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THE CONDITION OF ENGLISH MATHEMATICS FROM 1750 TO 1850

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With the passing of Sir Isaac Newton and the group of scholars associated with him, science in general and mathematics in particular reached a low ebb in England. This period of mathematical and scientific stagnation continued for about 100 years—that is, from about 1750 until approximately 1850 when the efforts of the members of the Analytical Society, formed at Cambridge University, began to be felt. The causes of this slackening of interest in mathematics among the English during this period are many and varied, and their relationships are interlocked to such an extent that it is not always possible to determine the exact part played by each circumstance. However, we shall attempt not only to point out the fundamental causes producing this slackening of interest but also to trace some of the factors which brought about a revival of interest in mathematics during the first half of the nineteenth century.

The first major factor in the deterioration of mathematical scholarship in eighteenth century England was the controversy over the priority of the calculus. This contention brought about not only a cleavage in the body of working mathematicians of England and the continent, but also fortified by a definite insular trend in English thought,[1] isolated mathematical thinking to such an extent that, as E. T. Bell states, " the obstinate British practically rotted mathematically for all of a century after the death of Newton ".[2]

One of the first results of the quarrel between the English and continental mathematicians was a definite difference in the approach to mathematical and scientific research. For instance in setting up the principles of the subject and in the demonstrating of most of the proofs of the theorems

[1] A. N. Whitehead, *Adventures of Ideas*, New York, p. 22.

[2] E. T. Bell, *Men of Mathematics*, New York, 1937, p. 114.

of the *Principia*, Newton followed the approach of classical geometric methods. Newton's fondness for purely synthetic methods may be observed by reading the following statement in his treatment of the quadrature of a curve:

> After the area of a curve has been found and constructed, we should consider the demonstration of the construction, that, laying aside all algebraic calculations as much as may be, the theorem may be adorned and made elegant, so as to become fit for public view.[3]

On the continent, on the other hand, Leibniz and his followers developed the calculus and carried on their problems of research along analytical methods. The jealousy of the English over their hero, Newton, resulted in their having little or no intercourse with the continental mathematicians; and as a result, they continued for years after Newton's death to work along purely synthetic methods. The use of synthetic methods did not necessarily mean " without algebraic treatment ", but it did imply that the solution of a problem be first set down and then be demonstrated true. This method may be excellent in conveying truths to the average man, but it is not a logical method to use in investigations of scientific problems nor in the communication of truths to the scientific world.

While the British were working with synthetic methods, the progressive Euler, Lagrange and other French mathematicians were frantically applying the methods of Leibniz to the problems of scientific research. Considering the foundation that Newton had laid and the heritage that he had left his countrymen, the larger share of this work should have been done by the British themselves. However, any attempt to adopt analytical methods or to reform the study of mathematics at the University—Cambridge—was regarded by the professional body of the University Senate as a national dishonour and a sin against Newton.[4]

A second result of the controversy over the priority of the calculus was the adoption in England of the clumsy Newtonian notation of dots and pricks instead of the flexible dy/dx notation of Leibniz. In England the Newtonian notation continued in use during part of the first half of the nineteenth century; while on the continent the Libnizian notation was adopted by 1730 under the influence of John Bernoulli (1667–1748) and his distinguished pupil Leonhard Euler (1707–1783).

The English adhered not only to the Newtonian notation but to much of Newton's terminology. An example typical of this slavish adherence is found in the term *fluxion*. While Newton used the term as a velocity, later

[3] Newton's *Fluxions*, Colson's Translation, London, Sec. 107, p. 116.

[4] W. W. R. Ball, *A History of the Study of Mathematics at Cambridge*, Cambridge, 1889, p. 117.

writers confused it with the differential and some, Harris Hayes for example, spoke of fluxions of fluxions. As a result of this confusion, the way was opened for the attack of Bishop Berkley on the calculus as developed by Newton and his immediate followers, and a path was prepared for the controversy between Jurin and Robins. Furthermore, the concept of fluxions was so closely associated with geometry, to the exclusion of analysis, that some later writers held the view that fluxions were really a branch of geometry.[5]

At first Newton's students and commentators kept pace with one another but as the breach, due in part to the priority quarrel, widened, the continental mathematicians far surpassed the English, and the distance between the two groups increased in direct proportion to the number and importance of the physical and mathematical problems which were found to depend upon this type of investigation. Sir John Leslie has summed this up in the following:

> The habit of studying our own authors on these subjects, produced at first by our own admiration of Newton and our dislike for his rivals, and increased by a circumstance very insignificant in itself, the diversity of notation, prevented us from partaking in the pursuits of our neighbours and cut us off in a great measure from the vast fields in which the genius of France, of Germany and Italy, was exercised with so much activity and success.

He continues by stating:

> Our island after the decease of Maclauren produced none to compete with the great mathematicians of the continent except Thomas Simpson, whose native talent had struggled through indigence and a neglect of education. . . . For a long period afterwards the inventive genius of England seemed to slumber. The learned were content with merely commenting on the Principia, but rarely borrowing a few scattered lights abroad. The current of investigation was diverted into other channels or absolved among humbler objects.[6]

A third cause for the decline in interest in mathematics and other allied sciences in England was the conditions in the professional body of mathematicians and scientists and in their operating societies. First we notice that scientific knowledge barely existed among the upper classes of English society. The pursuit of science did not constitute a profession in England as it did in the continental countries. This condition was brought about by the appointment to important governmental scientific positions of men who were amateurs or who possessed only a small amount of scientific

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[5] Florin Cajori, *A History of the Conception of Limits and Fluxions in England from Newton to Woodhouse*, Chicago, 1919, pp. 150–250.

[6] *Encyclopædia Britannia*, Eighth Edition, Vol. I, p. 694.

knowledge or whose interest was primarily political rather than scientific. We are able to trace the cause of this state of affairs back to the educational institutions in England at this time, where the teachers of mathematics and science were frequently drawn from the clergy or the legal professions. In most cases the members of the teaching profession were so poorly paid that the teachers were compelled to exert their best efforts along some other line in order to make a modest living. On this point Charles Babbage states: "We thus by a distructive misapplication of talent which our institutions create, exchange a profound philosopher for but a tolerable lawyer."[7] He also points out that there was no encouragement nor demand from the English government for scientists, and that in order to obtain the best positions one must have a financial standing or play politics. He then contrasts this state of affairs with that in France, stating that the eminence of a great scientist was as great as that of Napoleon himself and that a knowledge of science and mathematics was a recommendation for appointment to public office.

An example of the difference between the attitude of the English kings and that of the continental ruling monarchs is shown in the course followed by Frederick the Great of Prussia, Catherine of Russia, the Kings of Sardinia, and Napoleon. With these rulers the demands of civil, naval, and military engineering made the study of mathematics a necessity. They were clear-sighted enough to see that the best way to obtain the services of the most distinguished mathematicians was to pay their living expenses, let them produce the needed mathematics and then leave them free to work along the lines they preferred. One has only to glance at the enormous amount of original work produced by Euler, Lagrange, Daniell Bernoulli, and others to realize the wisdom of this scheme. As evidence of the fact that these men of science were respected and well paid for their knowledge is found in an excerpt from the opening address of M. Alexander von Humbolt at a meeting of scientists in Berlin in September 1828. It states as follows:

> The taste of knowledge possessed by the ruling family, has made knowledge itself fashionable; and the severe sufferings of the Prussians previous to the war by which themselves and Europe were freed, have impressed on them so strongly the lesson that 'knowledge is power', that its effects are visible in every department of the government; and there is no country in Europe in which talents and genius so surely open, for their possessors, the road to wealth and distinction.[8]

In contrast to the Continental attitude toward the scientist, we find that the English were devoid of any great leaders in science and that they

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[7] Charles Babbage, *Reflections on the Decline of Science in England*, London, 1830, p. 37.

[8] *The Edinburgh Journal of Science*, April 1829, Vol. 10.

were too prejudiced against the Continentals to employ them in any of the government services. Babbage points out in his *Reflections*[9] that the Royal Society gave but little more encouragement to mathematicians and scientists than the government itself. Although the Royal Society was formed primarily for the promotion of scientific interest among its members, it had now reached the point where a large number of members were visibly lacking in any scientific interest. In many instances the nobilty held membership in either the parent society or one of its member societies for no other purpose than for the prestige it gave them, while, on the other hand, many of the Fellowships were held by persons who lived in the remote parts of England and were obtained for them by their agents who lived in London. The majority of these persons had no particular interest in science nor, as a matter of fact, in the Society itself. With this situation in mind it is interesting to compare the ratio of the membership of the Academies in France, Italy and Germany to the population of these countries with the ratio of the membership of the Royal Society to the population of England; for example, the ratios are as follows: France 1 to 427,000; Italy and Germany 1 to 300,000; the Royal Society, 1 to 32,000. In other words a seat in the Academy of Berlin was nine times more selective than one in the Royal Society. As a result of this policy of selecting its members, we would expect to find a similar policy in the selection of the officers of the Society. In most instances the President, Vice-President, and Secretary were chosen without regard for their interest or qualifications in mathematics or science and as a result the affairs of the Society were conducted by friends of these officers. These agents were usually more interested in politics and special favours than in the welfare of the society or its members, and naturally always opposed any and every suggestion of reform from the members and officers.

One of the first men to recognize the deplorable state of British mathematics was Robert Woodhouse (1773–1827). He not only recognized the condition to which mathematics had fallen but also realized that one of the great obstacles to reform in English mathematics at this time was the lack of suitable text-books which would give students a working knowledge of the differential notation for use in physics and astronomy. Although Woodhouse's publications were not suitable for use in the public examinations of the University, they did reach a group of young students at Cambridge and aroused in them a desire for reform. Woodhouse was a rather eccentric character and his writings are somewhat typical of the man himself. His style was crabbed and his works were difficult to understand on account of his complicated grammatical constructions; in consequence, his reputation

[9] P. 40, ff.

depends less on his productiveness than on the fact that he was able to impress a group of young students at Cambridge and to bring about a desire for reform in mathematics.[10] This group—consisting of Peacock, Herschel, Babbage, and Whewell—decided to form a club, later called the Analytical Society, in order to study and advocate at Cambridge University the use of analytical and differential methods which were then being used so successfully by the leading continental mathematicians and scientists. Herschel stated, somewhat later, that the whole movement of reform as suggested by Woodhouse would have come to naught except for the perseverance of Peacock. However, De Morgan gives us the following statement in regard to Woodhouse: " but the few who can appreciate what he did will always regard him as one of the most philosophical thinkers and useful guides of his times."[11]

The Analytical Society met on Sunday mornings at breakfast with the avowed purpose of studying the new continental methods of analysis. This group of students were interested in mathematics, physical research, and astronomy; and their aim was to develop ways and means of correcting the situation which then existed in England with respect to the study of these subjects. Of the group of four, George Peacock (1791–1858) seems to have exerted the greatest influence. Although his written contributions are meagre, he did have an extensive knowledge of his subject, he was an excellent lecturer, and he commanded the admiration and respect of both his colleagues and students. His knowledge of mathematics and his practical good sense made him the natural leader of the new movement. Charles Babbage (1792–1871) gave the name to the society and stated that the avowed purpose of the group was to advocate " the principles of pure *d-ism* as opposed to the *dot-age* of the University ". His most important contributions were the publication of the " Calculus of Functions " in the *Philosophical Transactions* and his labors in founding the Astronomical Society. The third member of the group was William Whewell (1794–1866), who probably exerted a greater influence on his contemporaries than on later mathematicians. Because his knowledge was so wide and discursive that it could not be too deep, his reputation as a profound thinker and scholar subsided as time went on. It shloud be borne in mind, however, that his *History of the Inductive Sciences* played an important role in the development of scientific thinking in England for a considerable length of time after its publication. The fourth member of the group was the illustrious astronomer Sir John Herschel (1792–1871). Although his contributions to mathematics

[10] I. Todhunter, *Dr. William Whewell, his Writings and Letters*, London, 1876, Vol. 2, p. 28 ff; see also Ball's *History*, p. 117 ff.

[11] Ball, *loc. cit.*, pp. 119–21.

are meagre he did publish a number of works on astronomy and he is usually credited with the founding of modern stellar astronomy.

The avowed purpose of the Analytical Society was to supplant the old fluxional methods in England by the continental methods of analysis. In 1817 Peacock was made moderator for the year at Trinity College and the symbols of differentiation of Leibniz were used for the first time in that year's senate-house examination. In 1818 Peacock stated:

> I assure you that I shall never cease to exert myself to the utmost in the cause of reform, and that I will never decline any office which will increase my power to affect it.... It is my silent perseverance only that we can hope to reduce the many-headed monster of prejudice and make the University (Cambridge) answer her character as the living mother of good learning and science.[12]

The members of the Society felt that it was the exclusive use of the fluxional methods that was so hampering to the development of mathematics in England. They were particularly offended by the sign of isolation in use, along with the practice of treating all problems by geometrical methods. These things offended them more than the inherent defects of the fluxional method itself. In general this program of reform was definitely opposed by the older men of the University. However, the Society never gave up hope and, as a matter of fact, was encouraged by the younger professors as well as students, who lent their support to the efforts of the members of the Society. The differential methods were used for the second time in the 1818 examinations and the following year they were employed by Whewell.

