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ON A CERTAIN RECTILINEAR CONGRUENCE

By P. JHA and V. R. CHARIAR

1. Let a singly infinite family of curves on a surface S (one member of the family passing through each point of S) be in one-to-one correspondence with the points of a twisted curve Γ, the curve of the family corresponding to a point P on Γ being denoted by $\gamma(P)$. If P be joined to points on $\gamma(P)$, a cone $C(P)$ is generated and the generators of the family of cones form a line congruence L. In this paper it is proposed to study some properties of L, particularly when it is normal. It may be noted that when the family of cones $C(P)$ forms the family of enveloping cones of S the congruence is the congruence of lines that intersect the given curve and touch the given surface. The notations followed are those of Weatherburn [2].

2. If the vector position of P on Γ be denoted by $\bar{r}(s)$, s being the arc length, and the vector position of Q on S by $\bar{R}(u, v)$ the equation of the family of curves $\gamma(P)$ is given by an equation of the form

$$f(u, v, s)\, du + g(u, v, s)\, dv = 0. \tag{1}$$

Let $\lambda = |\bar{r} - \bar{R}|$, so that the unit generator through Q of S, taken as the director surface, is given by $\bar{d} = (\bar{r} - \bar{R})/\lambda$ and the congruence by

$$\bar{\mathrm{R}} = \bar{R} + \rho\bar{d}.$$

3. It is evident that P is one of the foci on $\bar{d}$. If $\bar{D} = (\bar{d} + \delta\bar{d})/|\bar{d} + \delta\bar{d}|$ denotes the unit generator through P' and Q', consecutive points on Γ and S, the condition that PQ intersects $P'Q'$ is

$$[\bar{d}, \bar{t}, \delta\bar{d}] = 0. \tag{2}$$

Hence we have

THEOREM 1. *The focal planes through a generator are the two planes through the generator and the tangents to the curves* Γ *and* $\gamma(P)$.

The limiting position of O, the point of intersection of PQ and $P'Q'$, is the other focus. If $\sigma = \lim PO$ and θ and ϕ are the angles between the ray and the tangent and principal normal to Γ at P, θ being supposed to be a function of ϕ and s,

$$\sigma = \lim \delta s \left(\cos\theta\, \bar{t}.\bar{D} - \bar{d}.\bar{D}\right) / \left(\bar{t}.\bar{D} - \cos\theta\, \bar{d}.\bar{D}\right)$$

$$= \lim \delta s \left(\sin\theta - \cot\theta\, \bar{t}\,.\,\delta\bar{d}\right) / \left(\delta\theta + k\cos\phi\,\delta s\right)$$

$$= \lim \delta s \left(\sin\theta - \cot\theta\, \bar{t}\,.\delta\bar{d}\right) / \left(\frac{\partial\theta}{\partial s}\delta s + \frac{\partial\theta}{\partial\phi}\delta\phi + k\cos\phi\,\delta s\right).$$

Now from the equation (2) we get

$$\left[\bar{d}, \bar{t}, \frac{\partial\bar{d}}{\partial\phi}\right]\delta\phi + \left[\bar{d}, \bar{t}, \frac{\partial\bar{d}}{\partial s}\right]\delta s = 0;$$

hence

$$\sigma = \frac{\sin\theta}{\left(\frac{\partial\theta}{\partial s} + k\cos\phi\right) - \frac{\partial\theta}{\partial\phi}\frac{[\bar{d}, \bar{t}, \bar{d}_1]}{[\bar{d}, \bar{t}, \bar{d}_2]}}$$

If θ is independent of ϕ, so that the cones are right circular,

$$\sigma = \sin\theta / (\theta_1 + k\cos\phi) = \rho, \text{say}.$$

It will be seen that the congruence is normal in this case.

Now, $\bar{d} = t\cos\theta + n\sin\theta\cos\phi + \bar{b}\sin\theta\sin\phi$.

Therefore

$[\bar{d}, \bar{t}, \bar{d}_2] = -\sin^2\theta$, and $[\bar{d}, \bar{t}, \bar{d}_1] = -\sin\theta(\tau\sin\theta - k\cos\theta\sin\phi)$.

Hence the distance of the other focus from Γ (along the ray) is $\sin^2\theta / \{\rho^{-1}\sin^2\theta - \theta_2(\tau\sin\theta - k\cos\theta\sin\phi)\}$.

Hence we get

THEOREM 2. *For such a congruence the distance of the second focus from the curve is* $\sin^2\theta / \{\rho^{-1}\sin^2\theta - \theta_2(\tau\sin\theta - k\cos\theta\sin\phi)\}$, *where* θ *is the angle between curve and the ray and* ϕ *the longitude of the ray and* ρ *the focal distance when the congruence is normal.*

4. It can be seen that the distance x of the feet of the common perpendicular between $\bar{d}$ and $\bar{D}$ is given by

$$x = -(\bar{t}\,.\,\delta\bar{d})\,\delta s/(\delta\bar{d})^2$$
$$= (\rho^{-1}\sin^2\theta\,\delta s^2 + \sin\theta\,\theta_2\,\delta\phi\,\delta s)/(e\,\delta s^2 + 2f\,\delta s\,\delta\phi + g\,\delta\phi^2),$$

where,

$$e = \bar{d}_1^2 = \sin^2\theta/\rho^2 + (\tau\sin\theta - k\cos\theta\sin\phi)^2 = \sin^2\theta/\rho^2 + A^2,$$
$$f = \bar{d}_1\,.\,\bar{d}_2 = \theta_2\sin\theta/\rho + A\sin\theta,$$
$$g = \bar{d}_2^2 = \theta_2^2 + \sin^2\theta,$$
$$h^2 = (\rho^{-1}\sin^2\theta - A\,\theta_2)^2.$$

The extreme values of x—the distance of the limits from Γ—are the roots of the equation

$$4\,x^2(\rho^{-1}\sin^2\theta - A\,\theta_2)^2 - 4\,x\sin^2\theta\,(\rho^{-1}\sin^2\theta - A\,\theta_2) - \theta_2^2\sin^2\theta = 0.$$

It follows that

$$x_1 + x_2 = \sin^2\theta/(\rho^{-1}\sin^2\theta - A\,\theta_2) = \text{the focal distance;}$$

and

$$x_1\,x_2 = -\,\theta_2^2\sin^2\theta/4\,(\rho^{-1}\sin^2\theta - A\,\theta_2)^2.$$

Evidently when $\theta_2 = 0$, i.e. θ is a function of s alone and the cones are right circular cones, one limit coincides with P, one focus, and therefore the other limit coincides with the other focus and the congruence is normal. Hence we have

THEOREM 3. *If the cones are right circular cones, the congruence is normal.*

5. The parameter of distribution of the rays $\bar{d}$ and $\bar{D}$ is easily seen to be given by

$$\beta = [\delta\,\bar{r},\,\delta\,\bar{d},\,\bar{d}]/(\delta\,d)^2$$
$$= (-\,A\sin\theta\,\delta s^2 - \sin^2\theta\,\delta\phi\,\delta s)/(e\,\delta s^2 + 2f\,\delta s\,\delta\phi + g\delta\phi^2).$$

The stationary values of β, β_1 and β_2 are the roots of the equation

$$4\,h^2\,\beta^2 + 4\,\beta\,(Ag\sin\theta - f\sin^2\theta) - \sin^4\theta = 0.$$

Hence

$$\beta_1 \beta_2 = -\sin^4 \theta/4\,h^2 = -\sigma^2/4.$$

The result contained in the above equation that "*Four times the product of the principal parameters of distribution for a ray is negative of the square of the focal distance*" is true for all rectilinear congruences† (as can be easily seen). The congruence is clearly hyperbolic. It can also be verified that for an elliptic ray the foci are imaginary although the limits are real. *When the congruence is normal*, i.e. when the cones are right circular cones *the principal parameters of distribution are equal to half the focal distance.*

6. It has been seen above that when θ is a function of s alone, the congruence is normal. This can also be proved as follows. If the congruence is normal, the variations of

$$\bar{R} = \bar{r} + \mu\,\bar{d},$$

represent displacements perpendicular to $\dot{d}$. Therefore

$$\bar{d}.\,(t\,\delta s + \mu_1\,\delta s\,\bar{d} + \mu_2\,\delta\phi\,\bar{d} + \mu\,\delta\bar{d}) \equiv 0.$$

Therefore

$$\mu_1 + \cos\theta = 0 \text{ and } \mu_2 = 0,$$

i.e. μ and therefore θ is a function of s alone and

$$\mu = -\int \cos\theta\,ds.$$

Hence we have

THEOREM 4. *The congruence L is normal only when the cones corresponding to points on the given curve are right circular cones with the tangents to the curve as axes and the surface normal to the rays is obtained by measuring a distance $-\int\cos\theta\,ds$ along the rays of the congruence.*

Clearly the points on the orthogonal surface (of the normal congruence) corresponding to the rays through P on Γ lie on a circle.

† The result appears to be new

Hence the surface orthogonal to the rays is generated by a family of circles. Evidently these circles form one set of lines of curvature—normals intersecting on Γ*—and being plane curves are geodesics.*

The fundamental magnitudes of the first and second order are given by

$$E = \sin^2\theta + e\,B^2 + 2\,B\sin^2\theta/\rho, \quad \text{where } B = \int \cos\theta\, ds,$$

$$F = fB^2,\ G = gB^2,\ H^2 = B^2\sin^4\theta\,(1 + B/\rho)^2;$$

$$L = eB + \sin^2\theta/\rho,\ M = fB,\ N = gB.$$

As $FN - GM = 0$, the curves $s =$ const., i.e. the circles form one family of lines of curvature, as already noted. The equation of the other family of lines of curvature is

$$f\,ds + g\,d\phi = 0.$$

As $LN - M^2 = \rho^{-1}\,B\,\sin^4\theta\,(1 + B/\rho)$, and one principal radius of curvature is clearly B, the other principal radius of curvature is $\rho + B$.

