NOTICE

Members and Subscribers who have not already paid their subscriptions for the current year are requested to do so early by Money order or by cheque (including Bank commission for places other than Madras) payable to the undersigned. Those whose subscriptions are in arrears will be materially helping the Society by bringing their subscriptions up to date.

A. NARASINGA RAO
Treasurer, Indian Math. Society,
ANNAMALAINAGAR P. O. Madras Presidency.

THE MATHEMATICS STUDENT

Volume XIII] SEPTEMBER 1945 [No. 3

PRECESSION OR AYANAMSA

BY

R. KRISHNAMURTI, *Hyderabad (Deccan)*

It is a pity that present-day Indians have not a correct knowledge of the teachings of the ancient Rishis about Precession or Ayanamsa. Precession phenomena have been described by the Rishis in the Rig-Samhita itself, the oldest spoken word of man, and the correct value of the rate of precession is also given in the same text. This phenomenon is discussed both in the Brahmanas and in the Upanishads. Later Samhitas (astronomical treatises) and Siddhantas give a full account of the phenomenon and the various constants based on it. But during recent times, a confusion in thought, has been caused by the imperfect and uncharitable translations and interpretations of our texts by oriental scholars and the imperfect understanding of some of our Pandits.

Vedic words convey different meanings as also Vedic sentences. The language and grammar are very old and several words have lost their original meaning and have also acquired a new and different meaning. Agni and Indra in the Sruti have got several meanings. As astronomical terms they stand for the vernal and autumnal equinoxes. This idea of the Rishis could be understood when we see that the presiding deity for *Krittika* is *Agni* and that of *Visakha*, the diametrically opposite star, is *Indragni*. *Chakshur*, (*Indra*), *Mitra*, *Agni*, and *Varuna* correspond to the solstices and equinoxes. If Agni is the deity of *Aswayuja* then Indra (*Twashta*) is the deity for *Chitra*. (*Taittiriya Samhita*). Thus *Agni* and *Indra* stand for the two diametrically opposite points on the Ecliptic, which constitute the two equinoxes.

Rig-Samhita 6-47-18 reads "*Rupam rupam pratirupo babhuva, tadasya rupam pratichakshanaya: Indro mayabhih pururupa iyate, yukta hyasya harayah sata dasa.*" The astronomical meaning of this verse is that the autumnal equinox reaches one nakshatra after another and obtains every nakshatra. He is in the West (*Pratichi*) and on the equator (*Akshanaya*). *Indra* is a Mayavi and takes many forms. Thousand horses are yoked (this refers to the decimal division of the ecliptic). Sruti gives the meaning of the word "*Rupam*" as "*Nakshtrani Rupam*; *Asvinau Vyattam*"; that *rupam* means nakshatra and *Asvini* stands first in the series. Rig-Samhita 3-53-8 "*Rupam rupam maghava bobhaviti*: *mayah krinvanastanvam pari swam*," conveys the same idea. Similarly it talks about *Agni* as the carrier of the oblations to gods wherever they may be, the gods being the nakshatras from *Asvini* to *Chitra*. Rig Samhita 10–5–3 reads "*Rutayini mayini sandadhate, mitva sisum jajnaturvardhayanti*: *visvasya nabhim carato dhruvasya kavescit antum manasa viyantah*." This means that the infant *Agni* (the Vernal equinox), is born as the point of conjunction of the parts of the ecliptic above and below the equator and that the celestial pole moves round the pole of the ecliptic (*Viswa*). The ecliptic is always spoken of as *Viswa* in the Srutis, as it constitutes all the twenty-seven nakshatras. Aryabhatta, in his Aryabhatiya mentions five important great circles of the celestial sphere, namely, the meridian circle, the equator circle, the prime vertical circle, the horizon circle and the ecliptic circle. The opening hymn of the *Purusha Sukta* speaks of these five circles as *Sirsha* circle, *Pata* circle, *Aksha* circle, *Bhumi* circle, and *Viswa* circle and says that a *dasa angula purushah atyatishtat*; i.e. that the positions of the celestial bodies are measured with the help of a vertical rod of ten angulas. (This is the decimal system; but in the sexagesimal system the rod is of twelve angulas) Aryabhatta gives the value of the obliquity of the ecliptic as 22½ degrees (Aryabhatiya, Gola, Verse 14) Purusha Sukta says "*Padosya Viswa Bhutani*" i-e the distance of *Viswa* (Ecliptic) from *Bhutani* (Equator) is one-fourth of a right-angle (*Pada*).

Thus in the Rig-Samhita we have definite statements that the celestial pole moves round the pole of the ecliptic, and this causes the precession motion of *Agni* (Vernal Equinox), and *Indra* (Autumnal Equinox) and other astronomical references.

The study of this phenomenon in the Brahmanas and Aranyakas was called *Pravragya Vidya*. The year being from *Asvini* to *Asvini*, the original Vishu or Equinox position, and as sacrifices were being performed with the help of Vishu position, the sacrifice to be performed in the interval between the Vishu position of the Sun and the Asvini position of the Sun was called pravargya. So annual sacrifices which were based on the lengths of the sidereal year had to be performed in the shape of the sacrifice for the tropical solar year and an additional sacrifice called pravargya. Taittiriya Aranyaka, First Prasna Fourth

Anuvaka, second panchati, reads "*Sa pravargyo Abhavat; Tasmadyassapravargyena yagnena yajate*". Sruti says "*Yagnovai samvatsarah*", that Yagna is the year. In the same Anuvaka the phenomenon is described in the following terms. "*Naiva deva na martyah na raja varuno vibhuh nagnir nendro napavamanah matrikkaccana vidyate: Divyasyaika dhanurartnih prithivya maparasritah tasyendro vamri rupena dhanurjya machinathsvayam*". This describes the celestial equator as a circle having no permanent divisions as deva part and asura part on account of its frequent change of positions on the celestial sphere. As the earth's axis is constantly wobbling, the terrestrial equatorial plane cuts the celestial sphere in constantly varying circles. Further it is stated that Indra cuts off the chord joining the ecliptic semicircle, very slightly, as a white-ant does. This describes the slight change in the position of the equinox. The same rik continues that this could be seen on the celestial sphere. The rik continues that the head of one of the bows of the ecliptic is crushed and this is the cause of pravargya. The phenomenon cannot be explained more clearly than this.

The Rig-Samhita gives the period of precession also, from which we could easily derive the rate of precession. Rig-Samhita (8-96-13), reads "*Ava drapso amsumatimatishtat iyanah krishno dasabhih sahasraih; avat tamindrah*". This says that the equinox point goes down to the Prithvi part of the ecliptic and Indra goes round the ecliptic 15 × 10 × 1,000 times, (in a period of Kalpa i.e. 4,320,000,000 years). From the above values we see that the time taken for one complete revolution is 28,800 years. This is the time taken by the equinox to gain a right ascension of 360 degrees. Thus the annual gain of right ascension is given as 45 seconds of arc and this by a simple mathematical transformation would give the annual change in celestial longitude 50·25 seconds of arc. The period for precession given by westerners as their latest is 25,867 years; this they give by dividing 360 degrees by the mean annual precession in longitude, whereas the Indian Rishis gave the period of precession by dividing 360 degrees by the mean annual precession in Right Ascension. Readers can judge for themselves which is the more accurate of the two.

The Indian Rishis were also aware that this phenomenon of revolution of the equinoxes round the ecliptic gave rise to the relative motion of the origin point Asvini with respect to the equinox point on the ecliptic. It is common knowledge that as a result of the motion of the Sun round the ecliptic in the course of a year, the Sun which rises due East on the Vishu or equinox day rises a few days after the equinox day at a point on the horizon of the observer a little deviated towards the North and on the summer solstice day the maximum distance towards the North from the East point. From here the point of sun-rise on the horizon turns towards the East and on the autumnal equinox day, it again rises exactly due East. After that the sun-rise takes place more and more towards the South of the East point and on the winter solstice day the Sun rises at the maximum distance towards the South from the East and this distance is also equal to the maximum distance gained towards the North. After that it comes back to its original position in the East. Similarly the Asvini origin point which moves relative to the equinoctial point around the ecliptic, oscillates on the horizon of the observer, finishing the complete oscillation in a period of 28,800 years. This phenomena of oscillation of the Asvini point on the horizon of an observer is given in Rig-Samhita as follows. "*Drapsascaskanda prithvimanudyam imamcha yonim anuyascu purvah; samanam yonim anusancarantam drapsam juhom anu sapta hotrah*" (R. V. 10-17-11.). This says that the Asvini point moves from east to north, travels back to its old position, goes back an equal distance on the other side and completes the whole round.

From the Rig-Samhita rik quoted previously it is seen that the number of complete revolutions in a chatur-yuga period of 4,320,000, years is 150. As each revolution gives rise to four different movements of Asvini origin on the horizon 1. from East to North 2. from Noth to East 3. from East to South and 4. South to East, the total number of movements are given as 150 × 4 or 600 parilambhanams. Rig-Sambita recognises the oscillation of the ascending point of the ecliptic on the horizon in 24 hours, of the rising Sun in the course of a year and of the Asvini Star in a period of 28,800 years.

So our later Samhitas and Siddhantas give either 150 or 600 as the fundamental value for the phenomenon of precession. Surya Siddhanta says "Trimsatkrityo Yuge" i.e 30 × 20 quarter oscillations in a chatur-yuga; Bhaskara's quotation of Surya Siddhanta "*Tad bhaganah sauroktu vyasta ayuta trayam kalpe*" i.e. 30,000 times, multiplied by 20, in a kalpa, viz: 600,000 quarter oscillations in a 1,000 chatur-yuga, which works out to 600 quarter oscillations in a chatur-yuga. Marichi says "*Khabhra swabhra agnayah kalpe kranti pata viparyayah Vyasta.*" i.e 30,000 multiplied by 20, in a Kalpa viz: 600 quarter oscillations in a Mahayuga. Sakalya Samhita also gives 600 quarter oscillations in a Mahayuga. Vasishta Siddhanta says "*Abdah khakhartubhr bhajyastaddostrighna dasadhrutah*" i.e 600 quarter oscillations in a Mahayuga.

On account of precession the extremity of the mid-day shadow of Asvini-Sun changes its position on the dial of the "Sanku" in the course of years and this was called *Agrayana*. As Agrayana gave the position of the vernal equinox and as in Amarakosa's time the vernal equinox was in Mrigasirsha nakshatra, Agrayana in Amarakosa is made to mean the nakshatra Mrigasirsha. Readers of Tilak's *Orion* may recollect how much trouble this interpretation of Amarakosa caused the author. All the allied phenomena caused by the fundamental precession are also called by the same name of precession. We have so far seen precession in Right Ascension, precession in celestial longitude, the oscillation of Asvini star on the horizon of an observer, the movement of the extremity of the shadow of the Asvini mid-day Sun etc. The angle through which the Asvini star oscillates from the East point depends upon the latitude of the observer, and it varies from 23½ degrees at the equator to 27 degrees at the Tropic of Cancer. Each writer stresses the importance on the particular aspect of precession he is describing. The western system stresses on the annual rate of change of celestial longitude, the Indian system primarily on the annual rate of change of Right Ascension as also Palachayai or the change in Agra. $\frac{1}{4} \times 54$ seconds of arc is the horizontal annual precession for the latitude of Rohitaka.

Maitreya Upanishad of Sama Veda says "*Dhruvasya prachlanam sthanam va tarunam nibhajvanam prithivya sthanadapusaranam suranam*" (ch. 1, Rik 2.). The positions of Dhruva (Pole) and trees change. Earth is drowned. The Suras run away leaving their places. The position of pole changing is due to precession and the deva stars (Suras) forsaking their places is also due to precession. So precession was known in India from Vedic times. The Bhagavata Purana deals with the same in the form of the story of Dadhichi and Viswarupa (Skanda 6, ch. 6–14), this being an elaboration of the Vedic idea. As Jyothisha is one of the six angas of the Veda Purusha, being the very eye of the Purusha, this important phenomenon formed the fundamental topic of discussion in all Vedic, Vedantic and Siddhantic literatures of India. So to say that precession was discussed by this writer first or that writer first, or by this book or that book, will be a blasphemy of the Vedas.

[*Krishna* stands for *Krishna-paksha* a word number for 15; *Vyasta* = *vi* (vimsati + *astha* (gunita)].

AN ASYMPTOTIC EXPANSION

BY

K. SANKARA PILLAI, *Central Research Institute, Trivandrum*

In connection with a problem in Statistics, the author had to study the asymptotic behaviour of the series defined by $f(x) = \sum (x^n/n!)^3$. The analysis required was a bit heavy and subsequently, it has been possible to derive the asymptotic expansion of $f(x)$ by using the method of Steepest Descent. The present paper is concerned with this study.

