THE MATHEMATICS STUDENT

Volume XIII] JUNE 1945 [No. 2

AN IDENTITY AND SOME DEDUCTIONS

BY

AMRIT SAGAR PURI, *Lahore*

1. Let t_n stand for the n^{th} term of the Fibonacci's series 1, 1, 2, 3, 5, 8, 13, ... and u_n for that of any other series $u_1, u_2, u_3, u_4, \ldots, u_n, \ldots$ We define two more series by their general terms v_n, w_n in the following way:

$$v_n = u_1 t_{n-2} + u_2 t_{n-1}, \qquad w_n = u_{n+2} - u_{n+1} - u_n.$$

It is easily seen that v_n represents the n^{th} term of the recurring series whose first two terms are u_1 and u_2 and the law of whose formation is that any term (from the third onwards) is equal to the sum of the preceding two terms

2. We prove

THEOREM 1. For $n \geqslant 3$,

$$u_n = v_n + \sum_{r=1}^{n-2} t_r \, w_{n-r-1}.$$

We shall prove the theorom by induction.

It is plain that

$$u_{n+2} - v_{n+2} = (u_n - v_n) + (u_{n+1} - v_{n+1}) + w_n$$

Now suppose the theorem to be correct for u_n and u_{n+1}; then

$$\begin{aligned} u_{n+2} - v_{n+2} &= t_1 w_{n-2} + t_2 w_{n-3} + \cdots\cdots + t_{n-3} w_1 \\ &\quad + t_1 w_{n-1} + t_2 w_{n-2} + t_3 w_{n+3} + \cdots\cdots + t_{n-1} w_1 + w_n \\ &= t_1 w_n + t_2 w_{n-1} + t_3 w_{n-2} + \cdots\cdots + t_n w_1. \end{aligned}$$

Thus the theorem is true for u_{n+2} also. But the theorem is seen to be true for u_3 and u_4 and so it holds generally.

3. Let z be any number, real or complex we consider the series

$$0, 0, 1, z, z^2, z^3, \cdots\cdots, z^{n-1},$$

n being a positive integer. By theorem 1 we have

$$\begin{aligned} z^{n-1} &= t_n + (z-1)\, t_{n-1} + (z^2 - z - 1) \sum_{r=1}^{n-2} t_r z^{n-r-2} \\ &= z\, t_{n-1} + t_{n-2} + (z^2 - z - 1) \sum_{r=1}^{n-2} t_r z^{n-r-2} \end{aligned}$$

Thus the expression $z^{n-1} - z\ t_{n-1} - t_{n-2}$ is divisible by $z^2 - z - 1$.

Hence the equation $x^n - xt_n - t_{n-1} = 0$ $(n > 2)$ is always reducible in the domain of rational numbers and is satisfied by

$$x = \frac{1}{2}(1 + \sqrt{5}),\ x = \frac{1}{2}(1 - \sqrt{5})$$

For instance the equation $x^{11} - 89x - 55 = 0$ has the roots

$$\frac{1 + \sqrt{5}}{2},\ \frac{1 - \sqrt{5}}{2}.$$

5. Now taking z to be an integer a, we have from section, 3 the congruence formula

$$a^{n-1} \equiv v_n \pmod{a^2 - a - 1},$$

where v_n stands for the n^{th} term of the recurring series 1, a, $1 + a$, $1 + 2a$, $2 + 3a$, ... , the law of formation being that any term (after the second) is equal to the sum of the preceding two. Now, since a and $(a^3 - a - 1)$ are coprimes, it follows that the least positive residues (mod. $a^2 - a - 1$) of the series 1, a, $1 + a$, $1 + 2a$, $2 + 3a$, ... will fall in periods, which sets in from the very first residue, and the number of residues, which is also the number of incongruent residues, of powers of a, in each period is a factor of $\phi\ (a^2 - a - 1)$, where ϕ is Euler's totient function.

Thus taking $a = 5$, $a^2 - a - 1 = 19$, we have

$5^{n-1} \equiv v_n$ (mod 19), where v_n as before, is the n^{th} term of the series 1, 5, 6, 11, 17, 28 . . . Residues (mod. 19) of the terms of the series are 1, 5, 6, 11, 17, 9, 7, 16, 4, 1, 5, 6, 11, 17, 9, 7, 16, 4. Thus the residues fall into periods and the length of each period is 9 which is a factor of ϕ (19) = 18.

Again if we put $a = 7$, we shall have $7^{n-1} \equiv v_n$ (mod. 41), where v_n is the general term of the series

1, 7, 8, 15, 23, 38, ...

The residues in this case fall into periods of 40 residues in each period and occur as below:

1, 7, 8, 15, 23, 38, 20, 17, 37, 13, 9, 22, 31, 12, 2, 14, 16, 30, 5, 35, 40, 34, 33, 26, 18, 3, 21, 24, 4, 28, 32, 19, 10, 29, 39, 25, 27 11, 36, 6.

Here 7 is a primitive root of 17.

FEUERBACH'S THEOREM

BY

T. NARAYANAMURTHY, *Mangalore.*

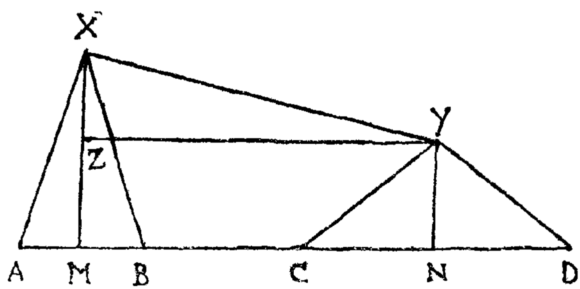
1. The object of this note is to give a proof of Feuerbach's Theorem, by using the following.

LEMMA: If two circles of radii r_1 and r_2 cut off on any line chords AB, CD († in the order ABCD as in the figure) respectively subtending angles 2α, 2β at the centres, then the two circles touch externally if

$$AD \cdot BC = 4r_1 r_2 \cos^2 \frac{\alpha+\beta}{2} \text{ or † } 4 r_1 r_2 \sin^2 \frac{\alpha-\beta}{2}$$

according as the centres of the two circles lie on the same side or on opposite sides of the straight line; and conversely.

Proof: Let X, Y be the centres and M, N the middle points of AB, CD. Draw YZ parallel to AB to meet XM at Z.



The necessary and sufficient codition for external contact is

$$(r_1+r_2)^2 = XY^2 = XZ^2 + ZY^2 = (XM \mp YN)^2 + (MB + BC + CN)^2$$

$$= (r_1 \cos\alpha \mp r_2 \cos\beta)^2 + (r_1 \sin\alpha + r_2 \sin\beta)^2 + BC^2 + 2BC\,(MB+CN)$$

$$= r_1^2 + r_2^2 \mp 2r_1 r_2 \cos(\alpha \pm \beta) + BC\,(AM + MB + BC + CN + ND)$$

the upper or lower signs being taken according as X, Y are on the same side or on opposite sides of the line.

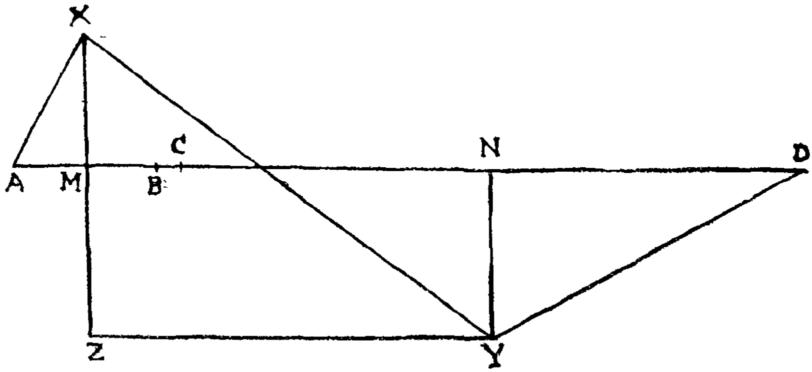
Hence, $AD \cdot BC = 4\, r_1\, r_2 \cos^2 \frac{\alpha+\beta}{2}$ or † $4\, r_1 r_2 \sin^2 \frac{\alpha-\beta}{2}$.

† The statements immediately following the mark † are due to the referee.

Similarly, the condition for internal contact can be shown to be

$$AD \cdot BC = 4r_1\, r_2 \sin^2 \frac{\alpha+\beta}{2} \text{ or } 4\, r_1\, r_2 \cos^2 \frac{\alpha-\beta}{2},$$

provided both D and C lie between A and B.



Cor: If two circles of radii r_1, r_2 touch and a chord AB of one, subtending an angle 2α at the centre is tangent to the other at C, then (i) $AC \cdot CB = 4r_1\, r_2 \sin^2 \frac{\alpha}{2}$, † when the contact is internal with centres on the same side or external with centres on opposite sides of the tangent chord; and † (ii) $AC \cdot CB = 4r_1\, r_2 \cos^2 \frac{\alpha}{2}$, when the contact is external with centres on the same side or internal with centres on opposite sides of the tangent chord.

The converse of this is also true, provided (C, D) do not separate (A, B).

2. We can now prove Feuerbach's Theorem that the nine-points circle touches the incircle and the excircles.

Let D, E, F be the mid-points, P, Q, R the feet of the altitudes to the sides BC, CA, AB; X, X_1 the points of contact with BC of the incircle and excircle opposite to A.

