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BABYLONIAN LOGARITHMS

BY

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In the *Mathematical Cuneiform Texts*, edited by O. Neugebauer and A. Sachs, page 35 (1945), the authors note that "we now have an Old-Babylonian tablet which answers the question to what power must a certain number a be raised to yield a given number? This problem is identical with finding the *logarithm* to the base a of a given number." This remark is followed at the noted place by several problems which are similar to those appearing in some of our modern text-books on elementary algebra used by students who are beginning the study of the subject of logarithms. For instance, the problems to prove that $\frac{1}{4}$ is the logarithm of 2 to the base 16 and that $\frac{3}{4}$ is the logarithm of 8 to the same base. It should be emphasized that these problems involve also the use of fractional exponents and that the first use of such exponents has frequently been credited to Oresme, about 1360 A.D. *Cf.*, J. Tropfke, *Geschickte der Elementar—Mathematik*, Vol. 3, page 199 (1937).

It should also be emphasized in this connection that there is no evidence that the ancient Babylonians had tables of logarithms such as we have at the present time and that the practical use of logarithmic tables is as old as the ancient Babylonians. What is clearly established now is that the fundamental *theory* of logarithms was known by these ancient people and that it was based by them on the law of exponents just as we now commonly do since the time of L. Euler (1707–1783) and not on the theories of arithmetic and geometric progressions as was done at the time of John Napier (1550-1617) to whom the invention of logarithms is credited even in such a modern and useful work even to mathematicians as Webster's *Biographical Dictionary* (1943) as well as in hundreds of other works of reference of recent date.

The theory of logarithms which the ancient Babylonians understood emphasizes the great absurdity of the following quotation from Volume 2, page 512 (1925), of the history of elementary mathematics by the late D. E. Smith (1861–1944) of Columbia University: "The invention of logarithms came on the world as a bolt from the blue. No previous work had led up to it, foreshadowed it or heralded its arrival. It stands isolated, breaking in upon human thought abruptly without borrowing from the work of other intellects or following known lines of mathematical thought." It should be emphasized that this statement was made in 1914 in Edinburgh at the Tersentary of the publication of the earliest volume on logarithms by John Napier and that it was quoted approvingly by D. E. Smith at the noted place. It would be difficult to find in the history of mathematics a more misleading statement by one who was supposed to have made an extensive study of this history and who published widely on it during many years.

It might have been thought that 300 years would be sufficient time to establish the merits of an individual as regards his contributions towards the development of such a widely known and important subject of elementary mathematics as that of logarithms, especially at a meeting of specialists from many different countries. Hence the quotation noted in the preceding paragraph is very illuminating since there are few subjects in whose development one can so easily see steps forwarced before the subjects appeared in their modern forms, and the tracing of such steps is one of the most useful features of the work of the mathematical historian who aims to give a clear insight into the development of the many closely related branches of our subjects constitutes one of the most useful features of the modern history of mathematics and tends to illuminate the subject itself.

At least three different lines of development in mathematics before the time of John Napier had somewhat similar objectives as his work on logarithms. The earliest of these seems to have been that of the ancient Babylonians to which reference was already made and which proceeded along the lines of exponents. This was probably rediscovered independently by L. Euler in the eighteenth century and was revived by him in its modern form and greatly popularized by his extensive other writings. The second was that which proceeded along the line of arithmetic and geometric progressions and was greatly stimulated by the writings of Archimedes who is commonly regarded as the greatest mathematician of antiquity and whose writings were widely known in Europe before the time of John Napier. Unfortunately the early writings on logarithms were frequently unnecessarily complicated by a failure to adopt the exponential point of view instead of that of arithmetic and geometric progressions.

A third well known line of mathematical development, which had an objective somewhat similar to that of John Napier and preceded his work, is the subject known as prosthaphæresis and proceeded along trigonometric lines. Its development at that time exhibits clearly that the notion of logarithms was then in the minds of various European mathematicians and hence it helps to establish more clearly that the foolish quotation from D. E. Smith's history of elementary mathematics noted above is entirely unfounded and it naturally warns the reader of this history that its statements should be verified by consulting other sources of historical information. D. E. Smith's history of elementary mathematics in two volumes (1923–25) is the most extensive history of elementary mathematics in the English language, and naturally inspires considerable confidence by its size and the positiveness of its style. Unfortunately, its author failed in many cases to make use of the most recent sources of information relating to the subjects under consideration and he was greatly handicapped by a lack of mathematical insight into some of the subjects which he attempted to explain.

It is possible that the appropriateness of the expression Babylonian logarithms will be questioned by some readers on the ground that the known developments of the ancient Babylonians along this line were very limited and have little in common with what the modern student of mathematics understands by the term logarithms. The enormous amount of labour involved in constructing the modern logarithmic tables is not reflected in the known work of the ancient Babylonians and the practical usefulness of these tables may not have been known to them. It must however be admitted that they made a good beginning along modern lines and that Newton and Leibniz, who are commonly regarded as founders of modern calculus, also knew very little of the vast subject of modern calculus. The mathematical historian is often compelled to assume an intelligent interpretation on the part of his readers in order to avoid too much prolixity.

PROOF OF A THEOREM OF LERCH AND P. KESAVA MENON

BY

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Lerch[1] and P. Kesava Menon[2] have proved the

THEOREM.—*Let p denote an odd prime. Then*

$$1^{p-1} + 2^{p-1} + \dots + (p-1)^{p-1} \equiv p + (p-1)! \pmod{p^2}.$$

We give a new proof, based on a recent note of Daljit Singh.[3]

Proof.—From (1·6) and (3·2) of Daljit Singh's paper we have

(A) $$\sum_{S=1}^{n} S^r = \sum_{t=1}^{r} f_{r,t}\, {}^{n+1}C_{t+1}$$

where

(B) $$f_{m,n} = \sum_{t=0}^{n} {}^nC_t (-1)^t (n-t)^m, \; f_{m,m} = (m)!$$

and ${}^nC_r = n!/r!(n-r)!$. From (B) for $n \leqslant p-2$,

(C) $$f_{p-1,n} \equiv (-1)^n \pmod{p}.$$

Hence putting $n = r = p-1$ in (A) we get (the a's are integers)

$$\sum_{S=1}^{p-1} S^{p-1} = \sum_{t=1}^{p-2} {}^pC_{t+1} \{(-1)^t + p\, a_t\} + (p-1)!$$

$$\equiv \sum_{t=1}^{p-2} (-1)^t\, {}^pC_{t+1} + (p-1)! \pmod{p^2}$$

$$\equiv p + (p-1)! \pmod{p^2}.$$

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CARATHEODORY'S INEQUALITY AND ALLIED RESULTS (II)

BY

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IN the notation of my earlier note [4] with the same title, let $f(z)$ be analytic in the circle $|z| \leqslant R$ and let $M(r) = \max\limits_{|z|=r \leqslant R} |f(z)|$, $A(r) = \max\limits_{|z|=r \leqslant R} \mathbf{R}\, f(z)$. Then, for $0 \leqslant |z| = r < R$, the inequalities in question can be written:

$$\text{I.}\qquad |f(z)| \leqslant |f(0)| + \frac{2r}{R-r}\{A(R) - \mathbf{R}\, f(0)\}.$$

$$\text{I } a.\qquad \mathbf{R}\, f(z) \leqslant \mathbf{R}\, f(0) + \frac{2r}{R+r}\{A(R) - \mathbf{R}\, f(0)\}.$$

$$\text{I } b.\qquad |\mathbf{I}\, f(z) - \mathbf{I}\, f(0)| \leqslant \frac{2Rr}{R^2 - r^2}\{A(R) - \mathbf{R}\, f(0)\}.$$

$$\text{II.}\qquad \frac{|f^{(\nu)}(0)|}{\nu\,!} \leqslant \frac{2}{R^\nu}\{A(R) - \mathbf{R}\, f(0)\},\ \nu \geqslant 1.$$

$$\text{III.}\qquad \frac{|f^{(\nu)}(z)|}{\nu\,!} \leqslant \frac{2R}{(R-r)^{\nu+1}}\{A(R) - \mathbf{R}\, f(0)\},\ \nu \geqslant 1.$$

I b is not as a rule explicitly stated, but it emerges simultaneously with I a, from the usual proof of I by means of Schwarz's lemma. This proof further establishes II in the case $\nu = 1$ whence the general case $\nu > 1$ follows immediately [3, Chap. I, Theorem 110, Proof (iv)] and can be used to obtain III [2, Chap. III, Theorem E]. It is shown in this note that $M(r)$-analogues of II, III, involving $M(R)$ and $|f(0)|$ instead of $A(R)$ and $\mathbf{R}\, f(0)$, can be obtained in the same way after we have established the $M(r)$-analogue of I [1, Satz II] by Schwarz's lemma.

It may be observed incidentally that *each of the inequalities* I, II, III *implies the other two*. For, while I $\rightarrow$ II $\rightarrow$ III as already noticed, III obviously includes II and II can be used in the power series for $f(z)$ to obtain I.

What I have called the $M(r)$-analogues of I, II, III are the following inequalities obtained under the same conditions as I, II, III, I′. $$|f(z)| \leqslant M(R)\frac{rM(R)+R|f(0)|}{r|f(0)|+RM(R)}.$$

II′. $$\frac{|f^{(\nu)}(0)|}{\nu!} \leqslant \frac{1}{R^{\nu}}\frac{[M(R)]^2-|f(0)|^2}{M(R)}, \ \nu \geqslant 1.$$

III′. $$\frac{|f^{(\nu)}(z)|}{\nu!} \leqslant \frac{R}{(R-r)^{\nu+1}}\frac{[M(R)]^2-|f(0)|^2}{M(R)}, \ \nu \geqslant 1.$$

I′ is a well-known result; but II′ and III′ seem to be known only in their cruder forms with $|f(0)|$ omitted, which for this very reason cannot be regarded as the $M(r)$-analogues of II and III.

First, it can be shown that the usual proof of I′ (reproduced below for the sake of completeness) yields II′ in the case $\nu = 1$. The transformations

$$Z = f(z), \ w = M\frac{Z-f(0)}{\overline{f(0)}\,Z-M^2} \text{ where } M \equiv M(R),$$

represent the z-area $\mathfrak{D}$: $|z| \leqslant R$ conformally on a Z-area $\subset \mathscr{E}$: $|Z| \leqslant M$ and also on a w-area $\subset \Delta$: $|w| \leqslant 1$; making $z = 0$ correspond to $Z = f(0) = ce^{i\gamma}$ (say) and to $w = 0$. Applying Schwarz's lemma to w (*qua* function of z), we see that a z-area $\subset \mathfrak{D}'$: $|z| \leqslant r$ is represented on a w-area $\subset \Delta'$: $|w| \leqslant r/R$ and consequently on a Z-area $\subset \mathscr{E}'$ bounded by the Z-circle corresponding to the w-circle $|w| = r/R$. This w-circle passes through the points $w = \pm(r/R)e^{i\gamma}$, cutting orthogonally the straight line am $w = \gamma$; and so, remembering that

$$Z = M\frac{Mw-f(0)}{\overline{f(0)}\,w-M},$$

it is easy to see that the corresponding Z-circle passes through the points

$$Z = Z_1 \equiv M\frac{M\frac{r}{R}+c}{c\frac{r}{R}+M}e^{i\gamma}, \ Z = Z_2 \equiv M\frac{M\frac{r}{R}-c}{c\frac{r}{R}-M}e^{i\gamma}.$$

and cuts orthogonally the straight line am $Z = \gamma$. In other words the frontier of $\mathscr{E}'$ is the circle having the join of $Z = Z_1$ and $Z = Z_2$ for a diameter and therefore points in the closed region $\mathscr{E}'$ satisfy

$$|Z| \leqslant \max(|Z_1|, |Z_2|) = |Z_1|,$$

which is I′.

Another result of applying Schwarz's lemma to $w(z)$ is $|dw/dz|_{z=0} \leqslant 1/R$ which, when used in the special case $z = 0$ of the relation:

$$f'(z) = \frac{dZ}{dz} = M\frac{|f(0)|^2-M^2}{(\overline{f(0)}\,w-M)^2}\frac{dw}{dz},$$

leads at once to the case $\nu = 1$ of II′.

To prove II′ in the general case $\nu = k > 1$, we have merely to apply the special case $\nu = 1$, to the function

$$F(\zeta) = F(z^k) = \sum_{n=0}^{\infty} \frac{f^{(nk)}(0)}{nk\,!} z^{nk} = \frac{1}{k} \sum_{r=0}^{k-1} f(e^{2\pi i r/k} z),$$

noticing that the application is justified by the facts $|F(\zeta)| \leqslant M \; (|\zeta| = R^k)$, $F(0) = f(0)$.

Finally, we see that III′ is a simple deduction from II′ since

$$f^{(\nu)}(z) = \sum_{n=\nu}^{\infty} \frac{f^{(n)}(0)}{n\,!} n(n-1)\ldots(n-\nu+1)\, z^{n-\nu}$$

and therefore, in consequence of II′,

$$|f^{(\nu)}(z)| \leqslant \frac{[M(R)]^2 - |f(0)|^2}{R^\nu M(R)} \sum_{n=\nu}^{\infty} n(n-1)\ldots(n-\nu+1)\left(\frac{r}{R}\right)^{n-\nu}$$

$$= \frac{[M(R)]^2 - |f(0)|^2}{R^\nu M(R)} \cdot \frac{\nu\,!}{\left(1-\frac{r}{R}\right)^{\nu+1}}.$$

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A GENERALIZATION OF BINOMIAL, LEXIAN AND POISSON DISTRIBUTIONS

BY

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Consider the distribution

$$f = (p_1 + q_1)^{\alpha} (p_2 + q_2)^{\beta} \dots (p_n + q_n)^{\eta}$$

where $\alpha, \beta, \dots \eta$ are positive integers, and $\alpha + \beta + \dots + \eta = n$.

It is obvious that if $\alpha = \beta = \dots = \eta = 1$, this reduces to the Lexian distribution, and if in addition $p_1 = p_2 = \dots = p_n$, then it reduces to the Binomial distribution.

I give below, the various constants of this distribution, the generating function of which is

$$\phi(t) = (q_1 + p_1 e^{it})^{\alpha} (q_2 + p_2 e^{it})^{\beta} \dots (q_n + p_n e^{it})^{\eta}$$

so that its characteristic function is

$$\psi(t) = \alpha \log (q_1 + p_1 e^{it}) + \beta \log (q_2 + p_2 e^{it}) + \dots + \eta \log (q_n + p_n e^{it}).$$

Now for any distribution

$$k_r = (-i)^r [D_t^r \psi(t)]_{t=0}.$$

$\therefore$ For this distribution we have

$$k_1 = (-i)[\alpha i p_1 + \beta i p_2 + \dots + \eta i p_n]$$
$$= (\alpha p_1 + \beta p_2 + \dots + \eta p_n)$$
$$k_2 = (\alpha p_1 q_1 + \beta p_2 q_2 + \dots + \eta p_n q_n)$$

Similarly

$$k_3 = \alpha p_1 q_1 (q_1 - p_1) + \beta p_2 q_2 (q_2 - p_2) + \dots + \eta p_n q_n (q_n - p_n)$$
$$k_4 = \alpha p_1 q_1 (1 - 6 p_1 q_1) + \beta p_2 q_2 (1 - 6 p_2 q_2) + \dots + \eta p_n q_n (1 - 6 p_n q_n)$$

Now $k_1 = \mu_1' = \alpha p_1 + \beta p_2 + \dots + \eta p_n$

$$k_2 = \mu_2 = \alpha p_1 q_1 + \beta p_2 q_2 + \dots + \eta p_n q_n$$
$$k_3 = \mu_3 = \alpha p_1 q_1 (q_1 - p_1) + \beta p_2 q_2 (q_2 - p_2) + \dots + \eta p_n q_n (q_n - p_n)$$
$$\mu_4 = k_4 + 3k_2^2 = 3(\alpha p_1 q_1 + \beta p_2 q_2 + \dots + \eta q_n p_n)^2 + \alpha p_1 q_1 (1 - 6 p_1 q_1) + \beta p_2 q_2 (1 - 6 p_2 q_2) + \dots + \eta p_n q_n (1 - 6 p_n q_n).$$

ON RAMANUJAN FUNCTION $\tau(n)$.

