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ERRATA

Page	*Line*	*For*	*Read*
28	11 from the bottom	we must have h odd	we must have h odd for $\lambda > 1$
29	18, 19, 20	Replace by the following :— $j = 4, 8, p^\lambda, 2p^\lambda$, T, 2T, 4J or 8J; where p is an odd prime, $\lambda \geq 1$, J and T are odd and ≥ 3; and (i) for each prime p which divides J, $\text{pot}_2\, \phi(p) = 1$; (ii) for each prime p, which divides T, $\text{pot}_2\, \phi(p)$ is the same; $\text{pot}_2\,(i)$ denotes as usual the index of the highest power of 2 which divides i. (Suggested by a note of Ramanathan)	
167	12	Herman Weyl	Hermann Weyl
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HOMOMORPHISMS AND CONGRUENCES IN GENERAL ALGEBRA

BY

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1. General Algebra and Modern Algebra.

Algebra started with the study of numbers, and number-based systems, like rational numbers, real numbers, polynomials, matrices, and functions of real variables. When, in the later part of the Ninteenth Century, Mathematicians began to consider the abstract axiomatic approach to the different branches of the subject, a study of the formal properties of the structural patterns given by these number based systems was initiated. This was the beginning of the theory now known as 'Modern Algebra'.

These structures can all be reduced to depend on one of the three fundamental structures, the group, the ring and the field. *A group* is defined to be an abstract set G, closed[1] for a binary, associative operation, +, relative to which there is an *identity element* O - satisfying $O+x=x+O=x$, for every x in G, and an *inverse* $-x$, to each element x in G satisfying $(-x)+x=x+(-x)=O$. When the *group-operation*, +, is also commutative, the group is called an '*Abelian Group*' (and also a '*Module*' if, as here, the operation is denoted by +). *A ring* is defined to be a module, R, closed for a second binary, associative operation, •, which is distributed by + [so that, for any x, y, z from R, $x\cdot(y+z)=x\cdot y+x\cdot z$ and $(y+z)\cdot x=y\cdot x+z\cdot x$]. When • is also commutative the ring is '*commutative*'. Finally *a field* is a double group, being defined as a ring, F, in which the set of elements other than O, form a group with • as group operation, with an identity element, 1, and an inverse x^{-1} to each element x, $(x\neq 0)$. If, as a ring, F is commutative, it is a *commutative field*. Otherwise it is a *skew field*.

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1. A set K is closed for a binary operation + if with each pair of elements (x, y) of K is associated a unique element, $(x+y)$, of K. The operation + is associative if $x+(y+z)=(x+y)+z$, for all x, y and z; it is commutative if $x+y=y+x$, for all x, y.

Since all the structures of 'Modern Algebra' are special cases of the above three, and each of these is a group for at least one operation, we may designate the structures of Modern Algebra generally as '*the group-based structures*'.

The only logical concept, besides that of the 'number', which is fundamental for the developement of Mathematics is the 'set-concept'. In 'set theory' one meets with a different class of structures for which the fundamental operations are those of set-union, set-intersection, and set-complementation. These set-algebras are not group-based structures. For neither of the basic binary operations $\cup$ or $\cap$—of set-union and set-intersection—can be a group-operation for a family, F, of sets. If $\cup$ is to be a group operation for F, there must be a unique solution in F for the equation $A \cup X = B$ whatever sets A,B be chosen from F. But unless A is the null set and B a one-element set, the equation has either no solutions or more than one solution (according as $A \subset B$ is untrue or true). In the same way $\cap$ cannot be a group-operation. To obtain a formal characterisation of these set-algebras, we take note of a remarkable feature of the set-operations, namely that they can all be defined solely in terms of a fundamental binary, (or two term), relation—the '*set inclusion*' relation, '$\subset$'. Thus the union of a family of sets C_r is characterised as that set C for which: each $C_r \subset C$ and, whenever each $C_r \subset$ some set D, then $C \subset D$. The formal properties of this relation '$\subset$'—namely that $A \subset A$ for every A, $A \subset B$ and $B \subset C$ imply $A \subset C$, and $A \subset B$, $B \subset A$ imply $A = B$—are embodied in the definition of '*an ordering relation*' $<$ on an abstract set K, as a reflexive, transitive relation—connecting certain pairs of K— such that $x < y$ and $y < x$ imply $x = y$, for all x,y from K. A set K, with such an ordering relation associated with it, is a '*partially ordered set*'. [2] In such a (partially) ordered set '*order operations*' (similar to the operations of set-algebra) can be defined in terms of the ordering relation. And ordered sets which are closed for selections of these operations are of special interest, and have special properties. All these structures we shall designate by the term '*order-based structures*'; they are also sometimes called 'lattice structures'. As important examples of these, we may consider the additive system, the multiplicative system and the lattice. Given two elements x,y of an ordered set K, the sum $(x+y)$, and product, $(x \cdot y)$, of these are defined to be elements which satisfy the conditions: $x < (x+y)$, $y < (x+y)$, and if $x < z$, $y < z$ for a

2. For a detailed study of partially ordered sets reference may be made to H. M. Mac Neille's paper on "Partially Ordered Sets", Trans. Am. Math. Soc., Vol. 42.

z of K, then $(x+y)<z$; $x \cdot y<x$, $x \cdot y<y$, and if $z<x$, $z<y$ for a z of K, then $z<(x \cdot y)$. Such a sum or product is unique if it exists. An additive system is an ordered set closed for finite additions—i.e., one containing the sum for every pair of elements. A multiplicative system contains the product for every pair of its elements. While a lattice contains both the sum and the product of every two of its elements. Obviously, the family E of all subsets of a set, including the nullset as a subset, forms such a lattice, with set inclusion as the ordering relation; $A \cup B$ and $A \cap B$ form the sum and the product of the two subsets A, B of E.

The group—, and order—based structures are, in a sense, complementary types of structures; the only known structures belonging to both these types are the number systems, and the Boolean rings defined and studied by M. H. Stone.[3] In order to be able to include structures belonging to both these types under a single theory we shall designate by '*General Algebra*', the study of all structural patterns, in a formal manner. The most important structures of 'General Algebra' can then be classified broadly into 'the group based structures' and 'the order based structures'.

2. The Method in General Algebra

In order to analyse and study the characteristic features of the abstract structures occurring in General Algebra, the method adopted is to compare structures of the same nature. This comparison is effected by considering the '*homomorphisms*' between two similar structures; where by a '*homomorphism*', (or *homomorphic map*), of one structure k on a similar structure k' we mean a map, f, of k on k' — [which maps each element x of k on a unique element $f(x)$ of k', in such a way that each x' of k' is the map of at least one x of k], which carries over all—or a selection of—the structural features of k—on to corresponding structural features of k'. [Thus f carries an operation θ on [4] k on to the corresponding operation θ' on k' if $f[\theta(c)]=\theta'\{f(c)\}$, for any subset c of k provided $\theta(c)$ exists; and f carries a relation R on k onto a corresponding relation R$'$ on k' if $x\text{R}y$ in k implies $f(x)\text{R}'f'(y)$. The *homomorphism* f is said to be *relative to* θ, *or* R, if it carries θ, or R, of k into a corresponding θ' or R$'$, of k'].

Any such homomorphic map f, of k on another similar structure k', determines an equivalence relation—i.e., a reflexive, symmetric and

[3] *Cf.* his paper on 'The theory of Representations for Boolean Algebras', Trans. Am. Math. Soc, Vol. 40.

[4] An operation θ on a set k is a method of associating, to certain subsets of k, single elements of k; the element associated thus with a subset c is denoted by $\theta(c)$.

transitive relation $E=E(f)$, on k; namely the relation E defined by the condition: xEy, for x, y of k, if and only if $f(x)=f(y)$. [Since $f(x)=f(x)$, $f(x)=f(y)$ implies $f(y)=f(x)$ and $f(x)=f(y)$, $f(y)=f(z)$ imply $f(x)=f(z)$, for all x, y, z from k, the relation E is reflexive, symmetric and transitive.] The relation E, $=E(f)$, is an equivalence relation for any map f of k on k'. The extra condition that f is a homomorphic map usually imposes further conditions on the associated equivalence $E(f)$; so we shall call these equivalences corresponding to homomorphic maps of k on other similar structures '*the homomorphic equivalences*', (or *homomorphic equivalence relations*). The characterisation of these among all equivalences can be undertaken if we know the nature of the structures k, k' and the nature of the structural features relative to which the homomorphisms are considered. Without knowing these, however, we can state one general result. It is:

2·1. if θ, θ' are corresponding operations defined on two similar structures k, k' respectively, and if a homomorphic map, f, of k on k' carries θ into θ', the corresponding equivalence $E(f)$ on k is a *congruence relative to* θ; where, by a congruence relative to θ, we mean an equivalence relation E on k satisfying the condition: when (c_i, d_i) is a family of pairs of elements of k each of which satisfies $c_i \to E\, d_i$, $[\theta\{c_i\}]E[\theta\{d_i\}]$ also holds, provided $\theta(c_i)$ and $\theta(d_i)$ exist in k.

This result is easily proved; for if $\theta\{c_i\}$, $\theta\{d_i\}$ exist in k, $\theta'\{f(c_i)\}$' $\theta'\{f(d_i)\}$ exist in k', being equal, respectively, to $f[\theta\{c_i\}]$ and $f[\theta\{d_i\}]$. And $c_i \, E \, d_i$ for each (c_i, d_i) implies $f(c_i)=f(d_i)$ for each (c_i, d_i), and so $\theta'\{f(c_i)\}$ must be equal to $\theta'\{f(d_i)\}$. Hence also $f[\theta\{c_i\}]=f[\theta\{d_i\}]$ or $[\theta\{c_i\}] \, E \, [\theta\{d_i\}]$ follows.

But the converse is not generally true. There may exist congruences relative to θ which do not equal the homomorphic equivalence defined by any homomorphism relative to θ.

The theory of congruences and homomorphisms takes different paths for the group—,and the order—based structures. It is the purpose of this paper to compare and contrast the salient features of the theory as it affects structures belonging to these two types. We shall begin by considering the group-based structures first.

3. Homomorphisms and Congruences for the group-based structures.

For the group-based structures the theory of congruences and homomorphisms depends ultimately on that for the groups. The group being defined in terms of a single operation, +, the homomorphisms are those relative to this operation. The two main results regarding group-congruences and homomorphisms are:

3·1. the homomorphic equivalences on a group G coincide with the congruences on G relative to the group operation, +;

3·2. every congruence, E, on G relative to this group operation +, is completely determined by a single class of mutually congruent elements of G; namely, by the '*normal subgroup*' of elements of G congruent to the identity element, O.

By the general result (2·1) we have only to show that every congruence E, on G relative to + is a homomorphic equivalence, to prove the result (3·1). Now given E, if we define '*the coset* $E(x)$ *relative to* E, containing x as the set of elements y of G for which $x\mathrm{E}y$ is true, it is seen that $x \varepsilon \mathrm{E}(x)$, $\mathrm{E}(x)=\mathrm{E}(y)$ if, and only if, $x\,\mathrm{E}y$ is true and if $\mathrm{E}(x)\neq\mathrm{E}(y)$, $\mathrm{E}(x)$ and $\mathrm{E}(y)$ are disjoint. Now these cosets combine by the elementwise operation +, as $\mathrm{E}(x+y)=\mathrm{E}(x'+y')$ if $\mathrm{E}(x)=\mathrm{E}(x')$ and $\mathrm{E}(y)=\mathrm{E}(y')$ - or, if $x\mathrm{E}x'$ and $y\mathrm{E}y'$ are true [this follows, since E is a congruence relative to +]. So the coset $\mathrm{E}(x+y)$, which depends only on $\mathrm{E}(x)$ and $\mathrm{E}(y)$ and not on x or y, may be denoted by $\mathrm{E}(x) \mp \mathrm{E}(y)$. Then this operation $\mp$ on the family $\overline{\mathrm{G}}$ of the distinct cosets of G relative to E, is seen to be associative, [and commutative], since + is such. And it is a group operation on $\overline{\mathrm{G}}$, since $\mathrm{E}(x) \mp \mathrm{E}(o)=\mathrm{E}(o) \mp \mathrm{E}(x)=\mathrm{E}(x)$ and $\mathrm{E}(-x) \mp \mathrm{E}(x)=\mathrm{E}(x) \mp \mathrm{E}(-x)=\mathrm{E}(o)$. And the map, g, of G on $\overline{\mathrm{G}}$ defined by putting $g(x)=\mathrm{E}(x)$, for each x in G, is a homomorphism from G to $\overline{\mathrm{G}}$, which is such that the homomorphic equivalence $\mathrm{E}(g)$ it defines on G is the same as the congruence E that we started with, [for $g(x)=g(y)$ or $\mathrm{E}(x)=\mathrm{E}(y)$ if, and only if, $x\mathrm{E}y$ is true]. So this completes the proof of (3·1).

