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ON PERIODIC INTEGRAL FUNCTIONS

BY

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[Received 30 July, 1940]

I

Introduction and general theorems

1. It is well known* that a periodic integral function $f(z)$ with period λ can be expressed in the form

$$f(z) = \sum_{n=-\infty}^{\infty} a_n \exp\left(\frac{2n\pi iz}{\lambda}\right), \qquad (1)$$

where

$$\chi(u) = \sum_{-\infty}^{\infty} a_n u^n \qquad (2)$$

is a function having, in general, two essential singularities at $u=0$ and $u=\infty$. In this paper I propose to give a representation of periodic integral functions in terms of composite integral functions†. I shall prove that a periodic integral function is the product of at most four composite integral functions. In Part II, I discuss the case when the periodic function has only a finite number of zeros in a periodic strip. I prove that the only functions of finite order having this property are functions of order one and finite type. I also find a necessary and sufficient condition that a function of the form

$$\sigma(z) = \prod_{1}^{\infty}\left(1 - \frac{z^2}{\lambda_n^2}\right) \qquad (3)$$

* E. Goursat, *Course of Analysis*, (Trans. by Hedrick and Dunkel), Vol. II. part 1, 145-7.

† A composite integral function is an integral function of an integral function.

should be periodic in terms of the properties of the sequence $\{\lambda_n\}$. In Part III, I consider the general case of functions of finite order and show that the auxiliary functions which appear in the general representation can be chosen to be functions of order zero.

2. We can suppose without loss of generality that the period in question is a positive number. The general theorem runs as follows:—

THEOREM I. *Let $f(z)$ be a periodic integral function with period $\lambda(>0)$. Let $\{z_n^+\}, \{z_n^0\}, \{z_n^-\}$ be the zeros of $f(z)$ in the strip $0 \leqslant x < \lambda, (z = x+iy)$ whose imaginary parts are positive, zero, and negative respectively. Let*

$$u_n^+ = \exp\left[-\frac{2\pi i z_n^+}{\lambda}\right], \quad u_n^0 = \exp\left[\frac{2\pi i z_n^0}{\lambda}\right], \quad u_n^- = \exp\left[\frac{2\pi i z_n^-}{\lambda}\right].$$

Let $g^+(u)$, $g^0(u)$, $g^-(u)$ be integral functions whose zeros are $\{u_n^+\}$, $\{u_n^0\}$, $\{u_n^-\}$ respectively. Then

$$f(z) = e^{h(z)} g^+ \exp\left(\frac{-2\pi i z}{\lambda}\right) g^0 \exp\left(\frac{2\pi i z}{\lambda}\right) g^- \exp\left(\frac{2\pi i z}{\lambda}\right), \quad (4)$$

where $h(z)$ is an integral function such that $h'(z)$ has the period λ. It is to be noted that the set $\{z_n^0\}$ is finite so that $g^0(u)$ can always be taken as a polynomial whose zeros are at $\{u_n^0\}$.

PROOF. If $\{z_n^+\}$ contains an infinity of terms, the imaginary part of $z_n^+ \to +\infty$ as $n \to \infty$ so that $|u_n^+| \to \infty$. Hence an integral function $g^+(u)$ always exists having its zeros at $\{u_n^+\}$. A similar statement can be made regarding the existence of $g^-(u)$. Regarding $g^0(u)$, there can at most be a finite number of zeros of $f(z)$ on the real axis in the strip $0 \leqslant x < \lambda$. Hence $\{u_n^0\}$ is a finite set so that $g^0(u)$ can be chosen to be a polynomial. Now the zeros of $g^+ \exp\left(\frac{-2\pi i z}{\lambda}\right)$ are the set $\{z_n^+ + k\lambda\}$, $k = 0, \pm 1, \pm 2, \ldots,$

$n=1, 2, \ldots$ and these are exactly the zeros* of $f(z)$ in the half-plane $y>0$. Similar statements hold for $g^0(u)$ and $g^+(u)$. Hence a relation of the form (4) holds where $h(z)$ is an integral function such that $\exp[h(z)]$ is periodic with period λ. Therefore, for each z there is an integer k such that

$$h(z+\lambda)-h(z) = 2k\pi i. \tag{5}$$

If (5) holds for a $z=z_0$ and a certain $k=k_0$, (5) holds with the same $k=k_0$ in the neighbourhood of $z=z_0$ and since $h(z)$ is an integral function, it follows that (5) holds for the same $k=k_0$ and for all z. Hence $h'(z+\lambda) = h'(z)$. This proves the theorem.

2.1. When the function $f(z)$ is an evèn or odd function, the representation given in (4) can be considerably simplified. We shall prove

THEOREM I A. *Let $f(z)$ be an even integral function with period λ. Let D be the portion of the z-plane defined by*

$$\left.\begin{array}{l} 0 \leqslant x < \lambda,\ y > 0 \\ 0 < x \leqslant \dfrac{\lambda}{2},\ y = 0. \end{array}\right\}$$

Let $\{z_n\}$ be the zeros of $f(z)$ in D and let it have a zero of order $2p$ at $z=0$. Let

$$u_n = \sin^2\frac{\pi z_n}{\lambda}$$

and let $g(u)$ be an integral function having $\{u_n\}$ for its zeros. Then

$$f(z) = e^{h(z)}\sin^{2p}\frac{\pi z}{\lambda}\, g\left(\sin^2\frac{\pi z}{\lambda}\right), \tag{6}$$

where $h(z)$ is an even integral function with period λ.

* Multiple zeros are counted according to their multiplicity. If z_0 is a zero of order k for $f(z)$, $z_0+m\lambda$ is also a zero of order k for any integer m.

PROOF. If $\{z_n\}$ is an infinite set, $|u_n| \to \infty$ and so an integral function $g(u)$ exists having $\{u_n\}$ for its zeros. Now the roots of the equation

$$\sin^2 \frac{\pi z_n}{\lambda} = \sin^2 \frac{\pi z}{\lambda}, \tag{7}$$

where z_n lies in the domain $0 \leqslant x < \lambda$, $y > 0$ consist of the set $\{\pm z_n + k\lambda\}$, $k = 0, \pm 1, \pm 2, \ldots$ If z_n lies in $0 < x < \lambda/2$, $y = 0$, $-z_n + \lambda$ lies in $\lambda/2 < x < \lambda$, $y = 0$. If $z_n = \lambda/2$ the roots of (7) are exactly the set $\{\pm\lambda/2 + k\lambda\}$. Hence the coefficient $e^{h(z)}$ in (6) has exactly the same zeros as $f(z)$, multiple zeros being counted according to their multiplicity. Hence an equation of the form (6) holds, $h(z)$ being an integral function. It is easily seen that $h(z)$ is even and so it has the period λ since $h'(z)$ has the period λ, as may be proved in the same manner as in § 2.

2.2. When $f(z)$ is odd and has period λ, we see that $z = 0$ and $z = \lambda/2$ are zeros. Also $f(z)/\sin\frac{2\pi z}{\lambda}$ is an even periodic function whose zeros are the same as those of $f(z)$ except the set $\{k\lambda\}$ and $\{\lambda/2 + k\lambda\}$, $k = 0, \pm 1, \pm 2, \ldots$. Hence we can state

THEOREM 1 B. *Let $f(z)$ be an odd function with period λ having a zero of order $2p+1$ at $z = 0$ and of order $q \geqslant 1$ at $z = \lambda/2$. Let D denote the portion of the z-plane as in Theorem 1 A. Let $\{z_n\}$ be the zeros of $f(z)$ in D, the zero at $z = \lambda/2$ being counted $q - 1 \geqslant 0$ times. Let*

$$u_n = \sin^2 \frac{\pi z_n}{\lambda}$$

and $g(u)$ an integral function with zeros at $\{u_n\}$. Then

$$f(z) = e^{h(z)} \sin \frac{2\pi z}{\lambda} \sin^{2p} \frac{\pi z}{\lambda}\, g\left(\sin^2 \frac{\pi z}{\lambda}\right), \tag{8}$$

where $h(z)$ is an even integral function with period λ.

II

3. In this part we confine ourselves to periodic functions having only a finite number of zeros in the periodic strip $0 \leqslant x < \lambda$. In this case the sets $\{z_n^+\}$, $\{z_n^0\}$ and $\{z_n^-\}$ are all finite so that the functions $g^+(u)$, $g^0(u)$, $g^-(u)$ in Theorem 1 can all be taken as polynomials having $\{u_n^+\}$, $\{u_n^0\}$, $\{u_n^-\}$ for their zeros. So we can state

THEOREM 2. *Let $f(z)$ be an integral function with period λ and having m, n and p zeros whose imaginary parts are positive, zero and negative respectively in the strip $0 \leqslant x < \lambda$. Then*

$$f(z) = e^{h(z)} g_m \exp\left(-\frac{2\pi i z}{\lambda}\right) g_n \exp\left(\frac{2\pi i z}{\lambda}\right) g_p \exp\left(\frac{2\pi i z}{\lambda}\right) \quad (9)$$

where $g_m(u)$, $g_n(u)$, $g_p(u)$ are polynomials of degrees m, n, p respectively.

3.1. When $f(z)$ is of finite order*, $e^{h(z)}$ in (9) is of finite order so that $h(z)$ is a polynomial. Since $h'(z)$ is periodic we must have $h'(z) = \alpha$, a constant. So $h(z) = \alpha z + \beta$ and $\alpha = \frac{2\pi i k}{\lambda}$ since $e^{h(z)}$ has period λ. Moreover it is evident that $g_m(u)$, $g_n(u)$, $g_p(u)$ can be so chosen as to have the value one at $u = 0$. So we can state

THEOREM 2A. *Let $f(z)$ be a function of finite order and period λ. Let it have m, n and p zeros in the strip $0 \leqslant x < \lambda$ whose imaginary parts are positive, zero and negative respectively. Then*

$$f(z) = A \exp\left(\frac{2k\pi i z}{\lambda}\right) g_m \exp\left(-\frac{2\pi i z}{\lambda}\right) g_n \exp\left(\frac{2\pi i z}{\lambda}\right) g_p \exp\left(\frac{2\pi i z}{\lambda}\right), \quad (10)$$

where k is an integer and $g_m(u)$, $g_n(u)$, $g_p(u)$ are polynomials of degrees m, n, p respectively and are such that

$$g_m(0) = g_n(0) = g_p(0) = 1, \quad (11)$$

A being a constant.

* For the definition of order, see III.

4. Next we shall find a necessary and sufficient condition that a periodic function of finite order has only a finite number of zeros in a periodic strip.

THEOREM 2 B. *Let $f(z)$ be a function of finite order and period λ. Let $M(r, f) = \max_{|z| \leqslant r} |f(z)|$. A necessary and sufficient condition that $f(z)$ has only a finite number of zeros in the strip $0 \leqslant x < \lambda$ is that*

$$\underline{\lim}_{r\to\infty} \frac{\log M(r,f)}{r} < \infty. \qquad (12)$$

PROOF. If $f(z)$ be of finite order with period λ and having only a finite number of zeros in $0 \leqslant x < \lambda$, the representation (10) holds so that we have

$$\overline{\lim}_{r\to\infty} \frac{\log M(r,f)}{r} < \infty$$

and, *a fortiori*, (12) holds. If (12) holds, we find by using Jensen's formula that

$$\underline{\lim}_{r\to\infty} \frac{n(r)}{r} < \infty, \qquad (13)$$

where $n(r)$ is the number of zeros of $f(z)$ in $|z| \leqslant r$. Hence there is a constant B and a sequence $r_1 < r_2 .. < r_n \to \infty$ such that

$$n(r_n) < Br_n. \qquad (14)$$

We can inscribe the square $x = \pm r_n/\sqrt{2}$, $y = \pm r_n/\sqrt{2}$ inside the circle $|z| = r_n$. If p_n be the number of zeros of $f(z)$ in the rectangle $x = 0$, $x = \lambda$, $y_n = \pm r_n/\sqrt{2}$ we see that the product of p_n and the integral part of $2r_n/\lambda$ cannot exceed $n(r_n)$, and hence by (14), Br_n. So p_n cannot exceed a finite number and since $r_n \to \infty$ the required result follows.