Now ND is parallel to SA, and

$$\tfrac{1}{2}\, D\hat{N}P = S\hat{A}P = C - B, \ (C > B).$$

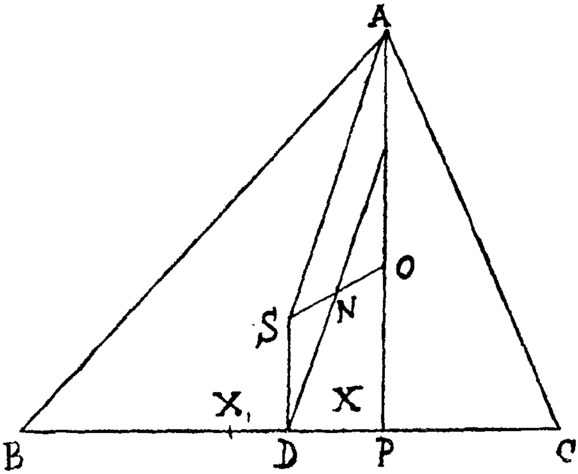
Taking BC as the tangent chord the first condition for tangency requires

$$DX \cdot XP = 2Rr \sin^2 \frac{C-B}{2}$$

$$DX_1 \cdot X_1P = 2Rr_1 \sin^2 \frac{C-B}{2}$$

Now, $X_1D = DX = \frac{1}{2}(BX - XC) = R(\sin C - \sin B)$, if $c > b$

$$DP = SA \sin \hat{SAP} = R \sin (C - B),$$



so that
$$XP = 2R \sin \frac{C-B}{2} \cdot \left\{ \cos \frac{C-B}{2} - \cos \frac{C+B}{2} \right\}$$
$$= 4R \sin \frac{C-B}{2} \sin \frac{B}{2} \sin \frac{C}{2};$$
$$X_1P = 2R \sin \frac{C-B}{2} \cdot \left\{ \cos \frac{C-B}{2} + \cos \frac{C+B}{2} \right\}$$
$$= 4R \sin \frac{C-B}{2} \cos \frac{B}{2} \cos \frac{C}{2},$$

and the conditions for tangency are readily verified for both the incircle and the excircle opposite to A.

Similarly the second condition for tangency will show that nine-points circle touches the two excircles touching BC produced.

Govt. College, Mangalore.

T. NARAYANAMURTHY.

Remarks by A. A. Krishnaswami Ayyangar

Mr. Narayanamurthy's lemma may be considered in some respects more important than its application to Feuerbach's theorem. It deserves some consideration on its own account. The immediate predecessor of the lemma is the well-known distance formula

$$a^2 = b^2 + c^2 + d^2 + 2bc \cos (bc) + 2bd \cos (bd) + 2cd \cos (cd) \quad \ldots \quad (1)$$

where a, b, c, d are the sides of a closed quadrilateral, plane or otherwise and (bc), (bd), (cd) denote respectively the angles between the sides indicated by the letters within brackets. This result is obviously capable of extension to any closed polygon.

Applying (1) to the quadrilateral XYDA of Fig. (1), where X, Y are the centres of circles which may be considered as cutting each other at angle θ, we get

$$r_1^2 + r_2^2 - 2r_1 r_2 \cos \theta = \mathrm{XY}^2$$

$$= r_1^2 + r_2^2 + \mathrm{AD}^2 - 2r_1 r_2 \cos (\alpha + \beta) - 2\mathrm{AD}\,(r_1 \sin \alpha + r_2 \sin \beta)$$

so that $\mathrm{AD \cdot BC} = 2r_1 r_2 \{\cos (\alpha + \beta) + \cos \theta\}$. ... (2)

Similarly $\mathrm{AC \cdot BD} = 2r_1 r_2 \{\cos (\alpha - \beta) - \cos \theta\}$. ... (3)

From (2) and (3), $(\mathrm{AB, CD}) = \dfrac{\cos (\alpha - \beta) - \cos \theta}{\cos (\alpha + \beta) - \cos \theta}$ ... (4)

When $\theta = 0$ or π, the circles touch and

$$\left.\begin{aligned} \mathrm{AD \cdot BC} &= -4 r_1 r_2 \sin^2 \frac{\alpha+\beta}{2} \text{ or } 4 r_1 r_2 \cos^2 \frac{\alpha+\beta}{2} \\ \mathrm{AC \cdot BD} &= -4 r_1 r_2 \sin^2 \frac{\alpha-\beta}{2} \text{ or } 4 r_1 r_2 \cos^2 \frac{\alpha-\beta}{2} \\ (\mathrm{AB, CD}) &= \sin^2 \frac{\alpha-\beta}{2} \Big/ \sin^2 \frac{\alpha+\beta}{2} \text{ or } \cos^2 \frac{\alpha-\beta}{2} \Big/ \cos^2 \frac{\alpha+\beta}{2} \end{aligned}\right\} \quad (5)$$

and conversely, when any one of the criteria (5) is satisfied, the circles touch. The cross-ratio condition fails when α or β is zero. It is important to note that α, β may be interchanged without altering the above results. Hence a number of theorems can be inferred.

The above criteria apply also for spheres, since the quadrilateral XYDA need not be a plane one.

An infinity of proofs for Feuerbach's theorem or any other contact theorem becomes thus available, based on lines which cut two circles at known angles

RESIDUAL TYPES OF PARTITIONS OF "0" INTO FOUR CUBES

BY

A. K. SRINIVASAN, *Student, Maharaja's College, Mysore*

It is fairly well-known that when a, b, c, d, are four integers positive or negative such that

$$a^3+b^3+c^3+d^3=0,$$

then $a+b+c+d$ is always a multiple of 6. This is obvious if we recollect that $x^3 \equiv x$ (mod. 6), when $x=0, 1, 2, 3, 4, 5$ and $\therefore$ $a^3+b^3+c^3+d^3 \equiv a+b+c+d$ (mod. 6).

We will now enquire the types of residues of a, b, c, d whose sum is zero. We exclude naturally those cases where a, b, c, d have a common factor. We regard the residual types of a, b, c, d and of $-a, -b, -c, -d$ as equivalent. Under the above convention, the following different cases have to be considered:

(i) All residues different:—The only possible sets are (0, 1, 2, 3) and (1, 2, 4, 5), since (0, 3, 4, 5) is by our convention equivalent to (0, 3, 2, 1).

(ii) All residues alike:—This case is impossible.

(iii) Three alike, one different:—There is only one case of this kind, (1, 1, 1, 3).

(iv) Two pairs of like residues:—There are only two cases, (1, 2, 1, 2) and (1, 5, 1, 5).

(v) Two alike and two different:—There are five cases of this kind, (0, 0, 1, 5); (0, 1, 1, 4); (1, 3, 3, 5); (2, 2, 3, 5); (2, 3, 3, 4).

Using index symbolism for repeated parts, we write below the only possible types of partitions, just *ten* in number, enumerated in dictionary order:

TYPES	ILLUSTRATIONS
I $(0^2\ 1\ 5)$	$12^3+19^3+53^3+(-54)^3=0$
II $(0\ 1\ 2\ 3)$	$46^3-37^3-3^3=6^6$*
This type happens to be of most frequent occurrence.	
III $(0\ 1^2\ 4)$	No example available
IV $(1^3\ 3)$	No example available
V $(1^2\ 2^2)$	$4^3+17^3+22^3+(-25)^3=0$
VI $(1^2\ 5^2)$	$13^3+65^3+121^3+(-127)^3=0$

* These examples are due to Ramanujan, (Vide *Collected Papers*, p 331).

TYPES		ILLUSTRATIONS
VII	(1 2 4 5)	$791^3+812^3=1010^3-1$*
VIII	(1 3^2 5)	$11^3+15^3+27^3+(-29)^3=0$
IX	(2^2 3 5)	No example available
X	(2 3^2 4)	$33^3+70^3+92^3+(-105)^3=0$

From the above table we may infer the following result:

THEOREM:—If the sum of four cubes vanishes, then at least one of them is of the form $(6m \pm 1)$ or $(6m+3)$, where m is an integer.

N. B.—The absence of illustrations for the Types III, IV and IX even after a careful search† among the existing examples makes one feel that they are probably non-existent. Will any interested reader probe more deeply into these cases?

In conclusion I thank Prof. A. A. Krishnaswami Ayyangar for his guidance in this investigation.

† I have tested the 'Tables of Partition' of Russel and Gwyther in vain for the missing Types. (*Math. Gaz.* Vol. XXI, pp. 34, 35)

REVERSIBLE PRIME—PAIRS

BY

GOPAL LAL MATHUR, *Jodhpur*.

A reversible prime pair' is a set of two primes such that one is obtained by reversing the digits of the other, e. g., (1583,3851).

'A similar reversible pair' is of the type (787,787) in which both members are identical otherwise it is a 'dissimilar pair'.

There are 16 similar pairs and 49 dissimilar pairs below 5000 (both members < 5000), making a total of 65. They are:—

(11, 11)*;	(13, 31);	(17, 71);	(37, 73);	(79, 97);
(101, 101)*;	(107, 701);	(113, 311);	(131, 131)*;	(149, 941);
(151, 151)*;	(157, 751);	(167, 761);	(179, 971);	(181, 181)*;
(191, 191)*;	(199, 991);	(313, 313)*;	(337, 733);	(347, 743);
(353, 353);	(359, 953);	(373, 373)*;	(383, 383)*;	(389, 983);
(709, 907);	(727, 727)*;	(739, 937);	(757, 757)*;	(769, 967);
(787, 787)*;	(797, 797)*;	(919, 919)*;	(929, 929)*;	(1021, 1201);
(1031, 1301);	(1033, 3301);	(1061, 1601);	(1091, 1901);	(1103, 3011);
(1151, 1511);	(1153, 3511);	(1181, 1811);	(1193, 3911);	(1213, 3121);
(1223, 3221);	(1231, 1321);	(1283, 3821);	(1381, 1831);	(1453, 3541);
(1471, 1741);	(1523, 3251);	(1583, 3851);	(1723, 3271);	(1733, 3371);
(1753, 3571);	(1913, 3191);	(1933, 3391);	(3023, 3203);	(3083, 3803);
(3163, 3613);	(3343, 3433);	(3373, 3733);	(3463, 3643);	(3583, 3853);

We observe that there are two sets of 2 consecutive similar pairs and one set of 4 consecutive ones.

The question is—Is these an infinity of such pairs?

A FURTHER NOTE ON INTUITIONISTIC SET-THEORY

BY

K. CHANDRASEKHARAN, *Madras*

The object of this note is to clarify Brouwer's idea of set presented in an earlier paper of mine. (*Math. Student* Vol. 9, p. 143). This clarification was pointed out to me as necessary by Prof. Alonzo Church of Princeton, and is essentially a result of studying the books of* Heyting and Black as well as the correspondence I have had with Prof. Church.

1. The infinite sequence ζ of integers 1, 2, 3, 4, . . . is taken as fundamental. Any sequence constructed by successive arbitrary choices of an integer from ζ is called a *choice sequence.* (Wahlfolge).