To prove (3·2), let E be, as before, any congruence on G relative to +. Then the coset E (O) is a 'subgroup' of G - i.e., contains $x+y$ with x,y and $(-x)$ with x, [since $\mathrm{OE}x$, $\mathrm{OE}y$ imply $(o+o)\ \mathrm{E}\ (x+y)$ or $\mathrm{OE}(x+y)$: and $\mathrm{OE}x$ implies $[o+(-x)]\mathrm{E}[x+(-x)]$ or $(-x)$ EO which implies $\mathrm{OE}(-x)$]. This 'subgroup' is also 'normal' - i.e., contains with x also $-t+x+t$, whatever t be from G, [since $\mathrm{OE}x$ implies $(-t+\mathrm{O}+t)\ \mathrm{E}\ (-t+x+t)$ or $\mathrm{OE}\ (-t+x+t)$]. The congruence E is completely defined by $\mathrm{E}(o)$ since $x\mathrm{E}y$ is true if, and only if, $[x+(-x)]\ \mathrm{E}\ [y+(-x)]$ or $\mathrm{OE}(y-x)$ or $(y-x)\,\varepsilon\mathrm{E}(o)$ holds. It can also be verified that given any normal subgroup N of G a congruence, E relative to + is determined if we write $x\mathrm{E}y$ when, and only when,

$(y-x)\,\varepsilon N$. We call E *the congruence modulo N* and write $x \equiv y$ (mod. N) for xEy. For this congruence E, E(O) is evidently = N. So for a group the homomorphic equivalences, the congruences relative to +, and the congruences modulo normal subgroups are all coincident.

For a ring it can be readily verified that the subgroup E(O) corresponding to any congruence E relative to + and •, is *a two-sided ideal*[5] *i.e.*, a subgroup containing with x also $x \cdot z$ and $z \cdot x$, for all z from the ring, [for $z \cdot O = O \cdot z = O$, for all z]. And the congruence E is that modulo this ideal, considered as a subgroup. Conversely also, the congruence modulo a subgroup, N, is also a congruence for •, (besides being one for +), if the subgroup is a two sided ideal, [as $x \equiv x'$ (mod N), $y \equiv y'$ (mod N) or $(x'-x)\varepsilon N$, $(y'-y)\varepsilon N$ imply $x' \cdot (y'-y)$, $(x'-x) \cdot y$ and $x' \cdot y' - x \cdot y = x' \cdot (y'-y) + (x'-x) \cdot y$ are in N, or $x \cdot y \equiv x' \cdot y'$ (mod. N)]. Hence for a ring the homomorphic equivalences, congruences relative to +, •, and congruences modulo two sided-ideals get identified.

For a field there are only two homomorphic equivalences or congruences, since there are only two distinct two sided ideals, namely, the whole field, and the one element set (O).

4. Homomorphisms and congruences for order-based structures.

Since for order based structures, there is a fundamental ordering relation, we have to define a homomorphism between two such structures to be, in the first place, relative to the ordering relation. It may also be relative to certain of the order operations. Since the homomorphisms have to carry over the ordering relation, the characterisation of the homomorphic equivalences for order based structures becomes somewhat complicated. But such a difficulty does not present itself when dealing with the three structures we defined earlier, namely the additive system, the multiplicative system and the lattice. We shall confine our attention to these only, and even here omit the treatment of multiplicative systems, since they behave exactly similar, and dual, to the additive systems.[6]

The simplicity in treatment for these structures arises from the fact that these can be characterised in terms of their basic operations only, without making use of a fundamental ordering relation. For instance an additive system can be characterised as an abstract

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[5] *This subgroup should be normal also, by our earlier remarks. But for a ring, as* it is an Abelian group for +, all subgroups are normal.

[6] I have considered in detail these homomorphisms and congruences for order based structures in a thesis recently accepted for the M.Sc. Degree by the Madras University. For a more detailed study, vide 'The theory of Homomorphisms and Congruences for Partially Ordered sets' in the Proc. Ind. Acad. Sci. Vol. 21, No. 6.

set k closed for a binary operation $+$ which is commutative associative and *tautological* (*i.e.,* satisfies $x+x=x$ for all x), while a lattice can be characterised as a set closed for two such operations $+$, $\cdot$ which are mutually related by the condition $x+y=y$ if, and only if, $x \cdot y=x$. For these conditions are true of $+$ in an additive system and of $+$ and $\cdot$ in a lattice. While when a set k is closed for such an operation, (or pair of operations), an ordering relation '$<$' can be defined for k by saying $x<y$ if, and only of, $x+y=y$, [and also $x \cdot y=x$, for a lattice]. Under this ordering relation, it can be immediately verified that $x+y$, $(x \cdot y)$ denotes the sum, (product), in k of x and y, and so k is an additive system, (or lattice).

Also this connection of $<$ to $+$ ensures that any homomorphism, f, from one additive system, k, (or lattice), to another, k', relative to $+$, ($+$ and $\cdot$), is also relative to the ordering relation $<$, since $x<y$ in k implies $x+y=y$ and so $f(x)+f(y)=f(x+y)=f(y)$ or $f(x)<f(y)$ in k'. So we can now omit altogether the statement that the homomorphisms are relative to the ordering relation, and need only consider those relative to the basic operations, as for groups or rings.

We are now able to assert a result similar to (3·1) for these systems.—i.e., the homomorphic equivalences identify with the congruences. One part is proved by using (2·1), while the other, that every congruence E is a homomorphic equivalence, is proved as in the case of the group (in proving (3·1)). Thus, for an additive system, the cosets E(x) are shown to combine by the element wise operations $+$, and so the family $\bar{k}$ of these cosets is closed for a binary operation $\bar{+}$, which is commutative, associative and tautological, since $+$ is such. Hence $\bar{k}$ is an additive system, and there is homomorphism from k to $\bar{k}$,— mapping each x of k on E(x) of $\bar{k}$, which defines on k the homomorphic equivalence E. So the congruence E relative to $+$ is a homomorphic equivalence. A similar proof can be constructed for the lattice.

But the analogue of result (3·2) is not true for the congruences on additive systems or lattices. All congruences cannot be defined in terms of a single coset, (or a finite selection of cosets) of mutually congruent elements. But there are certain special congruences–which I call '*regular congruences*'—which are so determined. These we now proceed to consider. [As before, we omit the regular multiplicative congruences on multiplicative systems, which are the duals of the regular additive congruences we treat of here.]

Let k be an additive system with a zero, O—i.e., an element which is $<$ every element of k. Then if E is any additive congruence on k [i.e., a congruence relative to +], the coset E (o) is a μ · *ideal*—i.e., a subset containing $x+y$, with x and y, and every $z<x$, with x. [For $o\text{E}x$ and $o\text{E}y$ imply $o\text{E}(x+y)$; while $o\text{E}x$ and $z<x$ imply $(o+z)$ $\text{E}(x+z)$ or $z\text{E}x$; so $x\text{E}z$, $o\text{E}x$ and $o\text{E}z$ follow, as E is transitive.]

Now a *congruence*—relative to + — can be defined *modulo any* μ· *ideal C of k* by saying that two elements x,y of k are congruent, $x\equiv y$ (mod. C), when and only when $x+t=y+t$, for some t of C. [It is a congruence relative to + ; for, whatever elements x,y,z, x',y', be taken from k, if t,t', and so $t''=t+t'$, are in C, then $x+t=x+t$; $x+t=y+t$ implies $y+t=x+t$; $x+t=y+t$, and $y+t'=z+t'$ imply $x+t''=y+t''=z+t''$; and $x+t=y+t$, and $x'+t'=y'+t'$ imply $(x+x')+t''=(y+y')+t''$.]

Starting from a congruence E on k (relative to +), if C=E(o), in general E is not identical with the congruence modulo C. When it is, we say E is a '*regular additive cangruence*'. In general we can say that $x\equiv y$ (mod C) implies $x\text{E}y$, [since $x+t=y+t$ and $t\text{E}o$ imply $(x+t)\text{E}(x+o)$, $(x+t)\text{E}(y+t)$ and $(y+t)\text{E}(y+o)$ or $x\text{E}(x+t)$, $(x+t)\text{E}(y+t)$ and $(y+t)\text{E}y$. Hence $x\text{E}y$ follows as E is transitive.] Thus for a regular congruence E, the converse, that $x\text{E}y$ implies $x\equiv y$ (mod C), where C=E(o), must be true. By its very definition, a regular additive congruence is the congruence modulo a μ· ideal. The converse is also true. The congruence modulo any μ· ideal C of k is 'regular additive', [since for this congruence E(o)=C itself, as $x\equiv o$ (mod C) if, and only if, for same t in C, $x+t=o+t$ or $x+t=t$ or $x<t$, which implies, and is implied by, $x\varepsilon$C].

For a lattice, L, with O and, 1 – where O$<$ every element of L$<$1 – certain 'regular lattice congruences' can be similarly defined which depend only on the cosets E(o), E(1) consisting of elements congruent to O and 1 respectively. [As before E(o) is a μ· ideal; E(1) is an α· ideal – i.e., a subset containing $x\cdot y$, with x, y, and all z of L$>x$, with x.] [7]

But the point to be noted is that all congruences are not 'regular' for these structures. There are certain contrasting types—like 'the irreducible', additive, (or lattice), congruences for which E(o) contains only the element O, (and E(1) contains only 1). Evidently if a congruence E is regular and irreducible, E(x)=(x) for all x. Corresponding to these, we have 'regular' and 'irreducible' homomorphisms for which the homomorphic equivalence defined by them is a 'regular' or 'irreducible' congruence.[7]

We shall conclude with a few examples of these types of congruences. We consider a sub-lattice M of the lattice L of all subsets of the set (a, b, c, d, e)—these subsets being partially ordered by the set inclusion relation. The sub-lattice M consists of the subsets: (a,b,c,d,e), (a,b,c,d), (a,b,c), (b,c,d), (b,c), (b) and the null set, O. The congruences are denoted by the corresponding partition of the lattice M into the cosets of mutually congruent elements;[8] thus:

$$E_1: [O, (b), (b,c), (a,b,c)]\ [(b,c,d), (a,b,c,d)]\ [(a,b,c,d,e)]$$
$$E_2: [O]\ [(b), (b,c)]\ [(a,b,c), (b,c,d), (a,b,c,d), (a,b,c,d,e)]$$
$$E_3: [O, (b), (b,c)]\ [(a,b,c), (b,c,d), (a,b,c,d), (a,b,c,d,e)]$$
$$E_4: [O, (b), (b,c), (a,b,c)]\ [(b,c,d), (a,b,c,d), (a,b,c,d,e)]$$
$$E_5: [O]\ [(b) (b,c), (a,b,c)]\ [(b,c,d) (a,b,c,d),]\ [(a,b,c,d,e,)]$$
$$E_6: [O, (b)], [(b,c), (a,b,c), (b,c,d), (a,b,c,d)], [(a,b,c,d,e)]$$

E_1, E_2, E_3 are examples of additive congruences of which E_1 is regular, E_2 is irreducible and E_3 is neither regular nor irreducible. Only E_1 is also a lattice congruence.

E_4, E_5, E_6 are lattice congruences of which E_4 is regular, E_5 is irreducible and E_6 is neither regular nor irreducible; E_4 is not a *regular* additive congruence; neither is E_6 while E_5 is, necessarily, also an irreducible additive congruence.

Before concluding, I wish to express my sincere gratitude to Dr. R. Vaidyanathaswamy, Head of the Mathematics Department for his invaluable help in the preparation of this paper.

[7] For a consideration of these types of congruences and homomorphisms in more detail refer to my paper mentioned in footnote 6.

[8] For example, besides connecting each element of M with itself E_5 relates only (a, b, c,) with (b, c,) and (a, b, c, d,) with (b, c, d,).

GLEANING

" Mathematics, like all the other sciences, opens its doors to those only who knock long and hard. No more damaging evidence can be adduced to prove the weakness of character than for one to have aversion to mathematics. For whether one wishes so or not, it is nevertheless true; that to have aversion for mathematics means to have aversion to accurate, painstaking, and persistent hard study, and to have aversion to hard study is to fail to secure a liberal education and thus fail to compete in that fierce and vigorous struggle for the highest and the truest and the best in life which only the strong can hope to secure."

B. F. FINKEL

ON THE ARITHMETIC AND THE GEOMETRIC MEANS FROM A TYPE III POPULATION

BY

S. JANARDANA AIYER, *Department of Statistics, Travancore University.*

1. One of the most popular alternatives to the normal distribution is the Gamma-Function distribution, commonly referred to as the Type III curves in the Pearsonian system and given by

$$P(X) = m^p X^{p-1} e^{-mX} / \Gamma(p). \qquad (0 \leqslant X)$$

The object of the present paper is to study in detail a few properties of the arithmetic and the geometric means from this population.