4.1. We can now get a more precise result for the increase of periodic functions satisfying (12). For such functions the representation (10) holds. We shall prove

THEOREM 2 C. *Let $f(z)$ be a function satisfying* (12) *and having a period* λ. *Let m, n, p be defined as in Theorem 2 A. Then there exists an integer k such that the following relations hold :*

$$\text{(i)} \lim_{r\to\infty} \frac{\log |f(re^{i\theta})|}{r} = (-k+m)\frac{2\pi}{\lambda}\sin\theta$$

for $0 < \theta < \pi$;

$$\text{(ii)} \lim_{r\to\infty} \frac{\log |f(re^{i\theta})|}{r} = -(k+n+p)\frac{2\pi}{\lambda}\sin\theta$$

for $\pi < \theta < 2\pi$;

$$\text{(iii)} \overline{\lim_{r\to\infty}} \frac{\log |f(\pm r)|}{r} = 0;$$

$$\text{(iv)} \lim_{r\to\infty} \frac{\log M(r,f)}{r} = \frac{2\pi}{\lambda} q,$$

where q is the greater of the two numbers $(m-k, p+n+k)$.

PROOF. It is known* that (12) along with the fact that $f(z)$ is bounded along the real axis is enough to ensure that

$$\overline{\lim_{r\to\infty}} \frac{\log M(r,f)}{r} < \infty,$$

so that $f(z)$ is of order one and finite type. Hence by Theorem 2 B, the conditions of Theorem 1 A are satisfied so that the representation (10) holds. Moreover if

$$u = e^{\frac{2\pi i z}{\lambda}},$$

we have $\log |u| = -\frac{2\pi}{\lambda} r\sin\theta$. Hence $\log |u| \to -\infty$ for $0 < \theta < \pi$ and $\to +\infty$ for $\pi < \theta < 2\pi$. These facts along with (11) and the property that

$$\lim_{|u|\to\infty} \frac{P(u)}{u^n}$$

exists and is not zero for a polynomial $P(u)$ of degree n give (i) and (ii). The relation (iv) follows from (i) and (ii) while (iii) is a consequence of the periodicity of $f(z)$.

* See, R. Nevanlinna, *Eindeutige Analytische Funktionen*, (1936), 43-4.

4.2. In Theorem 2 C, the numbers m, n, p are non-negative integers. If $m=n=p=0$, it follows that $k\neq 0$ unless $f(z)$ is a constant. If m, n, p are all not zero, $m+n+p$ is a positive integer. So $m-k$, $n+p+k$ cannot be both negative or both zero or one negative and the other zero, since their sum is $m+n+p$. Hence in this case one of them at least is a positive integer. Hence in all cases one at least of $m-k, n+p+k$ is a positive integer unless $f(z)$ is a constant. This leads to the following known result.*

THEOREM 2 D. *If $f(z)$ be a function with period λ satisfying the condition*

$$\varlimsup_{r\to\infty}\frac{\log M(r,f)}{r}<\frac{2\pi}{\lambda},$$

then $f(z)$ must be a constant.

PROOF. Obviously (12) holds and if $f(z)$ is not a constant we get from Theorem 2 C, that

$$\varlimsup_{r\to\infty}\frac{\log M(r,f)}{r}=\lim_{r\to\infty}\frac{\log M(r,f)}{r}\geqslant\frac{2\pi}{\lambda},$$

which contradicts the hypothesis.

5. Let

$$0<\lambda_1<\lambda_2\ldots<\lambda_n<\ldots \tag{15}$$

be a sequence tending to ∞ such that $\Sigma\lambda_n^{-2}$ converges. Then it is known that

$$\sigma(z)=\prod_1^\infty\left(1-\frac{z^2}{\lambda_n^2}\right) \tag{16}$$

is an integral function of order two at most. We now propose to find the condition that (16) should be a periodic function. We shall say that two infinite arithmetic progressions (A.P's) $\{a+nd\}$ and $\{b+nd\}$, $n=0$, ± 1, ± 2, . . ., with the same common difference d are complementary when one consists of the negatives of the

* J. M. Whittaker, *Interpolatory Function Theory*, Camb. Tracts, 33 (1935), 86, L_{602}.

terms in the other. If two complementary A.P's are identical, each is called a self-complementary A.P. We can now prove the following

THEOREM 2 E. *A necessary and sufficient condition that* (16) *should be a periodic function with period* λ *is that the sequence* $\{\pm\lambda_n\}$ *must be capable of being split up into an even number of distinct A. P's which can be grouped into complementary pairs, the common difference for each A. P. being* λ.

PROOF. $\sigma(z)$ is an even function having no zero at the origin and it is of order two at most. If it has the period λ, the number of zeros in the strip $0 \leqslant x < \lambda$ are all real and so finite in number. Let $\lambda_1, \lambda_2, .., \lambda_p$ be the zeros in $0 < x \leqslant \lambda/2$. Then a representation of the form (6) holds where $h(z)$, being an even and periodic polynomial, must be a constant. Hence we have, by Theorem 1 A

$$\sigma(z) = \prod_{n=1}^{p} \left\{ 1 - \frac{\sin^2 \frac{\pi z}{\lambda}}{\sin^2 \frac{\pi\lambda_n}{\lambda}} \right\}, \tag{17}$$

from which we easily see that $\{\pm\lambda_n\}$ can be split up into an even number of distinct A.P's which can be grouped into complementary pairs. In fact, the zeros of each factor of (17) give a complementary pair. So the condition is necessary. To prove that it is sufficient we choose from one of the sequences of each complementary pair one number. Thus we obtain a finite set of numbers $\lambda_1, \lambda_2, \ldots, \lambda_p$. Let

$$H(z) = \prod_{n=1}^{p} \left\{ 1 - \frac{\sin^2 \frac{\pi z}{\lambda}}{\sin^2 \frac{\pi\lambda_n}{\lambda}} \right\}.$$

Then it is easily seen that $H(z)$ has the same zeros as $\sigma(z)$ and has the period λ. So

$$\sigma(z) = e^{h(z)} H(z),$$

where $h(z)$ is a polynomial of degree one at most, since the exponent of convergence of $\{\pm\lambda_n\}$ is one so that $\sigma(z)$ is of order one. Since $h(z)$ is also even and $\sigma(0) = H(0) = 1$, we get $e^{h(z)} \equiv 1$. Hence $\sigma(z) = H(z)$ and so has period λ.

5.1. Similarly we can prove by using Theorem 1 B

THEOREM 2 F. *Let $\{\lambda_n\}$ be as in § 5. Let*

$$\sigma(z) = z \prod_{1}^{\infty} \left(1 - \frac{z^2}{\lambda_n^2}\right). \tag{18}$$

A necessary and sufficient condition that $\sigma(z)$ should be periodic with period λ is that $\{\pm\lambda_n\}$ must be capable of being split up into an even number of distinct A. P's with common difference λ, two of which are self-complementary while the remaining can be grouped into complementary pairs.

III

6. In this section we consider the general case of functions of finite order. Let $f(z)$ be an integral function. The order ρ is defined by

$$\overline{\lim_{r\to\infty}} \frac{\log \log M(r, f)}{\log r} = \rho.$$

If $0 \leqslant \rho < \infty$, the function is said to be of finite order. If $\rho > 0$, the type of $f(z)$ is defined by

$$\overline{\lim_{r\to\infty}} \frac{\log M(r, f)}{r^\rho} = d.$$

If $\rho = 0$, the function is said to be of order zero. In this case we require a finer distinction. We shall define the logarithmic order (l-order) of a function of order zero and the l-type by the limits

$$\left.\begin{aligned} \overline{\lim_{r\to\infty}} \frac{\log \log M(r, f)}{\log \log r} &= \rho \\ \overline{\lim_{r\to\infty}} \frac{\log M(r, f)}{(\log r)^\rho} &= d \end{aligned}\right\}. \tag{19}$$

It is easily seen that the l-order $\rho \geqslant 1$ unless $f(z)$ is a constant. For a polynomial the l-order is one, though the converse is not true. But it is necessary and sufficient for $f(z)$ to be a polynomial that $f(z)$ should be of l-order one and finite type. We shall define $m(r, f)$ by the relation

$$m(r, f) = \min_{|z|=r} |f(z)|.$$

6.1. We shall show that when $f(z)$ is of finite order and is periodic, the functions $g^{+}(u)$ and $g^{-}(u)$ in (4) can be taken as functions of order zero and of finite logarithmic order. For this purpose we require some preliminary lemmas. A function $\phi(u)$ of order zero with $\phi(0) = 1$ can be written in the form

$$\phi(u) = \prod_{n=1}^{\infty} \left(1 - \frac{u}{u_n}\right). \tag{20}$$

Let $|u_n| = r_n$. We consider along with $\phi(u)$ the function

$$\phi_1(u) = \prod_{n=1}^{\infty} \left(1 + \frac{u}{r_n}\right). \tag{21}$$

We shall refer to the exponent of convergence of any sequence $\{z_n\}$ with $|z_n| \to \infty$ as its order. The order of $\{u_n\}$ in (20) is zero. We first quote the following known lemma*.

LEMMA 1. *For the function $\phi(u)$ of order zero given by (20), we have the inequalities*

$$\int_0^r \frac{n(t)}{t} dt \leqslant \log M(r, \phi) \leqslant \int_0^r \frac{n(t)}{t} dt + r \int_r^{\infty} \frac{n(t)}{t^2} dt, \tag{22}$$

where $n(r)$ is the number of u_n's in $|u| \leqslant r$.

6.2. LEMMA 2. *Let $\phi(u)$ defined by (20) be such that there is a $\lambda > 0$ for which*

$$\varlimsup_{r \to \infty} \frac{\log M(r, \phi)}{(\log r)^{\lambda}} \geqslant d > 0.$$

Then

$$\varlimsup_{r \to \infty} \frac{\log m(r, \phi)}{(\log r)^{\lambda}} \geqslant d.$$

* G. Valiron, *Lectures on Integral Functions*, (1923), 132.

PROOF. Let $0 < \delta < d$. Consider the function

$$H(u) = \phi_1(u) e^{-(d-\delta)} (\log u)^\lambda.$$

The function $H(u)$ is of order zero in the region

$$-\pi \leqslant \text{amp } (u) \leqslant \pi, \ |u| \geqslant r_0 > 0, \tag{22}$$

and so by a well-known theorem* if $H(u)$ is bounded on the boundary of (22), it is bounded in the whole region (22). Then we must have

$$\log \phi_1(r) \leqslant (d-\delta)(\log r)^\lambda + O(1), \tag{23}$$

as $r \to \infty$. But

$$\phi_1(r) \geqslant |\phi(u)|$$

on $|u| = r$ so that

$$\log \phi_1(r) = \log M(r, \phi_1) \geqslant \log M(r, \phi),$$

from which we conclude that (23) contradicts the hypothesis. Hence $H(u)$ cannot be bounded on the boundary of (22) so that we conclude that

$$\overline{\lim_{r\to\infty}} \frac{\log m(r, \phi)}{(\log r)^\lambda} \geqslant \overline{\lim_{r\to\infty}} \frac{\log |\phi_1(-r)|}{(\log r)^\lambda} \geqslant d - 2\delta,$$

and as δ can be chosen arbitrarily small, the lemma follows.

6.3. LEMMA 3. *A necessary and sufficient condition that*†

$$f(z) = \phi(e^{az}), \ a \neq 0$$

should be of order ρ *is that* $\phi(u)$ *is of l-order* ρ.

PROOF. Let σ be the l-order of $\phi(u)$. Then it is easily seen that $\rho \leqslant \sigma$. Let $0 < \alpha < \sigma$. Then, by definition

$$\overline{\lim_{r\to\infty}} \frac{\log M(r, \phi)}{(\log r)^\alpha} = \infty > 1 > 0.$$

Hence by Lemma 2,

$$\overline{\lim} \frac{\log m(r, \phi)}{(\log r)^\alpha} \geqslant 1. \tag{24}$$

Let $z = x + iy$ so that $|u| = r = |e^{az}| = e^{ax}$, if $a > 0$ as may be supposed without loss of generality. Hence

* G. Valiron, l.c. p. 125.

† It is to be noted that $\rho \geqslant 1$, otherwise such a representation is not possible.

$$M\left(\frac{\log r}{a}, f\right) \geqslant |f(x)| = |\phi(r)| \geqslant m(r, \phi)$$

which, in conjunction with (24), gives

$$\varliminf_{r\to a} \frac{\log M\left(\frac{\log r}{a}, f\right)}{(\log r)^{\alpha}} \geqslant 1$$

so that $\rho \geqslant \alpha$, and since α can be chosen as near to σ as we please, we get $\rho \geqslant \sigma$. Hence $\rho = \sigma$.

