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ON THE LEAST PRIME IN AN ARITHMETICAL PROGRESSION

By S. Chowla.

Let k and l be integers prime to each other, of which k is positive. Dirichlet proved that there are infinitely many primes in the arithmetical progression $kx+l$. If $P(k, l)$ is the least prime of this form it is probable that $P(k, l) < k^{1+\epsilon}$ for every positive ϵ and all large k. We are very far at present even from being able to show that $P(k, l) < k^m$ where m is any fixed positive constant independent of k, which is large. However if we assume the truth of the so-called "extended Riemann hypothesis" we can[1] show that $P(k, l) < k^{2+\epsilon}$ for $k > k_0(\epsilon)$.

Taking the special case $l=1$ we can show that $P(k, 1) < e^{Ak}$ where A is a positive constant independent of k, in the special case when k is a prime as follows. We know that every prime factor of 2^k-1 is $\equiv 1 \pmod{k}$ when k is a prime. Hence $P(k, 1) < 2^k$ and the result stated follows. It is improbable that such a simple argument exists for general l.

In what follows I show that

$$\text{(I)} \qquad P(k,l) < e^{A k^{\frac{3}{2}} \log^6 k}$$

for primes $k \equiv 3 \pmod 4$. It is likely that my proof of (I) can be extended without difficulty to general k. I have not bothered to do this since much more can be proved by an appropriate revision of the proof of the generalized Prime Number Theorem giving the number of primes $\equiv l \pmod{k}$ not exceeding x. The results of this investigation will appear elsewhere.

§. 1. *Proof of (I)*: Throughout what follows k is a prime $\equiv 3 \pmod 4$, $\chi(n)$ is a Dirichlet's character (mod k). χ_0 is the principal character, χ_1 denotes any non-real character, χ_2 denotes (there is only one such when k is a prime) a real non-principal

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[1] This follows from Theorem 6 of Titchmarsh's paper "A divisor problem", *Rendiconti del Circolo Matematico di Palermo,* Tomo LIV, 1930, 414-429.

character. Numerals in thick print are references to the equations of this paper.

Lemma 1: We have(2)

(1) $$\sum_{a}^{b} \chi(n) = O(\sqrt{k}\log k).$$

Lemma 2: *We have*(3)

(2) $$\frac{1}{L_{\chi_1}(1)} = O(\log^5 k)$$

Lemma 3:

(3) $$\frac{1}{L_{\chi_2}(1)} = O(\sqrt{k})$$

Proof: We have

$$L_{\chi_2}(1) = \pi \frac{\Sigma\,(b-a)}{k\sqrt{k}} \geqq \frac{\pi}{\sqrt{k}}$$

(where b runs through the quadratic non-residues of k and a through the residues of k) since, for primes k

$$\Sigma b \equiv \Sigma a \equiv 0 \pmod{k}$$

and $L_{\chi_2}(1) \neq 0$.

For $\chi \neq \chi_0$ we have(4)

$$\sum_{n\leqq x} \frac{\chi(n)\log n}{n} = \sum_{p^m\leqq x} \frac{\chi(p^m)\log p}{p^m} \sum_{n\leqq \frac{x}{p^m}} \frac{\chi(n)}{n}$$

$$= \sum_{p^m\leqq x} \frac{\chi(p^m)\log p}{p^m} \left(L_\chi(1) + O\left(\frac{\sqrt{k}\log k\, p^m}{x}\right)\right) \text{(1)}$$

$$= L_\chi(1) \sum_{p\leqq x} \frac{\chi(p)\log p}{p} + O(\sqrt{k}\log k),\ \sum_{p^m\leqq x} \frac{\log p}{x}$$

(4) $$= L_\chi(1) \sum_{p\leqq x} \frac{\chi(p)\log p}{p} + O(\sqrt{k}\log k),$$

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[2] Landau, *Vorlesungen uber Zahlentheorie*, Bd. 3, S. 178..

[3] Landau, *Math. Annalen*, 70, 1911, 69—78.

[4] Landau, *Primzahlen*, 447—8.

$$\sum_{\substack{p\equiv l (\mathrm{mod}\ k)\\ p\leqslant x}} \frac{\log p}{p} = \frac{1}{S(k)} \sum_{\chi=1}^{S(k)} \frac{1}{\chi_\chi(l)} \sum_{p\leqslant x} \frac{\chi_\chi(p)\log p}{p}$$ [5]

$$= \frac{1}{S(k)} \frac{1}{\chi_0(l)} \sum_{p\leqslant x} \frac{\chi_0(p)\log p}{p} + \frac{1}{k-1} \sum_{\chi=\chi_1} \frac{1}{\chi(l)} \sum_{p\leqslant x} \frac{\chi(p)\log p}{p}$$

$$+ \frac{1}{k-1} \frac{1}{\chi_2(l)} \sum_{p\leqslant x} \frac{\chi_2(p)\log p}{p}$$

$$(5) \quad = \frac{\log x + O(1)}{k-1} + O(\sqrt{k}\log^6 k) \quad (1, 2, 3).$$

(I) follows immediately from (5).

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[5] $S(k)$ is the number of positive integers not exceeding k and prime to it.

ON A CERTAIN POLYNOMIAL ANALOGOUS TO LOMMEL'S POLYNOMIAL.

By Dr. S. C. Mitra, ph.d.

1. In a previous paper(1) the author has shown that the parabolic cylinder function $D_n(x)$ satisfies the relation

$$D_{m+n}(x)+(-1)^m(n+1)\ R_{n,\,m-2}(x)\ D_n(x)+(-1)^m R_{n-1,\,m-1}(x)\ D_{n+1}(x)=0,$$

where $R_{n,\,m}(x)$ is a polynomial analogous to that of Lommel in the Theory of Bessel Functions. The differential equation satisfied by $R_{n,\,m}(x)$ has been given in a subsequent paper(2) by Mr. R. S. Varma. The object of the present note is to show that a similar relation holds between the parabolic cylinder functions $D_{-(m+n+1)}(ix)$. $D_{(n+1)}(ix)$ and $D_{-(n+2)}(ix)$ and to obtain an interesting property of these functions.

2. Suppose n is a positive integer.

The functions $D_{-(n+1)}(\pm ix)$ satisfy the recurrence formulæ.

$$D_{-n}(ix)-ix\ D_{-(n+1)}(ix)-(n+1)\ D_{-(n+2)}(ix)=0,$$

and

$$D_{-n}(-ix)+ix\ D_{-(n+1)}(-ix)-(n+1)\ D_{-(n+2)}(-ix)=0. \qquad (1)$$

From these formulae, we get

$$\left\{D_{-(n+2)}(ix)\ D_{-(n+1)}(-ix)+D_{-(n+2)}(-ix)\ D_{-(n+1)}(ix)\right\}$$

$$=\frac{1}{n+1}\left\{D_{-(n+1)}(ix)\ D_{-n}(-ix)+D_{-(n+1)}(-ix)\ D_{-n}(ix)\right\}$$

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[1] "On the properties of a certain polynomial analogous to Lommel's polynomial," *Indian Physico-Mathematical Journal*, Vol. 3, No. I (1932), pp. 9-15.

[2] "On a certain polynomial analogous to Lommel's polynomial", *Journal of the Indian Mathematical Society*, Vol. 19, No. 12, December, (1932), p. 274.

$$=\frac{1}{(n+1)\,n\,(n-1)\,\ldots 1}\left\{ D_{-1}(ix)D_0(-ix)+D_{-1}(-ix)D_0(ix)\right\}$$

$$=\frac{\sqrt{2\pi}}{\Gamma(n+2)} \tag{2}$$

The functions $D_{-(m+n+1)}(\pm ix)$ satisfy the differential equation

$$\frac{d^2y}{dx^2}+\left(m+n+\tfrac{1}{2}-\tfrac{1}{4}x^2\right)y=0. \tag{3}$$

As a solution of (3), let us assume

$$y=D_{-(n+1)}(ix)\;A(x)+i\,D_{-(n+2)}(ix)\;B(x).$$

Substituting in (3), we get, after simplification, the equations

$$A''(x)+xA'(x)+mA(x)+2B'(x)=0,$$

and

$$B''(x)-xB'(x)+(m-1)\,B(x)-2(n+1)A'(x)=0. \tag{4}$$

From these we find that $A(x)$ and $B(x)$ satisfy the differential equations,

$$A^{\mathrm{IV}}(x)+(4n+2m+4-x^2)\quad A''(x)-3x\,A'(x)+m(m-2)$$
$$A(x)=0, \tag{5}$$

and

$$B^{\mathrm{IV}}(x)+(4n+2m+2-x^2)\;B''(x)-3x\;B'(x)$$
$$+(m^2-1)\;B(x)=0. \tag{6}$$

The equation (5) can be obtained from (6) by writing $m-1$ and $n+1$ for m and n respectively. So we shall solve equation (6) only.

To solve this equation, let us assume

$$B(x)=\Sigma C_r\;H_r(x).$$

We find

$$B(x)=C_0\Big[H_0(x)-\frac{(m-1)\,(m+1)}{2!\,(4n+2m-2)}H_2(x).$$
$$+\frac{(m-1)\,(m-3)\,(m+1)\,(m+3)}{4!\,(4n+2m-2)\,(4n+2m-6)}H_4(x)-\ldots\Big]$$
$$+C_1\Big[H_1(x)-\frac{(m-2\,(m+2)}{3!\,(4n+2m-4)}H_3(x)$$
$$+\frac{(m-2)\,(m-4)\,(m+2)\,(m+4)}{5!\,(4n+2m-4)\,(4n+2m-8)}H_5(x)-\ldots\Big] \tag{7}$$

From this we deduce

$$A(x)=A_0\Big[H_0(x)-\frac{m\,(m-2)}{2!\,(4n+2m)}H_2(x)$$

$$+\frac{m(m+2)\ (m-2)\ (m-4)}{4!(4n+2m)\ (4n+2m-4)}H_4\ (x)-\ldots\Big]$$

$$+\ C_1\Big[H_1(x)-\frac{(m-2)\ (m+2)}{3!\ (4n+2m-4)}H_3(x)$$

$$+\frac{(m-2)\ (m-4)\ (m+2)\ (m+4)}{5!\ (4n+2m-4)\ (4n+2m-8)}H_5\ (x)-\ldots\Big] \tag{8}$$

3. Let m be a positive integer. Leaving aside the details of actual calculations, we can easily prove that when m is an odd integer,

$$D_{-(m+n+1)}(ix)=A_1\Big[H_1\ (x)-\frac{(m-3)\ (m+1)}{3!\ (4n+2m-2)}H_3\ (x)$$

$$+\frac{(m-3)\ (m-5)\ (m+1)\ (m+3)}{5!\ (4n+2m-2)\ (4n+2m-6)}H_5\ (x)-\ldots\Big]D_{-(n+1)}(ix)$$

$$+iC_0\Big[H_0\ (x)-\frac{(m-1)\ (m+1)}{2!\ (4n+2m-2)}H_2(x)$$

$$+\frac{(m-1)\ (m-3)\ (m+1)\ (m+3)}{4!\ (4n+2m-2)\ (4n+2m-6)}H_4\ (x)-\ldots\Big]D_{-(n+2)}(ix),$$

and when m is an even integer,

$$D_{-(m+n+1)}(ix)=A_0\ \Big[H_0(x)-\frac{(m-2)m}{2!\ (4n+2m)}H_2(x)$$

$$+\frac{(m-2)\ (m-4)m(m+2)}{4!(4n+2m)\ (4n+2m-4)}-\ldots\Big]D_{-(n+1)}(ix)$$

$$+iC_1\Big[H_1(x)-\frac{(m-2)\ (m+2)}{3!\ (4n+2m-4)}H_3(x)$$

$$+\frac{(m-2)\ (m-4)\ (m+2)\ (m+4)}{5!\ (4n+2m-4)\ (4n+2m-8)}H_5(x)-\ldots\Big]D_{-(n+2)}(ix), \tag{A}$$

where

$$A_0=\frac{\Gamma(n+2)}{\Gamma(m+n+1)}\ \frac{(m-2)!\ (4n+2m)\ (4n+2m-4)\ldots(4n+8)}{(m-2)\ (m-4)\ldots 2m(m+2)\ldots(2m-4)}$$

$$C_0=-i\frac{\Gamma(n+2)}{\Gamma(m+n+1)}\ \frac{(m-1)!(4n+2m-2)\ (4n+2m-6)\ldots(4n+4)}{(m+1)\ (m+3)\ldots(2m-2)\ (m-1)\ (m-3)\ldots 2}$$

$$A_1=-i\frac{\Gamma(n+2)}{\Gamma(m+n+1)}\ \frac{(m-2)!(4n+2m-2)\ (4n+2m-6)\ldots(4n+8)}{(m+1)\ (m+3)\ldots(2m-4)\ (m-3)\ (m-5)\ldots 2}$$

$$C_1=-\frac{\Gamma(n+2)}{\Gamma(m+n+1)}\ \frac{(m-1)!(4n+2m-4)\ (4n+2m-8)\ldots(4n+4)}{(m-2)\ (m-4)\ldots 2(m+2)\ (m+4)\ldots(2m-2)}$$

4. Writing $-x$ for x, we get exactly similar relations involving $D_{-(m+n+1)}(-ix)$, $D_{-(n+1)}(-ix)$ and $D_{-(n+2)}(-ix)$. We now deduce the following interesting relations from (A).

When m is an odd integer,

$$D_{-(m+n+1)}(ix)\, D_{-(n+1)}(-ix) + D_{-(m+n+1)}(-ix)\, D_{-(n+1)}(ix)$$

$$=\frac{\sqrt{2\pi i}\, b_0}{\Gamma(n+2)}\Big[H_0(x)-\frac{(m-1)(m+1)}{2!\,(4n+2m-2)}H_2(x)$$

$$+\frac{(m-1)(m-3)(m+1)(m+3)}{4!\,(4n+2m-2)(4n+2m-6)}H_4(x)-\dots\Big],$$

$$D_{-(m+n+1)}(ix)\, D_{-(n+2)}(-ix) - D_{-(m+n+1)}(-ix)\, D_{-(n+2)}(ix)$$

$$=\frac{\sqrt{2\pi}}{\Gamma(n+2)}A_1\Big[H_1(x)-\frac{(m-3)(m+1)}{3!\,(4n+2m-2)}H_3(x)$$

$$+\frac{(m-3)(m-5)(m+1)(m+3)}{5!\,(4n+2m-2)(4n+2m-6)}H_5(x)-\dots\Big]. \qquad \text{(B).}$$

When m is an even integer,

$$D_{-(m+n+1)}(ix)\, D_{-(n+1)}(-ix) - D_{-(m+n+1)}(-ix)\, D_{-(n+1)}(ix)$$

$$=\frac{\sqrt{2\pi}}{\Gamma(n+2)}\, i\, C_1\Big[H_1(x)-\frac{(m-2)(m+2)}{3!(4n+2m-4)}H_3(x)$$

$$+\frac{(m-2)(m-4)(m+2)(m+4)}{5!\,(4n+2m-4)(4n+2m-8)}H_5(x)-\dots\Big],$$

$$D_{-(m+n+1)}(ix)\, D_{-(n+2)}(-ix) + D_{-(m+n+1)}(-ix)\, D_{-(n+2)}(ix)$$

$$=\frac{\sqrt{2\pi}}{\Gamma(n+2)}A_0\Big[H_0(x)-\frac{(m-2)m}{2!(4n+2m)}H_2(x)$$

$$+\frac{(m-2)(m-4)m(m+2)}{4!(4n+2m)(4n+2m-4)}H_4(x)-\dots\Big]. \qquad \text{(C).}$$

The results are true for all positive values of n.

REARRANGEMENT OF COMPLEX SERIES

by

V. GANAPATHY IYER

I. This is an attempt at an exhaustive classification of infinite series whose terms are complex numbers, according to the manner of distribution of the limiting values of the series when the terms of the series are re-arranged so as to form another series. By the limiting values of the series are meant the limit points of the sequence obtained by taking the sum to first n terms of the series in any particular rearrangement. A definite arrangement of the terms of the series being presupposed, the series converges or diverges according as the sequence has a unique finite or infinite limit; otherwise the series oscillates. In the case of complex series, infinity is said to be the unique limit if the *modulus* of the sum to first n terms of the series diverges to $+\infty$. No distinction is made between $\pm\infty$ or infinity in any other direction. This paper contains a generalisation of a theorem given by Steinitz.[1] The method of proof is practically the same as that given in the paper quoted. A simple geometric proof of the result in Lemma (v) below is given without introducing general considerations on the properties of convex sets of points.

II. Real series: Let $A=\sum_{i=1}^{\infty} a_i$ be a real series. If it converges and no rearrangement alters the convergence or the sum, the series is said to be unconditionally convergent. It is known that the necessary and sufficient condition for unconditional convergence is absolute convergence *i.e.* $\sum_{1}^{\infty} | a_i |$, converges. So we may speak of the two as equivalent. If a series converges but not absolutely then it is said to converge non-absolutely or conditionally. Similarly if A diverges to $\pm\infty$ and no rearrangement disturbs this divergence, the series is said to be absolutely

(1) Steinitz, Bedingt konvergente Reihen und konvexe Systeme. *J. fur die reine und angewandte. Mathematik*, B and 143. (pp. 128—175) § 14.

divergent.(2) With these definitions we may arrive at the classification of real series as follows. It is to be noted that nothing is assumed about the convergence of A in any rearrangement:—

Let $A=\sum_{1}^{\infty} a_n$ be any real series where $a_n \rightarrow 0$ as $n \rightarrow \infty$. Let $B=\Sigma b_n$ and $C=\Sigma c_n$ be the series formed from A by grouping together the positive and negative terms together. Here c_n denotes the numerical value of the corresponding negative term in A. Then three cases can arise:

(i) B and C both converge. Then obviously $\Sigma |a_n|$ converges and A is absolutely convergent.

(ii) One of B and C diverges while the other converges. Then the series diverges to $\pm\infty$(3) and no rearrangement can alter this fact. Hence this series is absolutely divergent.

(iii) Both B and C diverge. In this case the series could be rearranged so as to converge to any real sum or diverge or oscillate finitely or infinitely(4). In this case the series converges conditionally. The usual proof given assumes the convergence of the series A in some rearrangement but this is not essential. The proof(4) depends only on the fact that B and C diverge and $a_n \rightarrow 0$.

III. Complex Series: (1) In the following Greek letters $\alpha, \beta, \gamma \ldots$ will denote complex numbers and $a, b, c \ldots$ real numbers.

Let $\alpha_1, \alpha_2, \alpha_3 \ldots$ be a sequence of complex numbers. By a partial sum A of these numbers is meant the sum of any finite number of these terms picked out arbitrarily. The system of such partial sums will be denoted by Λ. Suppose in any particular arrangement of the sequence, say $\alpha_1, \alpha_2 \ldots, \sum_{1}^{\infty} \alpha_n$ converges. If no rearrangement alters the convergence or the sum, the series is said to be absolutely or unconditionally convergent. Otherwise it is said to be non-absolutely or conditionally convergent. It is easily seen by splitting $\Sigma \alpha_n$ into its real and imaginary parts that the necessary and sufficient condition for absolute or unconditional

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(2) This conception of absolute divergence, which secures completeness in the statement of results, was suggested by Dr. R. Vaidyanathaswami.

(3) A will diverge to $\pm\infty$ if B diverges and C converges; and to $-\infty$ if the reverse be the case.

(4) For a proof, see Bromwich, Infinite series (1908) (pp. 68—69), § 28

convergence is that $\Sigma \mid a_n \mid$ should converge. Similarly if a series diverges *i.e.* the series has infinity as a unique limit as explained in I and no rearrangement alters this fact the series is said to be absolutely divergent.(2)

(2) *Directions of absolute convergence and divergence*:—Any complex number can be regarded as a vector in a plane, its real and imaginary parts being the resolved parts or resolutes of the vector along the axes of reference. Instead of resolving along these axes, one might as well resolve them along any two perpendicular directions. If $a=\rho e^{i\lambda}$ and θ the angle which a st. line makes with the positive direction of the axis of reals, the resolutes in that direction and the perpendicular one are $e^{i\theta} \rho \cos (\lambda-\theta)$ and $i e^{i\theta} \rho \sin (\lambda-\theta)$. If all the terms of a series Σa_n be replaced by two such components we get two new series whose sum is the original series and each of these series is a real series multiplied by a complex factor depending on the direction of resolution. These latter do not affect the behaviour of the series and may be omitted in the discussion of convergence or divergence. Now suppose that the terms of a series be resolved in any direction and that the corresponding resolved series converges or diverges absolutely. Then that direction may be called a direction of absolute convergence or divergence(2) respectively. With this conception, the various cases that can occur may be analysed as follows:—

(i) There may be a direction of absolute divergence.

(ii) There may be two distinct directions of absolute convergence. In this case the series converges absolutely and every direction is a direction of absolute convergence.

(iii) There is only one direction of absolute convergence and no direction of absolute divergence.(5) In this case, if the series converges, the resolved part in the direction perpendicular to the direction of absolute convergence, is a conditionally convergent series and so could be rearranged by II (iii) so as to have different sums or oscillate.

(iv) There is no direction of absolute convergence or divergence. This is the case that will be examined in the sequel.

(3) *The nature of the partial sums along different directions*:—Let $a_1, a_2, \ldots$ be resolved along any direction and partial sums (in the sense explained in *III* (1)) of these resolved parts

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(5) That absolute convergence and divergence can occur together along different directions is shown below by en example (see *III* (6) (i))

be formed; the following cases corresponding to the four discussed in *III* (2) (i)—(iv), may be distinguished:—

(i) There is a direction of absolute divergence. Using *II* (ii) we see that the partial sums have a bound in one way and no bound in the opposite way along this direction of absolute divergence,

(ii) When there are two distinct directions of absolute convergence, the partial sums along these are bounded either way by *II* (*i*) and mereover since the series converges absolutely along every direction, the partial sums are bounded either way along every direction.

(iii) If there is no direction of absolute divergence but one of absolute convergence, then the partial sums along the latter are bounded either way but the partial sums along every other direction are unbounded both ways.

(iv) When there is no direction of absolute convergence or divergence, the partial sums are unbounded both ways along every direction (in virtue of *II* (iii)). In this case the partial sums of Σa_n are said to be totally unbounded. It means that if any direction be taken along with the perpendicular one as the axes of reference, there are partial sums having arbitrarily large positive and negative x- or y- co-ordinates. Or if any st. line be drawn in the complex plane there are partial sums of Σa_n lying on either side of that line, This is the case that occurs in the hypothesis of the main theorem.

(4) *The main theorem*:—Let $a_1, a_2, a_3 \ldots\ldots$ be a sequence of complex numbers. Let $m = \underset{n \to \infty}{Lim} \mid a_n \mid$. Let there be no direction of absolute convergence or divergence for the series $\overset{\infty}{\Sigma} a_n$. Let σ be any point of the complex plane and C_m (σ) a circle of radius km and centre σ. Then there is a particular rearrangement of Σa_n such that all its limit points lie within or on C_m (σ). Here k is an absolute constant an upper bound for which is 10.

It is to be noted that if $m = o$, the theorem becomes that such a series could be rearranged so as to converge to σ since then C_m (σ) is merely the point σ. Here, as in II for real series, nothing is assumed as regards the convergence of Σa_n in any rearrangement.

4 (1) The proof of the main theorem depends on a series of Lemmas(6) which depend *only* on the nature of a complex number as a two dimensional vector.

Lemma (i): If $\gamma = a_1\alpha_1 + \ldots + a_p\alpha_p$ $(o<a_i<1,\ i=1,\ 2;\ p)$, then γ could be expressed by a relation of the same form in which not more than two of the co-efficients $a_1, \ldots a_p$ are different from 0 or 1.

Proof: Let $\delta = a_1\alpha_1 + a_2\alpha_2 + a_3\alpha_3$.

Since $\alpha_1, \alpha_2, \alpha_3$ are three two-dimensional vectors we have a relation,

$$b_1\,\alpha_1 + b_2\,\alpha_2 + b_3\,\alpha_3 = o \text{ where the } b\text{'s are real.}$$

If one of a_1, a_2, a_3 be 0 or 1 then δ is already in the form in which not more than two are different from zero or one. So we may suppose $0<a_i<1$ $(i=1, 2, 3)$. Then

$$\delta = (a_1 - b_1 x)\,\alpha_1 + (a_2 - b_2 x)\,\alpha_2 + (a_3 - b_3 x)\,\alpha_3.$$

If now x be equal to the least positive or the greatest negative of the 6 numbers $\frac{a_i}{b_i}$, $\frac{a_i - 1}{b_i}$ $(i=1,\ 2,\ 3)$ which are all different from zero, we have one of the co-efficients zero while all the others lie between 0 or 1.

Now if in $\gamma = a_1\,\alpha_1 + \ldots + a_p\,\alpha_p$ there are h co-efficients $(2<h\leqslant p)$ different from zero or 1, we may suppose that these are $a_1, \ldots, a_h$ without loss of generality. Reduce $a_1\,\alpha_1 + a_2\,\alpha_2 + a_3\,\alpha_3$ to a form in which one co-efficient is zero or one. This is possible by what has been proved above. Then in γ, h becomes $h-1$. So we can arrive at the desired result in a finite number of steps.

Lemma (ii) If $a_1\alpha_1 + \ldots + a_p\alpha_p = 0$ $(0\leqslant a_t \leqslant 1)$ and not more than four of the co-efficients be different from 1, then there is a similar relation (if $p>4$) in which one of the numbers $\alpha_1, \ldots \alpha_p$ is absent.

We may suppose without loss of generality that a_1, a_2, a_3, a_4 are these co-efficients different from unity. If any of these be zero, the lemma is proved. So we may suppose $0<a_i<1$, $(i=1, 2, 3, 4)$. Then by the previous *Lemma* $a_1\alpha_1 + a_2\alpha_2 + a_3\alpha_3 + a_4\alpha_4$ could be reduced to a form in which not more than two are different from 0 or 1. If one of the remaining coefficients

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(6) Lemmas (i) to (iv) are contained in Steinitz's paper quoted in (1) Refer; § 15 (pp. 167), Hilfssatze: 1, 2, 4 and II. Lemma (v) has no analogue in Steinitz's paper though the result is deducible from various theorems established there. Lemma (vi) is his Hilfssatze 3.

be actually 0 then the Lemma is proved. Otherwise all the co-efficients excepting two are different from 1. So we may suppose in the original relation itself that not more than two are different from 1. Let now $\gamma = a_1\alpha_1 + a_2\alpha_2 + a_3\alpha_3 + a_4\alpha_4$, where $a_i > 0$.

Since $\alpha_1 - \alpha_4$, $\alpha_2 - \alpha_4$, $\alpha_3 - \alpha_4$ are three complex numbers, there is a relation

$$b_1(\alpha_1 - \alpha_4) + b_2(\alpha_3 - \alpha_4) + b_4(\alpha_3 - \alpha_4) = 0, \quad b\text{'s real}$$

or $b_1\alpha_1 + b_2\alpha_2 + b_3\alpha_3 + b_4\alpha_4 = 0.$ Where $\sum_1^4 b_i = 0.$

Therefore $\gamma = \sum_{i=1}^{4} (a_i - b_i\, x)\, \alpha_i.$

Since $\sum_1^4 b_i = 0$ and all the b_i's are not zero, there are positive as well as negative b's. If x be equal to the least value of $\frac{a_i}{b_i}$ where only such b_i's as are positive are counted we see that in γ one co-efficient is zero while the others are not negative, and the sum of the co-efficients remains unaltered.

Now the original relation may without loss of generality be written as $a_1\alpha_1 + a_2\alpha_2 + a_3\alpha_3 + a_4\alpha_4 + a_5\alpha_5 + \ldots + a_p\alpha_p = 0.$ When $0 < a_i < 1$ $(i = 1, 2)$ while $a_i = 1$, $(2 < i < p.)$

Dividing by $\sum_1^p a_i$ we get $\sum_1^p c_i\alpha_i = 0$ where $\sum_1^p c_i = 1$, $c_i > 0.$

Now by what has been proved above, this can be successively reduced till only 3 of $(\alpha_1, \ldots \alpha_p)$ are left in which the co-efficient are all non-negative while their sum is still $\sum_1^p c_i = 1.$

We may without loss of generality suppose that this relation is $c_3\alpha_3 + c_4\alpha_4 + c_5\alpha_5 = 0$, $c_i \geqslant 0$ while $c_3 + c_4 + c_5 = 1$ so that $0 \leqslant c_i \leqslant 1$, $i = 3, 4, 5.$

Therefore we have

$$a_1\alpha_1 + a_2\alpha_2 + (a_3 - c_3 x)\alpha_3 + (a_4 - c_4 x)\alpha_4 + (a_5 - c_5 x)\alpha_5 + a_6\alpha_6 + \ldots\ldots\ldots + a_p\alpha_p = 0.$$

If now x be equal to the least among $\frac{a_3}{c_3}, \frac{a_4}{c_4}, \frac{a_5}{c_5}$ (the fraction corresponding to any $c_i = 0$ being omitted), we see that one of the the co-efficients of $\alpha_3, \alpha_4, \alpha_5$ vanish and consequently not more than 4 are different from 1 while one of the numbers $\alpha_1, \ldots \alpha_p$ is absent. Hence the Lemma is proved.

Lemma (iii). Let $a_1+a_2+a_3+\ldots\ldots+a_p=0$. Then a rearrangement $\beta_1,\ldots\beta_p$ of the a's can be found such that the successive sums $|\beta_1|$, $|\beta_1+\beta\ |,\ldots|\beta_1+\ldots+\beta_p|$ are all less than $4\ m$ where m is the upper bound of the numbers $a_1,\ldots a_p$.

Proof: All the co-efficients in $a_1+\ldots+a_p=0$ are 1. Hence by *Lemma* (ii) a similar relation could be found in which one, say β_p is absent, while not more than four are different from 1 and these four lie between 0 and 1. If $p-1>4$ we can proceed in this manner and pick out $\beta_p,\ \ldots\beta_5$ such that if $\beta_1,\beta_2,\beta_3,\beta_4$ be these left we have $a_{k_1}\beta_1+a_{k_2}\beta_2+a_{k_3}\beta_3+a_{k_4}\beta_4+\beta_5+\ldots+\beta_k=0$ where $k=5,6,\ldots p$ and $0\leqslant a_{k_i}\leqslant 1$, $i\leqslant 1\ldots 4$.

$$\text{Hence } |\beta_1+\ldots+\beta_k|\ \leqslant\sum_{i=1}^{4}(1-a_{k_i})\ |\beta_i|\ \leqslant 4m;\ k=1,2,$$

.. p. So the Lemma is proved.(7)

Lemma (iv). If $\gamma=a_1+\ldots+a_p$ and $|\gamma|\leqslant h\ m$ where h is an integer and m the upper bound of $|ai|$ $(i=1\ldots p)$, then a rearrangement of $a_1\ldots a_p$ say $\beta_1,\ldots\beta_p$ exists such that $|\beta_1|$, $|\beta_1+\beta_2|$, $\ldots|\beta_1+\ldots+\beta_p|$ are all less than $(h+4)\ m$.

Proof: We have $a_1+\ldots+a_p-\frac{\gamma}{h}-\frac{\gamma}{h}-\ldots$ (h times) $=0$. Therefore by *Lemma* (iii) we can rearrange these $p+h$ quantities so that the absolute value of the successive sums is less than $4m$. But each of these sums, besides the a's will involve at most h numbers each equal to $-\frac{a}{h}$. Hence if these are omitted the remaining sums are a rearrangement of $a_1,\ldots a_p$ such that their absolute values are less than $4\ m+\left|\frac{\gamma}{h}\right|\ h$ which is less than $(h+4)\ m$.

Lemma (v). Let $a_1,\ a_2,\ a_3\ldots$ be a sequence of complex numbers. Let there be no direction of absolute convergence or divergence for Σa_n so that the partial sums are totally unbounded *i.e.* both the half-planes into which any st. line divides the plane contain partial sums from the system Λ(8). Then the system of numbers γ where $\gamma=a_1a_1+a_2a_2+\ldots+a_pa_p$ $(0\leqslant a_i\leqslant 1,\ i=1,2,\ldots p)$ where $p=1,2,3\ldots$—include all points of the complex plane.

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(7) The $\beta_1,\beta_2,\beta_3,\beta_4$ which are those left over at each stage whose co-efficients may not be one, need not be the same for all $k=1,2,3,\ldots p$.

(8) Refer III (1).

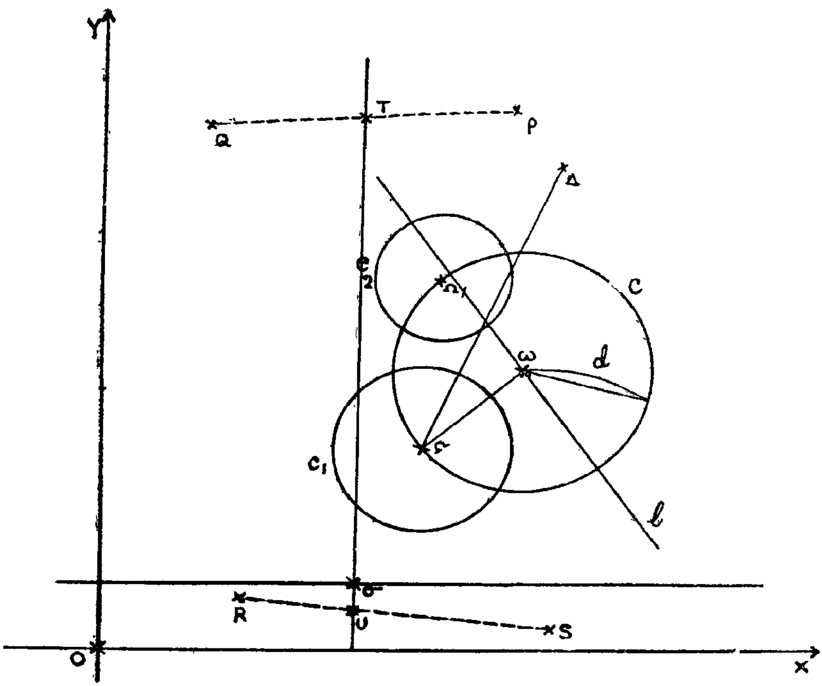

FIG. I.

Proof: This Lemma contains the kernel of the proof of the main theorem. First we may note a property of the system of numbers γ. Let γ_1 and γ_2 be any two numbers of the system. Then any point on the segment of the line joining (γ_1, γ_2) is also a point of the system. For Let

$$\left.\begin{array}{l}\gamma_1 = a_1\alpha_1 + \ldots + a_p\alpha_p \\ \gamma_2 = b_1\alpha_1 + \ldots + b_k\alpha_k\end{array}\right\} \quad \left.\begin{array}{l}(0 \leqslant a_i \leqslant 1)\ (i=1, 2 \ldots p) \\ 0 \leqslant b_i \leqslant 1 \quad (i=1, 2 \ldots k)\end{array}\right\}$$

and say $p \geqslant k$. Let $0 < h < 1$. Then

$$h\gamma_1 + (1-h)\gamma_2 = \sum_{i=1}^{k} [a_i h + b_i(1-h)]\alpha_i + \sum_{k+1}^{p} h a_i \alpha_i$$

and since $0 \leqslant a_i h + b_i(1-h) \leqslant h + 1 - h = 1$ and $0 \leqslant h\, a_i \leqslant 1$, we see that $h\gamma_1 + (1-h)\gamma_2$ is a point of the system. But as h describes $(0, 1)$, $h\gamma_1 + (1-h)\gamma_2$ describes the segment joining γ_1 to γ_2.

Now suppose, if possible, that σ is a point not belonging to γ (*see* Fig. I). Draw parallels to the x and y axes through σ. Then at least one of the quadrants into which the plane is thus divided must contain no point of the system γ. For if P, Q, R, S are such points in the first,... fourth quadrant respectively, we see that PQ, RS will cut the parallel to the Y axis in T and U

which will include σ between them. But T and U are points of γ by the property proved above and so σ itself would be a point of γ which is against hypothesis. Let Q_1 be the quadrant free from points of γ. If ω be any point interior to Q_1, then the distance, d, of ω from points of γ is a positive number. Let C be a circle of radius d and centre ω. Then C will have on its circumference at least one point Ω which is either a point of γ or a limit point of points of γ. There cannot be another such point Ω_1; for if there were, we could draw small circles C_1 and C_2 round Ω and Ω_1 such that the line joining any two points, one in each of these circles, will always cut C. Since C_1 and C_2 will contain points of γ however small their radii be, it would follow that C contains points of γ which is impossible since d is the lower bound of the distances of ω from points of γ. So there cannot be another point Ω_1. Let now l be a line perpendicular to the line $(\Omega\omega)$ and passing through ω. Then there cannot be a point of γ on that side of the line l which does not contain Ω. For if Δ is such a point, we can draw a circle C_1 round Ω so small that the join of any point inside C_1 to Δ cuts C and since C_1 contains points of γ however small its radius be, it would follow as above that C contains a point of γ which is again impossible. Hence there cannot be any such point. But this contradicts the hypothesis of total unboundedness of the partial sum of Λ since obviously any partial sum in Λ is a point of the system γ. Hence there cannot be such a point as σ. So the system γ include all points of the plane.

Lemma (vi). Let $a_1, a_2, \ldots\ldots$ be a sequence satisfying the condition of *Lemma* (v). Then given any point σ in the complex plane, there exists a partial sum A from Λ such that $|\sigma - A| \leqslant 2m$ where m is the upper bound of $|a_i|$ $(i=1, 2\ldots.)$. Further this partial sum could be so chosen as to include any given number from $a_1, a_2\ldots\ldots$

Proof: By *Lemma* (v), there is a p such that

$$\sigma = \alpha_1 a_1 + \ldots\ldots + \alpha_p a_p \ (0 \leqslant \alpha_i \leqslant 1).$$

By *Lemma* (i) there is a similar relation for σ in terms of $(a_1, \ldots a_p)$ in which not more than two co-efficients differ from zero or one. If these two numbers whose co-effieients are not zero or one be omitted, the remaining is a partial sum A of Λ and we have

$$|\sigma - A| \leqslant 2m.$$

If any particular element a_i say, should be included, we omit it from the original sequence. The rest $a_2, a_3\ldots$ also satisfy the condition of the *Lemma*. And so there is a partial sum A_1

such that $|(\sigma - a_1) - A^1| \leqslant 2m$. If $A = a_1 + A^1$ we get the desired results

(4·2). *Proof of the main theorem*: By hypothesis, the sequence $a_1, a_2 \ldots$ satisfy the condition of *Lemma* (v) or (vi). If a finite number of the a's be omitted, the remaining also satisfy the condition. Let m_k be the upper bound of $|a_k|, |a_{k+1}| \ldots$ so that $m_k \rightarrow m$ as $k \rightarrow \infty$ and $m_k \geqslant m_{k+1}$.

In order to avoid the multiplicative axiom, we arrange the partial sums of Λ as follows. Let A be any partial sum. If a_k occurs in it, substitute 2^k in its place and add up for all the terms occurring in A. The result is an integer which may be associated with A and be called the rank of A. Since any integer could be expressed in one and only one way in the form $a_0\, 2^p + \ldots\ldots + a_p$ where the a's are 0 or 1, every partial sum has only one rank and every integer is the rank of one and only one of the partial sums of Λ. Thus we may properly speak of the n^{th} partial sum.

By *Lemma* (vi) we can final a partial sum A which includes a_1 such that $|\sigma - A| \leqslant 2\, m_1$ Let π, be the first partial sum having this property. Then $|\sigma - \pi_1| \leqslant 2\, m_1$. Let, now, the terms of π_1 be omitted from $a_1, a_2, \ldots$ and let the partial sums of the remaining a's be arranged as before. The upper bound of the remaining terms is less than or equal to m_2 and let π_2 be the first partial sum which contains the first term of these remaining a's and has the property $|(\sigma - \pi_1) - \pi_2| \leqslant 2\, m_2$. In this way we can continue and arrive at the result that $|\sigma - (\pi_1 + \ldots + \pi_k)| \leqslant 2\, m_k$ where π_k is the first partial sum which includes the first term of the a's when the terms contained in $\pi_1, \ldots \pi_{k-1}$ are omitted from them. Now

$$|\pi_k| = \left| \left\{ \sigma - (\pi_1 + \ldots + \pi_{k-1}) \right\} - \left\{ \sigma - (\pi_1 + \ldots + \pi_k) \right\} \right|$$
$$\leqslant 2\, m_{k-1} + 2\, m_k \leqslant 4\, m_{k-1}.$$

Let π_k contain p_k terms say $a_{k_1}, a_{k_1}, \ldots a_{kp_k}$. Since $|\pi_k| \leqslant 4\, m_{k-1}$ we can by *Lemma* (iv) so rearrange these a's such that the successive sums are all less than $8\, m_{k-1}$ in absolute value. Here also we may consider the $p_k\,!$ re-arrangements and take the first which has the property stated. We may suppose that terms of π_k so rearranged are $a_{k_1} + a_{k_2} + \ldots + a_{kp_k}$. Then we obtain a series

$$a_{1_1}+\ldots\ldots+a_{1_{p_1}}+a_{2_1}+\ldots\ldots+a_{2_{p_2}}+\ldots\ldots+a_{k_1}+\ldots+a_{k_{p_k}}+\ldots$$

where if S_n denotes the sum to n terms and $k_1 \leq n \leq k_{p_k}$,

$$|\sigma - S_n| \leq |\sigma - (\pi_1+\ldots+\pi_{k-1})| + |a_{k_1}+a_{k_2}+\ldots+a_n|$$
$$\leq 2\, m_{k-1} + 8\, m_{k-1} = 10\, m_{k-1}.$$

And as $n \to \infty$, $k \to \infty$ and $m_{k-1} \to m$ we have

$\overline{\lim\limits_{n\to\infty}} |\sigma - S_n| \leq 10\, m$ from which the result stated in the main theorem follows.

(5) *The case $m=0$*:—If $m=0$, that is the terms of the series form a null sequence, the main theorem states that the series, under the conditions of the theorem, could be rearranged so as to converge to any number σ in the complex plane. This result enables us to give a complete classification of complex series whose terms tend to zero:—

(i) There is a direction of absolute divergence in any particular rearrangement. Then however rearranged this direction will continue to be one of absolute divergence and the complex series has the unique limit infinity. The complex series is absolutely divergent.[2]

(ii) There is no direction of absolute divergence[9] but two distinct directions of absolute convergence. Here the series converges absolutely, every direction is one of absolute convergence and the series has a unique finite limit however rearranged.

(iii) There is no direction of absolute divergence[10] but one and only one direction of absolute convergence. Here for all rearrangements this direction of absolute convergence is preserved, and since there is no direction of absolute divergence and no second direction of absolute divergence, the complementary part of the series perpendicular to this direction of absolute convergence must be a real series (except for a constant complex factor) which falls under case (iii), II. So this part could be rearranged so as to have any number in that direction as sum. Hence the limit points of the series for all rearrangement lie on a st. line perpendicular to the direction of absolute convergence at a distance along the latter direction equal to the sum of the absolutely convergent part. Here it is to be noted that since no rearrangement whatever affects the absolutely convergent part, not only convergent rearrangements of the complex series but also all possible rearrangements will have all their limit points on the

(9) This is excluded by the hypothesis of absolute convergence in two directions and so is superfluous.

(10) *See* foot-note (5).

st. line specified. In this case the series may be said to have a st. line sum.

(iv) There is no direction of absolute convergence or divergence. This is the case occurring in the hypothesis of the main theorem and here $m=0$. Consequently the complex series could be rearranged so as to converge to any number in the complex plane. It is unnecessary to state that for all rearrangements the limit points lie in the complex plane. What is to be noted is that each point is the sum of a particular rearrangement of the terms of the given sequence. In this case the series may be said to have a plane sum.

These four cases are mutually exclusive and thus give a comprehensive classification of series formed out of a null sequence.

(5·1) If $m \neq 0$ but finite, the main theorem states that there is a particular rearrangement of the series all of whose limit points lie within a circle of radius k_m round any given point σ of the complex plane. Here the hypothesis is that there is no direction of absolute convergence or divergence. Proceeding to the general case when $m \neq 0$, *firstly* it is evident that the series can never be rearranged so as to have a unique finite limit. *Next* it may happen that finite limit points lie on a st. line. But we cannot assert that the limit points fill up the plane. All the same, if all the limit points lie on a st. line it is obvious that the direction perpendicular to this line is one of absolute convergence and conversely. In this case the limit points may or may not fill up the whole line. *Thirdly* the limit points may not be confined to a single line. In this case if there are finite limit points, there cannot be a direction of absolute convergence or divergence and so by the main theorem the limit points must be distributed all over the plane and cannot be confined to even a finite number of st. lines. But here also we cannot assert that the limit points fill up the plane, They may or may not. Thus an exhaustive classification of the type given for $m=0$ is impossible when $m \neq 0$. Examples to illustrate the various possibilities are given below. It is also shown that in some cases when $m=\infty$, the main theorem may be utilised by proper grouping of the terms of the series to give some information regarding their limit points.

(6) (i) Example to show that directions of absolute convergence and divergence may occur together:—Let $a_n = 1 + \frac{i}{n^2}$.

The real part $1+1+1+\ldots\ldots$ is absolutely divergent while the other $\Sigma \frac{1}{n^2}$ is absolutely convergent.

(ii) Example to show that the limit points, when $m \neq 0$ may all lie on a line but need not fill it:—$a_n = (-1)^{n-1} + \frac{i}{n^2}$. Here $m=1$. Also $\Sigma \frac{1}{n^2}$ is absolutely convergent and all the limit points lie on the line $y = \sum_1^\infty \frac{1}{n^2} = \frac{\pi^2}{6}$. But the real part is $1-1+1-1+\ldots\ldots$ If in any particular rearrangement S_n, the sum to n terms, contain p_n and q_n positive and negative terms, $S_n = p_n - q_n$ and so S_n can take only integral values or the value zero. And by properly choosing p_n and q_n we can make any given integer a limit point of S_n. Hence the limit points are $\lambda + \frac{\pi^2}{6} i$ where λ is an integer and so do not fill up the whole line $y = \frac{\pi^2}{6}$.

(iii) Example to show that limit points may lie on a line and may fill up the whole line:—Let $a_{2n-1} = 1 + \frac{i}{(2n-1)^2}$, $a_{2n} = -\theta + \frac{i}{(2n)^2}$ where θ is a positive irrational number. Here also all the limit points lie on the line $y = \frac{\pi^2}{6}$. But the real part is $1-\theta+1-\theta+\ldots$ and using the notation of the previous example $S_n = p_n - q_n\,\theta$. But if θ is irrational it is known that by proper choice of the sequence p_n and q_n, $p_n - q_n\,\theta$ may be made to tend to any real number as limit. Hence all points of the line $y = \frac{\pi^2}{6}$ are limit points.

(iv) Example to show that the limit points do not lie on a line but do not fill up the plane:—$a_n = (-1)^{n-1}\left(1+\frac{i}{n}\right)$. Here $m=1$. There is no direction of absolute convergence or divergence. For the resolved part along the direction θ is, but for a constant factor, equal to $\Sigma (-1)^{n-1}\left(\cos\theta + \frac{i \sin\theta}{n}\right)$. If $\theta \neq \frac{\pi}{2}$, the positive and negative parts diverge separately and so there are not directions of absolute divergence or convergence.

If $\theta=\frac{\pi}{2}$, $\Sigma \frac{(-1)^{n-1}}{n}$ converges but not absolutely. Hence there is no direction of absolute convergence or divergence. Hence by the main theorem the limit points do not all lie on a line but are distributed all over the plane. But the real part is $1-1+1-1+\ldots.$ which can only have an integer as a limit point. Hence all the limit points lie on the lines $x=\lambda$ where λ is zero or an integer and so do not fill up the whole plane.

(v) Example to show that the limit points even if $m\neq 0$, may fill up the whole plane:—

First let $a_n=(-1)^{n-1}\left(\frac{1}{n}+\frac{i}{\log n}\right)$. Here $|a_n|\to 0$ and it is easily verified as in (iv) that there is no direction of absolute convergence or divergence. Hence by III (5) (iv), the series could be rearranged so as to have any point σ for its sum. Let now $\beta_n-\beta_{n-1}=(-1)^{n-1}\left(\frac{1}{n}+\frac{i}{\log n}\right)=a_n$. Then $\beta_n=1-\left(\frac{1}{2}+\frac{i}{\log 2}\right)+\ldots\ldots+(-1)^{n-1}\left(\frac{1}{n}+\frac{i}{\log n}\right)$ if $\beta_1=1$. So β_n tends to a finite limit which is obviously not zero. Consider now the series $C_1+C_2+\ldots\ldots+C_n\ldots\ldots$ where $C_{2n}=-\beta_n$, $C_{2n-1}=\beta_{n+1}$ so that $C_{2n-2}+C_{2n-3}=a_n$. Now writing $\Sigma\, C_n$ in the form $\Sigma\,(C_{2n-3}+C_{2n-2})=\Sigma\,(\beta_n-\beta_{n-1})=\Sigma\, a_n$ and rearranging $\Sigma\, a_n$ so as to converge to σ we see that if now a_n be replaced by $C_{2n-3}+C_{2n-2}$, a rearrangement of ΣC_n is obtained for which σ is a limit point and since $\overline{\operatorname{Lim}}_{n\to\infty}|C_n|$ is the same as $\overline{\operatorname{Lim}}_{n\to\infty}|\beta_n|$ which is not zero we have the desired example.

(vi) By the method of grouping employed in example (v), we can sometimes obtain information regarding the limit points of series whose terms are not bounded. For example, consider the series

$$1-2+3-4+5-6+\ldots.$$

Here $a_n=(-1)^{n-1}n$ and $|a_n|=n\to\infty$ as $n\to\infty$. Let $\beta_{2n-1}=a_{3n-2}+a_{3n-1}$ and $\beta_{2n}=a_{6n-3}+a_{6n}$, $n=1, 2,\ldots\ldots.$

Then $\beta_{2n-1}=(-1)^n$, $\beta_{2n}=-3$. Here $m=3$ for the series $\Sigma\,\beta_n$. Let in any rearrangement of $\Sigma\,\beta_n$, there be p positive and q

negative terms in S_n, the sum to first n terms. Let there be further r ones and s threes in q. Then $S_n = p - (\gamma + 3s)$. By proper choice of the sequence p, r, s, S_n may be made to have any integer for its limit and if in that rearrangement the β's be replaced by α's we have a rearrangement of the original series for which any given integer is a limit point. Further for the given series there cannot be any other limit points than integers. Hence $1-2+3-4+5-6+\ldots\ldots$ could be rearranged so as to have any integer for one of its limits. Hence given any integer, it can be expressed in an infinity of ways as the difference of two sums, one sum those of even and the other those of odd integers, the numbers of these odd and even integers exceeding any given limit whatever.

GENERALISATION OF JACOBI'S θ—FUNCTION FORMULAE.

By K. VENKATACHALIENGAR.

Jacobi's well known formulae on the multiplication of θ—functions can be generalised by making use of general orthogonal linear substitution. The generalised theorem runs as follows:—

If variables $(l, m, n, \ldots\ldots p)$ and $(l_1, m_1, n_1, \ldots\ldots p_1)$ are connected by means of the relations

$$\text{(i)}\begin{cases} a_{1_1} l + a_{1_2} m + \ldots\ldots a_{1_p} p = a_{1_1} l_1 + a_{2_1} m_1 + \ldots a_{p_1} p_1 \\ a_{2_1} l + a_{2_2} m + \ldots\ldots a_{2_p} p = a_{1_2} l_1 + a_{2_2} m_1 + \ldots a_{p_2} p_1 \\ \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \\ a_{p_1} l + a_{p_2} m + \ldots\ldots a_{p_p} p = a_{1_p} l_1 + a_{2_p} m_1 + \ldots a_{p_p} p_1 \end{cases}$$

where $a_{i_k} = - a_{k_i}$. Then the relation between the sets of variables is orthogonal. Moreover let the coefficients involved be all integers positive or negative. Let k be the value of the determinant

$\left| a_{1_1}, a_{2_2} \ldots a_{p_p} \right|$ and let $k \neq 2$ and let the orthogonal relation be written in the form

$$l = a_{1_1} l_1 + a_{2_1} m_1 + a_{3_1} n_1 + \ldots\ldots + a_{p_1} p_1$$
$$m = a_{1_2} l_1 + \ldots\ldots\ldots\ldots\ldots\ldots + a_{p_2} p_1$$
$$p = a_{1_p} l_1 + \ldots\ldots\ldots\ldots\ldots\ldots + a_{p_p} p_1$$

Then let the complete systems of solutions of the set of linear equations

$$\text{(A)} \quad a_{r_1} l + a_{r_2} m + a_{r_3} n + \ldots a_{r_p} p = o \quad [mod\ k]$$

$r = 1, 2, 3 \ldots\ldots p$, be given by
$(\lambda_1, \lambda_2, \lambda_3 \ldots \lambda_p)$, $(\mu_1, \mu_2, \ldots \mu_p)$ etc., where
$l = \lambda_1 \ (mod\ k), m = \lambda_2 \ (mod\ k), \ldots p = \lambda_p \ (mod\ k).$

and

$$l = \mu_1 \ (mod\ k),\ m = \mu_2 \ (mod\ k), \ldots p = \mu_p \ (mod\ k)$$

and so on.

And let the corresponding systems of solutions of

(B) $a_{1_r} l_1 + a_{2_r} m_1 + a_{3_r} n_1 + \ldots\ldots + a_{p_r} p_1 = o \quad [\mathit{mod}\ k]$

$r = 1, 2, \ldots\ldots p$, be given by

$(\lambda'_1, \lambda'_2, \lambda'_3 \ldots\ldots \lambda'_p)$, $(\mu'_1, \mu'_2, \mu'_3 \ldots\ldots \mu'_p)$ and etc.,

and let

$$f_{\lambda_1}(x) = \sum_{r=-\infty}^{r=+\infty} q^{l^2/k^2} e^{\frac{2ilx}{k}}, \quad f_{\lambda_2}(x) = \sum_{r=-\infty}^{r=+\infty} q^{m^2/k^2} e^{\frac{2imx}{k}},$$

and etc.,

where $l = rk + \lambda_1$, $m = rk + \lambda_2, \ldots\ldots$ and etc.,

and $q = e^{i\pi\tau}$ where the imaginary part of τ is $+ve$ and let

$$x_1 = a_{1_1} x + a_{2_1} y + a_{3_1} z + \ldots\ldots$$
$$y_1 = a_{1_2} x + a_{2_2} y + a_{3_2} z + \ldots\ldots$$
$$z_1 = a_{1_3} x + a_{2_3} y + a_{3_3} z + \ldots\ldots$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

then my generalisation takes the form

$$\sum_{(\lambda, \mu, \ldots\ldots)} f_{\lambda_1}(x) \quad f_{\lambda_1}(y) \ldots\ldots\ldots\ldots = \sum_{(\lambda', \mu', \ldots\ldots)} f_{\lambda'_1}(x_1) \quad f_{\lambda'_2}(y) \ldots\ldots$$

and it is also shown in the paper how each of these functions can be expressed in terms of θ-functions. One of the particular results takes the very elegant form viz., if $x_1 = \frac{1}{3}(-x+2y+2z)$, $y_1 = \frac{1}{3}(2x-y+2z)$, $z_1 = \frac{1}{3}(2x+2y-z)$,

then

$$\sum_{r=0}^{2} q^{r^2/3} e^{2rix} \theta_3\left(x + \frac{\pi r \tau}{3}\right) \theta_3\left(y + \frac{\pi r \tau}{3}\right) \theta_3\left(z + \frac{\pi r \tau}{3}\right)$$

remains unaltered by the substitution of x_1, y_1, z_1 in the place of x, y, z.

Now first of all it is necessary to show that the substitution (i) is orthogonal.

Now multiply the equations by the minors $\left[A_{1_1}, A_{2_1}, \ldots A_{p_1}\right]$ respectively of $\left[a_{1_1}, a_{2_1}, \ldots\ldots a_{p_1}\right]$ respectively in the determinant $\left| a_{1_1}, a_{2_2}, \ldots\ldots a_{p_p} \right|$ and adding we obtain

$$kl = \left(a_{1_1} A_{1_1} + a_{1_2} A_{2_1} + \ldots\ldots a_{1_p} A_{p_1}\right) l_1$$
$$+ \left(a_{2_1} A_{1_1} + a_{2_2} A_{2_1} + \ldots\ldots a_{2_p} A_{p_1}\right) m_1 \ldots\ldots$$

$= (2a_{1_1}A_{1_1} - k)\, l_1 + 2a_{2_2}A_{2_1}\, m_1 + 2a_{3_3}A_{3_1} n_1 + \ldots + 2a_{p_p}A_{p_1}\, p_1$

with corresponding expressions for $m, n, \ldots p$ and if we multiply by $(A_{1_1}, A_{1_2}, \ldots A_{1_p})$ and add we obtain similarly $kl_1 = (2a_{1_1}\, A_{1_1} - k)\; l + 2\, a_{2_2}\; A_{1_2} m \ldots 2\, A_{1_p}\; p\, a_{p_p}$ and with corresponding expressions for $m_1, n_1, \ldots\ldots p_1$. Therefore the relation is orthogonal.

Let k be an integer other than 2 and all the a's denote positive or negative integers less than k in absolute value.

Let the orthogonal relation obtained in the preceding paragraph be written in the form

$$l = a_{1_1}\; l_1 + a_{2_1}\; m_1 + \ldots\ldots a_{3_1}\; n_1 + \ldots\ldots$$
$$m = a_{1_2}\; l_1 + a_{2_2}\; m_1 + \ldots\ldots a_{3_2}\; n_1 + \ldots\ldots$$
$$\vdots$$

Now, let l_1, m_1, $n_1 \ldots\ldots$ and etc. take the complete sets of integral values positive or negative satisfying the relations

$$(A)\begin{cases} a_{1_1}\; l + a_{1_2}\; m + \ldots\ldots a_{1_p}\; p = o \pmod k \\ a_{2_1}\; l + a_{2_2}\; m + \ldots\ldots a_{2_p}\; p = o \pmod k \\ \vdots \\ a_{p_1}\; l + a_{p_2}\; m + \ldots\ldots a_{p_p}\; p = o \pmod k \end{cases}$$

and then $(l_1, m_1, n_1 \ldots .)$ will also be integers satisfying

$$(B)\begin{cases} a_{1_1}\; l_1 + a_{2_1}\; m_1 + \ldots\ldots a_{p_1}\; p_1 = o \pmod k \\ a_{1_2}\; l_1 + a_{2_2}\; m_1 + \ldots\ldots a_{p_2}\; p_1 = o \pmod k \\ \vdots \\ a_{1_p}\; l_1 + a_{2_p}\; m_1 + \ldots\ldots a_{p_p}\; p_1 = o \pmod k \end{cases}$$

and *vice versa.*

It is also obvious that if $(l, m, n \ldots .)$ take all integral values positive or negative consistent with relations (A) then $(l_1, m_1, n_1, \ldots .)$ take all the integral values consistent with relations (B) and *vice versa*. And corresponding to one set of values $(l, m, n \ldots . p)$ there is one and only one set of values $(l_1, m_1, n_1 \ldots . p_1)$.

Let the complete system of values $(l, m, n, \ldots . p)$ satisfying the relations (A) be denoted by $(\lambda_1, \lambda_2, \lambda_3 \ldots .)$, $(\mu_1, \mu_2, \mu_3 \ldots)$ and etc., where

$$l = \lambda_1 \ (mod\ k)\ ; \quad m = \lambda_2 \ (mod\ k) \ldots\ldots \quad p = \lambda_p \ (mod\ k)$$
$$l = \mu_1 \ (mod\ k)\ ; \quad m = \mu_2 \ (mod\ k) \ldots\ldots p = \mu_p \ (mod\ k)$$

and the complete system of values consistent with the relations (B) be denoted by the same letters with dashes, viz., $(\lambda'_1, \lambda'_2\, \lambda'_3 \ldots . \lambda'_p)$, $(\mu'_1, \mu'_2, \mu'_3 \ldots . \mu'_p) \ldots\ldots$etc.

Next consider the functions given by

$$f_{\lambda_1}(x) = \sum_{r=-\infty}^{r=+\infty} q^{l^2/k^2}\, e^{\frac{2ilx}{k}}, \quad f_{\lambda_1}(x) = \sum_{r=-\infty}^{r=+\infty} q^{m^2/k^2}\, e^{\frac{2imk}{k}} \quad \text{and etc.}$$

where $l = rk + \lambda_1$, $m = rk + \lambda_2$, $n = rk + \lambda_3 \ldots\ldots$and $kq = e^{i\pi\tau}$ where the imaginary part of τ is positive then it is easily seen that the series considered are absolutely convergent. Hence by Cauchy's theorem we can multiply any number of such series and rearrange them in any order we like. Now

$$(a) \ldots . \sum f_{\lambda_1}(x)\ f_{\lambda_2}(y)\ f_{\lambda_3}(z) = \sum_{r=-\infty}^{r=+\infty}\sum\sum . . q^{\frac{l^2+m^2+n^2 \ldots . p^2}{k^2}}\, e^{\frac{2i}{k}\sum lx}$$

where the sign of summation on the left represents that it has to run through $(\lambda, \mu, \ldots .$etc.)

Take new variables $(x, y, z, \ldots .)$ which are connected with $(x_1, y_1, z_1 \ldots .)$ by the relations

$$x_1 = a_{1_1}\ x + a_{2_1}\ y + a_{3_1}\ z + \ldots .$$
$$y_1 = a_{1_2}\ x + a_{2_2}\ y + a_{3_2}\ z + \ldots .$$
$$z_1 = a_{1_3}\ x + a_{2_3}\ y + a_{3_3}\ z + \ldots .$$

Now by the property of the orthogonal determinant if one set of variables $(l, m, n, \ldots . p)$ corresponds to $(l_1, m_1, n_1 \ldots\ldots)$. $\sum l^2 = \sum l_1{}^2$ and $\sum l\, x = \sum l_1 x$.

Now consider

$$\sum f_{\lambda'_1}(x_1)\ f_{\lambda'_2}(y_1)\ f_{\lambda'_3}(z_1) \ldots\ldots..$$
$$= \sum\sum\sum\ q^{\frac{\sum l_1^2}{k^2}}\, e^{2i \sum l_1 x_1}$$

where the sign of summation on the left runs through $\lambda', \mu' \ldots\ldots$etc.

We have already proved that corresponding to one set of variables (l, m, n....etc.); there corresponds one and only one set of values (l_1, m_1, n_1....etc.) consistent with relations (A) and (B) respectively. Hence, to each term of the series of the right hand side of (a) there corresponds one term which is equal to it in a series on the right hand side of (b) and *vice versa.* And therefore the two series considered are identical. Hence

$$\sum_{(\lambda, \mu, \ldots \text{etc.})} f_{\lambda_1}(x)\, f_{\lambda_2}(y)\, f_{\lambda_3}(z)\cdots = \sum_{(\lambda', \mu', \ldots \text{etc.})} f_{\lambda'_1}(x_1)\, f_{\lambda'_2}(y_1)\, f_{\lambda'_3}(z_1)\ldots$$

This is the extension of Jacobi's formulae. For each of the functions f can be expressed in terms of Θ functions in the following way. [The notation used here is that used in Whittaker and Watson's Modern Analysis.]

$$\sum_{-\infty}^{+\infty} q^{\left(\frac{rk+\lambda}{k}\right)^2} e^{2ix\left(r+\frac{\lambda}{k}\right)} = \left[q^{\lambda^2/k^2}\right] e^{\frac{2ix\lambda}{k}} \sum_{-\infty}^{+\infty} q^{r^2} e^{2ir\left[x+\frac{\pi t\lambda}{k}\right]}$$

$$= q^{\lambda^2/k^2} e^{\frac{2ix\lambda}{k}}\, \Theta_3\left(x+\frac{\pi t\lambda}{k}\right)$$

Hence the theorem enunciated above becomes

$$\sum_{(\lambda, \mu, \ldots)} q^{\frac{\Sigma \lambda^2_1}{k^2}} e^{\frac{2i}{k}\left\{\lambda_1 x+\lambda_2 y\ldots\right\}} \cdot \Theta_3\left(x+\frac{\pi t\lambda_1}{k}\right)\Theta_3\left(y+\frac{\pi t\lambda_2}{k}\right)$$

$$=\sum_{(\lambda', \mu' \ldots)} q^{\frac{\Sigma\lambda_1'^2}{k^2}} e^{\frac{2i}{k}\left\{\lambda_1' x+\lambda_2' y\ldots\right\}}\, \Theta_3\left(x_1+\frac{\pi t\lambda'_1}{k}\right)\Theta_3\left(y_1+\frac{\pi t\lambda'_2}{k}\right)$$

This theorem is of theoretical interest only and becomes complicated in the general case. I have discussed here a simple, symmetrical transformation which bears a resemblance to Jacobi's theorem.

Let

$$x_1 = \tfrac{1}{3}[-x+2y+2z]$$
$$y_1 = \tfrac{1}{3}[2x-y+2z]$$
$$z_1 = \tfrac{1}{3}[2x+2y-z]$$

and let (l, m, n) be connected with (l_1, m_1, n_1) by the same relations. Now this relation is orthogonal and as $(l_1-m_1) = -(l-m)$ and so on, we get that if (l, m, n) runs through all the integral values $+ve$ or $-ve$ which are such that $(l-m) = o$ (*mod 3*) and $(m-n) = o$ (*mod 3*), then (l_1, m_1, n_1) run through the same system of values as (l, m, n). Hence the theorem becomes

$$\sum_{r=0}^{r=2} f_r(x) f_r(y) f_r(z) = \sum_{r=0}^{r=2} f_r(x)(y_1) f_r(z_1)$$ or written in

terms of Θ functions it becomes

$$\sum_{r=0}^{r=2} q^{r^2/3} e^{2ri\Sigma x} \Theta_3\left(x+\frac{\pi rt}{3}\right) \Theta_3\left(y+\frac{\pi rt}{3}\right) \Theta_3\left(z+\frac{\pi rt}{3}\right)$$

$$= \sum_{r=0}^{r=2} q^{r^2/3} e^{2ri\Sigma x} \Theta_3\left(x_1+\frac{\pi rt}{3}\right) \Theta_3\left(y_1+\frac{\pi rt}{3}\right) \Theta_3\left(z_1+\frac{\pi rt}{3}\right)$$

In a similar manner one can discuss the transformation

$$x_1 = \frac{1}{n}\left\{-x+(n-1)\,y+(n-1)\,z+\ldots\text{ etc.}\ldots\right\}$$

$$y_1 = \frac{1}{n}\left\{(n-1)\,x-y+(n-1)\,z+\ldots\text{ etc.}\ldots\right\}.$$

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In another paper I propose to consider further examples of third order transformations and the relations obtained from them in the case of the Jacobian elliptic functions and σ functions.

ON THE EXPANSION OF BESSEL FUNCTIONS IN A SERIES OF MATHIEU FUNCTIONS AND ON A PROPERTY OF MATHIEU FUNCTIONS.

BY N. A. SHASTRI.

I. *Introduction.*

The expansions of Mathieu Functions in terms of Bessel Functions have been investigated by many writers, notable amongst them being Dr. Dhar † Dr. J. Dougall* and Goldstein,‡ but uptill now no writer has investigated the problem of expressing Bessel Functions in terms of Mathieu Functions. This latter method has its own advantages in as much as, it will enable us to obtain some of the properties of Mathieu Functions. Sections 2—4 deal with preliminary Lemmas and the main problem is discussed in sections 5—7. In the last section is obtained the relation between $ce_0(\theta, q)$, and $ce_2(\theta, q)$.

II. Let us first discuss the uniform convergence of the series,

$$\sum_{m=0}^{\infty} \frac{(-)^m (\tfrac{1}{2}kh\cos\theta)^{\mu+\nu+2m}\,\Gamma(\mu+\nu+2m+1)}{\Gamma(m+1)\,\Gamma(\mu+\nu+m+1)\,\Gamma(\mu+m+1)\,\Gamma(\nu+m+1)} \dots (1)$$

μ and ν being positive integers.

Consider the series,

$$\sum_{m=0}^{\infty} \frac{(\tfrac{1}{2}kh)^{\mu+\nu+2m}\,\Gamma(\mu+\nu+2m+1)}{\Gamma(m+1)\,\Gamma(\mu+\nu+m+1)\,\Gamma(\mu+m+1)\,\Gamma(\nu+m+1)} \dots (2)$$

It can be very easily shown that for this series,

$$\frac{u_{n+1}}{u_{n+2}} = \frac{2n^2}{(kh)^2}\left[1+O\left(\frac{1}{n^2}\right)\right]$$

$$\therefore \underset{n\to\infty}{Lt} \frac{u_{n+1}}{u_{n+2}} = \infty.$$

$\therefore$ The series (2) is convergent. Again $(-)^m (\cos\theta)^{\mu+\nu+2m}$ is always less than unity.

Therefore the series (1) is uniformly convergent †.

† Dhar: The Journal of the Ind. Math. Soc. Vol. XVI.

* Dougall; Proc. Edin. Math. Soc. Vol. XXXIV.

‡ Goldstein; Trans. Camb. Phil. Soc. Vol. XXIII, No. XI.

† Bromwich—Infinite Series, Chap. VII.

III. The value of $\int_0^{2\pi} J_\mu(kh\cos\theta)\, J_\nu(kh\cos\theta)\, d\theta$ will now be investigated.

Now*

$$J_\mu(kh\cos\theta)\, J_\nu(kh\cos\theta) =$$

$$\sum_{m=0}^{\infty} \frac{(-)^m \left(\tfrac{1}{2}kh\cos\theta\right)^{\mu+\nu+2m} \Gamma(\mu+\nu+2m+1)}{m!\,\Gamma(u+\nu+m+1)\,\Gamma(\mu+m+1)\,\Gamma(\nu+m+1)}.$$

Now the series on the right hand side is proved to be uniformly cunvergent in II and hence it can be integrated term by term. Therefore,

$$\int_0^{2\pi} J_\mu(kh\cos\theta)\, J_\nu(kh\cos\theta)\, d\theta =$$

$$\sum_{m=0}^{\infty} \frac{(-)^m \Gamma(\mu+\nu+2m+1)\left(\tfrac{1}{2}kh\right)^{\mu+\nu+2m}}{\Gamma(m+1)\,\Gamma(\mu+\nu+m+1)\,\Gamma(\mu+m+1)\,\Gamma(\nu+m+1)} \times$$

$$\times \int_0^{2\pi} (\cos\theta)^{\mu+\nu+2m}\, d\theta$$

$$= \sum_{m=0}^{\infty} \frac{(-)^m \Gamma(\mu+\nu+2m+1)\left(\tfrac{1}{2}kh\right)^{\mu+\nu+2m}}{\Gamma(m+1)\,\Gamma(\mu+\nu+m+1)\,\Gamma(\mu+m+1)\,\Gamma(\nu+m+1)}.$$

$$2\left[1+(-1)^{\mu+\nu}\right] \int_0^{\frac{\pi}{2}} (\cos\theta)^{\mu+\nu+2m}\, d\theta$$

$$= \sum_{m=0}^{\infty} \frac{(-)^m \Gamma(\mu+\nu+2m+1)\left(\tfrac{1}{2}kh\right)^{\mu+\nu+2m}}{\Gamma(m+1)\,\Gamma(\mu+\nu+m+1)\,\Gamma(\mu+m+1)\,\Gamma(\nu+m+1)}.$$

$$\frac{2\left[1+(-1)^{\mu+\nu}\right]\frac{\sqrt{\pi}}{2}\,\Gamma\left(\frac{\mu+\nu+2m+1}{2}\right)}{\Gamma\left(\frac{\mu+\nu+2m+2}{2}\right)}$$

$$= \begin{cases} O \text{ if } \mu+\nu \text{ is odd.} \\ \sum_{m=0}^{\infty} \left[\dfrac{2(-)^m (kh)^{\mu+\nu+2m} \left\{ \Gamma\left(\frac{\mu+\nu+2m+1}{2}\right)\right\}^2}{\Gamma(m+1)\,\Gamma(\mu+m+1)\,\Gamma(\nu+m+1)\,\Gamma(\mu+\nu+m+1)} \right] \\ \qquad \text{if } \mu+\nu \text{ is even.} \end{cases}$$

$= 2\,\beta_{\mu,\nu}$ (say) if $\mu+\nu$ is even.

It is very easy to show that the series represented by $\beta_{\mu,\nu}$ is convergent and so $\beta_{\mu,\nu}$ is finite.

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* Watson—Bessel Functions, Chap. V.

IV. Consider the series $\sum_{r=0}^{\infty} \phi_{2n}(r) J_{2r}(kh\cos\theta)$, where $\phi_{2n}(r)$ occurs as the co-efficient of $\cos 2r\theta$ in

$$\dagger\; ce_{2n}(\theta, q) = a_{0,n} \sum_{r=0}^{\infty} \phi_{2n}(r)\cos 2r\theta.$$

Now $a_{0,n}\,\phi_{2n}(r)$ is bounded for small values of q.

$$\therefore \quad |\, a_{0,n}.\ \phi_{2n}(r)\,| < K.$$

Hence. $\sum_{r=0}^{\infty} \phi_{2n}(r)\ J_{2r}(kh\cos\theta) < K \sum_{r=0}^{\infty} J_{2r}(kh\cos\theta)$.