It has been frequently pointed out that one may judge the interest and type of teaching in a given subject by examining the text-books that are used and written during the period in question. This fact is certainly borne out for the period under discussion. It is natural to expect that there had been practically no texts written in English using the differential notations, but on the contrary the text-books using the fluxional methods were few in number and far from satisfactory. One of the obstacles in the way of the reformers was a rule of the University that no question in a new subject, which had not been previously discussed in some treatise suitable and available for Cambridge students, should be set for a mathematical tripos. Since there had been no text-books written in English on the new analysis, it is needless to say that it was impossible for the reformers to bring these subjects to the attention of the students until the texts had been prepared. The major part of this work had to be done by the members of the Analytical Society; and it should be pointed out that a major part of the time and energy of

[12] Ball, *loc. cit.*, p. 120.

these scholars was consumed in writing elementary texts of a routine nature. One of the first texts in English to give the new analysis was Woodhouse's *Analytical Trigonometry* published in 1810. The needs of an analytical exposition of the calculus using the differential notation were met by the publication of a translation of La Croix's *Differential Calculus* in 1816. This publication was followed somewhat later by *Examples of the Illustrative Use of the Differential Calculus.* After the groundwork had been completed by a few translations and tracts, by the efforts of the members of the Analytical Society, other writers began to add to the list and to expand the range of their approach to the task. Noteworthy in this list is the *Principles of Analytical Geometry* by Henry Parr Hamilton, published in 1826, in which conic sections are defined by an equation of the second degree. This publication, along with John Hyme's publication of an *Analytical Geometry of Three Dimensions*, opened the way for the application of algebra and the rectangular system of co-ordinates to the study of geometry in England.[13] In algebra, Peacock in his *Algebra*, published in 1830, did much to reform the teaching of algebra, to establish algebra on a firm foundation, and to develop the subject as an abstract system of symbols that may be combined according to operations that conform with pre-assigned postulates. In the physical and applied sciences noteworthy additions were produced by Herschel, Whewell and George B. Airy (1801–1892).

By 1830 the introduction of new text-books was well on its way but, it is needless to say, that the efforts of these men were met with opposition by the friends and advocates of the fluxional methods of the Newtonian school. A typical example is found in a quotation from William Hales's *Analysis Fluxionum* published in Mesere's *Scriptores Logarithmici*, Vol. 5, where he makes mention of a review by a member of the Analytical Society of La Croix's "Differential Calculus" in the *Edinburgh Monthly Review*. He writes:

> How was it possible that the eyes of the Monthly Reviewers could be so holden .... as to assert, that Newton himself was not perfectly satisfied of the stability of the grounds on which he established the Methods of Fluxions?

In the same work he vents his wrath on D'Alembert by considering him a hostile critic of Newton, as a "philosophizing infidel", and as "one of the original conspirators against Christianity, and at once the glory and disgrace of the French Academy of Sciences."[14] The above remarks are typical of the campaign carried on by the adherents of the old fluxional methods against the reforms and reformers of mathematics in England.

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[13] Ball, *loc. cit.*, p. 117 ff.

[14] *London Monthly Review*, London, Vol. 32, 1801, pp. 176–182.

With the opening of the third decade of the nineteenth century, the reforms of the Analytical Society were well under way and the new material in the form of text-books had been completed for the preparation of students at the universities for their tripos examinations. Likewise the ground had been cleared and the foundation laid for the brilliant works of Sir William R. Hamilton, D. F. Gregory and George Boole. After the appearance of these men and with the beginning of the second half of the nineteenth century, English mathematics and theoretical science again took its rightful place in the scientific world.

A THEOREM ON RESIDUES AND ITS BEARING ON MULTIPLICATIVE FUNCTIONS WITH A MODULUS

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1. INTRODUCTION.—Hardy and Wright in their "Introduction to the Theory of Numbers" have shown the genesis of Ramanujan's function[1] $C_M(N)$ from $e^{2\pi i N/M}$ and also its multiplicative nature in the argument M by using the following theorem on residues: "If $(m, m') = 1$, a runs through a complete set of residues prime to m, a' runs through a complete set of residues prime to m', then $am' + a'm$ runs through a complete set of residues prime to mm'" (Theorem 60, p. 53).

In this paper, I first generalise the above theorem on residues and then show how from a certain class of quasi-multiplicative functions of two arguments we can derive modular multiplicative functions[2] of two arguments by a process which is quite general and similar to the one which has given rise to $C_M(N)$ from $e^{2\pi i N/M}$. Incidentally, we shall also see the multiplicative character of $C_M(N)$ in both the arguments, a fact which does not seem to have received much attention.

2. I will now briefly explain the class division of integers (mod M) by Dr. Vaidyanathaswamy[3] on which the generalisation of the theorem on residues is based.

Let $t_1 = 1, t_2, t_3, \ldots, t_r = M$, be the distinct divisors of the integer M. Then the M numbers $1, 2, \ldots, M$ considered as the representatives of the M distinct residue classes (mod M) can be divided into r mutually exclusive classes $c_1, c_2, \ldots, c_r$, where c_t consists of the numbers whose greatest common divisor with M is t_t. The class c_t evidently consists of $\phi\left(\frac{M}{t_t}\right)$ numbers. For, let $(a, M) = d$, where a is any one of the numbers $1, 2, \ldots, M$ and d (the greatest common divisor of a and M) is necessarily one of the numbers $t_1, t_2, \ldots, t_r$. Now, if $b \leqslant \frac{M}{d}$, then clearly $(bd, M) = (d, M)$ if and only if b is prime to $\frac{M}{d}$. Hence the number of numbers not exceeding

* I am indebted to Dr. R. Vaidyanathaswamy for his help in the preparation of this paper.

M and possessing the g.c.d d with M is precisely $\phi\left(\frac{M}{d}\right)$. Since $\sum_{d|M} \phi\left(\frac{M}{d}\right)$ $= \sum_{d|M} \phi(d) = M$, the classes exhaust the M numbers.

We shall refer to the $\phi\left(\frac{M}{d}\right)$ numbers of the class c_d or any set of $\phi\left(\frac{M}{d}\right)$ numbers which are congruent (mod M) to the $\phi\left(\frac{M}{d}\right)$ numbers of c_d, as a class (mod M) specified by the divisor d of M.

In particular, a complete set of residues prime to M is a class (mod M) specified by unity.

3. We will now proceed to the generalisation of the theorem referred to.

THEOREM A.—*If* $(M, M') = 1$, *a runs through a class (mod M) specified by the divisor d of M, a' runs through a class (mod M') specified by the divisor d' of M', then aM' + a'M runs through the class (mod MM') specified by the divisor dd' of MM'.*

Proof.—Clearly there are $\phi\left(\frac{M}{d}\right)\phi\left(\frac{M'}{d'}\right) = \phi\left(\frac{MM'}{dd'}\right)$ numbers $aM' + a'M$. They are distinct (mod MM'), for if

$$a_1M' + a_1'M \equiv a_2M' + a_2'M \pmod{MM'},$$

then $\quad a_1M' \equiv a_2M' \pmod{M}$

and so $\quad a_1 \equiv a_2 \pmod{M}$ since $(M, M') = 1$.

Similarly $\quad a_1' \equiv a_2' \pmod{M'}$.

But this contradicts the hypothesis that a_1, a_2 are distinct (mod M) and a_1', a_2' are distinct (mod M'). Hence the $\phi\left(\frac{MM'}{dd'}\right)$ numbers $aM' + a'M$ are distinct mod (MM').

Also $(a, M) = d$, $\quad (a', M') = d'$ by hypothesis.

$\therefore \quad (aM', M) = d, \quad (a'M, M') = d'$

$\therefore \quad (aM' + a'M, M) = d, \quad (a'M + aM', M') = d'$.

Hence $(aM' + a'M, MM') = dd'$.

Thus $aM' + a'M$ runs through a class (mod MM'), specified by the divisor dd' of MM'.

4. We will next show how from a certain class of quasi-multiplicative functions, we can generate multiplicative functions with a modulus:—

THEOREM B.—*Let $f(M, N)$ be an Arithmetic function with the modulus M possessing a quasi-multiplicative property in M, namely, $f(M, N)$ $f(M', N') = f(MM', NM' + N'M)$, whenever M is prime to M'.*

Then $F(M, N) = \Sigma f(M, R)$ {where R varies over a class (mod M) specified by the divisor g of M, g being the g.c.d. of M, N} is a multiplicative function of M, N with the modulus M.

Proof.—Let M, N, M', N' be integers such that $(MN, M'N') = 1$.

Suppose $(M', N') = g'$.

$$\begin{aligned} \text{Then } F(M, N)\, F(M', N') &= [\Sigma f(M, R)]\, [\Sigma f(M', R')] \\ &= \Sigma f(MM', RM' + R'M) \\ &= F(MM', NN') \text{ by Theorem A and definition of F.} \end{aligned}$$

Thus F (M, N) is multiplicative in M, N. That it has the modulus M follows from the fact that $(M, N) = g = (M, N + \lambda M)$ where λ is any integer.

5. Let $F(M, N) = \Sigma f(M, Nr)$ where r runs through a complete set of residues prime to M. Then we have the following interesting theorem which is slightly different from Theorem B.

THEOREM C.—*$F(M, N) = \dfrac{\phi(M)}{\phi\left(\dfrac{M}{g}\right)} \Sigma f(M, T)$ where T varies over a class (mod M) specified by $g = (N, M)$, and is multiplicative in M, N with the modulus M.*

Proof.—To prove the first part we have to use the following theorem of Dr. Vaidyanathaswamy.[4]

"If t is a divisor of N, the $\phi(N)$ numbers prime to N fall into $\phi(t)$ sets, each set consisting of $\dfrac{\phi(N)}{\phi(t)}$ numbers equal to each other mod t."

Applying this result, we see immediately that

$$F(M, N) = \frac{\phi(M)}{\phi\left(\frac{M}{g}\right)} \Sigma f(M, N\delta),$$

where δ runs through a complete set of residues prime to M/g. If we write $N\delta = T$, clearly T varies over a class (mod M) specified by the divisor $g = (M, N)$ so that

$$F(M, N) = \frac{\phi(M)}{\phi\left(\frac{M}{g}\right)} \Sigma f(M, T).$$

The second part follows from Theorem B.

6. Ramanujan's function $C_M(N)$ referred to in the introduction is a typical example of Theorem C. The function $f(M, N) = e^{2\pi i N/M}$ satisfies the relation $f(M, N) f(M', N') = f(MM', NM' + N'M)$ not only when M is prime to M′, but for all values of M, M′. Thus it is not merely quasi-multiplicative in M but quasi-linear in M.

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3. R. Vaidyanathaswamy, *Proc. Ind. Acad. Sci.*, 1937, 5, 63–70.
4. R. Vaidyanathaswamy, „ „ 66.

ON SOME VIBRATIONAL PROBLEMS

BY

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In a recent paper Christopherson[1] has discussed the solutions of the vibrational equation,

$$\nabla^2 z = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) z = -k^2 z \tag{1}$$

subject to the boundary conditions

$$z = 0, \tag{2·1}$$

and

$$\frac{\partial z}{\partial \nu} = 0, \tag{2·2}$$

$\partial \nu$ being an element of the normal drawn to the boundary.

He has obtained the known solution[2] due to Lame for an equilateral boundary and has shown how the same can be used for a regular hexagonal boundary. But in both the cases, the solution is incomplete as it contains only one arbitrary parameter, the system possessing two degrees of freedom. A similar drawback exists in solutions obtained by B. Sen[3] for equilateral boundaries.

From the boundary condition (2·1), which amongst other problems relates to the transverse vibration of membranes. Christopherson finds a solution of the type

$$z = 2 \sin m\pi x/a \cos m\pi y\sqrt{3}/a - \sin 2m\pi x/a \tag{3}$$

the sides of the equilateral triangle being $x = a$, $y = \pm x/\sqrt{3}$, and that of the regular hexagon being $x = \pm a$, $y = \pm x/\sqrt{3} \pm 2a/\sqrt{3}$. If the condition (2·2) is utilized, and this relates to a number of problems in Hydrodynamics and conduction of heat, the solution given in (3) is changed into

$$z = 2 \cos m\pi x/a \cos m\pi y\sqrt{3}/a + \cos 2m\pi x/a. \tag{4}$$

The complete solution corresponding to (3) has already been given by Seth[4] in a recent paper where other triangular boundaries are also discussed. In fact, the complete solution corresponding to (3) is

$$\begin{aligned} z = {} & 2 \sin (m - n) \pi x/a \cos (m + n) \pi y\sqrt{3}/a \\ & - 2 \sin (2m + n) \pi x/a \cos n\pi y\sqrt{3}/a \\ & + 2 \sin (2n + m) \pi x/a \cos m\pi y\sqrt{3}/a \end{aligned} \tag{5·1}$$

and corresponding to (4) it is

$$z = 2\cos(m-n)\pi x/a \cos(m+n)\pi y\sqrt{3}/a$$
$$+ 2\cos(2m+n)\pi x/a \cos n\pi y\sqrt{3}/a$$
$$+ 2\cos(2n+m)\pi x/a \cos m\pi y\sqrt{3}/a. \qquad (5\cdot2)$$

It may be mentioned that (5) also holds good for a rhombus containing an angle of 120° with sides given by

$$x=0,\ x=a,\ y=x/\sqrt{3},\ y=x/3+2a/\sqrt{3}.$$

Exact solutions corresponding to the boundary condition (2·2) are not given in Seth's paper mentioned above, and it will therefore be not out of place to give them here.

For a right-angled isosceles triangle, the solution is

$$z = \sin(2m+1)\pi x/2a \sin(2n+1)\pi y/2a$$
$$- \sin(2n+1)\pi x/2a \sin(2m+1)\pi y/2a, \qquad (6)$$

the sides being $x=a,\ y=\pm x$.

For an isosceles triangle containing an angle of 120°, we find

$$z = 2\cos(m-n)\pi x/a \cos(m+n+1)\pi y\sqrt{3}/a$$
$$- 2\sin\left(2m+1+\frac{2n+1}{2}\right)\pi x/a \sin(2n+1)\pi y\sqrt{3}/2a$$
$$+ 2\sin\left(2n+1+\frac{2m+1}{2}\right)\pi x/a \sin(2m+1)\pi y\sqrt{3}/2a \qquad (7)$$

the sides being $x=a,\ y=x\sqrt{3}$, and $y+x\sqrt{3}=2a/\sqrt{3}$

The same solution holds good for a right angled triangle containing an angle of 60°, the sides being $x=a,\ y=a/\sqrt{3},\ y=x\sqrt{3}$.

