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AN AFFINE ANALOGUE OF SINGER'S THEOREM

BY

R. C. BOSE.

INTRODUCTION.

1. James Singer[4] by using the Finite Projective Geometry $PG(2, p^n)$, proved the following theorem of the 'theory of numbers': Given an *integer* $m \geqq 2$ *of the form* p^n (p *being a prime*) *we can find* $m+1$ *integers*

$$d_0, d_1, d_2, \ldots, d_m \qquad (0.10)$$

such that among the $m(m+1)$ *differences* $d_i - d_{i'}$ ($i, i' = 0, 1, 2, \ldots, m, i \neq i'$) *reduced modulo* m^2+m+1, *the integers* $1, 2, 3, \ldots, m^2+m$ *occur exactly once.* Conversely *a set with the 'difference property' of this theorem, can be used to give a compact combinatorial representation of a Plane Finite Projective Geometry.*

2. In the present paper, I have by using the Finite Affine Geometry $EG(2, p^n)$ proved the following theorem which may be regarded as the 'affine analogue' of Singer's theorem: *Given an integer* $m \geqq 2$ *of the form* p^n (p *being a prime*) *we can find* m *integers*

$$d_1, d_2, \ldots, d_m \qquad (0.20)$$

such that among the $m(m-1)$ *differences* $d_i - d'_i (i, i' = 1, 2, \ldots, m, i \neq i')$ *reduced modulo* m^2-1, *all the positive integers less than* m^2-1 *and not divisible by* $q = m+1$, *occur exactly once.*

Any set $d_1, d_2, \ldots, d_m$, possessing the property envisaged in the above theorem may be called a '*difference set.*' Conversely *any difference set can be used to give a compact representation of a Finite Affine Geometry with m points on every line.* Let the integers $0, 1, 2, \ldots, m^2-2$, together with the adjoined number ∞ be regarded as the m^2 points of a

Finite Affine Geometry. Then the sets of collinear points are given by

$$(A)\ \infty,\ i,\ q+i,\ 2q+i, \ldots, (m-2)q+i \qquad (i=0,\ 1,\ 2, \ldots, q-1=m) \qquad (0.21)$$

$$(B)\ d_1+j,\ d_2+j, \ldots, d_m+j \qquad (j=0,\ 1,\ 2, \ldots, m^2-2), \qquad (0.22)$$

where the integers in (A) and (B) are to be reduced modulo m^2-1. The sets (0.21) yield $m+1$ lines, and the sets (0.22) yield m^2-1 lines, thus making up together the m^2+m lines of the Geometry.

3. It is now well known that the $p^{2n}-1$ degrees of freedom involved in the contrasts between p^{2n} objects, can be split up into p^n+1 independent sets of p^n-1 degrees of freedom each, each set representing comparisons among p^n groups of p^n objects. This splitting is usually done by using the properties of 'complete sets of orthogonal Latin Squares.' Using the properties of 'difference sets' and the geometry associated to any such set, a neat solution of this problem can be obtained.

The set (0.20) can be reduced to a standard form, in which when the integers $d_1, d_2, \ldots, d_m$ are reduced modulo $m+1=q$, we get the integers $1,\ 2, \ldots, m$ exactly once. When the set is written in this form, the parallel pencils of the corresponding affine geometry can be readily separated. In fact the ith pencil consists of the line

$$\infty,\ i,\ q+i,\ 2q+i, \ldots, (m-2)q+i \qquad (0.30)$$

together with the $(m-1)$ lines,

$$d_1+i+kq,\ d_2+i+kq, \ldots, d_m+i+kq \qquad (k=0,\ 1,\ 2, \ldots, m-2). \qquad (0.31)$$

Putting $i=0,\ 1,\ 2, \ldots, m$ we get all the $m+1=q$ pencils.

If now the $p^{2n}=m^2$ objects are identified with the points of our geometry, then the contrasts among p^n groups, each group consisting of the p^n objects corresponding to the points on a line of a fixed parallel pencil, give p^n-1 degrees of freedom. The degrees of freedom corresponding to the different parallel pencils are independent. The desired splitting up has thus been achieved, and the contrasts involved can be at once written down from (0.30), (0.31).

I

1. Consider the Finite Geometry $EG(2,p^n)$ whose points consist of the ordered pairs (x, y), where x, y are elements of the Galois field $GF(p^n)$, and whose lines are represented by the linear equations

$$ax+by+c=0, \tag{1.10}$$

the coefficients a, b, c being elements of $GF(p^n)$. We shall set

$$p^n = m,\ p^n+1 = m+1 = q. \tag{1.11}$$

Let us extend $GF(p^n)$ by a primitive element λ of $GF(p^{2n})$. Then λ does not belong to $GF(p^n)$. All the non zero elements of $GF(p^{2n})$ are

$$\lambda^0 = 1,\ \lambda,\ \lambda^2, \ldots, \lambda^{p^{2n}-2} = \lambda^{(m-1)q-1}. \tag{1.12}$$

Now $\lambda^{p^{2n}-1} = 1$ whence

$$\mu^{p^n-1} = 1 \text{ where } \mu=\lambda^{p^n+1} = \lambda^q. \tag{1.13}$$

The above equation shows that μ belongs to $GF(p^n)$. Consequently μ^2, μ^3,..belong to $GF(p^n)$. Hence among the $mq-1$ elements (1.12), those elements ($m-1$ in number) whose exponents are divisible by q belong to $GF(p^n)$. Clearly these are all the non-zero elements of $GF(p^n)$.

Now λ must satisfy an irreducible quadratic equation whose coefficients belong to $GF(p^n)$, say

$$\lambda^2 = a_1\lambda+a_2, \tag{1.14}$$

where a_1, a_2 are elements of $GF(p^n)$.

Every element α of $GF(p^{2n})$ can be expressed in the form

$$\alpha = x\lambda+y, \tag{1.15}$$

where x and y are elements of $GF(p^n)$. For

$$0 = 0\lambda+0 \tag{1.16}$$

$$1 = 0\lambda+1 \tag{1.17}$$

$$\lambda = 1\lambda+0 \tag{1.18}$$

whereas λ^u, $2 \leqslant u \leqslant p^{2n}-2$ can be expressed in the desired form by the help of (1.15). α is uniquely determined by (1.15). Conversely α being given, x and y are uniquely determined. If not, let there be two different ways of expressing α, say $\alpha = x\lambda + y = x'\lambda + y'$. We may suppose $x \neq x'$ (as otherwise $y = y'$ and the two ways are the same). This makes λ an element of $GF(p^n)$, which is absurd.

We now let the point (x, y) of our geometry correspond to the element α of $GF(p^{2n})$ given by (1.15). In particular the points (0, 0), (0, 1), (1, 0) and (a_1, a_2) correspond to 0, $1 = \lambda^0$, λ, λ^2 respectively. Since there is a (1, 1) correspondence between the points of $EG(p^{2n})$ and elements of $GF(p^{2n})$, the point (x, y) can also be called the point (α), α being determined by (1.15).

The points collinear with (x_1, y_1), (x_2, y_2) are

$$(\theta x_1 + \phi x_2,\ \theta y_1 + \phi y_2), \tag{1.190}$$

where $\theta + \phi = 1$ and ϕ is an element of $GF(p^n)$.

Now let

$$\alpha_1 = x_1\lambda + y_1,\ \alpha_2 = x_2\lambda + y_2, \tag{1.191}$$

hence
$$\theta\alpha_1 + \phi\alpha_2 = (\theta x_1 + \phi x_2)\lambda + (\theta y_1 + \phi y_2). \tag{1.192}$$

Thus *the m points collinear with* (α_1) *and* (α_2) *are given by* $(\theta\alpha_1 + \phi\alpha_2)$, *where* $\theta + \phi = 1$ *and* ϕ *is an element of* $GF(p^n)$ *i.e. has one of the values* $0, \lambda^0, \lambda^q, \lambda^{2q}, \ldots, \lambda^{(m-2)q}$.

2. Let the point (λ^t) be the point (x, y). Then

$$\lambda^t = x\lambda + y \tag{1.20}$$

$$\begin{aligned}\lambda^{t+1} &= x\lambda^2 + y\lambda \\ &= x(a_1\lambda + a_2) + y\lambda \text{ from } (1.14) \\ &= (a_1 x + y)\lambda + a_2 x.\end{aligned} \tag{1.21}$$

Hence (λ^{t+1}) is the point (x', y') where

$$x' = a_1 x + y \tag{1.22}$$

$$y' = a_2 x. \tag{1.23}$$

But the equations (1.22), (1.23) give an affine transformation T, which is independent of t. This shows that T

transforms the points of our geometry, other than the origin (0,0) according to the cycle

$$(\lambda^0)\to(\lambda)\to(\lambda^2)\to\ldots\to(\lambda^{p^{2n}-2}) = \lambda^{(m-1)q-1}\to(\lambda^{p^{2n}-1}) = (\lambda^0). \quad (1.24)$$

T^i transforms (λ^t) to (λ^{t+i}). $T^{p^{2n}-1} = T^{(m-1)q}$ is clearly the identical transformation. The origin (0, 0) obviously remains unchanged by T.

Lines through the origin will be transformed by T into lines through the origin. One line through the origin is the line determined by the points (0) and (1). The points on this line are (see end of para 1)

$$(0),\ (\lambda^0) = (1),\ (\lambda^q),\ (\lambda^{2q}),\ldots,\ (\lambda^{(m-2)q}). \quad (1.25)$$

Applying in succession the transformations T^0 (the identity,) $T, T^2,\ldots, T^{q-1} = T^m$ to (1.25) we get the $m+1$ lines through the origin as

$$\left.\begin{array}{lllllll}
(0), & (\lambda^0), & (\lambda^q), & (\lambda^{2q}) & \ldots & \ldots & (\lambda^{(m-2)q}) \\
(0), & (\lambda), & (\lambda^{q+1}), & (\lambda^{2q+1}), & \ldots & \ldots & (\lambda^{(m-2)q+1}) \\
(0), & (\lambda^2), & (\lambda^{q+2}), & (\lambda^{2q+2}), & \ldots & \ldots & (\lambda^{(m-2)q+2}) \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
(0), & (\lambda^{q-1}), & (\lambda^{2q-1}), & (\lambda^{3q-1}) & \ldots & \ldots & (\lambda^{(m-1)q-1})
\end{array}\right\} \quad (I)$$

points on the same line occupying the same row of scheme (I). Since the elements (1.12) are all distinct, it is clear that the lines of (I) are all distinct. The lines of the scheme (I) therefore give all the $(m+1)$ lines through the origin (0).

The line joining (1), (λ) does not pass through the origin for otherwise it would be coincident with the first line of (I) which is impossible, since (λ) is not a point on that line. Let the points on the line joining (1), (λ) be

$$(\lambda^{d_1}),\ (\lambda^{d_2}),\ldots,\ (\lambda^{d_m}),\ \text{where } d_1 = 0,\ d_2 = 1. \quad (1.26)$$

We shall first show that $d_i - d_j$ $(i \neq j)$ is not divisible by q. For every pair of points (λ^{d_i}), $(\lambda^{d_{i'}})$ for which

$d_i - d_{i'}$ is divisible by q occurs in one of the lines of (I). But as already shown, the line (1.26) does not pass through the origin.

Applying the transformation $T^0, T, T^2, \ldots, T^{(m-1)q-1} = T^{m^2-2}$ in succession to (1.26), we get m^2-1 lines

$$\left.\begin{array}{llll} (\lambda^{d_1}), & (\lambda^{d_2}), & \ldots\ \ldots & (\lambda^{d_m}) \\ (\lambda^{d_1+1}), & (\lambda^{d_2+1}), & \ldots\ \ldots & (\lambda^{d_m+1}) \\ (\lambda^{d_1+2}), & (\lambda^{d_2+2}), & \ldots\ \ldots & (\lambda^{d_m+2}) \\ \ldots, & \ldots & \ldots\ \ldots & \ldots \\ (\lambda^{d_1+m^2-2}), & (\lambda^{d_2+m^2-2}) & \ldots\ \ldots & (\lambda^{d_m+m^2-2}) \end{array}\right\} \text{(II)},$$

points on the same line occupying the same row of scheme (II). We can of course reduce the exponents mod (m^2-1). We shall first show that these lines are all distinct.

First, the two lines whose points are given by the first rows must be distinct, for otherwise a multiplication by λ merely permutes the elements

$$\lambda^{d_1}, \lambda^{d_2}, \ldots, \lambda^{d_m}$$

among themselves. This means that every line of the scheme (II) consists of the same points (except for the order). Hence there would occur in the scheme (II) only m distinct elements of $GF(p^{2n})$ whereas in every column there occur all the m^2-1 non-zero elements. Hence $m^2-1 \leqq m$ which is absurd, since $m = p^n \geqq 2$.

The first two lines of (II) cannot have more than one point in common. They obviously have the common point $(\lambda^{d_2}) = (\lambda^{d_1+1})$ i.e. the point λ (remembering $d_1 = 0$, $d_2 = 1$). From this it follows that in the first row occurs only one pair of exponents d_v, d_u such that $d_v - d_u = 1$ mod (m^2-1), viz. d_1, d_0. For if possible let $d_v - d_u = 1$ mod (m^2-1), where $u \neq 0$ mod (m^2-1) and consequently $v \neq 1$ mod (m^2-1). Then

$$d_v = d_u + 1 \text{ mod } (m^2-1).$$

Hence the first two lines of the scheme (II) have the common point (λ^{d_v}) besides (λ), which is absurd.

We can now prove that no two lines of the scheme (II) are identical. Suppose the $(i+1)$th and $(i'+1)$th lines are identical. Then

$$d_1+i = d_u+i' \bmod (m^2-1)$$
$$d_2+i = d_v+i' \bmod (m^2-1),$$

where $d_v \neq 1 \bmod (m^2-1)$ since $i \neq i'$. Hence

$$d_v-d_u = d_2-d_1,\ d_v \neq 1 \qquad \bmod (m^2-1)$$

which has been already proved to be impossible.