BY

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Dacca

Ramanujan* defines the function $\tau(n)$ as the coefficient of x^n in the expansion of

$$x\{(1-x)(1-x^2)(1-x^3)\ldots\ldots\}^{24}.$$

He conjectured that

$$\tau(m.n)=\tau(m)\,\tau(n) \text{ if } (m,n)=1.$$

This was proved by Mordell,† who gave the formula

$$\tau(p^\lambda)=\tau(p)\,\tau(p^{\lambda-1})-p^{11}\,\tau(p^{\lambda-2}),\ (\lambda\geqslant 2). \tag{1}$$

The author of this note proposes to express $\tau(p^\lambda)$ as a polynomial in $\tau(p)$. The proposed formula is given by

$$\tau(p^n)=\{\tau(p)\}^n-\binom{n-1}{1}p^{11}\{\tau(p)\}^{n-2}+\binom{n-2}{2}p^{22}\{\tau(p)\}^{n-4}$$
$$-\binom{n-3}{3}p^{33}\{\tau(p)\}^{n-6}+\ldots\ldots+(-)^r\binom{n-r}{r}p^{11r}\{\tau(p)\}^{n-2r}$$
$$+\ldots\ldots+(-)^{\left[\frac{n}{2}\right]}\binom{\left[\frac{n+1}{2}\right]}{\left[\frac{n}{2}\right]}p^{11\cdot\left[\frac{n}{2}\right]}\{\tau(p)\}^{n-2\left[\frac{n}{2}\right]}. \tag{2}$$

The proof is obtained by Induction. We have from (1)

$$\tau(p^2)=\{\tau(p)\}^2-p^{11}$$
$$\tau(p^3)=\{\tau(p)\}^3-2p^{11}\,\tau(p).$$

Both these satisfy the formula (2).

Suppose now that the formula (2) is true for

$$n=1,2,3,\ldots,2m-1.$$

Then, from (1) we have

$$\tau(p^{2m})=\tau(p)\,\tau(p^{2m-1})-p^{11}\,\tau(p^{2m-2}). \tag{3}$$

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* Ramanujan, *Collected Papers.*

† Mordell, *Proceedings of the Cambridge Philosophical Society*, 1919, **19**, 117–24.

In formula (3) we put for $\tau(p^{2m-1})$ and $\tau(p^{2m-2})$ their appropriate polynomial expressions in $\tau(p)$. We have then

$$\tau(p^{2m})=\tau(p)\Big[\{\tau(p)\}^{2m-1}-\binom{2m-2}{1}p^{11}\{\tau(p)\}^{2m-3}+\binom{2m-3}{2}p^{22}\{\tau(p)\}^{2m-5}$$

$$-\ldots+(-)^r\binom{2m-1-r}{r}p^{11r}\{\tau(p)\}^{2m-1-2r}$$

$$+\ldots+(-)^{m-1}\binom{m}{m-1}p^{11(m-1)}\tau(p)\Big]$$

$$-p^{11}\Big[\{\tau(p)\}^{2m-2}-\binom{2m-3}{1}p^{11}\{\tau(p)\}^{2m-4}+\binom{2m-4}{2}p^{22}\{\tau(p)\}^{2m-6}$$

$$-\ldots+(-)^r\binom{2m-2-r}{r}p^{11r}\{\tau(p)\}^{2m-2-2r}$$

$$+\ldots+(-)^{m-1}p^{11(m-1)}\Big]$$

$$=\Big[\{\tau(p)\}^{2m}-\Big\{\binom{2m-2}{1}+1\Big\}p^{11}\{\tau(p)\}^{2m-2}$$

$$+\Big\{\binom{2m-3}{2}+\binom{2m-3}{1}\Big\}p^{22}\{\tau(p)\}^{2m-4}-\ldots$$

$$+(-)^r\Big\{\binom{2m-1-r}{r}+\binom{2m-1-r}{r-1}\Big\}p^{11r}\{\tau(p)\}^{2m-2r}+\ldots+(-)^m p^{11m}\Big]$$

$$=\{\tau(p)\}^{2m}-\binom{2m-1}{1}p^{11}\{\tau(p)\}^{2m-2}+\binom{2m-2}{2}p^{22}\{\tau(p)\}^{2m-4}$$

$$+\ldots+(-)^r\binom{2m-r}{r}p^{11r}\{\tau(p)\}^{2m-2r}+\ldots\ldots+(-)^m p^{11m}.$$

So, we have proved that if the formula (2) be true for $n=1, 2, 3, \ldots, 2m-1$, then it is also true for $n=2m$.

On the other hand we can now show that if the formula (2) be true for $n=1, 2, 3, \ldots, 2m$ it is also true for $n=2m+1$.

It follows therefore that the formula (2) holds for all positive integral values of n.

ON THE APPEARANCE OF PRIME FACTORS IN THE SEQUENCE ASSOCIATED WITH FIBONACCI'S SEQUENCE

BY

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Let $u = \{1, 1, 2, 3, 5, \ldots\}$ and $v = \{1, 3, 4, 7, 11, \ldots\}$ denote Fibonacci's sequence and the sequence associated with it, in both of which each term is the sum of the two preceding terms. We shall say that a prime p appears as a factor in u (or v) if p divides some term of u (or v). It is known that every prime appears as a factor in u[1], while the primes $\equiv 3, 7, 11, 19 \bmod 20$ appear[2] and the primes $\equiv 13, 17 \bmod 20$ do not appear as factors[3] in v. The question which and how many of the primes $\equiv 1, 9 \bmod 20$ appear as factors in v has not been discussed so far and it is the purpose of this note to contribute to an answer for the primes $\equiv 1 \bmod 20$.

Let p be any prime factor of u_n (or v_n) such that p does not divide any u_m (or v_m) with $m < n$. Then p is called a *primitive* factor of u_n (or v_n). It is known that any primitive factor of u_{2n} is also a primitive factor of v_n and conversely.[4] Hence a prime p appears as a factor in v if and only if p is a primitive factor of a term of u with *even* index.

We proceed to prove the following:

THEOREM 1.—*Every primitive prime factor of $u_{5(2k+1)}$, where k is any positive integer, is $\equiv 1$ mod 20.* The proof is based on the following two propositions:—

A. *For odd n, every odd prime factor of u_n is $\equiv 1$ mod 4.*[5]

B. *For any positive integer n, every primitive prime factor p of u_n is $\equiv \left(\frac{5}{p}\right)$ mod n, where $\left(\frac{5}{p}\right)$ is Legendre's symbol.*[6]

Noting that $\left(\frac{5}{p}\right) = 1$ for $p \equiv \pm 1 \bmod 10$ and $\left(\frac{5}{p}\right) = -1$ for $p \equiv \pm 3 \bmod 10$, we deduce from B:

B′. *For odd n, every primitive prime factor $p > 5$ of u_n is $\equiv \left(\frac{5}{p}\right)$ mod $2n$.*

Now let p be a primitive prime factor of $u_{5(2k+1)}$. Then $p > 5$ (since $u_5 = 5$) and by B′ we have: $p \equiv \left(\frac{5}{p}\right) \bmod 10$, that is $p \equiv \pm 1 \bmod 10$.

By the above remark we have $\left(\frac{5}{p}\right) = 1$, whence $p \equiv 1 \bmod 10$, or $p \equiv 1, 11 \bmod 20$. But by A, $p \equiv 1, 9, 13, 17 \bmod 20$, whence $p \equiv 1 \bmod 20$.

THEOREM 2.—*Every primitive prime factor p of v_{10k}, where k is any positive integer, is* $\equiv 1$ *mod* 40.

The proof is based on the following two propositions:

C. *For* $n \equiv 2 \bmod 4$, *every odd prime factor of* v_n *is* $\equiv 1, 3, 9, 27 \bmod 40$. *For* $n \equiv 0 \bmod 4$, *every odd prime factor of* v_n *is* $\equiv 1, 7, 9, 23 \bmod 40$.[7]

D. *Every primitive prime factor p of* v_n *is* $\equiv \left(\frac{5}{p}\right) \bmod 2n$. (This follows immediately from B and the preliminary remark about the primitive prime factors of v_n and u_{2n}).

Now let p be a primitive prime factor of v_{10k}. Then by D we have: $p \equiv \left(\frac{5}{p}\right) \bmod 20$, whence we deduce, as in the proof of Theorem 1, that $p \equiv 1 \bmod 20$, or $p \equiv 1, 21 \bmod 40$. Combining this result with C we have $p \equiv 1 \bmod 40$.

THEOREM 3.—*There exists an infinitude of primes* $\equiv 1$ *mod* 20 *which do not appear as factors in v, as well as an infinitude of primes* $\equiv 1$ *mod* 40 *which appear as facotrs in v.*

The proof follows by the preliminary remark immediately from Theorems 1 and 2 and from the following theorem: *Every* u_n *with* $n > 12$ *and every* v_n *with* $n > 6$ *has at least one primitive prime factor.*[8]

It would appear that among the primes $\equiv 9 \bmod 20$, too, there exists an infinitude of numbers which do not appear in v, as well as an infinitude of numbers which appear in v. If this turned out to be correct, the following question would arise: Is there a number m such that among the primes p with $p \equiv 1, 9 \bmod 20$ those belonging to certain classes modulo m appear as factors in v, while those belonging to the remaining classes do not appear as factors in v?

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S-QUADRICS AND ISOTOMIC POLARS

BY

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Rajahmundry

In this paper, we define S-Quadrics and Isotomic Polars* and obtain some results in n-space, using the barycentric system of co-ordinates.

The main result, *viz.*, Theorem 4 identifies—for example, in 3-space, the polar of a point w.r.t. a tetrahedron with the polar of an allied point w.r.t. a *quadric* designated Mean S-Quadric, which, when the tetrahedron is regular, is a sphere with centre at the C.G. and of radius equal to the mean proportional between the circum- and in- radii of the tetrahedron.

2. *Preliminaries.*—The barycentric co-ordinates of a point P w.r.t. a reference simplex (A_i) in n-space, are the numbers p_i $(i = 1, 2, \cdots, n+1)$, such that P is the centroid of masses proportional to p_i placed at the vertices A_i. In what follows, we consider the sum of the co-ordinates of every point as 1. Thus each co-ordinate of G, the centroid of (A_i) is $\frac{1}{n+1}$.

We denote the symmetrics of $P \equiv (p_i)$, $Q \equiv (q_i)$ relative to G by $P' \equiv (p_i')$, $Q' \equiv (q_i')$; and the isotomic conjugate of P w.r.t. (A_i) by P_1.

The following results will be useful.

(1) If G, P, Q are collinear and $GQ = m.\ GP$, then

$$\frac{1}{n+1} = \frac{q_i - mp_i}{1-m} = \frac{p_i + p_i'}{2}$$

(2) $P_1 \equiv \left(\frac{1}{p_i}\right) \Big/ \Sigma \left(\frac{1}{p_i}\right)$

(3) $\sum_{r=1}^{n+1} p_r x_r = 0$ is the polar of P_1 w.r.t. (A_i).

(4) We define

$$S(\lambda) \equiv \sum_{r,s=1,2,\ldots,n+1}' (x_r - x_s)^2 - \lambda (\Sigma x_r)^2 = 0$$

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* In 3-space, for example, w.r.t a tetrahedron $A_1A_2A_3A_4$,

(i) P, P_1 are said to be isotomic conjugates when their joins to each vertex cut the opposite face in points which are isotomic conjugates w.r.t. that face.

(ii) the polar of P is the plane through the four lines such as the meet of the plane $A_2A_3A_4$ and the plane through the meets of A_2P, $A_3A_4A_1$; A_3P, $A_4A_1A_2$; A_4P, $A_1A_2A_3$.

(iii) the isotomic polar of P is the polar of the isotomic conjugate of P w.r.t. $A_1A_2A_3A_4$.

as *S-Quadrics*, where in Σ', each squared difference occurs once only and λ is an arbitrary parameter.

We shall call S (1) the *Mean S-Quadric*.

(5) The polar of (p_i) w.r.t. S (λ) is

$$\Sigma\left(p_r - \frac{\lambda+1}{n+1}\right) x_r = 0$$

which becomes $\Sigma p_r x_r = 0$ or $\Sigma p_r' x_r = 0$ when $\lambda = -1$ or $+1$.

(6) If a line through G cut S (λ), S (μ) in P, Q then GP : GQ = $\sqrt{\lambda} : \sqrt{\mu}$.

3. The following theorems are easily proved with the help of the above results.

THEOREM 1. *All S-Quadrics are concentric and similar and similarly placed relative to G.*

THEOREM 2. *$S(\lambda)$, $S\left(\frac{1}{\lambda}\right)$ are polar reciprocals w.r.t. $S(1)$ as well as w.r.t. $S(-1)$.*

For, if a line through G cut S (λ), S $\left(\frac{1}{\lambda}\right)$ in P, Q, the polars of P w.r.t. S (1), S (-1) are respectively the polars of Q, Q′ w.r.t. S $\left(\frac{1}{\lambda}\right)$.

THEOREM 3. *For $n = 2$ (and $n = 3$), $S(n)$ is the ellipse (ellipsoid) of minimum area (volume) circumscribing (A_i); and $S\left(\frac{1}{n}\right)$ is the quadric of maximum area (volume) inscribed in* (A_i).

When (A_i) is regular, S (λ) are circles (spheres) and it is easily seen that the radii of the circum-, mean-, in- S-Quadrics are as

$$\sqrt{n} : 1 : \sqrt{\frac{1}{n}}.$$

THEOREM 4. *The isotomic polar of* P *w.r.t.* (A_i) *is*

(1) *the polar of* P′ *w.r.t.* $S(1)$

(2) *the polar of* P *w.r.t.* $S(-1)$.

THEOREM 5. *A given point possesses a* common *isotomic polar w.r.t. all triangles (tetrahedra) of maximum area (volume) inscribed in a given ellipse (ellipsoid).*

For, in 2-space, for example, the given ellipse is the common S (n) for all the triangles; and so S (1) is common to all of them.

Cor.—The isotomic polar of a given point w.r.t. an equilateral triangle (regular tetrahedron) is unaltered in whatever manner the traingle (tetrahedron) is rotated about its C.G., provided that, in the case of the triangle the rotation is in its own plane.

THEOREM 6. *(a) All points got from* (p_i) *by permuting the co-ordinates lie on an S-Quadric of* (A_i).

(b) All lines (planes) got from $\Sigma p_r x_r = 0$, *by permuting the coefficients, touch the polar reciprocal of the quadric in (a) w.r.t.* $S\,(1)$.

These quadrics also contain respectively all the points and lines (planes) got by permuting likewise the subscripts of (p_i') and of the coefficients of $\Sigma p_r' x_r = 0$.

A NOTE ON SELF-CONJUGATE LATIN SQUARES OF PRIME DEGREE

By

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In a paper entitled 'The enumeration of the Latin rectangle of depth three by means of a formula of reduction, with other theorems relating to non-clashing substitutions and Latin squares", S. M. Jacob* defines a "complete cycle" Latin square as below:

"Let us operate on any group of degree n by a cyclic substitution of degree n of the same letters. The Latin square produced by $(n - 1)$ such operations in succession will be spoken of as a "complete-cycle Latin square".

