1}{e^z-1}, \text{ at the origin.}$$

Suppose x positive and less than unity. Then $z f(z) \to 0$ uniformly as $z \to \infty$ and hence the sum of the residues of $f(z)$ at all its poles on the plane is zero.

The residue at $(2\pi ir)$ is

$$\frac{e^{2\pi irx}-1}{(2\pi ir)^n\, e^{2\pi ir}}$$

and the residue at $-2\pi ir$ is

$$\frac{e^{-2\pi irx}-1}{(-2\pi ir)^n\, e^{-2\pi ir}}.$$

If n is even, say $2m$, the sum of the two residues

$$= (-1)^m \frac{2\cos 2\pi rx - 2}{(2\pi r)^{2m}}.$$

Hence the sum of the residues at all the poles

$$= (-1)^m\, 2\sum \frac{\cos 2\pi rx - 1}{(2\pi r)^{2m}}$$

$$= (-1)^m\, 2\sum_1^\infty \frac{\cos 2\pi rx}{(2\pi r)^{2m}} + (-1)^{m+1}\, 2\sum \frac{1}{(2\pi r)^{2m}}.$$

If n is odd, say $2m+1$, the sum of the residues at $\pm 2\pi ir$

$$= (-1)^n \frac{2\sin 2\pi rx}{(2\pi r)^{2n+1}}$$

Hence we obtain

$$\phi_{2n}(x)=(-1)^{n+1}2\ (2n)!\ \sum_{1}^{\infty}\frac{\cos 2\pi rx}{(2\pi r)^{2n}}+(-1)^n\ B_n$$

$$\phi_{2n+1}(x)=(-1)^{n+1}2\ (2n+1)!\ \sum_{1}^{\infty}\frac{\sin 2\pi rx}{(2\pi r)^{2n+1}}.$$

[A trigonometric method of proving these results is suggested in Ex. 16, page 256, Bromwich.]

Similarly we know that

$$\frac{e^{x}z}{e^{z}+1}=\sum_{0}^{\infty}\psi_n\ (x).\frac{z^n}{n!}\qquad\text{(See Bromwich, page 240).}$$

Proceeding as before, we obtain two analogous formulæ, *viz.*

$$\psi_{2n}\ (x)=(-1)^n\ 2(2n)\ !\ \sum_{0}^{\infty}\frac{\sin\ (2r+1)\ \pi x}{[(2r+1)\ \pi]^{2n+1}}$$

$$\psi_{2n+1}\ (x)=(-1)^{n+1}\ 2\ (2n+1)!\sum_{0}^{\infty}\frac{\cos\ (2r+1)\ \pi x}{[(2r+1)\ \pi]^{2n+2}}.$$

§ 9. We shall next pass on to the problems of Mr. Ramanujan (Q. 387) and Mr. Bhimasena Rao (Q. 609) referred to in the Introduction.

Consider the function $f\ (z)=\dfrac{z^{4n+1}}{\left(e^{2\pi z}-1\right)\left(e^{-2\pi iz}-1\right)}$

Its poles are $\pm r,\ \pm ri.$ Consider $\int f\ (z)\ dz$ taken round the indented quadrant of a circle with the origin as center and semi-circles round the poles. It is zero since the function has no poles inside the contour. Along a semicircle round α, write $z=\alpha+\delta e^{i\theta}$.

$$\int\ f\ (z)\ dz=\int_{\pi}^{0} f\left(\alpha+\delta e^{i\theta}\right)\ i\delta e^{i\theta}d\theta.$$

Making $\delta\to o$ and noting that $f(\alpha+\delta e^{i\theta})\ \delta e^{i\theta}\to\alpha'$, uniformly as $\delta\to 0$ (Whittaker, p. 117, § 6.23), we see that

$$\int f\ (z)\ dz=-\pi i\alpha',\quad [\alpha' \text{ being the residue at } \alpha].$$

For the integral along the semi circle round 'ir.' the upper and lower limits are $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ and the integral similarly equals $-\pi ia'$ (a' being the residue at ir).

Along the quadrants, the integrals are zero since $z\,f(z)\to 0$ uniformly as $z\to\infty$ and as $z\to 0$. Making the radii of the semi circles tend to zero, and the radii of the quadrants zero and infinity, we obtain

$$\int_{O'A}+\int_{AB}+\int_{BO''} f(z)\,dz-\pi i\varepsilon a'+\int_{o''o'} f(z)\,dz=o.$$

The residues at 'r' and 'ri' are equal and

$$=-\frac{1}{2\pi i}\,\frac{r^{4n+1}}{e^{2\pi r}-1}.$$

Along OA, z is real and $=x$.

$$\int_{OA'}=\int_{o}^{\infty}\frac{x^{4n+1}}{\left(e^{2\pi x}-1\right)\left(e^{-2\pi ix}-1\right)}dx.$$

Similarly $$\int_{BO'}=\int_{\infty}^{o}\frac{(ix)^{4n+1}}{\left(e^{2\pi ix}-1\right)\left(e^{2\pi x}-1\right)}idx$$

$$=\int_{o}^{\infty}\frac{x^{4n+1}}{\left(e^{2\pi x}-1\right)\left(e^{2\pi ix}-1\right)}dx.$$

Hence the above equation, from the theory of residues gives

$$\int_{o}^{\infty}\frac{x^{4n+1}}{e^{2\pi x}-1}dx\left[\frac{1}{e^{2\pi ix}-1}+\frac{1}{e^{-2\pi ix}-1}\right]=-\,\Sigma\frac{r^{4n+1}}{e^{2\pi r}-1}.$$

i. e. $$\Sigma\frac{r^{4n+1}}{e^{2\pi r}-1}=\int_{o}^{\infty}\frac{x^{4n+1}}{e^{2\pi x}-1}dx \qquad \text{(after simplication)}$$

$$=\frac{B_{m+1}}{4\,(2n+1)}.$$

In the case of Mr. Ramanujan's Q. 387, which is evidently a particular case of this, there is an additional term $-\frac{1}{8\pi}$ on the right. This is because, for the function $f(z)$ the origin is a pole and the residue at this pole is zero if n is any positive integer, and $-\frac{1}{4\pi^2 i}$ if n is zero.