2. A *set* is a law which correlates (or makes correspond) groups of signs to *some* of all the possible choice-sequences obtained from ζ, in the following manner. Given a particular choice-sequence, the law may correlate to the first integer of that sequence a group of signs (called, *the first stage*); or alternatively, it may specify that there is no such group. If there *is* a first stage, the set-law may specify it as the final stage; if it is not the final stage, we consider the second integer of the choice-sequence. Here again there are three cases. The law may correlate to the second integer either (a) nothing, or (b) a second and final stage, or (c) a second stage which is not final. In carrying on this procedure, if case (a) arises at any stage, the process of correlation is said to be *blocked;* if case (b) arises at any point, the process *terminates,* or *is ended*; if case (c) arises always, the process is *unending.* The sequence of groups of signs (Zeichenreihen) thus correlated to a given choice-sequence is called *an element of the set.* For any particular choice-sequence, if the process of correlation gets *blocked* at some stage, then, there is *no element* corresponding to that choice-sequence; if the process terminates at a particular stage, the element is *finite*; if the process is unending, the element is *infinite.*

---

* Max Black, *The Nature of Mathematics,* London and New York, 1934. A. Heyting, Mathematische Grundlagenforschung—"Intuitionismus—Beweistheorie" (*Ergebnisse der Mathematik und ihrer Grenzgebiete* vol. 3, no. 4) Berlin, 1934.

The set-law described above is subject to one further restriction; for each $n > 1$, if there is a choice-sequence P for which there is a non-final $(n-1)^{th}$ stage, then there is a choice-sequence Q which has the first $n-1$ integers the same as in P, and has some definite n^{th} stage correlated to its n^{th} integer.

The fore-going statement has been made, for the sake of easier intelligibility, from the classical or non-intuitionistic point of view. It must be understood that it is impossible for an intuitionist to say "given a particular choice-sequence", as we have done above. Rather, at any moment there is given only an initial segment of the choice-sequence, as determined by arbitrary choices made up to that time, and it is always possible to continue this initial segment in various ways by further arbitrary choices; the set-law serves to determine a correlated group of signs as each successive arbitrary choice is made, so long as the successive arbitrary choices are continued and so long as the process is not blocked or terminated.

In applying this definition of a set it is also necessary to remember that according to Brouwer, the integers 1, 2, 3, 4, . . . , in particular, are groups of signs, or "Zeichenreihen"; e.g., the integer 1944 *is* the finite sequence of digits 1, 9, 4, 4.

3. The sets used in my earlier paper are the following:

(i) A, consisting of the integers of ζ.

A is a set intuitionistically since the set-law is: "Every choice-sequence has a first stage which is final, and is, for each such sequence, the integer which comes first in that sequence.

(ii) M, consisting of all infinite binary decimals. The set-law is: "Every choice-sequence has for its first stage the decimal sign; and for the n^{th} stage, where $n > 1$, it has 0 or 1 according as the n^{th} integer of the choice-sequence is odd or even; no stage being final."

(iii) The species M″ of all positive integers n which make $x^n + y^n = z^n$ impossible to solve in positive integers cannot be proved to be intuitionistically a set in the present state of knowledge. Hence it should have been referred to in my earlier paper as a species rather than as a set.

(iv) The set M′ of all positive integers n which make $x^n + y^n = z^n$ soluble in positive integers is given by (or *is*) the following set-law: "If the integer which comes first in the choice-sequence is not of the form $2^x 3^y 5^z 7^u$ where x, y, z, u, are positive integers, or if it is of this form and $x^n + y^n \neq z^n$, then the process of correlation is blocked. If the integer which comes first in the choice-sequence is of the form $2^x\,3^y\,5^z\,7^n$ where x, y, z, n, are positive integers, and $x^n + y^n = z^n$, then the first stage is the integer n, and this is the final stage." In order to see that this is intuitionistically a set, it is not necessary to solve the Fermat Problem, or even to suppose that the Fermat Problem is solvable. But in the present state of knowledge it cannot be determined whether this set has other elements than the integers 1 and 2.

(v) The set M of all infinite binary decimals in which we know how many 0's occur before the first 1, or there is no 1 at all, is given by the following set-law. "If the integer which comes first in the choice squence is 1, then the first stage is the decimal sign, and the n^{th} stage, where $n > 1$, is 0 (regardless of what is the n^{th} integer of the choice-sequence). If the integer which comes first in the choice-sequence is 2, then the first stage is the decimal sign, the second stage is 1, and the n^{th} stage, where $n > 2$ is 0 or 1, according as the n^{th} integer of the choice-sequence is odd or even. If the integer a which comes first in the choice-sequence is greater than 2, then the first stage is the decimal sign, each stage from the second to the $(a-1)^{th}$ inclusive is 0, the a^{th} stage is 1, and the n^{th} stage, where n is greater than a, is 0 or 1 according as the n^{th} integer of the choice sequence is odd or even.

For the remaining examples of sets in my paper, the reader will now be able to construct the set-law without difficulty.

Each of these examples, it should be noted, is also an example of a species; a set being, according to Brouwer, a special case of a species of the first order.

On page 151, lines 6, 11, 12, 13, for the word 'set' or 'sets' should be substituted 'species', and for 'subset' should be substituted 'subspecies'. (*Math. Student* vol. 9).

Every man is a hero to his son till he tries to help the boy in his maths, homework.

Illustrated Weekly of India.

ON THE EQUATION $x_1^3+x_2^3=y_1^3+y_2^3$

BY

P. KESAVA MENON, *Madras Christian College.*

The general solution of the Diophantine equation

$$x_1^3+x_2^3=y_1^3+y_2^3 \qquad \ldots \quad (1)$$

is known to be*

$$x_1=-(a-3b)(a^2+3b^2)+1, \qquad x_2=(a+3b)(a^2+3b^2)-1,$$

$$y_1=-(a^2+3b^2)^2+(a+3b), \qquad y_2=(a^2+3b^2)^2-(a-3b).$$

Ramanujan gave the solution†

$$x_1=3a^2+5ab-5b^2 \qquad x_2=4a^2-4ab+6b^2$$

$$y_1=3b^2+5ab-5a^2 \qquad y_2=4b^2-4ab+6a^2$$

which, though not general, is remarkable for the fact that the values are homogeneous quadratic expressions in a, b. Another feature of this solution is that y_1 and y_2 are obtained from x_1, x_2 by the interchange of a and b. The object of this paper is to discuss a general principle of obtaining solutions of the type given by Ramanujan.

Let us consider the form

$$f(x,\ y)=(a_1x^2+\alpha_1xy-b_1y^2)^3+(a_2x^2+\alpha_2xy-b_2y^2)^3$$

where a_1, a_2, b_1, b_2 are numbers satisfying the relation

$$a_1^3+a_2^3+b_1^3+b_2^3=0 \qquad \ldots \quad (2)$$

If we could choose α_1, α_2 so as to make $f(x, y)$ symmetric in x, y, then a solution of equation (1) will be given by

$$x_1=(a_1\ x^2+\alpha_1\ xy-b_1\ y^2) \qquad x_2=(a_2\ x^2+\alpha_2\ xy-b_2\ y^2)$$

$$y_1=(a_1\ y^2+\alpha_1\ xy-b_1\ x^2) \qquad y_2=(a_2\ y^2+\alpha_2\ xy-b_2\ x^2),$$

since $x_1^3+x_2^3=f(x,\ y)=f(y,\ x)=y_1^3+y_2^3$.

---

* Hardy and Wright: *Theory of Numbers* p. 199.

† " " " p. 201,

In order that $f(x, y)$ shall be symmetric we must have, besides (2), the relations

$$\left.\begin{array}{l}(a_1^2-b_1^2)\ \alpha_1+(a_2^2-b_2^2)\ \alpha_2=0\\(\alpha_1^2-a_1\ b_1)(a_1+b_1)+(\alpha_2^2-a_2\ b_2)(a_2+b_2)=0\end{array}\right\}\quad\ldots\quad(3)$$

The two equations in (3) will reduce to one if α_1, α_2 are to take any of the following sets of values:

$$\begin{array}{llll}\text{(i)} & \alpha_1=\varepsilon a_1,\ \alpha_2=\varepsilon a_2 & \text{(iii)} & \alpha_1=\varepsilon a_1,\ \alpha_2=\varepsilon b_2\\ \text{(ii)} & \alpha_1=\varepsilon b_1,\ \alpha_2=\varepsilon b_2 & \text{(iv)} & \alpha_1=\varepsilon b_1,\ \alpha_2=\varepsilon a_2,\end{array}$$

where $\varepsilon=\pm1$. We shall consider only the first set of values since the others could be obtained from it by interchanging one or both sets of corresponding a's and b's in $f(x, y)$.

Substituting in (3) from (i) we get

$$a_1^3+a_2^3=a_1\ b_1^2+a_2\ b_2^2 \quad\ldots\quad(4)$$

By means of (2) we can reduce (4) to

$$\frac{a_1}{b_1}+\frac{a_2}{b_2}=1 \quad\ldots\quad(5)$$

Thus we get

THEOREM 1. *If a_1, a_2, b_1, b_2 are numbers satisfying* (2) *and* (5), *then a solution of equation* (1) *is given by*

$$\begin{array}{ll}x_1=a_1\ x^2+\varepsilon a_1\ xy-b_1\ y^2 & x_2=a_2\ x^2+\varepsilon a_2\ xy-b_2\ y^2\\ y_1=a_1\ y^2+\varepsilon a_1\ xy-b_1\ x^2 & y_2=a_2\ y^2+\varepsilon a_2\ xy-b_2\ x^2,\end{array}$$

$\varepsilon=\pm1$.

Taking $\varepsilon=-1$, $a_1=5$, $b_1=3$, $a_2=4$, $b_2=-6$ we get a solution equivalent to Ramanujan's.

It is possible to replace (5) by a weaker condition. From the first of equations (3) we have

$$\frac{\alpha_1}{a_2^2-b_2^2}=\frac{\alpha_2}{-(a_1^2-b_1^2)}=\frac{1}{\lambda},\ \text{say}.$$

Substituting in the second of (3) and simplifying with the help of (2) we get

$$\lambda^2=-(a_1+b_1)(a_2+b_2).$$

Since λ must necessarily be rational in order that α_1, α_2 be rational it follows that

$$-(a_1+b_1)(a_2+b_2)$$

must be a perfect square. When this condition is satisfied

$$\alpha_1=\varepsilon\ (a_2-b_2)\ \sqrt{-\left(\frac{a_2+b_2}{a_1+b_1}\right)},\qquad \alpha_2=\varepsilon\ (a_1-b_1)\ \sqrt{-\left(\frac{a_1+b_1}{a_2+b_2}\right)}.$$

We could, of course, make α_1, α_2 integral, if they are not already so, by replacing a_1, a_2, b_1, b_2 by ka_1, ka_2, kb_1, kb_2, (k being a suitable integral multiplier.)