It may be shown that the series for $f(x)$ is the term independent of t in the product

$$e^{x/t^2} \cdot I_0(2xt)$$

where $I_0(x)$ is the Bessel Function of order zero and of purely imaginary argument. Hence

$$2\pi i\, f(x) = \int_c e^{x/z^2} \cdot I_0(2xz) .\, dz/z$$

where c is any circle round $z=0$. For convenience let the radius of c be unity. By substituting $z=e^{iu}$ and using the integral formula for $I_0(2xz)$, $f(x)$ reduces to $4\pi^2 f(x) =$

$$\int_0^{2\pi} e^{x \cos 2u}\, du \int_0^{2\pi} e^{2x \cos\theta \cos u} \cos(x \sin 2u - 2x \sin u \cos\theta)\, d\theta$$

The interval 0 to 2π is now changed to 0 to π and

$$f(x) = \frac{1}{\pi^2} \int_0^{\pi} e^{x \cos u}\, du \int_0^{\pi} e^{2x \cos\theta \cos u} \cos(x \sin 2u - 2x \sin u \cos\theta)\, d\theta.$$

The integrand contains the exponential factor $e^{x(\cos 2u + 2\cos u \cos\theta)}$ and this for positive x has its maximum value e^{3x}. Hence $e^{x(\cos 2u + 2\cos u\cos\theta - 3)}$ tends uniformly to zero as x tends to infinity except at $u=\theta=0$ and $u=\theta=\pi$. Hence writing

$$e^{-3x} . f(x) =$$

$$\frac{1}{\pi^2} \int_0^{\pi} \int_0^{\pi} e^{x(\cos 2u + 2\cos\theta\cos u)} \cos(x \sin 2u - 2x \sin u \cos\theta)\, du .\, d\theta$$

the integral tends to zero as x tends to infinity everywhere except in the regions

$$\text{(i)}\quad 0 \leqslant u \leqslant \varepsilon, \quad 0 \leqslant \theta \leqslant \delta \ (R_1)$$

$$\text{(ii)}\quad \pi-\eta \leqslant u \leqslant \pi, \quad \pi-\rho \leqslant \theta \leqslant \pi \ (R_2)$$

Thus as $x \to \infty$,

$$f(x)\cdot e^{-3x} \to \int_{R_1} + \int_{R_2}$$

Denoting $\int_{R_1}$ by I (ε, δ), a simple transformation shows that the integral for R_2 is equal to I (η, ρ).

To evaluate I (ε, δ), using Taylor's Theorem with remainder, we write

$$\cos 2u + 2 \cos u \cos \theta - 3 = -(Au^2 - 2H\, u\theta + B\theta^2)$$

where $A \to 3$, $H \to 0$ and $B \to 1$ as ε and δ tend to zero, and are independent of u and θ. Hence

$$\pi^2.\ I\ (\varepsilon, \delta) = \int_0^\varepsilon du \int_0^\delta e^{-(Au^2-2H.\,u.\,\theta+B\theta^2).\,x} \cos\ (.\ .\ .\ .)\ d\theta$$

Applying the Mean value theorem to this integral, it is seen that

$$\pi^2.\ I\ (\varepsilon, \delta) = \cos \mu. \int_0^\varepsilon \int_0^\delta e^{-(Au^2-2H.\,U.\,\theta+B\theta^2).\,x}\, du\, d\theta$$

where $\cos \mu$ tends to unity as ε and δ tend to zero. Making the substitutions $x.\ u=v$ and $x.\ \theta=w$

$$I\ (\varepsilon, \delta) = \frac{\cos \mu}{\pi^2\ x} \int_0^{x.\,\varepsilon} \int_0^{x.\,\delta} e^{-(Av^2-2H.\,v.\,w.\,+Bw^2)}$$

As x tends to infinity the double integral tends to $\frac{1}{4}\pi.\ \sqrt{(AB-H^2)}$ and hence to $\pi\sqrt{3}/4$. Hence, noting that $\cos \mu$ tends to unity

I (ε, δ) and I (η, ρ) both tend to $I/4\pi.\ \sqrt{3}.\ x$ as x tends to infinity. Thus we have

$$f(x) \sim e^{3x}/2\pi x\ \sqrt{3}.$$

The complete asymptotic expansion can now be found by noting that $f(x)$ satisfies the differential equation

$$(x.\ d/dx)^3 z = 27.\ x^3.\ z.$$

Assuming that $z = e^{3x}.\ (\Sigma c_n/x^n)/x^{\rho+1}$ it is seen that $\rho=0$ and that the coefficients satisfy the recurrence relation

$$(n+1)^3.\ c_n + 27\ (n+2).\ c_{n+2} = 3\ (3n^2+9n+7)\ c_n$$

We may calculate the c's from the above formula and since c_0 has already been determined, the solution is complete. The first few terms are as follows

$$f(x) \sim \frac{e^{3x}}{2.\ \pi x.\ \sqrt{3}}\ (1+1/9x+2/81x^2+\ \cdots)$$

It may be pointed out that the asymptotic expansion can in general be obtained from different methods, but the constant associated with the function that denotes the nature of the infinity is very often difficult to get. In problems of the type discussed above the Method of Steepest Descent will prove useful. In particular, the dominant term in the asymptotic expansion of $I_0\ (2x)$ has been obtained by this method and the analysis is by no means heavy.

ON QUINTIC EQUATIONS SOLUBLE BY RADICALS

BY

S. CHOWLA, *Government College, Lahore.*

It seems to be rare for a quintic equation with integral cœfficients to be soluble by radicals.

More precisely, if we denote by f (N) the number of cases in which the quintic equation

$$ax^5+bx^4+cx^3+dx^2+ex+f=0$$

(where a, b, c, d, e, f are integers whose absolute value does not exceed N) is soluble by radicals, then the ratio of f (N) to $(2N+1)^6$ tends to 0 as $N \to \infty$.

The following theorems support, but do not prove the above conjecture.

Theorem 1. If $x^5+cx+d=0$ is an irreducible equation with integral coefficients, it is insoluble by radicals in the following cases:—

(*i*) d is fixed but c is a sufficiently large prime (positive or negative);

(*ii*) c is fixed but d is sufficiently large (positively or, negatively).

Theorem 2. If $f(N)$ denotes the number of cases in which the equation (with integral coefficients) $x^5+cx+d=0$ is soluble by radicals, where $|d| \leqslant N$, $|c| \leqslant N^{20}$, then the ratio of $f(N)$ to

$(2N+1)\ (2N^{20}+1)$ tends to 0 as $N \to \infty$,

PROBLEMS IN THE TEACHING OF MATHEMATICS†

BY

W. W. SAWYER, *Mathematics Department, College of Technology Leicestor, England*

Probably there is no subject which offers such possibilities for misunderstanding between teacher and pupil as mathematics does. The teacher stands at the blackboard. It is perfectly clear to him what the symbols mean, and what conclusions can be drawn from them. It is completely otherwise with many of the pupils. What the symbols are meant to represent, how the teacher knows what is right and what is wrong, what is the object of the whole business anyway—all this is wrapped in mystery. The great majority of the students say to themselves, "We shall never understand this stuff, but we want to get through the exam. We will have to learn it by heart."

This is not a satisfactory state of affairs. Learning by heart not only imposes a quite unnecessary strain on the student, it is also quite useless. It gives neither an understanding of the subject, nor the power to apply mathematics to practical life.

The more we can see things from the pupil's point of view the better teachers we shall be. And the first question in the pupil's mind is "Why do we have to do this at all ?" When I was at school, the boys were always asking this—and they never got a satisfactory answer. The teachers made up kinds of answers, but they were none of them very convincing. The fact is, I think, that mathematics is taught because it is *the custom* to teach it.

Now a custom may be a good custom or a bad custom, or more often, partly good and partly bad. It often helps in judging a custom if we know its history how it began and how it developed; and I should like to consider,—the history of Mathematics.

Man has been on the earth for something like half-a-million years. Mathematics is not nearly so old—at the outside perhaps 20,000 years. Our early ancestors had no schools, no mathematics. What led them to depart from that happy state ? Still more, how were they able to get away from it ? How was it possible for them, without text books and without teachers, to discover mathematical truths for themselves ?

† Paper read at the Delhi Conference during the discussion on the Teaching of mathematics in Schools and Colleges.

It seems fairly clear that mathematics first arose in purely practical questions; arithmetic in connection with trade, the gathering of taxes, the reckoning of the calendar, etc.: geometry in connection with building and land-surveying.

They are two points to note in these early developments: first, there was a clear and definite *purpose* in the minds of the men concerned: second, they were thoroughly *familiar* with the materials concerned. I do not believe there can be satisfactory reasoning unless these two conditions are satisfied and in many schoolrooms today they are not satisfied.

One marked feature of these practical problems is that they are self-checking. If a man was trying to build a house, and he found that his bricks would not fit together and there were holes for the wind to blow through he knew that he had not solved his problem. (I have in mind that students often do not know whether they have solved a problem or not. They ask, "Is this the answer? Is this what is wanted?)

An interesting example of such a problem is that of finding the length of the year. This problem was solved by a series of approximations, over a period of hundreds, or even thousands, of years. The early guesses were inaccurate, and the consequence of using incorrect values was that months gradually moved from their correct position, until July was in the middle of the winter and December in the middle of the summer. The question was, of course, of practical importance in order that seed might be sown at the correct season.

I believe that children, *at any rate when beginning to learn*, should follow something of the same path. They should not begin by learning arithmetic for arithmetic's sake. They should set out to do or to make some real thing: they should not be sitting at desks, but moving about and handling actual objects: and they should be able to judge for themselves whether or not they have achieved their goal. This stage is all-important. The early years are decisive. If a child does not learn to feel the actual meaning of a wall containing 300 bricks, of a stick being 6⅜th inches long, of a sack of sand weighing 50 pounds, it will always feel mystified by more advanced mathematics.

I may mention two examples in arithmetic, which I have sometimes found useful. One is the making of models to scale. Boys in England are interested in aircraft; so they make or draw aeroplanes to scale. If they make a mistake, they soon recognise the fact, and seek it out for themselves. Accuracy is even more necessary if they want to understand some complicated machine—for instance, the sliding valve gear on a locomotive. If the parts are not made properly to scale, the apparatus will not demonstrate the effect it is intended to.

The second example is deciphering simple codes. The letter which occurs most frequently in English is "*e*". Hence the symbol which occurs most often is likely to stand for "*e*". And so on for the other most frequent letters. (See Edgar Allen Poe, "The Gold Bug", for further details.)

These examples I mention mainly because (from the child's point of view) there is some *purpose* in them. Indian boys and girls may have special interests of their own. The principle is the same; we should ask, "What do they want to do? Can it be done without calculation?"

The most pointless subject, judging by the text book examples, is algebra. A man buys various articles; then he forgets what he paid for them, and has to work out the prices by simultaneous equations. Does anything remotely like this ever happen in real life? If so, does it occur often enough to justify wasting the precious hours of ones youth on it?

Algebra is in fact a most important instrument of modern life, though not in such applications as that just mentioned. It is a subject which we in Europe owe to Indian mathematicians.

The main problem (it seems to me) which led to algebra was the making of trigonometrical tables, which in turn arose from the needs of astronomy and (more particularly in Europe) from navigation. The interest in tables of square roots and the solving of cubic equations derived mainly from equations such as

$$\cos 2x = 2 \cos^2 x - 1$$
$$\cos 3x = 4 \cos^3 x - 3 \cos x.$$

Such equations were used to find the sines and cosines of angles in the well-known way, starting from 90°, 45° or 30°, and then bisecting or trisecting the angles.

There was of course a good deal of work required to calculate the tables, and any way of shortening the work was welcome. Any one who works for a long time computing tables is bound to notice certain properties. There is the famous identity discovered in this way

$$\sin a - \sin b = \sin b - \sin c - \frac{\sin b}{225}$$

in which c, b, a are angles which follow each other in a table with intervals 3° 45′. This identity was used by mathematicians in India to lessen the work of computation.

Just as geometrical discoveries are easily made by one who is familiar with the shapes of things, algebraic results are easily discovered by one familiar with the handling of figures.

Suppose, for instance, one had to make a table of reciprocals. One starts off working by long division a fairly laborious process. One obtains the results.

Number	Reciprocal
1·000	1·000
1·001	·999
1·002	·998
1·003	·997

But here a property leaps to the eye; as the number rises by ·001, the reciprocal falls by ·001. This is a far simpler way of getting the next few results than long division would be, provided (and this is where algebra comes in) one can find out how long this rule can be trusted to give the correct result.

This point should be stressed in teaching mathematics—that algebra was developed *with the object of avoiding long arithmetical calculations.*

This is an object with which most children will sympathise. There were very many results similar to this one discovered by the early computers of logarithmic tables. The "table of differences" at the side of the logarithmic and trigonometric tables is a case in point.

Children should be given a chance to find such rules for themselves and encouraged to find labour-saving devices in their work, so that they gradually discover algebra for themselves.

It cannot be too much emphasised that each mathematical subject grew gradually out of the previously known subjects.

Algebraic symbolism in particular developed gradually. For too many algebra books spring suddenly into full symbolism, which is quite unintelligible to learners.

The process by which symbolism arose is very natural and easily understood by children. When children are writing notes on history, they get tired of writing some long phrase such as "William the

Conqueror" and they shorten it to something like "W. the Conq." or "W. T. C.".

And that is all algebra is. The first general statements about Arithmetic were written out in grammatical language (in verse, even). Then, in the Bakhshali Manuscript "phalam" (equals) was abbreviated to "pha" and "yuta" (addition) to "yu". Bit by bit this process of abbreviation was continued until it reached our present form of symbols.

An interesting phase of this process is Brahmagupta's practice of indicating numbers by colours. He would have given the formula for the area of a rectangle something like this—"Measure the height of the rectangle, and write this number down in black ink. Measure the length and write it down in red ink. Multiply the black number by the red number. This gives the area". This idea might be useful in teaching children to-day.

While algebraic development was stimulated by the need for trigonometrical tables, this particular application is less important to-day. We have the tables all calculated and finished, and we do not need to calculate for ourselves. In any case, we should use series and interpolation by finite differences, not cubic equations, if we had to re-calculate the tables.

The most widespread elementary use of algebra in the modern world is the application of technical formulae. A mathematician reads a formula at a glance. The ordinary student does not. It is important that a fairly long time should be devoted to enabling learners to *feel* the significance of a formula. This includes, for instance the ability to distinguish between increasing functions, such as $y=x^2$ and decreasing functions, such as $y=1/x$. (x positive)

For instance, there is a formula in the design and repairing of roofs (and in other applications) for the weight that can safely be put on a beam of length l, breadth b, depth d,

$$W=kbd^2/l$$

The strength of the beam increases with increased breadth and depth, but the longer the beam is the less weight can it support. This is reasonable. It should be second nature with students to apply such general checks to formulae. Books often contain misprints, and the effect of a misprint nearly always is to make the formula quite absurd.