7. If the rays of the congruence touch S, the cones $C(P)$ envelope S and for a normal congruence, S has a family of right circular enveloping cones with vertices on Γ and axes along the tangents to Γ. The equation to $C(P)$ is

$$\{(\bar{\mathrm{R}} - \bar{r}).\bar{t}\}^2 = \lambda^2\cos^2\theta,$$

θ being a function of s alone and $\lambda = |\bar{\mathrm{R}} - r|$.

The characteristic is

$$\{(\bar{\mathrm{R}} - \bar{r}).\bar{t}\}\,\{-1 + (\bar{\mathrm{R}} - \bar{r}).k\bar{n}\} = -(\bar{\mathrm{R}} - \bar{r}).\bar{t}\cos^2\theta - \\ - \lambda^2\sin\theta\cos\theta\,\theta_1,$$

or

$$\lambda\cos\theta\,\{-1 + \lambda\,k\,\bar{d}.\bar{n}\} = -\lambda\cos^3\theta - \lambda^2\sin\theta\cos\theta\,\theta_1,$$

or

$$\lambda = \sin\theta/(\theta_1 + k\cos\phi) = \rho.$$

Taking $\bar{t}$, $\bar{n}$, $\bar{b}$ as the axes of x, y, z it can be seen that the characteristic lies in the plane whose equation is

$$\theta_1 x \tan\theta + ky = \sin^2\theta,$$

and therefore it is an ellipse, its centre being on the tangent. S is thus generated by a family of ellipses whose planes are parallel to the binormals to Γ. S is clearly the limit-focal surface. Hence we have

THEOREM 5. *If a normal congruence intersects a given curve and touches a given surface, the surface is the limit-focal surface and generated by a family of ellipses whose planes are parallel to the binormals to the curve.*

If all the ellipses coincide, the surface S degenerates into the ellipse E. In this case, congruence intersects the curves Γ and E and is normal. It has been proved [1] that a congruence which intersects two curves is a normal congruence only when the curves are a circle and its axis and therefore *E is a circle and Γ its axis.*

8. When Γ is a straight line, taken as the x-axis, the equation of the right circular cone is

$$y^2 + z^2 = (x - a)^2 \tan^2\theta,$$

where θ is a function of a and the characteristic curves are given by

$$(x - a)\frac{d\theta}{da} - \sin\theta\cos\theta = 0,$$

that is

$$x - a = \tfrac{1}{2}\sin 2\theta \frac{da}{d\theta},$$

which gives a (and therefore θ) as a function of x alone and therefore the characteristics are circles and the surface is one of revolution, unless $-a = \frac{1}{2}\sin 2\theta\, da/d\theta$ when the congruence consists of lines intersecting a circle and its axis. Hence we have

THEOREM 6. *If the congruence of tangents to a surface from points of a straight line forms a normal congruence, the surface is one of revolution with the given line as the axis of revolution.*

If $a = \operatorname{cosec} \theta$, the surface is the unit sphere with the line as a diameter and when $a = \tan \theta$, the surface is obtained by revolving a parabola about the tangent at the vertex.

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ON THE EQUATION $y^2 = x^3 - 3\lambda\mu x - \lambda^3 a - \mu^3 a^{-1}$

By P. KESAVA MENON

1. The equation

$$y^2 = x^3 - 3\,\lambda\mu x - \lambda^3 a - \mu^3 a^{-1} \tag{1}$$

has no known solutions in rational values of x and y for arbitrarily given rational values of a, λ, μ. Nevertheless from one known solution we can in general find others and even an infinity of them by the tangent and chord process or by the process of parametrization by means of the elliptic function $\wp(u)$, both methods leading to the same solutions. Here it is not proposed to solve the general problem but we are concerned with a curious transformation of (1) which arises in connection with another line of approach to the same problem.

2. The equation (1) can, of course, be written in the form

$$y^2 = \Pi_\rho\,(x - \lambda\,\rho - \mu\,\rho^{-1}), \tag{2}$$

where ρ runs through the cube roots of a.

Naturally we set in (2)

$$x - \lambda\rho - \mu\,\rho^{-1} = (\nu - X\rho - Y\rho^{-1})^2, \tag{3}$$

which gives

$$\left.\begin{aligned} x &= \nu^2 + 2XY \\ y &= \nu^3 - 3\,\nu\,XY - X^3\,a - Y^3\,a^{-1} \end{aligned}\right\} \tag{4}$$

and

$$\left.\begin{aligned} \lambda &= 2\,\nu X - Y^2\,a^{-1} \\ \mu &= 2\,\nu\,Y - X^2\,a \end{aligned}\right\} \tag{5}$$

Eliminating ν between the two relations (5) we get

$$aX(\mu + X^2\,a) = Y(\lambda\,a + Y^2). \tag{6}$$

Solutions of (6) in X, Y when substituted in (4) and (5) give rise to solutions of (1).

3. But what is more significant is that the equation (6) is actually equivalent to (1) in the sense that we can go from either to the other by a linear transformation. In fact the substitution

$$\left. \begin{aligned} X &= (x\lambda + \mu^2 a^{-1})/y \\ Y &= (x\mu + \lambda^2 a)/y \end{aligned} \right\}, \tag{7}$$

transforms (6) into (1) as may easily be verified.

As an immediate consequence we get the following

THEOREM 1. *If x_1, y_1 is a solution of*

$$y^2 = x^3 - 3\lambda\mu x - \lambda^3 a - \mu^3 a^{-1}$$

then another solution is given by

$$x = \nu^2 + 2XY, \quad y = \nu^3 - 3\nu xy - X^3 a - Y^3 a^{-1},$$

where

$$\nu = (\lambda + Y^2 a^{-1})/2X,$$

and

$$X = (x_1 \lambda + \mu^2 a^{-1})/y_1, \quad Y = (x_1 \mu + \lambda^2 a)/y_1.$$

We have also the dual

THEOREM 2. *If X_1, Y_1 is a solution of*

$$a X(\mu + X^2 a) = Y(\lambda a + Y^2),$$

then another solution is given by

$$X = \frac{\lambda(\nu^2 + 2x_1 y_1) + \mu^2 a^{-1}}{\nu^3 - 3\nu x_1 y_1 - x_1^3 a - y_1^3 a^{-1}},$$

$$Y = \frac{\mu(\nu^2 + 2x_1 y_1) + \lambda^2 a}{\nu^3 - 3\nu x_1 y_1 - x_1^3 a - y_1^3 a^{-1}},$$

where

$$\nu = \frac{\lambda + y_1^2 a^{-1}}{2x_1}.$$

4. Another transformation of (6) is obtained by writing

$$X = r/q, \; Y = p/s, \; a = q/s,$$

when it takes the homogeneous form

$$pq(p^2 + \lambda qs) = rs\,(r^2 + \mu\, qs). \tag{8}$$

If further we replace λ by λa and μ by $\mu\, a^{-1}$, equation (8) becomes

$$pq\,(p^2 + \lambda\, q^2) = rs\,(r^2 + \mu\, s^2). \tag{9}$$

The forms (8) and (9) are striking, especially the latter, and for $\lambda = \mu = 1$ we know that (9) is equivalent to the equation

$$A^4 + B^4 = C^4 + D^4. \tag{10}$$

We can, of course restate Theorem 2 for equations (8) and (9).

5. Let us consider a particular case. Let

$$\lambda = a + \beta, \quad \mu = a^{-1} + \beta$$

in Theorem 2. Then we see that $X = 1$, $Y = 1$ is a particular solution. Hence we get from Theorem 2 the following

THEOREM 3. *The equation*

$$X\,(1 + a^2 X^2 + a\,\beta) = Y\,(a^2 + Y^2 + a\,\beta) \tag{11}$$

has a solution given by

$$\left.\begin{aligned} X &= \frac{(a + \beta)\,(\nu^2 + 2) + a^{-1}\,(a^{-1} + \beta)^2}{\nu^3 - 3\nu - (a + a^{-1})}, \\ Y &= \frac{(a^{-1} + \beta)\,(\nu^2 + 2) + a\,(a + \beta)^2}{\nu^3 - 3\nu - (a + a^{-1})}, \end{aligned}\right\} \tag{12}$$

where $\nu = (a + a^{-1} + \beta)/2$.

Writing

$$Z = 2a\,(X^3 - Y) + \beta\,(X - Y), \tag{13}$$

equation (11) takes the form

$$Z^2 - \beta^2\,(X - Y)^2 = 4\,(X^3 - Y)\,(Y^3 - X). \tag{14}$$

Hence we get

Corollary 1. *Equation* (14) *has a parametric solution given by* (12) *and* (13).

Again, writing

$$X = \frac{v + y}{x - u}, \quad Y = \frac{x + u}{v - y}, \quad \alpha = \frac{x - u}{v - y},$$

in (11), it takes the form

$$x^4 + y^4 - u^4 - v^4 = \beta (x - u)(v - y)(x^2 + y^2 - u^2 - v^2). \quad (15)$$

Solving for x, y, u, v in terms of X, Y, α, we have

$$\left.\begin{array}{l} x = k(Y + \alpha), y = k(\alpha X - 1) \\ u = k(Y - \alpha), v = k(\alpha X + 1) \end{array}\right\} \quad (16)$$

Hence we get

Corollary 2. *A parametric solution of* (15) *is given by* (16) *and* (12).