Again $J_{2r}(kh\cos\theta) < \sum_{m=0}^{\infty} \left(\frac{kh}{2}\right)^{2m+2r} / (2r+m)!\ 2m! = H_{2r}(kh)$ say,

$$\therefore \quad \sum_{r=0}^{\infty} \phi_{2n}(r)\ J_{2r}(kh\cos\theta) < K \sum_{r=0}^{\infty} H_{2r}(kh).$$

Now it can be very easily shown that

$$\frac{H_{2r}(kh)}{H_{2r}+2(kh)} = \frac{2^4 - r^2}{(kh)^2}\left[1 + O\left(\frac{1}{r^2}\right)\right].$$

$$\therefore \quad \underset{r\to\infty}{Lt} \frac{H_{2r}(kh)}{H_{2r+2}(kh)} > 1.$$

Hence $\sum_{r=0}^{\infty} H_{2r}(kh)$ is convergent. Therefore both the series $\sum_{r=0}^{\infty} \phi_{2n}(r)\ J_{2r}(kh\cos\theta)$ and $\sum_{r=0}^{\infty} (-)^r\ \phi_{2n}(r)\ J_{2r}(kh\cos\theta)$ are uniformly convergent.

V. The expansion of $ce_{2r}(\theta, q)$ in terms of Bessel's Functions have been found out by Dr. Dhar† and Goldstein* by two methods (i) by using integral equations (ii) by transforming the Mathieu's equation into Bessel's differential equation by suitable substitutions. In this section the possibility of expanding $J_{2n}(kh\cos\theta)$ in terms of Mathieu Functions is discussed. The method followed is that of solving infinite equations.

Now we have.‡

$$ce_{2r}(\theta, q) = ce_{2r}\left(\frac{\pi}{2}, q\right) \sum_{n=0}^{\infty} (-)^n\ \phi_{2r}(n)\ J_{2n}(kh\cos\theta).$$

for all r.

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† Dongall: Loe. cit.

† Dhar—Loe. cit.

* Goldstein—Loe. cit.

‡ Dhar—Loe. cit., p. 55.

Hence solving the infinite equations for $r=0, 1, \ldots\ldots$

$$ce_0(\theta, q)=ce_0\left(\frac{\pi}{2}, q\right) [\phi_0(o)\ J_0(kh\cos\theta) - \phi_0(1)\ J_2(kh\cos\theta)$$
$$\ldots\ldots + (-)^n\ \phi_0(n)\ J_{2n}(kh\cos\theta)\ldots\ldots\ldots\ldots]$$

$$ce_{2r}(\theta, q)=ce_{2r}\left(\frac{\pi}{2}, q\right) [\phi_{2r}(o)\ J_0(kh\cos\theta) - \ldots\ldots\ldots\ldots\ldots$$
$$+(-)^n\ \phi_{2r}(n)\ \ J_{2n}(kh\cos\theta) + \ldots\ldots\ldots\ldots]$$

we have, writing J_{2n} for $J_{2n}\ (kh\cos\theta)$.

$$\frac{J_0}{D_0}=\frac{-J_2}{D_1}=\frac{J_4}{D_2}= = =\ldots=\frac{(-)^n J_{2r}}{D_n}= =\ldots=\frac{-1}{\Delta}$$

where $D_0, D_1\ldots\ldots$ and Δ are infinite determinate and

$$\Delta=\left|\begin{array}{lll} ce_0\left(\frac{\pi}{2}, q\right) \phi_0(o), & ce_0\left(\frac{\pi}{2}, q\right) \phi_0(1), & \ldots\ldots\ldots\ldots \\ ce_2\left(\frac{\pi}{2}, q\right) \phi_2(o), & ce_2\left(\frac{\pi}{2}, q\right) \phi_2(1), & \ldots\ldots\ldots\ldots \\ \ldots\ldots & \ldots\ldots & \ldots\ldots \\ \ldots\ldots & \ldots\ldots & \ldots\ldots \\ ce_{2r}\left(\frac{\pi}{2}, q\right) \phi_{2r}(o), & \ldots\ldots\ldots\ldots ce_{2r}\left(\frac{\pi}{2}, q\right) \phi_{2r}(r) & \ldots \\ \ldots\ldots & \ldots\ldots & \ldots\ldots \\ \ldots\ldots & \ldots\ldots & \ldots\ldots \end{array}\right|$$

But $a_{0,\,r}\ \phi_{2r}\ (n) = a_{n,\,r}$.

$$\text{Hence. } \Delta = \frac{ce_0\left(\frac{\pi}{2}, q\right) ce_2\left(\frac{\pi}{2}, q\right)\ldots\ldots cc_{2r}\left(\frac{\pi}{2}, q\right)\ldots}{a_{0,0}\cdot a_{0,1}\ldots\ldots\ldots\ldots a_{0,r}\ldots\ldots\ldots} \times$$

$$\left|\begin{array}{llllllll} a_{0,0}\cdot & a_{1,0}\cdot & a_{2,0}\ldots & \ldots & \ldots & \ldots & \ldots \\ a_{0,1}\cdot & a_{1,1}\cdot & a_{2,1}\ldots & \ldots & & & \\ a_{0,2}\cdot & a_{1,2}\cdot & a_{2,2}\ldots & \ldots & & & \\ \ldots & \ldots & \ldots\ a_{3,3} & \ldots & & & \\ \ldots & \ldots & \ldots & \ldots & & & \\ \ldots & \ldots & \ldots & \ldots & \ldots & & \\ a_{0,r}\cdot & a_{1,r} & \ldots & \ldots & \ldots & a_{r,r} & \ldots \\ \ldots & \ldots & \ldots & \ldots & & & \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \end{array}\right|$$

Now in the expansion of $ce_{2r}\ \ (\theta, q)$ the co-efficient of $\cos 2r\,\theta$ is unity. Hence

$$a_{r,\,r}=1 \text{ for all values of } r.$$

Hence the product of all the diagonal elements is unity. The sum of the non-diagonal elements can be shown to be convergent. The sum of the elements of the r^{th} row except the term in the leading diagonal is equal to

$$\sum_{s=0}^{\infty} a_{s,\, r-1}. = ce_{2r}\ (o,\, q) - 1.$$

Now $ce_{2r}(o, q) \rightarrow 1$ as $q \rightarrow 0$.

Hence $\mid ce_{2r}(o, q) - 1 \mid < \dfrac{\epsilon}{2^{r+1}}$ when $q < \eta$.

And this is true for every other row.

Hence the sum of the non-diagonal elements is less than

$$\sum_{r=0}^{\infty}{}' \frac{\epsilon}{2^{r+1}} = \epsilon. \text{ (a finite quantity).}$$

The absolute convergence of the series follows easily by a slight change in the above argument. The multiplying factor of the determinant is

$$\prod_{n=0}^{\infty} \left\{ \frac{ce_{2n}\left(\frac{\pi}{2}, q\right)}{a_{0,\, n}} \right\}.$$

But for all values of n, $\dfrac{ce_{2n}\left(\frac{\pi}{2}, q\right)}{a_{0,\, n}}$ is bounded. Hence we can choose a finite quantity A, so that

$$\left| \frac{ce_{2n}\left(\frac{\pi}{2}, q\right)}{a_{0,\, n}} \right| < A^{\frac{1}{2^{n+1}}} \text{ for all } n \text{ and for } q \text{ sufficiently small.}$$

$$\text{Hence } \prod_{n=0}^{\infty} \left\{ \frac{ce_{2n}\left(\frac{\pi}{2}, q\right)}{a_{0,\, n}} \right\} < A^{\sum_{0}^{\infty} \frac{1}{2^{n+1}}} = A.$$

Hence Δ is finite.|| Similarly it can be shown that each of $D_0, D_1, \ldots\ldots D_n, \ldots\ldots$ is convergent. The determinant D_n will contain in one of its columns the elements $ce_0(\theta, q), ce_2(\theta, q) \ldots\ldots ce_{2r}(\theta, q) \ldots$ etc. If we expand D_n in terms of the elements of this column, we get

$$J_{2n}(kh \cos\theta) = (-)^{n+1} \frac{D_n}{\Delta}$$

|| Whittaker and Watson—Modern Analysis, p. 2·8.

$$= \tfrac{1}{2} A_0\, ce_0(\theta, q) + A_1\, ce_2(\theta, q) + \ldots + A_r\, ce_{2n}(\theta, q) \ldots$$

where A_0, $A_1 \ldots \ldots$ etc. are coefficients which are functions of q and n.

VI. To bring out the dependence of the co-efficients of the expansion in the previous section we write the expansion as

$$J_{2n}(kh \cos\theta) = \tfrac{1}{2} A_{0,\,2n}\, ce_0(\theta, q) + \sum_{m=1}^{\infty} A_{m,\,2n}\, ce_{2m}(\theta. q). (1)$$

To determine the coefficients $A_{m,\,2n}$, we multiply both sides by $ce_{2m}(\theta, q)$ and integrate term by term with respect to θ from 0 to 2π. Therefore,

$$\int_0^{2\pi} J_{2n}(kh \cos\theta).\ ce_{2m}(\theta, q) = \tfrac{1}{2} A_{0,\,2n} \int_0^{2\pi} ce_0(\theta, q) \times$$

$$ce_{2m}(\theta, q)\, d\theta + \sum_{r=1}^{\infty} A_{r,\,2n} \int_0^{2\pi} ce_{2r}(\theta, q).\ ce_{2m}(\theta, q)\, d\theta.$$

Now the Mathieu Functions of Whittaker type, $ce_{2n}(\theta, q)$ are orthogonal*. Therefore,

$$\int_0^{2\pi} ce_{2m}(\theta, q).\ ce_{2n}(\theta, q)\, d\theta = 0 \text{ if } m \neq n.$$

and $\int_0^{2\pi} [ce_{2n}(\theta, q)]^2\, d\theta = \text{constant if } m = n.$

If following Goldstein† we adopt the convention that

$$\int_0^{2\pi} [ce_{2n}(\theta, q)]^2\, d\theta = \pi \quad n \neq 0.$$

and $\int_0^{2\pi} [ce_0(\theta, q)]^2\, d\theta = 2\pi \quad n = 0.$

we get, $\pi A_{m,\,2n} = \int_0^{2\pi} J_{2n}(kh \cos\theta).\ ce_{2m}(\theta, q)\, d\theta.$

But

$$ce_{2m}(\theta, q) = ce_{2m}\left(\frac{\pi}{2}, q\right) \sum_{r=0}^{\infty} (-)^r \phi_{2m}(r).\, J_{2r}(kh \cos\theta) \,..(2)$$

Where $\phi_{2m}(r)$ has the same meaning as in section IV. But the series (2) is already proved to be uniformly convergent in section IV. Hence both sides of (2) be multiplied by a bounded function $J_{2m}(kh \cos\theta)$ and integrated term by term Therefore,

$$\pi A_{m,\,2n} = \int_0^{2\pi} ce_{2m}(\theta, q) \cdot J_{2n}(kh \cos\theta)\, d\theta.$$

$$= \int_0^{2\pi} ce_{2m}\left(\frac{\pi}{2}, q\right) \sum_{r=0}^{\infty} (-)^r \phi_{2m}(r) J_{2r}(kh \cos\theta) J_{2n}(kh \cos\theta)\, d\theta.$$

$$= \sum_{r=0}^{\infty} ce_{2m}\left(\frac{\pi}{2}, q\right) (-)^r \phi_{2m}(r) \int_0^{2\pi} J_{2r}(kh \cos\theta)\, J_{2n}(kh \cos\theta) d\theta.$$

$$= 2\, ce_{2m}\left(\frac{\pi}{2}, q\right) \sum_{r=0}^{\infty} (-)^r \phi_{2m}(r) B_{2r,\,2n} \cdots\cdots\cdots (3).$$

Hence

$$J_{2n}(kh \cos\theta) = \tfrac{1}{2} A_{0,\,2n}\, ce_0(\theta, q) + \sum_{m=1}^{\infty} A_{m,\,2n}\, ce_{2m}(\theta, q) \text{ where}$$

$$A_{m,\,2n} = \frac{2}{\pi}\, ce_{2m}\left(\frac{\pi}{2}, q\right) \cdot \sum_{r=0}^{\infty} (-)^r \phi_{2m}(r) \quad B_{2r,\,2n}$$

and $B_{2r,\,2m}$ is given by section III.

VII. We are now in a position to justify term by term the integration of the series (1) of the preceding section. It will be enough to prove the uniform convergence of the series where $A_{m,\,2n}$, is given by (3).

The sum of series (1) is less than the sum of

$$\sum_{m=0}^{\infty} | A_{m,\,2n} | \; | ce_{2m}(\theta, q) |.$$

Now all $ce_{2m}(\theta, q)$ are bounded for sufficiently small values of q.

Hence $|ce_{2m}(\theta, q)| < k$ for all m.

Hence the sum of the series (1) is less than that of

$$K \sum_{m=0}^{\infty} | A_{m,\,2n} | < \frac{2k}{\pi} \sum_{m=0}^{\infty} \Big| \sum_{r=0}^{\infty} ce_{2m}\left(\frac{\pi}{2}, q\right) \phi_{2m}(r) \quad B_{2r,\,2n} \Big|$$

$$< \frac{2k}{\pi} \sum_{m=0}^{\infty} \sum_{r=0}^{\infty} \Big| ce_{2m}\left(\frac{\pi}{2}, q\right)\Big| \; | \phi_{2m}(r) | \; | B_{2r,\,2n}|.$$

To discuss the convergence of the double series† on the right hand side, it is sufficient to show that the limit of the sum of the terms in the square formed by the first m rows and m columns is finite. If $S_{m,\,m}$ denotes the required sum,

---

† Bromwich Infinite Series, Chap. V.

$$S_{m,\,m} = \left| ce_0\left(\tfrac{\pi}{2}, q\right) \phi_0(0) \right| \, |B_{0,\,2n}| + \left| ce_0\left(\tfrac{\pi}{2}, q\right) \right| \, |\phi_0(1)| \, |B_{2,\,2n}| + \ldots\ldots$$

$$+ \left| ce_2\left(\tfrac{\pi}{2}, q\right) \phi_2(0) \right| \, |B_{0,\,2n}| + \left| ce_2\left(\tfrac{\pi}{2}, q\right) \right| \, |\phi_2(1)| \, |B_{2,\,2n}| + \ldots\ldots$$

$$= |B_{0,\,2n}| \left[\left| ce_0\left(\tfrac{\pi}{2}, q\right) \right| |\phi_0(0)| + \left| ce_2\left(\tfrac{\pi}{2}, q\right) \right| |\phi_2(0)| \ldots\ldots\ldots m \text{ terms} \right]$$

$$+ |B_{2,\,2n}| \left[\left| ce_0\left(\tfrac{\pi}{2}, q\right) \right| \, |\phi_0(1) + \left| ce_2\left(\tfrac{\pi}{2}, q\right) \right| |\phi_2(1)| + \ldots\ldots\ldots \quad ,, \right]$$

Each term within the brackets is less than K a finite quantity.

Hence $S_{m,\,m} < K \sum_{m=0}^{m} m \, |B_{2m,\,2n}|$

$$\therefore \operatorname*{Lt}_{m \to \infty} S_{m,\,m} < K \sum_{m=0}^{\infty} m \, |B_{2m,\,2n}|.$$

Now $\dfrac{u_m}{u_{m+1}}$ for the series on the right hand side

$$= \frac{m}{m+1} \cdot \frac{|B_{2m,\,2n}|}{|B_{2m+2,\,2n}|}$$

If we compare the corresponding terms of $|B_{2m,\,2n}|$ and $|B_{2m+2,\,2n}|$, it can be very easily seen that every term of $|B_{2m,\,2n}|$ is greater than the corresponding term of $|B_{2m+2,\,2n}|$ Hence

$$\frac{|B_{2m,\,2n}|}{|B_{2m+2,\,2n}|} > 1 \text{ for all } m.$$

$$\therefore \operatorname*{Lt}_{m \to \infty} \frac{u_m}{u_{m+1}} > 1.$$

Hence the series on the right hand side of $S_{m,\,m}$ is convergent and so $\operatorname*{Lt}_{m \to \infty} S_{m,m}$ is finite. Therefore, the series (1) is uniformly convergent.

VIII. The expansion investigated in the last section will now be used to obtain a property of Mathieu Functions.

Now $e^{\frac{1}{2}z\left(t-\frac{1}{t}\right)} = J_0(z) + \sum_{n=1}^{\infty} \left\{ t^n + (-)^n t^{-n} \right\} J_n(z),$

Put $t = \pm e^{i\theta}$ we get after simplification;

$$e^{\pm iz\sin\theta} = J_0(z) + 2\sum_{n=1}^{\infty} J_{2n}(z)\cos 2n\theta \pm 2i\sum_{n=0}^{\infty} J_{2n+1}(z).\sin(2n+1)\theta.$$

Adding the two results and putting $\theta=0$ we have

$$1 = J_0(z) + 2\sum_{n=1}^{\infty} J_{2n}(z) \quad \ldots\ldots\ldots\ldots (1).$$

Substitute $kh\cos\theta$ for z. (1) reduces to

$$1 = J_0(kh\cos\theta) + 2\left\{ J_2(kh\cos\theta) + J_4(kh\cos\theta) + \ldots \right\} \ldots\ldots (2).$$

Expanding each of J_{2n} in terms of Mathieu functions by (1) and (3) of section V.

$$\begin{aligned} 1 = {} & \tfrac{1}{2} A_{00}\, ce_0(\theta,q) + A_{1,0}\, ce_2(\theta,q) + A_{2,0}\, ce_4(\theta,q) + \ldots\ldots \\ & + 2\,[\tfrac{1}{2} A_{0,2}\, ce_0(\theta,q) + A_{1,2}\, ce_2(\theta,q) + A_{2,2}\, ce_4(\theta,q) + \ldots. \\ & + \tfrac{1}{2} A_{0,4}\, ce_0(\theta,q) + A_{1,4}\, ce_2(\theta,q) + A_{2,4}\, ce_4(\theta,q) + \ldots \\ & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots] \ldots. (3) \end{aligned}$$

If q be taken so small that its powers higher than 3 be neglected, we have,

$$\left.\begin{aligned} B_{0,0} &= \pi[1 + 2^3 q + 3^2\cdot 2^2\, q^2 + \frac{5^2.2^5}{3^2} q^3 \ldots] \\ B_{0,2} &= B_{2,0} = \pi\,[-2q - 2^4 q^2 - 50q^3 \ldots] \\ B_{2,2} &= \pi\,[6q^2 + \frac{5\cdot 2^4}{3} q^3 \ldots\ldots] \\ B_{4,0} &= B_{0,4} = \pi\,[q^2 + 8q^3 \ldots\ldots] \\ B_{4,2} &= -\pi\,\frac{10}{3} q^3 \ldots\ldots\ldots] \\ B_{0,6} &= -\pi\,\frac{2}{9} q^3 \ldots\ldots, \end{aligned}\right\} \ldots. (4)$$

Again if we take the expansions of $ce_0(\theta, q)$, $ce_2(\theta, q)$, $ce_4(\theta, q)$ as $ce_0(\theta,q) = 1 + [4q - 28q^3 + O(q^5)]\cos 2\theta + [2q^2 + O(q^4)]\cos 4\theta + \ldots\ldots\ldots$

$$ce_2(\theta,q) = [-2q + \frac{4\theta}{3} q^3 + O(q^5)] + \cos 2\theta + \Big[\frac{2q}{3} +$$

$\frac{516}{3^2(6)^2} q^3 + O(q^5)]$ $\cos 4\theta + \ldots\ldots\ldots$

$$ce_4(\theta, q) = \left[\frac{1}{3} q^2 + O(q^4)\right] - \left[\frac{2q}{3} + \frac{14}{135} q^3 + \ldots\right] \cos 2\theta$$

$$+ \cos 4\theta + \left[\frac{2}{5} q + O(q^3)\right] \cos 6\theta + O(q^2) \cos 8\theta + \ldots\ldots$$

for which the values of a in the Mathieu's equation

$$\frac{d^2u}{d\theta^2} + (a + 16q \cos 2\theta)\, u = O.$$

are (1) $-32q^2 + 224q^4 + O(q^6)$.

(2) $4 + \frac{80}{3} q^2 - \frac{6104}{27} q^4 + O(q^6)$

(3) $16 + \frac{32}{15} q^2 + O(q^4)$.

we get

$$\left.\begin{aligned}
a_{0,0}\ \phi_0(0) &= 1 \\
a_{0,0}\ \phi_0(1) &= 4q - 28q^3 + \ldots\ldots \\
a_{0,0}\ \phi_0(2) &= 2q^2 \ldots\ \ldots \\
a_{0,1}\ \phi_2(0) &= -2q + \frac{40}{3} q^3 \ldots\ldots\ldots \\
a_{0,1}\ \phi_2(1) &= 1 \\
a_{0,1}\ \phi_2(2) &= \frac{2q}{3} + \ldots\ldots \\
a_{0,2}\ \phi_4(0) &= \frac{1}{3} q^2 \ldots\ldots\ldots \\
a_{0,2}\ \phi_4(1) &= -\frac{2}{3} q - \frac{14}{135} q^3 \ldots\ldots \\
a_{0,2}\ \phi_4(2) &= 1 \\
&\text{etc.}
\end{aligned}\right\} \ldots\ldots (5).$$

Using (4) and (5) we find the values of $A_{n, 2m}$ for different values of n and m.

$$\begin{aligned}
A_{0,0} &= 2\, ce_0\left(\frac{\pi}{2}, q\right) \left[1 + 8q \times 44q^2 + \frac{2^5 . 43}{3^2} q^3 \ \ldots\right]. \\
A_{0,2} &= ce_0\left(\frac{\pi}{2}, q\right) \left[-4q - 2^5 q^2 - 2^2 . 37 q^3 \ldots\ldots\ldots\right] \\
A_{0,4} &= ce_0\left(\frac{\pi}{2}, q\right) \left[2q^2 + 2^4 q^3 \ldots\ldots\ldots\ldots\ldots\right] \\
A_{0,6} &= ce_0\left(\frac{\pi}{2}, q\right) \left[-\frac{2^2}{3^2} q^3 \ldots\ldots\ldots\ldots\ldots\ldots\right]
\end{aligned}$$

$$\left.\begin{aligned}
A_{1,0} &= \frac{ce_2\left(\frac{\pi}{2}, q\right)}{a_{0,1}} \left[-2^4 q^3 \ldots\ldots\ldots\ldots\ldots\right] \\
A_{1,2} &= \frac{ce_2\left(\frac{\pi}{2}, q\right)}{a_{0,1}} \left[-2^2 q^2 \times \frac{2^5}{3} q^3 \ldots\ldots\ldots\ldots\right] \\
A_{1,4} &= \frac{ce_2\left(\frac{\pi}{2}, q\right)}{a_{0,1}} \left[\frac{2^3}{3} q^3 \ldots\ldots\ldots\ldots\ldots\ldots\right] \\
A_{2,0} &= \frac{ce_4\left(\frac{\pi}{2}, q\right)}{a_{0,2}} \left[O(q^4) \ldots\ldots\ldots\ldots\ldots\ldots\right] \\
A_{2,2} &= \frac{ce_4\left(\frac{\pi}{2}, q\right)}{a_{0,2}} O(q^4) \ldots\ldots\ldots\ldots\ldots\ldots]
\end{aligned}\right\} \ldots (6)$$

Rewriting (3) in a different form, we have

$$\begin{aligned}
1 = {} & ce_0(\theta, q) \left[\tfrac{1}{2} A_{0,0} + A_{0,2} + A_{0,4} + A_{0,6} + \ldots\right] \\
& + ce_2(\theta, q) \left[A_{1,0} + 2A_{1,2} + 2A_{1,4} \ldots\ldots\right] \\
& + ce_4(\theta, q) \left[A_{2,0} + 2A_{2,2} \ldots\ldots\ldots\ldots\right] \ldots (7)
\end{aligned}$$

Now it is easy to show all the terms in the square brackets except those that occur in (6) begin with powers higher than 3, and hence neglecting them we get

$$\text{coefficient of } ce_0(\theta, q) = ce_0\left(\frac{\pi}{2}, q\right) \left[1 + 4q + 14q^2 \times \frac{184}{9} q^3\right]$$

$$\text{coefficient of } ce_2(\theta, q) = \frac{ce_2\left(\frac{\pi}{2}, q\right)}{a_{0,1}} \left[-8q^2 + \frac{2^5 q^3}{3}\right]$$

other coefficients are to be neglected as they contain q^4 etc. Hence (7) reduces to.

$$1 = \left[1 + 4q + 14q^2 + \frac{184}{9} q^3\right] ce_0\left(\frac{\pi}{2}, q\right) ce_0(\theta, q)$$

$$+ \left[-8q^2 + \frac{32}{3} q^3\right] \frac{ce_2\left(\frac{\pi}{2}, q\right)}{a_{0,1}} ce_2(\theta, q) \ldots\ldots\ldots (8)$$

a relation between $ce_0(\theta, q)$ and $ce_2(\theta, q)$ when q is sufficiently small.

Equation (8) can be put in a more elegent form, if the values of $a_{0,1}$, $ce_0\left(\frac{\pi}{2}, q\right)$ and $ce_2\left(\frac{\pi}{2}, q\right)$ are substituted and powers of q higher than 3 are neglected we will then get.

$$1 = ce_0(\theta, q) - \left[4q + \frac{16}{3} q^2 + 22q^3\right] ce_2(\theta, q) \ldots\ldots (9)$$

It is interesting to note that the limit of the right hand side is unity as q tends to zero.

It is possible to establish relations between ce_0, ce_2, ce_4 .... etc. by this method, but the calculations are very tedious. Similar and other relations and expansions of other types are under investigation and I hope to communicate the results soon. In conclusion, I must thank Dr. Dhar for his valuable suggestions.

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ON ABUNDANT NUMBERS

By S. Chowla.

§ 1. We say that the positive integer n is abundant, when

$$\sigma(n) = \sum_{d \mid n} d \geqslant 2n.$$

The probability that a positive integer is abundant is less than $\frac{1}{2}$. More precisely if $A(x)$ is the number of solutions of

$$\sigma(n) \geqslant 2n \qquad (1 \leqslant n \leqslant x)$$

then*

$$0 < \frac{A(x)}{x} < \frac{1}{2}$$

for all $x > x_0$.

In this paper we prove that †

$$(1) \qquad \lim_{x \to \infty} \frac{A(x)}{x} = c \ (>0),$$

a result recently conjectured by Felix Behrend. In fact (1)‡ is a simple consequence of a theorem proved by I. Schoenberg in *Mathematische Zeitschrift*, 28, 1928, 171—199.

§ 2. Let

$$(2) \qquad x_{1n}, x_{2n}, \ldots, x_{nn} \qquad (n = 1, 2, 3, \ldots)$$

be n numbers all belonging to the interval $0 \leqslant x \leqslant 1$.

In the paper referred to above Schoenberg has proved that: *Fur die Zahlen* $x_{1n}, x_{2n}, \ldots, x_{nn}$ (2) *der strecke* 0,....1 *mogen die Grenzwerte,*

$$(3) \qquad \lim_{n \to \infty} \frac{x_{1n}^k + x_{2n}^k + \ldots\ldots + x_{nn}^k}{n} = \mu_k$$

fur $k = 1, 2, 3, \ldots\ldots$ *vorhanden sein.*

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* Felix Behrend, S.—*B Preuss. Akad. Wiss*, H. 21/23, 322—328 (1932).

† *Ibid.*, 1933.

‡ (1) has been proved independently by H. Davenport, *ibid.*, 830—837 (1933).

Es existiert eine eindeutig bestimmte für $R(s) > 0$ reguläre und beschränkte Funktion $\Phi(s)$, welche den Bedingungen $\Phi(n) = \mu_n$ fur $n = 1, 2, 3, \ldots$ genugt. Wenn $z(t)$ die Lösung des Momentenproblems,

$$(4) \quad \int_0^1 t^k \, dz(t) = \mu_k \qquad (k = 0, 1, 2, \ldots; \quad \mu_0 = 1)$$

bedeutet, so ist

$$\Phi(s) = \int_0^1 t^s \qquad dz(t), \quad R(s) > 0$$

eine Darstellung der Funktion $\Phi(s)$.

Es existiert

$$\lim_{\sigma \to 0} \Phi(\sigma + i\lambda) = \Phi(i\lambda) \qquad (s = \sigma + i\lambda)$$

gleichmassig fur $-\infty < \lambda < \infty$.

Die Zahlen (2) verteilen sich gewiss asymptotisch stetig, wenn

$$(5) \quad \Phi(0) = 1, \text{ und } \lim_{x \to \infty} \frac{1}{x} \int_0^x |\Phi(i\lambda)| \, d\lambda = \theta.$$

Die entsprechende Verteilungsfunktion ist $z(t)$, welches nach den Bedingungen (5) notwendig stetig ist.

§ 3.

Lemma 1: *For every complex value of s, we have*

$$(6) \quad \lim_{n \to \infty} \frac{\left\{ \frac{1^a}{\sigma_a(1)} \right\}^s + \left\{ \frac{2^a}{\sigma_a(2)} \right\}^s + \ldots + \left\{ \frac{n^a}{\sigma_a(n)} \right\}^s}{n} = \Phi(s)$$

and

$$(7) \quad \Phi(s) = \prod_p \left\{ \left(1 - p^{-1}\right) \left((1 + p^{-1})(1 + p^{-a})^{-s} + p^{-2} \right. \right.$$
$$\left. \left. (1 + p^{-a} + p^{-2a})^{-s} + \ldots \ldots \right) \right\}$$

where p runs through all primes.

Proof: For $R(s) > 1$ we have

$$\sum_{n=1}^{\infty} \left\{ \frac{n^a}{\sigma_a(n)} \right\}^k n^{-s} = \prod_p \left\{ 1 + p^{-s} (1 + p^{-a})^{-k} \right.$$
$$\left. + p^{-2s} \left(1 + p^{-a} + p^{-2a}\right)^{-k} + \ldots \ldots \right\}$$

$$=\zeta(s).\prod_p\left\{(1-p^{-s})\left(1+p^{-s}(1-p^{-a})^{-k}+p^{-2s}(1+p^{-a}+p^{-2a})^{-k}+\ldots\ldots\right)\right\}$$

$$=\zeta(s).\quad F(s)$$

where $\zeta(s)$ is Riemann's zeta function and $F(s)$ is absolutely convergent for $\sigma>$ Max $(\frac{1}{2}, 1-a)$. From the latter fact it follows by well-known methods that

$$\lim_{x\to\infty}\frac{\sum_{n\leqslant x}\left\{n^a/\sigma_a(n)\right\}^k}{x}=F\ (1).$$

Lemma 2: If $\Phi(s)$ *is defined by* (7) *then*

(i) $|\Phi(s)| \leqslant 1$ *for* $\sigma>0$,

(ii) $\Phi(0)=1$,

Proof: From (7),

$$\Phi(0)=\prod_p\left\{(1+p^{-1})\,(1+p^{-1}+p^{-2}+\ldots..)\right\}=1$$

and, for $\sigma>0$,

$$|\ \Phi(s)\ |\ <\ \prod_p\left\{(1-p^{-1})\,(1+p^{-1}+p^{-2}+\ldots.)\right\}=1.$$

Lemma 3: If $\Phi\ (s)$ *is defined by* (7), *then for* $a=1$,

(8) $$\lim_{x\to\infty}\frac{1}{x}\int_0^x|\ \Phi\ (i\lambda)|\ d\lambda=0,$$

Proof: We have

$$\Phi(i\lambda)=\prod_p\left\{(1-p^{-1})\left(1+p^{-1}(1+p^{-a})^{-i\lambda}+p^{-2}(1+p^{-a}+p^{-2a})^{-i\lambda}+\ldots\ldots\right)\right\}$$

$$\log|\ \Phi(i\lambda)|=R\sum_p\left\{\log(1-p^{-1})+\log\left(1+p^{-1}(1+p^{-a})^{-i\lambda}+p^{-2}(1+p^{-a}+p^{-2a})^{-i\lambda}+\ldots.\right)\right\}$$

$$=R\sum_p\left\{-\frac{1}{p}+\frac{(1+p^{-a})^{-i\lambda}}{p}+O(p^{-2})\right\}$$

$$=-2\sum_p\frac{1}{p}\sin^2\left\{\frac{\lambda}{2}\log\ (1+p^{-a})\right\}+O\ (1),$$

$$(9)\qquad |\Phi(i\lambda)| = e^{-2\sum_p \frac{1}{p}\sin^2\left\{\frac{\lambda}{2}\log\left(1+p^{-a}\right)\right\}+O(1)},$$

where the constant implied in O is independent of λ. If we now show that

$$(10)\qquad \lim_{x\to\infty}\frac{1}{x}\int_0^x e^{-2\sum_p \frac{1}{p}\sin^2\left\{\frac{\lambda}{2}\log\left(1+p^{-a}\right)\right\}}\,d\lambda=0$$

then (9) shows that (8) is proved.

The proof of (10) is exactly similar to the proof of equation (37) of Schoenberg's paper. Here we have to show that the numbers

$$(11)\qquad \log\left(1+p^{-a}\right)$$

are linearly independent* in Schoenberg's sense. This is easily seen to be true for $a=1$, and lemma 3 is proved. From the *satz* of § 2 and lemmas 1—3 it now follows that

If $0\leqslant\theta\leqslant 1$ *and* $A(x,\theta)$ *is the number of solutions of*

$$\frac{n}{\sigma(n)}\leqslant\theta$$
$$1\leqslant n\leqslant x$$

then

$$\lim_{x\to\infty}\frac{A(x,\theta)}{x}$$

exists and is a continuous function $z(\theta)$ *of* θ.

§ 4. Putting $\theta=1/2$ in the last result we get the result stated in § 1.

3rd September, 1933.

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* It seems likely that the numbers (11) are linearly independent for any $a>0$. The case $a=1$ is proved as in § 19 of Schoenberg's paper.

ON INTEGRAL EQUATION ASSOCIATED WITH PARABOLIC CYLINDER FUNCTIONS

By S. S. Shukrey, M.Sc.,

Research Scholar.

Introduction:

Parabolic cylinder functions have of late been studied by various workers.* The object of the present paper is to study them by the help of the integral equation which these functions satisfy and to study some of their properties. In Art. (1) the integral equation has been obtained, in Art. (2) the recurrence formulæ have been obtained from them. Articles (3) and (4) deal with several expressions of these functions. All the expressions that have been obtained are believed to be new.

I take this opportunity to express my best thanks to Professor S. C. Dhar for his kind help and guidance.

§ 1. Consider the differential equation,

$$(cx^2+b)\frac{d^2y}{dx^2}+cx\frac{dy}{dx}+(bk^2x^2+A)\,y=0 \tag{1}$$

where b, c, A and K are all constants.

We will show that,

$$\phi(x)=\lambda\int\frac{e^{\theta kx}}{(c\theta^2+b)^{\frac{1}{2}}}\,\phi(\theta)\,d\theta$$

satisfies the above equation taken within limits which will be obtained in the course of the analysis. Substituting $\phi(x)$ for y in the above we have

$$\lambda\left\{(cx^2+b)\frac{d^2}{dx^2}+cx\frac{d}{dx}+(bk^2x^2+A)\right\}\int\frac{e^{\theta kx}}{(c\theta^2+b)^{\frac{1}{2}}}\,\phi(\theta)\,d\theta=0$$

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* Lon. Math. Soc. 2, VIII, 1909—10.

Lon. Math. Soc. 2, XVI, 1916—1917.

Jour. of the Ind. Math. Soc., Vol. XIX, No. 8, April, 1932.

Dr. Gorakh Prasad, Proc. Benares Math. Soc., Vol. VIII (1925—28), pp. 47—48.

Whittaker and Watson, Modern Analysis [confluence Hypergeometric functions].

$$\lambda\int\left\{(cx^2+b)k^2\theta^2+cxk\theta+(b^2k^2x^2+A)\right\}\frac{e^{\theta kx}}{(c\theta^2+b)^{\frac{1}{2}}}\phi(\theta)\,d\theta=0$$

$$\lambda\int\left\{(c\theta^2+b)k^2x^2+(c\theta)kx+(bk^2\theta^2+A)\right\}\frac{e^{\theta kx}}{(c\theta^2+b)^{\frac{1}{2}}}\phi(\theta)\,d\theta=0$$

Integrating the left hand side by parts,

$$\lambda\int(c\theta)kx\frac{e^{\theta kx}}{(c\theta^2+b)^{\frac{1}{2}}}\phi(\theta)\,d\theta=\lambda\left[(c\theta^2+b)^{\frac{1}{2}}kx\;e^{\theta kx}\phi(\theta)\right]$$

$$-\lambda\int(c\theta^2+b)^{\frac{1}{2}}kx\left\{kx\;e^{\theta kx}\phi(\theta)+e^{\theta kx}\phi'(\theta)\right\}d\theta.$$

Further integrating by parts the second part of right hand side we have

$$-\lambda\int kx(c\theta^2+b)^{\frac{1}{2}}\left[kx\;\phi(\theta)+\phi'(\theta)\right]e^{\theta kx}\,d\theta=$$

$$\lambda\left\{-e^{\theta kx}\;(c\theta^2+b)^{\frac{1}{2}}\phi'(\theta)\right\}$$

$$+\lambda\int e^{\theta kx}\left[\frac{c\theta}{(c\theta^2+b)^{\frac{1}{2}}}\phi'(\theta)+(c\theta^2+b)^{\frac{1}{2}}\phi''(\theta)\right]d\theta.$$

Hence the total expression will be

$$\lambda\left\{kx(c\theta^2+b)^{\frac{1}{2}}\phi(\theta)\;e^{\theta kx}-(c\theta^2+b)^{\frac{1}{2}}\phi'(\theta)\;e^{\theta kx}\right\}$$

$$+\lambda\int\left[-k^2x^2(c\theta^2+b)^{\frac{1}{2}}\phi(\theta)\;e^{\theta kx}\right.$$

$$\left.+\frac{kx(c^2\theta^2)\phi'(\theta)+(c\theta^2+b)\phi''(\theta)\,e^{\theta kx}}{(c\theta^2+b)^{\frac{1}{2}}}\right]d\theta=0.$$

On simplifying and rearranging we have,

$$\lambda\left[kx\;\phi(\theta)-\phi'(\theta)\right]e^{\theta kx}\;(c\theta^2+b)^{\frac{1}{2}}+$$

$$\lambda\int\left\{(c\theta^2+b)\phi''(\theta)+c\theta\phi'(\theta)+(b\theta^2k^2+A)\phi(\theta)\right\}\frac{e^{\theta kx}}{(c\theta^2+b)^{\frac{1}{2}}}d\theta$$

$$=0.$$

Now ϕ is the same function of θ as it is of x so the expression under the integral sign in above vanishes in virtue of (1). Now in order that the first part may vanish we must have, either $(c\theta^2+b)$ equal to zero, which determines the limits for the integral or $[kx\;\phi(\theta)-\phi'(\theta)]$ equal to zero.

Hence either

$$\phi(x)=\lambda\int_{-\sqrt{bi/c}}^{\sqrt{bi/c}} \frac{e^{\theta kx}}{(c\theta^2+b)^{\frac{1}{2}}}\,\phi(\theta)\,d\theta \tag{2}$$

satisfies equation (1) or

$$\phi(x)=\lambda\int_{k_1}^{k_2} \frac{e^{\theta kx}}{(c\theta^2+b)^{\frac{1}{2}}}\,\phi(\theta)\,d\theta, \tag{3}$$

where k_1 and k_2 are the values of θ for which $\left\{ kx\,\phi(\theta)-\phi'(\theta) \right\}$ vanishes.

Now in equation (1) put $c=\theta$, $b=1$, $A=n+\frac{1}{2}$ and $k=-\frac{1}{4}$ then we have

$$\frac{d^2y}{dn^2}+\left(n+\tfrac{1}{2}-\tfrac{1}{4}x^2\right)y=\theta$$

which is the equation of the parabolic cylinder function (Weber). The integral equation satisfying above will be given by

$$\phi(x)=\lambda\int_{-\infty}^{\infty} e^{\frac{1}{2}i\theta x}\,\phi(\theta)\,d\theta \tag{4}$$

To find out λ we may use Hermite's result

$$D_n(x)=(-1)^n\, e^{\frac{1}{4}x^2}\,\frac{d^n}{dx^n}\left(e^{-\frac{1}{2}x^2}\right)$$

$$D_1(x)=\lambda\int_{-\infty}^{\infty} e^{\frac{1}{2}i\theta x}\,D_1(\theta)\,d\theta$$

$$D_1(x)=\lambda\int_{-\infty}^{\infty} e^{\frac{1}{2}i\theta x-\frac{1}{4}\theta^2}\,\theta\,d\theta$$

$$=\lambda\, e^{-\frac{1}{4}x^2}\int_{-\infty}^{\infty} e^{-\frac{(\theta-xi)^2}{4}}\,\theta\,d\theta$$

$$=\lambda\, 2\,xi\,\sqrt{\pi}\; e^{-\frac{1}{4}x^2}$$

Comparing this value from Hermite's result we have $\lambda=\dfrac{1}{2i\sqrt{\pi}}$; similarly working out the results for $D_2(n)$, $D_3(n)$, $D_4(n)$ ......

we have λ respectively equal to $\frac{1}{2\sqrt{\pi}(i)^2}$, $\frac{1}{2\sqrt{\pi}(i)^3}$ etc.

Hence

$$D_n(x) = \frac{1}{2\sqrt{\pi}(i)^n} \int_{-\infty}^{\infty} e^{\frac{1}{2}i\theta x} \, D_n(\theta) \, d\theta *$$

where n is an integer.

§ 2. Starting from the above integral equation it is easy to obtain the recurrence formulæ.

$$D_n(x) = \frac{1}{2\sqrt{\pi}(i)^n} \int_{-\infty}^{\infty} e^{\frac{i\theta x}{2}} \, D_n(\theta) \, d\theta$$

$$D_n(x) = \frac{(-1)^n}{2\sqrt{\pi}\,(i)^n} \int_{-\infty}^{\infty} e^{\frac{i\theta x}{2}+\frac{1}{4}\theta^2} \, \frac{d^n}{d\theta^n}\left(e^{-\frac{1}{2}\theta^2}\right) d\theta$$

$$D_{n+1}(x) = \frac{(-1)^{n+1}}{2\sqrt{\pi}(i)^{n+1}} \int_{-\infty}^{\infty} e^{\frac{i\theta x}{2}+\frac{1}{4}\theta^2} \, \frac{d^{n+1}}{d\theta^{n+1}}\left(e^{-\frac{1}{2}\theta^2}\right) d\theta$$

$$= \frac{(-1)^{n+1}}{2\sqrt{\pi}(i)^{n+1}} \int_{-\infty}^{\infty} e^{\frac{i\theta x}{2}+\frac{1}{4}\theta^2} \left\{ \frac{d^n}{d\theta^n}\left(e^{-\frac{1}{2}\theta^2}\right) (-\theta) \right\} d\theta$$

$$= \frac{(-1)^{n+1}}{2\sqrt{\pi}(i)^{n+1}} \int_{-\infty}^{\infty} e^{\frac{i\theta x}{2}+\frac{1}{4}\theta^2} \left\{ -\theta \frac{d^n}{d\theta^n}\left(e^{-\frac{1}{2}\theta^2}\right) - n \frac{d^{n-1}}{d\theta^{n-1}}\left(e^{-\frac{1}{2}\theta^2}\right) \right\} d\theta$$

$$= \frac{(-1)^{n+2}}{2\sqrt{\pi}(i)^{n+1}} \int_{-\infty}^{\infty} e^{\frac{i\theta x}{2}+\frac{1}{4}\theta^2} \, \theta \frac{d^n}{d\theta^n}\left(e^{-\frac{1}{2}\theta^2}\right) d\theta \, -$$

$$\frac{(-1)^{n+1}}{2\sqrt{\pi}(i)^{n+1}} \, n \int_{-\infty}^{\infty} e^{\frac{i\theta x}{2}+\frac{1}{4}\theta^2} \left\{ \frac{d^{n-1}}{d\theta^{n-1}}\left(e^{-\frac{1}{2}\theta^2}\right) \right\} d\theta$$

$$= \frac{(-1)^{n+2}}{2\sqrt{\pi}(i)^{n+1}} \int_{-\infty}^{\infty} e^{\frac{i\theta x}{2}+\frac{1}{4}\theta^2} \, \frac{d^n}{d\theta^n}\left(e^{-\frac{1}{2}\theta^2}\right) d\theta + x \, D_{n-1}(x) \tag{5}$$

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* I am indebted to Prof. Dhar for this general result of λ.

Also

$$D_n'(x)=\frac{(-1)^n}{2\sqrt{\pi}(i)^n}\int_{-\infty}^{\infty} e^{\frac{i\theta n}{2}+\frac{1}{4}\theta^2}\left\{\frac{d^n}{d\theta^n}\left(e^{-\frac{1}{2}\theta^2}\right)\right\}\frac{i\theta}{2}\,d\theta$$

multiplying the above by -2 and substituting in (6)

we have $D_{n+1}(x)=-2D'_n(x)+nD_{n-1}(x)$

or $D_{n+1}(x)-n\ D_{n+1}(x)+2D'n(x)=o,$ (7)

where n is an integer.

But by Watson's* formulae, viz.,

$$D_n(\theta)=e^{-\frac{1}{4}\theta^2}\sum_{m=a}^{\infty}\frac{\Gamma(\frac{1}{2}m-\frac{1}{2}n)}{m!\,\Gamma-n}(\sqrt{2})^{m-n-2}(-\theta)^m$$

Also

$$D_n(x)=\frac{1}{2\sqrt{\pi}(i)^n}\int_{-\infty}^{\infty} e^{\frac{i\theta x}{2}}\ D_n(\theta)\ d\theta$$

$$D_n(x)=\frac{1}{2\sqrt{\pi}(i)^n}\int_{-\infty}^{\infty} e^{\frac{i\theta x}{2}-\frac{1}{4}\theta^2}\sum_{m=o}^{\infty}\frac{\Gamma\left(\frac{m}{2}-\frac{n}{2}\right)}{m!\,\Gamma-n}(\sqrt{2})^{m-n-2}$$

$$(-\theta)^m\,d\theta.$$

$$D'_n(x)=\frac{1}{2\sqrt{\pi}(i)^n}\int_{-\infty}^{\infty}\frac{i\theta}{2}\,e^{\frac{i\theta x}{2}-\frac{1}{4}\theta^2}\sum_{m=o}^{\infty}\frac{\Gamma(\frac{1}{2}m-\frac{1}{2}n)}{m!\,\Gamma-n}(\sqrt{2})^{m-n-2}$$

$$(-\theta)^m\,d\theta.$$

§ 3.

$$Dn(x)=\lambda\int_{-\infty}^{\infty} e^{\frac{i\theta x}{2}}\ Dn(\theta)\ d\theta$$

$$=\lambda\int_{-\infty}^{\infty} e^{\frac{i\theta x}{2}-\frac{1}{2}\left(\frac{i\theta}{2}\right)^2-\frac{1}{4}x^2}\quad e^{-\frac{\theta^2}{8}+\frac{x^2}{4}}\ D_n(\theta)\ d\theta.$$

Also $\quad e^{xt-\frac{1}{2}t^2-\frac{1}{4}x^2}=\sum_{m=o}^{\infty}\frac{D_m(x)}{m!}\,t^m$ †

So $\quad D_n(x)=\lambda\, e^{\frac{x^2}{4}}\int_{-\infty}^{\infty}\sum_{m=o}^{\infty}\frac{D_m(x)}{m!}\left(\frac{i\theta}{2}\right)^m e^{\frac{-\theta^2}{8}}\ D_n(\theta)\ d\theta.$

* Watson Lon. Math. Soc. 2 VIII, 1909—1910.

† Gorakh Prasad, Proc. Benares Math Soc., Vol. VIII (1925-26), pp. 47, 48.

$$= \lambda e^{x^2/4} \sum_{m=0}^{\infty} \frac{D_m(x)}{m!} \left(\frac{i}{2}\right)^m \int_{-\infty}^{\infty} e^{-\theta^2/8} \theta^m D_n(\theta)\, d\theta^*$$

But $\int_0^\infty e^{(\frac{1}{4}-a)x^2} x^m D_n(x)\, dx = \frac{\sqrt{\pi}\, 2^{\frac{1}{2}n-m-1}\, \Gamma(m+1)}{a^{\frac{1}{2}(m+1)}\, \Gamma(\frac{1}{2}m-\frac{1}{2}n+1)}$

$$F\left\{-\frac{n}{2}, \frac{m+1}{2}\, ;\, \frac{m-n}{2}+1, 1-\frac{1}{2a}\right\}$$

putting $a=\frac{3}{8}$ in the above we have

$$\int_0^\infty e^{-\frac{x^2}{8}} x^m D_n(x)dx = \frac{\sqrt{\pi}\, 2^{\frac{1}{2}n-m-1}\, \Gamma(m+1)}{(\sqrt{3})^{m+1}(\sqrt{2})^{-3(m+1)}\, \Gamma(\frac{1}{2}m-\frac{1}{2}n+1)}\, .$$

$$F\left\{-\frac{n}{2}, \frac{m+1}{2}\, ;\, \frac{m-n}{2}+1, -\frac{1}{3}\right\}$$

Now $\int_{-\infty}^{\infty} e^{-\frac{\theta^2}{8}} \theta^m D_n(\theta)\, d\theta = \int_0^\infty e^{-\frac{\theta^2}{8}} \theta^m D_n(\theta)\, d\theta +$

$$\int_{-\infty}^{0} e^{-\frac{\theta^2}{8}} \theta^m D_n(\theta)\, d\theta$$

$$= \int_0^\infty e^{-\theta^2/8} \theta^m D_n(\theta)\, d\theta + \int_0^\infty e^{-\frac{\theta^2}{8}} (-\theta)^m D_n(-\theta) d\theta$$

$$= \left\{1+(-1)^{m+n}\right\} \int_0^\infty e^{-\frac{\theta^2}{8}} \theta^m D_n(\theta)\, d\theta.$$

$$\therefore D_n(x) = \frac{1}{2\sqrt{\pi}(i)^n} e^{x^2/4} \sum_{m=0}^{\infty} \frac{D_m(x)}{m!} \left(\frac{i}{2}\right)^m \left\{1+(-1)^{m+n}\right\} \ldots$$

$$\frac{\sqrt{\pi}\, 2^{\frac{1}{2}n-m-1}\, \Gamma(m+1)}{(\sqrt{3})^{m+1}\, (\sqrt{2})^{-3(m+1)}\, \Gamma(\frac{1}{2}m-\frac{1}{2}n+1)} F\left\{-\frac{n}{2}, \frac{m+1}{2}\, ;\, \frac{m-n}{2}+1, -\frac{1}{3}\right\}$$

$$D_n(x) = e^{x^2/4} \sum_{m=0}^{\infty} \frac{D_m(x)}{m!}$$

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* Watson, Proc. Lond. Math. Soc. (2), VIII, 1910.

$$\frac{(i)^{m-n}\left\{1+(-1)^{m+n}\right\}\Gamma(m+1)}{(\sqrt{3})^{m+1}(\sqrt{2})^{-(n-m-1)}\Gamma(\frac{1}{2}m-\frac{1}{2}n+1)}$$

$$F\left\{-\frac{n}{2},\frac{m+1}{2};\frac{m-n}{2}+1,-\frac{1}{3}\right\} \quad (8)$$

§ 4. We have

$$e^{xt-\frac{1}{2}t^2-\frac{1}{4}x^2}=\sum_{m=o}^{\infty}\frac{D_m(x)}{m!}t^m$$

$$e^{kxt-\frac{1}{2}k^2t^2-\frac{1}{4}x^2}=\sum_{m=o}^{\infty}\frac{D_m(x)}{m!}k^m\,t^m$$

$$e^{kxt-\frac{1}{2}t^2-\frac{1}{4}x^2k^2}=\sum_{m=o}^{\infty}\frac{D_m(kx)}{m!}t^m$$

$$\therefore e^{-\frac{1}{4}x^2(k^2-1)}\,e^{-\frac{1}{2}t^2(1-k^2)}\sum_{m=o}^{\infty}\frac{D_m(x)}{m!}k^m\,t^m=\sum_{m=o}^{\infty}\frac{D_m(kx)}{m!}t^m$$

put $x=r$ and $k=e^{\theta i}$ then

$$e^{-\frac{1}{4}r^2(e^{2\theta i}-1)}\,e^{-\frac{1}{2}t^2(1-e^{2\theta i})}\sum_{m=o}^{\infty}\frac{D_m(r)}{m!}e^{m\theta i}\,t^m=$$

$$\sum_{m=o}^{\infty}\frac{D_m(re^{\theta i})}{m!}t^m$$

$$\therefore\sum_{m=o}^{\infty}\frac{D_m(re^{\theta i})}{m!}t^m=e^{-\frac{1}{4}n^2(e^{2\theta i}-1)}\left\{\sum_{n=o}^{\infty}\frac{\left\{-\frac{1}{2}t^2(1-e^{2\theta i})\right\}^n}{n!}\right\}$$

$$\sum_{m=o}^{\infty}\frac{D_m(r)}{m!}e^{m\theta i}\,t^m$$

Equating the co-efficient of t^p on either side we have

$$\frac{D_p(re^{\theta i})}{p!}=e^{-\frac{1}{4}r^2(e^{2\theta i}-1)}\sum_{m=o}^{p}\frac{(-\frac{1}{2})^{\frac{p-m}{2}}}{(\frac{p-m}{2})!}(1-e^{2\theta i})^{\frac{p-m}{2}}e^{m\theta i}\frac{D_m(r)}{m!}$$

put $\theta=\frac{\pi}{2}$ and we have

$$\frac{D_p(ri)}{p!}=e^{r^2/2}\sum_{m=o}^{p}\frac{(-\frac{1}{2})^{\frac{p-m}{2}}}{(\frac{p-m}{2})!}2^{\frac{p-m}{2}}e^{m\pi i}\frac{D_m(r)}{r!}$$

$$=e^{\frac{r^2}{2}}\sum_{m=o}^{p}\frac{(-1)^{\frac{p-m}{2}}}{(\frac{p-m}{2})!}e^{m\pi i}\frac{D_m(r)}{m!}$$

$$=e^{r^2/2}\left\{\frac{(-1)^{\frac{p}{2}}}{\left(\frac{p}{2}\right)!}\frac{D0(r)}{1!}+i\frac{(-1)^{\frac{p-1}{2}}}{\left(\frac{p-1}{2}\right)!}\frac{D1(r)}{1!}+(-1)\frac{(-1)^{\frac{p-2}{2}}}{\left(\frac{p-2}{2}\right)!}\cdot\cdot\right.$$

$$\left.\frac{D2(r)}{2!}+i(-1)\frac{(-1)^{\frac{p-3}{2}}}{\left(\frac{p-3}{2}\right)!}\frac{D3(r)}{3!}+\ldots\right\}$$

$$\therefore\frac{Dp(ri)}{p!}=e^{\frac{r^2}{2}}\left\{\frac{(-1)^{\frac{p}{2}}}{\left(\frac{p}{2}\right)!}\frac{D_0(r)}{1!}+i\frac{(-1)^{\frac{p-1}{2}}}{\left(\frac{p-1}{2}\right)!}\frac{D_1(r)}{1!}\right.$$

$$+i^2\frac{(-1)^{\frac{p-2}{2}}}{\left(\frac{p-2}{2}\right)!}\frac{D_2(r)}{2!}+i^3\frac{(-1)^{\frac{p-3}{2}}}{\left(\frac{p-3}{2}\right)!}\frac{D_3(r)}{3!}+\ldots\ldots$$

$$\left.+(i)^p\frac{Dp(r)}{r!}\right\}.$$

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ON SOME EXPANSIONS AND INTEGRALS INVOLVING THE PARABOLIC CYLINDER FUNCTIONS*

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In Art. 1 of the present paper certain indefinite integrals involving $D_n(z)$ are evaluated, in Art. 2 the expansion of the product $D_\rho(z)\ D_m(z)\ D_n(z)$ is obtained in series of the functions $D_n(z)$, in Art, 3 the integral $\int_{-\infty}^{\infty} D_\sigma(z)\ D_p(z)\ D_m(z)\ D_n(z)\, dz$ is evaluated and in Art. 4 an Addition theorem is found.

1. *Some indefinite integrals.* $D_n(y)$ is known to satisfy the following formulæ:—

$$\left.\begin{aligned} D'_n(y)+\tfrac{1}{2}y\, D_n(y)-n\, D_{n-1}(y)&=o \\ D_{n+1}(y)-y\, D_n(y)+n\, D_{n+1}(y)&=o \end{aligned}\right\}\dagger \qquad (1{\cdot}1)$$

$$\left.\begin{aligned} \frac{d}{dy}\left[e^{-\frac{1}{4}y^2} D_n(y)\right] &= -e^{-\frac{1}{4}y^2} D_{n+1}(y) \\ \frac{d}{dy}\left[y^{n+1} e^{-\frac{1}{4}y^2} D_n(y)\right] &= -y^n e^{-\frac{1}{4}y^2} D_{n+2}(y) \end{aligned}\right\}\ddagger \qquad (1{\cdot}2)$$

From (1·1) we have

$$\frac{d}{dy}\left[e^{-ay^2} y^\rho D_m(y)\, D_n(y)\right]$$
$$=\left[\rho y^{\rho-1}+(1-2a)\, y^{\rho+1}\right] e^{-ay^2} D_m(y)\ D_n(y)$$
$$-e^{-ay^2} y^\rho\left[D_m(y)\ D_{n+1}(y)+D_n(y)\ D_{m+1}(y)\right]$$

and

$$\frac{d}{dy}\left[e^{-ay^2} y^\rho\, D_{m+1}(y)\ D_{n+1}(y)\right]$$
$$=\left[\rho y^{\rho-1}-(1+2a)\, y^{\rho+1}\right] e^{-ay^2} D_{m+1}(y)\ D_{n+1}(y)$$
$$-e^{-ay^2} y^\rho\left[(m+1)\, D_m(y)\ D_{n+1}(y)+(n+1)\, D_n(y)\ D_{m+1}(y)\right]$$

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*I am indebted to Dr. P. L. Srivastava and Dr. Gorakh Prasad for their kind help and encouragement in the preparation of the paper.

†Whittaker and Watson, Modern Analysis, 4th Edition, p. 350.

‡The formulæ of (1·2) can be deduced from those of (1·1).

It follows therefore that

$$\int e^{-ay^2} y^\rho \left[m\, D_m(y)\, D_{n+1}(y) + n\, D_n(y)\, D_{m+1}(y) \right] dy$$
$$+ \int \left[\rho y^{\rho-1} + (1-2a)\, y^{\rho+1} \right] e^{-ay^2} D_m(y)\, D_n(y)\, dy$$
$$- \int \left[\rho y^{\rho-1} - (1+2a)\, y^{\rho+1} \right] e^{-ay^2} D_{m+1}(y)\, D_{n+1}(y)\, dy$$
$$= e^{-ay^2} y^\rho \left[D_m(y)\, D_n(y) - D_{m+1}(y)\, D_{n+1}(y) \right]^* \quad (1{\cdot}3)$$

From (1·2) we have

$$\frac{d}{dy} \left[e^{-\frac{1}{2}y^2} y^{m+n+\rho+2} D_m(y)\, D_n(y) \right]$$
$$= y^{m+n+\rho+1} e^{-\frac{1}{2}y^2} \times$$
$$\left[\rho D_m(y)\, D_n(y) - D_m(y)\, D_{n+2}(y) - D_n(y)\, D_{m+2}(y) \right] \quad (1{\cdot}4)$$

and

$$\frac{d}{dy} \left[e^{-\frac{1}{2}y^2} y^{n+\rho+1} D_m(y)\, D_n(y) \right]$$
$$= y^{n+\rho} e^{-\frac{1}{2}y^2} \times$$
$$\left[\rho D_m(y)\, D_n(y) - D_m(y)\, D_{n+2}(y) - y\, D_n(y)\, D_{m+1}(y) \right] \quad (1{\cdot}5)$$

whence

$$\int e^{-\frac{1}{2}y^2} y^{m+n+\rho+1} \left[\rho D_m(y)\, D_n(y) - D_m(y)\, D_{n+2}(y) - D_n(y)\, D_{m+2}(y) \right] dy$$
$$= e^{-\frac{1}{2}y^2} y^{m+n+\rho+2} D_m(y)\, D_n(y) \quad (1{\cdot}6)$$

and

$$\int e^{-\frac{1}{2}y^2} y^{n+\rho} \left[\rho D_m(y)\, D_n(y) - D_m(y)\, D_{n+2}(y) - y\, D_n(y)\, D_{m+1}(y) \right] dy$$
$$= e^{-\frac{1}{2}y^2} y^{n+\rho+1} D_m(y)\, D_n(y) \quad (1{\cdot}7)$$

As a special case, let us put $\rho = o$ in (1·3), (1·6) and (1·7) then

$$\int e^{-ay^2} \left[m\, D_m(y)\, D_{n+1}(y) + n\, D_n(y)\, D_{m+1}(y) \right] dy$$
$$+ \int y\, e^{-ay^2} \left[(1-2a)\, D_m(y)\, D_n(y) + (1+2a)\, D_{m+1}(y)\, D_{n+1}(y) \right] dy$$
$$= e^{-ay^2} \left[D_m(y)\, D_n(y) - D_{m+1}(y)\, D_{n+1}(y) \right] \quad (1{\cdot}8)$$

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* For the special case $a = o$. *Cf.* R. S. Varma Tohoku Math. Journal, Vol. 34, Part I, 1931.

$$\int e^{-\frac{1}{2}y^2} y^{n+m+1} \left[D_m(y)\, D_{n+2}(y)+D_n(y)\, D_{m+2}(y)\right] dy$$
$$=-e^{-\frac{1}{2}y^2}\, y^{m+n+2}\, D_m(y)\, D_n(y) \tag{1·9}$$

and

$$\int e^{-\frac{1}{2}y^2} y^n \left[D_m(y)\, D_{n+2}(y)+y\, D_n(y)\, D_{m+1}(y)\right] dy$$
$$=-e^{-\frac{1}{2}y^2}\, y^{n+1}\, D_m(y)\, D_n(y) \tag{1·10}$$

Further if m be equal n we obtain from (1·8), (1·9) and (1·10)

$$2n\int e^{-ay^2} D_n(y)\, D_{n+1}(y)\, dy$$
$$+\int y\, e^{-ay^2} \left[(1-2a)\, D^2{}_n(y)+(1+2a)\, D^2{}_{n+1}(y)\right] dy$$
$$=e^{-ay^2}\left[D^2{}_n(y)-D^2{}_{n+1}(y)\right] \tag{1·11}$$

$$\int e^{-\frac{1}{2}y^2} y^{2n+1}\, D_n(y)\, D_{n+2}(y)\, dy$$
$$=-\tfrac{1}{2}\, e^{-\frac{1}{2}y^2}\, y^{2n+2}\, D^2{}_n(y) \tag{1·12}$$

and

$$\int e^{-\frac{1}{2}y^2} y^n \left[D_n(y)\, D_{n+2}(y)+y\, D_n(y)\, D_{n+1}(y)\right] dy$$
$$=-e^{-\frac{1}{2}y^2}\, y^{n+1}\, D^2{}_n(y) \tag{1·13}$$

If $a=\frac{1}{2}$ or $-\frac{1}{2}$, (1·11) gives us

$$\int e^{-\frac{1}{2}y^2} \left[2n\, D_n(y)\, D_{n+1}(y)+2y\, D^2{}_{n+1}(y)\right] dy$$
$$=e^{-\frac{1}{2}y^2}\left[D^2{}_n(y)-D^2{}_{n+1}(y)\right] \tag{1·14}$$

$$\int e^{\frac{1}{2}y^2} \left[2n\, D_n(y)\, D_{n+1}(y)+2y\, D^2{}_n(y)\right] dy$$
$$=e^{\frac{1}{2}y^2}\left[D^2{}_n(y)-D^2{}_{n+1}(y)\right] \tag{1·15}$$

2. *The expansion of $D_o(z)\, D_m(z)\, D_n(z)$.*

It has been shown that*

$$D_o(z)\, D_{2n}(z)=(-1)^n\, (2\pi)^{-1/2}\, e^{\frac{1}{4}z^2}\times$$
$$\sum_{m=o}^{\infty} (-1)^m\, \Gamma\left(\frac{2n+2m+1}{2}\right) \frac{D_{2m}(z)}{\lfloor 2m} \tag{2·1}$$

and

$$D_o(z)\, D_{2n+1}(z)=(-1)^n\, (2\pi)^{-1/2}\, e^{\frac{1}{4}z^2}\times$$
$$\sum_{m=o}^{\infty} (-1)^m\, \Gamma\left(\frac{2n+2m+3}{2}\right) \frac{D_{2m+1}(z)}{\lfloor 2m+1} \tag{2·2}$$

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* R. S. Varma, Proc. Benares Math. Soc., Vol. X, pp. 17-18.

Again, if $p \not> n$, then*

$$D_p(z)\ D_n(z) = D_o(z)\ \left[D_{n+p}(z) + p.nD_{n+p-2}(z) + \frac{p.(p-1)}{1\cdot 2}\ n(n-1)\ D_{p+n-4}(z) + \ldots\ \ldots\right] \quad (2\cdot 3)$$

If we substitute the expansions of each of the terms on the right side of (2·3) and collect the co-efficients of similar terms together, we arrive at the following expansions:—

If $p+n$ is even

$$D_o(z)\ D_p(z)\ D_n(z) = (-1)^n\ (2\pi)^{-1/2} \times \sum_{m=o}^{\infty} (-1)^m\ a_m\ \frac{D_{2m}(z)}{\lfloor 2m} \quad (2\cdot 4)$$

Where, $a_m = \sum_{r=o}^{p} \frac{(p)!\ (n)!}{r!\ (p-r)!\ (n-r)!} \Gamma\left(\frac{n+p+2m-2r+1}{2}\right)$

If $p+n$ is odd

$$D_o(z)\ D_p(z)\ D_n(z) = (-1)^n\ (2\pi)^{-1/2} \times \sum_{m=o}^{\infty} (-1)^m\ b^m\ \frac{D_{2m+1}(z)}{\lfloor 2m+1} \quad (2\cdot 5)$$

Where, $b^m = \sum_{r=o}^{p} \frac{(p)!\ (n)!}{r!\ (p-r)!\ (n-r)!} \Gamma\left(\frac{n+p+2m-2r+3}{2}\right)$

3. We can now evaluate the definite integral

$$\int_{-\infty}^{\infty} D_o(z)\ D_p(z)\ D_n(z)\ D_m(z)\ dz$$

If $m+n+p$ is odd

$$\int_{-\infty}^{\infty} D_o(z)\ D_p(z)\ D_n(z)\ D_m(z)\ dz = o \quad (3\cdot 1)$$

If $m+n+p$ is even, we have by virtue of the two equations†

$$\int_{-\infty}^{\infty} D_m(z)\ D_n(z)\ dz = o \qquad m \neq n \quad (3\cdot 2)$$

and $$\int_{-\infty}^{\infty} \left\{ D_n(z) \right\}^2 dz = (2\pi)^{1/2}\ n! \quad (3\cdot 3)$$

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* Gorakh Prasad, Proc. Benares Math Soc., Vol. II, p. 18 (1920).

† Whittaker and Watson, *Loc. cit*, pp. 350-51.

the relation

$$\int_{-\infty}^{\infty} D_0(z)\ D_p(z)\ D_n(z)\ D_m(z)\ dz$$

$$=(-1)^{\frac{m+2n}{2}} \sum_{r=o}^{p} \frac{(p)!\ (n)!}{r!\ (p-r)!\ (n-r)!}\ \Gamma\left(\frac{n+p+m-2r+1}{2}\right)$$

(for the case $p+n$ even)

$$=(-1)^{\frac{2n+m-1}{2}} \sum_{r=o}^{p} \frac{(p)!\ (n)!}{r!\ (p-r)!\ (n-r)!}\Gamma\left(\frac{n+p+m-2r+2}{2}\right)$$

(for the case $p+n$ odd) (3·4)

4. *Addition theorem.*

It is known* that

$$e^{ut-\frac{t^2}{2}-\frac{u^2}{4}}=\sum_{o}^{\infty} \frac{D_n(u)}{n!}\ t^n \tag{4·1}$$

From (4·1) we obtain

$$\sum_{o}^{\infty} D_n(u \sin a+v \cos a)\ \frac{t^n}{n!}$$

$$=e^{(u \sin a+v \cos a)\ t-\frac{t^2}{2}-\frac{(u \sin a+v \cos a)^2}{4}}$$

$$=e^{ut \sin a-\frac{t^2 \sin^2 a}{2}-\frac{u^2}{4}} \times$$

$$e^{vt \cos a-\frac{t^2 \cos^2 a}{2}-\frac{v^2}{4}}\ e^{-\frac{(u \sin a+v \cos a)^2}{4}+\frac{u^2+v^2}{4}}$$

or

$$\sum_{o}^{\infty} D_n(u \sin a+v \cos a)\ \frac{t^n}{n!}=e^{-\frac{(u \sin a+v \cos a)^2}{4}+\frac{u^2+v^2}{4}} \times$$

$$\left\{\sum_{o}^{\infty} D_n(u)\ \frac{t^n \sin^n a}{n!}\right\} \left\{\sum_{o}^{\infty} D_n(v)\ \frac{t^n \cos^n a}{n!}\right\}$$

Equating the co-efficients of t^n on both the sides we have

$$D_n\ (u \sin a+v \cos a)=e^{\frac{(u \cos a-v \sin a)^2}{4}} \times$$

$$[D_o(u)\ D_n(v)\ \cos^n a+n\ D_1(u)\ D_{n-1}(v)\ \sin a \cos^{n-1} a$$

$$+\frac{n(n-1)}{2}\ D_2(u)\ D_{n-2}(v)\ \sin^2 a\ \cos^{n-2} a+\ldots\ldots\ldots]$$

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* Hari Shanker, Proc. Benares Math. Soc., Vol. VI (1924) For a direct proof, *see* Gorakh Prasad, Proc. Benares Math. Soc., Vol. VII—VIII (1925-26).

or

$$D_n(u \sin a + v \cos a) = e^{\frac{(u \cos a - v \sin a)^2}{4}} \times$$

$$\sum_{m=0}^{n} {}^nC_m \; D_m(u) \; D_{n-m}(v) \sin^m a \cos^{n-m} a \quad (4{\cdot}2)$$

If we put

$\sin a = \cos a = \frac{1}{\sqrt{2}}$, $u = \sqrt{2}\, x$ and $v = \sqrt{2}\, y$ in (4·2), we have*

$$D_n(x+y) = 2^{-n/2} \, e^{\frac{(x-y)^2}{4}} \times$$

$$\sum_{m=0}^{n} {}^nC_m \; D_m(\sqrt{2}\, x) \; D_{n-m}(\sqrt{2}\, y) \quad (4{\cdot}3)$$

* Hari Shanker, Proc. Benares Math. Soc., Vol. VI, p. 14 (1924).

ON THE REDUCIBILITY OF THE GENERAL ELLIPTIC INTEGRAL INTO LOGARITHMS

By K. VENKATACHALIENGAR.

In the following paper I have considered the question of the reduction of the general elliptic integral into logarithms. Abel[1] has considered the problem in an algebraic manner and has given some sufficient conditions. Special cases were considered by Dolbnia[2], Goursat[3] and Halphen[4] and others. My aim in the following paper is to give a set of conditions which do not involve relations with the elliptic transcendentals, namely η, ω, etc. I give in the following paper a set of sufficient conditions for the reducibility, not involving the elliptic transcendentals in any way, which are believed to be new.

For the formation of my conditions the knowledge of the location of the logarithmic critical points is necessary. Consistent with the previous results obtained by others in connection with particular results, these logarithmic critical points are situated at points given by

$$a_r = \frac{2p\omega_1 + 2q\omega_2}{S}$$

where p, q, S are integers and $2\omega_1$, and $2\omega_2$ are the periods of the elliptic integral. Although this condition appears to be involving the transcendentals it is known that the condition can be given in a rationalized form. After this condition, I have given two other conditions, one of them being necessary, *viz.*, the co-efficient of the integral function corresponding to the sum of all the integrals of the second kind. The other condition is towards creating the sufficiency. The last two conditions, after the knowledge of the location of the logarithmic critical points, are formed with the co-efficients of the function that is to be integrated, and radicals involving the logarithmic critical points. This is, however, the general case. If some more conditions which are given in the body of the paper, as well as the conditions set forth below are satisfied, then the integral reduces itself into logarithms.

1. Oevres.
2. Oevres.
3. Series of papers published in Bull. de la Soc. Maths. (1880-1900).
4. Functions Elliptiques.

The conditions are—

Let the logarithmic critical points be given by

$$\left\{ a_1, a_2, a_3, \ldots\ldots\ldots\ldots a_n \right\}.$$

Then the set of conditions are

$$(1)\ldots\left\{\begin{array}{l} l_1\, a_1 + l_2\, a_2 + \ldots\ldots\ldots\ldots\ldots + l_r\, a_r = \dfrac{2p_1\,\omega_1 + 2q_1\,\omega_2}{S_1} \\ l_{r+1}\, a_{r+1} + \ldots\ldots\ldots\ldots\ldots + l_t\, a_t = \dfrac{2p_2\,\omega_1 + 2q_2\,\omega_2}{S_2} \\ l_{t+1}\, a_{t+1} + \ldots\ldots\ldots\ldots\ldots + l_x\, a_x = \dfrac{2p_3\,\omega_1 + 2q_3\,\omega_2}{S_3} \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ l_s\, a_s \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots + l_n\, a_n = \dfrac{2p_k\,\omega_1 + 2q_k\,\omega_2}{S_k} \end{array}\right.$$

where all the letters except the a's are integers, positive or negative.