Thus we get

THEOREM 2. *If a_1, a_2, b_1, b_2 be numbers satisfying the relations*

$$a_1^3+a_2^3+b_1^3+b_2^3=0 \qquad \dots \quad (2)$$

and $\quad -(a_1+b_1)(a_2+b_2)$ *is a perfect square,*

a solution of the equation $x_1^3+x_2^3=y_1^3+y_2^3$ *is given by*

$$x_1=a_1\,x^2+\alpha_1\,xy-b_1\,y^2 \qquad x_2=a_2\,x^2+\alpha_2\,xy-b_2\,y^2$$

$$y_1=a_1\,y^2+\alpha_1\,xy-b_1\,x^2 \qquad y_2=a_2\,y^2+\alpha_2\,xy-b_2\,x^2$$

where $$\alpha_1=\varepsilon\,(a_2-b_2)\sqrt{\{-(a_2+b_2)/(a_1+b_1)\}},$$

$$\alpha_2=\varepsilon\,(a_1-b_1)\sqrt{\{-(a_1+b_1)/(a_2+b_2)\}}$$

and $\quad \varepsilon=\pm 1.$

For instance, let $a_1=1$, $a_2=12$, $b_1=-9$, $b_2=-10$. Then $\alpha_1=11$, $\alpha_2=20$ for $\varepsilon=1$, so that we get the following solution of (1):

$$x_1=x^2+11xy+9y^2 \qquad x_2=12x^2+20xy+10y^2$$

$$y_1=y^2+11xy+9x^2 \qquad y_2=12y^2+20xy+10x^2$$

Writing,

$$\left.\begin{array}{ll} A_1=a_1\,x^2+\alpha_1\,xy-b_1\,y^2 & B_1=b_1\,x^2+\alpha_1\,xy+a_1\,y^2 \\ A_2=a_2\,x^2+\alpha_2\,xy-b_2\,y^2 & B_2=b_2\,x^2+\alpha_2\,xy+a_2\,y^2 \end{array}\right\} \qquad \dots \quad (6)$$

we get $\quad A_1+B_1=(x^2-y^2)(a_1+b_1), \qquad A_2+B_2=(x^2-y^2)(a_2+b_2).$

It follows that if $-(a_1+b_1)(a_2+b_2)$ is a perfect square so is $-(A_1+B_1)(A_2+B_2)$ and by Theorem 2

$$A_1^3+A_2^3+B_1^3+B_2^3=0.$$

Thus A_1, A_2, B_1, B_2 satisfy the conditions of theorem 2 and we get

THEOREM 3. *If a_1, a_2, b_1, b_2 are as in Theorem 2, then*

$$x_1=A_1\,\xi^2+\beta_1\,\xi\eta-B_1\,\eta^2 \qquad x_2=A_2\,\xi^2+\beta_2\,\xi\eta-B_2\,\eta^2$$

$$y_1=A_1\,\eta^2+\beta_1\,\xi\eta-B_1\,\xi^2 \qquad y_2=A_2\,\eta^2+\beta_2\,\xi\eta-B_2\,\xi^2$$

is also a solution of equation (2), A_1, A_2, B_1, B_2 *being given by* (6) *and*

$$\beta_1=\varepsilon'\,(A_2-B_2)\sqrt{-(a_2+b_2)/(a_1+b_1)}$$

$$\beta_2=\varepsilon'\,(A_1-B_1)\sqrt{-(a_1+b_1)/(a_2+b_2)}$$

$$\varepsilon'=\pm 1.$$

GENERALISATION OF A CERTAIN DEFINITE INTEGRAL

BY

U. S. NAIR, *University College, Trivandrum,*

AND

G. S. MAHAJANI, *Fergusson College, Poona.*

1. The object of this note is to evaluate the definite integral

$$I=\int_0^\infty \frac{\sin a_1x}{x}\cdot\frac{\sin a_2x}{x}\cdots\frac{\sin a_nx}{x}\cdot\cos b_1x\cdot\cos b_2x\cdots\cos b_\mu x\,dx.$$

The case when $|a_1|>|a_2|+\cdots+|a_n|+|b_1|+\cdots+|b_\mu|$ is found as example 6, p. 122 of *Modern Analysis*, by Whittaker and Watson, where the value of I is given to be $\frac{\pi}{2}a_2a_3\cdots a_n$ under that condition.

2. Let $s=a_1+a_2+\cdots+a_n$

$s_1=s$ with one term negative,

$s_2=s$ with two terms negative,

and so on. The number of terms of the type s_r is ${}^nC_r$. Now it is easily proved by induction that

$$\cos a_1x\ \cos a_2x\cdots\cos a_nx$$
$$=\frac{1}{2^{n-1}}\left\{\cos sx+\sum\cos s_1x+\sum\cos s_2x+\cdots\right\}\qquad\ldots\quad(1)$$

where the last term is $\frac{1}{2}\sum\cos s_m$ or $\sum\cos s_m$ according as $n=2m$ or $2m+1$. By differentiating with respect to $a_1, a_2,\cdots a_n$ successively, we get

$$\sin a_1x\cdot\sin a_2x\cdots\sin a_nx$$
$$=\frac{(-1)^n}{2^{n-1}}\left\{\cos sx-\sum\cos s_1x+\sum\cos s_2x-\cdots(-1)^m\frac{1}{2}\sum\cos s_nx\right\}\ldots\quad(2)$$

and

$$\sin a_1x\cdot\sin a_2x\cdots\sin a_{2n+1}x$$
$$=\frac{(-1)^n}{2^{n-1}}\left\{\sin sx-\sum\sin s_1x+\sum\sin s_2x-\cdots(-)^n\sum\sin s_nx\right\}\quad\ldots\quad(3)$$

3. By using (2), (3), and the known result that

$$\int_0^\infty\frac{\sin ax}{x}\,dx=\frac{\pi}{2}\,\mathrm{sgn}\,(a)$$

where sgn $(a)=1$, 0 or -1 according as $a>0=0$ or <0, we get, on integrating by parts $(n-1)$ times

$$\int_0^\infty \frac{\sin a_1x}{x}\frac{\sin a_2x}{x}\cdots\frac{\sin a_nx}{x}dx$$

$$=\frac{\pi}{2^n(n-1)!}\left\{s^{n-1}\operatorname{sgn}(s)-\sum s_1^{n-1}\operatorname{sgn}(s_1)+\sum s_2^{n-1}\operatorname{sgn}(s_2)\cdots\right\}\ldots \quad (4)$$

the last term in the bracket being

$$(-1)^m\cdot\tfrac{1}{2}\sum s_m^{n-1}\cdot\operatorname{sgn}(s_m) \text{ or } (-1)^m\sum s_m^{n-1}\cdot\operatorname{sgn}(s_m)$$

according as $n=2m$ or $2m+1$.

4. Now the general value of I could be obtained by using (4) from the formula

$$I=\frac{\partial^m}{\partial b_1\partial b_2\cdots\partial b_\mu}\int\frac{\sin a_1x}{x}\cdot\frac{\sin a_2x}{x}\cdots\frac{\sin a_nx}{x}\cdot\frac{\sin b_1x}{x}\cdots\frac{\sin b_\mu x}{x}dx$$

If $a_1=a_2=\cdots=a_n=1$ in (4) we get, as a special case,

$$\int_0^\infty\left(\frac{\sin x}{x}\right)^n dx=\frac{\pi}{2^n(n-1)!}\left\{n^{n-1}-n\cdot(n-2)^{n-1}+\frac{n(n-1)}{2}(n-4)^{n-1}+\cdots\right\}$$

It does not appear easy to prove directly from the formula for I that its value is equal to $\frac{\pi}{2}a_2a_3\cdots a_n$ when $|a_1|>|a_2|+\cdots+|a_n|+|b_1|+\cdots+|b_\mu|$. But a proof by induction can be given. The result is easily verified when $\mu=n=1$. Suppose, now, that the result has been proved for a given n and for *all* values of μ from 1 up to μ. Then the result for $m+1$ and the same n is obtained by writing $2\sin a_1x\cos b_{m+1}x=\sin(a_1+b_{m+1})x+\sin(a_1-b_{m+1})x$ which gives the value of the required integral to be $\frac{1}{2}\times 2\cdot\frac{\pi}{2}(a_2\cdots a_n)=\frac{\pi}{2}a_2\cdots a_n$. So the result is true for *that* n and for *all* μ. Next suppose that the result has been proved from 1 up to n and for *any* μ. Then integrating

$$\int_0^\infty\frac{\sin a_1x}{x}\cdot\frac{\sin a_2x}{x}\cdots\frac{\sin a_nx}{x}\cdot\cos b_1x\cdots\cos b_nx\cdot\cos tx\,dx$$

with respect to t in $(0, a_{n+1})$ we see that the result is true for $n+1$ and *any* m. Now the proof is completed by induction if we note that if

$$|a_1|>|a_2|+\cdots+|a_n|+|b_1|+\cdots+|b_\mu|$$

then

$$|a_1|>|a_2|+\quad+|a_r|+|b_1|+\cdots+|b_s|$$

for $r\leqslant n$ and $s\leqslant\mu$.

ON NUMBERS WHICH ARE THE SUM OR DIFFERENCE OF TWO CUBES

BY

B. SURYANARAYANA RAO, *Yelahanka*

1. Let $$N = x^3 + y^3 \qquad \dots \quad (1)$$

where x, y are integers. We write

$$x + y = m, \qquad x^2 - xy + y^2 = n \qquad \dots \quad (2)$$

so that $N = m \times n$. From (2) we have

$$xy = \frac{m^2 - n}{3} \qquad \dots \quad (3)$$

$$x - y = \sqrt{(4n - m^2)/3} = k \text{ say.} \qquad \dots \quad (4)$$

There is no loss of generality in taking the positive value of the square root (i.e. $k > 0$), since x may be supposed algebraically greater than y.

$$\therefore \qquad x = (m + k)/2, \qquad y = (m - k)/2 \qquad \dots \quad (5)$$

From (4) we have $4n > m^2$, and m and n must be both multiples of 3 or both prime to 3. Also since $n > m^2/4$, we have $n > m$ provided $n > 4$. If $m > k$, then x and y are both positive and N is the *sum* of two cubes. In this case $m^2 > n$ from (3) so that m^2 lies between n and $4n$. If $m = k$, N is a perfect cube while if $m < k$, y is negative, and N is the difference of two cubes.