Since the transformation T changes lines not passing through the origin, into lines not passing through the origin and the first line of scheme (II) does not pass through the origin, none of the m^2-1 lines of the scheme (II) passes through the origin. Together with the $m+1$ lines of the scheme (I), all of which pass through the origin, they make the m^2+m lines of our geometry.

We may observe that the transformation T cyclically permutes the $(m+1)$ lines of (I), and the m^2-1 lines of (II). The lines of (I) are left invariant by $T^q = T^{m+1}$.

3. There are just m lines of the scheme (II) viz.

$$(\lambda^{d_1-d_{i'}}),\ (\lambda^{d_2-d_{i'}}), \ldots,\ (\lambda^{d_m-d_{i'}})\ (i'=1, 2, \ldots, m) \quad (1.30)$$

which together with the line

$$(0),\ (\lambda^0)=(1),\ (\lambda^q),\ (\lambda^{2q}), \ldots,\ (\lambda^{(m-2)q}) \qquad (1.25)$$

joining (0) and (1) make up the pencil of $m+1$ lines through the point $(\lambda^0)=(1)$. Each of these $m+1$ lines contains beside the point (λ^0) just $m-1$ points. These $(m+1)(m-1)$ points must be all the m^2-1 points of the plane other than (λ^0). But in (1.25) occurs the point (0) and all points corresponding to powers of λ divisible by $q=m+1$. Hence in the m lines (1.30) must occur, [besides (λ^0)] all points corresponding to the powers of λ not divisible by q, each point occurring just once. Thus among the $m(m-1)$ numbers d_i-d_i' $(i, i'=1, 2, \ldots, m, i \neq i')$

reduced modulo m^2-1 must occur all the positive integers less than m^2-1 and not divisible by $q=m+1$, just once. We thus get the following theorem of the theory of numbers.

THEOREM 1. *Given an integer* $m \geqslant 2$, *of the form* p^n (*p being a prime*), *we can find m integers*

$$d_1, d_2, \ldots, d_m \tag{1.31}$$

such that among the $m(m-1)$ *differences* $d_i - d_{i'}$ $(i, i' = 1, 2, \ldots, m,$ $i \neq i')$ *reduced modulo* m^2-1, *all the positive integers less than* m^2-1 *and not divisible by* $q=m+1$ *occur exactly once.*

4. If the integers $d_1, d_2, \ldots, d_m$ satisfy the conditions of Theorem I, so will the integers

$$d'_1 = d_1 + x,\ d'_2 = d_2 + x, \ldots,\ d'_m = d_m + x. \tag{1.40}$$

Let $k_1, k_2, \ldots, k_m$ be the m integers satisfying

$$k_i = d_i \bmod (q=m+1),\ 0 \leqslant k_i < m \ (i = 1, 2, \ldots, m). \tag{1.41}$$

Since $d_i - d_{i'} \neq 0 \bmod (q)$, $(i, i' = 1, 2, \ldots, m, i \neq i')$, the m integers k_i given by (1.41) are all different. Hence there is just one integer t, $0 \leqslant t \leqslant m$ missing from the set $k_1, k_2 \ldots, k_m$. Taking $x = q - t$ in (1.40), we see that $d'_1, d'_2, \ldots, d'_m$ when reduced mod (q) give just the m different integers $1, 2, \ldots, m$.

A set of integers $d_1, d_2, \ldots, d_m$ satisfying the difference property involved in Theorem I may be called a '*difference set*'. A 'difference set' may be said to be in a '*standardised form*' if the integers of the set reduced modulo $m+1$ give just the m different integers $1, 2, \ldots, m$. We can now state the following:

A given 'difference set' can be reduced to a 'standardised form' by adding a suitable fixed integer x to every integer of the set. A difference set in the standardised form, consisting of $m \geqslant 2$ *integers can always be obtained corresponding to any value of m, which is a prime power.*

II

1. We shall now prove the following converse theorem:

THEOREM 2. *If the m integers* $(m \geqslant 2)$,

$$d_1, d_2, \ldots, d_m \tag{2.10}$$

form a 'difference set', then the integers $0, 1, 2, \ldots, m^2-2$ *together with the adjoined number* ∞, *may be regarded as the* m^2 *points of a Finite Affine Geometry, the sets of collinear points being given by*

$$(A)\quad \infty, i, q+i, 2q+i, \ldots, (m-2)q+i \qquad (i=0, 1, 2, \ldots, q-1=m)$$

$$(B)\quad d_1+j, d_2+j, \ldots, d_m+j \quad \Big(j=0, 1, 2, \ldots, (m-1)q-1=m^2-2\Big),$$

where the integers in (A) and (B) are to be reduced modulo m^2-1.

To prove the theorem, we have to verify that (*a*) *there are m points on every line,* (*b*) *through every point pass* $m+1$ *lines,* (*c*) *any two points lie on one and only one line and* (*d*) *two lines have not more than one point in common.*

The property (*a*) obviously holds. To prove (*b*), we notice that through ∞ there pass the $m+1$ lines (*A*). Let l be any other point. Let $l=tq+i_0$, $0 \leqslant i_0 < q$. Then l lies on only one line of (*A*), namely the line for which $i=i_0$. Also l lies on just m lines of (*B*), namely the lines given by values of j satisfying

$$j=l-d_1, j=l-d_2, \ldots, j=l-d_m \bmod (m^2-1). \tag{2.11}$$

Thus through l, there pass just $m+1$ lines and (*b*) is proved. The points ∞ and l occur together only on one line, namely the line $i=i_0$ of (*A*). Let l' be any other point, $l'=t'q+i'_0$, $0 \leqslant i'_0 < q$. If $i_0=i'_0$, then l and l' lie on just one line of (*A*) namely the line $i=i_0=i'_0$ and on no line of (*B*) since $d_u-d_v \neq 0 \bmod (q)$, $(u, v=1, 2, \ldots, m, u \neq v)$. If however $i_0 \neq i'_0$, then since $l \neq l' \pmod q$ in this case, they do not occur together on a line of (A). The necessary and sufficient condition for their occurring together on the line $j=j_0$ of (B) is that there exist u, $1 \leqslant u \leqslant m$ and v, $1 \leqslant v \leqslant m$ such that

$$l = d_u + j_0,\ l' = d_v + j_0 \mod (m^2-1). \qquad (2.12)$$

This gives $d_u - d_v = l - l' \mod (m^2-1)$ so that from the difference property of the set (2.10), d_u and d_v are uniquely determined. Hence j_0 is uniquely determined, so that the points lie on just one line of (B). This completes the proof of (c).

Instead of proving (d), we shall prove a more definite result, which will be later useful to us. We first notice that the result of adding the same integer x to every integer of the 'difference set' (2.10) is merely to permute the lines (B) among themselves, while the lines (A) remain unchanged. We can therefore without loss of generality consider the difference set (2.10) to be in the 'standardised form'. With this supposition we shall prove the following:

(d'). The m^2+m lines (A) and (B) can be divided into parallel pencils, each pencil consisting of m lines, such that two lines belonging to different pencils have just one point in common, whereas two lines belonging to the same pencil have no point in common.

The lines of the ith pencil consist of the line A_i

$$\infty,\ i,\ q+i,\ 2q+i, \ldots, (m-2)q+i \qquad (2.13)$$

and the $m-1$ lines B_{ik}

$$d_1+i+kq,\ d_2+i+kq, \ldots, d_m+i+kq \ (k=0, 1, 2, \ldots, q-1=m-2). \qquad (2.14)$$

Putting $i = 0, 1, 2, \ldots, m$, we get all the pencils.

If the line A_i has the point l in common with the line B_{ik} we must have for some t, $0 \leqq t \leqq m-2$ and some u, $1 \leqq u \leqq m$,

$$l = tq+i = d_u + i + kq \mod (m^2-1).$$

This gives $d_u = 0 \pmod q$, which is impossible since the difference set (2.10) is in the standard form.

If the lines B_{ik} and $B_{ik'}$, $k \neq k'$, have the point l in common, we must have for some u, $1 \leq u \leq m$ and some v, $1 \leq v \leq m$

$$d_u + i + kq = d_v + i + k'q \bmod (m^2 - 1)$$

or $d_u - d_v = 0 \bmod (q)$ which is impossible since none of the differences $d_u - d_v$ is divisible by q.

We have now shown that two lines of the same pencil have no point in common. Let us consider two lines belonging to different pencils. Clearly the lines A_i and $A_{i'}$ $(i \neq i')$ have only the point ∞ in common. The necessary and sufficient condition for the lines A_i and $B_{i'k}$ $(i \neq i')$ to have the point l in common is that we can find some t, $0 \leq t \leq m-2$ and some u, $1 \leq u \leq m$ such that

$$l = tq + i = d_u + i' + kq \bmod (m^2 - 1)$$

i. e. $$d_u = i - i' \bmod (q).$$

Since our difference set is in the standard form, this uniquely determines d_u and hence l. Hence A_i and $B_{i'k}$ $(i \neq i')$ have just one point in common. Finally consider the lines B_{ik} and $B_{i'k'}$ $(i \neq i')$. The necessary and sufficient condition for these lines to have the point l in common is that we can find u, $1 \leq u \leq m$ and v, $1 \leq v \leq m$ such that

$$l = d_u + i + kq = d_v + i' + k'q \bmod (m^2 - 1)$$

i. e. $$d_v - d_u = i - i' + q\,(k - k') \bmod (m^2 - 1).$$

Since $i \neq i'$ from the difference property of the set (2.10), d_v and d_u are uniquely determined. Hence l is uniquely determined, so that the lines B_{ik} and $B_{i'k}$ have just one common point. This completes the proof of (d') and of our theorem.

3. If $m = p^n$ and $d_1, d_2, \ldots, d_m$ are the exponents in 1.25), the geometry corresponding to Theorem 2 is

isomorphic with the geometry $EG(2, p^n)$, for the correspondence

$$\infty = (0),\ 0 = (\lambda^0),\ 1 = (\lambda), \ldots, m^2-2 = (\lambda^{m^2-2}) \quad (2.30)$$

between the points, takes over the $m+1$ lines (A) into the $m+1$ lines of scheme (I), and the m^2-1 lines (B) into the m^2-1 lines of scheme (II).

The geometry $EG(2, p^n)$ is Desarguesian, but no proof exists to show that the geometry corresponding to any arbitary 'difference set' is Desarguesian. When the geometry corresponding to a difference set is Desarguesian, we may call it a '*Desarguesian difference set*'. The following theorem will be found useful for quickly writing down Desarguesian difference sets for different values of m.

THEOREM 3. *If* λ *is a primitive root of* $GF(p^n)$, $q = m+1$, *and*

$$1 + \lambda^{u+kq} = \lambda^{d_{k+2}},\ k = 0, 1, 2, \ldots, m-2 \quad (2.31)$$

u being a fixed integer, not divisible by q, *then* $d_1 = 0, d_1, \ldots, d_m$ *is a Desarguesian difference set.*

The exponents appearing in any row of the scheme (II) of § 1, clearly constitute a Desarguesian difference set. Since u is not divisible by q, λ^u is not an element of $GF(p^n)$. Hence $1+\lambda^u$ is not an element of $GF(p^n)$. Let $1+\lambda^u = \lambda^t$, then t is not divisible by q. Now the line L joining (1) and (λ^t) does not pass through the origin (0), for otherwise (λ^t) would be on the line (1.25) joining (0) and (1), and hence t would be divisible by q contrary to what has been proved. Hence the line L is one of the lines of the scheme (II). The points on this line are $(\theta + \phi\lambda^t)$, where $\theta + \phi = 1$ and $\phi = 0, \lambda^0, \lambda^q, \ldots, \lambda^{(m-2)q}$ (cf. I § 1).

The points on L are therefore the point $(1) = (\lambda^0)$ together with the $m-1$ points

$$(1 - \lambda^{kq} + \lambda^{kq+t}) = (1 + \lambda^{u+kq}) = (\lambda^{d_{k+2}}),\ k = 0, 1, \ldots, m-2 \quad (2.32).$$

Hence the exponents in the row of the scheme (II) of I, corresponding to the line L are $d_1 = 0, d_2, \ldots, d_m$. This proves our theorem.

A difference set obtained by using the above theorem is not in the standard form, but as we have seen, it can be easily standardised by the addition of a suitable integer to every member of the set. We tabulate below 'standardised Desarguesian difference sets' calculated by using the above theorem for the values $m = 2, 3, 4, 5, 7, 8, 9$ and 11. Only one set for each value of m has been given.

TABLE I.

Standardised difference set.

$m = 2$	1, 2
$m = 3$	1, 6, 7
$m = 4$	1, 3, 4, 12
$m = 5$	1, 3, 16, 17, 20
$m = 7$	1, 2, 5, 11, 31, 36, 38
$m = 8$	1, 6, 8, 14, 38, 48, 49, 52
$m = 9$	1, 13, 35, 48, 49, 66, 72, 74, 77
$m = 11$	1, 27, 55, 58, 65, 66, 71, 80, 98, 100, 117.

III

1. It is well known that the $m^2 - 1$ degrees of freedom involved in the contrasts between m^2 objects, can be split up into $m+1$ independent sets of $m-1$ degrees of freedom each, each set representing comparisons among m groups of m objects; provided that we can construct a 'complete set of orthogonal $m \times m$ Latin squares'. It is well known that this latter problem is solvable for any prime m. Yates and Fisher[2,3] showed its solvability for the values $m = 8$ and 9. Later, Stevens[5] and the author[1] independently demonstrated that a 'complete set of orthogonal Latin Squares can be constructed, when m is any prime power. Hence using orthogonal Latin squares

the splitting up of the m^2-1 degrees of freedom mentioned above can always be done when m is a prime power. But the result (d') of II § 1, regarding the pencils of the Finite Affine Geometry generated by a standardised difference set, enables us to do this splitting up in a neat and direct manner.