Suppose we consider the integral of the form

$$(2)\ \ldots \int \frac{a_m x^m + a_{m-1}\, x^{m-1} + \ldots + a_o}{b_n\, x^m + \ldots\ldots\ldots\ldots + b_o} \cdot \frac{dx}{\sqrt{4x^3 - g_2 x - g_3}}.$$

Then dividing the numerator by the denominator (if $m > n$) and substituting $x = p\ (u,\ g_2,\ g_3)$ we obtain two integrals

$$(3)\ldots\ldots \int (k_o + k_1\, p + \ldots\ldots\ldots\ldots\ldots + k_q\, p^q)\, du$$

$$(4)\ldots\ + \int \left(\frac{c_o + c_1\, p + c_2\, p^2 + \ldots\ldots\ldots + c_s\, p^s}{(p - a_1)^{\lambda_1}\, (p - a_2)^{\lambda_2} \ldots\ldots (p - a_r)^{\lambda_r}} \right) du$$

where $a_r = p\ (\alpha_r)$, $(r = 1, 2, \ldots\ldots\ldots\ldots n)$.

First of all, it is easily seen that the condition for the reducibility of the integral into logarithms is that the integral should consist of a number of logarithms of elliptic functions and terms expressed rationally in terms of the elliptic functions used, *i.e.*,

$$p(u) \text{ and } p'(u)$$

Hence in order to find a criterion for the reducibility it is enough to consider the terms that are not periodic. Such terms will be of the following three forms:—

(a) u, (b) $\zeta(u)$, (c) $\operatorname{Log}\left\{ \sigma(u + \alpha_r) \right\}$.

Now first of all $\zeta(u)+du$, where d is a constant can never represent a doubly periodic function; for, if it does, then

$$2\,d\omega_1 - \eta_1 = 0$$
$$\text{and} \quad 2\,d\omega_2 - \eta_2 = 0;$$

This is certainly inconsistent with the relation

$$\eta_1\,\omega_2 - \eta_2\,\omega_1 = \pi i$$

and $\zeta(u)$ combined with terms in (a) and in (c) can never be represented as a number of logarithms of elliptic functions, for, $e^{\zeta(u)}$ has an isolated essential singularity at $u=0$ which can never be made regular by the addition of any number of terms of the form (a) and (c). Hence for the reducibility it is necessary that the co-efficient of $\zeta(u)$ should vanish. This co-efficient is evidently rationally expressible in terms of the a_r's and the co-efficients of the various terms in the original integral (2), and $\sqrt{4a^3_r - g_2\,a_r - g_3}$'s.

On returning to our original integral (2) I determine the co-efficients of $\zeta(u)$ and u. We know that the polynomial in integral (3) can be written in the form

$$d_o+d_1\,p(u)+d_2\,p'(u)+\ldots\ldots+d_k\,p^{(k)}(u)\,;$$

hence, we see that the co-efficients of u and $\zeta(u)$ in integral (3) are d_o and d_1 respectively. Next as regards (4), on expressing the fraction in the integral in partial fractions, we arrive at expressions of the form

$$\int\frac{du}{[p(u)-p(a)]^r}\,;$$

now we can calculate this integral by means of known recurrence formulae. As the form of one term in it is wanted for my conditions I propose to obtain it in this way.

$$\frac{p'(a)}{p(u)-p(a)}=\zeta(u+a)-\zeta(u-a)-2\zeta(a).$$

Differentiating this successively with respect to a, we get

$$\frac{1}{[p(u)-p(a)]^r}=f[p(a),\,p'(a)]\times M+\ldots\ldots\ldots\ldots$$
$$\ldots\ldots+A\,[p(a),\,p'(a)]\,\{\,p(u+a)+p(u-a)\,\}$$
$$+\Psi\,[p(a),\,p'(a)]$$

Where

$$M=\zeta(u+a)-\zeta(u-a)-2\zeta(a)\,;$$

and the functions f, Ξ and Ψ are rational functions of $p(a)$ and $p'(a)$.

And finally as $\zeta(u+a)-\zeta(u)$ is an elliptic function we can write the expression after integrating the integral (4), in the following form:—

$$\sum_a f_a \log e^{-2\zeta(a)} \frac{\sigma(u+a)}{\sigma(u-a)} + e_0 u + e_1 \zeta(u).$$

+ terms which can be expressed rationally in terms of $p(u)$, and $p'(u)$. Hence our integral (2) is pseudo-elliptic if the following function, *viz.*,

$$\sum_a f_a \log e^{-2\zeta(a)} \frac{\sigma(u+a)}{\sigma(u-a)} + (e_0+d_0)\, u + (e_1+d_1)\, \zeta(u)$$

is expressible as a finite combination of logarithms of elliptic functions.

Now I propose to find a set of sufficient conditions for the reducibility.

The first necessary condition that is obtained is, of course,

(A) .... $(e_1+d_1)=0$.

Now suppose we write the other terms in the following form

$$k\, u + \sum_a f_a \log e^{[-2\zeta(a)-\lambda_a]u} \times \frac{\sigma(u+a)}{\sigma(u-a)}$$

where the λ_a's are arbitrary which are fixed later on, and k is expressible in terms of λ's, a_r's and $\sqrt{4\, a_r^3 - g_2\, a_r - g_3}$'s rationally.

So also is the condition $e_1+d_1=0$.

Now we shall find the condition for the expression

$$f_a \log e^{-[2\zeta(a)+\lambda_a]u} \times \frac{\sigma(u+a)}{\sigma(u-a)}$$

to be the logarithm of an elliptic function [of course, I always mean, with the same periods] writing this in the form

$$\frac{f_a}{R_a} \log e^{-R_a[2\zeta(a)+\lambda_a]u} \times \left[\frac{\sigma(u+a)}{\sigma(u-a)}\right]^{R_a}$$

where R_a is an integer.

The condition for

$$e^{-R_a[2\zeta(a)+\lambda_a]u} \times \left[\frac{\sigma(u+a)}{\sigma(u-a)}\right]^{R_a}$$

to be an elliptic function with $2\omega_1$ and $2\,\omega_2$ as periods is obtained easily, *viz.*,

$$-\omega_1\, [2\zeta(a)+\lambda_a] + \eta_1\, a = \frac{r\,\pi\, i}{R_a}$$

$$-\omega_2\,[2\zeta(a)+\lambda_a]+\eta_2\,a=\frac{s\,\pi\,i}{R_a}$$

where r and s are integers. Using the fact that

$\eta_1\omega_2-\eta_2\omega_1=\pi i$, we obtain the condition for this to be

$$a=\frac{2\,r\omega_1+2\,s\omega_2}{R_a}$$

$$\zeta\,(a)+\frac{\lambda_a}{2}=\frac{2\,r\eta_1+2\,s\eta_2}{R_a}$$

and corresponding conditions for the remaining terms. Here the second condition by means of an artifice can be put in a rational form in the following way.

The condition is

$$\frac{R_a\,\lambda_a}{2}=r\eta_1+s\eta_2-R_a\,\zeta\left[\frac{2\,r\omega_1+2\,s\omega_2}{R_a}\right]$$

Now we know that

$Z(a)=\zeta[R_a-1)a]-(R_a-1)\ \zeta(a)$ can be expressed rationally in terms of $p(a)$ and $p'(a)$. Now this function is evidently equal to

$$-\zeta\,(a)-(R_a-1)\ \ \zeta\,(a)+r\eta_1+s\eta_2$$

$$=-R_a\,\zeta\,(a)+r\eta_1+s\eta_2=\frac{R_a\lambda_a}{2}$$

Hence the λ's are determined rationally in terms of $[p(a_r)]$'s and $[p'(a_r)]$'s; and therefore k is also expressible in terms of the above functions rationally.

The other condition for reducibility after the condition that are indicated for the a's are satisfied is that k, which depends upon the λ's should be equal to zero. Hence this is the other condition which is certainly rational in

(a_r)'s and $[\sqrt{4\,a_r^3-g_2\,a_r\,-\,g_3}\,]$'s.

In order to obtain a rational condition in place of the condition that are to be satisfied in the case of the a's. We may proceed in the following way. Now from the expression of $(p(a), p(2a), p(3a), \ldots$ which are formed rationally in terms of

a_r, and $\sqrt{4\,a_r^3-g_2\,a_r-g_3}$;

If this series is such [and it is obvious that if we have $a=\dfrac{2r\omega_1+2s\omega_2}{R_a}$, then $p[R_a-1)a]=p(a)]$ that after a certain stage we get one term in the series which is equal to $p(a)$, and the next term becomes infinite, and the other terms once again become equal to $p(a)$, $p(2a)$, . . . then $a=\frac{2r\omega_1+2s\omega_2}{R_a}$. Hence the conditions that the a's should satisfy can be put in the above form.

Now this is a general condition, but in particular cases it may so happen that the criterion is widened. For illustration, I have taken the case where there are only five logarithmic critical points, *viz.*, a, β, γ, δ, ϵ. But the general case can also be dealt with similarly.

Suppose that integers l, m, n, P, Q exist such that

$$\frac{f_a}{l}=\frac{f_\beta}{m}=\frac{f_\gamma}{n} \text{ and } \frac{f_\delta}{P}=\frac{f_\epsilon}{Q}$$

Then we can combine the terms in the following way

$$\frac{f_a}{lR_a}\log\left\{e^{-[2l\zeta(a)+2m\zeta(\beta)+2n\zeta(\gamma)+\lambda_a]R_a u}\times\left[\frac{\sigma(u+a)}{\sigma(u-a)}\right]^{lR_a}\times\left[\frac{\sigma(u+\beta)}{\sigma(u-\beta)}\right]^{lR_\beta}\times\left[\frac{\sigma(u+\gamma)}{\sigma(u-\gamma)}\right]^{lR_\gamma}\right\}$$

$$+\frac{f_\delta}{PR_\delta}\log\left\{e^{-[2P\zeta(\delta)+2Q\zeta(\epsilon)+\lambda_\delta]R_\delta u}\times\left[\frac{\sigma(u+\delta)}{\sigma(u-\delta)}\right]^{PR_\delta}\times\left[\frac{\sigma(u+\epsilon)}{\sigma(u-\epsilon)}\right]^{QR_\epsilon}\right\}$$

Now the conditions in this case are

$$la+m\beta+n\gamma=\frac{2r\omega_1+2s\omega_2}{R_a}$$

$$\text{and}\quad P\delta+Q\epsilon=\frac{2p\omega_1+2q\omega_2}{R_\delta};$$

which again can be put in the proper rational form in a similar manner

and two other conditions which are

$$\lambda_a=-l\zeta(a)-m\zeta(\beta)-n\zeta(\gamma)+\frac{r\eta_1+s\eta_2}{R_a}$$

$$\lambda_\delta=-P\zeta(\delta)-Q\zeta(\epsilon)+\frac{p\eta_1+q\eta_2}{R_\delta}$$

Here also by means of a similar artifice we can express the λ's rationally in terms of

(a_r)'s and $[\sqrt{4a_r^3-g_2a_r-g_3}]$'s.

Now

$\zeta(l\alpha+m\beta+n\gamma)-l\zeta(\alpha)-m\zeta(\beta)-n\zeta(\gamma)$ can be expressed rationally in terms of $p(\alpha)$, $p'(\alpha)$, $p(\beta)$, $p'(\beta)$, etc. Hence λ_α can be expressed in terms of the above quantities, and so is λ_δ expressible. Therefore the condition corresponding to $k=0$ is also expressed rationally in terms of

$$(a_r)\text{'s and } [\sqrt{4\,a_r^3-g_2\,a_r-g_3}\,]\text{'s.}$$

As a corollary the following interesting results can be obtained *viz.*:

If a general elliptic integral which consists only of integrals of the first and second kinds [in that case only it is reducible to logarithms] is such that the places at which the integral has logarithmic critical points, namely $[a_1, a_2, a_3, \ldots a_n]$ are such that

$$a_r=\frac{2\,p_r\omega_1+2\,q_r\omega^2}{k_r},\ (r=1, 2, 3, \ldots n,)$$ then by adding an integral of the form

$$k\int\frac{dx}{\sqrt{4\,x^3-g_2\,x-g_3}}$$ it is reducible into logarithms.

HEILBRONN'S CLASS-NUMBER THEOREM

By S. Chowla.

This note contains a slightly modified version of Heilbronn's proof of his class-number theorem.[1] In particular my proof is independent of the theory of ideals.

1. The notation used here is the same as in Heilbronn's paper ("On the class-number in imaginary quadratic fields," to be published shortly). The following additional ideas are introduced.

Let $\chi_1(n)$ denote a real primitive character $(mod\ m_1)$ $(m_1 > o)$ such that

$$\sum_{n=1}^{\infty} \chi_1(n)\, n^{-s} = O$$

for some $s=\rho$ in the half-plane $\sigma > \frac{1}{2}$.

Let $p_1, p_2, p_3 \ldots\ldots$ denote the primes in ascending order of magnitude which are not contained in m_1. We now choose m so that

(1) $m = m_1\, p_1\, p_2\, p_3 \ldots\ldots p_r$

where r will be defined later.

We define $\chi(n)$ a character $(mod\ m)$ so that

$$\chi(n) = \chi_1(n) \qquad [(n, m)=1],$$

$$\chi(n) = O \qquad [(n, m)>1].$$

It follows that

$$L_o(s) = L(s, \chi) = \sum_1^{\infty} \chi(n)\, n^{-s}$$

vanishes for $s=\rho$ where

(2) $\rho = \theta + i\,\phi \qquad (\frac{1}{2} < \theta < 1).$

A is an absolute positive constant[2] such that for $1 \leqslant l_2 \leqslant m$, $\frac{1}{2} < \sigma < 1$, we have

(3) $m^{-2\sigma}\, \zeta\left(2s-1, \frac{l_2}{m}\right) = O\,(mA-2), \quad [A>2].$

---

(1) Equation (11) of this paper. $H=h\,(d)$ is the number of primitive classes of binary quadratic forms of negative discriminant—d.

(2) It is easy to prove the existence of such an A.

We define r as the least positive integer satisfying

$$(4) \quad |d|^{\frac{\theta}{4}-\frac{1}{6}} < mA < |d|^{\frac{\theta}{2}-\frac{1}{3}}$$

where[3] $\theta > \frac{3}{4}$. Since m_1 is fixed it follows from (1) and (4) that

$$(5) \quad r \to \infty \text{ as } -d \to \infty.$$

The constants implied in our O symbols depend only on m_1, s and ρ but they are independent of r, d and H.

2. *Proof of the theorem.*—Following Heilbronn we prove that (for details see the concluding section 3).

$$(6) \quad O = Z(2\rho) \prod_{p/m} (1-p^{-2\rho}) \sum_{a} \chi(a)\, a^{-\rho} + O\left(HmA\,|d|^{\frac{1}{4}-\frac{\theta}{2}}\right) + O\left(HmA\,|d|^{-\frac{1}{2}\theta}\right).$$

Further, since $\chi(a)$ is O unless a is prime to m, it follows that

$$(7) \quad \left|\sum_{a} \chi(a)\, a^{-\rho}\right| \geqslant 1 + O\left(\sum_{\substack{a \\ a>k}} a^{-\rho}\right)$$

where k is the least positive integer prime to m. From (1) and (5) we obtain:

$$(8) \quad k \to \infty \text{ as } -d \to \infty.$$

Since

$$(9) \quad \sum_{\substack{a \\ a>k}} a^{-\rho} = O\left(Hk^{-\theta}\right)$$

it follows from (9), (8) and (7) that if H is bounded, then

$$(10) \quad \left|\sum_{a} \chi(a) a^{-\rho}\right| \geqslant \tfrac{1}{3}.$$

If H is bounded the equations (4), (6) and (10) contain a contradiction as $-d \to \infty$, since the moduli of the terms

$$Z(2\rho) \text{ and } \prod_{p/m} (1-p^{-2\rho})$$

are greater than absolute constants $(\theta > \frac{3}{4})$.

Hence

$$(11) \quad H = h(d) \to \infty \text{ as } -d \to \infty,$$

the desired result.

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(3) If $\theta \leqslant \frac{3}{4}$ for all m_1, then by a theorem of Hecke, $H = h(d) \to \infty$ as $-d \to \infty$.

3. From (3) and the proof of lemma 9 of Heilbronn's paper it follows that[4]

$$(12)\quad \psi(s)=O\left(mA-2\ |\ d\ |^{\frac{1}{4}-\frac{\sigma}{2}}+|\ d\ |^{-\frac{1}{2}\sigma}\right).$$

From (12) and the proof of lemma 10 of Heilbronn's paper it follows that[5]

$$(13)\quad L_o(s)\ L_2(s)=Z(2s)\ \prod_{p/m}(1-p^{-2s})\ \sum_a \chi(a)\ a^{-s}$$

$$+O(HmA\ |\ d\ |^{\frac{1}{4}-\frac{1}{2}\sigma}+Hm2\ |\ d\ |^{-\frac{1}{2}\sigma}),$$

whence putting $s=\rho$ we get

$$(14)\quad 0=Z(2\rho)\ \prod_{p/m}(1-p^{-2\rho})\ \sum_a \chi(a)\ a^{-\rho}$$
$$+O(HmA\ |\ d\ |^{\frac{1}{4}-\frac{1}{2}\theta}+Hm2\ |\ d\ |^{-\frac{1}{2}\theta})$$

and this is the same as (6) since $A>2$.

---

(4) $\psi(s)=\sum_{\substack{y=1\\ y\equiv l_2(m)}}^{\infty}\ \sum_{\substack{x=-\infty\\ x\equiv l_1(m)}}^{\infty}(ax^2+bxy+cy^2)^{-s}$ for $\sigma>1$.

(5) For $\sigma>\frac{1}{2}$, $s\neq 1$.

COLLINEATIONS IN PATH-SPACE

BY D. D. KOSAMBI (*Poona, India*).

A geometry attached to systems of second order differential equations of the generic type

$$(1) \quad \ddot{x}^i + a^i(x, \dot{x}, t) = 0 \quad \dot{x}^i = \frac{dx^i}{dt} \text{ etc.} \quad (i = 1 \ldots n)$$

has been discussed elsewhere[1]. Curves representing solutions of (1) can be regarded as the generalized autoparallel lines or *paths* of a space, and the intrinsic differential geometry thereof is developed from two main assumptions: (*a*) the tensor invariance of all fundamental equations, including (1), and (*b*) the existence of a vectorial operator, the vanishing of which defines a parallelism making solutions of (1) autoparallel lines.

I here attempt to investigate a special type of path-space which allows continuous groups of deformations carrying paths into paths,

Let $u^i(x)$ be a vector field representing an infinitesimal transformation of such a group by means of the "small displacement"

$$\bar{x}^i = x + u^i \, \delta\xi$$

Then the functions u^i must satisfy the equations of variation of (1):

$$(2) \quad \ddot{u}^i + a^i_{;r} \dot{u}^r + a^i_{,r} u^r = 0$$

As usual, a repeated index denotes summation; moreover, $f_{;k} = \frac{df}{d\dot{x}^k}$ and $f_{,k} = \frac{df}{dx^k}$. Inasmuch as the operator for total differentiation with respect to t is

$$\frac{d}{dt} = -a^r \frac{d}{d\dot{x}^r} + \dot{x}^r \frac{d}{dx^r} + \frac{d}{dt}$$

we find that (2) reduce to

$$(2') \quad u^i_{,m,r} \dot{x}^m \dot{x}^r - a^r u^i_{,r} + a^i_{;j} u^j_{,r} \dot{x}^r + a^i_{,j} u^j = 0.$$

Let it be further assumed that a^i has the form of a polynomial in $\dot{x}$:

$$(3) \quad a^i = A^i + A^i_h \dot{x}^h + \Gamma^i_{hr} \dot{x}^h \dot{x}^r + \ldots + A^i_{h_1 \ldots h_m} \dot{x}^{h_1} \ldots \dot{x}^{h_m} + \ldots$$

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(1) D. D. Kosambi *Rendiconti della Reale Accademia dei Linceit* pp.

D. D. Kosambi *Math. Zeitschrift*, *Bd.* 37, (1933), pp. 608, 618.

The coefficients A, Γ are functions of $\dot{x}$ alone, symmetric in all subscripts; the letter Γ has been used for the quadratic terms only, for reasons that will be apparent later.

In the previous papers referred to, as well as in a remarkable exposition by M. Cartan[2], it was shown that

$$a^i - \tfrac{1}{2}\dot{x}^r a^i{}_{,r} \quad \text{as also} \quad a^i{}_{,l,m;n}$$

and their further partial derivatives with respect to $\dot{x}$ are tensors of the rank expressed by the indices. It follows, since (1) are tensor invariant, that:

In a polynomial a^i of the form (3), the terms of any degree except two have tensor co-efficients ($A^i \ldots$). The coefficients of the second degree terms ($\Gamma^i{}_{jk}$) have the same laws of transformation as those of a symmetric affine connection.

We can, therefore, obtain a covariant differentiation with respect to the Γ's alone, by the usual rules:

$$(4) \qquad \lambda^i{}_{|h} = \lambda_{,h} + \lambda^r \, \Gamma^i{}_{hr}$$

and so on for tensors of any rank.

The equations (2′) also represent polynomials in $\dot{x}$, which must vanish identically, as our infinitesimal transformations form vector fields independent of the particular paths chosen. We thus obtain, from terms not of the second degree in $\dot{x}$,

$$(5) \quad u^m{}_{|r}\big[A^i{}_{mh_2 \ldots h_j}\,\delta^r{}_{h_1} + A^i{}_{h_1 m \ldots h_j}\,\delta^r{}_{h_2} + \cdots + A^i{}_{h_1 h_2 \ldots m}\,\delta^r{}_{h_j} - \delta_{im}\, A^r{}_{h_1 h_2 \ldots h_j}\big] + u^m A^i{}_{h_1 h_2 \ldots h_j/m} = o.$$

where the vertical bar before a subscript denotes covariant differentiation with respect to $\Gamma^i{}_{jk}$ as defined in (4); $\delta^i{}_j$ are the usual Kronecker symbols, zero or unity in value as the two indices are different or coincident. The second degree terms, however, give:

$$(6') \quad u^i{}_{|j|k} + u^i{}_{|k|j} = u^l\,[R^i{}_{jkl} + R^i{}_{kjl}].$$

$R^i{}_{jkl}$ being the curvature tensor for the Γ's. But with the following identities:—

$$R^i{}_{jkl} + R^i{}_{jlk} = 0$$
$$(7) \qquad R^i{}_{jkl} + R^i{}_{klj} + R^i{}_{ljk} = 0$$
$$u^i{}_{|j|k} - u^i{}_{|k|j} = -R^i{}_{hjk}\, u^h$$

we can reduce this to the normal form

$$(6) \qquad u^i{}_{|j|k} = R^i{}_{jkl}\, u^l .$$

(2) E. Cartan *Math. Zeitschrift, Ibid.*, pp. 619, 622.

We have thus broken up the equations of variation into one system of partial differential equations of the second order, and several of the first order, all being tensorial in form.

The problem of determining whether any solutions of (5) and (6) exist is reducible to one of algebra[3], though not explicitly soluble as a rule. The general solution, if any exist, can be expressed in terms of p independent fundamental solutions $(p \leqq n^2+n)$ as a linear combination of these with constant coefficients. But (6) has a further very important property, easily proved by means of its compatibility conditions and the identities (7). That is, if u^i, v^i be any two distinct solutions, the alternant or Poisson bracket

$$(u, v)^i \equiv u^r v^i{}_{,r} - v^r u^i{}_{,r} \equiv u^r v^i{}_{|r} - v^r u^i{}_{|r}$$

is also a solution of (6). Thus our independent infinitesimal transformations generate a group. It does not by any means follow that the common solutions of [5] and [6] generate a Lie group. This is the case, however, when none or only one such common solution exists, apart from this trivial case, the most general conditions can again be reduced to a problem of algebra, and in fact to the discussion of the independence of a series of linear or bilinear forms. One might consider the possibility of [5] being a consequence of the compatibility conditions of [6], or, of the equations [5] themselves possessing the group property. In general, it would not seem that such multiparameter groups exist when the a^i contain terms of degree higher than two in $\dot{x}$. The main point is that there exists a covariant derivation as for the affine connections, and that the general operation of the *derivate* or *biderivate* which I have elsewhere defined, can be replaced by a known and familiar type. The analytic connections are not more general than those of the form $a^i = \Gamma^i{}_{jk}\, \dot{x}^j \dot{x}^k + E^i{}_l\, \dot{x}^l + \omega^i$ which, by the way, are the only ones that are analytic in the space of $n+1$ dimensions wherein t is taken as one of the x's.

The same discussion for the most general form of a^i has no meaning, but is easily extensible to a^i that are analytic in $\dot{x}$ and sufficiently differentiable in x to allow a discussion of compatibility conditions. Even more, convergence of the infinite series can be ignored if merely an expansion of the prescribed form exists. Formally, each power of $\dot{x}$ in the expansion yields just one equation, independent of all other terms except those of the second degree. Apart from the question of solving an infinite set of differential equations (present also in the analytic case) the only difficulty possible would be that of the absence of uniqueness of

(3) L. P. Eisenhart *Non-Riemannian Geometry* (1927), pp. 126, 132.

expansion. But in this last case, if it can occur at all, we may regard the various forms as given by the use of different ways of describing the same space; or as different spaces that are feasible for the same paths. Similarly, asymmetric components, corresponding to the torsion tensor and the like can be introduced in the various coefficients, though they will not appear in the actual equations (1) or (3).

The question of collineations (path-preserving continuous groups of transformations) in path-spaces for which the a^i *possess a formal expansion by polynomials homogeneous in* $\dot{x}$, *can be dealt with by methods similar to those used for manifolds with asymmetric affine connection The particular connection, moreover, is represented by the coefficients of the quadratic terms in the expansion.*

References.

(1) D. D. Kosambi. Math Zeitschrift Bd. 37 (1933), pp. 608, 618.
(2) E. Cartan. Ibid, pp. 619, 622.
(3) L. P. Eisenhart. Non-Riemannian Geometry (1927), pp. 126, 132.

TAUBERIAN THEOREMS ON GENERALISED LAMBERT'S SERIES

BY

V. GANAPATHY IYER.

I. Introduction.

1. Let $o<\lambda_1<\lambda_2 \ldots <\lambda_n \ldots$, $\lambda_n \to \infty$ and $\alpha>0$. We consider the series

$$\sum_1^\infty \frac{a_n e^{-\lambda_n s}}{1-e^{-\alpha\lambda_n s}} \ldots\ldots\ldots\ldots (1)$$

where $s=\sigma+it$. The series (1) may be called the Generalised Lambert's Series which, when $\lambda_n=n$, $\alpha=1$, $e^{-s}=x$, reduces to the ordinary Lambert's series. It can be easily proved that the series (1) converges, if it converges at all, in a half-plane $\sigma \geqslant \sigma_0 > 0$[1]. In what follows it is supposed that the half-plane of convergence is $\sigma>0$ and that $s=\xi \to 0$ through positive real values. The object of this paper is to prove for the series (1) the analogue of the following theorems for Dirichlets' series:—

Theorem (A)[2].—Let $f(s)=\Sigma a_n e^{-\lambda_n s}$ converge for $R(s)>0$ and let $f(s)\to A$ as $s\to o$ through positive real values. Let, further, $a_n=O\left(\frac{\lambda_n-\lambda_{n-1}}{\lambda_n}\right)$. Then $\sum_1^\infty a_n=A$.

Theorem (B)[3]. — Let $a_n \geqslant 0$ and $f(s)=\Sigma a_n e^{-\lambda_n s} \sim A s^{-\alpha} L\left(\frac{1}{s}\right)$ as $s \to 0$ through positive real values, where $a \geqslant 0$

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1. It can be proved, by partial summation, that the two series $\Sigma \frac{a_n e^{-\lambda_n s}}{1-e^{-\mu_n s}}$ and $\Sigma a_n e^{-\lambda_n s}$ converge at the same time for $a\sigma>o$. Here $\mu_1<\mu_2 \cdots$ and $\mu_n \to \infty$. If $\mu_n=a\lambda_n$, this series becomes that considered in (1).

2. J. E. Littlewood, "The Converse of Abel's Theorem on power series", Proc. L.M.S., (2), 9 (1910), 434—48. The proof given in this paper assumes $\frac{\lambda_n}{\lambda_{n-1}} \to 1$ as $n\to\infty$. For a proof without any restriction on λ_n, see K. Ananda Rau, "On the converse of Abel's Theorem", Journal of the L.M.S., Vol. 3, part 3, 200—205.

3 Hardy and Littlewood, "Tauberian Theorems concerning power series and Dirichlet's series whose co-efficients are positive". Proc. L.M.S., 2 (13), 1913, 174—191.

and $L(u)=(l_1 u)^{\alpha_1} \ldots\ldots (l_r u)^{\alpha_r}$, where $l_r u=$[log log...... (r times)] (u) and the first $\alpha_k \neq 0$ is positive. Then $A_n=\sum_1^n a_r \sim \frac{A}{\Gamma(\alpha+1)} \lambda n^{\alpha} L(\lambda n)$.

2. In order to prove analogous theorems for the series (1), the following theorem, due to N. Wiener[4], is used in the sequel:—

Theorem (*C*)[5].—Let $f(\omega)$ be a function bounded in (o, ∞) Let $\omega N(\omega)$ be a function belonging to L_1 in $(0, \infty)$ such that

$$h(u)=\frac{1}{\sqrt{2\pi}} \int_0^\infty N(\omega)\, \omega^{iu}\, d\omega$$

does not vanish for real values of u. Let

$$\underset{\xi \to 0}{Lt} \int_0^\infty N(\xi\omega)\, f(\omega)\, d\omega = A \int_0^\infty N(\omega)\, d\omega.$$

Then, if $\omega M(\omega)$ be any function belonging to L_1 in $(0, \infty)$, we have,

$$\underset{\xi \to 0}{Lt} \int_0^\infty M(\xi\omega)\, f(\omega)\, d(\omega) = A \int_0^\infty M(\omega)\, d\omega$$

3. The analogue of theorem (*A*) is proved in section II and that of theorem (*B*) in section III. Finally some examples to illustrate these theorems are given.

II

4. *Theorem* (*1*).—Let

$$f(\xi)=\alpha\xi \sum_1^\infty \frac{a_n e^{-\lambda_n \xi}}{1-e^{-\alpha\lambda_n \xi}} \ldots\ldots\ldots\ldots\ldots \quad (2)$$

converge for every $\xi > 0$, and let

$f(\xi) \to A$, as $\xi \to 0$ through positive values.

Let

$$\zeta(s, a)=\frac{1}{a^s}+\frac{1}{(a+1)^s}+\frac{1}{(a+2)^s}+\ldots\ldots\ldots\ldots \quad (3)$$

4. N. Wiener, "Tauberian Theorems", Annals of Mathematics, second series, Vol. 33, No. 1, pp. 1—100, Jan. 1932. See page 25, theorem VIII. Theorem (C) quoted above is obtained from the last theorem by a simple variable transformation.

5. This theorem can be stated by saying that for a bounded sequence if an average of a particular kind, with some restriction, exists, then averages of all kinds, where the averaging function belongs to L_1, also exist.

have no zeroes on the line $R(s)=1$ when $\alpha=\frac{1}{\alpha}$. And let

$$a_n=O\ (\lambda_n-\lambda_{n-1}) \quad \text{.......................} (4)$$

Then $\sum\limits_1^\infty \frac{a_n}{\lambda_n}$ converges to the value A. In particular if $a=1$, (3) does not vanish on $R(s)=1$ and so the theorem is true. Conversely, let $\sum\limits_1^\infty \frac{a_n}{\lambda_n}$ converge to A. Then, without any further assumption, (2) converges for $\xi>0$ and $f(\xi)\to A$ as $\xi\to 0$.

Proof.—The proof is based on the following lemmas:—

Lemma (1-a).—Let $h(\xi)=\dfrac{\alpha\,\xi\, e^{-\xi}}{1-e^{-\alpha\,\xi}}-1$. Then, there are positive constants $\eta=\eta(\alpha)$, $\kappa=\kappa(\alpha)$ depending on α only such that

$$|h(\xi)|\leqslant \kappa(\alpha)\xi,$$

for $0\leqslant\xi\leqslant\eta$.

Proof.—Let $\beta>0$ and $0\leqslant\xi\leqslant 1$. Then,

$$h(\xi)-\beta\,\xi=\frac{1}{1-e^{-\alpha\xi}}\left\{\alpha\,\xi\, e^{-\xi}-(1+\beta\,\xi)\ (1-e^{-\alpha\xi})\right\}$$

$$=\frac{\alpha\,\xi^2}{1-e^{-\alpha\,\xi}}\left\{\left(\frac{\alpha}{2}-1-\beta\right)+\xi\,\kappa(\xi,\alpha,\beta)\right\}$$

where $|\kappa(\xi,\alpha,\beta)|\leqslant\alpha\, e+e^{\alpha}(1+\beta)$ for $0\leqslant\xi\leqslant 1$. Hence if $\beta=|\frac{\alpha}{2}-1|+1$, $h(\xi)-\beta\xi\leqslant 0$ for $0\leqslant\xi\leqslant\dfrac{1}{\alpha e+e^{\alpha}(1+\beta)}=\eta(\alpha)$

Similarly by considering $h(\xi)+\beta\xi$, we see that, by taking $\beta=|\frac{\alpha}{2}-1|+1$, $h(\xi)+\beta\xi\geqslant 0$ for $o\leqslant\xi\leqslant\dfrac{1}{\alpha e+e^{\alpha}(1+\beta)}=\eta(\alpha)$.

Hence,

$$|h(\xi)|\leqslant\beta\,\xi=\left(\left|\frac{\alpha}{2}-1\right|+1\right)\xi=\kappa(\alpha)\xi$$

for $0\leqslant\xi\leqslant\eta(\alpha)$. So the lemma is proved.

Lemma (1-b).—Let $A(\omega)=\sum\limits_{\lambda_n\leqslant\omega}\frac{a_n}{\lambda_n}$, $A(\omega)=0$ $0\leqslant\omega\leqslant\lambda_1$.

If $f(\xi)=O(1)$ as $\xi\to 0$ and a_n satisfies the condition (4), then $A(\omega)=O(1)$ as $\omega\to\infty$.

Proof.—There is a C such that

$$|a_n|\leqslant C(\lambda_n-\lambda_{n-1}).$$

Now, we have,

$$f(\xi)-A(\omega)=\sum_{\lambda_n\leqslant\omega}\frac{a_n}{\lambda_n}\left\{\frac{\alpha\,\lambda_n\xi\, e^{-\lambda_n\xi}}{1-e^{-\alpha\lambda_n\xi}}-1\right\}+\sum_{\lambda_n>\omega}\frac{a_n\alpha\,\xi\, e^{-\lambda_n\xi}}{1-e^{-\alpha\lambda_n\xi}}$$

$$=S_1+S_2\text{, say} \quad \text{.........................} (5)$$

By using Lemma (1-a), we see that, if $\omega\,\xi=\eta(\alpha)$,

$$|S_1| \leqslant \sum_{\lambda_n \leqslant \omega} C \left(\frac{\lambda_n-\lambda_{n-1}}{\lambda_n}\right) \kappa(\alpha)\, \lambda_n \xi$$

$$\leqslant C\, \xi\omega\, \kappa(\alpha) = C\, \eta(\alpha)\, \kappa(a) \ldots\ldots\ldots\ldots\ldots\ldots \quad (6)$$

Also

$$|S_2| \leqslant C\, \alpha\, \xi \sum_{\lambda_n > \omega} \frac{(\lambda_n-\lambda_{n-1})\, e^{-\lambda_n \xi}}{1-e^{-\alpha\lambda_n\xi}}$$

$$\leqslant C\, \alpha\, \xi \sum_{\lambda_n > \omega} \lambda_n \left| \Delta\left(\frac{e^{-\lambda_n\xi}}{1-e^{-\lambda_n\xi}}\right)\right|$$

$$\leqslant C\, \alpha\, \xi \sum_{\lambda_n > \omega} \int_{\lambda_n}^{\lambda_{n+1}} x \left| \frac{d}{dx}\, \frac{e^{-x\xi}}{1-e^{-\alpha x\xi}} \right| dx$$

$$\leqslant C\, \alpha\, \xi \int_{\omega}^{\infty} (x\xi) \left| \frac{d}{d(x\xi)} \left(\frac{e^{-x\xi}}{1-e^{-\alpha x\xi}} \right) \right| dx$$

$$= C\, \alpha \int_{\eta(\alpha)}^{\infty} x \left| \frac{d}{dx}\left(\frac{e^{-x}}{1-e^{-\alpha x}}\right)\right| dx \ldots\ldots\ldots\ldots \quad (7)$$

Combining (6) and (7), we have from (5), $|f(\xi)-A(\omega)| \leqslant H(\alpha)$ where $\omega\,\xi=\eta(\alpha)$. Hence letting $\omega\to\infty$ through arbitrary values and $\xi\to o$ through the values $\frac{\eta(\alpha)}{\omega}$ we see that since $f(\xi)=O(1)$ as $\xi\to o$, $A(\omega)=O(1)$ as $\omega\to\infty$. So the lemma is proved.

Proof of theorem (1).—Since $f(\xi)\to A$ as $\xi\to 0$, and a_n satisfies (4), we have by Lemma (1-b) that

$$A(\omega)=O(1) \text{ as } \omega\to\infty.$$

Therefore,

$$f(\xi) = a\, \xi \sum_{1}^{\infty} \frac{a_n}{\lambda_n}\, \frac{\lambda_n\, e^{-\lambda_n\xi}}{1-e^{-\alpha\lambda_n\xi}}$$

$$= a\, \xi \int_0^{\infty} A(\omega)\, \frac{d}{d\omega}\left(\frac{\omega e^{-\omega\xi}}{e^{-\alpha\omega\xi}-1}\right) d\omega$$

$$= \xi \int_0^{\infty} A(\omega)\, N(\omega\xi)\, d\omega$$

$$\to A \text{ as } \xi\to 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (8),$$

where, $$N(\omega) = \alpha\, \frac{d}{d\omega}\left(\frac{\omega e^{-\omega}}{e^{-\alpha\omega}-1}\right) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (9).$$

Here we use theorem C. Obviously $N(\omega)$ belongs to L_1.

Also

$$h(u) = \frac{a}{\sqrt{2\pi}} \int_0^\infty \frac{d}{d\omega}\left(\frac{\omega e^{-\omega}}{e^{-\alpha\omega}-1}\right) \omega^{iu}\, d\omega$$

$$= \underset{\lambda \to 0}{Lt} \frac{a}{\sqrt{2\pi}} \int_0^\infty \frac{d}{d\omega}\left(\frac{\omega e^{-\omega}}{e^{-\alpha\omega}-1}\right) \omega^{iu+\lambda}\, d\omega$$

$$= \underset{\lambda \to 0}{Lt} \frac{\alpha(\lambda+iu)}{\sqrt{2\pi}} \int_0^\infty \frac{\omega^{iu+\lambda} e^{-\omega}}{1-e^{-\alpha\omega}}\, d\omega$$

$$= \underset{\lambda \to 0}{Lt} \frac{\alpha(\lambda+iu)}{\sqrt{2\pi}} \int_0^\infty \omega^{iu+\lambda}.\left[e^{-\omega}+e^{-\omega(1+\alpha)}+..\right] d\omega$$

$$= \underset{\lambda \to 0}{Lt} \frac{(\alpha\lambda+iu)}{\sqrt{2\pi}\,\alpha^{1+\lambda+iu}} \Gamma(1+\lambda+iu)\, \zeta\left(1+\lambda+iu, \frac{1}{\alpha}\right)$$

$$= \frac{iu}{\sqrt{2\pi}\,\alpha^{iu}} \Gamma(1+iu)\, \zeta\left(1+iu, \frac{1}{\alpha}\right)$$

$$\neq 0$$

by hypothesis. The positive quantity λ is introduced to make term by term integration possible. Also $\int_0^\infty N(\omega)\, d\omega = 1$. Hence all the conditions of theorem (C) are satisfied when $N(u)$ has the value (9). So we can replace $N(u)$ by any function $N(u)$ in (8) where $u\,N(u)$ belongs to L_1.

Let

$$M(x) = \begin{cases} 1, & 0 \leqslant x \leqslant 1 \\ 0, & x > 1. \end{cases}$$

Then we get from (8), putting $\xi = \frac{1}{\omega}$, that

$$\frac{1}{\omega}\int_0^\omega A(t)\, dt \to A, \text{ as } \omega \to \infty \quad \ldots\ldots\ldots\ldots\ldots (10),$$

But the left side of (10) is the Riesz's mean (type λ_n) of the first order for the series $\sum \frac{a_n}{\lambda_n}$ and (10) says that this mean exists. But $\frac{a_n}{\lambda_n} = O\left(\frac{\lambda_n - \lambda_{n-1}}{\lambda_n}\right)$ by (4). So by a well-known theorem. $A(\omega) \to A$ as $\omega \to \infty$, that is $\sum_1^\infty \frac{a_n}{\lambda_n} = A$.

Conversely let $\sum_1^\infty \frac{a_n}{\lambda_n} = A$. Then $A(\omega) \to A$ as $\omega \to \infty$.

Therefore (10) is valid and can be written as

$$\xi \int_{0}^{\infty} M(\xi \omega)\ A(\omega)\ d\omega \to A \text{ as } \xi \to o, \qquad \dots\dots\dots (11)$$

where $$M(x) = \begin{cases} 1, & 0 \leqslant x \leqslant 1 \\ 0, & x > 1. \end{cases}$$

Also, $\frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} M(x)\ x^{iu}\ dx = \frac{1}{\sqrt{2\pi}(1+iu)} \neq 0$ for any real u.

So by theorem (C) we can replace $M(x)$ in (11) by $N(u)$ given by (9). So the conclusion stated by the relation (8) is valid which proves the converse part of theorem (1).

5. *Remark on the converse part of theorem* (1).—The converse part of theorem (1) is the usual Abelian theorem for the series (2). It is not necessary to use theorem (C) to prove it. An alternative method is given below :—

We have, $A(\omega) \to A$ by hypothesis. So,

$$f(\xi) - A = \alpha\, \xi \int_{\omega}^{\infty} [A(\omega) - A]\ \frac{d}{d\omega} \left(\frac{\omega e^{-\omega\xi}}{e^{-\alpha\omega\xi} - 1} \right) d\omega$$

$$= \int_{\omega}^{\infty} \left[A\left(\frac{\omega}{\xi}\right) - A \right] N(\omega)\ d\omega$$

$$= \int_{0}^{\sqrt{\xi}} + \int_{\sqrt{\xi}}^{\infty}$$

$$= O(1)\ \sqrt{\xi} + 0(1)\ O(1), \text{ as } \xi \to 0$$

$$= o(1) \text{ as } \xi \to 0$$

because, $A\left(\frac{\omega}{\xi}\right) = O(1)$ in $0 \leqslant \omega \leqslant \sqrt{\xi}$ and $A\left(\frac{\omega}{\xi}\right) = 0\ (1)$ in $\sqrt{\xi} \leqslant \omega < \infty$ as $\xi \to 0$ and $\int_{0}^{\infty} |\ N(u)\ |\ du$ is finite. Hence $f(\xi) \to A$ as $\xi \to 0$ as is to be proved.

III

6. *Theorem* (2).—Let $a_n \geqslant o$ and $A(\omega) = \sum\limits_{\lambda_n \leqslant \omega} \frac{a_n}{\lambda_n}$. Let

$$f(\xi) = a\xi \sum_{1}^{\infty} \frac{a_n\, e^{-\lambda_n \xi}}{1 - e^{-\alpha\lambda_n \xi}} \sim A\xi^{-\beta}\ L\left(\frac{1}{\xi}\right) \qquad \dots\dots\dots\dots (12)$$

as $\xi \to 0$. Here $\beta \geqslant 0$ and

$$L(u) = (l_1\, u)^{\alpha_1}\ (l_2\, u)^{\alpha_2} \dots (l_r\, u)^{\alpha_r} \qquad \dots\dots\dots\dots\dots (13)$$

where $l_k(u)$ = the k^{th} repeated logarithm of u and $(\alpha_1, \ldots . \alpha_r)$ are *any constants* whatever.[6] Let

$$\zeta\left(1+\beta+iu, \frac{1}{\alpha}\right) \neq 0 \ldots\ldots\ldots\ldots\ldots\ldots \quad (14)$$

for real u, which is satisfied when (*i*) $\alpha=1, \beta \geqslant 0$ and (*ii*) β greater than some positive constant depending on α when $\alpha \neq 1$. Then,

$$A(\omega) \sim \frac{A\alpha^{\beta}\,\omega^{\beta}\,L(\omega)}{\beta\Gamma(\beta+1)\,\zeta\left(\beta+1, \frac{1}{\alpha}\right)} \ldots\ldots\ldots\ldots\ldots\ldots (15)$$

as $\omega \rightarrow \infty$. Conversely, if (15) be true, then (12) is true without any further assumption on a_n or on the non-vanishing of (14).

Proof.—We shall, first, prove several lemmas:—

Lemma (2-a).—Under the hypothesis of theorem (2),

$$A(\omega) = O(\omega^{\beta} L(\omega)) \text{ as } \omega \rightarrow \infty.$$

Proof.—It can be shown by differentiation that $\dfrac{\alpha x e^{-x}}{1-e^{-\alpha x}}$ is a monotone increasing or decreasing function of x according to the value of α in a certain range $0 \leqslant x \leqslant \eta(\alpha)$, where $\eta(\alpha)$ is a positive constant depending on α. So there exists a constant $C(\alpha)$ such that, if $\omega\xi = \eta(\alpha)$, we have,

$$o \leqslant A(\omega) \leqslant C(\alpha)\, \alpha\xi \sum_{\lambda_n \leqslant \omega} \frac{a_n e^{-\lambda_n \xi}}{1-e^{-\alpha\lambda_n\xi}}$$

$$= O\left[f\left(\frac{\eta}{\omega}\right) \right]$$

$$= O\left(\omega^{\beta} L(\omega)\right)$$

since $a_n \geqslant 0$. So the lemma is proved.

Lemma (2-b)[7].—Let η_r be such that $l_r(\eta_r)=1$, so that $l_k(u) \geqslant 1$, $(k=1, 2, \ldots r)$ for $u \geqslant \eta_r$. Let $\overline{L(u)}$ denote the same function as in (13) with $\alpha_1, \ldots \alpha_r$ replaced by their moduli. Let

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6. Usually it is supposed that in (13) (*i*) $\alpha_1 = \alpha_2 = \ldots \alpha_r = 0$ or (*ii*) the first $\alpha_k \neq 0$ is positive. But this restriction *is not necessary*. The same remark holds as regards theorem (*B*) which could also be proved by the methods of this paper

7. The formula (13) does not give real values of $L(u)$ for sufficiently small values of u. Since $L(u)$ is considered only for large values of u we suppose that it is defined to take a continuous set of values lying between two positive bounds for $0 \leqslant u \leqslant \eta_r$ so that $L(u)$ as well as $\dfrac{1}{L(u)}$ is bounded for these values of u.

h be a positive constant and u_0 a sufficiently great positive quantity. Then,

$$(i)\quad \frac{L\left(\frac{u}{\xi}\right)}{L\left(\frac{1}{\xi}\right)} = O\left\{\overline{L\left(\frac{1}{\xi}\right)}\right\} \text{ uniformly for } 0<u\leqslant \xi\eta_r.$$

$$(ii)\quad \frac{L\left(\frac{u}{\xi}\right)}{L\left(\frac{1}{\xi}\right)} = O\left\{\overline{L\left(\frac{1}{\xi}\right)}\right\}^2 \text{ uniformly for } \xi\eta_r \leqslant u \leqslant \frac{1}{\left(\log\frac{1}{\xi}\right)^h}$$

$$(iii)\quad \frac{L\left(\frac{u}{\xi}\right)}{L\left(\frac{1}{\xi}\right)} \to 1 \text{ uniformly for } \frac{1}{\left(\log\frac{1}{\xi}\right)^h} \leqslant u \leqslant \left(\log\frac{1}{\xi}\right)^h$$

as $\xi \to 0$.

$$(iv)\quad \frac{L\left(\frac{u}{\xi}\right)}{L\left(\frac{1}{\xi}\right)} = O\left\{\overline{L(u)}\right\} \text{ for } u \geqslant u_0 \text{ uniformly.}$$

Here the O-signs refer to the variable ξ as it tends to zero.

Proof.—(*i*) This follows from the convention adopted in footnote (7).

(*ii*) We have, $\Theta \leqslant L\left(\frac{u}{\xi}\right) \leqslant \overline{L\left(\frac{1}{\xi\left(\log\frac{1}{\xi}\right)^h}\right)} \leqslant \overline{L\left(\frac{1}{\xi}\right)}$

for $\xi\eta_r \leqslant u \leqslant \frac{1}{\left(\log\frac{1}{\xi}\right)^h}$ and $\frac{1}{L\left(\frac{1}{\xi}\right)} \leqslant \overline{L\left(\frac{1}{\xi}\right)}$. So (*ii*) is true.

(*iii*) We have, in $\frac{1}{\left(\log\frac{1}{\xi}\right)^h} \leqslant u \leqslant \left(\log\frac{1}{\xi}\right)^h$,

$$l_1\left(\frac{u}{\xi}\right) = l_1\left(\frac{1}{\xi}\right) + l_1(u)$$

$$= l_1\left(\frac{1}{\xi}\right)\left\{1+\theta_1\right\}, \text{ where } |\theta_1| \leqslant \frac{h l_2\frac{1}{\xi}}{l_1\left(\frac{1}{\xi}\right)},$$

$$l_2\left(\frac{u}{\xi}\right) = l_2\left(\frac{1}{\xi}\right)\left\{1+\theta_2\right\}, \text{ where } |\theta_2| = O\left(\frac{l_2\frac{1}{\xi}}{l_1\frac{1}{\xi}\, l_2\frac{1}{\xi}}\right)$$

and so on. Since $\theta_1, \theta_2, \ldots$ tend to zero uniformly as $\xi \to 0$ we get (*iii*) whatever $\alpha_1, \alpha_2, \ldots \alpha_r$ be.

(*iv*) Let $u_0 > \eta_r$. We have

$$l_1\left(\frac{u}{\xi}\right) = l_1(u) + l_1\left(\frac{1}{\xi}\right), \text{ so that}$$

$$1 \leqslant \frac{l_1\left(\frac{u}{\xi}\right)}{l_1\left(\frac{1}{\xi}\right)} \leqslant 2\, l_1(u), \text{ if } \xi \text{ is sufficiently small.}$$

Hence,

$$1 \leqslant \frac{l_2\left(\frac{u}{\xi}\right)}{l_2\left(\frac{1}{\xi}\right)} \leqslant l_2(u) + \log 2 \text{ and so on.}$$

Therefore in all cases,

$$\frac{L\left(\frac{u}{\xi}\right)}{L\left(\frac{1}{\xi}\right)} = O\left[\overline{L(u)}\right]$$

uniformly for all $u \geqslant u_0$. So the lemma is proved.

Lemma (2-*c*).—Let $g(\xi) = \xi \int_0^{\infty} \left| \frac{L(x)}{L\left(\frac{1}{\xi}\right)} - 1 \right| H(\xi x)\, dx$

where $H(u)$ is

(*i*) bounded over any finite range

(*ii*) $\Theta\,(e^{-u\lambda})$, $\lambda > 0$ as $u \to \infty$.

Then $g(\xi) = 0(1)$ as $\xi \to 0$. In particular the lemma is true if $H(u) = u^{\beta} \left| \frac{d}{du}\left(\frac{u\, e^{-u}}{e^{-\alpha u} - 1}\right)\right|$ where $\beta \geqslant 0$.

Proof.—Let $h = 1 + 2\,|\alpha_1|$ in Lemma (2-*b*). We have,

$$g(\xi) = \int_0^{\infty} \left| \frac{L\left(\frac{x}{\xi}\right)}{L\left(\frac{1}{\xi}\right)} - 1 \right| H(x)\, dx$$

$$= \int_0^{\xi\eta_r} + \int_{\xi\eta_r}^{\left(\log\frac{1}{\xi}\right)^{-h}} + \int_{\left(\log\frac{1}{\xi}\right)^{-h}}^{u_0} + \int_{u_0}^{\infty}$$

$$= S_1 + S_2 + S_3 + S_4, \text{ say.}$$

By Lemma (2-*b*) (*iv*) and the second condition imposed on $H(u)$, we can find u_0 so that $|S_4| < \frac{\epsilon}{2}$. Let u_0 be so chosen and fixed. Now, by Lemma (2-*b*) (*i*), (*ii*) and (*iii*) we have if ξ is sufficiently small.

$$S1=O\left[\xi\,\overline{L\left(\frac{1}{\xi}\right)}\right],$$

$$S2=O\left[\frac{\left\{\overline{L\left(\frac{1}{\xi}\right)}\right\}^2}{\left(\log\frac{1}{\xi}\right)^{1+2|\alpha_1|}}\right]$$

$$S3=\underset{\left(\log\frac{1}{\xi}\right)^{-h}\leqslant x\leqslant u_0}{\text{Max}:}\left|\frac{L\left(\frac{x}{\xi}\right)}{L\left(\frac{1}{\xi}\right)}-1\right|.\quad O(u_0)$$

$$=0(1)\ O(u_0).$$

Hence we can find a $\xi 0$ such that for $0<\xi\leqslant\xi 0$.

$$S_1+S_2+S_3<\epsilon/2.$$

So $0\leqslant g(\xi)<\epsilon$ for $0<\xi\leqslant\xi_0$. So the lemma is proved.

Lemma (*2-d*).—$\lambda(\omega)=\frac{1}{\omega}\int_0^{\omega}\left|\frac{L(x)}{L(\omega)}-1\right|d\omega\to 0$ as $\omega\to\infty$.

Proof.— In lemma (2-*c*), take $H(x)=1$, $o\leqslant x\leqslant 1$ and $H(x)=0$ for $x>1$ and put $=\xi\ \frac{1}{\omega}$.

Lemma (*2-e*)[8]—If $f(x)$ he such that $x\,f'(x)$ increases as x increases to infinity and $\phi(x)$ be a positive increasing function such that $\frac{\phi(x)}{x\phi'(x)}\to 1$, then $f(x)\sim\phi(x)$ involves $f'(x)\sim\phi'(x)$. In particular we can take $\phi(x)=A\omega^{\beta+1}L(\omega)$, $\beta\geqslant 0$.

Proof.—For a proof, reference is made to the memoir cited in footnote (8).

Proof of theorem (2).—By Lemma (2-*a*), $A(\omega)=O(\omega^{\beta}L(\omega)|$. Let $B(\omega)=\frac{A(\omega)}{\omega^{\beta}L(\omega)}$, so that $B(\omega)$ is bounded. Then, the relation (12) could be written as

$$\underset{\xi\to 0}{Lt}\ \frac{\alpha\,\xi}{L\left(\frac{1}{\xi}\right)}\int_0^{\infty}B(\omega)\ L(\omega)\ (\omega\xi)^{\beta}\ \frac{d}{d\omega}\left(\frac{\omega e^{-\omega\xi}}{e^{-\alpha\omega\xi}-1}\right)=A\ldots\ldots(16)$$

Since $B(\omega)$ is bounded, we see by using Lemma (2-*c*) that

$$\underset{\xi\to 0}{Lt}\ \xi\int_0^{\infty}B(\omega)\ N(\omega\xi)=A\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(17)$$

where $N(\omega)=a\omega^{\beta}\frac{d}{d\omega}\left(\frac{\omega\,e^{-\omega}}{e^{-\alpha\omega}-1}\right)\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(18)$

8. See the paper quoted in footnote (3).

Now we apply theorem (c) to (17). Here,

$$\int_0^\infty N(\omega)\ \omega^{iu} d\omega = \alpha \int_0^\infty \omega^{iu+\beta} \frac{d}{d\omega}\left(\frac{\omega\, e^{-\omega}}{e^{-\alpha\omega}-1}\right) d\omega$$

$$\alpha(\beta+iu)\int_0^\infty \omega^{\beta+iu} \frac{e^{-\omega}}{1-e^{-\alpha\omega}}\, d\omega$$

$$= \frac{(\beta+iu)\ \Gamma(1+\beta+iu}{\alpha^{\beta+iu}}\ \zeta(1+\beta+iu, \tfrac{1}{\alpha})$$

$$\neq 0$$

by hypothesis for any real u. Now (17) could be written as

$$\xi \int_0^\infty B(\omega)\ N(\omega\xi)\ d\omega \to \frac{A\alpha^\beta}{\beta\,\Gamma(\beta+1)\ \zeta\left(\beta+1, \frac{1}{\alpha}\right)} \int_0^\infty N(\omega)\ d\omega \quad \ldots\ldots (19)$$

as $\xi \to 0$. Since $B(\omega)$ is bounded. we get by theorem (c), by taking

$$M(x) = \begin{cases} x^\beta, & 0 \leqslant x \leqslant 1 \\ 0, & x > 1 \end{cases}$$

that, $$\xi \int_0^{\frac{1}{\xi}} B(\omega)\ \ (\omega\xi)^\beta\ d\omega \to \frac{A\alpha^\beta}{\beta(\beta+1)\ \Gamma(\beta+1)\ \zeta\left(\beta+1, \frac{1}{\alpha}\right)}$$

$$= \frac{K}{\beta+1}, \text{ say} \ldots\ldots\ldots\ldots (20)$$

Putting $\xi = \frac{1}{\omega}$, we get

$$\frac{1}{\omega^{\beta+1}} \int_0^\omega \frac{A(t)}{L(t)}\ dt \to \frac{K}{\beta+1} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (20')$$

since $A(t) = B(t)\ t^\beta\ L(t)$.

Now, $$\left| \frac{1}{\omega^{\beta+1}\, L(\omega)} \int_0^\omega A(t)\ dt - \frac{1}{\omega^{\beta+1}} \int_0^\omega \frac{A(t)}{L(t)}\ dt \right|$$

$$\leqslant \frac{1}{\omega^{\beta+1}\, L(\omega)} \left| \int_0^\omega A(t) \right| \frac{L(\omega)}{L(t)} - 1 \Bigg|\ dt$$

$$\leqslant \frac{C}{\omega} \int_0^\omega \left| \frac{L(t)}{L(\omega)} - 1 \right|\ dt, \text{ since } |A(t)| \leqslant c\, t^\beta\, L(t)$$

$= 0\ (1)$, as $\omega \to \infty$, by Lemma $(2\text{-}d)$.

So, $$\int_0^\omega A(t)\ dt \sim \frac{K}{\beta+1}\ \omega^{\beta+1}\ L(\omega) \quad \ldots\ldots\ldots\ldots\ldots\ldots (21)$$

Since $A(t)$ is increasing function of (t), we see that $\omega A(\omega)=\omega. \frac{d}{d\omega}\int_0^\omega A(t)\,dt$ is increasing. Hence by Lemma (2-e), we get

$$A(\omega)\sim K\,\omega^\beta\,L(\omega)\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(15)$$

as is to be proved.

Conversely if (15) holds, so does (21) and by Lemma (2-d), (20′) and hence (20). The relation (20) could be written as

$$\xi\int_0^\infty B(\omega)\,M(\xi\omega)\,d\omega\to K\int_0^\infty M(\omega)\,d\omega$$

and $\int_0^\infty M(\omega)\,\omega^{iu}\,d\omega=\frac{1}{1+\beta+iu}\neq 0$ for any real u.

Hence we can replace $M(\omega)$ by $N(\omega)$ given by (18) and conclude that (17) is valid. By using Lemma (2-c) we see that (16) holds which is nothing but the relation (12).

7. *Remark.*—By using the Lemmas (2-c) and (2-d), the converse part of theorem (2) could be proved, as in § 5, without the use of theorem (c). It is to be noted that the hypothesis $A(\omega)\sim C\omega^\beta L(\omega)$ is not equivalent to $A(\lambda_n)\sim C\lambda_n^\beta L(\lambda_n)$ unless some more restriction is placed on the sequence (λ_n). The relation $A(\omega)\sim C\omega^\beta L(\omega)$ implicitly contains the assumption that

(i) in case $\beta>0$, $\frac{\lambda_n}{\lambda_{n-1}}\to 1$ as $n\to\infty$,

(ii) in case $\beta=0$, $\alpha_1=\ldots=\alpha_{\kappa-1}=0$, $\alpha_\kappa\neq 0$, $\frac{l_\kappa\,\lambda_n}{l_\kappa\,\lambda_{n-1}}\to 1$ as $n\to\infty$.

Since, in the direct part of theorem (2), the relation (15) is proved to be a consequence of (12) without any hypothesis on λ_n, it follows that (12) involves—whenever $a_n\geqslant 0$ and (14) is valid—the relation stated in (i) or (ii) above.

8. *Examples.*—(i) The familiar relations

$$\text{(a)}\quad \sum_1^\infty\frac{\mu(n)}{n}=0. \qquad \text{(b)}\quad \sum_1^\infty\frac{\lambda(n)}{n}=0\ldots\ldots\ldots\ldots(22)$$

can be obtained by applying theorem (1) to the known results

$$\text{(a}'\text{)}\quad \xi\sum_1^\infty\frac{\mu(n)\,e^{-n\xi}}{1-e^{-n\xi}}=\xi e^{-\xi}. \qquad \text{(b}'\text{)}\quad \xi\sum_1^\infty\frac{\lambda(n)\,e^{-n\xi}}{1-e^{-n\xi}}$$

$$=\xi\sum_1^\infty e^{-n^2\xi},\ \ldots\ldots\ldots(22')$$

where $\mu(n)$ is the Möbius function and $\lambda(n)=(-1)^{\rho}$, ρ being the number of prime factors of n counted according to multiplicity and $\lambda(1)=1$.

(*ii*). Let $\phi(n)$ be the number of numbers less than n and prime to it. From $\sum_{d/n} \phi(d)=n$, we get

$$\xi \sum_{1}^{\infty} \frac{\phi(n)\, e^{-n\xi}}{1-e^{-n\xi}}=\xi\, e^{-\xi}\,(1-e^{-\xi})^{-2} \sim \frac{1}{\xi} \text{ as } \xi \to o \quad \text{.. (23)}$$

Since $\phi(n) \geqslant 0$, we get, by applying theorem (2) with $\lambda_n=n$. $\beta=\alpha=1$, that

$$A(\omega)=\sum_{n \leqslant \omega} \frac{\phi(n)}{n} \sim \frac{\omega}{\zeta(2)}=\frac{6\omega}{\pi^2} \quad \text{.................... (23')}$$

From (23') we get

$$\begin{aligned} B(\omega)&=\sum_{n\leqslant\omega} \phi(n) \\ &=\sum_{n\leqslant\omega} \frac{\phi(n)}{n}\, n \\ &=\frac{6\omega^2}{\pi^2}\left\{1+0(1)\right\}-\sum_{n\leqslant(\omega-1)} \frac{6n}{\pi^2}\left\{1+0(1)\right\} \\ &=\frac{3\omega^2}{\pi^2}\left\{1+0(1)\right\} \\ &\sim\frac{3\omega^2}{\pi^2} \quad \text{.................................... (23'')} \end{aligned}$$

(*iii*) Let $p_1, p_2, p_3 \ldots\ldots$ be the successive primes, Let $\Pi(x)$ denote the number of primes not exceeding x. Let $c(n)$ denote a positive function of n and let $\tau(n)$ be defined by the relation

$$\tau(n)=\sum_{pd/n} c(d) \quad \text{............................ (24).}$$

Then the following results are known:—

$$\xi \sum_{1}^{\infty} \frac{c(n)\, e^{-p_n\xi}}{1-e^{-p_n\xi}}=\xi \sum_{1}^{\infty} \tau(n)\, e^{-n\xi} \quad \text{............ (24')}$$

$$\left.\begin{aligned} &(\alpha) \quad \Pi(x) \sim \frac{x}{\log x} \\ &(\beta) \quad n \sim \frac{p_n}{\log p_n} \\ &(\gamma) \quad p_n \sim n \log n \end{aligned}\right\} \quad \text{.......................... (24'')}$$

(*iii-a*) Let $c(n)=1$. Then $\tau(n)=$the number of distinct prime factors of n. Now,

$$A(\omega)=\sum_{p_n\leqslant\omega}\frac{1}{p_n}\sim\log\log\omega\qquad(25)$$

by (β) and (γ) of (24″). So by the converse part of theorem (2), we get by (24′) that

$$\xi\sum_{1}^{\infty}\tau(n)\,e^{-n\xi}\sim\log\log\frac{1}{\xi},$$

or

$$\sum_{1}^{\infty}\tau(n)\,e^{-n\xi}\sim\xi^{-1}\log\log\frac{1}{\xi}\qquad(25')$$

Hence by theorem (B), since $\tau(n)\geqslant 0$, we get

$$B(\omega)=\sum_{n\leqslant\omega}\tau(n)\sim\omega\log\log\omega\qquad(25'')$$

(*iii-b*) Let $c(n)=n$. Then $\tau(n)=$the sum of the ranks of the prime divisors of n. Here

$$A(\omega)=\sum_{p_n\leqslant\omega}\frac{n}{p_n}\sim\frac{\omega}{(\log\omega)^2}\qquad(26)$$

by (β) and (γ) of (24″). Hence, as before,

$$\sum_{1}^{\infty}\tau(n)\,e^{-n\xi}\sim\zeta(2)\,\xi^{-2}\frac{1}{(\log\frac{1}{\xi})^2}\qquad(26')$$

So by theorem (B) (See footnote 6),

$$B(\omega)=\sum_{n\leqslant\omega}\tau(n)\sim\frac{\pi^2}{12}\frac{\omega^2}{(\log\omega)^2}\qquad(26'')$$

since $\zeta(2)=\sum\frac{1}{n^2}=\frac{\pi^2}{6}$.

(*iii-c*) Let $c(n)=p_n$. Then $\tau(n)=$the sum of the distinct prime divisors of n. We have,

$$A(\omega)=\sum_{p_n\leqslant\omega}\frac{p_n}{p_n}=\sum_{p_n\leqslant\omega}1\sim\frac{\omega}{\log\omega}\qquad(27)$$

by (α), (24″). So we have by (24′)

$$\sum_{1}^{\infty}\tau(n)\,e^{-n\xi}\sim\zeta(2)\,\xi^{-2}\left(\log\frac{1}{\xi}\right)^{-1}\qquad(27')$$

Hence by theorem (B),

$$B(\omega)=\sum_{n\leqslant\omega}\tau(n)\sim\frac{\pi^2}{12}\frac{\omega^2}{\log\omega}\qquad(27'')$$

9. *Note.* The conclusion of theorem (1) still remains true when the condition (4) on a_n is replaced by the following more general one:—

$$\left.\begin{aligned} &\sum_{\nu=1}^{n} \left|\frac{a_\nu}{\lambda_\nu}\right|^p \left(\frac{\lambda_\nu}{\lambda_\nu-\lambda_{\nu-1}}\right)^p (\lambda_\nu-\lambda_{\nu-1})=O(\lambda_n) \\ \text{that is, } &\sum_{\nu=1}^{n} \left|\frac{a_\nu}{\lambda_\nu-\lambda_{\nu-1}}\right|^p (\lambda_\nu-\lambda_{\nu-1})=O(\lambda_n) \end{aligned}\right\} \ldots (X_p)$$

where p is a positive quantity *greater than unity.* It is easy to prove that if (4) holds, then (X_p) is true for all $p \geqslant 1$ but not conversely. Hence (X_p) is a more general condition than (4). Theorem (1) is valid even when $p=1$, provided in that case the O-sign in X_p ($\equiv X_1$) be replaced by 0-sign.

AN EXTENSION OF HEILBRONN'S CLASS-NUMBER THEOREM

BY

S. CHOWLA.

Let $h(d)$ denote the number of primitive classes of binary quadratic forms of negative discriminant d. Heilbronn[1] has recently proved that

Theorem I.

$$h(d) \to \infty$$
$$\text{as} \quad -d \to \infty.$$

By a slight modification of Heilbronn's argument I show that

Theorem II.

$$\frac{h(d)}{2^t} \to \infty$$
$$\text{as} \quad -d \to \infty,$$

where t *is the number of different prime factors of* d.

Both these results were conjectured by Gauss.[2]

Theorem II is equivalent to

Theorem III.

$$p(d) \to \infty$$
$$\text{as} \quad -d \to \infty,$$

where p(d) *is the number of primitive classes in the principal genus.*

We shall write

$$h(d) = H, \; p(d) = P.$$

We assume, with Heilbronn, that there is[3] an $m > 0$ and a character $\chi(n)$ (mod m) such that

$$L_0(s) = L(s, \chi) = \sum_{n=1}^{\infty} \chi(n)\, n^{-s}$$

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1. *Quart. J. of Math.*, (Oxford), 5 (1934), 150—160. Hereafter referred to as Heilbronn. I am very much indebted to Dr. Heilbronn for an advance copy of his manuscript.

2. *Disquisitiones Arithmeticae* (1801), Art. 303.

3. If there is no such m, then, by a theorem of Hecke,

$$h(d) > c\,\frac{\sqrt{|d|}}{\log |d|} \quad (d < -1),$$

so that Theorem II is then certainly true.

vanishes for $s=\rho$ where

(1) $\rho=\theta+i\phi \ (\frac{1}{2}<\theta<1)$.

We further suppose, as we obviously may, that

$$m\equiv O(2)$$

so that

(2) $\chi(n)=O$ for $n\equiv O(2)$.

We follow the notation of Heilbronn except that the constants implied in our O symbols are independent of H and t.

Lemma I. *If*

(3) $a \mid (x^2-d),\ (a, 2d)=1 \qquad [a>1]$

and if

(4) $a_s x^2 + b_s xy + c_s y^2 \ (1\leqslant s\leqslant P)$

are the P *classes in the principal genus then there is an* s *with* $1\leqslant s\leqslant P$ *such that*

(5) $a^{2P}=a_s x^2+b_s xy+c_s y^2 \ (y\neq O)$.

Proof. Now a^{2P} can be represented by the P classes of the principal genus and not by any of the other $H-P$ classes. The number of representations of a^{2P} by these P forms, is, by a well known theorem,[1] exactly

$$2\ (2P+1)=4\ P+2.$$

Now $a_s x^2+b_s xy+c_s y^2$ can represent a^{2P} with $y=O$ in at most 2 ways. Hence the P classes (4) can represent a^{2P} with $y=O$ in at most $2P$ ways. It follows that a^{2P} must have at least one representation by

$$a_s x^2 + b_s xy + c_s y^2$$

with $y\neq O$ for some s in $1\leqslant s\leqslant P$.

Lemma II. *If*

(6) $(a, 2d)=1,\ a \mid (x^2-d) \qquad [a>1]$

then

(7) $$a^{2P} \geqslant \sqrt{\frac{3|d|}{16}}$$

Proof. From Lemma I there is an s $(1\leqslant s\leqslant P)$ such that

$$a^{2P}=a_s x^2+b_s xy+c_s y^2 (y\neq O)$$

or

(8) $4a_s . a^{2P}=(2a_s x+b_s y)^2-dy^2 \ (y\neq O)$.

1. *See*, for example, Landau, *Varlesungen uber Zahlentheorie*, satz. 204.

Further

$$(9) \qquad 1 \leqslant a_s \leqslant \sqrt{\frac{|d|}{3}}$$

From (8) and (9) we obtain (7).

In what follows a runs through the minima of the H forms of discriminant d.

Lemma III.[2]

(10) *If* $a|d^k$ *for some* $k>0$, *then* $\mu(a)\neq 0$, $a|d$.

Lemma IV. *If*

$$(11) \qquad a \nmid d, \quad a \equiv 1 \ (2)$$

then

$$(12) \qquad a^{2P} \geqslant \sqrt{\frac{3|d|}{16}}.$$

Proof. From (11) and (10), $a \nmid d^k$ for any $k>0$. Hence $(a_1, d)=1$ for some a_1 with $a_1>1$ such that $a_1|a$. These give $(a_1, 2d) = 1$.

Further since $a_1 \mid (x^2-d)$, we obtain from Lemma II,

$$(13) \qquad a_1^{2P} \geqslant \sqrt{\frac{3|d|}{16}}.$$

(12) follows from (13) since $|a| \geqslant |a_1|$.

Lemma V.[3] *If*

$$A>0, \ A|d, \ \mu(A)\neq 0$$

then there is at most one form of discriminant d *with minimum* A.

Lemma VI.[4] *If under the assumptions of Lemma* V,

$$A \leqslant \left|\sqrt{\frac{d}{4}}\right|$$

then there is at least one form of discriminant d *with minimum* A.