It may be mentioned that both for the hexagon and for the isosceles triangle containing an angle of 120° the solution given in (5) and (7) only gives 'symmetric vibrations'.

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SOME DIOPHANTINE EQUATIONS

BY

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Given a solution of the system of equations

$$A_1x_1^r+A_2x_2^r+\ldots+A_nx_n^r=0 \qquad (r=0,1,2,\ldots,m-1) \quad (1)$$

let us try to obtain a solution of the extended system

$$A_1x_1^r+A_2x_2^r+\ldots+A_nx_n^r=0 \quad (r=0,1,2,\ldots,m-1,m+1) \quad (2)$$

We shall require two lemmas, which are easily proved:

LEMMA 1.—If $x_i=a_i\,(i=1,2,\ldots,n)$ be a solution of either of the systems (1) or (2), then $x_i=ka_i\,(i=1,2,\ldots,n)$ will also be a solution of that system.

LEMMA 2.—If $x_i=a_i\,(i=1,2,\ldots,n)$ be a solution of the system (1), then $x_i=x+a_i\,(i=1,2,\ldots,n)$ will also be a solution of the system for all x.

Corollary.—If $x_i=a_i\,(i=1,2,\ldots\ldots,n)$ be a solution of (1), then $x_i=a_i-a_n\,(i=1,2,\ldots,n-1)$ will be a solution of the system

$$A_1x_1^r+\ldots+A_{n-1}x_{n-1}^r=0 \qquad (n=1,2,\ldots,m-1)$$

$$A_1+A_2+\ldots+A_{n-1}=-A_n.$$

Let us now try to satisfy the system (2) by the values $x_i=x+a_i$ $(i=1,2,\ldots,n)$ where $x_i=a_i$ is a solution of (1), by choosing x properly. The equation

$$\Sigma A_i(x+a_i)^{m+1}\equiv x^{m+1}\Sigma A_i+\binom{m+1}{1}x^m\Sigma A_i a_i+\ldots+\Sigma A_i a_i^{m+1}=0$$

reduces to

$$(m+1)x\Sigma A_ia_i^m+\Sigma A_i a_i^{m+1}=0.$$

Hence if we take $x=-\dfrac{\Sigma A_i a_i^{m+1}}{(m+1)\Sigma A_i a_i^m}$, then $x_i=x+a_i\ (i=1,2,\ldots,n)$ will satisfy the system (2). Using Lemma 1 we obtain

THEOREM 1. *If $x_i=a_i\,(i=1,2,\ldots,n)$ be a solution of the system* (1), *then*

$$x_i=(m+1)a_i(\Sigma A_ia_i^m)-(\Sigma A_ia_i^{m+1})$$

will be a solution of the extended system (2).

Taking $m = 1$ in Theorem 1 we get the following:

THEOREM 2. *If* $\sum_{i=1}^{n} A_i = 0$, *then*

$$x_i = 2a_i\left(\sum_{j=1}^{n} A_j a_j\right) - \sum_{j=1}^{n} A_j a_j^2$$

is a solution of $A_1 x_1^2 + A_2 x_2^2 + \dots + A_n x_n^2 = 0$ *for all values of* $a_1, a_2, \dots, a_n$.

In Theorem 2 take $n = 2k$, $A_1 = \dots = A_k = 1$, $A_{k+1} = A_{k+2} = \dots = A_n = -1$, and write

$$\left.\begin{aligned} a_{k+i} &= -b_i \\ x_{k+i} &= \ \ y_i \end{aligned}\right\} \quad (i = 1, 2, \dots, k).$$

Then we get

THEOREM 3. *A solution of the equation*

$$x_1^2 + \dots + x_k^2 = y_1^2 + \dots + y_k^2$$

is given by

$$x_i = 2a_i\,(a_1 + \dots + a_k + b_1 + \dots + b_k) - (a_1^2 + \dots + a_k^2 - b_1^2 - \dots - b_k^2)$$
$$y_i = 2b_i\,(a_1 + \dots + a_k + b_1 + \dots + b_k) - (b_1^2 + \dots + b_k^2 - a_1^2 - \dots - a_k^2)$$
$$(i = 1, 2, \dots, k).$$

Let us take $m = 2$, $n = 2k$, $A_1 = \dots = A_k = -A_{k+1} = - \dots = -A_n = 1$ in Theorem 1. Then we get

THEOREM 4.—*A solution of the system*

$$x_1 + x_2 + \dots + x_k = y_1 + y_1 + \dots + y_k$$
$$x_1^3 + x_2^3 + \dots + x_k^3 = y_1^3 + y_2^3 + \dots + y_k^3$$

is given by

$$x_i = 3a_i\,(a_1^2 + \dots + a_k^2 - b_1^2 - \dots - b_k^2) - (a_1^3 + \dots + a_k^3 - b_1^3 - \dots - b_k^3)$$
$$y_i = 3b_i\,(a_1^2 + \dots + a_k^2 - b_1^2 - \dots - b_k^2) - (a_1^3 + \dots + a_k^3 - b_1^3 - \dots - b_k^3)$$
$$(i = 1..\ 2, \dots, k)$$

where $a_1 + a_2 + \dots + a_k = b_1 + b_2 + \dots + b_k$.

Let us now consider another problem. We suppose that a solution of the system

$$A_1 x_1^r + \dots + A_n x_n^r = 0 \quad (r = 0, 1, 2, \dots, m-2, m+1) \tag{3}$$

is known, say $x_i = a_i$ $(i = 1, 2, \dots, n)$. Then

$$\sum_{i=1}^{n} A_i (x + a_i)^{m+1} = x^{m+1} \Sigma A_i + \binom{m+1}{1} x^m \Sigma A_i a_i + \dots + \Sigma A_i a_i^{m+1},$$
$$= \binom{m+1}{2} (\Sigma A_i a_i^{m-1})\, x^2 + \binom{m+1}{1} (\Sigma A_i a_i^m)\, x.$$

This will be zero if

$$\frac{m}{2}(\Sigma A_i a_i^{m-1})x + \Sigma A_i a_i^m = 0.$$

Thus we get

THEOREM 5.—*If* $x_i = a_i$ $(i = 1, 2, \ldots, n)$ be *a solution of the system*

$$\Sigma A_i x_i^r = 0 \; (r = 0, 1, 2, \ldots, m-2, m+1), \tag{4}$$

then $x_i = ma_i \; (\Sigma A_j a_j^{m-1}) - 2\,\Sigma A_j a_j^m \; (i = 1, 2, \ldots\ldots, n)$ *will also be a solution.*

Taking $m = 2$, $n = 2k$, $A_1 = \ldots = A_k = -A_{k+1} = -\ldots = -A_n = 1$, and writing $x_{k+i} = y_i$, $a_{k+i} = b_i$ $(i = 1, 2, \ldots, k)$, we get

THEOREM 6.—*If* $x_i = a_i$, $y_i = b_i$ $(i = 1, 2, \ldots, k)$ satisfy

$$x_1^3 + x_2^3 = \ldots + x_k^3 = y_1^3 + y_2^3 + \ldots + y_k^3,$$

then $x_i = a_i(a_1 + \ldots + a_k - b_1 - \ldots - b_k) - (a_1^2 + \ldots + a_k^2 - b_1^2 - \ldots - b_k^2)$

$y_i = b_i(a_1 + \ldots + a_k - b_1 - \ldots - b_k) - (a_1^2 + \ldots + a_k^2 - b_1^2 - \ldots - b_k^2)$

will also satisfy the equation.

Let us write $ma_i\left(\sum_{j=1}^{n} A_j a_j^{m-1}\right) - 2\sum_{j=1}^{n} A_j a_j^m = \alpha_i$. Theorem 5 may then be read as follows: If $x_i = a_i$ $(i = 1, 2, \ldots, n)$ satisfies equations (4), then $x_i = \alpha_i$ $(i = 1, 2, \ldots, n)$ will also satisfy those equations. It follows for the same reason, that

$$x_i = m\alpha_i\left(\sum_{j=1}^{n} A_j \alpha_j^{m-1}\right) - 2\sum_{j=1}^{n} A_j \alpha_j^m$$

will also satisfy (4). Writing $m\,\Sigma A_j a_j^{m-1} = p$, $-2\,\Sigma A_j a_j^m = q$, we have $\alpha_i = pa_i + q$,

$$\Sigma A_j \alpha_j^{m-1} = \Sigma A_j (pa_j + q)^{m-1} = (\Sigma A_j a_j^{m-1})p^{m-1} = \frac{p^m}{m},$$

$$\Sigma A_j \alpha_j^m = \Sigma A_j (pa_j + q)^m = p^m\,\Sigma A_j a_j^m + mp^{m-1}q\,\Sigma A_j a_j^{m-1}$$

$$= -\frac{p^m q}{2} + p^m q = \frac{p^m q}{2},$$

since $\Sigma A_j a_j^n = 0$ for $(n = 0, 1, 2, \ldots, m-2)$, so that

$$m\alpha_i\,(\Sigma A_j \alpha_j^{m-1}) - 2\,\Sigma A_j \alpha_j^m = m(pa_i + q)\frac{p^m}{m} - 2\cdot\frac{p^m q}{2} = p^{m+1} a_i.$$

It follows that, effectively,

$$x_i = m\alpha_i\,\Sigma A_j \alpha_j^{m-1} - \Sigma A_j \alpha_j^m \qquad (i = 1, 2, \ldots, n)$$

is the same as the original solution. Thus we get

THEOREM 7.—*The two solutions*

$$x_i = a_i,\; x_i = ma_i\,\Sigma A_j a_j^{r-1} - 2\,\Sigma A_j a_j^m \qquad (i = 1, 2, \ldots, n)$$

of the system $\Sigma A_i x_i^r = 0 \qquad (r = 0, 1, 2, \ldots, m-2, m+1)$

have a mutually reciprocal character in the sense that the process by which the second is derived from the first, gives back the first from the second.

Let us now consider the system of equations

$$\Sigma x_{i_1} x_{i_2} \ldots . x_{i_j} = \Sigma y_{i_1} y_{i_2} \ldots . y_{i_j} \qquad (j = 1, 2, \ldots, m-1)$$

where the summations are over all combinations, j at a time of the n suffixes $1, 2, \ldots, n$. Let

$$x_i = a_i,\ y_i = b_i \qquad (i = 1, 2, \ldots, n)$$

be a solution of this system and let us write for shortness

$$\Sigma a_{i_1} a_{i_2} \ldots . a_{i_j} = A_j,\ \Sigma b_{i_1} b_{i_2} \ldots . b_{i_j} = B_j \qquad (j = 1, 2, \ldots .)$$

Then $\quad \Sigma (t + a_{i_1})(t + a_{i_2}) \ldots (t + a_{i_j}) - \Sigma (t + b_{i_1}) \ldots . (t + b_{i_j})$

$$= \binom{n-1}{j-1} (A_1 - B_1) t^{j-1} + \binom{n-2}{j-2} (A_2 - B_2) t^{j-2} + \ldots + (A_j - B_j)$$

$$= 0 \text{ if } 1 \leq j \leq m-1$$

$$= (n-m)(A_m - B_m) t + (A_{m+1} - B_{m+1}) \text{ if } j = m+1.$$

If $\quad t = -\left(\dfrac{(A_{m+1} - B_{m+1})}{(n-m)(A_m - B_m)}\right)$, then

$$\Sigma \{ (t + a_{i_1}) \ldots . (t + a_{i_j}) \ldots . (t + a_{i_{m+1}}) - (t + b_{i_1}) \ldots . (t + b_{i_{m+1}}) \} = 0$$

so that we have

THEOREM 8.—*If* $x_i = a_i,\ y_i = b_i\ (i = 1, 2, \ldots, n)$ *be a solution of the system of equations*

$$\Sigma x_{i_1} x_{i_2} \ldots . x_{i_j} = \Sigma y_{i_1} y_{i_2} \ldots . y_{i_j} \qquad (j = 1, 2, \ldots, m-1)$$

then

$$x_i = (n-m) a_i (A_m - B_m) - (A_{m+1} - B_{m+1})$$

$$y_i = (n-m) b_i (A_m - B_m) - (A_{m+1} - B_{m+1})$$

will be a solution of the extended system

$$\Sigma x_{i_1} \ldots . x_{i_j} = \Sigma y_{i_1} \ldots . y_{i_j} \qquad (j = 1, 2, \ldots, m-1, m+1).$$

In a similar manner, analogous to Theorem 5, we get

THEOREM 9.—*If* $x_i = a_i,\ y_i = b_i \quad (i = 1, 2, \ldots, n)$ *be a solution of the system*

$$\Sigma x_{i_1} \ldots . x_{i_j} = \Sigma y_{i_1} \ldots . y_{i_j} \qquad (j = 1, 2, \ldots, m-2, m+1)$$

then

$$x_i = (n-m+1)(A_{m-1} - B_{m-1}) a_i - 2(A_m - B_m)$$

$$y_i = (n-m+1)(A_{m-1} - B_{m-1}) b_i - 2(A_m - B_m)$$

is also a solution of the system.

ON INTEGER CUBE-ROOTS OF THE UNIT MATRIX

BY

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In a paper in the October, 1927 issue of the *Journal of the London Mathematical Society*, Prof. Turnbull proved that the integer matrix of order n

$$X_n = [x_{ij}],\ x_{ij} = (-1)^{n-j}\binom{n-j}{i-1}$$

has the property $X_n^3 =$ the unit matrix E_n.