Let $\bar{X}$ and g be the arithmetic and the geometric means respectively in the sample, $X_1, X_2, \ldots \ldots \ldots X_n$.

The characteristic function of $\bar{X}$ and g is

$$\phi(t_1, t_2) = E\left[e^{it_1\bar{X}} \cdot g^{t_2}\right]$$

$$= \frac{m^{np}}{\Gamma^n(p)} \int \cdots \int e^{it_1\bar{X}} \cdot g^{t_2} \cdot (X_1 \cdots X_n)^{p-1} e^{-m(X_1 + \cdots + X_n)} dX_1 \cdots dX_n.$$

$$= \frac{m^{np}}{\Gamma^n(p)} \left[\int_0^\infty X^{p+\frac{t_2}{n}-1} \cdot e^{-X\left(m-\frac{it_1}{n}\right)} dX\right]^n$$

$$= \frac{\Gamma^n\left(p+\frac{t_2}{n}\right)}{\Gamma^n(p) \cdot m^{t_2}\left(1-\frac{it_1}{mn}\right)^{np+t_2}} \qquad \ldots (1)$$

The simultaneous sampling distribution of $\bar{X}$ and g, $P(g, \bar{X})$, therefore is

$$= \frac{1}{4\pi^2 i \cdot \Gamma^n(p)} \int_{-\infty}^{\infty} dt_1 \cdot \int_{-i\infty}^{i\infty} \frac{e^{-it_1\bar{X}} \cdot g^{-t_2-1}}{m^{t_2}} \cdot \frac{\Gamma^n\left(p+\frac{t_2}{n}\right)}{\left(1-\frac{it_1}{mn}\right)^{np+t_2}} \cdot dt_2$$

$$= \frac{(mn\bar{X})^{np} \cdot e^{-mn\bar{X}}}{(g\bar{X}) \cdot \Gamma^n(p)} \cdot \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \left(\frac{n\bar{X}}{g}\right)^{t_2} \cdot \frac{\Gamma^n\left(p+\frac{t_2}{n}\right)}{\Gamma(np+t_2)} \cdot dt_2$$

$$= \frac{(mn\bar{X})^{np} \cdot e^{-mn\bar{X}}}{(g\bar{X})\ \Gamma^n(p)} \times (I) \qquad \ldots (2)$$

where $\mathrm{I} = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \left(\frac{n\bar{\mathrm{X}}}{g}\right)^{t_2} \cdot \frac{\Gamma^n\left(p + \frac{t_2}{n}\right)}{\Gamma(np+t_2)} \, dt_2$ ... (3)

The integral I cannot be evaluated in finite terms of elementary functions. A solution however exists as an infinite series which may be obtained either by the method of differential equations or of complex integration. The results established, herein, are based upon the values of $p\,(g, \bar{\mathrm{X}})$ and $\phi\,(t_1, t_2)$ given by the relations (2) and (1) respectively.

2. Approximation for P (g, $\bar{\mathrm{X}}$).

Let the size of the sample n, be so large that terms of the order of $(1/n^2)$ may be neglected in comparison with terms of lower orders. From (3)

$$\mathrm{I} = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \left(\frac{n\bar{\mathrm{X}}}{g}\right)^{t_2} \cdot \frac{\Gamma^n\left(p + \frac{t_2}{n}\right)}{\Gamma(np+t_2)} \cdot dt_2$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\frac{n\bar{\mathrm{X}}}{g}\right)^{it} \frac{\Gamma^n\left(p + \frac{it}{n}\right)}{\Gamma(np+it)} \, dt$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{it \log \left(\frac{n\bar{\mathrm{X}}}{g}\right) + n.\log\Gamma\left(p + \frac{it}{n}\right) - \log\Gamma(np+it)} dt$$

The exponential factor within the integral, after expansion in descending powers of n gives,

$$\mathrm{I} = \frac{\Gamma^n(p)\cdot(np)^{np-\frac{1}{2}}.e^{-\frac{1}{12np}+np}}{\sqrt{2\pi}} \times$$

$$exp.\ \frac{1}{2\pi} \int_{-\infty}^{\infty} \Big\{ it\left[\psi\,(p) + \log\frac{n^2\bar{\mathrm{X}}p}{g} + \frac{1}{2np} + \frac{1}{12n^2p^2}\right]$$

$$+ \frac{(it)^2}{n}\left[\frac{\psi'\,(p)}{2\,!} - \frac{1}{2p} - \frac{1}{4np^2}\right]$$

$$+ \frac{(it)^3}{n^2}\left[\frac{\psi''\,(p)}{3\,!} + \frac{1}{6p^2}\right] + \mathrm{O}\left(\frac{1}{n^3}\right)\Big\} dt \qquad \text{... (4)}$$

where $\psi\,(p) = (d/dp)\log\Gamma(p)$

$\psi'\,(p) = (d/dp)\,[\psi\,(p)]$, and so on, If n is so large that terms of order $(1/n^2)$, may be neglected, then (4) gives,

$$I = \frac{e^{-\frac{1}{12np}+np} \cdot \Gamma^n(p) \cdot (np)^{np-\frac{1}{2}}}{(2\pi)^{3/2}} \int_{-\infty}^{\infty} e^{it\theta - \frac{t^2\phi}{n}} dt.$$

$$= \frac{e^{-\frac{1}{12np}+np-\frac{\theta^2 n}{4\phi}} \cdot \Gamma^n(p) \cdot (np)^{np-\frac{1}{2}}}{(2\pi)\left(\frac{n}{\phi}\right)^{\frac{1}{2}}} \quad \ldots \quad (5)$$

where $\theta = \psi(p) + \log\left(\frac{n^2\bar{X}p}{g}\right) + \frac{1}{2np}$ ... (6)

and $\phi = \frac{1}{2}\left[\psi'(p) - \frac{1}{p}\right]$ ... (7)

From (2) and (5), therefore, we may write,

$$P(g, \bar{X}) = \frac{(mn^2p)^{np} \cdot e^{np - \frac{1}{12np}}}{2\pi \cdot (p\phi)^{\frac{1}{2}}} \frac{\bar{X}^{np-1}}{g} \cdot e^{-mn\bar{X} - \frac{\theta^2 n}{4\phi}} \times \left[1 + O\left(\frac{1}{n^2}\right)\right] \ldots \quad (8)$$

so that for large values of n,

$$P(g, \bar{X}) \sim \frac{(mn^2p)^{np} \; e^{np - \frac{1}{12np}}}{2\pi \cdot \left[\frac{p\psi'(p) - 1}{2}\right]^{\frac{1}{2}}} \cdot \frac{\bar{X}^{np-1}}{g}$$

$$\times \; e^{-mn\bar{X} - \frac{np}{2[p\psi'(p)-1]}\left[\psi(p) + \log\frac{n^2\bar{X}p}{g} + \frac{1}{2np}\right]^2} \left[\cdots\right] \ldots \quad (9)$$

In problems of testing hypothesis, when a large sample is available and if it may be assumed from a priori considerations that the sample variates might have been drawn possibly from a Type III population, then the above expression for $P(g, \bar{X})$ is helpful in prescribing probability levels for p and m for rejection or acceptance of a hypothesis.

3. Properties of the characteristic function of $\bar{X}$ and g.

Though the form for $P(g, \bar{X})$ is highly complicated, it is possible to study a large number of the properties of $\bar{X}$ and g with the help of the characteristic function, $\phi(t_1, t_2)$. It generates the moment functions and renders easy the evaluation of quantities like the β coefficients, the correlation coefficient etc., which involve them.

$$\phi(t_1, t_2) = E\left(e^{it_1\bar{X}} \cdot g^{t_2}\right)$$

$$= \iint e^{it_1\bar{X}} \cdot g^{t_2} \cdot P(g \cdot \bar{X}) \cdot d\bar{X} \cdot dg.$$

$$= \mu_{0 \cdot t_2} + (it_1)\,\mu_{1 \cdot t_2} + \frac{(it_1)^2}{2!}\mu_{2 \cdot t_2} + \cdots + \frac{(it_1)^j}{j!}\mu_{j \cdot t_2} + \cdots + \text{ad inf.}$$

where $\mu_{j \cdot t_2}$ stands for the moment function given by

$$E(\bar{X}^j \cdot g^{t_2}) = \iint \bar{X}^j \cdot g^{t_2} \cdot P(g \cdot \bar{X}) \cdot dg \cdot d\bar{X} \quad \ldots \quad (10)$$

Also from (1),

$$\phi(t_1, t_2) = \frac{\Gamma^n\left(p + \frac{t_2}{n}\right)}{\Gamma^n(p) \cdot m^{t_2}} \left\{ 1 + (np + t_2)\left(\frac{it_1}{mn}\right) + \frac{(np+t_2)(np+t_2+1)}{2!}\left(\frac{it_1}{mn}\right)^2 + \cdots + \cdots + \frac{(np+t_2)\cdots\cdots(np+t_2+j-1)}{j!}\left(\frac{it_1}{mn}\right) + \cdot \text{ ad inf.} \right\} \quad \ldots \quad (11)$$

Equating coefficients of like powers of t_1, in (10) and (11), it is seen that

$$\mu_{j \cdot t} = \frac{(np+t)(np+t+1)\cdots(np+t+j-1)}{m^{t+j} \cdot n^j} \frac{\Gamma^n\left(p + \frac{t}{n}\right)}{\Gamma^n(p)} \quad \ldots \quad (12)$$

4. The β Coefficients for g.

If m_t stands for the tth moment of the geometric mean about its central value,

$$\beta_1 = m_3^2/m_2^3 \cdot \text{ and } \beta_2 = m_4/m_2^2 \cdot \quad \ldots \quad (13)$$

But,

$$\left.\begin{aligned} m_2 &= \mu_{0\cdot 2} - \mu_{0\cdot 1}^2 \\ m_3 &= \mu_{0\cdot 3} - 3\mu_{0\cdot 1}\,\mu_{0\,2} + 2\mu^3_{0\cdot 1} \\ m_4 &= {}_{0\cdot 4} - 4\mu_{0\cdot 3}\,\mu_{0\cdot 1} + 6\mu_{0\cdot 2}\,\mu^2_{0\cdot 1} - 3\mu_{0\cdot 1}^4 \end{aligned}\right\} \quad \ldots \quad (14)$$

Also from (12),

$$\mu_{0\,t} = \frac{1}{m^t} \frac{\Gamma^n\left(p + \frac{t}{n}\right)}{\Gamma^n(p)} \quad \ldots \quad (15)$$

From (13), (14) and (15), β_1 and β_2, may be calculated, but they are not simple and neat. However, when n is large such that terms of orders $(1/n^2)$ and higher may be neglected, β_1 and β_2 reduce to simpler forms. Then

$$\mu_{0\,t} = \frac{1}{m^t}\, e^{t\psi(p) + \frac{t^2}{2n}\psi'(p)}$$

and

$$\beta_1 = \left[e^{\frac{\psi'(p)}{n}} - 2\right] \cdot \left[e^{\frac{\psi'(p)}{n}} - 1\right] \quad \ldots \quad (16)$$

$$\beta_2 = e^{\frac{4\psi'(p)}{n}} + 2\,e^{\frac{3\psi'(p)}{n}} + 3\,e^{\frac{2\psi'(p)}{n}} - 3 \quad \ldots \quad (17)$$

If n is large enough so that terms of order $(1/n)$ may be neglected, then $\beta_1 = 0$, and $\beta_2 = 3.$, thereby agreeing with the central limit theorem that the sampling distribution of g when the size of the sample is too large is approximately normal.