Wherever possible, formulae should not merely be worked out and discussed, but actually applied to the design of real apparatus. For instance, the formula for the expansion of a steel bar due to increase of temperature can be used to design a temperature-measuring instrument. This involves various geometrical calculations also, as the expansion is small, and some means has to be found of magnifying it.

Many other experiments are possible, and can even be done inside a classroom on a small scale—for instance, the formula above for the strength of a beam can be tested by putting weights on small pieces of sticks. It is best if this arises naturally in the course of designing something, e.g. a weighing machine or a small bridge across a stream. Scientific laws appeal more to boys if they can see what knowledge of these laws enables one to do.

One can find endless examples simply by looking at a good Engineer's Reference Book.

I believe the main difficulty in algebra is right at the beginning. If learners understand algebraic symbolism, and if they are familiar with and interested in arithmetical calculations, they should not find any great difficulty in the later parts of the subject.

Just as algebra tends to be sharply divided from arithmetic, calculus is also often presented to students as a completely new and different subject. Very few students realise that the *typical problems of calculus were attacked and solved before calculus had been invented.*

For instance, one may take Mercator's projection of the globe. This is, of course, not a "projection" in a geometrical sense like central projection or cylindrical projection. It starts from the problem: assuming the meridians to be represented on a map by evenly spaced vertical lines, how must the parallels of latitude be shown on the map if the track of any ship steering a fixed course is to appear as a straight line?

In modern notation, this would be expressed by the equation

$$-d\,\theta = k \sin\theta.\ dy$$

where y is the height on the map at which the colatitude θ is marked. This we should solve by integration of $d\,\theta/\sin\theta$. Mercator however, working before calculus was known, did the whole work by arithmetic. In modern language we should say that he approximated by taking $d\,\theta$ not as an infinitesimal but as a finite small number, say ·001, and then working out the space "dy" which should appear on the map between the two lines of latitude θ and $\theta - \cdot 001$. It was then noticed that Mercantor's table for the position of lines of latitude was the same as the table for tan $\theta/2$.

Thus this isolated result was known *before* calculus existed as a systematic subject.

Calculus, like algebra, began on a set of *experimentally observed rules* which could be used to shorten long arithmetical calculations.

The old fashioned method of presenting calculus, beginning with differentiation of x^2 by means of $\frac{(x+dx)^2 - x^2}{dx}$ nearly always puts students off.

They do not feel that it would ever have occurred to them to do this. The procedure based on the historical development, i.e. from a definite problem, through arithmetical approximation, to an empirical law, afterwards verified by algebra, seems to offer less difficulty.

A more detailed study of the practical problems of the 16th and 17th centuries will be found very helpful by anyone who desires to teach calculus well. These problems covered navigation, map-making, the design of clocks and telescopes, ship construction, and mechanics generally.

Some problems of this period, which are not of great interest to-day, have become "fossilized" in calculus text-books. These can be a source of perplexity to students, who ask, "What is the object of this section?"

To be honest with our students, I think we teachers of mathematics ought to go through the mathematics syllabus and classify its contents under the following headings:—

1. Methods of great generality and mathematical importance.
2. Results which are beautiful and interesting in themselves.
3. Methods and results of practical and technical value.
4. Work of historical interest only.
5. Things which we cannot find any reason for including at all.

Lack of space prevents me from developing these ideas as I should like to do, but I hope some of them will provoke discussion.

GREGORY'S SERIES IN THE MATHEMATICAL LITERATURE OF KERALA*

BY

K. MUKUNDA MARAR AND C. T. RAJAGOPAL,

Madras Christian College, Tambaram.

1. This essay presents in modern garb a few of the mathematical contents of a Malayāḷam manuscript entitled *Yukti-Bhāṣa* supposed to be some three hundred years old. The contents seem to be older still since we find their gist, in the form of bald mathematical rules, in the Sanskrit (*manuscript*) work called *Tantrasaṅgraha*, produced (to judge from the internal evidence) about the year 1500 A.D., by one Nīlakaṇṭha, a writer well known throughout the Kerala country, for his erudite expositions of Jyotiṣa, Mīmāṁsa, Vyākaraṇa and Vadānta.

The first section of the essay gives the proof, substantially as it stands in *Yukti-Bhāṣa*, of Gregory's series for the inverse tangent. The second is devoted exclusively to the method given in *Yukti-Bhāṣa*, for transforming Gregory's series for π, into other alternating series which are more rapidly convergent. This method, stated in the language of modern mathematics, is seen to be sufficiently general to convert Gregory's series for π into another in which the general term is $O(1/n^{2p+1})$ or $O(1/n^{2p+2})$ as $n \to \infty$, p being any positive integer.

I

2. Gregory's series for the inverse tangent,

$$\text{arc tan } t = t - \frac{1}{3} t^3 + \frac{1}{5} t^5 - \cdot \cdot \cdot \; (|\text{arc tan } t \,| \leqslant \pi/4),$$

is established in *Yukti-Bhāṣa*, by means of the two lemmas which follow.

2·1. The first of the lemmas is obtained geometrically from similar triangles and the second is a deduction from the first.

---

* The two sections of this paper, with slight alterations form part of an article. "On the Hindu Quadrature of the circle," published in the *Journal of the Bombay Branch of the Royal Asiatic Society*. N. S., Vol. 20, 1944. They are reproduced here by kind permission of the B. B. R. A. S.

LEMMA 1. Let P, Q be points on the tangent at A to a circle of unit radius whose centre is O; let OP meet the circle in p. If m is the foot of the perpendicular from p to OQ, pm is given by

$$pm = \frac{PQ}{OP \cdot OQ}$$

LEMMA 2 If, in Lemma 1, PQ is small, the arc pq of the circle intercepted between OP and OQ is given by

$$\text{arc } pq = \frac{PQ}{OP^2} \quad \text{or} \quad \frac{PQ}{1+AP^2},$$

correct to a first approximation.

PROOF OF GREGORY'S SERIES. Let the point B, on the tangent at A, be such that $A\hat{O}B \leqslant 45°$. Write $\tan A\hat{O}B = AB = t \;(\leqslant 1)$. Divide AB into n equal parts, denoting the points of division and the end points by P_0 $(=A)$, P_1, P_2, . . , P_{n-1}, P_n $(=B)$, in the order in which they occur from A to B. Let OB meet the circle in b. Then we find, from Lemma 2,

$$\text{arc } Ab = \lim_{n\to\infty} \sum_{r=0}^{n-1} \frac{P_r P_{r+1}}{1+AP_r^2} = \lim_{n\to\infty} \sum_{r=0}^{n-1} \frac{t/n}{1+(rt/n)^2}$$

$$= \lim_{n\to\infty} \sum_{r=0}^{n-1} \frac{t}{n} \left\{ 1 - \left(\frac{rt}{n}\right)^2 + \cdots + (-1)^{\nu-1}\left(\frac{rt}{n}\right)^{2\nu-2} + \frac{(-1)^\nu \left(\frac{rt}{n}\right)^{2\nu}}{1+\left(\frac{rt}{n}\right)^2} \right\}.$$

Now making use of the result $\lim_{n\to\infty} \frac{1}{n^{p+1}} \sum_{r=0}^{n-1} r^p = \frac{1}{p+1}$, we obtain

$$\text{arc } Ab = t - \frac{t^3}{3} + \cdots + (-1)^{\nu-1} \frac{t^{2\nu-1}}{2\nu-1} + (-1)^\nu R_\nu,$$

where $0 < R_\nu < 1/(2\nu+1)$; whence, letting $n \to \infty$, we are led to Gregory's series for $0 < t \leqslant 1$. The validity of the series for $-1 \leqslant t < 0$ becomes obvious when we change the sign of t.

II

3. Issuing from Gregory's series for π,

(1) $$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \cdots,$$

there is an endless sequence of approximations to $\pi/4$ in which every approximation is better than the preceding. For large values of n, the first three approximations are:

(2a) $$\frac{\pi}{4} \doteqdot 1 - \frac{1}{3} + \frac{1}{5} - \cdots \pm \frac{1}{n} \mp \frac{1}{2(n+1)},$$

(3a) $$\frac{\pi}{4} \doteqdot 1 - \frac{1}{3} + \frac{1}{5} - \cdots \pm \frac{1}{n} \mp \frac{(n+1)/2}{(n+1)^2+1},$$

(4a) $$\frac{\pi}{4} \doteqdot 1 - \frac{1}{3} + \frac{1}{5} - \cdots \pm \frac{1}{n} \mp \frac{\frac{(n+1)^2}{4}+1}{\{(n+1)^2+4+1\}\frac{n+1}{2}}.$$

These approximations give rise to the following alternating series whose sums involve π linearly and in which the nth term is $O(1/n^{2p+1})$ as $n \to \infty$, $p=1, 2, 3$.

(2) $$\frac{\pi-3}{4} = \frac{1}{3^3-3} - \frac{1}{5^3-5} + \frac{1}{7^3-7} - \cdots,$$

(3) $$\frac{\pi}{16} = \frac{1}{1^5+4\cdot 1} - \frac{1}{3^5+4\cdot 3} + \frac{1}{5^5+4\cdot 5} - \cdots,$$

(4) $$\left(\frac{\pi}{4} - \frac{7}{9}\right)\Big/36 = \frac{1}{(3^3-3)(2^2+5)(4^2+5)} - \frac{1}{(5^3-5)(4^2+5)(6^2+5)} + \cdots$$

In the concluding portion of a section bearing the name *Paridhi Vyāsam* in *Yukti-Bhāṣa* we find the method of deriving from (1) first (2a), (3a) and thence (2), (3). This method is explained in general terms in §§4, 4. 1 below.

3·1. A slight modification of the method leads to a second sequence of approximations to π which can be derived from the alternating series

(1′) $$\frac{\pi-2}{4} = \frac{1}{2^2-1} - \frac{1}{4^2-1} + \frac{1}{6^2-1} - \cdots,$$

in which the general term is $O(1/n^2)$. The derivation of this series itself is simple. In (1), grouping together two consecutive terms beginning with the first and then two consecutive terms beginning with the second, we obtain successively

$$\frac{\pi}{8} = \frac{1}{2^2-1} + \frac{1}{6^2-1} + \frac{1}{10^2-1} + \cdots,$$

$$\frac{4-\pi}{8} = \frac{1}{4^2-1} + \frac{1}{8^2-1} + \frac{1}{12^2-1} + \cdots,$$

whence by subtraction (1′) follows. The first two members of the sequence of approximations which can be derived from (1′), in much the same way as (2a) or (3a) can be derived from (1), are

(2′a) $$\frac{\pi-2}{4} \doteq \frac{1}{2^2-1} - \frac{1}{4^2-1} + \cdots \pm \frac{1}{n^2-1} \mp \frac{1}{2(n+1)^2}$$

(3′a) $$\frac{\pi-2}{4} \doteq \frac{1}{2^2-1} - \frac{1}{4^2-1} + \cdots \pm \frac{1}{n^2-1} \mp \frac{1}{2\{(n+1)^2+2\}}$$

in the series corresponding to which the nth term will be shown to $O(1/n^{2p+2})$, $p=1$, 2 respectively (§5 *infra*).

(4) In (1), let us denote the partial sum of the first $(2m+1)$ terms and that of the first $2m$ terms by $S\left(\frac{n+1}{2}\right)$ and $S\left(\frac{n-1}{2}\right)$ respectively, where $n=4m+1$. These partial sums are in fact approximate values of $\frac{\pi}{4}$ when n is large and can be improved upon by applying to them 'corrections' $-f(n+1)$ and $f(n-1)$ respectively, the corrected values being

$$T\left(\frac{n+1}{2}\right) = S\left(\frac{n+1}{2}\right) - f(n+1),$$

$$T\left(\frac{n-1}{2}\right) = S\left(\frac{n-1}{2}\right) + f(n-1).$$

It will be noticed that the sign prefixed to the correction which is applied to any partial sum is the same as the sign of the total remainder following the partial sum in question; also, that the corrected value $T\left(\frac{n+1}{2}\right)$ of $S\left(\frac{n+1}{2}\right)$ is the partial sum of $\frac{n+1}{2}$ terms of a new series for $\frac{\pi}{4}$ more rapidly convergent than (1). Denoting by u_n the $\frac{n+1}{2}$th term of the new series, we have by subtraction

(5) $$u_n = \frac{1}{n} - f(n+1) - f(n-1).$$

In (5) let us change n successively to $n-2$, $n-4$, . . . , 3 and eliminate $f(n-1)$, $f(n-3)$, ... $f(4)$ between the relations thus obtained and (5). Then we get

$$-u_3 + u_5 - \cdots + u_n = -\frac{1}{3} + \frac{1}{5} - \cdots + \frac{1}{n} + f(2) - f(n+1).$$

Hence if $f(n)$ satisfies the preliminary condition $f(n) = O(1)$ $(n\to\infty)$ we have the following series for π more rapidly convergent than (1)

(6) $$\frac{\pi}{4} = 1 - f(2) - u_3 + u_5 - \cdots + u_n - \cdots$$

4·1. Now if we are to have $u_n = O(n^{-(2p+1)})$ $(n \to \infty)$, it is readily seen from (5) that a condition to be satisfied by f, more comprehensive than the one already mentioned, is $f(x) = O(1/x)$ $(x \to \infty)$. Therefore, in testing the suitability of various functions f in (5), we may confine ourselves to those of the type

$$(7) \qquad 2f(x) = \frac{a_1}{x} + \frac{a_2}{x^2} + \frac{a_3}{x^3} + \cdots \quad (x \geqslant x_0).$$

With such a choice of f, (5) can be written:

$$(8) \qquad \frac{1}{n} = 2f(n) + 2\frac{f''(n)}{\underline{|2}} + 2\frac{f^{iv}(n)}{\underline{|4}} + \cdots + \frac{1}{\underline{|2p}}\{f^{(2p)}(n-\theta_1) + f^{(2p)}(n+\theta_2)\} + u_n$$

where $0 < \theta_1, \theta_2 < 1$ and the successive terms of the right-hand member (excluding u_n) are $O\left(\frac{1}{n}\right)$, $O\left(\frac{1}{n^3}\right)$, $\cdots$, $O\left(\frac{1}{n^{2p+1}}\right)$ respectively. Now introducing the condition $u_n = O\left(\frac{1}{n^{2p+1}}\right)$ in (8), we find that

$$(9) \qquad \frac{1}{n} + O\left(\frac{1}{n^{2p+1}}\right) = 2f(n) + 2\frac{f''(n)}{\underline{|2}} + \cdots + \frac{2}{\underline{|2p-2}} f^{(2p-2)}(n).$$

Employing (7) in (9) we can calculate $a_1, a_2, \cdots, a_{2p}$ and verify that $a_2 = a_4 = \cdots = a_{2p} = 0$. Any function of the type (7) with the first $2p$ co-efficients having these calculated values, can be used in (5) to get a series like (6) for π in which $u_n = O\left(\frac{1}{n^{2p+1}}\right)$.