At the end of his paper, Jacob cautiously gives expression to the following conjecture: "Every self-conjugate Latin square of prime degree is a complete-cycle Latin square".

The following self-conjugate Latin square of degree 7 is not a complete-cycle Latin square.

A	B	C	D	E	F	G
B	C	A	E	F	G	D
C	A	B	F	G	D	E
D	E	F	G	A	B	C
E	F	G	A	D	C	B
F	G	D	B	C	E	A
G	D	E	C	B	A	F

Since 7 is prime, Jacob's conjecture is false.

* *Proc. Lond. Math. Soc.*, (2), 31, 1930, 329–34.

ON THE SELF-INVERSE MODULE

BY

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Prof. Levi[1] has considered certain properties of a self-inverse module. Here I shall consider a few more new properties of a self-inverse module and its multiplicators and divisors. I thank Dr. Levi for helpful criticisms.

The module A will be called self-inverse if whenever a is an element of A, $1/a$ is also an element of A. Let M (A) be the multiplicators of A which form a field.[2]

Prop. 1. *If a be an element of A, then the even powers of a are elements of M (A).*

Let a, b be the elements of A, then[3] a^2b is an element of A. Then $a - \frac{1}{a^2b} = \frac{a^3b-1}{a^2b}$ will be an element of A. $\frac{a^2b}{a^3b-1}$ will be an element of A. Hence $\frac{a^2b}{a^3b-1} - \frac{1}{a} = \frac{1}{a^4b-a}$, $a^4b - a$, a^4b will be elements of A. Hence a^4 is an element of M (A). Let $a^{2n}b$ be an element of A, then $a - \frac{1}{a^{2n}b} = \frac{a^{2n+1}b-1}{a^{2n}b}$, $\frac{a^{2n}b}{a^{2n+1}b-1} - \frac{1}{a} = \frac{1}{a^{2n+2}b-a}$, $a^{2n+2}b - a$, $a^{2n+2}b$ will be elements of A. Hence a^{2n+2} will be an element of M (A). Since a^2, a^4 are elements of M (A), $a^6, a^8, \ldots$ are elements of M (A).

Prop. 2. *If a, b are elements of A, then $a^n b^m$ will be an element of A if $n+m$ be odd and an element of M (A) if $n+m$ be even except when $n \neq 1$, $m = 1$ or $n = 2\nu+1$, $m = 2\mu+1$.*

If $n+m$ is odd then $a^nb^m = a \cdot a^{2\nu}b^{2\mu}$ (or $ba^{2\nu}b^{2\mu}$) $= a\alpha$ (or $b\alpha$) where α belongs to M (A). Hence a^nb^m belongs to A if $n+m$ be odd.

If $n+m$ be even then $a^nb^m = a^{2\nu}b^{2\mu}$ or $a^{2\nu+1}b^{2\mu+1}$. Now $a^{2\nu}b^{2\mu}$ belongs to M (A). $a^{2\nu+1}b^{2\mu+1} = ab\alpha$ where α belongs to M (A).

Now $a^{2\nu+1}b^{2\mu+1}$ will belong to M (A) if the characteristic of the field $\neq 2$, since[4] ab belongs to M (A) if the field's characteristic $\neq 2$.

Prop. 3. *The divisors of a self-inverse module A form a field* Q (A).

Let M (A) be the field of multiplicators. Since M (A) is a field every multiplicator $\neq 0$ is a divisor and conversely. Hence divisors form a field Q (A) = M (A).

PROP. 4. *If a, b be elements of a self inverse module in a field of characteristic* $\neq 2$, *then ab belongs to* Q (A) *if* $ab \neq 0$.

Since ab belongs to M (A), $ab \cdot \frac{1}{c}$ belongs to A where c is an element of A..

Hence $\frac{c}{ab}$ belongs to A. Hence ab belongs to Q (A).

PROP. 5. *If a, b be elements of a self-inverse module in a field of characteristic* $\neq 2$ *then* $\frac{a}{b}, \frac{b}{a}, a^2, b^2$ *belong to* Q (A) *if* $ab \neq 0$.

Let c belong to A. Then ac belongs to Q (A). Hence $\frac{ac}{b}$ belongs to A.

Hence $\frac{a}{b}$ belongs to Q (A). Similarly $\frac{b}{a}, a^2, b^2$ will belong to Q (A).

PROP. 6. *If a, b be the elements of A in a field of characteristic* $\neq 2$ *and* $ab \neq 0$, *n then* $ab \pm n$, $a^2b^2 - n^2$ *will belong to* Q (A).

Since ab belongs to M (A), $ab\frac{1}{c} \pm \frac{1}{c}$, $ab\frac{1}{c} \pm \frac{2}{c}$, ..., $ab\frac{1}{c} \pm \frac{n}{c}$ belong to A. Hence $\frac{c}{ab \pm n}$ belongs to A. Hence $ab \pm n$ belongs to Q (A).

Since $\frac{c}{ab+n}, \frac{c}{ab-n}$ belong to A, then $\frac{2nc}{a^2b^2 - n^2}$ will belong to A. Hence $\frac{a^2b^2 - n^2}{2n}$ will belong to Q (A).

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A NOTE ON QUASI-MONOTONE SERIES

BY

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1. A sequence (a_n) of positive numbers is called *quasi-monotone* if for some $\alpha \geqslant 0$, $a_{n+1} \leqslant a_n \left(1 + \frac{\alpha}{n}\right)$, $(n = 1, 2, \ldots.)$. A series is called quasi-monotone if its terms form a quasi-monotone sequence. A function $f(x)$ is called quasi-monotone if $f(x) \geqslant 0$, $f(x+h) \leqslant f(x) \left(1 + \frac{\alpha}{x}\right)$ for some $\alpha \geqslant 0$, $x > x_0$ and for all h such that $0 \leqslant h \leqslant 1$.

Otto Szasz has in a brief abstract [1] defined quasi-monotone series and has stated that Abel's Theorem (If $\Sigma a_n < \infty$, then $na_n \to 0$) and Cauchy's condensation test (the series Σa_n and $\Sigma 2^n a_{2^n}$ are both convergent or both divergent) hold for such series. In this note we prove

THEOREM.—*Let $f(x)$ be continuous and positive, $f(x+h) \leqslant f(x) \left(1 + \frac{\alpha}{x}\right)$ for some $\alpha \geqslant 0, x > x_0$ and for all h such that $0 \leqslant h \leqslant 1$, then*

(i) *the series $\sum_{N}^{\infty} f(n)$ and the integral $\int_{N}^{\infty} f(t)\, dt$ are both convergent or both divergent.*

(ii) *The series $\Sigma f(n)$ and $\Sigma a^n f(a^n)$, where $a > 1$ is a constant, are both convergent or both divergent.*

(iii) *The convergence of $\int_1^{\infty} x^{\mu} f(x)\, dx$ implies $\lim_{x \to \infty} x^{\mu+1} f(x) = 0$.*

In §3 we construct two series to show that α cannot be replaced by $\log n$ in Abel's Theorem or Cauchy condensation test for quasi-monotone Series.

§2. Proof of (i). Let $n > 1 + x_0$ and consider

$$I(n) = \int_{n-1}^{n} f(x)\, dx = f(\xi), \text{ where } n - 1 \leqslant \xi \leqslant n.$$

Now $f(\xi) \leqslant f(n-1)\left(1 + \frac{\alpha}{n-1}\right) \leqslant (1+\alpha)\, f(n-1)$.

Further $f(n) \leqslant f(\xi)\left(1 + \frac{\alpha}{\xi}\right) \leqslant (1+\alpha) f(\xi)$.

Hence

$$\frac{f(n)}{1+a} \leqslant f(\xi) = \mathrm{I}(n) \leqslant (1+a) f(n-1),$$

for $n \geqslant \mathrm{N} > 1 + x_0$ and so

$$\frac{1}{1+a}\{f(\mathrm{N}) + f(\mathrm{N}+1) + \ldots + f(n)\} \leqslant \int_{\mathrm{N}-1}^{n} f(x)\, dx$$

$$\leqslant (1+a)\{f(\mathrm{N}-1) + \ldots + f(n-1)\}$$

and the integral test follows.

(ii) Let $n > x_0$ and consider

$$\mathrm{I} = \int_{a^n}^{a^{n+1}} f(x)\, dx = (a^{n+1} - a^n) f(\xi)$$

where $a^n \leqslant \xi \leqslant a^{n+1}$. Let X be a point such that $0 \leqslant \xi - \mathrm{X} \leqslant 1$, $a^n \leqslant \mathrm{X} \leqslant a^{n+1}$ and $\mathrm{X} \equiv a^n \pmod 1$. Then

$$f(\xi) \leqslant f(\mathrm{X})\left(1 + \frac{a}{\mathrm{X}}\right)$$

$$\leqslant f(\mathrm{X}-1)\left(1 + \frac{a}{\mathrm{X}-1}\right)\left(1 + \frac{a}{\mathrm{X}}\right)$$

$$\leqslant f(a^n)\left(1 + \frac{a}{a^n}\right)\left(1 + \frac{a}{1+a^n}\right) \cdots \left(1 + \frac{a}{\mathrm{X}}\right)$$

$$\leqslant f(a^n)\left(1 + \frac{a}{a^n}\right) \cdots \left(1 + \frac{a}{\theta}\right) = f(a^n)\,\Pi, \text{ say}$$

where $\theta - 1 < a^{n+1} \leqslant \theta$ and $\theta \equiv a^n \pmod 1$.

Now $\Pi \leqslant e^{a\left(\frac{1}{a^n} + \frac{1}{1+a^n} + \cdots + \frac{1}{\theta}\right)}$

$$\leqslant e^{a\left(a + \frac{1}{\theta}\right)} < e^{2aa} = k \text{ say}.$$

Further let y be a point such that $0 \leqslant y - \xi \leqslant 1$, $a^n \leqslant y \leqslant a^{n+1}$, $a^{n+1} \equiv y \pmod 1$. Then

$$f(y) \leqslant f(\xi)\left(1 + \frac{a}{\xi}\right) \leqslant (1+a) f(\xi).$$

Hence

$$f(a^{n+1}) \leqslant f(a^{n+1} - 1)\left(1 + \frac{a}{a^{n+1}-1}\right)$$

$$\leqslant f(y)\left(1 + \frac{a}{y}\right) \cdots \left(1 + \frac{a}{a^{n+1}-1}\right) = f(y)\,\Pi_1, \text{ say}$$

and

$$\Pi_1 < \left(1 + \frac{a}{\theta_1}\right)\left(1 + \frac{a}{1+\theta_1}\right) \cdots \left(1 + \frac{a}{a^{n+1}-1}\right) = \Pi_2, \text{ say}$$

where $\theta_1 \leqslant a^n < \theta_1 + 1$ and $a^{n+1} \equiv \theta_1 \pmod 1$.

Now $\mathrm{II}_2 < e^{a\left(\frac{1}{\theta_1}+\dots+\frac{1}{a^{n+1}-1}\right)} < e^{a\left(\frac{1}{\theta_1}+a\right)} < k.$

Hence

$$f(a^{n+1}) \leqslant k\, f(y) \text{ and so}$$

$$f(\xi) \geqslant \frac{f(y)}{1+a} \geqslant \frac{1}{(1+a)k} f(a^{n+1})$$

which gives

$$\frac{a^{n+1}(a-1)f(a^{n+1})}{a(1+a)k} \leqslant \int_{a^n}^{a^{n+1}} f(t)\,dt \leqslant (a-1)\, ka^n f(a^n)$$

from which the result (ii) follows.

(iii) Consider $\mathrm{I} = \int_{\mathrm{X}}^{2\mathrm{X}} x^{\mu} f(x)\,dx = \mathrm{X}\xi^{\mu} f(\xi)$ where $\mathrm{X} \leqslant \xi \leqslant 2\mathrm{X}$.

Now $f(2\mathrm{X}) \leqslant f(y)\left(1+\frac{a}{y}\right)\dots\dots\left(1+\frac{a}{2\mathrm{X}-1}\right)$

where $0 \leqslant y-\xi \leqslant 1$, $2\mathrm{X} \equiv y \pmod 1$. Hence

$$f(2\mathrm{X}) \leqslant f(\xi)\left(1+\frac{a}{\xi}\right)\left(1+\frac{a}{y}\right)\dots\left(1+\frac{a}{2\mathrm{X}-1}\right)$$

$$\leqslant (1+a) f(\xi)\, e^{a\left(\frac{1}{y}+\dots+\frac{1}{2\mathrm{X}-1}\right)}$$

$$\leqslant (1+a) f(\xi)\, e^{a\frac{2\mathrm{X}-1-y+1}{y}} \leqslant (1+a) f(\xi)\, e^{2a}.$$

Hence

$$\mathrm{I} = \mathrm{X}\xi^{\mu} f(\xi) \geqslant \mathrm{X}^{1+\mu} \frac{f(2\mathrm{X})}{(1+a)\,e^{2a}}.$$

Since the integral is convergent,

$$\mathrm{X}^{1+\mu}\frac{f(2\mathrm{X})}{(1+a)\,e^{2a}} \leqslant \mathrm{I} < \epsilon \text{ for } \mathrm{X} > \mathrm{X}_0 \text{ and so } \lim_{x\to\infty} x^{1+\mu} f(x) = 0.$$

§3. Ex. 1. Let $n_1, n_2, \dots$ be a rapidly increasing sequence of positive integers such that $\sum_{k=1}^{\infty} \frac{1}{\log n_r}$ is convergent, and let $k_r = \left[\frac{n_r}{\log n_r}(l_2 n_r + 2 l_3 n_r)\right]$.*

For instance, we may take

$$n_r = (r+3)!^{(r+3)!^{(r+3)!}}, \quad (r = 1, 2, \dots). \text{ Let}$$

$$f(n) = \frac{1}{n ln\,(l_2 n)^2},\quad e^e < n \leqslant n_1$$

$$= \frac{1}{n_1 (ln_1)(l_2 n_1)^2}\left(1+\frac{ln_1}{n_1}\right)\left(1+\frac{l(n_1+1)}{n_1+1}\right)\dots\left(1+\frac{l(n_1+r)}{n_1+r}\right)$$

for $n = n_1 + r + 1$, $r = 0, 1, 2, \dots, k_1$.