5. The coefficient of correlation between g and $\bar{X}$.

It has been proved that $\bar{X}$ and g calculated from a Type III population are not independently distributed. The coefficient of correlation r between them may now be obtained. If as before $m_{j,t}$ stands for the corresponding central moment, then

$$r = \frac{m_{1\cdot1}}{\sqrt{m_{0\cdot2}\cdot m_{2\cdot0}}} = \frac{\mu_{1\cdot1}-\mu_{1\cdot0}\cdot\mu_{0\cdot1}}{\sqrt{(\mu_{0\cdot2}-\mu_{0\cdot1}^2)\cdot(\mu_{2\cdot0}-\mu_{1\cdot0}^2)}}$$

$$= \frac{\Gamma^n\left(p+\frac{1}{n}\right)}{(np)^{\frac{1}{2}}\left\{\Gamma^n(p)\cdot\Gamma^n\left(p+\frac{2}{n}\right)-\Gamma^{2n}\left(p+\frac{1}{n}\right)\right\}^{\frac{1}{2}}} \qquad \ldots (18)$$

When n is large, so that as before, terms of the order of $(1/n^2)$ may be neglected then,

$$r \sim \frac{1}{(np)^{\frac{1}{2}}\left\{e^{\frac{\psi'(p)}{n}}-1\right\}^{\frac{1}{2}}} \qquad \ldots (19)$$

If n is so large that terms of order $(1/n)$ may also be neglected then

$$r \sim \frac{1}{\sqrt{p\psi'(p)}} \qquad \ldots (20)$$

6. The regression of $\bar{X}$ on g.

$$\bar{X}_g = \int_0^\infty \bar{X}\cdot P(\bar{X}/g)^* \, d\bar{X}$$

$$= \frac{1}{P(g)}\int_0^\infty \bar{X}P(\bar{X}, g)\cdot d\bar{X}.$$

$$= \frac{(mn)^{np}}{\Gamma^n(p)\cdot g\cdot P(g)}\cdot\frac{1}{2\pi i}\int_0^\infty\int_{-i\infty}^{i\infty}\bar{X}^{np+t}\, e^{-mn\bar{X}}\left(\frac{n}{g}\right)^t\frac{\Gamma^n\left(p+\frac{t}{n}\right)}{\Gamma(np+t)}\,d\bar{X}\cdot dt$$

(applying 2.)

$$= \frac{1}{(mn)\cdot\Gamma^n(p)\cdot g\cdot P(g)}\;\frac{1}{2\pi i}\int_{i\infty}^{i\infty}\frac{(np+t)\cdot\Gamma^n(p+\frac{t}{n})}{(mg)^t}.\,dt \qquad \ldots (21)$$

Now the characteristic function of g is

$$E(g^t) = \frac{1}{m^t}\cdot\frac{\Gamma^n\left(p+\frac{t}{n}\right)}{\Gamma^n(p)}$$

and therefore the distribution $P(g)$ of g is

$$\frac{1}{\Gamma^n(p)\cdot g}\cdot\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}(mg)^{-t}\,\Gamma^n\left(p+\frac{t}{n}\right).\,dt \qquad \ldots (22)$$

From (21) and (22) it may be easily seen that

$$\bar{X}_g = \frac{1}{(mn)}\left[(np-1) - \frac{gP'(g)}{P(g)}\right] \qquad \dots \quad (23)$$

If now the value of P(g) in series, viz.,

$$P(g) = \frac{n \cdot m^{np} \cdot g^{np-1}}{\Gamma(n)\,\Gamma^n(p)} \sum_{r=0}^{\infty} (-1)^{n+nr+1} \left[\frac{d^{n-1}}{dz^{n-1}} \frac{(mg)^{nz}}{\Gamma^n(1+z)}\right]_{z=r}$$

be substituted in (23) we obtain,

$$\bar{X}_g = \frac{1}{m} \frac{\sum_{r=0}^{\infty} (-1)^{n+nr} \left[\frac{d^{n-1}}{dz^{n-1}} \frac{z \cdot (mg)^{nz}}{\Gamma^n(1+z)}\right]_{z=r}}{\sum_{r=0}^{\infty} (-1)^{n+nr+1} \left[\frac{d^{n-1}}{dz^{n-1}} \frac{(mg)^{nz}}{\Gamma^n(1+z)}\right]_{z=r}} \qquad \dots \quad (24)$$

Even in the simple case when $n=2$, or $n=3$, this expression for $\bar{X}_g$ is by no means elementary.

7. The regression of g on $\bar{X}$.

Unlike the regression of $\bar{X}$ on g the regression of g on $\bar{X}$ is linear. The regression of g on $\bar{X}$,

$$G_{\bar{x}} = \int_0^\infty g\, P(g/\bar{X}) \cdot dg.$$

$$= \int_0^\infty g \cdot \frac{P(g \cdot \bar{X})}{P(\bar{X})} \cdot dg$$

$$= \bar{X} \cdot \int_0^1 L \cdot P(L) \cdot dL.$$

where $L = (g/\bar{X})$.

The characteristic function of L,

$$\phi(t) = E(L^t) = \int_0^1 L^t P(L) \cdot dL.$$

$$= \frac{\Gamma(np)\Gamma^n\left(p + \frac{t}{n}\right)}{\Gamma^n(p) \cdot \Gamma(np+t)} \cdot n^t.$$

so that

$$G_{\bar{x}} = \phi(1)\,\bar{X} = \frac{\Gamma^n\left(p + \frac{1}{n}\right)}{p\,\Gamma^n(p)}\,\bar{X} \qquad \dots \quad (25)$$

The same regression equation is obtained by fitting a polynomial in $\bar{X}$ to $G_{\bar{x}}$, the constants being determined by the method of least squares.

In this connection it is interesting to point out that even though the distribution of g and $\bar{X}$ is definitely non-normal the same linear regression equation for g on $\bar{X}$ is obtained by fitting a polynomial equation, the constants being determined by the method of least squares.

INDEFINITE INTEGRATION BY MEANS OF RESIDUES

BY

E. H. NEVILLE

1. If $f(z)$ is a single valued function that is real for real values of z, the evaluation of the integral

$$\int_0^\infty f(x^2)\, dx$$

as a sum of residues by means of a semicircular contour is a process which we are all taught. Why are we left to discover for ourselves the far less obvious process for the evaluation of the integral of the same function with an arbitrary positive real lower limit c?

The integrand by which the evaluation is effected

$$f(z^2) \log \frac{c+z}{c-z}.$$

The contour, before the limit is taken, is a semicircular one, whose base extends along the imaginary axis from $i\mathrm{R}$ to $-i\mathrm{R}$ and whose central radius, along the positive half of the real axis, is slit from c to R. Within the region bounded by this contour, the logarithm is single valued, and its imaginary part may be taken to be between $-\pi$ and π; with this convention, the logarithm is real for real values of z between 0 and c, and the logarithms corresponding to conjugate values of z, themselves have conjugate values. The extreme values of the imaginary part of the logarithm are attained on the edges of the cut, and the sum of the integrals inwards along the lower edge of the cut and outwards along the upper edge is

$$\int_{\mathrm{R}}^{c} f(x^2) \left\{ \log \frac{c+x}{c-x} - i\pi \right\} dx + \int_c^{\mathrm{R}} f(x^2) \left\{ \log \frac{c+x}{c-x} + i\pi \right\} dx,$$

that is,

$$2\pi i \int_c^{\mathrm{R}} f(x^2) dx.$$

On the imaginary axis, the logarithm is imaginary and $f(-y^2)$ by hypothesis is real, and therefore the integral is real. It follows that if the integral along the semicircle tends to zero as $\mathrm{R}\to\infty$, and if $f(z^2)$ has no poles on the imaginary axis, then

The value of

$$\int_c^\infty f(x^2) dx$$

is the sum of the real parts of the residues of the function

$$f(z^2)\log\frac{c+z}{c-z}$$

at those poles of $f(z^2)$ whose real parts are positive.

2. We consider first the adaptation of this result when the poles of $f(z^2)$ are all simple, for if the residue of $f(z^2)$ at a simple pole a is A, the residue there of $f(z^2)\log\{(c+z)/(c-z)\}$ is

$$A\log\{(c+a)/(c-a)\}.$$

(i) If the pole a is on the real axis between 0 and c, the residue A is real, and the logarithm has its real value.

For example,

$$\int_c^\infty \frac{dx}{x^2-a^2} = \frac{1}{2a}\log\frac{c+a}{c-a},$$

without decomposition into partial fractions, the denominator $2a$ arising simply as the value when $z=a$ of the derivative of z^2-a^2.

(ii) If a has the complex value $a'+ia''$, with $a''>0$, and if the residue A is $A'+iA''$, the conjugate point $\bar{a}$, that is, $a'-ia''$, also is a pole, and the residue at $\bar{a}$ is $A'-iA''$. The logarithms of $(c+a)/(c-a)$, $(c+\bar{a})/(c-\bar{a})$ are conjugate numbers $\mu'\pm i\mu''$. By addition,

$$\mu'=\tfrac{1}{2}\log\frac{(c+a)(c+\bar{a})}{(c-a)(c-\bar{a})}=\tfrac{1}{2}\log\frac{c^2+|a|^2+2ca'}{c^2+|a|^2-2ca'}$$

$$=\operatorname{arg\,tanh}\frac{2ca'}{c^2+|a|^2};$$

and μ'' is the angle, between 0 and π, from the radius joining a to to the radius joining $-c$ to a; hence $\pi-\mu''$ is the angle at a in the triangle whose vertices are a and $\pm c$, and since the area of this triangle is ca'',

$$\tan\mu''=-\frac{4ca''}{|c+a|^2+|c-a|^2-4c^2}=\frac{2ca''}{c^2-|a|^2}.$$

Thus the sum of the residues of $f(z^2)\log\{(c+z)(c-z)\}$ at a and $\bar{a}$, which is $2A'\mu'-2A''\mu''$, is explicitly

$$2A'\operatorname{arg\,tanh}\frac{2ca'}{c^2+|a|^2}-2A''\arctan\frac{2ca''}{c^2-|a|^2},$$

with the inverse tangent between 0 and π.

For example, the function $1/(z^4+1)$ has only one pole in the first quadrant, and if this pole is a, the residue there is $1/4a^3$, that is $-a/4$; since $a=(1+i)/\sqrt{2}$, we have

$$2\sqrt{2}\int_c^\infty\frac{dx}{x^4+1}=\arctan\frac{c\sqrt{2}}{c^2-1}-\operatorname{arg\,tanh}\frac{c\sqrt{2}}{c^2+1}.$$

If $f(z^2)=(z^2+1)/(z^4-2z^2\cos 2\alpha+1)$, with $0<\alpha<\frac{1}{2}\pi$, the only relevant pole is cis α, and

$$A=\frac{a^2+1}{4a(a^2-\cos 2\alpha)}=\frac{2a\cos\alpha}{4ia\sin 2\alpha}=-\frac{i}{4\sin\alpha};$$

hence

$$\int_c^\infty \frac{(x^2+1)dx}{x^4-2x^2\cos 2\alpha+1}=\frac{1}{2\sin\alpha}\text{arc tan}\frac{2c\sin\alpha}{3c^2-1}.$$

(iii) A pole on the imaginary axis can be avoided by an indent, and since the value of the integral along the evading semicircle tends in the limit to half the value round a complete circumference, the residue contributes to the integral to be evaluated exactly as if the pole was literally cut in half by the axis. If $a''>0$, the logarithm has the value $i\,\{\pi-2$ arc tan $(c/a'')\}$, that is, $2i$ arc tan (a''/c), with the inverse tangent between 0 and $\frac{1}{2}\pi$; this agrees with the general result in (ii) above, for if a' is zero, tan μ'' becomes $2ca''/(c^2-a''^2)$.

Remembering that the residue is to be halved, we derive from the pair of poles $\pm ia''$ a term $-2A''$ arc tan (a''/c) for the ultimate integral. The elementary formula

$$\int_c^\infty \frac{dx}{x^2+a^2}=\frac{1}{a}\text{arc tan}\frac{a}{c}$$

is included here.

Collecting the results, we have, if the poles of $f(z^2)$ are all simple and if the semicircular integral of $f(z^2)\log\{(c+z)/(c-z)\}$ tends to zero,

$$\frac{1}{2}\int_c^\infty f(x^2)dx=\sum\left\{A'\text{ arc tanh}\frac{2ca'}{c^2+|a|^2}-A''\text{ arc tan}\frac{2ca''}{c^2-|a|^2}\right\}$$

$$+\sum_r A'\text{ arg tanh}\frac{a'}{c}-\sum_i A''\text{ arc tan}\frac{a''}{c},$$

where the sum $\sum$ extends to the poles of $f(z^2)$ which are strictly inside the first quadrant of the plane, the sum $\sum_r$ to the poles on the real axis between 0 and c, and the sum $\sum_i$ to the poles on the positive half of the imaginary axis; the inverse tangents are all between 0 and π.

3. If a pole a is not simple, there is no change in principle, but the residue required is no longer merely a multiple of the

residue of $f(z^2)$. Near $z=a$ we have

$$\log\frac{c+z}{c-z}=\log\frac{c+a}{c-a}+\left\{\frac{1}{(c-a)}+\frac{1}{(c+a)}\right\}(z-a)+$$

$$\left\{\frac{1}{(c-a)^2}-\frac{1}{(c-a)^2}\right\}\frac{(z-a)^2}{2}+\left\{\frac{1}{(c-a)^3}+\frac{1}{(c+a)^3}\right\}\frac{(z-a)^3}{3}+\ldots,$$

and if the principal part of $f(z^2)$ in this neighbourhood is

$$\frac{A_r}{(z-a)^r}+\frac{A_{r-1}}{(z-a)^{r-1}}+\cdots+\frac{A_2}{(z-a)^2}+\frac{A_1}{(z-a)},$$

the residue of $f(z^2)\log\{(c+z)/(c-z)\}$ is

$$A_1\log\frac{(c+a)}{(c-a)}+A_2\frac{(a+c)-(a-c)}{(c^2-a^2)}+A_3\frac{(a+c)^2-(a-c)^2}{2(c^2-a^2)^2}$$

$$+\cdots+A_r\frac{(a+c)^{r-1}-(a-c)^{r-1}}{(r-1)(c^2-a^2)^{r-1}}.$$

Combination of residues from conjugate poles presents no fresh problems, for the additional terms are all rational functions.