Taking $\beta = 0$ we get Euler's well-known solution of (10).

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ON THE ANALYTIC CONTINUATION OF NEWTON SERIES

By NIRMALA PANDEY

The object of this note is to prove a theorem on the analytic continuation of Newton series.

THEOREM. *Let*

$$\Omega(z) = \sum_{n=0}^{\infty} (-1)^n e^{Ain} g(n)(z-1)(z-2)\dots(z-n),\ 0 < A < 2\pi, \quad (1)$$

be any factorial series which has a finite abscissa λ *of convergence. Let the coefficient* $g(n)$, *when considered as a function* $g(\omega)$ *of* $\omega(=x+iy)$, *satisfy the following conditions :*

(*a*) *it is regular in some half-plane* $x \geqslant h$, *the values of* $g(k)$, $k = 0, 1, \dots, [h]$, *being finite numbers ;*

(*b*) *uniformly for* $x \geqslant h$,

$$\limsup_{|y|\to\infty} \frac{1}{|y|} \log |[g(x+iy)/g(x)]| \leqslant d,$$

where d *is the smaller of the two numbers* $A - \pi/2$ *and* $3\pi/2 - A$.

Then the function $\Omega(z)$ *defined by the series* (1) *when* $\operatorname{Re} z > \lambda$, *represents an integral function of* z *and will be defined, for all large non-integral* $h_1 > 0$, *by the expression*

$$\sum_{0}^{[h_1]} (-1)^n e^{Ain} g(n)(z-1)(z-2)\dots(z-n) +$$

$$+ \frac{\Gamma(z)\sin \pi z}{\pi} \int_{-\infty}^{\infty} e^{Ai(h_1+iy)} g(h_1+iy) \frac{\Gamma(h_1+1-z+iy)}{\{\exp 2\pi i(h_1+iy)-1\}} dy, \quad (2)$$

in the half-plane $\operatorname{Re} z \leqslant h_1$.

Let the series (1) be written in the form

$$\Omega(z) = \frac{1}{\pi} \sum_{n=0}^{\infty} e^{Ain} g(n)\, \Gamma(z)\, \Gamma(n+1-z) \sin \pi z. \quad (3)$$

Let $\Gamma(h_1, j)$ denote a region in the ω-plane formed by the lines $\omega = h_1 + iy$, $\omega = h_1 + n + iy$ and $\omega = x \pm ij$, where n is a positive integer and j is an arbitrarily large positive number.

Let C be any (fixed) closed bounded region of the z-plane containing an interval of the positive real axis for which $z = r \geqslant [\lambda + \delta]$, $\delta > 0$, in its interior.

If now h_1 is taken sufficiently large, the equations $\omega + 1 - z = L$, $L = 0, -1, -2, \ldots$ will not be satisfied by ω in $\Gamma(h_1, j)$, if z lies in C. Hence for a fixed z in C,

$$\frac{1}{\pi} e^{Ai\omega} g(\omega) \Gamma(z) \Gamma(\omega + 1 - z) \sin \pi z$$

is a regular function of ω in $\Gamma(h_1, j)$.

Let us suppose, for a moment, that z lies on a closed bounded interval of the positive real axis contained in C for which Re $z \geqslant [\lambda + \delta]$, $\delta > 0$. Then by means of the calculus of residues we may write

$$\frac{1}{\pi} \int_{\Gamma(h_1, j)} \frac{e^{Ai\omega} g(\omega) \Gamma(z) \Gamma(\omega + 1 - z) \sin \pi z}{\exp(2\pi i \omega) - 1} d\omega$$

$$= \sum_{[h_1]+1}^{[h_1]+n} (-1)^n e^{Ain} g(n) (z-1)(z-2) \ldots (z-n). \qquad (4)$$

Let

$$D(z, \omega) = \frac{e^{Ai\omega} g(\omega) \Gamma(z) \Gamma(\omega + 1 - z) \sin \pi z}{\pi \{\exp(2\pi i \omega) - 1\}}.$$

Now integrate the function $D(z, \omega)$ about that side of $\Gamma(h_1, j)$ upon which $\omega = x + ij$. Let B_1 be the contribution arising from integrating $D(z, \omega)$ along this side. Then

$$B_1 = \frac{\Gamma(z) \sin \pi z}{\pi} \int_{h_1+n}^{h_1} \frac{e^{Ai(x+ij)} g(x + ij) \Gamma(x + 1 - z + ij)}{\{\exp 2\pi i (x + ij) - 1\}} dx.$$

It is a well-known property of the Gamma function that if α and β are real, then we may write

$$|\alpha + i\beta| = (2\pi)^{1/2} |\alpha + i\beta|^{\alpha - 1/2} e^{-\alpha - \beta \tan^{-1}(\beta/\alpha)} (1 + \delta), \qquad (5)$$

where δ approaches zero as either α or β becomes infinite.

Also

$$|\exp\{2\pi i (x + ij)\} - 1| = O(1). \qquad (6)$$

By application of the condition (*b*), (5) and (6) we find that the absolute value of the integrand in B_1 vanishes to as high an order as that of $e^{-\epsilon j}\ |x+1-r+ij|^{x+1/2-r}$, $\epsilon > 0$, as $j \to \infty$. Hence we have $\lim_{j\to\infty} B_1 = 0$.

Similarly using the inequality

$$|\exp\{2\pi i(x-ij)\}-1|^{-1} < k_1 c^{-2\pi j}, \tag{7}$$

and the conditions (*b*) and (5), we find that the absolute value of the integrand in B_2 arising from integrating $D(z, w)$ along that side upon which $w = x - ij$, vanishes to as high an order as that of $e^{-\epsilon j}\,|x+1-r-ij|^{x+1/2-r}$ as $j \to \infty$. Hence we have $\lim_{j\to\infty} B_2 = 0$.

Let B_3 be the contribution arising from integrating $D(z, w)$ along that side of $\Gamma(h_1, j)$ upon which $w = n + h_1 + iy$. Then

$$B_3 = \frac{\Gamma(z)\sin\pi z}{\pi} \times$$

$$\times \int_{-\infty}^{\infty} \frac{e^{Ai(n+h_1+iy)}\, g(n+h_1+iy)\,\Gamma(n+h_1+1-z+iy)}{\{\exp 2\pi i(n+h_1+iy)-1\}}\, dy. \tag{8}$$

It can be easily seen that the absolute value of the integrand in B_3 vanishes to as high an order as that of

$$e^{-\epsilon|y|}\,|n+h_1+1-z\pm iy|^{n+h_1+1/2-z}$$

as $y \to \pm\infty$. Hence it follows that the improper integral in (8) exists.

Moreover, B_3 can be written in the form

$$B_3 = \frac{g(n)\,\Gamma(n+1-z)}{\Gamma(1-z)} \int_{-\infty}^{\infty} e^{Ai(n+h_1+iy)}\, \frac{g(n+h_1+iy)}{g(n)} \times$$

$$\times \frac{\Gamma(n+h_1+1-z+iy)}{\Gamma(n+1-z)\,\{\exp 2\pi i(n+h_1+iy)-1\}}\, dy.$$

The series (3) being assumed convergent for $\operatorname{Re} z > \lambda$, we have at once

$$\lim_{n\to\infty} g(n)\,\Gamma(z)\,\Gamma(n+1-z)\sin\pi z = 0.$$

Again, from (5), we may write

$$\left|\frac{\Gamma(n+1+h_1-z+iy)}{\Gamma(n+1-z)}\right| < \frac{|n+1+h_1-z+iy|^{n+1/2+h_1-z}}{|n+1-z|^{n+1/2-z}} \times$$

$$\times \exp\left\{-y\tan^{-1}\frac{y}{(n+1+h_1-z)}\right\}$$

$$< e^{(\pi/2+\epsilon/2)|y|}$$

Hence taking account of the condition (*b*) of the hypothesis we at once have $\lim_{n\to\infty} B_3 = 0$.

Hence for $z = r$ on a closed bounded interval of the positive real axis contained in C for which $z = r \geqslant [\lambda+\delta]$, $\delta > 0$, we may from (4) write

$$\sum_{[h_1]+1}^{\infty} (-1)^n e^{Ain} g(n)(z-1)(z-2)\ldots(z-n)$$

$$= \frac{\Gamma(z)\sin\pi z}{\pi}\int_{-\infty}^{\infty}\frac{g(h_1+iy)\,e^{Ai(h_1+iy)}\,\Gamma(h_1+1-z+iy)}{\{\exp 2\pi i(h_1+iy)-1\}}\,dy. \quad (9)$$

We have so far restricted z to lie only on a closed bounded interval of the positive real axis contained in C for which $z = r \geqslant [\lambda+\delta]$, $\delta > 0$. Suppose now that z is allowed to take on any value, real or complex, in the region C. Denoting by B_4 the integral on the right hand side of (9) we have

$$B_4 = \int_{-\infty}^{\infty}\frac{e^{Ai(h_1+iy)}g(h_1+iy)\Gamma(h_1+1-z+iy)}{\{\exp 2\pi i(h_1+iy)-1\}}\,dy,$$

which may be written in the form

$$B_4 = \int_{-\infty}^{-p} + \int_{-p}^{p} + \int_{p}^{\infty}\frac{e^{Ai(h_1+iy)}g(h_1+iy)\,\Gamma(h_1+1-z+iy)}{\{\exp 2\pi i\,(h_1+iy)-1\}}\,dy$$

$$= I_1 + I_2 + I_3, \text{ say,}$$

where p may be chosen sufficiently large but independent of z.