Lemma VII. *For* $\sigma > \frac{1}{2}$,

$$(14) \qquad \left|\sum_a \chi(a)\, a^{-s}\right| \geqslant \frac{1}{4H^2} + O\left(H|d|^{-\frac{\sigma}{2}}\right) + O\left(\frac{H}{|d|^{\sigma/4P}}\right)$$

---

2 Lemma XI of Heilbronn.

3. Lemma XII of Heilbronn.

4. Lemma XIII of Heilbronn.

Proof. By (2) and (10),

$$\left| \sum_{a} \chi(a) a^{-s} - \sum_{n \mid d} \chi(n) \mu^2(n) n^{-s} \right|$$

$$= \left| \sum_{a \dagger d} \chi(a) a^{-s} - \sum_{\substack{n \mid d \\ n \neq a}} \chi(n) \mu^2(n) n^{-s} \right|$$

$$(15) \quad = \left| \sum_{\substack{a \dagger d \\ a \equiv 1(2)}} \chi(a) a^{-s} - \sum_{\substack{n \mid d \\ n \neq a}} \chi(n) \mu^2(n) n^{-s} \right|$$

Now by (11) and (12),

$$(16) \quad \sum_{\substack{a \\ a \dagger d \\ a \equiv 1(2)}} \chi(a) a^{-s} = O\left(\frac{H}{|d|^{\sigma/4P}} \right)$$

By Lemmas V and VI,

$$\sum_{\substack{n \mid d \\ n \neq a}} \chi(n) \mu^2(n) n^{-s} = O\left(2^t |d|^{-\frac{\sigma}{2}} \right)$$

$$(17) \quad = O\left(H |d|^{-\frac{\sigma}{2}} \right)$$

Further, as in the proof of Lemma X of Heilbronn,

$$(18) \quad \left| \sum_{n \mid d} \chi(n) \mu^2(n) n^{-s} \right| \geqslant \frac{1}{4H^2}.$$

(14) follows from (15), (16), (17), (18).

Lemma VIII. *For* $\sigma_m < \sigma < 2$, $s \neq 1$,

$$(19) \quad L_o(s)\, L_2(s) = \zeta(2s) \prod_{p \mid m} \left(1 - p^{-2s}\right) \sum_{a} \chi(a) a^{-s}$$

$$+ O\left\{ \left(|s| + \frac{1}{|s-1|} \right) \left(H |d|^{\frac{1}{4} - \frac{\sigma}{2}} + H |d|^{-\frac{\sigma}{2}} \right) \right\}.$$

Proof. When $\sigma_m < \sigma < 2$, $s \neq 1$ we have for $\phi(s)$, the expression on the top of page 156 of Heilbronn,

$$(20) \quad \phi(s) = O\left\{ \left(|s| + \frac{1}{|s-1|} \right) \left(|a^\sigma| \,.\, |d|^{-\sigma} + |a^{\sigma-1}| \,.\, |d|^{\frac{1}{2}-\sigma} \right) \right\}$$

Now

$$(21) \quad |a^\sigma d^{-\sigma}| = \left| \frac{a^2}{d} \right|^{\frac{\sigma}{2}} . \, |d|^{-\frac{\sigma}{2}},$$

(22) $\left|a^{\sigma-1}d^{\frac{1}{2}-\sigma}\right|=a^{-\frac{1}{2}}\left|\frac{a^2}{d}\right|^{\frac{\sigma}{2}-\frac{1}{4}}.|d|^{\frac{1}{4}-\frac{\sigma}{2}}$,

(23) $a\geqslant 1,\ 3a^2\leqslant|d|$.

(19) follows from (20), (21), (22), (23) and the proof of Lemma X in Heilbronn.

Proof of the main result.

We put $s=\rho$ in Lemma VIII. Then we get

(24) $$O=\zeta(2\rho)\prod_{p\mid m}\left(1-p^{-2\rho}\right)\sum_a\chi(a)\,a^{-\rho}$$
$$+O\left(H|d|^{\frac{1}{4}-\frac{\theta}{2}}+H|d|^{-\frac{\theta}{2}}\right)$$

From Lemma VII,

(25) $$\left|\sum_a\chi(a)a^{-\rho}\right|\geqslant\frac{1}{4H^2}+O\left(H|d|^{-\frac{\theta}{2}}\right)+O\left(\mathrm{H}|d|^{-\theta/4P}\right).$$

Now

$$\zeta(2\rho)\prod_{p\mid m}\left(1-p^{-2\rho}\right)$$

is absolutely greater than a positive constant independent of d. Hence unless we assume

(26) $P\to\infty$ as $-d\to\infty$,

(24) and (25) contradict each other for $-d\to\infty$. Hence (26) is true. This proves our result for negative discriminants d where d or $\frac{d}{4}$ is quadratfrei. The result for general $d<O$ follows now from Lemma I of Heilbronn.

We could also have proved our Theorem II by using the following theorem due to Pepin[1]:

If $(a,2d)=1,\ a\mid(x^2-d)$ $[a>1]$

then $a^{2p}=x^2-dy^2\ (y\neq O)$

where p is the number of classes in the principal genus of reduced primitive non-equivalent binary quadratic forms of negative determinant d.

This result is quoted by Dickson in his History of the theory of numbers, Vol. 3, page 58, without, however, the necessary restriction that a should be prime to d.

July 25, 1934,
WALTAIR.

1. *Atti Accad Pont Nuovi Lincei,* 33, 1879—1880, 50—59. This paper has been inaccessible to me.

A LIST OF SELF-RECIPROCAL FUNCTIONS*

BY

BRIJ MOHAN MEHROTRA.

1. The equations

$$f(x)=\sqrt{\frac{2}{\pi}}\int_0^\infty f(y)\cos xy\,dy,$$

$$f(x)=\sqrt{\frac{2}{\pi}}\int_0^\infty f(y)\sin xy\,dy,$$

or, more generally,

$$f(x)=\int_0^\infty \sqrt{xy}\,J_\nu(xy)\,f(y)\,dy, \qquad (1.1)$$

where $J_\nu(x)$ is a Bessel function of order $\nu \geqslant -\frac{1}{2}$, have recently engaged the attention of several mathematicians. A function satisfying (1.1) is said to be self-reciprocal for J_ν transforms while a function satisfying (1.1) with the sign of one side changed, is said to be skew-reciprocal for J_ν transforms.

Several solutions of these equations have been given, a few of which might be mentioned: one was given by Bailey† in 1930; two others were given, during the same year, by Hardy and Titchmarsh.‡ Two more solutions were given by Prof. Titchmarsh in an unpublished manuscript a little later.§

Self-reciprocal functions have been appearing from time to time in different papers by different writers.⁑ A few such functions have recently been discovered by the present writer.‖ The object of this note is two-fold: (1) to collect the more

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* This note formed the last chapter of my Ph. D. Thesis submitted to the University of Liverpool in Oct., 1933.

† Bailey (1).

‡ Hardy and Titchmarsh (6).

§ For a brief account of these solutions, *see* Mehrotra (10). A detailed discussion thereof is shortly to appear in the press.

⁑ A history of these functions has recently been sent for publication by me.

‖ These were suggested to me by the work of Glaisher (3).

important of all such functions under one head, (2) to show that from the formal point of view, at least, all these functions fall under one or the other of the five solutions mentioned above.

Following Hardy and Titchmarsh I will say that a function is R_ν if it is self-reciprocal for J_ν transforms, and it is—R_ν if it is skew-reciprocal for J_ν transforms; also, for $R_{\frac{1}{2}}$ and $R_{-\frac{1}{2}}$ I will write R_s and R_c respectively.

2. Before proceeding to the examples, it seems desirable to state, very briefly, the five solutions of (1.1) mentioned above. All these formulæ are subject to appropriate conditions in each case.

(i) If $f(x)$ is $\pm R_\nu$,

$$f(x)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} 2^{\frac{1}{2}s}\,\Gamma(\tfrac{1}{4}+\tfrac{1}{2}\nu+\tfrac{1}{2}s)\psi(s)x^{-s}\,ds, \tag{A}$$

where $$\psi(s)=\pm\psi(1-s), \tag{2.1, 2.2}$$

Here, by Mellin's Inversion Formula,

$$2^{\frac{1}{2}s}\,\Gamma(\tfrac{1}{4}+\tfrac{1}{2}\nu+\tfrac{1}{2}s)\psi(s)=\int_0^\infty x^{s-1}f(x)\,dx.* \tag{2.3}$$

(ii) If $f(x)$ is $\pm R_\nu$,

$$f(x)=\frac{x^{\frac{1}{2}-\nu}}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{\frac{1}{2}x^2 s}\,s^{-\frac{1}{2}(\nu+1)}\,\mu(s)\,ds, \tag{B}$$

where $$\mu(s)=\pm\mu\left(\frac{1}{s}\right). \tag{2.4, 2.5}$$

By Mellin's Inversion Formula

$$\mu(s)=s^{\frac{1}{2}(\nu+1)}\int_0^\infty x^{\frac{1}{2}+\nu}\,e^{-\frac{1}{2}sx^2}\,f(x)\,dx.\dagger \tag{2.6}$$

(iii) The function

$$f(x)=x^{\frac{1}{2}+\nu}\int_0^\infty \omega^\nu\,e^{-\frac{1}{2}x^2\omega^2}\,F(\omega)\,d\omega, \tag{B'}$$

* Hardy and Titchmarsh (6), § 1.4.
† Hardy and Titchmarsh (6), § 1.5.

where $$F(\omega) = F\left(\frac{1}{\omega}\right), \tag{2.7}$$

is R_ν. *

(iv) If $f(x)$ is R_ν,

$$f(x) = \tfrac{1}{2} x^{\frac{5}{2}} \int_0^\infty \sqrt{t}\, J_{\frac{1}{2}\nu}\left(\tfrac{1}{2} x^2 t\right) \psi(t)\, dt, \tag{C}$$

where $$\psi(t) = \psi\left(\frac{1}{t}\right)^{\dagger}.$$

(v) If $f(x)$ is R_ν,

$$f(x) = \tfrac{1}{2} \sqrt{x} \int_0^\infty \frac{1}{\sqrt{t}}\, J_{\frac{1}{2}\nu}\left(\tfrac{1}{2} x^2 t\right) \Psi(t)\, dt, \tag{D}$$

where $$\Psi(t) = \Psi\left(\frac{1}{t}\right)^{\dagger}.$$

3. *Examples*:

(i) Putting $F(\omega) = \omega^{a-\frac{1}{2}} + \omega^{\frac{1}{2}-a}$ $(o < a < 1)$,
in (B′) we get

$$f(x) = 2^{\frac{1}{2}\nu - \frac{3}{4}} \left\{ 2^{\frac{1}{2}a} \Gamma\left(\tfrac{1}{4} + \tfrac{1}{2}\nu + \tfrac{1}{2}a\right) x^{-a} + 2^{\frac{1}{2} - \frac{1}{2}a} \Gamma\left(\tfrac{3}{4} + \tfrac{1}{2}\nu - \tfrac{1}{2}a\right) x^{a-1} \right\}.^{\ddagger}$$

In particular, when $a = \frac{1}{2}$,

$$F(\omega) = 2, \qquad f(x) = 2^{\frac{1}{2}(\nu+1)} \Gamma\left(\tfrac{1}{2}\nu + \tfrac{1}{2}\right) \cdot \frac{1}{\sqrt{x}}.^{\|}$$

(ii) Putting in (C)

$$\psi(t) = \frac{2^\nu}{\sqrt{\pi}} \Gamma\left(\tfrac{1}{2} + \tfrac{1}{2}\nu\right) \left(t + \frac{1}{t}\right)^{-\frac{1}{2}\nu - \frac{1}{2}}$$

and using a formula given by Watson,‡ we get

$$f(x) = x^{\frac{1}{2}+\nu}\, e^{-\frac{1}{2}x^2}\ \S$$

We get the same result by putting in (D)

$$\Psi(t) = \frac{2^{\nu+2}}{\sqrt{\pi}} \Gamma\left(\tfrac{3}{2} + \tfrac{1}{2}\nu\right) \left(t + \frac{1}{t}\right)^{-\frac{1}{2}\nu - \frac{3}{2}}.$$

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* Bailey (1). *See also* Bailey (2) and Mehrotra (9) § 5 (iii).
† *See* Mehrotra (10), §§ 8—9.
‡ *See* Hardy and Titchmarsh (6), § 3 (10).
‖ *See* Weber (13) 230.
‡ Watson (11), § 13.6 (2).
§ *See* Weber (13) 228.

(iii) Let $f(x)=x^{\frac{1}{2}+\nu}\cos\left(\tfrac{1}{2}x^2-\frac{\nu+1}{4}\pi\right)$. *

Calculating $\mu(s)$ from (2.6) we get

$$\mu(s)=2^{\nu-1}\,\Gamma(\nu+1)\left\{\left(e^{\frac{\pi i}{4}}\sqrt{s}+\frac{1}{e^{\frac{\pi i}{4}}\sqrt{s}}\right)^{-\nu-1}+\left(\frac{e^{\frac{\pi i}{4}}}{\sqrt{s}}+\frac{\sqrt{s}}{e^{\frac{\pi i}{4}}}\right)^{-\nu-1}\right\},$$

which satisfies (2.4).

If $f(x)=x^{\frac{1}{2}+\nu}\sin\left(\tfrac{1}{2}x^2-\frac{\nu+1}{4}\pi\right)$, *

which is $-R_\nu$,

$$\mu(s)=2^{\nu-1}\,\Gamma(\nu+1)\left\{\left(e^{\frac{\pi i}{4}}\sqrt{s}+\frac{1}{e^{\frac{\pi i}{4}}\sqrt{s}}\right)^{-\nu-1}-\left(\frac{e^{\frac{\pi i}{4}}}{\sqrt{s}}+\frac{\sqrt{s}}{e^{\frac{\pi i}{4}}}\right)^{-\nu-1}\right\},$$

which satisfies (2.5).

(iv) If, in (A), we take $\psi(s)=P(\tfrac{1}{2}-s)$ where $P(u)$ is an even polynomial, or an even integral function of order less than 1, we find that

$$f(x)=2\sum_{0}^{\infty}\frac{\left(-\tfrac{1}{2}x^2\right)^n}{\underline{|n}}\,P(2n+\tfrac{1}{2})$$

is R_c. If $P(u)$ is a polynomial,

$$f(x)=e^{-\frac{1}{2}x^2}\,Q(x),$$

where $Q(u)$ is a polynomial. The simplest example, after $P(u)=1$, is $P(u)=u^2$, when

$$f(x)=2\,e^{-\frac{1}{2}x^2}\left(x^4-3x^2+\tfrac{1}{4}\right).\ \dagger$$

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* *See* Bailey (1), § (4.1) and (4.2),

† Hardy and Titchmarsh (6), § 3 (2).

(v) If $f(x)=\sqrt{x}\, J_{\frac{1}{2}\nu}(\tfrac{1}{2}x^2)$,

using a formula given by Watson*, we get, from (2.3),

$$\psi(s)=\frac{2^{-\frac{1}{2}\nu-\frac{3}{4}}\sqrt{\pi}}{\Gamma(\frac{5}{8}+\frac{1}{4}\nu+\frac{1}{4}s)\,\Gamma(\frac{7}{8}+\frac{1}{4}\nu-\frac{1}{4}s)},$$

where $0<R(\frac{1}{4}+\frac{1}{2}\nu+\frac{1}{2}s)<R(\frac{1}{2}\nu)+\frac{3}{2}$. Obviously, $\psi(s)$ satisfies (2.1).†

(vi) If $f(x)=x^{\frac{5}{2}} J_{\frac{1}{2}\nu}(\tfrac{1}{2}x^2)$,

using the same formula, we get, from (2.3),

$$\psi(s)=\frac{2^{\frac{5}{4}-\frac{1}{2}\nu}\sqrt{\pi}}{\Gamma(\frac{1}{8}+\frac{1}{4}\nu+\frac{1}{4}s)\,\Gamma(\frac{8}{8}+\frac{1}{4}\nu-\frac{1}{4}s)},$$

where $0<R(\frac{5}{4}+\frac{1}{2}\nu+\frac{1}{2}s)<R(\frac{1}{2}\nu)+\frac{3}{2}$. Here too $\psi(s)$ satisfies (2.1).‡

(vii) Let $f(x)=x^{\frac{1}{2}-\nu}(x^2-b^2)^{\frac{1}{4}(\nu-1)}\, J_{\frac{1}{2}(\nu-1)}(b\sqrt{x^2-b^2})$ $\quad (x>b>0)$,

$=0 \qquad (0<x<b)$,

Using a formula given by Watson,‖ we get, from (2.6),

$$\mu(s)=b^{\frac{1}{2}(\nu-1)}\, e^{-\frac{1}{2}b^2(s+\frac{1}{s})},$$

which satisfies (2.4).§

(viii) Let $f(x)=x^{\frac{1}{2}+\nu}(x^2+a^2)^{-\frac{1}{4}(\nu+1)} K_{\frac{1}{2}(\nu+1)}(a\sqrt{x^2+a^2})$ $\quad (a>0)$.

That this function is R_ν, results by elementary substitutions, from a formula given by Watson.

From (2.6), we get

$$2^{\frac{1}{2}s}\,\Gamma(\tfrac{1}{4}+\tfrac{1}{2}\nu+\tfrac{1}{2}s)\,\psi(s)=\int_0^\infty x^{s+\nu-\frac{1}{2}}\,\frac{K_{\frac{1}{2}(\nu+1)}(a\sqrt{x^2+a^2})}{(x^2+a^2)^{\frac{1}{4}(\nu+1)}}\,dx.$$

Evaluating this integral by another formula given by Watson,⁂ we get

$$\psi(s)=2^{-\frac{5}{4}}\left(\frac{2}{a}\right)^{\frac{1}{2}(\nu+1)} K_{\frac{1}{2}(s-\frac{1}{2})}(a^2),$$

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* Watson (11), § 13.24 (1).

† Hardy and Titchmarsh (6), § 3 (8).

‡ Titchmarsh—unpublished manuscript.

‖ Watson (11), § 13.3 (4).

§ Hardy and Titchmarsh (6), § 3 (7).

‡ Watson (11). § 13.47 (2).

⁂ Watson (11), § 13.47 (6).

where $0 \leqslant -\frac{1}{2}-\nu < R(s)$. As $K_{\mu}(x)$ is an even function of μ, it follows that $\psi(s)$ satisfies (2.1).

(ix) The function $f_1(x) = \frac{x}{\sqrt{2\pi}} - \left[\frac{x}{\sqrt{2\pi}}\right]$, $f_1(x) = \frac{1}{2}$, according as x is not or is a multiple of $\sqrt{2\pi}$, is R_c. In this case the integral in (2.3) has to be interpreted as a Stieltjes Integral. We have

$$2^{\frac{1}{2}s}\,\Gamma(\tfrac{1}{2}s)\psi(s) = \lim_{N\to\infty}\int_0^{(N+\frac{1}{2})a} x^{s-1}\,df_1(x) \qquad (a=\sqrt{2\pi})$$

$$= \lim_{N\to\infty}\left[\frac{1}{a}\int_a^{(N+\frac{1}{2})a} x^{s-1}\,dx - \sum_1^N (na)^{s-1}\right]$$

$$= \lim_{N\to\infty} a^{s-1}\left[\frac{(N+\frac{1}{2})^s}{s} - \sum_1^N n^{s-1}\right] = -(2\pi)^{\frac{1}{2}s-\frac{1}{2}}\zeta(1-s).$$

Hence $\psi(s) = -\dfrac{1}{\sqrt{2\pi}}\dfrac{\pi^{\frac{1}{2}s}}{\Gamma(\frac{1}{2}s)}\zeta(1-s)$

satisfies (2.1).*

(x) Let $r(n)$ be the number of representations of n as a sum of two squares and let

$$\overline{P}(x) = \sum_{0\leqslant n\leqslant x}{}' r(n) - \pi x, \qquad f(x) = x^{-\frac{3}{2}}\left[\overline{P}\left(\frac{x^2}{2\pi}\right) - 1\right],$$

the dash implying the insertion of a factor $\frac{1}{2}$ in the last term of the sum when x is an integer. Then $f(x)$ is R_2. From (2.3),

$$2^{\frac{1}{2}s}\,\Gamma(\tfrac{5}{4}+\tfrac{1}{2}s)\psi(s) = \tfrac{1}{2}(2\pi)^{\frac{1}{2}s-\frac{3}{4}}\frac{Z(\frac{3}{4}-\frac{1}{2}s)}{\frac{3}{4}-\frac{1}{2}s},$$

where $Z(s)$ is the function defined for $\sigma > 1$ by

$$Z(s) = \sum \frac{r(n)}{n^s} = \zeta(s)\,L(s)$$

$$= \zeta(s)\,\left(1^{-s} - 3^{-s} + 5^{-s} - 7^{-s} + \ldots\ldots\right).$$

Hence $\psi(s) = 2^{-\frac{7}{4}}\pi^{\frac{1}{2}s-\frac{3}{4}}\dfrac{\zeta(\frac{3}{4}-\frac{1}{2}s)\,L(\frac{3}{4}-\frac{1}{2}s)}{(\frac{3}{4}-\frac{1}{2}s)\,\Gamma(\frac{5}{4}+\frac{1}{2}s)}$,

which satisfies (2.1).†

(xi) Let $f(x) = x^{\frac{1}{2}+\nu}\,e^{-\frac{1}{2}x^2}\displaystyle\sum_{r=0}^{n}\frac{(-1)^{n-r}\,x^{2r}}{\lfloor r\,\lfloor n-r\,\Gamma(r+\nu+1)}$. ‡

* Hardy and Titchmarsh (6), § 5.4.

† *See* Hardy (5) and Hardy and Titchmarsh (6), § 3 (9).

‡ This is a trivial transformation of the function given by Wilson (14).

This function is $\pm R_\nu$ according as n is an even or odd integer. From the formal point of view, at least, we get, from (2.6),

$$\mu(s)=\sum_{r=o}^{n}\frac{(-1)^{n-r}\,s^{\frac{1}{2}\nu+\frac{1}{2}}\,2^{r+\nu}}{\lfloor r\,\lfloor n-r\,(s+1)^{r+\nu+1}}$$

$$=\frac{(-1)^n\,2^\nu}{\lfloor n\left(\sqrt{s}+\frac{1}{\sqrt{s}}\right)^{\nu+1}}\sum_{r=o}^{n}(-1)^n\,{}^nC_r\left(\frac{2}{s+1}\right)^r$$

$$=\frac{(-1)^n\,2^\nu}{\lfloor n\left(\sqrt{s}+\frac{1}{\sqrt{s}}\right)^{\nu+1}}\left(1-\frac{2}{s+1}\right)^n=\frac{(-1)^n\,2^\nu}{\lfloor n\left(\sqrt{s}+\frac{1}{\sqrt{s}}\right)^{\nu+1}}\left(\frac{s-1}{s+1}\right)^n.$$

Here, $\mu(s)$ clearly satisfies the equation

$$\mu(s)=(-1)^n\,\mu\left(\frac{1}{s}\right)$$

(xii) Let $f(x)=2\sum_{n=1}^{\infty}\left(nx\sqrt{\frac{\pi}{2}}\right)^{\frac{1}{2}\nu+\frac{1}{4}}K_{\frac{1}{2}\nu+\frac{1}{4}}(nx\sqrt{2\pi})-\frac{\Gamma(\frac{1}{2}\nu+\frac{3}{4})}{x\sqrt{2}}.$*

Here, from (2.3),

$$2^{\frac{1}{2}s}\Gamma(\tfrac{1}{4}+\tfrac{1}{2}\nu+\tfrac{1}{2}s)\psi(s)=2\sum_{n=1}^{\infty}\int_0^\infty x^{s-1}\left(nx\sqrt{\frac{\pi}{2}}\right)^{\frac{1}{2}\nu+\frac{1}{4}}$$

$$K_{\frac{1}{2}\nu+\frac{1}{4}}(nx\sqrt{2\pi})\,dx-\frac{\Gamma(\frac{1}{2}\nu+\frac{3}{4})}{\sqrt{2}}\int_0^\infty x^{s-2}dx.$$

Evaluating the first term on the right-hand side by a formula given by Watson,† we get

$$\pi^{-\frac{1}{2}s}\,2^{\frac{1}{2}s-1}\,\Gamma(\tfrac{1}{2}s)\,\Gamma(\tfrac{1}{4}+\tfrac{1}{2}\nu+\tfrac{1}{2}s)\,\zeta(s).$$

The integral in the second term is not convergent, but by an appropriate adaptation of Riesz's logarithmic means,‡ it becomes

$$-\frac{\Gamma(\frac{1}{2}\nu+\frac{3}{4})}{\sqrt{2}}\lim_{\lambda\to\infty}\frac{1}{\log\lambda}\int_1^\lambda\frac{dw}{w}\int_{\frac{1}{w}}^{w}x^{s-2}\,dx,$$

which tends to 0 unless $s=1$, when it tends to infinity.

Hence $\qquad\psi(s)=\dfrac{\Gamma(\frac{1}{2}s)}{2\pi^{\frac{1}{2}s}}\,\zeta(s),$

which satisfies (2.1).

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* This is a trivial deviation from the result given by Watson (12).

† Watson (11), § 13.21 (8).

‡ *See* Hardy and Titchmarsh (6), § 3 (10).

For the particular case $\nu = \frac{1}{2}$. this function becomes a constant multiple of the familiar R_s function

$$\frac{1}{e^{x\sqrt{2\pi}}-1} - \frac{1}{x\sqrt{2\pi}}.^{*} \tag{3.1}$$

(xiii) Let $f(x) = \sum_{n=1}^{\infty} (-1)^{n-1} \left[(2n-1)x \right]^{\frac{1}{2}\nu + \frac{3}{4}}$

$$K_{\frac{1}{2}\nu - \frac{1}{4}} \left[(2n-1)x\sqrt{\frac{\pi}{2}} \right].^{\dagger}$$

Here, $\psi(s) = 2^{\frac{3}{4}\nu - \frac{7}{8}} . \pi^{-\frac{1}{4}\nu - \frac{3}{8}} . \left(\frac{4}{\pi}\right)^{\frac{1}{2}s} \Gamma(\frac{1}{2}+\frac{1}{2}s) \; L(s)$

where $L(s) = \frac{1}{1^s} - \frac{1}{3^s} + \frac{1}{5^s} - \frac{1}{7^s} + \ldots\ldots$

That $\psi(s)$ satisfies (2.1), follows from the functional equation of $L(s)$.

For the particular case $\nu = -\frac{1}{2}$, the function becomes

$$2^{-\frac{5}{4}} \pi^{\frac{1}{4}} \frac{1}{\cosh\left(x\sqrt{\frac{\pi}{2}}\right)}.^{\ddagger} \tag{3.2}$$

(xiv) Let $f(x) =$

$$\sum_{r=0}^{\infty} (-1)^r \left\{ (4r+1)x \right\}^{\frac{1}{2}\nu + \frac{3}{4}} K_{\frac{1}{2}\nu - \frac{1}{4}} \left\{ (4r+1)\frac{x\sqrt{\pi}}{2} \right\}$$

$$+ \sum_{r=0}^{\infty} (-1)^r \left\{ (4r+3)x \right\}^{\frac{1}{2}\nu + \frac{3}{4}} K_{\frac{1}{2}\nu - \frac{1}{4}} \left\{ (4r+3)\frac{x\sqrt{\pi}}{2} \right\}.$$

Here. $\psi(s) = \frac{2^{\nu - \frac{1}{2}}}{\pi^{\frac{1}{4}\nu + \frac{3}{8}}} . \frac{2^{\frac{3}{2}s}}{\pi^{\frac{1}{2}s}} . \Gamma(\frac{1}{2}+\frac{1}{2}s) \; L_1(s),$

where $L_1(s) = \sum_{r=0}^{\infty} (-1)^r \left[\frac{1}{(4r+1)^s} + \frac{1}{(4r+3)^s} \right]$

$$= \frac{1}{1^s} + \frac{1}{3^s} - \frac{1}{5^s} - \frac{1}{7^s} + \frac{1}{9^s} + \frac{1}{11^s} - \ldots\ldots$$

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* The value of the integral in (2.3) when $f(x)$ denotes the function (3.1) has been given by Gram. (4).

† I remark that examples (*xiii*)—(*xvii*) have occured to me only recently. In what follows I have proved, only from the formal point of view, that they are R_ν.

‡ Hutchinson (7)

Hence we are required to prove that $\psi(s)$ satisfies (2.1); in other words, that

$$2^{3s-\frac{3}{2}}\,\Gamma(\tfrac{1}{2}+\tfrac{1}{2}s)\ \mathrm{L}_1(s)=\pi^{s-\frac{1}{2}}\,\Gamma(1-\tfrac{1}{2}s)\ \mathrm{L}_1(1-s). \quad (3.3)$$

Hutchinson* has proved recently that if

$$Z(a,b,s)=Z(s)=\sum_{r=0}^{\infty}\frac{1}{(ar+b)^s},$$

then

$$Z_b+Z_{a-b}=\frac{2\pi^{s-\frac{1}{2}}}{a^s}\cdot\frac{\Gamma(\frac{1}{2}-\frac{1}{2}s)}{\Gamma(\frac{1}{2}s)}\sum_{n=1}^{\infty}\frac{\cos\frac{2bn\pi}{a}}{n^{1-s}}, \quad (3.4)$$

$$Z_b-Z_{a-b}=\frac{2\pi^{s-\frac{1}{2}}}{a^s}\cdot\frac{\Gamma(1-\frac{1}{2}s)}{\Gamma(\frac{1}{2}+\frac{1}{2}s)}\sum_{n=1}^{\infty}\frac{\sin\frac{2bn\pi}{a}}{n^{1-s}}. \quad (3.5)$$

Now, $\mathrm{L}_1(s)$ may be thrown into the form

$$\left[\sum_{r=0}^{\infty}\frac{1}{(8r+1)^s}-\sum_{r=0}^{\infty}\frac{1}{(8r+7)^s}\right]+\left[\sum_{r=0}^{\infty}\frac{1}{(8r+3)^s}-\sum_{r=0}^{\infty}\frac{1}{(8r+5)^s}\right].$$

Putting in (3.5) $b=1$, $a=8$ and $b=3$, $a=8$, successively, we get

$$\mathrm{L}_1(s)=\frac{2\pi^{s-\frac{1}{2}}}{8^s}\,\frac{\Gamma(1-\frac{1}{2}s)}{\Gamma(\frac{1}{2}+\frac{1}{2}s)}\left[\sum_{n=1}^{\infty}\frac{\sin\frac{n\pi}{4}}{n^{1-s}}+\sum_{n=1}^{\infty}\frac{\sin\frac{3n\pi}{4}}{n^{1-s}}\right]$$

$$=\frac{\pi^{s-\frac{1}{2}}}{2^{3s-\frac{3}{2}}}\cdot\frac{\Gamma(1-\frac{1}{2}s)}{\Gamma(\frac{1}{2}+\frac{1}{2}s)}\,\mathrm{L}_1(1-s),$$

after a little algebra. This equation is equivalent to (3.3).

For the particular case $\nu=-\frac{1}{2}$, the function becomes

$$\pi^{\frac{1}{4}}\,\frac{\cosh\left(\frac{x\sqrt{\pi}}{2}\right)^*}{\cosh\,(x\sqrt{\pi})}. \quad (3.6)$$

(xv) Let $f(x)=\sum_{r=0}^{\infty}\Big[(3r+1)x\Big]^{\frac{1}{2}\nu+\frac{3}{4}}K_{\frac{1}{2}\nu-\frac{1}{4}}\Big[(3r+1)x\sqrt{\frac{2\pi}{3}}\Big]$

$$-\sum_{r=0}^{\infty}\Big[(3r+2)x\Big]^{\frac{1}{2}\nu+\frac{3}{4}}K_{\frac{1}{2}\nu-\frac{1}{4}}\Big[(3r+2)x\sqrt{\frac{2\pi}{3}}\Big].$$

From (2.3),

$$\psi(s)=2^{\frac{1}{4}\nu-\frac{13}{8}}\left(\frac{3}{\pi}\right)^{\frac{1}{4}\nu+\frac{3}{8}+\frac{1}{2}s}\Gamma(\tfrac{1}{2}+\tfrac{1}{2}s)\ \mathrm{L}_2(s),$$

* Hutchinson (7).

where $L_2(s)=\sum_{r=o}^{\infty}\left[\frac{1}{(3r+1)^s}-\frac{1}{(3r+2)^s}\right]$

$$=\frac{1}{1^s}-\frac{1}{2^s}+\frac{1}{4^s}-\frac{1}{5^s}+\frac{1}{7^s}-\frac{1}{8^s}+\ldots\ldots$$

We are now to prove that $\psi(s)$ satisfies (2.1), or that

$$3^{s-\frac{1}{2}}\,\Gamma(\tfrac{1}{2}+\tfrac{1}{2}s)\;L_2(s)=\pi^{s-\frac{1}{2}}\,\Gamma(1-\tfrac{1}{2}s)\;L_2(1-s). \qquad (3.7)$$

Putting $b=1$, $a=3$ in (3.5) we get

$$L_2(s)=\frac{2\pi^{s-\frac{1}{2}}}{3^s}\,\frac{\Gamma(1-\frac{1}{2}s)}{\Gamma(\frac{1}{2}+\frac{1}{2}s)}\sum_{n=1}^{\infty}\frac{\sin\frac{2n\pi}{3}}{n^{1-s}}$$

$$=\frac{\pi^{s-\frac{1}{2}}\,\Gamma(1-\frac{1}{2}s)}{3^{s-\frac{1}{2}}\,\Gamma(\frac{1}{2}+\frac{1}{2}s)}\;L_2(1-s),$$

which is equivalent to (3.7).*

For the particular case $\nu=-\frac{1}{2}$ the function becomes

$$\left(\frac{3\pi}{8}\right)^{\frac{1}{4}}\frac{\sinh\left(x\sqrt{\frac{\pi}{6}}\right)}{\sinh\left(x\sqrt{\frac{3\pi}{2}}\right)}=\frac{\left(\frac{3\pi}{8}\right)^{\frac{1}{4}}}{1+2\cosh\left(x\sqrt{\frac{2\pi}{3}}\right)}. \qquad (3.8)$$

(xvi) Let $f(x)=\sum_{r=o}^{\infty}(-1)^r\Big[(4r+1)x\Big]^{\frac{1}{2}\nu+\frac{1}{4}}K_{\frac{1}{2}\nu+\frac{1}{4}}\Big[(4r+1)\frac{x\sqrt{\pi}}{2}\Big]$

$$-\sum_{r=o}^{\infty}(-1)^r\Big[(4r+3)x\Big]^{\frac{1}{2}\nu+\frac{1}{4}}K_{\frac{1}{2}\nu+\frac{1}{4}}\Big[(4r+3)\frac{x\sqrt{\pi}}{2}\Big],$$

From (2.3),

$$\psi(s)=\frac{2^{\nu-\frac{3}{2}}}{\pi^{\frac{1}{4}\nu+\frac{1}{8}}}\left(\frac{8}{\pi}\right)^{\frac{1}{2}s}\Gamma(\tfrac{1}{2}s)\;L_8(s),$$

where $L_8(s)=\sum_{r=o}^{\infty}(-1)^r\left[\frac{1}{(4r+1)^s}-\frac{1}{(4r+3)^s}\right]$

$$=\frac{1}{1^s}-\frac{1}{3^s}-\frac{1}{5^s}+\frac{1}{7^s}+\frac{1}{9^s}-\ldots.$$

If $\psi(s)$ is to satisfy (2.1), we must have

$$2^{3s-\frac{3}{2}}\,\Gamma(\tfrac{1}{2}s)\;L_8(s)=\pi^{s-\frac{1}{2}}\Gamma(\tfrac{1}{2}-\tfrac{1}{2}s)\;L_8(1-s). \qquad (3.9)$$

Now, $L_8(s)$ may be thrown into the form

$$\left[\sum_{r=o}^{\infty}\frac{1}{(8r+1)^s}+\sum_{r=o}^{\infty}\frac{1}{(8r+7)^s}\right]-\left[\sum_{r=o}^{\infty}\frac{1}{(8r+3)^s}+\sum_{r=o}^{\infty}\frac{1}{(8r+5)^s}\right].$$

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* Equations (3.3) and (3.7) have also been established by Malmsten (8).

Putting in (3.4) $b=1, a=8$ and $b=3, a=8$ successively, we get

$$L_3(s)=\frac{2\pi^{s-\frac{1}{2}}}{8^s}\frac{\Gamma(\frac{1}{2}-\frac{1}{2}s)}{\Gamma(\frac{1}{2}s)}\left[\sum_{n=1}^{\infty}\frac{\cos\frac{n\pi}{4}}{n^{1-s}}-\sum_{n=1}^{\infty}\frac{\cos\frac{3n\pi}{4}}{n^{1-s}}\right]$$

$$=\left(\frac{\pi}{8}\right)^{s-\frac{1}{2}}\frac{\Gamma(\frac{1}{2}-\frac{1}{2}s)}{\Gamma(\frac{1}{2}s)}L_3(1-s),$$

which is equivalent to (3.9).

For the particular case $\nu=\frac{1}{2}$, the function becomes

$$\pi^{\frac{1}{4}}\frac{\sinh\left(\frac{x\sqrt{\pi}}{2}\right)}{\cosh\,(x\sqrt{\pi})}. \tag{3.10}$$

(xvii) Let $f(x)=\sum_{r=0}^{\infty}(-1)^r\left[(6r+1)x\right]^{\frac{1}{2}\nu+\frac{1}{4}}K_{\frac{1}{2}\nu+\frac{1}{4}}\left[(6r+1)x\sqrt{\frac{\pi}{6}}\right]$

$$-\sum_{r=0}^{\infty}(-1)^r\left[(6r+5)x\right]^{\frac{1}{2}\nu+\frac{1}{4}}K_{\frac{1}{2}\nu+\frac{1}{4}}\left[(6r+5)x\sqrt{\frac{\pi}{6}}\right].$$

From (2.3),

$$\psi(s)=2^{\frac{3}{4}\nu-\frac{13}{8}}\left(\frac{3}{\pi}\right)^{\frac{1}{4}\nu+\frac{1}{8}}\cdot\left(\frac{12}{\pi}\right)^{\frac{1}{2}s}\Gamma(\tfrac{1}{2}s)\,L_4(s),$$

where $L_4(s)=\sum_{r=0}^{\infty}(-1)^r\left[\frac{1}{(6r+1)^s}-\frac{1}{(6r+5)^s}\right]$

$$=\frac{1}{1^s}-\frac{1}{5^s}-\frac{1}{7^s}+\frac{1}{11^s}+\frac{1}{13^s}-\frac{1}{17^s}-\frac{1}{19^s}+\ldots$$

If $\psi(s)$ is to satisfy (2.1) we must have

$$12^{s-\frac{1}{2}}\Gamma(\tfrac{1}{2}s)\,L_4(s)=\pi^{s-\frac{1}{2}}\Gamma(\tfrac{1}{2}-\tfrac{1}{2}s)\,L_4(1-s). \tag{3.11}$$

Now, $L_4(s)$ may be thrown into the form

$$\left[\sum_{r=0}^{\infty}\frac{1}{(12r+1)^s}+\sum_{r=0}^{\infty}\frac{1}{(12r+11)^s}\right]-\left[\sum_{r=0}^{\infty}\frac{1}{(12r+5)^s}+\sum_{r=0}^{\infty}\frac{1}{(12r+7)^s}\right]$$

Putting in (3.4) $b=1$, $a=12$ and $b=5$, $a=12$ successively, we get

$$L_4(s)=\frac{2\pi^{s-\frac{1}{2}}}{12^s}\frac{\Gamma(\frac{1}{2}-\frac{1}{2}s)}{\Gamma(\frac{1}{2}s)}\left[\sum_{n=1}^{\infty}\frac{\cos\frac{n\pi}{6}}{n^{1-s}}-\sum_{n=1}^{\infty}\frac{\cos\frac{5n\pi}{6}}{n^{1-s}}\right]$$

$$=\left(\frac{\pi}{12}\right)^{s-\frac{1}{2}}\frac{\Gamma(\frac{1}{2}-\frac{1}{2}s)}{\Gamma(\frac{1}{2}s)}L_4(1-s),$$

which is equivalent to (3.11).

For $\nu = \frac{1}{2}$, the function becomes

$$\left(\frac{3\pi}{2}\right)^{\frac{1}{4}} \frac{\sinh\left(x\sqrt{\frac{2\pi}{3}}\right)}{\cosh\left(x\sqrt{\frac{3\pi}{2}}\right)} = \left(\frac{3\pi}{2}\right)^{\frac{1}{4}} \frac{2\sinh\left(x\sqrt{\frac{\pi}{6}}\right)}{2\cosh\left(x\sqrt{\frac{2\pi}{3}}\right)-1}.^{*} \tag{3.12}$$

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* That the functions (3.2), (3.6), (3.8) are R_c and the functions (3.10) and (3.12) are R_s, follows independently from formulae given by Glaisher (3).

"ON THE PRODUCT OF PARABOLIC CYLINDER FUNCTIONS"

BY

S. C. DHAR, D.SC.

§ 1. *Introduction.*

In a recent paper I have, by a method of transformation of integrals, obtained expressions for the product of two parabolic cylinder functions with different arguments in the form

$$D_n(X)D_m(x)=\frac{1}{(\sqrt{2})^{m+n}}\sum_{r=0}^{n+m}B_{n+m-r}D_{n+m-r}\left(\frac{X+x}{\sqrt{2}}\right)D_r\left(\frac{X-x}{\sqrt{2}}\right)$$

where B_{n+m-r} are certain co-efficients and m and n positive integers. In fact it has been proved by me that this is but a particular form of a more general expression given by

$$D_n(X)\ D_m(x)=$$

$$\sum_{r=0}^{n+m} A_{n+m-r}\ D_{n+m-r}\ (X\cos\alpha+x\sin\alpha)\ D_r(X\sin\alpha-x\cos\alpha),$$

where

$$A_{n+m-r}=\sin^{m+r}\alpha\cos^{n-r}\alpha\ \frac{n(n-1)\ldots.(n-r+1)}{r!}$$

$$\times F\left(-r,-m\,;\,n-r+1\,;\,-\frac{1}{\tan^2\alpha}\right)$$

In this paper an attempt is made to study the case when m and n are not integers.

§ 2. It was proved by Watson* that the parabolic cylinder functions $D_{-n-1}(-iz)$ is given by the integral

$$D_{-n-1}(-iz)=\frac{2^{n+1}}{\Gamma(n+1)}\ e^{\frac{1}{4}z^2}\int_{-\infty}^{\infty}e^{-2u^2+2izu}\ u^n\ du,\ldots..(1)$$

when arg. $u=0$ and $n>-1$.

From this it is easy to show that when $n>-1$

$$D_n(z)=(-1)^{\frac{1}{2}n}\ 2^{n+1}(2\pi)^{-\frac{1}{2}}\ e^{\frac{1}{4}z^2}\int_{-\infty}^{\infty}e^{-2u^2+2izu}\ u^n\ du\ (2)$$

* Watson: Proc. Lon. Math. Soc., Vol. VIII, p. 404.

Hence when $n>-1$ and $m>-1$ and X and x are any two quantities, the product of two parabolic cylinder functions can be written as

$$D_n(X)\,D_m(x)$$

$$=(-1)^{\frac{1}{2}(n+m)}\frac{2^{n+m+1}}{\pi}e^{\frac{1}{4}(X^2+x^2)}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{-2u^2-2t^2+2iXu+2ixt}\,u^n\,t^m\,du\,dt$$

$$=(-1)^{\frac{1}{2}(n+m)}\frac{2^{n+m+1}}{\pi}e^{\frac{1}{4}(X^2+x^2)}\,\underset{R\to\infty}{Lt}\iint_{S_R}e^{-2u^2-2t^2+2iXu+2ixt}\,u^n\,t^m\,du.dt\;\ldots\ldots(3)$$

where S_R denote the square

$$-R\leqslant u\leqslant R$$
$$-R\leqslant t\leqslant R$$

§ 3. Now, transform the integral by the following substitution

$$\left.\begin{array}{l}u\sin\alpha-t\cos\alpha=T\\ u\cos\alpha+t\sin\alpha=U\end{array}\right\}\;\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4)$$

with the special value $\alpha=\frac{\pi}{4}$, which is unnecessary for the present discussion.

The double integral (3) integrated over the area given by

$$t=0,\ u=R\ \text{and}\ u\sin\alpha+t\cos\alpha=R\sqrt{2}\ \left(\text{when}\ \alpha=\frac{\pi}{4}\right)$$

gives

$$\int_0^R e^{-2t^2+2ixt}\,t^m\,dt\int_R^{2R-t}e^{-2u^2+2iXu}\,u^n\,du.$$

Now,

$$\int_R^{2R-t}e^{-2u^2+2iXu}\,u^n\,du=\left[\frac{1}{2iX}e^{-2u^2+2iXu}\,u^n\right]_R^{2R-t}$$
$$-\frac{1}{2iX}\int_R^{2R-t}e^{2iXu}\frac{d}{du}\left[e^{-2u^2}\,u^n\right]du$$
$$=O\left(e^{-2R^2}\right)+O\left(e^{-2R^2}\right)$$

and the double integral over the triangle is less in modulus than

$$O\left(e^{-2R^2}\right)\int_0^R e^{-2t^2}\,t^m\,dt=O(1).$$

Similarly the double integral over the area

$$u=0,\ t=R \text{ and } u\sin\alpha+t\cos\alpha=R\sqrt{2}$$

will give the same result.

Hence if we denote by T_R the area enclosed by

$$-R\sqrt{2}<U<R\sqrt{2}$$
$$-R\sqrt{2}<T<R\sqrt{2}$$

the integral (3) becomes

$$D_n(X)\,D_m(x)$$
$$=(-1)^{\frac{1}{2}(n+m)}\frac{2(\sqrt{2})^{n+m}}{\pi}e^{\frac{1}{4}(X^2+x^2)}\underset{R\to\infty}{Lt}\iint_{T_R}F\,d\bar{U}\,dT$$

$$\left.\begin{aligned}&=(-1)^{\frac{1}{2}(n+m)}\frac{2(\sqrt{2})^{n+m}}{\pi}e^{\frac{1}{4}(X^2+x^2)}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}F\,d\bar{U}\,dT\\&\text{where}\\&F=(U+T)^n(U-T)^m e^{-2(U^2+T^2)+\sqrt{2}i(X+x)U+\sqrt{2}i(X-x)T}\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots\ldots\ldots(5)$$

§ 4. Take the special case when $X=x$, then the integral (5) reduces to

$$D_n(x)\,D_m(x)$$
$$=(-1)^{\frac{1}{2}(n+m)}\frac{2(\sqrt{2})^{n+m}}{\pi}e^{\frac{1}{2}x^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(U+T)^n(U-T)^m$$
$$\times e^{-2(U^2+T^2)+2\sqrt{2}ixU}\,dU\,dT\ldots\ldots(6)$$

Now

$$\left.\begin{aligned}&(U+T)^n(U-T)^m\\&=\sum_{r=0}^{\infty}B_{n+m-r}\cdot T^{n+m-r}U^r,\\&\text{where}\\&B_{n+m-r}=(-1)^m\frac{n(n-1)\ldots(n-r+1)}{r!}\\&\qquad F(-r,-m;n-r+1;-1)\end{aligned}\right\}\ldots(7)$$

Hence from (6) and (7) we get

$$D_n(x)\,D_m(x)$$
$$=(-1)^{\frac{1}{2}(n+m)}\frac{2(\sqrt{2})^{n+m}}{\pi}\sum_{r=0}^{\infty}B_{n+m-r}$$
$$\times\int_{-\infty}^{\infty}e^{-2U^2+2\sqrt{2}ixU+\frac{1}{2}x^2}U^r\,dU\int_{-\infty}^{\infty}e^{-2T^2}T^{n+m-r}\,dT$$

(term by term integration is justifiable)

$$=(-1)^{\frac{1}{2}(n+m)}\frac{(\sqrt{2})^{n+m+1}}{\sqrt{\pi}}\sum_{r=0}^{\infty}\frac{(-1)^{-\frac{1}{2}r}}{2^r}B_{n+m-r}\,D_r\,(x\sqrt{2})$$

$$\times\int_{-\infty}^{\infty}e^{-2T^2}\,T^{n+m-r}\,dT\,.\,.\,.\,.\,.\,.\,.\,.\,.\quad(8)$$

Now, the integral

$$\int_{-\infty}^{\infty}e^{-2T^2}\,T^{n+m-r}\,dT$$

$$=\frac{2^{r-n-m-1}\,(-1)^{-\frac{1}{2}(n+m)}}{(n+m-r+1)\,(n+m-r+3)\,.\,.\,(n+m-1)}\sqrt{2\pi}\,D_{n+m}(0),$$

(when r is even)

$$=\frac{2^{r-n-m-1}\,(-)^{-\frac{1}{2}(n+m+1)}}{(n+m-r+1)\,(n+m-r+3)\,.\,.\,(n+m)}\sqrt{2\pi}\,D_{n+m+1}(0)$$

(when r is odd)

Hence from (8) we get,

$$D_n(x)\,D_m(x)=D_{n+m}(0)\,\frac{1}{(\sqrt{2})^{n+m}}\sum_{0=r}^{\infty}$$

$$\frac{(-1)^{-r}}{(n+m-2r+1)\,(n+m-2r+3)\,.\,.(n+m-1)}\,B_{n+m-2r}\,D_{2r}(x\sqrt{2})$$

$$+D_{n+m+1}(0)\,\frac{1}{(\sqrt{2})^{n+m}}\sum_{r=0}^{\infty}$$

$$\frac{(-1)^{-r-1}}{(n-m-2r)\,(n+m-2r+2)\,.\,.(n+m)}\,B_{n+m-2r-1}\,D_{2r+1}(x\sqrt{2})$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\quad(9)$$

Now, if n and m are both positive integers and are equal, then since $D_{2n+1}(0)=0$, we get

$$D_n^2(x)=\frac{1}{2^n}\sum_{r=0}^{n}\frac{n!\,(2r-1)\,(2r-3)\,.\,.\,.3\,.\,1}{r!\,(n-r)!}\,D_{2n-2r}(x\sqrt{2})\qquad(10)^*$$

on simplification from (9).

August, 1934.

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* *Cf.* Mitra: Edin. Math. Soc., Vol. IV, p. 32.

A GENERAL FORMULA FOR THE MOMENTS OF THE HYPERGEOMETRICAL SERIES

BY

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Mysore.

§ 1. In 1924, Prof. Pearson gave a recurrence formula* for the successive moments of the Hypergeometrical series about its mean and suggested the possibility of deriving a general expression for a moment of any order by determinants.† In this paper we obtain such an expression which is perhaps more useful for direct numerical calculation than Prof. Romanovsky's formula.‡

We start with Pearson's formula which may be written

$$-(n-s+1)\mu_s+\sum_{t=2}^{s-1} c_{s,t}\,\mu_{s-t+1}+B=0 \qquad (1)$$

where $A=nq+r(p-q)$, $B=pqr(n-r)$

$$C_{s,2}=\binom{s-1}{2}-A\binom{s-1}{1}$$

$$C_{s,t(>2)}=\binom{s-1}{t}-A\binom{s-1}{t-1}+B\binom{s-1}{t-2}$$

r is the size of the sample drawn from a population of n containing pn marked and qn unmarked individuals and rq the mean of unmarked individuals, about which μ_s the s^{th} moment is calculated.

Changing s to $s-1, s-2, \ldots\ldots 2$ in (1) and remembering that $\mu_1=0$, we get $(s-1)$ linear equations in all, from which we eliminate the $(s-2)$ variables $\mu_2, \mu_3, \ldots \mu_{s-1}$ and obtain

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* Biometrika, 1924 Vol. XVI, p. 159.

† Biometrika, 1924, Vol. XVI, p. 162. Here Prof. Pearson remarks "I have not however, so far succeeded in reaching any useful general result. I should be glad to hear of progress in this direction, even for example, in the simple case of the binomial. It would enable us to deal more effectively than at present with some of the probable errors of random sampling from limited populations."

‡ Biometrika, 1925, Vol. XVII, p. 59.

$$\begin{vmatrix} c_{s,2} & c_{s,3} & c_{s,4}\cdots\cdots c_{s,s-2} & c_{s,s-1} & B-\mu_s(n-s-1) \\ -(n-s+2) & c_{s-1,2} & c_{s-1,3}\cdots c_{s-1,s-3} & c_{s-1,s-2} & B \\ 0 & -(n-s+3) & c_{s-2,2}\cdots c_{s-2,s-4} & c_{s-2,s-3} & B \\ 0 & 0 & -(n-s+4)\cdots c_{s-3,s-5} & c_{s-3,s-4} & B \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0\cdots\cdots c_{5,3} & c_{5,4} & B \\ 0 & 0 & 0\cdots\cdots c_{4,2} & c_{4,3} & B \\ 0 & 0 & 0\cdots\cdots -(n-2) & c_{3,2} & B \\ 0 & 0 & 0\cdots\cdots 0 & (-n-1) & B \end{vmatrix}$$

$=0$. This simplifies to

$$\mu_s(n-s+1)\ (n-s+2)\cdots(n-1)=B\Delta_s \qquad (2)$$

where Δ_s is a determinant of the $(s-1)^{th}$ order in which all the elements to the left of the minor diagonal bordering on the principal diagonal are zeroes;

$$\Delta_s=\begin{vmatrix} c_{s,2} & c_{s,3}\cdots\cdots\cdots c_{s,s-1} & 1 \\ -(n-s+2) & c_{s-1,2}\cdots\cdots c_{s-1,s-2} & 1 \\ 0 & -(n-s+3)\cdots\cdots c_{s-2,s-3} & 1 \\ \cdots & \cdots & \cdots \\ 0 & 0\cdots\cdots\cdots c_{3,2} & 1 \\ 0 & 0\cdots\cdots\cdots -(n-1) & 1 \end{vmatrix}$$

The number of terms in the expanded form of Δ_s is 2^{s-2}, while in Romanovsky's formula

$\mu_s=L_{0,s}+L_{1,s}\,U_1+\cdots+L_{s,s}\,U_s$, where $L_{h,s}(h>0)$ is a polynomial of $(s-h+1)$ terms, the total number of terms in the expression for μ_s is $\frac{s(s+1)}{2}+1$. Since $2^{s-2}<\frac{s(s+1)}{2}+1$ when $s<7$, it ought to be much easier to work with formula (2) than Romanovsky's, at least for the first *six* moments. Beyond the sixth moment, a recurrence formula will be decidedly advantageous.

Formula (2) has the unique property of being self-sufficient and does not require any auxiliary tables. Applying it to the test example* of Pearson to obtain μ_5, given $n=100$, $p=\frac{1}{10}$, $q=\frac{9}{10}$

$r=10$, we find $A=82$, $B=81$ and $\mu_5+\frac{81\Delta_5}{96.97.98.99}=-\frac{215532}{52283}$

$$\text{Since } \Delta_5=\begin{vmatrix} 6-4A & 4-6A+4B & 1-4A+6B & 1 \\ -(n-3) & 3-3A & 1-3A+3B & 1 \\ 0 & -(n-2) & 1-2A & 1 \\ 0 & 0 & -(n-1) & 1 \end{vmatrix}=-128\times 35922$$

The corresponding formula for the Binomial distribution is obtained by taking the limit of μ_s is in (2) when $n\rightarrow\infty$. It is readily written down by retaining only the co-efficients of the highest powers of n in both the numerator and denominator of μ_s.

Hence, for the binomial distribution,

$$\mu_s=pqr\begin{vmatrix} c_{s,2} & c_{s,3} \cdots\cdots & c_{s,s-1} & 1 \\ -1 & c_{s-1,2} \cdots\cdots & c_{s-1,s-2} & 1 \\ 0 & -1 \cdots\cdots & c_{s-2,s-3} & 1 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 \cdots\cdots & c_{3,2} & 1 \\ 0 & 0 \cdots\cdots & -1 & 1 \end{vmatrix} \qquad (3)$$

where $c_{s,2}=-q(s-1)$

and $c_{s,t(>2)}=-q\binom{s-1}{t-1}+pqr\binom{s-1}{t-2}$

$$\text{For example } \mu_5=pqr\begin{vmatrix} -4q & -6q+4pqr & -4q+6pqr & 1 \\ -1 & -3q & -3q+3pqr & 1 \\ 0 & -1 & -2q & 1 \\ 0 & 0 & -1 & 1 \end{vmatrix}$$

$$=pqr\begin{vmatrix} 2-4q & 4pqr & -1+2q \\ -2 & -3q & 1-3q+3pqr \\ 0 & -1 & 1-2q \end{vmatrix}$$

$$=pqr(1-2q)(2-6q+6pqr-1+4pqr-6q+12q^2($$
$$=pqr(p-q)(1+2pq.\overline{5r-6})$$

Again, for the moments of the Poisson-series make $q\rightarrow 0$, $p\rightarrow 1$, $rq\rightarrow m$ in (3) and we obtain

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* Biometrika, Vol. XVII, p. 60.

$$\mu_s = m \begin{vmatrix} 0 & m\binom{s-1}{1} & m\binom{s-1}{2} \dots & m\binom{s-1}{s-3} & 1 \\ -1 & 0 & m\binom{s-2}{1} \dots & m\binom{s-2}{s-4} & 1 \\ \dots & \dots & \dots & \dots\dots & \\ 0 & 0 & 0 & -1 & 1 \end{vmatrix} \quad \dots\dots (4)$$

§ 2, A recurrence formula for Δ_s is easily obtained by merely expanding the determinant in terms of the elements of the first row. We have

$$\Delta_s = c_{s,2}\,\Delta_{s-1} + (n-s+2)\,c_{s,3}\,\Delta_{s-2} + (n-s+2)(n-s+3). \\ c_{s,4}\,\Delta_{s-3} + \dots + (n-s+2)(n-s+3)\dots(n-2)\Delta_2\,c_{s,s-1} \\ +(n-s+2)(n-s+3)\dots(n-1). \qquad (5)$$

which is readily identified with (1), if we write

$$\Delta_s = \frac{\mu_s}{B}(n-s+1)\dots(n-1), \text{ etc.}$$

This enables us to expand Δ_s in powers of n without taking into account the fact that $c_{s,t}$ contains n.

The highest power of n in Δ_s is easily seen to be n^{s-2} and the term independent of n is

$$\begin{vmatrix} c_{s,2} & c_{s,3} \dots c_{s,s-1} & 1 \\ s-2 & c_{s-1,2} \dots c_{s-1,s-2} & 1 \\ 0 & s-3 \dots c_{s-2,s-3} & 1 \\ \dots & \dots & \dots \\ 0 & 0 \quad 1 & 1 \end{vmatrix} = (-1)^{\frac{(s-1)(s-2)}{2}} \begin{vmatrix} 1 & c_{s,s-1} \dots c_{s,3} & c_{s,2} \\ 1 & c_{s-1,s-2} \dots c_{s-1,2} & s-2 \\ 1 & c_{s-2,s-3} \dots s-3 & 0 \\ \dots & \dots & \dots \\ 1 & 1 \quad \dots 0 & 0 \end{vmatrix}$$

$$= (-1)^{\frac{(s-1)(s-2)}{2}} \begin{vmatrix} \binom{s-1}{s-1} & \binom{s-1}{s-2} \dots\dots & \binom{s-1}{2} & \binom{s-1}{1} \\ \binom{s-2}{s-2} & \binom{s-2}{s-3} \dots\dots & \binom{s-2}{1} & 0 \\ \dots\dots & \dots\dots\dots\dots & & \dots\dots \\ \binom{1}{1} & 0 \quad \dots\dots & 0 & 0 \end{vmatrix} \times E$$

$$= (s-1)!\; D_{s-2}$$

where (1) D_{s-2} is a continuant of the $(s-2)^{th}$ order whose principal diagonal elements are all equal to $(-A)$ bordered on the right by a minor diagonal whose elements are all B and on the left by another whose elements are all 1, the remaining elements being zeroes, and (ii) E is this continuant bordered on the top and to the left by

$$1, 0, 0, 0, \ldots\ldots 0, \quad (\overline{s-2} \text{ zeroes})$$
$$1, 1, 0, 0, \ldots\ldots 0, \quad (\overline{s-3} \text{ zeroes})$$

respectively.

$$\left[E \equiv \begin{vmatrix} 1 & 0 & 0 & 0 \ldots 0 & 0 & 0 \\ 1 & -A & B & 0 \ldots 0 & 0 & 0 \\ 0 & 1 & -A & B \ldots 0 & 0 & 0 \\ \multicolumn{7}{c}{\ldots\ldots\ldots\ldots\ldots} \\ 0 & 0 & 0 & 0 \ldots 1 & -A & B \\ 0 & 0 & 0 & 0 \ldots 0 & 1 & -A \end{vmatrix} \right]$$

The coefficients of the other powers of n are too complicated to be of any practical use for computation. Equating the coefficients of powers of n in (5), we have, after much labour, obtained the coefficients of n^{s-3} and n^{s-4} so that they may be of use in calculating as far as the fifth moment.

If we write

$$\Delta_s = n^{s-2} + P_{s,1}\, n^{s-3} + P_{s,2}\, n^{s-4} + \ldots + (s-1)!\, D_{s-2}$$

and similar expressions for Δ_{s-1}, Δ_{s-2}, etc. in (5), we find on equating the coefficients of n^{s-3} and n^{s-4}

$$P_{s,1} = \sum_{t=2}^{s-1} c_{s,t} - \sum_{t=1}^{s-2} t$$

$$= \sum_{2}^{s-1} \binom{s-1}{t} - A \sum_{1}^{s-2} \binom{s-1}{t} + B \sum_{1}^{s-3} \binom{s-1}{t} - \frac{(s-1)(s-2)}{2}$$

$$= \tfrac{1}{2}(2^s - s^2 + s - 2) - A(2^{s-1} - 2) + B(2^{s-1} - s - 1) \quad \ldots (6)$$

and

$$P_{s,2} = \sum_{t=2}^{s-1} c_{s,t}\, P_{s-t+1,1}$$

$$- \sum_{t=3}^{s-1} c_{s,t}\, (\overline{s-2} + \overline{s-3} + \ldots + \overline{s-t+1})$$

$$+ \Sigma\, tu\, (t \neq u;\ t, u = 1, 2, 3, \ldots s-2).$$

$$= \sum_{t=2}^{s-1} c_{s,t}\, P_{s-t+1,1} + \tfrac{1}{2} \sum_{t=3}^{s-1} (t-2)(t-2s+1)\, c_{s,t}$$

$$+ \tfrac{1}{24}(s-1)(s-2)(s-3)(3s-4).$$

$$= A^2(3^{s-1} - 3.2^{s-1} + 3) + \tfrac{1}{2}B^2(3^{s-1} - S.2^{s-1} - 2^s + s^2 + s + 1)$$

$$+\tfrac{1}{2}AB(2^{s-1}+5.2^{S}-3^{S}-6s-7)$$
$$A(2^{s-2}\overline{s^2-s+12}-3^{s}-s^{2}+s-3)$$
$$+\tfrac{1}{2}B(5.3^{s-1}-2^{s-1}\overline{s^2+12}+s^{3}+s+9)$$
$$+1-2^{s-2}(s^{2}-s+6)+2.3^{s-1}+\tfrac{1}{24}s(s-1)(3s^{2}-7s+14). \qquad \ldots\ldots\ldots\ldots\ldots(7)$$

Using the above formulas we find

$$\mu_5=\frac{B(n^{3}+n^{2}.\overline{5-14A+10B}+n.\overline{36A^{2}-20AB-10A-24B+48AB-24A^{3}})}{(n-1)(n-2)(n-3)(n-4)}$$

which agrees with the determinental form of the result used on p. 3 (*supra.*)

"ON A CERTAIN POLYNOMIAL ANALOGOUS TO LOMMEL'S POLYNOMIAL"

BY

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In my previous paper* bearing this title, I investigated the differential equation satisfied by the polynomial $R_{n,m}(x)$, introduced by Dr. S. C. Mitra,† and gave the following expressions for this polynomial in a series of Hermite's polynomial $H_n(x)$:

$$(-1)^m R_{n,\,m-2}(x)=$$

$$\frac{(m-2)!\,(2m+4n)\,(2m+4n-4)\ldots\ldots(2m+4n-2m+8)}{m(2+m)\ldots(m-4+m)\,(2-m)\,(4-m)\ldots(m-2-m)}\times$$

$$\left[H_0(x)+\frac{m(2-m)}{2!\,(2m+4n)}H_2(x)+\frac{m(2+m)\,(2-m)\,(4-m)}{4!(2m+4n)\,(2m+4n-4)}H_4(x)+\ldots\ldots\right],$$

when m is an even integer and

$$(-1)^m R_{n,\,m-2}(x)=$$

$$\frac{(m-2)!\,(2m+4n-2)\;2m+4n-6)\ldots\ldots(2m+4n-2m+8)}{(1+m)\,(3+m)\ldots(m-4+m)\,(3-m)\,(5-m)\ldots(m-2-m)}\times$$

$$\left[H_1(x)+\frac{(1+m)\,(3-m)}{3!\,(2m+4n-2)}H_3(x)+\frac{(1+m)\,(3+m)\,(3-m)\,(5-m)}{5!\,(2m+4n-2)\,(2m+4n-6)}H_5(x)+\ldots\ldots\right],$$

when m is an odd integer.

The object of this paper is to evaluate some integrals involving $R_{n,m}(x)$ and to give a summation formula.

**Journal of the Indian Mathematical Society*, Vol. 19 (1932), pp. 274—278.

† "On the properties of a certain polynomial analogous to Lommel's polynomial", *Indian Physico-Mathematical Journal*, Vol. 3 (1932), pp. 9—15. *See also* his paper, bearing the present title, in the *Journal of the Indian Mathematical Society*, Vol. 1 (1934), pp. 4—7.

1. It follows easily from the above expansions that

$$R_{n,\,m}(x)=\sum_{r=0}^{r=\frac{m}{2}}(-\tfrac{1}{2})^{r+\frac{m}{2}}\Big\{(m-2r+2)\,(m-2r+4)\,..\,(m+2r)\Big\}\times$$

$$\Big\{(m+2n-2r+2)\,(m+2n-2r)\ldots(2n+4)\Big\}\frac{H_{2r}(x)}{(2r)\,!},\ldots\ldots(1)$$

when m is even, and

$$R_{n,\,m}(x)=\sum_{r=0}^{r=\frac{m-1}{2}}(-\tfrac{1}{2})^{r+\frac{m+1}{2}}\Big\{(m-2r+1)(m-2r+3)\,..\,(m+2r+1)\Big\}\times$$

$$\Big\{(m+2n-2r+1)\,(m+2n-2r-1)\,..\,(2n+4)\Big\}\frac{H_{2r+1}(x)}{(2r+1)\,!},\,..\,(2)$$

when m is odd.

We know* that for two different integral values of m and n the equation of orthogonality holds

$$\int_{-\infty}^{\infty}e^{-\frac{1}{2}x^2}\,H_m(x)\;H_n(x)\;dx=0 \qquad (3)$$

and for $m=n$

$$\int_{-\infty}^{\infty}e^{-\frac{1}{2}x^2}\Big\{H_n(x)\Big\}^2\,dx-\sqrt{2\pi}\;(n)\,! \qquad (4)$$

Multiply both sides of (1) by $e^{-\frac{1}{2}x^2}\,H_{2r}(x)$, where r is given by

$$0\leqslant r\leqslant\frac{m}{2}, \qquad (5)$$

and integrate, between the limits $-\infty$ and ∞, with respect to x; we obtain, by the help of (3) and (4),

$$\int_{-\infty}^{\infty}e^{-\frac{1}{2}x^2}\,H_{2r}(x)\;R_{n,\,m}(x)\;dx$$

$$=(-\tfrac{1}{2})^{r+\frac{m}{2}}\sqrt{2\pi}\Big\{(m-2r+2)\,(m-2r+4)\ldots(m+2r)\Big\}\times$$

$$\Big\{(m+2n-2r+2)\,(m+2n-2r)\ldots.(2n+4)\Big\},$$

* P. Appell and I. Kampede Feriet, 'Fonctions Hypergeometriques e Hyperfpheriques, Polynomes D'Hermıta' (Gauthier-Villars, Paris, 1926) p. 348.

true for even integral values of m and for values of r satisfying the relation (5)

2. Following the method of § 1, we deduce the integrals:

$$\int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2} H_{2r+1}(x)\ R_{n,m}(x)\ dx=0,$$

$$m \text{ even and } 0 \leqslant r < \frac{m}{2}$$

$$\int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2} H_{2r+1}(x)\ R_{n,m}(x)\ dx$$

$$=(-\tfrac{1}{2})^{r+\frac{m+1}{2}}\sqrt{2\pi}\left\{(m-2r+1)(m-2r+3)\ldots(m+2r+1)\right\}\times$$

$$\left\{(m+2n-2r+1)(m+2n-2r-1)\ldots(2n+4)\right\},$$

$$m \text{ odd and } 0 \leqslant r \leqslant \frac{m-1}{2}.$$

and

$$\int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2} H_{2r}(x)\ R_{n,m}(x)\ dx=0,$$

$$m \text{ odd and } 0 \leqslant r \leqslant \frac{m-1}{2}.$$

3. The polynomial $R_{n,m}(x)$ satisfies the recurrence relation*

$$R_{n,m}(x)+x\,R_{n,m-1}(x)+(m+n+1)\ R_{n,m-2}(x)=0.$$

It follows that

$$-x\ R_{n,m-1}(x)=R_{n,m}(x)+(m+n+1)\ R_{u,m-2}(x)\ldots(6)$$

and

$$-y\ R_{n,m-1}(y)=R_{n,m}(y)+(m+n+1)\ R_{n,m-2}(y)\ldots(7)$$

Multiply (6) by $R_{n,m-1}(y)$ and (7) by $R_{n,m-1}(x)$ and subtract: we obtain

$$(y-x)\frac{R_{n,m-1}(y)\ R_{n,m-1}(x)}{(m+n+1)!}$$

$$=\frac{1}{(m+n+1)!}\Big[R_{u.m}(x)\ R_{n,m-1}(y)-R_{n,m}(y)\ R_{n,m-1}(x)\Big]$$

* *See* Dr. Mitra's first paper quoted above.

$$+\frac{1}{(m+n)!}\Big[R_{n,m-2}(x)\,R_{n.m-1}(y)-R_{n,m-2}(y)\,R_{n,m-1}(x)\Big] \quad \ldots\ldots(8)$$

Putting $m=1, 2, 3, \ldots$ in (8) successively and adding we obtain

$$(y-x)\sum_{p=1}^{m}\frac{R_{n,p-1}(y)\,R_{n,p-1}(x)}{(p+n+1)!}$$

$$=\frac{1}{(m+n+1)!}\Big[R_{n,m}(x)\,R_{n,m-1}(y)-R_{n,m}(y)\,R_{n,m-1}(y)\Big]$$

$$+\frac{1}{(n+1)!}\Big[R_{n,-1}(x)\,R_{n,0}(y)-R_{n,-1}(y)\,R_{n,0}(x)\Big] \ldots\ldots(9)$$

Since $R_{n,0}(x)-1$ and $R_{n,1}(x)=-x$, we may, by virtue of the recurrence relation, take $R_{n,-1}(x)=0$. We have therefore

$$(y-x)\sum_{p=1}^{m}\frac{R_{n,p-1}(y)\,R_{n,p-1}(x)}{(p+n+1)!}$$

$$=\frac{1}{(m+n+1)!}\Big[R_{n,m}(x)\,R_{n,m-1}(y)-R_{n,m}(y)\,R_{n,m-1}(y)\Big] \quad \ldots\ldots(10)$$

In particular, putting $y=x+t$ in (10) and equating the coefficient of t on either side, we obtain

$$\sum_{p=1}^{m}\frac{\{R_{n,p-1}(x)\}^2}{(p+n+1)!}\;\frac{1}{(m+n+1)!}\Big[R_{n,m}(x)\,R'_{n,m-1}(x)-R'_{n,m}(x)\,R_{n,m-1}(x)\Big]$$

where dashes denote differentiation with respect to x.

"ON SOME INTEGRALS INVOLVING BESSEL FUNCTIONS".

BY

M. ZIA-UD-DIN AND N. G. SHABDE.

Introduction.

Prof. G. N. Watson* has recently revaluated the integral

$$\int_0^\infty J_\mu(at)\, J_\nu(bt)\, e^{-ct}\, dt$$

This integral has an important application in finding the generating function of Jacobi Polynomials†. It may, therefore, be worth while to consider in this note the following similar and more general integrals:

$$(i)\quad \int_0^\infty J_\mu(at)\, J_\nu(bt)\, e^{-ct}\, \frac{dt}{t} \qquad ; \mu+\nu>0$$

$$(ii)\quad \int_0^\infty J_\mu(at)\, J_\nu(bt)\, e^{-ct}\, t^{\mu+\nu}\, dt \qquad ; R(\mu+\nu)>-\tfrac{1}{2}$$

and

$$(iii)\quad \int_0^\infty J_\mu(at)\, J_\nu(bt)\, e^{-ct}\, t^{\mu+\nu+1}\, dt$$

Finally we also obtain the integral

$$(iv)\quad \int_0^\infty (\tfrac{1}{2}\rho)^\rho \frac{\sqrt{(\mu+\rho)}}{(a^2+\rho^2)^{\frac{1}{2}(\mu+\rho)}\sqrt{(\rho+1)}}\; {}_2F_1\left(\frac{\rho+\mu}{2}, \frac{1-\mu+\rho}{2}, \rho+1, \frac{\rho^2}{a^2+\rho^2}\right) J_\rho(\rho\lambda)\, d\rho$$

$$=\frac{\lambda^\mu}{\lambda^2-1}\, e^{-a\lambda}\ ; \qquad \begin{cases} \lambda\geqslant 0 \\ \mu>0 \end{cases}$$

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* *Journal London Math. Society*; Vol. 9, Part 1, Jan. 1934, p. 16; "An infinite integral involving Bessel functions".

† G. N. Watson, *J. L. M. S.*, Vol. 9, Part 1. p. 22; "Notes on generating functions of polynomials—(4) Jacobi Polynomials".

which in the particular case of $a=0$, $\mu=1$, reduces to the known result

$$(v) \qquad \int_0^\infty J_\rho(\rho\lambda)\, d\rho = \frac{\lambda}{\lambda^2-1}$$

§ 1.