In this particular case, we have at the origin

$$\int_{o''o'} f(z)\, dz = \int_{\frac{1}{2}\pi}^{o} f(\delta e^{i\theta})\ i\delta e^{i\theta} d\theta.$$

$$= -\frac{\pi}{2} i\, a',$$

where a' is the residue at the origin.

Hence the additional term on the right ; and we obtain thus Mr. Ramanujan's problem

$$\sum \frac{r}{e^{2\pi r}-1} = \frac{1}{24} - \frac{1}{8\pi}.$$

To obtain the second of Mr. Bhimasena Rao's results, examine the function

$$\int \frac{z^{4n+1}}{(e^{\pi z}+1)(e^{-\pi i z}+1)} dz.$$

taken round the same contour with semi circles round the poles $(2r+1)$, $(2r+1)i$,

Residue at $(2r+1)$ = residue at $(2r+1)i$

$$= \frac{(2r+1)^{4n+1}}{\pi i \{ e^{(2r+1)\pi} + 1 \}}.$$

Hence we obtain that the sum of the integrals round all the semi circles is equal to

$$= -2. \frac{(2r+1)^{4n+1}}{\{ e^{(2r+1)\pi} + 1 \}}.$$

$$\int_{OA} + \int_{AB} + \int_{BO} = 2 \sum \frac{(2r+1)^{4n+1}}{e^{(2r+1)\pi} + 1}$$

Now $\int_{AB} \to 0$ since $z f(z) \to 0$ as $z \to \infty$,

$$\int_{OA} + \int_{BO} = \int_{0}^{\infty} \frac{x^{4n+1}}{e^{\pi x}+1} \left[\frac{1}{e^{-\pi i x}+1} + \frac{1}{e^{\pi i x}+1} \right] dx$$

$$=\int_0^{\infty} \frac{x^{4n+1}}{e^{\pi x}+1}\,dx$$

$$=B_{2n+1}\frac{(2^{4n+1}-1)}{2(2n+1)} \text{ (see § 3 above).}$$

Hence the second result.

§ 9*a*. A very interesting result, which I think is highly probable is obtained by examining $\int f(z)dz$ round the same coutour, when

$$f(z)=\frac{z^n}{(e^{2\pi z}-1)(e^{-2\pi iz}-1)}$$

Residue at 'r' + residue at 'ri'

$$=-\frac{1}{2\pi i}\frac{r^n}{e^{2\pi r}-1}\left[1+(-1)^{\frac{n-1}{2}}\right].$$

Exactly as before

$$\int_{OA}=\int_0^{\infty}\frac{x^n}{(e^{2\pi x}-1)(e^{-2\pi ix}-1)}\,dx$$

$$\int_{BO}=i^{n-1}\int_0^{\infty}\frac{x^n}{(e^{2\pi ix}-1)(e^{2\pi x}-1)}\,dx.$$

Hence we obtain

$$-\frac{(1+i^{n-1})}{2}\sum\frac{r^n}{e^{2\pi r}-1}=\int_0^{\infty}\frac{x^n}{(e^{2\pi x}-1)(e^{-2\pi ix}-1)}\,dx$$

$$+i^{n-1}\int_0^{\infty}\frac{x^n}{(e^{2\pi ix}-1)(e^{2\pi x}-1)}\,dx.$$

Putting $n=4m+1$, we obtain the above result.

Putting $n=4m$, we get

$$-\frac{(1-i)}{2}\sum\frac{r^{4m}}{e^{2\pi r}-1}=\int_0^\infty\frac{x^{4m}}{(e^{2\pi n}-1)\,(e^{-2\pi ix}-1)}dx$$

$$-i\int_0^\infty\frac{x^{4m}}{(e^{2\pi ix}-1)\,(e^{2\pi x}-1)}dx$$

$$=\int_0^\infty\frac{x^{4m}}{e^{2\pi ix}-1}dx\left[-\frac{1}{2}+\frac{i}{2}\cot\ \pi x-i\left(-\frac{1}{2}-\frac{i}{2}\cot\ \pi x\right)\right].$$

$$=\int_0^\infty\frac{x^{4m}}{e^{2\pi x}-1}\,dx.\ (i-1)\ \tfrac{1}{2}\ (1+\cot\ \pi x).$$

Whence we obtain

$$\sum\frac{r^{4n}}{e^{2\pi r}-1}=\int_0^\infty\frac{x^{4m}}{e^{2\pi x}-1}\ (1+\cot\ \pi x)\ dx.$$

Putting $n=4m-1$, since $1+i^{n-1}=1+i^{4m-2}=0$, we get

$$\int_0^\infty\frac{x^{4m-1}}{(e^{2\pi x}-1)\,(e^{-2\pi ix}-1)}\ dx\ =\ \int_0^\infty\frac{x^{4m-1}}{(e^{2\pi xi}-1)\,(e^{2\pi x}-1)}dx$$

i.e.

$$\int_0^\infty\frac{x^{4m-1}}{e^{2\pi x}-1}\ dx\ \left[-\frac{1}{2}+\frac{i}{2}\cot\ \pi x\right]$$

$$=\int_0^\infty\frac{x^{4m-1}}{e^{2\pi x}-1}\ dx\ \left[-\frac{1}{2}-\frac{i}{2}\cot\ \pi x\right];$$

Whence we obtain the result

$$\int_0^\infty\frac{x^{4m-1}}{e^{2\pi x}-1}\ \cot\ \pi x=0.$$

Putting $n=4m+2$, we obtain exactly as before

$$\sum \frac{r^{4m+2}}{e^{2\pi r}-1}=\int_0^\infty \frac{x^{4m+2}}{e^{2\pi n}-1}(1+\cot \pi x)dx.$$

Putting $n=4m-2$, we obtain the same result with $4m-2$ substituted for $4m+2$.