2. *Let the m^2 objects $(m=p^n)$ be denoted by*

$$\infty,\ 0,\ 1,\ 2, \ldots, m^2-2.$$

Take a standardised difference set $d_1, d_2, \ldots, d_m$ and form $q = m+1$ sets, each set containing m groups of m objects in the following manner:—The first group in the i-th set is

$$[\infty, i, q+i, 2q+i, \ldots, (m-2)q+i] \qquad (3.10)$$

the other $m-1$ groups being

$$[d_1+i+kq, d_2+i+kq, \ldots, d_m+i+kq] \quad (k=0, 1, 2, \ldots, m-2) \qquad (3.11)$$

the integers in (3.11) *being reduced modulo m^2-1. The $m+1$ sets are obtaining by putting $i = 0, 1, 2, \ldots, m$.*

Clearly all the m^2 objects occur in every set. The contrasts between the groups of the same set represent $m-1$ degrees of freedom. Now our objects can be identified with the m^2 points of the Finite Affine Geometry of Theorem II. It follows from (d') that the objects of a given group in the ith set, correspond to the points on a certain line of the ith pencil. Now two lines belonging to different pencils have just one point in common. Hence if $i \neq i'$, the m objects of any given group in the ith set are distributed one each among the groups of the i'th set. Hence the $m-1$ degrees of freedom corresponding to the contrasts between the m groups of the i th set are orthogonal to the $m-1$ degrees of freedom corresponding to the contrasts between the m groups of the i' th set. The m^2-1 degrees of freedom corresponding to the contrasts between the m^2 objects have thus been split up into $m+1$ sets of $m-1$ degrees of freedom each.

For example the 6 sets in the case of 25 objects obtained by using the difference set for $m=5$ in Table I, are shown below:—

1st set.

[∞,	0,	6,	12,	18]
[1,	3,	16,	17,	20]
[7,	9,	22,	23,	2]
[13,	15,	4,	5,	8]
[19,	21,	10,	11,	14]

2nd set.

[∞,	1,	7,	13,	19]
[2,	4,	17,	18,	21]
[8,	10,	23,	0,	3]
[14,	16,	5,	6,	13]
[20,	22,	11,	12,	15]

3rd set.

[∞,	2,	8,	14,	20]
[3,	5,	18,	19,	22]
[9,	11,	0,	1,	4]
[15,	17,	6,	7,	10]
[21,	23,	12,	13,	16]

4th set.

[∞,	3,	9,	15,	21]
[4,	6,	19,	20,	23]
[10,	12,	1,	2,	5]
[16,	18,	7,	8,	11]
[22,	0,	13,	14,	17]

5th set.

[∞,	4,	10,	16,	22]
[5,	7,	20,	21,	0]
[11,	13,	2,	3,	6]
[17,	19,	8,	9,	12]
[23,	1,	14,	15,	18]

6th set.

[∞,	5,	11,	17,	23]
[6,	8,	21,	22,	1]
[12,	14,	3,	4,	7]
[18,	20,	9,	10,	13]
[0,	2,	15,	16,	19]

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ON THE ZEROS AND CLOSURE OF ORTHOGONAL FUNCTIONS

BY

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This note deals primarily with infinite sets of continuous functions $\phi_n(x)$ uniformly bounded and orthonormal over (0,1), i.e. the sequence being $\phi_1, \phi_2, \ldots, \phi_n$,

$$\int_0^1 \phi_i \phi_j dx = \delta_{ij}, \ |\phi_n| \leqslant M \text{ for all } n, \ 0 \leqslant x \leqslant 1. \qquad (1)$$

An elementary knowledge of the subject, such as is contained in a standard textbook like vol. I of Courant-Hilbert: *Methoden d. Math. Physik* is assumed for the sake of brevity. Only main details of proofs are given, and results obviously extensible to L-integrable functions are stated only for continuous functions in order to keep the treatment relatively simple.

Without further explicit statement, we use the following well-known relationship between ordinary convergence and convergence in the mean. A boundedly convergent sequence $\{ f_n(x) \}$ also converges in the mean almost everywhere to the same limit function, and for a sequence that c.i.m., a subsequence can be found to converge almost everywhere to a summable function.

1. We can state at once, a simple but not quite trivial theorem about the average behavior of $\{ \phi \}$.

THEOREM 1. *No sequence* $\{ \phi \}$ *can converge (c.i.m.) over* (0,1); *nor to any value except zero over any subinterval of* (0,1). *But every such sequence is summable in the mean to zero by most regular methods of summability over* (0,1) *or any subinter-*

val thereof.

PROOF. Convergence implies $\int_0^1 (\phi_n - \phi_m)^2 dx \to 0$. But for distinct orthonormal functions, the integral always has the value 2. For the second part, let (a, b) be any subinterval of $(0,1)$ and consider $\sigma_n = \frac{\phi_1 + \phi_2 + \ldots + \phi_n}{n}$.

Now
$$\int_a^b \sigma_n^2 dx \leqslant \int_0^1 \sigma_n^2 dx = \frac{1}{n} \to 0.$$

This extends to any Toeplitz Matrix where the sum of squares of elements in the ith row tends to 0 as $i \to \infty$. If $\{\phi\}$ converged over a subinterval, it would be summable to the same limit function.

As a corollary, we see that no $\{\phi\}$ can be uniformly continuous in n over the whole of $(0,1)$, nor have a difference quotient uniformly bounded for all n. Or else, a subsequence could be found to converge over $(0,1)$. That is, while every infinite set of orthogonal functions averages out to zero for any permissible method of averages, the randomness increases in general with the index.

From Bessel's inequality, $\sum c_n^2$ converges, $c_n = \int_0^1 f\phi_n dx$. The same is true if $c_n = \int_a^b f\phi_n dx$ for every (a, b) in $(0,1)$, as is seen by taking $f(x)$ as zero outside (a, b). By taking $f > 0$, in (a, b) we see that the necessary condition $c_n \to 0$ cannot be satisfied unless $\phi_n \to 0$ or the ϕ's change sign, for every (a, b) in $(0,1)$.

2. It is easy to show that *for closure, infinitely many of the functions ϕ_n must have at least one change of sign each in every subinterval (a, b) however small.* From the Parseval equality, taking the functions as zero outside the proper interval, it follows that *for a closed $\{\phi\}$ and any integrable f, $\sum c_n c_n' = \int f^2 dx$, where $c_n = \int_a^b f\phi_n dx, c_n' = \int_{a'}^{b'} f\phi_n dx$ and the integral of f^2 above is taken over the region common to the two*

intervals (a, b), (a', b'). Now suppose that only a finite number of the ϕ_n have any change of sign in some sub-interval, whence a still smaller interval could be found such that no ϕ changes sign therein. Let this be sub-divided into two, (a'', ξ), (ξ, b''). For a positive f, the associated coefficients c_n, c'_n must both have the same sign. But by the above lemma, $\Sigma c_n c'_n = 0$, which establishes a contradiction and proves our theorem.

If $\phi_n \to 0$ over some (a, b), then this tendency must not be too strong if $\{\phi\}$ is to be closed. In fact, suppose $|\phi_n| \leqslant \delta_n \to 0$ over (a, b). Taking $f = 1$ in (a, b), $f = 0$ outside, we have $\Sigma c_n^2 = b - a \leqslant (b-a)^2 \Sigma \delta_n^2$. This leads to $1 \leqslant (b-a) \Sigma \delta_n^2$. Decreasing the interval does not increase any δ_n, hence, if $\Sigma \delta_n^2$ converges, the right side can be made arbitrarily small. *For closure of* $\{\phi\}$, $\Sigma \delta_n^2$ *and* $\sum \int_a^b \phi_n^2 dx$ *must both diverge.* The divergence of the series of integrals follows from Schwarz's inequality $\left(\int_a^b f^2 dx\right) \times \left(\int_a^b \phi_n^2 dx\right) \geqslant c_n^2$ and from the closure property $\sum c_n^2 = \int_a^b f^2 dx$. It is a consequence of these that $\sum \int_a^b \phi_n^2 dx \geqslant 1$. Now, if the series of integrals converge for some (a, b) the remainder after a sufficiently large but fixed number of terms may be made arbitrarily small; the initial terms, no matter how many, being fixed in number, can also be made to have an arbitrarily small sum by reducing the interval—which does not increase the remainder. Thus, the left side could be made as small as desired, but would still remain greater than unity, which leads to a contradiction in case $\sum \int_a^b \phi_n^2 dx$ converges.

It would seem intuitively probable that the number of changes of sign must not be too great, if $\{\phi\}$ is to be closed. Let μ_n be defined as the number of changes of sign of ϕ_n in $(0,1)$. For the common orthogonal functions of mathematical physics, we usually have $\mu_n = n-1$. This can be proved directly from the differential equations, which have boundary conditions making the end points of the interval conjugate. For the nth function of the set, the previous $n-1$ conjugate points move inside the fundamental interval and furnish precisely $n-1$ changes of sign.

While this is too strong a result for the general type of ϕ, something like it holds for polynomials, at least of a particular kind. For those constructed for interpolation, it is again usually true that $\mu_n = n-1$. For a more general case, we can take, according to a classic theorem of Muntz, any set of powers $\{x^{p_n}\}$, which is closed if (and only if) $\Sigma 1/p_n$ diverges. The set of orthonormal functions constructed from these by the usual process will also be closed. But by Descartes's rule of signs, the nth of these functions (polynomial with n terms) cannot have more than $n-1$ positive roots, hence $\mu_n \leqslant n-1$ even here.

3. Before proceeding to the general case, we consider a set of functions that operate by change of sign alone, without the restriction of orthogonality.

Let $\{x_n\}$ be a set of distinct points everywhere dense in $(0,1)$, with $0 < x_n < 1$ for all n. Let $\gamma_r(x)$ be defined over $(0,1)$ by $\gamma_r = 1$, $0 \leqslant x \leqslant x_r$ and $\gamma_r = 0$, $x_r < x \leqslant 1$. The set $\{\gamma\}$ is closed over $(0,1)$ as $\int_0^1 f\gamma_n dx = 0$ for all n implies that the integral of f vanishes for a set of points everywhere dense in $(0,1)$, i.e. that the function itself vanishes almost everywhere in the fundamental interval.

We now consider a set $\{\psi\}$ constructed as follows: $0 < x_{i_1} < x_{i_2} < \ldots < x_{i_r} < 1$ being distinct points of the set $\{x_n\}$ above, ψ_i assumes alternately the values ± 1 over the intervals (inclusive of the right hand end point for each) marked by 0, x_{i_1}, $x_{i_2}, \ldots,$ x_{i_r}, 1. That is,

$$\psi_i = 1+2(\gamma_{i_1}-\gamma_{i_2})+2(\gamma_{i_3}-\gamma_{i_4})+\cdots+\begin{cases}2(\gamma_{i_{r-1}}-\gamma_{i_r}), & r \text{ even.}\\ 2(\gamma_{i_r}-1), & r \text{ odd.}\end{cases} \tag{2}$$

Finally, let it be required by hypothesis that any finite number of the ψ's are linearly independent as also the γ's and that every point of $\{x_n\}$ is ultimately used for the construction of the ψ's. We can then state

THEOREM 2. *A necessary and sufficient condition for the closure of* $\{\psi\}$ *is that the number of distinct points of the set* $\{x_n\}$ *needed for the construction of* n *functions of* $\{\psi\}$ *should be equal to* $n-1$ *infinitely often.*

Formula (2) shows that each ψ is composed of the function 1 and as many of the γ's as it has changes of sign. Because of linear independence, no set of n of the ψ's can be composed of less than n of the component functions 1, $\{\gamma\}$. From this, the condition of the theorem is seen to be sufficient for closure, as it implies that n of the component functions can be solved for in terms of the ψ's infinitely often, and as every x_n is used up, every γ can ultimately be expressed as a linear combination of the ψ's.

The condition is also necessary, as a contradiction would follow if $\{\psi\}$ were closed without it. The least unfavourable case would be that $\psi_1, \psi_2, \psi_3, \ldots, \psi_n$ would need, for all large n, at least $n+1$ of the component functions. To solve for the component functions would then ultimately need all the ψ's and at least one extra function ψ_0. That is, the set $\{\psi\}$ cannot be closed without the inclusion of at least one more function, ψ_0.

The function 1 is not itself included in the set considered in Theorem 2. If this be included, we have a corollary, that a necessary and sufficient condition for the closure of the enlarged set is that n of the ψ's, with the exclusion of 1, should require just n points of $\{x_n\}$ infinitely often. This covers the case of the orthonormal system constructed (Kaczmarz, Walsh, Paley) by multiplication of the proper number of Rademacher functions. There, $\psi_0 = 1$, and $\psi_1, \psi_2, \psi_3, \ldots, \psi_n$ have changes of sign at points all included in the points dividing $(0,1)$ into $n+1$ equal sub-intervals, for $n = 2^k - 1$.

4. The preceding theorem rests on the lemma that if n functions of one set can be expressed as linearly independent linear combinations of n of another set *infinitely often,* the two sets must be closed together. For the general case, we need something similar, with the substitution of approximation for exact equality. This is given by

THEOREM 3. *If $\{\phi\}$ is closed, there exists at least one closed set of powers $\{x^{p_n}\}$ such that it is possible to approximate simultaneously to r functions of $\{\phi\}$ by means of exactly r powers of the set for any given degree of approximation, and infinitely many values of r.*

One such process of simultaneous approximation can be built up as follows. Take a set of positive, distinct, decreasing constants λ_i such that $\Sigma\lambda_i^2$ converges and $\Sigma\lambda_i\phi_i(s)\phi_i(t)$ converges uniformly in the unit square. Then the series defines the symmetric kernel $K(s,t)$ of the integral equation

$$\int_0^1 K(s,t)\phi(t)\,dt = \lambda\phi(s).$$

The kernel is positive definite by construction, without any multiple characteristic values, provided, of course, $\{\phi\}$ is closed as in our present hypothesis. Our desired process of approximation is now derived by approximating to $K(s,t)$ by symmetric polynomials $K_n(s,t)$. The degenerate kernels K_n then have characteristic functions and characteristic values of λ which approximate to functions of $\{\phi\}$ and the corresponding λ_i. Moreover, for any kernel, the (approximating) characteristic polynomials are orthonormal.