4·11. Clearly a function $f(n)$ satisfying our requirements can be expressed in the form of a continued fraction,

$$(10) \qquad 2f(n) = \frac{b_1}{n+}\,\frac{b_3}{n+}\,\frac{b_5}{n+}\cdots\frac{b_{2p-1}}{n},$$

where the b's can be calculated by expressing them in terms of the a's.

4·2. The method explained above gives as particular cases all the results mentioned in §3.

First, $p=1$ in (9) makes $2f(n) = 1/n$ and leads us to (2a). Now calculating u_n from (5) and employing it in (6), we get (2).

Next, $p=2$ in (9) makes $a_1 = 1$, $a_3 = -1$, $(a_2 = a_4 = 0)$ and consequently $b_1 = b_3 = 1$. Hence (10) leads to $2f(n) = \frac{1}{n+}\,\frac{1}{n}$ and thence to (3a).

Making use of this expression in (5), we obtain a value for u_n which substituted in (6) gives rise to the following series equivalent to (3).

$$\left(\frac{\pi}{4}-\frac{1}{5}\right)\Big/ 4 = -\frac{1}{3(2^2+1)(4^2+1)}+\frac{1}{5(4^2+1)(6^2+1)}-\ \cdots$$

Finally, when $p=3$, we obtain from (9), $a_1=1$, $a_3=-1$, $a_5=5$, $(a_2=a_4=a_6=0)$ and hence $b_1=b_3=1$, $b_5=4$. Therefore $2f(n)$ $=\frac{1}{n+}\ \frac{1}{n+}\ \frac{4}{n}$ whence (4a) follows. The corresponding value of u_n, calculated from (5) and used in (6) gives us (4).

5. We proceed to explain how, by slightly modifying the method in §4·1, we can convert (1′) into another alternating series in which the nth term is $O(1/n^{2p+2})$. Denoting the partial sum of the first $n/2$ or $(2m+1)$ terms of the series (1′) by $S(n/2)$ and the corresponding partial sum of the new series by $T(n/2)$, let us write

$$T(n/2)=S(n/2)-f(n+1),$$

where $f(n)$ is to be determined so that u_n, the $n/2$nd term of the new series, is $O(1/n^{2p+2})$. This leads to the relation

(5′) $$O(1/n^{2p+2})=u_n=\frac{1}{n^2-1}-f(n+1)-f(n-1).$$

Since we have as a preliminary condition $f(n)=O(1/n^2)$, we can take

(7′) $$2f(x)=\frac{1}{x}\left\{\frac{a_1}{x}+\frac{a_2}{x^2}+\ \cdots\right\} \qquad (x \geqslant x_0).$$

This choice of f makes (5′) equivalent to

(9′) $$\frac{1}{n^2-1}+O\left(\frac{1}{n^{2p+2}}\right)=\sum_{r=0}^{p-1}\frac{2}{\lfloor 2r}f^{(2r)}(n),$$

whence we can calculate $a_1, a_2, \ldots, a_{2p}$ (verifying, as in §4. 1, that $a_3=a_4=\cdots=a_{2p}=0$). Thus, the correction function $f(n)$ which relates a partial sum of (1′) to the corresponding partial sum of the new series derived from (1′) is indeterminate save for its first $2p$ coefficients. It is however uniquely determined if restricted to the form

(10′) $$2f(n)=\frac{1}{n}\left\{\frac{b_1}{n+}\ \frac{b_3}{n+}\ \cdots\ \frac{b_{2p-1}}{n}\right\}.$$

5·1. Illustrations of §5 are furnished by (2′a) and (3′a).

First consider the case $p=1$. Then (9′) gives $2f(n)=1/n^3$ which establishes (2′) and, in conjunction with (5′), determines an alternating series with the same sum as (1′) and $u_n=O(1/n^4)$.

If $p=2$ in (9′) we find that $a_1=1$, $a_3=-2$, ($a_2=a_4=O$) and hence in (10′) $b_1=1$, $b_3=2$. This establishes at one stroke (3′a) and an alternating series having the same sum as (1′) and $u_n=O\ (1/n^6)$.

The number of such illustrations is of course infinite.

6. Before concluding we may point out that a series for π in which all the terms are of the same sign and the nth term is $O(1/n^{2p+2})$ or $O(1/n^{2p+3})$ can be obtained by grouping together consecutive terms in the alternating series derived from (1) or (1′) which have the nth term $O(1/n^{2p+1})$ or $O(1/n^{2p+2})$. Thus from (2), grouping together consecutive terms beginning with the first, we get the series

$$\frac{\pi-3}{6}=\frac{1}{(2\cdot2^2-1)^2-2^2}+\frac{1}{(2\cdot4^2-1)^2-4^2}+\frac{1}{(2\cdot6^2-1)-6^2}+\cdots$$

Readers who wish to inquire into the origin, in India, of the series for π discussed in this paper, will find references likely to be of help to them in the *JBBRAS* article already mentioned.

PROOFS OF SOME WELL KNOWN THEOREMS IN CONTINUED FRACTIONS

BY

HUKAMCHAND, *S. D. College, Lâhore.*

1. In this note, I give proofs of some well-known theorems of continued fractions. The proof of theorem I is suggested by Charve's proof (1) of Lagrange's general periodicity theorem. The closest approach to the proof of theorems II and III is the discussion in (2).

Let α and β be conjugate irrational roots of quadratic equation with integral co-efficients, then α is called a *reduced quadratic irrational* if $\alpha>1>-\beta>0$

2. The first theorem has been proved by Weber (3 - i) and Bachmann (4 - i) and runs as follows:—

THEOREM I. *Every quadratic irrational is equivalent to a reduced quadratic irrational.*

Let α be an irrational root of the quadratic equation with integral co-efficients

$$ax^2+2bx+c=0 \qquad \dots \quad (2\cdot1)$$

Suppose that the continued fraction for α is given by

$$\alpha=\{a_1\ a_2 \cdot \cdot \cdot\ a_n \mid t\}$$

where t is the complete quotient at the $(n+1)^{th}$ stage. It follows that $t>1$ and with the usual notation

$$\alpha=\frac{p_n\,t+p_{n-1}}{q_n\,t+q_{n-1}} \qquad \dots \quad (2\cdot2)$$

Since $p_n\,q_{n-1}-p_{n-1}\,q_n=(-1)^n$, it follows that α and t are equivalent.

It is to be proved that t is reduced.

Substituting in (2·1) from (2·2), it is seen that t is a root of the equation

$$F(x)\equiv A_n x^2+2\,B_n x+C_n=0 \qquad \dots \quad (2\cdot3)$$

where $A_n=ap_n^2+2b\,p_n\,q_n+cq_n^2=q_n^2\left[a\left(\frac{p_n}{q_n}\right)^2+2b\left(\frac{p_n}{q_n}\right)+c\right]$

$B_n=ap_n\,p_{n-1}+b\,(p_n\,q_{n-1}+p_{n-1}\,q_n)+cq_n\,q_{n-1}$

$C_n=a\,p^2_{n-1}+2b\,p_{n-1}\,q_{n-1}+c\,q^2_{n-1}=q^2_{n-1}\left[a\left(\frac{p_{n-1}}{q_{n-1}}\right)^2+2b\frac{p_{n-1}}{q_{n-1}}+c\right]$

The numbers A_n and C_n are different from zero for otherwise the equation (2·1) will have rational roots which is excluded by hypothesis. One of the numbers p_{n-1}/q_{n-1}, p_n/q_n belongs to the odd sequence of convergents and the other to the even and since $(p_n/q_n)\to\alpha$ as $n\to\infty$, we can find a value n_0 of n such that the interval $(p_{n-1}/q_{n-1},\ p_n/q_n)$, $n\geqslant n_0-2$ contains only the root α of (2·1). Hence A_n and C_n have opposite signs and there is no loss of generality in supposing that $A_n>0$ and $C_n<0$. This implies that n is even.

To prove that t is reduced. it has already been remarked that $t>1$. It remains to prove that its conjugate τ satisfies the relation $-1<\tau<0$.

Now $F(-\infty)$ is positive.

$F(-1)=(q_n-q_{n-1})^2\ (a\lambda^2+2b\lambda+c)$ where $\lambda=(p_n-p_{n-1})/(q_n-q_{n-1})$

With the supposition that n is even,

$$\frac{p_{n-2}}{q_{n-2}}\geqslant\lambda>\frac{p_n}{q_n}.$$

Since p_n/q_n, p_{n-2}/q_{n-2} belong to the same sequence of convergents, no root of (2·1) lies between them for sufficiently high values of n and therefore between p_n/q_n and λ. Hence $F(-1)$ has the sign of A_n.

$$\therefore\quad F(-1)>0.$$

And $F(0) = C_n < 0$.

Hence $-1 < \tau < 0$.

3. THEOREM II. *The simple continued fraction which represents a reduced quadratic irrational α is purely periodic.*

If α and β are conjugate and

$$\alpha = [\overline{a_1 a_2 \cdots a_n}]$$

then $$-1/\beta = [\overline{a_n a_{n-1} \cdots a_1}] \qquad \text{(Galois)}$$

The periodicity is a consequence of Lagranges' periodicity theoren (1). This may also be inferred from the theorem that reduced numbers with a given discriminant are finite in number (3 - ii, 4 - ii)

Since α is reduced, the complete quotients in its development are also reduced. There is no loss of generality in supposing that

$$\alpha = [a\ \overline{a_1\ a_2 \cdots a_n}] \qquad \ldots \quad (3{\cdot}1)$$

If α_n is the complete quotient at the $\overline{n+1}$ th step, then

$$\alpha = a + \frac{1}{\alpha_1} = \frac{p_{n+1}\ \alpha_1 + p_n}{q_{n+1}\ \alpha_1 + q_n}$$

Eliminating α_1 from these equations, α is found to be a root of

$$F(x) \equiv q_n x^2 + (q_{n+1} - aq_n - p_n)\, x - (p_{n+1} - ap_n) = 0 \qquad \ldots \quad (3{\cdot}2)$$

One root of this equation is $\alpha > 1$. It is necessary to find the condition that $-1 < \beta < 0$.

Now $F(-\infty) > 0$ and therefore $F(-1) > 0$, $F(0) < 0$.

But $F(-1) = q_n - q_{n+1} + aq_n + p_n - p_{n+1} + a\ p_n$

$= (1 + a - a_n)(p_n + q_n) - (q_{n-1} + p_{n-1})$

is a positive integer

$\therefore\ 1 + a - a_n \geqslant 1$ or $a \geqslant a_n$

Also $F(0) = -(p_{n+1} - a\,p_n) = (a - a_n)\, p_n - p_{n-1}$

is a negative integer which demands that $a \leqslant a_n$

$$\therefore\ a = a_n.$$

Thus a forms a part of the period.

$$\therefore\ \alpha = [\overline{a_1 a_2 \cdots a_n}]$$

and is the positive root of the equation

$$q_n x^2 - (p_{n-1} - q_{n-1})\, x - p_{n-1} = 0 \qquad \ldots \quad (3{\cdot}3)$$

where p_n/q_n is the nth convergent of α.