It thus appears that $\int_0^\infty \frac{x^{4m-1}}{e^{2\pi x}-1} \cot \pi x \, dx=0$

and we are led to the general conclusion that

$$\sum_1^\infty \frac{r^n}{e^{2\pi r}-1}=\int_0^\infty \frac{x^n}{e^{2\pi x}-1}. \quad \{1+f(x)\}\, dx,$$

where

$f(x)=0$ if $n=4m+1$, and $=\cot \pi x$. if $n=4m$ or $4m \pm 2$,

except when $n=1$, which is Mr. Ramanujan's problem.

§ 10. The above method will now be utilized for obtaining some further results.

(*a*) Examine $\int \frac{z^{4n+1}}{\left(e^{\pi z}-e^{-\pi z}\right)\left(e^{\pi i z}-e^{-\pi i z}\right)} dz$ taken round an infinite circle with the origin as centre.

Residue at each of the points r, ri, $-r$, $-ri$ is

$$(-1)^r \frac{r^{4n+1}}{e^{\pi r}-e^{-\pi r}} \quad \frac{1}{2\pi i}.$$

Residue at the origin is as before zero, if n is any positive integer and $+\frac{1}{4\pi^2 i}$, if n is zero.

Hence we obtain

$$\sum_1^\infty (-1)^r \frac{r^{4n+1}}{e^{\pi r}-e^{-\pi r}}=0;$$

$$\sum_1^\infty (-1)^r \frac{r}{e^{\pi r}-e^{-\pi r}}=-\frac{1}{8\pi}.$$

(*b*) Consider $\int \frac{f(z)}{\sin \pi z}\, dz$ taken over a circle of radius $R \to \infty$.

The poles are $\pm r$, 0, and the poles of $f(z)$.

Residue at $r = (-1)^r \frac{f(r)}{\pi}$.

Residue at $-r = (-1)^r \frac{f(-r)}{\pi}$.

Residue at $0 = \frac{f(0)}{\pi}$.

$$\therefore \quad \sum_1^\infty (-1)^r \frac{f(r)+f(-r)}{\pi} + \frac{f(0)}{\pi} = \frac{1}{2\pi i}\int \frac{f(z)}{\sin \pi z}dz - \mathrm{R}\frac{f(z)}{\sin \pi z}$$

where the last term on the right means that we should take the sum of the residues of the function with respect to the poles of $f(z)$.

If in particular $f(z)$ is such that $z\,f(z) \to 0$ as $z \to \infty$, we get

$$\frac{f(0)}{\pi} + \sum_1^\infty (-1)^r \frac{f(r)+f(-r)}{\pi} = -\mathrm{R}\frac{f(z)}{\sin \pi z}.$$

Ex. In particular let $f(z)$ be an even function. We get

$$f(0) + 2\sum_1^\infty f(r) = -\pi \mathrm{R}\frac{f(z)}{\sin \pi z}.$$

Put $f(z) = \frac{x}{z^2 - x^2}$. We get

$$-\frac{1}{x} + 2\sum_1^\infty (-1)^r \frac{x}{r^2 - x^2} = -\mathrm{R}\frac{\pi f(z)}{\sin \pi z} = -\frac{\pi}{\sin \pi x}$$

i.e.
$$\pi \operatorname{cosec} \pi x = \frac{1}{x} + 2\sum_1^\infty (-1)^{r-1} \frac{x}{r^2 - x^2}.$$

(Ex. 19, page 190, Bromwich)

(*c*) Consider similarly $\int \frac{f(z)}{\cos \pi z}dz$.

We obtain if $z\,f(z) \to 0$ as $z \to \infty$.

$$\sum_0^\infty (-1)^{r+1}\left[f\left(\frac{2r+1}{2}\right) - f\left(-\frac{2r+1}{2}\right)\right] = -\pi\,\mathrm{R}\frac{f(z)}{\cos \pi z}.$$

If $f(z)$ is an odd function, this gives

$$\sum_{0}^{\infty} (-1)^{r+1} \quad 2f\left(\frac{2r+1}{2}\right) = -\pi \, \mathrm{R} \frac{f(z)}{\cos \pi z}$$

Ex. Let $f(z) = \frac{z}{z^2 - x^2}$. We get

$$\sum_{0}^{\infty} (-1)^{r+1} \frac{2r+1}{\left(\frac{2r+1}{2}\right)^2 - x^2} = \pi \sec \pi x \quad \text{(Page 190, Bromwich).}$$

(*d*) Consider $\int \frac{f(z)}{\tan \pi z} dz$. We get

$$\sum_{-\infty}^{\infty} \quad f(r) + \pi \, \mathrm{R} \frac{f(z)}{\tan \pi z} = 0.$$

Ex. 1. Let $f(z) = \frac{a}{(x-z)(a-z)}$. We get

$$\sum_{-\infty}^{\infty} \frac{a-x}{(x-n)(a-n)} = \pi (\cot \pi x - \cot \pi a).$$

Ex 2. Let $f(z) = \frac{x}{z(x-z)}$.

We see here that the origin is a pole for $f(z)$ as well as for $\cot \pi z$.