R. Vaidyanathaswamy in the April, 1928 issue of the same *Journal* obtained an expression for integer rth-roots of the unit matrix. As a special case he obtained $\begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix}$ as a cube root of the unit matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

In this note I prove that* the necessary and sufficient conditions for the integer matrix $X_2 = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ to be a cube root of $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ are

$$(1)\ a + d = -1;\quad (2)\ ad - bc = 1$$

except when $a = d = 1$ and $b = c = 0$.

Proof.—Consider the transformation

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}(x, y) = \{ax + by,\ cx + dy\}.$$

It can be easily verified that

$$A^3 = \{(a^3 + 2abc + bcd)\,x + b\,(a^2 + ad + d^2 + bc)\,y,$$
$$c\,(a^2 + ad + d^2 + bc)\,x + (d^3 + 2bcd + abc)\,y\}$$

---

* The study of R. Vaidyanathaswamy's paper, referred to above, has led S. Chowla and me to conjecture that all the $(p-1)$th order integer matrices $[X_{p-1}]$ satisfying $[X_{p-1}]^p =$ the unit matrix E_{p-1} where p is a prime can be expressed as $\Delta^{-1} M_{p-1} \Delta$ where Δ is an integer matrix of determinant ± 1 and

$$M_{p-1} = \begin{bmatrix} -1 & -1 & \cdot & \cdot & \cdot & \cdot & -1 \\ 1 & 0 & \cdot & \cdot & \cdot & \cdot & 0 \\ 0 & 1 & \cdot & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot & \cdot & 1 & 0 \end{bmatrix}$$

We have used the theorem, proved in this note, to prove the conjecture for $p = 3$.

Therefore the necessary and sufficient conditions for X_2 to be a cube root of E_2 are

$$a^3 + 2abc + bcd - 1 = 0 \qquad \text{(I)}$$

$$a^2 + ad + d^2 + bc = 0 \qquad \text{(II)}$$

$$d^3 + 2bcd + abc - 1 = 0 \qquad \text{(III)}$$

It is easily seen that (III) = (I) + $(d - a)$ (II). Therefore writing k for bc we obtain the necessary and sufficient conditions as

$$a^3 + (2a + d)\,k = 1$$

$$a^2 + ad + d^2 + k = 0.$$

Eliminating k we have

$$(a + d)^3 = -1$$

or

$$(a + d) = -1$$

as a necessary condition.

Also

$$bc = k = -(a^2 + d^2 + ad) = ad - (a + d)^2 = ad - 1$$

Therefore

$$ad - bc = 1$$

is another necessary condition.

That the two conditions are sufficient can be easily verified by substituting these in (I) and (II).

ON THE REMAINDER IN TAYLOR'S THEOREM

BY

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The first of the theorems which appear below gives a formula for the remainder after n terms of the Taylor series of a function differentiable n times. The formula explains the genesis of all the known forms of the remainder. The second theorem gives inequalities satisfied by the remainder after n terms when the function is differentiable $(n-1)$ times and the $(n-1)$th derivative is continuous but not differentiable. Young's form of Taylor's theorem turns out to be a special case of the second theorem.

THEOREM I.—*Suppose that*

(i) $\left.\begin{cases} f(x),\ f'(x), \ldots, f^{(n-1)}(x), \\ g(x),\ g'(x), \ldots, g^{(p-1)}(x), \\ h(x),\ h'(x), \ldots, h^{(q-1)}(x) \end{cases}\right\}$ *are continuous when* $a \leqslant x \leqslant b$;

(ii) $f^{(n)}(x),\ g^{(p)}(x),\ h^{(q)}(x)$ *exist when* $a < x < b$.

If we write

$$R_n(f) = f(b) - f(a) - (b-a)f'(a) - \ldots - \frac{(b-a)^{n-1}}{(n-1)!} f^{(n-1)}(a),$$

$$R_p(g) = g(b) - g(a) - (b-a)g'(a) - \ldots - \frac{(b-a)^{p-1}}{(p-1)!} g^{(p-1)}(a),$$

$$R_q(h) = h(b) - h(a) - (b-a)h'(a) - \ldots - \frac{(b-a)^{q-1}}{(q-1)!} h^{(q-1)}(a),$$

then there is a ξ *such that* $a < \xi < b$ *and*

$$\begin{vmatrix} f(b), & g(b), & h(b), \\ R_n(f), & R_p(g), & R_q(h), \\ \frac{(b-\xi)^{n-1}}{(n-1)!} f^{(n)}(\xi), & \frac{(b-\xi)^{p-1}}{(p-1)!} g^{(p)}(\xi), & \frac{(b-\xi)^{q-1}}{(q-1)!} h^{(q)}(\xi) \end{vmatrix} = 0$$

Proof.—The special case $n = p = q = 1$ of the theorem is well known. The general case follows at once from the special case when we replace $f(x)$, $g(x)$, $h(x)$ by $F_n(x)$, $G_p(x)$, $H_q(x)$ respectively, where

$$F_n(x) = f(x) + (b-x)f'(x) + \ldots + \frac{(b-x)^{n-1}}{(n-1)!} f^{(n-1)}(x),$$

$$G_p(x) = g(x) + (b-x)g'(x) + \ldots + \frac{(b-x)^{p-1}}{(p-1)!} g^{(p-1)}(x),$$

$$H_q(x) = h(x) + (b-x)h'(x) + \ldots + \frac{(b-x)^{q-1}}{(q-1)!} h^{(q-1)}(x),$$

DEDUCTIONS.—

I (A): $h(x) = 1$ gives, for a ξ such that $a < \xi < b$,

$$\begin{vmatrix} R_n(f), & R_p(g) \\ \frac{(b-\xi)^{n-1}}{(n-1)!} f^{(n)}(\xi), & \frac{(b-\xi)^{p-1}}{(p-1)!} g^{(p)}(\xi) \end{vmatrix} = 0.$$

This is Mahajani's form for $R_n(f)$ [1].

I (B): Take $h(x) = 1$, $g(x) = (x-a)^m$, $m \geqslant p$ and set $b = a + k$, $\xi = a + \theta k$ $(0 < \theta < 1)$. Then

$$\frac{R_n(f)}{\mathrm{B}(p, m-p+1)} = k^n \frac{(1-\theta)^{n-p}}{\theta^{m-p}} \frac{f^{(n)}(a+\theta k)}{(n-1)!}.$$

This is a form of the remainder given by Edwards in his *Differential Calculus*, concluding Miscellaneous Examples, Ex. 52.

I (C): If $m = p$ in I (B) we have the Schlömilch-Roche form of the remainder:

$$R_n(f) = k^n \frac{(1-\theta)^{n-p}}{p} \frac{f^{(n)}(a+\theta k)}{(n-1)!}.$$

THEOREM II.—*In Theorem* I, *drop the hypothesis* (ii). *Denote by* $D\phi(x)$ *either any one of the derivates of* $\phi(x)$ *on the right, viz.,* $D^+\phi(x)$, $D_+\phi(x)$, *or any one of the derivatives on the left, viz.,* $D^-\phi(x)$, $D_-\phi(x)$. *Then there are* ξ_1, ξ_2 *such that* $a < \xi_1, \xi_2 < b$ *and the determinant*

$$\begin{vmatrix} f(b), & g(b), & h(b), \\ R_n(f), & R_p(g), & R_q(h), \\ (b-x)^{n-1}\frac{D f^{(n-1)}(x)}{(n-1)!}, & (b-x)^{p-1}\frac{D g^{(p-1)}(x)}{(p-1)!}, & (b-x)^{q-1}\frac{D h^{(q-1)}(x)}{(q-1)!} \end{vmatrix}$$

is non-positive for $x = \xi_1$ *non-negative for* $x = \xi_2$.

Proof.—First we establish the special case $n = p = q = 1$ of our theorem by appealing to Pollard's extension of Rolle's theorem to continuous non-differentiabe functions [2, §3]. Then we have merely to replace $f(x)$, $g(x)$, $h(x)$ by $F_n(x)$, $G_p(x)$, $H_q(x)$ respectively as in the proof of Theorem I.

DEDUCTIONS from Theorem II corresponding to I (A), I (B), I (C) are the following:

II (A):

$$\begin{vmatrix} R_n(f), & R_p(g) \\ (b-\xi_1)^{n-1}\frac{D f^{(n-1)}(\xi_1)}{(n-1)!}, & (b-\xi_1)^{p-1}\frac{D g^{p-1}(\xi_1)}{(p-1)!} \end{vmatrix} \leqslant 0$$

$$\leqslant \begin{vmatrix} R_n(f) & R_p(g) \\ (b-\xi_2)^{n-1}\frac{D f^{(n-1)}(\xi_2)}{(n-1)!}, & (b-\xi_2)^{p-1}\frac{D g^{(p-1)}(\xi_2)}{(p-1)!} \end{vmatrix}$$

$$(a < \xi_1 < \xi_2 < b)$$

II (B):

$$k^n \frac{(1-\theta_1)^{n-p}}{\theta_1^{m-p}} \frac{Df^{n-1}(a+\theta_1 k)}{(n-1)!} \geqslant \frac{R_n(f)}{\mathrm{B}(p, m-p+1)}$$
$$\geqslant k^n \frac{(1-\theta_2)^{n-p}}{\theta_2^{m-p}} \frac{Df^{(n-1)}(a+\theta_2 k)}{(n-1)!}$$

$$(m \geqslant p; \ 0 < \theta_1, \theta_2 < 1)$$

II (C):

$$k^n \frac{(1-\theta_1)^{n-p}}{p} \frac{Df^{(n-1)}(a+\theta_1 k)}{(n-1)!} \geqslant R_n(f)$$
$$\geqslant k^n \frac{(1-\theta_2)^{n-p}}{p} \frac{Df^{(n-1)}(a+\theta_2 k)}{(n-1)!}$$

$$(0 < \theta_1, \theta_2 < 1)$$

II (D). II (C), with the additional condition that $f^{(n)}(a)$ exists, yields

$$\lim_{k \to 0} \frac{R_n(f)}{k^n/n!} = f^{(n)}(a)$$

which is Young's form of Taylor's theorem.

To prove this we put $p = n$ in II (C) obtaining

$$Df^{(n-1)}(a+\theta_1 k) \geqslant \frac{R^n(f)}{k^n/n!} \geqslant Df^{(n-1}(a+\theta_2 k)$$

We then choose $\delta > 0$ so that for $0 < k \leqslant \delta$,

$$f^{(n)}(a) + \epsilon \geqslant \frac{f^{(n-1)}(a+k) - f^{(n-1)}(a)}{k} \geqslant f^{(n)}(a) - \epsilon,$$

whence, recalling that the derivates and incrementary ratios of a continuous function have the same bounds in any interval, we establish that, for all sufficiently small k

$$f^{(n)}(a) + \epsilon \geqslant Df^{(n-1)}(a) + \theta_1 k$$
$$\geqslant \frac{R_n(f)}{k^n/n!} \geqslant Df^{(n-1)}(a+\theta_2 k) \geqslant f^{(n)}(a) - \epsilon.$$

This involves to the conclusion sought.

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CONCERNING RECIPROCAL SCREWS

BY

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1. F. J. Noronha's note "On Reciprocal Screws", *Maths. Student*, 1944, **12**, 73–4, has a central idea which can be presented in a more familiar form, more familiar at any rate to readers acquainted with the usual discussions of 'Forces in Three Dimensions'. In this form the idea can be embodied in a theorem which is nothing but a purely statical description of the relationship between reciprocal screws generally defined by our text-books in statico-kinematical terms.

THEOREM.—*For two screws to be reciprocal, a necessary and sufficient condition is that a wrench on one of the screws should have a pair of conjugate lines which are nul lines for a wrench on the other screw.*

The proof of this result depends on two others which (because they are well known) I state without their proofs.

(a) '*In the usual notation, let wrenches F, F' have the components*

$$(X, Y, Z; \ L, M, N), \tag{1}$$

$$(X', Y', Z'; \ L', M', N') \tag{2}$$

respectively, referred to an arbitrarily chosen set of rectangular axes. Then the screws on which the wrenches act are reciprocal (according to the usual definition) if and only if

$$\Sigma(LX' + L'X) = 0. \qquad (3)^*$$

(b) *For the wrench F', defined as in (2), the line l_k having (homogeneous) co-ordinates*

$$(X_k, Y_k, Z_k; \ L_k, M_k, N_k), \quad L_kX_k + M_kY_k + N_kZ_k = 0 \tag{4}$$

is a nul line if and only if

$$\Sigma(L_kX' + L'X_k) = 0. \tag{5}$$

Proof of Theorem.—Let us start with conjugate lines l_1, l_2 of F. Then if l_1 is given by co-ordinates

$$(X_1, Y_1, Z_1; \ L_1, M_1, N_1),$$

* The condition (3) for reciprocal screws, although it may not find a prominent place (or any place at all) in the commonly used text-books, is nearly three-quarters of a century old; its first expression being in Klein's paper in *Math. Annalen*, 1871, **4**, 403–15.

l_2 will be given by

$$(X - \lambda X_1, Y - \lambda Y_1, Z - \lambda Z_1;\ \ L - \lambda L_1, M - \lambda M_1, N - \lambda N_1)$$

λ being a constant determined by F and l_1.

To prove the necessity of the given condition, we have to show that if (3) is given and if (5) is satisfied with the co-ordinates of l_1 in place of the co-ordinates of l_k, then (5) is also satisfied with the coordinates of l_2 instead of those of l_k. This is obvious.

To prove the sufficiency of the given condition, we have to show that if (5) is satisfied with the coordinates of l_1, and also with those of l_2, then (3) follows. This is also obvious.

2. An immediate deduction from our theorem is

COROLLARY 1. *Given a tetrahedron ABCD, it is necessary and sufficient for two wrenches F, F′ to act on reciprocal screws that AB, CD should be conjugate lines of F and AC, BD should be conjugate lines of F′.*

This is the first of the two results in §2 of Noronha's note and leads at once to the second result. It needs no demonstration after our theorem since clearly *AB*, *CD* which are conjugate lines for *F* are nul lines for *F′*.