An important property of this process is that a subsequence at least of the orthonormal polynomials obtained for K_n always tends to the appropriate limit as $\lambda_i^{(n)} \to \lambda_i$ and $K_n \to K$. There are as many mutually orthogonal polynomials as there are distinct powers of the variable in the degenerate kernel.

If, now, the theorem is assumed false, we can again show a contradiction. Suppose that for all large n, $\phi_1, \phi_2, \phi_3, \ldots, \phi_n$ need at least $n+1$ powers for the purpose of simultaneous approximation. Then there exists at least one extra characteristic function for each of the approximating kernels, and a subsequence of these can be chosen so as to tend to a limiting function. But this extra function cannot tend to any fixed ϕ_i, as there would then be two distinct function sequences tending to the same limit, yet mutually orthogonal. The extra function cannot approximate successively to some ϕ_{n+p} because this extra function (in a sub-sequence) tends to a limit, which is impossible for any sub-sequence of $\{\phi\}$ by Theorem 1. So, there would exist a function in L^2 orthogonal to all the ϕ's but not equivalent to zero, as it has a unit norm. This contradicts the hypothesis of closure, and so, the theorem is established.

This result can be applied to suitably restricted $\{\phi\}$ in order to put a limit on μ_n. We introduce a new definition:

DEFINITION: *A set ϕ is said to be strongly random if (a) for any (a,b) in $(0,1)$ lim inf $\int_a^b \phi_n^2 dx \geqslant k(b-a)$, with $k>0$, and (b) for all large n, max $|\phi_n| \geqslant K > 0$ between any two consecutive changes of sign of ϕ_n.*

It must be kept in mind that $\{\phi\}$ is a uniformly bounded continuous, closed, orthonormal set. The present definition gives restrictions that make the graph of ϕ_n arch away from the x-axis, preventing convergence to zero. It makes $\mu_n \to \infty$, forbids vacuoles (rectangle with sides parallel to the axes such that no graph of any ϕ passes through it; their absence is a fair substitute for almost-everywhere denseness of the values of $\{\phi\}$) in the rectangle formed in the x, y plane by $0 \leqslant x \leqslant 1$, $-K \leqslant y \leqslant K$. The behaviour of such functions is, then, close to that of the functions ψ of Theorem 2, in a general way. The arching away from the x-axis enables us to represent all the changes of sign in the approximating polynomials. If the approximating polynomial failed to follow one of the "arches", across the axis of x, then by the second condition for strong randomness, it would fail to be a good approximation in the ordinary sense. The first condition would also prevent it from being a good approximation in the mean. This would lead to a contradiction if the set admits simultaneous approximation. We now apply Theorem 3 and Descartes's rule of signs as before, to get our final result:

THEOREM 4. *A necessary condition for the closure of a strongly random set of orthonormal functions is that* $\mu_n \leqslant n-1$ *infinitely often.*

That is, the set cannot be closed if $\mu_n \geqslant n$ for all large n. In particular, as n/μ_n does not tend to zero even if the μ_n be in non-decreasing order, a necessary condition for the closure of $\{\phi\}$ is that $\sum \frac{1}{\mu_n}$ diverge. Also, if no two ϕ's of a closed set have the same number of changes of sign, $\mu_n = n-1$ with a suitable rearrangement of $\{\phi\}$ if necessary. Clearly, no condition of this type can possibly be sufficient.

As the discussion was meant to be elementary, no attempt has been made to obtain "best possible" results. For a strongly random set, μ_n is a fair measure of the "randomness". The content of Theorem 4, is that this, which we might call the entropy of the functions, cannot increase too rapidly if the functions are to enable us to represent all possible states.

[I apologise to the reader if circumstances beyond my control should make it impossible for me to prevent the simultaneous publication abroad of this note.]

OPERATIONAL IMAGES OF SELF-RECIPROCAL FUNCTIONS

BY

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The object of this paper* is to investigate methods for finding the operational images of functions which are self-reciprocal. The results are given in two theorems. The first gives the operational image of a function which is self-reciprocal in the Hankel-transform of order ν, and the second the image of a function which is self-reciprocal in the sine-transform.

THEOREM 1. *If* $\phi(p) \doteq f(x)$, *then*

$$\phi(p) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} 2^{\frac{1}{2}s}p^s\psi(s)\Gamma(1-s)\Gamma(\tfrac{1}{2}\nu+\tfrac{1}{2}s+\tfrac{1}{4})\, ds$$

provided that $f(x)$ *is* R_ν *and belongs to the class* $A(\omega, a)$ *and* $\psi(s)=\psi(1-s)$.

PROOF: We know from Hardy and Titchmarsh† that a necessary and sufficient condition that a function $f(x)$ of $A(\omega, a)$ should be its own Hankel-transform of order ν is that it should be of the form

$$f(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} 2^{\frac{1}{2}s}\Gamma(\tfrac{1}{2}\nu+\tfrac{1}{2}s+\tfrac{1}{4})\psi(s)x^{-s}\, ds, \tag{1}$$

where $0<c<1$, and

$$\psi(s)=\psi(1-s). \tag{2}$$

* I am indebted to Dr. R. S. Varma for suggestions and help in the preparation of this paper.

† *Quart. Jour. Math.* (Oxford), 1 (1930), 196-231.

By definition, $\phi(p) \doteqdot f(x)$ if

$$\phi(p) = p\int_0^\infty e^{-px} f(x)\, dx. \tag{3}$$

Hence using (1) and (3), we have

$$\phi(p) = p\int_0^\infty e^{-px}\, dx.\ \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} 2^{\frac{1}{2}s}\Gamma(\tfrac{1}{2}\nu+\tfrac{1}{2}s+\tfrac{1}{4})\psi(s)x^{-s}ds$$

$$= \frac{p}{2\pi i}\int_{c-i\infty}^{c+i\infty} 2^{\frac{1}{2}s}\psi(s)\Gamma(\tfrac{1}{2}\nu+\tfrac{1}{2}s+\tfrac{1}{4})ds \int_0^\infty e^{-px}x^{-s}dx, \tag{4}$$

assuming that it is permissible to change the order of integration.

Evaluating the second integral in (4), we have

$$\phi(p) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} 2^{\frac{1}{2}s}p^s\psi(s)\Gamma(1-s)\Gamma(\tfrac{1}{2}\nu+\tfrac{1}{2}s+\tfrac{1}{4})ds.$$

COR. 1. *If $f(x)$ is R_c we get, on putting $\nu = -\frac{1}{2}$ in the above theorem,*

$$\phi(p) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} 2^{\frac{1}{2}s}\, p^s\psi(s)\Gamma(1-s)\Gamma(\tfrac{1}{2}s)ds.$$

COR. 2. *If $f(x)$ is R_s, then $\nu = \frac{1}{2}$ and the above theorem gives*

$$\phi(p) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} 2^{\frac{1}{2}s}p^s\psi(s)\Gamma(1-s)\Gamma(\tfrac{1}{2}+\tfrac{1}{2}s)ds.$$

EXAMPLE 1.

For the function

$$f(x) = 2^{\frac{3}{4}-\frac{1}{2}\nu}x^{\nu+\frac{1}{2}}e^{-\frac{1}{2}x^2},\ R(\nu) > -1,$$

which is R_ν, we know* that $\psi(s) = 1$. Taking this value of $\psi(s)$ in Theorem 1, we get

$$\phi(p) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} 2^{\frac{1}{2}s}p^s\Gamma(1-s)\Gamma(\tfrac{1}{2}\nu+\tfrac{1}{2}s+\tfrac{1}{4})\, ds$$

$$= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} 2^{\frac{1}{2}s}p^s\, \frac{2^{-s}\Gamma(\frac{1}{2}-\frac{1}{2}s)\Gamma(1-\frac{1}{2}s)}{\sqrt{\pi}}\, \Gamma(\tfrac{1}{2}\nu+\tfrac{1}{2}s+\tfrac{1}{4})ds$$

* E. C. Titchmarsh, *Theory of Fourier Integrals* (Oxford, 1937), p. 260.

$$= \frac{1}{\pi^{\frac{3}{2}} i} \int_{\frac{1}{2}c - i\infty}^{\frac{1}{2}c + i\infty} \Gamma(\tfrac{1}{2}-t)\Gamma(1-t)\Gamma(\tfrac{1}{2}\nu+t+\tfrac{1}{4}) \left(\frac{p^2}{2}\right)^t dt,$$

by a change of the variable.

If C be defined to be the semi-circle of radius ρ on the right of the line $x = \frac{1}{2}c$ with centre at $(\frac{1}{2}c, 0)$, it is readily seen by using the asymptotic formula*

$$\log \Gamma(z+a) = (z+a-\tfrac{1}{2}) \log z - z + \tfrac{1}{2} \log (2\pi) + o(1),$$

that the integrand is of the order

$$O\left[|t|^{\frac{1}{2}\nu+\frac{1}{4}-|t|} \exp\{ -2\pi |I(t)| \} \right]$$

as $t \to \infty$ on the line $x = \frac{1}{2}c$ or on C.

Hence the integral converges.

If $\frac{1}{2} < c < 1$, the condition that $R(\nu) > -1$ ensures that the poles of $\Gamma(\frac{1}{2}\nu+t+\frac{1}{4})$, i.e. the points $t = -\frac{1}{2}\nu - \frac{1}{4} - n$, for $n = 0, 1, 2, \ldots$ lie on the left of the path of integration and the poles of $\Gamma(\frac{1}{2}-t)\Gamma(1+t)$, viz. the points $t = \frac{1}{2}+n;\ 1+n$; $(n = 0, 1, 2, \ldots)$ lie on the right of the path.

Following Barnes' method, the integral is equal to minus $2\pi i$ times the sum of the residues of the integrand at the points $t = \frac{1}{2}+n;\ 1+n$; for $n = 0, 1, 2.$ Hence

$$\phi(p) = -\frac{2}{\sqrt{\pi}} \text{ times the sum of the residues}$$

$$= -\frac{2}{\sqrt{\pi}} \left\{ \sum_{n=0}^{\infty} \frac{-\pi\Gamma(\frac{1}{2}\nu+\frac{3}{4}+n)}{n!\Gamma(\frac{1}{2}+n)} \left(\frac{p^2}{2}\right)^{n+\frac{1}{2}} \right.$$

$$\left. + \sum_{n=0}^{\infty} \frac{\pi\Gamma(\frac{1}{2}\nu+\frac{5}{4}+n)}{n!\Gamma(\frac{3}{2}+n)} \left(\frac{p^2}{2}\right)^{n+1} \right\}.$$

This gives after some simplification

$$\phi(p) = \left\{ 2^{\frac{1}{2}} p \Gamma(\tfrac{1}{2}\nu+\tfrac{3}{4})\, {}_1F_1(\tfrac{1}{2}\nu+\tfrac{3}{4};\ \tfrac{1}{2};\ \tfrac{1}{2}p^2) \right.$$

$$\left. + \frac{\Gamma(-\frac{1}{2})p^2\Gamma(\frac{1}{2}\nu+\frac{5}{4})}{\Gamma(\frac{1}{2})}\, {}_1F_1(\tfrac{1}{2}\nu+\tfrac{5}{4};\ \tfrac{3}{2};\ \tfrac{1}{2}p^2) \right\}. \tag{5}$$

* Whittaker and Watson, *Modern Analysis*, (1935), p. 279.

If we use the formula*

$$D_n(x) = \frac{\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{2}-\frac{1}{2}n)} 2^{\frac{1}{2}n} e^{-\frac{1}{4}x^2} {}_1F_1(-\tfrac{1}{2}n; \tfrac{1}{2}; \tfrac{1}{2}x^2)$$

$$+ \frac{\Gamma(-\frac{1}{2})}{\Gamma(-\frac{1}{2}n)} 2^{\frac{1}{2}n-\frac{1}{2}} x e^{-\frac{1}{4}x^2} {}_1F_1(\tfrac{1}{2}-\tfrac{1}{2}n; \tfrac{3}{2}; \tfrac{1}{2}x^2)$$

then (5) can be written in the form

$$\phi(p) = 2^{\frac{3}{4}-\frac{1}{2}\nu}\Gamma(\nu+\tfrac{3}{2}) p e^{\frac{1}{4}p^2} D_{-\nu-\frac{3}{2}}(p),$$

a result due to R. S. Varma.†

EXAMPLE 2. We know‡ that

$$x^{\frac{1}{2}} I_n(\tfrac{1}{4}x^2) K_n(\tfrac{1}{4}x^2) = \frac{1}{2\pi i}\int_{2k-\frac{1}{2}-i\infty}^{2k-\frac{1}{2}+i\infty} 2^{\frac{1}{2}s-\frac{7}{4}}$$

$$\times \frac{\Gamma(\frac{1}{2}s+\frac{1}{4})\Gamma(n+\frac{1}{8}+\frac{1}{4}s)\Gamma(\frac{3}{8}-\frac{1}{4}s)x^{-s}ds}{\Gamma(\frac{5}{8}+\frac{1}{4}s)\Gamma(\frac{7}{8}+n-\frac{1}{4}s)}$$

where $\frac{1}{4}<k<\frac{3}{4}$.