6.4. By a similar argument we can prove

LEMMA 4. *A necessary and sufficient condition that*

$$f(z) = \phi(e^{az})$$

should be of order ρ *and type* d *is that* $\phi(u)$ *is of l-order* ρ *and l-type* $d|a|^{-\rho}$ *where* d *is supposed to be positive.*

7. The next lemma connects the order of increase of the function $n(r)$ of the zeros of $\phi(u)$ with its l-order.

LEMMA 5. *Let* $n(r)$ *be the number of zeros of* $\phi(u)$ *defined by* (20) *in* $|u| \leqslant r$. *Let*

$$\sigma = \varlimsup_{r\to\infty} \frac{\log n(r)}{\log\log r}.$$

Then the l-order ρ *of* $\phi(u)$ *is equal to* $1+\sigma$.

PROOF. By Lemma 1

$$\int_0^{r^\alpha} \frac{n(t)}{t}\, dt \leqslant \log M(r^\alpha, \phi).$$

Let $\alpha > 1$. Then the above inequality gives, by the definition of l-order,

$$n(r)(\alpha-1)\log r = O[(\log r^\alpha)^{\rho+\varepsilon}]$$

for every $\varepsilon > 0$. Hence $\sigma \leqslant \rho - 1$. Next, by the last inequality in Lemma 1, we get

$$\log M(r, \phi) \leqslant \int_0^r \frac{n(t)}{t}\, dt + r\int_r^\infty \frac{n(t)}{t^2}\, dt. \qquad (24)$$

By definition

$$n(r) = O[\log r)^{\sigma+\varepsilon}]$$

for every $\varepsilon > 0$. Using this in (24) we easily see that $\rho \leqslant \sigma + 1$ by noting that

$$r\int_r^{\infty}\frac{n(t)}{t^2}\,dt=\int_1^{\infty}\frac{n(rv)}{v^2}\,dv=O\Big[\int_1^{\infty}\frac{(\log r+\log v)^{\sigma+\varepsilon}}{v^2}\,dv\Big]$$
$$=O[(\log r)^{\sigma+\varepsilon}].$$

Hence $\rho=\sigma+1$.

7.1. LEMMA 6. *Let* $\{z_n\}$, $z_n=x_n+iy_n$, $n=0,1,2,..$ *be a sequence in the strip* $0\leqslant x<\lambda$ *such that* $|z_n|\to\infty$ *as* $n\to\infty$. *Then the exponent of convergence* σ *of* $\{z_n\}$ *and the exponent of convergence* ρ *of the set* $\{z_n+m\lambda\}$, $m=0,\pm1,\pm2,..$ *are related by the equation* $\rho=1+\sigma$.

PROOF. It is evident that $\{z_n+m\lambda\}$ and $\{iy_n+m\lambda\}$ have the same exponent of convergence since $0\leqslant x_n\leqslant\lambda$. It is evident that $\rho\geqslant1$. If $p>1$, we have

$$\sum_{m=0}^{\infty}\frac{1}{|m\lambda+iy_n|^p}=\sum_{m=0}^{\infty}\frac{1}{(y_n^2+m^2\lambda^2)^{p/2}}$$
$$=O\left(\frac{1}{|y_n|^p}\right)+\int_0^{\infty}\frac{dt}{(y_n^2+t^2\lambda^2)^{p/2}}$$
$$=O\left(\frac{1}{|y_n|^p}\right)+\frac{1}{\lambda|y_n|^{p-1}}\int_0^{\infty}\frac{dv}{(1+v^2)^{p/2}}. \tag{25}$$

Hence if $\sum\frac{1}{|y_n|^{p-1}}$ converges, so does the double series

$$\sum_{(n)}\sum_{(m)}\frac{1}{|iy_n+m\lambda|^p}. \tag{26}$$

Hence we conclude that $\rho\leqslant\sigma+1$. Similarly if (26) converges, so does the series on the left side of (25), and the relation (25) holds. So $\sum\frac{1}{|y_n|^{p-1}}$ converges. Hence $\sigma\leqslant\rho-1$ or $\rho\geqslant\sigma+1$. Therefore $\rho=\sigma+1$.

7.2. LEMMA 7. *Let* $z_n=x_n+iy_n$, $y_n>0$ *be a sequence of order* ρ *in the strip* $0\leqslant x_n<\lambda$. *Let*

$$u_n=\exp\left(-\frac{2\pi iz_n}{\lambda}\right)$$

and $n(r)$ *be the number of* u_n*'s in* $|u|\leqslant r$. *Then*

$$\rho = \overline{\lim_{r\to\infty}} \frac{\log n(r)}{\log \log r}.$$

PROOF. We have

$$|u_n| = \exp\left(\frac{2\pi y_n}{\lambda}\right)$$

so that

$$y_n = \frac{\lambda}{2\pi} \log |u_n|$$

from which we conclude that

$$N\left[\frac{\lambda}{2\pi}\log r+\lambda\right] \geqslant n(r) \geqslant N\left[\frac{\lambda}{2\pi}\log r-\lambda\right], \tag{27}$$

where $N(t)$ is the number of $\{z_n\}$ in $|z| \leqslant t$. The relation (27) gives the required result, since it is known that*

$$\overline{\lim_{r\to\infty}} \frac{\log N(r)}{\log (r)} = \rho.$$

8. We can now prove that if $f(z)$ is of finite order ρ and has period λ, then $g^+(u)$ and $g^-(u)$ in (4) can be chosen so as to be of order zero and that, then, $h(z) = \exp(2k\pi iz/\lambda)$ where k is an integer. First we note that $\rho \geqslant 1$ and if the number of zeros of $f(z)$ in a periodic strip be more than finite then, by Theorems 2B and 2D

$$\lim_{r\to\infty} \frac{\log M(r,f)}{r} = \infty. \tag{28}$$

Since we have considered the case of a finite number of zeros in section II, we shall suppose that the number of zeros in a periodic strip is infinite so that $\rho \geqslant 1$ and (28) holds. We shall prove

THEOREM 3. *Let $f(z)$ be of order ρ and period λ. Then the functions $g^+(u)$ and $g^-(u)$ in (4) can be chosen so as to be of order zero and l-order ρ at the most. One at least of $g^+(u)$, $g^-(u)$ will have the precise l-order ρ. The function $h(z) = \frac{2k\pi iz}{\lambda} + B$,*

* R. Nevanlinna, l. c., 207-8.

where B is a constant and k an integer. The explict expressions for $g^+(u)$ *and* $g^-(u)$ *are*

$$g^+(u)=\prod_1^\infty\left(1-\frac{u}{u_n^+}\right) \text{ and } g^-(u)=\prod_1^\infty\left(1-\frac{u}{u_n^-}\right). \quad (29)$$

The function $g^0(u)$ *can always be chosen as a polynomial as already stated in Theorem* 1.

PROOF. The order of the zeros of $f(z)$ cannot exceed ρ. Let it be σ. Then the order of the zeros of $f(z)$ in the strip $0 \leqslant x < \lambda$ is $\sigma - 1$ in virtue of Lemma 6. Then by using Lemmas 7 and 5 we find that the l-order of $g^+(u)$ and $g^-(u)$ defined by 29 cannot exceed σ and one at least is of l-order σ. Hence by Lemma 4 the function

$$H(z) = g^+\exp\left(-\frac{2\pi iz}{\lambda}\right)g^0\exp\left(\frac{2\pi iz}{\lambda}\right)g^-\exp\left(\frac{2\pi iz}{\lambda}\right)$$

is of order σ. Hence

$$f(z)=e^{h(z)}H(z),$$

where $h(z)$, being of finite order, must be a polynomial. Since $h'(z)$ is periodic we have $h(z)=\frac{2k\pi iz}{\lambda}+B$. Now if $\sigma < \rho$, $f(z)$ must be of order one and moreover

$$\overline{\lim_{r\to\infty}}\frac{\log M(r, f)}{r} < \infty$$

so that if (28) holds we must have $\sigma=\rho$. This proves the theorem.

8.1. It is well-known* that if $\phi(u)$ is a function of order zero then

$$\overline{\lim_{r\to\infty}}\frac{\log m(r, \phi)}{\log M(r, \phi)} = 1.$$

Using this in conjunction with Theorem 3 and noting that if $y \to \infty$, $u = \exp(2\pi iz/\lambda) \to 0$ and if $y \to -\infty$, $|u| \to \infty$, we easily see the truth of the following

* G. Valiron, l.c., 136.

THEOREM 4. *Let $f(z)$ be a periodic function of finite order satisfying* (28). *Let*

$$m(\xi) = \min |f(z)|, \quad M(\xi) = \max |f(z)|$$

as z varies on the line $y = \xi$ where $z = x + iy$. Then

$$\overline{\lim_{|y|\to\infty}} \frac{\log m(y)}{\log M(y)} = 1.$$

8.2. We conclude with a theorem to show that the assumption of periodicity in a function of finite order imposes very heavy restrictions. It is well-known that a function of order one that is bounded along two intersecting lines must be a constant. For periodic functions, any line parallel to the direction of the period is a line of boundedness. It can now be shown that a periodic function of any finite order cannot be bounded along any other line intersecting the direction of periodicity. This follows from Theorem 4 and the periodic property since $\log M(y) \to \infty$ as $|y| \to \infty$. We can state this result in the following form.

THEOREM 4A. *Let $f(z)$ be a function of finite order and period $\lambda > 0$. Then $f(z)$ cannot be bounded along any whole line intersecting the real axis without reducing to a constant.*

ON A FEW RECURRENTS

BY

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1. In this paper are given three recurrents of which the first one, viz. (5) has been derived from two earlier Theorems (1) and (2); the second, viz. (16), which is the main result in this paper is the general case of (5), obtained by generalization. In proving (16), Theorems (6), (9) and (11) have been obtained. The third recurrent (18) has been deduced from Theorem (11), the recurrence-formula (19) being the same as (11), expressed in a suitable form.

Some notations used in earlier papers* are also used here, viz.

(i) $\begin{bmatrix} n \\ c \end{bmatrix} = (a^n - 1)\ (a^{n-1} - 1)\ (a^{n-2} - 1) \ldots (a^c - 1),$

(ii) $\begin{bmatrix} n \\ c \end{bmatrix}_2 = (a^n - 1)\ (a^{n-2} - 1)\ (a^{n-4} - 1) \ldots (a^c - 1)$

(iii) ${}^nS_x =$ sum of the products of n factors $1, a, a^2, \ldots, a^{n-1}$ taken x at a time.

For the sake of references, four theorems published in earlier communications* are given below:—

$$\sum_{x=0}^{k} (-)^x \begin{bmatrix} 2k - 1 - 2x \\ 1 \end{bmatrix}_2 {}^{2k}S_{2x} = (-)^k \tag{1}$$

$$\sum_{x=0}^{k} (-)^x \begin{bmatrix} 2k - 1 - 2x \\ 1 \end{bmatrix}_2 {}^{2k+1}S_{1+2x} = (-)^k \tag{2}$$

$$\frac{\begin{bmatrix} n \\ 1 \end{bmatrix}}{\begin{bmatrix} x \\ 1 \end{bmatrix}\begin{bmatrix} n-x \\ 1 \end{bmatrix}} {}^xS_x = {}^nS_x \tag{3}$$

* Chakrabarti, S. C., On a few Algebraic Identities, *Bull. Calcutta Math. Soc.*, 27 (1935), 37-44.

and $$\sum_{x=0}^{n} (-)^x u_x \, {}^nS_x \tag{4}$$

is the first element of the nth order of differences obtained from the series

$$u_0, u_1, u_2, u_3, \ldots,$$

by using $1, a, a^2, a^3, \ldots,$ *as the successive multipliers.*

1.1. THEOREM. *Denote*

$$A_6 = \begin{vmatrix} {}^7S_1 & {}^7S_2 \ldots {}^7S_5 & a\,{}^6S_1 + (-)^6 \\ 1 & {}^6S_1 \ldots {}^6S_4 & a\,{}^5S_1 \\ \ldots & \ldots & \ldots \\ & 1 \; {}^3S_1 & a\,{}^2S_1 \\ & 1 & a \end{vmatrix}_6$$

Then the general recurrent of the same type as A_6 *is*

$$A_n = (-)^{\frac{n-2}{2}} \begin{bmatrix} n+1 \\ 3 \end{bmatrix}_2 \text{ or } (-)^{\frac{n-1}{2}} \begin{bmatrix} n \\ 1 \end{bmatrix}_2 \tag{5}$$

according as n is even or odd.