The proof of our theorem also leads to

COROLLARY 2.—*It is given that all but one of a finite number of forces act along nul lines of a wrench F′. Then a necessary and sufficient condition for the wrench F equivalent to the given forces to act on a screw reciprocal to that of F′ is that the excluded force should also act along a nul line of F′.*

ON THE EQUATION $ax^2 \pm by^2 = cz^2$

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1. If one solution in integers of the equation

$$ax^2 - by^2 = 1 \tag{1}$$

is known, we could at once write down explicitly an infinity of solutions of the same equation by the following simple device.

Let $x = m$, $y = n$ be a solution of equation (1). Write $am^2 = \cosh^2 \theta$. Then $bn^2 = \sinh^2 \theta$. If we write $ax^2 = \cosh^2 p\theta$, then $by^2 = \sinh^2 p\theta$ so that

$$x = m\frac{\cosh p\theta}{\cosh \theta}, \; y = n\frac{\sinh p\theta}{\sinh \theta} \tag{2}$$

satisfy equation (1). If p is odd, both $\frac{\cosh p\theta}{\cosh \theta}$ and $\frac{\sinh p\theta}{\sinh \theta}$ are polynomials in $\cosh^2 \theta$ and $\sinh^2 \theta$ so that (2) will give us integral solutions of (1) for all odd values of p. Thus we get

THEOREM 1.—*If $x = m$, $y = n$ be a solution in integers of equation* (1), *then*

$$x = \frac{(m\sqrt{a} + n\sqrt{b})^{2q-1} + (m\sqrt{a} - n\sqrt{b})^{2q-1}}{2\sqrt{a}},$$

$$y = \frac{(m\sqrt{a} + n\sqrt{b})^{2q-1} - (m\sqrt{a} - n\sqrt{b})^{2q-1}}{2\sqrt{b}}$$

are integral solutions of (1) *for all positive integral values of q.*

Corollary 1.—Taking $q = 2$ we see, after a little reduction, that if $x = m$, $y = n$ satisfy (1), then $x = m(4bn^2 + 1)$, $y = n(4am^2 - 1)$ will also be a solution.

Corollary 2. Take $b = a - 1$ in equation (1). It becomes

$$a(x^2 - y^2) = 1 - y^2 \tag{3}$$

This is obviously satisfied by $x = 1$, $y = 1$ for all a. It follows that

$$x = \frac{(\sqrt{a} + \sqrt{a-1})^{2q-1} + (\sqrt{a} - \sqrt{a-1})^{2q-1}}{2\sqrt{a}},$$

$$y = \frac{(\sqrt{a} + \sqrt{a-1})^{2q-1} - (\sqrt{a} - \sqrt{a-1})^{2q-1}}{2\sqrt{a-1}}$$

are solutions of (3) for all integral values of q.

In the solution (2), the restriction to odd values of p is not necessary if $a = 1$. In this case, equation (1) reduces to

$$x^2 - by^2 = 1. \tag{4}$$

If $x = m$, $y = n$ be a solution of (4), then writing $m = \cosh\theta$ we see that $x = \cosh p\theta$, $y = n\dfrac{\sinh p\theta}{\sinh\theta}$ also satisfy (4). Cosh $p\theta$ and $\dfrac{\sinh p\theta}{\sinh\theta}$ being both polynomials in $\cosh\theta$ for all integral values of p we get

THEOREM 2.—*If $x = m$, $y = n$ be a solution of* (4), *then*

$$x = \frac{(m+\sqrt{m^2-1})^p + (m-\sqrt{m^2-1})^p}{2},$$

$$y = n \cdot \frac{(m+\sqrt{m^2-1})^p - (m-\sqrt{m^2-1})^p}{2\sqrt{m^2-1}}$$

are also solutions for all integral values of p.

Corollary 1.—Taking $p = 2$ we see that if $x = m$, $y = n$ be a solution of (4) then $x = 2m^2 - 1$, $y = 2mn$ is also a solution.

Corollary 2.—Taking $b = c^2 - 1$ in (4) it becomes

$$x^2 + y^2 = 1 + (cy)^2 \tag{5}$$

This is obviously satisfied by $x = c$, $y = 1$. Therefore

$$x = \frac{(c+\sqrt{c^2-1})^p + (c-\sqrt{c^2-1})^p}{2}$$

$$y = \frac{(c+\sqrt{c^2-1})^p - (c-\sqrt{c^2-1})^p}{2\sqrt{c^2-1}}$$

are solutions of (5) for all integral values of p.

2. Let us now consider the equation

$$ax^2 + by^2 = cz^2 \tag{6}$$

and let $x = m$, $y = n$, $z = p$ be a solution in integers. Writing $am^2 = cp^2\cos^2\theta$ we have $bn^2 = cp^2\sin^2\theta$. Hence if we write $ax^2 = cz^2\cos^2 q\theta$, then the value of y which satisfies (6) will be given by $by^2 = cz^2\sin^2 q\theta$;

i.e., $$x = z \cdot \frac{m}{p}\,\frac{\cos q\theta}{\cos\theta},\quad y = z \cdot \frac{n}{p}\,\frac{\sin q\theta}{\sin\theta}.$$

If q is odd both $\dfrac{\cos q\theta}{\cos\theta}$ and $\dfrac{\sin q\theta}{\sin\theta}$ are polynomials in $\cos^2\theta$ and $\sin^2\theta$ so that $\dfrac{x}{y}$ and $\dfrac{y}{z}$ are both rational.

If we take $z = c^{\frac{q-1}{2}} p^q$, then both x and y will be integral. Thus we get

THEOREM 3.—*If $x = m$, $y = n$, $z = p$ satisfy equation* (6), *then the equation*

$$ax^2 + by^2 = (cz^2)^q \tag{7}$$

q being an odd integer, will be satisfied by

$$x = a^{\frac{q-1}{2}} m^q - \binom{q}{2} a^{\frac{q-3}{2}} bm^{q-2} n^2 + \binom{q}{4} a^{\frac{q-5}{2}} b^2 m^{q-4} n^4 - \ldots.$$

$$y = \binom{q}{1} a^{\frac{q-1}{2}} m^{q-1} n - \binom{q}{3} a^{\frac{q-3}{2}} bm^{q-3} n^3 + \binom{q}{5} a^{\frac{q-5}{2}} b^2 m^{q-5} n^5 - \ldots$$

$$z = p.$$

Corollary.—Let us take $c = a + b$ in equation (6). It becomes

$$ax^2 + by^2 = (a + b) z^2 \tag{8}$$

This is obviously satisfied by $x = 1$, $y = 1$, $z = 1$. It follows that the equation

$$ax^2 + by^2 = (a + b)^q \tag{9}$$

q being odd, is satisfied by

$$x = a^{\frac{q-1}{2}} - \binom{q}{2} a^{\frac{q-3}{2}} b + \binom{q}{4} a^{\frac{q-5}{2}} b^2 - \ldots.$$

$$y = \binom{q}{1} a^{\frac{q-1}{2}} - \binom{q}{3} a^{\frac{q-3}{2}} b + \binom{q}{5} a^{\frac{q-5}{2}} b^2 - \ldots.$$

If in equation (6) $a = c = 1$, so that it becomes

$$x^2 + by^2 = z^2, \tag{10}$$

and $x = m$, $y = n$, $z = p$ is a solution of (10), we may write $m = p \cos \theta$, $bn^2 = p^2 \sin^2 \theta$ from which it readily follows that

$$x = z \cos q\theta,\ y = z \cdot \frac{n}{p} \cdot \frac{\sin q\theta}{\sin \theta} \tag{11}$$

satisfy (10). For all integral values of q, $\cos q\theta$ and $\frac{\sin q\theta}{\sin \theta}$ are polynomials in $\cos \theta$, so that (11) gives rational values of $\frac{x}{z}$ and $\frac{y}{z}$ which satisfy (10). Moreover taking $z = p^q$, both x and y become integral. Thus we get

THEOREM 4.—*If $x = m$, $y = n$, $z = p$ satisfy equation (10), then a solution of the equation*

$$x^2 + by^2 = z^{2q} \tag{12}$$

will be given by

$$x = m^q - \binom{q}{2} bm^{q-2} n^2 + \binom{q}{4} b^2 m^{q-4} n^4 - \ldots,$$

$$y = \binom{q}{1} m^{q-1} n - \binom{q}{3} bm^{q-3} n^3 + \binom{q}{5} b^2 m^{q-5} n^5 - \ldots.$$

$$z = p,$$

for all integral values of q.

3. In an exactly similar manner, using hyperbolic functions instead of circular functions, we get

THEOREM 5. *If $x = m$, $y = n$, $z = p$ be a solution of*

$$ax^2 - by^2 = cz^2 \tag{13}$$

then the equation

$$ax^2 - by^2 = (cz^2)^q \tag{14}$$

will be satisfied (for odd values of q if $a \neq c$ and for all values of q if $a = c = 1$) by

$$x = a^{\frac{q-1}{2}} m^q + \binom{q}{2} a^{\frac{q-3}{2}} bm^{q-2} n^2 + \ldots .$$

$$y = \binom{q}{1} a^{\frac{q-1}{2}} m^{q-1} n + \binom{q}{3} a^{\frac{q-3}{2}} bm^{q-3} n^3 + \ldots .$$

$$z = p.$$

Corollary.—Writing $c = a - b$ we see that the equation

$$ax^2 - by^2 = (a - b)^q \tag{15}$$

q being odd, is satisfied by

$$x = a^{\frac{q-1}{2}} + \binom{q}{2} a^{\frac{q-3}{2}} b + \binom{q}{4} a^{\frac{q-5}{2}} b^2 + \ldots .$$

$$y = \binom{q}{1} a^{\frac{q-1}{2}} + \binom{q}{3} a^{\frac{q-3}{2}} b + \binom{q}{5} a^{\frac{q-5}{2}} b^2 + \ldots$$

COLLEGIATE SECTION

A Mean-Value Theorem and its Applications

1. *Differentiation of a determinant.*—Let

$$\phi = \begin{vmatrix} u_{11} & u_{12} & \dots & u_{1n} \\ u_{21} & u_{22} & \dots & u_{2n} \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ u_{n1} & u_{n2} & \dots & u_{nn} \end{vmatrix}$$

be a determinant of order n where u_{ij} are functions of x. Then ϕ' is the sum of n determinants obtained by differentiating the elements in each column and leaving the other columns unaltered. Instead of columns we can also differentiate the rows. Thus

$$\phi' = \sum_{j=1}^{n} \begin{vmatrix} u_{11} & \cdot\cdot & u'_{1j} & \cdot\cdot & u_{1n} \\ u_{21} & \cdot\cdot & u'_{2j} & \cdot\cdot & u_{2n} \\ \cdot\cdot & & & & \\ \cdot\cdot & & & & \\ \cdot\cdot & & & & \\ u_{n1} & \cdot\cdot & u'_{nj} & \cdot\cdot & u_{nn} \end{vmatrix} = \sum_{i=1}^{n} \begin{vmatrix} u_{11} & \dots & u_{1n} \\ u_{21} & \dots & u_{2n} \\ u'_{i1} & \dots & u'_{in} \\ u_{n1} & \dots & u_{nn} \end{vmatrix}$$

2. The object of this note is to prove the following general mean value theorems and indicate some of their applications. It turns out that many of the examples on mean value theorems usually found in text-books are special cases of Theorem 1 and its generalisation given in Theorem 2.

THEOREM 1. *Let $f(x)$ and its first $(n-1)$ derivatives be continuous in the closed interval (a, b) and let $f^{(n)}(x)$ exist in $a < x < b$. Let $x_0 < x_1 < x_2 \dots < x_n$ be $(n+1)$ distinct points in (a, b). Then*

$$\begin{vmatrix} f(x_0) & f(x_1) & \dots & f(x_n) \\ x_0^{n-1} & x_1^{n-1} & \dots & x_n^{n-1} \\ \cdot\cdot & \cdot\cdot & \dots & \cdot\cdot \\ \cdot\cdot & \cdot\cdot & \dots & \cdot\cdot \\ \cdot\cdot & \cdot\cdot & \dots & \cdot\cdot \\ 1 & 1 & & 1 \end{vmatrix} = \frac{f^{(n)}(\xi)}{n!} \begin{vmatrix} x_0^{n} & x_1^{n} & \dots & x_n^{n} \\ x_0^{n-1} & x_1^{n-1} & \dots & x_n^{n-1} \\ \cdot\cdot & \cdot\cdot & \dots & \cdot\cdot \\ \cdot\cdot & \cdot\cdot & \dots & \cdot\cdot \\ \cdot\cdot & \cdot\cdot & \dots & \cdot\cdot \\ 1 & 1 & & 1 \end{vmatrix}$$

for some ξ in $a < \xi < b$.