* $ln = \log n$, $l^2 n = \log\log n$ etc.

$$= \frac{1}{n\,(ln)\,(l_2n)^2} \text{ for } n_1 + k_1 + 2 \leqslant n \leqslant n_2.$$

$$= \frac{1}{n_2 l n_2\,(l_2 n_2)^2}\left(1+\frac{ln_2}{n_2}\right)\cdots\left(1+\frac{l(n_2+r)}{n_2+r}\right)$$

$$\text{for } n = n_2 + r + 1,\ r = 0, 1, 2, \ldots, k_2$$

and so on.

$$\Sigma f(n) = \Sigma_1 + \Sigma_2, \text{ where } \Sigma_1 < \Sigma \frac{1}{nln\,(l_2n)^2} < \infty.$$

Writing $n_r = n$ and $k_r = k$ we have

$$\frac{1}{nln\,(l_2n)^2}\left\{\left(1+\frac{ln}{n}\right)+\left(1+\frac{ln}{n}\right)\left(1+\frac{l(1+n)}{1+n}\right)\right.$$

$$\left.+\ \ldots + \left(1+\frac{ln}{n}\right)\left(1+\frac{l(1+n)}{1+n}\right)\cdots\left(1+\frac{l(n+k)}{n+k}\right)\right\}$$

$$< \frac{1}{nln\,(l_2n)^2}\left\{\left(1+\frac{ln}{n}\right)+\ \ldots\ +\left(1+\frac{ln}{n}\right)^{k+1}\right\}$$

$$\sim \frac{1}{nln\,(l_2n)^2}\left\{\frac{\left(1+\frac{ln}{n}\right)^{k+1}-1}{ln/n}\right\}\left\{1+\frac{ln}{n}\right\}$$

$$\sim \frac{1}{(ln)^2\,(l_2n)^2}\, e^{k\log\left(1+\frac{ln}{n}\right)}$$

$$\sim \frac{1}{(ln)^2\,(l_2n)^2}\, e^{\frac{ln}{n}\frac{n}{ln}(l_2n+2l_3n)} \sim \frac{1}{\log n}$$

and hence $\Sigma_2 < c\,\Sigma \dfrac{1}{\log n_r}$ and so $\Sigma f(n)$ is convergent. Further

$$(n+k+1)f(n+k+1) \sim \left\{n+\frac{n}{\log n}(l_2n+2l_3n)\right\}$$

$$\times \frac{1}{nln\,(l_2n)^2}\left\{\left(1+\frac{ln}{n}\right)\left(1+\frac{l(n+1)}{n+1}\right)\cdots\left(1+\frac{l(n+k)}{n+k}\right)\right\}$$

$$> \frac{1}{ln\,(l_2n)^2}\left\{1+\frac{l(n+k)}{n+k}\right\}^k \to 1 \text{ as } n = n_r \to \infty.$$

Hence $\overline{\lim}\ nf(n) = 1$.

Further $f(n+1) = a_{n+1} \leqslant a_n\left(1+\dfrac{\log n}{n}\right)$ for all $n > e^e$.

Ex. 2. Let $m_1, m_2, \ldots$ be a rapidly increasing sequence of integers such that if $2^{m_r} = n_r$ and $k_r = \left[\dfrac{n_r}{\log n_r}(l_2n_r + 3\,l_3n_r)\right]$, then $\Sigma \dfrac{k_r}{n_r}$ is convergent.

Let $f(n) = \frac{1}{n\,ln\,(l_2 n)^2} \qquad e^e < n \leqslant n_1 - k_1 - 1$

$$= \frac{1}{n_1 \left\{1 + \frac{l(n_1-1)}{n_1-1}\right\} \cdots \left\{1 + \frac{l(n_1-k_1)}{n_1-k_1}\right\}}, \qquad n = n_1 - k_1$$

$$= \frac{1}{n_1 \left\{1 + \frac{l(n_1-1)}{n_1-1}\right\} \cdots \left\{1 + \frac{l(n_1-k_1+1)}{n_1-k_1+1}\right\}}, \qquad n = n_1 - k_1 + 1$$

$$= \ldots\ldots$$

$$= \frac{1}{n_1 \left\{1 + \frac{l(n_1-1)}{n_1-1}\right\}}, \; n = n_1 - 1$$

$$= \frac{1}{n_1}, \qquad n = n_1$$

$$= \frac{1}{n\,ln\,(l_2 n)^2} \qquad 1 + n_1 \leqslant n \leqslant n_2' - k_2 - 1$$

and so on.

Then $\Sigma f(n) = \Sigma_1 + \Sigma_2$, where $\Sigma_1 < \Sigma \frac{1}{n\,ln\,(l_2 n)^2} < \infty$

and $\Sigma_2 < \Sigma \frac{1 + k_r}{n_r} < 2\Sigma \frac{k_r}{n_r} < \infty$.

Further let

$$\Pi(n_r) = \left\{1 + \frac{\log(n_r - 1)}{n_r - 1}\right\} \cdots \left\{1 + \frac{\log(n_r - k_r)}{n_r - k_r}\right\}$$

$$> \left\{1 + \frac{\log(n_r - k_r)}{n_r - k_r}\right\}^{k_r} \sim (ln_r)\,(l_2 n_r)^3.$$

Choose N so large that

$$\left\{1 + \frac{\log(n-1)}{n-1} \cdots \left\{1 + \frac{\log(n-k)}{n-k}\right\} > (ln)\,(l_2 n)^2\right.$$

for all $n \geqslant N$. Here $k = \left[\frac{n}{\log n}(l_2 n + 3 l_3 n)\right]$.

Let n_1 be so large that $n_1 - k_1 - 1 > 2N$. Then if we write $f(n) = a_n$, we have $a_{n+1} \leqslant a_n \left(1 + \frac{\log n}{n}\right)$, for $n > e^e$. Further $\Sigma f(n)$ is convergent but $\Sigma\, 2^n f(2^n)$ is divergent, since for an infinity of m

$$2^m f(2^m) = 2^m \frac{1}{2^m} = 1.$$

Post Script.—Since this note was sent to the press, a paper by Otto Szasz has appeared in *American Journal of Mathematics*, LXX, 203–06, containing Theorem 1 (his Theorems 1 and 2). The proofs are different and the remaining part of this paper is believed to be of sufficient interest, to justify the publication of this paper.

S. M. SHAH.

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ON A PROBLEM OF ADDITIVE THEORY OF NUMBERS

BY

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1. In this paper all Latin letters denote positive integers. We write $(m)^k = (n)^k$ when

$$\sum_{s=1}^{s=m} x_s^k = \sum_{t=1}^{t=n} y_t^k \tag{1}$$

has a non-trivial solution, *i.e.*, a solution in positive integers

$$x_1, x_2, \ldots, x_m, y_1, y_2, \ldots, y_n \text{ with } (x_1, x_2, \ldots, x_m, y_1, y_2, \ldots, y_n) = 1$$

and if $m = n$, the y's are not merely a permutation of the x's. We use $\beta(k)$ to denote the least value of n such that (1) has a non-trivial solution with $m < n$, and $N(k)$ to denote the least value of n such that (1) has a non-trivial solution with $m = n$.

2. The following results are proved here:—

(i) $\beta(10) = 11$, (ii) $\beta(14) = 23$, (iii) $\beta(15) = 24$, and (iv) $\beta(16) = 29$.

3. If $a_1^h + a_2^h + \ldots + a_j^h = b_1^h + b_2^h + \ldots + b_j^h$, $(1 \leqslant h \leqslant k)$

then we write $[a_1, a_2, \ldots, a_j]_k = [b_1, b_2, \ldots, b_j]_k$. (2)

Tarry* has observed that (2) implies

$$[a_1, a_2, \ldots, a_j, b_1 + d, b_2 + d, \ldots, b_j + d]_{k+1} = [b_1, b_2, \ldots, b_j, a_1 + d, a_2 + d, \ldots, a_j + d]_{k+1}. \tag{3}$$

It is also known that (2) implies

$$[z + a_1, z + a_2, \ldots, z + a_j]_k = [z + b_1, z + b_2, \ldots, z + b_j]_k \tag{4}$$

4. We have $[2, 5]_1 = [3, 4]_1$ (5)

Applying (3) to (5) with $d = 3, 5, 7, 8, 13, 11, 9, 19, 17$ and 25 in succession, we obtain

$$[2, 6, 12, 22, 28, 29, 43, 49, 53, 55, 59, 65, 69, 71, 75, 81, 95, 96, 102, 112, 118, 122]_{11}$$

$$= [3, 4, 15, 19, 27, 31, 46, 47, 50, 56, 62, 62, 68, 74, 77, 78, 93, 97, 105, 109, 120, 121]_{11}. \tag{6}$$

* See Dickson: *History of the Theory of Numbers*, II, p. 710.

Again applying (4) to (6) with $z = -62$, we have

$$[-60, -56, -50, -40, -34, -33, -19, -13, -9, -7, -3, 3, 7, 9, 13, 19, 33, 34, 40, 50, 56, 60]_{11}$$
$$= [-59, -58, -47, -43, -35, -31, -16, -15, -12, -6, 0, 0, 6, 12, 15, 16, 31, 35, 43, 47, 58, 59]_{11}$$

From this we deduce

$$59^{10} + 58^{10} + 47^{10} + 43^{10} + 35^{10} + 31^{10} + 16^{10} + 15^{10} + 12^{10} + 6^{10} = 60^{10} + 56^{10} + 50^{10} + 40^{10} + 34^{10} + 33^{10} + 19^{10} + 13^{10} + 9^{10} + 7^{10} + 3^{10}$$

i.e., $(10)^{10} = (11)^{10}$. Hence $\beta(10) \leqslant 11$.

Earlier result $\beta(10) \leqslant 14$.†

5. Applying (3) to (5) with $d = 3, 5, 7, 8, 13, 11, 9, 19, 17, 6, 4, 1, 10$ and 2 in succession, we get

$$[2, 5, 5, 5, 12, 16, 19, 20, 21, 27, 28, 28, 30, 33, 34, 42, 42, 43, 48, 48, 51, 58, 59, 63, 64, 71, 74, 74, 79, 80, 80, 88, 89, 92, 94, 94, 95, 101, 102, 103, 106, 110, 117, 117, 117, 120]_{15}$$
$$= [3, 3, 4, 8, 11, 15, 18, 22, 24, 25, 26, 29, 29, 32, 38, 40, 40, 45, 47, 50, 52, 54, 61, 61, 68, 70, 72, 75, 77, 82, 82, 84, 90, 93, 93, 96, 97, 98, 100, 102, 107, 111, 114, 118, 119, 119]_{15}. \quad (7)$$

Now applying (4) to (7) with $z = -61$, we have

$$[-59, -56, -56, -56, -49, -45, -42, -41, -40, -34, -33, -33, -31, -28, -27, -19, -19, -18, -13, -13, -10, -3, -2, 2, 3, 10, 13, 13, 18, 19, 19, 27, 28, 31, 33, 33, 34, 40, 41, 42, 45, 49, 56, 56, 56, 59]_{15}$$
$$= [-58, -58, -57, -53, -50, -46, -43, -39, -37, -36, -35, -32, -32, -29, -23, -21, -21, -16, -14, -11, -9, -7, 0, 0, 7, 9, 11, 14, 16, 21, 21, 23, 29, 32, 32, 35, 36, 37, 39, 43, 46, 50, 53, 57, 58, 58]_{15}.$$

From this we deduce

$$2 \cdot 58^{14} + 57^{14} + 53^{14} + 50^{14} + 46^{14} + 43^{14} + 39^{14} + 37^{14} + 36^{14} + 35^{14} + 2 \cdot 32^{14} + 29^{14} + 23^{14} + 2 \cdot 21^{14} + 16^{14} + 14^{14} + 11^{14} + 9^{14} + 7^{14}$$
$$= 59^{14} + 3 \cdot 56^{14} + 49^{14} + 45^{14} + 42^{14} + 41^{14} + 40^{14} + 34^{14} + 2 \cdot 33^{14} + 31^{14} + 28^{14} + 27^{14} + 2 \cdot 19^{14} + 18^{14} + 2 \cdot 13^{14} + 10^{14} + 3^{14} + 2^{14}.$$

† Moessner and Schulz : *Math. Zeit*, Band **41**, p. 340.

i.e., $(22)^{14} = (23)^{14}$. Hence $\beta(14) \leqslant 23$.

Earlier result $\beta(14) \leqslant 33$.*

6. Again applying (3) to (5) with $d = 3, 5, 7, 8, 13, 11, 9, 19, 17, 6, 4, 1, 10, 7$ and 15 in succession, we get

$$[3, 3, 8, 12, 14, 17, 25, 26, 26, 28, 35, 37, 37, 40, 40, 49, 51, 51, 53, 57, 62, 63, 72, 74, 78, 78, 83, 86, 90, 95, 98, 99, 101, 104, 106, 109, 109, 115, 119, 121, 124, 124, 132, 135, 137, 140]_{16}$$

$$=[2, 5, 7, 10, 18, 18, 21, 23, 27, 33, 33, 36, 38, 41, 43, 43, 47, 52, 56, 59, 64, 64, 68, 70, 79, 80, 85, 89, 91, 91, 93, 102, 102, 105, 105, 107, 114, 116, 116, 117, 125, 128, 130, 134, 139, 139]_{16}. \qquad (8)$$

Now applying (4) to (8) with $z = -71$, we obtain

$$[-68, -68, -63, -59, -57, -54, -46, -45, -45, -43, -36, -34, -34, -31, -31, -22, -20, -20, -18, -14, -9, -8, 1, 3, 7, 7, 12, 15, 19, 24, 27, 28, 30, 33, 35, 38, 38, 44, 48, 50, 53, 53, 61, 64, 66, 69]_{16}$$

$$=[-69, -66, -64, -61, -53, -53, -50, -48, -44, -38, -38, -35, -33, -30, -28, -27, -24, -19, -15, -12, -7, -7, -3, -1, 1, 8, 9, 14, 18, 20, 20, 22, 31, 31, 34, 34, 36, 43, 45, 45, 46, 54, 57, 59, 63, 68, 68]_{16}.$$

From this we deduce

$$2\cdot 68^{15} + 63^{15} + 59^{15} + 57^{15} + 54^{15} + 46^{15} + 2\cdot 45^{15} + 43^{15} + 36^{15} + 2\cdot 34^{15} + 2\cdot 31^{15} + 22^{15} + 2\cdot 20^{15} + 18^{15} + 14^{15} + 9^{15} + 8^{15}$$

$$= 69^{15} + 66^{15} + 64^{15} + 61^{15} + 2\cdot 53^{15} + 50^{15} + 48^{15} + 44^{15} + 2\cdot 38^{15} + 35^{15} + 33^{15} + 30^{15} + 28^{15} + 27^{15} + 24^{15} + 19^{15} + 15^{15} + 12^{15} + 2\cdot 7^{17} + 3^{15} + 1^{15}$$

i.e., $(22)^{15} = (24)^{15}$. Hence $\beta(15) \leqslant 24$.

Earlier result $\beta(15) = 25$.†

7. Applying (3) to (5) with $d = 3, 5, 7, 8, 13, 11, 9, 19, 17, 6, 4, 16, 10, 23, 25$ and 16 in succession, we get

$$[2, 7, 9, 19, 20, 21, 23, 28, 33, 33, 40, 42, 47, 48, 49, 50, 54, 55, 62, 62, 69, 76, 81, 82, 83, 89, 91, 95, 103, 107, 109, 115, 116, 117, 122, 129, 136, 136, 143, 144, 148, 149, 150, 151, 156, 158, 165, 165, 170, 175, 177, 178, 179, 189, 191, 196]_{17}$$

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* Moessner and Schulz : *Ibid., loc. cit.*

† Hansraj Gupta, *Proc. of the Indian Academy of Sciences*, Vol. IV, p. 574.

$$=[3, 4, 15, 17, 17, 18, 25, 32, 32, 37, 39, 39, 44, 46, 46, 53, 58, 58, 59, 65, 66, 70, 72, 73, 79, 87, 97, 98, 99, 99, 100, 101, 111, 119, 125, 126, 128, 132, 133, 139, 140, 140, 145, 152, 152, 154, 159, 159, 161, 166, 166, 173, 180, 181, 181, 183, 194, 195]_{17}. \qquad (9)$$

Now applying (4) to (9) with $z = -99$, we have

$$[-97, -92, -90, -80, -79, -78, -76, -71, -66, -66, -59, -57, -52, -51, -50, -49, -45, -44, -37, -37, -35, -30, -23, -18, -17, -16, -10, -8, -4, 4, 8, 10, 16, 17, 18, 23, 30, 35, 37, 37, 44, 45, 49, 50, 51, 52, 57, 59, 66, 66, 71, 76, 78, 79, 80, 90, 92, 97]_{17}$$

$$=[-96, -95, -84, -82, -82, -81, -74, -67, -67, -62, -60, -60, -55, -53, -53, -46, -41, -41, -40, -34, -33, -29, -27, -26, -20, -12, -2, -1, 0, 0, 1, 2, 12, 20, 26, 27, 29, 33, 34, 40, 41, 41, 46, 53, 53, 55, 60, 60, 62, 67, 67, 74, 81, 82, 82, 84, 95, 96]_{17}.$$

From this we deduce

$$96^{16}+95^{16}+84^{16}+2\cdot 82^{16}+81^{16}+74^{16}+2\cdot 67^{16}+62^{16}+2\cdot 60^{16} + 55^{16}+2\cdot 53^{16}+46^{16}+2\cdot 41^{16}+40^{16}+34^{16}+33^{16}+29^{16} + 27^{16}+26^{16}+20^{16}+12^{16}+2^{16}+1^{16}$$

$$=97^{16}+92^{16}+90^{16}+80^{16}+79^{16}+78^{16}+76^{16}+71^{16}+2\cdot 66^{16} + 59^{16}+57^{16}+52^{16}+51^{16}+50^{16}+49^{16}+45^{16}+44^{16} + 2\cdot 37^{16}+35^{16}+30^{16}+23^{16}+18^{16}+17^{16}+16^{16}+10^{16} + 8^{16}+4^{16}.$$

i.e., $(28)^{16} = (29)^{16}$. Hence (iv) $\beta(16) \leqslant 29$.