Hence we should examine the residue at the origin of $\frac{f(z)}{\tan \pi z}$ and substitute this for $f(0)$.

$$\frac{f(z)}{\tan \pi z} = \frac{x}{z(x-z)} \frac{\cos \pi z}{\sin \pi z}$$

$$= \frac{1}{z} \left(1 + \frac{z}{x} + \ldots\right) \left(1 - \frac{\pi^2 z^2}{\underline{|2}} + \ldots\right) \frac{1}{\pi z}\left(1 + \frac{\pi^2 z^2}{\underline{|3}} - \ldots\right)$$

$$= \frac{1}{\pi x z} + \ldots\ldots \text{terms containing other powers.}$$

$\therefore$ Residue at the origin $= \frac{1}{\pi x}$.

Residue at $z = x$ is $-\cot \pi x$.

$$\therefore \qquad -\sum_{\infty}^{\infty}{}' \left(\frac{1}{x-n} + \frac{1}{n}\right) + \frac{1}{x} = \pi \cot \pi x.$$

Ex. 3. $f(z)=\frac{1}{(x-z)^2}$. We get

$$\sum_{-\infty}^{\infty}\frac{1}{(x-n)^2}=-\pi\,\mathrm{R}\frac{\cot\pi z}{(x-z)^2}$$

We have to find the residue of $\frac{\cot\pi z}{(x-z)^2}$ at $z=x$. Now let

$$\phi(x)=\cot\pi x.$$

Write $\cot\pi z=\cot[\pi x-\pi(x-z)]$

$$=\phi(z+y),\ \text{where}\ y=-x+z.$$

$$=\phi(x)+y\,\phi'(x)+\frac{y^2}{\lfloor 2}\phi''(x)+\frac{y^3}{\lfloor 3}\phi'''(x)+\dots\dots$$

Again

$$\phi(x)=\cot\pi x$$

$$\phi'(x)=-\pi\operatorname{cosec}^2\pi x$$

$$\phi''(x)=2\pi^2\operatorname{cosec}^2\pi x\cot\pi x$$

$$\phi'''(x)=+2\pi^3[3\operatorname{cosec}^4\pi x-2\operatorname{cosec}^2\pi x].$$

Hence we obtain

$$\frac{\cot\pi z}{(x-z)^2}=\frac{\phi(x-y)}{y^2}.$$

$\therefore$ Residue at $z=x$ is $-\phi'(x)$.

$$\therefore\quad \sum_{-\infty}^{\infty}\frac{1}{(x-n)^2}=\pi^2\operatorname{cosec}^2\pi x.$$

For $\frac{\cot\pi z}{(x-z)^3}$, residue at $z=x$ is $-\frac{1}{2!}\phi''(x)$.

$$\therefore\quad \sum_{-\infty}^{\infty}\frac{1}{(x-n)^3}=+\frac{\pi}{\lfloor 2}\phi''(x)$$

$$=+\pi^3\operatorname{cosec}^2\pi x\cot\pi x.$$

For $\frac{\cot\pi z}{(x-z)^4}$, residue at $z=x$ is $\frac{1}{3!}\phi'''(x)$.

$$\therefore\quad \sum_{-\infty}^{\infty}\frac{1}{(x-n)^4}=\pi^4\left(\operatorname{cosec}^4\pi x-\frac{2}{3}\operatorname{cosec}^2\pi x\right).$$

Here put $x=\frac{1}{2}-\frac{1}{2n}$. We get Q. 1000, first part. The Second part is similarly obtained.

Put $x=\frac{1}{2n}$, we get Q. 1001, first part. The other parts are similarly obtained.

More generally, we obtain

$$\sum_{-\infty}^{\infty} \frac{1}{(x-n)^r} = \frac{\pi}{\lfloor r-1}. \quad (-1)^r. \quad \frac{d^{r-1}}{dx^{r-1}} \cot \pi x.$$

[See Bromwich, Ex. 17, Page 190].

Ex. 4 Let $f(z) = \frac{x}{z^2 - x^2}$. $f(z)$ is an even function.

We get

$$-\frac{1}{x} + \sum_{1}^{\infty} \frac{2x}{n^2 - x^2} = -\pi \, \mathrm{R} \, \frac{x}{z^2 - x^2} \cot \pi z.$$

$$= -\frac{\pi}{\tan \pi x}.$$

$$\therefore \quad \pi \cot \pi x = \frac{1}{x} + \sum_{1}^{\infty} \frac{2x}{x^2 - n^2}. \quad \text{(a well known result).}$$

§ 11. Next consider the residues of functions of the type $f(x) \frac{\sin ax}{\sin \pi x}$, etc. We require the following well-known theorem in the theory of residues; viz. If C is a circle of radius $R \to \infty$, and $f(z)$ is a function such that $z f(z) \to k$ as $z \to \infty$, we have

$$\int f(z)\, dz = \int z f(z) \frac{dz}{z}.$$

$$= k \int \frac{dz}{z} = 2\pi i \, k.$$

(See: Fortyth, *Theory of Functions*).

§ 12. Consider the function $f(x) \frac{\sinh ax}{\sinh \pi x}$.

The poles are $\pm ri$, 0. Residue at the origin is 0 since $\sinh 0 = 0$.

The sum of the residues at 'ri' and 'ri' is

$$\frac{f(ri)}{\pi} \frac{i \sin ra}{(-1)^r} - \frac{f(-ri)\, i \sin ra}{(-1)^r \pi}$$

$$= (-1)^r \frac{1}{\pi} i \sin ra \; \{ f(ri) - f(-ri) \}.$$

$$\therefore \quad \sum_{1}^{\infty} (-1)^r \, i \, [f(ri) - f(-ri)] \sin ra + \mathrm{R} \, f(z) \frac{\sinh az}{\sinh \pi z} = 0.$$

Ex. 1. Let $f(z)=\dfrac{1}{x-z}$.

$$z\,f(z)\to -1 \text{ as } z\to\infty.$$

Hence we obtain

$$\frac{\sinh ax}{\sinh x\pi}=\Sigma\ (-1)^{r-1}\frac{\sin ar}{\pi}\,\frac{2r}{x^2+r^2}.$$

[Ex. 23, Page 257 Bromwich].