2. Since x and y are integers, k must be an integer, and from (5) m and k must be both odd or both even.

Taking $m = 2p - 1$, $k = 2q - 1$ where p and q are integers, we get
$$n = (m^2 + 3k^2)/4 = (1 + p^2 - p + 3q^2 - 3q).$$

We have thus the identity
$$(2p - 1)(1 + p^2 - p + 3q^2 - 3q) = (p + q - 1)^3 + (p - q)^3 \qquad \dots \quad (6)$$
giving the most general solution of (1) when N has an odd factor $(2p - 1)$.

Accordingly if we construct the matrix whose (p, q)th element is $1 + p(p - 1) + 3q(q - 1)$, then the product of every element in the pth row by the multiplier $2p - 1$ is the sum of two cubes $(p > q)$ or the difference of two cubes according as the element occurs to the right or left of the leading diagonal $(p = q)$.

Multiplier					
1	1	7	19	37	...
3	3	9	21	39	...
5	7	13	25	43	...
7	13	19	31	49	...
...	...	...	...	...	

The diagonal element in the pth row is $(2p-1)^2$ and the element r steps to its right exceeds it by $3(r^2-r+2pr)$.

3. Now taking the case when both $m=2p$ and $k=2q$ are both even, we have the identity

$$2p\,(p^2+3q^2)=(p+q)^3+(p-q)^3$$

giving the most general solution of (1) with an even factor $2p$ for N. We have again a matrix whose $(p,\,q)$th element is p^2+3q^2 with the property that any element in the pth row, multipled by the corresponding multiplier $2p$ is the sum or difference of two cubes.

Multiplier			Matrix			
2	1	4	13	28	...	...
4	4	7	16	31	...	...
6	9	12	21	36	...	...
...	...	...	...	...	...	...

Combining the two matrices into one, we have the scheme of multipliers and matrix

1	1	7	19	37	61	91	
2	1	4	13	28	49	76	
3		3	9	21	39	63	
4		4	7	16	31	52	
5		7	7	13	25	43	
6		9	12	21	36	57	
7		13	13	19	31	49	73

It will be seen that the leading diagonal consists of square terms, and the product of any number in a row by the multiplier of that row is the sum or difference of 2 cubes according as it occurs to the right or left of the leading diagonal. All numbers which are the sum or difference of two cubes find themselves in the list.

Remarks by the Editor

It will be seen that the principle underlying the above consolidated table is the formula

$$(m+r)^3+(-r)^3=m(m^2+3mr+3r^2).$$

The multiplier in thick type on the left is m, while the diagonal element of this row (mth row) is m^2 and the element r steps to its right is $m^2+3mr+3r^2$.

ON THE SERIES Σs^r

BY

DALJIT SINGH, *Delhi.*

The summation of the series $\sum_{s=1}^{n} s^r$ is made to depend upon a class of numbers which can be built up successively by a simple law. In this respect it is felt that this method is simpler than that involving the use of Bernoulli's Numbers.

1. The sum of the first n natural numbers is known to be equal to $\frac{1}{2} n(n+1)$. This may be put in the form

$$\sum_{s=1}^{n} s = {}_{n+1}C_2 \quad \ldots \quad (1{\cdot}1)$$

It is easy to show that

$$s^2 = 1.\ {}_sC_1 + 2.\ {}_sC_2 \quad \ldots \quad (1{\cdot}2)$$

By a well known theorem

$$\sum_{s=1}^{n} {}_sC_r = {}_{(n+1)}C_{(r+1)};\ s \geqslant r > 0 \quad \ldots \quad (1{\cdot}3)$$

Hence
$$\sum_{s=1}^{n} s^2 = 1.\ {}_{(n+1)}C_2 + 2.\ {}_{(n+1)}C_3 \quad \ldots \quad (1{\cdot}4)$$

Similarly
$$\sum_{s=1}^{n} s^3 = 1.\ {}_{(n+1)}C_2 + 6.\ {}_{(n+1)}C_3 + 6.\ {}_{(n+1)}C_4 \quad \ldots \quad (1{\cdot}5)$$

In general
$$\sum_{s=1}^{n} s^r = f_{r,1}\ {}_{(n+1)}C_2 + f_{r,2}\ {}_{(n+1)}C_3 + \cdots + f_{r,r-1}\ {}_{(n+1)}C_r \ldots \quad (1{\cdot}6)$$

2. The following table gives a few values of $f_{m,n}$

$m \downarrow$ $n \rightarrow$	1	2	3	4	5	6
1	1					
2	1	2				
3	1	6	6			
4	1	14	36	24		
5	1	30	150	240	120	
6	1	62	540	1560	1800	720

It may be proved by induction that the law connecting the numbers $f_{m,n}$ is given by the equation

$$f_{m,n} = n\{f_{m-1,n-1} + f_{m-1,n}\} \qquad \dots \quad (2\cdot1)$$

3. If we divide each column by its uppermost number we get another table of $\left\{ \begin{matrix} m-n \\ n \end{matrix} \right\} = f_{m,n}/n\,!$

The numbers in this table are formed in accordance with the law

$$\left\{ \begin{matrix} m \\ n \end{matrix} \right\} = \left\{ \begin{matrix} m \\ n-1 \end{matrix} \right\} + n \left\{ \begin{matrix} m-1 \\ n \end{matrix} \right\}; \quad \left\{ \begin{matrix} 0 \\ n \end{matrix} \right\} = 1 \qquad \dots \quad (3\cdot1)$$

It can easily be shown that

$$\left\{ \begin{matrix} m \\ n \end{matrix} \right\} = \frac{1}{n\,!} (n^{m+n} - {}_nC_1\,(n-1)^{m+n} + {}_nC_2\,(n-2)^{m+n} \cdots + (-1)^{n-1}\,{}_nC_{n-1}) \dots \quad (3\cdot2)$$

4. A property of the $f_{m,n}$ table is that the difference between the sums of alternate digits in any row is unity,

$$i.e. \quad f_{r\,r} - f_{r\,r-1} + f_{r\,r-2} \cdots + (-1)^{r-1} f_{r,1} = 1 \qquad \dots \quad (4\cdot1)$$

The above property may be used in summation of a binomial series. By using (3·2) and (3·3) in (4·1) it may be shown that

$$r^r - {}_{r+1}C_1 \cdot (r-1)^r + {}_{r+1}C_2 \cdot (r-2)^r \cdots + (-1)^r\; {}_{r+1}C_{r-1} = 1. \qquad \dots \quad (4\cdot2)$$

5. **Bernoulli's Numbers and $f_{m,n}$.**

The well-known formula for the summation of the rth powers of the first n natural numbers in terms of Bernoulli's Numbers is

$$\sum_1^n s^r = \frac{n^{r+1}}{r+1} + \frac{1}{2} n^r + B_1 \frac{r}{2\,!} n^{r-1} - B_3 \frac{r(r-1)(r-2)}{4\,!} n^{r-3} + \cdots \qquad \dots \quad (5\cdot1)$$

Between Bernoulli's Numbers and $f_{m,n}$ holds the relations

$$(-1)^{m+1} B_{2m+1} = \frac{f_{2m+2,\,2m+2}}{(2m+2)(2m+3)} - \frac{f_{2m+2,\,2m+1}}{(2m+1)(2m+2)} + \frac{f_{2m+2,\,2m}}{2m\,(2m+1)} \cdots \quad (5\cdot2)$$

It is also easy to prove from these that

$$\frac{f_{2m+1,\,2m+1}}{(2m+1)(2m+2)} - \frac{f_{2m+1,\,2m}}{2m\,(2m+1)} + \frac{f_{2m+1,\,2m-1}}{(2m-1)\,2m} \cdots = 0 \qquad \dots \quad (5\cdot3)$$

A MULTIPLICATORY FORMULA FOR THE GENERAL RECURRING SEQUENCE OF ORDER 2

BY

D. JUZUK AND TH. MOTZKIN, *Jerusalem*

We consider the general recurring sequence of order 2, defined by

(1) $$w_l = a\, w_{l-2} + b\, w_{l-1}, \qquad (l = 0, \pm 1, \pm 2, \ldots)$$

with arbitrary complex $a \neq 0$, b, w_0, w_1, and also the special case u_l with the same a, b and

(2) $$u_0 = 0, \;\; u_1 = 1.$$

Our purpose is to establish the formula

(3) $$w_{kl} = \sum_{i=0}^{k} U_{k,l,i}\; w_i, \text{ where } U_{k,l,i} = \binom{k}{i} (a\, u_{l-1})^{k-i} u_l^i, \;\; (k = 0, 1, 2, \ldots)$$

which, for a constant $k > 0$, expresses the values of $w_l^1 = w_{kl}$ in terms of $w_0, \ldots, w_k$.

Putting $w_k^{(m)} = w_{k+m}$, $m = 0, \pm 1, \pm 2, \ldots$, which sequence belongs to the same a, b, we have the equivalent formula

(3′) $$w_{kl+m} = \sum_{i=0}^{k} U_{k,l,i}\; w_{i+m}, \qquad (m = 0, \pm 1, \pm 2, \ldots)$$

and, in particular,

(3″) $$u_{kl} = \sum_{i=0}^{k} U_{k,l,i}\; u_i, \qquad u_{kl+1} = \sum_{i=0}^{k} U_{k,l,i}\; u_{i+1}.$$

The formulae (3″) were given by H. Siebeck (*Jour. für Math.* 33, 1846, 71–76), who, however, neither considered other sequences w_l than u_l and $u_l^{(1)}$, nor general a, b (which he supposes to be relatively prime integers) — his proof being founded on the theory of continued fractions.

We give two direct proofs. While the first proof is based on induction, the second one is heuristic and leads to a generalization of (3) for recurring sequences of any order, to be published separately.

As a common base for both proofs we need the formula

(4) $$w_{k+l} = a\, u_{l-1}\, w_k + u_l\, w_{k+1},$$

which is easily verified by induction with regard to l.

As w_l can be obtained as a linear combination of any two non-proportional sequences with the same a, b, it is sufficient to prove (3) for any two such sequences, e.g. to prove (3″).