The equation whose roots are

$$\tau = -\frac{1}{\alpha},\ t = -\frac{1}{\beta} > 1$$

is
$$p_{n-1}x^2 - (p_n - q_{n-1})\,x - q_n = 0$$
$$\therefore\quad p_{n-1}t^2 - (p_n - q_{n-1})\,t - q_n = 0$$
$$\therefore\quad t = \frac{p_n\,t + q_n}{p_{n-1}\,t + q_{n-1}}$$

Hence $\frac{p_n}{p_{n-1}},\ \frac{q_n}{q_{n-1}}$ are respectively the n th and the $(n-1)$ th convergents of C. F for t and the complete quotient is t. ... (5)

But
$$\frac{p_n}{p_{n-1}} = [a_n\,a_{n-1} \cdots a_1]$$
$$\therefore\quad -\frac{1}{\beta} = [a_n\,a_{n-1} \cdots a_1 | t] = [\overline{a_n\,a_{n-1} \cdots a_1}]$$

4. THEOREM III.

If $\alpha = \sqrt{d} > 1$ *be irrational, then*
$$\alpha = [a\ \overline{a_1\,a_2 \cdots a_n}]$$
is such that $a_n = 2a$,
$$a_i = a_{n-i} \quad (i = 1,\ 2,\ 3 \cdots \overline{n-1})$$

If $\alpha = \sqrt{d}$, $\beta = -\sqrt{d}$ and $a = [\sqrt{d}]$, then if $\alpha = a + \frac{1}{\alpha_1}$
$$\beta = a + \frac{1}{\beta_1}$$

It is easy to see that α_1 is reduced. Thus the C. F for α is of the form (3·1) and therefore the positive root of (3·2). But the necessary and sufficient condition that the equation (3·2) be of the form $x^2 - d = 0$, $d > 0$, is
$$q_{n+1} - aq_n - p_n = 0$$
that is
$$\frac{q_{n+1}}{q_n} = a + \frac{p_n}{q_n}$$
that is
$$[a_n\,a_{n-1} \cdots a_1] = [2a\ a_1\ a_2 \cdots a_{n-1}]$$
$$\therefore\quad a_n = 2a,\quad a_{n-1} = a_1,\quad a_{n-2} = a_2 \cdots$$

The converse may be proved by retracing the steps.

References

1. (*a*) O. Perron—*Die Lehre von den Kettenbruchen* pp. 77–79.
   (*b*) Hardy and Wright—*Theory of numbers* page 143.
2. E. Cahen—*Theorie des nombres* page 194.
3. Weber—*Lehrbuch der Algebra* Bd. I (French translation by Griess (i) page 452; (ii) page 454.
4. Bachmann—*Grundlehren der neueren Zahlentheorie.* (i) pp 131–133 (ii) page 133
5. 1 (*a*) page 63; 1 (*b*) page 139.

NOTES AND DISCUSSIONS

A Note on the sign of the Perpendicular

1. At the very threshold of analytical geometry, there is a scandalous dispute about the orientation of a straight line and the sign to be attached to the perpendicular to it from a point not on the line (*Math. Gaz.* Vol. XXVI, Pp. 132, 133 and Vol. XXVII, pp. 82–83). According to Salmon's *Conic Sections*, if we were only concerned with one perpendicular we should only look to its absolute magnitude and it would be unmeaning to prefix any sign.

Prof. Neville thinks that attaching a definite sign to the perpendicular is misapplied labour. At this rate it should be meaningless and misapplied labour for a historian to qualify his dates with B. C. or A.D. or to give the time of day with the specification A.M. or P.M.. Signs are employed in mathematics for clarity, precision and unification of results. To disregard them is a retrograde step. Given a point and a line, or two lines one must be in a position to indicate mathematically on which side of the line the given point lies and also to specify which pair of vertically opposite angles has a certain magnitude. These points are inadequately discussed in text-books and require fuller treatment.

2. If we take the line $ax+by+c=0$, we have to fix upon a standard form for it. There are six ways open to us according as a, b or c is made positive or negative in the standard form. It is but natural to select the first term a and fix the standard form with a positive a. This convention immediately fixes up the positive side of the line if we bring in another natural principle that a point is on the positive side if its co-ordinates substituted in the standard form gives a positive value. Now, if a is positive and we substitute a sufficiently large positive value (x_1) for x and 0 for y in $ax+by+c$, we get a positive result and therefore all points on the same side of the line $ax+by+c=0$ as this point $(x_1, 0)$ are on the positive side, which works out to saying that the right side i.e., the side along which the x-co-ordinate increases as we move to ∞ is the positive side of the line. The exceptional case occurs when $ax+by+c=0$ is parallel to the X-axis, i.e., $a=0$ in which case, we have to modify the conventions in keeping with that for the X-axis itself and state that all

points above the line (the standard form now being that in which $b>0$) should be deemed to be on the positive side and those below the line on the negative side. With these conventions settled, we can fix up with greater precision the position of any point with respect to any line than we could if we knew only the distance of the point and do not care to attach any sign to it. Other uses of this convention are in determining the position of a curve with respect to a set of new axes of reference. We give a few illustrations:

Example 1:—$(2x-3y-1)^2 = -3x-2y-5$.

How is this parabola situated with respect to the lines $2x-3y-1=0$, $3x+2y+5=0$ written in standard form? The positive sides of these lines being the right side, we can say at once that the curve is on the negative side or the left side of the straight line $3x+2y+5=0$. Our text-books feel helpless in tackling such questions without finding actually a point on the curve and thereby determining the lie of the curve with respect to the axes of reference.

Example 2:—How is the hyperbola $(2x-3y-1)(3x-2y-5)+7=0$, situated with respect to the asymptotes?

Here we note that a point on the curve must be on the right side of one line and the left side of the other or *vice versa* and hence understand the position of the curve. Another question suggested in this case is how to determine the angle between the asymptotes. An allied problem is to determine the angle between the tangents to a conic, from a given point.

Example 3:—Given the sides of a triangle in the form $a_i x+b_i y+c_i=0$ $(i=1, 2, 3)$ find the bisectors of the exterior angles.

Such problems are solved even to-day only with the help of a figure. This is certainly misapplied labour, since the calculation of a cross-ratio solves the difficulty of choosing the proper bisector quite simply.

3. The topological property of two lines OA, OB separating OC, OD implies that the cross-ratio O (AB, CD) is negative. The opposite condition indicates non-separation. Thus it follows that two points A, B lie in a pair of vertically opposite angles or in a pair of adjacent angles formed by two lines COC′, DOD′ according as O (AB, CD) is positive or negative.

THEOREM: The point (x_1, y_1) lies in the acute or obtuse angle between the lines

$$f(x, y) \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0, \ (a + b \neq 0)$$

according as

$$(a + b)(ax_1^2 + 2hx_1y_1 + by_1^2 + 2gx_1 + 2fy_1 + c)$$

is negative or positive.

Let $ax^2 + 2hxy + by^2 + 2gx + 2fy + c \equiv (lx + my + n)(l'x + m'y + n')$ and the point of intersection of the lines be (p, q) so that $ll' + mm' = a + b$, and $lp + mq + n = 0$ and $l'p + m'q + n' = 0$; transferring the origin to O, (p, q) and taking new axes parallel to the original directions, we get the equations for the lines thus; say OP : $lx + my = 0$, OQ : $l'x + m'y = 0$. Now the line, say, OS : $ly - mx = 0$ is perpendicular to OP and is always in the obtuse angle between the given lines and the equation of the line joining the new origin to $(x_1 - p, y_1 - q)$, T say, is $x(y_1 - q) - y(x_1 - p) = 0$: obviously (x_1, y_1) is in the acute or the obtuse angle between the lines according as O(PQ, TS) is negative or positive;

$$\text{i.e.,} \quad \left(-\frac{l}{m} - \frac{y_1 - q}{x_1 - p}\right)\left(-\frac{l'}{m'} - \frac{m}{l}\right) \Big/ \left(-\frac{l}{m} - \frac{m}{l}\right)\left(-\frac{l'}{m'} - \frac{y_1 - q}{x_1 - p}\right)$$

is negative or positive

i.e., $(lx_1 + my_1 - lp - mq)\ (ll' + mm') \ / \ (l^2 + m^2)(lx_1 + m'y_1 - l'p - m'q)$ is negative or positive.

i.e., $(lx_1 + my_1 + n)(l'x_1 + m'y_1 + n)(ll' + mm') = (a + b) f(x_1 y_1)$ is negative or positive.

Cor. 1. The angle between the lines which contain (x_1, y_1) is $\tan^{-1}\left(\pm \dfrac{2\sqrt{h^2 - ab}}{a + b}\right)$ where the sign of the root is opposite to that of $f(x_1, y_1)$.

Cor. 2. The angle between the tangents from any point to a conic is just the angle which contains the centre and so can be easily to be obtained by Theorem I, Cor. 1.

Cor. 3. The angle between the tangents from (x_1, y_1) to the conic

$f(x, y) \equiv \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - 1$ is $\tan^{-1} \dfrac{2ab\sqrt{\dfrac{x_1^2}{a^2} + \dfrac{y_1^2}{b^2} - 1}}{x_1^2 + y_1^2 - a^2 - b^2}$, because the sign of the root opposite to that of $f(0, 0)$ is positive.

The angle between the asymptotes of the conic

$$ax^2+2\,hxy+by^2+2gx+2fy+c=0$$

(meaning thereby the angle which contains the curve) is $\tan^{-1}\left(\pm\frac{2\,(h^2-ab)^{\frac{1}{2}}}{a+b}\right)$ where the sign of the root is that of $\frac{\Delta}{C}$ or $-\Delta$, since the equation of the asymptotes is

$$ax^2+2hxy+by^2+2gx+2fy+c=\frac{\Delta}{C}$$

Hence the angle between the asymptotes is obtuse or acute according as $(a+b)\Delta$ is positive or negative.

4. In this connection, we wish to suggest that the introduction of the idea of a pseudo-slope may be convenient and useful in the calculation of cross-ratios. For, if we define λ, as the pseudo-slope of the line $u=\lambda v$ with respect to $u=0$ and $v=0$ on the analogy of 'm' in $y=mx$ being called the slope *w. r. t.*, $y=0$ and $x=0$, the cross ratio of $\{u=0,\ v=0;\ u=\lambda_1\ v,\ u=\lambda_2\ v,\}$ is $\frac{\lambda_1}{\lambda_2}$. The λ-slope of $mx-ly=0$ w.r.t. $lx+my=0$ and $l'x+m'y=0$ is $\frac{l^2+m^2}{ll'+mm'}$. Hence the bisector of the acute angle between the lines $lx+my+n=0$ and $lx+m'y+n'=0$ is $\frac{lx+my+n}{l'x+m'y+n'}=\pm\sqrt{\frac{l^2+m^2}{l'^2+m'^2}}$ the upper or lower sign being taken according as $ll'+mm'$ is negative or positive. Similarly, if the sides BC, CA, AB of a triangle are in order $a_i\,x+b_i\,y+c_i=0$ $(i=1,\ 2,\ 3)$, the bisector of the exterior angle at A is

$$\frac{a_2x+b_2y+c_2}{a_3x+b_3y+c_3}=\pm\sqrt{\frac{a_2^2+b_2^2}{a_3^2+b_3^2}},$$

the upper or lower sign being taken according as

$$(a_2\,b_1-a_1\,b_2)/(a_3\,b_1-a_1\,b_3)$$

is positive or negative, since the parallel through A to BC lies obviously in the exterior angle at A and has a pseudo-slope $(a_2\,b_1-a_1\,b_2)\,/\,(a_3\,b_1-a_1\,b_3)$ with respect to $a_2\,x+b_2\,y+c_2=0$ and $a_3\,x+b_3\,y+c_3=0$.

Maharaja's College, Mysore. A. A. KRISHNASWAMI AYYANGAR.

Notes on the Ellipse (II)

This note is a continuation of the previous note which appeared in vol. X, pp. 176 of *The Mathematics Student.* It gives simple constructions of a few more lengths and angles associated with a point of an ellipse and the confocal (and orthogonal) hyperbola through the point. The symbols of the previous note have been retained and a few new symbols introduced.

Let λ and λ' denote the transverse and the conjugate axes of the confocal hyperbola at P

Then $$r_1 - r_2 = 2\lambda$$

But $$r_1 - r_2 = \text{AN} - \text{NB} = 2\,\text{ON}$$

$$\therefore \quad \lambda = \text{ON} = a\cos\omega \qquad \dots \quad (6).$$

From the invariant relation,

$$\lambda^2 + \lambda'^2 = a^2 - b^2$$

we have $$\lambda'^2 = a^2 - a^2\cos^2\omega - b^2 = a^2\sin^2\omega - b^2$$

$$= f^2 - b^2 \qquad \text{by (A) of previous note}$$

$$= f^2 - f^2\sin^2\alpha \qquad \text{by (4)}$$

$$\therefore \quad \lambda' = f\cos\alpha \qquad \dots \quad (7)$$

It will now be manifest that in the diagram of the previous note Q L represents λ', L being the point of contact.

Also from (2) of last note it is clear that f is the length of the conjugate semi-diameter at P,

so that $$r f \sin \chi = a \cdot b \qquad \dots \quad (8)$$

or $$p = r. \sin \chi \qquad \dots \quad (9)$$

In relation (8) above χ denotes the angle between the conjugate diameters at P.

We also observe $$r^2 = p^2 + p'^2 \qquad \dots \quad (10)$$

where p' denotes the length of the perpendicular from the centre upon the normal to the ellipse or the tangent of the hyperbola.

Analogous to relations (1) and (5) of the former note, we have (for the hyperbola)

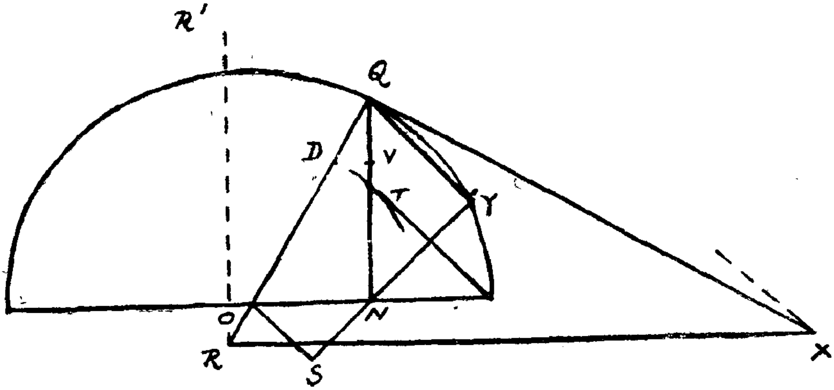
$$\lambda\cos\alpha = p' \qquad \dots \quad (11)$$

and $$\rho' = \frac{f^3}{\lambda\lambda'} \qquad \dots \quad (12)$$

ρ' being the radius of curvature of the hyperbola at P.