Ex. 2. If we similarly examine $f(x)\,\dfrac{\cosh az}{\sinh \pi z}$, we obtain

$$\mathrm{R}\,f(z)\frac{\cosh az}{\cosh \pi z}+f(o)+\Sigma(-1)^r\frac{f(ri)+f(-ri)}{\pi}\cos ar=0.$$

Put $f(z)=\dfrac{1}{x-z}$. We get

$$\frac{\cosh ax}{\sinh \pi x}=\frac{1}{x}+\Sigma\ (-1)^r\ \frac{2x\cos ar}{\pi(x^2+r^2)}.$$

[Ex. 23, page 257 Bromwich].

§ 13. Similarly by taking $f(x)\,\dfrac{\sin ax}{\sin \pi x}$ and $f(x),\ \dfrac{\cos ax}{\sin \pi x}$ we get

$$\sum_1^\infty(-1)^r\ \{f(r)-f(-r)\}\sin ar=-\pi\,\mathrm{R}\frac{\sin az}{\sin \pi z}f(z)$$

$$f(0)+\sum_1^\infty(-1)^r\{f(r)+f(-r)\}\ \cos ar=-\pi\,\mathrm{R}\frac{\cos az}{\sin \pi z}f(z).$$

If $f(x)=\dfrac{1}{x-z}$, we get

$$\sum_1^\infty\ (-1)^r\frac{\sin ar}{x^2-r^2}=\frac{\pi}{2}\,\frac{\sin ax}{\sin \pi x}.$$

$$\sum_1^\infty\ (-1)^r\frac{\cos ar}{x^2-r^2}=-\frac{1}{2x^2}+\frac{\pi}{2x}\,\frac{\cos ax}{\sin \pi x}.$$

ASTRONOMICAL NOTES.

1. *Comets.* A comet (1919 *f*) was recorded on two plates taken on December 10, 1919 at Hamburg by Dr. Baade. It is probably identical with Holmes's Comet whose return to perhelion should have happened about November 30.

Another comet (1919 *g*) was discovered by Mr. J. F. Skjellerup at the Cape of Good Hope on December 18. It was also observed at the Royal Observatory by Mr. Woodgate. The position of the comet March 11 will be R.A. 20H. 36M. Decl. 8°. 7 N. It should be visible early in the morning during this month.

A new object of about the 10th magnitude (provisionally designated 1920 *a*) was discovered by Senor Comas Sola at Barcelona. It was first reported to be a comet but now appears to be a minor planet.

2. *Perturbation of Neptune.* In Harward Circular 215 Prof. W.H. Pickering draws the attention of Astronomers to the fact that Neptune is now gradually deviating from its computed position in the manner that it should do, if disturbed by an unknown outer planet. The deviation so far observed amounts to a little over 2″ and if this is due to the perturbing action of an outer planet it is expected to increase to about 15″ in the course of a few decades. The unknown planet is believed to be small, about terrestrial size and to be travelling in a highly elliptical orbit, the present distance from the Sun being about 68 times that of the earth. Prof. Pickering states that it should be located at present in R. A. 6H. 35M. and Dec. 23° N., Mag. about 15 and as it is surrounded by numerous brighter stars in the Milky Way, identification will be difficult.

3. *Eclipses.* In the year 1920, there will be four eclipses, the first two of which will occur during the month of May.

i. A total Eclipse of the Moon, May 3, 1920: the beginning of the eclipse will be generally visible in India.

The circumstances of the eclipse are as under :—

Moon enters Penumbra May 3	4—19 A.M.	Indian Standard Time.
Moon enters Umbra	5—31 ,,	
Total eclipse begins	6—45 ,,	
Total eclipse ends	7—57 ,,	
Moon leaves Umbra	9—11 ,,	
Moon leaves Penumbra	10—23 ,,	

Magnitude of the eclipse will be 1·224 (the moon's diameter being taken as unity).

ii. A partial Eclipse of the Sun, May 18, 1920, invisible in India.: the eclipse will be generally visible in Australia and the southern part of the Indian Ocean.

NIZAMIAH OBSERVATORY, HYDERABAD. T. P. Bhaskara Sastri.

SOLUTIONS.

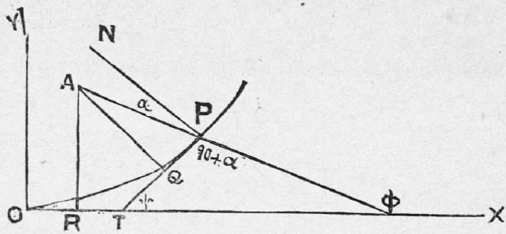
Question 819.

(K. APPUKUTTAN ERADY, M.A.) :—If ρ, ρ' be the radii of curvature at corresponding points of a curve and its α—evolute, where α is a function of the arc measured from a fixed point on the curve, show that

$$\rho'\left(\frac{1}{\rho}+\frac{d\alpha}{ds}\right)+\sin\alpha=\frac{d}{ds}\left\{\frac{\cos\alpha}{\frac{1}{\rho}+\frac{d\alpha}{ds}}\right\}.$$

Solution by Martyn M. Thomas, K. B. Madhava and K. R. Rama Iyer.

Choose the tangent and normal at the fixed point O, as fixed axes of coordinates ; let A be a point on the α—evolute corresponding to P on the given curve.



Let the coordinates of A be (X, Y) referred to the fixed axes, and (u, v) referred to the tangent and normal at P. Let P be (x, y).

From A drop perpendiculars AR, AQ on OX and the tangent PT.