Consider first the integral I_3. The absolute value of the integrand, for sufficiently large values of y, is less than

$$e^{-\epsilon y}\,|h_1+1+z_1+iy|^{z_1+1/2+h_1},$$

where z_1 stands for the maximum absolute value of z in the region C. But

$$e^{-\epsilon y}|h_1+1+z_1+iy|^{z_1+1/2+h_1}=e^{-\epsilon y/2}e^{-\epsilon y/2}|h_1+1+z_1+iy|^{z_1+1/2+h_1}$$
$$<k_1' e^{-\epsilon y/2},$$

where k_1' is a constant. Hence we have

$$|I_3|<k_1'\int_p^\infty e^{-\epsilon y/2}\,dy=k_1'\frac{2}{\epsilon}e^{-\epsilon p/2},$$

so that I_3 can be made less in absolute value than any arbitrarily small positive number η by a proper choice of p. The same is also true of I_1 if p is chosen sufficiently large. Also it can be seen easily that I_2 represents an analytic function of z in C. Hence the uniform convergence of the improper integral B_4 for $z=re^{i\theta}$ in C is established. Also since C contains an interval of the positive real axis for which (9) holds, it follows that the right-hand side of (9) provides the analytic continuation of

$$\Omega(Z)-\sum_{n=0}^{[h_1]}(-1)^n e^{Ain}g(n)(z-1)(z-2)\dots(z-n)$$

over the region C. The equation (9) therefore persists for all values of z in C. But as the region C may be chosen to contain any bounded portion of the complex plane in its interior, $\Omega(z)$ is an integral function of z, which, by virtue of the equation (9), is defined by the expression on the right-hand side of (2) in the half-plane $\operatorname{Re} z\leqslant h_1$.

REMARK. As a particular case of the above theorem we prove that

$$\Omega(z)=\sum_{n=0}^{\infty}\frac{(z-1)(z-2)\dots(z-n)}{1.2.\dots n}$$

represents an integral function of z. This also follows as a special case of a theorem proved by V. F. Cowling [1].

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ON THE ANALYTIC CONTINUATION OF CERTAIN SERIES

By NIRMALA PANDEY

1. In this note I am concerned with the study of the analytic continuation of the following series :

$$\sum_{1}^{\infty} e^{Ain^{\beta}-sn^{\alpha}} \frac{\Gamma(n)}{\Gamma(z+n)}, \; A > 0, \; 0 < \alpha < \beta < 1, \tag{1.1}$$

$$\sum_{2}^{\infty} e^{Ai(\log n)^{\beta}-s\log n} \frac{\Gamma(n)}{\Gamma(z+n)}, \; \beta > 2, \; A > 0. \tag{1.2}$$

To begin with I consider the series

$$H(s, z) = \sum_{1}^{\infty} e^{Ain^{\beta}-sn^{\alpha}} \frac{\Gamma(n)}{\Gamma(z+n)}, \; A > 0, \; 0 < \alpha < \beta < 1,$$

defined initially for $R(s) > 0$ and $R(z) > 1$ and prove that it represents an integral function of both s and z, it being well known that

$$f(s) = \sum_{1}^{\infty} e^{Ain^{\beta}-sn^{\alpha}} \quad A > 0, \; 0 < \alpha < \beta < 1,$$

represents an integral function of s, and that

$$\Omega(z) = \sum_{1}^{\infty} \frac{\Gamma(n)}{\Gamma(z+n)}$$

is an analytic function of z at least in the half-plane $\mathrm{Re}(z) > 1$.

2. I prove the following theorem.

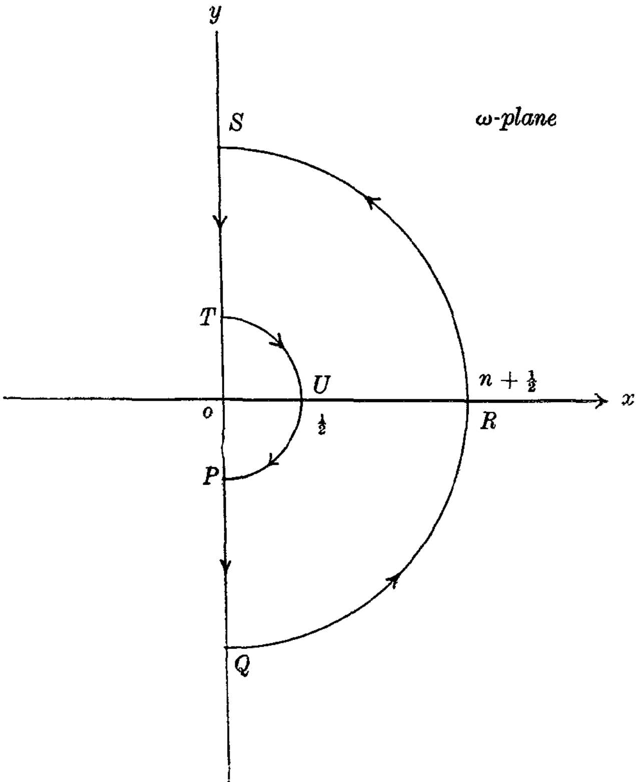
THEOREM 1. *Let*

$$H(s, z) = \sum_{1}^{\infty} e^{Ain^{\beta}-sn^{\alpha}} \frac{\Gamma(n)}{\Gamma(z+n)}, \; A > 0, 0 < \alpha < \beta < 1; \tag{2.1}$$

then $H(s, z)$ is an integral function of both s and z.

The series (2.1) is absolutely and uniformly convergent for $\mathrm{Re}\, s \geqslant \delta > 0$ and $\mathrm{Re}\, z > 1$. Let us therefore take for a moment s

real and positive and greater than δ (positive and arbitrarily small) and z also real and $z \geqslant 1+\epsilon > 1$. Take also a contour in the ω-plane bounded by two semi-circular arcs drawn with centre at the origin and radii $n+\frac{1}{2}$ and $\frac{1}{2}$ respectively, and also by the parts of the imaginary axis as shown in the figure.



Then, by Cauchy's theorem, we may write

$$\sum_{1}^{n} e^{Ain^{\beta}-sn^{a}} \frac{\Gamma(n)}{\Gamma(z+n)} = \int_{PQRSTUP} e^{Ai\omega^{\beta}-s\omega^{a}} \frac{\Gamma(\omega)}{\Gamma(z+\omega)\,(e^{2\pi i\omega}-1)}\, d\omega, \tag{2.2}$$

where ω^β and ω^α are defined as exp $(\beta \log \omega)$ and exp $(\alpha \log \omega)$ and $\log \omega$ has its principal value.

Put $\omega = \rho e^{i\psi}$ and consider the integral over the arc QR. Now

$$\int_{QR} e^{Ai\omega^\beta - s\omega^\alpha} \frac{\Gamma(\omega)}{\Gamma(z+\omega)(e^{2\pi i\omega}-1)}\, d\omega$$

$$= \int_{-(\pi/2)}^{0} \frac{ie^{Ai\rho^\beta(\cos\beta\psi + i\sin\beta\psi) - s\rho^\alpha(\cos\alpha\psi + i\sin\alpha\psi)}\Gamma(\rho e^{i\psi})\,\rho e^{i\psi}\,d\psi}{\Gamma(z+\rho e^{i\psi})\left\{\exp 2\pi i\,\rho(\cos\psi + i\sin\psi) - 1\right\}},$$

and changing ψ into $-\psi$ and taking the modulus of the integral, we have

$$\left|\int_{QR}\right| \leqslant k_1 \int_0^{\pi/2} e^{A\rho^\beta \sin\beta\psi - s\rho^\alpha\cos\alpha\psi - 2\pi\rho\sin\psi}\,\rho^{-z+1}\,d\psi,$$

where k_1 is some constant, since

$$\left|\frac{1}{e^{2\pi i\omega}-1}\right| = O(e^{-2\pi\rho|\sin\psi|})$$

and

$$\frac{\Gamma(\rho e^{i\psi})}{\Gamma(z+\rho e^{i\psi})} = O(|\rho e^{i\psi}|^{-z}) = O(\rho^{-z}).$$

We therefore have

$$\left|\int_{QR}\right| \leqslant k_1\,\rho^{1-z}\,e^{-\delta\rho^\alpha\cos\alpha\pi/2}\int_0^{\pi/2} e^{A\rho^\beta\sin\psi - 2\pi\rho\sin\psi}\,d\psi$$

$$\leqslant \quad ,, \quad ,, \quad ,, \int_0^{\pi/2} e^{-\rho(2\pi - A\rho^{\beta-1})\sin\psi}\,d\psi$$

$$\leqslant \quad ,, \quad ,, \quad ,, \int_0^{\pi/2} e^{-\rho(2\pi-\delta_1)\sin\psi}\,d\psi,$$

where δ_1 is sufficiently small for ρ sufficiently large. Therefore

$$\left|\int_{QR}\right| \leqslant k_1\,\rho^{1-z}\,e^{-\delta\rho^\alpha\cos\alpha\pi/2}\int_0^{\pi/2} e^{-\rho(2\psi/\pi)(2\pi-\delta_1)}\,d\psi$$

$$\leqslant \quad ,, \quad ,, \quad ,, \left[\frac{e^{-\rho(2\psi/\pi)(2\pi-\delta_1)}}{-(2\rho/\pi)(2\pi-\delta_1)}\right]_0^{\pi/2}$$

$$\leqslant k_1'\,\rho^{1-z}\,e^{-\delta\rho^\alpha\cos\alpha\pi/2}\,\frac{1}{\rho}\left[1 - e^{-\rho(2\pi-\delta_1)}\right],$$

so that the integral along the arc $\to 0$ as $\rho \to \infty$.