Remark.—Those who know the elements* of the Calculus of Finite Differences will see that the value of $\frac{f^{(n)}(\xi)}{n!}$ in the above equation is the *divided difference* of $f(x)$ with respect to the $(n+1)$ points $(x_0, \ldots, x_n)$. The above result is known, but the proof below is simpler than the usual proofs.

$$\textit{Proof.}\text{—Let } \phi(x) = \begin{vmatrix} f(x) & f(x_0) & \ldots & f(x_n) \\ x^n & x_0^{\,n} & \ldots & x_n^{\,n} \\ x^{n-1} & x_1^{\,n-1} & \ldots & x_n^{\,n-1} \\ \cdot\cdot & \cdot\cdot & \ldots & \cdot\cdot \\ \cdot\cdot & \cdot\cdot & \ldots & \cdot\cdot \\ 1 & 1 & \ldots & 1 \end{vmatrix}$$

Then $\phi(x)$ has its first $(n-1)$ derivatives continuous in (a, b) and $\phi^{(n)}(x)$ exists in $a < x < b$. Now

$$\phi(x_0) = \phi(x_1) = \ldots = \phi(x_n) = 0.$$

Hence by Rolle's theorem there exist n points $(\xi_{11}, \xi_{12}, \ldots \xi_{1n})$

$$[x_0 < \xi_{11} < x_2 < \xi_{12} < x_3 \ldots < x_{n-1} < \xi_{1n} < x_n]$$

such that $\qquad \phi'(\xi_{11}) = \ldots = \phi'(\xi_{1n}) = 0.$

Again applying Rolle's theorem to $\phi'(x)$, there exist $(n-1)$ points $(\xi_{21}, \xi_{22}, \ldots \xi_{2\overline{n-1}})$ such that

$$\phi''(\xi_{21}) = \ldots = \phi''(\xi_{2,\overline{n-1}}) = 0.$$

Repeating the argument we see that there exists at least one point $\xi_{n1} = \xi$ $(a < \xi < b)$ such that $\phi^{(n)}(\xi) = 0$. Now using the rule for differentiation given in § 1 and noting that only the elements of the first column are functions of x, we get that

$$\phi^{(n)}(x) = \begin{vmatrix} f^{(n)}(x) & f(x_0) & \ldots & f(x_n) \\ n! & x_0^{\,n} & \ldots & x_n^{\,n} \\ 0 & x_0^{\,n-1} & \ldots & x_n^{\,n-1} \\ \cdot\cdot & \cdot\cdot & \ldots & \cdot\cdot \\ \cdot\cdot & \cdot\cdot & \ldots & \cdot\cdot \\ 0 & 1 & \ldots & 1 \end{vmatrix}.$$

Putting $x = \xi$ and expanding in terms of the first column we get the required result.

3. In Theorem 1 we have supposed that the points $(x_0, \ldots x_n)$ are distinct. In Theorem 2 we give the general result when some points coincide.

* See Milne-Thompson : *Calculus of Finite Differences.*

THEOREM 2.—*Let $f(x)$ satisfy the hypothesis of Theorem* 1. *Let $(x_1, \ldots, x_p)$ be p distinct points in (a, b). Let x_1 be counted k_1 times, x_2 be counted k_2 times and so on. Let $k_1 + k_2 + \ldots + k_p = (n+1)$. Then*

$$\begin{vmatrix} f(x_1) & f'(x_1) & \cdots & f^{(k_1-1)}(x_1) & f(x_2) & f'(x_2) & \cdots & f^{(k_2-1)}(x_2) & \cdots \\ x_1^{n-1} & (n-1)_1 x_1^{n-2} & \ldots & (n-1)_{k_1-1} x_1^{n-k_1} & x_2^{n-1} & (n-1)_1 x_2^{n-2} & \ldots & (n-1)_{k_2-1} x_2^{n-k_2} & \ldots \\ \cdot\cdot & \cdot\cdot & & \cdot\cdot & \cdot\cdot & \cdot\cdot & & \cdot\cdot & \\ \cdot\cdot & \cdot\cdot & & \cdot\cdot & \cdot\cdot & \cdot\cdot & & \cdot\cdot & \\ \cdot\cdot & \cdot\cdot & & \cdot\cdot & \cdot\cdot & \cdot\cdot & & \cdot\cdot & \\ 1 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & \cdots \end{vmatrix}$$

$$= \frac{f^{(n)}(\xi)}{n!} \begin{vmatrix} x_1^n & n_1 x_1^{n-1} & \cdots & n_{k_1-1} x_1^{n-k_1+1} & x^{2n} & n_1 x_2^{n-1} & \cdots & n_{k_2-1} x_2^{n-k_2+1} \ldots \\ x_1^{n-1} & (n-1)_1 x_1^{n-2} & \cdots & \cdot\cdot & x_2^{n-1} & (n-1)_1 x_2^{n-2} & \cdots & \cdot\cdot \\ \cdot\cdot & \cdot\cdot & & \cdot\cdot & \cdot\cdot & & & \cdot\cdot \\ \cdot\cdot & \cdot\cdot & & \cdot\cdot & \cdot\cdot & & & \cdot\cdot \\ \cdot\cdot & \cdot\cdot & & \cdot\cdot & \cdot\cdot & & & \cdot\cdot \\ 1 & 0 & \cdots & \cdot\cdot & 1 & 0 & \cdots & 0 \end{vmatrix}$$

where $n_t = n(n-1)\ldots(n-t+1)$ and $a < \xi < b$:

Proof.—Let

$$\phi(x) = \begin{vmatrix} f(x) & f(x_1) & f'(x_1) & \cdots & f^{(k_1-1)}(x_1) & f(x_2) & \cdots & f^{(k_2-1)}(x_2) & \cdots \\ x^n & x_1^n & n_1 x_1^{n-1} & \cdots & n_{k_1-1} x_1^{n-k_1+1} & x_2^n & \cdots & n_{k_2-1} x_2^{n-k_2+1} & \ldots \\ \cdot\cdot & & & & & & & & \\ \cdot\cdot & & & & & & & & \\ \cdot\cdot & & & & & & & & \\ 1 & 1 & 0 & \ldots & 0 & 1 & \cdots & 0 & \cdots \end{vmatrix}$$

We see that

$$\phi(x_1) = \ldots = \phi^{(k_1-1)}(x_1) = 0; \ \phi(x_2) = \ldots = \phi^{(k_2-1)}(x_2) = 0; \ldots$$

so that $\phi(x) = 0$ at $(n+1)$ points, *viz.*, at x_1 repeated k_1 times; at x_2 repeated k_2 times and so on. Hence $\phi'(x)$ vanishes at n points, *viz.*, at x_1 repeated $(k_1 - 1)$ times; at x_2 repeated $(k_2 - 1)$ times and so on *and* at $(p-1)$ other points $(\xi_{11}, \xi_{12}, \ldots \xi_{1\overline{p-1}})$. We can repeat the argument and conclude that $\phi''(x)$ is zero at $(n-1)$ points and finally that $\phi^{(n)}(x)$ vanishes at some point ξ in $a < \xi < b$. Writing out $\phi^{(n)}$ and putting $x = \xi$ we get the result required.

Remark.—If we examine the proof of Theorem 1, it will be seen that if $f(x)$ is continuous in (a, b) and $f'(x)$ exists in $a < x < b$, then n points $(\xi_{11}, \xi_{12}, \ldots \xi_{1n})$ exist in $a < x < b$ such that $\phi'(\xi_{11}) = \ldots = \phi'(\xi_{1n}) = 0$.

We can now continue the argument if we merely suppose that $f''(x)$ exists in $a < x < b$. Hence Theorem 1 holds even if we suppose only that $f(x)$ is continuous in (a, b) while $f^{(i)}(x)$ exists for $a < x < b$ $(i = 1, 2, \ldots, n)$. But this cannot be supposed in Theorem 2, since if $x_1 = a$ repeated twice, we require the existence $f'(x)$ at $x = a$.

4. In applying the above results the following will be useful:

$$\begin{vmatrix} x_0^n & x_1^n & \ldots & x_n^n \\ x_0^{n-1} & x_1^{n-1} & \ldots & x_n^{n-1} \\ \cdot\cdot & \cdot\cdot & & \cdot\cdot \\ \cdot\cdot & \cdot\cdot & & \cdot\cdot \\ 1 & 1 & \ldots & 1 \end{vmatrix} = (x_0-x_1)\ldots(x_0-x_n)\times(x_1-x_2)\ldots(x_1-x_n)\times \cdots \times(x_{n-1}-x_n)$$

and

$$\begin{vmatrix} x_1^n & n_1 x_1^{n-1} & \cdots n_{k_1-1} x_1^{n-k_1+1} & x_2^n & n_1 x_2^{n-1} & \cdots n_{k_2-1} x^{n-k_2+1} \ldots \\ \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot \\ \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot \\ 1 & 0 & 0 & 1 & 0 & 0 \end{vmatrix}$$

$$= (-1)^{\lambda} A (x_1 - x_2)^{k_1 k_2} (x_1 - x_3)^{k_1 k_3} \ldots (x_1 - x_p)^{k_1 k_p} \times (x_2 - x_3)^{k_2 k_3} \ldots (x_2 - x_p)^{k_2 k_p} \times \ldots \times (x_{p-1} - x_p)^{k_{p-1} k_p},$$

where $\lambda = \Sigma \frac{1}{2} k_i (k_i - 1)$ and $A = \prod_{i=1}^{p} \{1!\,2! \ldots (k_i - 1)!\}$.

The first one is well known and the second is obtained from the first by differentiating the second, third, ... k_1th columns once with respect to x_1, twice with respect to $x_2, \ldots, (k_1 - 1)$ times with respect to x_{k_1} respectively and afterwards replace the x's in the first k_1 columns by x_1; and similarly dealing with the next k_2 columns and so on.

4·1. *Special case* 1.—Let $x_0 = x_1 = \ldots = x_{n-1} = a$ and $x_n = b$. This gives (by Theorem 2) that*

$$f(b) = f(a) + (b-a) f'(a) + \ldots + \frac{(b-a)^{n-1}}{(n-1)!} f^{n-1}(a) + (b-a)^n \frac{f^{(n)}(\xi)}{n!}$$

$a < \xi < b$. This is the usual Taylor's theorem.

Special case 2.—$x_0 = x_1 = a$; $x_2 = x_3 = b$. This gives the result

$$f(b) = f(a) + \frac{b-a}{2} \{f'(a) + f'(b)\} - \frac{(b-a)^3}{12} f^{(3)}(\xi),$$

* The calculation in this as well as some of the following cases are a bit long. But they are merely exercises on determinants and often the details of calculations would become evident by examining special cases and by the use of the two results in § 4.

$a < \xi < b$. Written in the form

$$\int_a^b \phi(x)\,dx = \frac{b-a}{2}[\phi(a)+\phi(b)] - \frac{(b-a)^3}{12}\phi''(\xi)$$

the result is found in Hardy's *Pure Mathematics* (H.P.M.), page 329 (7th Edition, 1938).

Special case 3.—Let $x_0 = x_1 = a$; $x_2 = x_3 = \frac{a+b}{2} = c$; $x_4 = x_5 = b$.

We get

$$f(b) = f(a) + \frac{b-a}{6}\{f'(a) + 4f'(c) + f'(b)\} - \frac{(b-a)^5}{2880} f^{(5)}(\xi).$$

Written in the form

$$\int_a^b \phi(x)\,dx = \frac{b-a}{6}\{\phi(a) + 4\phi(c) + \phi(b)\} - \frac{(b-a)^5}{2880}\phi^{(4)}(\xi)$$

this gives the usual Simpson's rule for approximate evaluation of an area (see H.P.M.: l.c. p. 329).

Special case 4.—Let $x_0 = x_1 = x_2 = a$; $x_3 = x_4 = b$.

Then we get

$$f(b) = f(a) + \frac{b-a}{3}[2f'(a) + f'(b) + \frac{(b-a)^2}{6} f''(a) - \frac{(b-a)^4}{72} f^{(4)}(\xi).$$

This is found as an example in one of the older editions of H.P.M.

Special case 5.—We conclude with one more illustration. Let $x_0 = a$, $x_1 = a + h, \ldots, x_n = a + nh = b$.

Here we get

$$\Delta^n_h\,\phi(a) = \sum_{r=0}^{n} (-1)^r \phi(a+rh) = (-h)^n \phi^n(\xi)$$

where $a < \xi < b$ (See H.P.M., p. 333).

5. Special cases can be multiplied. It may be noted that all the special cases can be proved directly and often more easily. But all these proofs depend on proper choice of auxiliary functions to which Rolle's theorem is to be applied and in many of these cases the choice of the auxiliary functions is not evident. It is remarkable that all these special cases and many others are particular renderings of the general results in Theorems 1 and 2.

Annamalainagar. V. GANAPATHY IYER.

Generalization of the Law of the Mean

Goursat quotes the following form due to Stieltjes:—

If $f(x)$, $\phi(x)$ and $\psi(x)$ are continuous in the closed interval (a, b) and are differentiable in the open interval (a, b), then

$$\begin{vmatrix} f(a) & f(b) & f(c) \\ \phi(a) & \phi(b) & \phi(c) \\ \psi(a) & \psi(b) & \psi(c) \end{vmatrix} = \tfrac{1}{2}(b-c)(c-a)(a-b) \begin{vmatrix} f(a) & f'(\xi) & f''(\eta) \\ \phi(a) & \phi'(\xi) & \phi''(\eta) \\ \psi(a) & \psi'(\xi) & \psi''(\eta) \end{vmatrix}$$

where ξ, η lie between the least and the greatest of a, b, c.

This result has been proved by me in my "Lessons in Elementary Analysis" (3rd edition, page 115) by a method which is easily extended to give the most general form as under.