Now, X = OR = Projection of OA on X axis

= algebraical sum of the projections of OP, PQ, QA on X—axis.

$$=x-u\cos\psi-v\sin\psi.$$

Similarly, $Y=y-u\sin\psi+v\cos\psi$.

$$\therefore \quad \frac{dX}{ds}=\cos\psi-\frac{du}{ds}\cos\psi+u\sin\psi\frac{1}{\rho}-\frac{dv}{ds}\sin\psi-v\cos\psi\frac{1}{\rho}$$

$$=\cos\psi\left(1-\frac{du}{ds}-\frac{v}{\rho}\right)+\sin\psi\left(\frac{u}{\rho}-\frac{dv}{ds}\right)\quad\ldots\quad\ldots\quad(1)$$

$$\frac{dY}{ds}=\sin\psi\left(1-\frac{du}{ds}-\frac{v}{\rho}\right)-\cos\psi\left(\frac{u}{\rho}-\frac{dv}{ds}\right)\quad\ldots\quad\ldots\quad(2)$$

$$\therefore\quad\left(\frac{dS}{ds}\right)^2=\left(\frac{dX}{ds}\right)^2+\left(\frac{dY}{ds}\right)^2=\left(1-\frac{du}{ds}-\frac{v}{\rho}\right)^2+\left(\frac{u}{\rho}-\frac{dv}{ds}\right)^2\quad\ldots\quad(3)$$

Also $\tan(90+\alpha+\psi)=\dfrac{dY}{dX}=\dfrac{\left(1-\dfrac{du}{ds}-\dfrac{v}{\rho}\right)-\cot\psi\left(\dfrac{u}{\rho}-\dfrac{dv}{ds}\right)}{\cot\psi\left(1-\dfrac{du}{ds}-\dfrac{v_1}{\rho}\right)+\left(\dfrac{u}{\rho}-\dfrac{dv}{ds}\right)}$

$$\therefore \quad \cot\alpha=\frac{\dfrac{u}{\rho}-\dfrac{dv}{ds}}{1-\dfrac{du}{ds}-\dfrac{v}{\rho}} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (4)$$

$\therefore$ Substituting in (3) from (4),

$$\left(\frac{dS}{ds}\right)^2=\left(\frac{u}{\rho}-\frac{dv}{ds}\right)^2\left(\tan^2\alpha+1\right)$$

$$\therefore \quad \frac{dS}{ds}=\pm\sec\alpha\left(\frac{u}{\rho}-\frac{dv}{ds}\right) \quad \ldots \quad \ldots \quad \ldots \quad (5)$$

Since $u=v\tan\alpha$, $\dfrac{du}{ds}=\tan\alpha.\dfrac{dv}{ds}+v\sec^2\alpha.\dfrac{d\alpha}{ds}$.

$\therefore$ Substituting for u and $\dfrac{du}{ds}$ in (4), we have,

$$1=v\sec^2\alpha\left(\frac{1}{\rho}+\frac{d\alpha}{ds}\right)$$

$$\therefore \quad v=\frac{\cos^2\alpha}{\dfrac{1}{\rho}+\dfrac{d\alpha}{ds}} \;\&\; u=\frac{\sin\alpha\cos\alpha}{\dfrac{1}{\rho}+\dfrac{d\alpha}{ds}}.$$

Taking the lower sign in (5),

$$\frac{dS}{ds}=\sec\alpha\left(\frac{dv}{ds}-\frac{u}{\rho}\right)$$

$$\therefore \quad \frac{dS}{d\phi}.\frac{d\phi}{ds}=\sec\alpha.\frac{d}{ds}\left\{\frac{\cos^2\alpha}{\dfrac{1}{\rho}+\dfrac{d\alpha}{ds}}\right\}-\frac{1}{\rho}.\frac{\sin\alpha}{\dfrac{1}{\rho}+\dfrac{d\alpha}{ds}}.$$

where $\phi=90+\alpha+\psi$.

$$\therefore \quad \rho'.\frac{d\alpha+d\psi}{ds}=\cos\alpha.\frac{d}{ds}\left\{\frac{1}{\dfrac{1}{\rho}+\dfrac{d\alpha}{ds}}\right\}-\frac{2\sin\alpha}{\dfrac{1}{\rho}+\dfrac{d\alpha}{ds}}.\frac{d\alpha}{ds}-\frac{1}{\rho}.\frac{\sin\alpha}{\dfrac{1}{\rho}+\dfrac{d\alpha}{ds}}$$

$$\therefore \quad \rho'\left(\frac{d\alpha}{ds}+\frac{1}{\rho}\right)=\cos\alpha.\frac{d}{ds}\left\{\frac{1}{\dfrac{1}{\rho}+\dfrac{d\alpha}{ds}}\right\}-\frac{\sin\alpha}{\dfrac{1}{\rho}+\dfrac{d\alpha}{ds}}.\frac{d\alpha}{ds}-\sin\alpha$$

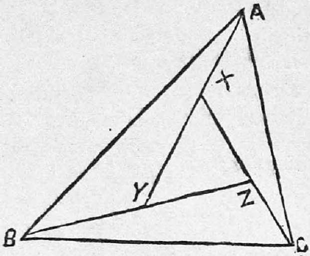
$$\therefore \quad \rho\left(\frac{d\alpha}{ds}+\frac{1}{\rho}\right)+\sin\alpha=\frac{d}{ds}\left\{\frac{\cos\alpha}{\dfrac{1}{\rho}+\dfrac{d\alpha}{ds}}\right\}.$$

Question 821.

(M. K. Kewalramani.):—If through A, B, C lines AXY, BYZ, CZX are drawn so as to make the same angle Θ with AB, BC, CA respectively and form the triangle XYZ, prove that $\rho=2\sigma\sin(\omega-\Theta)$ where ω is the Brocard angle of the triangle ABC, σ the radius of the first Lemoine Circle of ABC, and ρ is the radius of the Cosine Circle of XYZ.