Also taking the modulus of the integral over the arc RS we have

$$\left|\int_{RS}\right| \leqslant k_2 \int_0^{\pi/2} e^{-A\rho^\beta \sin\beta\psi - s\rho^\alpha \cos\alpha\psi}\, \rho^{1-z}\, d\psi,$$

since, on the arc RS,

$$\frac{1}{(e^{2\pi i\omega}-1)} = O(1).$$

We therefore have

$$\left|\int_{RS}\right| \leqslant \frac{k_2}{\beta}\, e^{-\delta\rho^\alpha \cos\alpha\pi/2}\, \rho^{1-z} \int_0^{\beta\pi/2} e^{-A\rho^\beta \sin\phi}\, d\phi$$

$$\leqslant \;,,\;\;,,\;\;,, \int_0^{\beta\pi/2} e^{-A\rho^\beta . 2\phi/\pi}\, d\phi$$

$$\leqslant \;,,\;\;,,\;\;,, \left[\frac{e^{-A\rho^\beta 2\phi/\pi}}{-A\rho^\beta .\ 2/\pi}\right]_0^{\pi/2}.$$

Hence the integral along the arc $RS \to 0$ as $\rho \to \infty$.

If therefore we take s to be real and $\geqslant \delta > 0$ and z also to be real and $\geqslant 1+\epsilon > 1$, we have, from (2.2),

$$H(s,z) = \sum_1^\infty e^{Ain^\beta - sn^\alpha} \frac{\Gamma(n)}{\Gamma(z+n)}$$

$$= \int_{TUP} + \int_{PQ} - \int_{TS} e^{Ai\omega^\beta - s\omega^\alpha} \frac{\Gamma(\omega)}{\Gamma(z+\omega)\,(e^{2\pi i\omega}-1)}\, d\omega, \quad (2.3)$$

as Q and S move to infinity along PQ and TS respectively.

As the integrand is an integral function of both s and z on the path of integration TUP which is finite, the integral over this semi-circular arc represents an integral function of both s and z.

We shall next show that the integrals over the straight paths PQ and TS when Q and S move towards infinity, represent integral functions of both s and z.

Consider the integral over PQ. Putting $\omega = i\rho$ and $z = x+iy$,

$$\left|\int_{PQ}\right| \leqslant \left|k_3 \int_{1/2}^\infty e^{Ai\rho^\beta(\cos\beta\pi/2 - i\sin\beta\pi/2) - (\sigma+it)(\cos\pi\alpha/2 - i\sin\pi\alpha/2)\rho^\alpha} \times\right.$$

$$\times\ (\rho e^{-i\pi/2})^{x+iy}\, e^{-2\pi\rho}\, (-i)\, d\rho$$

$$\leqslant k_3 \int_{1/2}^{\infty} e^{A\rho^\beta \sin\beta\pi/2 - \rho^\alpha(\sigma\cos\alpha\pi/2 + t\sin\alpha\pi/2) - 2\pi\rho + \pi y/2} . \rho^x \, d\rho ,$$

which shows that the integral converges absolutely and uniformly for all finite values of σ, t, x and y like the integral $\int^\infty e^{-\Delta\rho} \, d\rho$, $\Delta > 0$.

Similarly

$$\left| \int_{ST} \right| \leqslant k_4 \int_{1/2}^{\infty} e^{-A\rho^\beta \sin\beta\pi/2 - \rho^\alpha(\sigma\cos\alpha\pi/2 - t\sin\alpha\pi/2) - \pi/2y} . \rho^x \, d\rho ,$$

which again shows that the integral over ST converges absolutely and uniformly for all finite values of σ, t, x and y like the integral $\int^\infty e^{-\Delta\rho^\beta} \, d\rho$, $\Delta > 0$.

Hence the three integrals on the right of (2.3) represent integral functions of s and z. This proves the desired result.

REMARKS. (*a*) The result remains true if $\beta = 1$ and $0 < A < 2\pi$. (*b*) If $s = 0$, then

$$H(z) = \sum_1^\infty e^{Ain^\beta} \frac{\Gamma(n)}{\Gamma(z+n)} \quad A > 0, \, 0 < \beta < 1,$$

is an integral function of z, so that the factorial series

$$\sum_1^\infty e^{Ain^\beta} \frac{1.2 \ldots n}{z(z+1) \ldots (z+n)}$$

represents an analytic function in the whole z-plane excepting the points $z = 0, -1, -2, -$. The case $p = 1$ has been considered by V. F. Cowling [1].

3. THEOREM 2. *Let*

$$H(s, z) = \sum_2^\infty e^{Ai(\log n)^\beta - s \log n} \frac{\Gamma(n)}{\Gamma(z+n)}, \quad A > 0, \, \beta > 2; \qquad (3.1)$$

then $H(s, z)$ represents an integral function of both s and z.

The proof of this theorem is analogous to that of the previous one.

REMARKS. (*a*) A similar result holds for the series

$$\sum_{2}^{\infty} e^{Ai(\log n)^{\beta} - s(\log n)^{\alpha}} \frac{\Gamma(n)}{\Gamma(z+n)}, \quad \alpha > 1, \beta > \alpha + 1,$$

so that by putting $z = 0$, we prove that

$$\sum_{2}^{\infty} e^{Ai(\log n)^{\beta} - s(\log n)^{\alpha}}$$

represents an integral function of s if $\alpha > 1$ and $\beta > \alpha + 1$. Dr. Srivastava [2] has shown that this result is true even if $\beta > \alpha > 1$.

(*b*) Putting $s = 0$, we prove that the function represented by the series

$$H(z) = \sum_{2}^{\infty} e^{Ai(\log n)^{\beta}} \frac{\Gamma(z)}{\Gamma(z+n)} \quad A > 0, \beta > 2,$$

is an integral function of z, so that the factorial series

$$\sum_{2}^{\infty} e^{Ai(\log n)^{\beta}} \frac{1.2 \ldots n}{z(z+1) \ldots (z+n)}$$

represents an analytic function of z in the whole finite plane excepting the points $z = 0, -1, -2, \ldots$.

Thanks are due to Prof. P. L. Srivastava, D. Phil. (Oxon) for his guidance in the preparation of this note.

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1. V. F. COWLING : Analytic continuation of factorial series, *American Jour. Math.* 71 (1949), 283-286.

2. P. L. SRIVASTAVA : On the analytic continuation of functions represented by a class of Dirichlet's series, *Jour. Indian Math. Soc.* 17 (1) (1927-28), 103-152.

MATHEMATICAL NOTES

A simple proof of a formula in the theory of functions

By ALEXANDER DINGHAS, *New York*

1. Let $f(z)$ be a function analytic inside a semicircular domain and on its boundary H. More than seventeen years ago I gave a simple formula [*Preuss. Akad. Wiss. Mathem. Physik Klasse*, (1935) Bd. 33, S. 576-596. *Bull. Soc. Math.* 64 (1936), 78-86], which expresses the value of $f(z)$ at any point inside H, in terms of the values of its real part on H, in a similar way as the classical Poission's formula for a circle. The proof given there by means of Green's function was rather complicated. In the following, I am giving a proof which furnishes the desired result in a few lines.

2. Let the function $f(z)$ be regular in the semicircular domain

$$|z| < R, \quad x \geqslant 0, \quad (z = x + iy), \tag{2.1}$$

and let ζ denote a point of the boundary H of (2.1).

According to Cauchy's Residue-theorem we get for every point inside (2.1),

$$2f(z) - f(0) = \frac{1}{2\pi i}\int_H f(\zeta)\left\{\frac{\zeta+z}{\zeta-z} - \frac{R^2 - z\zeta}{R^2 + z\zeta}\right\}\frac{d\zeta}{\zeta}, \tag{2.2}$$

and

$$f(0) = \frac{1}{2\pi i}\int_H f(\zeta)\left\{\frac{R^2 + \bar{z}\zeta}{R^2 - \bar{z}\zeta} - \frac{\zeta - \bar{z}}{\zeta + \bar{z}}\right\}\frac{d\zeta}{\zeta}, \tag{2.3}$$

where $\bar{z}$ denotes the conjugate number to z and the integrals being taken in Cauchy's sense get the point $\zeta = 0$.

Now, for any point ζ on the periphery $|\zeta| = R$, we have $\zeta\bar{\zeta} = R_2$. Similarly we have for any ζ on the segment $-R < y < R$, $\bar{\zeta} = -\zeta$. Therefore we obtain from (2.3)

$$\overline{f(0)} = \frac{1}{2\pi i}\int_H \overline{f(\zeta)}\left\{\frac{\zeta+z}{\zeta-z} - \frac{R^2 - z\zeta}{R^2 + z\zeta}\right\}\frac{d\zeta}{\zeta},$$

and combining with (2.2),

$$f(z) = i \,\mathrm{Im}\, f(0) + \frac{1}{2\pi i} \int_H Re\, f(\zeta) \left\{ \frac{\zeta + z}{\zeta - z} - \frac{R^2 - z\zeta}{R^2 + z\zeta} \right\} \frac{d\zeta}{\zeta}$$

This is the desired result.