Proof of (3″). (3″) is evidently true for $k=0$. Let (3″) be true for a certain $k \geqslant 0$; then we shall prove (3″) for $k+1$. Indeed, noting that $u_0 = \binom{k}{k+1} = \binom{k}{-1} = 0$, we have, by (4) and (1),

$$
\begin{aligned}
u_{(k+1)l} &= u_{kl+l} \\
&= au_{l-1}\, u_{kl} + u_l\, u_{kl+1} \\
&= au_{l-1} \sum_{i=0}^{k} u_{k,l,i}\, u_i + u_l \sum_{i=0}^{k} U_{k,l,i}\, u_{i+1} \\
&= \sum_{i=0}^{k+1} \binom{k}{i} (au_{l+1})^{k-i+1} u_l^i\, u_i + \sum_{i=0}^{k+i} \binom{k}{i-1} (au_{l-1})^{k-i+1} u_l^i\, u_i \\
&= \sum_{i=0}^{k+1} U_{k+1,l,i}\, u_i ;
\end{aligned}
$$

$$
\begin{aligned}
u_{(k+1)l+1} &= u_{kl+(l+1)} \\
&= au_l\, u_{kl} + u_{l+1}\, u_{kl+1} \\
&= au_l \sum_{i=0}^{k} U_{k,l,i}\, u_i + u_{l+1} \sum_{i=0}^{k} U_{k,l,i}\, u_{i+1} \\
&= u_l \sum_{i=1}^{k+1} U_{k,l,i-1} (u_{i+1} - bu_i) + (au_{l-1} + bu_l) \sum_{i=0}^{k+1} U_{k,l,i}\, u_{i+1} \\
&= \sum_{i=1}^{k+1} \binom{k}{i-1} (au_{l-i})^{k-i+1} u_l^i u_{i+1} - b \sum_{i=1}^{k+1} \binom{k}{i-1} (au_{l-i})^{k-i+1} u_l^i\, u_i \\
&\quad + \sum_{i=0}^{k+1} \binom{k}{i} (au_{l-i})^{k-i+1} u_l^i\, u_{i+1} + b \sum_{i=0}^{k+1} \binom{k}{i} (au_{l-1})^{k-i} u_l^{i+1}\, u_{i+1} \\
&= \sum_{i=0}^{k+1} U_{k+1,l,i}\, u_{i+1}.
\end{aligned}
$$

Alternative proof of (3). We consider the two recurring sequences α^l and β^l belonging to the same a, b, whence α, β satisfy the equation

(5) $$x^2 = a + bx.$$

By (4), with $k=0$, we have

(6) $$\alpha = au_{l-1} + u_l\,\alpha.$$

Raising both sides of (6) to the power k we obtain (3) for $w\equiv\alpha$. The same is true for β, which, if $4a+b^2\neq 0$, † i.e. $\alpha\neq\beta$, proves (3).

As an immediate consequence of the first formula of (3″)

$$u_{kl} = u_l \sum_{i=1}^{k} \binom{k}{i} (au_{l-1})^{k-i}\, u_l^{i-1}\, u_i$$

it follows that, *in case a, b are integers and $l \geqslant 0$, u_{kl} is divisible by u_l*.

The last result, and the main formula (3), for $l \geqslant 0$, (obtained by the first proof), are seen to be true in an arbitrary number—ring or abstract ring containing 1. provided that $ab=ba$; if 1 is divisible by a, they hold also for $l>0$.

The Einstein Institute of Mathematics,
Hebrew University, Jerusalem.

---

†If $4a+b^2=0$, i.e. $\alpha=\beta$, we can use the sequences α^l and $l\alpha^l$. Indeed, by (4), $k=0$, we have

(7) $$l\alpha^l = \alpha u_l,$$

whence by (6) and $k\binom{k-1}{i-1}=i\binom{k}{i}$,

$$kl\alpha^{kl} = k\alpha\, u_l\,(au_{l-1}+u_l\,\alpha)^{k-1}$$
$$-k\sum_{i=1}^{k}\binom{k-1}{i-1}(au_{l-1})^{k-i}\,u_l^i\,\alpha^i$$
$$=\sum_{i=0}^{k}\binom{k}{i}(au_{l-1})^{k-i}\,u_l^i\,i\,\alpha^i.$$

We can also say that (3) considered as an algebraical identity for the variable a, with constant b, k, l, w_0, w_1 ($w_2, \ldots, w_k, u_{l-1}, u_l$ having been expressed by a, b, w_0, w_1), holds always, since it holds for $a\neq -b^2/4$.

NOTES AND DISCUSSIONS

Gaskin's theorem and the orthoptic constant.

It is not every central conic that has a director circle, and Mr. Lakshmanamurthi's "Analytical Proof" (*Math. Student*, 12, p. 105) actually establishes a more general theorem than Gaskin's.

Let p be a fixed diameter of a conic whose centre is C, and let Q be a variable point on the curve; let the tangent at Q cut p in T, and let the ordinate to p from Q cut p in V. Then the product CT, CV is independent of the position of Q, and is a number ω to be associated with p. If ω is positive, then p cuts the curve in the two points whose distance from C is $\sqrt{\omega}$; if ω is negative, then p does not cut the curve, but ω is even more important for that very reason. I call ω the *radial measure* of the diameter p.

The sum of the radial measures of a variable pair of conjugate diameters is a constant for the conic; we can call it the *orthoptic constant.* If the orthoptic constant is positive, the conic has an orthoptic circle, and the constant is the square of the radius of the circle; if the constant is negative, no circle exists, but theorems traditionally enunciated in terms of the circle may survive as true theorems in real metrical geometry if they are properly enunciated in terms of the constant.

Gaskin's theorem is a case in point. If the equation of a circle is

$$x^2+y^2+2gx+2fy+c=0,$$

then c is the power of the origin for the circle. What Mr. Lakshmanamurthi really proves is that

If a triangle is self-conjugate for a conic, the power of the centre of the conic for the circumcircle of the triangle is the orthoptic constant of the conic.

Another theorem of the same kind is a favourite of mine to illustrate the generalisation of enunciations. The altitudes AD, BE, CF of a triangle ABC meet in the orthocentre H, and the products HA·HD, HB·HE, HC·HF have a common value ρ. If the triangle is obtuse-angled, ρ is positive, and the circle with centre H and radius $\sqrt{\rho}$ is a circle for which the triangle is selfconjugate, the *polar circle* of the triangle. But whether or not the circle exists, ρ exists and can be called the *polar constant* of the triangle. With this definition, we can prove that

If a triangle is circumscribed to a conic, the square of the distance between the centre of the conic and the orthocentre of the triangle is the sum of the orthoptic constant of the conic and the polar constant of the triangle.

If we say, as is usual, that the polar circle of the triangle is orthogonal to the orthoptic circle of the conic, we accept an interpretation which is not valid if the conic is an obtuse hyperbola or if the triangle is acute-angled. We can enunciate two distinct partial theorems:

If a triangle circumscribes a central conic that is not an obtuse hyperbola, the power of the orthocentre of the triangle for the orthoptic circle of the conic is the polar constant of the triangle;

If an obtuse-angled triangle circumscribes a central conic, the power of the centre of the conic for the polar circle of the triangle is the orthoptic constant of the conic.

These two theorems together cover the ground, for a tangent triangle to an obtuse hyperbola is necessarily obtuse-angled, but the conditions of the two theorems are not mutually exclusive, and the one theorem that embraces them both is much the most satisfying.

E. H. NEVILLE

Application of Contour Integrals to a problem in electrostatics.

1. In Hydro-dynamics, while seeking the solution of problems of cylinders kept in a flow of liquid, the forces and couples on the cylinder are found in terms of contour integrals calculated round the boundary of the cylinders. Similar integrals can be found in the case of two dimensional problems in Electrostatics. Thus take the following problem:

A cylindrical conductor is kept in an electrostatic field. It is required to calculate the force on the conductor.

We know that the force on each surface element of a conductor is $2\pi\sigma^2$ for unit area, normal to the element, σ being the surface density of charge. Let ds be an element of a cross-section of the cylinder and ψ be the angle which the tangent makes with the

x-axis. Then the components X, Y of the force on the conductor along the axes are given by

$$X = \int_c 2\pi\sigma^2\, ds \sin\psi = \int_c 2\pi\sigma^2\, dy,$$
$$Y = -\int_c 2\pi\sigma^2\, ds \cos\psi = -\int_c 2\pi\sigma^2\, dx,$$

where $\frac{dx}{ds} = \cos\psi$, $\frac{dy}{ds} = \sin\psi$ and $\int_c$ denotes the integral taken round the closed curve c the cross-section of the cylinder.

We combine these components as follows :

$$X - iY = 2\pi \int_c \sigma^2 (dy + i dx)$$
$$= 2\pi i \int_c \sigma^2 (dx - i dy)$$
$$= \frac{i}{8\pi}\int_c (E_x^2 + E_y^2) \frac{dx - idy}{dx + idy}(dx + idy) \text{ since } \sigma = \frac{\sqrt{E_x^2 + E_y^2}}{4\pi}$$
$$= \frac{i}{8\pi} \int_c (E_x^2 + E_y^2) \frac{E_x - iE_y}{E_x + iE_y}(dx + idy) \text{ since } \frac{dx}{E_x} = \frac{dy}{E_y}$$
$$= \frac{i}{8\pi} \int_c (E_x - iE_y)^2 \cdot (dx + idy)$$
$$= \frac{i}{8\pi} \int_c \left(\frac{dw}{dz}\right)^2 dz$$

since if u is the potential function due to the electrical distribution and $w = u + iv$ and $z = x + iy$, then

$$\frac{dw}{dz} = u_x + iv_x = u_x - iu_y = -E_x + iE_y$$

Hence $X - iY = \frac{i}{8\pi}\int_c \left(\frac{dw}{dz}\right)^2 dz$ where c is any closed curve (simply connected) outside the boundary of the cylinder, not enclosing any other singularities of the field. This is obvious by Cauchy's theorem.

2. *To calculate the couple G on the cylinder*

$$G = \int (2\pi\sigma^2\, ds \sin\psi \cdot y + 2\pi\sigma^2\, ds \cos\psi \cdot x)$$
$$= 2\pi \int_c \sigma^2 (ydy + xdx)$$
$$= R \cdot 2\pi \int_c \sigma^2 (x + iy)(dx - idy)$$

where R denotes real part of what follows.

$$\therefore G = R \cdot \frac{1}{8\pi} \int_c (E_x^2 + E_y^2)(x + iy)(dx - idy)$$
$$= R \cdot \frac{1}{8\pi} \int_c (E_x - iE_y)^2 z dz \qquad \text{as above}$$
$$= \frac{1}{8\pi} R \int_c \left(\frac{dw}{dz}\right)^2 z d z \cdot$$
$$= \frac{1}{8\pi} R \int_c \left(\frac{dw}{dz}\right)^2 z dz \cdot$$

Fergusson College,
Poona.