ON EQUAL SUMS OF LIKE POWERS

BY

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1. In this paper all small italic letters denote positive integers unless otherwise mentioned.

A non-trivial solution of the equations

$$x_1^k + \ldots + x_n^k = y_1^k + \ldots + y_n^k = z_1^k + \ldots + z_n^k \tag{1}$$

in positive integers is one in which no x is equal to any y or z and no y is equal to a z, and

$$(x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n, z_1, z_2, \ldots, z_n) = 1.$$

If integers $a_1, a_2, \ldots, a_j, b_1, b_2, \ldots, b_j, c_1, c_2, \ldots, c_j$ exist such that no a is equal to any b or c, nor any b equal to a c, and if

$$a_1^l + \ldots + a_j^l = b_1^l + \ldots + b_j^l = c_1^l + \ldots + c_j^l, \; (l = 1, 2, \ldots, k) \tag{2}$$

we write

$$[a_1, \ldots, a_j]_k = [b_1, b_2, \ldots, b_j]_k = [c_1, c_2, \ldots, c_j]_k. \tag{3}$$

Solutions of (2) with all the c's zero are well known but solutions of (2) with a's, b's and c's not all zero are not known for $l > 2$. We give here a method for finding solutions of such equations. For $l = 2$ and $l = 3$ the results are interesting, for solutions of (2) exist with $j = l + 1$.

2. *LEMMA 1. *Tarry has observed that*

$$[a_1, a_2, \ldots, a_j]_k = [b_1, b_2, \ldots, b_j]_k \tag{4}$$

implies that

$$\begin{aligned} &[a_1, a_2, \ldots, a_j, b_1 + d, b_2 + d, \ldots, b_j + d]_{k+1} \\ &= [b_1, b_2, \ldots, b_j, a_1 + d, a_2 + d, \ldots, a_j + d]_{k+1} \end{aligned} \tag{5}$$

We observe here that corresponding results hold for sets of integers > 2.

In particular, we have

LEMMA 2.

$$[a_1, a_2, \ldots, a_j]_k = [b_1, b_2, \ldots, b_j]_k = [c_1, c_2, \ldots, c_j]_k \tag{6}$$

* Dickson : *History of the Theory of Numbers*, Vol. 2, p. 710.

implies that

$$[a_1, a_2, \ldots, a_j, b_1 + x, \ldots, b_j + x, c_1 + y, \ldots, c_j + y]_{k+1}$$
$$= [a_1 + x, \ldots, a_j + x, b_1 + y, \ldots, b_j + y, c_1, c_2, \ldots, c_j]_{k+1}$$
$$= [a_1 + y, \ldots, a_j + y, b_1, b_2, \ldots, b_j, c_1 + x, \ldots, c_j + x]_{k+1}. \quad (7)$$

3. Applying Lemma 1 to

$$[0, a + b]_1 = [a, b]_1 \quad (8)$$

with $d = a + b$, $a + 2b$, in succession (the order is relevant), we have

$$[0, 2a + b, a + 3b, 3a + 4b]_3 = [a, b, 3a + 3b, 2a + 4b]_3. \quad (9)$$

Set in (9)

$$\left.\begin{array}{l} 2a + b = x, a + 3b = y \\ \text{so that } a = \dfrac{3x - y}{5}, b = \dfrac{2y - x}{5}. \\ \text{Now taking } x = 5m, y = 5n, \text{ we get} \\ a = 3m - n, b = 2n - m, 3a + 4b = 5(m + n). \end{array}\right\} \quad (10)$$

Again set in (10)

$$\left.\begin{array}{ll} & a + 3b = x = 5m, \\ & 2a + b = y = 5n \\ \text{so that} & a = 3n - m, b = 2m - n, \\ \text{and} & 3a + 4b = 5(m + n) \end{array}\right\} \quad (11)$$

From (9), (10) and (11), it follows that

$$\left.\begin{array}{l} [0, 5m, 5n, 5(m + n)]_3 \\ = [3m - n, 2n - m, 6m + 3n, 2m + 6n]_3 \\ = [3n - m, 2m - n, 6n + 3m, 6m + 2n]_3. \end{array}\right\} \quad (12)$$

We note the following particular cases of (12):

$m = 2$, $n = 3$,

$$[0, 10, 15, 25]_3 = [3, 4, 21, 22]_3 = [1, 7, 18, 24]_3 \quad (13)$$

$m = 4$, $n = 3$,

$$[0, 15, 20, 35]_3 = [2, 9, 26, 33]_3 = [5, 5, 30, 30]_3 \quad (14)$$

$m = 5$, $n = 3$

$$[0, 15, 25, 40]_3 = [1, 12, 28, 39]_3 = [4, 7, 33, 36]_3. \quad (15)$$

4. Again applying Lemma 1 to (9) with $d = 2a + 3b$ $a + 3b$, in succession, we get

$$[0, 2a + b, 5a + 6b, a + 4b, 4a + 9b, 6a + 10b]_5 = [a, b, 4a + 4b, 2a + 6b, 6a + 9b, 5a + 10b]_5. \quad (16)$$

Now we proceed as in para (3) setting in (16)

in one case $2a + b = 7m$, $a + 4b = 7n$

so that $a = 4m - n$, $b = 2n - m$ (17)

and in the other $a + 4b = 7m$, $2a + b = 7n$

so that $a = 4n - m$, $b = 2m - n$ (18)

and we get

$$[0, 7m, 7n, 14m + 7n, 7m + 14n, 14m + 14n]_5$$
$$= [4m - n, 2n - m, 12m + 4n, 2m + 10n, 15m + 12n, 10m + 15n]_5$$
$$= [4n - m, 2m - n, 12n + 4m, 2n + 10m, 15n + 12m, 10n + 15m]_5. \quad (19)$$

We note the following particular cases.

$m = 2, n = 3,$

$$[0, 14, 21, 49, 56, 70]_5 = [4, 5, 34, 36, 65, 66]_5 = [1, 10, 26, 44, 60, 69)_5 \quad (20)$$

$m = 4, n = 3,$

$$[0, 21, 28, 70, 77, 98]_5 = [2, 13, 38, 60, 85, 96]_5 = [5, 8, 46, 52, 90, 93]_5. \quad (21)$$

From (19) with the repeated application of Lemma 2 we can derive parametric solutions of (2) with $l = 6, j = 18, l = 7, j = 54$, etc.

Also (6) implies that

$$[a_1 + z, \ldots, a_j + z]_k = [b_1 + z, \ldots, b_j + z]_k = [c_1 + z, \ldots, c_j + z)]_k \quad (22)$$

z being any integer, positive or negative.

Now from (22) and (19) with $z = -(7m + 7n)$ we obtain

$$\left.\begin{aligned}&[-(7m+7n),\ -7n,\ -7m,\ 7m,\ 7n,\ 7m+7n]_6\\ &=[-(8n+3m),\ -(8m+5n),\ -(5m-3n),\\ &\qquad (8n+3m),\ (8m+5n),\ (5m-3n)]_6\\ &=[-(8m+3n),\ -(8n+5m),\ -(5n-3m),\ (8m+3n),\\ &\qquad (8n+5m),\ (5n+3m)]_6\end{aligned}\right\} \tag{28}$$

which implies that

$$\left.\begin{aligned}&(7m)^4+(7n)^4+(7m+7n)^4\\ &=(8n+3m)^4+(8m+5n)^4+(5m-3n)^4\\ &=(8m+3n)^4+(8n+5m)^4+(5n-3m)^4\end{aligned}\right\} \tag{24}$$

which proves that $^*r^1_{4,3}(N) \geq 3$ for an infinite number of values of N.

We note finally the following particular cases of (24):

$m = 2,\ n = 3$

$$14^4+21^4+35^4=31^4+30^4+1^4=25^4+34^4+9^4=1,733,522 \tag{25}$$

$m = 4,\ n = 3$

$$28^4+21^4+49^4=36^4+47^4+11^4=41^4+44^4+3^4 \tag{26}$$

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* $r^1_{k,s}(N)$ stands for the number of primitive representations of N as the sum of positive kth powers.

COLLEGIATE SECTION

A Note on Linear Differential Equations

The following determination of a particular integral of the linear differential equation

$$F(D)y = \phi(x),$$

where $F(D)$ has constant coefficients and $\phi(x)$ is a polynomial in x, has features of interest.

Let $f(D)$ be a polynomial in D which has D for a factor, let $\phi(x)$ be a polynomial in x of degree p, and form the sequence of functions $y_0, y_1, y_2, \ldots$ from the initial term $y_0 = \phi(x)$ by the recurrence formula

$$y_n = \phi(x) + f(D)\, y_{n-1}.$$

Then

$$y_{n+1} - y_n = f(D)\,\{y_n - y_{n-1}\}.$$

But, y_0 is a polynomial of degree p, and therefore, since $f(D)$ has D for a factor, $y_1 - y_0$ is a polynomial of degree not greater than $p - 1$, and $y_2 - y_1, y_3 - y_2, \ldots$ are polynomials of steadily diminishing degree; $y_p - y_{p-1}$ is a constant. Hence $y_{p+1} - y_p = 0$, that is, $y_p = \phi(x) + f(D)\, y_p$. In other words, y_p is a solution of the equation

$$\{1 - f(D)\}\, y = \phi(x).$$

Since

$$y_1 - y_0 = f(D)\,\phi(x),\ y_2 - y_1 = \{f(D)\}^2 \phi(x),\ y_3 - y_2 = \{f(D)\}^3 \phi(x), \ldots$$

the solution appears as

$$y_p = [1 + f(D) + \{f(D)\}^2 + \{f(D)\}^3 + \ldots \{f(D)\}^p]\, \phi(x),$$

but terms of degree higher than p in D may be omitted from all the operators $f(D)$, $\{f(D)\}^2$, $\{f(D)\}^3, \ldots$. When this is done, the series

$$1 + f(D) + \{f(D)\}^2 + \{f(D)\}^3 + \ldots + \{f(D)\}^p$$

becomes simply the polynomial which is found by dividing $1 - f(D)$ into unity as far as the term in Dp.

If $F(D)$ is a polynomial with a non-zero constant term a_0, we may write $F(D)$ as $a_0\{1 - f(D)\}$ and solve the equation $\{1 - f(D)\}\, y = \phi(x)/a_0$. If the term of lowest degree in $F(D)$ is $a_k D^k$, repeated integration of $\phi(x)$ produces a polynomial $\psi(x)$, of degree $p + k$ such that $D^k \psi(x) = \phi(x)$, and writing $F(D)$ as $a_k D^k \{1 - fD\}$, we solve the equation $\{1 - f(D)\}\, y = \psi(x)/a_k$.

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*On a Problem of V. G. Iyer**

Problem.—If $f(x)$ is defined and has derivatives of the first n orders in $(a < x < b)$, and a number θ exists such that

(1) θ is fixed, *i.e.*, independent of x and h; and

(2) $f(x+h) - f(x) - \sum_{r=1}^{n-1} \frac{h^r}{r!} f^r(x) = \frac{h^n}{n!} f^n(x + \theta h)$ for x and h such that

(3) $a < x < x + h < b$,

then, $f^{n+1}(x)$ exists in $a < x < b$, and either (*a*) $f^{n+1}(x) = 0$ in $a < x < b$, or (*b*) $f^{n+1}(x) = c \neq 0$ in $a < x < b$; and so $f(x)$ is a polynomial of degree $\leqslant n + 1$.

Proof.—[In the following proof, $h > 0$, and x and h are such that x and $x + h$ satisfy (3)].

For fixed h, the derivatives of the l. h. s. exists in virtue of the hypothesis for every x; and hence f^{n+1} exists at $x + \theta h$, and so at every x, since every number can be put in the form $x + \theta h$ by choice of x and $x + h$ as close to it as desired. Hence follows by differentiation with respect to x, that (2) under the condition (3) is true with $f'(x)$ in place of $f(x)$. Hence, by induction, the derivative of every order exists at every x and is continuous. Now, differentiating (2) $n + 1$ times and $n + 2$ times w. r. t. h and making $h \to +0$, follows by Leibniz's Theorem and continuity, that

(4) $f^{n+1}(x) = (n+1)\,\theta f^{n+1}(x)$; *i.e.*, $f^{n+1}(x)\,[1 - (n+1)\,\theta] = 0$

and

(5) $f^{n+2}(x) = \dfrac{2 + n(n+1)}{2}\,\theta^2 f^{n+2}(x)$; *i.e.*,

$$f^{n+2}(x)\left[1 - \frac{\theta^2 (n+1)(n+2)}{2}\right] = 0.$$

Now, either (*a*) $f^{n+1}(x) = 0$ every where in $a < x < b$,

or (*b*) $f^{n+1}(x) = c \neq 0$ for *some* x.

Then (4) give

(6) $\theta = \dfrac{1}{n+1}$,

and (5) and (6) give

(7) $f^{n+2}(x) = 0$; *i.e.*, $f^{n+1}(x) = c \neq 0$ for every x.

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* A somewhat complicated proof for the case $n = 1$ and the interval $(-\infty, \infty)$ was given in the March-June 1946 issue of the *Math. Student*, by V. G. Iyer on the hypothesis of the truth of (1) for *all x and h*. In the course of correspondence recently opened by me, he has sent me a proof for general n and (a, b), simpler than his published one under reference for $n = 1$ and $(-\infty, \infty)$. The proof below, which differs from his after the establishing of the existence of $f^{n+1}(x)$, but has been accepted by him, does not require any information by way of hypothesis for negative h.

Remarks on the Discussion of the Sequence

$$u_{n+1} = \sqrt{\left(\frac{u_n^2 + ab^2}{a+1}\right)},\ a > 0;\ b^2 \geqslant 0,$$

(*vide* p. 41 of *the Mathematics Student* of March-June, 1946.)

It is not necessary to consider two relations. The first one, *viz.*,

$$u_{n+1}^2 - b^2 = \frac{u_n^2 - b^2}{a+1}$$

contains all the results. For, from it follow

$$u_n^2 - b^2 = \frac{u_{n-1}^2 - b^2}{a+1},\ \text{and so on giving}$$

$$u_{n+1}^2 - b^2 = \frac{u_1^2 - b^2}{(a+1)^n}$$

whether *from above* or *from below* $u_1, u_2, u_3, \ldots u_n, \ldots$ go on approaching b, and as $\frac{1}{(a+1)^n} \to 0$ as $n \to \infty$, the limit of u_{n+1} is b; etc., etc.