Putting $n=\frac{1}{4}\nu$, this gives, if $0<c<1$,

$$x^{\frac{1}{2}} I_{\frac{1}{4}\nu}(\tfrac{1}{4}x^2) K_{\frac{1}{4}\nu}(\tfrac{1}{4}x^2) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} 2^{\frac{1}{2}s-\frac{7}{4}} x^{-s}$$

$$\times \frac{\Gamma(\frac{1}{2}s+\frac{1}{4})\Gamma(\frac{1}{4}\nu+\frac{1}{8}+\frac{1}{4}s)\Gamma(\frac{3}{8}-\frac{1}{4}s)}{\Gamma(\frac{5}{8}+\frac{1}{4}s)\Gamma(\frac{7}{8}+\frac{1}{4}\nu-\frac{1}{4}s)},$$

$$= \frac{2^{-(\frac{7}{4}+\frac{1}{2}\nu)}}{2\pi i}\int_{c+i\infty}^{c+i\infty} 2^{\frac{1}{2}s} \frac{\Gamma(\frac{1}{2}\nu+\frac{1}{2}s+\frac{1}{4})\Gamma(\frac{1}{4}s+\frac{1}{8})\Gamma(\frac{3}{8}-\frac{1}{4}s)x^{-s}ds}{\Gamma(\frac{5}{8}+\frac{1}{4}\nu+\frac{1}{4}s)\Gamma(\frac{7}{8}+\frac{1}{4}\nu-\frac{1}{4}s)} \quad (6)$$

where we have used the duplication formula

$$\Gamma(2z) = \frac{2^{2z-1}\Gamma(z)\Gamma(z+\frac{1}{2})}{\sqrt{\pi}}. \quad (7)$$

It follows from (6) that the function

$$x^{\frac{1}{2}} I_{\frac{1}{4}\nu}(\tfrac{1}{4}x^2) K_{\frac{1}{4}\nu}(\tfrac{1}{4}x^2)$$

* Whittaker and Watson, *loc. cit.* p. 347.

† R. S. Varma, *Proc. Camb. Phil. Soc.* 33 (1937), 211.

‡ B. Mohan, *Proc. Edin. Math. Soc.* (2), 6 (1938), 93.

is R_ν, provided that $R(\nu) > -1$, a result proved otherwise by Erdelyi.*

We see from (6) by comparing it with (1) that

$$\psi(s) = 2^{-(\frac{7}{4}+\frac{1}{2}\nu)} \frac{\Gamma(\frac{1}{4}s+\frac{1}{8})\Gamma(\frac{3}{8}-\frac{1}{4}s)}{\Gamma(\frac{5}{8}+\frac{1}{4}\nu+\frac{1}{4}s)\Gamma(\frac{7}{8}+\frac{1}{4}\nu-\frac{1}{4}s)}$$

satisfies the condition (2).

Using this value of $\psi(s)$ in Theorem 1, we find that the image of $x^{\frac{1}{2}}I_{\frac{1}{4}\nu}(\frac{1}{4}x^2)K_{\frac{1}{4}\nu}(\frac{1}{4}x^2)$ is

$$\phi(p) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} 2^{\frac{1}{2}s}p^s\, 2^{-(\frac{7}{4}+\frac{1}{2}\nu)} \times \frac{\Gamma(1-s)\Gamma(\frac{1}{4}s+\frac{1}{8})\Gamma(\frac{3}{8}-\frac{1}{4}s)\Gamma(\frac{1}{2}\nu+\frac{1}{2}s+\frac{1}{4})ds}{\Gamma(\frac{5}{8}+\frac{1}{4}\nu+\frac{1}{4}s)\Gamma(\frac{7}{8}+\frac{1}{4}\nu-\frac{1}{4}s)}.$$

Using (7), we get, by a change of the variable

$$\phi(p) = \frac{1}{4\pi^3 i}\int_{\frac{1}{4}c-i\infty}^{\frac{1}{4}c+i\infty} \frac{\Gamma(\frac{1}{8}+t)\Gamma(\frac{1}{4}-t)\Gamma(\frac{3}{8}-t)\Gamma(\frac{1}{2}-t)\Gamma(\frac{5}{8}-t)\Gamma(1-t)\Gamma(\frac{1}{8}+\frac{1}{4}\nu+t)}{\Gamma(\frac{7}{8}+\frac{1}{4}\nu-t)}\left\{\frac{p^4}{16}\right\}^t dt.$$

If C be defined to be the semi-circle of radius ρ on the right of the line $x=\frac{1}{4}c$, with centre at $(\frac{1}{4}c, 0)$, it is found, as in the previous example, that the integrand is of the order

$$O\left[|t|^{-2|t|-\frac{7}{8}}\exp\{-4\pi|I(t)|\}\right]$$

as $t\to\infty$ on the line $x=\frac{1}{4}c$ or on C.

Hence the integral converges.

We suppose that $0<c<1$. This ensures that the poles of $\Gamma(\frac{1}{8}+\frac{1}{4}\nu+t)$, viz. $t=-\frac{1}{8}-\frac{1}{4}\nu-n$, $(n=0, 1, 2, \ldots)$ lie on the left of the path for $R(\nu)>-1$. It also follows that the poles of $\Gamma(\frac{1}{4}-t)\Gamma(\frac{3}{8}-t)\Gamma(\frac{1}{2}-t)\Gamma(\frac{5}{8}-t)\Gamma(1-t)$ viz. the points $t=\frac{1}{4}+n, \frac{3}{8}+n, \frac{1}{2}+n, \frac{5}{8}+n, 1+n$; for $n=0, 1, 2, \ldots$ lie on the right of the path of integration.

As in the previous example, we have

* A. Erdelyi, *Jour. Lond. Math. Soc.* 13 (1938), 153.

$\phi(p) = -\frac{1}{2\pi i}$ times the sum of the residues at the poles of the integrand.

Evaluating these residues, we get, after simplification,

$$\phi(p) = \frac{1}{2\pi}\{\psi_1(p)+\psi_2(p)+\psi_3(p)+\psi_4(p)+\psi_5(p)\},$$

where

$$\psi_1(p) = \frac{p}{\sqrt{2}} \frac{\Gamma(\frac{3}{8})\Gamma(\frac{1}{8})\Gamma(\frac{3}{8})\Gamma(\frac{3}{8}+\frac{1}{4}\nu)}{\Gamma(\frac{5}{8}+\frac{1}{4}\nu)}\, {}_3F_4\left\{\begin{matrix}\frac{3}{8}-\frac{1}{4}\nu,\ \frac{3}{8},\ \frac{3}{8}+\frac{1}{4}\nu; \\ \frac{1}{4},\ \frac{5}{8},\ \frac{3}{4},\ \frac{7}{8};\end{matrix}\ \frac{p^4}{16}\right\}$$

$$\psi_2(p) = \left(\frac{p}{2}\right)^{\frac{3}{2}} \frac{\Gamma(-\frac{1}{8})\Gamma(\frac{1}{8})\Gamma(\frac{1}{4})\Gamma(\frac{5}{8})}{\sqrt{\pi}}\, {}_3F_4\left\{\begin{matrix}\frac{1}{2}-\frac{1}{2}\nu,\ \frac{1}{2};\ \frac{1}{2}+\frac{1}{2}\nu; \\ \frac{3}{8},\ \frac{5}{4},\ \frac{7}{8},\ \frac{9}{8}\end{matrix}\ \frac{p^4}{16}\right\}$$

$$\psi_3(p) = \frac{p^2}{4} \frac{\Gamma(-\frac{1}{4})\Gamma(-\frac{1}{8})\Gamma(\frac{1}{8})\Gamma(\frac{5}{8})\Gamma(\frac{5}{8}+\frac{1}{4}\nu)}{\sqrt{\pi}\ \Gamma(\frac{3}{8}+\frac{1}{4}\nu)}$$

$$\times\, {}_3F_4\left\{\begin{matrix}\frac{5}{8}-\frac{1}{4}\nu,\ \frac{5}{8},\ \frac{5}{8}+\frac{1}{4}\nu; \\ \frac{1}{2},\ \frac{7}{8},\ \frac{9}{8},\ \frac{5}{4}\end{matrix}\ \frac{p^4}{16}\right\}$$

$$\psi_4(p) = -\left(\frac{p}{2}\right)^{\frac{5}{2}} \frac{\Gamma(-\frac{3}{8})\Gamma(-\frac{1}{4})\Gamma(-\frac{1}{8})\Gamma(\frac{3}{8})\Gamma(\frac{3}{4})\Gamma(\frac{3}{4}+\frac{1}{4}\nu)}{\pi\Gamma(\frac{1}{4}+\frac{1}{4}\nu)}$$

$$\times\, {}_3F_4\left\{\begin{matrix}\frac{3}{4}-\frac{1}{4}\nu,\ \frac{3}{4},\ \frac{3}{4}+\frac{1}{4}\nu; \\ \frac{5}{8},\ \frac{9}{8},\ \frac{5}{4},\ \frac{11}{8}\end{matrix}\ \frac{p^4}{16}\right\}$$

$$\psi_5(p) = \frac{p^4}{16\pi} \frac{\Gamma(-\frac{3}{4})\Gamma(-\frac{5}{8})\Gamma(-\frac{1}{2})\Gamma(-\frac{3}{8})\Gamma(\frac{9}{8})\Gamma(\frac{9}{8}+\frac{1}{4}\nu)}{\Gamma(\frac{1}{4}\nu-\frac{1}{8})}$$

$$\times\, {}_3F_4\left\{\begin{matrix}\frac{9}{8}-\frac{1}{4}\nu,\ \frac{9}{8},\ \frac{9}{8}+\frac{1}{4}\nu; \\ \frac{11}{8},\ \frac{3}{2},\ \frac{13}{8},\ \frac{7}{4}\end{matrix}\ \frac{p^4}{16}\right\}.$$

THEOREM 2. *If* $\phi(p) \doteq f(x)$ *then*

$$\phi(p) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} p \int_0^\infty \frac{y f(y)}{y^2+p^2}\, dy;$$

provided that $f(x)$ *is* R_s, *and* $R(p) > 0$.

PROOF: Since $f(x)$ is R_s,

$$f(x) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^\infty \sin xy f(y)\, dy. \tag{8}$$

Hence, using (3) and (8), we have

$$\phi(p) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} p \int_0^\infty e^{-px} dx \int_0^\infty \sin xy\, f(y)\, dy$$

$$= p \int_0^\infty y^{\frac{1}{2}} f(y)\, dy \int_0^\infty e^{-px} J_{\frac{1}{2}}(xy)\, x^{\frac{1}{2}} dx. \tag{9}$$

Since*

$$\int_0^\infty e^{-at} J_\nu(bt) t^{\mu-1} dt = \frac{b^\nu \Gamma(\mu+\nu)}{2^\nu (a^2+b^2)^{\frac{1}{2}(\mu+\nu)} \Gamma(\nu+1)}$$

$$\times {}_2F_1\left(\frac{\mu+\nu}{2}, \frac{1-\mu+\nu}{2}; \nu+1; \frac{b^2}{a^2+b^2}\right),$$

$$R(\mu+\nu)>0,\ R(a)>0,\ R(a\pm ib)>0.$$

we obtain from (9)

$$\phi(p) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} p \int_0^\infty \frac{y f(y)}{y^2+p^2}\, dy.$$

EXAMPLE 1. The function† $D_{2n+1}(x\sqrt{2})$ is $\pm$ R_s according as n is even or odd.

Using this function

$$\phi(p) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} p \int_0^\infty \frac{y}{y^2+p^2} D_{2n+1}\,(2^{\frac{1}{2}} y)\, dy$$

$$= \frac{(2n+1)!}{2^n} p \int_0^\infty \frac{y^2 e^{-\frac{1}{2}y^2}}{y^2+p^2} \sum_{r=0}^{n} \frac{(-1)^r y^{2n-2r}}{r!\,(n-r)!\,\Gamma(n-r+\frac{3}{2})} dy.$$

Integrating term by term by the help of the result‡

$$2\int_0^\infty \frac{x^{m-n} e^{-\frac{1}{2}x^2}}{(x^2+a^2)^n} dx = 2^{\frac{1}{2}m-n} \Gamma(\tfrac{1}{2}m-n)\, {}_1F_1(n, 1+n-\tfrac{1}{2}m; \tfrac{1}{2}a^2)$$

$$+ a^{m-2n} \frac{\Gamma(n-\frac{1}{2}m)\Gamma(\frac{1}{2}m)}{\Gamma(n)}\, {}_1F_1(\tfrac{1}{2}m; 1-n+\tfrac{1}{2}m; \tfrac{1}{2}a^2);\ R(m) \geqslant 0$$

* Watson, *Theory of Bessel Functions*, p. 385.

† E. C. Titchmarsh, *loc. cit.* p. 261,

‡ R. S. Varma, *loc. cit.*

we get

$$\phi(p) = \frac{(2n+1)!}{2^{n+1}} p \sum_{r=0}^{n} \frac{(-1)^r \theta^r}{r!(n-r)!\Gamma(n-r+\frac{3}{2})},$$

where $\theta_r = 2^{\frac{1}{2}+n-r}\Gamma(\frac{1}{2}+n-r)\,{}_1F_1(1;\, \frac{1}{2}+r-n;\, \frac{1}{2}p^2)$

$$+\, p^{2n-2r+1}\Gamma(-\tfrac{1}{2}+r-n)\Gamma(\tfrac{3}{2}+n-r)\,{}_1F_1(\tfrac{3}{2}+n-r;\tfrac{3}{2}+n-r,\, \tfrac{1}{2}p^2).$$

An interesting case is obtained by taking $n=0$ in this result. We find that the image of $2^{\frac{1}{2}}xe^{-\frac{1}{2}x^2}$ is given by

$$\phi(p) = \frac{p}{2\Gamma(\frac{3}{2})}\Big\{ 2^{\frac{1}{2}}\Gamma(\tfrac{1}{2})\,{}_1F_1(1;\, \tfrac{1}{2};\, \tfrac{1}{2}p^2)$$

$$+\, p\Gamma(-\tfrac{1}{2})\Gamma(\tfrac{3}{2})\,{}_1F_1(\tfrac{3}{2};\, \tfrac{3}{2},\, \tfrac{1}{2}p^2)\Big\}$$

$$= 2^{\frac{1}{2}} p e^{\frac{1}{4}p^2} D_{-2}(p),$$

a particular case of the first example in Theorem 1.