The recurrence formula in this case is

$$\sum_{x=0}^{n-1} (-)^x (A_{n-x} + 1)\, {}^{n+1}S_x = (-)^{n-1} \, {}^nS_1 a.$$

[Let us consider the particular case of this formula when $n=6$, then the left side becomes

$$\sum_{x=0}^{5} (-)^x A_{6-x} \, {}^7S_x - {}^7S_6 + {}^7S_7, \text{ for } \sum_{x=0}^{7} (-)^x \, {}^7S_x = 0$$

$$= \frac{a^7-1}{a-1} \sum_{x=0}^{3} (-)^x \begin{bmatrix} 5-2x \\ 1 \end{bmatrix}_2 {}^6S_{2x} - \sum_{x=0}^{3} (-)^x \begin{bmatrix} 5-2x \\ 1 \end{bmatrix}_2 {}^7S_{1+2x}$$

$$= -\frac{a^7-1}{a-1} + 1, \text{ by (1) and (2)}$$

$$= -{}^6S_1 a.$$

The general case may be similarly proved.]

2. THEOREM.

$$\sum_{x=0}^{n-1} (-)^x \frac{a^x}{{}^xS_x}\, {}^{n-1}S_x = (-)^{\frac{n-1}{2}} \left[\begin{matrix} n-2 \\ 1 \end{matrix}\right]_2 \text{ or } (-)^{\frac{n}{2}} \left[\begin{matrix} n-1 \\ 1 \end{matrix}\right]_2 \qquad (6)$$

according as n is odd or even.

PROOF. First we establish the following two relations:—

If M_n denotes the left side of (6), then

(i) $M_n - M_{n-1} = 0$, if n is odd (7)

and (ii) $M_n + (a^{n-1} - 1) M_{n-1} = 0$, if n is even. (8)

Write out M_n and M_{n-1} in full, then in the first case by (3), we have

$$M_n - M_{n-1} = -a^{n-1}\left\{1 - \frac{a^{n-2}-1}{a-1} + \ldots + \frac{a^{n-2}-1}{a-1} - 1\right\},$$

where in the expression on the right side the terms equidistant from the beginning and the end are equal in magnitude but opposite in sign. As there are an even number of terms in the expression, it vanishes.

In the second case

$$M_n - M_{n-1} = -a^{n-1}\left\{1 - \frac{a^{n-2}-1}{a-1} + \ldots - \frac{a^{n-2}-1}{a-1} + 1\right\}$$

$$= -a^{n-1}\phi \text{ (say)}$$

and

$$M_{n-1} - \phi = -(a^{n-2}-1)\left\{1 - \frac{a^{n-3}-1}{a-1} + \ldots + \frac{a^{n-3}-1}{a-1} - 1\right\}.$$

In the last equation, the expression on the right side vanishes for the same reason as in the first case. So the relation (8) is established.

It is evident that the relations (7) and (8) are also true if M_n denotes the right side of (6). So with the help of (7) and (8), Theorem (6) may be proved by induction.

3. Let

$$(r, p)_n = 1 + \frac{a^r - 1}{a^2 - 1} a^p + \frac{(a^r - 1)(a^{r+2} - 1)}{(a^2 - 1)(a^4 - 1)} a^{2p} + \frac{(a^r - 1)(a^{r+2} - 1)(a^{r+4} - 1)}{(a^2 - 1)(a^4 - 1)(a^6 - 1)} a^{3p} + \ldots n \text{ terms};$$

$(r, p)_n = 1$, if n is 1;

$(r, p)_n = 0$, if n is zero or negative.

Then we have

THEOREM.

$$\sum_{x=0}^{n-1} (-)^x \frac{R_{n-x}}{\begin{bmatrix} n+r-x \\ n-x \end{bmatrix}} {}^{n-1}S_x = (-)^{n-1} \frac{{}^{r+1}S_{r+1} - {}^{n}S_n}{(a^{n+r} - 1)\begin{bmatrix} r \\ 1 \end{bmatrix}}, \qquad (9)$$

where

$$\left.\begin{aligned}
R_n &= (-)^{\frac{n-1}{2}} \begin{bmatrix} n+r-1 \\ r \end{bmatrix}_2 (n+1, r)_{\frac{r+1}{2}}, \text{ if } n \text{ and } r \text{ are both odd;} \\
R_n &= (-)^{\frac{n-1}{2}} \begin{bmatrix} n+r \\ r+1 \end{bmatrix}_2 (n+1, r+1)_{\frac{r}{2}}, \text{ if } n \text{ is odd and } r \text{ even;} \\
R_n &= (-)^{\frac{n-2}{2}} \begin{bmatrix} n+r \\ r+2 \end{bmatrix}_2 (n, r+2)_{\frac{r+1}{2}}, \text{ if } n \text{ is even and } r \text{ odd:} \\
R_n &= (-)^{\frac{n-2}{2}} \begin{bmatrix} n+r-1 \\ r+1 \end{bmatrix}_2 (n, r+1)_{\frac{r+2}{2}}, \text{ if } n \text{ and } r \text{ are both even.}
\end{aligned}\right\} \qquad (10)$$

PROOF. First we establish the following

LEMMA.

$$\sum_{x=0}^{n-1} \frac{(a^{n+r-x} - 1)a^x}{{}^{x}S_x} k_{n-x} \; {}^{n-1}S_x$$

$$= (-)^{\frac{n-1}{2}} \frac{\begin{bmatrix} n-2 \\ 1 \end{bmatrix}_2}{\begin{bmatrix} r \\ 1 \end{bmatrix}} {}^{r+1}S_{r+1} \text{ or } (-)^{\frac{n-2}{2}} \frac{\begin{bmatrix} n-1 \\ 1 \end{bmatrix}_2}{\begin{bmatrix} r \\ 1 \end{bmatrix}} {}^{r+1}S_{r+1} \qquad (11)$$

according as n is odd or even, k_n being the left or right side of (9), $n > 1$.

(i) To establish (11), when k_n denotes the left side of (9), write out R_n and R_{n-1} in full when n and r both are odd. Then in the first case of (10), we have

$$R_n - R_{n-1} = (-)^{\frac{n-1}{2}} \begin{bmatrix} n+r-1 \\ r+2 \end{bmatrix} \left\{ t_1 + t_2 + t_3 + \ldots \frac{r+1}{2} \text{ terms} \right\},$$

where

$$t_1 = a^r,\ t_2 = \frac{a^{n+1}-1}{a^2-1} a^{2r} - a^r,$$

$$t_3 = \frac{(a^{n+1}-1)(a^{n+3}-1)}{(a^2-1)(a^4-1)} a^{3r} - \frac{a^{n+1}-1}{a^2-1} a^{2r},$$

and so on.

Hence adding all the t's we have

$$R_n - R_{n-1} = (-)^{\frac{n-1}{2}} \begin{bmatrix} n+r-1 \\ r+2 \end{bmatrix}_2 \frac{\begin{bmatrix} n+r-2 \\ n+1 \end{bmatrix}_2}{\begin{bmatrix} r-1 \\ 2 \end{bmatrix}_2} a^{\frac{1}{2}r(r+1)}$$

$$= (-)^{\frac{n-1}{2}} \begin{bmatrix} n-2 \\ 1 \end{bmatrix}_2 a^r\, {}^{n+r-1}S_r. \qquad (12)$$

In like manner in each of the other three cases of (10), a result either the same as or similar to that of (12) may be obtained. Then from these four results equivalent to $R_n - R_{n-1}$, we have

$$R_n - R_{n-1} = (-)^{\frac{n-1}{2}} \begin{bmatrix} n-2 \\ 1 \end{bmatrix}_2 a^r\, {}^{n+r-1}S_r$$

$$\text{or } (-)^{\frac{n-2}{2}} \begin{bmatrix} n-1 \\ 1 \end{bmatrix}_2 a^r\, {}^{n+r-1}S_r \qquad (13)$$

according as n is odd or even.

Again write out k_n and k_{n-1} in full, multiply k_{n-1} by ${}^{n-1}S_1/{}^1S_1$ and add the product to k_n. In the result so obtained the term containing R_{n-1} will disappear. Then to this result if the product $\frac{{}^{n-1}S_2}{{}^2S_2} k_{n-2}$ be added, the term containing R_{n-2} will also be removed. Proceeding in this manner we shall arrive at the relation

$$\sum_{x=0}^{n-1} \frac{k_{n-x}}{{}^{x}S_{x}} {}^{n-1}S_{x} = \frac{R_n}{\begin{bmatrix} n+r \\ n \end{bmatrix}}. \tag{14}$$

Now write out R_n and R_{n-1} as obtained from (14) in full, and then we have

$$R_n - R_{n-1} = \begin{bmatrix} n+r-1 \\ n \end{bmatrix} \sum_{x=0}^{n-1} \frac{(a^{n+r-x}-1)a^x}{{}^{x}S_{x}} k_{n-x} {}^{n-1}S_{x}. \tag{15}$$

So from (13) and (15), the relation (11) is obtained when k_n denotes the left side of (9).

(ii) Now suppose k_n denotes the right side of (9). Then substituting in the left side of (11), the value of k_{n-x} as obtained from the right side of (9), we have

$$\sum_{x=0}^{n-1} \frac{(a^{n+r-x}-1)a^x k_{n-x}}{{}^{x}S_{x}} {}^{n-1}S_{x}$$

$$= (-)^{n-1} \frac{{}^{r+1}S_{r+1}}{\begin{bmatrix} r \\ 1 \end{bmatrix}} \sum_{x=0}^{n-1} (-)^x \frac{a^x}{{}^{x}S_{x}} {}^{n-1}S_{x}$$

$$+ (-)^n \frac{{}^{n}S_{n}}{\begin{bmatrix} r \\ 1 \end{bmatrix}} \sum_{x=0}^{n-1} (-)^x a^{(2-n)x} {}^{n-1}S_{x}.$$

In the last equation the second summation vanishes, because it is the first element of the $n-1$th order of differences obtained from the series

$$1,\ a^{2-n},\ a^{2(2-n)},\ a^{3(2-n)},\ \ldots$$

by using $1, a, a^2, a^3, \ldots$ as the successive multipliers. The first summation is the left side of (6). So the relation (11) is also true if k_n denotes the right side of (9). So the relation (11) between $k_n, k_{n-1}, \ldots, k_1$ is fully established in both cases when k_n denotes the right and the left side of (9).

Now it is evident that the relation (9) holds good in the case of k_1, i.e. when $n=1$. Then from the relation between k_1 and k_2 obtained from (11), we can show that (9) holds good in the case of k_2. Similarly it can be shown that (9) holds good in the case of k_3, k_4 etc. Thus with the help of (11), Theorem (9) may be easily proved by induction.

4. Denote

$$C_2 = \begin{vmatrix} {}^{r+2}S_1 & {}^{r+1}S_r a^r + (-)^2 \\ 1 & {}^{r}S_r a^r \end{vmatrix}_2,$$

$$C_3 = \begin{vmatrix} {}^{r+3}S_1 & {}^{r+3}S_2 & {}^{r+2}S_r a^r + (-)^3 \\ 1 & {}^{r+2}S_1 & {}^{r+1}S_r a^r \\ & 1 & {}^{r}S_r a^r \end{vmatrix}_3.$$

Recurrents C_4, C_5 etc., may be similarly expressed. Then we have

THEOREM. $C_n = R_n$. (16)

In this case the recurrence formula is

$$\sum_{x=0}^{n-1} (-)^x (R_{n-x}+1)\ {}^{r+n}S_x = (-)^{n-1}\ {}^{r+n-1}S_r a^r. \tag{17}$$

[The left side of (17)

$$= \sum_{x=0}^{n-1} (-)^x R_{n-x}\ {}^{r+n}S_x + (-)^{n-1}\ {}^{r+n-1}S_{n-1} a^{n-1}$$

for $\sum_{x=0}^{k-1} (-)^x\ {}^{n}S_x = (-)^{k-1}\ {}^{n-1}S_{k-1} a^{k-1}$

$$= \frac{\begin{bmatrix} r+n \\ 1 \end{bmatrix}}{\begin{bmatrix} n-1 \\ 1 \end{bmatrix}} \sum_{x=0}^{n-1} (-)^x \frac{R_{n-x}}{\begin{bmatrix} n+r-x \\ n-x \end{bmatrix}}\ {}^{n-1}S_x + (-)^{n-1}\ {}^{r+n-1}S_{n-1} a^{n-1},$$

$= (-)^{n-1}\ {}^{r+n-1}S_r a^r$, by (9).

Thus the formula (17) is proved.]

As an illustrative example of the last recurrent when $r=7$ and $n=4$, we have

$$\begin{vmatrix} {}^{11}S_1 & {}^{11}S_2 & {}^{11}S_3 & {}^{10}S_7a^7+(-)^4 \\ 1 & {}^{10}S_1 & {}^{10}S_2 & {}^{9}S_7a^7 \\ & 1 & {}^{9}S_1 & {}^{8}S_7a^7 \\ & & 1 & {}^{7}S_7a^7 \end{vmatrix}_4 = -\begin{bmatrix} 11 \\ 9 \end{bmatrix}_2 (4,9)_4.$$

Corollary (1). *If* $r=1$, *the recurrent* C_n *reduces to* A_n.