As in the Watson's paper, if $R(\mu+\nu+1) > 0$,

$J_\mu(at)\, J_\nu(bt) =$

$$\frac{a^\mu b^\nu}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{(\mu-\nu)i\theta}\left(\frac{e^{\theta i}+e^{-\theta i}}{a^2e^{\theta i}+b^2e^{-\theta i}}\right)^{\frac{1}{2}(\mu+\nu)}$$

$$\times J_{\mu+\nu}\left\{t\left(e^{\theta i}+e^{-\theta i}\right)^{\frac{1}{2}}\left(a^2e^{\theta i}+b^2e^{-\theta i}\right)^{\frac{1}{2}}\right\}d\theta \qquad \ldots\ldots\ldots\ldots\ldots\ldots(1{\cdot}1)$$

Using the Pincherle's formula

$$\int_0^\infty e^{-ct} J_\rho(\beta t)\,\frac{dt}{t} = \frac{1}{\rho}\left\{\frac{\beta}{c+\sqrt{c^2+\beta^2}}\right\}^\rho \ldots\ldots\ldots\ldots(1{\cdot}2)$$

we get if $\mu+\nu>0$,

$$\int_0^\infty J_\mu(at)\, J_\nu(bt)\, e^{-ct}\,\frac{dt}{t} =$$

$$\frac{a^\mu b^\nu}{\pi(\mu+\nu)}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{(\mu-\nu)i\theta}\left[\frac{e^{\theta i}+e^{-\theta i}}{c+\sqrt{c^2+(e^{\theta i}+e^{-\theta i})(a^2e^{\theta i}+b^2e^{-\theta i})}}\right]^{\mu+\nu} d\theta \ldots\ldots(1{\cdot}3)$$

Following the ideas in Watson's paper and suppressing a heavy algebra we have (1·3) equal to

$$\frac{2\,a^\mu b^\nu}{\pi\, c^{\mu+\nu-2}.(\mu+\nu)}\int_0^\xi \frac{(\zeta R)^\mu\,(\zeta/R)^\nu\cos(\mu-\nu)\omega\,.\,d\zeta}{\sqrt{\{(c^2-a^2\zeta^2)(c^2-b^2\zeta^2)-c^4\zeta^2\}}}$$

$$+\frac{2\,a^\mu b^\nu(a^2+b^2)}{\pi.\,c^{\mu+\nu-2}.(\mu+\nu)}\int_0^\xi \frac{(\zeta R)^\mu\,(\zeta/R)^\nu\cos(\mu-\nu)\omega\,.\,\zeta^2 d\zeta}{(c^2-a^2\zeta^2)\sqrt{(c^2-a^2\zeta^2)(c^2-b^2\zeta^2)-c^4\zeta^2}}$$

$$+\frac{2(a^2-b^2)\,a^\mu b^\nu}{\pi\, c^{\mu+\nu}\,(\mu+\nu)}\int_0^\xi \frac{(\zeta/R)^\nu\,(\zeta R)^\mu\cos(\mu-\nu)\omega\,.\,\zeta\, d\zeta}{(c^2-a^2\zeta^2)}$$

where ξ, ζ, R, ω have the same meaning as in Watson's paper, that is,

$$R = \sqrt{\frac{c^2-b^2\,\zeta^2}{c^2-a^2\,\zeta^2}}$$

$$\frac{\sqrt{\{(c^2-a^2\zeta^2)(c^2-b^2\zeta^2)-c^4\zeta^2\}}}{c^2\zeta}=\tan\omega \qquad \left(0\leqslant\omega\leqslant\frac{\pi}{2}\right)$$

and $\dfrac{2c^2}{\xi^2}=c^2+a^2+b^2+\sqrt{(c^2+a^2+b^2)^2-4a^2b^2}$

If $X\equiv(c^2\sec^2\omega+a^2+b^2)^2-4a^2b^2$; (1·3) finally becomes equal to

$$\frac{a^\mu b^\nu c^2}{\pi(\mu+\nu)}\int_0^{\frac{\pi}{2}}\left(\frac{2c\sec\omega}{c^2\sec^2\omega+b^2-a^2+\sqrt{X}}\right)^\mu\left(\frac{2c\sec\omega}{c^2\sec^2\omega+a^2-b^2+\sqrt{X}}\right)^\nu \times\frac{\cos(\mu-\nu)\omega\,.\,d\omega}{\sqrt{X}}$$

$$+\frac{a^\mu b^\nu c^2(a^2+b^2)}{\pi(\mu+\nu)}\int_0^{\frac{\pi}{2}}\left(\frac{2c\sec\omega}{c^2\sec^2\omega+b^2-a^2+\sqrt{X}}\right)^\mu \left(\frac{2c\sec\omega}{c^2\sec^2\omega+a^2-b^2+\sqrt{X}}\right)^\nu\times\frac{\sec^2\omega\,d\omega}{\sqrt{X}}$$

$$\times\cos(\mu-\nu)\omega\left[\frac{4c^4\sec^2\omega}{\left\{c^2(c^2\sec^2\omega+b^2-a^2+\sqrt{X})(c^2\sec^2\omega+a^2-b^2+\sqrt{X})-4a^2c^4\sec^2\omega\right\}}\right]$$

$$+\frac{8(a^2-b^2)a^\mu b^\nu c^6}{\pi(\mu+\nu)}\times\int_0^{\frac{\pi}{2}}$$

$$\frac{\left(\dfrac{2c\sec\omega}{c^2\sec^2\omega+b^2-a^2+\sqrt{X}}\right)^\mu\left(\dfrac{2c\sec\omega}{c^2\sec^2\omega+a^2-b^2+\sqrt{X}}\right)^\nu\sec^4\omega\,.\,\tan\omega\,.\,d\omega}{\sqrt{X}\,.\left[c^2(c^2\sec^2\omega+b^2-a^2+\sqrt{X})(c^2\sec^2\omega+a^2-b^2+\sqrt{X})-4a^2c^4\sec^2\omega\right]}$$

§ 2.

Proceeding as in § 1 and using the Pincherle's formula

$$\int_0^\infty e^{-ct}J_\nu(\beta t)\,t^\nu\,dt=\frac{(2\beta)^\nu\,\Gamma(\nu+\frac{1}{2})}{(c^2+\beta^2)^{\frac{1}{2}}\sqrt{\pi}(c^2+\beta^2)^\nu}\quad\ldots\ldots\ldots\ldots(2\text{·}1);$$

where $R(\mu+\nu)>-\frac{1}{2}$, we get

$$\int_0^\infty J_\mu(at)\,J_\nu(bt)\,e^{-ct}\,t^{\mu+\nu}\,dt$$

$$=\frac{2^{\mu+\nu}\,a^{\mu}\,b^{\nu}\,\sqrt{(\mu+\nu+\frac{1}{2})}}{\pi^{\frac{3}{2}}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{(\mu-\nu)i\theta}$$

$$\left[\frac{e^{\theta i}+e^{-\theta i}}{c^2+(e^{\theta i}+e^{-\theta i})(a^2e^{\theta i}+b^2e^{-\theta i})}\right]^{\mu+\nu}\frac{d\theta}{\sqrt{\left\{c^2+(e^{\theta i}+e^{-\theta i})(a^2e^{\theta i}+b^2e^{-\theta i})\right\}}}$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (2{\cdot}2)$$

$$=\frac{2^{\mu+\nu}a^{\mu}b^{\nu}\sqrt{(\mu+\nu+\frac{1}{2})}}{\pi^{\frac{3}{2}}\;c^{\mu+\nu-1}}\Bigg[\int_{0}^{\xi} R^{\mu-\nu}e^{-(\mu-\nu)i\omega}\zeta^{\mu+\nu}$$

$$\frac{d\,\zeta}{\sqrt{c^2+(e^{\theta i}+e^{-\theta i})(a^2e^{\theta i}+b^2e^{-\theta i})}}$$

$$\times\frac{c^{\mu+\nu}}{\zeta^{\mu+\nu}}\left\{\frac{e^{\theta i}+e^{-\theta i}}{c^2+(e^{\theta i}+e^{-\theta i})(a^2e^{\theta i}+b^2e^{-\theta i})}\right\}^{\mu+\nu}$$

$$-\int_{\xi}^{0}\frac{R^{\mu-\nu}e^{(\mu-\nu)i\omega}\,\zeta^{\mu+\nu}\,d\omega}{\sqrt{c^2+(e^{\theta i}+e^{-\theta i})(a^2e^{\theta i}+b^2e^{-\theta i})}}\,\frac{c^{\mu+\nu}}{\zeta^{\mu+\nu}}$$

$$\left\{\frac{e^{\theta i}+e^{-\theta i}}{c^2+(e^{\theta i}+e^{-\theta i})(a^2e^{\theta i}+b^2e^{-\theta i})}\right\}^{\mu+\nu}\Bigg]$$

$$=\frac{2^{2(\mu+\nu)+1}\,c\,a^{\mu}b^{\nu}\sqrt{(\mu+\nu+\frac{1}{2})}}{c^{-2(\mu+\nu)}\;\pi^{\frac{3}{2}}}\int_{0}^{\frac{\pi}{2}}\frac{\sec^2\omega\,d\omega}{\sqrt{X}}$$

$$\frac{\cos\,(\mu-\nu)\omega\,.\,R^{\mu+\nu}\,.\,\zeta^{\mu+\nu}}{\left[c^6+2c^2\zeta(a^2-b^2)\sqrt{c^4\zeta^2-(c^2-a^2\zeta^2)(c^2-b^2\zeta^2)}+a^4(a^2+c^2)\zeta^4-c^2a^4\zeta^2+2b^2c^4\zeta^2-a^2b^4\zeta^4+b^4c^2\zeta^2\right]^{\mu+\nu}}$$

§ 3.

Similarly we have

$$\int_{0}^{\infty} J_{\mu}(at)\;J_{\nu}(bt)\;e^{-ct}\,.\,t^{\mu+\nu+1}\,dt$$

$$=\;\frac{2^{\mu+\nu+1}\,a^{\mu}\,b^{\nu}\,c\sqrt{(\mu+\nu+\frac{3}{2})}}{\pi^{\frac{3}{2}}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{(\mu-\nu)\theta i}$$

$$\left[\frac{(e^{\theta i}+e^{-\theta i})}{c^2+(e^{\theta i}+e^{-\theta i})(a^2e^{\theta i}+b^2e^{-\theta i})}\right]^{\mu+\nu}$$

$$\frac{d\theta}{\left[c^2+(e^{\theta i}+e^{-\theta i})(a^2e^{\theta i}+b^2e^{-\theta i})\right]^{\frac{3}{2}}}$$

and can be expressed as an integral in the variable ω as in § 1 and § 2.

§ 4.

We have*

$$\int_0^\infty e^{-at}\, J_\nu(bt)\, t^{\mu-1}\, dt$$

$$=(\tfrac{1}{2}b)^\nu \frac{\surd(\mu+\nu)}{(a^2+b^2)^{\frac{1}{2}(\mu+\nu)}\surd(\nu+1)}\, {}_2F_1\left(\frac{\mu+\nu}{2}, \frac{1-\mu+\nu}{2}, \nu+1; \frac{b^2}{a^2+b^2}\right) \qquad \ldots\ldots\ldots\ldots (4{\cdot}1)$$

where $R(\mu+\nu)>0$, $|b|<|a|$, $a>0$, $b\geqslant 0$.

Hence

$$\int_0^\infty e^{-at}\, J_m(mt)\, t^{\mu-1}\, dt$$

$$=(\tfrac{1}{2}m)^m \frac{\surd(\mu+m)}{(a^2+m^2)^{\frac{1}{2}(\mu+m)}\surd(m+1)}\times$$

$${}_2F_1\left(\frac{m+\mu}{2}, \frac{1-\mu+m}{2}, m+1, \frac{m^2}{a^2+m^2}\right),\ R(\mu)>0,\ m\geqslant 0 \ldots\ldots\ldots (4{\cdot}2)$$

Applying to (4·2) the converse of MacRobert's† Fourier Bessel Integral Theorem that

$$\text{If } f(\lambda)=\int_p^q \phi(\rho)\, J_\rho(\rho\lambda)\, \rho\, d\rho, \qquad 0\leqslant p<q$$

then

$$\int_0^\infty \left(\lambda-\frac{1}{\lambda}\right) f(\lambda)\, J_m(m\lambda)\, d\lambda = \begin{cases} \frac{1}{2}\left\{\phi(m+0)+\phi(m-0)\right\}; & p<m<q \\ \text{or}, & 0<m<p \text{ or } m>q \end{cases} \qquad \ldots\ldots\ldots (4{\cdot}3)$$

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* Watson's *Theory of Bessel Functions*, p. 385.

† "Fourier Integrals", *Proc. Royal Soc. Edin.*, 51 (1930—31), p. 121.

we get

$$\int_0^\infty (\tfrac{1}{2}\rho)^\rho \frac{\sqrt{(\mu+\rho)}}{(a^2+\rho^2)^{\frac{1}{2}(\mu+\rho)}\sqrt{(\rho+1)}} \times$$

$$2F_1\left\{\frac{\rho+\mu}{2}, \frac{1-\mu+\rho}{2}, \rho+1, \frac{\rho^2}{a^2+\rho^2}\right\} J_\rho(\rho\lambda)\ \rho\ d\rho$$

$$= \frac{\lambda^\mu}{\lambda^2-1} e^{-a\lambda} \qquad \left.\begin{matrix}\lambda \geqslant 0 \\ \mu \geqslant 0\end{matrix}\right\} \ldots\ldots(4{\cdot}4)$$

If in (4·4) we take $\mu=1$ and $a=0$, we get

$$\int_0^\infty J_\rho(\lambda\rho)\ d\rho = \frac{\lambda}{\lambda^2-1} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4{\cdot}5)$$

which is a well-known result*

* Mac Robert: *l,c.*

"BOUNDARY PROBLEMS IN NON-LINEAR PARABOLIC EQUATIONS"

BY

RAZIUDDIN SIDDIQI

(Hyderabad-Deccan).

§ 1. In a previous paper published in the Math. Zeitschrift[1], I have proved that one and only one regular solution of the equation

$$\frac{\partial^2 u}{\partial x^2}-\frac{\partial u}{\partial t}=P(x, t, u)=\sum^{\infty} p_r(x, t)\, u^r$$

exists which satisfies the conditions

$$u(0, t)=u(\pi, t)=0 \text{ for all } t\geqslant 0$$
$$u(x, 0)=f(x) \text{ for } 0\leqslant x\leqslant \pi.$$

In this paper, I consider the same problem for a general non-rectangular domain bounded by the x-axis, and the two curves Γ_1 and Γ_2 with the equations $x=h_1(t)$ and $x=h_2(t)$. It is proved that on transforming to new variables, the problem is reduced to one in a rectangular domain.

In the second part, I consider the mixed boundary problem

$$u=0 \text{ for } x=0;\ \frac{\partial u}{\partial x}=0 \text{ for } x=\tfrac{\pi}{2} \text{ for all } t\geqslant 0.$$

It is proved that by continuing the functions $u(x, t)$ and $f(x)$ in $\frac{\pi}{2}\leqslant x\leqslant\pi$ according to the method of images, the solution of the mixed boundary problem can be brought to depend on that of the first problem treated in the previous paper.

§ 2. Consider a non-rectangular domain D bounded by the x-axis between the points 0 and π, and the two curves Γ_1 and Γ_2 with the equations

(1) $(\Gamma_1)\ x=h_1(t), \qquad (\Gamma_2)\ x=h_2(t).$

Let

(2) $h_1(0)=0, \qquad h_2(0)=\pi, \qquad h_2(t)>h_1(t)$ for all t,

and suppose that the function

(3) $p(t)\equiv h_2(t)-h_1(t)$

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1. M. R. Siddiqi: Math. Zeitschrift, Vol. 35 (1932), pp. 464—484.

is continuous along with its derivatives of the first and second order for all t.

The problem is to determine the solution $u(x, t)$ of the non-linear partial differential equation

(4) $$\frac{\partial^2 u}{\partial x^2}-\frac{\partial u}{\partial t}=u^2$$

which is regular in D and which satisfies the boundary conditions:

(5) $u=f(x)$ for $t=0$,
$u=0$ on Γ_1 and Γ_2.

We shall prove that on transforming to new variables, the problem can be reduced to the determination of the solution of

$$\frac{\partial^2 u}{\partial x^2}-\frac{\partial u}{\partial t}=q(t)u^2$$

in the rectangular domain $0\leqslant x\leqslant\pi$, $0\leqslant t$.

Our integration method as developed in the previous paper would then be applicable.

First of all we make the transformation

(6) $$x=h_1(t)+\frac{x'}{\pi}\,p(t),\quad t=t.$$

To the domain D of the $(x,\ t)$ plane would then correspond the rectangular domain

(7) $0\leqslant x'\leqslant\pi,\qquad 0\leqslant t$

of the $(x',\ t)$ plane. The differential equation (4) is here transformed to the equation

(8) $$\frac{\partial^2 u}{\partial x'^2}-\frac{p^2(t)}{\pi^2}\frac{\partial u}{\partial t}=\frac{p^2(t)}{\pi^2}\,u^2$$

If we transform further

(9) $$t'=\pi^2\int_0^t\frac{dt}{p^2(t)},$$

and write

(10) $u'(x', t')=u(x, t)$, $p'(t')=p(t)$, $f'(x')=f(x)$,

then we see that u' satisfies the equation

(11) $$\frac{\partial^2 u'}{\partial x'^2}-\frac{\partial u'}{\partial t'}=\frac{p'^2(t')}{\pi^2}\,u'^2$$

and the boundary conditions become

(12) $u'=f'(x')$ for $t'=0$ and for all x' in $0\leqslant x'\leqslant\pi$
$u'=0$ for $x'=0$ and $x'=\pi$ for all $t'\geqslant 0$.

Thus the proposed reduction has been obtained.

§ 3. Let us now try to determine a solution of

$$(1)\quad \frac{\partial^2 u}{\partial x^2}-\frac{\partial u}{\partial t}=u^2$$

in the domain

$$(2)\quad 0\leqslant x\leqslant \tfrac{\pi}{2},\quad 0\leqslant t$$

which satisfies the mixed boundary conditions

$$(3)\quad u(0,t)=0, \frac{\partial u(0,\frac{\pi}{2})}{\partial x}=0 \text{ for all } t\geqslant 0$$

and

$$(4)\quad u(x,0)=f(x) \text{ in } 0\leqslant x\leqslant \tfrac{\pi}{2}.$$

This problem can be brought to depend upon the first boundary problem considered in the previous paper by continuing the functions $u(x, t)$ and $f(x)$ in the interval $\frac{\pi}{2}\leqslant x\leqslant\pi$ according to the method of images.

Thus we take for all t:

$$(5)\quad \begin{aligned} u(\pi-x,t)&=u(x,t)\\ f(\pi-x)&=f(x)\end{aligned}$$

where x varies in the interval $0\leqslant x\leqslant\frac{\pi}{2}$. Now we define two functions $u_1(x,t)$ and $f_1(x)$ in the interval $0\leqslant x\leqslant\pi$ as follows:

$$(6)\quad \begin{aligned} u_1(x,t)&=u(x,t) \text{ in } 0\leqslant x\leqslant\tfrac{\pi}{2}\\ &=u(\pi-x,t) \text{ in } \tfrac{\pi}{2}\leqslant x\leqslant\pi\end{aligned}$$

and

$$(7)\quad \begin{aligned} f_1(x)&=f(x) \text{ in } 0\leqslant x\leqslant\tfrac{\pi}{2}\\ &=f(\pi-x) \text{ in } \tfrac{\pi}{2}\leqslant x\leqslant\pi.\end{aligned}$$

We see then that $u_1(x,t)$ is a regular solution of equation (1) in the domain

$$(8)\quad 0\leqslant x\leqslant\pi,\quad 0\leqslant t$$

which satisfies the following boundary conditions:

$$(9)\quad \begin{aligned} u_1(0,t)&=u(0,t)=0\\ u_1(\pi,t)&=u(0,t)=0.\end{aligned}$$

and

$$u_1(x,0)=u(x,0)=f(x) \text{ in } 0\leqslant x\leqslant\tfrac{\pi}{2},$$
$$u_1(x,0)=u(\pi-x,0)=f(\pi-x) \text{ in } \tfrac{\pi}{2}\leqslant x\leqslant\pi,$$

so that

$$(10)\quad u_1(x,0)=f_1(x) \text{ in } 0\leqslant x\leqslant\pi.$$

Therefore, $u_1(x,t)$ is a unique solution of the equation (1), as proved in the previous paper.

Now we define another function $u_2(x, t)$, in the domain (8) by the relation:

$$(11) \quad u_2(x, t) = u_1(\pi - x, t).$$

Obviously, for all x and t in (8):

$$(12) \quad \frac{\partial^2 u_2(x, t)}{\partial x^2} - \frac{\partial u_2(x, t)}{\partial t} = \frac{\partial^2 u_1(\pi - x, t)}{\partial x^2} - \frac{\partial u_1(\pi - x, t)}{\partial t}$$

$$= u_1^2(\pi - x, t)$$

$$= u_2^2(x, t),$$

so that $u_2(x, t)$ satisfies the differential equation (1).

Moreover, we get

$$(13) \quad u_2(0, t) = u_1(\pi, t) = 0,$$

$$u_2(\pi, t) = u_1(0, t) = 0.$$

and

$$(14) \quad u_2(x, 0) = u_1(\pi - x, 0) = f_1(\pi - x) = f_1(x).$$

Thus we find that $u_2(x, t)$ satisfies also the same boundary conditions as $u_1(x, t)$, and consequently it must be identical with $u_1(x, t)$, for the solution $u_1(x, t)$ was shown to be unique.

Therefore we have for all x in $0 \leqslant x \leqslant \pi$, and all $t \geqslant 0$.

$$(15) \quad u_2(x, t) \equiv u_1(x, t)$$

i. e.

$$(16) \quad u_1(\pi - x, t) \equiv u_1(x, t).$$

This can be true only when

$$(17) \quad \frac{\partial u_1(\frac{\pi}{2}, t)}{\partial x} = 0.$$

Thus, the function $u_1(x, t)$ is a solution of our original problem (3) and (4) in the domain (2).

SOME INTEGRAL REPRESENTATIONS OF THE K—FUNCTION.

BY

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1. The K—function which is a solution of the differential equation

$$x\frac{d^2y}{dx^2}=(x-n)y \qquad \text{(1·1)}$$

is connected with the confluent hypergeometric function $W_{n,\frac{1}{2}}(x)$ by the relation*

$$\Gamma(1+n)K_{2n}(x)=W_{n,\frac{1}{2}}(2x) \qquad \text{(1·2)}$$

Using the contour integral † for $W_{n,\frac{1}{2}}(2x)$ we get

$$K_{2n}(x)=-\frac{1}{2\pi i}\frac{e^{-x}}{n}\int_{\infty}^{(0+)}e^{-t}(-t)^{-n}(2x+t)^n dt \qquad \text{(1·3)}$$

where the integrand is rendered one valued by taking $|arg(-t)|\leqslant\pi$. The contour is one which starts at infinity on the real axis encircles the origin once in the positive direction and returns to the starting point and it is so chosen that the point $t=-2x$ lies outside it. Changing t into $-u$ we get

$$K_{2n}(x)=\frac{e^{-x}}{2\pi in}\int_{-\infty}^{(0+)}e^{u}u^{-n}(2x-u)^n du \qquad \text{(1·4)}$$

In this paper various transformations of the contour integrals are obtained. The particular cases of these transformations give very elegant integrals. In view of the importance of the K—function in certain branches of Physics, the following study of the integral representations seems to be justifiable.

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* Bateman—*Pro. Nat. Aca. of Sci.* Vol. 17 (1931) p. 689 (3).

† Whittaker and Watson.—*Modern Analysis.*—Chap. 16.

2. When n is an integer we get by deforming the contour

$$K_{2n}(x)=\frac{e^{-x}}{2\pi i n}\int^{(0+)} e^{u} u^{-n}(2x-u)^{n}\, du$$

$$=\frac{1}{2\pi i n}\int^{(x+)} e^{u}\left[\frac{x-u}{x+u}\right]^{n} du \ldots\ldots\ldots\ldots\ldots (2.1)$$

by transferring the origin to the point x.

Now using the transformation $u=x\frac{t^2-1}{t^2+1}$ which is a combination of two transformations $u=x\frac{z-1}{z+1}$ and $z=t^2$ we get the unit circle round the point x in the u—plane transformed into a unit circle at the origin in the t—plane and for every description of the unit circle in the u—plane t describes its contour twice. Hence

$$K_{2n}(x)=\frac{x}{n\pi i}\int^{(0+)} e^{x\frac{t^2-1}{t^2+1}}\frac{t^{-2n-1}}{(1+t^2)^2}\, dt$$

$$=\frac{1}{2\pi i}\int^{(0+)} e^{x\frac{t^2-1}{t^2+1}} t^{-2n-1}\, dt+\frac{1}{4n\pi i}\left[t^{-2n}\, e^{x\frac{t^2-1}{t^2+1}}\right]_{(0+)}$$

as can be easily seen by integrating the first member by parts. The second member being an analytic function vanishes.

Therefore

$$K_{2n}(x)=\frac{1}{2\pi i}\int^{(0+)} e^{x\frac{t^2-1}{t^2+1}}\, t^{-2n-1} dt \ldots\ldots\ldots\ldots\ldots (2{\cdot}2)$$

a result obtained by Shabde* starting with Bateman's definition. Substituting $t=e^{-i\theta}$ in and simplifying we get

$$K_{2n}(x)=\frac{2}{\pi}\int_0^{\frac{\pi}{2}}\cos\,(2n\theta-x\tan\theta)\, d\,\theta \ldots\ldots\ldots\ldots (2{\cdot}3)$$

which is Bateman's† definition of $K_{2n}(x)$.

3. Let us take a contour in (1·4) consisting of the real axis from $-\infty$ to -1, the circle $|u|=1$ and back on the real axis from -1 to $-\infty$. Restricting ourselves to $-\pi\leqslant arg\; u\leqslant\pi$ on the first path we have $u=\rho e^{-i\pi}$, on the circle $u=e^{i\theta}$, and on the third path $u=\rho e^{i\pi}$

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* Shabde—*Bull. Cal. Math. Soc.* Vol. 24 (1932), p. 109.

† Bateman—*Trans. Amer. Math. Soc.* Vol. 33, p. 817, (1·1).

Hence

$$K_{2n}(x)=\frac{e^{-x}}{2\pi in}\int_{-\infty}^{(0+)} e^{u}u^{-n}(2x-u)^{n}du$$

$$=\frac{e^{-x}}{2\pi in}\left\{\int_{-\infty}^{-1}+\int_{c}+\int_{-1}^{-\infty}\right\}e^{u}u^{-n}(2x-u)^{n}du$$

$$=-\frac{e^{-x}\sin(n-1)\pi}{n\pi}\int_{1}^{\infty}e^{-t}t^{-n}(2x+t)^{2n}dt+\frac{e^{-x}}{\pi n}\int_{0}^{\pi}r^{n}e^{\cos\theta}$$

$$\cos[\sin\theta-(n-1)\theta-n\alpha]d\theta \qquad \ldots\ldots\ldots\ldots\ldots\ldots(3\cdot1)$$

where $r^{2}=1+4x^{2}-4x\cos\theta$ and $\tan\alpha=\dfrac{\sin\theta}{2x-\cos\theta}$ .... (3·2)

In the particular case when n is an integer this reduces to

$$K_{2n}(x)=\frac{e^{-x}}{\pi n}\int_{0}^{\pi}r^{n}e^{\cos\theta}\cos[\sin\theta-(n-1)\theta-n\alpha]\,d\theta\ldots.(3\cdot3)$$

where r and α are given by (3·2).

The equation (3·1) is analogous to Schlafli's modification of Bessel's integral.

4. Now take the integral (1·4) along the parabola whose focus is at the origin. The equation* to the parabola may be taken as

$$u=\frac{\cos\theta+i\sin\theta}{1+\cos\theta},\text{ whence } du=\frac{-\sin\theta+i(1+\cos\theta)}{(1+\cos\theta)^{2}}\,d\theta.$$

We have

$$\int_{-\infty}^{(0+)}e^{u}u^{-n}(2x-u)^{n}du=i\int_{0}^{\pi}e^{\frac{1}{2}[1-\tan^{2}\theta/2]}\sec^{3}\theta/_2\, r^{n}$$

$$\cos[\tan\theta/_2-(n-\tfrac{1}{2})\theta-n\phi]\,d\theta$$

where $r^{2}=(1+4x\cos^{2}\theta/_2)^{2}-16x\cos^{4}\theta/_2$

and $\tan\phi=\dfrac{\sin\theta}{2x+(2x-1)\cos\theta}$

Putting $\tan\frac{\theta}{2}=t$ we get

$$K_{2n}(x)=\frac{e^{-x}}{\pi n}\int_{0}^{\infty}e^{\frac{1}{2}(1-t^{2})}(1+t^{2})^{\frac{1}{2}}r^{n}\cos[t-(2n-1)$$

$$\text{arc}\tan t-n\phi]dt\ldots\ldots\ldots\ldots(4\cdot1)$$

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* Bourguet,—*Acta Math.* 2 Vol. I. pp. 295-296; 363-367.

where $r^2 = \dfrac{[4x-1+t^2]^2+4t^2}{(1+t^2)^2}$

and $\tan\phi = \dfrac{2t}{4x-1+t^2}$.

In the particular case when $x=0$ we have

$$K_{2n}(0) = \frac{1}{\pi n}\int_0^\infty e^{\frac{1}{2}(1-t^2)}\,(1+t^2)^{\frac{1}{2}} \cos\,[t+\text{arc}\tan t]\,dt.$$

Further when n is an integer $K_{2n}(0)=0$. Hence

$$\int_0^\infty e^{\frac{1}{2}(1-t^2)}(1+t^2)^{\frac{1}{2}} \cos\;(t+\text{arc}\tan t) = 0 \ldots\ldots\ldots\ldots (4{\cdot}2)$$

5. Finally take the contour as the parabola

$$u = -\frac{\cos\theta}{\sin^2\theta}(\cos\theta + i\sin\theta) = -\cot^2\theta - i\cot\theta$$

whose vertex is at the origin, with a small circle enclosing the origin. When $R(n)<0$ the integral (1·4) along this circle vanishes. Now $\dfrac{du}{d\theta} = \dfrac{2\cos\theta + i\sin\theta}{\sin^3\theta}$

Therefore $\displaystyle\int_{-\infty}^{(0+)} e^u u^{-n}(2x-u)^n du$

$$= -2i\int_0^{\frac{\pi}{2}} e^{-\cot^2\theta}(-\cot\theta)^{-n}\sin^{n-2}\theta\; r^n[1+4\cot^2\theta]^{\frac{1}{2}}$$

$$\sin\,[\cot\theta + n\theta - n\phi - \psi]\,d\theta$$

where $r^2 = \cot^2\theta + (2x+\cot^2\theta)^2$, $\tan\phi = \dfrac{\cot\theta}{2x+\cot^2\theta}$

and $\tan\psi = \frac{1}{2}\tan\theta$

Hence

$$K_{2n}(x) = \frac{e^{-x}}{\pi n}\int_0^\infty e^{-t^2}(-t)^{-n}(1+t^2)^{-\frac{n}{2}}\; r^n[1+4t^2]^{\frac{1}{2}}$$

$$\sin\,[\text{arc}\cot 2t - t - n\alpha]\,dt \ldots\;\ldots\ldots\ldots (5{\cdot}1)$$

where $\cot\alpha = \dfrac{t(2x+1+t^2)}{2x}$ and $r^2 = t^2 + (2x+t^2)^2$

In the particular case when $x=0$ and n a negative integer we get $K_{2n}(0)=0$ and therefore,

$$\int_0^\infty e^{-t^2}\,[1+4t^2]^{\frac{1}{2}}\sin\,[\text{arc}\cot 2t - t]\,dt = 0 \ldots\ldots\ldots\ldots (5.2)$$

SOME SELF-RECIPROCAL FUNCTIONS.

BY

BRIJ MOHAN MEHROTRA, M.A., PH.D.

1. In a recent paper* I have proved the theorem:—

If $f(x)$ is self-reciprocal for J_μ transforms, the function

$$g(x)=\frac{1}{x}\int_0^x Q\left(\log\frac{x}{y}\right) f(y)\, dy$$

is self-reciprocal for J_ν transforms, provided that

$$Q(x)=\frac{1}{2\pi i}\int_{k-i\infty}^{k+i\infty} \Gamma(\tfrac{1}{4}+\tfrac{1}{2}\mu+\tfrac{1}{2}s)\,\Gamma(\tfrac{3}{4}+\tfrac{1}{2}\nu-\tfrac{1}{2}s)\,\lambda(s)e^{xs}\, ds \quad (x>0)$$

$$=0 \qquad (x<0),$$

where k is any positive number and $\lambda(s)$ satisfies the equations

$$\lambda(s)=\lambda(1-s). \tag{1·1}$$

If, in this theorem, we put $\mu=\nu$, we get our Corollary I:—
If $f(x)$ is self-reciprocal for J_ν transforms, the function

$$g(x)=\frac{1}{x}\int_0^x Q\left(\log\frac{x}{y}\right) f(y)\, dy$$

is self-reciprocal for J_ν transforms, provided that

$$Q(x)=\frac{1}{2\pi i}\int_{k-i\infty}^{k+i\infty} e^{xs}\,\lambda(s)\, ds \qquad (x>0)$$

$$=0 \qquad (x<0),$$

where $\lambda(s)$ satisfies (1·1).

Here I give a few interesting examples of this theorem.

(i) Let $\lambda(s)=\dfrac{1}{s(s-1)}$,

where $R(s)>1$. Then

$$Q(u)=e^u-1 \qquad (u>0)$$
$$=0 \qquad (u<0).$$

Hence $g(x)=\int_0^x \frac{f(y)}{y}\, dy=\frac{1}{x}\int_0^x f(y)\, dy.$

* B. M. Mehrotra: Theorems connecting different classes of self-reciprocal Functions—*Proc. Edin. Math. Soc.* II 4 (1934), pp. 53—56.

(*ii*) Let

$$\lambda(s)=\frac{(\frac{1}{2}a)^n\,\Gamma(m+n)}{\left[(s-\frac{1}{2})^2+a^2\right]^{\frac{1}{2}(m+n)}\,\Gamma(n+1)}\;{}_2F_1\left\{\frac{m+n}{2},\frac{1-m+n}{2};n+1;\frac{a^2}{(s-\frac{1}{2})^2+a^2}\right\},$$

where $R(s-\frac{1}{2}\pm ia)$ are both positive. Then, by a formula given by Watson,* we get

$$Q(u)=u^{m-1}\,e^{\frac{1}{2}u}\,J_n(au) \qquad (u>0)$$
$$=0 \qquad (u<0).$$

Hence $$g(x)=\frac{1}{\sqrt{x}}\int_0^x\frac{1}{\sqrt{y}}\left(\log\frac{x}{y}\right)^{m-1}J_n\left(a\log\frac{x}{y}\right)f(y)\,dy.$$

As a particular case, putting $m=1$ we get

$$g(x)=\frac{1}{\sqrt{x}}\int_0^x\frac{1}{\sqrt{y}}\,J_n\left(a\log\frac{x}{y}\right)f(y)\,dy.$$

Other examples could be added by making use of similar formulæ given by Watson.

2. For the particular cases $\mu=\mp\frac{1}{2}$, $\nu=\pm\frac{1}{2}$, the above theorem takes the following simpler forms:—

Corollary II. If $f(x)$ is self-reciprocal for cosine transforms the function

$$g(x)=\frac{1}{x}\int_0^x Q\left(\log\frac{x}{y}\right)f(y)\,dy$$

is self-reciprocal for sine-transforms, provided that

$$Q(x)=\frac{1}{2i}\int_{k-i\infty}^{k+i\infty}\frac{1}{\sin\frac{s\pi}{2}}\,\lambda(s)\,e^{xs}\,ds \qquad (x>0)$$
$$=0 \qquad (x<0),$$

where $\lambda(s)$ satisfies (1.1)

Corollary III. If $f(x)$ is self-reciprocal for sine-transforms, the function

$$g(x)=\frac{1}{x}\int_0^x Q\left(\log\frac{x}{y}\right)f(y)\,dy$$

is self-reciprocal for cosine-transforms provided that

$$Q(x)=\frac{1}{2i}\int_{k-i\infty}^{k+i\infty}\frac{1}{\cos\frac{s\pi}{2}}\,\lambda(s)\,e^{xs}\,ds \qquad (x>0)$$
$$=0 \qquad (x<0),$$

where $\lambda(s)$ satisfies (1.1).

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*G. N. Watson; *Theory of Bessel Functions*, Cambridge 1922, § 13·2 (3).

A SIGNIFICANT INTEGRAL INVARIANT IN THE THEORY OF RECTILINEAR CONGRUENCES.

BY

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1. If we take a closed curve C and through points of it draw lines, then the surface generated by these lines is a ruled surface.

Let the equations of the ruled surface be $\xi=x+uX$, $\eta=y+uY$, $\zeta=z+uZ$ where (x, y, z) are the co-ordinates of a point O of the curve C and are functions of v where v is the arc of C measured from some fixed point on it up to 0, u is the distance along the generator, measured from 0, and X, Y, Z are the direction cosines of the generator.

The equation of the orthogonal trajectories of the generators is

$$u=\text{const.}-\int\left(X\frac{dx}{dv}+Y\frac{dy}{dv}+Z\frac{dz}{dv}\right)dv$$

or

$$u=\text{const.}-\int(Xdx+Ydy+Zdz).*$$

The orthogonal trajectories are closed curves if

$$\int_C(Xdx+Ydy+Zdz)=0$$

where the line integral is taken over the boundary of the closed curve C.

Hence the distance between the two points where an orthogonal trajectory cuts the generator through 0, which we shall call for convenience, 'the pitch (p) of the pencil' is given by

$$p=\int_C(Xdx+Ydy+Zdz).$$

This distance is the same for any orthogonal trajectory.

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* This integral invariant has been considered by E. Cartan. *See* 'Le Principe de Dualite et certaines integrales multiples de l' espace, tangential et de l' espace regle; *Bull. Soc. Math. de France*, t. 24 (1896), pp, 140-177, but the periodical was not accessible.

The object of this paper is to obtain various properties of this integral invariant. The results arrived at have been obtained under the help and guidance of Prof. C. H. Rowe of Trinity College, Dublin to whom my sincere thanks are due.

2. Consider a thin pencil formed by rays adjacent to a ray 'l' of the congruence given by $\xi=x+tX$, $\eta=y+tY$, $\zeta=z+tZ$, where x, y, z; and X, Y, Z are functions of two parameters u and v, we have

$$p=\int_C (Xdx+Ydy+Zdz)$$

where C is the closed curve on the surface of reference which forms the boundary of the area dS on it cut off by the pencil.

$$=\int_C \Sigma Xx_1\, du+\Sigma Xx_2\, dv$$

where the subscripts 1 and 2 denote differentiation with regard to u and v respectively.

$$=\int\int\left\{\frac{\partial}{\partial u}(\Sigma Xx_2)-\frac{\partial}{\partial v}(\Sigma Xx_1)\right\} du\, dv$$

by Green's formula*, where the double integral is extended over the area dS bounded by C.

$$=\int\int\left\{(\Sigma X_1x_2+\Sigma Xx_{12})-(\Sigma X_2x_1+\Sigma Xx_{12})\right\} du\, dv$$

$$=\int\int(\Sigma X_1x_2-\Sigma X_2x_1)du\, dv=\int\int(\Sigma X_1x_2-\Sigma X_2x_1)\cdot\frac{d\sigma}{\sqrt{EG-F^2}}$$

where $d\sigma$ is the area of the spherical representation of dS;

$$E=\Sigma X_1^2,\ F=\Sigma X_1X_2,\ G=\Sigma X_2^2.$$

Denoting the limiting value of p when $d\sigma$ shrinks to a point by $\frac{dp}{d\sigma}$ we have

$$\frac{dp}{d\sigma}=\frac{\Sigma X_1x_2-\Sigma X_2x_1}{\sqrt{EG-F^2}}=\frac{f-f'}{\sqrt{EG-F^2}},\ \text{where } f\equiv\Sigma X_1x_2,\ f'\equiv\Sigma X_2x_1.$$

$$=\sqrt{(t_1-t_2)^2-(\rho_1-\rho_2)^2}\ \dagger$$

where t_1, t_2 are the distances of the two limit points of the line 'l' from 0 and ρ_1, ρ_2 those of the two focal points of 'l'.

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* Goursat, *Mathematical Analysis*, Vol. II, p. 263.
† Eisenhart, *Differential Geometry*, p. 399.

$$=(\rho_1-\rho_2)\cdot\frac{\sqrt{(t_1-t_2)^2-(\rho_1-\rho_2)^2}}{\rho_1-\rho_2}$$

$$\therefore\ \frac{dp}{d\sigma}=2\rho\ .\ \cot\theta$$

where 2ρ *is the distance between the focal points, and* θ *is the angle between the focal planes.*

Hence the value of $\frac{dp}{d\sigma}$ *depends only upon the line 'l' and is the same for all the pencils of the congruence. containing 'l'. Its vanishing is the condition that the congruence be normal.*

Also it can be easily verified that

$$\frac{f-f'}{\sqrt{EG-F^2}}=\frac{E\delta''+G\delta-F\ .\ 2\delta'}{EG-F^2},$$ where δ, δ', δ'' are the co-efficients of Sannia's Second quadratic differential form;*

=Mean parameter† of the Congruence by definition.

Hence the value of $\frac{dp}{d\sigma}$ *at* l *is equal to the mean parameter of the congruence.*

N.B.—This result also follows from Bianchi's formula $h=\sqrt{d^2-\lambda^2}$‡ where d and λ are distances between limit points and focal points respectively and h is the mean parameter of the congruence.

Cor. 1. *When the ruled surface is deformed, the generators remaining straight, the pitch of the pencil remains unaltered.*

Cor. 2. From the relation

$$p=\int_C \Sigma X x_1\,du+\Sigma X x_2\,dv,$$
$$=\iint_{dS}\left(\Sigma X_1 x_2-\Sigma X_2 x_1\right)du\,dv,$$

it follows that *the pitch of the pencil is always zero, if* $\Sigma X_1x_2-\Sigma X_2x_1=0$, *i.e., if the congruence is normal. This condition is also necessary* because if the congruence is not normal, *i.e.* if

* Bianchi, *Lezioni di geometria differenziale,* Vol. I, p. 495.

† Sannia, *Annali di Matematica* 1908, p. 152; or *Mathematische Annalen* 1910, p. 411.

‡ Bianchi, 'Sulle Congruenze rettilinee W a parametro medio Constante', *Annali di Matematica,* 1913, p. 263.

$\Sigma X_1 x_2 - \Sigma X_2 x_1$ is not identically zero in the area dS, we can find a region so small that it will be of constant sign inside this region, it being assumed that $\Sigma X_1 x_2 - \Sigma X_2 x_1$ is a continuous function. But in this case the line integral taken along C will not be zero, *i.e.*, the pitch will not be zero.

3. Let the rays of a congruence be the tangents to a family of ∞' curves on a sheet of the focal surface and let these curves be taken as the parametric curves v=const. and their orthogonal trajectories as the parametric curves u=const.

p=pitch of a pencil at a line l of the congruence

$$=\int\int(\Sigma X_1 x_2 - \Sigma X_2 x_1)\, du\, dv = \int\int(\Sigma X_1 x_2 - \Sigma X_2 x_1) . \frac{dS}{\sqrt{EG-F^2}}.$$

Denoting the limiting value of p when dS shrinks to a point by $\frac{dp}{dS}$ we have

$$\frac{dp}{dS} = \frac{1}{\sqrt{EG-F^2}} . \ (\Sigma X_1 x_2 - \Sigma X_2 x_1).$$

Now in this case the direction cosines X, Y, Z are given by $X=x'$, $Y=y'$, $Z=z'$ where dashes denote differentiation with regard to s, the arc of the curve v=const. Since x, y, z are functions of u only,

$$x_2 = y_2 = z_2 = x_{12} = y_{12} = z_{12} = 0 ; \ ds^2 = E du^2,$$

$$\therefore X = x' = x_1 u' = x_1 . \frac{1}{\sqrt{E}}, \ X_2 = -\frac{x_1 E_2}{2 E^{3/2}},$$

$$\Sigma X_1 x_2 = 0 ; \ -\Sigma X_2 x_1 = \Sigma x_1^2 . \frac{E_2}{2 E^{3/2}} = \frac{E_2}{2 E^{1/2}}.$$

$\therefore$ Since $F=0$ we get

$$\frac{dp}{dS} = \frac{1}{\sqrt{EG}} . \left(\frac{E_2}{2E^{1/2}}\right)$$

$$\therefore \frac{dp}{dS} = \frac{1}{\rho_{gu}},$$ *the geodesic curvature of the curve* v = *const.*

If the family of curves on a sheet of the focal surface be a family of geodesics, then $\frac{1}{\rho_{gu}}=0$, $\therefore p=0$, *i.e.*, the Congruence is normal, which agrees with the well known result, *viz.* "A necesssary and sufficient condition that the tangents to a family of curves form a normal congruence is that the curves be geodesics".

4. Consider the congruence formed by the lines that meet two curves C_1 and C_2 which are such that along C_1, u is constant and along C_2, v is constant. Let P, $\{ x(v), y(v), z(v) \}$ be a point on C_1 and Q, $\{ x'(u), y'(u), z'(u) \}$ be a point on C_2. Also let $PQ=r$ and let the direction cosines of PQ be X, Y, Z.

Pitch of a pencil of the congruence at the ray PQ is given by

$$p=\iint\left(\Sigma X_1 x_2-\Sigma X_2 x_1\right)du\, dv$$

$$=\iint \Sigma X_1 x_2\, du\, dv, \text{ since } x_1=y_1=z_1=0 \ldots\ldots\ldots (4.1)$$

Now we have

$$r^2\equiv PQ^2=\Sigma(x'-x)^2$$

$$\therefore\ 2rr_1=\Sigma 2(x'-x)x_1'$$

$$\therefore rr_{12}+r_1r_2=-\Sigma x_1'x_2, \text{ since } x_2'=y_2'=z_2'=0 \ldots\ldots\ldots\ldots (4.2)$$

But

$$x'=x+rX, \ldots\ldots, \ldots\ldots\ldots$$

Differentiating with regard to u and v we get

$$x_1'=rX_1+Xr_1, \ldots\ldots\ldots, \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4.3)$$

and

$$0=x_2+rX_2+Xr_2, \ldots\ldots\ldots, \ldots\ldots\ldots\ldots\ldots\ldots\ldots (4.4)$$

$$\therefore rr_{12}+r_1r_2=-\Sigma x_2\left(rX_1+Xr_1\right) \text{ from (4.2) and (4,3)}$$

$$=-r\,\Sigma x_2X_1-r_1\,\Sigma Xx_2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4.5)$$

Multiplying both sides of the three similar equations of (4.4) by X, Y, Z respectively and adding up we get

$$0=\Sigma Xx_2+r\,\Sigma XX_2+r_2\,\Sigma X^2$$

$$=\Sigma Xx_2+r\,.\,0+r_2$$

$$\therefore\ r_2=-\Sigma X x_2.$$

Putting this value of $\Sigma\ \ x_2$ in (4.5) we get

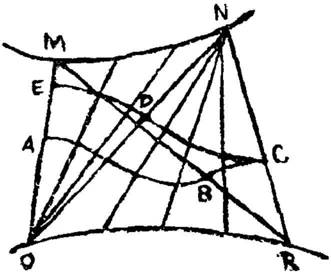
$$rr_{12}+r_1r_2=-r\,\Sigma x_2X_1+r_1r_2$$

$$\therefore\ r_{12}=-\Sigma x_2X_1 \quad \because\ r\neq 0.$$

Hence

$$p=-\iint\frac{\partial^2 r}{\partial u\,\partial v}\,du\,dv \qquad \text{(from 4.1)}$$

This result can also be obtained in another way as follows:—



Let MQ be a line of the congruence joining the points 'u' and 'v' and let NR be a consecutive ray joining the points '$u+du$' and '$v+dv$'.

Consider the pencil of the congruence as shown in the figure. Let $ABCDE$ be an orthogonal trajectory.

Then the pitch at the ray MQ is AE which is required to be found.

Let $MQ \equiv r(u, v)$ and $NR \equiv r(u+du, v+dv)$.
Also let $MB = x$.

Then $RB = r(u, v+dv) - x = RC$.

$\therefore NC \equiv NR - RC = r(u+du, v+dv) - \{ r(u, v+dv) - x \} = ND$.

$\therefore QD \equiv NQ - ND = r(u+du, v) - \Big[r(u+du, v+dv)$

$- \{ r(u, v+dv) - x \} \Big] = QE.$

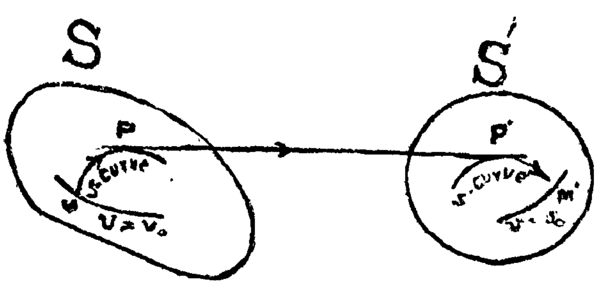
$\therefore$ pitch at $MQ \equiv AE \equiv QE - QA \equiv QE - \{ r(u, v) - x \}$
$= QE + x - r(u, v)$
$= - \{ r(u+du, v+dv) - r(u+du, v) - r(u, v+dv) + r(u, v) \}$

$\therefore$ in the limit when $r(u+du, v+dv)$ coincides with $r(u, v)$ we get

$$\text{pitch} = -\iint \frac{\partial^2 r}{\partial u \, \partial v} \, du \, dv.$$

GENERALISATION FOR ANY CONGRUENCE.

Let the congruence be referred to its developables, *i.e.*, u=const. and v=const. give the developables.



Then if (u, v) are taken as co-ordinates of the point P on the sheet S of the focal surface where the ray (u, v) touches S, the u—curves on S are the curves that the rays touch, and the v—curves are their conjugates. On the second sheet S' a similar statement holds, u and v being interchanged.

Let $u=u_0$ be a fixed v—curve on S so that u alone increases from M to P, and let $v=v_0$ be a fixed u—curve on S' so that v alone increases from P' to M'; and let the curves that the ray PP' touches meet these fixed curves in M and M'.

Also let $\theta\equiv \text{arc } MP+PP'+\text{arc } P'M'$.

On S let $ds^2=Edu^2+2Fdu\,dv+Gdv^2$,

on S' let $ds'^2=E'du^2+2F'du\,dv+G'dv^2$, so that

$$\theta=\int_{u_0}^{u}\sqrt{E}\,du+r+\int_{v}^{v_0}\sqrt{G'}\,dv, \qquad \text{where } r=PP'.$$

Let (x, y, z) and (x', y', z') be the co-ordinates of P and P' and let X, Y, Z be the direction consines of PP'.

Now pitch of any pencil of the congruence at the ray PP' is given by

$$p=\iint\left(\Sigma X_1x_2'-\Sigma X_2x_1'\right)du\,dv$$

$$=-\iint\left\{\frac{\partial}{\partial v}(\Sigma Xx_1')-\frac{\partial}{\partial u}(\Sigma xX_2')\right\}du\,dv\,\ldots\ldots\ldots(4.6)$$

But

$$r^2=\Sigma(x-x')^2$$

$$\therefore\ 2rr_1=2\Sigma(x-x')\,(x_1-x_1')$$

$$\therefore\ r_1=\Sigma\,\frac{x-x'}{r}(x_1-x_1')=\Sigma-X(x_1-x_1')$$

$$=-\,\Sigma Xx_1+\Sigma Xx_1'$$

$$\therefore\ r_{12}=-\frac{\partial}{\partial v}(\Sigma Xx_1)+\frac{\partial}{\partial v}(\Sigma Xx_1')$$

Also

$$\frac{x_1}{\sqrt{E}}=X=\frac{x_2'}{\sqrt{G'}},$$

$$\therefore\ \Sigma Xx_1=\frac{\Sigma x_1^2}{\sqrt{E}}=\sqrt{E}\qquad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4.7)$$

and

$$\Sigma Xx_2'=\frac{\Sigma x_2'^2}{\sqrt{G'}}=\sqrt{G'}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4.8)$$

$$\therefore\ r_{12} = -\frac{\partial}{\partial v}(\sqrt{E}) + \frac{\partial}{\partial v}(\Sigma X x_1') \text{ from (4.7)}$$

or

$$\frac{\partial}{\partial v}(\Sigma X x_1') = r_{12} + \frac{\partial}{\partial v}(\sqrt{E}) \text{ from (4.9)}$$

$\therefore$ (4.6) becomes

$$p = -\iint \left\{ r_{12} + \frac{\partial}{\partial v}(\sqrt{E}) - \frac{\partial}{\partial u}(\sqrt{G'}) \right\} du\, dv \text{ from (4.8) and (4.9)}$$

$$= -\iint \frac{\partial^2 \theta}{\partial u\, \partial v}\, du\, dv.$$

Hence the pitch of any pencil of the congruence

$$= -\iint \frac{\partial^2 \theta}{\partial u \partial v}\, du\, dv, \text{ where } \theta = \text{arc } MP + PP' + \text{arc } P'M'.$$

THE THEORY OF THE PEDAL LINE

By R. VAIDYANATHASWAMY, M.A., D.Sc.

The author has studied in previous papers[1] the correspondences on a circle related to the pedal line theory of the triangle, by the symbolic methods of the invariant theory of binary forms. Langley's[2] pedal line chain shews however that the method must only be a special case of a uniform theory applicable to the n—point inscribed in a circle, for any value of n. This general theory is obtained in this paper, which from the point of view of invariant theory may be considered as a study of certain simple covariant forms of a binary n—ic and the quadratic representing the circular points.

§ 1. *The Pedal Angle.*

Consider two n—points $A_1 A_2 \ldots A_n$, $B_1 B_2 \ldots B_n$ inscribed in a fundamental circle centre O. Let OX be a direction of reference and let angle $XOA_i = \alpha_i$, $XOB_i = \beta_i$. We define the *pedal angle* between the two n—points to be

$$\tfrac{1}{2}(\beta_1+\beta_2+\ldots+\beta_n-\alpha_1-\alpha_2-\ldots-\alpha_n).$$

The points to be noted in regard to this definition are

(1) the pedal angle depends only on the two n—points and is independent of the direction of reference.

(2) The pedal angle changes sign when the two n—points are interchanged.

(3) While vectorial angles are known modulo 2π, the pedal angle is defined only modulo π.

Two inscribed n—points are said to be *pedo-parallel* or *pedo-perpendicular* according as their pedal angle is equal to 0 or $\frac{\pi}{2}$ (mod π).

1. Vaidyanathaswamy, The General (m, n) correspondence (*Proc. Camb. Phil. Soc.* Vol. 23, pp. 109-119), The (2, 1) Correspondence (Ibid, pp. 233-61), The Pedal Correspondence (Ibid, pp. 631-48).

2. Langley's chain is as follows: If x, 1, 2, 3, 4, 5....are on a circle, the feet of the perpendiculars from x on its pedal lines *w. r. t.* the triangles 234, 134, 124, 123 lie on a line which is defined to be the pedal line of x *w. r. t.* the four-point 1234; the feet of the perpendiculars from x on its pedal lines *w. r. t.* the five four-points comprised in 12345, lie on a line which is defined to be the pedal line of x *w. r. t.* 12345; and so on, *ad infinitum.*

An n—point $A_1 A_2 .. A_n$ possesses a unique pedo-parallel regular n—point, whose vertices have the vectorial angles $\frac{1}{n}(\alpha_1+\alpha_2+..+\alpha_n)+\frac{2m\pi}{n}$ $(m=0, 1,.., n-1)$.

If i, j denote the circular points on the circle, the angles which the directions of i, j make with OX are respectively $\tan^{-1}\sqrt{-1}$, $-\tan^{-1}\sqrt{-1}$. From this it may be easily shewn that (1) the only n—points pedo-parallel to $A_1 A_2 .. A_{n-1}\ i$ are those of the form $B_1 B_2 .. B_{n-1}\ i$, where the B's are arbitrary points on the circle, (2) the n—points of the form $A_1 A_2 .. A_{n-2}\ i\ j$ are pedo-parallel to *every* n—point.

§ 2. *Pedo-parallel systems.*

The totality of n—points pedo-parallel to a given n—point (or making a known pedal angle with a given n—point) constitute a *pedo-parallel system*. It is clear that the vectorial angles of the n—points of a pedo-parallel system satisfy a relation of the form $\alpha_1+\alpha_2+..+\alpha_n=k$. This shews that the pedo-parallel system is a linear system whose n-ple points constitute a regular n—point with the vectorial angles given by $n\theta=k$ or $\theta=\frac{k}{n}+\frac{2m\pi}{n}$ $(m=0, 1,... n-1)$. The sum of these n values of θ is equal to $k+\pi\ (n-1)$ (mod 2π). Thus the n—ple points (which constitute the n—point apolar to every n—point of the pedo-parallel system) form the regular n—point contained in the pedo-parallel or pedo-perpendicular system, according as n is odd or even. We have therefore the

THEOREM :—*The unique regular n-point apolar to a given n—point is pedo-parallel or pedo-perpendicular to it according as n is odd or even.*

As an illustration take $n=2$; the pedo-perpendicular regular 2—point of $(\alpha_1\ \alpha_2)$ is the pair of extremities of the diameter bisecting $\alpha_1\ \alpha_2$ namely $(\frac{1}{2}(\alpha_1+\alpha_2),\ \frac{1}{2}(\alpha_1+\alpha_2)+\pi)$. The apolarity of these two point-pairs is an evident property of the circle.

We may note that every pedo-parallel system contains all the n—points which comprise $(i\ j)$.

§ 3. *The pedal direction.*

We define *the pedal direction* of the point x with respect to the n—point $\alpha_1\ \alpha_2 .. \alpha_n$ to be the direction of the chord whose extremities constitute the quadratic polar of x with respect to the pedo-parallel regular n—point of $\alpha_1\ \alpha_2 .. \alpha_n$.

The vertices θ of the pedo-parallel regular n—point are given by

$$n\,\theta=\alpha_1+\alpha_2+\ldots+\alpha_n+\pi\ (n-1)\ (\text{mod } 2\pi)$$

Hence the quadratic polar $(y_1\ y_2)$ of x are given by

$$2\,y+(n-2)\,x=\alpha_1+\alpha_2+\ldots+\pi\ (n-1).$$

Hence the direction $\frac{1}{2}\,(y_1+y_2+\pi)$ of the chord $y_1\ y_2$ is equal to

$$\tfrac{1}{2}\,(\alpha_1+\ldots+\alpha_n+\pi(n-1)-(n-2)x).$$

From this expression for the pedal direction of x, it folllows that

(1) If any of the angles α increases by θ, the pedal direction increases by $\frac{\theta}{2}$.

(2) If x increases by θ, its pedal direction increases by $-(n-2)\,\frac{\theta}{2}$.

(3) The pedal directions of x *w. r. t.* the $(n+1)$—point $(x\ \alpha_1\ \alpha_2\ldots\alpha_n)$ and *w. r. t.* the n—point $(\alpha_1\ \alpha_2\ldots\alpha_n)$ are mutually perpendicular.

(4) The angle between the pedal directions of x, *w. r. t* two n—points is independent of x and equal to the pedal angle between them.

(5) The vertices of a regular $(n-2)$—point have the same pedal direction *w. r. t.* any n—point.

THEOREM. *If through each point x we draw the chord xy parallel to the pedal direction of x, w. r. t. the n—point $(\alpha_1\ \alpha_2\ldots\alpha_n)$, the correspondence $(x\rightarrow y)$ is the $(n-1, 1)$ polar correspondence of the pedo-perpendicular regular n—point of $(\alpha_1\ \alpha_2\ldots\alpha_n)$.*

For, by hypothesis,

$$\tfrac{1}{2}(x+y+\pi)=\tfrac{1}{2}(\Sigma\alpha+\pi(n-1)-(n-2)x)\ (\text{mod } \pi)$$
$$\therefore\ x+y=\Sigma\alpha+\pi n-(n-2)x\ (\text{mod } 2\pi)$$
$$\text{or } (n-1)\ x+y=\Sigma\alpha+\pi n\ (\text{mod } 2\pi)$$

Hence the correspondence $(x\rightarrow y)$ is the $(n-1, 1)$ polar correspondence of the regular n—point whose vertices θ are given by

$$n\theta=\Sigma\alpha+\pi n\ (\text{mod } 2\pi).$$

The sum of the values of θ determined by this equation is equal to $\Sigma\alpha+\pi n+\pi(n-1)=\Sigma\alpha+\pi$ (mod 2π). Hence this n—point is the regular pedo-perpendicular n—point of $(\alpha_1\alpha_2\ldots\alpha_n)$.

If $n=1$, the pedal direction of x, *w. r. t.* α is perpendicular to αx, and the correspondence $(x\rightarrow y)$ carries x to the diametrically opposite point of α, which is the pedo-perpendicular *1*—point of α. If $n=2$, the pedal direction of x is the direction of $\alpha_1\alpha_2$, and the correspondence $(x\rightarrow y)$ is the involution whose double points are the extremities of the diameter bisecting $(\alpha_1\ \alpha_2)$.

§ 4. *Algebraic Formulae.*

We represent henceforth the points on the circle by binary symbols, the circular points being represented by i, j. Let the vertices of the n—point be the roots of the equation $f(x)=0$. By means of the identity $(ij)x=(ix)j-(jx)\,i$, the equation $f(x)=0$ can be written in the form

$$(ix)^n f(j)-\binom{n}{1}(ix)^{n-1}(jx)f_i(j)+\ldots+(-1)^n (jx)^n f(i)=0.$$

Now a regular n—point is given by an n—ic of the form $A(ix)^n+B(jx)^n$. Hence the apolar regular n—point of $f(x)$ is given by

$$(ix)^n f(j)-(jx)^n f(i).$$

Hence by the theorem in § 2, the pedo-parallel regular n—point of $f(x)$ is $(ix)^n f(j)+(-1)^n (jx)^n f(i)$, and the pedo-perpendicular regular n—point is $(ix)^n f(j)+(-1)^{n-1} (jx)^n f(i)$.

THEOREM. *The pedal angle between the regular n—points $A\,(ix)^n+B\,(jx)^n$, $A'\,(ix)^n+B'(jx)^n$, is given by*

$$e^{2\sqrt{-1}\psi}=\frac{AB'}{A'B}.$$

For the pedal angle is evidently independent of the parametric representation chosen. Let us choose the tangent of half the vectorial angle of any point on the circle as its parameter; the parameters of i, j are then $\sqrt{-1}, -\sqrt{-1}$. Hence the vectorial angles of the two given regular n—points are then determined by

$$\left(\frac{\sqrt{-1}-\tan\frac{\theta}{2}}{-\sqrt{-1}-\tan\frac{\theta}{2}}\right)^n=-\frac{B}{A} \quad \text{or} \quad e^{n\sqrt{-1}\theta}=(-1)^{n-1}\frac{B}{A}$$

$$e^{n\sqrt{-1}\phi}=(-1)^{n-1}\frac{B'}{A'}$$

Hence the required pedal angle $\psi=\frac{n\phi-n\theta}{2}$ is given by

$$e^{2\sqrt{-1}\psi}=\frac{AB'}{A'B}.$$

COR. The pedal angle ψ between two n—points $f(x)$, $\phi(x)$, being equal to the pedal angle between their pedo-parallel regular n—points, is given by

$$e^{2\sqrt{-1}\psi}=\frac{f(j)\,\phi(i)}{f(i)\,\phi(j)}.$$

Hence the conditions for pedo-parallelism and pedo-perpendicularity of $f(x)$, $\phi(x)$ are respectively

$$f(j)\ \phi(i)-f(i)\ \phi(j)=0,$$
$$f(j)\ \phi(i)+f(i)\ \phi(j)=0.$$

If $n=2$, we note that these are the conditions for the parallelism and perpendicularity of the chords corresponding to f and ϕ.

§ 5. *The auto-correspondence of an n—point.*

An n—point $f(t)\equiv f_n(t)=0$ determines uniquely a $(n-2, 2)$ correspondence $A(x, y)=0$ satisfying the conditions

(1) the n fixed points of the correspondence are given by $f_n(t)=0$,

(2) when $y=i$ or j, the $n—2$ corresponding values of x coalesce at j or i respectively.

These amount to $3n—4$ independent conditions. The number of effective constants in the $(n—2, 2)$ correspondence is $3n—4$ both when $n=3$ and when $n>3$. Hence the correspondence is determined uniquely by these conditions. We call $A(x, y)$ the auto-form and the correspondence $A(x, y)=0$, the auto-correspondence of the n—point $f(t)$. If $(y_1\ y_2)$ correspond to x in $A\ (x, y)=0$, we call $y_1\ y_2$ the auto-chord of x *w.r.t.* the n—point.

To obtain the expression for $A(x, y)$, we note that the equation $f(x)=0$, which determines the n—point, can be written in the form

$$(ix)^n f(j)-\binom{n}{1}(ix)^{n-1}(jx)\ f_i(j)+\ldots\ldots+(-1)^n(jx)^n f(i)=0$$

or $(ix)^n f(j)+(-1)^n(jx)^n f(i)+f_{n-2}(x)\ (ix)\ (jx)=0$, where

$$f_{n-2}(x)=-\binom{n}{1}\ f_i(j)\ (ix)^{n-2}+\binom{n}{2}(ix)^{n-3}(jx)f_{ii}(j)-\ldots\ldots$$

We call the $(n—2)$—point determined by $f_{n-2}\ (x)$ *the residual $(n—2)$—point of $f(x)$.*

Consider now the correspondence

$$A(x, y)\equiv(ix)^{n-2}(iy)^2 f(j)+(-1)^n(jx)^{n-2}(jy)^2 f(i)+ f_{n-2}(x)\ (iy)\ (jy)=0.$$

It evidently satisfies the conditions (1) and (2) and is hence the auto-correspondence of $f(x)$.

From this expression for the auto-form, it is clear that if x is a root of $f_{n-2}(x)=0$, its auto-chord passes through the centre of the circle. Thus

The auto-chords w. r. t. $f_n(x)$ of any vertex of its residual $(n—2)$—point pass through the centre.

From the form of $A(x, y)$ the auto-chord of x is parallel to the chord $(ix)^{n-2}(iy)^2 f(j)+(-1)^n (jx)^{n-2}(jy)^2 f(i)$; but since this is the quadratic polar of x *w. r. t.* $(ix)^n f(j)+(-1)^n (jx)^n f(i)$ —the pedo-parallel regular n—point of $f(x)$—, it defines the pedal direction of x *w. r. t.* the n—point f.

Hence

The auto-chord of any point is parallel to its pedal direction.

The concurrency system of the auto-form is the system of sets of $(n-2)$ points whose auto-chords are concurrent. It is clearly the net determined by $(ix)^{n-2}$, $(jx)^{n-2}$ and $f_{n-2}(x)$. Hence

The concurrency system of the auto-correspondence is the net determined by the residual $(n-2)$—point and the pencil of regular $(n-2)$—points.

§. 6 *The orthocentre of the n—point.*

In the case of the inscribed triangle it is known that the orthocentre is the intersection of the auto-chords of i, j. We define the orthocentre of the n—point $f(x)$ to be the intersection of the auto-chords *w. r. t.* f of i, j. These auto-chords are given by the quadratics

$$(ij)^{n-2}(jy)^2 f(i)+f_{n-2}(i)\,(iy)\,(jy),$$
$$(ij)^{n-2}(iy)^2 f(j)+f_{n-2}(j)\,(iy)\,(jy).$$

The orthocentre is therefore represented by the quadratic apolar to these two, which is readily seen to be

$$f(j)\,f_j(i)\,(iy)^2+f(i)\,f_i(j)\,(jy)^2-\frac{2}{n}f(i)\,f(j)\,(iy)\,(jy).$$

The centroid of the n—point is easily shewn to correspond to the quadratic

$$f(j)\,f_j(i)\,(iy)^2+f(i)f_i(j)\,(jy)^2-2\,f(i)\,f(j)\,(iy)\,(jy).$$

It follows that *if we join the centre O to the centroid G and produce OG to $(n-1)$ times its length we reach the orthocentre.*

§ 7. *The pedal correspondence.*

The pedal correspondence *w. r. t.* a triangle $f_3(x)=(\alpha x)(\beta x)(\gamma x)$ is known to have the form

$$(ij)^2 f_3(x)\,(iy)\,(jy)-2\,(xy)\,\{f(j)\,(ix)^2(iy)$$
$$+f(i)\,(jx)^2(jy)\}=0.$$

The fixed points of this correspondence are $\alpha, \beta, \gamma, i, j$. The correspondence

$$f(j)\ (ix)^2(iy)+f(i)\ (jx)^2(jy)=0,$$

is the (2, 1) polar correspondence of the pedo-perpendicular regular 3—point of $f_3(x)$; this by §5 is parallel to the pedal direction of x *w. r. t.* f_3. Thus the second term in the pedal form makes correspond to each x, the extremities of the chord through x parallel to the pedal direction of x *w. r. t.* f_3. Since the first term is a multiple of $(iy)(jy)$, the pedal line of x is always parallel to the pedal direction of x.

In the case of an n—point f_n (*i.e.*, given by $f_n(t)=0$), we define the pedal correspondence as an $(n, 2)$ correspondence between x and an extremity y of 'the pedal line of x *w. r. t.* the n—point f_n' which satisfies the following two conditions

(1) its $n+2$ fixed points are i, j and f_n,

(2) the pedal line of x is to be always parallel to the pedal direction of x *w. r. t.* the n—point.

These two conditions determine the form of the pedal correspondence to be

$$F\equiv(ij)^{n-1} f_n(x)\ (iy)\ (jy)-k(xy)\left\{f_n(j)\ (ix)^{n-1}(iy)+ (-1)^{n-1} f(i)\ (jx)^{n-1}\ (jy)\right\}=0,$$

where the parameter k is still undetermined, To determine k we shall adopt a method suggested by Langley's chain. Consider the $(n+1, 2)$ correspondence

$$F'\equiv(ij)^n f_n(x)\ (\alpha x)\ (iy)\ (jy)-l(xy)\left\{f_n(j)\ \alpha j)\ (ix)^n (iy)+(-1)^n f(i)\ (\alpha i)\ (jx)^n(jy)\right\}=0,$$

which is formed *w. r. t.* the $(n+1)$—point $f_n(x)(\alpha x)$ in the same way as F, but with a different parameter l. Let x correspond to $y_1\ y_2$ in F and to $y_3\ y_4$ in F' and let $y_1 y_2$, $y_3 y_4$ interest in z. We shall shew that *the condition $l=2k$ suffices to ensure that xz is perpendicular to $y_1 y_2$ for all positions of x.*

For the other extremity of the chord through x parallel to $y_1\ y_2$ is determined by the linear form

$$f_n(j)\ (ix)^{n-1}(iy)+(-1)^{n-1} f(i)\ (jx)^{n-1}(jy) \ldots\ldots (1)$$

The other extremity of the chord (xz) is given by the linear form

$$\frac{F'-(ij)\ (\alpha x)\ F}{(xy)}\equiv k(ij)\ (\alpha x)\left[f(j)\ (ix)^{n-1}\ (iy)+(-1)^{n-1} f(i)\ (jx)^{n-1}\ (jy\right]-l\left[f(j)\ (\alpha j)\ (ix)^n(iy)\right..$$

$$+(-1)^n f(i)\ (\alpha i)\ (jx)^n (jy)]$$
$$\equiv f(j)\ (ix)^{n-1}(iy)\left\{k(ij)\ (\alpha x)-l(\alpha j)\ (ix)\right\}$$
$$+(-1)^{n-1}f(i)\ (jx)^{n-1}(jy)\ \left\{k(ij)\ (\alpha x)\right.$$
$$\left.+l(\alpha i)\ (jx)\right\} \qquad \ldots.(2)$$

The condition that (1) and (2) may represent diametrically opposite points, is

$$k(ij)\ (\alpha x)+l(\alpha i)\ (jx)+k(ij)\ (\alpha x)-l(\alpha j)\ (ix)\equiv(2\,k-l)\ (ij)\ (\alpha x)=0, \text{ or } l=2k.$$

Hence from the form of the pedal correspondence for $n=3$, we get by Langley's chain definition of the pedal line, the pedal correspondence of the n—point $f(x)$ in the form

$$\pi(x, y)\equiv(ij)^{n-1}\, f(x)\ (iy)\ (jy)-2^{n-2}(xy)$$
$$\times\left\{f(j)\ (ix)^{n-1}(iy)+(-1)^{n-1}f(i)\ (jx)^{n-1}\ (jy)\right\}=0.$$

We call $\pi\ (x, y)$ *the pedal form* of $f(x)$, and may denote by $\pi_\kappa(x, y)$ the form F above, so that $\pi_\kappa(x, y)$ reduces to $\pi\ (x, y)$ for $k=2^{n-2}$

§ 8. *The Pedal form and the Auto-form.*

The form $\pi_\kappa(x, y)\equiv(ij)^{n-1}\ f(x)\ (iy)\ (jy)-k(xy)$ $\times\left\{f(j)(ix)^{n-1}(iy)+(-1)^{n-1}f(i)(jx)^{n-1}(jy)\right\}$ becomes divisible by $(ix)\ (jx)$ when $k=1$. Hence

$$\pi_1(x, y)\equiv(ix)\ (jx)\ A(x, y),$$

where $A\ (x, y)$ is a certain $(n—2, 2)$ form. Since the fixed points of π_1, are the $n+2$ points i, j, f, it follows that the fixed points of A are the n points f. Also, the n values of x which correspond to $y=i$ in π_κ are i, and j repeated $(n—1)$ times. Hence the $(n—2)$ values of x which correspond to $y=i$ in $A(x, y)$, all coalesce at j, which shews that $A\ (x, y)$ is a numerical multiple of the auto-form of $f(x)$.