2. *The Result.*—If $f_1(x), f_2(x), f_3(x), \ldots f_n(x)$ are continuous in the closed interval (a, k), and differentiable in the open interval (a, k), then

$$\begin{vmatrix} f_1(a) & f_1(b) & f_1(c) & \ldots f_1(k) \\ f_2(a) & f_2(b) & f_2(c) & \ldots f_2(k) \\ \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ f_n(a) & f_n(b) & f_n(c) & \ldots f_n(k) \end{vmatrix} \div \begin{vmatrix} 1 & 1 & 1 & \ldots 1 \\ a & b & c & \ldots k \\ a^2 & b^2 & c^2 & \ldots k^2 \\ \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ a^{n-1} & b^{n-1} & c^{n-1} & \ldots k^{n-1} \end{vmatrix}$$

is equal to

$$\begin{vmatrix} f_1(a) & f_1'(a) & f_1''(\beta) & \ldots f_1^{n-1}(\lambda) \\ f_2(a) & f_2'(a) & f_2''(\beta) & \ldots f_2^{n-1}(\lambda) \\ f_3(a) & f_3'(a) & f_3''(\beta) & \ldots f_3^{n-1}(\lambda) \\ \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ f_n(a) & f_n'(a) & f_n''(\beta) & \ldots f_n^{n-1}(\lambda) \end{vmatrix} \div \begin{vmatrix} 1 & 0 & 0 & & \ldots 0 \\ a & 1 & 0 & & \ldots 0 \\ a^2 & 2a & 2\cdot 1 & & \ldots 0 \\ a^3 & 3a^2 & 3\cdot 2\beta & 3\cdot 2\cdot 1 & \ldots 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ a^{n-1} & \overline{n-1}a^{n-2} & \ldots & \ldots & (n-1)! \end{vmatrix}$$

Thus we get the final result:

$$\begin{vmatrix} f_1(a) & f_1(b) & f_1(c) & \ldots f_1(k) \\ f_2(a) & f_2(b) & f_2(c) & \ldots f_2(k) \\ f_3(a) & f_3(b) & f_3(c) & \ldots f_3(k) \\ \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ f_n(a) & f_n(b) & f_n(c) & \ldots f_n(k) \end{vmatrix} =$$

$$\frac{1}{(1\,!)\,(2\,!)\,(3\,!)\ldots(\overline{n-1}\,!)}\,\Pi(b-c)\begin{vmatrix} f_1(a) & f_1'(\alpha) & f_1''(\beta) & \ldots f_1^{n-1}(\lambda) \\ f_2(a) & f_2'(\alpha) & f_2''(\beta) & \ldots f_2^{n-1}(\lambda) \\ f_3(a) & f_3'(\alpha) & f_3''(\beta) & \ldots f_3^{n-1}(\lambda) \\ \ldots & \ldots & \ldots & \ldots \\ f_n(a) & f_n'(\alpha) & f_n''(\beta) & \ldots f_n^{n-1}(\lambda) \end{vmatrix}$$

where $\Pi(b-c)$ denotes the product of the differences of the n numbers $a, b, c \ldots . k$ and $\alpha, \beta, \gamma \ldots . \lambda$ denote $(n-1)$ numbers between the least and the greatest of those numbers.

DR. G. S. MAHAJANI.

A Note on Continued Fractions

If $\alpha = \dot{a}_1 + \frac{1}{a_2 +} \cdots + \frac{1}{\dot{a}_n}$ then a is the positive root of a quadratic equation with rational coefficients, and if β be the negative root of the equation then

$$-\frac{1}{\beta} = \dot{a}_n + \frac{1}{a_{n-1} +} \ldots\ldots + \frac{1}{\dot{a}_1}.$$

[*Vide*: Charles Smith's "Algebra," p. 462.]

The following is an alternative proof of the above proposition:—

Let $\frac{p_{n-1}}{q_{n-1}}, \frac{p_n}{q_n}$ be the last two convergents of the first period.

Then $$\alpha = \frac{\alpha\, p_n + p_{n-1}}{\alpha\, q_n + q_{n-1}}$$

Hence α is a root of the equation

$$x^2 q_n + x\,(q_{n-1} - p_n) - p_{n-1} = 0. \qquad (1)$$

The roots of this are obviously of different signs and the positive root is the value of the continued fraction. Let the negative root be β. Then α and β are conjugate surds. The same arguments apply to the kth period so that we have equally

$$x^2 q_{kn} + x\,(q_{kn-1} - p_{kn}) - p_{kn} = 0. \qquad (2)$$

The sequence of equations (2) obtained by putting $k = 1, 2, 3 \ldots .$ have a common root α and have also the conjugate surd β, since the coefficients are rational. Therefore they have all the same roots.

Hence $\alpha\beta = \frac{-p_{kn-1}}{q_{kn}}$ where k is any positive integer.

Let $$k \to \infty ; \quad \frac{-p_{kn-1}}{q_{kn}} = \frac{-p_{kn-1}}{q_{kn-1}} \times \frac{q_{kn-1}}{q_{kn}}$$

$$\therefore \lim_{K \to \infty} \frac{-p_{kn-1}}{q_{kn}} = \lim_{k \to \infty} \frac{-p_{kn-1}}{q_{kn-1}} \times \frac{q_{kn-1}}{q_{kn}}$$

$$\therefore \alpha\beta = -\alpha \times \lim_{k \to \infty} \frac{q_{kn-1}}{q_{kn}}$$

That is $\beta = \lim_{k \to \infty} \frac{-q_{kn-1}}{q_{kn}}$, and since $\beta \neq 0$, we get

$$-\frac{1}{\beta} = \lim_{k \to \infty} \frac{q_{kn}}{q_{kn-1}} = a_n + \frac{1}{a_{n-1} +} \cdots\cdots + \frac{1}{a_1}.$$

Matunga, Bombay. K. A. VISWANATHAN.

Problems connected with Courses of Study in Higher Mathematics in Indian Universities

I

This note is in response to the kind invitation of the Editor of *The Mathematics Student* to state my views on some of the problems relating to Courses of Studies in Mathematics in Indian Universities. In this note I confine myself to some aspects of these problems in so far as they concern advanced courses of studies in mathematics. Though in some places a reference is made to the post-intermediate honours degree course of three years' duration the remarks made will apply to all advanced courses in mathematics. It is my object to make out that some of the problems are peculiar to the subject of mathematics; for instance, an advanced mathematical course intended for the very bright ones among those that have passed the intermediate examination is as unsuitable for the mediocre students as a post-graduate course is unsuitable to one who has merely passed the matriculation examination. Some of the serious defects from which the present courses suffer arise from the fact that this is not fully realized by those who are in charge of making admissions to these courses of studies; no one can blame a non-mathematician if he argues by analogies and thinks that a student who has secured 60% marks in mathematics in the intermediate examination is necessarily a keen mathematician who deserves to be encouraged by being admitted to the honours course. They know not what they do when they admit students on such considerations. Some heads of departments of mathematics recommend the admission of mediocre students also but in this note I do not propose to go into the reasons for such recommendations. In Section IV, I discuss the harmful effects of such admissions; in Section V, I draw attention to the easily avoidable but considerable waste of energy that occurs when students attend lectures but have no books or cyclostyled (or printed) notes of the teacher to consult before, during and after each lecture. I shall discuss in a later article some other aspects of the problem of propogation and creation of higher mathematical knowledge.

II

In the Madras University the honours course of three years' duration was started in the year 1911. The chief object was to have fairly advanced courses of study in various subjects. Syllabuses were framed for such courses of study in each subject and the courses were intended to be taken exclusively by the very bright among those that had passed the intermediate examination. A student could appear for the honours examination once only and was not permitted to appear for it after four years of his passing the intermediate examination. I have been told that in one of the colleges where upto about the year 1920 the admissions were made strictly according to certain rules of their own devising a student was not ordinarily admitted to the honours course in mathematics unless he had secured in the intermediate examination the following minimum marks*: mathematics 90%; English 50%; physics 60%. It was thought that the course was and should be such that only the best talents could cope with it.

It cannot be said that at present the original object is strictly adhered to and, in my opinion, it is high time that the whole matter is gone into in detail. I think that there is a strong case for a three-year course in mathematics meant exclusively for the very brightest among those who have passed the intermediate examination but if the Government and Universities do not favour such an exclusive course then, so far as mathematics is concerned, that fact should be stated explicitly and a course of study planned suitably.

III

The abstract nature of the subject and the frequent use of difficult, complex, novel or subtle arguments make the study of higher mathematics unsuitable to those who have not the requisite aptitude, desire and facilities. An excellent lecture in mathematics *may almost wholly be unintelligible* to a student either because he is not quick enough to follow the flow of reasoning in the lecturer's arguments

* A good reason for insisting upon high marks in other subjects too such as English and Physics was that those marks implied that the g. q. i. (general quotient of intelligence) of the student was quite satisfactory. There is a really important reason as to why when making admissions to the honours course in mathematics one should, in general, take into account the g. q. i. of students. The reason is that the kind of mathematics that is taught to an honours student is so different from the intermediate mathematics that high marks in mathematics alone in the intermediate examination do not usually imply the fitness of the student to take the honours course.

or because he is not fully acquainted with the results proved in earlier lectures, which results are being used in the course of the latest lecture. When a series of lectures are delivered on a single connected topic in mathematics all but the bright and industrious students fail to follow what is being said after the first two or three lectures. This is because a student has to be bright and smart to follow the earlier lectures, and unless he has regular habits of study he does not succeed in consolidating what he has been taught. And a thorough acquaintance with much of what has preceded is essential for an understanding of the later lectures; indeed this is true of most mathematical lectures. Such considerations may not apply in the same degree to lectures in other subjects.

If the syllabus covers a lot of mathematics, and the time prescribed for the course is comparatively short then students of a certain calibre only can be admitted to the course. If the admissions are to be made so as to include a wider class of students then either the syllabus should be lightened considerably or the period of study for the course be increased suitably, say, by one year, or, what is undesirable, we should reconcile ourselves to a large number of failures or lower standards in examinations.

IV

Many evil consequencies follow if there is a great disparity between the actual (low) average calibre of students admitted to a course of study and the calibre envisaged at the time of divising the course. Some of these evils are described in the next few paragraphs.

When an honours class in mathematics contains a large number of students who are not fit to take up the honours course the teachers find that to suit the needs of weaker pupils, they (the teachers) have to water down the contents of each lecture and omit the more difficult and subtle points of proofs without the students being made aware of the incompleteness of such proofs (or a brief and unintelligible note is dictated to cover the points). A common but sad experience of the teachers is that often students give proofs with gaping omissions; this shows that the student not only does not know how to prove the theorem on hand but has no idea of what a proof is. And nothing can be more deplorable than this at the honours stage. The inner meaning and the deeper significances of theorems and proofs are not discussed because such discussions will be unintelligible to most students and the teacher does not want to lose his reputation of being a good expositor.

Since the lectures have to be diluted there arises a need for a larger number of lectures; moreover the teacher does in the class-room much of the work that competent students should do for themselves. Much that is important, interesting and even essential is not referred to at all either because all the available time is needed for teaching the more elementary parts of the subject or because such discussions will not be understood by a majority of students. There is no time to discuss generalizations, converse theorems, the general background and, naturally, no references are made to outstanding unsolved problems* which lie pretty near to the topic under discussion. This lack of time for essential discussions is there in spite of the fact that in some colleges and universities students have to attend every week much more than twenty lectures, each of one hour's duration. This is a kind of barley-and-glucose-water feeding and does harm to good students who are thus sacrificed to the imagined interests of weak students. Moreover the physical and other strains of listening to so many mathematical lectures on different subjects, though the lectures are not concentrated ones, make it difficult if not undesirable for the students to attempt to read books for themselves and to learn to think independently. Having become accustomed to have every petty little detail explained to them they find that even an article in a text-book is difficult to understand unless the teacher has previously lectured on it.† Few if any among the honours students ever dream of reading a paper in a mathematical journal.

This state of affairs has its reactions on the standard of examinations, the syllabuses, the method of recruitment to the teaching staff and on the quality of teachers available. In course of time most of the recruitments to teaching posts have to be made from among students trained in the above manner. The teacher who is appreciated by some colleges and universities is the man who has (worldly) commonsense and has no nonsense in him by way of learning, critical scholarship, love of research or capacity to enthuse his students. Universities may for show purposes recruit one man with some reputation to each department of study but colleges often act on an unwritten but firm policy of excluding from their staff any one with a scientific reputation. This is not because that colleges are run by men who hate knowledge but because the aim of many colleges is to secure good results in university examinations in spite of having admitted many students below the normal calibre expected of those that take up such courses of studies. Under the circumstances it stands to reason that special devices should be adopted to secure results; one device is to ensure that examinations are of a narrow fixed pattern where no searching questions can be set except perhaps as an alternative to a routine question. In course of time the syllabuses are either framed to suit the weaker students or interpreted in such a way as to lower its content. I know of at least one university where it is usual to set 12 questions, mostly book propositions, all of which the candidates are free to attempt. Each question carries 16 marks and the total marks for the paper is fixed at 100.

* References to unsolved problems, especially when the statement of the problem is such as can be well understood, have a stimulating effect in many ways except on those who have no sense of curiosity, and think that any discussion is a wasteful digression if no questions are likely to be set on it in the university examinations.

† This is probably one of the reasons for the reluctance on the part of students to buy text-books; they are unintelligible any way, and why bother when it is known that the teacher will give the contents of each article with many additional explanations so that, for the moment even those that have forgotten much relevant mathematics can follow what is being said in the class. Incidentally this encourages students to come to lectures unprepared.

Many a student secures 150 or more marks, and those that secure 100 or more marks are shown as having secured 100% marks. But even among the best of Indian universities the examinations are such that mediocre but industrious students with good memory can do very well, especially if they are coached and drilled. This coaching and drilling are, obviously, unavoidable if weak students with no liking for mathematics are to be enabled to pass examinations. Even the better type of students get higher marks in such examinations if they are coached and drilled. Since new intelligent questions are not set in examinations it does not 'pay' for the good students too to take any trouble to look out beyond what is done in the class-room. Initiative and originality are at a disadvantage so far as concerns the securing of a rank in such an examination.

The advantages of coaching and drilling are of a kind that are easily demonstrable, nay, visible. They bear fruit within a short time and the fruits (passing in examinations) are such as are much sought after. The disadvantages are not so easily discerned though they constitute a national disaster. There are more universities in India than in any one of such countries as France, Italy, Canada, Sweden, etc. Yet what is the status of India in the scientific world as compared with any of the abovementioned countries? Reforms are urgently needed but to the inherent difficulties in the situation are to be added the opposition of vested interests.