Corollary (2). *If* $r=0$ *and* n *is an odd number the recurrent* C_n *vanishes.* C_n now becomes a recurrent in which the first element of the last column is zero and every other element of that column is 1.

Corollary (3). *If* $r=0$ *and* n *is an even number, the recurrent* C_n *is equal to*

$$(-)^{\frac{n-2}{2}}\begin{bmatrix} n-1 \\ 1 \end{bmatrix}_2.$$

5. THEOREM. *Denote*

$$D_5=\begin{vmatrix} \frac{a^4-1}{a-1}a & \frac{(a^4-1)(a^3-1)}{(a^2-1)(a-1)}a^2 & \frac{a^4-1}{a-1}a^3 & a^4 & -\begin{bmatrix} 3 \\ 1 \end{bmatrix}_2 \\ 1 & \frac{a^3-1}{a-1}a & \frac{a^3-1}{a-1}a^2 & a^3 & \begin{bmatrix} 3 \\ 1 \end{bmatrix}_2 \\ & \cdots\cdots & \cdots\cdots & & \cdots\cdots \\ & & 1 & a & -\begin{bmatrix} 1 \\ 1 \end{bmatrix}_2 \\ & & & 1 & \lambda-1 \end{vmatrix}_5$$

then $$D_n={}^nS_n\,\lambda-1. \qquad (18)$$

In D_n the element in the rth row and kth column is

$$\frac{{}^{n-r}S_{k-r+1}}{{}^{k-r+1}S_{k-r+1}}a^{k-r+1}$$

and the rth element in the last column is $(-)^{\frac{n-r+1}{2}}\begin{bmatrix} n-r \\ 1 \end{bmatrix}_2$, if one of n and r be even and the other odd, and $(-)^{\frac{n-r+2}{2}}\begin{bmatrix} n-r-1 \\ 1 \end{bmatrix}_2$, if n and r both are even or both odd. $R.$ F in this case is

$$\sum_{x=0}^{n-1} (-)^x ({}^{n-x}S_{n-x}\lambda - 1) \frac{{}^{n-1}S_x a^x}{{}^xS_x} = (-)^{\frac{n+1}{2}} \begin{bmatrix} n-2 \\ 1 \end{bmatrix}_2$$

or $(-)^{\frac{n+2}{2}} \begin{bmatrix} n-1 \\ 1 \end{bmatrix}_2$ according as n is odd or even. (19)

[The left side of (19) is

$$\lambda \sum_{x=0}^{n-1} (-)^x \frac{{}^{n-x}S_{n-x} a^x}{{}^xS_x} {}^{n-1}S_x - \sum_{x=0}^{n-1} (-)^x \frac{a^x}{{}^xS_x} {}^{n-1}S_x.$$

The second summation is the left side of (6) and the first part

$$= \lambda \sum_{x=0}^{n-1} (-)^x a^{\frac{1}{2}n(n-1)-x(n-2)} \, {}^{n-1}S_x$$

$$= \lambda \, {}^nS_n \sum_{x=0}^{n-1} (-)^x a^{(2-n)x} \, . \, {}^{n-1}S_x$$

$$= 0.$$

So the formula is proved.]

Corollary (1). *If* $\lambda = 1/{}^nS_n$, D_n *vanishes.*

Corollary (2). *If* $\lambda = 0$, $D_n = -1$. This corollary may be obtained direct, taking (6) as *R. F.*

SOME PROPERTIES OF GENERALISED COMBINATORY FUNCTIONS

BY

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1. Throughout this paper, p denotes an odd prime; all other small letters except x, denote integers $\geqslant 0$, unless stated otherwise. ϕ is Euler's function. The symbol "$\dot{<}$" is read "less than and prime to."

If $p^\alpha | m$ but $p^{\alpha+1} \nmid m$ we say that m is α-potent in p, and write $\text{pot}_p\, m = \alpha$.

If $\phi(p^\beta) | m$ but $\phi(p^{\beta+1}) \nmid m$, we say that m is β-piquant in p, and write $\text{piq}_p\, m = \beta$.

2. Let

$$(x+1)(x+a)(x+a^2)\ldots(x+a^{n-1}) = \sum_{r=0}^{n} \begin{bmatrix} n \\ r \end{bmatrix}_a x^{n-r};\ a > 0. \quad (1)$$

Then $\begin{bmatrix} n \\ 0 \end{bmatrix}_a = 1, \begin{bmatrix} n \\ 1 \end{bmatrix}_a = \dfrac{a^n - 1}{a-1}$, and $\begin{bmatrix} n \\ n \end{bmatrix}_a = a^{n(n-1)/2}$.

We shall take $\begin{bmatrix} n \\ r \end{bmatrix}_a = 0$ when $r > n > 0$.

Putting -1 for x in (1), we readily obtain

$$\sum_{r=0}^{n} (-1)^r \begin{bmatrix} n \\ r \end{bmatrix}_a = 0. \quad (2)$$

The function $\begin{bmatrix} n \\ r \end{bmatrix}_a$ has properties very similar to those possessed by the combinatory function $\binom{n}{r}$ to which it reduces when $a = 1$. The object of this note is to prove a few of these and to apply them in proving a congruence property of primitive roots. The idea was suggested by one of Professor S. C. Chakrabarti's papers.*

3. We have

$$(x+1)(x+a)(x+a^2)\ldots(x+a^{k-1})(x+a^k)\ldots(x+a^{k+l-1})$$
$$=(x+1)(x+a)(x+a^2)\ldots(x+a^{k-1})(x+a^k.1)(x+a^k.a)(x+a^k.a^2)\ldots(x+a^k.a^{l-1})$$
$$=\sum_{s=0}^{k}\begin{bmatrix}k\\s\end{bmatrix}_a x^{k-s}.\ \sum_{t=0}^{l}\begin{bmatrix}l\\t\end{bmatrix}_a a^{kt}x^{l-t}.$$

Hence
$$\begin{bmatrix}k+l\\r\end{bmatrix}_a=\sum_{s+t=r}\begin{bmatrix}k\\s\end{bmatrix}_a\cdot\begin{bmatrix}l\\t\end{bmatrix}_a\cdot a^{kt}. \tag{3}$$

In particular, replacing k and l in turn by n, and taking l or k equal to unity, we obtain

$$\begin{bmatrix}n+1\\r\end{bmatrix}_a=\begin{bmatrix}n\\r\end{bmatrix}_a+a^n\begin{bmatrix}n\\r-1\end{bmatrix}_a; \tag{4}$$

and also
$$=a^r\begin{bmatrix}n\\r\end{bmatrix}_a+a^{r-1}\begin{bmatrix}n\\r-1\end{bmatrix}_a. \tag{5}$$

From the first of these results, we have

$$\begin{bmatrix}n+1\\r\end{bmatrix}_a=\sum_{m=r-1}^{n}a^m\begin{bmatrix}m\\r-1\end{bmatrix}_a. \tag{6}$$

Putting $r=2, 3, 4,\ldots$, in succession in this result, we can show that

$$\begin{bmatrix}n\\r\end{bmatrix}_a=\frac{(a^n-1)(a^n-a)(a^n-a^2)\ldots(a^n-a^{r-1})}{(a-1)(a^2-1)(a^3-1)\ldots(a^r-1)}, \tag{7}$$

$$=\frac{(a^n-1)(a^{n-1}-1)(a^{n-2}-1)\ldots(a^{n-r+1}-1)}{(a-1)(a^2-1)(a^3-1)\ldots(a^r-1)}a^{r(r-1)/2}, \tag{8}$$

$$=\frac{n]_a}{r]_a\cdot(n-r)]_a}\begin{bmatrix}r\\r\end{bmatrix}_a,\quad r\geqslant 1, \tag{9}$$

* S. C. Chakrabarti, On a few algebraic identities, *Bull. Calcutta Math. Soc.* 27 (1935), 37-44.

where $i]_a = (a^i - 1)(a^{i-1} - 1)(a^{i-2} - 1)\ldots(a-1)$, $i \geqslant 1$; and $0]_a = 1$.

We can now define $\begin{bmatrix} x \\ r \end{bmatrix}_a$ for all values of x by the relation

$$\begin{bmatrix} x \\ r \end{bmatrix}_a = \frac{(a^x - 1)(a^{x-1} - 1)(a^{x-2} - 1)\ldots(a^{x-r+1} - 1)}{r]_a} \begin{bmatrix} r \\ r \end{bmatrix}_a ; \; r \geqslant 1.$$

We take $\begin{bmatrix} x \\ 0 \end{bmatrix}_a = 1$ for all values of x including zero.

Thus it is easy to show that

$$\begin{bmatrix} -1 \\ r \end{bmatrix}_a = (-a)^{-r},$$

and
$$\begin{bmatrix} -k \\ r \end{bmatrix}_a = (-1)^r \begin{bmatrix} k+r-1 \\ r \end{bmatrix}_a \cdot a^{-r(2k+r-1)/2}.$$

I believe that these symbols are more suggestive than those employed by Prof. Chakrabarti.

4. In (7) replace a by b^k, b being a primitive root of p^{u+v} and $u \geqslant 1$, $v = \text{pot}_p\, k$, k any integer $\geqslant 1$. Also let $n = \phi(p^u)$, and $(k, p-1) = g$; so that

$$k = p^v g k_1 \text{ and } (p-1) = gh,$$

where
$$(h, k_1) = 1 = (k_1, p).$$

Then
$$\begin{bmatrix} n \\ r \end{bmatrix}_{b^k} = \frac{b^{kn} - 1}{b^{kr} - 1} \cdot \prod_{t=1}^{r-1} \frac{b^{kn} - b^{kt}}{b^{kt} - 1} ; \; n \geqslant r > 1;$$

and
$$= \frac{b^{kn} - 1}{b^k - 1} \text{ when } r = 1.$$

We proceed to prove

THEOREM 1. *If* $\phi(p^a) \leqslant g(r-1) < \phi(p^{a+1})$, $u \geqslant a \geqslant 0$;

and
$$\text{piq}_p gr = \beta, \; u \geqslant \beta \geqslant 0;$$

then
$$\begin{bmatrix} n \\ r \end{bmatrix}_{b^k} \equiv (-1)^{r-1} \frac{b^{kn} - 1}{b^{kr} - 1} \pmod{p^{2u-a-\beta}}.$$

When $r=1$, the result stated in the theorem is trivial, we can suppose, therefore, that $n \geqslant r \geqslant 2$. We have

$$b^{kn}-b^{kt} \equiv 1-b^{kt} \pmod{p^{u+v}},\ t=1, 2, \ldots, r-1.$$

Since $\phi(p^a) \leqslant g(r-1) < \phi(p^{a+1})$, for some value t_1 of $t \leqslant (r-1)$, $\text{piq}_p gt = \alpha$; and for no value of $t \leqslant (r-1)$ is $\text{piq}_p gt > \alpha$. Hence $\text{piq}_p kt_1 = v+\alpha$ or 0, according as $\alpha > 0$ or $\alpha = 0$.

Therefore* $\dfrac{b^{kn}-b^{kt}}{b^{kt}-1} \equiv -1 \pmod{p^{u-\alpha} \text{ or } p^{u+v}}$, $t=1, 2, \ldots, r-1$; according as $\alpha > 0$ or $\alpha = 0$.

Again since $\text{piq}_p gr = \beta$, we must have

$$\text{piq}_p kr = v+\beta \text{ or } 0, \text{ according as } \beta > 0 \text{ or } \beta = 0.$$

Therefore $\dfrac{b^{kn}-1}{b^{kr}-1} \equiv 0 \pmod{p^{u-\beta} \text{ or } p^{u+v}}$,

according as $\beta > 0$ or $\beta = 0$.

Thus $$\begin{bmatrix} n \\ r \end{bmatrix}_{b^k} \equiv (-1)^{r-1} \frac{b^{kn}-1}{b^{kr}-1} \pmod{p^\delta},$$

where
$$\begin{aligned} \delta &= 2u-\alpha-\beta, \text{ when } \alpha > 0 \text{ and } \beta > 0; \\ &= 2u+v-\beta, \text{ when } \alpha = 0 \text{ and } \beta > 0; \\ &= 2u-\alpha+v, \text{ when } \alpha > 0 \text{ and } \beta = 0; \\ &= 2(u+v), \text{ when } \alpha = 0 \text{ and } \beta = 0. \end{aligned}$$

It may be noted that $\alpha = 0$ only if $(r-1) < h$, and $\beta = 0$ only if $h \nmid r$. Moreover $\delta \geqslant 1$, except when $\alpha = \beta = u$, i.e. when $r = hp^{u-1}r'$, $g \geqslant r' \geqslant 2$.