We must not overlook the one new possibility, of a pole at the origin. The principal part must be of the form

$$\frac{A'_{2k}}{z^{2k}}+\frac{A'_{2k-2}}{z^{2k-2}}+\cdots+\frac{A'_4}{z^4}+\frac{A'_2}{z^2},$$

where the numerators are all real; near $z=0$,

$$\log\frac{c+z}{c-z}=2\left\{\frac{z}{c}+\frac{z^3}{3c^3}+\frac{z^5}{5c^5}+\cdots\right\}.$$

Thus the half-residue required is

$$\frac{A'_2}{c}+\frac{A'_4}{3c^3}+\cdots+\frac{A'_{2k-1}}{(2k-1)c^{2k-1}}.$$

For an example with multiple poles, take $f(z^2)=1/Z^2$, where $Z=z^4-2z^2\cos 2\alpha+1$. Writing $a=\operatorname{cis}\alpha$, $z-a=t$, we have

$$Z=\lambda t+\mu t^2+O(t^3),$$

$$f(z^2)=\frac{1}{\lambda^2t^2}\left\{1-\frac{2\mu}{\lambda}t\right\}+O(1),$$

where

$$\lambda=4a^3-4a\cos 2\alpha=4ia\sin 2\alpha,\quad \mu=6a^2-2\cos 2\alpha;$$

the residue of $f(z^2)\log\{(c+z)/(c-z)\}$ is

$$A_1\log\frac{c+a}{c-a}+\frac{2A_2c}{c^2-a^2},$$

where

$$A_1 = -\frac{2\mu}{\lambda^3} = \frac{3a^2 - \cos 2\alpha}{16ia^3 \sin^3 2\alpha}, \quad A_2 = \frac{1}{\lambda^2} = -\frac{1}{16a^2 \sin^2 2\alpha}$$

Using the indentities

$$\cos 3\alpha = \cos \alpha\ (2\cos 2\alpha - 1),\ \sin 3\alpha = \sin \alpha\ (2 \cos 2\alpha + 1),$$

we have

$$A_1 = -\frac{i}{16 \sin^3 2\alpha}\left\{\cos \alpha(3 + \cos 2\alpha - 2\cos^2 2\alpha) - i \sin \alpha(3 - \cos 2\alpha - 2\cos^2 2\alpha)\right\}$$

$$= \frac{3 + 2\cos 2\alpha}{64 \cos^3 \alpha} - i\frac{3 - 2\cos 2\alpha}{64 \sin^3 \alpha}$$

$$= \frac{1 + 4\cos^3 \alpha}{64 \cos^2 \alpha} - i\frac{1 + 4\sin^2 \alpha}{64 \sin^3 \alpha},$$

and since

$$2\mathrm{R}l\, \frac{2A_2 c}{c^2 - a^2} = -\mathrm{R}l\, \frac{c\bar{a}^2(c^2 - \bar{a}^2)}{4(c^2 - a^2)(c^2 - \bar{a}^2)\sin^2 2\alpha},$$

we have finally

$$\int_c^\infty \frac{dx}{(x^4 - 2x^2 \cos 2\alpha + 1)^2} = \frac{1 + 4\cos^2 \alpha}{32 \cos^3 \alpha} \arg\tanh \frac{2c \cos \alpha}{c^2 + 1}$$

$$+ \frac{1 + 4\sin^2 \alpha}{32 \sin^3 \alpha} \arctan \frac{2c \sin \alpha}{c^2 - 1} - \frac{c(c^2 \cos 2\alpha - \cos 4\alpha)}{4(c^4 - 2c^2 \cos 2\alpha + 1)\sin^2 2\alpha}$$

4. If the lower limit c of the integral does not occur parametrically in $f(z^2)$, the function obtained, regarded as a function of c, is simply the negative of an indefinite integral. The particular results we have given can all be obtained otherwise: see for instance Bromwich's *Elementary Integrals* (1911), p. 15, Ex. 34, where the elegant parallel use of inverse hyperbolic and circular tangents is introduced. It is true also that the evaluation of the principal part of any function near a pole is fundamentally the same operation as the determination of a group of partial fractions. Ultimately our work is equivalent to the familiar elementary processes, but it is none the less true that without burdening the memory we do avoid a vast amount of algebraic detail.

5. Professor Hardy prefers, he tells me, to approach these evaluations differerently. Since

$$\int_c^\infty f(x)dx = \int_0^\infty f(x + c)dx,$$

the generality in the lower limit is somewhat spurious, and theoretically there is only the one problem, of evaluation from 0 to ∞, to solve. For this purpose we multiply the complex integrand $f(z)$ by a logarithm and slit the plane along the positive half of the real axis; there is no restriction on $f(z)$ to be even. In the slit plane, no branch of $\log z$ assumes conjugate values for conjugate values of z, but $\log(-z)$ can be taken to be real for negative real values of z, and then has its imaginary part negative or positive according as the imaginary part of z is positive or negative; we therefore use the integrand $f(z)\log(-z)$. The contour integral consists in the first place of the sum of

$$-2\pi i \int_0^R f(x)dx$$

and an integral round the whole circumference of $|z| = R$, and therefore if the latter integral tends to zero, then
the value of

$$\int_0^\infty f(x)dx$$

is the negative of the sum of the residues of the function $f(z)\log(-z)$ at all the poles of $f(z)$.

For example, for arbitrary complex values of p and q, provided that $q \neq 0$ and that $(z+p)^2 - q^2$ has no real zero that is not strictly negative,

$$\int_0^\infty \frac{dx}{(x+p)^2+q^2} = \frac{1}{2q}\left\{\log(p+q) - \log(p-q)\right\},$$

where each logarithm has its imaginary part between $-\pi$ and π. Here we are not using partial fractions, but determining the residue of $\log(-z)/\{(z+p)^2 - q^2\}$ when $z + p = \pm q$ as the value of

$$\log(-z)/\{2(z+p)\}.$$

If $f(z)$ is real for real values of z, the required integral is real and the imaginary parts of residues are irrelevant. At a negative real pole of $f(z)$, the residue of $f(z)\log(-z)$ is real; if $f(z)$ has a complex pole a, then $f(z)$ has the conjugate point $\bar{a}$ also for a pole, and the sum of the residues of $f(z)\log(-z)$ at a and $\bar{a}$ is twice the real part of the residue at one of these two points.

Thus if p and q are real with $p > q > 0$, then

$$\int_0^\infty \frac{dx}{(x+p)^2 - q^2} = \frac{1}{2q}\log\frac{p+q}{p-q} = \frac{1}{q}\ \text{arg tanh}\,\frac{q}{p}.$$

Near $z = -(p-q)$,

$$\frac{1}{(z+p+q)^2} = \frac{1}{\{2q+(z+p-q)\}^2} = \frac{1}{4q^2} - \frac{z+p-q}{4q^3} + O(z+p-q)^2,$$

$$\log(-z) = \log\{(p-q)-(z+p-q)\} = \log(p-q) - \frac{z+p-q}{p-q} + O(z+p-q)^2;$$

hence the residue of $\log(-z)/\{(z+p)^2-q^2\}^2$ at $-p+q$ is

$$-\frac{1}{4q^3}\log(p-q) - \frac{1}{4q^2(p-q)},$$

and therefore

$$\int_0^\infty \frac{dx}{\{(x+p)^2-q^2\}^2} = \frac{1}{4q^3}\log(p-q) + \frac{1}{4q^2(p-q)} - \frac{1}{4q^3}\log(p+q) + \frac{1}{4q^2(p+q)}$$

$$= \frac{p}{2q^2(p^2-q^2)} - \frac{1}{2q^3}\arg\tanh\frac{q}{p}.$$

We can deduce this last formula from the preceding formula by differentiation with respect to q, but since the range of integration is infinite the deduction is not quite trivial.

If p and q are both real and positive, the residue of $\log(-z)/\{(z+p)^2+q^2\}$ at $-p+iq$ is the value there of $\log(-z)/\{2(z+p)\}$, and therefore the real part of this residue is $(1/2q)\,\mathrm{I}m\{\log(p-iq)\}$; hence

$$\int_c^\infty \frac{dx}{(x+p)^2+q^2} = \frac{1}{q}\arctan\frac{q}{p}.$$

Since the residue of $\log(-z)/\{(z+p)^2+q^2\}^2$ at this same point is

$$\frac{1}{4iq^3}\log(p-iq) + \frac{1}{4q^2(p-iq)},$$

we have

$$\int_0^\infty \frac{dx}{\{(x+p)^2+q^2\}^2} = -2\mathrm{I}m\left\{-\frac{1}{4q^3}\log(p-iq)\right\} - 2\mathrm{R}l\,\frac{1}{4q^2(p-iq)}$$

$$= \frac{1}{2q^3}\arctan\frac{q}{p} - \frac{p}{2q^2(p^2+q^2)},$$

again a result obtainable by differentiation.

6. On one aspect of this work I am in complete agreement with Professor Hardy. The initial restriction to even functions and a semicircular contour is a mistake, for the process is not in any sense an extension of the simple process of evaluating the integral of $f(x)$ from $-\infty$ to $+\infty$ by integrating the function $f(z)$ itself round a contour. But I am reluctant to follow Professor Hardy in removing the parameter c from the limit of integration to the integrand, for this transformation obscures, if it does not conceal, the essential character of the result, the expression of an indefinite integral by the help of a complete contour. In the examples in §5, we need only substitute $c+p$ for p in the evaluated result to obtain the value of the integral from c to ∞, but this looks too much like

an accidental advantage of the form in which the integrands have been taken; a quadratic factor $x^2+2x\cos\alpha+1$ would not lend itself at all naturally to the same use.

Let us then slit the real axis as before from c to $+\infty$, but let us take for contour the complete circumference of $|z|=R$ together with the real axis described from R to c and back again; let the integrand now be $f(z)\log(c-z)$, when $f(z)$ is not restricted to be even and $\log(c-z)$ has real values for real values of z between $-\infty$ and c. The contribution of the real axis to the contour integral is

$$-2\pi i\int_c^R f(x)\,dx,$$

and therefore if the contribution of the circumference tends to zero, *the value of*

$$\int_c^\infty f(x)\,dx$$

is the negative of the sum of the residues of $f(z)\log(c-z)$ *at all the poles of* $f(z)$.

If $f(z)$ is real for real values of z, the residue of $f(z)\log(c-z)$ at a real pole is real, and complex poles occur in conjugate pairs with conjugate residues. Hence for example

$$\int_c^\infty \frac{dx}{x^2-2x\cos\alpha+1} = -2Rl\,\frac{\log(c-\operatorname{cis}\alpha)}{2(\operatorname{cis}\alpha-\cos\alpha)}$$

$$= -\frac{1}{\sin\alpha}\,Im\log(c-\operatorname{cis}\alpha)$$

$$= \frac{1}{\sin\alpha}\arctan\frac{\sin\alpha}{c-\cos\alpha}.$$

If $f(z)$ is real for imaginary values as well as for real values of z, then $f(z)$ is an even function, and the residue of $f(z)\log(c-z)$ at $-a$ is the negative of the residue of $f(z)\log(c+z)$ at a. Thus the sum of the negatives of the residues of $f(z)\log(c-z)$ at a and $-a$ is the residue of the one function

$$f(z)\{\log(c+z)-\log(c-z)\}$$

at the one pole a, and we are brought back to §1.

7. Obsession with even functions suggests a queer development. Any function $f(z)$ can be written as $f_1(z^2)+2zg_1(z^2)$, where

$$2f_1(z^2)=f(z)+f(-z),\quad 4zg_1(z^2)=f(z)-f(-z),$$

and making the obvious change in the independent variable in one integral we have

$$\int_c^\infty f(x)\,dx = \int_c^\infty f_1(x^2)\,dx + \int_{c^2}^\infty g_1(x)\,dx.$$

Repeating the dissection we have, for any value of n.

$$\int_c^\infty f(x)\,dx = \int_{c_0}^\infty f_1(x^2)\,dx + \int_{c_1}^\infty f_2(x^2)\,dx + \cdots +$$

$$\int_{c_{n-1}}^\infty f_n(x^2)\,dx + \int_{c_n}^\infty g_n(x)\,dx,$$

where $c_r = c^{2^r}$. To put the matter differently, we can express the arbitrary function $f(z)$ in the form

$$f_1(z^2) + 2zf_2(z^4) + 2^2 z^3 f_3(z^8) + \cdots + 2^{r-1} z^{\frac{1}{2}N-1} f_r(z^N) + 2^r z^{N-1} g_r(z^N),$$

where $N = 2^r$, and the integral of each term except the last is in effect the integral of an even function.