EXAMPLE 2. We know that the function

$$f(x) = \frac{\sinh(x\sqrt{\frac{2}{3}\pi})}{\cosh(x\sqrt{\frac{3}{2}\pi})}$$

is R_s.

sing this function in Theorem 2, we find that

$$\phi(p) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} p \int_0^\infty \frac{x\sinh(x\sqrt{\frac{2}{3}\pi})}{(x^2+p^2)\cosh(x\sqrt{\frac{3}{2}\pi})}\, dx.$$

To evaluate this integral, we consider

$$I = \int_\Gamma \frac{z\sinh(z\sqrt{\frac{2}{3}\pi})}{(z^2+p^2)\cosh(z\sqrt{\frac{3}{2}\pi})}\, dz,$$

where Γ is a rectangular contour with vertices at $\pm R$, $\pm R+iR$.

It is easy to see that

$$\phi(p) = \left(\frac{\pi}{2}\right)^{\frac{1}{2}} p\left[4\sum_{n=0}^{\infty} \frac{(-)^n(2n+1)\sin(\frac{2}{3}n\pi+\frac{1}{3}\pi)}{\pi(2n+1)^2-6p^2} - \frac{\sin(p\sqrt{\frac{2}{3}\pi})}{\cos(p\sqrt{\frac{3}{2}\pi})}\right].$$

ON SUPERPOSABILITY

BY

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In a previous paper* we have developed the idea of superposability in Cartesian coordinates. The treatment in cylindrical and spherical polar coordinates gives rise to certain new results which we propose to discuss in the present paper.† Superposability being a physical property, it is needless to establish the theory in these two coordinate systems, and we state without proof two results which we shall employ in our analysis.

(*a*) If the vortex lines of a motion coincide with its stream lines the motion is self-superposable as well as superposable on another motion possessing the same property. In cylindrical coordinates the velocity components (u, v, w) of such a motion satisfy the equations

$$\left.\begin{aligned} (\nabla^2+\lambda^2)u-\frac{u}{r^2}-\frac{2}{r^2}\frac{\partial v}{\partial \varphi} &= v\frac{\partial \lambda}{\partial z}-\frac{w}{r}\frac{\partial \lambda}{\partial \varphi} \\ (\nabla^2+\lambda^2)v-\frac{v}{r^2}+\frac{2}{r^2}\frac{\partial u}{\partial \varphi} &= w\frac{\partial \lambda}{\partial r}-u\frac{\partial \lambda}{\partial z} \\ (\nabla^2+\lambda^2)w &= \frac{u}{r}\frac{\partial \lambda}{\partial \varphi}-v\frac{\partial \lambda}{\partial r} \end{aligned}\right\} \qquad \text{(I)}$$

and in spherical polar coordinates they satisfy the equations

* Superposable Fluid Motions, *Proc. Benares Math. Soc.* (1940).

† I am indebted to Professor J. A. Strang for his keen interest and helpful suggestions in this investigation.

$$\left.\begin{aligned}
&(\nabla^2+\lambda^2)\,u = \\
&\frac{v}{r\sin\theta}\frac{\partial\lambda}{\partial\varphi}-\frac{w}{r}\frac{\partial\lambda}{\partial\theta}+\frac{2u}{r^2}+\frac{2\cot\theta}{r^2}v+\frac{2}{r^2}\frac{\partial v}{\partial\theta}+\frac{2}{r^2\sin\theta}\frac{\partial w}{\partial\varphi} \\
&(\nabla^2+\lambda^2)\,v = \\
&w\frac{\partial\lambda}{\partial r}-\frac{u}{r\sin\theta}\frac{\partial\lambda}{\partial\varphi}-\frac{2}{r^2}\frac{\partial u}{\partial\theta}+\frac{v}{r^2\sin^2\theta}+\frac{2\cot\theta}{r^2\sin\theta}\frac{\partial w}{\partial\varphi} \\
&(\nabla^2+\lambda^2)\,w = \\
&\frac{u}{r}\frac{\partial\lambda}{\partial\theta}-v\frac{\partial\lambda}{\partial r}-\frac{2}{r^2\sin\theta}\frac{\partial u}{\partial\varphi}-\frac{2\cot\theta}{r^2\sin\theta}\frac{\partial v}{\partial\varphi}+\frac{w}{r^2\sin^2\theta},
\end{aligned}\right\}\quad\text{(II)}$$

where λ is the proportionality factor between the vorticity components ξ, η, ζ and the velocity components.*

(*b*) The values of u, v, w satisfying the equations of motion, the equation of continuity and the equations $\xi=\lambda u$, $\eta=\lambda v$, $\zeta=\lambda w$ are given by

$$\lambda(u,\ v,\ w) = (\varphi_1,\ \varphi_2,\ \varphi_3)\ e^{-\nu\lambda^2 t},$$

where λ is supposed to be a constant and $\varphi_1, \varphi_2, \varphi_3$ are three functions depending only on the space variables and satisfying the equations

$$\left.\begin{aligned}
\lambda\varphi_1 &= \frac{1}{r}\frac{\partial\varphi_3}{\partial\varphi}-\frac{\partial\varphi_2}{\partial z} && \text{(i)} \\
\lambda\varphi_2 &= \frac{\partial\varphi_1}{\partial z}-\frac{\partial\varphi_3}{\partial r} && \text{(ii)} \\
\lambda r\varphi_3 &= \frac{\partial}{\partial r}(r\varphi_2)-\frac{\partial\varphi_1}{\partial\varphi} && \text{(iii)} \\
0 &= \frac{\partial}{\partial r}(r\varphi_1)+\frac{\partial\varphi_2}{\partial\varphi}+r\frac{\partial\varphi_3}{\partial z}, && \text{(iv)}
\end{aligned}\right\}\quad\text{(III)}$$

or

$$\left.\begin{aligned}
\lambda\varphi_1 &= \frac{1}{r}\frac{\partial\varphi_3}{\partial\theta}-\frac{1}{r\sin\theta}\frac{\partial\varphi_2}{\partial\varphi}+\frac{\varphi_3}{r}\cot\theta && \text{(i)} \\
\lambda\varphi_2 &= \frac{1}{r\sin\theta}\frac{\partial\varphi_1}{\partial\varphi}-\frac{\partial\varphi_3}{\partial r}-\frac{\varphi_3}{r} && \text{(ii)} \\
\lambda\varphi_3 &= \frac{\partial\varphi_2}{\partial r}+\frac{\varphi_2}{r}-\frac{1}{r}\frac{\partial\varphi_1}{\partial\theta} && \text{(iii)} \\
0 &= \frac{1}{r^2}\frac{\partial}{\partial r}(\varphi_1 r^2)+\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\varphi_2\sin\theta)+\frac{1}{r\sin\theta}\frac{\partial\varphi_3}{\partial\varphi}, && \text{(iv)}
\end{aligned}\right\}\quad\text{(IV)}$$

* These equations are obtained by using the relations $\xi=\lambda u$, $\eta=\lambda v$, $\zeta=\lambda w$ and the equation of continuity.

according as we work in cylindrical or in spherical polar coordinates.

We proceed to solve the systems (III) and (IV) in sections A and B respectively by supposing one of the functions φ_1, φ_2, φ_3 to be identically zero.

SECTION A

1. If $\varphi_1 = 0$, the values of φ_2 and φ_3 as obtained from equations (ii) and (iii) are

$$\varphi_2 = -\alpha J_0'(\lambda r) - \beta Y_0'(\lambda r) \text{ and } \varphi_3 = \alpha J_0(\lambda r) + \beta Y_0(\lambda r),$$

where α and β are functions of φ and z only and

$$Y_0(\lambda r) = J_0(\lambda r) \log \lambda r + 2\left\{ J_2(\lambda r) - \tfrac{1}{2} J_4(\lambda r) + \tfrac{1}{3} J_6(\lambda r) - \ldots \right\}.$$

Substituting the values of φ_2 and φ_3 in (i) and (iv) we get

$$\frac{\partial \alpha}{\partial \varphi} J_0(\lambda r) + \frac{\partial \beta}{\partial \varphi} Y_0(\lambda r) + r\frac{\partial \alpha}{\partial z} J_0'(\lambda r) + r\frac{\partial \beta}{\partial z} Y_0'(\lambda r) = 0$$

$$r\frac{\partial \alpha}{\partial z} J_0(\lambda r) + r\frac{\partial \beta}{\partial z} Y_0(\lambda r) - \frac{\partial \alpha}{\partial \varphi} J_0'(\lambda r) - \frac{\partial \beta}{\partial \varphi} Y_0'(\lambda r) = 0.$$

Now $Y_0(\lambda r)$ and $Y_0'(\lambda r)$ contain terms involving $\log \lambda r$ and in order that the above may be true we must have

$$\frac{\partial \beta}{\partial \varphi} = \frac{\partial \beta}{\partial z} = 0 \text{ and } \frac{\partial \alpha}{\partial \varphi} = \frac{\partial \alpha}{\partial z} = 0,$$

i.e. α and β must be absolute constants. The solution is

$$\begin{aligned} u &= 0 \\ \lambda v &= -[\alpha J_0'(\lambda r) + \beta Y_0'(\lambda r)]e^{-\nu\lambda^2 t} \\ \lambda w &= [\alpha J_0(\lambda r) + \beta Y_0(\lambda r)]e^{-\nu\lambda^2 t}. \end{aligned}$$

On a fixed boundary v and w must also vanish, i.e. we must have

$$\frac{\alpha}{\beta} = -\frac{Y_0'(\lambda r)}{J_0'(\lambda r)} = -\frac{Y_0(\lambda r)}{J_0(\lambda r)}.$$

These have no common solution and the above values of u, v and w are valid only for an unbounded fluid extending to infinity. The stream lines and the vortex-lines are cylindrical helices. The motion decays exponentially with time and the decay is slow for fluids of low kinematic viscosity and rapid for fluids of high kinematic viscosity.

2. If $\varphi_2 = 0$, the values of φ_1 and φ_3 as obtained from equations (i) and (iii) are

$$\varphi_1 = \delta \cos(\lambda r\varphi) - \gamma \sin(\lambda r\varphi), \quad \varphi_3 = \gamma \cos(\lambda r\varphi) + \delta \sin(\lambda r\varphi),$$

where γ and δ are functions of r and z only.

These satisfy (iii) if

$$\left(\frac{\partial\delta}{\partial z} - \frac{\partial\gamma}{\partial r}\right)\cos(\lambda r\varphi) - \left(\frac{\partial\gamma}{\partial z} + \frac{\partial\delta}{\partial r}\right)\sin(\lambda r\varphi)$$
$$+\lambda_\varphi \gamma \sin(\lambda r\varphi) - \delta\lambda_\varphi \cos(\lambda r\varphi) = 0,$$

i.e. if $\gamma = \delta = 0$; so that there exists no solution in this case.

3. If $\varphi_3 = 0$, we get the result already obtained in the previous paper referred to above.

4. Passing on to the case where λ is a function of one space variable and t alone, we need consider the variables ϕ and r and omit z, which has already been considered.*

(i) λ cannot be a function of φ and t alone; for the equations $\xi = \lambda u$, etc. and the equation of continuity give

$$u\frac{\partial\lambda}{\partial r} + \frac{v}{r}\frac{\delta\lambda}{\delta\varphi} + w\frac{\partial\lambda}{\partial z} = 0, \tag{4.1}$$

which requires that v must identically vanish if $\lambda_\phi \neq 0$, and it is easy to verify that there does not exist a solution satisfying $\xi = \lambda u$ etc. with $v \equiv 0$.

(ii) If λ is a function of r and t alone, the treatment becomes difficult. But λ cannot be a function of r alone.

If $\lambda_r \neq 0$ and λ_φ and λ_z are zero, $u \equiv 0$ from (4·1).

* Self-Superposable Fluid Motions of the type $\xi = \lambda u$, etc. *Proc, Benares Math. Soc.* (1940).

The equations $\xi = \lambda u$ etc. reduce to

$$\frac{1}{r}\frac{\partial w}{\partial \varphi} - \frac{\partial v}{\partial z} = 0 \tag{4.2}$$

$$\lambda v = -\frac{\partial w}{\partial r} \tag{4.3}$$

$$\lambda r w = \frac{\partial}{\partial r}(rv). \tag{4.4}$$

The equation of continuity is

$$\frac{\partial v}{\partial \varphi} + r\frac{\partial w}{\partial z} = 0. \tag{4.5}$$

Using (I) we can write the equations of motion as

$$\frac{\partial \chi'}{\partial r} = 0, \quad -\frac{1}{r}\frac{\partial \chi'}{\partial \varphi} = \frac{\partial v}{\partial t} + \nu(\lambda^2 v - w\lambda_r),$$

$$-\frac{\partial \chi'}{\partial z} = \frac{\partial w}{\partial t} + \nu(\lambda^2 w + v\lambda_r),$$

where $\chi' = \frac{p}{\rho} + \frac{1}{2}q^2 + \Omega$.