Solution by N. Sankara Aiyar.

$\angle YXZ=\angle XAC+\angle XCA=A$. Hence the triangles ABC & XYZ are similar.



Now, $\frac{CX}{CA}=\frac{\sin(A-\Theta)}{\sin A}$ *i.e.* $CX=\frac{b\sin(A-\Theta)}{\sin A}$ & $CZ=\frac{a\sin\Theta}{\sin C}$

$$\therefore\quad XZ=2R\left\{\frac{\sin B\sin(A-\Theta)}{\sin A}-\frac{\sin A\sin\Theta}{\sin C}\right\}$$

$$=2R\left\{\sin B\cos\Theta-\sin B\cot A\sin\Theta-\frac{\sin(B+C)\sin\Theta}{\sin C}\right\},$$

$$=2R\sin B\ \{\cos\Theta-\sin\Theta(\cot A+\cot B+\cot C)\},$$

$$=2R\sin B\ \{\cos\Theta-\sin\Theta\cot\omega\},$$

$$=2R\sin B.\frac{\sin(\omega-\Theta)}{\sin\omega}.$$

The ratio of similitude is therefore $\frac{\sin(\omega-\Theta)}{\sin\omega}$.

In any triangle $\rho=2\sigma\sin\omega$ and hence $\rho=2\sigma\sin(\omega-\Theta)$ for Δs ABC & XYZ.

Question 827.

(A. C. L. Wilkinson.) :—If a skew surface is defined by

$$x=z,\ y=bx+\beta,\ z=cx+\gamma$$

where b, c, β, γ are functions of t, and if the axis of x is the generator corresponding to $t=0$, the origin the central point of this generator, and $z=0$ the tangent plane at the origin; then the hyperboloid of closest contact along the generator $t=0$ is given by

$$2c' \beta'^2 z = c' \gamma'' y^2 + 2c'^2 \beta' xy - b'' \beta' z^2 - (c' \beta'' - c'' \beta') yz$$

where the values of the differential co-efficients of b, c, β, γ are for $t=0$.

Solution by K. R. Rama Aiyar.

The hyperboloid of closest contact has three consecutive generators in common with the surface; and so the axis of x is a generator and the origin is the central point. So the equation to the hyperboloid assumes the form

$$\lambda y^2 + \mu z^2 + \nu z + f yz + h xy = 0. \qquad \ldots \qquad \ldots \qquad (A)$$

Evidently when $t=0$, $b=\beta=c=\gamma=0'$ and the generator adjacent to the x axis is given by $y=(b' x+\beta') \delta t$, $z=\delta t (c' x+\gamma')$ where b', β'. c', γ' are values corresponding to $t=0$. Since the central point is the origin, the x axis, a generator and the z axis, the normal at the origin the S. D between x axis and the adjacent generator is the y axis. Since, therefore, $x=0$, $z=0$ both intersects and is perpendicular to $y=dt (b' x+\beta')$, $z=dt (c' x+\gamma')$ we find

$$b' = \gamma' = 0.$$

Since the surface has triple contact with the hyperboloid along the x axis we have the following equations satisfying A

(1) $y=0$, $z=0$

(2) $y=\beta' \delta t$, $z=\delta t. c' x$.

(3) $y=(\beta''+b'' x)\frac{\delta t^2}{2}+\beta' \delta t$

$z=(\gamma''+c'' x)\frac{\delta t^2}{2}+c' x \delta t$

Hence we find omitting higher powers of δt than necessary

$$(h\beta'+\nu c') x \delta t = 0$$

and $x \delta t (h\beta'+\nu c')+\delta t^2 \left\{ x^2 (h\frac{b''}{2}+\mu c'^2)+x (h\frac{\beta''}{2}+\nu\frac{c''}{2}+f\beta' c') + \lambda\beta'^2+\nu\frac{\gamma''}{2} \right\} = 0$

for all values of x.

$\therefore \quad h \beta'+\nu c'=0, \quad$ or $\quad -\frac{h}{c'}=\frac{\nu}{\beta'}=\kappa$ (say)

$$h \frac{\beta''}{2}+\mu c'^2=0,$$

$$h \frac{\beta''}{2}+\nu \frac{c''}{2}+f \beta' c'=0$$

$$\lambda \beta'^2 + \nu \frac{\gamma''}{2} = 0.$$

$$\therefore \quad \nu = \kappa\beta',\ h = -\kappa c',\ \mu = \kappa\frac{\beta''}{2c'}\ \lambda = -\kappa\ \frac{\gamma}{2\beta'}$$

$$f = -\frac{\kappa}{\beta' c'}\left\{\frac{c''\ \beta'}{2} - \frac{c'\ \beta''}{2}\right\}.$$

Hence the equation (A) reduces to

$$2c'\ \beta'^2\ z = c'\ \gamma''\ y^2 + 2c'^2\ \beta'\ xy - b''\ \beta'\ z^2 - (c'\ \beta'' - c''\ \beta')\ yz$$

where the values of the differential co-efficients of b, c, β, γ are for $t=0$

Question 850.

(A. C. L. Wilkinson):—Prove that

$$\int_0^\infty \frac{\sin x \sinh x}{\cosh x + \cos x}\ \frac{dx}{x} = \frac{\pi}{4},$$

and

$$\int_0^\infty \frac{\cos x \sinh x}{\cosh x + \cos x}\ \frac{dx}{x} = 0.$$

Solution by K. B. Madhava.