Now we give the construction for ρ', p' and χ and incidentally obtain new constructions for α and λ'.

In the diagram A, B, N, O, V, D, Q are the same as in the previous diagram. BT touches the circle, with centre O and radius OD$=p$, at T. OS and QY are perpendiculars on the perpendicular to BT through N.



$$\text{Sin } \angle \text{OBT} = \frac{\text{OT}}{\text{OB}} = \frac{p}{a} = \sin\alpha$$

$$\therefore \qquad \angle \text{OBT} = \alpha \qquad \ldots \text{(vii)}$$

$$\text{SN} = \text{ON } \cos\alpha = \lambda\cos\alpha = p' \qquad \ldots \text{(viii)}$$

$$\text{QT} = \text{QN}\cdot\cos\alpha = f\cos\alpha = \lambda' \qquad \ldots \text{(ix)}$$

$$\text{TS}^2 = \text{OT}^2 + \text{OS}^2 = p^2 + p'^2 = r^2$$

$$\therefore \quad \angle \text{OST} = \sin^{-1}\frac{\text{OT}}{\text{TS}} = \sin^{-1}\frac{p}{r} = \chi \qquad \ldots \text{(x)}$$

Produce QO to R where QR$=\rho$ (R can be constructed as in previous note). Draw tangent at Q cutting the parallel through R at X.

then $$\text{RX} = \rho\cdot\sec\omega = \frac{f^3}{ab\cos\omega} = \frac{f^3}{\lambda\cdot b} = \frac{\lambda'}{b}\cdot\frac{f^3}{\lambda\lambda'} = \frac{\lambda'}{b}\cdot\rho'.$$

but from (7) $$\lambda' = f\cdot\cos a = \frac{b}{\sin\alpha}\cdot\cos\alpha$$

$$\therefore \quad \lambda' = b\cdot\cot\alpha \qquad \ldots (13)$$

Hence if we draw XR′ and RR′ parallel to TB and perpendicular to XR respectively, to meet in R′ then

$$\text{RR}' = \text{RX}\cdot\tan\alpha = \text{RX}\cdot\frac{b}{\lambda'}$$

$$\therefore \qquad \text{RR}' = \rho' \qquad \ldots \text{(xi)}$$

68 Rash Behari Avenue
Calcutta

GAGANBIHARI BANDYOPADHYAY.

Lecture Notes

1. Criteria for an Ellipse.

In the December 1941 issue of the *Mathematics Student*, Mr. V. Balachandran gave the following criteria for a real ellipse.

The equation

$$ax^2+2hxy+by^2+2gx+2fy+c=0 \quad \dots \quad (1)$$

represents a real ellipse, if and only if the roots λ_1 and λ_2 of the quadratic

$$\lambda_2-(a+b)\ \lambda+(ab-h^2)=0$$

have a sign opposite to that of Δ, where a, b, c, f, g, h are real and Δ has the usual meaning.

Referring (1) to its centre, we get as usual

$$ax^2+2hxy+by^2+\frac{\Delta}{c}=0;$$

or

$$(ax+hy)^2+(ab-h^2)\ y^2=-\frac{a\Delta}{c}. \quad \dots \quad (2)$$

This will represent an ellipse if and only if

$$c=ab-h^2>0.$$

It will further contain real points if and only if

$$\frac{a\Delta}{c}<0, \text{ i.e. } a\Delta<0.$$

These criteria are more easily applied.

Balachandran's criteria reduce to these if we notice that λ_1, λ_2 are both positive if

$$c=ab-h^2>0 \text{ and } a>0.$$

They are both negative when $c>0$ and $a<0$; hence in either case $c>0$ and $a\Delta<0$.

If $c>0$, $a\Delta>0$ the ellipse is imaginary.

It is interesting to note that for a proper parabola $a\Delta<0$, while for a hyperbola $a\Delta$ can have any value whatsoever.

HANSRAJ GUPTA
A. A. KRISHNASWAMI AYYANGAR

2. *The reduction of the General Equation of the second degree.*

In the case of a central conic, after changing to parallel axes through the centre as usual, we may *directly* change to the principal axes, avoiding polar considerations altogether. In the case of a parabola, after finding the axis and the tangent at the vertex as usual, we may again *directly* change to these as axes.

The central conic. It is sufficient to consider the equation

$$ax^2+2hxy+by^2=1. \quad \dots \quad (2\cdot1)$$

The principal axes of this are given by

$$\frac{x^2-y^2}{a-b}=\frac{xy}{h} \quad \dots \quad (2\cdot2)$$

In other words, if

$$\frac{x^2-y^2}{a-b}-\frac{xy}{h}\equiv(px+qy)(qx-py) \quad \dots \quad (2\cdot3)$$

then k, l exist so that

$$ax^2+2hxy+by^2\equiv k\ (px+qy)^2+l\ (qx-py)^2 \quad \dots \quad (2\cdot4)$$

Now (2·1) takes the form

$$k\,(p^2+q^2)\left[\frac{px+qy}{\sqrt{p^2+q^2}}\right]^2+l\,(p^2+q^2)\left[\frac{qx-py}{\sqrt{p^2+q^2}}\right]^2=1 \quad \dots \quad (2\cdot5)$$

So, we see that whenever (2·1) is a real locus, k, l cannot both be <0 and that we may, without loss of generality, take the cases which arise as: case 1. k, l both >0, $k<l$; case 2. $k>0$, $l<0$.

Now set $\quad X=\dfrac{px+qy}{\sqrt{p^2+q^2}},\ Y=\dfrac{qx-py}{\sqrt{p^2+q^2}} \quad \dots \quad (2\cdot6)$

$$\frac{1}{A^2}=k(p^2+q^2),\ \frac{1}{B^2}=|l\,(p^2+q^2)|\text{ the } \textit{numerical} \text{ value of } l\,(p^2+q^2)\dots \quad (2\cdot7)$$

Obviously X, Y, are themselves co-ordinates of the point (x, y); the axes being respectively $qx-py=0$; $px+qy=0$.

Now (2·5) and hence (2·1) takes the standard form

$$\frac{X^2}{A^2}+\frac{Y^2}{B^2}=1 \text{ in case 1; and } \frac{X^2}{A^2}-\frac{Y^2}{B^2}=1 \text{ in case 2.}$$

The parabola. We reduce $S=0$, as usual to the form

$$(Lx+My+k)^2=m\ (Mx-Ly+l).$$

5

We then re-write this as $(Lx+My+k)^2 = np\ (Mx - Ly + l)$, where $n = |m|$. Hence p is ± 1.

Then we set

$$X = \frac{(Mx - Ly + l)\,p}{\sqrt{L^2+M^2}},\quad Y = \frac{Lx+My+k}{\sqrt{L^2+M^2}},\quad 4A = \frac{n}{\sqrt{L^2+M^2}} \quad \dots \quad (3{\cdot}1)$$

Now $S=0$ reduces to the standard form $Y^2 = 4Ax$.

Example: $S \equiv 2x^2 - 4xy - y^2 + 8x - 2y + 4 = 0$. This is a hyperbola with centre at $(-1, 1)$. Set $x' = x+1$, $y' = y-1$. Transferring to parallel axes through the centre, we have $S \equiv 2x'^2 - 4x'y' - y'^2 = 1$.

The principal axes are given by $\dfrac{x'^2 - y'^2}{3} = \dfrac{x'y'}{-2}$ or $2x' - y' = 0$, $x' + 2y' = 0$.

Now $S \equiv \dfrac{3}{5}(2x' - y')^2 - \dfrac{2}{5}(x' + 2y')^2 = 1$ or $3\left(\dfrac{2x' - y'}{\sqrt{5}}\right)^2 - 2\left(\dfrac{x' + 2y'}{\sqrt{5}}\right)^2 = 1$.

$\therefore$ Setting $X = \dfrac{2x' - y'}{\sqrt{5}}$, $Y = \dfrac{x' + 2y'}{\sqrt{5}}$, $A^2 = \dfrac{1}{3}$, $B^2 = \dfrac{1}{2}$ we have the standard form.

The length of the transverse axis and the eccentricity are

$$2A = 2/\sqrt{3} \text{ and } e = \sqrt{\frac{A^2+B^2}{A^2}} = \sqrt{\frac{5}{2}}.$$

The equations of the transverse and conjugate axes are $Y=0$, $X=0$ i.e. $x + 2y - 1 = 0$, $2x - y + 3 = 0$.

The foci are the points $X = \pm Ae$, $Y = 0$ i.e. the roots of the equations $\dfrac{2x - y + 3}{\sqrt{5}} = \pm\sqrt{\dfrac{5}{6}}$, $x + 2y - 1 = 0$.

The directrices are the lines $X = \pm\dfrac{A}{e}$ i.e. $\dfrac{2x - y + 3}{\sqrt{5}} = \pm\sqrt{\dfrac{2}{15}}$.

Govt. Training College
Rajahmundry.

K. SATYANARAYANA.

---

* For discussions on this topic Vide *Mathematical Gazette*: January 1925; Oct. 1925; Feb. 1934; Feb. 1935; Oct. 1940; Dec. 1940; Dec. 1943.

* The factors on the right side are easily got by factorisation.

SOME INSTRUCTIVE PROBLEMS AND THEIR SOLUTIONS—I

[Under this heading, we shall publish from time to time, problems of special interest to students preparing for examinations, together with solutions and comments on them.—EDITOR]

Analysis

PROBLEM. (1) Show that $\int_0^x \sin \frac{1}{t}\, dt$ has a derivative at $x=0$.

(2) If $f(x)$ is continuous in (0, 1) except at the points of the sequence $\left(\frac{1}{n}\right)$; show that $f(x)$ is integrable.

(3) If $f(x)=0$ in (0, 1) except at the points of a sequence (α_n) and $f(\alpha_n)=\frac{1}{n}$, $(n=1, 2, \ldots)$ prove that $f(x)$ is integrable.

Solution and remarks by V. Ganapati Iyer.

(1) Theoretically, differentiation and integration (in any sense, e.g., Reimann, Lebesgue, etc.) are not exact inverses of each other. How far they are so is itself a main problem in any theory of integration. For Reimann integration the main result in this connection runs as follows:—

If $f(x)$ is bounded and integrable in (a, b) and $F(x)=\int_0^x f(t)\, dt$ then (i) $F(x)$ is continuous in (a, b) and (ii) $F'(x)=f(x)$ at any point of continuity of $f(x)$.

But it does not follow that $F'(x)$ does not exist at a point of discontinuity of $f(x)$. For instance, if $f(x)\to l$ as $x\to x_0$ but $f(x)$ is not continuous at x_0 (i.e., $f(x)$ has a removable discontinuity at x_0) then $F'(x_0)=l$. A more complicated case is the one in the problem.

(1) Here we have

$$F(x)= \int_0^x \sin \frac{1}{t}\, dt=\int_0^x t^2 \cdot \sin \frac{1}{t} \frac{dt}{t^2}\, dt$$

$$= \left[t^2 \cos \frac{1}{t}\right]_0^x - 2\int_0^x t \cos \frac{1}{t}\, dt$$

$$= x^2 \cos \frac{1}{x} + O(x^2)$$

$$= O(x^2) \text{ since } \cos \frac{1}{t} \text{ is bounded for } |t|>0$$

Hence $\lim\limits_{x\to 0} \dfrac{F(x)}{x} = 0$. So $F'(0)=0$ since $F(0)=0$.

The conclusion in this problem holds good in some general cases. Suppose $f(x)$ is bounded and integrable in $(0, \delta)$, $\delta>0$ but not continuous at $x=0$. Let $F(x)=\int_0^x f(t)\,dt$. Then $F'(+0)$ exists and equals zero if either (i) $\int_0^x \frac{f(t)}{t}\,dt$ converges at $x=0$ or (ii) there exists some $\alpha>1$ such that $\int_\varepsilon^x \frac{f(t)}{t^\alpha}\,dt$ is bounded for $0<\varepsilon<x\leqslant\delta$. The problem under discussion comes under (ii) with $\alpha=2$. The proof under condition (ii) is similar to the special problem above. If (i) holds we have $\int_0^x f(t)\,dt = x\phi(x) - \int_0^x \phi(t)\,dt = \int_0^x [\phi(x)-\phi(t)]\,dt = o(x)$ where $\phi(x) = \int_0^x \frac{f(t)}{t}\,dt$. Hence $F'(+0)=0$. For instance we can take $f(x)=\cos\frac{1}{x^n}$ or $\sin\frac{1}{x^n}$ $(n\neq 0.)$

PART (2) Let $\varepsilon>0$ be given. Then the number of points of the sequence $\left(\frac{1}{n}\right)$ outside the interval $\left(0, \frac{\varepsilon}{2}\right)$ is finite. Such points can be enclosed in a *finite number* of intervals of total length not exceeding $\varepsilon/2$. Along with $(0, \varepsilon/2$ we) get a *finite number* of intervals of total length not exceeding ε such that these enclose all points of discontinuity of $f(x)$ in $(0, 1)$. Hence by a known theorem, $f(x)$ is integrable in $(0, 1)$.