The conditions of integrability of these require that

$$\frac{\partial^2 \chi'}{\partial \varphi\,\partial z} = \frac{\partial^2 \chi'}{\partial z\,\partial \varphi}; \quad \frac{\partial^2 \chi'}{\partial r\,\partial \varphi} = 0 \text{ and } \frac{\partial^2 \chi'}{\partial r\,\partial z} = 0.$$

The first of these is identically satisfied and we get from the last two by using (4.3) and (4.4)

$$v = \frac{Be^{-(a+c)t}}{3\nu r\lambda\lambda_r}\left\{ r\lambda c + \nu\lambda_r \right\} - \frac{A\,e^{(c-a)t}}{3\,\nu r\lambda\lambda_r}\left\{ r\lambda c - \nu\lambda_r \right\}$$

and $w = Ae^{(c-a)t} + Be^{-(a+c)t}$, where

$$a = \frac{\nu\lambda^3 r - \nu r\lambda_{rr}}{r\,\lambda}, \quad b = \frac{1}{r^2\lambda^2}\left\{ (\nu\lambda^3 r - \nu r\lambda_{rr})^2 - \nu^2\lambda_r^2 + 9\nu^2 r^2\lambda^2\lambda_r^2 \right\}$$

$c = \frac{\nu\lambda_r}{r\lambda}(1 - 9r^2\lambda^2)^{\frac{1}{2}}$ and A, B are functions of r, φ, z alone.

Substituting the values of v and w in (4.3) we get

$$\frac{\lambda A\,e^{(c-a)t}}{3\nu r\lambda\lambda_r}(r\lambda c - \nu\lambda_r) - \frac{\lambda B\,e^{-(a+c)t}}{3\nu r\lambda\lambda_r}(r\lambda c + \nu\lambda_r) =$$

$$\frac{\partial A}{\partial r}e^{(c-a)t} + \frac{\partial B}{\partial r}e^{-(a+c)t} + At\left(\frac{\partial c}{\partial r} - \frac{\partial a}{\partial r}\right)e^{(c-a)t} + Bt\left\{-\frac{\partial c}{\partial r} - \frac{\partial a}{\partial r}\right\}e^{-(a+c)t}$$

so that $\frac{\partial a}{\partial r} = 0$, $\frac{\partial c}{\partial r} = 0$, i.e. a and c are constants and

$$\frac{\partial A}{\partial r} = \frac{\lambda A(r\lambda c - \nu\lambda_r)}{3\nu r\lambda\lambda_r} \text{ and } \frac{\partial B}{\partial r} = -\frac{\lambda B(r\lambda c + \nu\lambda_r)}{3\nu r\lambda\lambda_r}.$$

Substituting the values of v and w in (4.4) and making use of the above results we get

$$(r\lambda c - \nu\lambda_r)^2 + 9\nu^2 r^2 \lambda^2 \lambda_r^2 + 9\nu^2 \lambda r \lambda_r^2 \frac{\partial}{\partial r}\left(\frac{r\lambda c - \nu\lambda_r}{3\nu\lambda\lambda_r}\right) = 0. \quad (4.6)$$

and $$(r\lambda c + \nu\lambda_r)^2 + 9\nu^2 r^2 \lambda^2 \lambda_r^2 - 9\nu^2 \lambda r \lambda_r^2 \frac{\partial}{\partial r}\left(\frac{r\lambda c + \nu\lambda_r}{3\nu\lambda\lambda_r}\right) = 0. \quad (4.7)$$

Adding and substituting the value of c^2, we get on integrating $\lambda = e/r^{2/3}$, where e is a constant.

Subtracting (4.7) from (4.6) we get on simplification $\lambda_r = 3r\lambda_{rr}$ which is not satisfied by the value of λ just obtained.

But if λ is a constant, (4.6) and (4.7) are identically satisfied and a and c reduce to constants. In that case, however, u need not be zero. But if u is zero, the solution is as obtained in §1.

SECTION B

Here we are concerned with the treatment in spherical polars only.

1. If $\varphi_1 = 0$, the values of φ_2 and φ_3 as obtained from the system (IV) are

$$\varphi_2 = -\frac{1}{r\sin\theta}\frac{\partial\chi}{\partial\varphi}\sin\lambda r + \frac{1}{r}\frac{\partial\chi}{\partial\theta}\cos\lambda r,$$

$$\varphi_3 = \frac{1}{r\sin\theta}\frac{\partial\chi}{\partial\varphi}\cos\lambda r - \frac{1}{r}\frac{\partial\chi}{\partial\theta}\sin\lambda r,$$

where χ is a function of θ and φ only, satisfying

$$\sin\theta\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\chi}{\partial\theta}\right) + \frac{\partial^2\chi}{\partial\varphi^2} = 0$$

so that

$$u = 0,\ \lambda v = \left(\frac{1}{r \sin\theta} \frac{\partial \chi}{\partial \varphi} \sin \lambda r + \frac{1}{r} \frac{\partial \chi}{\partial \theta} \cos \lambda r \right) e^{-\nu \lambda^2 t}$$

$$\lambda w = \left(\frac{1}{r \sin\theta} \frac{\partial \chi}{\partial \varphi} \cos \lambda r - \frac{1}{r} \frac{\partial \chi}{\partial \theta} \sin \lambda r \right) e^{-\nu \lambda^2 t}.$$

The above is motion on concentric spheres. The boundaries for a non-viscous fluid may be any two concentric spheres. In the case of a viscous fluid the velocities v, w cannot vanish simultaneously on any boundary $r = a$.

2. If $\varphi_2 = 0$, (i) and (iii) give

$$(1 - \mu^2) \frac{\partial^2 \varphi_1}{\partial \mu^2} - 2\mu \frac{\partial \varphi_1}{\partial \mu} + \lambda^2 r^2 \varphi_1 = 0,$$

where $\mu = \cos\theta$.

If we put $\lambda^2 r^2 = n\ (n+1)$, the equation becomes the well-known Legendre equation and the consideration of the various possible cases determined according to the value of n shows that no solution is possible, because in satisfying the two remaining equations of (IV) we have to differentiate with respect to r, which involves terms in $\log \cos\theta$, which cannot be made to disappear unless $\varphi_1 = 0$ and $\varphi_3 = 0$.

3. If $\varphi_3 = 0$, there exists no solution as is obvious from § 2 of section A.

4. We now propose to discuss if λ can be a function of t, and only one of the variables r, θ and φ. For this purpose we deduce the equation

$$u \frac{\partial \lambda}{\partial r} + \frac{v}{r} \frac{\partial \lambda}{\partial \theta} + \frac{w}{r \sin\theta} \frac{\partial \lambda}{\partial \varphi} = 0 \tag{4.1}$$

with the help of the relations $\xi = \lambda u$ etc. and the equation of continuity.

(i) If λ is a function of r and t alone and $\lambda_r \neq 0$, $u = 0$ from (4.1) and the solution of $\xi = \lambda u$ etc. satisfying the equation of continuity is

$$u = 0$$

$$v = \frac{1}{r \sin\theta} \chi_\phi \sin\omega + \frac{1}{r} \chi_\theta \cos\omega$$

$$w = \frac{1}{r \sin\theta} \chi_\phi \cos\omega - \frac{1}{r} \chi_\theta \sin\omega,$$

where $\omega = \int \lambda dr$ and χ is a function of θ, φ and t alone satisfying $\sin\theta \frac{\partial}{\partial\theta}(\sin\theta\chi_\theta) + \chi_{\phi\phi} = 0$.

The values of u, v and w have yet to satisfy the equations of motion. Proceeding as in a previous paper* we conclude that λ must be an absolute constant.

(ii) λ cannot be a function of φ and t alone. This is obvious from § 4(i) of section A.

(iii) If λ is a function of θ and t alone, $v = 0$ from (4.1) and the equations $\xi = \lambda u$, etc. and the equation of continuity reduce to

$$\frac{1}{r}\frac{\partial w}{\partial\theta} + \frac{w}{r}\cot\theta = \lambda u, \quad \frac{\partial u}{\partial\varphi} - \sin\theta\frac{\partial}{\partial r}(rw) = 0, \quad -\frac{1}{r}\frac{\partial u}{\partial\theta} = \lambda w$$

and $\frac{1}{r^2}\frac{\partial}{\partial r}(ur^2) + \frac{1}{r\sin\theta}\frac{\partial w}{\partial\varphi} = 0.$

Using (II), the equations of motion can be written as

$$\frac{\partial u}{\partial t} + \nu\left(\lambda^2 u + \frac{w}{r}\lambda_\theta\right) = -\frac{\partial\chi'}{\partial r}, \quad 0 = -\frac{1}{r}\frac{\partial\chi'}{\partial\theta},$$

$$\frac{\partial w}{\partial t} + \nu\left(\lambda^2 w - \frac{u}{r}\lambda_\theta\right) = -\frac{1}{r\sin\theta}\frac{\partial\chi'}{\partial\varphi}.$$

We must have $\frac{\partial^2\chi'}{\partial\varphi\partial r} = \frac{\partial^2\chi'}{\partial r\partial\varphi}$, which means that $u = 0$ and hence $w = 0$, so that there exists no solution in this case.

* Self-Superposable Fluid Motions of the type $\xi = \lambda u$ etc. *Proc. Benares Math. Soc.* (1940).

STUDIES IN FOURIER ANSATZ AND PARABOLIC EQUATIONS*

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CHAPTER V

The Non-Linear Equation

1. We shall consider in this chapter the non-linear equation,

$$\delta u = \frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial t} = f\left(x, t, u, \frac{\partial u}{\partial x}\right) \tag{5.1}$$

for the homogeneous boundary condition

$$u(0, t) = u\ (2\pi, t) \tag{5.2}$$

$$\frac{\partial u}{\partial x}(0, t) = \frac{\partial u}{\partial x}(2\pi, t) \tag{5.3}$$

or

$$\int_{t_1}^{t_2} \frac{\partial u}{\partial x}(0, t) dt = \int_{t_1}^{t_2} \frac{\partial u}{\partial x}(2\pi, t)\ dt, \tag{5.4}$$

$$u(x, 0) = u_0(x) \tag{5.5}$$

and discuss the existence of a solution of (5·1). The crux of the method consists in reducing the above problem, to a problem in continuous transformation of an abstract space into itself and then applying Fixpunktsatz of Schauder. We write as in the last Chapter § 4

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial t} = \rho(x, t), \tag{5.6}$$

then u and $p(x, t)$ are linear transformations of $\rho(x, t)$, viz.

$$u = L_1(\rho) \tag{5.7}$$

$$p = L_2(\rho). \tag{5.8}$$

*Continued from the *Jour. Madras University*, 14 (1942), 73-142. These chapters form Part I of the thesis approved for the D. Sc. degree of the Madras University.

Then (5.1) can be put symbolically in the form

$$\rho = f[x, t, L_1(\rho), L_2(\rho)]. \tag{5.9}$$

Now if $\rho(x, t)$ belongs to a certain complete space S_1 then $u = L_1(\rho)$ and $p = L_2(\rho)$ will belong to two other spaces S_1 and S_2. By proper choice of hypotheses on the function f on the right of (5·1), we recognise the transformation

$$\bar{\rho} = f[x, t, L_1(\rho), L_2(\rho)] \tag{5.10}$$

as a continuous transformation in the space S_1, with its range and domain in S_1.

Using the inequalities of the last chapter we can prove the existence of a solution. In general (5·10) will be simply a continuous—or a weakly continuous—transformation, and then we require ρ to belong to a separable space, if we were to apply Lemma 4 of Chapter I. If however the function on the right of (5.1) is independent of $\partial u/\partial x$, we can recognise (5.10) as a completely continuous transformation and then we can apply Lemma 3 of Chapter I.

2. We shall first consider the case where the function on the right of (5.1) is independent of $\partial u/\partial x$, that is, the equation

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial t} = f(x, t, u). \tag{5.11}$$

We know already that the space of bounded measurable functions is not separable. Now let $\rho(x, t)$ defined in (5.6) belong to the space B defined in the rectangle $0 \leqslant x \leqslant 2\pi$, $0 \leqslant t \leqslant T$ and let $f(x, t, u)$ be a bounded measurable function of the three variables in every bounded domain $0 \leqslant x \leqslant 2\pi$, $0 \leqslant t \leqslant T$, $|u| \leqslant K$ uniformly continuous with respect to u. Let there exist a positive monotonic increasing function $\phi(r)$ with $\phi(0)=0$ such that

$$|f(x, t, u)| \leqslant \phi(t+|u|). \tag{5.12}$$

Since $f(x, t, u)$ is uniformly continuous with respect to u it is easily observed that $f[x, t, L_1(\rho)]$ is a completely continuous transformation of the space B into itself; for it, ρ_n converges weakly to ρ, $u_n(x, t)$ converges strongly to $u(x, t)$ and then $f(x, t, u_n)$ converges strongly to $f(x, t, u)$ in the space B.

Now let the function $u_0(x)$ defined in (5.5) satisfy the inequality $|u_0(x)| < N$ and let $|\rho(x, t)| \leqslant M$. Then, by Chapter IV, $|u(x, t)| < N + Mt$. (5.16)

In the space B, the hypersphere

$$\|\rho(x,t)\| \leqslant M$$

is a closed convex set, which is weakly compact and the completely continuous transformation

$$\begin{aligned}\bar{\rho} &= f[x, t, L_1(\rho)] \\ &= F(\rho) \quad \text{say}\end{aligned} \tag{5.17}$$

transforms the sphere (5.16) into the sphere

$$\begin{aligned}\|\bar{\rho}\| &\leqslant \phi(t+N+Mt) \\ &= \phi(N+\overline{1+M}t).\end{aligned} \tag{5.18}$$

Let N be sufficiently small so that

$$\phi(N) \leqslant M, \tag{5.19}$$

or we can choose M so as to satisfy inequality (5.19) and then t can be chosen so small, say $t=\delta$, that

$$\phi(N+\overline{1+M}\,t) \leqslant M \tag{5.20}$$

for $t \leqslant \delta$. For such values of t, (5.17) is a completely continuous transformation transforming the sphere (5.16) into itself. Therefore by Lemma 3 of Chapter I, there exists an invariant point of the transformation (5.17) that is to say, there exists a function $\rho(x, t) \in B$ so that

$$\rho = F(\rho) = f[x, t, L_1(\rho)].$$

This means that (5·11) has a solution fulfilling the boundary conditions (5·2), (5·3) and (5·5).