3. The classical Poisson's formula for a circle $C : |z| \leqslant R$ can of course also be proved in the same way.

We have to use here the two integrals

$$2f(z) - f(0) = \frac{1}{2\pi i} \int_C f(\zeta) \frac{\zeta + z}{\zeta - z} \cdot \frac{d\zeta}{\zeta}, \tag{3.1}$$

and

$$f(0) = \frac{1}{2\pi i} \int_C f(\zeta) \frac{R^2 + \bar{z}\zeta}{R^2 - \bar{z}\zeta} \cdot \frac{d\zeta}{\zeta}. \tag{3.2}$$

Again, as before, we have $\zeta \bar{\zeta} = R^2$ and we obtain

$$\overline{f(0)} = \frac{1}{2\pi i} \int_C \overline{f(\zeta)} \frac{\zeta + z}{\zeta - z} \cdot \frac{d\zeta}{\zeta}.$$

An addition to (3.1) gives the desired result.

Partitions of zero into 4 cubes

By B. V. Ramasarma, *Waltair*

In a note entitled 'Residual types of partitions of "0" into four cubes' [*Math. Student*, 13 (1945), 47-8], A. K. Srinivasan has investigated the possible types of residues (mod 6) of a, b, c, d satisfying

$$a^3 + b^3 + c^3 + d^3 = 0. \tag{1}$$

If $0 \leqslant a_1, b_1, c_1, d_1 < 6$ and $a \equiv a_1$, $b \equiv b_1$, $c \equiv c_1$, $d \equiv d_1$ (mod 6) he has shown that there are only 10 different types of quadruples (a_1, b_1, c_1, d_1). For 7 of these types he gives examples of integers a, b, c, d satisfying (1) and having given residues mod 6. For the three types (1, 1, 1, 3), (2, 2, 3, 5) and (0, 1, 1, 4) he gives no examples and believes that such types cannot exist. We prove here that such types *do not* exist.

For instance take (1, 1, 1, 3). Let a, b, c, d satisfy (1) and

$$a \equiv 1, b \equiv 1, c \equiv 1, d \equiv 3 \pmod 6, \tag{2}$$

then $a = 6x + 1$, $b = 6y + 1$, $c = 6z + 1$, $d = 6w + 3$, x, y, z, w being integers. Thus

$$\begin{aligned} 0 &= a^3 + b^3 + c^3 + d^3 \\ &= 216\,(x^3 + y^3 + z^3 + w^3) + \\ &\quad + 18\,(6x^2 + x + 6y^2 + y + 6z^2 + z + 6w^2 + 3w) + 30. \end{aligned} \tag{3}$$

The first two terms on the right of (3) are divisible by 18 but not the last showing that (2) cannot hold. The others may be dealt with similarly.

CLASSROOM NOTES

A summation problem

By HANSRAJ GUPTA, *Punjab University College, Hoshiarpur*

1. In a letter to Dr. A. Narasinga Rao, Martin G. Beumer of Holland posed the problem of evaluating the sum

$$S_n = \sum_{r=0}^{n} \frac{n(n-1)(n-2)\ldots(n-r+1)}{2n(2n-1)(2n-2)\ldots(2n-r+1)} 2^r.$$

In the following solution the only property of the combinatory functions used is

$$\binom{m}{r} + \binom{m}{r-1} = \binom{m+1}{r}.$$

We readily have

$$S_n = \sum_{r=0}^{n} \frac{n!}{(n-r)!} \cdot \frac{(2n-r)!}{(2n)!} 2^r$$

$$= \sum_{r=0}^{n} \frac{n!\, n!}{(2n)!} \cdot \frac{(2n-r)!}{n!\,(n-r)!} 2^r;$$

so that

$$\binom{2n}{n} S_n = \sum_{r=0}^{n} \binom{2n-r}{n} 2^r = u_n, \text{ say,}$$

then

$$u_n = \binom{n}{n} 2^n + \binom{n+1}{n} 2^{n-1} + \binom{n+2}{n} 2^{n-2} + \ldots + \binom{2n-1}{n} 2 + \binom{2n}{n}. \quad \text{(A)}$$

With this notation

$$\frac{u_{n+1}}{2} = \binom{n+1}{n+1} 2^n + \binom{n+2}{n+1} 2^{n-1} + \binom{n+3}{n+1} 2^{n-2} + \ldots + \binom{2n}{n+1} 2 + \binom{2n+1}{n+1} + \frac{1}{2}\binom{2n+2}{n+1}. \quad \text{(B)}$$

Subtracting, we have

$$\frac{u_{n+1}}{2} - u_n = \binom{n+1}{n+1} 2^{n-1} + \binom{n+2}{n+1} 2^{n-2} + \dots + \binom{2n}{n+1} + \frac{1}{2}\binom{2n+2}{n+1}$$

$$= \frac{u_{n+1}}{4},$$

because

$$\binom{2n+1}{n+1} = \binom{2n+1}{n} = \frac{1}{2}\cdot\frac{2n+2}{n+1}\binom{2n+1}{n} = \frac{1}{2}\binom{2n+2}{n+1}.$$

Hence

$$u_{n+1} = 4u_n.$$

But

$$u_1 = 4.$$

Therefore

$$u_n = 4^n.$$

Hence

$$S_n = \frac{4^n}{\binom{2n}{n}}.$$

2. The following is another method for evaluating u_n.

We have

$$2\binom{n}{n} + \binom{n+1}{n} = \binom{n+2}{n+2} + \binom{n+2}{n+1},$$

$$2^2\binom{n}{n} + 2\binom{n+1}{n} + \binom{n+2}{n} = 2\binom{n+2}{n+2} + 2\binom{n+2}{n+1} + \binom{n+2}{n}$$

$$= \binom{n+3}{n+3} + \binom{n+3}{n+2} + \binom{n+3}{n+1}.$$

Proceeding in this manner, we get

$$2^k\binom{n}{n} + 2^{k-1}\binom{n+1}{n} + 2^{k-2}\binom{n+2}{n} + \dots + \binom{n+k}{n}$$

$$= \binom{n+k+1}{n+k+1} + \binom{n+k+1}{n+k} + \binom{n+k+1}{n+k-1} + \dots + \binom{n+k+1}{n+1}.$$

In particular for $k = n$,

$$u_n = \binom{2n+1}{2n+1} + \binom{2n+1}{2n} + \dots + \binom{2n+1}{n+1} = \frac{1}{2}\sum_{r=0}^{2n+1}\binom{2n+1}{r} = 2^{2n}.$$

A note

By R. R. Sharma

The problem is to evaluate

$$u_n = \sum_{r=0}^{n}\binom{2n-r}{n} 2^r.$$

Now

$$u_n + \binom{2n}{n} = \binom{2n}{n} + (2-1)\sum_{r=0}^{n}\binom{2n-r}{n} 2^r$$

$$= \sum_{r=0}^{n}\left\{\binom{2n-r}{n} - \binom{2n-r-1}{n}\right\} 2^{r+1}$$

$$= \sum_{r=0}^{n}\binom{2n-r-1}{n-1} 2^{r+1}.$$

Similarly

$$u_n + \binom{2n}{n} + \binom{2n-1}{n} 2 = \sum_{r=0}^{n}\binom{2n-r-2}{n-2} 2^{r+2}.$$

Proceeding thus, after n steps, we get

$$u_n + \binom{2n}{n} + \binom{2n-1}{n} 2 + \binom{2n-2}{n} 2^2 + \dots + \binom{n+1}{n} 2^{n-1}$$

$$= \sum_{r=0}^{n} \binom{n-r}{0} 2^{n+r} = 2^{2n+1} - 2^n .$$

Hence

$$2u_n = 2^{2n+1} \text{ or } u_n = 2^{2n}.$$

BOOK REVIEWS

Introduction to the foundations of mathematics. By Raymond L. Wilder, John Wiley and Sons, Inc., New York, and Chapman and Hall, Ltd., London, 1952, xiv + 305 pp.

This is an introduction to some of the fundamental concepts of mathematics and to modern developments in the foundations of mathematics. The content is that of a course which the author has been giving for more than twenty years to students at the undergraduate or first-year graduate level, intended less for those who would become research mathematicians than for prospective teachers, actuaries, statisticians, and others who had specialized in undergraduate mathematics.

Part I of the book first introduces the student to the informal axiomatic method, i.e. to the older form of the axiomatic method in which the logical apparatus employed is a "taken-for-granted" "natural" logic (the quotation marks are the author's), and only the terms specific to a particular mathematical theory (as distinguished from those of logic) are listed among the undefined terms, and only specifically mathematical propositions (as distinguished from logical) are set down as axioms. The treatment then proceeds partly on such axiomatic basis, and partly on the still more informal basis of direct development of the underlying "natural" logic. There is an excellent account of the notions of consistency, independence, completeness, and categoricalness, with several simple systems of axioms introduced as examples. Then follows a chapter on the theory of sets, with discussion of the Russell antinomy; the usual elementary operations on sets; the definition of an infinite set as a set which has no one-to-one correspondence with a section of the natural numbers; the Peirce-Dedekind definition of an infinite set; proof of the equivalence of the two definitions of the infinite; and a very clear statement of the role of the axiom of choice in this proof, and of the axiom of choice itself and its significance.

The next chapter treats the notion of enumerability, the diagonal procedure, and the transfinite cardinal numbers. Here an interesting feature is a discussion of the notion of an *effective* definition and of an *effective* enumeration, based on a process of iterated application of the diagonal procedure which (though not the same) resembles in some essential respects Hardy's famous fallacious proof* that $\aleph_1 \leqslant c$.

There follows a chapter on well-ordered sets and ordinal numbers, in which, in particular, transfinite induction is introduced; the well-ordering theorem is proved as a consequence of the axiom of choice; Burali-Forti's antinomy is stated; and the continuum problem (i.e. the problem whether $\aleph_1 = c$) is discussed. The next chapter has an axiomatic treatment of the real number system. And then Part I closes with a chapter on groups and their significance for the foundations.