G. L. CHANDRATREYA

On the Limit of a Series of variable Terms

(A) Let $$F(n) = \sum_{r=1}^{N} u_r(n)$$

where $u_r(n)$ *is a function of* n; N *is either* ∞ *or tends to* ∞ *with and for every fixed* r, $\underset{n\to\infty}{Lt}\, u_r(n) = v_r$,

We enquire as to under what conditions we may conclude that

(B) $\underset{n\to\infty}{\text{Lt}}\, F(n) = \sum_{r=1}^{\infty} v_r$, *provided this series on the right converges.*

A necessary and sufficient condition for this to hold is given in the following

THEOREM. *If* $F(n)$ *is defined as in* (A), *the necessary and sufficient condition for* (B) *to hold is, that corresponding to every arbitrary pre-assigned* ε *(however small) and positive integer* M, *(however large,) there exists a positive integer* $m > M$, *but independent of* n, *such that the inequality*

$$\left| \sum_{r=m+1}^{N} u_r(n) \right| < \varepsilon \text{ holds for every } n > \text{some } n_0$$

We first prove that **the condition is necessary.**

$$\text{Suppose } \underset{n\to\infty}{\text{Lt}}\, \mathrm{F}(n) = \sum_{r=1}^{\infty} v_r \quad \dots \quad (1)$$

If ε and M are pre-assigned, we can determine a positive integer $n_1(\varepsilon)$ such that

$$\left| \mathrm{F}(n) - \sum_{1}^{\infty} v_r \right| = \left| \sum_{r=1}^{N} u_r(n) - \sum_{r=1}^{\infty} v_r \right| < \varepsilon/3 \text{ for every } n > n_1.$$

Again since $\sum_{r=1}^{\infty} v_r$ is convergent, we can determine a positive integer $m > \mathrm{M}$, of course independent of n_1 so large that

$$\left| \sum_{r=m+1}^{\infty} v_r \right| < \varepsilon/3.$$

Now, n_2 is selected such that $\mathrm{N} > m$ for $n > n_2$.

Since $u_r(n) \to v_r$, we can determine n_3 such that

$$\sum_{r=1}^{m} [u_r(n) - v_r] < \varepsilon/3 \text{ for } n > n_3.$$

$\therefore$ for $n > max(n_1, n_2, n_3) = n_0$ (say),

$$\left| \sum_{r=m+1}^{\infty} u_r(n) \right| \leqslant \left| \sum_{r=1}^{N} u_r(n) - \sum_{r=1}^{\infty} v_r \right| + \left| \sum_{r=1}^{m} [u_r(n) - v_r] \right| + \left| \sum_{r=m+1}^{\infty} v_r \right|$$

$$< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon.$$

$\therefore$ we can determine $m > M(n)$ such that

$$\left| \sum_{r=m+1}^{N} u_r(n) \right| < \varepsilon \text{ for } n > n_0$$

We shall now prove that **the condition is sufficient.**

Since $\sum^{\infty} v_r$ is convergent, we can determine M such that

$$\left| \sum_{r=k+1}^{\infty} v_r \right| < \varepsilon/3 \text{ for every } k > M.$$

From hypothesis, $\left| \sum_{r=m+1}^{N} u_r(n) \right| < \varepsilon/3$ holds for some $m > M$ and $n > n_1$

Since $u_r(n) \to v_r$, we can determine n_2, so large that

$$\left| \sum_{r=1}^{m} [u_r(n) - v_r] \right| < \varepsilon/3 \text{ for every } n > n_2.$$

$\therefore$ for every $n >$ maximum $(n_1, n_2) = n_0'$ (say)

$$\left| \sum_{r=1}^{N} u_r(n) - \sum_{r=1}^{\infty} v_r \right| \leqslant \left| \sum_{r=1}^{m} [u_r(n) - v_r] \right| + \left| \sum_{r=m+1}^{N} u_r(n) \right| + \left| \sum_{r=m+1}^{\infty} v_r \right|$$

$$< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon$$

$$\therefore \operatorname*{Lt}_{n \to \infty} F(n) = \sum_{r=1}^{\infty} v_r.$$

Tannery's Theorem (Bromwich *Infinite Series*) is a special case of the above result.

MEHAR SINGH

The Perimeter of an Ellipse

The perimeter of an ellipse of axes $2a$, $2b$ and eccentricity e is given by

$$2\pi a\left\{1-\frac{e^2}{2^2}-\frac{1^2\cdot 3}{2^2\cdot 4^2}e^4-\frac{1^2\cdot 3^2\cdot 5}{2^2\cdot 4^2\cdot 6^2}e^6-\dots\right\}$$

The Indian mathematician Mahavira in his *Ganita Sāra Sangraha,** written in 850 A.D. gives the formula $\sqrt{(24b^2+16a^2)}$ for the perimeter of the ellipse. This works out as $2a\sqrt{10(1-\frac{3}{5}e^2)}$. Mahavira's method of getting this result is not known, but $\sqrt{10}$ is probably taken as approximately equal to π, as was often done in early Indian mathematics. Replacing $\sqrt{10}$ by π, the value $2\pi a\sqrt{1-\frac{3}{5}e^2}$ is not a bad approximation for the perimeter. The binomial theorem gives $2\pi a\ (1-\frac{3}{10}e^2-\frac{9}{200}e^4)$. as far as e^4, while the series above gives $2\pi a\ (1-\frac{1}{4}e^2-\frac{3}{64}e^4)$. At any rate, the formula $2\pi a\sqrt{1-\frac{3}{5}e^2}$ is a much better result than the formula $2\pi\sqrt{\frac{a^2+b^2}{2}}$, which will be found in some of our mathematical tables (e. g. Clark's), and which is quite inaccurate when e is not small.

"BANGALORE"

M. Cay's Extension of Feuerbach's Theorem

The extended theorem reads as follows:—

If two isogonal conjugates with respect to a triangle are collinear with the circumcentre, then their pedal circle touches the 9-points circle of the triangle.

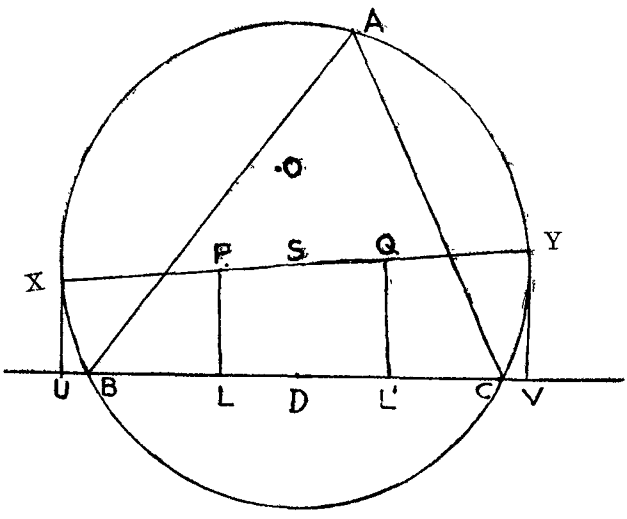
Here is a simple proof. Let P, Q be two isogonal conjugate points and S the circumcentre of the triangle. Let LL′, MM′, NN′ be the feet of the perpendiculars from P, Q on the sides, and let the line PQ• cut the circumcircle at X, Y. Since, by hypothesis, XY is a diameter, the pedal lines of X and Y meet at right angles at say O.

* English translation by M. Rangacharya, Chapter VII, Stanza 63. Vide also the translator's remarks.

Let U and V be the feet of the perpendiculars from X and Y on BC. Then from the right angled triangle UOV we have DU = DO = DV where D is the mid-point of BC. Let E, F be the mid-points of CA and AB.

From the figure we see that

$$\frac{SX}{SP} = \frac{DU}{DL} ; \frac{SX}{SQ} = \frac{DU}{DL'}$$



whence by multiplication and putting R for the circumradius

$$\frac{R^2}{SP \cdot SQ} = \frac{DU^2}{DL \cdot DL'} = \frac{DO^2}{DL \cdot DL'} = \frac{EO^2}{EM \cdot EM'} = \frac{FO^2}{FN \cdot FN'}$$

Hence the 9 - points circle, the common pedal circle of P and Q, and the point circle at O are coaxal. But O is on the 9 - points circle. Hence the two circles touch at O.

Loyola College
Madras

K. SIVARAJ
Class 5 Hons.

Some Limit Theorems

The object of this paper is to evaluate certain limits employing Riemann integration.

Let $f(x)=a$ when x is irrational

$$=\psi\,(q) \text{ when } x=p/q \quad (p \text{ being prime to } q)$$

where $\psi\,(n)$ tends to a as n tends to infinity through integral values. Also let $f(0)=0$.

It is clear that $f(x)$ is continuous in the interval $(0, 1)$ except at the rational points, and so is integrable—R.

Since $f=a$ except at an enumerable set we have

$$\int_0^1 F(x)\,f(x)dx=a\int_0^1 F(x)dx,$$

where F is another function integrable—R in the same interval.

But $$\int_0^1 F(x)\,f(x)dx=\text{Lt. } \sum f(\xi_i)F(\xi_i)\Delta x_i \qquad \ldots \quad (1)$$

following the usual notation ξ_i being any point in (x_{i-1}, x_i). More generally we can also write it as

$$\text{Lt. } \sum f(\xi_i)F(\eta_i)\Delta x_i \qquad \ldots \quad (2)$$

where ξ_i, η_i are in the same interval (x_{i-1}, x_i).

Let $\Delta x_i=\dfrac{1}{n}$ and p_i/q_i the fraction with the least denominator in $\left(\dfrac{i-1}{n}, \dfrac{i}{n}\right)$.

From (1) we get, taking $\xi_i=p_i/q_i$,

$$\text{Lt } \frac{1}{n}\sum_{i=1}^{n} \psi(q_i)\,F\,(p_i/q_i)=a\int_0^1 F(x)dx.$$

Further, we may take $\xi_i=p_i/q_i$ and $\eta_i=i/n$ in (2).