V

I am strongly of the view that much time and energy is wasted on account of

(i) the students not having suitable text-books,

and (ii) lack of cyclostyled (or printed) notes of lectures.

If a student knows what will precisely be the topics of lectures on any day and comes to the college prepared for them, having already read up the relevant articles in books and notes, and again after the lectures goes home and consolidates what he has learnt during the day by further reading and writing out his own notes and solving exercises and problems then he will find it easy to remember what he has been taught, and what is no less important, will be in a position to follow intelligently the succeeding lectures. To obtain an equivalent result without following such a procedure is not possible. But unfortunately it is usual to see that just when students need books most, *viz.*, in the beginning of the course, they are without books. There should be a rule that if a student is well to do then he should deposit along with his fees at the time of his admission the cost of text-books which he is expected to possess. If he is poor but it is considered that he deserves to be admitted to the course then he should be given a set of text-books. This seems to be the proper thing to do if we aim at a democratic and efficient system of education. To say that it is the student's business to have or not to have books is manifestly wrong. It is even more wrong to have unexceptionable rules about this matter and never enforce them.

We might as well attempt to give modern military training without the recruit handling a single weapon.

I give below a quotation about the usefulness of distributing copies of printed (or cyclostyled) notes beforehand. Even if text-books are prescribed these notes could contain much extra matter and form a supplement to the book. Moreover the notes will indicate the order in which the topics will be taken, and give a list of problems to be done along with lectures, and at home at each stage. It will minimize the amount of notes that a student has to take during each lecture and thereby enable him to concentrate on the lecture. It will help the external examiners to have a precise idea of the standard aimed at. The quotation referred to above now follows. It is from the preface to a printed edition of notes of lectures by J. E. Littlewood, M.A., F.R.S. [*The Elements of the Theory of Real Functions*, Cambridge (England), W. Heffer & Sons Ltd., 1946].

" ... The subject calls for great precision of statement, and experience has taught me, when lecturing upon it, to dictate word for word all enunciations and proofs. Rather more than half the time has probably been spent in this, the remainder being devoted to explanation and comment. It became an obvious course to print the matter formerly dictated, and I carried out the experiment of lecturing from the first edition of printed notes in the Michaelmas term of 1925. The present second edition is slightly enlarged and is intended to be intelligible independently of the lectures.

I hope, however, that my best pupils in the University of Cambridge will not too hastily assume that the existence of the notes makes attending the lectures themselves entirely valueless. I am one of those who believe that lectures can have great value, and particularly at a certain moderately advanced stage of mathematical education. The modern standard of conciseness and lucidity in original papers and advanced text-books is on the whole a high one, but the style is one for expert only. We may demand two things of an original paper, a complete and accurate exposition on the one hand, and on the other that it should convey what is the real 'point' of the subject-matter. For various reasons, . . ., the second demand is invariably sacrificed to the first. A lecture, however more particularly when it is supported by a complete exposition in print, is the very place for the provisional nonsense that the second generally calls for. This would appear ridiculous if enshrined in print, and its real function is to disappear when it has served its turn. . . . The infinitely greater flexibility of speech enables me here to do without a blush what I shrink from doing in print.

I wish, finally, to commend for more general use the practice of providing lecture notes in advance. Among obvious advantages the chief is economy of time and energy: my course formerly consisted of 22 lectures; now, when it is fuller and more discursive, it consists of 15. It is possible that the art of lecturing has not yet recognized the full importance of the younger invention of printing."

Finally I would add the following recommendation: If a small library of mathematical books be kept in a separate reading room near the lecture rooms so that the students could easily refer to books and journals mentioned in the course of a lecture then that would be truly helpful. The habit of looking into reference books and journals is not without value. The library may be in charge of a teacher of the Department, and should be made available throughout the day and the earlier part of the night (in residential institutions); this does not imply that the teacher be in the library all the time.

Andhra University,
Waltair.

T. VIJAYARAGHAVAN.

Some Kinetic Equivalents of a Right Circular Cone

The following systems of kinetic equivalents to a uniform right circular cone may be of interest to the student of Dynamics:—

(i) A hollow cone of mass M is equivalent to a uniform circular ring of mass $M/3$ along the circumference of the base and another of mass $2M/3$ and radius half that of the base, through the mid-points of the generators.

(ii) A solid cone of mass M is equivalent to a mass $M/20$ at the vertex, another $M/20$ at the centre of the base, a uniform circular ring of mass $M/10$ along the circumference of the base and another of mass $4M/5$ and radius half that of the base with centre at the centre of inertia of the body and in a plane parallel to that of the base.

(iii) If a is the radius of the base, a solid cone of mass M is also equivalent to a mass $M/10$ at the mid-point of the axis, a uniform circular ring of mass $M/10$ and radius a and another of mass $M/5$, radius $a/2$ in the base, one of mass $M/5$, radius $a/2$ through the mid-points of the generators and another of mass $2M/5$, radius $a/2$ through the centre of inertia of the body, the centres being along the axis and the planes perpendicular to it.

These systems may be obtained by considering the hollow cone as composed of an infinite number of triangles and the solid one as composed of an infinite number of tetrahedra.

Department of Labour,
Government of India.

M. V. SESHAGIRI RAO.

REVIEWS

Perpetual Ephemeris of the Planetary Cycles

BY L. NARAYANA RAO, M.A., Calicut

(Pp. 587+xxxvii+vi, Price Rs. 8-8-0, 1941)

This book of Tables is the result of unaided and single-handed effort by an ardent, enthusiastic and painstaking mathematical astronomer who has laboured *six* years at it in an environment not quite congenial to such enterprises. It is the duty of every Indian in independent India to recognise the work of a fellow Indian scholar and give him all encouragement. Several of us, Indians, are secretly, if not openly, astrologically minded and should therefore welcome and support the achievement of Mr. Narayana Rao in having provided us with precise data for astrological calculations, *viz.*, the geocentric longitudes of the Sun and the Moon and the planets Mars, Mercury, Jupiter, Venus and Saturn.

The author has followed apparently the Nandi Nadi Vakyams for planetary positions and adopted the cycles of 79, 46, 83, 8 and 59 years for Mars, Mercury, Jupiter, Venus and Saturn respectively. The epoch of the text is 1st January 1854 and tables run from this date right up to the end of the above cycles for the respective planets. Thus Saturn's cycle runs from 1st January 1854 to 8th January 1913, *i.e.*, about 59 years.

At the end of the book there is an Appendix purporting to explain to the working astronomers or astrologer how the geocentric positions can be calculated for other dates not given in the tables. If the author had only mentioned the authoritative sources of his astronomical data as is done in the Nautical Almanac, the readers' faith in the Ephimerides would have been strengthened. For the interested readers, the reviewer would like to mention the following works which deal with planetary positions:

(1) *Planetary Co-ordinates for the years* 1800–1940 *referred to the equinox of* 1950·0 (prepared by H.M. Nautical Almanac Office), 12*s*. 6*d*. net (1933).;

(2) *An Indian Ephemeris*, Vol. I, Part I, by L. D. Swamikannu Pillai, I.S.O. (1922);

(3) *Planetary Tables from* 3200 *B.C. to* 3100 *A.D.*, by C. G. Rajan, B.A., Rs. 5 (1933);

of which the first gives the heliocentric coordinates and the third the raw data for calculating geocentric coordinates, while the second alone provides the mean geocentric longitudes. But the present work goes deeper than the second of the above in the matter of geocentric longitudes and gives more accurate results.

It is therefore recommended as an important book to be owned by all educational and religious institutions in India where astrology is studied and practised.

The Ephemeris deserves a speedy second edition with fewer typographic errors.

Mysore A. A. K.

The Poetry of Mathematics and Other Essays

BY DAVID EUGENE SMITH

(Scripta Mathematica, Yeshiva University, pp. 6+90, Price £1·25, 1947)

This is the second printing—the first printing was in 1934—of five essays by the late D. E. Smith who was a well-known collector of books, documents, autograph letters, instruments, portraits, medals, counters and so on. He was the founder of the History of Science Society and the founder of the journal *Scripta Mathematica*, and for one year (1920–1921) the President of the Mathematical Association of America. He has been a Vice-President (1908–1920), President (1928–1932) and Honorary President (1932) of the International Commission on the Teaching of Mathematics. He was the departmental editor in mathematics of the 14th edition of the *Encyclopædia Britannica* He was a writer of elementary mathematical texts which sold in millions and some of which were translated into Spanish, Arabic and Chinese. For 25 years (1901–1926) he was Professor of Mathematics at Teachers' College, Columbia University.

One may expect that a collection of essays by D. E. Smith will be pleasant to read and so indeed are the essays reviewed here. The book contains many fine quotations and what D. E. Smith himself writes will make excellent quotations.

This book contains five essays on (i) The Poetry of Mathematics, (ii) The Call of Mathematics, (iii) Religio Mathematici, (iv) Thomas Jefferson and Mathematics, and (v) Gaspard Monge, Politician. The essays will interest a wide class of readers. The articles are not themselves mathematical but they relate to lines of interest which mathematics suggests. Some of the essays concern the spirit in which the teacher may feel it desirable to approach the subject with his classes. Not all may understand the call of higher mathematics, but every pupil may be led to hear, even if indistinctly, the call of the lower phases of the subject, as set forth in the second essay. The call of mathematics is not only to our physical wellbeing as is well understood but also to our spiritual wellbeing.

This is the kind of book which one would like to see in larger numbers in our school and college libraries.

T. V.

Portraits of Eminent Mathematicians

With Biographical Sketches Portfolio 1

BY DAVID EUGENE SMITH

(New Deluxe Edition. Price 5·00. Scripta Mathematica, Yeshiva University, 1946)

The Scripta Mathematica has three lines of publications all of which should be of the utmost utility to those who are interested in Mathematics in its more general and cultural aspects—an excellent quarterly of the same name devoted to the history and philosophy of Mathematics, about half a dozen booklets, and a series of portfolios of portraits of which the first two are devoted to eminent mathematicians, the third to Philosophers who were also Mathematicians, and the last to famous Physicists. Each consists of about a dozen portraits printed on art paper 10″×13″ along with a descriptive folder of double the size with autographs and other material relating to the mathematician in question. The portlolio under review is a reprint of the portfolio issued in 1936 and is devoted to Archimedes, Copernicus, Viete, Galileo, Napier, Descartes, Newton, Leibniz, Lagrange, Gauss. Lobachevsky and Sylvester.

The teaching of mathematics becomes lifeless unless taught in the historical background, and these portraits and descriptive material together with books like Bell's *Men of Mathematics* and *Development of Mathematics* should be in every college and University Library to inspire young men and women. The portraits are of a size that may be conveniently framed and made permanently available for display in the mathematics class rooms. They are warmly recommended as an absolute necessity in the reorganization of mathematical teaching which is being planned in renascent India.

A. N. RAO.

The Mathematical Discoveries of Newton

BY H. W TURNBULL, M.A , F.R S

(Blackie & Son Ltd., London, pp. vi+88, Price 5 *sh.*, 1945)

The book contains the substance of two lectures delivered by the author. Without going into too much detail the author has tried to explain—"as far as the work of geniuses can be explained"—what led Newton to these discoveries. No attempt is made to deal with the controversies associated with the discovery of the differential calculus, but "the positive interest afforded by contemplating the wonderful range covered by his early mathematical work provides an adequate theme for this short study".

The book is very readable and sheds some new light too on the oft told story of Newton's achievements; none but an expert on the subject could have written this book. The section headings are 12 in number: (1) Early Influences, (2) The Binomial Theorem, (3) The Method of Fluxions, (4) The *De Analysi*,

(5) The *De Quadratura*, (6) The *Geometria Analytica*, (7) The Solid of Least Resistance and the Curve of Quickest Descent, (8) Angular Sections, (9) Interpolation and Finite Differences, (10) The *Arithmetica Universalis*, (11) Cubic Curves, and (12) Geometry in the *Principia*. The book contains a chronological table, a bibliography and an index.

T. V.

Fermagoric Triangles

BY PEDRO A PIZA

(Imprenta Soltero, Santura, P.R., 1945, Price not given)

This book of about 150 pages is Publication Number 1 of the Polytechnic Institute of Puerto Rico. The President of the Institute, Jarvis S. Morris has written an introduction to the book and from it we learn that Mr. Pedro Antonio Piza is a successful and progressive business man of San Juan, that this is his first book, that he has a love—almost a passion—for numbers and their significance, and that numbers are like living things to him and he has the time of his life with them." The author says (p. 147) that the preparation of the book was a prolonged labour of sheer joy; he firmly believes that Fermat did have a proof of his last theorem and that any attempt to solve the problem, in order to have possibilities of success, "should particularly avoid and evade the complicated theories of Kummer and his followers", and that those theories have only succeeded in unnecessarily entangling the problem more and more every day (p. 10). The author says that he believes 'to have unearthed some apparently new and sparkling facets in the gem that is Fermat's equation' (p. 12); his work is not based on any previous modern investigation known to him; his occasional consultations of professional specialists had mostly a negative or indifferent response.

A fermagoric triangle of *n*th degree is one whose sides a, b, c satisfy the equation $a^n+b^n=c^n$. Thus a triangle whose sides are 12, $\sqrt{93}-3$, $\sqrt{93}+3$ is a fermagoric triangle of degree 3; similarly the triangle whose sides are 4, $\sqrt{(-3+\sqrt{136})}+1$, $\sqrt{(-3++136)}-1$ is a fermagoric triangle of degree 4; the sides of a fermagoric triangle of degree 5 is given by 20, $\sqrt{(-25+\sqrt{64500})}-5$, $\sqrt{(-25+\sqrt{64500})}+5$. Plainly these triangles can be constructed by ruler and compasses. The book is easy to read and will interest sympathetic souls.

T. V.

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