Whether any important applications can be made of these expansions remains to be seen, but an example shows that they are not altogether barren. Let

$$f(x, \alpha) = \frac{1}{x^2 - 2x\cos\alpha + 1} = \frac{x^2+1}{x^4 - 2x^2\cos 2\alpha + 1} + \frac{2x\cos\alpha}{x^4 - 2x^2\cos 2\alpha + 1}.$$

Then

$$\int_c^\infty f(x, \alpha)\,dx = \frac{\phi(c, \alpha)}{2\sin\alpha} + \cos\alpha \int_{c^2}^\infty f(x, 2\alpha)\,dx,$$

where

$$\phi(c, \alpha) = \text{arc tan}\,\frac{2c\sin\alpha}{c^2-1}, \quad 0 < \phi(c, \alpha) < \pi;$$

that is,

$$\sin\alpha \int_c^\infty f(x, \alpha)\,dx = 2^{-1}\phi(c_0, \alpha) + 2^{-1}\sin 2\alpha \int_{c_1}^\infty f(x, 2\alpha)\,dx,$$

if c_r denotes as before c^{2^r}. Repeating, we have

$$\sin\alpha \int_c^\infty f(x, \alpha)\,dx = 2^{-1}\phi(c_0, \alpha) + 2^{-2}\phi(c_1, 2\alpha) + \cdots + 2^{-r}\phi(c_{n-1}, 2^{r-1}\alpha)$$

$$+ 2^{-r}\sin 2^r\alpha \int_{c_r}^\infty f(x, 2^r\alpha)\,dx.$$

The integral in the last term is bounded, and therefore the function on the left is given by the infinite series $\sum 2^{-r}\phi(c_{r-1}, 2^{r-1}\alpha)$.

We have in fact seen in §6 that

$$\sin\alpha \int_c^\infty f(x, \alpha) = \text{arc tan}\,\frac{\sin\alpha}{c - \cos\alpha}.$$

Thus we have come on an expansion

$$\arctan\frac{\sin\alpha}{c-\cos\alpha}=\frac{1}{2}\arctan\frac{2c\sin\alpha}{c^2-1}+\frac{1}{2^2}\arctan\frac{2c^2\sin 2\alpha}{c^4-1}+\frac{1}{2^3}\arctan\frac{2c^4\sin 4\alpha}{c^8-1}+\cdots,$$

easy to verify from the elementary identities

$$\arctan\frac{\sin\alpha}{c-\cos\alpha}+\arctan\frac{\sin\alpha}{c+\cos\alpha}=\arctan\frac{2c\sin\alpha}{c^2-1},$$

$$\arctan\frac{\sin\alpha}{c-\cos\alpha}-\arctan\frac{\sin\alpha}{c+\cos\alpha}=\arctan\frac{\sin 2\alpha}{c^2-\cos 2\alpha},$$

but by no means obvious until we have been put on a track leading to it.

CONGRUENCE PROPERTIES OF $\sigma(n)$.

BY

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§1. Let $a_1, a_2, a_3, \ldots, a_k$; $k=\phi(j)$; be the set of positive integers less than and prime to j. Then, if $w_i \geqslant 1$ be the least prime such that $w_i \equiv a_i$ (mod. j), the set of integers $w_1, w_2, w_3, \ldots, w_k$ shall be spoken of as the "prime residue set" modulo j.

If $m=\Pi p^\beta$, where p's are different primes and β's integers > 0, then the number M obtained by replacing each p on the right side by its least prime residue modulo j, shall be called the replica of m modulo j.

In what follows, w's shall denote special primes, p's any primes, and q's quadratfrei integers. All other letters shall denote positive integers unless otherwise stated.

The following simple Lemmas regarding $\sigma(n)$ – the sum of the divisors of n – are easily proved.

LEMMA 1. $\sigma(p) \mid \sigma(p^\beta)$, if and only if β is odd.

LEMMA 2. If $q_1 \mid q$, then $\sigma(q_1) \mid \sigma(q)$.

LEMMA 3. If M be the replica of m modulo j,

then $m \equiv \mathrm{M}$ (mod. j),

and $\sigma(m) \equiv \sigma(\mathrm{M})$ (mod. j).

§2. Ramanathan* has recently proved that for every $n > 0$,

$$\sigma(jn-1) \equiv 0 \pmod{j} \qquad \dots \quad (1)$$

when $j = 3, 4, 6, 8, 12$ or 24. He further states that these *appear* to be the only moduli for which (1) holds.

The object of this note is to prove the

THEOREM: $\sigma(jn-1) \equiv 0 \pmod{j}$, $j \geqslant 3$,
for every, $n > 0$, if and only if
$$a^2 \equiv 1 \pmod{j}$$
for every a (less than and) prime to j. ... (2)

It can be easily shown that 3, 4, 6, 8, 12 and 24 are the only values of $j \geqslant 3$ for which (2) holds. Our theorem would therefore remove the doubt expressed in Ramanathan's statement.

§3. *Proof of the Theorem.*

Let j be any one of the integers: 3, 4, 6, 8, 12 and 24; and let $m = jn - 1$. Then, if M be the replica of m modulo j; we have $\sigma(m) \equiv \sigma(M) \pmod{j}$.

Let $M = g^2 q$, then from Lemma 1, it follows that
$\sigma(q) \mid \sigma(M)$.

Hence $j \mid \sigma(m)$ if $j \mid \sigma(q)$.

Since $M \equiv m \equiv -1 \pmod{j}$

while $g^2 \equiv 1 \pmod{j}$, we must have
$$q \equiv -1 \pmod{j}. \qquad \dots \quad (3)$$

The number of q's, satisfying (3), which can be formed with the members: $w_1, w_2, w_3, \dots, w_k$ of the complete prime residue set modulo j, is limited, and it can be easily verified that for every such q, $j \mid \sigma(q)$.

Hence $j \mid \sigma(m)$, i.e., $\sigma(jn-1) \equiv 0 \pmod{j}$; $j = 3, 4, 6, 8, 12, 24$.

[Thus when $j = 24$, the "prime residue set" modulo j, is
1, 5, 7, 11, 13, 17, 19, 23.

Since $\sigma(5) = 6$, $\sigma(7) = 8$, $\sigma(11) = 12$, $\sigma(13) = 14$, $\sigma(17) = 18$, $\sigma(19) = 20$ and $\sigma(23) = 24$.

Therefore $24 \mid \sigma(q)$, except when $q = 1, 5, 7, 11, 13, 17, 19, 65, 85, 91, 133, 221$ or 247; and none of these is congruent to $-1 \pmod{24}$.

* *Mathematics Student*, 11 (1943), 33-35.

Hence $\sigma(24n-1)\equiv 0 \pmod{24}$ for every $n > 0$.]

Conversely, let j be any positive integer, such that

$$\sigma(jn-1)\equiv 0 \pmod{j} \text{ for every } n > 0,$$

then $a^2\equiv 1 \pmod{j}$, for every a less than and prime to j.

If $a_1, a_2, a_3, \ldots, a_k$, $k=\phi(j)$ be the set of positive integers less than and prime to j, arranged in ascending order of magnitude, and for some $i < k$,

$$a_i^2 \not\equiv 1 \pmod{j}.$$

then, without loss of generality, we may suppose that $i \leqslant k/2$.

Let $a_i\, a_v\equiv 1 \pmod{j}$, $v\neq i$;

then $a_i\, a_u\equiv -1 \pmod{j}$, $u=k-v+1$.

Let p_i and p_u be any primes such that

$$p_i\equiv a_i \pmod{j}, \text{ and } p_u\equiv a_u \pmod{j}.$$

Then $p_i\, p_u\equiv -1 \pmod{j}$, while

$$\begin{aligned}\sigma(p_i\, p_u) &= (1+p_i)(1+p_u),\\ &\equiv a_i + a_u \pmod{j},\\ &\equiv a_i - a_v \pmod{j},\\ &\not\equiv 0 \pmod{j};\end{aligned}$$

for if $u \leqslant \frac{k}{2}$, then $0 < a_i + a_u < j$;

and if $u > \frac{k}{2}$, then $0 < |a_i - a_v| < \frac{j}{2}$.

The theorem is thus completely proved.

§4. The Lemmas of §1 hold good even when σ is replaced by σ_h, where $\sigma_h(n)$ denotes the sum of the h-th powers of the divisors of n.

We proceed to obtain conditions for the existence of a congruence relation of the type

$$\sigma_h(jn+l)\equiv 0 \pmod{j} \qquad \text{(A)}$$

true for every $n \geqslant 0$, a given $j > 1$, and an $l > 1$ prime to j.

If $1, w_1, w_2, w_3, \ldots, w_{k-1}$; $k=\phi(j)$; be the complete prime residue set modulo j, and

$$l\equiv w_i \pmod{j},$$

then in order that we may have

$$\sigma_h(w_i) \equiv 0 \pmod{j},$$

we must have

$$l^h \equiv w_i^h \equiv -1 \pmod{j}. \qquad (1)$$

Again let $l \equiv w_\alpha\, w_\beta \pmod{j}$, (B)

then since $\sigma_h(w_\alpha\, w_\beta) = (1 + w_\alpha^h)(1 + w_\beta^h)$,

$$\equiv (1 + w_\alpha^h + w_\beta^h + l^h) \pmod{j},$$

$$\equiv w_\alpha^h + w_\beta^h \pmod{j};$$

we must have also

$$w_\alpha^h \equiv -w_\beta^h \pmod{j}.$$

Therefore $\qquad w_\alpha^{2h} \equiv -(w_\alpha\, w_\beta)^h \equiv 1 \pmod{j}.$

Giving different values to α in (B), we get the second necessary condition *viz.*,

For every w, $\qquad w^{2h} \equiv 1 \pmod{j}. \qquad (2)$

If $j = p^\lambda$ J ; (J, p) = 1 ; and $\lambda \geqslant 1$; p an odd prime ; then conditions (1) and (2) require h to be an odd multiple of $\frac{1}{2}\,\phi(p^\lambda)$. (3)

If $j = 2^\lambda$ J, where J is odd and $\lambda \geqslant 1$; then

since $\qquad l^h \equiv -1 \pmod{j}$,

$$\equiv -1 \pmod{2^\lambda};$$

while $\qquad w^{2h} \equiv 1 \pmod{2^\lambda}$, for every w ;

we must have h odd ;

$$l \equiv -1 \pmod{2^\lambda} \text{ but } \not\equiv 1 \pmod{j}; \text{ and } \lambda \leqslant 3. \qquad (4)$$

To show that these conditions are sufficient, we write m for $jn + l$.

If M be the replica of m modulo j, and

$$M = g^2\, q,$$

then $\qquad q \neq 1. \qquad (5)$

Let $\qquad q = w_\alpha\, w_\beta\, w_\gamma \ldots w_\rho. \qquad$ (C)

Then $\qquad q^h \equiv l^h \equiv -1 \pmod{j}.$

Since for every w,

$$w^{2h} \equiv 1 \pmod{j};$$

therefore

$$w^h \equiv 1 \text{ or } -1 \pmod{\mu},$$

where μ is j or $\frac{1}{2} j$ according as j is odd or even.

Therefore, of the w's on the right of (C), at least one, say w_α, must satisfy the relation:

$$w^h \equiv -1 \pmod{\mu}.$$

Thus $j \mid \sigma_h(q)$ and therefore also $\sigma_h(m)$, when j is odd.

When j is even, then

$$\sigma_h(q) = (1+w_\alpha^h)(1+w_\beta^h) \cdots (1+w_\rho^h)$$

$$\equiv 0 \pmod{j},$$

provided there are at least two w's on the right of (C),

when $q = w_\alpha \equiv l \pmod{j}$,

$$\sigma_h(q) = 1 + w_\alpha^h \equiv 1 + l^h \equiv 0 \pmod{j}.$$

The conditions obtained are thus seen to be sufficient.

§5. It may be remarked that in view of conditions (3) and (4), congruences of the type (A) can and do exist only when

$$j = 4,\ 8,\ p^\lambda,\ J,\ 2J,\ 4J \text{ or } 8J\,;$$

where p is an odd prime; $\lambda \geqslant 1$; and $J \geqslant 3$ is odd with no prime factor of the form $4t+1$.