From this relation, we immediately have the

THEOREM: *The pedal line of x w. r.t. an n—point lies between x and its auto-chord w. r. t. the same n—point; and the distance of x from its pedal line is $\frac{1}{2^{n-2}}$ of its distance from the auto-chord.*

§ 9. *The parabola with focus at a point x on the circle.*

For later work it is necessary to have certain binary formula relating to a parabola whose focus is x. Any line in the plane cuts out on the circle two points whose binary parameters are the roots of a quadratic which can be expressed in the form

$$L(iy)(jy)+M(xy)(iy)+N(xy)(jy).$$

We take L, M, N as the homogeneous co-ordinates of the line; the triangle of reference for this co-ordinate system has evidently x, j, i for its vertices. The equation to a parabola with focus at x is therefore of the form

$$\frac{p}{L}+\frac{q}{M}+\frac{r}{N}=0.$$

We may refer to this as the parabola (p, q, r) with focus at x. The point of contact of the parabola with the line at infinity has co-ordinates (o, r, q) and therefore the axis of the parabola has co-ordinates $(o, q, -r)$.

Hence the other extremity y' of the chord of the circle which is the axis of this parabola corresponds to the linear form $q(iy)-r(jy)$. The point y'' determined by the linear form $q(iy)-re^{2\sqrt{-1}\theta}(jy)$ has evidently the property that $y'y''$ subtends an angle θ (mod π) at any point on the circle. In fact the pedal angle ψ between y' y'' is given by

$$e^{2\sqrt{-1}\psi}=\frac{qr\, e^{2\sqrt{-1}\theta}}{qr}=e^{2\sqrt{-1}\theta} \text{ or } \psi=\theta \pmod{\pi}.$$

If now $\left(L, q, -r\, e^{2\sqrt{-1}\theta}\right)$ be the co-ordinates of the tangent of the parabola which is inclined at an angle θ (mod π) to its axis, we have

$$\frac{p}{L}+1-e^{2\sqrt{-1}\theta}=0 \text{ or } L=\frac{p}{e^{-2\sqrt{-1}\theta}-1}.$$

By a θ—*tangent* of a parabola, we mean the tangent which is inclined at an angle θ (mod π) to its axis. It is a well-known property of the parabola that the locus of intersection of any tangent to it with the line from the focus which makes an angle θ (mod π) with it, is the θ-*tangent*. We have thus proved that *in our binary co-ordinate-system the co-ordinates of the θ-tangent of the parabola (p, q, r) with focus at x, are* $\left\{p, q(e^{-2\sqrt{-1}\theta}-1), r(e^{2\sqrt{-1}\theta}-1)\right\}$. In particular the tangent at the vertex of the parabola has the co-ordinates $(p, -2q, -2r)$.

§ 10. *The generalised pedal line.*

The $(n+2, 2)$ correspondence determined *w. r. t.* an n—point $f(x)$ by the equation

$$p(ij)^{n-1}\ f(x)(iy)(jy) - (xy)\Big\{ q\, f(j)(ix)^{n-1}(iy) + (-1)^{n-1}\, r\, f(i)(jx)^{n-1}(jy) \Big\} = 0,$$

may be called the generalised pedal correspondence of $f(x)$, with the (homogeneous) parameters (p, q, r). We proceed to prove a fundamental property of this correspondence.

Write $f(x)(\alpha x) = f_{n+1}(x)$; the above correspondence can be written in the form

$$\frac{p(ij)^n f_{n+1}(x)(iy)(jy)}{(ij)(\alpha x)} + \frac{q(ix)^n f_{n+1}(j)(xy)(iy)}{(j\alpha)(ix)} + \frac{(-1)^n r\, f_{n+1}(i)(jx)^n (xy)(jy)}{(\alpha i)(jx)} = 0.$$

Since $(ij)(\alpha x) + (j\alpha)(ix) + (\alpha i)(jx) = 0$, identically, we see that the (p, q, r)—pedal line of x *w. r. t.* the n—point $f(x)$, touches the parabola of parameters $\Big\{ p(ij)^n f_{n+1}(x), q(ix)^n f_{n+1}(j), (-1)^n f_{n+1}(i)(jx)^n \Big\}$. From symmetry it follows that this parabola touches the (p, q, r)—pedal line of x *w. r. t.* every n—point comprised in $f_{n+1}(x)$. Also the correspondence between x and the extremities of the chord which is the θ—tangent of this parabola, is

$$p(ij)^n f_{n+1}(x)(iy)(jy) - (xy)\Big\{ q(1-e^{-2\sqrt{-1}\theta}) \times f_{n+1}(j)(ix)^n(iy) + (-1)^n r\,(1-e^{2\sqrt{-1}\theta}) \times f_{n+1}(i)(jx)^n(jy) \Big\} = 0.$$

By our definition this is the $\Big\{ p, q(1-e^{-2\sqrt{-1}\theta}), r(1-e^{2\sqrt{-1}\theta}) \Big\}$—pedal line of x *w. r. t.* the $(n+1)$—point $f_{n+1}(x)$. We have thus reached the

THEOREM : *The intersections of the* $(n+1)$ (p, q, r)*—pedal lines of* x *w. r. t. the n—points comprised in* $f_{n+1}(x)$, *with the respective lines through* x *making the angle* θ *with them, lie on the* $\left\{ p, q(1-e^{-2\sqrt{-1}\theta}), r(1-e^{2\sqrt{-1}\theta}) \right\}$ *—pedal line of* x *w. r. t. the* $(n+1)$*—point* $f_{n+1}(x)$.

§ 11. *The representation of the generalised pedal correspondence by angular parameters.*

The pedal correspondence *w. r. t.* the *1*—point α is by § 7,

$$(\alpha x)(iy)(jy) - \tfrac{1}{2}(xy)\left\{ (\alpha j)(iy) + (\alpha i)(jy) \right\} \qquad (1)$$

This is easily seen to reduce to a multiple of $\left\{ (\alpha y)(jx)(iy) + (jy)(ix) \right\}$, so that the pedal line of x *w. r. t.* α is the line through α perpendicular to αx. We define the θ—pedal correspondence of the *1*—point α to be that which makes correspond to x the extremeties of the chord, which is the θ—tangent of the parabola whose focus is x, and the tangent at the vertex, the pedal line of x *w. r. t.* α. Since (1) is the tangent at the vertex, it follows from § 9 that the parameters of the parabola are $\left((\alpha x), \frac{(\alpha j)}{4}, \frac{(\alpha i)}{4}\right)$, and therefore the co-ordinates of its θ—tangent are $\left\{ (\alpha x), \frac{(\alpha j)}{4}(e^{-2\sqrt{-1}\theta}-1), \frac{(\alpha i)}{4}(e^{2\sqrt{-1}\theta}-1) \right\}$.

Hence the θ—pedal correspondence of the *1*—point α is

$$(\alpha x)(iy)(jy) + (xy)\left\{ \frac{(\alpha j)}{4}(e^{-2\sqrt{-1}\theta}-1)(iy) + \frac{(\alpha i)}{4}(e^{2\sqrt{-1}\theta}-1)(jy) \right\} = 0.$$

In other words, the homogeneous parameters of the θ—pedal correspondence of α are $(4, 1-e^{-2\sqrt{-1}\theta}, 1-e^{2\sqrt{-1}\theta})$. We may now by the theorem of § 10 define successively the $(\theta_1, \theta_2, .., \theta_n)$ —pedal line of x *w. r. t.* the n—point $f(x)$; namely, it is the line which by § 10 contains the intersections of the $(\theta_1, \theta_2, .., \theta_{n-1})$— pedal lines of x *w. r. t.* the $(n—1)$—points comprised in $f(x)$ with the respective lines drawn through x so as to make the angle θ_n with them. Since the homogeneous parameters of the θ_1—pedal correspondence *w. r. t.* a *1*—point have been shewn to be $(4, 1-e^{-2\sqrt{-1}\theta_1}, 1-e^{2\sqrt{-1}\theta_1})$, it follows by successive applications of the theorem of §10, that the homogeneous parameters of the $(\theta_1, \theta_2, .., \theta_n)$—pedal correspondence *w. r. t.* an n—point are

$$\left\{4, \prod_{k=1}^{n} (1-e^{-2\sqrt{-1}\theta_k}), \prod_{k=1}^{n} (1-e^{2\sqrt{-1}\theta_k})\right\}.$$

In particular it follows that the correspondence is unaffected by permuting the angular parameters $\theta_1, \theta_2, .., \theta$

OPERATIONAL METHODS AND THE k—FUNCTION

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I. In a recent paper the author has investigated some properties of the k—function with the help of the operational methods. In this paper these methods are used to get some more properties. This paper illustrates that the operational calculus is an effective short-hand for most of the complicated methods in analysis. Most of the results obtained in this paper are believed to be new.

We will call 'the image' $f(p)$ the operational representation of 'the original' function $h(x)$ if

$$f(p)=p\int_0^\infty e^{-px}\,h(x)\,dx \tag{1·1}$$

provided the integral converges, and will denote it by

$$f(p)\doteq h(x) \text{ or by } h(x)\doteq f(p)$$

both meaning that $h(x)$ transformed operationally gives $f(p)$ and $f(p)$ interpreted backwards gives $h(x)$. For the general discussion of the following rules and theorems the paper of Drs. B. Pol and Niessen[1] may be referred to. We will also use Lerche's theorem[2] that if $f_1(p)\doteq h_1(x)$ and $f_2(p)\doteq h_2(x)$ then $f_1(p)\equiv f_2(p)$ implies $h_1(x)\equiv h_2(x)$.

$$p\,f(p)\doteq\frac{d}{dx}\,h(x), \quad \text{if } h(o)=0, \tag{1·2}$$

$$p\left[-\frac{d}{dp}\right]^n\frac{f(p)}{p}\doteq x^n\,h(x) \quad n>0 \tag{1·3}$$

$$\frac{p}{p+\alpha}\,f(p+\alpha)\doteq e^{-\alpha x}\,h(x) \tag{1·4}$$

$$\int_0^\infty\frac{f(p)}{p}\,dp=\int_0^\infty\frac{h(x)}{x}\,dx. \tag{1·5}$$

If $f_1(p)\doteq h_1(x)$ and $f_2(p)\doteq h_2(x)$, then

$$\frac{1}{p}f_1(p)\,f_2(p)\doteq\int_0^x h_1(\xi)\,h_2(x-\xi)\,d\xi$$

$$\text{or } \int_0^x h_2(\xi)\,h_1(x-\xi)\,d\xi. \tag{1·6}$$

1. Pol & Niessen.—*Phil. Mag.* (7) 13 (1932) 537-77.
2. Lerch.—*Acta. Math.* 27 (1903) 339-51.

This theorem will be called the 'Product Theorem'.

In what follows the following operational representations will be used.

$$\frac{1}{p^n} \doteqdot \frac{x^n}{\Gamma(n+1)} \tag{1·7}$$

$$\frac{p}{1+p^2} \doteqdot \text{Sin } x \tag{1·8}$$

$$\frac{p^2}{1+p^2} \doteqdot \text{Cos } x \tag{1·9}$$

$$\left(1-\frac{L}{p}\right)^n \doteqdot \frac{1}{\Gamma(1+n)} L_n(x) \tag{1·10}$$

where $L_n(x)$ is a Laguerre polynomial of degree n.

$$\frac{2p(1-p)^{n-1}}{(1+p)^{n+1}} \doteqdot K_{2n}(x) \qquad n>0 \tag{1·11}$$

$$\frac{p}{1+p} \doteqdot K_0(x) \tag{1·12}$$

The first four have been obtained by Pol and the last two by the author.

2. *Recurrence Formulae.* By (1·2) and (1·11) we have

$$K'_{2n}(x) \doteqdot \frac{2p^2(1-p)^{n-1}}{(1+p)^{n+1}}. \tag{2·1}$$

Now

$$\frac{2p(1-p)^{n-2}}{(1+p)^{n+1}}\left\{(1+p)-(1-p)\right\} = \frac{2p^2(1-p)^{n-2}}{(1+p)^n}\left\{1+\frac{1-p}{1+p}\right\}.$$

Therefore

$$\frac{2p(1-p)^{n-2}}{(1+p)^n} - \frac{2p(1-p)^{n-1}}{(1+p)^{n+1}} = \frac{2p^2(1-p)^{n-2}}{(1+p)^n} + \frac{2p^2(1-p)^{n-1}}{(1+p)^{n+1}}.$$

Hence after interpretation with the help of (1·11) we get

$$K_{2n-2}(x) - K_{2n}(x) = K'_{2n-2}(x) + K'_{2n}(x). \tag{2·2}$$

Again by (1·3) and (1·11) we have

$$4x\, K_{2n}(x) \doteqdot -4p\frac{d}{dp}\left\{\frac{2(1-p)^{n-1}}{(1+p)^{n+1}}\right\}$$

$$= \frac{16(n-p)\, p(1-p)^{n-2}}{(1+p)^{n+2}}$$

$$=\frac{2p(1-p)^{n-2}}{(1+p)^{n+2}}\Big\{(2n-2)(1+p)^2+4n(1-p^2)$$

$$+(2n+2)(1-p)^2\Big\}$$

$$=(2n-2)\frac{2p(1-p)^{n-2}}{(1+p)^{n}}+4n\frac{2p(1-p)^{n-1}}{(1+p)^{n+1}}$$

$$+(2n+2)\frac{2p(1-p)^{n}}{(1+p)^{n+2}}$$

After interpreting the right hand member with the help of (1·11) we obtain

$$4xK_{2n}(x)=(2n-2)K_{2n-2}(x)+4nK_{2n}(x)$$

$$+(2n+2)K_{2n+2}(x). \quad (2{\cdot}3)$$

Finally using (1·3) with (2·1) we get

$$4xK'_{2n}(x)\doteq 4p-\left(\frac{d}{dp}\right)\frac{2p(1-p)^{n-1}}{(1+p)^{n+1}}$$

$$=\frac{2p(1-p)^{n-2}}{(1+p)^{n+2}}\Big[(2n-2)(1+p)^2-(2n+2)(1-p)^2\Big]$$

which gives after interpretation the relation

$$4xK'_{2n}(x)=(2n-2)K_{2n-2}(x)-(2n+2)K_{2n+2}(x). \qquad (2{\cdot}4)$$

3. Now from (1·11) we get

$$\sum_1^\infty 2mK_{2m}(x)\doteq\sum_1^\infty\frac{4mp(1-p)^{m-1}}{(1+p)^{m+1}}$$

$$=\frac{4p}{(1+p)^2}\sum_1^\infty m\left[\frac{1-p}{1+p}\right]^{m-1}$$

$$=-2p\sum_1^\infty\frac{d}{dp}\left[\frac{1-p}{1+p}\right]^{m}$$

$$=-2p\frac{d}{dp}\sum_1^\infty\left[\frac{1-p}{1+p}\right]^{m}$$

$$=-p\frac{d}{dp}\left[\frac{1}{p}-1\right]$$

$$=\frac{1}{p}$$

$$\doteq x$$

after interpretation with the help of (1·7). Hence

$$x=\sum_1^\infty 2mK_{2m}(x) \tag{3·1}$$

Again

$$\sum_1^\infty (2m)^2 K_{2m}(x) \doteqdot \frac{8p}{(1+p)^2}\sum_1^\infty m^2\left[\frac{1-p}{1+p}\right]^{m-1}$$

$$=-4p\sum_1^\infty m\frac{d}{dp}\left[\frac{1-p}{1+p}\right]^m$$

$$=2p\frac{d}{dp}\left[(1-p^2)\frac{d}{dp}\sum_1^\infty\left(\frac{1-p}{1+p}\right)^m\right]$$

$$=p\frac{d}{dp}\left[1-\frac{1}{p^2}\right]$$

$$=\frac{2}{p^2}$$

$$\doteqdot x^2$$

by (1·7). Hence

$$x^2=\sum_1^\infty (2m)^2 K_{2m}(x). \tag{3·2}$$

Finally

$$\sum_1^\infty (-)^n K_{2n}(x) \doteqdot \frac{2p}{(1+p)^2}\sum_1^\infty (-)^n\left[\frac{1-p}{1+p}\right]^{n-1}$$

$$=-\frac{p}{1+p}$$

$$\doteqdot -K_0(x)$$

by (1·12). Hence

$$\sum_0^\infty (-)^n K_{2n}(x)=0 \tag{3·3}$$

a result due to Bateman.[3]

Now by (1·5)

$$\int_0^\infty \frac{K_{2n}(x)}{x}dx=2\int_0^\infty \frac{(1-p)^{n-1}}{(1+p)^{n+1}}dp$$

3. Bateman.—*Trans. Amer. Math. Soc.* 33. 817-31.

$$=-\int_0^\infty\left[\frac{2}{1+p}-1\right]^{n-1}d\left[\frac{2}{1+p}-1\right]$$

$$=\frac{1}{n}\left[1+(-1)^{n+1}\right].$$

Hence

$$\left.\begin{aligned}&\int_0^\infty \frac{K_{4n}(x)}{x}dx=0\\ \text{and}\quad&\int_0^\infty \frac{K_{4n+2}(x)}{x}dx=\frac{2}{2n+1}.\end{aligned}\right\}\tag{3·4}$$

4. We have from (1·8) and (1·3)

$$x\sin x \doteqdot p\left(-\frac{d}{dp}\right)\frac{1}{1+p^2}$$

$$=\frac{2p^2}{(1+p^2)^2}\tag{4·1}$$

But

$$\sum_{n=o}^{\infty}(-)^n(2n+1)K_{4n+2}(x)\doteqdot\sum_{o}^{\infty}(-)^n(2n+1)\frac{2p(1-p)^{2n}}{(1+p)^{2n+2}}$$

$$=p\frac{d}{dp}\left[\sum_{o}^{\infty}(-)^n\left\{\frac{p-1}{p+1}\right\}^{2n+1}\right]$$

$$=\tfrac{1}{2}p\frac{d}{dp}\left[\frac{p^2-1}{p^2+1}\right]$$

$$=\frac{2p^2}{(1+p^2)^2}.$$

Hence interpreting with the help of (4·1) we get

$$x\sin x=\sum_{n=o}^{\infty}(-)^n(2n+1)K_{4n+2}(x).\tag{4·2}$$

Similarly from (1·9) and (1·3) we have

$$x\cos x\doteqdot\frac{p(p^2-1)}{(1+p^2)^2}\tag{4·3}$$

and

$$\sum_{1}^{\infty}(-)^n 2nK_{4n}(x)\doteqdot+p\sum_{1}^{\infty}(-)^{n+1}\frac{d}{dp}\left[\frac{1-p}{1+p}\right]^{2n}$$

$$=-p\frac{d}{dp}\left[\sum_1^\infty(-)^n\left(\frac{1-p}{1+p}\right)^{2n}\right]$$

$$=\frac{p(p^2-1)}{(1+p^2)^2}.$$

Hence interpreting backwards we obtain

$$x\cos x=\sum_1^\infty(-)^n 2n\, K_{4n}(x). \tag{4·4}$$

Combining (4·1) with (1·3) we get

$$x^2\sin x \doteq \frac{2p(3p^2-1)}{(1+p^2)^3}.$$

Hence

$$x^2\sin x - x\cos x \doteq \frac{2p(3p^2-1)}{(1+p^2)^3}-\frac{p(p^2-1)}{(1+p^2)^2}$$

$$=\frac{p(6p^2-p^4-1)}{(1+p^2)^3}. \tag{4·5}$$

But

$$\sum_{n=o}^\infty(-)^n(2n+1)^2 K_{4n+2}(x)\doteq\sum_o^\infty(-)^n(2n+1)^2\frac{2p(1-p)^{2n}}{(1+p)^{2n+2}}$$

$$=-p\sum_0^\infty(2n+1)\,(-)^n\frac{d}{dp}\left(\frac{1-p}{1+p}\right)^{2n+1}$$

$$=\tfrac{1}{2}p\frac{d}{dp}\left[(1-p^2)\frac{d}{dp}\sum_0^\infty(-)^n\left(\frac{1-p}{1+p}\right)^{2n+1}\right]$$

$$=\frac{p(6p^2-p^4-1)}{(1+p^2)^3}.$$

Hence after interpretation we have

$$x^2\sin x - x\cos x = \sum_{n=0}^\infty(-)^n(2n+1)^2 K_{4n+2}(x). \tag{4·6}$$

Similarly we can prove that

$$x^2\cos x + x\sin x \doteq \frac{4p^2(p^2-1)}{(1+p^2)^3}$$

$$\doteq\sum_1^\infty(-)^n 4n^2 K_{4n}(x). \tag{4·7}$$

Equations (4·6) and (4·7) can also be written in the form

$$x^2\sin x - x\cos x = \sum_{n=0}^\infty n^2 K_{2n}(x)\sin\frac{n\pi}{2} \tag{4·8}$$

$$x^2 \cos x + x \sin x = \sum_{n=0}^{\infty} n^2 K_{2n}(x) \cos \frac{n\pi}{2}. \tag{4·9}$$

Multiply the first equation by $\sin x$ and the second by $\cos x$ and add we obtain

$$x^2 = \sum_{0}^{\infty} n^2 \cos\left[x - n\frac{\pi}{2}\right] K_{2n}(x). \tag{4·10}$$

Similarly by multiplying by $\cos x$ and $\sin x$ respectively and subtracting we get

$$x = \sum_{0}^{\infty} n^2 \sin\left[x - \frac{n\pi}{2}\right] K_{2n}(x). \tag{4·11}$$

The term by term differentiations in this and the previous articles can be easily justified.

5. Now

$$\sum_{n=o}^{\infty} \frac{(a+n)\Gamma(2a+n)}{\Gamma(1+n)} K_{2a+2n}(x)$$

$$\doteqdot \sum_{n=o}^{\infty} \frac{(a+n)\,\Gamma(2a+n)}{\Gamma(1+n)} \, \frac{2p(1-p)^{a+n-1}}{(1+p)^{a+n+1}}$$

$$= \frac{2p(1-p)^{a-1}}{(1+p)^{a+1}} \left\{ a \sum_{n=o}^{\infty} \frac{\Gamma(2a+n)}{\Gamma(1+n)} \left(\frac{1-p}{1+p}\right)^n + \frac{1-p}{1+p} \sum_{n=o}^{\infty} \frac{\Gamma(2a+n+1)}{\Gamma(n+1)} \left(\frac{1-p}{1+p}\right)^n \right\}$$

$$= \frac{2p(1-p)^{a-1}}{(1+p)^{a+1}} \left\{ a\Gamma(2a)\left(1 - \frac{1-p}{1+p}\right)^{-2a} + \Gamma(2a+1)\frac{1-p}{1+p}\left(1 - \frac{1-p}{1+p}\right)^{-2a-1} \right\}$$

$$= 2^{-2a}\,\Gamma(1+2a)\,\frac{1}{p^2}\left(1 - \frac{1}{p^2}\right)^{a-1}$$

$$= 2^{-2a}\,\Gamma(1+2a)\,\frac{1}{p}\left(1 - \frac{1}{p}\right)^{a-1} \frac{1}{p}\left(1 + \frac{1}{p}\right)^{a-1}. \tag{5·1}$$

We will now find "the original" of the right-hand side of (5·1). We have from (1·10)

$$\left(1 - \frac{1}{p}\right)^{a-1} \doteqdot \frac{1}{\Gamma(a)} L_{a-1}(x). \tag{5·2}$$

Let us assume

$$\frac{1}{p}\left(1+\frac{1}{p}\right)^{a-1} \doteqdot f(x).$$

Hence by (1·4) we have

$$\frac{p^a}{(p-1)^{a+1}} \doteqdot e^x f(x). \tag{5·3}$$

But by applying (1·4) to

$$\frac{1}{p^a} \doteqdot \frac{x^a}{\Gamma(1+a)}$$

we get

$$\frac{p}{(p-1)^{a+1}} \doteqdot \frac{e^x x^a}{\Gamma(1+a)}.$$

Hence by the repeated application of (1·3) we have

$$\frac{p^a}{(p-1)^{a+1}} \doteqdot \frac{1}{\Gamma(1+a)}\left(\frac{d}{dx}\right)^{a-1} (e^x x^a). \tag{5·4}$$

Therefore from (5·3) and (5·4) we have by Lerche's Theorem

$$f(x) = \frac{1}{\Gamma(1+a)} e^{-x}\left(\frac{d}{dx}\right)^{a-1} (e^x x^a). \tag{5·5}$$

Therefore with the help of the Product Theorem (1·6) and (5·2) and (5·5) we get 'the original' of the right hand side of (5·1) as

$$\frac{2^{-2a}\Gamma(1+2a)}{\Gamma(a)\Gamma(1+a)} \int_0^x L_{a-1}(x-t)e^{-t}\left(\frac{d}{dt}\right)^{a-1} (e^t t^a)dt.$$

Hence

$$\sum_{n=0}^{\infty} \frac{(a+n)\Gamma(2a+n)}{\Gamma(1+n)} K_{2a+2n}(x) = \frac{2^{-2a}\Gamma(1+2a)}{\Gamma(a)\Gamma(1+a)}$$

$$\times \int_0^x L_{a-1}(x-t)e^{-t}\left(\frac{d}{dt}\right)^{a-1} (e^t t^a)\, dt. \tag{5·6}$$

But

$$L_{a-1}(x-t) = \sum_{r=0}^{a-1} (-)^{a-r-1}\left[\frac{\Gamma(a)}{\Gamma(a-r)}\right]^2 \frac{(x-t)^{a-r-1}}{\Gamma(r+1)}$$

[Courant and Hilbert—*Methoden der Mathematischen Physik*, p. 78.]

$$\left(\frac{d}{dt}\right)^{a-1}(e^t t^a)=\Gamma(a)\,\Gamma(a+1)\,e^t\sum_{p=o}^{a-1}\frac{t^{a-p}}{\Gamma(1+p)\Gamma(a-p)\Gamma(a+1-p)}$$

and

$$\int_0^x (x-t)^{a-r-1}\,t^{a-p}\,dt=\frac{\Gamma(a-r)\,\Gamma(a-p+1)}{\Gamma(2a-r-p+1)}\,x^{2a-r-p}.$$

Making use of these three equations, (5·6) can be written as

$$\sum_{n=o}^{\infty}\frac{(a+n)\Gamma(2a+n)}{\Gamma(1+n)}\,K_{2a+2n}(x)=2^{-2a}\Gamma(1+2a)\,[\Gamma(a)]^2$$

$$\times\sum_{r=o}^{a-1}\sum_{p=o}^{a-1}\frac{(-)^{a-r-1}\,x^{2a-r-p}}{\Gamma(1+r)\Gamma(1+p)\Gamma(a-r)\Gamma(a-p)\Gamma(2a-r-p+1)}.\quad(5{\cdot}7)$$

We can express the left hand member in a different form. For it is equal to

$$\sum_{n=o}^{\infty}\frac{(a+n)\,(2a+n-1)\ldots.(a+n)\,(a+n-1)\ldots.\ldots(n+1)\Gamma(n+1)}{\Gamma(n+1)}K_{2a+2n}(x)$$

$$=\sum_{n=a}^{\infty}n^2(n^2-1^2)\ldots\ldots[n^2-(a-1)^2]\;K_{2n}(x),$$ changing $(n+a)$ into n.

$$=\sum_{n=a}^{\infty}\left\{\prod_{r=o}^{a-1}(n^2-r^2)\right\}K_{2n}(x).$$

Hence

$$\sum_{n=a}^{\infty}\left\{\prod_{r=o}^{a-1}(n^2-r^2)\right\}K_{2n}(x)=2^{-2a}\Gamma(1+2a)\,[\Gamma(a)]^2$$

$$\times\sum_{r=o}^{a-1}\sum_{p=o}^{a-1}\frac{(-)^{a-r-1}\,x^{2a-r-p}}{\Gamma(1+r)\Gamma(1+p)\Gamma(a-r)\Gamma(a-p)\Gamma(2a-r-p+1)}.\quad(5{\cdot}8)$$

The following four particular cases of (5·8) may be noted.

(1) $a=1$ R.H.S.$=\frac{x^2}{4}$.

Therefore $$\sum_{n=1}^{\infty}n^2\,K_{2n}(x)=\frac{x^2}{4}\quad(5{\cdot}9)$$

a result obtained by a different method in § 3.

(2) $a=2$ R.H.S.$=\frac{x^2}{16}(12-x^2)$.

Hence $$\sum_{2}^{\infty} n^2(n^2-1^2)\, K_{2n}(x)=\frac{x^2}{16}(12-x^2). \tag{5·10}$$

(3) $a=3$ R.H.S. $=\frac{x^2}{64}[x^4-60x^2+360]$.

Hence $$\sum_{3}^{\infty} n^2(n^2-1^2)(n^2-2^2)\, K_{2n}(x)=\frac{x^2}{64}[x^4-60x^2+360]. \tag{5·11}$$

(4) $a=4$ R.H.S. $=\frac{x^2}{2^8}[20160-5040x^2+168x^4-x^6]$.

Hence

$$\sum_{n=4}^{\infty} n^2(n^2-1^2)(n^2-2^2)(n^2-3^2)\, K_{2n}(x)$$

$$=\frac{x^2}{2^8}[20160-5040x^2+168x^4-x^6]. \tag{5·12}$$

THE NUMBER OF PARTICLES IN THE PATH OF A RAY OF LIGHT TRAVERSING THE EARTH'S ATMOSPHERE

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§ 1. *Introductory.*

It is useful in several astronomical problems to have some knowledge of the number of particles of a gas in the earth's atmosphere, that are encountered by a ray of light proceeding from a stellar source and reaching the observer. For instance, in stellar photometry, it is necessary to know the diminution in the intensity of light from a star due to absorption by the earth's atmosphere; this is easily found, if the number of particles of the absorbing gas and its coefficient of absorption are known. Or again, to deduce the laws of intensity-distribution of the telluric lines in a spectrum such as that of the sun, the number of particles of the gas producing those lines needs to be ascertained. The following is an attempt to arrive at an approximate formula for the number of such particles; which, while taking account of atmospheric refraction, is *simple* enough for ready calculation.

§ 2. *Historical.*

In connection with the determination of the diminution of light-intensity of a stellar source, several formulae have been obtained[1] for the "air-mass" absorbing the light or for closely related quantities. Lambert, taking into account the variation of density of the atmosphere with height, arrived at an expression for the extinction of light:

$$\log J_0 - \log J_z = A(1-\sec z) + \frac{1}{2} B \sec z \tan^2 z - \frac{1\cdot 3}{2\cdot 4} \times C \sec z \tan^4 z + \ldots\ldots$$

1. For a summary of the work done in this connection, and for full references to the original papers, cf. *Handbuch der Astrophysik*, Bd. II., I Half, II part, pp. 171 et seq.

where J_z = intensity of light at apparent zenith distance z,

$$A=k\int_0^Y \frac{\rho y\,dy}{\sqrt{R^2+y^2}},\quad B=k\int_0^Y \frac{\rho y^3\,dy}{(R^2+y^2)^{\frac{3}{2}}},\quad C=k\int_0^Y \frac{\rho y^5\,dy}{(R^2+y^2)^{\frac{5}{2}}}, \text{ etc.,}$$

k is a constant factor depending upon the medium,

R is the radius of the earth,

ρ is the density of air at height h,

y is defined by the equation $y^2=h^2+2\,Rh$,

and Y is the value of y for $h=H$, the height of the atmosphere, so that $Y^2=H^2+2\,RH$.

Here A, B........can be determined by observation at a series of definite zenith distances and their values for any given zenith distance computed by interpolation.

While the coefficients of Lambert's series are empirical, Bouger obtained an expression for the air-mass $F(z)$, by assuming the hypothesis $\frac{B}{B_0}=\frac{\rho}{\rho_0}=n$, where B is the pressure of the air coirespond-ing to the density ρ_0. His series for $F(z)$ converges for values of z between 0 and χ, where χ is the positive root of the equation

$$\chi^2+2\,R\chi-R^2\cos^2 z=0.$$

And this part of $F(z)$, is integrated with the help of the above-mentioned hypothesis, yielding the formula

$$F_0(z)=J_0\sec z-\frac{J_1}{R}\sec z\tan^2 z+\frac{3}{2}\frac{J_2}{R^2}\sec^3 z\tan^2 z-\ldots.$$

where $J_0=1-e^{-\chi/l_0}$

$J_1=l_0J_0-\chi e^{-\chi/l_0}$

$J_2=2\,l_0J_1-\chi^2 e^{-\chi/l_0}$

....

....

$$J_n=nl_0J_{n-1}-\chi^n e^{-\chi/l_0}=\frac{\chi^{n+1}}{(n+1)l_0}e^{-\chi/l_0}\left\{1+\frac{1}{n+2}\frac{\chi}{l_0}+\frac{1}{(n+2)(n+3)}\frac{\chi^2}{l_0^2}+\ldots\ldots\right\}$$

where again, l_0 amounts to the height of the equivalent homogeneous atmosphere.

Bemporad discusses this as well as the remaining part of $F(z)$, viz. $F_1(z)=\int_\chi^H dF$, and arrives at the form

$$F(z)=\sec z-\frac{l_0}{R}\sec z\tan^2 z+\frac{3\,l_0^2}{R^2}\sec^3 z\tan^2 z-\ldots\ldots;$$

while Bouger himself renders it as

$$F(z)=\sec z-\frac{l_0}{2R}\sec z\tan^2 z+(l_0-\tfrac{1}{3}R\cos^2 z)\frac{l_0\tan^2 z}{2R^2\cos^2 z}-\ldots.$$

For $z>85^\circ$, these series are barely convergent, and Bemporad gives a modification. The hypothesis of Bouger clearly involves the assumption that the atmosphere is at a constant temperature. Bemporad, indeed, introduces corrections to Bouger's results necessitated by the variation of temperature with height in our atmosphere.

Laplace struck a different note in attempting to express the air-mass in the path of a ray in terms of the atmospheric refraction itself. He obtained the equation $dF=1/(c\rho_0\mu_0)\times\frac{d(\text{Refraction})}{\sin z}\mu^3\frac{r}{R}$, which, after some approximation and integration, yields his "extinction-formula", $F(z)=K\frac{\text{Refraction}}{\sin z}$, where $K=1/c\rho_0\mu_0$ is a constant.

The atmospheric refraction is directly observed; or it can be expressed as a series in powers of tan z. In the latter case, it is possible to write

$$F(z)\sim K\frac{\alpha}{1-\alpha}\sec z\left\{\left(1+\frac{1}{2}\alpha-\frac{l_0}{R}\right)-\left(\frac{l_0}{R}-\frac{\alpha}{2}\right)\tan^2 z\right\},$$

where α is a constant of refraction.

Here also the calculations are mainly for an atmosphere at constant temperature, though some allowance for the variation of temperature is implied by the equation $c=c_0+\alpha t$.

It will be noticed that all these formulæ consist of terms depending on direct observation, or else of power series involving l_0, the observed height of the homogeneous atmosphere. In either case the results are more or less of an empirical character. We propose to proceed from *first principles* and try to obtain an approximate formula involving mainly the physical constants of a gas—so that it is applicable not merely to the " air-mass " taken as a whole, but even to any constituent of the atmosphere, such as oxygen or water-vapour. Moreover, we shall not merely confine our attention to an isothermal atmosphere (cf. Bouger) and then introduce corrections for changes of temperature (cf. Bemporad), but shall include in our scope an atmosphere consisting of a stratum at constant temperature and one in which the temperature changes. However, for the sake of completeness as well as clarity we shall consider successively the cases of a homogeneous atmosphere, an isothermal atmosphere, and one in which the temperature changes (but adiabatic conditions hold), and finally combine these to discuss our actual atmosphere.

§ 3. *Homogeneous Atmosphere*

If the atmosphere were a homogeneous layer of thickness H, the path of a ray of light inside it will be a straight line of length $H \sec z$, where z is the apparent zenith distance of the source. The particles encountered by the ray will be those enclosed in a cylinder of this length and of cross-section $\pi\sigma^2$, where σ is the average diameter of a particle. If ρ be the density of air, the air-mass encountered $= \pi\sigma^2\rho H \sec z$. If M be the weight of a particle of the substance, the number of particles $N=(\pi\sigma^2/M) \times \rho H \sec z$. Or, putting $\pi\sigma^2/M=K$, we have $N=K\rho H \sec z$.

If the pressure of air and gravity at the surface of the earth be denoted by p_0, g_0*, then $p_0=g_0 \rho H$,

and hence
$$N=K\frac{p_0}{g_0} \sec z. \qquad (1)$$

§ 4. *Isothermal Layer*

But, since the atmosphere is not homogeneous, the stellar ray is subjected to increasing refraction as it penetrates the atmosphere and the path is bent into a curve. First, let us suppose the temperature to be constant throughout, and the atmosphere to be divided into a number of concentric strata which are so thin that the density may be taken as uniform within each stratum.

Let a ray strike a layer of density ρ and refractive index μ at the point P. The tangent (PT) to the curved path at P will be the direction of the refracted ray at P. If O be the centre of the earth, $\angle TPO=\phi$ is the angle of refraction.

The segment of the ray inside the layer$=dl=dr \sec\phi$, where dr is the diminutive increment of the radial distance r. The number of particles in this layer is

$$dn=k\rho\, dr \sec \phi.$$

Also,
$$\mu r \sin \phi=\mu_0\, a \sin z,$$

where μ_0, a, z are the values of μ, r, ϕ at the lowest layer, *i.e.* at the surface of the earth; thus a is the radius of the earth, and z the apparent zenith distance.

* The suffixed notation is employed here, for the sake of convenience in comparing the results that follow.

Eliminating ϕ between these two equations,

we get $$dn = K\rho \frac{\mu r\, dr}{\sqrt{\mu^2 r^2 - \mu_0^2 a^2 \sin^2 z}}. \qquad (2)$$

We may take the height of the atmosphere to be small compared, with the radius of the earth, and hence we may put* $r/a = 1+f$ where f is so small that its powers higher than the first may be neglected.

Then, (2) becomes

$$dn = K\rho \frac{\mu\,(1+f)\, a df}{\left\{\mu^2 - \mu_0^2 \sin^2 z + 2\mu^2 f\right\}^{\frac{1}{2}}}$$

$$= K\rho \frac{\mu\,(1+f)\, a df}{(\mu^2 - \mu_0^2 \sin^2 z)^{\frac{1}{2}}} \left\{1 + \frac{2f\,\mu^2}{\mu^2 - \mu_0^2 \sin^2 z}\right\}^{-\frac{1}{2}}. \qquad (2\cdot1)$$

The last factor may be expanded as a binomial series involving powers of f, provided $\left|\frac{2f\mu^2}{\mu^2 - \mu_0^2 \sin^2 z}\right| < 1$,

for the whole range of values of μ, *i.e.* for $1 < \mu \leqslant \mu_0$,

which will be true if $1 - \frac{\mu_0^2}{\mu^2} \sin^2 z > \pm 2f$, or $\sin^2 z < \frac{1-2f}{\mu_0^2}$.

Taking $\mu_0 = 1\cdot000294$, and $f \sim 10^{-3}$, (for the atmosphere of oxygen and nitrogen), $z < 87°5'$. For $z < 75°$, $\frac{2f\mu^2}{\mu^2 - \mu_0^2 \sin^2 z} < \frac{1}{3}$, so that the 2nd and higher powers of this term may certainly be neglected.

Under these conditions, (2·1) may be written as

$$dn = K\rho\, a\mu \left\{ \frac{1}{(\mu^2 - \mu_0^2 \sin^2 z)^{\frac{1}{2}}} - \frac{\mu_0^2 \sin^2 z\, f}{(\mu^2 - \mu_0^2 \sin^2 z)^{\frac{3}{2}}} \right\} df. \quad (2\cdot2)$$

Hence the total number of particles in the path of the ray is

$$N = \int dn = Ka \int \frac{\rho\mu}{(\mu^2 - \mu_0^2 \sin^2 z)^{\frac{1}{2}}} df - Ka \int \frac{\mu_0 \sin^2 z\, \rho\mu f}{(\mu^2 - \mu_0^2 \sin^2 z)^{\frac{3}{2}}} df. \qquad (3)$$

The limiting values for integration are

$$\left.\begin{array}{lll} & \text{from } f = f' \text{ (say)} & \text{to } f = o \\ i.e. & \text{from } \rho = o & \text{to } \rho = \rho_0 \\ & \text{and } \mu = 1 & \text{to } \mu = \mu_0. \end{array}\right\} \qquad (3\cdot1)$$

*This method of approach is suggested by the one given in Ball *Spherical Astronomy*, p. 123.

The three variables ρ, μ, f are not independent. The first two are connected by Gladstone and Dale's law,

$$\mu-1=c\rho \tag{4}$$

while ρ and f are related, as we have assumed isothermal equilibrium, by the law

$$\frac{\rho}{\rho_0}=e^{-\frac{Mgaf}{kT}} \tag{5}$$

where k=Boltzmann's constant and T=Absolute Temperature.

Also $$g=g_0a^2/r^2=g_0(1-2f). \tag{6}$$

Hence $$Mgaf/kT=Mg_0(1-2f)af/kT$$
$$=Mg_0af/kT$$
to the 1st order of small quantities.

Putting $$Mg_0\,a/kT=\text{constant}=b \tag{7*}$$

(6) becomes $$\frac{\rho}{\rho_0}=e^{-bf}. \tag{8}$$

Hence $$d\rho/\rho=-bdf. \tag{8·1}$$

Also from (4), $$d\mu=cd\rho. \tag{4·1}$$

From (8·1) and (4.1),[1] $$\rho df=-\frac{1}{bc}\,d\mu. \tag{9}$$

Hence the integral of the 1st term in (3), viz.,

$$Ka\int_0^{f'}\frac{\rho\mu}{(\mu^2-\mu_0^2\sin^2 z)^{\frac{1}{2}}}\,df,$$

can be written as

$$-\frac{Ka}{bc}\int_{\mu_0}^{1}\frac{\mu d\mu}{\sqrt{\mu^2-\mu_0^2\sin^2 z}}+\frac{Ka}{bc}\left\{\mu_0\cos z-\sqrt{1-\mu_0^2\sin^2 z}\right\}. \tag{10}$$

This may be simplified further, by denoting the small quantity (μ_0-1) by x and expanding the second term by Maclaurin's theorem, *i. e.* put

$$\sqrt{1-\mu_0^2\sin^2 z}=\left\{1-(1+x)^2\sin^2 z\right\}^{\frac{1}{2}}\equiv f(x).$$

By the above-mentioned theorem, omitting higher powers of x

$$f(x)=f(o)+xf'(o),$$
$$=\cos z-x\frac{\sin^2 z}{\cos z}.$$

*$b=Mg_0a/kT=mg_0a/RT$, where R=gas-constant, and m=weight of a grm.—atom of the substance. Taking $R=826\times10^7$, $T\sim290^\circ$, $a=6{\cdot}37\times10^8$ cm; $m=\frac{1{\cdot}293}{0{\cdot}090}=14{\cdot}367$, $g_0=981$ cm/sec^2, we get $b\sim375$.

Hence the integral$=\frac{Ka}{bc}\left\{\mu_0 \cos z-\cos z+(\mu_0-1)\frac{\sin^2 z}{\cos z}\right\}$

$$=\frac{Ka}{bc}(\mu_0-1)\sec z. \qquad (10{\cdot}1)$$

Again, $$\frac{\mu_0-1}{c}=\rho_0=Mn_0$$

if n_0 be the number ot atoms per c.c. at the surface of the earth.

Also, $$p_0=n_0\,kT.$$

Hence $$\frac{\mu_0-1}{c}=\frac{M}{kT}p_0=\frac{b}{g_0 a}\,p_0.$$

Hence the integral $$=K\frac{p_0}{g_0}\sec z \qquad (10{\cdot}2),$$

which is the same as (1).

Therefore, the next term is the correction to be added to this, when taking account of refraction. To evaluate

$$Ka\,\mu_0^2\sin^2 z\int_0^{f'}\frac{\rho\mu f}{(\mu-\mu_0^2\sin^2 z)^{\frac{3}{2}}}\,df,$$

as this involves f as a factor, we may put $\mu=\mu_0=1$ since the error introduced will be of the order of $f(\mu_0-1)$, which may be neglected. The integral then takes the form

$$Ka\frac{\sin^2 z}{\cos^3 z}\int_0^{f'}\rho f\,df. \qquad (11)$$

But $$\int_0^{f'}\rho f\,df=-\frac{1}{b}\int_{\rho_0}^{0}f\,d\rho, \text{ [on using the relation (8·1)]}$$

$$=\frac{1}{b}\int_0^{f'}\rho\,df, \text{ [on integration by parts.]} \qquad (12)$$

Hence (11) becomes $$\frac{Ka}{b}\frac{\sin^2 z}{\cos^3 z}\int_0^{f'}\rho\,df. \qquad (13)$$

Also, $$p_0=\int_0^{f'}g\rho a\,df=\int_0^{f'}g_0(1-2f)\rho\,a\,df$$

$$=ag_0\left[\int_0^{f'}\rho\,df-2\int_0^{f'}\rho f\,df\right]$$

$$=ag_0\left(1-\frac{2}{b}\right)\int_0^{f'}\rho\,df. \qquad \text{[using (12)]}$$

Hence $$\frac{a}{b}\int_0^{f'} \rho\, df = \frac{p_0}{g_0(b-2)}.$$

Substituting in (13), we get for the integral

$$K\frac{p_0}{g_0(b-2)}\ \frac{\sin^2 z}{\cos^3 z}. \tag{14}$$

Hence the total number of particles

$$N = K\frac{p_0}{g_0}\sec z - K\frac{p_0}{g_0(b-2)}\tan^2 z \sec z$$

$$= K\frac{p_0}{g_0(b-2)}\sec z\left\{(b-2)-(\sec^2 z-1)\right\}$$

$$= K\frac{p_0}{g_0}\frac{b-1}{b-2}\sec z\left\{1-\frac{1}{b-1}\sec^2 z\right\}. \tag{15}$$

Or, N is of the form* $C \sec z + D \sec^3 z$, (15·1)

where $C = K\, p_0(b-1)/g_0(b-2)$ and $D = -K\, p_0/g_0(b-2)$.

In the case of the atmosphere, b is very large compared with 1 and 2, so that (15) can be written approximately as

$$N = K\frac{p_0}{g_0}\sec z\left\{1-\frac{1}{b}\sec^2 z\right\}\dagger. \tag{15·2}$$

§ 5. *Adiabatic Equilibrium*

The assumption of constant temperature is again one that is not true of a good part of our atmosphere, and it is desirable to inquire what happens when the temperature also varies. For a general discussion we have to restrict the variation of temperature to conditions obeying the adiabatic law.

Here, instead of (5), we have the relations

$$\rho^\gamma = \text{constant}\times p = \text{constant}\times e^{-Mg_0 af/kT} \tag{16.1}$$

and $$\rho\times T = \text{constant}\times p. \tag{16·2}$$

From (16·1) and (16·2), $\rho^{\gamma-1}/T = \text{constant}$. (16·3)

From (16·1) and (16·3),

$$\left.\begin{aligned} \gamma\frac{d\rho}{\rho} &= -\frac{Mg_0\, a}{k}\left(\frac{df}{T}-\frac{f}{T^2}dT\right) \\ (\gamma-1)\frac{d\rho}{\rho} &= \frac{dT}{T}. \end{aligned}\right\} \tag{16·4}$$

*cf. the form of the expression for Atmospheric refraction $A \tan z + B \tan^3 z$.

†Though the length of the path of the refracted ray is actually greater than that considered in (1), viz. $af \sec z$, the number of particles becomes smaller This is of course due to the variation of the density along the path of the ray.

Eliminating $\frac{dT}{T}$, we get $\left\{(\gamma-1)f-\frac{k\gamma T}{Mg_0 a}\right\}\frac{d\rho}{\rho}=df.$ (16·5)

Also, since $d\rho=d\mu/c,$

(16·5) becomes $\rho\, df=\frac{d\mu}{c}\left\{(\gamma-1)f-\frac{k\gamma T}{Mg_0 a}\right\}.$ (16·6)

But $T=\text{const.}\times e^{-\left(\frac{\gamma-1}{\gamma}\right)\frac{Mg_0 af}{kT}}$

or $T\sim T_0-\frac{\gamma-1}{\gamma}\;\frac{Mg_0 af}{k}.$

Hence substituting in (16·6), we get

$$\rho df=-\frac{d\mu}{c}\;\frac{k\gamma\, T_0}{Mg_0\, a} \qquad (16·7)$$

$$=-\frac{1}{b_1 c}\, d\mu \qquad (16·8)$$

where $b_1=\frac{Mg_0 a}{k\gamma\, T_0}=\frac{b}{\gamma}.$ (16·9)

A comparison of (16·8) and (16·9) with (9) and (7) respectively, will shew, that the integrals will be the same as in the previous case, with the difference that b_1 is written instead of b. Thus, for the first integral, we have instead of (10·1)

$$\frac{Ka}{b_1 c}(\mu_0-1)\ \sec z. \qquad (17)$$

Since $\frac{\mu_0-1}{c}=\frac{M}{kT_0}p_0=\gamma\frac{b_1}{ag_0}p_0$

the integral becomes $K\gamma\,\frac{p_0}{g_0}\sec z.$ (17·1)

For the second integral, we have instead of (14),

$$K\,\frac{p_0}{g_0(b_1-2)}\;\frac{\sin^2 z}{\cos^3 z}, \qquad (18)$$

which may be written as $K\frac{p_0}{g_0}\;\frac{\gamma}{b-2\gamma}\tan^2 z\sec z.$ (18·1)

Hence $N=K\gamma\,\frac{p_0}{g_0}\sec z-\frac{K\gamma}{b-2\gamma}\;\frac{p_0}{g_0}\sec z\ (\sec^2 z-1)$

$$=\frac{K\gamma}{b-2\gamma}\,\frac{p_0}{g_0}\sec z\left\{(b-2\gamma)-(\sec^2 z-1)\right\}$$

$$=K\gamma\,\frac{b+1-2\gamma}{b-2\gamma}\;\frac{p_0}{g_0}\sec z\left\{1-\frac{1}{b-(2\gamma-1)}\sec^2 z\right\} \qquad (19)$$

Again, since 2γ and $2\gamma-1$ are very small compared with b, we may approximately put

$$N=K\gamma \frac{p_0}{g_0} \sec z\left(1-\frac{1}{b} \sec^2 z \right). \tag{20}$$

§ 6. *The Earth's Atmosphere and the Law of Variation of air-mass*

Comparing (15·2) and (20), we find that in an isothermal layer as well as in a layer of variable temperature the air-mass varies with the zenith distance according to the same law, viz.:

$$N \propto \sec z\left(1-\frac{1}{b} \sec^2 z \right). \tag{21}$$

Hence, if we ignore the effects of convection and other mass-movements and consider the earth's atmosphere in equilibrium as consisting of a stratosphere in which the temperature is constant fitting on to the lower troposphere in which the temperature varies with height, we may legitimately infer the same law of variation to hold.

It is noteworthy that the law of variation contains only the physical constants of the gas (and the radius of the earth $= b = mg_0a/RT_0$), and does not involve other factors like the height of the homogeneous atmosphere. This is of advantage, if we have to determine the variation of mass of a particular constituent of the atmosphere, provided, of course, that this constituent obeys the adiabatic laws, and again that it is not too intimately mixed up with the rest of the air; in the latter case it will have to be considered along with the air-mass as a whole.

To determine the constant of variation for air, let us denote the "air-mass" in a tube of unit cross-section around the path of the ray of light by the equation

$$\overline{N}=A \sec z\left(1-\frac{1}{b} \sec^2 z \right),$$

where A is some constant. At $z=0$, $\overline{N}_0=A\ (1-1/b)$. But since ρ_0 and H are the density and height of the equivalent homogeneous atmosphere, we have at $z=0$, $\overline{N}_0=\rho_0 H$.

Hence $$A=\rho_0 H/(1-1/b).$$

Hence, we may write $$\overline{N}=\rho_0 HF(z) \tag{22}$$

where $$F(z)=\frac{1}{1-1/b} \sec z\left(1-\frac{1}{b} \sec^2 z \right). \tag{23}$$

(23) may also be written in the alternative forms

$$F(z)=\frac{b}{b-1}\sec z-\frac{1}{b-1}\sec^3 z \tag{23·1}$$

and $$F(z)=\sec z-\frac{1}{b-1}\sec z\tan^2 z.* \tag{23·2}$$

Substituting the value $b\sim 375$, for air, (23.1) becomes

$$F(z)=\frac{375}{374}\sec z-\frac{1}{374}\sec^3 z. \tag{23·3}$$

The values of $F(z)$ are calculated with the help of this equation, for different values of z at intervals of 6° up to 84°, and given in Table I, 2nd row. In the third row is shown the values computed by the more complicated formula of Bemporad.† It is seen that the results yielded by our simpler formula are in close agreement with those of Bemporad up to $z=84°$.

§ 7. *Darkening of Stars by the Earth's Atmosphere*

We may apply this formula to deduce the darkening effect on stellar images caused by absorption of a part of the light by our atmosphere, and compare the results with some of the direct observations.

If I be the intensity of light coming from a star, and $\overline{k}$ the mean absorption coefficient for air, the apparent luminosity of the star at zenith distance z is

$$I_z=I\,e^{-\overline{k}\,\overline{N}}=I\,e^{-\overline{k}\rho_0 HF(z)}. \tag{24}$$

Putting the constant $$e^{-\overline{k}\rho_0 H}=p \tag{25}$$

this becomes $$I_z=I\,p^{F(z)}‡.$$

Hence $$\frac{I_z}{I_0}=\frac{Ip^{F(z)}}{Ip^{F(0)}}=p^{F(z)-F(0)}=p^{F(z)-1}. \tag{26}$$

If m_z, m_0 be the apparent magnitudes of the star at zenith distance z and at the zenith respectively, the magnitude-difference is given by the equation

$$0{\cdot}4(m_z-m_0)=\log_{10} I_0/I_z=(-\log_{10} p)\,(F(z)-1). \tag{27}$$

*This is of the same *form* as the expressions due to Bouger and Bemporad up to the second term. The coefficients of the second term are of course different in the three formulæ.

†These values are taken from Table XIIa. (p. 268) of *Handbuch der Astrophysik*, vol. cit.

‡p is thus a measure of the transmission of the Incident Intensity per atom, and may be called the "transmission-coefficient" of the medium.

Thus, for a given zenith distance the darkening of a star relative to its brightness at zenith can be found, if we know the air-mass $F(z)$ at that zenith distance and the transmission-coefficient p for the observing station.

Such calculations with the help of our formula for $F(z)$ are shown in Tables II (a) and II (b), row 4, and compared with the results of direct observation (row 2) and with computations by the formula of Bemporad*. Table II (a) shows the figures for Potsdam; the value of p used in Bemporad's calculation, and here adopted by us for the sake of comparison is $p=0{\cdot}835$.

We find that our results compare very well with Bemporad's up to 60°, and both of these agree fairly closely with Potsdam observations up to 48°, after which divergences begin. At 60° where the two computations agree, the observed value is 18% greater than Bemporad's figure; at 66° and 72° it is more than 20% greater than Bemporad's value. If we bear in mind this degree of agreement, the difference between our results and the other two will not be considered too large even at 84° where our value differs most from Bemporad's, it is smaller than the Potsdam figure by less than 18%. The divergence between the computations and the observations may to a large extent be accounted for by the presence of dust-particles in the lower layers of our atmosphere, which "scatter" a fraction of the light and increase the darkening effect †.

In Table II (b), the observations at Sāntis are compared with the computations by the aid of Bemporad's formula and our formula for $F(z)$. The value of p adopted here for Sāntis (height=2500 metres) is obtained by interpolating between values deduced from direct photometric observation at M. Grigna (height 2·15 km.) and Weissmies (height 3·5 km.)‡.

* Bemporad's values are given in Table XI(a) of *Handbuch der Astrophysik*, vol cit, for intervals of 5° up to 50°, and then for every degree; values of $F(z)$ for angles within the intervals (*e g.* 36°) are interpolated here. The results of observations at Potsdam and Sāntis are extracted from G. Muller's work in Table X-a (Ibid.) to four decimal places under the column "Logarithmen" [$=0{\cdot}4\ (m_z - m_0)$] and to two decimal places under the column of "Gross Extinctions"; here, the values are obtained from the "Logarithmen" by multiplying by 2·5

† It is possible to allow for this greater darkening by choosing a suitable constant b' instead of b in the law of variation. The last row of Table II (a) exhibits the values computed for $b'-1=1000$: the results are in very close agreement with Potsdam observations, being indeed slightly better than Bemporad's calculations right up to 87°.

‡These figures are taken from *Handbuch der Astrophysik*. vol. cit., p. 198

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		M. Grigna.	Weissmeis.	Säntis.
	Height ..	2150 *m.*	3500 *m.*	2500 *m.*
p:	Bemporad.	0·871	0·953	
	Zipler ..	0·847	0·873	
	Average.	0·859	0·913	$\cdot 859 \times \left(\frac{\cdot 913}{\cdot 859} \right)^{\frac{350}{1350}}$ $= 0\cdot 873$

Comparison shows that the observations and computations tally with one another; the differences are not much, and, if at all, our results are slightly better than Bemporad's—though this is probably due to the choice of p.

§ 8. *Conclusion*

Thus we may express the variation of the number of particles of a gas in our atmosphere, in the path of a ray of light from a star of zenith distance z, by the law

$$N \propto \sec z \left(1 - \frac{1}{b} \sec^2 z \right),$$

which is sufficiently accurate (up to $z=84°$) for all practical purposes and which, moreover, is adapted to *ready* calculation. We have found that this approximate formula derived from first principles by purely mathematical methods and involving only the physical constants of a gas compares fairly well with formulæ based on empirical measures and with results of direct observation. Finally, we may draw attention to the fact that the expression can take a form $(C \sec z + D \sec^3 z)$, analogous to the law of atmospheric refraction, of the same order of approximation, $(A \tan z + B \tan^3 z)$—a fact which may yield some satisfaction from the aspect of pure mathematical æstheticism.

TABLE I.

Values of $F(z)$.

z.	Bemporad.	Above Formula.	z.	Bemporad	Above Formula.
6°	1·005	1·005	48°	1·492	1·490
12°	1·022	1·022	54°	1·698	1·693
18°	1·052	1·052	60°	1·995	1·984
24°	1·094	1·094	66°	2·447	2·425
30°	1·154	1·154	72°	3·209	3·154
36°	1·235	1·234	78°	4·716	4·425
42°	1·344	1·343	84°	8·900	7·251

TABLE II.

VALUES OF $(m_z - m_0)$.

(a) *Potsdam Observations.*

z.	Potsdam.	Bemporad.	Above Formula	(b′—1) =1000	z.	Potsdam.	Bemporad.	Above Formula.	(b′—1) =1000
6°	0·000	·001	·001		54°	·156	·137	·136	·137
12°	·002	·005	·004		60°	·230	·195	·193	·195
18°	·007	·011	·010		66°	·341	·283	·279	·283
24°	·015	·019	·018		72°	·519	·432	·422	·432
30°	·028	·030	·030		78°	·825	·728	·671	·725
36°	·046	·046	·046		84°	1·490	1·547	1·224	1·508
42°	·070	·068	·067	·068	86°	2·041	2·240	..	2·037
48°	·106	·097	·096	·097	87°	2·482	2·813	..	2·178

(b) *Sāntis Observations.*

z.	Sāntis.	Bemporad.	Above Formula.	z.	Sāntis.	Bemporad.	Above Formula.
6°	0·000	·001	·001	48°	·069	·073	·072
12°	·003	·003	·003	54°	·097	·103	·102
18°	·007	·008	·008	60°	·139	·147	·145
24°	·013	·014	·014	66°	·202	·213	·210
30°	·022	·023	·023	72°	·307	·326	·317
36°	·033	·035	·034	78°	·514	·548	·505
42°	·048	·051	·050	84°	1 069	1·165	·922

ON THE OSCULATING SPACES OF A RATIONAL NORM CURVE, WHICH CUT A LINEAR COMPLEX IN NULL-VARIANT COMPLEXES

By B. RAMAMURTI, Annamalai University

The lines of a linear complex L in $[n]$ which lie in a sub-space $[r]$ form a linear complex in that sub-space, which may be denoted by $L_{[r]}$. If $L_{[r]}$ has at least one singular point, then it is said to be null-variant. Since every linear complex in $[2k]$ has at least one singular point, every $[2k]$ cuts L in a null-variant complex. In general a linear complex in $[2k+1]$ has no singular point and if it has one singular point, then it has at least a line of singular points.

If we take a rational norm curve R_n in $[n]$, any linear complex L determines, and is determined by, the skew-symmetric double-binary form, given symbolically by $a_x^n\, b_y^n$ where

$$a_1^r\, a_2^{n-r}\, b_1^s\, b_2^{n-s} = -a_1^s\, a_2^{n-s}\, b_1^r\, b_2^{n-r} \qquad (1\cdot1)$$

which, when equated to zero, gives the (n, n) correspondence between the points x and y on R_n, which are conjugate with respect to the null-system defined by L. It is shown in a previous paper*, that there are $2k(n-2k+1)$ points on R_n the osculating $[2k-1]$'s at which cut L in null-variant complexes. Hence if $T_k(x)$ gives parametrically these points on R_n, $T_k(x)$ must be a binary covariant of the double-binary form $a_x^n\, b_y^n$ in the cogradient variables x and y. The object of this note is to obtain a symbolical form of this covariant. In the special case, when the linear complex L is outpolar to the norm curve R_n, so that the corresponding double-binary form is of the type $(xy)\, a_x^{n-1}\, a_y^{n-1}$, the covariant $T_k(x)$ is identified to be $(a_1\, a_2)^4 \ldots (a_1\, a_k)^4 \ldots (a_{k-1}\, a_k)^4\, a_{1,x}^{2n-4k+2} \ldots a_{k,x}^{2n-4k+2}$ where $a_1, a_2, \ldots a_k$, are symbols equivalent to a in a_x^{2n-2}.

Let the equation of L in line co-ordinates $\pi_{ik} = x_i\, y_k - x_k\, y_i$, be

$$\sum A_{ik}\, \pi_{ik} = 0 \qquad (i, k = 0, 1, \ldots, n).$$

* Linear complexes related to a rational norm curve, *Proc. Indian Academy of Sciences*, Vol. 1. No. 7. (1935).

If the rational norm curve R_n be given parametrically by

$$x_r = {}^nC_r t^r \qquad (r=0, 1, \ldots, n)$$

the points x and y on R_n are conjugate with respect to the null-system defined by L if

$$\sum A_{rs}\, {}^nC_r\, {}^nC_s\, (x^r y^s - x^s y^r) = 0. \qquad (1\cdot2)$$

(1·2) must be identical with (1·1), when in the latter we replace $\frac{x_1}{x_2}, \frac{a_1}{a_2}$ etc., by x, a etc. Comparing, we have

$$A_{rs} = a^r b^s = -a^s b^r = \tfrac{1}{2} \begin{vmatrix} a^r & b^r \\ a^s & b^s \end{vmatrix}. \qquad (1\cdot3)$$

If the linear complex L be given symbolically by

$$\pi_A^2 = (\pi_0 A_0 + \pi_1 A_1 + \ldots + \pi_n A_n)^2 = 0 \qquad (1\cdot4)$$

where the A's are complex symbols so that $A_r A_s = -A_s A_r$, the $[2k-1]$'s cutting L in null-variant complexes belong to the linear complex Γ_k given symbolically* by

$$\pi^2_{A_1} \pi^2_{A_2} \ldots \pi^2_{A_k} = 0, \qquad (1\cdot5)$$

where π^{2k} are the complex co-ordinates of a $[2k-1]$ and $A_1^2, A_2^2 \ldots A_k^2$ are complex symbols equivalent to A^2. The covariant $T_k(x)$ is obtained by substituting in Γ_k, the co-ordinates of the osculating $[2k-1]$ of R_n. The highest degree of the parameter x, which occurs in the above co-ordinates is $2k\,(n-2k+1)$ and this occurs only in the co-ordinate $\pi^{2k}_{n, n-1, \ldots, n-2k+1}$. The coefficient of the above co-ordinate in Γ_k, which is then the coefficient of the leading term in $T_k(x)$ is

$$\left(A_1^2 A_2^2 \ldots A_k^2\right)_{n, n-1, \ldots, n-2k+1}$$

i. e.

$$\begin{vmatrix} A_{1,n} & A_{1\, n-1} & \ldots & A_{1, n-2k+1} \\ A_{1,n} & A_{1, n-1} & \ldots & A_{1, n-2k+1} \\ A_{2,n} & A_{2, n-1} & \ldots & A_{2, n-2k+1} \\ A_{2,n} & A_{2, n-1} & \ldots & A_{2, n-2k+1} \\ \cdot & \cdot & \ldots & \cdot \\ \cdot & \cdot & \ldots & \cdot \\ A_{k,n} & A_{k, n-1} & \ldots & A_{k, n-2k+1} \\ A_{k,n} & A_{k, n-1} & \ldots & A_{k, n-2k+1} \end{vmatrix}. \qquad (1\cdot6)$$

Since $\begin{vmatrix} A_{r_1} & A_{r_2} \\ A_{r_1} & A_{r_2} \end{vmatrix} = 2\, A_{r_1} A_{r_2} = \begin{vmatrix} a^{r_1} & b^{r_1} \\ a^{r_2} & b^{r_2} \end{vmatrix}$

* Weitzenböck, *Complex-symbolik*, (1908) pp. 166-72.

(1·6) is equal to the following determinant in ordinary symbols,

$$\begin{vmatrix} a_1^n & a_1^{n-1} & \dots & a_1^{n-2k+1} \\ b_1^n & b_1^{n-1} & \dots & b_1^{n-2k+1} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ a_k^n & a_k^{n-1} & \dots & a_k^{n-2k+1} \\ b_k^n & b_k^{n-1} & \dots & b_k^{n-2k+1} \end{vmatrix}. \tag{1·7}$$

It is obvious that if two of the symbols $a_1, b_1, \ldots, a_k, b_k$ are identical, (1·7) vanishes identically. Hence (1·7) has as factors, all the bracket factors

$$(a_1 b_1)\ (a_1 a_2) \ldots (a_1 a_k)\ (a_1 b_k)\ \ldots\ (a_k b_k) \tag{1·8}$$

obtained by taking any two of the symbols $a_1, b_1, \ldots, a_k, b_k$. Further the least index of any of the symbols in (1·7) is $(n-2k+1)$ so that

$$a_1^{n-2k+1}\ b_1^{n-2k+1}\ \ldots\ a_k^{n-2k+1}\ b_k^{n-2k+1}, \tag{1·9}$$

is also a factor of (1·7). In the product of (1·8) and (1·9) we have the requisite degree namely n, in each of the symbols and hence (1·7) is, but for a numerical factor, equal to $(a_1 b_1)\ (a_1 a_2) \ldots (a_1 b_k) \ldots (a_k b_k)\ a_1^{n-2k+1} \ldots b_k^{n-2k+1}$. This is then the coefficient of the leading term in $T_k(x)$ and it is obvious from its form that it is a semi-invariant, for linear transformations of the cogradient variables x and y in $a_x^n\, b_y^n$. Thus we have

$$T_k(x) = (a_1 b_1)\ (a_1 a_2) \ldots (a_1 b_k) \ldots (a_k b_k)\ a_{1,x}^{n-2k+1} \ldots b_{k,x}^{n-2k+1}.$$

If we take the linear complex L in $[n]$ determined, with reference to a rational norm curve R_n by a skew-symmetric double-binary form $a_x^n\, b_y^n$, the covariant $T_k(x)$, giving parametrically the points on R_n the osculating $[2k-1]$'s at which cut L in null-variant complexes is symbolically

$$(a_1 b_1)\ (a_1 a_2) \ldots (a_1 b_k)\ \ldots\ (a_k b_k)\ a_{1,x}^{n-2k+1}\ b_{1,x}^{n-2k+1} \ldots a_{k,x}^{n-2k+1}\ b_{k,x}^{n-2k+1}$$

where $a_1, a_2, \ldots, a_k$ and $b_1, b_2 \ldots, b_k$ are symbols equivalent to a and b respectively.

ON SOME RELATIONS IN MATHIEU FUNCTIONS

By M. ZIA-UD-DIN, M.A., University College, Swansea, Wales

1. Recurrence formulae connecting

$$ce'_n\, i\,\xi,\; ce_n\, i\,\xi \text{ with } ce_{n-1}\, i\,\xi \text{ and } ce_{n+1}\, i\,\xi,$$

have been constructed by Prof. Whittaker*.

In this note relations connecting

(1) $Se'_n\, i\,\xi$, $Se_n\, i\,\xi$ with $ce_{n-1}\, i\,\xi$ and $ce_{n+1}\, i\,\xi$

(2) $ce'_n\, i\,\xi$, $ce_n\, i\,\xi$ with $Se_{n-1}\, i\,\xi$ and $Se_{n+1}\, i\,\xi$

have been obtained.

2. The typical fundamental solution of the partial differential equation

$$\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}+V=0 \qquad (A)$$

in terms of elliptic coordinates (ξ, η) defined by

$$x+i\,y=h\cosh(\xi+i\,\eta)$$

is

$$V=Se_n\, i\,\xi\; Se_n\,\eta.$$

If V is a solution of the equation (A) then $\partial V/\partial y$ is also a solution, and since

$$\frac{\partial}{\partial y}=\frac{1}{h(\cosh^2\xi-\cos^2\eta)}\left(\cosh\xi\sin\eta\frac{\partial}{\partial\xi}+\sinh\xi\cos\eta\frac{\partial}{\partial\eta}\right)$$

therefore the expression

$$\frac{1}{\cosh^2\xi-\cos^2\eta}(i\cosh\xi\sin\eta Se'_n i\; Se_n\eta+\sinh\xi\cos\eta\; Se_n i\,\xi Se'_n\eta)$$

is a solution of (A) and it can be expressed as a sum or series of the fundamental solutions.

Considering evenness etc., the equation thus obtained will be of the form

$$\frac{1}{\cosh^2\xi-\cos^2\eta}(i\cosh\xi\sin\eta\; Se'_n i\,\xi\; Se_n\eta+\sinh\xi\cos\eta\; Se_n i\,\xi\; Se'_n\eta)$$

* E. T. Whittaker, *Journal L.M.S.* 4, (1929), 88-96.

$$=\sum_{\rho} a_\rho \, ce_\rho i\xi \, ce_\rho \eta, \tag{B}$$

where ρ takes only values which differ from n by an odd number.

Multiply (B) by $ce_{n-1}\eta$ and integrate with respect to η between $-\pi$ and π.

Considering the orthogonal properties

$$\int_{-\pi}^{\pi} ce_m \eta \, ce_l \eta \, d\eta = 0 \quad (m \neq l),$$

all the terms on the right hand will vanish except that for which $\rho = n-1$.

Hence we obtain

$$i \cosh \xi \, Se'_n i\xi \int_{-\pi}^{\pi} \frac{ce_{n-1}\eta \, Se_n \eta \sin \eta \, d\eta}{\cosh^2 \xi - \cos^2 \eta}$$

$$+ \sinh \xi \, Se_n i\xi \int_{-\pi}^{\pi} \frac{\cos \eta \, Se'_n \eta \, ce_{n-1}\eta \, d\eta}{\cosh^2 \xi - \cos^2 \eta}$$

$$= a_{n-1} \, ce_{n-1} i\xi \int_{-\pi}^{\pi} ce^2_{n-1} \eta \, d\eta,$$

or
$$i \bar{\varpi}_n(\xi) \, Se'_n i\xi + \bar{\sigma}_n(\xi) \, Se_n i\xi = \bar{\mu}_n \, ce_{n-1} i\xi. \tag{2·1}$$

Similarly multiplying (B) by ce_{n+1} and integrating, we have

$$i \bar{\rho}_n(\xi) \, Se'_n i\xi + \bar{\lambda}_n(\xi) \, Se_n i\xi = \bar{\nu}_n \, ce_{n+1} i\xi, \tag{2·2}$$

where $\bar{\mu}_n$ and $\bar{\nu}_n$ are constants, and

$$\bar{\varpi}_n(\xi) = \frac{\cosh \xi}{2\pi} \int_{-\pi}^{\pi} \frac{ce_{n-1}\eta \, Se_n \eta \sin \eta \, d\eta}{\cosh^2 \xi - \cos^2 \eta},$$

$$\bar{\sigma}_n(\xi) = \frac{\sinh \xi}{2\pi} \int_{-\pi}^{\pi} \frac{ce_{n-1}\eta \, Se'_n \eta \cos \eta \, d\eta}{\cosh^2 \xi - \cos^2 \eta},$$

$$\bar{\rho}_n(\xi) = \frac{\cosh \xi}{2\pi} \int_{-\pi}^{\pi} \frac{ce_{n+1}\eta \, Se_n \eta \sin \eta \, d\eta}{\cosh^2 \xi - \cos^2 \eta},$$

$$\bar{\lambda}_n(\xi) = \frac{\sinh \xi}{2\pi} \int_{-\pi}^{\pi} \frac{ce_{n+1}\eta \, Se'_n \eta \cos \eta \, d\eta}{\cosh^2 \xi - \cos^2 \eta}.$$

3. Following the above method consider the solution $V = ce_n \times i\xi ce_n \eta$ of (A) with $\partial/\partial y$, then we obtain the relations connecting

$ce'_n i\,\xi$, $ce_n i\,\xi$ with $Se_{n-1} i\,\xi$ and $Se_{n+1} i\,\xi$.

$$i\,\bar{\varpi}_n(\xi)\; ce'_n i\,\xi + \bar{\sigma}_n(\xi)\; ce_n i\,\xi = \bar{\mu}_n\, Se_{n-1} i\,\xi, \tag{3·1}$$

$$i\,\bar{\rho}_n(\xi)\; ce'_n i\,\xi + \bar{\lambda}_n(\xi)\; ce_n i\,\xi = \bar{\nu}_n\, Se_{n+1} i\,\xi, \tag{3·2}$$

where $\bar{\mu}_n$, $\bar{\nu}_n$ are constants,

$$\bar{\varpi}_n(\xi) = \frac{\cosh\xi}{2\pi}\int_{-\pi}^{\pi} \frac{Se_{n-1}\eta\; ce_n\eta\; \sin\eta\; d\eta}{\cosh^2\xi - \cos^2\eta},$$

and similar values for $\bar{\sigma}_n(\xi)$, $\bar{\rho}_n(\xi)$ and $\bar{\lambda}_n(\xi)$.