Note.—The above is given as example No. 15 on p. 67 of my "*Lessons in Elementary Analysis*" (Third edition).

Jaipur. G. S. MAHAJANI.

A Proof of Morley and Peterson's Theorem

STATEMENT.—*If l_1, l_2, l_3, l_4 are four lines such that the shortest distance between l_1 and l_2 cuts the S.D between l_3 at l_4 at right angles and the S.D between l_1 and l_3 cuts the S.D between l_2 and l_4 at right angles, then the S.D between l_1 and l_4 cuts the S.D. between l_2 and l_3 at right angles.*

PROOF.—Let $l_{1,2}$ (*S.D* between l_1 and l_2) be taken as the X-axis and $l_{3,4}$ (*S.D* between l_3 and l_4) be taken as the Y axis.

Now, let the four lines be

$$(l_1)\quad \frac{x - x_1}{0} = \frac{y}{1} = \frac{z}{m_1}$$

$$(l_2)\quad \frac{x - x_2}{0} = \frac{y}{1} = \frac{z}{m_2}$$

$$(l_3)\quad \frac{x}{1} = \frac{y - y_3}{0} = \frac{z}{m_3}$$

and

$$(l_4)\quad \frac{x}{1} = \frac{y - y_4}{0} = \frac{z}{m_4}.$$

So, the direction cosines of $l_{1,3}$ are $m_3, m_1, -1$

and D.C's of $l_{2,4}$ are $m_4, m_2, -1$.

Since $l_{1,3}$ is $\perp^r$ to $l_{2,4}$, we have

$$1 + m_1m_2 + m_3m_4 = 0. \tag{A}$$

The condition that $l_{1,4}$ may be $\perp^r$ to $l_{2,3}$ is got from (A) by replacing 3 by 4 and 4 by 3, in the suffixes. Hence it is the same as (A).

Let $l_{1,3}$ cut l_1 at $(x_1, \lambda, \lambda m_1)$, so that the equation of $l_{1,3}$ is

$$\frac{x - x_1}{m_3} = \frac{y - \lambda}{m_1} = \frac{z - \lambda m_1}{-1}.$$

This intersects l_3.

$$\therefore \begin{vmatrix} x_1, & \lambda - y_3, & \lambda m_1 \\ m_3, & m_1, & -1 \\ 1, & 0, & m_3 \end{vmatrix} = 0$$

i.e., $m_1 m_3 x_1 + (y_3 - \lambda)(1 + m_3^2) - \lambda m_1^2 = 0.$

$$\therefore \quad \lambda = \frac{m_1 m_3 x_1 + y_3(1 + m_3^2)}{1 + m_1^2 + m_3^2} \tag{B}$$

Similarly, the equation of $l_{2,4}$ is

$$\frac{x - x_2}{m_4} = \frac{y - \lambda'}{m_2} = \frac{z - \lambda' m_2}{-1}$$

where

$$\lambda' = \frac{m_2 m_4 x_2 + y_4(1 + m_4^2)}{1 + m_2^2 + m_4^2} \tag{C}$$

Hence $l_{1,3}$ will intersect $l_{2,4}$ if

$$\begin{vmatrix} x_1 - x_2, & \lambda - \lambda', & \lambda m_1 - \lambda' m_2 \\ m_3, & m_1, & -1 \\ m_4, & m_2, & -1 \end{vmatrix} = 0.$$

i.e., if $(x_1 - x_2)(m_2 - m_1) - \lambda m_4(1 + m_1^2 + m_3^2)$
$\qquad - \lambda' m_3(1 + m_2^2 + m_4^2) = 0,$ using (A),

i.e, if $(x_1 - x_2)(m_2 - m_1) - m_4\{x_1m_1m_3 + y_3 + y_3m_3^2\}$
$\qquad - m_3\{x_2m_2m_4 + y_4 + y_4m_4^2\} = 0,$ (substituting for λ and λ'.)

i.e., the condition reduces to

$$(x_1 - x_2)(m_2 - m_1) - m_3m_4(m_1x_1 + m_2x_2) - (m_4y_3 + m_3y_4)$$
$$- m_3m_4(m_3y_3 + m_4y_4) = 0. \tag{D}$$

(D) is the condition that $l_{1,3}$ should intersect $l_{2,4}$. The condition that $l_{1,4}$ should intersect $l_{2,3}$ is got from (D) by replacing 3 by 4 and 4 by 3. But since (D) is symmetrical in the suffixes 3 and 4, we get the same condition. Hence the theorem follows.

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COLLEGIATE SECTION

*Ptolemy's Theorem**

The following two proofs of Ptolemy's Theorem may be worthy of notice. The second proof avoids the use of similar triangles.

1. Let $ABCD$ be a cyclic quadrilateral. Draw AL, AM parallel respectively to BC, CD to meet the circumcircle of the quadrilateral in L and M. Then it is readily seen that chord AB equals chord CL, chord AD equals chord CM and chord BL equals chord AC, while DL becomes parallel to BM so that the perpendiculars to BM and DL from C are one and the same. If the feet of these perpendiculars be X and Y and the diameter of the circle $ABCD$ be $2R$, we have

$$2R \cdot XY = 2R \cdot CX + 2R \cdot CY$$

i.e., $BD \cdot BL = BC \cdot CM + CD \cdot CL$, from the triangles BDL, BCM and CDL;

i.e., $BD \cdot AC = BC \cdot AD + CD \cdot AB$.

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2. $ABCD$ is a cyclic quadrilateral. Along AB, AC, AD are set off respectively lengths AE, AG, AF equal to CD, BD and BC in order, EH and FK are drawn parallel to the tangent at A to the circle $ABCD$ to meet AC in H and K. Now the following pairs of triangles are congruent:

$$\triangle BCD \equiv \triangle GEA$$
$$\triangle AKF \equiv \triangle GHE$$

so that $BC = EG = AF$, $DB = AG$, $AK = GH$, and $A\hat{G}E = D\hat{B}C = D\hat{A}C$.

Further E, B, H, C are concyclic as also C, D, F, K since $E\hat{B}C = CDF = A\hat{K}F = E\hat{H}C$.

Hence $AB \cdot AE = AC \cdot AH$,

$$AD \cdot AF = AC \cdot AK = AC \cdot GH$$
$$\therefore AB \cdot AE + AD \cdot AF = AC(AH + GH)$$
$$= AC \cdot AG$$
$$\therefore AB \cdot CD + AD \cdot BC = AC \cdot BD.$$

A. A. K.

† For three other proofs of the same theorem, see *Math. Gazette*, Vol. XI, No. 162, pp. 236, 237 (January, 1923).

A Note on Casey's Theorem on Four Circles Tangential to a Fifth

LEMMA.—*Let A and B be the centres of two circles of radii a and b. Let MN be a common tangent to them. And let O be the centre of a circle of radius r, touching the circles A and B at P and Q. Join AM, BN, AB, OA, OB and PQ. From B draw BL parallel to MN meeting AM at L. Then denoting the common tangent MN by the symbol* $\overline{ab}$, *it is required to shew that*

$$\frac{PQ^2}{MN^2} = \frac{r^2}{(r+a)(r+b)}.$$

PROOF.—Now, $MN^2 = BL^2 = AB^2 - (OA - OB)^2$

$$= 2OA \cdot OB - (OA^2 + OB^2 - AB^2)$$

$$= 2\, OA \cdot OB\,(1 - \cos\theta), \text{ where } \theta \text{ is the angle } AOB.$$

$$= 4\, OA \cdot OB \sin^2 \frac{\theta}{2} = 4\,(r+a)(r+b)\sin^2\frac{\theta}{2}.$$

Again since $OP = OQ$, $PQ = 2r \sin \frac{\theta}{2}$.

Hence
$$\frac{PQ^2}{MN^2} = \frac{r^2}{(r+a)(r+b)}.$$

THEOREM.—Let C and D be centres of two other circles touching the circle O at R and S.

Then the lemma gives $\dfrac{PQ}{\overline{ab}} = \dfrac{r}{\sqrt{(r+a)(r+b)}}$, $\dfrac{QR}{\overline{bc}} = \dfrac{r}{\sqrt{(r+b)(r+c)}}$.

$$\frac{RS}{\overline{cd}} = \frac{r}{\sqrt{(r+c)(r+d)}}, \quad \frac{PS}{\overline{ad}} = \frac{r}{\sqrt{(r+a)(r+d)}}, \quad \frac{PR}{\overline{ac}} = \frac{r}{\sqrt{(r+a)(r+c)}}$$

$$\frac{QS}{\overline{bd}} = \frac{r}{\sqrt{(r+b)(r+d)}}.$$

Hence, observing that in the circle O, Ptolemy's theorem gives,

$$PQ \cdot RS + PS \cdot QR = PR \cdot QS.$$

We get, by substitution,

$$\overline{ab} \cdot \overline{cd} + \overline{ad} \cdot \overline{bc} = \overline{ac} \cdot \overline{bd}.$$

which is Casey's celebrated theorem.

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Mathematics at Oxford

Somehow, mathematics at Oxford becomes definitely an art and is content not to be seduced into the fold of the sciences even if an inviting phrase like the "Queen of the Sciences" is thrown at it. Perhaps the tradition of Oxford in art has something to do with it and symbolically the famous High Street of Oxford is as mathematically pleasing and continuous as it is artistically appealing.

The late Sadlerian Professor Love, the author of the classical work on "Elasticity" made his lectures more of poetry of form and rhythm than a prose of science and deduction. Elegance and form were to him the very breath of Oxford mathematics and he would avoid even short methods in order to preserve a certain harmony and life in the solution of the problems.

Oxford retains its weakness for Geometry—a branch which always affords such fruity and choice solutions; Dr. Newbolt of Merton while lecturing to the students used to stop a few seconds before an elegant result pleasing both as a solution and an art-form. Apart from these examples the pervading spirit of the mathematical men at Oxford was and is always a leaning towards the sense of the beauty and the rhythm both in the problem and the subsequent solution.

Oxford has its mathematical society which is called "The Invariant Society" named with a sense of humour as it is obvious that being a student society its members are never the same. And yet the society flourishes under the name and the papers read there betray the essential traditions of all things Oxford—a leaning towards forms of art.

An attempt is made in the mathematical museum to model in chalk the cubic and the quartic surfaces. One notices in the workmanship the balance of the forces of the science and the art. Mathematics tends always to lie hidden in things more manifest and yet we find the pure charm of it in such models to represent the surfaces.

Among mathematical libraries at Oxford is an excellent collection of works at the Magdalean college which contains some original works. The famous Radcliffe science library has a whole floor for mathematics and adjoins the professors' rooms and the museum. The whole looks down on a garden in the strict Oxford tradition.

Mathematical lectures usually begin with a full room during the earlier part of the term. Its subsequent career depends entirely on the usefulness of the lectures as relevant to the syllabus. But the student usually never misses the informal classes or the seminars which take him beyond the syllabus and initiate him to research. These informal classes even more reveal the subtle but very real difference in the development of the mathematics. For Oxford will always lean towards the form and art of mathematics and look with lingering eyes on elegant solutions.

G. A. DHARMARAJAN.

Series and Their Regions of Convergence

[At appropriate stages in every course of mathematical study, it becomes necessary to deliver "review lectures" dealing with well-defined topics. These are intended to be of help in reviewing and consolidating the knowledge gained so far, and in appreciating its relation to cognate fields of study. Such lectures afford the teacher an opportunity of inserting such loose ends as may be difficult to fit elsewhere, to point out interesting and difficult problems beyond the immediate purview, and to weave individual results into a harmonious pattern. They call for some scholarship and sympathetic understanding of the subject with which they deal. We have great pleasure in giving below, the skeleton of such a review lecture by one who has always sought to make his class-room lectures otner than routine performances.]

A. N. R., *Editor.*

I. DOMAIN OF CONVERGENCE

When a function $f(z)$ is defined as the sum of an infinite series of functions of z, say, $f(z) = \Sigma u_n(z)$, one of the first questions which arises is: For what values of z does the series converge? For instance the series

(i) $\sum_{n=0}^{\infty} z^n$

(ii) $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$,

(iii) $\left(1 - \frac{1}{2^{s-1}}\right) \zeta(s) = \sum_{n=1} \frac{(-1)^{n+1}}{n^s} = 1 - \frac{1}{2^s} + \frac{1}{3^s} - \frac{1}{4^s} + \ldots$

(iv) $\sum_{n=0}^{\infty} e^{nz} z^n$

(v) $\sum_{n=0}^{\infty} (z + z^2)^{2n}$

(vi) $\sum_{n=1}^{\infty} \frac{\cos nx}{n}$

converge respectively in the following domains :—

(1) $|z| < 1$, *i.e.*, the interior of the unit circle,

(2) $\sigma > 1$, where $s = \sigma + it$, *i.e.*, the half-plane to the right of $\sigma = 1$,

(3) $\sigma > 0$, *i.e.*, the half plane to the right of $\sigma = 0$,

(4) $|ze^z| < 1$, *i.e.*, the region bounded on the right by the curve $y^2 = e^{-2x} - x$, where $z = x + iy$,

(5) $|z + z^2| < 1$, *i.e.*, in the interior of the curve $|z(z+1)| = 1$, *i.e.*, $(x^2 + y^2)\{(x+1)^2 + y^2\} = 1$,

(6) x real and not equal to any integral multiple of 2π.

It is interesting to know that the domain of convergence of a power series is, unlike in (6,) a single connected domain and that too always a circle. This is a theorem of Abel. A fairly easy argument shows that

(A) *if* $\Sigma a_n z^n$ *is convergent for* $z = \xi$, *then the series is convergent, indeed absolutely, whenever* $|z| < |\xi|$.

Either the series converges for all values of z (*e.g.*, $\Sigma z^n/n!$) or for $z = 0$ only (*e.g.*, $\Sigma n!\, z^n$) or, in view of (A) and Dedekind's theorem on sections of real numbers, there exists a finite positive number R such that the series converges whenever $|z| < R$ and diverges whenever $|z| > R$; when $|z| = R$ the discussion of convergence or divergence is apt to be difficult. We refer to the circle $|z| = R$ as the circle of convergence and its interior as the domain of convergence of the series. We may compare (*A*) to the following result relating to Dirichlet series*:

(B) *If* $\Sigma \frac{b_n}{n^s}$ *is convergent for* $s = s_0 = \sigma_0 + it_0$ *then the series is convergent, indeed uniformly, throughout the angular region in the s-plane defined by the inequality* $|\text{am}\,(s - s_0)| < \frac{1}{2}\pi - \delta$, *where* δ *is any positive number less than* $\frac{1}{2}\pi$.

Since in the above result δ is arbitrary, it follows that the series is convergent whenever $\sigma > \sigma_0$; it is then easy to deduce that the domain of convergence is a half-plane.

Exercise.—Find the domain of convergence of the series $\Sigma \frac{b_n}{n^s}$ *when* (i) $b_n = n!$, (ii) $b_n = 1/n!$, (iii) $b_n = (-1)^n n^{\frac{1}{2}}$.

[*Hint.*—Consider for what real values of s the series is convergent and then apply (*B*).]

For a power series the domain of convergence is also the domain of absolute convergence. For a Dirichlet series such need not be the case; thus the series $\Sigma \frac{(-1)^n}{n^s}$ converges whenever $\sigma > 0$ but the convergence is absolute if and only if $\sigma > 1$. The half-plane of convergence contains the half-plane of absolute convergence and an infinite vertical strip of unit breadth. For the series Σn^{-s} the half-plane of convergence coincides with that of absolute convergence.