Part II deals with questions and controversies about the nature of mathematics, and with developments in the foundations of mathematics which either require or lead to a formalization of logic (in addition to, or even instead of, specifically mathematical axioms). After a chapter about early developments, ending with a statement of Zermelo's axioms for set theory, there are chapters which treat, in order, the Frege-Russell thesis (basing mathematics on logic alone), the intuitionism of Brouwer, and the proof theory of Hilbert. Especially good is the sketch, in this last chapter, of the proof of Gödel's incompleteness theorem, which is based on lectures by Henkin.

The final chapter of the book is a reworking of the content of the author's address to the International Congress of Mathematicians

*In *The Quarterly Journal of Pure and Applied Mathematics*, vol. 35 (1903-4) pp. 87-94. Hardy's proof is fallacious only if understood as claiming to have obtained the result without use of the axiom of choice or to have defined a particular subset of the continuum of cardinal number $\aleph_1$ (see *Proceedings of the London Mathematical Society*, series 2, vol. 3, pp. 170-188, and series 2, vol. 4, pp. 10-17). And an analogous remark applies to Wilder's "fallacious theorem", 3.1.8.1.

of 1950 (published in the *Proceedings* of the Congress). Here the author for the first time abandons the impartiality which he has adopted in stating the case for each of the doctrines treated in Part II, and states his own view of the nature of mathematics. This is summed up by saying that "mathematics, like other cultural entities, is what it is as a result of collective human effort... what it becomes will not be determined by the discovery of 'mathematical truth' now hidden from us, but by what mankind, *via* cultural paths, makes it.... In short, mathematics is what we make it.... Until we make it, it fails to 'exist.' And, having been made, it may at some future time even fail to be 'mathematics' any longer."—Even the Platonic realist would no doubt admit the importance of cultural influences in determining what part of the whole body of mathematical truth is in fact discovered and explored at any particular time. But the author, as the quotation shows, makes use of the cultural determination of mathematics as an argument for a form of conceptualism, which differs from more usual forms in its emphasis on impermanence in the content of mathematics.

Of the two parts of the book, Part I is of uniformly high excellence, and certainly the best existing exposition for its purpose. The shorter Part II shows many of the merits of Part I but suffers from excessive condensation. For example, the development of the logic of quantifiers (pp. 223-224) and that of the intuitionistic propositional calculus (pp. 244-246) are not carried far enough to make these matters really clear to the student. And in the description of various forms of the theory of types on pages 225-228, so many different things are set forth in brief space without elaboration or illustration that the treatment becomes very difficult to follow.

There are some scattering minor errors which deserve correction, though they are unimportant in relation to the book as a whole.

There is a discrepancy between the statement of the Peano axioms for the natural numbers on page 66, where it is made a part of the second axiom that the successor of a number is unique, and the statement of the same axioms on page 149, where the second axiom asserts only the existence of a successor. Moreover the uniqueness of the successor does not follow from the axioms on page 149.

Again on page 149, the difficulty involved in using the mathematical induction principle to justify definition by recursion is glossed over (compare Landau, *Grundlagen der Analysis*). This would be ameliorated if the definition of $<$ on page 150 preceded the definitions of addition and multiplication.

On page 222, the ascription to Carnap of the opinion that strict implication should be understood as a relation between sentences is historically doubtful. On page 227 the definition of predicative function seems to be incorrect on the basis of the terminology that has just been introduced in the preceding sentences. On page 250 (last line), *modus ponens* is incorrectly stated in the form of a formula scheme—though elsewhere it appears correctly as a rule. On page 258 the example given in the third paragraph is incorrect. On page 259 the statement of Rosser's theorem is not quite accurate.

ALONZO CHURCH

A mathematician's miscellany. By J. E. Littlewood, Methuen, London, 1953, vii+136 pp. 15sh.

This is a peculiar book, rather like a diary than a book. It is a collection of notes and articles of great variety and interest, sufficiently light to appeal to the amateur and sufficiently intriguing to provoke the professional mathematician, with such titillating titles as *The Zoo, From Fermat's Last Theorem to the Abolition of Capital Punishment, Large Numbers, Lion and Man,* etc. It has the true Littlewood touch; there is nothing cheap or trivial. An autobiographical section describes the author's mathematical education. Four book reviews are reprinted, one of which is the famous review of the *Collected Papers of S. Ramanujan.* There is a rich fare of puzzles and paradoxes and jokes and curiosities, which confirm the author's dictum that "a good mathematical joke is better, and better mathematics, than a dozen mediocre papers.... Incidentally the joke is *in* the mathematics, not merely about it." There are also several high-brow pieces of technical mathematics.

The book opens with a highly interesting article giving an unusual set of examples of *Mathematics with minimum 'raw material'*, drawn from unexpected sources that range from feminine psychology to Marcel Riesz's double convexity theorem. In between the author asks and answers such questions as : What is the best stroke ever made in a game of billiards? Can a dissertation of 2 lines deserve and get a Fellowship in mathematics? Is every cipher breakable? There is an amusing section on *Cross-purposes, unconscious assumptions, howlers and misprints*, in which the author says, "I remember reading the description of co-ordinate axes in Lamb's *Higher Mechanics* : Ox and Oy as in 2 dimensions and Oz vertical. For me this is quite wrong ; Oz is horizontal (I work always in an armchair with my feet up)." An anecdote about Hardy turns up : "I read in the proof-sheets of Hardy on Ramanujan : 'as someone said, each of the positive integers was one of his personal friends.' My reaction was, "I wonder who said that : I wish I had." In the next proof-sheets I read (what now stands), 'it was Littlewood who said.........' [What happened was that Hardy received the remark in silence and with poker face, and I wrote it off as a dud. I later taxed Hardy with this habit ; on which he replied : "Well, what is one to do, is one always to be saying 'damned good'? To which the answer is yes.]" It is in the same book that Hardy says, in line with the same practice, "I owe more to Ramanujan than to anyone else in the world with one exception."

That the author is a great virtuoso in constructing "gegenbeispiel"'s is illustrated by *The Zoo* in which one finds, among other things, the conformal mapping of the mouth of a crocodile. Although Hardy once remarked, "even Littlewood could not make ballistics respectable; if he could not, who can?", there is an engaging section on *Ballistics*, followed by a brief but provocative essay on *The Dilemma of Probability Theory*. Could there be a chain of ideas from Fermat's last theorem to the abolition of capital punishment? "I think so, with some give and take", replies the author. Of particular interest to some Indian students will be the author's remark on Forsyth's *Theory of Functions of a Complex Variable*, that it 'was out of date when written (1893)', and the long quotation on the notion of a 'function' which has an 'obscurity as of midnight'. A long section on *Large Numbers* (reprinted from the *Mathematical Gazette*) begins with a consideration of numbers connected directly or indirectly with daily life, games of chance, coincidences and improbabilities, [" Improbabilities are apt to be over-estimated. It is true that I should have been surprised in the past to learn that Professor Hardy had joined the Oxford group. But one could not say the adverse chance was $10^6 : 1$. Mathematics is a dangerous profession; an appreciable proportion of us go mad, and then this particular event would be quite likely "], and works up to illuminating comments on the Skewes number.

Although the auto-biographical section ends somewhat abruptly with the sentence, " I soon began my 35-year collaboration with Hardy", it gives much interesting information of Littlewood as a student, of his 'first contact with a startled Hardy', of the 'authentic thrill' which the first volumes of the Borel series gave him, of his first longish paper on integral functions and the 'violently unfavourable' reception it got from one of the referees [" .... by the time I learned in later life who he was I had disinterestedly come to think of him a bit of an ass......... I have not since had trouble with papers, with the single exception that the Cambridge Philosophical Society once rejected (quite wrongly) one written in collaboration with Hardy "], of his tutor's 'heroic suggestion' that he should prove the Riemann hypothesis, a suggestion 'which was not without result', of his 'youthful views' on the Prime Number Theorem which illustrate 'the uncertainty of judgment and taste in a beginner in a field with no familiar landmarks', of his three years as Richardson lecturer at Manchester University where he had 'high pressure work on top of the low pressure mountain', and of his joining the Trinity staff in 1910 and his proof of the famous converse of Abel's theorem, which marked the end of his mathematical education. Of the Mathematical Tripos at Cambridge he would only say, "the old Tripos and its vices are dead horses; I will not flog them. I do not claim to have suffered high-souled frustration... I will say, however, that for me the thing to avoid, for doing creative work, is above all Cambridge life, with the constant bright conversation of the clever, the wrong sort of mental stimulus, all the goods in the front window". Out of these pages emerges a personality very different from that of Hardy's, which makes their extraordinarily long and fruitful collaboration all the more remarkable.

The author's aim is 'entertainment and there will be no uplift.' That aim has been admirably achieved. This is an enjoyable book.

K. Chandrasekharan

The printing of mathematics. By T. W. Chaundy, P. R. Barrett and Charles Batey, Oxford University Press, 1954, x + 105 pp. 15 *sh*.