Consider the integral $\int \frac{e^{z\,(1+i)}}{e^z + e^{iz}}\ \frac{dz}{z}$ over the contour consisting of (i) the x—axis from O to R (ii) the quadrant of the circle centre the origin and radius R, and (iii) the y—axis from R to O; and make R tend to infinity. On this contour there are no infinities of the integrand, for its only poles are given by $z = (2n+1)\ i\pi + iz$. *ie* the line $y+x=0$. Putting now on (i) $z=x$; on (ii) $z = \text{R}\,e^{i\theta}$ and on (iii) $z=iy$, we have as R becomes infinite

$$\int_0^\infty \frac{e^{x(1+i)}}{e^x + e^{ix}}\ \frac{dx}{x} + \int_0^{\frac{1}{2}\pi} \frac{e^{\text{R}e^{i\theta}}}{1 + e^{\text{R}(1-i)\,e^{i\theta}}}\ id\theta + \int_\infty^0 \frac{e^{ix\,(1+i)}}{e^{ix} + e^{-x}}\ \frac{dx}{x} = 0.$$

On the infinite quadrant, the second integral is in modulus $=1$, and therefore, collecting the first and the third integrals together, we have

$$\int_0^\infty \frac{e^{2ix}\ (e^x - e^{-x})}{e^{ix}\ (e^{ix} + e^{-ix} + e^x + e^{-x})}\ \frac{dx}{x} = \frac{\pi}{4}.$$

Hence separating the real and imaginary parts we have

$$\int_0^\infty \frac{\sinh x \cos x}{\cosh x + \cos x}\ \frac{dx}{x} = 0$$

and

$$\int_0^\infty \frac{\sinh x \sin x}{\cosh x + \cos x}\ \frac{dx}{x} = \frac{\pi}{4}.$$

Question 864.

(M. K. KEWALRAMANI):—Prove that the primitive of the differential equation

$$\frac{d^2 y}{dx^2} - a^2 y + \frac{p(1-p)}{x^2} y = 0$$

can be put into the form

$$y = x^p \left[A \int_0^1 \frac{\cosh arx}{(1-r^2)^{1-p}} dr + B \int_1^\infty \frac{e^{-arx}}{(r^2-1)^{1-p}} dr \right]$$

where p is always positive.

Solution by C. Krishnamachari and S. V. Venkatachala Iyer.

Let $$y = \int e^{-rt} x^m \, R \, dr,$$

where t is a function of x alone, and R a function of r alone. Differentiating and substituting in the equation, we easily obtain

$$-\int r \frac{d^2 t}{dx^2} x^m e^{-rt} R \, dr - 2m \int x^{m-1} e^{-rt} \frac{dt}{dx} R \, r dr$$
$$+ \int e^{-rt} R \, x^m \, dr \left\{ r^2 \left(\frac{dt}{dx}\right)^2 - a^2 \right\}$$
$$+ \{ m(m-1) + p(1-p) \} \int e^{-rt} x^{m-2} R \, dr = 0.$$

Put $m = p$ and $t = ax$.

The first and last terms vanish. The equation now reduces to

$$-2pa \int x^{p-1} e^{-arx} R \, r dr + \int e^{-arx} x^p R \, dr \, (r^2-1) \, a^2 = 0.$$

Integrating the second term by parts, we have

$$-2pa \int x^{p-1} e^{-arx} R \, r dr - \left[e^{-arx} x^{p-1} R \, (r^2-1) \, a \right]$$
$$+ \int e^{-arx} x^{p-1} a \frac{d}{dr} \{ R(r^2-1) \} \, dr = 0,$$

i.e. $$a \int e^{-arx} x^{p-1} R dr \left[\frac{d}{dr} \{ R(r^2-1) \} - 2p R r \right]$$
$$+ [e^{-arx} x^{p-1} R \, (r^2-1) \, a] = 0 \quad \ldots \quad (1)$$

Now R is found from the equation

$$\frac{d}{dr} \{ R(r^2-1) \} - 2p R k = 0$$

i.e. $$R = (r^2-1)^{p-1}.$$

The limits of integration should be so closen that the last term in (1) vanishes. The values of r are $\pm$ 1 and ∞.

$\therefore$ The primitive is

$$A x^p \int_{-1}^{1} \frac{e^{-arx}}{(1-r^2)^{1-p}} dr + B x^p \int_{1}^{\infty} \frac{e^{-arx}}{(r^2-1)^{1-p}} dr.$$

The first can be written

$$A_x p \left[\int_{-1}^{0} \frac{e^{-arx}}{(1-r^2)^{1-p}} dr + \int_{0}^{1} \frac{e^{-arx}}{(1-r^2)^{1-p}} dr \right].$$

Putting $-r$ for r in the first part, we get the required result.

QUESTIONS FOR SOLUTION.

1093. (Martyn M. THOMAS, M. A.) :—A fixed ray of light falls on a plane mirror revolving about an axis in its plane. Show that the locus, in the plane, of the point of incidence is a conic, and that the reflected rays generate a ruled conicoid.

1094. (S. R. RANGANATHAN) :—If the number of A's : the number of B's : the number of C's as $a : b : c$ and if a_1 per cent. and a_2 per cent. of the A's are also B's and C's respectively, discuss the limits between which the percentage of the B's that are also C's should lie.

1095. (R. S. NARASIMHAN) :—Seven thieves A, B, C, ... secure a sum of rupees, but are obliged to conceal it without counting it. A returns alone, divides the sum into 7 parts, finds that there are 6 rupees over, takes the 6 rupees and one-seventh part and departs. B returns alone and does the same with the diminished sum. He divides it into 7 parts, find that there are 5 rupees over, takes the 5 rupees and one-seventh part and departs. Each does the same the successive remainders being 6, 5, 4, 3, 2, 1, 0. Finally, they all come together and divide the remainder which is a multiple of 7, into 7 equal parts and each takes one part. Find the least number of rupees stolen and the amount that each gets.