THEOREM 1. *If the bounded measurable function $f(x, t, u)$ of three variables, is uniformly continuous with respect u, satisfying the inequality*

$$|f(x, t, u)| \leqslant \phi(t + |u|),$$

where $\phi(\gamma)$ is a positive monotonic increasing function vanishing at the origin, then the equation (5·1) *has a solution fulfilling the boundary conditions* (5·2), (5·3) *and* (5·5) *for sufficiently small values of t.*

The limit for t depends on the initial function $u_0(x)$.

If on the other hand

$$|f(x, t, u)| \leqslant \phi(t) \qquad (5·21)$$

so that the bound on the right is independent of u, then we can show that there exists a solution for all t. For whatever the rectangle R be, viz. $0 \leqslant x \leqslant 2\pi$, $0 \leqslant t \leqslant T$, if we choose $M \geqslant \phi(T)$ we see that the sphere (5·16) is transformed into itself and so there exists a solution of (5·1) for all finite t.

THEOREM 2. *If $f(x, t, u)$ satisfies the inequality* (5·21) *but otherwise the remaining hypotheses of Theorem* 1 *are satisfied, there exists a solution of* (5·1) *as in Theorem* 1, *for all finite t.*

3. We shall now proceed to the equation

$$\delta u = f(x, t, u, \partial u/\partial x).$$

Let us suppose that $\rho(x, t)$ belongs to the space H_α, $0 < \alpha \leqslant 1$. Then in (5.6) u and $\partial u/\partial x$ are continuous and $\partial^2 u/\partial x^2$ is bounded. Therefore $u(x, t)$ and $p(x, t) = \partial u/\partial x$, belong to the Lipschitz class i.e. H_1. Now let us assume that $f(x, t, r, s)$ is continuous with respect to all the variables and periodic with respect to x and belongs to H_α with respect to the variables x, r and s that is,

$$|f(x_1, t, r_1, s_1) - f(x_2, t, r_2, s_2)|$$
$$\leqslant c\{|x_1 - x_2|^\alpha + |r_1 - r_2|^\alpha + |s_1 - s_2|^\alpha\}, \qquad (5·22)$$

where c may depend on t, vanishing for $t = 0$. Let us further assume that

$$\left.\begin{matrix}|u_0(x)| \\ |u'_0(x)|\end{matrix}\right\} \leqslant N \tag{5.23}$$

and

$$||f(x, t, u, p)|| \leqslant \phi\ (t + |u| + |p|). \tag{5.24}$$

If

$$||\rho(x, t)|| \leqslant M \tag{5.25}$$

then

$$|u(x, t)| \leqslant N + Mt \tag{5.26}$$

$$|p(x, t)| \leqslant N + kMt^{\frac{1-\alpha}{2}}, \tag{5.27}$$

where k is some positive constant. Then the transformation (5.10) will transform the hypersphere (5.25) into the hypersphere

$$||\bar{\rho}|| \leqslant (\phi\ (t + 2N + Mt + kMt^{\frac{1-\alpha}{2}}) \tag{5.28}$$

$$\leqslant M \tag{5.29}$$

provided $\phi(2N) < M$ and t is sufficiently small. For such small values of t depending on the initial function $u_0(x)$, we see by applying Lemma 4 of Chapter I that there exists a solution of (5.1) for the boundary conditions (5.2), (5.3) and (5.5).

THEOREM 3. *If the function $f(x, t, r, s)$ is periodic with respect to x, belongs to H_α with respect to x, r and s and is continuous with respect to t, there exists a solution of* (5.1) *for small values of t, when f satisfies the condition* (5.24).

3.1 We can similarly extend the theorem to the space H^p_α. If $\rho(x, t) \in H^p_\alpha$ then we have seen that $\partial^2 u/\partial x^2 \in L^p$ so that $\partial u/\partial x$ being the integral of a L^p function belongs to H_α. Now let $f(x,t,r,s)$ be a bounded continuous function of the variables t, r and s belonging to H^p with respect to x and to the space H_1—i.e. Lipschitzian—with respect to r and s,

$$|f(x, t, r_1, s_1) - f(x, t, r_2, s_2)| \leqslant k(|r_1 - r_2| + |s_1 - s_2|). \tag{5.30}$$

Then

$$\left\{\int_0^{2\pi} |f[x+h, t, u(x+h, t)\, p(x+h, t)] - f[x, t, u(x, t)\, p(x, t)]|^p dx\right\}^{\frac{1}{p}}$$

$$\leqslant k\left[h^\alpha + \left\{\int_0^{2\pi} |u(x+h, t) - u(x, t)|^p dx\right\}^{\frac{1}{p}}\right.$$

$$\left. + \left\{\int_0^{2\pi} |p(x+h, t) - p(x, t)|^p\, dx\right\}^{\frac{1}{p}}\right]$$

$$\leqslant k(h^\alpha + h + h) \tag{5.31}$$

$$\leqslant k\, h^\alpha,$$

so that $f(x, t, u, p)$ belongs to H_α^p. Therefore if $f(x,t,r,s)$ satisfies (5.24), $\bar{\rho} = f[x, t, L_1(\rho), L_2(\rho)]$ will be a continuous transformation in the space H_α^p into itself and there will exist a solution for small values of t.

We can find a solution for all values of t if the bound for f is independent of u and p, that is if we have an inequality similar to (5·21) instead of (5·24).

4. We shall now pass to the space C' of continuous functions not necessarily periodic. Let $f(x, t, r, s)$ be a continuous function of all the variables, and let

$$|f(x, t, r, s)| \leqslant \phi(t + |r| + |s|). \tag{5.32}$$

Then the transformation

$$\bar{\rho} = f\,[x, t, L_1\,(\rho), L_2(\rho)] = F\,(\rho)$$

transforms the sphere $||\rho|| \leqslant M$ into the sphere

$$||\bar{\rho}|| \leqslant \phi(2N + t + Mt + KM\sqrt{t}\,) \leqslant M$$

provided $M > \phi(2N)$ and t is small. Though the space C' is separable, the sphere $||\rho|| \leqslant M$ is not weakly compact. We can however proceed thus. In the space of continuous functions consider the set H_M defined by

$$u = L_1(\rho),$$

where $\rho \varepsilon C'$ and $||\rho|| \leqslant M$. Since $L_1(\rho)$ is a completely continuous transformation, this set H_M is closed, convex and also compact. In this space, the transformation

$$f(x, t, u, \partial u/\partial x)$$

is continuous, since, if $\rho_n \to \rho$, $u_n \to u$ and $\partial u_n/\partial x \to \partial u/\partial x$ and f is continuous with respect to all the variables.

Now consider the functional equation

$$\bar{u} = L_1[f(x, t, u, \partial u/\partial x)].$$

This will transform the set H_M into itself provided $M \geqslant \phi\ (2N)$ and t is small. Then by Chapter I, Lemma 3 there exists an invariant point and hence a solution of (5.1). Hence we have

THEOREM 4. *If the function $f(x, t, r, s)$ is continuous with respect to all the variables, satisfying the inequality* (5.32), *then there exists a solution of* (5.1) *for the boundary condition* (5.2), (5.3) *and* (5.5) *for small values of t.*

If instead of (5.32), (5.21) is fulfilled a solution exists for all values of t.

5. We shall conclude this Chapter after making a few observations on the uniqueness of the solution defined in the previous section.

5.1 If the function $f(x, t, u, p)$ is Lipschitzian with respect to u and p, the transformation

$$\bar{\rho} = f[x, t, L_1(\rho), L_2(\rho)]$$

considered as a transformation in the space C or L^p will be such that

$$\begin{aligned} ||\bar{\rho}_1 - \bar{\rho}_2|| &\leqslant ||f[x, t, L_1(\rho_1), L_2(\rho_1)] - f[x, t, L_1(\rho_2), L_2(\rho_2)]|| \\ &\leqslant k\{ ||L_1(\rho_1) - L_1(\rho_2)|| + ||L_2(\rho_1) - L_2(\rho_2)|| \} \\ &\leqslant kt\, ||\rho_1 - \rho_2||, \end{aligned}$$

by Chapter IV, (4.31) and (4.34) with $\gamma = 0$. Therefore if $kt < 1$, that is if t is sufficiently small, we can apply the method of successive approximations and prove not only the existence but also uniqueness of the solution.

5.2 Now consider the equation

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial t} = f\left(x, t, u, \frac{\partial u}{\partial x}\right)$$

for the boundary conditions (5.2), (5.3) and (5.5). Let

$$u(x, t) \sim \tfrac{1}{2}\alpha_0(t) + \sum_{n=1}^{\infty} \Big(\alpha_n(t) \cos nx + \beta_n(t) \sin nx \Big) \quad (5.34)$$

$$f(x, t, u, p) \sim \tfrac{1}{2} F_0[t;\ \alpha_n(t);\ \beta_n(t)]$$

$$+ \sum_{r1}^{\infty} \Big\{ F_\nu[t;\ \alpha_n(t);\ \beta_n(t)] \cos \nu x + G_\nu[t;\ \alpha_n(t);\ \beta_n(t)] \sin \nu(x) \Big\}. \quad (5.35)$$

Then we have by Chapter IV

$$\left.\begin{aligned} \alpha_n(t) + n^2 \int_0^t a_n(s)\, ds &= a_n - \int_0^t F_n(s; \alpha_\nu(s); \beta_\nu(s)\, ds \\ \beta_n(t) + n^2 \int_0^t \beta_n(s)\, ds &= b_n - \int_0^t G_n[s; \alpha_\nu(s); \beta_\nu(s)]\, ds. \end{aligned}\right\} \quad (5.36)$$

If $v(x, t)$ is another solution of (5.1) with

$$v(x, t) \sim \tfrac{1}{2} \gamma_0(t) + \sum_{n=1}^{\infty} \Big(\gamma_n(t) \cos nx + \delta_n(t) \sin nx \Big) \quad (5.37)$$

then the Fourier coefficients $\gamma_n(t)$; $\delta_n(t)$ will satisfy relations analogous to (5.36).

Now consider the integral

$$\int_0^t \Big\{ \alpha_n(s) - \gamma_n(s) \Big\} \Big\{ F_n[s; \alpha_\nu(s); \beta_\nu(s)] - F_n[s;\ \gamma_\nu(s);\ \delta_\nu(s)] \Big\}\, ds$$

$$= \int_0^t (\alpha_n(s) - \gamma_n(s))\ d\theta_n(s), \quad (5.38)$$

where

$$\theta_n(t) = -\Big\{ [\alpha_n(t) - \gamma_n(t)] + n^2 \int_0^t [\alpha_n(s) - \gamma_n(s)]\, ds \Big\}.$$

Hence the right side of (5.38) is

$$-\int_0^t \Big\{ \alpha_n(s) - \gamma_n(s) \Big\} d[\alpha_n(s) - \gamma_n(s)] - n^2 \int_0^t [\alpha_n(s) - \gamma_n(s)]^2 ds$$

$$= -\tfrac{1}{2}[\alpha_n(t) - \gamma_n(t)]^2 - n^2 \int_0^t [\alpha_n(s) - \gamma_n(s)]^2 ds,$$

so that (5.38) leads to

$$[\alpha_n(t) - \gamma_n(t)]^2 = -2n^2 \int_0^t [\alpha_n(s) - \gamma_n(s)]^2 ds$$

$$-2 \int_0^t (\alpha_n - \gamma_n)[F_n(s;\ \alpha_\nu;\ \beta_\nu) - F_n(s;\ \gamma_\nu;\ \delta_\nu)]\, ds. \quad (5.39_1)$$

Similarly

$$[\beta_n(t)-\delta_n(t)]^2 = -2n^2\int_0^t [\beta_n(s)-\delta_n(s)]^2 ds$$

$$-2\int_0^t (\beta_n-\delta_n)\ [G_n(s;\ \alpha_\nu;\ \beta_\nu)-G_n(s;\ \gamma_\nu;\ \delta_\nu)]ds. \quad (5{\cdot}39_2)$$

Adding $(5{\cdot}39_1)$ and $(5{\cdot}39_2)$ and then summing from $n=$ 1 to ∞, we see

$$\int_0^{2\pi}(u-v)^2 dx = -2\int_0^t\int_0^{2\pi}\left(\frac{\partial u}{\partial x}-\frac{\partial v}{\partial x}\right)^2 dxdt$$

$$-2\int_0^t\int_0^{2\pi}(u-v)\ [f(x,\ t,\ u,\ p)-f(x,\ t,\ v,\ q)]\ dx\ dt$$

or

$$\int_0^{2\pi}(u-v)^2 dx+2\int_0^t\int_0^{2\pi}\left(\frac{\partial u}{\partial x}-\frac{\partial v}{\partial x}\right)^2 dxdt$$

$$=-2\int_0^t\int_0^{2\pi}(u-v)[f(x,\ t,\ u,\ p)-f(x,\ t,\ v,\ q)]dxdt. \quad (5.40)$$

The left side of (5.40) is always positive and the right side will be always negative if $f(x,\ t,\ u,\ \partial u/\partial x)$ is a monotonic function of u independently of the other variables. This contradiction will establish the fact that a solution of (5.1) if it exists must be unique.

THEOREM 5. *If the bounded measurable function $f\ (x,\ t,\ r,\ s)$ is a monotonic increasing function of r independently of the other variables, there cannot exist more than one solution of* (5.1).

Combining this condition of monotony with the other conditions defined in the previous sections we prove that there exist unique solutions.

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