The most difficult part of any job of mathematical printing is the composing of mathematics. This was done largely by hand until 1930, when the University Press, Oxford, in collaboration with Professors G. H. Hardy and R. H. Fowler, and the editors of the *Quarterly Journal of Mathematics*, and with the Monotype Corporation, made the first serious attempt to develop the resources of the Monotype machine for the composition of mathematics. By much adaptation and recutting of type faces a new system was evolved in which the technique of mechanical composition was geared to the demands of mathematical printing. The system has since been improved almost to the point of perfection, and no printing-house is entitled to greater praise for this advance than the University Press, Oxford. The code of rules which was drafted by them for the use of compositors at the Press and those authors and editors whose work was produced there, and which has served to establish the now familiar and justly famous Oxford style, is elaborated here in sufficient detail to be of great practical utility. In order to make these rules fully intelligible, and to make clear their motivation, an introductory chapter, of just twenty pages, on the mechanics of mathematical printing, has been written by Mr. Batey. This excellent exposition, remarkable alike for its simplicity and clarity, is followed by a long chapter of fifty three pages, by Mr. Chaundy and Mr. Barrett, in which detailed methods are recommended by which authors might ease the printer's task and their own. This chapter, written in a lively and engaging style, will be of great help to authors, editors, readers, and compositors. It may be considered as a modern version of the ' Notes on the Preparation of Mathematical Papers ' published by the London Mathematical Society in 1932 under G. H. Hardy's inspiration, such a version having become necessary on account of the new role of the machine compositor. The recommendations made here reflect the Oxford practice, which differs in some respects from that of other authorities, but the principles on which they are based are, almost always, unexceptionable. The book is superbly produced, and will certainly serve as an acceptable and ready work of reference.

The only typographical discrepancies that the reviewer has noticed occur on pages 15, 17 and 31. On page 15, the expression ' $(\phi = 0, 1, \ldots\ldots, n)$ ' occurs six times, and on each occasion the figures 0, 1 are in italic. This is not justified by the Oxford practice; if there is a special reason for this particular usage, it has not been made clear. On p. 17, line 23, the sentence, 'The practice of so disposing the pages of a book so that . . . ' has perhaps one 'so' too many. On p. 31, line 18, the suffix of the last ' a ' should not have been in Greek.

The reviewer is unable to appreciate the practice of using bold-face figures both for numbering the sections in a paper, and for numbering the references given at the end of a paper. It is true that the enumeration of the references is done in a smaller point, but they are indicated in the body of the paper by bold-face numerals of regular size, and on a page which contains a large number of references, a strident effect is produced. The practice of using ordinary numerals enclosed in brackets [ ] is perhaps preferable, especially because the brackets become indispensable, in any case, if the author wishes to cite the references by chapter and page. Again, the practice of inserting a rule in place of the author's name, whenever more than one work by the same author is listed in the references, not only upsets the aesthetic balance but creates a problem in case an author figures as a joint author in one instance and as the sole author in another. But these are minor points on which no reasonable person can be dogmatic.

K. CHANDRASEKHARAN

Analytical solid geometry. By M. V. Jambunathan, India Book Company, Bangalore, 1952, 113 pp. Rs. 2-4.

This nice booklet attempts, as the author claims, to serve as an introduction to the analytical geometry of three dimensions, having for its scope the study of the 'point', 'line' and 'plane'. Although the treatment follows the familiar lines of the standard English books on the corresponding topics, the author has brought to bear his experience as a teacher both in the methods of presentation, by aiming at clearer exposition, and in the selection of useful worked examples.

In spite of the fact that much care must have been bestowed in preparing a book of this kind, a slightly different treatment of the topics would have considerably increased the utility and the scope of the book. For example, an elementary discussion of vectors, projections, etc., and a somewhat modified presentation making use of vector ideas wherever necessary, would help the student to understand better the notion of the distance "of a point to a plane", or "to a line", "the direction cosines of a direction", etc., and furthermore to avoid the usual and somewhat tricky methods of changing signs of perpendicular distances, etc.

Again certain other details in this book also call for comment. For instance, the sign and range of r, θ, ϕ in the spherical polar coordinate system should be clearly indicated to the beginner and should not be left vague as we find in § 9 on p. 10. The idea conveyed in § 16, p. 20 relating to the definition of the cone is confusing in as much as a cone should be taken to extend on both sides of the vertex; further it would be instructive if the cylinder be introduced as the limiting form of a cone when the vertex recedes to infinity.

Moreover, the statement on p.20 relating to the vertical angle of the cone, viz. that " any plane through the vertex...cone " is not correct. There are also a few misleading, but presumably, printer's mistakes which might have been well avoided. Thus: on p. 33 "the direction cosines of AP are

$$\frac{x_1 - a}{r} = \frac{y_1 - \beta}{r} = \frac{z_1 - \gamma}{r}\text{''},$$

and on page 79, relating to the equation of the plane through (x_1, y_1, z_1) and the line

$$\frac{x - a}{l} = \frac{y - \beta}{m} = \frac{z - \gamma}{n},$$

" This can be expanded as the determinant... ".

On the whole the book would unhesitatingly serve as a constant companion to the students for whom it is designed.

K. R. Aiyer

NEWS AND NOTICES

The following members have been admitted to life-membership in the Society : V. A. Mahalingam, S. S. Lal Mathur, K. Padmavally, V. Rangachar and M. V. Subba Rao.

The following persons have been elected ordinary members of the Society : Nirmala Pandey and Darshan Singh.

The following members of the American Mathematical Society have been admitted as members of the Indian Mathematical Society under the reciprocity agreement : Stephen Hoffman and L. G. Hutchinson.

Dr. Ram Behari has been nominated to represent the Society on the Council of the Ramanujan Institute of Mathematics.

The Osmania University has kindly extended an invitation to hold the twentieth Conference of the Indian Mathematical Society at Hyderabad in December 1954. The invitation has been accepted.

The Society has received publication grants of Rs. 1,000 from the National Institute of Sciences of India, and Rs. 2,000 from the Government of India, for the year 1953-54.

It is understood that Professors H. Rademacher, L. Schwartz and P. A. M. Dirac will be in residence at the Tata Institute of Fundamental Research during the winter term of the academic year 1954-55.

Professor M. H. Stone, President of the International Mathematical Union, has appointed the following Commission on the World Directory of Mathematicians :

P. Belgodère (Paris), E. Bompiani (Rome), W. V. D. Hodge (Cambridge), H. Levy (London), M. H. Stone, Chairman (Chicago).

An arrangement has been reached with Butterworths Publications Limited, London, according to which Butterworths will prepare the material for the World Directory with the assistance of the named committee, which will furnish technical advice. The Union has also agreed to pay a contribution of $2000 towards the expenses of preparing the manuscript. Of this sum, $1000 will be paid in 1953 and $1000 in 1954.

International Congress of Mathematicians, Amsterdam, September 2-9, 1954.

The following mathematicians have accepted the invitation of the Organizing Committee to deliver a *one-hour* address at the Congress: K. Borsuk (Warsaw), R. Brauer (Cambridge Mass.), J. Dieudonné (Ann Arbor, Mich.), S. Goldstein (Haifa), Harish-Chandra (New York), B. Jessen (Copenhagen), A. Lichnerovicz (Paris), J. V. Neumann (Princeton, N.J.), J. Neyman (Berkeley), B. Segre (Rome), C. L. Siegel (Göttingen), E. Stiefel (Zürich), A. Tarski (Berkeley), E. C. Titchmarsh (Oxford), K. Yosida (Osaka).

The following mathematicians have accepted the invitation of the Organizing Committee to deliver a *half-hour* address at the Congress:

Section I (*Algebra and Theory of Numbers*). H. Davenport (London), P. Erdös (Los Angeles), E. Hlawka (Vienna), N. Jacobson (New Haven), H. Maass (Heidelberg), A. Neron, D. G. Northcott.

Section II (*Analysis*). H. Behnke (Münster, Westf.), F. Bureau (Liege), M. L. Cartwright (Cambridge, Eng.), L. Cesari (U.S.A.), K. Chandrasekharan (Bombay), A. Erdélyi (Pasadena), W. K. Hayman (Exeter), E. Hille (New Haven), P. J. Myrberg (Finland), C. Pauc (Nantes), Wazewski (Poland), A. Zygmund (Chicago).

Section III (*Geometry and Topology*). G. Ancochea (Madrid), W. L. Chow (Baltimore), H. S. M. Coxeter (Toronto), B. Eckmann (Switzerland), H. Freudenthal (Utrecht), D. Montgomery (Princeton), H. S. Ruse (Leeds), J. P. Serre (Paris).

Section IV (*Probability and Statistics*). J. L. Doob (U.S.A.), R. Fortet (France), M. G. Kendall (London).

Section V (*Mathematical Physics and Applied Mathematics*). L. Collatz (Hamburg), G. Fichera (Triest), M. R. Hestenes (Los Angeles), J. Kampé de Fériet (Lille), J. J. Stoker (New York).

Section VI (*Logic and Foundations*). P. Lorenzen (Bonn), A. Mostowski (Warsaw), J. B. Rosser (Ithaca).

Section VII (*Philosophy, History and Education*). C. T. Daltry (England), J. E. Hofmann (Bayern), K. Piene (Oslo).

ERRATUM

By M. K. Singal and Ram Behari

It is regretted that relation (2.3) in our paper in the *Mathematics Student*, XXII (1954), p.37 is not correct. The deduction of the results of §§ 4,5 is therefore not valid.

THE INDIAN MATHEMATICAL SOCIETY

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The Society publishes two periodicals, *The Journal of the Indian Mathematical Society* and the *Mathematics Student*. The annual subscription for the *Journal* is rupees twelve, and that for the *Student* is rupees six. Back volumes of both the periodicals are available except for a few numbers out of stock. The following publications of the Society are also available : (i) Memoir on cubic transformations associated with a desmic system, by Dr. R. Vaidyanathaswamy, pp. 92, Rs. 3, (ii) Tables of Partitions, by Dr. Hansraj Gupta, pp. 81, Rs. 5.

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