PART (3) Firstly we shall prove that $f(x)$ is continuous at a point α in $(0, 1)$ not belonging (α_n). Let $\varepsilon<0$ be given and let n_0 be an integer such that $\frac{1}{n_0}\leqslant\varepsilon$. Choose $\delta>0$ such that the interval $(\alpha-\delta, \alpha+\delta)$ does not contain any of the points $(\alpha_1, \alpha_2, \ldots, \alpha_{n_0})$. This is possible since α does not belong to (α_n). If x is in $(\alpha-\delta, \alpha+\delta)$, then either $f(x)=0$ or $|f(x)|\leqslant\frac{1}{n_0}\leqslant\varepsilon$. Hence $|f(x)|\leqslant\varepsilon$ if $|x-\alpha|\leqslant\delta$. So $f(x)$ is continuous at α. A similar argument will show that $f(x)$ is *not* continuous at any point of the sequence (α_n).

If (α_n) has only a finite number of limit points * we can use the argument used in (2) and conclude that $f(x)$ is integrable. In the general case we can give two proofs :—

(i) It is known that a necessary and sufficient condition that a bounded function $f(x)$ should be integrable in (a, b) is that the points of discontinuity of $f(x)$ in (a, b) form a null set, i.e., they have Lebesgue measure zero. Any enumerable set is a null set. Since, in the given case, $f(x)$ is continuous in (0, 1) except at the points (α_n), it follows that $f(x)$ is integrable.

(ii) In an elementary course it is not likely that the theorem mentioned in (i) is proved. In such cases we can give a direct proof. Let ω and σ be two positive numbers. Let n_0 be such that $\frac{1}{n_0} \leqslant \omega$. Enclose the points $(\alpha_1, \ldots, \alpha_{n_0})$ in a finite number of intervals of total length not exceeding σ. In any of the remaining intervals of (0, 1), the oscillation of $f(x)$ does not exceed ω since $f(x)=0$ or $0 < f(x) \leqslant \frac{1}{n_0} \leqslant \omega$ at any point of such an interval. Hence we get a subdivision of (0, 1) into a finite number of parts such that the sum of the lengths of sub-intervals in which the oscillation of $f(x) \geqslant \omega$ does not exceed σ. Hence by a known theorem $f(x)$ is integrable.

Geometry

PROBLEM.—Show that any triangle and its polar triangle with respect to any conic are perspective. Deduce Hesse's theorem that if two pairs of opposite sides of a quadrangle are conjugate with respect to a conic, so are the third pair. Calling such a quadrangle an apolar quadrangle of the conic, prove that:—

If S_2 is a conic circumscribed to a self-polar triangle of another conic S_1 then:—

(*a*) there exist ∞^1 self-polar triangles of S_1 inscribed in S_2,

(*b*) there exist ∞^3 apolar quadrangles of S_1 inscribed in S_2

(*c*) there exist ∞^1 triangles self-polar to S_2 and circumscribed to S_1.

Deduce that if a quadric cone has one set of three mutually perpendicular generators, then it has ∞^1 such sets and also ∞^3 sets of 4 generators such that the plane through any two of these generators of a set is perpendicular to the plane through the other two.

Solution and remarks by M. Venkataraman

Let ABC be any triangle and A′B′C′ the poles of BC, CA, AB with respect to any conic S. Let BC, B′C′ meet at P and let AA′ meet BC, B′C′ at X, X′ respectively.

Then

$(BCXP) = A\,(BCXP) = (C'B'PX')$ replacing each line by its pole
$= (B'C'X'P)$ interchanging in pairs

Since these two projective ranges have a common corresponding element P, it follows that BB′, CC′ and XX′ (= AA′) are concurrent at say O—the centre of perspective of triangles ABC, A′B′C′.

Let now ABCD be any quadrangle such that AB, CD and AC, BD are pairs of conjugate lines with respect to a conic S. As before let A′ B′ C′ be the poles of BC, CA, AB and let O be the centre of perspective of triangles ABC, A′ B′ C′. Since AB, CD are conjugate, C′ the pole of AB must lie on CD. (i.e.) CC′D are collinear. Similarly, BB′ D are collinear (i.e.) D is the point of inter-section of BB′ and CC′ (i.e.) D coincides with O. ∴ AA′ passes through D (i.e.) AD contains A′ the pole of BC. ∴ AD is conjugate to BC. This proves Hesse's theorem.

It is now obvious that any three vertices of such an apolar quadrangle can be chosen arbitrarily, and the fourth vertex will be uniquely fixed—as *the centre of perspective of the triangle* formed by these three points with its polar triangle. There is however, one notable case of exception namely when the three chosen points are the vertices of a self polar triangle—in which case, the fourth vertex of the apolar quadrangle can be any where. It is easily verified that the necessary and sufficient condition for such a degenerate case is that two of the vertices of the apolar quadrangle be conjugate.

LEMMA. If two apolar quadrangles ABC_1D_1, ABC_2D_2 of a conic S have two vertices in common, all the six vertices must lie on a conic.

For (AC_1, BD_1) (AD_1, BC_1) (AC_2, BD_2) and (AD_2, BC_2) are by the definition of an apolar quadrangle, pairs of conjugate lines through A and B. ∴ $A\,(C_1\,D_1\,C_2\,D_2) = B\,(D_1\,C_1\,D_2\,C_2)$
$= B\,(C_1\,D_1\,C_2\,D_2)$

∴ $A\,B\,C_1\,D_1\,C_2\,D_2$—the six vertices of the two quadrangles must lie on one and the same conic.

Let now S_1 S_2 be two conics such that there exists one apolar quadrangle ABCD of S_1 inscribed in S_2. Let A_1 B_1 C_1 be any three

Since the polar AA′ of P passes through O, we have O is conjugate to P and similarly to Q, R the intersections of CA, C′ A′ and AB, A′ B′ respectively. (i.e.) O, the centre of perspective and PQR—the axis of perspective of the polar triangles are pole and polar.

points on S_2. Then, complete in succession the apolar quadrangles A_1 B C D_2, A_1 B_1 C_2 D_2, A_1 B_1 C_1 D_1 of S_1. Now, since ABCD. A_1 B C D_2 are two apolar quadrangles of S_1 with two common vertices B and C, their six vertices $ABCDA_1D_2$ must lie on a conic (i.e.) D_2 must lie on the conic S_2 through the other points. Similary we get that C_2 and D_1 must lie on S_2. Thus we get that if there is one apolar quadrangle of S_1 incribed on S_2, an apolar quadrangle of S_1 can be constructed with three of its vertices at any three points of S_2; and having its fourth vertex also on S_2 (i.e.) there are ∞^3 apolar quadrangles of S_1 inscribed in S_2.

Let now S_1 S_2 be any two conics such that there exists one self polar triangle ABC of S_1 inscribed in S_2. Let D be any other point on S_2. Then A B C D is *one apolar quadrangle of* S_1 *inscribed in* S_2 and hence by the above, there are ∞^3 apolar quadrangles of S_1 inscribed in S.

Again let A′ be any point on S_2. Let its polar line with respect to S_1 meet S_2 at B′ and C′. Then from what precedes, an apolar quadrangle of S_1 (say) A′ B′ C′ D′ can be constructed such that D′ also lies on S_2. But since in this apolar quadrangles of S_1, A′ and B′ are conjugate with respect to S_1, the quadrangle splits into a self polar triangle and another point—easily seen to be A′ B′ C′ and D′ (i.e.,) a self-polar triangle of S_1, A′ B′ C′ can be constructed to have its three vertices on S_2 and having one of its vertices at any assigned point on S_2 (i.e.) there exist ∞^1 triangles self-polar to S_1 and inscribed in S_2. Result (*c*) follows from (*a*) by reciprocation.

Let now O be the vertex of a cone Γ such that it has one set of three mutually perpendicular generators say *a*, *b*, *c*.

Let the section of Γ by the plane π at infinity be say S_2 and let S_1 denote the circle at infinity. Let *a*, *b*, *c*, meet π at A, B, C respectively. We shall re-call here that two lines (planes) are said to be perpendicular when their sections by π are conjugate points (lines) with respect to S_1. Since, *a*, *b*, *c* are given to be a set of perpendicular generators of Γ, we have that A B C is a triangle inscribed in S_2 and self-polar *to* S_1 Hence, by what precedes, (*a*) there exist ∞^1 triangles A′ B′ C′ inscribed in S_1 and self polar to S_2 and (*b*) there exist ∞^3 quadrangles A_1 B_1 C_1 D_1 inscribed in S_1 and circumscribed to S_2. Joining O to all the points on S_2, and recalling the significance of perpendicular lines and planes, we have: there exist (*a*) ∞^1 sets of three generators of Γ namely OA′, OB′, OC′ such that these lines are mutually perpendicular and (*b*) there are ∞^8 sets of 4 generators of Γ (viz.) OA_1, OB_1, OC_1 and OD_1 such that the plane through any two of these lines say OA_1, OB_1 is perpendicular to the plane through the other two OC_1, OD_1 (since the section of these planes by π namely A_1B_1 and C_1D_1 are opposite sides of an apolar quadrangle of S_1 and so are conjugate with respect to it).

The introduction of co-ordinates clarifies the situation considerably.

We shall take the vertices of the quadrangle ABCD as $\pm 1 \pm 1 \pm 1$. Let the line equation of the conic S be $al^2+bm^2+cn^2+2fmn+2gnl+2hlm=0$.

The pairs of opposite sides of the quadrangle are

$$x+y=0 \qquad x-y=0$$
$$x+z=0 \qquad x-z=0$$
$$y+z=0 \qquad y-z=0.$$

If two pairs—say the first two—of these are given to be conjugate lines of S_1 we have $a-b=0$ and $a-c=0$. Hence we get $b-c=0$ which shows that the third pair of opposite sides are also conjugate.

Let $S_2=(a'b'c'f'g'h')\ (xyz)^2=0$ be a conic containing one polar quadrangle ABCD of S_1 We shall take the co-ordinates of ABCD as above. Since S_2 contains the points $\pm 1, \pm 1, \pm 1$, its equation is seen to reduce to $a'x^2+b'y^2+c'z^2=0$ subject to the condition $a'+b'+c'=0$. But this is seen to be the condition that the invariant $\Theta=aa'+bb'+cc'+2ff'+2gg'+2hh'$ of the conics S_1 and S_2 should vanish. This property we know reflects the conditions of S_2 containing triangles self-polar to S_1.

Let now A′B′C′ be any three points on S_2 and let D′ along with A′B′C′ constitute an apolar quadrangle of S. We shall now choose another triangle of reference such that A′B′C′D′ now have the co-ordinates $\pm 1 \pm 1 \pm 1$. [Let D′ be (1, 1, 1)]. Let S_1 be $(a\,b\,c\,f\,g\,h)\ (lmn)^2=0$ and S_2 be $(a'b'c'f'g'h')\ (xyz)^2=0$.

Since A′B′C′D′ is an apolar quadrangle of S, it follows easily that $a=b=c=k$ (say).

Since the Θ invariant of S, S′ vanishes

$$ka'+kb'+kc'+ff'+gg'+hh'=0.$$

Since S′ contains $(-1, 1, 1)$ $(1, -1, 1)$ and $(1, 1-1)$ we have:

$$a'+b'+c'+2f'-2g'-2h'=0.$$
$$a'+b'+c'-2f'+2g'-2h'=0.$$
$$a'+b'+c'-2f'-2g'+2h'=0.$$

Also $ka' + kb' + kc' + 2ff' + 2gg' + 2hh' = 0.$

$\therefore \quad f' = g' = h' = 0 = a' + b' + c'.$

$\therefore \quad a' + b' + c' + 2f' + 2g' + 2h' = 0.$

Hence S′ contains (1, 1, 1) also which shows that if there is one apolar quadrangle of S_1 inscribed in S_2, there are ∞^3 such apolar quadrangles—in which case there are ∞^1 self-polar triangles of S_1 inscribed in S_2.

FACILITIES FOR STATISTICAL STUDY OFFERED AT VARIOUS CENTRES IN INDIA

[There has been, in recent times, a strong demand for mathematicians with a good statistical training. We give below details regarding facilities for such training offered by institutions which have devoted special attention to statistical studies — EDITOR]

STATISTICAL LABORATORY, *Travancore University, Trivandrum.*

The Statistical Laboratory of the Travancore University is a Section in the Department of Research of the University. Here teaching and research go hand in hand. A Statistical Bureau is attached to the Laboratory and this serves to give advice to Government and recognised private organisations on Statistical technique, especially in conducting surveys and analysing the result. The Library has a good collection of recent books and journals dealing with Statistics. The laboratory has also a good collection of calculating Machines and Tables for statistical work.

Courses of Study

A. A two-year post graduate course in Statistics is conducted in the Laboratory. Admission to this course is limited to eight students. Students undergo tuition in the following subjects :—

1. *Mathematics.*

Modern Algebra, n dimensional Geometry, Mathematical Analysis, Differential and Difference Equations, Calculus of Variations and Integral Equations.

2. *Statistical Theory.*

3. *Applications of Statistics* to Agriculture, Biometry, Epidiomology, Economics and Mathematical Physics.

4. *Numerical Mathematics.*

The Students have to take six written papers and two practical tests including viva. Every candidate has to do field work for a period of about two months.

B. Facilities are also afforded for Research in Statistics for candidates who have already acquired a fairly good knowledge of Pure Mathematical Analysis. M. Sc. and Ph.D Degrees can be taken by such candidates.

C. Candidates specially deputed by Universities can work in the Statistical Laboratory for short periods—during which time, personal individual attention will be given both in theory and practical.

Further details may be had from the Director of Research, Travancore University, Trivandrum.