4. Whittaker has developed the properties of the corresponding coefficients ϖ_n, σ_n, ρ_n, λ_n, for recurrence formulæ.

A similar investigation for our coefficients appears difficult, as there is no obvious symmetry in the coefficients.

Still we shall obtain by actual calculation the values of $\bar{\varpi}_1$, $\bar{\sigma}_1$, $\bar{\rho}_1$ and $\bar{\lambda}_1$.

For the calculation the following integrals* will be used.

$$\frac{\sinh\xi}{2\pi}\int_{-\pi}^{\pi} \frac{\cos\eta\,\cos(2p+1)\,\eta}{\cosh^2\xi - \cos^2\eta}\,d\eta = e^{-(2p+1)\xi}. \tag{4·1}$$

$$\frac{\cosh\xi}{2\pi}\int_{-\pi}^{\pi} \frac{\sin\eta\,\sin(2p+1)\,\eta}{\cosh^2\xi - \cos^2\eta}\,d\eta = e^{-(2p+1)\xi}. \tag{4·2}$$

when the real part of ξ is zero.

Now $$\bar{\varpi}_1 = \frac{\cosh\xi}{2\pi}\int_{-\pi}^{\pi} \frac{\sin\eta\; ce_0\eta\; Se_1\eta\; d\eta}{\cosh^2\xi - \cos^2\eta}$$

$$= \frac{\cosh\xi}{2\pi}\int_{-\pi}^{\pi} \frac{\sin\eta\, d\eta}{\cosh^2\xi - \cos^2\eta}[1 + (4q - 28q^3 + \ldots)\cos 2\eta + (2q^2 - \ldots)$$
$$\times \cos 4\eta + \ldots\ldots][\sin\eta + q\sin 3\eta - q^2(\tfrac{1}{3}\sin 5\eta + \sin 3\eta) + \ldots\ldots]$$

*loc. cit. p. 92.

$$=\frac{\cosh\xi}{2\pi}\int_{-\pi}^{\pi}\frac{\sin\eta\, d\eta}{\cosh^2\xi-\cos^2\eta}\Big\{\sin\eta+(4q-28q^3+..)\frac{\sin 3\eta-\sin\eta}{2}$$
$$+q\sin 3\eta+q^2(\tfrac{1}{3}\sin 5\eta+\sin 3\eta)+\ldots\Big\}.$$

Applying (4·2) we get

$$\bar{\varpi}_1=e^{-\xi}[1-(2q-14q^3+..)+..]+e^{-3\xi}[3q+q^2-14q^3+..]$$
$$+e^{-5\xi}[\tfrac{1}{3}q^2+..]+\ldots$$

In the same way making use of (4·1) and (4·2) we shall have

$$\bar{\sigma}_1=e^{-\xi}[1+2q-14q^3+..]+e^{-3\xi}[5q+3q^2-14q^3+..]$$
$$+e^{-5\xi}[\tfrac{5}{3}q^2+..]+\ldots,$$

$$\bar{\rho}_1=e^{-\xi}(-\tfrac{1}{2}-2q+..)+e^{-3\xi}(\tfrac{1}{2}-2q^2+..)+\ldots,$$

$$\bar{\lambda}_1=e^{-\xi}(\tfrac{1}{2}-2q+..)+e^{-3\xi}(\tfrac{1}{2}-6q^2+..)+\ldots\quad.$$

Similar analysis will apply to the calculation of other coefficients.

AN EXTENSION OF THE DETERMINANT-CONCEPT BASED ON GROUP-CHARACTERS

By R. VAIDYANATHASWAMY, M.A., D.Sc.

The notion of determinant was extended in a satisfactory and general manner to arrays with many suffixes by Rice* and Lecat, and independently by the present writer†. The first condition for a satisfactory extension is the recognition of what I have called *character* of a suffix, or more explicitly, of the fact that each suffix of an array whose determinant is to be defined, may be assigned either of two characters, which, following a happy nomenclature of Rice may be called the *signant* and *non-signant* character, respectively. If

$$|a(m_1, m_2, .., m_r)| \qquad (m_1, .., m_r = 1, 2, ..., n)$$

be an r—dimensional array of the nth order, Rice's theory assigns a signant or non-signant character to the $(r-1)$ indices other than m_1, and defines the determinant of the array as

$$\sum \pm a(1, m_{21}, .., m_{r1})\, a(2, m_{22}, .., m_{r2}) \, .. a(n, m_{2n}, .., m_{rn}),$$

where the summation is for all permutations $(m_{i1} m_{i2} .. m_{in})$ of $1, 2, .., n (i=2, 3, .., r)$; the ambiguous sign is defined to be that of the product $\Pi\epsilon_i$, extended over those values of i other than 1 for which m_i is a signant index, where $\epsilon_i = +1$ or -1 according as $(m_{i1} m_{i2} .. m_{in})$ is an even or odd permutation of $(1, 2, .., n)$. In the present writer's theory on the other hand, characters were assigned to all the r indices without exception, and the determinant was defined to be

$$\frac{1}{n!} \sum \pm a(m_{11}, m_{21}, .., m_{r1}) \ldots a(m_{1n}, m_{2n}, .., m_{rn}),$$

the ambiguous sign being defined as before, but by means of all the indices without exception. In this theory it turns out that the determinant thus defined vanishes identically, unless the total number of indices with the signant character is even; from the point of view of this result, Rice's definition appears as a special case, in which the originally unspecified character of the first

* "P-way determinants with an application to transvectants", *Am. Jour. of Mathematics*, Vol. 40 (1918).

† "On mixed determinants", *Proc. Roy. Soc. of Edinburgh*, 1925.

index is so adjusted as to make the total number of signant indices even, and thereby lead to a determinant which does not vanish identically. The symmetry and generality of the latter definition as compared with Rice's, would appear to make it the more acceptable one*.

In the present paper the concept of character is extended, and a general kind of determinant is defined and shown to possess the typical properties which characterise the 'mixed determinants'—that is, the determinants whose indices have either the signant or the non-signant character. Though these new determinants may not be of much importance in practical algebra, they have a profound theoretical interest as they serve to exhibit the concept of 'character' in its starkest generality.

A similar extension is attempted in regard to the notion of 'symmetry' of a determinant; all problems concerning such symmetrical determinants lead ultimately to problems related to certain kinds of rectangular arrangements of n elements, about which little seems to be known.

I. Characters of an Abelian group†

Any numerical function $\chi(A)$ of the elements $A, B, \ldots,$ of an Abelian group is called a character, if $\chi(A) \neq 0$ for any element A, and $\chi(AB) = \chi(A)\chi(B)$ for any two elements A, B of the group. From this it follows that $\chi(E) = 1$ for the identity-element E of the group, and that $\chi(A)$ is an rth root of unity, if r is the order of A. If the Abelian group S be of order n, and is generated by the ν elements A_i of order a_i $(i=1, 2, \ldots, \nu)$, so that $n = a_1 a_2 \ldots a_\nu$, it follows that we can choose $\chi(A_i)$ to be a particular a_i—th root, ω_i, of unity, for $i = 1, 2, \ldots, \nu$, and that then the values of χ for all elements of the group are determined. It follows that there are precisely n characters of the Abelian group, which we may denote by $\chi_1 \chi_2 \ldots \chi_n$. The character which takes the value 1 for every element of the group is called the *principal character*, and may be denoted by χ_1. The product $\chi\chi'$ of two characters χ, χ' is defined to be the character whose value for any element is the product of the values of χ and χ' for that element. It follows easily that the n characters form an Abelian group under multiplication as thus defined, the principal character playing the role of the identity-

* *Cf.* Lecat, "Le determinant superior qu'est il exactement". etc. *Revue generale des Sciences*, 1929.

† For paras I and II of this paper, reference may be made to Weber, *Lehrbuch der Algebra* Vol. II, pp. 134, 196, or to Burnside, *Theory of Groups*.

element. It is easy to see that this Abelian group is simply isomorphic with our original group S.

The following relation between the n elements of the group and the n characters is of importance.

THEOREM I. *If χ is a fixed character and A runs through all the elements of the group, the sum $\Sigma\chi(A)$ has the value 0, except when χ is the principal character in which case it has the value n; if however A is a fixed element and χ runs through all the n characters, the same sum has the value 0, unless A is the identity-element of the group, in which case it has the value n.*

For if χ takes the value ω_i for the generating element A_i of the group ($i=1, 2, .., \nu$), the sum $\Sigma\chi(A)$ extended over all elements A is equal to

$$\prod_i (1+\omega_i+\omega_i^2+..+\omega_i^{a_i-1})..$$

If any ω_i is different from unity, the corresponding factor in this product vanishes; if all the ω_i's are unity, the product has the value $a_1a_2...a_\nu=n$. This proves the first part. If $A=A_1^{p_1}A_2^{p_2}...A_\nu^{p_\nu}$ is a fixed element, and χ runs over all the characters, the sum is equal to

$$\prod_i \left(\Sigma\,\omega_i^{p_i}\right),$$

where the Σ refers to all possible a_i-th roots ω_i of unity. The sum $\Sigma\omega_i^{p_i}$ is however equal to a_i or 0, according as p_i is or is not equal to 0 mod (a_i). This proves the second part.

II. ABELIAN CHARACTERS OF A GROUP

We define an Abelian character χ of any group S to be a numerical function of the elements $s_1s_2...$, of the group, satisfying the conditions, $\chi(ss')=\chi(s)\chi(s')$ for any two elements s, s' of S, and $\chi(e)\neq 0$, where e is the identity-element of S. It follows at once that $\chi(e)$ must be equal to 1. We proceed now to prove that any Abelian character χ of S may be regarded as a character of an Abelian group isomorphic with S.

Let P be the subset of elements of S for which the given Abelian character takes the value unity. It is clear that P is a sub-group of S; for if $\chi(s_1)=\chi(s_2)=1$ for two elements s_1, s_2 of S, it follows from the definition of Abelian character that $\chi(s_1s_2)=1$ and $\chi(s_1^{-1})=1$. Thus the product of any two elements of P, as well as the inverse of any element of P belongs to P, so that P is a subgroup of S. Also P is a self-conjugate subgroup; for if s is any element of S which does not belong to P, the elements sP are all the elements of S for which χ has the value $\chi(s)$. For since $\chi(st)=\chi(s)\chi(t)$, $\chi(st)=\chi(s)$ implies $\chi(t)=1$ or t belongs to P. The same reasoning shews that the set Ps has the same property. It follows that the sets Ps, sP are identical for any element s of S, or in other words, P is a self-conjugate subgroup. Also, if P, s_1P, $s_2P, \ldots$, be the distinct classes into which the elements of S fall in respect of the subgroup P, the above proof shews that the character χ takes the same value for elements belonging to the same class, and different values for elements belonging to different classes. Thus χ is a function of these classes, rather than of the individual elements of S. Now since P is a self-conjugate subgroup, these classes combine among themselves by multiplication according to the rule $s_1P.\ s_2P=s_1s_2PP=s_1s_2P$, and are in fact the elements of a group, namely the factor-group S/P. The set $s_1s_2P=s_1P.\ s_2P$ consists of all the elements of S for which the character χ takes the value $\chi(s_1s_2)=\chi(s_1)\,\chi(s_2)$. But evidently the set $s_2s_1P=s_2P.\ s_1P$ has the same property. Hence the two sets $s_1P.\ s_2P$, $s_2P.\ s_1P$ are identical, or the multiplication of the classes is commutative. In other words the factor-group S/P is Abelian, and it is clear from the above that χ may be any character of this Abelian group. Thus the Abelian characters of a group are the characters of Abelian groups isomorphic with it.

To determine all the Abelian characters of the group S, we must determine all the self-conjugate subgroups whose factor-group is Abelian. If P be a self-conjugate group with this property, the sets s_1s_2P, s_2s_1P are identical for any two elements s_1, s_2 of S, and therefore every *commutator* $s_1^{-1}s_2^{-1}s_1s_2$ of S belongs to P. Therefore P contains the commutator subgroup C—namely, the subgroup generated by all the commutators of S. Also, the commutator group C is itself a self-conjugate subgroup whose factor-group is Abelian. For let $s_1^{-1}s_2^{-1}s_1s_2=c$, $s_1^{-1}t^{-1}s_1t=c_1$ be two commutators of S; the commutator c_2 of s_1 and s_2t is then

$$c_2=s_1^{-1}t^{-1}s_2^{-1}s_1s_2t=c_1t^{-1}ct.$$

Hence $t^{-1}ct=c_1^{-1}c_2$ is an element of the commutator group; evidently the same statement will hold when c is replaced by a product of commutators *i.e.* by an element of the commutator group. This proves that the commutator group C is self-conjugate. Also, if s_1C, s_2C be two of the classes into which elements of S fall in respect of the self-conjugate subgroup C, the two product-classes $s_1C.\, s_2C = s_1s_2C$, and $s_2C.\, s_1C = s_2s_1C$ must be identical, since $s_1^{-1}s_2^{-1}s_1s_2$ being a commutator belongs to C. Hence the factor-group S/C is Abelian. Also since any self-conjugate subgroup P whose factor-group S/P is Abelian, contains C, it follows that S/P is a subgroup of S/C. Thus S/C is the *largest* Abelian group isomorphic with S, in the sense that any Abelian group isomorphic with S must be abstractly identical either with S/C or with one of its subgroups. It follows that the Abelian characters of S are identical with the characters of the Abelian group S/C.

Let now S be a group of order n, and C its commutator subgroup of order t. We may prove results which are the analogues of Theorem I for Abelian groups.

THEOREM II (a) *If χ is any Abelian character of S the sum $\Sigma\chi(s)$ extended over all the elements s of S is equal to n or 0, according as χ is or is not the principal character.*

(b) *If s is a fixed element of S, the sum $\Sigma\chi(s)$ extended over the n/t Abelian characters is equal to n/t or 0, according as s does or does not belong to the commutator subgroup.*

For let $C_1 = C, C_2, \ldots$, be the n/t classes into which the elements of S fall in respect of the subgroup C. Since these classes are the elements of the factor-group S/C, it follows from Theorem I that $\sum_i \chi(C_i) = n/t$ or 0 according as χ is or is not the principal character. Since $\Sigma\chi(s) = t \sum_i \chi(C_i)$ the first part follows. Again, if C_i be the class containing the fixed element s, it follows from Theorem I that $\Sigma\chi(C_i)$ summed for all the characters χ is equal to n/t or 0, according as C_i is or is not the same as C_1. Since $\Sigma\chi(C_i) = \Sigma\chi(s)$, it follows that the latter sum is equal to n/t or 0, according as s does or does not belong to the commutator subgroup.

THEOREM III. *The symmetric group on n symbols has only two Abelian characters, namely the principal (or non-signant) character, and the alternating (or signant) character. The former takes the value $+1$ for all permutations, the latter takes the value $+1$ for even, and -1 for odd permutations.*

For let $a_1a_2\ldots a_n$ be the n symbols. The cyclic substitution

$$(a_1a_2a_3) = (a_1a_2)(a_1a_3).$$

Hence the cyclic substitution $(a_1a_3a_2)$ can be put in the form

$$\begin{aligned}(a_1a_3a_2) &= (a_1a_2a_3)(a_1a_2a_3)\\ &= (a_1a_2)(a_1a_3)(a_1a_2)(a_1a_3)\end{aligned}$$

=a commutator of the symmetric group, since $(a_1 a_2)^{-1} = (a_1 a_2)$.

Thus every cyclic substitution on three of the n symbols is a commutator. It is easy to see that all such cyclic substitutions generate the alternating group; for $(a_1 a_2)(a_1 a_3) = (a_1 a_2 a_3)$, $(a_1 a_2)(a_3 a_4) = (a_1 a_2 a_3)(a_3 a_1 a_4)$ and therefore every even permutation can be expressed in terms of cyclic permutations on three of the n symbols. Thus the alternating group is the commutator subgroup and the symmetric group has only two Abelian characters as stated.

This fact is closely related to the determinant concept. From the definition to be given of the determinant, it will be seen that the equality of status of these two Abelian characters renders inevitable the extension of the determinant concept by Rice and others through the introduction of non-signant characters.

III. The definition of the generalised determinant

If each of the r indices $m_1 m_2 \ldots m_r$ ranges from 1 to n, the n^r elements

$$a(m_1, m_2, \ldots, m_r)$$

can be arranged in the form of a hyper-cube matrix in space of r dimensions; they form then *an r—dimensional matrix of order n.* The subset of elements of this matrix, for which k of the indices—say $m_1, m_2, \ldots, m_k$—have fixed values, form a $(r-k)$—dimensional matrix of order n, which we call the *subsection* of the given matrix corresponding to the given fixed values of $m_1 m_2 \ldots m_k$. The subsections corresponding to a single index are *prime sections* of the matrix.

We can associate with this matrix a single number, called *the generalised determinant of the matrix,* provided we are given a permutation-group G_ν of order ν on the n symbols $1\ 2\ 3 \ldots n$, and we associate with each of the r indices of the matrix an Abelian character of G_ν. The character thus associated with, or assigned to the index m_i may be denoted by χ_i; the r characters $\chi_1, \ldots, \chi_r$ need not of course be distinct. The generalised determinant $|a|$ of the matrix is now defined as

$$\frac{1}{\nu}\sum \chi_1(m_{11}, m_{12}, \ldots, m_{1n})\chi_2(m_{21}, m_{22}, \ldots, m_{2n})$$
$$\ldots a(m_{11}, m_{21}, \ldots, m_{r1}) a(m_{12}, m_{22}, \ldots, m_{r2}) \ldots$$
$$a(m_{1n}, m_{2n}, \ldots, m_{rn}),$$

where $(m_{11}, m_{12}, \ldots, m_{1n}) \ldots (m_{r1}, m_{r2}, \ldots, m_{rn})$ run independently over all permutations of G_ν and $\chi_i(m_{i1}, m_{i2}, \ldots, m_{in})$ is the value of the Abelian character χ_i for the permutation $(m_{i1}, m_{i2}, \ldots, m_{in})$ of G_ν. This expression for the determinant is a homogenous polynomial of order n in the elements a of the matrix, with ν^r terms; any term in the expression may be specified by writing down in rectangular form the indices of the n elements which occur in the term, namely,

$$\left|\begin{matrix} m_{11} & m_{21} & \ldots & m_{r1} \\ m_{12} & m_{22} & \ldots & m_{r2} \\ \vdots & & & \\ m_{1n} & m_{2n} & \ldots & m_{rn} \end{matrix}\right|$$

By our definition the r columns represent permutations of G_ν, and the numerical coefficient of this term is the product of the values of the character χ_i associated with the ith index for the permutation represented by the ith column, for $i=1, 2, \ldots, r$.

In the case in which G_ν is the symmetric group on n symbols, of order $\nu=n!$, we have seen that it has only two Abelian characters, the signant and the non-signant, and our definition reduces to the usual definition of determinant with indices possessing either of two characters, the signant or the non-signant. We shall call the determinants of this kind which arise when the reference group G_ν is the symmetric group, *special determinants.*

IV. FUNDAMENTAL PROPERTIES

THEOREM IV. *If the prime sections of the matrix corresponding to any index m_i are subjected to any permutation ω of G_ν, the value of the generalised determinant is thereby multiplied by $\chi_i(\omega^{-1})$.*

For in the numerical factor of the general term with the indicial matrix

$$\left|\begin{matrix} m_{11} & m_{21} & \ldots & m_{r1} \\ \vdots & & & \\ m_{1n} & m_{2n} & \ldots & m_{rn} \end{matrix}\right|$$

in the new determinant, we would have instead of $\chi_i(m_{i1}, m_{i2}, \ldots, m_{in})$, $\chi_i\{\omega^{-1}\times(m_{i1}, m_{i2}, \ldots, m_{in})\}$, since the permutation (belonging to G_ν) which carries the actual sequence of the prime sections corresponding to m_i in the new matrix, into $(m_{i1}, m_{i2}, \ldots, m_{in})$ is not $(m_{i1}, m_{i2}, \ldots, m_{in})$, but $\omega^{-1}\times(m_{i1}, m_{i2}, \ldots, m_{in})$. Thus the value of the determinant is multiplied by $\chi_i(\omega^{-1})$.

For the case of the special determinants, this theorem becomes the known result, that *the interchange of two prime sections of the matrix corresponding to an index m_i does or does not change the sign of the determinant, according as m_i is a signant or non-signant index.*

THEOREM V. *The generalised determinant vanishes identically unless the product $\chi_1\chi_2\ldots\chi_r$ of the characters of all its r indices is the principal character.*

For, if in the indicial matrix

$$\begin{vmatrix} m_{11} \; m_{21} \ldots m_{r1} \\ m_{12} \; m_{22} \ldots m_{r2} \\ \vdots \\ m_{1n} \; m_{2n} \ldots m_{rn} \end{vmatrix}$$

of the general term in the defining expansion of the determinant, we subject the n rows to any permutation ω of G_ν, we get another indicial matrix which corresponds to the same term. Thus every term occurs ν times in the defining expansion. If we write $\chi_1\chi_2\ldots\chi_r=\chi$, the total coefficient of $\prod_{i=1}^{n} a(m_{1i}, m_{2i}, \ldots, m_{ri})$ in the defining expansion is therefore

$$\prod_j \chi_j(m_{j1}, m_{j2} \ldots, m_{jn}) \left\{ \Sigma\chi(\omega) \right\},$$

where the summation is for all permutations ω of G_ν. By Theorem II, the sum $\Sigma\chi(\omega)$ vanishes, and hence the determinant vanishes identically, if χ is not the principal character.

For the special determinants, this theorem reduces to the result, that *a special determinant vanishes identically unless the total number of its signant indices is even.*

Another consequence of the theorem is that *a determinant of one dimension vanishes identically, unless its index has the principal character.* When the index has the principal character, it is easy to see from the definition, that the value of the determinant is the product of its elements.

V. Development by Linkage of Indices

The general term in the defining expansion of the determinant may be denoted by the corresponding indicial matrix:

$$\begin{vmatrix} m_{11} \; m_{21} \ldots m_{r1} \\ m_{12} \; m_{22} \ldots m_{r2} \\ \vdots \\ m_{1n} \; m_{2n} \ldots m_{rn} \end{vmatrix}$$

where the columns represent permutations of G_ν. Following Rice, we may call the permutation $(m_{i1}m_{i2}\ldots m_{in})$ the *locant of the index m_i for the term in question.* Denoting by l_i the locant of m_i for the general term, we may write the general term in the abbreviated form $a(l_1, l_2, \ldots, l_r)$; the numerical coefficient of this term is then $\chi_1(l_1)\chi_2(l_2)\ldots\chi_r(l_r)$. The definition of the determinant is then:

$$|a| = \frac{1}{\nu}\sum \chi_1(l_1)\chi_2(l_2)\ldots\chi_r(l_r)a(l_1, l_2, \ldots, l_r),$$

where in the summation $l_1, l_2, \ldots, l_r$ run independently over all the permutations of G_ν. Consider now any two indices, say m_1, m_2; since their locants l_1, l_2 for the general term are both permutations of G_ν, we can write $l_2 = l_1 . \Omega$, where Ω is a permutation of G_ν. We can now divide the ν^r terms of the determinant into ν groups, such that Ω is the same permutation of G_ν in each group, while it varies from group to group. It is clear that m_2 no longer functions as an index within any of these groups; for in the group specified by the value, say ω, of the linkage-permutation Ω, the locant l_2 of the index m_2 in any term, is determined from the locant l_1 of m_1 in the same term, by the equation $l_2 = l_1\omega$. We can shew easily that each of these groups differs only by a numerical factor from the expansion of a determinant of order n and $r-1$ dimensions (with the same reference-group G_ν). For, the division into groups may be exhibited by writing the determinant in the form

$$|a| = \frac{1}{\nu}\sum \chi_1(l_1)\chi_2(l_2)\ldots\chi_r(l_r)a(l_1, l_2, \ldots, l_r)$$

$$= \frac{1}{\nu}\sum_{\Omega}\sum_{l_1, l_3, \ldots, l_r} \chi_1(l_1)\chi_2(l_1\Omega)\ldots\chi_r(l_r)a(l_1, l_1\Omega, \ldots, l_r).$$

The group specified by the value ω of Ω is therefore

$$\frac{1}{\nu}\sum_{l_1, l_3, \ldots, l_r} \chi_1(l_1)\chi_2(l_1\omega)\chi_3(l_3)\ldots a(l_1, l_1\omega, \ldots, l_r).$$

$$= \frac{1}{\nu}\chi_2(\omega)\sum \chi_{12}(l_1)\chi_3(l_3)\ldots a(l_1, l_1\omega, \ldots, l_r),$$

where χ_{12} is the product of the characters χ_1, χ_2. By writing

$$a(l_1, l_1\omega, l_3, \ldots, l_r) = b(l_1, l_3, \ldots, l_r),$$

it becomes clear that this group of terms is $\chi_2(\omega)$ times the expansion of a determinant with the $(r-1)$ indices $m_1, m_3, \ldots m_r$, in which m_1 has the character χ_{12} and the remaining indices have the same character as in the original determinant. Thus by linking the index m_2 to the index m_1, we are able to express the r-dimensional determinant $|a|$ as a linear combination of $\nu(r-1)$—dimensional determinants of the same order and reference-group G_ν, in the form

$$|a| = \Sigma \chi_2(\omega)\Delta_\omega,$$

where in the sum on the right, the linkage permutation ω runs over the ν permutations of G_ν, and in each determinant Δ_ω the $(r-1)$ indices other than m_2 occur, the indices $m_3, m_1, \ldots, m_r$ having the same character in Δ_ω as in the original determinant, while the character of m_1 in Δ_ω is the product χ_{12} of the characters of m_1 and m_2 in the original determinant.

This process of development by linkage may evidently be applied to several indices simultaneously. Thus, by linking $m_2, m_3, \ldots, m_k$ with m_1, we could express the determinant $|a|$ as a linear combination of ν^{k-1} $(r-k+1)$—dimensional determinants of the same order and reference-group, in the form:

$$|a| = \sum \chi_2(\omega_2)\chi_3(\omega_3)\ldots\chi_k(\omega_k)\Delta(\omega_2, \omega_3, \ldots, \omega_k),$$

where on the right, the $k-1$ linkage permutation $\omega_2\omega_3\ldots\omega_k$ run independently over all the permutations of G_ν, and each Δ being a determinant with the indices m_1, $m_{k+1}, \ldots, m_r$, the indices $m_{k+1}, \ldots, m_r$ having the same character in Δ as in $|a|$, while the character of m_1 in Δ is the product of $\chi_1\chi_2\ldots\chi_k$.

In particular, if we develop $|a|$ by linking all but one of the indices with that one—say m_1, we thereby express it as a linear combination of ν^{r-1} one-dimensional determinants, in each of which the single index has the character $\chi_1\chi_2\ldots\chi_r$. If we replace here each of these one-dimensional determinants by its expansion, we would come back to the defining expansion of the determinant $|a|$. Since a one-dimensional determinant vanishes identically unless its character is the principal character, we have here a confirmation of the fact that the r-dimensional determinant $|a|$ vanishes identically unless the product of the characters of all its indices is the principal character.

VI. Product of Determinants

In my previous paper*, three stages were described in forming the product in determinant form, of two special determinants of the same order; namely, product by *conjunction*, product by *identification* or *element-multiplication*, and product by *contraction*

* "On Mixed Determinants." *loc cit.*

or *file-multiplication.* We shall now shew that the first two of these apply *mutatis mutandis* to the generalised determinant. The last is peculiar to the special determinant, that is, to the case in which the reference group G_ν is the symmetric group on n symbols.

Given two determinants $|a(m_1, m_2, \ldots, m_r)|$ and $|b(\mu_1, \mu_2, \ldots, \mu_s)|$, of the same order n, and the same reference group G_ν, we see immediately by multiplying together their defining expansions, that

$$|a| \times |b| = \frac{1}{\nu} |c(m_1, \ldots, m_r, \mu_1, \ldots, \mu_s)|,$$

where $c(m_1, \ldots, m_r, \mu_1, \ldots, \mu_s) = a(m_1, \ldots, m_r) \times b(\mu_1, \ldots, \mu_s)$, and the indices have the same character in the determinant on the right, as on the left. We call $\frac{1}{\nu}|c|$ the *product by conjunction* of the determinants $|a|$ and $|b|$.

On account of the special form of the elements c the determinant $|c|$ can be simplified, and will thereby lead us to the *product by identification.* In our abbreviated notation the determinant $|c|$ can be written in the form:

$$|c| = \frac{1}{\nu} \sum \chi_1(l_1) \ldots \chi_r(l_r) \chi'_1(\lambda_1) \ldots \chi'_s(\lambda_s) a(l_1, \ldots, l_r)\ b(\lambda_1, \ldots, \lambda_s),$$

where the l's and λ's are locants of the indices m and μ, and χ and χ' their characters. If we develop the determinant by linking up an index μ with an index m—say μ_1 with m_1, the group of terms corresponding to the linkage permutation ω is

$$\frac{1}{\nu} \chi'_1(\omega) \sum \chi_1 \chi'_1(l_1) \chi_2(l_2) \ldots \chi_r(l_r) \chi'_2(\lambda_2) \ldots \chi'_s(\lambda_s) \\ \times a(l_1, \ldots, l_r)\ b(l_1\omega, \lambda_2, \ldots, \lambda_s).$$

To shew that this group of terms remains the same when the linkage permutation ω varies, we shall shew that this is identical with the group corresponding to the identical linkage permutation, namely

$$\frac{1}{\nu} \sum \chi_1 \chi'_1(l_1) \chi_2(l_2) \ldots \chi_r(l_r) \chi'_2(\lambda_2) \ldots \chi'_s(\lambda_s) \\ \times a(l_1, \ldots, l_r)\ b(l_1, \lambda_2, \ldots, \lambda_s).$$

For in this expression each $b(l_1, \lambda_2, \ldots, \lambda_s)$ considered as a product of n elements b is identical $b(l_1\omega, \lambda_2\omega, \ldots, \lambda_s\omega)$. Now the product

$$a(l_1, i_2, \ldots, l_r)\ b(l_1\omega, \lambda_2\omega, \ldots, \lambda_s\omega)$$

occurs in the group of terms corresponding to the linkage permutation ω with the coefficient

$$\frac{1}{\nu}\chi'_1(\omega)\chi_1\chi'_1(l_1)\chi_2(l_2)\ldots\chi_r(l_r)\chi'_2(\lambda_2\omega)\ldots\chi'_s(\lambda_s\omega)$$

$$=\frac{1}{\nu}\chi'_1\chi'_2\ldots\chi'_s(\omega)\ \chi_1\chi'_1(l_1)\chi_2(l_2)\ldots\chi_r(l_r)\chi'_2(\lambda_2)\ldots\chi'_s(\lambda_s).$$

If the determinant $|b|$ does not vanish identically, the product of its s characters must be the principal character (Theorem V) and therefore, for any ω

$$\chi'_1\chi'_2\ldots\chi'_s(\omega)=1.$$

Thus the above coefficient is the same as the coefficient of the same product in the group of terms corresponding to the identical linkage permutation. Thus we see that if the determinant $|b|$ does not vanish identically, the ν groups of terms in the development by linkage of $|c|$ are equal to one another. The group of terms corresponding to the identical linkage permutation is the expansion of the determinant,

$$|d(m_1,\ldots, m_r,\mu_2,\ldots,\mu_s)|,$$

where $d(m_1,\ldots, m_r, \mu_2,\ldots, \mu_s)=a(m_1,\ldots, m_r)b(m_1, \mu_2,\ldots,\mu_s)$.

Accordingly $|c|$ is ν times the determinant $|d|$, and therefore

$$|a|\times|b|=\frac{1}{\nu}|c|=|d|.$$

It is clear from our reasoning that this equation is true provided one at least of the determinants $|a|$, $|b|$ does not vanish identically. We refer to $|d|$ as *the product by identification* of $|a|$ and $|b|$.

VII. Extension of the notion of symmetry

There may exist a permutation ω on r symbols, other than the identical permutation, which when performed on the r indices $(m_1, m_2,\ldots, m_r)$ of the general element $a(m_1, m_2,\ldots, m_r)$ of the determinant, results in the multiplication of the element by a definite constant k, for all values of $m_1,\ldots, m_r$. We call ω a *symmetry permutation* of the determinant. It is clear that the totality of symmetry permutations constitutes a permutation group T on the r indices, which we call the *symmetry-group* of the determinant. Further it is evident that the constant multiplier k associated with any symmetry-permutation ω is the value for the permutation ω of a definite Abelian character σ of the symmetry-group T; we call σ the *symmetry-character* of the determinant.

We shall say that the determinant possesses ***regular*** symmetry, when all the symmetry-permutations interchange among themselves the indices possessing the same character; *i.e.* when no symmetry permutation carries an index into another possessing a different character. In the contrary case, the symmetry is *irregular.*

THEOREM VI. *A determinant of order n possessing regular symmetry, vanishes identically, if the nth power of its symmetry-character* σ *is not the principal character.*

For it is obvious that the interchange of two indices possessing the same character does not alter the determinant. Hence when the determinant possesses regular symmetry, the determinant will be left unaltered when a symmetry permutation ω is performed on the indices of all the element. But since the effect of the symmetry permutation ω is to multiply each element by $\sigma(\omega)$, the determinant is thereby multiplied by $\{\sigma(\omega)\}^n = \sigma^n(\omega)$. Hence if $\sigma^n(\omega) \neq 1$ for each symmetry permutation ω—that is to say, if σ^n is not the principal character, the determinant must vanish identically.

For the special determinants, this theorem gives the known result that *a determinant which possesses skew-symmetry in two indices which are both signant or both non-signant, vanishes identically unless its order is even.*

For the special determinants a property relating to irregular symmetry is known*, namely, *a determinant possessing skew-symmetry in two indices, one of which is signant, and the other non-signant, vanishes identically whether its order is even or odd.*

The corresponding property relating to irregular symmetry of the generalised determinant appears difficult to discover.

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* "On Mixed Determinants" *loc. cit,*

ON THE RATIONAL NORM CURVE

By R. VAIDYANATHASWAMY AND B. RAMAMURTI

1, In the paper 'On the rational norm curve II'*, a one-to-one correspondence has been effected between quadric envelopes Q in a space S_n of n dimensions and linear line complexes L in a space S_{n+1} of $(n+1)$ dimensions in the following manner. Any simplex inscribed in a rational norm curve R_n in S_n may be specified by the binary $(n+1)$—ic, a_x^{n+1}, giving the parameters of its vertices. Let the simplex be made to correspond to the point of intersection of the osculating primes of a rational norm curve R_{n+1} in S_{n+1} at the points on it, given parametrically by a_x^{n+1}. It is well known that any quadric inpolar to R_n determines a pencil of simplexes inscribed in R_n and self-polar with respect to the quadric, and thereby a line in S_{n+1}. Then the lines corresponding to inpolar quadrics outpolar to a quadric envelope Q belong to a linear complex L. The object of this note is to study the above correspondence between Q and L, analytically and to prove that in this correspondence, the several spectra of quadric envelopes, and the several spectra of linear complexes, associated as in § 2, with R_n and R_{n+1} respectively, correspond to each other.

2. Any algebraic (n, n) correspondence between two variables x and y may be given by equating to zero an algebraic (n, n) form $a_x^n b_y^n$. If the form is symmetric in x and y, the above correspondence determines, and is determined by, apolarity with respect to a quadric locus P or a quadric envelope Q according as x and y correspond parametrically to the points on R_n or the osculating primes at these points. If however the double-binary form be the product of $(xy)^{2i}$ and apolar form $a_x^{n-2i} a_y^{n-2i}$, let the quadric locus P (the quadric envelope Q) be said to belong to the spectrum $P_i(Q_i)$. Since, from Gordon's Theorem any symmetric form $a_x^n\, b_y^n$ can be developed in a series of terms of

* Vaidyanathaswamy, "On the rational norm curve II" *Journ. Lond. Math. Soc.* (1932) 52-57.

the type $(xy)^{2i}a_x^{n-2i}a_y^{n-2i}$, it follows that any quadric $P(Q)$ can be expressed as a linear combination of quadrics taken one uniquely from each spectrum $P_i(Q_i)$.

Similarly if a $(n+1, n+1)$ form is skew-symmetric in x and y, it can be represented symbolically by $(xy)a_x^n b_y^n$. The correspondence defined by it, determines, and is determined by, a null-system defined by a linear line complex L or a linear S_{n-1}—complex M, in a S_{n+1} according as x and y correspond parametrically to the points of a R_{n+1} or the osculating primes to R_{n+1} at these points. If the double-binary form be of the type $(xy)^{2i+1}a_x^{n-2i}$ $\times a_y^{n-2i}$, let the associated linear line complex L (linear S_{n-1} complex M) be said to belong to the spectrum $L_i(M_i)$. Then any linear complex $L(M)$ can be expressed as a linear combination of linear complexes, taken one uniquely from each spectrum $L_i(M_i)$.

3. We shall now obtain the equation of the correspondence determined on R_n by a quadric locus P, which as an envelope is inpolar to R_n.

Let the pencil of simplexes inscribed in R_n and self-polar with respect to the quadric be given parametrically by $a_x^{n+1}+kb_x^{n+1}=0$. The points x and y on R_n are conjugate with respect to P if x and y belong to the same member of the above pencil. Hence

$$a_x^{n+1}+kb_x^{n+1}=0$$

$$a_y^{n+1}+kb_y^{n+1}=0.$$

Hence $$a_x^{n+1}b_y^{n+1}-a_y^{n+1}b_x^{n+1}=0.$$

Developing each term in Gordon's series in powers of (xy), we have, on removing the factor (xy), the equation of the correspondence determined by P to be

$$\sum_{r=0}^{k} C_{n+1,\,2r+1}\left\{(ab)^{2r+1}a_x^{n-2r}b_x^{n-2r}\right\}_{y^{n-2r}}(xy)^{2r}=0 \quad (3\cdot1)$$

where $$C_{n+1,\,i}=\frac{({}^{n+1}C_i)^2}{2(n+1)+1-{}^iC_i}$$

and $k=\frac{n}{2}$ or $\frac{n-1}{2}$ according as n is even or odd and the subscript of the bracket indicates the $(n-2r)$-th polar with respect to y. A quadric envelope Q and a quadric locus P determining respectively the correspondences on R_n given by

$$\sum_{r=0}^{k} C_{n,\,2r}\left(A_{r,\,x}^{2n-4r} \right)_{y^{n-2r}} (xy)^{2r}=0 \qquad (3{\cdot}2)$$

and
$$\sum_{r=0}^{k} C_{n,\,2r}\left(B_{r,\,x}^{2n-4r} \right)_{y^{n-2r}} (xy)^{2r}=0 \qquad (3{\cdot}3)$$

are apolar† if
$$\sum_{r=0}^{k} C_{n,\,2r}\left(A_r B_r \right)^{2n-4r}=0. \qquad (3{\cdot}4)$$

Hence the quadric locus (3·1) which as an envelope is inpolar to R_n is apolar to the quadric envelope (3·2) if

$$\sum_{r=0}^{k} C_{n+1,\,2r+1}(ab)^{2r+1}(aA_r)^{n-2r}(bA_r)^{n-2r}=0. \qquad (3{\cdot}5)$$

If binary forms of order $(n+1)$ are represented by the points in a S_{n+1}, with reference to a rational norm curve R_{n+1}, the coefficients of any such form, are linear functions of the co-ordinates of the corresponding point. Hence (3·5) is the equation of the linear complex L of lines corresponding to the pencils $a_x^{n+1}+kb_x^{n+1}$ determined by inpolar quadrics outpolar to the quadric envelope (3·2). The correspondence, which the linear complex L determines on R_{n+1} is then given by

$$\sum_{r=0}^{k} C_{n+1,\,2r+1}(xy)^{2r+1}\left(A_x^{2n-4r} \right)_{y^{n-2r}}=0. \qquad (3{\cdot}6)$$

4. Let us now obtain the linear complexes in S_{n+1} corresponding to the quadrics of the spectra Q_i.

If we take the inpolar quadric, determining as an envelope the polar correspondence $a_x^n a_y^n=0$, the corresponding linear complex L is given by $(xy)\, a_x^n a_y^n=0$. Hence L belongs to the spectrum L_0. Thus we have

To the quadric Q_0, inpolar to R_n in S_n, and touching the osculating primes of R_n at the points given parametrically by a_x^{2n}, corresponds the linear complex L_0 outpolar to R_{n+1} in S_{n+1}, and containing the tangents of R_{n+1} at the points given parametrically by a_x^{2n}. (4·1)

† Waelsch, "Uber binare formen und die correlationen Mehrdimensionaler Raume". *Monatshefte fur Mathematik and Physik.* (1895) § 17.

Similarly, if we take the quadric envelope Q_i determining on R_n the correspondence $(xy)^{2i} a_x^{n-2i} a_y^{n-2i}=0$ the corresponding linear complex L determines on R_{n+1} the correspondence $(xy)^{2i+1} \times a_x^{n-2i} a_y^{n-2i}$ Interpreting the spectra geometrically, we have

To a quadric envelope Q_i containing all the osculating S_{n-i}'s of R_n and the osculating S_{n-i-1}'s at the points a_x^{2n-4i}, and apolar to the quadric loci containing the osculating S_r's $(r \geqslant i)$ corresponds the linear complex L_i in S_{n+1} containing all the osculating S_i's of R_{n+1} and the osculating S_{i+1}'s at the points a_x^{2n-4i} and apolar to all the linear S_{n-1}—complexes containing the osculating S_{n-r-1}'s of R_{n+1} $(r \geqslant i)$. (4·2)

In (4·1) and (4·2) the double-binary equation of L is that of Q multiplied by (xy). In general it is not so. But as is obvious from (3·2) and (3·5), if Q is a linear combination of members taken one from each of the spectra $Q_{r_1}, Q_{r_2} \ldots Q_{r_k}$, the corresponding linear complex L is a linear but different combination of the corresponding linear complexes in the spectra $L_{r_1}, L_{r_2} \ldots L_{r_k}$.

MINIMAL SURFACES WITH REFERENCE TO THE LINE OF STRICTION OF THE ASYMPTOTIC LINES

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In a paper published in the *Mathematical Gazette* Vol. 13 (1926), Dr. C. E. Weatherburn has shown that a singly infinite family of curves on a surface possesses a line of striction—the locus of points where the geodesic curvature of the orthogonal trajectories of the family of curves vanishes. The object of the present paper is to obtain some minimal surfaces with reference to the line of striction of the asymptotic lines.

When the asymptotic lines on a minimal surface are taken as parametric curves, the line-element of the surface can be thrown in the form $ds^2=\rho(du^2+dv^2)$ where ρ denotes the absolute value of either of the principal curvatures of the surface. This function ρ satisfies Gauss's characteristic equation

$$\rho(\rho_{11}+\rho_{22})=2\rho+\rho_1^2+\rho_2^2$$

where as usual the suffixes 1, 2 denote differentiation with respect to u and v respectively. The geodesic curvatures of the curves v=constant and u=constant are respectively

$$-\rho_2/2\rho^{\frac{3}{2}} \text{ and } \rho_1/2\rho^{\frac{3}{2}}.$$

Hence the line of striction of the family u=constant is given by $\rho_2=0$ and the line of striction of the family v=constant is given by $\rho_1=0$.

§ 1. *Minimal surface for which the line of one system of asymptotic lines is a member of the system.*

The condition requires that ρ_2 and ρ_1 should respectively be functions of v and u alone *i.e.* $\rho_{12}=0$. Hence

$$\rho=f(u)+\phi(v).$$

Substituting this value of ρ in the characteristic equation we have

$$(f+\phi)(f''+\phi'')-f'^2-\phi'^2-2(f+\phi)=0,$$

or $$(ff''-f'^2-2f)+(\phi\phi''-\phi'^2-2\phi)+f\phi''+f''\phi=0. \qquad (1\cdot1)$$

Hence, because f and ϕ are functions of u and v alone, supposing f'' and ϕ'' are neither zero nor constants we have

$$ff''-f'^2-2f=0 \tag{1·2}$$

$$\phi\phi''-\phi'^2-2\phi=0 \tag{1·3}$$

$$f\phi''+f''\phi=0. \tag{1·4}$$

From (1·4) we have $\dfrac{f''}{f}=-\dfrac{\phi''}{\phi}$ and each of these must be a constant equal to ω^2, say.

Therefore $f=A \sinh \omega(u+\alpha)$ where A and α are constants
$=A \sinh \omega u$ by a suitable choice of the parameter.

Similarly $\phi=B \sin \omega v$.

Substituting the value of f in (1·2) we have

$$A^2\omega^2(\sinh^2\omega u-\cosh^2\omega u)-2A \sinh \omega u=0$$

which is impossible.

Similarly the value of ϕ is also impossible.

Hence either $f''=\phi''=0$ or else both are constants.

If f'' and ϕ'' are both zero f and ϕ are respectively αu and βv by suitable choice of the parameters, α and β being arbitrary constants. But these values also do not satisfy (1·2) and (1·3).

Hence $f''=\text{constant}=2\alpha$ say,

hence $f=\alpha(u+\beta')^2+\beta$, β, β' being constants
$=\alpha u^2+\beta$ by suitable choice of u.

Similarly $\phi=\gamma v^2+\delta$.

Substituting these values in (1·1) we get

$$(\alpha u^2+\beta)\alpha-2\alpha^2u^2-(\alpha u^2+\beta)+\gamma(\gamma v^2+\delta)-2\gamma^2v^2-(\gamma v^2+\delta)$$
$$+\gamma(\alpha u^2+\beta)+\alpha(\gamma v^2+\delta)\equiv 0.$$

Equating to zero the coefficients of u^2, v^2 and the constant we have

$$\alpha^2+\alpha-\alpha\gamma=0 \tag{1·5}$$

$$\gamma^2+\gamma-\alpha\gamma=0 \tag{1·6}$$

$$\alpha\beta-\beta+\gamma\delta-\delta+\beta\gamma+\alpha\delta=0. \tag{1·7}$$

From (1·5) and (1·6) we get

$$\alpha=0 \quad\text{or}\quad \alpha-\gamma+1=0,$$
$$\text{or } \gamma=0 \quad\text{or}\quad \gamma-\alpha+1=0.$$

$\alpha=\gamma=0$ gives $\beta+\delta=0$ *i.e.* there is no such surface. If $\alpha=0$, we get $\gamma=-1$ and $\beta+\delta=0$. Again the surface does not exist.

Similarly for all other combinations we get impossible values.

The only other possible solution is that either f or ϕ is zero (it is immaterial which). Supposing ϕ to be zero $ds^2=f(u)\times(du^2+dv^2)$. The surface therefore is one of revolution and hence is a right helicoid. The line of striction of the generators is an orthogonal trajectory of the generators. But the line of striction of the other system of asymptotic lines does not exist. Hence we have the

THEOREM. *The line of striction of the asymptotic lines of a minimal surface is an orthogonal trajectory of the asymptotic lines if and only if the surface is a right helicoid.*

§ 2. *The line of striction of one system of asymptotic lines is an oblique trajectory.*

The line of striction of the family $v=$const is given by $\rho_1=0$. It cuts the curve $v=$constant at an angle $\pi-\tan^{-1}\rho_{11}/\rho_{12}$. Hence the condition that it may be an oblique trajectory is that $\rho_{11}=c\rho_{12}$ where $c=\tan\alpha$ and is neither zero nor infinite. Hence we get

$$\rho=f(u+cv)+\phi(v).$$

Substituting in the characteristic equation we have

$$(f+\phi)(\sec^2\alpha f''+\phi'')=2(f+\phi)+\sec^2\alpha f'^2+\phi'^2+2cf'\phi'. \quad (2\cdot1)$$

As in the preceding section, it can be shown that the equation becomes impossible unless either f or ϕ is zero.

If f is zero we get the trivial solution—a right helicoid. If however $\phi=0$ we have,

$$ff''-f'^2-2\cos^2\alpha f=0. \quad (2\cdot2)$$

Writing $$f=\psi^2$$

we have $$\psi\psi''-\psi'^2=\cos^2\alpha. \quad (2\cdot3)$$

Differentiating, $$\psi\psi'''-\psi'\psi''=0$$

hence, $$\psi''=\omega^2\psi$$ ω being a constant.

Therefore $$\psi=A\cosh\omega(u+cv)$$

by a suitable choice of u, A being an arbitrary constant. Again because

$$\psi\psi''-\psi'^2=\cos^2\alpha,$$

we get $$\omega=\frac{\cos\alpha}{A},$$

or $$\psi=A\cosh\frac{u\cos\alpha+v\sin\alpha}{A}.$$

Hence the line-element of the surface—the asymptotic lines being taken as the parametric curves—can be put in the form

$$ds^2=A^2\cosh^2\frac{u\cos\alpha+v\sin\alpha}{A}\{du^2+dv^2\}$$

Changing the parameters by the relations,

$$u \cos \alpha + v \sin \alpha = U$$

and

$$u \sin \alpha - v \cos \alpha = V$$

we have
$$ds^2 = A^2 \cosh^2 \frac{U}{A}(dU^2 + dV^2).$$

Again putting
$$\xi = \int A \cosh \frac{U}{A} dU$$
$$= \sinh \frac{U}{A}$$

we get
$$ds^2 = d\xi^2 + (A^2 + \xi^2) dV^2.$$

Hence the surface is applicable to a right helicoid, the curves $u \cos \alpha + v \sin \alpha =$ constant— which are also geodesics—and their orthogonal trajectories corresponding to the generators and the orthogonal trajectories of the right helicoid.

The same result is also true if instead of asymptotic lines we write lines of curvature.

The asymptotic lines of the surface above obtained have peculiarly interesting properties analogous to those of the generators of a ruled surface.

(a) The line of striction of both the systems is the same curve $u \cos \alpha + v \sin \alpha = 0$. It cuts the asymptotic lines at a constant angle and is a geodesic—corresponding to Bonnet's Theorem for generators.

Conversely, if the lines of striction of the two systems coincide, $c\rho_1 = \rho_2$ and we obtain $\rho = f(u + cv)$ and hence, etc.

(b) Starting from the point of striction if we measure the angle of rotation θ of the tangent plane as the point of contact moves along the asymptotic line we get

$$\frac{d\theta}{ds} = -\tau = \sqrt{-k} = 1/A^2 \cosh^2 \frac{u \cos \alpha + v \sin \alpha}{A}.$$

Along the asymptotic line $v =$ constant

$$ds = A \cosh \frac{u \cos \alpha + v \sin \alpha}{A} du.$$

Hence
$$\frac{d\theta}{du} = \frac{1}{A} \operatorname{sech} \frac{u \cos \alpha + v \sin \alpha}{A}.$$

Integrating
$$\theta = \frac{1}{A} \int_{u \cos \alpha + v \sin \alpha = 0}^{u} \operatorname{sech} \frac{u \cos \alpha + v \sin \alpha}{A} du$$
$$= \frac{1}{\cos \alpha} \tan^{-1} \left\{ \sinh \frac{u \cos \alpha + v \sin \alpha}{A} \right\}$$

or
$$\tan (\theta \cos \alpha) = \sinh \frac{u \cos \alpha + v \sin \alpha}{A}.$$

Hence as the point of contact moves from one end of the asymptotic line to the other, the tangent plane rotates from $-\frac{\pi}{2}\sec\alpha$ to $+\frac{\pi}{2}\sec\alpha$.

(*c*) Also, if the length of the asymptotic line be measured from the point of striction we have

$$s = A^2 \sec\alpha \sinh\frac{u\cos\alpha + v\sin\alpha}{A}$$

hence
$$\tan(\theta\cos\alpha) = \frac{S}{A^2}\cos\alpha.$$

$$(cf.\ \tan\theta = \frac{u}{\beta}\ \text{for a ruled surface}).$$

(*d*) Again if **a** is the unit tangent to the asymptotic line $v =$ constant we have

$$\operatorname{div}\mathbf{a} = \rho_1/2\rho^{\frac{3}{2}}$$

$$2\frac{\partial}{\partial u}(\operatorname{div}\mathbf{a}) = \frac{\rho_{11}\rho^{\frac{3}{2}} - \frac{3}{2}\rho^{\frac{1}{2}}\rho_1^2}{\rho^3}.$$

The divergence will be stationary *i.e.* the geodesic curvature of the orthogonal trajectory is an extremum where

$$2\rho\rho_{11} - 3\rho_1^2 = 0,$$

or substituting for ρ we get

$$\sinh\phi = \pm 1$$

where
$$\phi = \frac{u\cos\alpha + v\sin\alpha}{A}.$$

Therefore $\tan(\theta\cos\alpha) = \pm 1$

or
$$\theta = \pm\frac{\pi}{4}\sec\alpha.$$

Hence the geodesic curvature of the orthogonal trajectory is stationary where the tangent plane is inclined at an angle $\pm\frac{\pi}{4}\sec\alpha$ to the tangent plane at the point of striction, the corresponding values for a ruled surface being $\pm\frac{\pi}{4}$.

(*e*) Again if the point moves with unit velocity along the asymptotic line, the angular velocity of rotation of the tangent plane is $\sqrt{-k} = \frac{1}{\rho}$. Hence because $ds = \sqrt{\rho}\,du$ the angular acceleration is

$$\frac{1}{\sqrt{\rho}}\frac{\partial}{\partial u}\frac{1}{\rho} = -\frac{\rho_1}{\rho^{\frac{5}{2}}}.$$

hence we find that the angular acceleration is stationary where

$$2\rho\rho_{11}-5\rho_1^2=0$$

i.e. $$\sinh\phi=\pm 1/\sqrt{3}.$$

Hence $$\theta=\pm\frac{\pi}{6}\sec\alpha.$$

Thus we find that the asymptotic lines of the surface obtained above possess properties which are exact analogues of the properties of the generators of a ruled surface. If the line of striction $\rho_1=0$ is also a line of curvature, $\rho_{11}=\pm\rho_{12}$, which gives $\alpha=\pm\frac{\pi}{4}$ and conversely. Other minimal surfaces with other properties of the line of striction are under investigation.

A BRIEF HISTORY OF SELF-RECIPROCAL FUNCTIONS*

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1. By means of Fourier's Cosine-Integral, a function is, subject to certain conditions, capable of being represented as a repeated integral

$$f(x)=\frac{2}{\pi}\int_0^\infty \cos ux\, du \int_0^\infty f(y) \cos uy\, dy$$

in the interval $(0, \infty)$. This may be expressed by the two equations

$$f(x)=\sqrt{\frac{2}{\pi}}\int_0^\infty g(y) \cos xy\, dy, \tag{1·1}$$

$$g(x)=\sqrt{\frac{2}{\pi}}\int_0^\infty f(y) \cos xy\, dy. \tag{1·2}$$

When these equations hold good, the functions $f(x)$ and $g(x)$ are said to be Fourier cosine-transforms of each other. Cauchy† described them as reciprocal functions‡. The integrals in the two equations may exist in the ordinary Cauchy sense or in some generalised sense, for example, in the mean-square sense.

In the same way, Hankel's formula

$$f(x)=\int_0^\infty uJ_\nu(ux)du\int_0^\infty y\, J_\nu(uy)\, f(y)dy,$$

where $J_\nu(x)$ is a Bessel Function of order $\nu\geqq-\frac{1}{2}$, may be expressed by the two equations

$$f(x)=\int_0^\infty \sqrt{xy}\, J_\nu(xy)\, g(y)dy,$$

$$g(x)=\int_0^\infty \sqrt{xy}\, J_\nu(xy)\, f(y)dy.$$

*This was the introductory chapter of my Ph. D. Thesis on 'Self-Reciprocal Functions' submitted to the University of Liverpool in October, 1933.

†See Cauchy (12).

‡'Of the first kind', those 'of the second kind' being Sine-transforms of each other.

Two functions so connected are said to be J_ν transforms of each other.

For the more important results of the theory of transforms, reference may be made to the work of Hobson, Plancherel and Titchmarsh*. A detailed discussion of the theory is outside the scope of the present work. Here we are concerned only with the special case when $f(x)=g(x)$, that is,

$$f(x)=\int_0^\infty \sqrt{xy}\, J_\nu(xy)\, f(y)dy. \tag{1·3}$$

In this case, $f(x)$ is said to be self-reciprocal for J_ν transforms.

When in equations (1·1) and (1·2), $f(x)=g(x)$, that is,

$$f(x)=\sqrt{\frac{2}{\pi}}\int_0^\infty f(y)\cos xy\, dy, \tag{1·4}$$

$f(x)$ is said to be self-reciprocal for cosine-transforms.

In the same way, a function satisfying the equation

$$f(x)=\sqrt{\frac{2}{\pi}}\int_0^\infty f(y)\sin xy\, dy, \tag{1·5}$$

is said to be self-reciprocal for sine-transforms.

Obviously enough (1·4) and (1·5) are particular cases of (1·3) for $\nu=-\frac{1}{2}$ and $\nu=\frac{1}{2}$ respectively.

A function satisfying (1·3) with the sign of one side changed, is said to be skew-reciprocal for J_ν transforms. Functions that are skew-reciprocal for cosine and sine-transforms may be defined in the same way.

Following Hardy and Titchmarsh, I will say that a function is R_ν if it is self-reciprocal for J_ν transforms and it is $-R_\nu$ if it is skew-reciprocal for J_ν transforms; for $R_{\frac{1}{2}}$ and $R_{-\frac{1}{2}}$ I will write R_s and R_c respectively.

2. The subject of self-reciprocal functions, in its present form, appears to be of recent origin. But these functions have been occurring in the work of various mathematicians for a long time though the authors may not always have been conscious of their existence as such. Thus, as early as 1811, Laplace† gave the formulæ

$$\int_0^\infty \frac{\cos x}{\sqrt{x}}\,dx=\sqrt{\frac{\pi}{2}}=\int_0^\infty \frac{\sin x}{\sqrt{x}}\,dx.$$

* See Hobson (21), Plancherel (43, 44, 45), Titchmarsh (53, 54, 55).

† Laplace (24) 365.

Obviously, these formulæ can be put into the form

$$\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\cos xy}{\sqrt{y}}\,dy = \frac{1}{\sqrt{x}} = \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\sin xy}{\sqrt{y}}\,dy^*, \qquad (2\cdot1)$$

which implies that the function $\frac{1}{\sqrt{x}}$ is R_c and R_s.

The same formulae have, since then, occurred in the work of various other mathematicians†.

On page 366 of the same work of Laplace of 1811 occurs the formula

$$\int_0^\infty e^{-a^2x^2}\cos rx\,dx = \frac{\sqrt{\pi}}{2a}e^{-\frac{r^2}{4a^2}}. \qquad (2\cdot2)$$

As can be easily seen by putting $a=\frac{1}{\sqrt{2}}$, this formula implies that the function

$$e^{-\frac{1}{2}x^2} \qquad (2\cdot3)$$

is R_c.

This formula too has occurred in the work of various other mathematicians‡.

In 1815 Cauchy|| also gave the formulae

$$\int_0^\infty \cos\omega^2\cos 2m\omega\,d\omega = \frac{\sqrt{\pi}}{2\sqrt{2}}(\cos m^2 + \sin m^2), \qquad (2\cdot4)$$

$$\int_0^\infty \sin\omega^2\cos 2m\omega\,d\omega = \frac{\sqrt{\pi}}{2\sqrt{2}}(\cos m^2 - \sin m^2). \qquad (2\cdot5)$$

Multiplying (2·5) by i and subtracting from (2·4) we get, after a slight change of the variable,

$$\sqrt{\frac{2}{\pi}}\int_0^\infty e^{-i\left(\frac{1}{2}y^2-\frac{\pi}{8}\right)}\cos xy\,dy = e^{i\left(\frac{1}{2}x^2-\frac{\pi}{8}\right)}.$$

* Pringsheim (46) actually puts the result in this form.

† In slightly different forms, they were given in 1815 by Cauchy (11), in 1822 by Fourier (13) 406—7, in 1843 by Boncompagni (8), in 1849 by Oettinger (40) 234—5, in 1875 by Laurent (25), in 1878 by Boussinesq (9) and in 1904 by Nielsen (39).

‡ For example, it was given in 1815 by Cauchy (10) 117, in 1822 by Fourier (13) 434, in 1837 by Kummer (23), in 1849 by Oettinger (40) 229 and in 1854 by Raabe (47).

||Cauchy (10) 118.

Equating real and imaginary parts we deduce that the functions

$$\cos\left(\frac{1}{2}x^2-\frac{\pi}{8}\right) \quad \text{and} \quad \sin\left(\frac{1}{2}x^2-\frac{\pi}{8}\right) \tag{2·6}$$

are R_c and $-R_c$ respectively.

Equations equivalent to (2·4) and (2·5) have also been given in a slightly different form by Fourier* in 1822.

On page 119 of the same work of Cauchy† occur the formulae

$$\int_0^\infty 2\omega \cos \omega^2 \sin 2m\, \omega = \sqrt{\frac{\pi}{2}}\, m\,(\sin m^2 - \cos m^2).$$

$$\int_0^\infty 2\omega \sin \omega^2 \sin 2m\, \omega = \sqrt{\frac{\pi}{8}}\, m\,(\sin m^2 + \cos m^2).$$

Adopting the same procedure as above, it can be shown that these formulae imply that the functions

$$x\cos\left(\frac{1}{2}x^2-\frac{3\pi}{8}\right) \quad \text{and} \quad x\sin\left(\frac{1}{2}x^2-\frac{3\pi}{8}\right) \tag{2·7}$$

are R_s and $-R_s$ respectively.

An interesting example, which is now familiar, may be found in the work of Legendre‡. In 1817 he gave the formula

$$\int_0^\infty \frac{\sin rx\, dx}{e^{2\pi x}-1} = \frac{1}{4}\frac{e^r+1}{e^r-1} - \frac{1}{2r}. \tag{2·8}$$

Changing the variable, re-arranging the terms, and using the well-known result

$$\int_0^\infty \frac{\sin rx}{x}\, dx = \frac{\pi}{2} \qquad (r>0),$$

we can re-write (2·8) as

$$\sqrt{\frac{2}{\pi}}\int_0^\infty \left(\frac{1}{e^{y\sqrt{2\pi}}-1} - \frac{1}{y\sqrt{2\pi}}\right)\sin xy\, dy = \frac{1}{e^{x\sqrt{2\pi}}-1} - \frac{1}{x\sqrt{2\pi}},$$

which shows that the function

$$\frac{1}{e^{x\sqrt{2\pi}}-1} - \frac{1}{x\sqrt{2\pi}} \tag{2·9}$$

is R_s.

* Fourier (13) 479.
† Cauchy (10).
‡ Legendre (26).

A formula equivalent to (2·8) was given in 1823 by Abel*.

In 1868 Weber† proved that

$$\int_0^\infty y^{\nu+1}\, e^{-p^2y^2}\, J_\nu(xy)\,dy = \frac{x^\nu}{(2p^2)^{\nu+1}}\, e^{-\frac{x^2}{4p^2}}.$$

Putting $p=\frac{1}{\sqrt{2}}$ we see that this formula implies that the function

$$x^{\frac{1}{2}+\nu}\, e^{-\frac{1}{2}x^2} \tag{2·10}$$

is R_ν. This function includes (2·3) as a particular case for $\nu=-\frac{1}{2}$.

In 1871 Meyer‡ obtained, by formal differentiation from an equation equivalent to (2·2), the formula

$$\int_0^\infty ye^{-ay^2} \sin 2xy\, dy = \frac{1}{2a}\sqrt{\frac{\pi}{a}}\, xe^{-x^2/a}.$$

By giving 'a' the particular value $\frac{1}{2}$ and putting x for $2x$ we arrive at the formula

$$\sqrt{\frac{2}{\pi}}\int_0^\infty ye^{-\frac{1}{2}y^2} \sin xy\, dy = xe^{-\frac{1}{2}x^2}.$$

This shows that the function $xe^{-\frac{1}{2}x^2}$ which is a particular case of (2·10) for $\nu=\frac{1}{2}$, is R_s.

In 1897 Glaisher‖ proved the formulae

$$\int_0^\infty \frac{\cos at}{\cosh \pi t}\, dt = \frac{1}{2\cosh\frac{a}{2}}\ §,$$

$$\int_0^\infty \frac{\cosh \pi t}{\cosh 2\pi t} \cos at\, dt = \frac{1}{2\sqrt{2}}\, \frac{\cosh\frac{a}{4}}{\cosh\frac{a}{2}},$$

$$\int_0^\infty \frac{\sinh \pi t}{\cosh 2\pi t} \sin at\, dt = \frac{1}{2\sqrt{2}}\, \frac{\sinh\frac{a}{4}}{\cosh\frac{a}{2}},$$

* Abel (1).

† Weber (58).

‡ Meyer (37).

‖ Glaisher (14).

§ A result equivalent to this has been obtained by Hardy (15) in 1902 by a different process.

$$\int_0^\infty \frac{\sinh 2\pi t}{\cosh 3\pi t} \sin at\, dt = \frac{1}{2\sqrt{3}} \frac{\sinh \frac{a}{3}}{\cosh \frac{a}{2}},$$

$$\int_0^\infty \frac{\sinh \pi t}{\sinh 3\pi t} \cos at\, dt = \frac{1}{2\sqrt{3}} \frac{\sinh \frac{a}{6}}{\sinh \frac{a}{2}}.$$

With slight changes of the variables it will be seen that these formulae imply that the functions

$$\frac{1}{\cosh\left(x\sqrt{\frac{\pi}{2}}\right)}, \quad \frac{\cosh\left(\frac{x\sqrt{\pi}}{2}\right)}{\cosh(x\sqrt{\pi})}$$

and
$$\frac{\sinh\left(x\sqrt{\frac{\pi}{6}}\right)}{\sinh\left(x\sqrt{\frac{3\pi}{2}}\right)} = \frac{1}{1+2\cosh\left(x\sqrt{\frac{2\pi}{3}}\right)}$$

are R_c, and the functions

$$\frac{\sinh\left(\frac{x\sqrt{\pi}}{2}\right)}{\cosh(x\sqrt{\pi})}$$

and
$$\frac{\sinh\left(x\sqrt{\frac{2\pi}{3}}\right)}{\cosh\left(x\sqrt{\frac{3\pi}{2}}\right)} = \frac{2\sinh\left(x\sqrt{\frac{\pi}{6}}\right)}{2\cosh\left(x\sqrt{\frac{2\pi}{3}}\right)-1}$$

are R_s*.

All the examples of self-reciprocal functions given in the preceding pages appear simple in form. Hereafter the functions begin to be a bit more complicated. The work of Hardy reveals two self-reciprocal functions involving infinite integrals. Thus in 1904 Hardy† brought out a fairly long note to show that the function

$$\int_x^\infty e^{\frac{1}{2}(x^2-t^2)}\, dt \qquad (2.11)$$

was R_s.

* I have now generalised these results. See Mehrotra (32).
† Hardy (16).

In 1908 Hardy* proved that if

$$\Psi_\alpha(\sqrt{c^2+x^2})=\int_0^\infty t^\alpha e^{-t^2-\frac{c^2+x^2}{t^2}}dt,$$

then

$$\int_0^\infty \Psi_\alpha(\sqrt{c^2+x^2})x^{\nu+1}J_\nu(2\mu x)\,dx=\mu^\nu\frac{c^{\alpha+2\nu+3}}{2}\Psi_{-\alpha-2\nu-4}(c\sqrt{1+\mu^2}).$$

Giving α the particular value $-\nu-2$, making the substitution $x=\frac{t\sqrt{c}}{\sqrt{2}}$ and finally putting x for $\mu\sqrt{2c}$, we get

$$\int_0^\infty \Psi_{-\nu-2}(\sqrt{c(c+\tfrac{1}{2}t^2)})\,t^{\nu+1}J_\nu(xt)\,dt=x^\nu\Psi_{-\nu-2}\left(\sqrt{c(c+\tfrac{1}{2}x^2)}\right).$$

This formula shows that the function

$$x^{\frac{1}{2}+\nu}\Psi_{-\nu-2}(\sqrt{c(c+\tfrac{1}{2}x^2)})=\frac{x^{\frac{1}{2}+\nu}}{c^{\frac{1}{2}+\frac{1}{2}\nu}}\int_0^\infty e^{-c\left(t^2+\frac{1}{t^2}\right)}e^{-\frac{x^2}{2t^2}}t^{-\nu-2}\,dt \tag{2·12}$$

is R_ν.

In 1914 Humbert† proved that

$$\frac{1}{\sqrt{x}}=\lambda\int_0^{\pm\infty} f(xy)\frac{dy}{\sqrt{y}}, \tag{2·13}$$

for a certain value of λ. Putting $f(t)=\sqrt{t}\,J_\nu(t)$ we get

$$\frac{1}{x}=\lambda\int_0^{\pm\infty} J_\nu(xy)\,dy \tag{2·14}$$

for a certain value of λ. This seems to be a particular case of the formula given by Watson‡

$$\int_0^\infty r^{q-n-1}J_n(xr)\,dr=2^{q-n-1}x^{n-q}\frac{\Gamma(\frac{1}{2}q)}{\Gamma(n-\frac{1}{2}q+1)},$$

where $0<R(q)<R(n)+\frac{3}{2}$. Putting $n=\nu$, $q=\nu+1$ in this formula, we get

$$\frac{1}{x}=\int_0^\infty J_\nu(xy)\,dy.$$

* Hardy (17.)

† Humbert (22).

‡ Watson (56) § 13·24 (1). This formula was first obtained by Weber (58) for integral values of ν. It was extended to general values of ν by Sonine (52) in 1880, but the completely general result was given by Schafheitlin (51) in 1887.

This shows that if we take the upper sign in the integral (2·14), $\lambda=1$. This formula which includes (2·1) as particular cases for $\nu=\mp\frac{1}{2}$, shows that the function $\frac{1}{\sqrt{x}}$ is R_ν.

During the same year, Milne* proved that if Weber's functions, of integral order, associated with the parabolic cylinder, were denoted by

$$D_n(x)=(-1)^n e^{\frac{1}{4}x^2}\frac{d^n}{dx^n}(e^{-\frac{1}{4}x^2}),$$

these functions satisfied the integral equations

$$\int_0^\infty D_{2n}(y)\cos(\tfrac{1}{2}xy)\,dy=(-1)^n\sqrt{\pi}\,D_{2n}(x),$$

$$\int_0^\infty D_{2n+1}(y)\sin(\tfrac{1}{2}xy)\,dy=(-1)^n\sqrt{\pi}\,D_{2n+1}(x).\dagger$$

These equations, as a slight change of the variables will show, indicate that the function

$$D_{2n}(x\sqrt{2}) \tag{2·15}$$

is $\pm R_c$ according as n is an even or odd integer, and the function

$$D_{2n+1}(x\sqrt{2}) \tag{2·16}$$

is $\pm R_s$ according as n is an even or odd integer.

In 1919 Ramanujan‡ stated the two formulae

$$\int_0^\infty \frac{\sin nx\,dx}{x+\frac{1}{x+}\frac{2}{x+}\frac{3}{x+}\cdots}=\frac{\sqrt{\frac{\pi}{2}}}{n+\frac{1}{n+}\frac{2}{n+}\frac{3}{n+}\cdots}, \tag{2·17}$$

$$\int_0^\infty \frac{\sin\frac{1}{2}\pi nx\,dx}{x+\frac{1^2}{x+}\frac{2^2}{x+}\frac{3^2}{x+}\cdots}=\frac{1}{n+}\frac{1^2}{n+}\frac{2^2}{n+}\frac{3^2}{n+}\cdots,$$

as questions for solution. With a change of the variable, the latter formula may be thrown into the form

$$\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\sin xy\,dy}{ay+\frac{1^2}{ay+}\frac{2^2}{ay+}\frac{3^2}{ay+}\cdots}=\frac{1}{ax+\frac{1^2}{ax+}\frac{2^2}{ax+}\frac{3^2}{ax+}\cdots}, \tag{2·18}$$

* Milne (38).
† Due in this form to Wilson (59).
‡ Ramanujan (48).

where $a=\sqrt{\frac{2}{\pi}}$. Formulæ (2·17) and (2·18) obviously indicate that the functions

$$\frac{1}{x+}\,\frac{1}{x+}\,\frac{2}{x+}\,\frac{3}{x+}\cdots, \tag{2·19}$$

$$\frac{1}{ax+}\,\frac{1^2}{ax+}\,\frac{2^2}{ax+}\,\frac{3^2}{ax+}\cdots, \tag{2·20}$$

where $a=\sqrt{\frac{2}{\pi}}$, are R_s.