Thus

$$\text{Lt. } 1/n \sum_{i=1}^{n} \psi\,(q_i)\; F(i/n)=a\int_0^1 F\,(x)\,dx.$$

In particular, if $F=1$ we have

$$\text{Lt. } 1/n \sum_{i=1}^{n} \psi\,(q_i)=a.$$

Taking $\psi(n)=1/n$

$$\text{Lt. } 1/n \sum_{i=1}^{n} 1/q_i = 0.$$

Putting $F(x)=x^s$ where s is greater than 1 and $\psi(n)=1/n$

$$\text{Lt. } 1/n \sum_{i=1}^{n} (i/n)^s \cdot \frac{1}{q_i} = 0$$

or

$$\sum_{i=1}^{n} i^s/q_i = o\,(n^{s+1})$$

Taking $\Delta\ x_i = 1/n$; $F=1$; $\xi_i = i/n$ in (1)

we get, after a little simplification

$$\sum \psi(d)\,\phi(d) \sim na$$

where $\phi(n)$ is Euler's function representing the number of integers not greater than n and prime to n, the summation being for all divisors d of n.

Madras Christian College, Tambaram. } C. V. KRISHNA REDDI

Correcting Watch Time by watch Vibrations.

A good watch may be made to run faster or slower by balancing it, face horizontal, on a bit of rubber about 1/32 inch thick cut from a live elastic rubber band.

A button of rubber ⅛ inch round or square will cause the watch to gain ten to twenty seconds over night or in about ten hours. It may require several attempts to balance the watch horizontally on such a small button. The guard or chain must be removed. The oscillations or shummey of the whole watch due to the reactions of its balance wheel should be plainly visible.

A similar rubber support about 3/16 inch square will cause the watch to lose about six seconds in ten hours.

Using the cut-and-try method with rubber supports, a good watch may be maintained with ten seconds of correct time indefinitely by repeating the correcting operations occasionally as required, without opening the case.

Opening the watch case for the usual methods of correcting watch time is objectionable as dust and moisture may be admitted. It is also difficult to adjust the registration between the second hand and the minute hand.

E. M. TINGLEY in *School Science and Mathematics.*

REVIEWS

Analysis of Functions of Real Variables

By B. SEETHARAMA SASTRY, M.A. (*Cal.*), *University of Mysore.*
(Satyasodhana Publishing House, Fort, Bangalore City, Price Rs. 12-8-0)

This text-book on Analysis gives in a compact form a connected story of the subject covering more or less the syllabus for the B.Sc. (Hons.) Degree courses in South Indian Universities. The subject matter is well-arranged, and treatment rigorous and clear. There are numerous examples, many of them being either completely worked out or accompanied by sufficient hints for solutions.

There are some slight mistakes which, it is hoped, will be removed in the next edition of the book. On page 25, the statement that an open set is one which is not closed is incorrect; an open set is the complement of a closed set. On page 54, in the definition of the limit of $f(x)$ at $x=c$, the inequalities should be $|f(x)-l|<\varepsilon$ if $0<|x-c|<\eta$ (*and not* if $|x-c|<\eta$); then alone the statement in note (i) following immediately will be correct. On pages 98-99, the statement that (B) is less comprehensive than (A) will be correct only if the second condition in (A) is replaced by "if $u_n^{1/n} \geqslant 1$ for an infinity of values of n, then $\sum u_n$ diverges".

A chapter on Fourier Series and one containing a detailed treatment of trigonometric series and products would have given more completeness to the book and increased its usefulness to the students.

The book is a welcome addition to the small number of Indian text-books on Analysis covering the syllabus for the Honours degree courses of the Indian Universities.

Annamalainagar V. G.

Students' Guide to Statistics.

By T. S. SANKARANARAYANA PILLAI, *Pachaiappa's College, Madras,*
(Universal Book Service, Madras, 1944 pp. 346, Price Rs. 5.)

In any Statistical enquiry there are four stages, viz. design, collection of data, their classification and tabulation, and inference. In section A the requisites of the first three stages are discussed, with illustrations taken from the recent census. Diagramatic representations are always appealing and easy to grasp. These are discussed in section B. Then comes the mathematical analysis of the data. "The Human mind is incapable of grasping in its entirety any large mass of quantitative data. They require to be specified by a relatively few constants." Hence the need for averages and measures of dispersion which are considered in section C. With these the analysis of the sample is complete. Next arises the question of the reliability of the inferences drawn from the sample. This will lead us on to problems of specification, of distribution and significance tests. A detailed study of these has been omitted as being beyond the scope of an elementary guide to the subject.

In the short space available, the author has introduced the calculus of probability, and followed it up by a discussion of the Binomial distribution. In as much as the normal probability curve can be regarded as the limiting form of the symmetric point binomial, it would have been better if Chapter XIV (Normal curve) came immediately after Chapter XII (Binomial). That would have enabled the students to appreciate the full significance of 'the Standard Error' introduced in Chapter XIII. It would have been in accordance with the spirit of the times if the term 'Probable Error' were left out and 'the standard error' and its significance emphasised. A discussion of fitting the normal curve to given data using the table of areas will enhance the utility of the book.

The last section deals with the application of the statistical method to mortality problems and index numbers.

The book should prove exceedingly useful to students of the B. A. and B.Sc. Classes whose requirements have been specially kept in view by the author.

Annamalainagar V. SEETHARAMAN.

Solid Geometry

By K. D. PANDAY, M.A., *Central Provinces Educational Service*

Nawal Kishore Press, Lucknow, pp. 138, Price Re. 1.

This booklet explains the properties of figures in space and applies them to the study of the parallelopiped, tetrahedron, prism, cylinder, cone and sphere. There are a large number of graded examples and, at the end, a collection of formulæ relating to mensuration and a collection of miscellaneous examples and of questions taken from the Nagpur University Intermediate papers and the U.P. and Rajaputana Boards. A feature of the book which makes for clarity is the arrangement of each argument and the conclusion derived therefrom in corresponding opposite columns. This should prove a very useful device for adoption both in the class room and in books for junior students.

The book is well written and should prove very useful in giving the students of the Intermediate classes their first experience of theoretical solid Geometry.

Annamalainagar A. N. R.

ANNOUNCEMENTS AND NEWS

The following persons have been admitted as members of the Society :-

K. M. Saxena Esq. Lecturer in mathematics, S. D. Intermediate College, Muzaffarnagar U. P.

H. F. Merchant Esq. 2372 139, Signal Training Centre, (India), Jubbalpore, C. P.

T. V. Viswanathan Esq., Asst. Professor, Presidency College, Madras.

G. L. Chandratreya Esq., Fergusson College, Poona.

Ratan Shanker Misra, Research Student, Delhi University, Delhi.

Shanti Narayan, M.A., Prof. of Mathematics, D. A. V. College Lahore.

Miss Lolita Bose, B.A., D.T., Teacher, Jubilee Girls School, c/o. N. N. Bose Esq., 33 Model House Lucknow.

Oudh Behari Shukla, Esq., M.Sc., Lecturer in mathematics, Durbar College, Rewa.

The following gentlemen have been admitted as life members of the Society.

K. Venkatachaliengar, D.Sc., Mysore University, Bangalore.

M. Abdulla Butt Esq., Lecturer, Muslim University, Aligarh.

Dr. V. Ganapati Iyer of the Annamalai University has been declared to be the first winner of the Narasinga Rao Medal for Mathematical Research. According to the terms of the endowment, the medal is to be awarded for the best solution of a problem to be proposed from time to time, and for the solution of which a sufficient time (about 18 months) is to be given. The first problem was proposed in The Mathematics Student Vol. XI p. 62 and the Journal of the Indian Mathematical Society Dec. 1943.

Dr. N. S. Nagendranath, Professor of Mathematical Physics at the Andhra University has accepted an appointment as Professor of Mathematics at the Patna University.

Dr. N. Sundararama Sastry, formerly of the Statistics Department, Madras University, has been appointed Statistician, Reserve Bank, Bombay.

The cause of higher studies in this country is materially strengthened by the establishment of the Tata Institute of Fundamental Research at Bombay where research in Pure Physics and Mathematics are to be carried on in the first instance. The institute works under the ægis of the Tata Trust and is financed by the Trust as well as by grants from the Government of India and the Government of Bombay. The Director is Prof. Bhabha and Prof. D. D. Kosambi of Poona is the only other Professor at present, but Prof. S. Chandrasekhar is expected to join early in 1947, while there will be visiting professors in theoretical physics and mathematics from India and other countries. It is hoped that a chair in experimental Physics will be instituted in due course. Students in limited numbers are taken for research if considered fit, but no terms or degrees are granted by the Institute which is not affiliated to any University, though several Universities have agreed to recognize work done in the Institute in connection with their research degrees. Facilities will also be provided, within limits, for advanced workers to carry on their work at the Institute. We wish the Institute every success and hope it will develop into the Princeton Institute of India.

Prof. Harold Davenport F. R. S. has been appointed to the Astor Chair of Mathematics in the University of London in succession to Prof. G. B. Jeffery who is now Director of the Institute of Education, University of London.

Prof. Wolfgang Pauli (Switzerland), Princeton University, has been awarded the Nobel Prize for Physics for 1945.

The Sir C. R. Reddi National Prize has been awarded to Prof. C. V. Chandrasekharan working at the Yerke's observatory in America.

BOOKS RECEIVED FOR REVIEW

J. L. COOLIDGE: A History of the conic sections and Quadric Surfaces, Oxford, Clarendon Press, 1945, pp 214, Price 21 s. Net.

H. W. TURNBULL. The Mathematical discoveries of Newton: Blackie and Sons Ltd., London and Glasgow, 1945, pp 68, Price 5s. net.

To find the age of the moon on any day

Mr. Forbes gives in his delightful book, "*The wonder and the glory of the stars*", the following rule for computing the age of the Moon on any day in any year. The error rarelv exceeds one day.

"Add together the year-number, the month-number and the day-number. Reduce the sum (if it exceeds 30) by subtracting multiples of 30. The remainder is the age of the Moon".

The year-number is obtained by subtracting 1930 from the year in question, multiplying the remainder by 11 and subtracting all multiples of 30. The month-numbers for Jan., Feb. etc., are respectively 0, 2, 0, 2, 2, 4, 4, 6, 7, 8, 9, 10. The day-number is the same as the date of the month.

From *First steps in astronomy.*