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* See, for instance, p. 289 (§ 9·11) of *The Theory of Functions*, by E. C. Titchmarsh.

There is another proof for the result that the domain of convergence of a power series is a circle and this proof yields a formula for the value of the radius of convergence. We guess from a consideration of particular cases such as

$$1 + x + x^2 + x^3 + x^4 + x^5 + \ldots, \qquad R = 1,$$

$$1 + 2x + 4x^2 + 8x^3 + 16x^4 + 32x^5 + \ldots, \qquad R = \tfrac{1}{2},$$

$$1 + 3x + 9x^2 + 27x^3 + 81x^4 + 243x^5 + \ldots, \qquad R = \tfrac{1}{3},$$

$$1 + 2x + x^2 + 8x^3 + x^4 + 32x^5 + \ldots, \qquad R = \tfrac{1}{2},$$

$$1 + 2x + 9x^2 + x^3 + 16x^4 + 243x^5 + \ldots, \qquad R = \tfrac{1}{3},$$

$$\Sigma\, 2^{\lambda_n} x^{\lambda_n} \qquad R = \tfrac{1}{2},$$

$$\Sigma\, (2^{\lambda_n} + 3^{\lambda_n})\, x^{\lambda_n}, \qquad R = \tfrac{1}{3},$$

where $\lambda_1, \lambda_2, \ldots$ is any strictly increasing sequence of positive integers, that perhaps the radius of convergence of the power series $\Sigma\, a_n x^n$ is equal to

$$\frac{1}{\overline{\lim\limits_{n \to \infty}}\, |\, a_n\,|^{1/n}}.$$

Once this guess is made in this or in some other way, it is easy to verify that it yields the correct result. We write α_n for $a_n{}^{1/n}$; it does not matter which nth root of a_n is denoted by α_n and then we have

$$\Sigma\, a_n z^n = \Sigma\, (\alpha_n z)^n.$$

Plainly the series on the right converges if $\overline{\lim}\, |\, \alpha_n z\,| < 1$ and diverges if $\overline{\lim}\, |\, \alpha_n z\,| > 1$. We deduce at once the Cauchy-Hadamard theorem:

(C) *The power series* $\Sigma\, a_n z^n$ *converges if* $|\, z\,| < R$ *and diverges if* $|\, z\,| > R$, *where* $R = 1/\overline{\lim}\, |\, \alpha_n\,|$.

We can also write $\log R = -\, \overline{\lim}\, \dfrac{\log |\, a_n\,|}{n}$. A corresponding formula for the Dirichlet series $\Sigma b_n n^{-s}$ gives the abscissa of convergence in terms not of b_n but of s where $s_n = b_1 + b_2 + \ldots + b_n$:

(D) *If* s_n *does not converge to a finite limit, then the abscissa of convergence is equal to*

$$\overline{\lim_{n \to \infty}}\, \frac{\log\, |\, s_n\,|}{\log\, n}.$$

EXERCISES.—(*a*) *Prove that if* $s_n = a_0 + a_1 + \ldots + a_{n-1}$, *then the radius of convergence of* $\Sigma\, a_n z^n$ *is also that of* $\Sigma\, s_n z^n$.

(*b*) *Prove that the series obtained by differentiating any power series* $\Sigma\, a_n z^n$ *term by term, viz.,* $\sum\limits_{n=1}^{\infty} n a_n z^{n-1}$ *has also the same radius of convergence as* $\Sigma\, a_n z^n$.

(c) *If* $0 < R_1 < R_2$, *and* R_1 *and* R_2 *be the radii of convergence of* $\Sigma a_n z^n$ *and* $\Sigma b_n z^n$, *then show that* $\Sigma (a_n + b_n) z^n$ *has the radius of convergence* R_1.

(d) *Find the radius of convergence of* $\Sigma n! x^{n^2}$, *and of* $\Sigma 2^{n \cos n\theta} x^n$, θ *being a real number.*

(e) *Show that the domain of convergence of* $\Sigma a_n z^n$ *is included in that of any subseries* $\Sigma a_{\lambda_n} z^{\lambda_n}$ *but that the domain of convergence of* $\Sigma (-1)^n n^s$ *is not included in that of the sub series*

$$\frac{1}{2^s} + \frac{1}{4^s} + \frac{1}{6^s} + \frac{1}{8^s} + \ldots = \Sigma (-1)^{2n} (2n)^{-s}.$$

II. CONVERGENCE OR DIVERGENCE ON THE CIRCLE OF CONVERGENCE

I stated earlier that it is apt to be difficult to settle questions of convergence and divergence on the circle of convergence. If $\Sigma a_n z^n$ is absolutely convergent for a value of z on the circle of convergence $|z| = R$ then plainly it is so for all z for which $|z| = R$. Such, for instance, is the case if $a_n = 1/n^2$, $n = 1, 2, 3, \ldots$. If $a_n R^n$ does not tend to zero, then too no interesting question arises since then the series can converge for no value of z on the circle of convergence; such is the case if $a_n = \cos n\theta$. The series for $\log (1 - z)$ and for $(1 - z)^c$, $c > -1$ converge at all points on the unit circle except $z = 1$. It is very much more difficult to show that there exists a power series for which $a_n R^n = 0$ as $n \to \infty$ and yet does not converge at one point even on the circle of convergence. N. Lusin* gave an example of a power series $\Sigma a_n x^n$ where $a_n \to 0$ but such that the series does not converge at one point even on the unit circle. W. Sierpinski considered the series

$$a_0 - a_0 x + a_1 x^2 - a_1 x^3 + a_2 x^4 - a_2 x^5 + \ldots = (1 - x) \Sigma a_n x^{2n},$$

where $\Sigma a_n x^n$ is the series given by Lusin. We see at once that Sierpinski's series converges at just one point on the circle of convergence, *viz.*, $x = 1$. G. H. Hardy† gave an example of a power series which on the entire circle of convergence converges uniformly but not absolutely.

EXERCISE.—*Show that the series*

$$\frac{z^3}{1} - \frac{z^6}{1} + \frac{z^9}{2} - \frac{z^{18}}{2} + \frac{z^{27}}{3} - \frac{z^{54}}{3} + \ldots = \sum_{n=1}^{\infty} \frac{z^{3^n} - z^{2 \cdot 3^n}}{n}$$

converges whenever $z^{3^n} = 1$ *and diverges whenever* $z^{3^n} = -1$ *and that such points are dense everywhere on the unit circle.*

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* 'Über eine Potenzreihe,' *Rendiconti del Circolo Matematico di Palermo*, **32** (1911), 386–90, see also E. Landau.

† "A Theorem concerning Taylor's Series," *The Quarterly Journal of Pure and Applied Mathematics*, **44** (1913), 147–60.

III. OVER-CONVERGENCE

The example (v) given in the first paragraph of this note is an interesting one. The series $\sum_{n=0}^{\infty} (z+z^2)^{3^n}$ is a series of polynomials, *viz.*, $(z+z^2)^{3^n}$, $n = 0, 1, 2, \ldots$, *i.e.*,

$z+z^2,\ z^3+3z^4+3z^5+z^6,\ z^9+9z^{10}+36z^{11}+\ldots+z^{18},\ z^{27}+27z^{28}+\ldots+z^{54}, \ldots, \ldots$

The nth polynomial is of degree $2\cdot 3^{n-1}$ and the $(n+1)$th polynomial begins with the term z^{3^n}; hence if we write

$$\sum_{n=0}^{\infty} (z+z^2)^{3^n} = \Sigma a_n z^n = z + z^2 + z^3 + 3z^4 + \ldots,$$

then the original series of polynomials is obtained from the power series on the right side by merely putting in suitable brackets; thus

$$z+z^2+z^3+3z^4+\ldots = (z+z^2)+(z^3+3z^4+\ldots)+\ldots = \Sigma(z+z^2)^{3^n}.$$

Now the domain of convergence of the power series is also the domain of absolute convergence of the power series and if we write $|z| = r$ then the absolute convergence of the power series implies (and is implied by) the convergence of the series $\Sigma(r+r^2)^{3^n}$. Hence $0 \leqslant r + r^2 < 1$, and we easily verify that the inequality is satisfied if and only if $r < \frac{1}{2}(\sqrt{5}-1)$. Hence the radius of convergence of the power series is $\frac{1}{2}(\sqrt{5}-1)$. But it is extremely interesting to know that the series of polynomials converges in a larger domain than the circle $|z| = \frac{1}{2}(\sqrt{5}-1)$; the larger domain is the interior of the curve $|z+z^2| = 1$, *i.e.*, $|z(z+1)| = 1$, *i.e.*, $(x^2+y^2)[(x+1)^2+y^2] = 1$. The domain plainly includes $z = -1$ and extends on the left along the real axis up to $-\frac{1}{2}(\sqrt{5}+1)$. Now let $s_n(z)$ denote $\sum_{m=0}^{n} a_m z^m$ where $\sum_{n=0}^{\infty} (z+z^2)^{3^n} = \Sigma a_n z^n$. The convergence of the power series means the convergence of the sequence $s_n(z)$ as $n \to \infty$, whereas the convergence of the series of polynomials means the convergence of the sequence $s_{2\cdot 3^n}(z)$ as $n \to \infty$. For the case under consideration the domain of convergence of the sequence $s_n(z)$ is the circle $|z| = \frac{1}{2}(\sqrt{5}-1)$ where as the domain of convergence of the subsequence $s_{2\cdot 3^n}(z)$ is the larger but non-circular domain $|z(z+1)| < 1$. Important researches have been made in connection with this phenomenon of over-convergence.* The following is a curious example of no importance; if

$$\sum_{n=1}^{\infty} \frac{z^{\frac{1}{2}n(n+1)}(z-1)(z-2)\ldots(z-n)}{n!} = \Sigma a_n z^n,$$

then the power series on the right converges if and only if $|z| < 1$ whereas the series of polynomials on the left converges in addition for $z = 1, 2, 3, \ldots$. The following examples are more interesting.

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* See, for instance, A. Ostrowski, "On Representation of Analytical Functions by Power Series," *Journal of the London Mathematical Society*, I, (1926), 251–63.

Let $Z_1 = z,\ Z_2 = z + \frac{z^2}{2\,!},\ Z_3 = z + \frac{z^2}{2\,!} + \frac{z^3}{3\,!}, \ldots$

so that $Z_n = \sum_{m=1}^{n} z^m/m\,! \to e^z - 1$ when $n \to \infty$.

Consider

$$\sum_{n=1}^{\infty} \left(\frac{Z_n}{2}\right)^{n!} = \sum_{n=1}^{\infty} b_n z^n;$$

it is easily verified that the expression on the left could be obtained by inserting suitable brackets in the power series on the right. Since $Z_n \to e^z - 1$ it follows that the series of polynomials on the left converges whenever $|\,e^z - 1\,| > 2$, and, therefore, in particular, for all values of z that lie to the left of the y-axis. The radius of convergence of the power series is obtained by considering for what real positive values of z do we have $e^z - 1 < 2$? Thus we see that the domain of convergence of the power series is the circle $|\,z\,| = \log 3$ whereas the domain of convergence of the bracketed series includes the half plane to the left of the y-axis.

EXERCISE.—*If*

$$\sin z = \sum_{m=1}^{\infty} a_m z^m,\ Z_n = \sum_{m=1}^{n} a_m z^m$$

$$f(z) = \sum_{n=1}^{\infty} (2Z_n)^{n!} = \sum_{n=1}^{\infty} b_n z^n,$$

then show that the domain of convergence of the power series $\Sigma\, b_n z^n$ *is the circle* $|\,z\,| < \sin h^{-1} \frac{1}{2}$ *and that the series of polynomials* $\Sigma\,(2Z_n)^{n!}$ *converges in a set of disconnected domains, viz., the interiors of the level curves* $|\,\sin z\,| = \frac{1}{2}$.

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On the Nature of θ in the Lagrange Remainder in the Taylor Series

1*. *Problem.*—In the Taylor formula

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!} f''(x) + \dots$$
$$+ \frac{h^{n-1}}{(n-1)!} f^{n-1}(x) + \frac{h^n}{n!} f^n(x+\theta h) \qquad (0 < \theta < 1) \qquad \text{(I)}$$

it is proposed to investigate under what conditions,

(1) θ is independent of both x and h;

(2) θ is independent of x and depends on h only;

(3) θ is independent of h and depends on x only.

2*. *Discussion.*—It is well known that the θ_n occurring in the Taylor-Maclaurin formula,

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \dots + \frac{x^n}{n!} f^n(\theta_n x) \qquad \text{(II)}$$

is given by

$$\theta_n = \frac{1}{n+1} + \frac{n}{2(n+1)^2(n+2)} \left\{ \frac{f^{n+2}(0)}{f^{n+1}(0)} + \epsilon_x \right\} x$$

where $\epsilon_x \to 0$, as $x \to 0$, with the conditions that $f(x)$ and its first $(n+2)$ derivatives are continuous and $f^{n+1}(0) \neq 0$. [See writer's *Elementary Analysis*, Third Edition, page 141.]

Applying the same procedure to the Taylor-formula I, we get

$$\theta = t_n = \frac{1}{n+1} + \frac{n}{2(n+1)^2(n+2)} \left[\frac{f^{n+2}(x)}{f^{n+1}(x)} + \epsilon_h \right] h \; . \qquad \text{(III)}$$

[The conditions to be satisfied are as before that $f(x)$ and its first $n+2$ derivatives are continuous and $f^{n+1}(x) \neq 0$.]

This result III gives us all the information we seek. Thus:

3*. (1) We see at once from III that as $h \to 0$, $\theta \to \frac{1}{n+1}$. Hence when θ is to be independent of x and h, it can have no other constant value than $\frac{1}{n+1}$.

A *necessary* condition for θ to be $\frac{1}{n+1}$, whatever x and h may be is that the coefficient of $h = 0$ (For sufficiency the coefficients of *all* powers of h in III must vanish). That is, we get

$f^{n+2}(x) = 0$, or $f(x)$ is of the form

$$A + Bx + Cx^2 + \dots + kx^{n+1}.$$

It is easy to verify that the condition is also sufficient and that with this form for $f(x)$, θ in I is actually $\frac{1}{n+1}$.

Note.—If, in I, $f^n(x+\theta h)$ is itself constant, it means that the relation I will be satisfied for all values of θ; this is the case when $f(x)$ is of the form

$$A + Bx + Cx^2 + \dots + lx^n.$$

(2) To consider the case when θ is to be independent of x only. A necessary condition from III is seen to be that the coefficient of h, *viz.*,

$$\frac{f^{n+2}(x)}{f^{n+1}(x)} = \text{constant}.$$

(The sufficient condition is that the coefficients of the various powers of h in III are constant.)

We get therefore

$$\log f^{n+1}(x) = \text{constant}$$

i.e., $f(x)$ is of the form $A + Bx^n + Cx^2 + \dots + lx^n + km^x$.

We now proceed to show that with this form for $f(x)$ the value of θ can be worked out and is easily seen to be independent of x.

Substituting

$$A + Bx + \dots + lx^n + km^x$$

in I, and observing that for the terms $(A + Bx + \dots + lx^n)$ the relation is satisfied for any value of θ identically, and noting that

$$\frac{d}{dr} m^x = (\log m)^r m^x$$

we get

$$k(m^{x+h} - m^x) =$$

$$km^x\left[h\log m + \frac{(h\log m)^2}{2!} + \dots + \frac{(h\log m)^{n-1}}{(n-1)!}\right] + k\frac{(h\log m)^n}{n!} m^{x+\theta h}.$$

Cancelling out km^x, we get

$$m^{\theta h} = \frac{m^h - \left[1 + h\log m + \frac{(h\log m)^2}{2!} + \dots + \frac{(h\log m)^{n-1}}{(n-1)!}\right]}{\frac{(h\log m)^n}{n!}}$$

i.e., $\theta h \log m =$

$$\log\left[\frac{e^{h\log m} - \left[1 + \frac{(h\log m)}{1!} + \frac{(h\log m)^2}{2!} + \dots + \frac{(h\log m)^{n-1}}{(n-1)!}\right]}{\frac{h^n}{n!}(\log m)^n}\right]$$

i.e., $$\theta = \frac{1}{h \log m} \log 1$$

and this does not contain x.