THEOREM 2. *If $r = hr_1$, $r_1 \geqslant 1$; then**

$$\frac{b^{kn}-1}{b^{kr}-1} + (-1)^{r_1} \binom{p^{u-1}g}{r_1} \equiv 0 \pmod{p^\lambda},$$

where $\lambda = u$ or δ whichever is smaller.

* Evidently, all fractions occurring in Theorems 1 and 2 must first be reduced to their lowest terms.

We have identically

$$\prod_{m<p^u}(x-m^k) \equiv \prod_{j=0}^{n-1}(x-b^{jk}) \pmod{p^u}.$$

Therefore*

$$(x^h-1)^{p^{u-1}g} \equiv \sum_{i=0}^{n}(-1)^i\begin{bmatrix} n \\ i \end{bmatrix}_{b^k} x^{n-i} \pmod{p^u}.$$

Hence

$$(-1)^{r_1}\binom{p^{u-1}g}{r_1} \equiv (-1)^r\begin{bmatrix} n \\ r \end{bmatrix}_{b^k} \pmod{p^u};$$

$$\equiv (-1)^{2r-1}\frac{b^{kn}-1}{b^{kr}-1} \pmod{p^\lambda};$$

and the theorem follows.

* M. Bauer, Zur Theorie der identischen Kongruenzen, *Bull. de la Soc. Phys-Math. de Kazan*, (3) 3 (1928).

SELF-RECIPROCAL FUNCTIONS INVOLVING APPELL'S FUNCTION

BY

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1. Recently, I have given* a few theorems on self-reciprocal functions. The object of this note is to add a few more theorems of the same type and to derive some new self-reciprocal functions therefrom.

I shall say that a function is $\pm R_\lambda$ according as it is self (skew)-reciprocal for J_λ transforms.

THEOREM I. *If*

$$P(x) = \frac{1}{2\pi i}\int_{b-i\infty}^{b+i\infty} 2^s\Gamma(\tfrac{1}{4}+\tfrac{1}{2}\mu+\tfrac{1}{2}s)\Gamma(\tfrac{1}{4}+\tfrac{1}{2}\nu+\tfrac{1}{2}s)\omega(s)x^{-s}ds, \qquad (1.1)$$

where $\omega(s)$ *satisfies the equation*

$$\omega(s) = \omega(1-s) \qquad (1.2)$$

in the strip

$$0 < b < 1, \qquad (1.3)$$

and

$$f(x) = \frac{1}{2\pi i}\int_{k-i\infty}^{k+i\infty} 2^{\frac{3}{2}s}\Gamma(\tfrac{1}{4}+\tfrac{1}{2}\mu+\tfrac{1}{2}s)\,\Gamma(\tfrac{1}{4}+\tfrac{1}{2}\nu+\tfrac{1}{2}s) \times\Gamma(\tfrac{1}{4}+\tfrac{1}{2}\lambda+\tfrac{1}{2}s)\,\chi(s)x^{-s}ds, \qquad (1.4)$$

where $\chi(\ )$ *satisfies* (1.2) *in the strip*

$$0 < k < 1, \qquad (1.5)$$

then the functi n

$$g(x) = \int_0^\infty P(y)f(xy)\,dy$$

i. R_λ.

The proof is similar to that of Theorem I given in the paper referred to.

* B. Mohan (formerly B. M. Mehrotra), On self-reciprocal functions, *Quart. J. Math.* (Oxford) 10 (1939), 252-60.

2. EXAMPLE (1). I start with the Weber-Schafheitlin integral*

$$\int_0^\infty x^{m-1} J_{n+p}(ax) J_{n-p-1}(ax)\,dx$$
$$= \frac{a^{-m}\Gamma(1-\frac{1}{2}m)\;\Gamma(n+\frac{1}{2}m-\frac{1}{2})}{2\Gamma(\frac{3}{2}+p-\frac{1}{2}m)\;\Gamma(\frac{1}{2}+n-\frac{1}{2}m)\;\Gamma(\frac{1}{2}-p-\frac{1}{2}m)},$$

where $m \neq 1$ and p is an integer or 0.

By Mellin's Inversion formula† we get

$$J_{n+p}(ax) J_{n-p-1}(ax)$$
$$= \frac{1}{4\pi i}\int_{d-i\infty}^{d+i\infty} \frac{a^{-m}\Gamma(1-\frac{1}{2}m)\;\Gamma(n-\frac{1}{2}+\frac{1}{2}m)}{\Gamma(\frac{3}{2}+p-\frac{1}{2}m)\;\Gamma(\frac{1}{2}+n-\frac{1}{2}m)\;\Gamma(\frac{1}{2}-p-\frac{1}{2}m)}\,x^{-m}dm.$$

Hence we have

$$x^{\alpha} J_{n+p}(ax) J_{n-p-1}(ax)$$
$$= \frac{1}{4\pi i}\int_{d-a-i\infty}^{d-\alpha+i\infty} \frac{a^{-\alpha-s}\Gamma(1-\frac{1}{2}\alpha-\frac{1}{2}s)\;\Gamma(n-\frac{1}{2}+\frac{1}{2}\alpha+\frac{1}{2}s)\;x^{-s}ds}{\Gamma(\frac{3}{2}+p-\frac{1}{2}\alpha-\frac{1}{2}s)\;\Gamma(\frac{1}{2}+n-\frac{1}{2}\alpha-\frac{1}{2}s)\;\Gamma(\frac{1}{2}-p-\frac{1}{2}\alpha-\frac{1}{2}s)}.$$

Putting $\alpha = \frac{1}{2}-\frac{1}{3}\nu,\;\; n = \frac{1}{2}+\frac{1}{3}\nu,\;\; p = \frac{1}{6}\nu-\frac{1}{2}\lambda-\frac{1}{2},\;\; a = 2^{-\frac{3}{2}}$, we get

$$x^{\frac{1}{2}-\frac{1}{3}\nu} J_{\frac{1}{2}\nu-\frac{1}{2}\lambda}(2^{-\frac{3}{2}}x) J_{\frac{1}{6}\nu+\frac{1}{2}\lambda}(2^{-\frac{3}{2}}x)$$
$$= \frac{2^{\frac{9}{4}-\frac{1}{2}\nu}}{2\pi i}\int_{d-\frac{1}{2}+\frac{1}{3}\nu-i\infty}^{d-\frac{1}{2}+\frac{1}{3}\nu+i\infty} \frac{2^{\frac{3}{2}s}\Gamma(\frac{3}{4}+\frac{1}{6}\nu-\frac{1}{2}s)\;\Gamma(\frac{1}{4}+\frac{1}{6}\nu+\frac{1}{2}s)}{\Gamma(\frac{3}{4}+\frac{1}{3}\nu-\frac{1}{2}\lambda-\frac{1}{2}s)\;\Gamma(\frac{3}{4}+\frac{1}{2}\nu-\frac{1}{2}s)}$$
$$\times \frac{x^{-s}ds}{\Gamma(\frac{3}{4}+\frac{1}{2}\lambda-\frac{1}{2}s)},$$

where $\nu > -1$, $\frac{2}{3} < d < 2$ and $\frac{1}{6}\nu-\frac{1}{2}\lambda-\frac{1}{2}$ is an integer or 0.

This integral is of the same form as (1.4) with $\frac{5}{6} < d < 2$, $-1 < \nu < \frac{3}{2}$, $\mu = \frac{2}{3}\nu-\lambda$, and

$$\chi(s) =$$
$$\frac{2^{\frac{9}{4}-\frac{1}{2}\nu}\Gamma(\frac{1}{4}+\frac{1}{6}\nu+\frac{1}{2}s)\;\Gamma(\frac{3}{4}+\frac{1}{6}\nu-\frac{1}{2}s)}{\Gamma(\frac{1}{4}+\frac{1}{3}\nu-\frac{1}{2}\lambda+\frac{1}{2}s)\Gamma(\frac{3}{4}+\frac{1}{3}\nu-\frac{1}{2}\lambda-\frac{1}{2}s)\Gamma(\frac{1}{4}+\frac{1}{2}\nu+\frac{1}{2}s)\Gamma(\frac{3}{4}+\frac{1}{2}\nu-\frac{1}{2}s)}$$
$$\times\left[\Gamma(\tfrac{1}{4}+\tfrac{1}{2}\lambda+\tfrac{1}{2}s)\;\Gamma(\tfrac{3}{4}+\tfrac{1}{2}\lambda-\tfrac{1}{2}s)\right]^{-1}.$$

* G. N. Watson, *Theory of Bessel Functions*, (1922), § 13-41 (3).

† G. H. Hardy, *Mess. Math.* 47 (1918), 178-84,

It follows that the function

$$x^{\frac{1}{2}-\frac{1}{3}\nu} J_{\frac{1}{2}\nu-\frac{1}{2}\lambda}(2^{-\frac{3}{2}}x) J_{\frac{1}{6}\nu+\frac{1}{2}\lambda}(2^{-\frac{3}{2}}x) \qquad (2.1)$$

may be taken as the function $f(x)$ in Theorem I where $\mu = \frac{2}{3}\nu - \lambda$, $-1 < \nu < \frac{3}{2}$ and $\frac{1}{6}\nu - \frac{1}{2}\lambda - \frac{1}{2}$ is an integer or zero. (2.2)

In 1932 I gave* the following three functions as examples of our functions $P(x)$ defined by (1.1):—

(a) $x^{\frac{1}{2}(\mu+\nu+1)} K_{\frac{1}{2}(\nu-\mu)}(x)$

(b) $x^{\frac{1}{2}(\nu-\mu+1)} J_{\frac{1}{2}(\mu+\nu)}(x)$

(c) $x^{\frac{1}{2}(\mu-\nu+1)} J_{\frac{1}{2}(\mu+\nu)}(x)$.

If we take (2.1) as $f(x)$ and function (a) as $P(x)$ in Theorem I, we obtain

$$\begin{aligned} g(x) &= \int_0^\infty y^{\frac{1}{2}+\frac{5}{6}\nu-\frac{1}{2}\lambda} K_{\frac{1}{6}\nu+\frac{1}{2}\lambda}(y) \\ &\qquad \times (xy)^{\frac{1}{2}-\frac{1}{3}\nu} J_{\frac{1}{2}\nu-\frac{1}{2}\lambda}(2^{-\frac{3}{2}}xy) J_{\frac{1}{6}\nu+\frac{1}{2}\lambda}(2^{-\frac{3}{2}}xy)\, dy \\ &= x^{\frac{1}{2}-\frac{1}{3}\nu} \int_0^\infty y^{1+\frac{1}{2}\nu-\frac{1}{2}\lambda} J_{\frac{1}{2}\nu-\frac{1}{2}\lambda}(2^{-\frac{3}{2}}xy)\, J_{\frac{1}{6}\nu+\frac{1}{2}\lambda}(2^{-\frac{3}{2}}xy) \\ &\qquad \times K_{\frac{1}{6}\nu+\frac{1}{2}\lambda}(y)\, dy, \end{aligned}$$

subject to the conditions (2.2).

Evaluating this integral by a formula given by Bailey† we get

$$\begin{aligned} g(x) &= \frac{x^{\frac{1}{2}-\frac{1}{3}\nu} 2^{\frac{1}{2}\nu-\frac{1}{2}\lambda} (2^{-\frac{3}{2}}x)^{\frac{1}{2}\nu-\frac{1}{2}\lambda} (2^{-\frac{3}{2}}x)^{\frac{1}{6}\nu+\frac{1}{2}\lambda}}{\Gamma(1+\frac{1}{2}\nu-\frac{1}{2}\lambda)\ \Gamma(1+\frac{1}{6}\nu+\frac{1}{2}\lambda)} \\ &\times \Gamma(1+\frac{1}{2}\nu-\frac{1}{2}\lambda)\ \Gamma(1+\frac{2}{3}\nu) \\ &\times F_4(1+\frac{1}{2}\nu-\frac{1}{2}\lambda,\ 1+\frac{2}{3}\nu;\ 1+\frac{1}{2}\nu-\frac{1}{2}\lambda,\ 1+\frac{1}{6}\nu+\frac{1}{2}\lambda;\ -\tfrac{1}{8}x^2;\ -\tfrac{1}{8}x^2), \\ &\qquad (\nu+2 > \lambda) \end{aligned}$$

where F_4 denotes Appell's function.