Conditions (1) and (2) were recently stated by Ramanathan in a letter to the writer.

Pythagorean Cosmology

At the centre of their cosmos they placed Hestia, the Central Fire, to shed light and warmth on the Sun and the planets. A sceptic who disbelieved in the gods could hardly accept the theological explanation of the invisibility of the Central Fire. To satisfy him, the Pythagoreans invented one of their most ingenious theories. The inhabited regions of the earth, they pointed out, were all on that side of the earth which is always turned away from the centre of its orbit So to view the Central Fire it would be necessary to go beyond India. As not even Pythagoras himself had travelled so far, it was unlikely that any one else would. But suppose someone did. Hestia would still be invisible to him, because between it and the earth, Antichthon would cut off the view. Could not the traveller wait till the Counter-Earth rolled by? He could not; earth and Counter-Earth kept even pace together as they revolved about the Central Fire. Not even the 19th century inventors of the space-filling ether imagined a more satisfying explanation of the impossibility of observing the unobservable.

E. T. BELL in *Scripta Mathematica*, March 1945.

CONGRUENCE PROPERTIES OF $\sigma_a(n)$

BY

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1. If $\sigma_a(n)$ is the sum of the a-th powers of the divisors of the integer n so that $\sigma(n) = \sigma_1(n)$ the sum of the divisors of n, then

(1·1) $\sigma(kn-1) \equiv 0 \pmod{k}$*

for every $n > 0$ and $k = 3, 4, 6, 8, 12$ or 24.

Recently I have been able to obtain more general results concerning $\sigma_a(n)$. The important theorems are:

THEOREM 1. If $k > 2$, $(k, l) = 1$ then a necessary condition that $\sigma_a(km+l) \equiv 0 \pmod{k}$ for every $m > 0$ is $l^a \equiv -1 \pmod{k}$

THEOREM 2. If $k > 2$, $(k, l) = 1$ and $l^a \equiv -1 \pmod{k}$ then a necessary and sufficient condition that $\sigma_a(km+l) \equiv 0 \pmod{k}$ for every $k > 0$ is that $\lambda^{2a} \equiv 1 \pmod{k}$ for every λ prime to k.

If $a = 1$ we deduce in particular that the congruences for $\sigma(n)$ can be only those in (1·1). Mr. Hansraj Gupta in the previous paper does not state the theorem 1 for $a = 1$ which, I think, is necessary.

It will be shown elsewhere that the values of 'a' depend upon k and that k itself cannot be arbitrary.

Examples. (i) $k = 65$. Then

$\sigma_{6a}(65m+l) \equiv 0 \pmod{65}$ for $l = 2, 7, 8, 18, 28, 32, 33, 37, 47, 57, 58$ and 63, a being any odd number.

(ii) $k = 72$. $\sigma_3(72m+l) \equiv 0 \pmod{72}$ for $l = 35, 47, 71$.

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* *The Mathematics Student*. 1943. (P. 33-35).

NOTES AND DISCUSSIONS

A Solution of the General Quartic

The following solution of the quartic would, I believe, interest the readers. The equation of the resolvent cubic is obtained directly here.

The general quartic is of the form:

$$x^4+2ax^3+bx^2+2cx+d=0,$$

or $$x^4+2ax^3=-(bx^2+2cx+d). \qquad \dots (1)$$

Now, for all values of y, we have

$$(x^2+ax+y)^2=x^4+2ax^3+(a^2+2y)x^2+2ayx+y^2. \qquad \dots (2)$$

Hence, making use of (1), we get

$$(x^2+ax+y)^2=(a^2+2y-b)x^2+2(ay-c)x+(y^2-d). \qquad \dots (3)$$

The right side of (3) will be a perfect square if

$$(ay-c)^2=(y^2-d)(2y+a^2-b). \qquad \dots (4)$$

This is the resolvent cubic for the quartic.

Any root y of (4) will make both sides of (3) perfect squares and the solution follows readily.

Consider for example the quartic

$$x^4-10x^3+44x^2-104x+96=0.$$

The resolvent cubic is

$$(-5y+52)^2=(y^2-96)(2y-19),$$

or $$y^3-22y^2+164y-440=0.$$

One root of this is easily seen to be 10.

The equation now takes the form $(x^2-5x+10)^2=(x+2)^2$.

Hence $x=2,\ 4,\ 2\pm 2i\sqrt{2}$.

The case when $a=0$ was considered by Hacke in *The American Math. Monthly*, 48, 1941, 327—8.

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On a simplified form of the Euler-Maclaurin Sum formula

S. S. Pillai has proved by a simple method the following results in vol. VII, p. 70 f.

$$\log(n!) = n \log n - n + \tfrac{1}{2} \log n + C + \theta/3(n-1) \qquad (|\theta|<1)$$

$$\sum_{1}^{n} \frac{1}{r} = \log n + \gamma + \theta/n \qquad (0<\theta<1)$$

It is worth while remembering that these results are included in a theorem which arises out of a consideration of the Euler-Maclaurin summation formula but which can be as simply proved as our two results. The theorem is given in Pólya and Szegö, *Aufgaben und Lehrsätze aus der Analysis*, II Abschn., Nr. 18:

THEOREM. *If $f(x)$ is differentiable for $x \geqslant 1$ and $f'(x)$ is negative, continuous, steadily increasing, and if*

$$s_n = \tfrac{1}{2}f(1) + f(2) + \cdots + f(n-1) + \tfrac{1}{2}f(n) - \int_1^n f(x)\,dx,$$

then

(i) $\lim\limits_{n\to\infty} s_n$ *exists* $= s$ *(say)*,

(ii) $0 < s - s_n < -\tfrac{1}{8}f'(n)$.

Taking successively $f(x) = -\log x,\ 1/x$, we have:

COROLLARY 1.

$$\log n! = (n+\tfrac{1}{2})\log n - n + 1 - s + \varepsilon_n \qquad (0 < \varepsilon_n < 1/8n),$$

$$\gamma_n - \gamma \begin{cases} < \dfrac{1}{2n} \\ > \dfrac{1}{2n} - \dfrac{1}{8n^2} \end{cases} \qquad \left(\gamma_n = \sum_{1}^{n} \frac{1}{r} - \log n\right).$$

COROLLARY 2. *If, in the theorem,* $\lim\limits_{x\to\infty} f(x)$ *is finite, then*

$\sum^{\infty}(s - s_n)$ *is convergent, since* $\sum^{\infty} -f'(n)$ *is so.*

COROLLARY 3. *If, in Corollary 2, we write* $\lim\limits_{x\to\infty} f(x) = f(\infty)$, *and if*

$$t_n = f(1) + f(2) + \cdots + f(n) - \int_1^n f(x)\,dx,$$

then t_n steadily decreases to the limit t (say), and $\sum^{\infty}(t_n - t)$ converges or diverges with $\sum^{\infty}\Big[f(n) - f(\infty)\Big]$.

The last result is proved by observing that

$$s_n = t_n - \tfrac{1}{2}f(1) - \tfrac{1}{2}f(n),$$

and consequently

$$s - s_n = t - t_n + \tfrac{1}{2}\,[f(n) - f(\infty)].$$

Corollary 3 appears in the Question Paper, Pure Mathematics III, Madras B.A. (Hons.), 1936, in the incorrect form: $\sum^{\infty}(t_n - t)$ converges or diverges with $\sum^{\infty} f(n)$. That this is wrong is shown by the case $f(n) = e^{1/n^2}$ in which $f(\infty) = 1$ and $\sum^{\infty} f(n)$ diverges, but $\sum^{\infty}[f(n) - f(\infty)]$ converges and consequently also $\sum^{\infty}(t_n - t)$.

T. K. RAGHAVACHARI

Remarks on Cauchy's Convergence Principle

These remarks are offered with a view to helping the student new to analysis to realize the precise connection between two notes of S. S. Pillai in this journal. The first of the notes is on the definition of oscillation (vol. IX, pp. 165-7) and the second on the sufficiency part of Cauchy's general theorem on convergence (vol. X, p. 91 f.). The proofs given are in a form applicable to any (real) function of a real variable and not only to a function of a positive integral variable. To secure continuity of thought, the argument used in the second of the two notes referred to above, is briefly restated in this form.

CAUCHY'S PRINCIPLE. *If corresponding to any* $\varepsilon > 0$, *there exists a* $\delta(\varepsilon) > 0$ *such that*

(C) $\quad |\phi(x_2) - \phi(x_1)| < \varepsilon$ for $0 < |x_2 - a| < |x_1 - a| \leqslant \delta(\varepsilon)$,

then $\lim_{x \to a} \phi(x)$ *exists.*

PROOF—Any real number ξ must be either 'superior' or 'inferior' or 'intermediate'—superior when $\phi(x) \leqslant \xi$ for every $x \neq a$ in some neighbourhood of a; inferior when $\phi(x) \geqslant \xi$ for every $x \neq a$ in some neighbourhood of a; intermediate when $\phi(x_1) < \xi < \phi(x_2)$ for an $x_1 \neq a$ and an $x_2 \neq a$ in every neighbourhood of a. It can be shown that, in virtue of (C), there is at least one superior number, at least one inferior number, and at most one intermediate number (say, k). Hence there is a section of the real numbers formed with the upper class consisting of superior numbers, the lower class consisting of either inferior numbers *and* the number k, or *only* inferior numbers, according as k, does or does not exist. In either case it is easy to prove that $\lim \phi(x)$ exists and corresponds to the section. In the first case the limit is k and approached both from above and from below; in the second case the limit is approached from one side only.

The proof outlined above establishes at one stroke Cauchy's principle and the complement to the principle which Pillai has given in the form of a positive definition of oscillation. The equivalence of this new definition and the usual negative definition can be stated in the form of a

COROLLARY TO CAUCHY'S PRINCIPLE. *A necessary and sufficient condition for $\phi(x)$ to oscillate (according to the usual definition) as x tends to a is that, there should be constants k_1, k_2, satisfying*

(D) $\phi(x_1)<k_1<k_2<\phi(x_2)$

for an $x_1 \neq a$ and an $x_2 \neq a$ in every neighbourhood of a.

Proof—In the first place, given (D) there are two intermediate numbers k_1, k_2. If $\phi(x)\to -\infty$ or $+\infty$ as $x\to a$, real numbers will be *either* all superior *or* all inferior. If $\phi(x)\to\alpha$ as $x\to a$, real numbers, with the exception of, at the most, one intermediate number, will be *both* superior *and* inferior. Thus (D), with either the divergence of $\phi(x)$ or the convergence of $\phi(x)$, leads to a contradiction. Hence (D) ensures that $\phi(x)$ shall oscillate as $x\to a$.

Next, if $\phi(x)$ oscillates, real numbers cannot be *either* all superior *or* all inferior, since this means $\phi(x)\to\pm\infty$ as $x\to a$; nor can they be *both* superior *and* inferior with, at the most, one exception which is intermediate, for this means $\phi(x)\to\alpha$ as $x\to a$. Hence among the real numbers there must be at least two—k_1, k_2 (say)—which are intermediate. That is, (D) is necessary for the oscillation of $\phi(x)$.

It will be seen that (D) is equivalent to the condition $\overline{\lim}\ \phi(x) > \underline{\lim}\ \phi(x)$ as $x\to a$, just as (C) is equivalent to the condition that $\phi(x)$ is bounded and $\overline{\lim}\ \phi(x)=\underline{\lim}\ \phi(x)$. In examples, however, (D) may be more convenient to use than its equivalent in terms of upper and lower limits. The following is a case in point.

DEDUCTION FROM (D). *If $\phi(x)$ is monotonic in a neighbourhood of $x=a$, which is wholly on one side of a, then $\phi(x)$ cannot oscillate as x approaches a from the side in question.*

Proof—To fix ideas, suppose $\phi(x)$ is monotonic in the open interval $[a, a+h]$ where $h>0$. If $\phi(x)$ oscillates as $x \to a+0$, we can find in $[a, a+h]$, first x_1', x_2', then x_1'', x_2'', satisfying the conditions:

$$\max(x_1'', x_2'') < \min(x_1', x_2'),$$
$$\phi(x_1') < k_1 < k_2 < \phi(x_2'),$$
$$\phi(x_1'') < k_1 < k_2 < \phi(x_2'').$$

Then $\phi(x_1') < \phi(x_2'')$ and $\phi(x_1'') < \phi(x_2')$. Either the first inequality or the second contradicts our hypothesis that $\phi(x)$ is monotonic in $[a, a+h]$, the first when $\phi(x)$ is increasing and the second when $\phi(x)$ is decreasing. Hence $\phi(x)$ cannot oscillate for the contemplated mode of approach of x to a.

Christian College, *Tambaram.* C. T. RAJAGOPAL.