Note.—If we put $n = 1$, we get Wolstenholme's result (example 1659), *viz.*, when $f(x) = A + Bx + Cm^x$, then θ in $f(x+h) = f(x) + hf'(x+\theta h)$ is equal to

$$\frac{1}{h \log m} \log \left[\frac{e^{h \log m} - 1}{h \log m}\right].$$

(3) Lastly III shows that it is impossible for θ to depend on x only. for if it is to be independent of h it is seen to have tne constant value $\frac{1}{n+1}$.

3*. *Remarks.*—Normally θ is a function of x and h. The present discussion shows that (1) if the coefficient of h is constant, θ is independent of x, and (2) that θ cannot depend only on x and be independent of h; it must be a constant in that case.

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FOUR FOURS

The problem of expressing whole numbers by the repeated use of a single digit is a most interesting mathematical amusement and has been tackled by many well-known puzzle makers, *viz.*, E. V. Lucas, Ozney, Henry Ernest Dudney. The last named expresses all the natural integers up to 112 with four fours (no more and no fewer) using the various arithmetical signs, *e.g.*, +, –, ×, ÷, √, ·, ! fourth power, etc. These were published by him in 1899. Unknown to him however, they were also published before this time in 1881 in the first volume of "Knowledge". Dudney states in his book "Modern Puzzles" (puzzle 58) that integers beyond 112 cannot be expressed with four fours. By using in addition the sign of summation, *viz.*, Σ in the sense $\Sigma 4$ equal to $1 + 2 + 3 + 4 = 10$, I have been able to express numbers up to 1,000 and beyond by the use of four fours and shall gladly communicate any of the expressions I have obtained to those who may be interested in them. A few specimens from my list are given below by way of illustration.

$$31 = 4! + \sqrt{4} + \frac{\sqrt{4}}{\cdot 4}$$

$$119 = \frac{4! \times \Sigma 4 - \sqrt{4}}{\sqrt{4}}$$

$$267 = (\Sigma\sqrt{4})^{\frac{\sqrt{4}}{\cdot 4}} + 4!$$

$$287 = 4^4 + \Sigma(\Sigma 4) - 4!$$

$$302 = \Sigma(4!) + 4 - 4 + \sqrt{4}$$

$$395 = \Sigma(4! + \sqrt{4}) + 44$$

$$442 = 444 - \sqrt{4}$$

$$483 = \Sigma(\Sigma 4 \times \Sigma\sqrt{4}) + 4! - \Sigma(\Sigma\sqrt{4})$$

$$494 = (4! - \sqrt{4})^{\sqrt{4}} + \Sigma 4$$

$$531 = 4! \times (4! - \sqrt{4}) + \Sigma\sqrt{4}$$

$$605 = \Sigma(4!) \times \sqrt{4} + \frac{\sqrt{4}}{\cdot 4}$$

$$643 = \Sigma(4! + \Sigma 4) + 4! \times \sqrt{4}$$

$$693 = \Sigma\{\Sigma(4 + 4)\} + 4! + \Sigma\sqrt{4}$$

$$734 = (4 + \sqrt{4})! + \Sigma 4 + 4$$

$$749 = \frac{\Sigma\{\Sigma(\Sigma 4)\}}{\sqrt{4}} - 4! + \Sigma\sqrt{4}$$

$$841 = \Sigma(\Sigma 4 \times 4) + 4! - \Sigma\sqrt{4}$$

$$862 = 4! \times \Sigma(4 + 4) - \sqrt{4}$$

$$883 = \Sigma(4!) \times \Sigma\sqrt{4} - \Sigma\{\Sigma(\Sigma\sqrt{4})\} + 4$$

$$935 = \Sigma 44 - \Sigma\left(\frac{4}{\cdot 4}\right)$$

$$1000 = \Sigma 4 \times (4! \times 4 + 4)$$

Though there is no general method for writing any integer as the sum of four fours, the following should succeed in many cases although the expressions obtained are not always the simplest.

The first point worth noticing is that all integers up to 32 (except 29) can be expressed by using only two fours. The principal part of the number is formed by writing a Σ before a suitable one of these, and then the required number can be formed by adding to or subtracting from the principal part any number less than 32 (except 29), *e.g.*,

$$158 = \Sigma 16 + 22 = \Sigma(4 \times 4) + 4! - \sqrt{4}$$

$$226 = \Sigma 21 - 5 = \Sigma(4! - \Sigma\sqrt{4}) - \frac{\sqrt{4}}{\cdot 4}$$

This method will give all numbers up to $\Sigma 32 + 28 = 556$.

Royal Institute of Science, Bombay. S. V. MAZUMDAR.

On a Property of the Median

To show that *the sum of the absolute values of the deviations is a minimum when the deviations are taken from the Median.*

A proof of the above result for unrepeated observations is to be found in text-books ana a proof for a frequency distribution has been attempted in the text-book by C. Jones. The present note supplies a more direct algebraical proof.

Consider the frequency distribution (x_i, f_i) $i = 1, \ldots . n$. Let M, the median lie between x_r and x_{r+1} and A be any number that lies between x_s and x_{s+1}.

Case i.—Let $s < r$ and uence $A < M$. Then

$$\sum_1^n f_i \mid x_i - M \mid = \sum_1^s f_i \mid x_i - A \mid - \sum_{s+1}^r f_i \mid x_i - A \mid$$

$$+ \sum_{r+1}^n f_i \mid x_i - A \mid + (M - A) [\sum_1^r f_i - \sum_{r+n}^n f_i].$$

$$< \sum_1^n f_i \mid x_i - A \mid + (M - A) [\sum_1^r f_i - \sum_{r+n}^n f_i].$$

Since $M > xr$, $\sum_1^r f_i < \frac{N}{2}$ and $\sum_{r+1}^n f_i > \frac{N}{2}$ where $N = \sum_1^n f_i$, so that the second term on the right is negative and hence

$$\sum_1^n f_i \mid x_i - M \mid < \sum_1^n f_i \mid x_i - A \mid.$$

Case ii.—Let $s > r$, so that $A > M$. Consider the aistribution (x_i', f_i) where $x_i' = -x_i$. The median of this aistribution is $M' = -M$. Choose $A' = -A$. Tnen $A' < M$. Applying the previous result, we get

$$\sum_1^n f_i \mid x_i' - M' \mid < \sum_1^n f_i \mid x'_i - A' \mid.$$

But $\quad \mid x_i' - M' \mid = \mid x_i - M \mid$ and $\mid x_i' - A' \mid = \mid x_i - A \mid$

Hence $\quad \sum_1^n f_i \mid x_i - M \mid < \sum_1^n f_i \mid x_i - A \mid.$

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REVIEWS

The Differential Geometry of Ruled Surfaces

By Ram Behari, University of Delhi

(Lucknow University Studies No. XVIII, issued October 1946)

The book owes its origin to extension lectures delivered at the University of Lucknow in 1942; a résumé of it can be found in the authors' Présidential Address delivered at the 33rd Indian Science Congress where also many historical hints can be found. The ruled surfaces are discussed by the help of classical methods of differential geometry the foundation of which was laid by Gauss. The author himself has given many contributions to the knowledge about ruled surfaces; the bibliography at the end of the book enumerates 21 of his papers on the subject, more than of any other one of the more of fifty authors mentioned there. This bibliography seems to be a remarkable piece of scholarly industry which will be very helpful to everybody who needs information about this particular subject.

The first chapter deals with some general facts of differential geometry. Purists will be shocked by remarks like: "The surface is developable if consecutive generators intersect" but, one may take such remarks as a "facon de parler" since the passage to a rigorous treatment can be made very easily for the problems considered here. The second chapter deals with the line of striction, the third chapter with properties of the generators. At the end of the third chapter, some results of Rangachariar are pointed out which are not to be found in the usual textbooks. At this place one may miss the classical theorem that when two sets of generating straight lines exist, the surface is necessarily a quadric or a plane. The chapters 4 to 6 contain much of the author's own work *e.g.*, in chapter 4 (curved asymptotic lines) the geometrical interpretation of Laguerre's function and a new proof of a theorem of Picard about the asymptotic lines which can be determined by quadratures. In the fifth chapter the deformation of ruled surfaces is discussed and the difference between deformation and applicability is pointed out. Furthermore particular classes of deformations, some invariants of deformation and infinitesimal deformations are discussed. The sixth chapter is dedicated to the ruled surfaces which are generated by linear congruences. Whereas in these chapters only methods of classical differential geometry are used, the concluding note (p. 81) mentions various authors who have applied tensor analysis to ruled surfaces. "Hlavaty has obtained a trivalent symmetrical tensor which, he claims, stands godfather to ruled surfaces". This sentence sounds very encouraging. One may expect that the ruled surfaces will sail out of the back waters of mathematics and join the trend of modern science. Unfortunately Hlavaty's book is written in the Czech language. An authorised German translation by Max Pinl which appeared in 1939 is hardly available in India at present. One cannot overdo emphasizing the necessity for restoring the broken ties between the scientists all over the world and of an unhampered flow of scientific literature from country to country.

F. W. Levi

Mathematics as a Culture Clue and Other Essays

BY CASSIUS JACKSON KEYSER

(Scripta Mathematica, New York, 1947), viii. + 277 pp., $ 3·75

The book is Vol. 1 of the collected works of C. J. Keyser, Adrian Professor Emeritus, Columbia University, New York. The publication has been undertaken by *The friends of Cassius Jackson Keyser.* The title of the book is that of the third essay which deals with a thesis formulated by Oswald Spengler in his book *Der Untergang des Abendlands* (The Decline of the West). The thesis is that

The type of Mathematics found in any major Culture is a clue, or key, to the distinctive character of the Culture taken as a whole.

The book under review contains in all twelve essays and the titles of the other eleven essays are (1) The Meaning of Mathematics, (2) The Bearings of Mathematics, (3) Scientists Teach Laymen, (4) The Nature of the Doctrinal Function and Its Role in Rational Thought, (5) Mathematics and the Science of Semantics, (6) A Glance at Some of the Ideas of Charles Sanders Peirce, (7) William Benjamin Smith, (8) Mathematics and the Dance of Life, (9) Three Great Synonyms: Relation, Transformation, Function, (10) Vilfredo Federico Damaso Pareto: Mathematician, Economist, Sociologist, (11) Panthetics. The author writes clearly and forcefully and his style reminds one of a philosophically minded fervent theologian who is as keen on stating and explaining true doctrines as on exposing the fallacies of heretics. The essays are thought provoking and will be of profit to those who can give a fair measure of disciplined attention to the reading of them.

T. V.

ANNOUNCEMENTS AND NEWS

The following persons have been elected members of the Indian Mathematical Society :—

(i) Dr. Miss Bina Ghose, M.A., D.Phil. (Oxon.), Dip.-in-Education (Oxford), Assistant Educational Adviser, Ministry of Education, New Delhi.

(ii) Prof. N. M. Basu, M.A., D.Sc., Professor of Mathematics, Dacca University.

(iii) Dr. H. Subramania Aiyar, M.A., Ph.D., Principal (Retd.), University College, Trivandrum.

(iv) Prof. R. P. Boas, Jr., Brown University, Providence 12, R.I., U.S.A.

(v) M. V. Subba Rao, M.A., M.Sc., Assistant Professor, Presidency College, Madras, 5.

(vi) Miss N. Padma, M.A., Research Student, Madras University, 5, Rangachary Road, Mylapore, Madras, 4.

(vii) Miss M. Susheela, M.A., Research Student, Madras University, 13, Marshall Road, Egmore, Madras, 8.

(viii) V. K. Balachandran, M.A., Research Student, Madras University, 11, Lauder's Gate Road, Vepery, Madras, 7.

(ix) B. K. Sen Gupta, M.A., Srikhanda, Bengal (West).

(x) H. H. Thanawalla, Student, B.Sc. Class, 439, China Bazaar, 'Sri Hari Niwas', 5th Floor, Bombay.

(xi) Prof. Patravali, Royal Institute of Science, Bombay.

(xii) R. Venkatachalam Iyer, Siven Koil Street, Karamana, Trivandrum.

The following gentlemen have been enrolled as Life-Members of the Society :—

Sir K. S. Krishnan, Director, National Physical Laboratory, University Buildings, Delhi.

Dr. S. S. Pillai, D.Sc., Department of Mathematics, Calcutta University.

Prof. G. C. Patni, Professor of Mathematics, Maharaja's College, Jaipur.

Dr. S. Minakshisundaram, who was invited to the Institute for Advanced Study at Princeton, returned back to the Andhra University in July 1948.

The well-known Polish Journal *Studia Mathematica* which was under suspension owing to the World War has been revived again.

The next International Congress of Mathematicians will be held in Cambridge, Massachusetts, U.S.A., from August 30th to September 6th, 1950, under the auspices of the American Mathematical Society. Owing to the outbreak of World War II in September 1939, there has been no International Congress of

Mathematicians since 1936 when it met at Oslo, Norway, and the American Mathematical Society hopes that the gathering in 1950 will be a truly international one. Harvard University will be the principal host institution.

A number of invited hour addresses by outstanding mathematicians have been planned. In addition, sectional meetings for the presentation of contributed papers not included in the conference programmes will be held in—I. Algebra and Theory of Numbers ; II. Analysis ; III. Geometry and Topology ; IV. Probability and Statistics, Actuarial Science, Economics ; V. Mathematical Physics and Applied Mathematics ; VI. Logic and Philosophy ; VII. History and Education.

The Chairman of the Organizing Committee is Prof. Garrett Birkhoff of Harvard University, and the Secretary Prof. J. R. Kline of the University of Pennsylvania, who is also the Secretary of the American Mathematical Society. The Organizing Committee hopes that it will be possible to furnish board and room without charge to all mathematicians from outside the North American Continent, who are members of the Congress. The membership fee will be announced later.

Communications regarding the conference should be addressed to the American Mathematical Society, 531, West 116th Street, New York City 27, U.S.A.

We are glad to announce that the Indian Mathematical Society has become one of the sponsoring organizations of *Mathematical Reviews*, an abstracting journal published by the American Mathematical Society. Under the terms of the sponsorship agreement, members of a sponsoring organization may subscribe to *Mathematical Reviews* at the rate of 6.50 dollars per year (one half of the list price of 13 dollars per year), including back volumes which are available. A printed leaflet and a subscription card have been mailed to each member of the Society for use in this connection. *Mathematical Reviews*, founded in 1940, is run with the help of specialists in each topic under both Pure Mathematics as well as Applied Mathematics and Mathematical Physics, and is indispensible for active workers, as well as high grade educational institutions.

Prize Problem.—The latest prize problem announced for the "Narasinga Rao Medal for Mathematical Research" related to non-desarguesian geometries. Details of the problem will be found on pp. 179 and 180 of Volume XIII of the *Mathematics Student*. The competition is open to all members of the Society of Indian domicile. The latest date fixed for the receipt of theses solving the problem was the 1st July 1948. Owing to the non-receipt of solutions, **it has been decided to extend the period by one year, till 1st July 1949.** Competitors who are unable to consult the particular issue of the *Mathematics Student* containing the prize problem (December 1945 issue) may write to the Editor Prof. A. Narasinga Rao, Andhra University, Waltair.

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