THE INDIAN STATISTICAL INSTITUTE, *Calcutta.*

Since 1932 special facilities for training in Statistics have been provided by the Indian Statistical Institute for (i) candidates with an adequate background of mathematics and/or economics, and (ii) officers on deputation from Government Departments, Universities and other institutions. The Training course is of one year's duration commencing from July; applications for admission must usually be received by the second week of May. Both theoretical and practical training is given. In the theoretical classes instruction is provided in the essentials of descriptive and mathematical statistics, as also in vital, economic and official statistics, and the applications of statistics to agriculture, phychology and education. Special stress is laid on the theory of sample surveys, and the use of sampling methods in the collection of statistics. The main aim of the practical classes is to teach the handling of statistical data. Special stress is laid on making the trainees quick and efficient computers.

Officers on deputation who come with their own problems, are put under the guidance of some senior worker. They are also allowed to attend the theoretical lectures and practical classes of the training section.

From the current year (1946) a special course of training in Industrial Statistics has been started under the auspices of the Board of Industrial and Scientific Research. This is also a one year's course with the session beginning in July.

The Institute possesses a fine up to date library, which the trainees can use. They have also unique opportunities for coming in contact with actual large scale statistical work carried on at the Institute. It is interesting in this connexion to quote the following observations from the recent note on the Statistical Institute which appeared in *Nature* 15 December 1945, p. 722 :

"The Institute, as it has now developed, has many facts: on the educational side equally as a training ground for computors and routine statisticians, and as a centre of postgraduate research in the most far-reaching branches of the mathematical theory of statistics and experimental design; as a professional institute and learned society bringing together all schools of thought in Indian statistics; as an agency employed by departments of Government and advisory bodies, in the essential work of collecting, scrutinizing and digesting the facts upon which administrative decisions must depend............... There is perhaps no other organization in which practical and theoretical work are more thoroughly integrated."

STATISTICS COURSES IN THE UNIVERSITY OF MYSORE, *Mysore*.

Instruction in Statistics in this University is given in three departments of Maharaja's College, Mysore.

1. Department of Mathematics and Statistics,
2. Department of Psychology,
3. Department of Economics and Politics.

Courses for B. Sc. (Pass), B. Sc. (Hons.) and for M. Sc. Degree examinations include Statistics as an optional subject, a Major, or Minor subject.

I. B. Sc. (Pass): Optional Subjects

Group 10. 'Experimental Psychology', 'Mathematical Statistics', 'Child Psychology and Educational Psychology'.

Group 11. 'Mathematics', 'Mathematical Statistics and Mathematical Economics', and either Economics or Sociology or Psychology.

The second subject of this group includes Mathematical Statistics, Mathematical Economics, Mathematics or Social Measurements the latter being offered by candidates who take Mathematics instead of Sociology. The syllabus for Statistics in this group is as follows :

Theory of attributes, Association of attributes, Standard frequency distributions, measures of location and dispersion, moments, correlation, curve fitting, sampling, χ^2 distribution, Interpolation.

Group 16. Mathematics, Statistics and Geography.

II. B. A. (Hons.)

Economics (Major) with one of the Minors: Elements of Statistics or Pure Mathematics with Applied Mathematics which includes General Statistics etc.

Philosophy (C) (Psychology Branch) (Major) with a Minor subject which includes Statistics and Scientific Method. The course of study in Statistics as Psychology Minor comprises:

Normal curve, Graphic representation of Frequency distributions, Measures of central tendency and variability, Reliability of statistical constants, Psycho-Physical Methods, Rank order, Paired comparisons, Spearman's rank correlation, Pearson's product-moment correlation, Yule's coefficient of association, Probable error, Coefficients of reliability and validity, Corrections of a coefficient of Correlation for Attenuation, Partial and multiple correlations, Tetrad equation.

The syllabus for Mathematics (Minor) corresponding to Economics (Major) for the Third Paper in Mathematics is as follows:

Collection and classification of Statistics, central tendencies, curve fitting, method of least squares, correlation, elementary theory of sampling, essentials of mental measurement, Weber's and Fechner's laws, constants of mental measurement, reliability measures.

III. B. Sc. (Hons.)

1. *Statistics* (Major) with two special subjects and Economics or Experimental Psychology or Physics as a minor subject.

2. *Experimental Psychology* (Major) with Mathematical Statistics included in one of the minor subjects.

3. *Economics* (Major) with Advanced Statistics, Mathematical Economics and Social Measurements as minor Subjects.

The course of study for B. Sc. (Hons.) with Statistics as major subject is as prescribed below:

(i) *Pure Mathematics* covering the ordinary Honours standard in Calculus, Geometry and Algebra with a special study of Legendre's and Bessel's functions and Orthogonal Polynomials.

(ii) *Statistics* covering the ground of Kendall's *Advanced Theory of Statistics* and Fisher's *Statistical Methods for Research Workers* and *Design of Experiments* in addition to the Theory of Probability, Interpolation, Graduation and Linear Difference Equations.

(iii) *Econometrics* dealing with the Mathematical Theory of economic problems: monopoly, taxation, wages, interest, money, and foreign exchange; and

(iv) two *special subjects* chosen one from each of the following groups of subjects:

A. (1) Modern Algebra, (2) Theory of Functions of a Complex Variable, (3) Calculus of Variations, (4) Higher Geometry, (5) Advanced Theory of Probability.

B. (1) Factor Analysis, (2) Biometrics, (3) Special statistical methods in agriculture, (4) Special statistical methods in Industry, (5) Special statistical methods in Psychology and Education, (6) Actuarial Science. The Special subjects now taught are A (5) and B (6).

IV. M. Sc. (Statistics)

This is a one year course open to Honours graduates of this University and those of other Universities possessing equivalent qualifications. The examination consists of four papers and a *Viva Voce* examination or a thesis and a *Viva Voce* examination. Four subjects for the four papers are chosen from the following list which is not exhaustive :—

Theory of Functions of a Complex Variable, Theory of Functions of a Real Variable, Modern Algebra, Theory of Groups, Difference Equations, Sampling theory, Design of Experiments, Genetic Studies, Advanced Econometrics, Quality Control and Specification Problems in Industry.

IMPERIAL COUNCIL OF AGRICULTURAL RESEARCH, *New Delhi*

These courses were started by the Imperial Council of Agricultural Research, in the year 1945, with a view to assist persons seeking statistical careers in the Agriculture and Animal Husbandry Departments of the country. Prior to the starting of these classes, the Council was imparting instruction on these lines to a select number of candidates, deputed for the purpose from time to time by their respective Departments and States. There has, however, been an ever growing demand for training of this type, and the Council accordingly decided to regularise the training and to institute special classes for the purpose.

The courses of training at present offered are :—

A. One Year's Course:

An Attempt is made in this course to teach statistical methods in agricultural and animal husbandry research and the associated theory of statistics. The course includes the design, analysis and interpretation of field and animal husbandry experiments, and the methods of collecting agricultural statistics of crops, livestock, fisheries etc. in vogue in India and abroad. A course of lectures on large scale sample surveys and their application in improving the methods of collecting agricultural statistics is also given.

This course is intended primarily to train persons for appointments as statistical assistants in agricultural, animal husbandry and veterinary Departments. A certificate is awarded to students satisfactorily completing the course.

B. Second Year: Advanced Course:

This course covers portions mentioned in 'A' above in greater detail. It includes advanced course in statistical methods in agricultural and animal husbandry research and the associated advanced theory of statistics. Regular courses of lecturers on the following subjects are also arranged:

Design of experiments, research methods in plant and animal breeding, statistical concepts of genetics. In addition, every student is required to undertake a statistical study of suitable agricultural data. This course is intended to train persons primarily for appointments as Statisticians in agricultural, animal husbandry and veterinary Departments. A diploma in statistics of the Imperial Council of Agricultural Research is awarded to students satisfactorily completing this course.

C. A Short Course for officers

This extends over a period of about 3 months and a half and is meant for officers on deputation from either the Council's schemes or otherwise sent by the Provinces and States. This course covers only the routine statistical methods ordinarily employed by research workers in Agriculture and Animal husbandry and hence in a way, is limited in scope.

D. In addition to the above three courses, a series of lectures on elementary Mathematics, fundamental and applied Biology and general agriculture are arranged for the benefit of those who did not undergo this training previously.

General. Extensive facilities regarding library and laboratory exist in the Section. There are a large number of calculating machines in the Section. The Section maintains a well-equipped sectional library where reprints of important publications are available. In addition, library facilities are available to the students at the Imperial Agricultural Research Institute through special arrangement with the Director of the Institute. Periodic seminars are held at which staff and students discuss problems of theoretical and practical interest bearing on the research work of the Section.

ANNOUNCEMENTS AND NEWS

The following gentlemen have been admitted as members of the Indian Mathematical Society :—

S. C. Oak Esq., B.E., A.M.I.E., 7 A, Vissonji Park, Naigaum Crossroad, Dadar, Bombay.

S. P. Bhatnagar Esq., M.A., M.Sc., Professor, Herberts College, Kotah (Rajaputana).

A. S. Apte Esq., M.A., Research Student, Tata Institute for Fundamental Research, Bombay.

B. Dutta Esq., Teacher, Chandranagore.

V. N. Amble Esq., M.Sc., Statistical Assistant, Imperial Council of Agricultural Research, New Delhi.

Brij Lall Esq., M.A., Lecturer, D. A. V. College, Hoshiarpur.

Om Prakash Malhotra, M. A., Professor, D. A. V. College, Hoshiarpur.

Ram Prakash Bambah, M.A., Research Student, Iswarnagar, Mughalpura.

M. Parameswara Iyer Esq., B.A., L.T., Board High School, Usilampatti, (Madura District).

S. C. Ghosal Esq., B.Sc., LL.B., Secretary to the Prime Minister, Alwar, (Rajaputana).

The Society acknowledges with gratitude the gift of Rs. 500 from the Rockfeller Foundation, distributed through the National Institute of Sciences, Calcutta.

We learn that Messrs. K. G. Ramanathan, and K. Chandrasekharan have been appointed to membership of the Institute for Advanced Study at Princeton for the Academic year 1946-47 with a stipend of 2100 Dollars each.

Prof. P. C. Mahalanobis has been appointed a member of the Statistical Commission established by the Economic and Social Council of the United Nations Organization.

The Indian Science Congress met at Bangalore in December 1945, with Dr. Ram Behari of the Delhi University as the President of the Mathematics Section. Over 30 papers were contributed to the section of which about 15 were read and discussed. Dr. Ram Behari's Presidential Address was devoted to an elaborate survey of Recent Advances in the differential geometry of ruled surfaces.

The next session of the Indian Science Congress will be held at Patna. Prof. D. D. Kosambi has been elected President of the section on Mathematics, and Dr. N. G. Shabde as the Recorder of the section.

The Sixth International Congress for Applied Mechanics is proposed to be held in Paris from September 22nd to September 29th, 1946. The Congress will meet at the Sarbonne, and will be divided into the following sections :—

1. Structures, Elasticity, Plasticity.
2. Hydro and Aerodynamics, Hydraulics.
3. Solid Dynamics ; Vibration and sound, Friction and Lubrication.
4. Thermodynamics, Heat transfer, Combustion Fundamentals of neuclear energy.

Those wishing to become members should inform the General Secretary (Maurice Roy, Institute Henri-Poincare, Paris) regarding their intention to attend the Congress and to read papers thereat with details. A payment of an inscription fee of 300 francs is required from members.

GLEANING

Oh hexagonal, triangular, colorless creature,
Essence of Geometry in every feature,
Your contours alone are glorified,
When your angular symmetry's magnified.
For you the baker or jeweller cares,
Your design he copies for his wares.

Along with the triangle, rectangle and prism,
Geometry books should teach "snowflakism".
With sisters and brothers of similar form,
You make the setting of a true winter's morn.
Highest of praises to you may be sung,
Without you "White Christmas" could never have come.

JOHN HOLLANDER in *The Mathematics Teacher*, Feb. 46.


18 JAN 1947
MADRAS

THE INDIAN MATHEMATICAL SOCIETY'S FOURTEENTH CONFERENCE, DELHI, 19TH DECEMBER, 1945.

1st row (sitting) S. S. Pillai, H. R. Gupta, B. R. Seth, A. N. Singh, D. D. Kapadia, Sir Shankar Lal, R. Vaidyanathaswami, F. W. Levi (*President*), Sir B. N. Rau (*President of the Inaugural Session*), R. B. Ram Kishore, A. Narasinga Rao, V. Ganapati Iyer, A. C. Banerji, A. A. Krishnaswami Ayyangar, S. M. Shah.

2nd row J. P. Jaiswal, K. Sambasiva Rao, Md. Khaja Mohideen, M. A. Hosain, R. S. Varma, K. D. Panday, N. N. Bose, S. Nageswaran, B. Ramamurti, Ram Behari (*Local Secretary*), S. Subramaniam, B. C. Chatterji, K. S. Sastri, S. Minakhshisundaram, D. R. Kaprekar, Kaushik, S. R. Das, T. Venkatarayudu, ——, N. D. Tewari.

3rd row Afzal Hussain, V. D. Thawani, Abbas Rizwi, M. L. Chandrafreya, S. A. Hamid, H. C. Chatruvedi, R. D. Misra, Abdulla Butt, K. D. Sayal, Hari Shanker, S. R. Gupta, H. C. Gupta, S. M. Karmalkar, Shanti Naryan, V. S. Krishnan, H. C. Saxsena, Banwari Lal, Ambikeshwar Sarma, J. N. Mitra.

4th row ——, B. B. Bagi, O. P. Jain, B. Bhownik, R. L. Gupta, Vijay Perkash, F. C. Auluck, G. R. Toshiwal, N. K. Saha, P. L. K., C. P. S. Menon, Srinivas Asthana, A. K. Srinivasan, R. S. Misra, P. D. Gupta.

5th row Brij Mohan, Sahib Ram Mandan, ——, ——, Kailash Behari, Karam Chand Dhawan, P. L. Bhatnagar, Bala Gangadharan, J. N. Kapoor, ——