It follows that the function

$$x^{\frac{1}{2}+\frac{1}{3}\nu} F_4(1+\frac{1}{2}\nu-\frac{1}{2}\lambda,\ 1+\frac{2}{3}\nu;\ 1+\frac{1}{2}\nu-\frac{1}{2}\lambda,\ 1+\frac{1}{6}\nu+\frac{1}{2}\lambda; -\tfrac{1}{8}x^2;\ -\tfrac{1}{8}x^2),$$

* *Proc. Lond. Math. Soc.* (2) 34 (1932), 231-40.

† Some infinite integrals involving Bessel functions, *Proc. Lond. Math. Soc.* (2) 40 (1935), 37-48, (2.1).

where $-1 < \nu < \frac{3}{2}$, $\nu + 2 > \lambda$ and $\frac{1}{6}\nu - \frac{1}{2}\lambda - \frac{1}{2}$ is an integer or 0, is R_λ.

3. EXAMPLE (2). If we take the function (*b*) as our $P(x)$ in Theorem 1 we get

$$g(x) = \int_0^\infty y^{\frac{1}{6}\nu+\frac{1}{2}\lambda+\frac{1}{2}} J_{\frac{5}{6}\nu-\frac{1}{2}\lambda}(y)\,(xy)^{\frac{1}{2}-\frac{1}{3}\nu} J_{\frac{1}{2}\nu-\frac{1}{2}\lambda}(2^{-\frac{3}{2}}xy) \times J_{\frac{1}{6}\nu+\frac{1}{2}\lambda}(2^{-\frac{3}{2}}xy)\,dy$$

$$= x^{\frac{1}{2}-\frac{1}{3}\nu}\int_0^\infty y^{1-\frac{1}{6}\nu+\frac{1}{2}\lambda} J_{\frac{1}{2}\nu-\frac{1}{2}\lambda}(2^{-\frac{3}{2}}xy) J_{\frac{1}{6}\nu+\frac{1}{2}\lambda}(2^{-\frac{3}{2}}xy) \times J_{\frac{5}{6}\nu-\frac{1}{2}\lambda}(y)\,dy.$$

Evaluating this integral by another formula given by Bailey* we obtain

$$g(x) = \frac{x^{\frac{1}{2}-\frac{1}{3}\nu}2^{\frac{1}{2}\lambda-\frac{1}{6}\nu}(2^{-\frac{3}{2}}x)^{\frac{1}{2}\nu-\frac{1}{2}\lambda}(2^{-\frac{3}{2}}x)^{\frac{1}{6}\nu+\frac{1}{2}\lambda}\,\Gamma(1+\frac{2}{3}\nu)}{\Gamma(1+\frac{1}{2}\nu-\frac{1}{2}\lambda)\,\Gamma(1+\frac{1}{6}\nu+\frac{1}{2}\lambda)\,\Gamma(\frac{1}{6}\nu-\frac{1}{2}\lambda)}$$
$$\times F_4(1+\tfrac{1}{2}\lambda-\tfrac{1}{6}\nu,\ 1+\tfrac{2}{3}\nu;\ 1+\tfrac{1}{2}\nu-\tfrac{1}{2}\lambda,\ 1+\tfrac{1}{6}\nu+\tfrac{1}{2}\lambda;\ \tfrac{1}{8}x^2,\ \tfrac{1}{8}x^2),$$

where $-2 < \lambda - \frac{1}{3}\nu < 1$.

It follows that the function

$$x^{\frac{1}{2}+\frac{1}{3}\nu}F_4(1+\tfrac{1}{2}\lambda-\tfrac{1}{6}\nu,\ 1+\tfrac{2}{3}\nu;\ 1+\tfrac{1}{2}\nu-\tfrac{1}{2}\lambda,\ 1+\tfrac{1}{6}\nu+\tfrac{1}{2}\lambda;\ \tfrac{1}{8}x^2,\ \tfrac{1}{8}x^2),$$

where $-1 < \nu < \frac{3}{2}$, $-2 < \lambda - \frac{1}{3}\nu < 1$, and $\frac{1}{6}\nu - \frac{1}{2}\lambda - \frac{1}{2}$ is an integer, is R_λ.

If we take the function (*c*) instead of (*b*) we arrive at the same R_λ function.

4. THEOREM 2. *If*

$$P(x) = \frac{1}{2\pi i}\int_{b-i\infty}^{b+i\infty} \Gamma(\tfrac{1}{4}+\tfrac{1}{2}\mu+\tfrac{1}{2}s)\ \Gamma(\tfrac{3}{4}+\tfrac{1}{2}\nu-\tfrac{1}{2}s)\ \omega(s)x^{-s}ds, \quad (4.1)$$

where $\omega(s)$ *satisfies* (1.2) *in the strip* (1.3), *and*

$$f(x) = \frac{1}{2\pi i}\int_{k-i\infty}^{k+i\infty} 2^{-\frac{1}{2}s}\,\Gamma(\tfrac{1}{4}+\tfrac{1}{2}\mu+\tfrac{1}{2}s)\ \Gamma(\tfrac{3}{4}+\tfrac{1}{2}\nu-\tfrac{1}{2}s) \times\Gamma(\tfrac{3}{4}+\tfrac{1}{2}\lambda-\tfrac{1}{2}s)\ \chi(s)x^{-s}ds,$$

where $\chi(s)$ *satisfies* (1.2) *in the strip* (1.5), *then the function*

* *loc. cit.* (7.1).

$$g(x) = \int_0^\infty P(xy) f(y)\,dy$$

is R_λ.

PARTICULAR CASE. When $\mu = \nu$,

$$P(x) = \frac{1}{2\pi i}\int_{b-i\infty}^{b+i\infty} \theta(s) x^{-s} ds,$$

where $\theta(s)$ satisfies (1.2). Hence

$$P(1/x) = \frac{1}{2\pi i}\int_{bi-\infty}^{b+i\infty} \theta(s) x^{s} ds$$

$$= \frac{1}{2\pi i}\int_{1-b-i\infty}^{1-b+i\infty} \theta(1-s) x^{1-s} ds = \frac{1}{2\pi i}\int_{1-b-i\infty}^{1-b+i\infty} \theta(s) x^{1-s} ds,$$

so that
$$P(1/x) = xP(x).$$

If we put $P(x) = x^{-\frac{1}{2}}F(x)$, this equation becomes

$$F(x) = F(1/x). \tag{4.2}$$

Hence, when $\mu = \nu$, Theorem 2 assumes the simpler form: *If*

$$f(x) = \frac{1}{2\pi i}\int_{p-i\infty}^{p+i\infty} 2^{\frac{1}{2}s}\Gamma(\tfrac{1}{4}+\tfrac{1}{2}\lambda+\tfrac{1}{2}s)\, \chi(s) x^{s-1} ds, \tag{4.3}$$

where $0 < p < 1$ *and* $\chi(s)$ *satisfies* (1.2), *and the function* $F(x)$ *satisfies* (4.2), *then the function*

$$g(x) = x^{-\frac{1}{2}}\int_0^\infty y^{-\frac{1}{2}}F(xy)\, f(y)\,dy$$

is R_λ.

It would be interesting to compare this theorem with the following theorem* proved by me in 1932 :—

If $f(x)$ *is* R_λ, *and* $F(x)$ *satisfies* (4.2), *the function*

$$g(x) = \int_0^\infty y^{-\frac{1}{2}}F(y)\, f(xy)\,dy$$

is R_λ.

* *Bull. Calcutta Math. Soc.* 25 (1933), 167-72, (3.5).

In the latter case, $f(x)$ satisfies the equation

$$f(x) = \frac{1}{2\pi i}\int_{q-i\infty}^{q+i\infty} 2^{\frac{1}{2}s}\Gamma(\tfrac{1}{4}+\tfrac{1}{2}\lambda+\tfrac{1}{2}s)\ \chi(s)x^{-s}ds$$

in place of (4.3).

THEOREMS 3 AND 4. *Theorems* 1 *and* 2 *remain true if* $\omega(s)$ *and* $\chi(s)$ *satisfy the equation*

$$\omega(s) = -\omega(1-s) \tag{4.4}$$

in place of (1 2).

THEOREMS 5 AND 6. *If one of the functions* $\omega(s)$, $\chi(s)$ *satisfies* (1.2) *and the other satisfies* (4.4), *the functions* $g(x)$ *in Theorems* 1 *and* 2 *are* $-R_\lambda$

ON A NEW CHAIN OF THEOREMS IN CIRCLE-GEOMETRY*

BY

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1. *Introduction.* Circle-geometry possesses several beautiful chain-theorems of successive incidence between points and circles, of which the Miquel-Clifford chain is probably the most well known. In this paper, I discuss what appear to be the beginnings of a new chain of incidence theorems, but it is not clear whether it could be extended indefinitely. The main theorem of this paper is equivalent under inversion to the following:

Given n circles $(2 \leqslant n \leqslant 7)$ *whose centres are concyclic, we associate with each odd value of n a point, and with each even value of n a circle in such a way that the circles or points associated with each value are incident with the points or circles associated with the next higher value.* (1.1)

To start the chain, we need only associate with two circles their radical axis.

2. It is well known that the ∞^3 circles of a plane π may be represented by the points of a projective space S_3, called the Circle-space, the ∞^2 point circles corresponding to points on a quadric Ω called the Absolute. Let P be any point in a plane π called the origin. Then given any two proper circles C_1, C_2 on π, the unique circle through P of the coaxal system of which C_1, C_2 are members is said to be *associated* with the circles C_1C_2 and is denoted by C_{12} as well as by C_{21}.

* My thanks are due to Prof. A. Narasinga Rao for guidance and criticism in the preparation of this paper.

3. Taking three circles C_i $(i = 1, 2, 3)$ the above definition gives us three circles C_{23}, C_{31}, C_{12}. That these circles have a common point C_{123} other than P is inversively equivalent to the theorem that *the radical axes of three circles taken two by two are concurrent at a point.* It is further seen that the points P and C_{123} are inverses each of the other with respect to the common orthogonal circle of the three circles $C_i (i = 1, 2, 3)$. The point C_{123} is defined to be the point *associated* with the set of three circles $C_i (i = 1, 2, 3)$.

4. Taking four circles C_i $(i = 1, 2, 3, 4)$ with respect to which the inverses of P lie on a circle, we get four such points, viz. C_{123}, C_{124}, C_{234}, C_{134}. I have shown elsewhere* that

If the inverses of a point P w.r.t. four circles C_i $(i=1,..,4)$ lie on a circle, then the inverses of P w.r.t. the four circles respectively orthogonal to sets of three chosen from C_i also lie on a circle. (4.1)

From the property of the points C_{ijk} mentioned in (3) and the Theorem (4.1), the result immediately follows that these four points lie on a circle C_{1234} which we define to be the circle *associated* with the tetrad of circles C_i.

5. Next, let C_i be five circles w.r.t. which the inverses of P lie on a circle. We now propose to show that *the five circles associated with the five sets of four circles chosen from C_i are concurrent at a point C_{12345} which is defined to be the point associated with the five circles $C_i (i = 1,\ldots, 5)$.*

In the circle-space S_3, let p be the tangent plane to the Absolute Quadric Ω at the point P and g_1, g_2 the generators of Ω at P. Let us represent, for convenience, by the same symbol both the circle and its representative

* On the inverses of a circle w.r.t. a tetrad of fixed circles and their orthogonal tetrad, *Proc. Indian Acad. Sci.* 9 (1937), 128-32.

point in S_3. Owing to the restriction on P in (1.1), the points C_i lie on a quadric cone Q with vertex at P, of which g_1, g_2 are a pair of generators. Let α be the unique conic on p inpolar to all the conics cut out on p by the system of quadrics through the five points $C_i(i=1,\ldots,5)$. It is well known that the joining lines and planes of any five points $C_i(i=1,\ldots,5)$ meet p in points and lines forming a Desargues configuration self-polar in regard to α.* Let Δ_i be the tetrahedron whose vertices are the four points other than C_i of the set $C_i(i=1,\ldots,5)$. Now, the pairs of of opposite edges of Δ_1 meet p in pairs of points conjugate in regard to α. Hence if the faces of Δ_1 meet p in the four lines $l_i(i=1,\ldots,4)$ every conic of the tangential pencil of conics on p defined by the four lines l_i is inpolar to α and in particular the complex conic on p of any tetrahedral complex which has Δ_1 for its fundamental tetrahedron is inpolar to α. Thus the complex conic S_1 on p of the tetrahedral complex Γ_1 which has Δ_1 for its fundamental tetrahedron and g_1, g_2, for lines of the complex touches g_1, g_2 and is inpolar to α. But by the definition of α, the line pair g_1, g_2 is outpolar to α, i.e. g_1, g_2 are conjugate lines with regard to α. Hence S_1 touches the polar of P w.r.t. α.