On Stirling's Approximation for $\Gamma(x)$

The object of this note is to give a proof of Stirling's approximation for $\Gamma(x)$. Several methods of deriving this important formula are available,* but the procedure adopted here appears novel. The proof is based on the method of characteristic functions.

Consider a stochastic variable z following the probability law

$$p(z) = [x^x/\Gamma(x)],\ z^{x-1}.\ e^{-xz} \text{ where } x > 0 \text{ and } z > 0.$$

The characteristic function of the distribution of z is

$$\phi\ (it) = \int_0^\infty e^{itz}\ p(z)\ dz = [1 - it/x]^{-x}.$$

$\text{Log } \phi\,(it) = it - t^2/2x + (it)^3/3x^2 \ldots$ x being large.

Hence

$$\phi\,(it) = e^{it - t^2/2x}\ [1 + (it)^3/3x^2 + \cdots]$$

By Fourier Transform,

$$p(z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itz}.\ \phi(it)\ dt$$

$$= \frac{1}{2\pi}\int e^{-itz+it-t^2/2x}\,[1 + (it)^3/3x^2 + \cdots]\ dt$$

$$= \frac{1}{2\pi}\int [1 + (-\mathrm{D})^3/3x^2 + \cdots].\ e^{-itz+it-t^2/2x}\ dt$$

where D denotes the operator d/dz.

$$= [1 - \mathrm{D}^3/3x^2 + \cdots]\ \mathrm{N}(z)$$

where

$$N(z) = \frac{1}{2\pi} \int e^{-it(z-1) - t^2/2x} \, dt$$

$$= (x/2\pi)^{1/2} . \; e^{-x(z-1)^2/2}$$

The process of differentiation after integration is clearly justified by the exponential form of the integrand in $N(z)$.

Hence

$$x^x/\Gamma(x). \; z^{x-1}. \; e^{-zx} = [1 + D^3/3x^2 + \cdots] \; (x/2\pi)^{1/2} . \; e^{-\frac{x(z-1)^2}{2}}$$

$$= [1 + F]. \; N(z)$$

where F is a function in positive integral powers of $(z-1)$. It will be observed that for all values of $z \neq 1$, the right side of the above equation tends to zero for large x. The only exceptional value is $z = 1$ and in this case,

$$e^{-x}. \; x^x/\Gamma(x) \sim (x/2\pi)^{1/2} \qquad \text{for } x \to \infty$$

Thus

$$\Gamma(x) \sim (2\pi)^{1/2} . \; x^{x-1/2} \; e^{-x}$$

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(MISS) ALEYAMMA GEORGE.

Some Asymptotic Values

1. The object of this note is to prove that the integrals

$$\text{(i)} \quad \int_0^\infty e^{-t} \left(1 + \frac{t}{n}\right)^n dt$$

and

$$\text{(ii)} \quad \int_0^n e^{t} \left(1 - \frac{t}{n}\right)^n dt$$

have the same asymptotic value $\sqrt{(n\pi/2)}$. These results are of the same depth as Stirling's asymptotic formula for the Γ-function, the latter being deducible from them.

It is readily seen that the integrand in (i) is monotone decreasing from 1 to 0 as t increases from 0 to ∞. Hence we may change the variable by the substitution

$$e^{-t} \left(1 + \frac{t}{n}\right)^n = 1 - x.$$

As t increases from 0 to ∞, x increases from 0 to 1.

We find $(1-x)dt = \left(\frac{n}{t}+1\right)dx$, and taking logarithms and expanding $\log(1+t/n)$ by Taylor's theorem with a remainder after two terms

$$\log\frac{1}{1-x} = t - n\left[\frac{t}{n} - \frac{t^2}{2n^2(1+\theta t/n)^2}\right] = \frac{nt^2}{2(n+\theta t)^2},$$

where $0 < \theta < 1$. This gives

$$\frac{n}{t}+\theta = \sqrt{\left(\frac{n}{2}\right)}\left(\log\frac{1}{1-x}\right)^{-\frac{1}{2}}$$

whence

$$\int_0^\infty e^{-t}\left(1+\frac{t}{n}\right)^n dt = \int_0^1\left(\frac{n}{t}+1\right)dx$$

$$= \int_0^1\sqrt{\frac{n}{2}}\log\left(\frac{1}{1-x}\right)^{-\frac{1}{2}}dx + \int_0^1(1-\theta)\,dx$$

$$= \sqrt{\frac{n\pi}{2}}+\omega \qquad (0<\omega<1)$$

COROLLARY. Let n be a positive integer. Then the value of the integral is easily seen to be

$$\text{(iii)}\quad 1+1+\left(1-\frac{1}{n}\right)+\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)$$

$$+\cdots+\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{n-1}{n}\right).$$

It follows that the expression (iii) lies between $\sqrt{(n\pi/2)}$ and $\sqrt{(n\pi/2)}+1$.

2. The integral (ii) is treated in the same manner as the integral (i). In fact if we change the variable by the substitution

$$e^t(1-t/n)^n = 1-x$$

we get, as before,

$$(1-x)\,dt = \left(\frac{n}{t}-1\right)dx,$$

$$\log\frac{1}{1-x} = \frac{nt^2}{2(n-\theta' t)^2},$$

$$\frac{n}{t}-\theta' = \sqrt{\left(\frac{n}{2}\right)}\left(\log\frac{1}{1-x}\right)^{-\frac{1}{2}}, \qquad (0<\theta'<1,)$$

whence $$\int_0^n e^t(1-t/n)^n\,dt = \sqrt{\frac{n\pi}{2}}-\omega'$$

where $0<\omega'<1$.

3. We shall now deduce Stirling's asymptotic formula for the Γ-function from the above results.

By changing variables suitably we readily get

$$\Gamma(n+1) = \int_0^\infty e^{-x}\, x^n\, dx = \int_0^n + \int_n^\infty$$

$$= n^n e^{-n} \int_0^n e^t \left(1-\frac{t}{n}\right)^n dt + n^n e^{-n} \int_0^\infty e^{-t} \left(1+\frac{t}{n}\right)^n dt$$

$$= n^n e^{-n} \left(\sqrt{\left(\frac{n\pi}{2}\right)} - \omega' + \sqrt{\left(\frac{n\pi}{2}\right)} + \omega\right)$$

$$= n^n e^{-n} \{\sqrt{(2n\pi)} + \omega - \omega'\}$$

$$= n^n e^{-n} \sqrt{(2n\pi)}\{1 + \eta/\sqrt{(2n\pi)}\}$$

where $\eta = \omega - \omega'$ lies between -1 and $+1$.

Finally it may be remarked that each of the integrals $\int_0^n$ and $\int_n^\infty$ is, in a sense, half of the integral $\int_0^\infty e^{-x}\, x^n\, dx$.

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P. KESAVA MENON.

GLEANING

Then there is the mysterious number 5040 which Plato gives in his Laws as the population of his Ideal City. Any one will recognise 5040 as the total number of different ways of arranging 7 things in a row, say, 7 books on a shelf. The number is 7! Written thus its numerological possibilities are embarrassingly evident. Even the super-sacred 7 occurs, to say nothing of the female 2, the male 3 the just 4 the 5 regular bodies and the perfect 6. Among its other claims to civic attention, 7 is the number of Plato's hills that must be surmounted to attain knowledge and wisdom. In fact 7 " is " actually these hills. But there is infinitely more concealed in this encyclopaedic number. Any Cosmic numerologist will observe that 5040 has exactly 60 divisers, while 60 has exactly 12, and 12 has exactly the perfect 6, and 6 has exactly the Just 4, while 4 has exactly 3, and 3 has exactly the female 2, which has exactly 2 and so on, 2—2—2—......for ever. From these facts it can be shown that the Ideal City is contained in the Nuptial Number and that it recurs eternally once it is firmly (" fourly ") established. The implications of the Zodiacal 12 are too obvious to need mention. The 3 epitomizes the Ideal Family of the City all through the Great Year.

E. T. BELL in *Scripta Mathematica*, March 1945.

ANNOUNCEMENTS AND NEWS

The Fourteenth Conference of the Society will be held in Delhi on December 21—24, at the invitation of the Delhi University. Papers intended for the Conference should be sent to Dr. A. Narasinga Rao, Annamalainagar P. O. along with two brief abstracts. Members of the Society are entitled to read papers or take part in discussions. Others may do so on registering themselves as members of the Conference on payment of Rs. 5 which should be remitted by money order to Dr. A. Narasinga Rao, Treasurer of the Society. Dr. Ram Behari, Head of the Mathematics Department, Delhi University has been appointed Local Secretary and should be addressed on all matters relating to accommodation, program etc.

The following persons have been admitted as members of the Society :—

George Abraham Esq., Lecturer in mathematics, Christian College, Tambaram.

Daljit Singh Esq., Statistical Section, Imperial Agricultural Research Institute, New Delhi.

T. K. Manickavasagam Pillai Esq., M.A., Lecturer, Annamalai University, Annamalainagar.

M. N. Ramakrishna Pillai Esq., B.A., B.L., Accountant, A. G's Office, Trivandrum.

D. W. Kerkar Esq., Professor, S. P. College, Poona.

J. K. Bhattacharya, M. Sc., D. I. P., Tripura State, Bengal.

Prof. D. D. Kosambi has been admitted as a Life Member of the Society on payment of the usual composition fee of Rs. 150.

The Society acknowledges with gratitude a gift of Rs. 450 from the Rockfeller Foundation, distributed through the National Institute of Sciences, Calcutta.

The University of Travancore and the Government of Travancore have been pleased to give a grant of Rs. 500 each as a donation to the Publication Fund of the Society.

A Committee consisting of Prof. F. W. Levi (Chairman), Dr. A. Narasinga Rao, Dr. Ram Behari and Dr. A. N. Singh, Prof. N. R. Sen and M. R. Siddiqi has been appointed to consider the question of a uniform syllabus in Mathematics at the various Universities, and to frame such a syllabus.

We offer our congratulations to Dr. Ram Behari, St. Stephen's College, Delhi, on whom the University of Dublin has conferred the degree of Doctor of Science (Sc. D.) in recognition of his work on the Differential Geometry of Ruled Surfaces.

The Statistical Section of the Imperial Council of Agricultural Research has instituted a 2 years course in Statistics as applied to Agriculture and

Animal Husbandry. The first year's course is elementary and is intended for those desiring to qualify themselves for appointment as statistical assistants in the Departments of Agriculture and Animal Husbandry while the second year's course is more advanced and qualifies for higher posts. The course begins on 26th July and extends up to the end of April each year. Full information may be had from the Statistical Adviser, Imperial Council of Agricultural Research, New Delhi.

Henri Leon Lebesgue, the great French Mathematician (born in 1875), died in 1941. The first statement of his brilliant idea of dividing the range of variation of $f(x)$ instead of the range of x is contained in a note "Sur une generalisation de l' integrale definie" published in *Comptes Rendus* in 1901, while his great *Thèse* with the full account of his work appeared in *Annali di Matematica* of 1902. His great book *Leçons sur l'integration* was published in 1904. A detailed obituary appears in the *Jour. Lond. Math. Soc.*, Jan. 1944.

Sir Arthur Eddington, Plumian Professor of Astronomy at the Cambridge University and well known through his lucid and popular exposition of Relativity died in 1944. A detailed obituary will appear in *The Mathematical Gazette.*

We regret to announce the death at Madras in the third week of July 1945 of N. Durairajan, M.A., B.E., Superintending Engineer, Public Works Department, Madras, an active member of the Society and a keen mathematician. In spite of heavy official work in important executive posts, Mr. Durairajan continued his mathematical investigations and wrote nearly 20 papers for the *Journal* and the *Student* dealing with such topics as the Biparabola, Nets of cubic curves, Complex Geometry, Desmic Tetrahedra, etc. The late V. Ramaswami Ayyar, Founder of the Indian Mathematical Society has named a point connected with a quadrangle, its "Durairajan point" †. (vide J. I. M. S. Question 1711).

† This is the point styled the "Bennet Point" by H. F. Baker in his *Principles of Geometry*, Vol. IV.

GLEANINGS

It is said that the Egyptians, and Lacedaemonians seldom elected any new kings but such as had some knowledge in the mathematics; imagining those who had not, to be men of imperfect judgements, and unfit to rule and govern.

Though Plato's censure that those who did not understand the 117th proposition of the 13th book of Euclid's Elements ought not to be ranked among rational creatures, was unreasonable and unjust, yet to give a man character of universal learning, who is destitute of a competent knowledge in the Mathematics, is no less so. From FRANKLIN: *Usefulness of Mathematics* 1735.

* * * *

There still remain three studies suitable for freemen. Calculation in Arithmetic is one of them; the measurement of length, surface, and depth is the second; and the third has to do with the revolutions of the stars in reference to one another.

From PLATO: *Republic* 350 B. C. (Tr. by Jowett)