Arguing similarly, it is readily seen that the five complex conics $S_i(i=1,\ldots,5)$, of the five tetrahedral complexes $\Gamma_i(i=1,\ldots,5)$ defined by the five tetrahedra $\Delta_i(i=1,\ldots,5)$ and for each of which g_1, g_2 are lines of the complex, have a common tangent line β which is the polar of P w.r.t. α. The triangle formed by g_1, g_2 and β is self-polar in regard to α. If Δ'_i be the polar tetrahedron of Δ_i in regard to the Absolute Ω, Q'_i the quadric cone circumscribing Δ'_i with vertex at P and having g_1, g_2 for a pair of generators, reciprocation w.r.t. Ω of the result established above gives that the five cones

* Baker, *Principles of Geometry*, Vol. IV, 10-11.

$Q'_i (i = 1, \ldots, 5)$ have besides g_1, g_2 a generator λ in common. From this, Theorem (5.1) follows immediately on considering the intersections with Ω of the cones $Q'_i (i = 1, \ldots, 5)$.

6. Taking next six circles $C_i (i = 1, \ldots, 6)$ in the circle space, let $a_i (i = 1, \ldots, 6)$ be the six conics and $\beta_i (i = 1, \ldots, 6)$ the six lines on p defined by the six sets of five points chosen from $C_i (i = 1, \ldots, 6)$. Let Δ be the triangle in the vertices of which the twisted cubic through the six points meets the plane p. Then it is known that Δ is self-polar w.r.t. all the conics a_i*. Further, since the line-pair g_1, g_2 is outpolar to each of the conics a_i, it is readily seen that they belong to a tangential pencil R(range) of conics. Since Δ and the triangle formed by g_1, g_2 and β_i are both self-polar in regard to a_i, their six sides touch a conic. But the conic γ touching g_1, g_2 and the sides of Δ is definite. Hence the six lines $\beta_i (i = 1, \ldots, 6)$ and the lines g_1, g_2 all touch the conic γ. γ is in fact the envelope of polars of P w.r.t. the conics of the tangential pencil R. Let $\lambda_i = (i = 1, \ldots, 6)$ be the six lines defined as in (5) by the six sets of five points chosen from C_i. Then, by reciprocation in regard to Ω of the result established above, we see that the six lines λ_i and g_1, g_2 are all generators of the same quadric cone Q with vertex at P. Considering the intersection of Q with Ω, the theorem follows immediately that

If the inverses of a point P w.r.t. six circles $C_i (i = 1, 2, \ldots, 6)$ lie on a circle, then the six points associated with the six sets of five circles chosen from C_i lie on a circle which is definea to be the circle associated with the six circles $C_i (i = 1, \ldots, 6)$. (6.1)

* Baker, *Principles of Geometry*, Vol. IV, 10-11.

7. We proceed next to consider the case of seven circles $C_i (i = 1, 2, ..., 7)$. In the circle-space let $R_i (i = 1, 2, ..., 7)$ be the seven ranges of conics on p defined by the seven sets of six points chosen from $C_i (i = 1, 2, ..., 7)$ and let γ_i be the envelope of polars of P w.r.t. the conics of the range R_i. Now all the quadrics through the seven points C_i pass through an eighth point and form a net. These cut out on p a net N of conics such that any conic of N is outpolar to any conic of any of the ranges R_i. The conics of the net N through any point and in particular through P form a pencil of conics. Since the seven points $C_i (i = 1, 2, ..., 7)$ lie on a quadric cone of vertex P having g_1, g_2 for generators, the line-pair g_1, g_2 is a conic of the above pencil. Hence the conics of such a pencil, as is easy to see, all touch at P and cut g_1, g_2 in two other points P_1, P_2. Let P_1P_2 be the line δ. Let S_i be the unique conic of the range R_i for which g_1 and δ are conjugate lines. Since g_1, g_2 are conjugate lines for S_i and since every conic of the pencil is outpolar to S_i it follows immediately that the triangle Δ formed by g_1, g_2 and δ is self-polar in regard to S_i. Thus there exists in each of the seven ranges of conics R_i a unique conic w.r.t. which the triangle formed by g_1, g_2 and δ is self-polar. Hence all the seven conics $\gamma_i (i = 1, 2, ..., 7)$ touch the line δ. If Q_i is the quadric cone which is the reciprocal of the conic γ_i w.r.t. Ω, we see that the seven quadric cones $Q_i (i = 1, 2, ..., 7)$ have, besides g_1, g_2 a common generator. Considering as before the intersections with the Absolute Ω of the cones Q_i, the theorem follows that

If the inverses of a point w.r.t. seven circles $C_i (i = 1, 2, ..., 7)$ lie on a circle, then the seven circles associated with the seven sets of six circles chosen from C_i are concurrent at a point which is defined to be the point associated with the seven circles $C_i (i = 1, 2, ..., 7)$.

8. It is well known that all quadrics through seven points pass through an eighth common point and that all quadrics through any seven of the set pass through the eighth point of the set. From this property and the results established already we easily deduce the following theorem

Given any seven circles $C_i (i = 1, 2, \ldots, 7)$ w.r.t. which the inverses of a point P lie on a circle Σ, a unique circle C_8 could be added to them in such a way that the inverse of P w.r.t. C_8 lies on Σ and that the point associated with any seven of the set of eight circles $C_i (i = 1, 2, \ldots, 8)$ is the same.

9. It is clear that these results belong to Moebius Geometry, that is to the invariant theory of the group of transformations of non-oriented circles which carry point circles into point circles. If P be identified with the "point at infinity" of the inversive plane, the chain deals with sets of circles whose centres (inverses of P) are concyclic as stated in § 1.

A NOTE ON CERTAIN SELF-RECIPROCAL FUNCTIONS

BY

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In a recent issue of this *Journal**, Dr. Dhar has investigated some types of functions, which are self-reciprocal in the Hankel-Transform. He has shown ' by the help of the infinite integral of the W-function ' that

$$x^{\nu-2k+\frac{1}{2}}e^{-\frac{1}{4}x^2}W_{k,\,k-\frac{1}{2}}(x^2/2)$$

and

$$x^{\nu-2k+\frac{1}{2}}e^{-\frac{1}{4}x^2}W_{k,\,-k+\frac{1}{2}}(x^2/2) \text{ are } R_\nu. \qquad (1)$$

The object of this note is to point out that these relations are already contained in Varma's integral†

$$\int_0^\infty y^{2l-n-1}e^{-\frac{1}{4}y^2}J_n(xy)\,W_{k,\,m}(y^2/2)\,dy$$

$$=\frac{2^{l-n-1}x^n\,\Gamma(l+m+\frac{1}{2})\,\Gamma(l-m+\frac{1}{2})}{\Gamma(n+1)\,\Gamma(l-k+1)}\,{}_2F_2\left[\begin{matrix} l+m+\frac{1}{2},\ l-m+\frac{1}{2}; \\ n+1,\ l-k+1;\ -\frac{1}{2}x^2\end{matrix}\right] \qquad (2)$$

and can be deduced from it as special cases.

The generalised hypergeometric function on the right side of (2) reduces to Kummer's function when $m=k-\frac{1}{2}$ or $-k+\frac{1}{2}$ and in that case we have the relation

$$\int_0^\infty y^{2l-n-1}e^{-\frac{1}{4}y^2}J_n(xy)\,W_{k,\,\pm(k-\frac{1}{2})}(y^2/2)\,dy$$

$$=\frac{2^{l-n-1}x^n\,\Gamma(l+k)}{\Gamma(n+1)}\,{}_1F_1(l+k;\ n+1;\ -x^2/2), \qquad (3)$$

* S. C. Dhar, *Jour. Ind. Math. Soc.* (2) 4 (1940), 91-6.

† R. S. Varma, *Jour. Ind. Math. Soc.* (2) 3 (1938), 25-33 and 54-5.

which for $l = n-k+1$, takes the form

$$\int_0^\infty y^{n-2k+1} e^{-\frac{1}{4}y^2} J_n(xy) W_{k,\pm(k-\frac{1}{2})}(y^2/2)\,dy$$
$$= \frac{x^n e^{-\frac{1}{2}x^2}}{2^k} = e^{-\frac{1}{4}x^2} x^{n-2k} e^{-\frac{1}{4}x^2}(x^2/2)^k. \tag{4}$$

Since*

$$W_{k,m}(z) = \frac{1}{\Gamma(c)} e^{-z/2} z^k \int_0^\infty e^{-t} t^{c-1} {}_2F_1(\tfrac{1}{2}-k+m, \tfrac{1}{2}-k-m; c; -\tfrac{t}{z})\,dt,$$

$R(c) > 0$, $|\arg z| < \pi$, $z \neq 0$, we have for $m = \pm(k-\frac{1}{2})$ and $z = x^2/2$

$$W_{k,\pm(k-\frac{1}{2})}(x^2/2) = e^{-\frac{1}{4}x^2}(x^2/2)^k \tag{5}$$

and consequently by (4)

$$\int_0^\infty (xy)^{\frac{1}{2}} J_n(xy) e^{-\frac{1}{4}y^2} y^{n-2k+\frac{1}{2}} W_{k,\pm(k-\frac{1}{2})}(y^2/2)\,dy$$
$$= e^{-\frac{1}{4}x^2} x^{n-2k+\frac{1}{2}} W_{k,\pm(k-\frac{1}{2})}(x^2/2), \tag{6}$$

which is the same as (1) quoted above.

With the help of (5), the result (6) can be put in the form

$$\int_0^\infty e^{-\frac{1}{2}y^2} J_n(xy) y^{n+1}\,dy = e^{-\frac{1}{2}x^2} x^n, \tag{7}$$

i.e. $\quad e^{-\frac{1}{2}x^2} x^{n+\frac{1}{2}}$ is R_n. (8)

* C. S. Meijer: *Nieuw Archief voor Wiskunde*, (2) 18 (1934).
or
A. Erdelyi: *Proc. Ben. Math. Soc.* (2) 1 (1939), 43.

ON BICIRCULAR QUARTICS

BY

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1. The bicircular quartic is the two-dimensional analogue of the general cyclide, some of whose properties were considered by the present writer in a previous paper.* It is needless to repeat the corresponding work for the bicircular quartic, but it is worth while to consider some special results regarding this curve.

The locus of a point, whose polar conic with respect to a bicircular quartic is a rectangular hyperbola, is a circle concentric with the focal conics of the curve. The intersections of this circle with the Hessian of the curve give the points for each of which the polar conic consists of a pair of perpendicular straight lines. The number of such points is twelve, since the Hessian is a sextic. Similarly we can obtain the points for each of which the polar conic is a pair of parallel lines, by considering the intersections of the Hessian with the central bicircular quartic which is the locus of a point whose polar conic is a parabola.

There are just four points whose polar conics with respect to the curve are circles. These are the foci of the focal conics and are the *double foci* of the bicircular quartic.

* H. Bagchi, A Note on Cyclides, *Jour. Ind. Math. Soc.* (2) 4 (1940), 120-4.

2. Consider a bicircular quartic which possesses a centre as well as a finite node. The equation of a central bicircular quartic can be written*

$$(x^2+y^2+k^2)^2 = ax^2+by^2.$$

If this possesses a finite node then $k = 0$, so that the node coincides with the centre. When $a+b = 0$, this reduces to the lemniscate of Bernoulli. Hence we get

If a bicircular quartic possesses a centre and a finite node, and has a rectangular hyperbola for its focal conic, the curve is a lemniscate.

The lemniscate is the envelope of a circle which intersects orthogonally a given rectangular hyperbola and passes through its centre.

3. Cassini's oval given by the bipolar equation $rr' = c^2$ can be expressed by the equation

$$\{(x+a)^2+y^2\}\{(x-a)^2+y^2\} = c^4.$$

This can also be written in the form*

$$(x^2+y^2+k^2)^2 = 4(l^2x^2+m^2y^2),$$

where $k^4 = a^4-c^4$, $2l^2 = k^2+a^2$, $2m^2 = k^2-a^2$.

The curve is therefore a central bicircular quartic whose focal conic and circle of inversion are respectively

$$\frac{x^2}{l^2}+\frac{y^2}{m^2} = 1, \text{ and } x^2+y^2 = k^2.$$

Since $k^2 = l^2+m^2$, the circle is the director circle of the conic. Hence,

Cassini's oval is a central bicircular quartic whose focal conic and circle of inversion are given by a central conic and its director circle.

The three-dimensional Cassini's oval evidently exists and possesses an analogous property.

* H. Bagchi, *loc. cit.*

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