The truth of these formulæ has since been established in 1928 by Rao and Aiyer* and in 1929 by Phillips†. The latter proved them by showing that (2·19) was the same as (2·11); and that

$$\frac{1}{x+}\,\frac{1^2}{x+}\,\frac{2^2}{x+}\,\frac{3^2}{x+}\cdots=\int_0^\infty \frac{e^{-xu}}{\cosh u}\,du,$$

so that (2·20) is only a constant multiple of

$$\int_0^\infty \frac{e^{-xt}\,dt}{\cosh\left(t\sqrt{\frac{\pi}{2}}\right)},$$

which may also be expressed in the form

$$2\left(\frac{1}{x+b}-\frac{1}{x+3b}+\frac{1}{x+5b}-\frac{1}{x+7b}+\cdots\right)^{\ddagger},$$

where $b=\sqrt{\frac{\pi}{2}}$.

Thus we see self-reciprocal functions appearing in the form of integrals, continued fractions and series. An interesting self-reciprocal function in the form of a series was given by Wilson|| in 1924. He proved that if $T_\nu^n(x)$ denoted Sonine's polynomial defined by

$$T_\nu^n(x)=\frac{(-1)^n e^x x^{-\nu}}{n!\,\Gamma(n+\nu+1)}\frac{d^n}{dx^n}\left(e^{-x}x^{n+\nu}\right)$$

$$=\sum_{r=0}^{n}\frac{(-1)^{n-r}x^r}{r!\,(n-r)!\,\Gamma(r+\nu+1)},$$

then the function

$$x^{\frac{1}{2}+\nu}e^{-\frac{1}{2}x^2}\,T_\nu^n(x^2) \tag{2·21}$$

* Rao and Aiyer (50).

† Phillips (41).

‡ *See* Hardy and Titchmarsh (19) 210.

|| Wilson (59).

was $\pm R_\nu$ according as n was an even or odd integer. As shown by Wilson, this function includes (2·15) and (2·16) as particular cases for $\nu=-\frac{1}{2}$ and $\nu=\frac{1}{2}$ respectively.

In 1926 Picard* showed that the function

$$e^{-ax}-\sqrt{\frac{2}{\pi}}\frac{a}{x^2+a^2}$$

was $-R_c$†.

In 1927 Hardy‡ proved that

If $r(n)$ is the number of representations of n as a sum of two squares, and

$$\bar{P}(x)=\sum_{0\leqslant n\leqslant x}{}' \, r(n)-\pi x,$$

where the dash implies that a factor $\frac{1}{2}$ is to be inserted in the last term of the sum when x is an integer, then

$$\int_0^\infty \frac{1}{y}\left\{\bar{P}\left(\frac{y^2}{2\pi}\right)-1\right\}J_2(xy)\,dy=\frac{1}{x^2}\left\{\bar{P}\left(\frac{x^2}{2\pi}\right)-1\right\}.$$

Obviously, this formula implies that the function

$$x^{-\frac{3}{2}}\left\{\bar{P}\left(\frac{x^2}{2\pi}\right)-1\right\}$$

is R_2.

A. L. Dixon has also proved the same formula‖.

A more general, but much less obvious, example of the same type, has been given in 1929 by Wilton§.

In 1930 Bailey‡ proved that the function

$$x^{\frac{1}{2}-\nu}\,e^{\frac{1}{4}x^2}\int_x^\infty t^{2\nu-1}\,e^{-\frac{1}{2}t^2}\,dt, \tag{2·22}$$

which, he showed⁂, could also be written in the form

$$\cfrac{x^{\nu-\frac{1}{2}}}{x+\cfrac{2-2\nu}{x+}\,\cfrac{2}{x+}\,\cfrac{4-2\nu}{x+}\,\cfrac{4}{x+}\cdots}, \tag{2·23}$$

was R_ν. Evidently, this function includes (2·19), or, which is the same thing, (2·11) as a particular case for $\nu=\frac{1}{2}$.

* Picard (**42**).
† *See* Hardy and Titchmarsh (**19**) 197.
‡ Hardy (**18**).
‖ *See* Hardy and Titchmarsh (**19**) 212: Second footnote.
§ Wilton (**60**).
‡ Bailey (**2**).
⁂ Bailey (**3**). *See also* Bailey (**5**).

During the same year, Bailey* gave, among others, two interesting examples of self-reciprocal functions:

(1) If $(p)_\alpha = p(p+1)(p+2)\ldots\ldots(p+\alpha-1)$ and $(p)_0=1$, the function

$$x^{\frac{1}{2}+\nu}\, e^{-\frac{1}{2}x^2} \sum_{r=0}^{n} \frac{(-n)_r\, x^{2r}}{2^r\, r!(1+\frac{1}{2}\nu-\frac{1}{2}n)_r}$$

is $\pm R_\nu$ according as n is an even or odd integer. By giving to n particular values and expanding the series it will be easily seen that this function is of practically the same type as (2·21).

(2) The function

$$x^{\frac{1}{2}+\nu} \int_a^{\frac{1}{a}} t^\nu\, F(t)\, e^{-\frac{1}{2}x^2t^2}\, dt, \tag{2·24}$$

where $F(t)$ is a function satisfying the equation

$$F(t) = F\left(\frac{1}{t}\right),$$

is R_ν†. This seems to have been the first example of a self-reciprocal function which could aim at any generality. It includes (2·22) as a particular case for

$$a=0, \qquad F(t) = \left(t+\frac{1}{t}\right)^{\nu-1}$$

From a formula given by Watson‡, Bailey also deduces the two formulae

$$\int_0^\infty t^{\nu+1}\, J_\nu(xt) \cos\left(\frac{u^2t^2}{2} - \frac{\nu+1}{4}\pi\right) dt = x^\nu\, u^{-2\nu-2} \cos\left(\frac{x^2}{2u^2} - \frac{\nu+1}{4}\pi\right),$$

$$\int_0^\infty t^{\nu+1} J_\nu(xt) \sin\left(\frac{u^2t^2}{2} - \frac{\nu+1}{4}\pi\right) dt = -x^\nu u^{-2\nu-2} \sin\left(\frac{x^2}{2u^2} - \frac{\nu+1}{4}\pi\right).$$

Putting $u=1$ in these formulae we find that the functions

$$x^{\frac{1}{2}+\nu} \cos\left(\frac{x^2}{2} - \frac{\nu+1}{4}\pi\right) \text{ and } x^{\frac{1}{2}+\nu} \sin\left(\frac{x^2}{2} - \frac{\nu+1}{4}\pi\right) \tag{2·25}$$

are R_ν and $-R_\nu$ respectively. These functions include (2·6) and (2·7) as particular cases for $\nu=-\frac{1}{2}$ and $\nu=\frac{1}{2}$ respectively.

* Bailey (4).

† This result has now been generalised by me. *See* Mehrotra (27) 236 and (29). *See also* Bailey (6).

‡ Watson (56) § 16·53.

3. Thus scattered examples of self-reciprocal functions have been occurring in the work of various mathematicians. But, so far, a systematic treatment of the subject was lacking. This deficiency was, to a great extent, made up by the memoir of Hardy and Titchmarsh on 'Self-Reciprocal Functions'† in 1930. They gave two very general solutions of equation (1·3). Before stating their results, it seems necessary to reproduce their definitions of classes of functions $A(\omega, a)$ and $A^*(\omega, a)$.

DEFINITION. 1. *$f(x)$ belongs to $A(\omega, a)$, where $0<\omega \leqslant \pi$, $a<\frac{1}{2}$, if (i) it is an analytic function of $x=re^{i\theta}$, regular in the angle A defined by $r>0$, $|\theta|<\omega$, and (ii) it is $O(|x|^{-a-\delta})$ for small x, and $O(|x|^{a-1+\delta})$ for large x, for every positive δ and uniformly in any angle $|\theta|\leqq\omega-\eta<\omega$.*

DEFINITION 2. *$f(x)$ belongs to $A^*(\omega, a)$. where $0<\omega\leqq\frac{1}{4}\pi$, $0<a<\frac{1}{2}$, if (i) it is an analytic function of $x=re^{i\theta}$, regular in the angle A^* defined by $r>0$, $|\theta| <\omega$, and (ii) it is $O\left(|x|^{-a-\frac{1}{2}-\delta}\right)$ for small x, and $O\left(|x|^{a-\frac{1}{2}+\delta}\right)$ for large x, for every positive δ and uniformly in any angle $|\theta| \leqq\omega-\eta<\omega$.*

Among their more important results may be mentioned the following:

THEOREM 1. *A necessary and sufficient condition that a function $f(x)$ of $A(\omega, a)$ should be $\pm R_\nu$ is that it should be of the form*

$$f(x)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} 2^{\frac{1}{2}s}\,\Gamma(\tfrac{1}{4}+\tfrac{1}{2}\nu+\tfrac{1}{2}s)\Psi(s)x^{-s}\,ds,$$

where $\Psi(s)$ is regular and satisfies

$$\Psi(s)=\pm\Psi(1-s) \tag{3·1}$$

in the strip

$$a<\sigma<1-a; \tag{3·2}$$

$$\Psi(s)=O\left\{e^{(\frac{\pi}{4}-\omega+\eta)|t|}\right\}$$

for every positive η and uniformly in any strip interior to (3·2) and c is any value of σ in (3·2).

This is the first solution of (1·3).

The following theorem which involves the use of Stieltjes Integrals:

THEOREM 2. *Suppose that $f_1(x)$ has bounded variation in any finite interval $(0, X)$ and*

$$f_1(0)=0; \qquad f_1(x)=\tfrac{1}{2}\left\{f_1(x+0)+f_1(x-0)\right\} \quad (x>0).$$

† Hardy and Titchmarsh (19).

Suppose also that (i) the integral

$$\int_1^\infty \frac{|df_1(x)|}{x^2}$$

is finite, (ii) the integral

$$\int_0^\infty x^{s-1}\, df_1(x) = \lim_{\delta\to 0,\, X\to\infty} \int_\delta^X x^{s-1}\, df_1(x)$$

is convergent for $|\sigma-\frac{1}{2}|<\alpha\leqslant\frac{1}{2}$; *and (iii) the integral*

$$\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\sin xy}{y}\, df_1(y) = \sqrt{\frac{2}{\pi}} \lim_{Y\to\infty}\int_0^Y \frac{\sin xy}{y}\, df_1(y)$$

is convergent and equal to $f_1(x)$ *for all positive* x.

Then

$$f_1(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} 2^{\frac{1}{2}s}\,\Gamma(\tfrac{1}{4}+\tfrac{1}{2}\nu+\tfrac{1}{2}s)\,\Psi(s)\,\frac{x^{1-s}}{1-s}\, ds$$

$$= \frac{1}{2\pi i}\lim_{T\to\infty}\int_{c-iT}^{c+iT} 2^{\frac{1}{2}s}\,\Gamma(\tfrac{1}{4}+\tfrac{1}{2}\nu+\tfrac{1}{2}s)\,\Psi(s)\,\frac{x^{1-s}}{1-s}\, ds,$$

where $|c-\frac{1}{2}|<\alpha$ *and* $\Psi(s)$ *is regular and satisfies* (3·1) *(with the upper sign) in* $|c-\frac{1}{2}|<\alpha$.

THEOREM 3. *A necessary and sufficient condition that a function* $f(x)$ *of* $A^*(\omega, a)$ *should be* R_c *is that it should be of the form*

$$f(x) = \frac{x}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{\frac{1}{2}x^2 s}\, s^{-\frac{1}{4}}\,\mu(s)\, ds, \tag{3·3}$$

where c *is any positive number, the integral is the limit of an integral over* $(c-iT, c+iT)$ *and* $\mu(s)$ *has the properties*

(i) $\mu(s)=\mu(\rho e^{i\phi})$ *is an analytic function of* s, *regular in the angle* $B(\omega, a)$ *defined by* $\rho>0$, $|\phi|<\frac{1}{2}\pi+2\omega$;

(ii) $\mu(s)$ *is* $O\left(|s|^{-\frac{1}{2}a-\delta}\right)$ *for small* s, *and* $O\left(|s|^{\frac{1}{2}a+\delta}\right)$ *for large* s, *for every positive* δ *and uniformly in any angle* $|\phi|\leqslant\frac{1}{2}\pi+2\omega-\zeta<\frac{1}{2}\pi+2\omega$;

(iii) $\mu(s)$ *satisfies the equation*

$$\mu(s)=\mu\left(\frac{1}{s}\right)$$

in $B(\omega, a)$.

The formula corresponding to (3·3) for general ν is

$$f(x)=\frac{x^{\frac{1}{2}-\nu}}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{\frac{1}{2}x^2 s}\, s^{-\frac{1}{2}(\nu+1)}\,\mu(s)\,ds.$$

This is the second solution of (1·3)*.

They have also given a list of R_ν functions. Some of these functions have already occurred in the foregoing pages. Among others, mention may be made of

(1) $f(x)=\sqrt{x}\,J_{\frac{1}{2}\nu}(\frac{1}{2}x^2)$;

(2) $f(x)=x^{\frac{1}{2}-\nu}(x^2-b^2)^{\frac{1}{4}(\nu-1)}\,J_{\frac{1}{2}(\nu-1)}(b\sqrt{x^2-b^2})\quad (x>b>0),$

$=0 \qquad (0<x<b).$

To illustrate the theorem involving Stieltjes Integrals given above, they have also given the very interesting example of an R_c function

$$f_1(x)=\frac{x}{\sqrt{2\pi}}-\left[\frac{x}{\sqrt{2\pi}}\right], \qquad f_1(x)=\tfrac{1}{2}, \tag{3·4}$$

according as x is not or is a multiple of $\sqrt{2\pi}$. This function has also been conventionally defined in the alternative form

$$f_1(x)=1.\ \infty \qquad f_1(x)=-(2\pi)^{-\frac{1}{2}}, \tag{3·4}$$

according as x is or is not a positive integral multiple of $\sqrt{2\pi}$.

In their memoir quoted above, Hardy and Titchmarsh were content to determine the various classes of self-reciprocal functions. But in 1931 they took up a new aspect of the problem, namely, how the various classes of self-reciprocal functions were connected with one another. In their paper "Formulæ connecting different classes of self-reciprocal functions"† they have given theorems to show that if $f(x)$ is R_μ, other functions, depending on $f(x)$, can be found such that they are R_ν. For example, if $f(x)$ is R_μ, the function

$$g(x)=x^{\frac{1}{2}(\mu+\nu+1)}\int_0^\infty y^{\frac{1}{2}(\mu+\nu+1)}\,K_{\frac{1}{2}(\mu-\nu)}(xy)\,f(y)\,dy$$

* For other solutions *see* Mehrotra (36). *See also* (35).

† Hardy and Titchmarsh (20).

is R_ν.† Another of their theorems may be stated as follows

If $f(x)$ *is* R_c, *the function*

$$h(x)=\int_0^\infty k(xt)\, f(t)\,dt$$

is R_s *provided that*

$$k(x)=\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} \Gamma(s)\,\lambda(s)\,x^{-s}\,ds,$$

where $\lambda(s)=\lambda(1-s)$‡.

They have given the following examples under the above theorem: For

$$\lambda(s)=1, \qquad \lambda(s)=\frac{\sqrt{3}}{2\sqrt{\pi}}\,\Gamma(s-\tfrac{1}{3})\,\Gamma(\tfrac{2}{3}-s),$$

$$\lambda(s)=\frac{\sqrt{\pi}}{\Gamma(\frac{1}{2}+\frac{1}{2}s)\,\Gamma(1-\frac{1}{2}s)}, \qquad \lambda(s)=\frac{2\sqrt{\pi}}{\Gamma(\frac{1}{2}s)\,\Gamma(\frac{1}{2}-\frac{1}{2}s)},$$

the corresponding values of $k(x)$ are

$$e^{-x}, \quad x^{-\frac{1}{6}}\, e^{\frac{1}{2}x}\, K_{\frac{1}{6}}(\tfrac{1}{2}x), \qquad J_0(x) \text{ and } x\,J_0(x).$$

In particular, putting

$$h(x)=\int_0^\infty e^{-xt}\,df_1(t),$$

where $f_1(x)$ is the function (3·4) we arrive at the familiar R_s function (2·9).

Towards the end of the same year, Watson gave an R_ν function which, with slight changes in the procedure, may be put into the form

$$2\sum_{n=1}^{\infty}\left(nx\sqrt{\frac{\pi}{2}}\right)^{\frac{1}{2}\nu+\frac{1}{4}} K_{\frac{1}{2}\nu+\frac{1}{4}}(nx\sqrt{2\pi})-\frac{\Gamma(\frac{1}{2}\nu+\frac{3}{4})}{x\sqrt{2}}.^{*} \qquad (3\cdot5)$$

† This result has been generalised by me. *See* Mehrotra (**27**) § 8. For similar other theorems *see* Mehrotra (**31, 33, 34**).

‡ For a generalisation of this result *see* Mehrotra (**27**) § 4.

* In fact Watson (57) proves that if

$$F_\nu(z)\equiv\tfrac{1}{2}\Gamma(\nu)+2\sum_{n=1}^{\infty}(\tfrac{1}{2}nz)^\nu K_\nu(nz)$$

For the particular case $\nu=\frac{1}{2}$, this function reduces to a constant multiple of (2·9).

In the year 1932 Bailey* started a new question. What kind of relation, if any, exists between definite integrals involving self-reciprocal functions? This question was suggested to him by a paper of Ramanujan† who showed a particular type of relation to exist between two particular infinite integrals. Bailey has generalised this result, his main theorem being:

If $f(x)$ is R_ν and real, and $-1<\nu<\frac{1}{2}$ and if

$$\phi(n)=\int_0^\infty x^{\nu+\frac{1}{2}} \cos \tfrac{1}{2}nx^2 f(x)\,dx,$$

$$\Psi(n)=\int_0^\infty x^{\nu+\frac{1}{2}} \sin \tfrac{1}{2}nx^2 f(x)\,dx,$$

then
$$\phi(n) \sin \frac{\nu+1}{2}\pi=\frac{1}{n^{\nu+1}}\Psi\left(\frac{1}{n}\right)+\Psi(n)\cos\frac{\nu+1}{2}\pi,$$

$$\Psi(n) \sin \frac{\nu+1}{2}\pi=\frac{1}{n^{\nu+1}}\phi\left(\frac{1}{n}\right)-\phi(n)\cos\frac{\nu+1}{2}\pi‡.$$

$$=\sqrt{\pi}\Gamma(\nu+\tfrac{1}{2})\, z^{2\nu}\left[\frac{1}{z^{2\nu+1}}+2\sum_{n=1}^{\infty}\frac{1}{(z^2+4n^2\pi^2)^{\nu+\frac{1}{2}}}\right],$$

then the function $F_{\frac{1}{2}\nu+\frac{1}{4}}(x\sqrt{2\pi})$ is R_ν. With certain changes in the various steps of the process it can be proved that if

$$F'_\nu(z)\equiv 2\sum_{n=1}^{\infty}(\tfrac{1}{2}nz)^\nu K_\nu(nz)-\frac{\sqrt{\pi}\,\Gamma(\nu+\frac{1}{2})}{z}$$

$$=2\sqrt{\pi}\,\Gamma(\nu+\tfrac{1}{2})z^{2\nu}\sum_{n=1}^{\infty}\frac{1}{(z^2+4n^2\pi^2)^{\nu+\frac{1}{2}}}-\tfrac{1}{2}\Gamma(\nu),$$

then the function $F'_{\frac{1}{2}\nu+\frac{1}{4}}(x\sqrt{2\pi})$, which is the same as (3·5) is R_ν.

* Bailey (7).

† Ramanujan (49).

‡ For a generalisation of this result *see* Mehrotra (28) § 6. *See also* in this connection Mehrotra (30).

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THE RANGE OF SAMPLES TAKEN FROM A RECTANGULAR POPULATION

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The range is defined as the difference in character between the largest and the smallest individuals of a sample. The problem of the range of samples arises as a special case of 'Galton's Difference Problem'(1) first given by Karl Pearson in 1902. It was proved by Pearson that if a random sample of n individuals is taken from a population of N members, which, when N is very large may be taken to obey some law of frequency which is expressed by the Curve $y=N\phi(x)$, $y\delta x$ being the total frequency of individuals with character or organs lying between x and $x+\delta x$, the average between the pth and $(p+1)$th individuals when the sample is arranged in order of magnitude of the character is given by the expression

$$\lambda_{n,p}=\frac{n!}{(n-p)!\,p!}\int_{-\infty}^{\infty} \alpha^{n-p}\,(1-\alpha)^p\,dx.$$

where

$$\alpha=\int_{-\infty}^{x}\phi(x)\,dx.$$

This result was extended by J. O. Irwin to find the average difference between the pth and the qth individuals(2). It was proved that

$$\text{mean }(x_p-x_q)=\int_{-\infty}^{\infty}\left\{I_\alpha(n-q+1,q)-I_\alpha(n-p+1,p)\right\}dx, \quad (1)$$

where $I_x(r,s)$ denotes the ratio of the incomplete Beta Function

$$\int_0^x (1-x)^{r-1}\,x^{s-1}\,dx,$$

to the complete Beta Function

$$\int_0^1 (1-x)^{r-1}\,x^{s-1}\,dx.$$

Now by substituting $p=1$ and $q=n$ in the above we get the mean range. The determination of the higher moments of the range for any general population is a very difficult matter. L. H. C. Tippet obtained expressions for higher moments of range for samples from a 'Normal population'(3).

The object of the present paper is to obtain expressions for the higher moments about the arithmetic mean of the range for samples taken from a rectangular population.

MOMENTS OF THE FREQUENCY DISTRIBUTION OF THE RANGE.

Suppose 0 to w represents the range of the original rectangular population from which samples of size n are taken. Then all the values of the variable between 0 and w are of equal frequency of occurrence in the population. Let x_1 be the character of the first individual and x_n that of the last in a sample of size n; and the intermediate values are $x_2, x_3 \ldots$, where

$$x_1 \leqslant x_2 \leqslant x_3 \ldots \leqslant x_n.$$

Then supposing that the original population is infinite the chance that we have one individual at x_1, one at x_n and $n-2$ between them is

$$p=\frac{n!}{(n-2)!}\left(\frac{dx_1}{w}\right)\left(\frac{dx_n}{w}\right)(x_n-x_1)^{n-2}/w^{n-2}$$

$$=\frac{n!}{(n-2)!}\frac{(x_n-x_1)^{n-2}}{w^n}dx_1\,dx_n. \tag{2}$$

Now x_n may be any where between 0 to w and then x_1 should be between 0 and x_n. Therefore if $\bar{w}$ is the mean range,

$$\bar{w}=\frac{n!}{(n-2)!\,w^n}\int_0^w\int_0^{x_n}(x_n-x_1)^{n-2}(x_n-x_1)\,dx_1\,dx_n,$$

$$=\frac{n-1}{n+1}w. \tag{3}$$

Suppose μ_m is the mth moment of the range taken about the mean then,

$$\mu_m=\frac{n!}{(n-2)!\,w^n}\int_0^w\int_0^{x_n}(x_n-x_1)^{n-2}(x_n-x_1-\bar{w})^m dx_1\,dx_n. \tag{4}$$

Consider the integral

$$I_{n-2,m}=\int_0^{x_n}(x_n-x_1)^{n-2}(x_n-x_1-\bar{w})^m\,dx_1.$$

Integrating by parts, we get,

$$I_{n-2,m}=\left[-(x_n-x_1-\bar{w})^m\frac{(x_n-x_1)^{n-1}}{n-1}\right]_0^{x_n}-\frac{m}{n-1}\int_0^{x_n}(x_n-x_1-\bar{w})^{m-1}$$
$$\times(x_n-x_1)^{n-1}\,dx_1$$

$$=(x_n-\bar{w})^m\frac{x_n^{n-1}}{n-1}-\frac{m}{n-1}I_{n-1,m-1}.$$

Similarly

$$I_{n-1,m-1}=(x_n-\bar{w})^{m-1}\frac{x_n^n}{n}-\frac{m-1}{n}I_{n,m-2}.$$

This process can be continued until finally we get,

$$I_{n-2,m}\equiv(x_n-\bar{w})^m\frac{x_n^{n-1}}{n-1}-\frac{m}{n-1}(x_n-\bar{w})^{m-1}\frac{x_n^n}{n}+\frac{m(m-1)}{(n-1)n}$$
$$\times(x_n-\bar{w})^{m-2}\frac{x_n^{n+1}}{n+1}+\dots\,. \tag{5}$$

Hence

$$\int_0^w\int_0^{x_n}(x_n-x_1)^{n-2}(x_n-x_1-\bar{w})^m\,dx_1\,dx_n=\int_0^w I_{n-2,m}\,dx_n,$$

and

$$\int_0^w(x_n-\bar{w})^m\frac{x_n^{n-1}}{n-1}dx_n=\left[(x_n-\bar{w})^m\frac{x_n^n}{(n-1)n}\right]_0^w-\frac{m}{n-1}$$
$$\times\int_0^w(x_n-\bar{w})^{m-1}\frac{x_n^n}{n}dx_n$$

$$=(w-\bar{w})^m\frac{w^n}{(n-1)n}-\frac{m}{n-1}\int_0^w(x_n-\bar{w})^{m-1}\frac{x_n^n}{n}dx_n. \tag{6}$$

The second term on the right side of (6) is identical with the corresponding term on the right side of (5). Now

$$\frac{2m}{n-1}\int_0^w(x_n-\bar{w})^{m-1}\frac{x_n^n}{n}dx_n=\frac{2m}{n-1}(w-\bar{w})^{m-1}\frac{w^{n+1}}{n(n+1)}$$
$$-\frac{2m(m-1)}{(n-1)n}\int_0^w(x_n-\bar{w})^{m-2}\frac{x_n^{n+1}}{n+1}dx_n. \tag{7}$$

The second term on the right side of (7) is identical with the third term on the right side of (5). Hence continuing the integration we get,

$$\int_0^w I_{n-2,m}\, dx_n = (w-\bar{w})^m \frac{w^n}{(n-1)n} - \frac{2m}{(n-1)}(w-\bar{w})^{m-1}\frac{w^{n+1}}{n(n+1)}$$

$$+\frac{3m(m-1)}{(n-1)n}(w-\bar{w})^{m-2}\frac{w^{n+2}}{(n+1)(n+2)}+\ldots \quad (8)$$

Putting $(w-\bar{w})=x$ and $w=y$, the above expression reduces to

$$\int_0^w I_{n-2,m} dx_n = \frac{w^n}{(n-1)n} x^m \left\{1-\frac{2m}{n+1}\frac{y}{x}+\frac{3m(m-1)}{(n+1)(n+2)}\left(\frac{y}{x}\right)^2-\ldots\right\}$$

$$= \frac{w^n}{(n-1)n} x^m F(-m, 2; n+1; y/x), \quad (9)$$

where $F(a, b; c; z) = 1+\frac{a.b}{1.c}z+\frac{a(a+1)b(b+1)}{1.2.c(c+1)}z^2+\ldots$.

But from (3) $\bar{w}=\frac{n-1}{n+1}w$

hence $x=w-\bar{w}=2w/(n+1)$

i.e. $y/x=\frac{1}{2}(n+1)$.

Substituting these values in (9), we get

$$\mu_m = \frac{(n-1)n}{w^n}\int_0^w I_{n-2,m} dx_n$$

$$= \left(\frac{2w}{n+1}\right)^m F\left(-m, 2; n+1; \frac{n+1}{2}\right). \quad (10)$$

Substituting $m=2$, 3, 4 and simplifying we have

$$\mu_2 = \frac{2(n-1)}{(n+1)^2(n+2)} w^2 \quad (11)$$

$$\mu_3 = \frac{-4(n-1)(n-3)}{(n+1)^3(n+2)(n+3)} w^3 \quad (12)$$

$$\mu_4 = \frac{24(n-1)(n^2-2n+3)}{(n+1)^4(n+2)(n+3)(n+4)} w^4 \quad (13)$$

$$\beta_1 = \frac{2(n+2)(n-3)^2}{(n+3)^2(n-1)} \quad (14)$$

$$\beta_2 = \frac{6(n+2)(n^2-2n+3)}{(n-1)(n+3)(n+4)}. \quad (15)$$

Hence all the above statistical constants of the range of samples from an infinite rectangular population, are expressed as functions of the size of the samples and the range of the original population.

It will be very interesting to study the change in the values of these constants of distribution of ranges, for samples of different sizes from an infinite rectangular population. The following table gives the values of the arithmetic mean, variance β_1, β_2 for samples of sizes, 2, 5, 10 and 20 from an infinite rectangular population, the unit in each case being $w/10$ i.e. ($1/10$ of the range of the original rectangular population).

Constants of the distribution of ranges.

(Unit$=w/10$)

Size of sample	Arithmetic mean	Variance	β_1	β_2
2	$0{\cdot}\dot{3}$	$0{\cdot}\dot{5}$	0·24	2·40
5	$0{\cdot}\dot{6}$	0·318	0·219	2·625
10	$0{\cdot}\dot{8}\dot{1}$	0·124	0·723	3·648
20	0·905	0·039	1·265	4·569

We find from this table that the arithmetic mean increases as the size of the sample increases and the variance decreases. Also the values of β_1, β_2 always fall in the type I region of the diagram of Karl Pearson's types of curves. This result is in accordance with E. S. Pearson's theory.

EXPERIMENTAL VERIFICATION

A population of 4000 was taken from Tippet's(5) random sampling numbers and samples of 20 were taken from this population. These were distributed into 10 classes and the key to a rectangular population was prepared. The actual range of each sample was found from the difference between the first and the last variates and this was expressed in terms of the class interval as unit. A frequency table of 20 classes was formed from these 200 values of the range. The following table gives the actual frequency distribution together with the theoretical frequency distribution given by E. S. Pearson(5), viz.

$$P_w = \{ nw - (n-1)W \} W^{n-1}/w^n.$$

The observed and theoretical values of arithmetic mean variance μ_2, μ_3 and β_1, β_2 are also given.

Frequency distribution of range of samples of 20.

Value of range.	Frequency				$(O-E)^2/E$
	Expected E		Observed O		
6·0—6·2	·08	5·9	—	6	—
6·2—6·4	·14		1		—
6·4—6·6	·23		—		—
6·6—6·8	·37		—		—
6·8—7·0	·60		1		—
7·0—7·2	·93		2		—
7·2—7·4	1·43		1		—
7·4—7·6	2·15		1		—
7·6—7·8	3·18		4		·2
7·8—8·0	4·61		4		·8
8·0—8·2	6·53		8		·35
8·2—8·4	9·06		6		1·1
8·4—8·6	12·26		13		—
8·6—8·8	16·12		16		—
8·8—9·0	20·53		15		1·5
9·0—9·2	25·04		22		·4
9·2—9·4	28·72		30		·1
9·4—9·6	29·96		34		·5
9·6—9·8	26·18	37·9	34	42	·4
9·8—10·0	11·76		8		

$n'=11$, $\lambda^2=5·4$, $P=0·8608$.

'*Frequency constants' for the range of samples of 20 from a rectangular population.* (Unit$=w/10$).

'Constant'	Value	
	Expected	Observed
Arithmetic mean	9·048 (0·061)	9·067
μ_2	0·396 (0·039)	0·414
μ_3	—0·278 (0·042)	—0·345
μ_4	0·701 (0·106)	0·847
β_1	1·265	1·68
β_2	4·569	4·89

NOTE: The values given in brackets are the probable errors of the respective constants.

The observed frequencies are in very close correspondence with the expected frequencies of the range. The frequency constants are in very close agreement with their theoretical values. Hence it is quite clear that the frequency distribution of the range in small samples from a rectangular population is of the type I curve; and the values of the constants for different values of n are those as given above.

CONCLUSION

We therefore conclude that the mth moment μ_m about the arithmetic mean of the range of samples of size n from an infinite rectangular population of range w is given by

$$\mu_m = \left(\frac{2w}{n+1}\right)^m F\left(-m, 2; n+1; \frac{n+1}{2}\right),$$

and that the frequency distribution of the range of samples follows the type I curve of Karl Pearson.

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ON SIMULTANEOUS OPERATIONAL CALCULUS

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1. Heaviside's operational calculus involving one parameter has been used by many writers to obtain many well-known properties and theorems in pure and applied mathematics. But, so far as is known, simultaneous symbolic calculus was used only by Dr. Balth van der Pol* to obtain certain integrals involving the Bessel functions. In the present note an attempt is made to use the simultaneous operational methods to obtain results which ordinarily would involve long and tedious calculations.

In this calculus, two variables x and y are treated operationally by means of two parameters p and q respectively

$$\frac{1}{p} \doteqdot x$$

$$\frac{1}{q} \doteqdot y$$

or in general

$$f_1(p) \doteqdot h_1(x)$$

$$f_2(q) \doteqdot h_2(y)$$

based upon

$$f_1(p) = p\int_0^\infty e^{-px} h_1(x)\,dx$$

$$f_2(q) = q\int_0^\infty e^{-qy} h_2(y)\,dy. \qquad (1 \cdot 1)$$

The following are the formulæ and the theorems used in this paper

$$f\left(\frac{p}{s}\right) \doteqdot h(sx) \qquad s = \text{const.} \qquad (1 \cdot 2)$$

$$\frac{p}{p+\alpha} f(p+\alpha) \doteqdot e^{-\alpha x} h(x) \qquad (1 \cdot 3)$$

* Balth van der Pol & Nissen, *Phil, Mag.* (7) II, (1931), p. 368.

$$\frac{1}{p^{2\alpha}} \doteqdot \frac{x^{2\alpha}}{\Gamma(1+2\alpha)} \quad \alpha > -1 \tag{1·4}$$

$$\frac{1}{p^n} e^{-\frac{1}{p}} \doteqdot x^{\frac{n}{2}} J_n(2\sqrt{x}). \tag{1·5}$$

For other results the papers of Dr. Pol may be referred to.

2. We have* by the definition of generalized Laguerre polynomials

$$(1-z)^{-1-\alpha} \exp\left[-\frac{xz}{1-z}\right] = \sum_{n=0}^{\infty} L_n^{(\alpha)}(x)\, Z^n,$$

where $L_n^{(\alpha)}(x)$ satisfies the following properties of orthogonality

$$\int_0^\infty e^{-t} t^\alpha L_m^{(\alpha)}(t)\, L_n^{(\alpha)}(t)\, dt = \begin{cases} 0 & m \neq n, \\ \dfrac{\Gamma(\alpha+n+1)}{\Gamma(n+1)} & m = n. \end{cases}$$

From these we get by putting $p = z/(1-z)$,

$$\int_0^\infty e^{-px} e^{-x} x^\alpha L_n^{(\alpha)}(x)\, dx = \frac{\Gamma(\alpha+n+1)}{\Gamma(n+1)} \frac{p^n}{(1+p)^{\alpha+n+1}}.$$

Hence from (1·1)

$$\frac{\Gamma(\alpha+n+1)}{\Gamma(n+1)} \frac{p^{n+1}}{(1+p)^{\alpha+n+1}} \doteqdot e^{-x} x^\alpha L_n^{(\alpha)}(x). \tag{2·1}$$

Again

$$e^{-x} x^\alpha L_n^{(\alpha)}(x) = \frac{1}{\Gamma(n+1)} \frac{d^n}{dx^n}(e^{-x} x^{\alpha+n}).$$

Let

$$\chi_n(x) = \left\{ \Gamma(n+1)\, \Gamma(2\alpha+n+1) \right\}^{-\frac{1}{2}} \frac{e^{\frac{1}{2}x}}{x^\alpha} \left(\frac{d}{dx}\right)^n (e^{-x} x^{2\alpha+n}).$$

Therefore

$$e^{-\frac{1}{2}x} x^\alpha \chi_n(x) = \chi_n'(x) \text{ (say)},$$

$$\doteqdot \left\{ \frac{\Gamma(n+1)}{\Gamma(2\alpha+n+1)} \right\}^{\frac{1}{2}} e^{-x} x^{2\alpha} L_n^{2\alpha}(x)$$

$$\doteqdot \left\{ \frac{\Gamma(2\alpha+n+1)}{\Gamma(n+1)} \right\}^{\frac{1}{2}} \frac{p^{n+1}}{(1+p)^{2\alpha+n+1}}.$$

Similarly

$$\chi_n'(y) \doteqdot \left\{ \frac{\Gamma(2\alpha+n+1)}{\Gamma(n+1)} \right\}^{\frac{1}{2}} \frac{q^{n+1}}{(1+q)^{2\alpha+n+1}}, \qquad \left(\frac{1}{q} \doteqdot y\right).$$

* Einar Hille, *Proc. Nat. Acad. Sci.*, Vol. 12, (1926), first note.

Hence

$$\sum_{n=0}^{\infty}(-t)^n \chi_n'(x)\,\chi_n'(y) \doteqdot \frac{pq}{\{(1+p)(1+q)\}^{2\alpha+1}}$$

$$\times \sum_{0}^{\infty} \frac{\Gamma(2\alpha+n+1)}{\Gamma(n+1)} \frac{\{-pqt\}^n}{\{(1+p)(1+q)\}^n}$$

$$= \frac{\Gamma(1+2\alpha).\,pq}{\{(1+p)(1+q)\}^{2\alpha+1}} \left\{1+\frac{pqt}{(1+p)(1+q)}\right\}^{-2\alpha-1}$$

$$= \frac{\Gamma(1+2\alpha).\,pq}{\{(1+p)(1+q)+tpq\}^{2\alpha+1}} \qquad |t|<1. \qquad (2{\cdot}2)$$

We will now find the 'original' of the right hand side of (2·2).

From (1·5) we have

$$\frac{1}{p^n}\,e^{-\frac{1}{p}} \doteqdot x^{\frac{n}{2}} J_n(2\sqrt{x}).$$

Therefore using (1·2)

$$\frac{(yt)^{2\alpha}}{\{(1+t)^2 p\}^{2\alpha}} \exp\left\{-\frac{yt}{p(1+t)^2}\right\} \doteqdot \left\{\frac{xyt}{(1+t)^2}\right\}^{\alpha} J_{2\alpha}\left\{\frac{2\sqrt{xyt}}{1+t}\right\}.$$

Now apply (1·3) we obtain after a slight simplification

$$\frac{y^{2\alpha}p}{\{p(1+t)+1\}^{2\alpha+1}} \exp\left\{-\frac{yt}{1+t+p(1+t)^2}\right\}$$

$$\doteqdot \frac{1}{1+t} \exp\left\{-\frac{x}{1+t}\right\} \left(\frac{xy}{t}\right)^{\alpha} J_{2\alpha}\left\{\frac{2\sqrt{xyt}}{1+t}\right\}.$$

Multiply both sides by $\exp\left(-\frac{y}{1+t}\right)$ and note that y is regarded as independent of x. We get

$$\frac{y^{2\alpha}p}{\{p(1+t)+1\}^{2\alpha+1}} \exp\left\{-y\left(\frac{1}{1+t}+\frac{t}{1+t+p(1+t)^2}\right)\right\}$$

$$\doteqdot \frac{1}{1+t} \exp\left\{-\frac{x+y}{1+t}\right\} \left(\frac{xy}{t}\right)^{\alpha} J_{2\alpha}\left\{\frac{2\sqrt{xyt}}{1+t}\right\}.$$

Now

$$y^{2\alpha} \doteqdot \frac{\Gamma(1+2\alpha)}{q^{2\alpha}}.$$

Therefore from (1·3)

$$y^{2\alpha}\exp\left\{-y\left(\frac{1}{1+t}+\frac{t}{1+t+p(1+t)^2}\right)\right\}$$

$$\doteqdot\frac{q\Gamma(1+2\alpha)}{\left\{q+\frac{1}{1+t}+\frac{t}{1+t+p(1+t)^2}\right\}^{2\alpha+1}}.$$

Hence

$$\frac{1}{1+t}\exp\left\{-\frac{x+y}{1+t}\right\}\left(\frac{xy}{t}\right)^{\alpha}J_{2\alpha}\left\{\frac{2\sqrt{xyt}}{1+t}\right\}$$

$$\doteqdot\frac{pq\Gamma(1+2\alpha)}{\{1+p(1+t)\}^{2\alpha+1}\left\{q+\frac{1}{1+t}+\frac{t}{1+t+p(1+t)^2}\right\}^{2\alpha+1}}$$

$$=\frac{pq\Gamma(1+2\alpha)}{\{(p+1)(q+1)(t+1)-(p+q+1)t\}^{2\alpha+1}}$$

$$=\frac{pq\Gamma(1+2\alpha)}{\{(p+1)(q+1)+tpq\}^{2\alpha+1}}. \tag{2·3}$$

Therefore from (2·2) and (2·3) we get

$$\sum_{n=0}^{\infty}(-t)^n\chi_n'(x)\chi_n'(y)=\frac{1}{1+t}\exp\left\{-\frac{x+y}{1+t}\right\}\left(\frac{xy}{t}\right)^{\alpha}J_{2\alpha}\left\{\frac{2\sqrt{xyt}}{1+t}\right\},$$

or

$$\sum_{n=0}^{\infty}(-t)^n\chi_n(x)\chi_n(y)=\frac{t^{-\alpha}}{1+t}\exp\left\{-\tfrac{1}{2}(x+y)\frac{1-t}{1+t}\right\}J_{2\alpha}\left\{\frac{2\sqrt{xyt}}{1+t}\right\}$$

$$|t|<1. \tag{2·4}$$

One particular case of (2·4) is worth noting. If we put $\alpha=0$ we get

$$\sum_{n=0}^{\infty}(-t)^n\psi_n(x)\psi_n(y)=\frac{1}{1+t}\exp\left\{-\tfrac{1}{2}(x+y)\frac{1-t}{1+t}\right\}J_0\left\{\frac{2\sqrt{xyt}}{1+t}\right\}$$

$$|t|<1,$$

where

$$\psi_n(x)=\frac{e^{-\frac{1}{2}x}}{\Gamma(n+1)}L_n(x),$$

$L_n(x)$ being a Laguerre polynomial of degree n. This result has been obtained by Hardy* by another method.

* G. H. Hardy, *Jour. Lon. Math. Soc.* 7 (1932).

3. Now

$$\frac{nJ_n(x)}{x} \doteqdot p[\sqrt{1+p^2}-p]^n$$

$$= p\alpha^n \tag{3·1}$$

and

$$J_n(y) \doteqdot \frac{q\beta^n}{\sqrt{1+q^2}}$$

where

$$\beta = \sqrt{1+q^2}-q.$$

Therefore

$$\sum_1^\infty \frac{nJ_n(x)\,J_n(y)}{x} \doteqdot \sum_1^\infty \frac{pq(\alpha\beta)^n}{\sqrt{1+q^2}}$$

$$= \frac{pq\,\alpha\beta}{\sqrt{1+q^2}\,(1-\alpha\beta)}, \tag{3·2}$$

p, q being arbitrary parameters and so they may be taken sufficiently small so that $\alpha\beta<1$. We will now find the 'original' of the right hand side. Assuming q as a constant for a moment we have

$$\frac{p\alpha}{1-\alpha\beta} = \frac{p(p-\sqrt{1+p^2}-\beta)}{\beta(\beta-1/\beta-2p)}$$

$$= \frac{p}{p+q}\left\{\tfrac{1}{2}+\frac{1}{2\beta}(-p+\sqrt{1+p^2})\right\}$$

$$\doteqdot \tfrac{1}{2}e^{-qx}+\frac{1}{2\beta}\int_0^x \frac{J_1(\xi)}{\xi}\,e^{-q(x-\xi)}\,d\xi,$$

using the relation

$$\frac{p}{p+q} \doteqdot e^{-qx}$$

and (3·1) with the product theorem of the operational calculus. Therefore

$$\frac{pq\,\beta\alpha}{\sqrt{1+q^2}(1-\alpha\beta)} \doteqdot \tfrac{1}{2}\frac{q\beta e^{-qx}}{\sqrt{1+q^2}}+\tfrac{1}{2}\int_0^{x-y}\frac{qe^{-q(x-\xi)}}{\sqrt{1+q^2}}\,\frac{J_1(\xi)}{\xi}\,d\xi$$

$$+\tfrac{1}{2}\int_{x-y}^{x}\frac{qe^{-q(x-\xi)}}{\sqrt{1+q^2}}\,\frac{J_1(\xi)}{\xi}\,d\xi,$$

$$= \tfrac{1}{2}\frac{q\beta e^{-qx}}{\sqrt{1+q^2}}+\tfrac{1}{2}\int_y^x \frac{qe^{-qv}}{\sqrt{1+q^2}}\,\frac{J_1(x-v)}{x-v}\,dv+\tfrac{1}{2}\int_0^y \frac{qe^{-qv}}{\sqrt{1+q^2}}\,\frac{J_1(x-v)}{x-v}\,dv,$$

$$= I_1+I_2+I_3 \text{ (say)}.$$

But

$$\frac{qe^{-vq}}{\sqrt{1+q^2}} \doteqdot J_0(y-v) \qquad y>v$$
$$\doteqdot 0 \qquad y<v$$

and

$$\frac{q\beta e^{-qx}}{\sqrt{1+q^2}} \doteqdot J_1(y-x) \qquad y>x$$
$$\doteqdot 0 \qquad y<x.$$

Now $v<y<x$. Hence the 'originals' of I_1 and I_2 with respect to q are zero and

$$I_3 = \tfrac{1}{2}\int_0^y J_0(y-v)\frac{J_1(x-v)}{x-v}\,dv. \tag{3·3}$$

Hence from (3·2) and (3·3) we have

$$\sum_{n=1}^{\infty} \frac{nJ_n(x)\,J_n(y)}{x} = \tfrac{1}{2}\int_0^y J_0(y-v)\frac{J_1(x-v)}{x-v}\,dv, \tag{3·4}$$

which is the Newman's Series.

COMPOSITE MEROMORPHIC FUNCTIONS

By K. S. SURYANARAYANAN, University of Madras

1. Polya* has considered the conditions under which an integral function of an integral function is an integral function of finite order. He proves the following results:

(1·1) the internal function is a polynomial and the external function is of finite order; or

(1·2) the internal function is an integral function of finite order in which case the external function is of zero order. In this paper it is proposed to extend these results to Meromorphic Functions and to investigate the order of the composite function, given the orders of the individual function.

2. The following results due to Nevanlinna† are used in the investigation:

Let f be a function at most meromorphic in the finite part of the plane. Let

$$\left.\begin{aligned} \log^+ a &= \log a \text{ when } \log a \geqslant 0 \\ &= 0 \quad\text{ when } \log a < 0 \end{aligned}\right\} \text{ where } a > 0;$$

$$m(r, z) = \frac{1}{2\pi}\int_0^{2\pi} \log^+ \left| \frac{1}{f(re^{i\phi}) - z} \right| d\phi, \text{ with the convention}$$

that $\dfrac{1}{f(re^{i\phi}) - z}$ is to be replaced by $f(re^{i\phi})$ when $z = \infty$;

$$N(r, z) = \int_0^r n(t, z)\frac{dt}{t}, \text{ where } n(t, z) \text{ is the number of}$$

roots of $f(x) = z$ in $|x| \leqslant t$;

and let $$T(r, f) = m(r, \infty) + N(r, \infty);$$

when f is an integral function we put $M(r) = \underset{|z| \leqslant r}{\operatorname{Max}} |f(z)|$.

With this notation we have,

(2·1) $$T(r, f) = m(r, \infty) + N(r, \infty) = m(r, z) + N(r, z) + O(1), \text{ for any } z;$$

* *Journal of the L.M.S.*, Vol. (1), (1926), pp. 12-16.

† *Acta Mathematica*, Band 46,(1925), pp. 1-9?.

(2·2) $T(r, fg) \leqslant T(r, f) + T(r, g) + O(1)$;

(2·3) $T(r, 1/f) = T(r, f) + O(1)$;

(2·4) $a_1, a_2, \ldots, a_q$ being arbitrary numbers, $q \geqslant 3$,

$$(q-2)\, T(r, f) \leqslant \sum_{\nu=1}^{q} N(r, a_\nu) + S(r), \text{ where}$$

(2·41) $S(r) = O[\log r + \log T(r, f)]$

except over a set of values of r of finite measure, and

(2·42) $S(r) = O(\log r)$

for *all* $r \to \infty$ when f is of finite order;

(2·5) when f is of finite order and is represented in the form f_1/f_2 where f_1, f_2 are integral functions expressed as Weirstrass's products with the least genus, then the order of f is the greater of the orders of f_1 and f_2;

(2·6) when f is an integral function,

$$T(r, f) \leqslant \log M(r) \leqslant 3T(2r, f).$$

The order of a meromorphic function is defined to be

$$\overline{\lim_{r \to \infty}} \frac{\log T(r, f)}{\log r}.$$

When there are several functions $f, g \ldots$, in the course of a discussion we shall use suffixes to denote the different functions to which N, n etc. belong, e.g. N_f, N_g, n_f, n_g etc.

3. Let now

$f(z) = g[h(z)] = g(\omega)$, $\omega = h(z)$, where f is meromorphic and g, h are at most meromorphic. The following two cases alone can occur:

(3·1) g has no essential singularity at infinity, that is g is a rational function. In this case h can be a meromorphic function.

(3·2) g has an essential singularity at infinity. In this case $h(z)$ must be an integral function for $f(z)$ to be meromorphic, since if $z = \alpha$ be a finite pole of $h(z)$, α would be an essential singularity for $f(z)$.

4. We now prove the following theorem:

THEOREM. *If $f(z)$ is a meromorphic function of finite order λ then*

(4·1) *$h(z)$ is of order λ in case g is a rational function;*

(4·2) *$g(z)$ is of order zero in case $h(z)$ is an integral function;*

(4·3) *$g(z)$ is of order $\leqslant \lambda/p$ if $h(z)$ is a polynomial of degree p.*

PROOF:—We first prove (4·1). Here $g(\omega)$ is a rational function. Let $\omega_1, \ldots, \omega_q$ be the poles of $g(\omega)$. The poles of $f(z)$ are the points where $h(z)=\omega_\nu$, $\nu=1, \ldots q$. Using (2·42), we get if $q \geqslant 3$*

$$(q-2)\ T(r, h) \leqslant \sum_{\nu=1}^{q} N_h(r, \omega_\nu)+S(r)$$

$$\leqslant T(r, f)+S(r) \quad \text{by (2·1)}.$$

Therefore,

$$(4·4) \qquad T(r, h) \leqslant K(r)\ T(r, f),$$

where $K(r)$ is bounded as $r \to \infty$. Again splitting the numerator and denominator of $g(\omega)$ into linear factors and using (2·2), (2·3) and the obvious property $T(f-a)=T(f)+O(1)$, we get

(4·5) $T(r, f) \leqslant cT(r, h)$ where $c>0$ is a constant.

Combining (4·4) and (4·5) we get (4·1).

Next we proceed to prove (4·2) and (4·3). Let

$$R = \text{Max } |h(z)| \text{ for } |z| \leqslant r.$$

Let the roots of the equation $g(\omega)=a$ in $|\omega| \leqslant R$ be $(\omega_1, \omega_2 \ldots)$, $n_g(R, a)$ in number. Then by (2·42) we get

$$n_g(R, a) \leqslant 2+\sum_{\nu=1}^{ng} \frac{N_h(r, \omega_\nu)}{T(r, h)}+O(1)$$

$$\leqslant \frac{N_f(R, a)}{T(r, h)}+O(1).$$

Therefore,

$$(4·6) \qquad n_g(R, a) \leqslant \frac{T(r, f)}{T(r, h)}+O(1).$$

Now h is an integral function. Hence we know $M(r)/r^m \to \infty$ as $r \to \infty$ for all integer m in case h is not a polynomial, and for $m<p$ if h is a polynomial of degree p. Hence in all cases we can find a $c>0$ such that

$$\frac{\log M(r)}{\log r} \geqslant c.$$

Hence by (2·6) and (4·6) we get

$$(4·7) \qquad n_g(R, a) \leqslant Ar^{\lambda+\varepsilon},$$

where A is a constant and $\varepsilon>0$ is as small as we please.

Let now

$$h(z)=\sum_{0}^{\infty} a_m z^m.$$

* If $q \leqslant 2$, (4·1) can be verified by direct calculation.

Then by Cauchy's inequalities,

$$|a_m|\, r^m \leqslant R.$$

So by (4·7),

$$(4{\cdot}8) \qquad n_g(|a_m|\, r, a) \leqslant A r^{\frac{\lambda+\varepsilon}{m}}$$

From (4·8) we get by the use of (2·42) with $q=3$,

$$(4{\cdot}9) \quad T(|a_m|\, r, g) \leqslant \sum_{\nu=1}^{3} N_g(|a_m|\, r, a_\nu) + S(r)$$

$$\leqslant \log(r) \left\{ \Sigma n_g(|a_m|\, r, a_\nu) \right\} + S(r)$$

$$\leqslant B r^{\frac{\lambda+\varepsilon}{m}} + S(r),$$

where B is a constant. If h were an integral function we can take m as large as we please and (4·9) shows that g is of order zero. If h were a polynomial of degree p, we can take $p=m$ in (4·9) and we deduce (4·3). *We can show, by using (2·5), that the order of g is actually* λ/p *in the case of* (4·3).

5. Let us now ascertain the order of $f(z)$, given the orders of g and h. The case when g is a rational function can be disposed of as in the proof of (4·1). Let us assume that h is an integral function of order λ and g a meromorphic function of order μ. Using the same notation as in § 4, we get by (4·6),

$$n_g(R, a) \leqslant \frac{T(r, f)}{T(r, h)} + O(1).$$

Now

$$R = \text{Max } |h(z)| \text{ for } |z| \leqslant r$$

and so

$$R > \text{Exp } (r^{\lambda-\varepsilon})$$

for a sequence (r_n) tending to infinity. Hence

$$n_g[\text{Exp } (r^{\lambda-\varepsilon}), a] \leqslant r^{-(\lambda-\varepsilon)}\, T(r, f) + O(1)$$

and so

$$N_g[\text{Exp}(r^{\lambda-\varepsilon}), a] \leqslant T(r, f) + O(r^{\lambda-\varepsilon}).$$

Therefore we get by using (2·42)

$$T[\text{Exp}(r^{\lambda-\varepsilon}), g] \leqslant T(r, f) + O(r^{\lambda-\varepsilon}) + S[\text{Exp}(r^{\lambda-\varepsilon})].$$

Since g is of order μ, we get finally

$$(5{\cdot}1) \qquad \text{Exp}[(\mu-\varepsilon) r^{\lambda-\varepsilon}] \leqslant T(r, f).$$

On the other hand by using (2·2), (2·3) and (2·5) we get, if $g = g_1/g_2$, g_1, g_2 being integral functions,

$$(5\cdot2) \quad T(r,f) \leqslant T(r,g_1)+T(r,g_2)+O(1) \leqslant \text{Exp}[(\mu+\varepsilon)r^{\lambda+\varepsilon}].$$

Combining (5·1) and (5·2) we get

$$(5\cdot3) \quad \text{Exp}[(\mu-\varepsilon)r^{\lambda-\varepsilon}] \leqslant T(r,f) \leqslant \text{Exp}[(\mu+\varepsilon)r^{\lambda+\varepsilon}]$$

where $\varepsilon>0$ is as small as we please and where the first part of the inequality holds for the sequence (r_n) tending to infinity, the latter part holding for $r\to\infty$ continuously. Hence f is of regular growth and its order is comparable with that of the function

$$\text{Exp}\left[\text{Exp}\{\mu r^{\lambda}\}\right].$$

6. Examples:—(a) Let $f(z)=\tan z^p$, $g(\omega)=\tan \omega$, $\omega=z^p$. It can be easily verified that the order of $f(z)$ is p and of $g(\omega)$ is one. Since $h(z)$ is a polynomial of degree p, this result illustrates (4·3).

(b) $f(z)=(1-\tan^2 z)/(1+\tan^2 z)$, $h(z)=\tan z$. Here $f(z)=\cos 2z$ and so is an integral function of order one. $h(z)=\tan z$ is a meromorphic function of order one, and $g(\omega)=(1-\omega^2)/(1+\omega^2)$ is a rational function. This illustrates (4·1).

(c) Let $f(z)=\tan e^z$, $g(\omega)=\tan \omega$, $\omega=h(z)=e^z$. Here g and h are of order one. Hence the order of $f(z)$ must be comparable with that of $\text{Exp}(e^r)$.

7. Note: It is to be noted that the result (2·4) is not used in general but along with (2·42), that is only when the residue $S(r)=O(\log r)$ for *all* $r\to\infty$ which is the case only when the function in question is of finite order. So in (4·4) h is assumed to be of finite order and in (4·6) and (4·9) g is assumed to be of finite order. These assumptions have not been proved in the course of the discussion since that would make the proofs complicated. So these assumptions are proved to be true below:

(7·1) We shall show that *when f is of finite order and g a rational function, h is of finite order.* With the notation of the proof of (4·1) we get by definition and by the use of (2·41) instead of (2·42),

$$(7\cdot11) \quad T(r,f) \geqslant N_f(r,\infty) = \sum_{\nu=1}^{q} N_h(r,\omega_\nu) \geqslant [1+O(1)]\, T(r,h),$$

except for a set of values of r of finite measure. Hence if h were of infinite order, (7·11) shows f is also of infinite order which contradicts the hypothesis. Here $q\geqslant3$. If $q\leqslant2$, (4·1) can be verified by direct calculation.

(7·2) We shall next show that *if f is of finite order and h an integral function, g is of finite order.* With the notation of the proof of (4·2) we get, using (2·41) again,

$$n_g(R, a) \leqslant \frac{T(r, f)}{T(r, h)} + O(1)$$

except for a set of values of r of finite measure. Hence

$$\begin{aligned} n_g(|a_m| r^m, a) &\leqslant n_g(R, a) \\ &\leqslant \frac{T(r, f)}{T(r, h)} + O(1), \end{aligned}$$

i.e.

$$\begin{aligned} N_g(|a_m| r^m, a) &\leqslant n_g(|a_m| r^m, a) \times (m \log r) \\ &\leqslant m \frac{\log r}{T(r, h)} T(r, f) + O(1). \end{aligned}$$

Therefore by another use of (2·41), we get

$$\begin{aligned} (7{\cdot}21) \quad T(r, f) &\geqslant \frac{T(r, h)}{m \log r} \left\{ \sum_{\nu=1}^{3} N_g(|a_m| r^m, a_\nu) + O(1) \right\} \\ &\geqslant \frac{T(r, h)}{m \log r} T(|a_m| r^m, g) \left\{ 1 + O(1) \right\}, \end{aligned}$$

except for a set of values of r of finite measure. (7·21) shows that if g were of infinite order f would also be of infinite order which again contradicts the hypothesis.

A NOTE ON THE VALUES OF AN ANALYTIC FUNCTION NEAR AN ESSENTIAL SINGULARITY*

By V. GANAPATHY IYER, Madras University

1. Let $\{f_n(z)\}$, $n=1, 2, \ldots$, be a family of functions holomorphic in the interior of a domain D. The family is said to be normal in D when every sub-sequence selected from the family contains another converging uniformly in any closed region lying in D. The family is said to be quasi-normal when any sub-sequence contains another which converges uniformly in any closed region D' contained in D, except may be at a finite number of points of D'; these points of non-uniform convergence, called irregular points, might vary with the sub-sequence chosen and with the region D'. If small circles round these points be excluded from D', the sub-sequence selected converges uniformly in the remaining portion of D'.

2. Let $f(z)$ be any function with an isolated essential singularity at O; we can suppose O to be at infinity without loss of generality. Then $f(z)$ is the sum of an integral function and another which tends to zero as $|z| \to \infty$. The latter part can be omitted without affecting the discussion below. Therefore we start with an integral function $f(z)$. The plane is divided into rings (Γ_n), $n=1, 2, \ldots$, the boundary of Γ_n being the circles with centre $z=0$ and radii $\frac{3}{4}\, 2^{n-1}$ and $\frac{3}{4}\, 2^n$, $n=1, 2, \ldots$.

Put $f_n(z)=f(2^n z)$, $n=0, 1, 2, \ldots$. When z varies over Γ_1, $2^n z$ varies over Γ_{n+1} and $f_n(z)$ takes in Γ_1 the values of $f(z)$ in Γ_{n+1}. With this notation the following unproved assertions have been made by P. Montel†:

(α) there cannot exist two different numbers a and b such that the numbers of zeros of $f(z)-a$ and $f(z)-b$ in Γ_n have a finite upper bound as $n \to \infty$;

(β) the family $\{f_n(z)\}$ cannot be quasi-normal in Γ_1.

* Communicated to the Ninth Conference of *The Indian Mathematical Society*, December 1935.

† See P. Montel, "Lecons sur les familles normales des fonctions analytiques", *Collection de Monographies de E. Borel*, pp. 80-81.

The object of this note is to disprove, by means of an example, both these statements. It is known that $\{ f_n(z) \}$ cannot be normal in Γ_1 and that (α) involves (β) though the converse is not true. An example to illustrate the latter also is given in § 4.

3. Consider the function

$$\phi(z) = \prod_{n=1}^{\infty} \left(1 - \frac{z}{2^n} \right);$$

this is an integral function with the zeros at $z = 2^n$, $n = 1, 2, \ldots$. Let C_n be a circle of centre $z = 2^n$ and radius $2^{n-1}\sigma$, $n = 0, 1, 2 \ldots$ where $0 < \sigma < \frac{1}{2}$ and σ can be as small as we please. These circles are non-overlapping and Γ_n contains just one circle C_{n-1} in its interior. Let z be any point outside these circles and p be such that $2^p < |z| < 2^{p+1}$. We have

$$\phi(z) = \prod_{n=1}^{p-1} \left(1 - \frac{z}{2^n} \right) \times \left(1 - \frac{z}{2^p} \right) \left(1 - \frac{z}{2^{p+1}} \right) \times \prod_{n=p+2}^{\infty} \left(1 - \frac{z}{2^n} \right)$$

$$= \phi_1 \times \phi_2 \times \phi_3, \text{ say.} \tag{1}$$

Now,

$$|\phi_3| \geqslant \prod_{n=p+2}^{\infty} \left(1 - \frac{2^{p+1}}{2^n} \right) = \prod_{k=1}^{\infty} \left(1 - \frac{1}{2^k} \right) = H, \text{ say}$$

where $0 < H < 1$;

also,

$$|\phi_2| \geqslant \frac{\sigma}{2} \cdot \frac{\sigma}{2} = \frac{\sigma^2}{4};$$

and

$$|\phi_1| \geqslant \prod_{n=1}^{p-1} \left(\frac{2^p}{2^n} - 1 \right) \geqslant (2^{p-1} - 1) \geqslant \left(\frac{|z|}{4} - 1 \right).$$

Therefore (1) gives

$$|\phi| \geqslant H \frac{\sigma^2}{4} \left(\frac{|z|}{4} - 1 \right). \tag{2}$$

The relation (2) shows that $|\phi(z)| \to \infty$ as $|z| \to \infty$, z remaining outside the circles C_n. Let $M > 0$ be any number and consider any number $a \neq 0$ and such that $|a| \leqslant M$. Choose r_0 such that $|\phi(z)| \geqslant M + 1$ for all $|z| \geqslant r_0$, z not being inside C_n. Then the zeros of $\phi(z) - a$ outside $|z| = r_0$ can lie only in these circles C_n lying outside $|z| = r_0$.

Now on the circumference of any such C_n, $|a/\phi(z)| \leqslant M/(M+1) < 1$ and so the number of zeros of $\phi(z)$ and $\phi(z) - a$ in C_n are the same, that is $\phi(z) = a$ just once in C_n for all $n \geqslant n_0$ where n_0 depends only on M. Hence for an infinity of values of a, the number of zeros of $\phi(z)-a$ is bounded in Γ_n, being just equal to one. This disproves the statement (α). Next consider the family $\{\phi_n(z)\}$, $\phi_n(z)=\phi(2^n z)$, $n=1, 2, \ldots$. By (2), we see that $|\phi_n(z)|\rightarrow\infty$ as $n\rightarrow\infty$ uniformly for all z in Γ_1 from which is excluded the circle C_0 of radius $\frac{1}{2}\sigma$ and centre $z=1$. Hence $\{\phi_n(z)\}$ is quasi-normal in Γ_1 with the only irregular point at $z=1$, this point being an effective irregular point since $\{\phi_n(z)\}$ cannot be normal in Γ_1. Hence the statement (β) cannot be true.

4. Consider the function

$$\psi(z)=\prod_{k=1}^{\infty}\left(1-\frac{z}{2^k}\right)^k,$$

which has $z=2^k$ as a zero of the kth order, $k=1, 2, \ldots$. We shall prove that $|\psi(z)|\rightarrow\infty$ as $|z|\rightarrow\infty$ outside the circles C_k, $k=0, 1, 2, \ldots$. Let z be any point outside C_k and p be such that $2^p<|z|<2^{p+1}$. We have,

$$\psi(z)=\prod_{k=1}^{p-1}\left(1-\frac{z}{2^k}\right)^k\times\left(1-\frac{z}{2^p}\right)^p\left(1-\frac{z}{2^{p+1}}\right)^{p+1}\times\prod_{k=p+2}^{\infty}\left(1-\frac{z}{2^k}\right)^k$$

$$=\psi_1\times\psi_2\times\psi_3, \text{ say.} \tag{3}$$

Now,

$$|\psi_3(z)|\geqslant\prod_{k=p+2}^{\infty}\left(1-\frac{2^{p+1}}{2^k}\right)^k$$

$$=\left\{\prod_{k=1}^{\infty}\left(1-\frac{1}{2^k}\right)\right\}^{p+1}\times\psi(1)$$

$$=H^{p+1}\psi(1), \text{ say, where } 0<H<1; \tag{4}$$

also,

$$|\psi_2(z)|\geqslant\left(\frac{\sigma}{2}\right)^{2p+1} \tag{5}$$

and

$$|\psi_1(z)|\geqslant(2^2-1)^{p-2}(2^3-1)^{p-3}\ldots(2^{p-1}-1). \tag{6}$$

Now we can find a constant λ, $0<\lambda<1$ such that

$$H^{p+1}\left(\frac{\sigma}{2}\right)^{2p+1}\psi(1)\geqslant\lambda^p.$$

Then choose h so that

$$(2^2-1)(2^3-1)\ldots(2^h-1)\,\lambda\geqslant\rho>1$$

where ρ is a fixed number >1.

Then, if $p>h$, we have by (3), (4), (5) and (6),

$$|\psi(z)|\geqslant\lambda^p(2^2-1)^{p-2}(2^3-1)^{p-3}\ldots(2^h-1)^{p-h}\times\left(\frac{|z|}{4}-1\right)$$

$$\geqslant\frac{\rho^p}{(2^2-1)\ldots(2^h-1)^h}\times\left(\frac{|z|}{4}-1\right). \tag{7}$$

As $|z|\to\infty$, $p\to\infty$; and h depends only on σ, H and ρ. Hence (7) shows that $|\psi(z)|\to\infty$ as $|z|\to\infty$, z not in C_n. Hence as before $\psi(z)-a$, for all a such that $|a|\leqslant M$, has just k zeros in C_k for all $k\geqslant k_0$, k_0 depending on M only. Therefore the number of zeros of $\psi(z)-a$ in Γ_k tends to infinity as $k\to\infty$ for an infinity of values of a. On the other hand the family $\{\psi_n(z)\}$ is quasi-normal in Γ_1 since by (7), $|\psi_n(z)|\to\infty$ as $n\to\infty$ uniformly for all z in Γ_1 from which the circle C_0 is excluded. Hence $\{\psi_n(z)\}$ is quasi-normal with the single irregular point at $z=1$; hence for this function (β) is false while (α) is true.

THE ASYMPTOTIC CURVES OF THE CUBIC AND QUARTIC SCROLLS*

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§ 1. The cubic scroll of the first type

The asymptotic curves of the cubic scroll of the first type are rational quartic curves which cut every generator in two points harmonically separating the intersections of the generator with the directrix lines.

This theorem was first proved by Wilczynski† who deduced it from his well-known differential equations of a ruled surface. It is the object of this section to employ the theory of correspondence to discuss the asymptotic curves.

Let any generator meet an asymptotic curve C of any scroll in k points. If the simple directrix line of the scroll (in its absence, any simple directrix curve) be denoted by D_2, a $(1, k)$ correspondence is set up between the points of D_2 and of C, and incidentally an involutory $(k-1, k-1)$ correspondence on the curve C. If p and π denote respectively the genera of the scroll and of C, and if η be the number of generators which touch C, we have‡

$$\eta = 2(\pi - 1) - 2k(p-1).$$

We shall prove in § 3 that the torsal generators are the only generators that can touch C, and conversely that every torsal generator will touch C. Hence η=number of torsal generators. In the case of the general scroll, this number is $2(n+2p-2)$, but the number can often be less for special cases.

For the cubic scroll of the first type, we have $p=0, \eta=2$. Hence $k=2-\pi$. The value of π is therefore either 0 or 1. If $\pi=1$, $k=1$, and a (1, 1) correspondence is then established between the points of C and the line D_2. This is impossible since the genus of D_2 is zero. Hence, we must have $\pi=0$, and $k=2$.

* Presented to the Ninth Conference of the *Indian Mathematical Society*, December, 1935.

† Wilczynski, *Projective Diff. Geom. of curves and surfaces*, p. 145. E. P. Lane, *Proj. Diff. Geom. of curves and surfaces*, p. 58. Ex. 3.

‡ W. L. Edge, *Theory of Ruled Surfaces*, § 17 or H. F. Baker, *Principles of Geometry*, Vol. VI. p. 25.

If m is the order of C, the degree of the scroll generated by the involution on $C=m-1$*. Hence $m-1=3$. We have thus proved that C is a rational quartic curve which meets every generator in two points.

Any plane through D_2 contains two generators meeting on the double line D_1 of the scroll. Hence D_2 cannot meet C at all in any point. Any plane through D_1 contains only one generator. Hence, every asymptotic curve C must meet D_1 in two points. These two points are, according to a result attributed to Snyder†, the unodes (pinch-points) of the scroll. We shall arrive at an independent proof of this, in the course of this discussion.

A skew quartic curve is rational either when it belongs to the "Second Species", i.e. when there is a unique quadric passing through the curve, or when the curve possesses an actual double point. In the latter case, through any point there will pass two chords of the curve which do not pass through the double point, in other words the curve has two apparent double points and one actual double point. Since, in the present case, through any point on D_1 there pass three distinct chords of C, C is a quartic of the second species.

A torsal generator is a locus of parabolic points, and the plane through it and D_2 touches the surface all along the generator. Hence the torsion of an asymptotic line at a point where it meets a torsal generator is zero, and this is the necessary and sufficient condition that the osculating plane to the curve, i.e. the tangent plane to the surface should have four-pointic contact with the curve. Since C is a quartic, it follows that a torsal generator cannot meet (or touch) C in two distinct points. It must touch C at a single point.

Besides the torsal generators, no other tangent of C can meet D_2, for if it does, the plane of the tangent and D_2 contains two generators of the scroll, and would therefore meet C altogether in 6 points. But the *rank* of the curve C is 6, while its *class* may be 4, 5 or 6 according to circumstances‡. Hence, each of the torsal generators must count as three tangents meeting D_2; in other words, the points of intersection of D_2 and the torsal generators (these points are the cubical points of the scroll)‖, will be *triple points* on the developable having C for its edge of regression.

* Edge, *loc. cit.* § 19 (foot-note).

† Wilczynski, *loc. cit.* p. 145. No proof or reference is given.

‡ Vide Sommerville, *Analytical Geometry of Three Dimensions*, § 14·725.

‖ Vide *Journal Ind. Math. Soc*, Vol. XIX, p. 47.

Again, through a general point on D_2 there pass the two osculating planes corresponding to the points where the generator through the point on D_2 meets C, and two others, viz. those corresponding to the points of C on the two torsal generators (if the osculating planes at these points be determinate). Not more than four osculating planes of C can pass through a general point on D_2, for if the osculating plane at any other point of C pass through the point on D_2, the tangent thereat would meet D_2.

We conclude from these considerations that C has two inflexions lying on the two torsal generators. A torsal generator being thus an inflexional tangent meets C in three coincident points. This point is necessarily at the intersection of the torsal generator with D_1, for otherwise the plane (constituting part of the Hessian) through D_1 and the torsal generator would meet C altogether in five points.

We have thus proved Snyder's result, and proved further that *the unodes are points of inflexion on all the asymptotic curves.*

Since the torsal generators have to be counted as trisecants to the curve C, they will be generators of the quadric through C. Taking the equation of the scroll in the form

$$x^2z=y^2w,$$

the torsal generators are

$$x=0,\ w=0, \text{ and } y=0,\ z=0.$$

Any quadric through these is of the form

$$xy+pxz+qyw+rzw=0,$$

p, q, r being constants. The residual quartic curve of intersection of the surface with the scroll meets a generator of the latter,

$$y=mx,\ z=m^2w$$

where

$$x^2+xw(pm+q)+rmw^2=0.$$

If the curve should be an asymptotic curve, the two points of intersection, i.e. the two values of x/w coincide only when $m=0$, and $m=\infty$. Hence $p=0$, $q=0$ for an asymptotic curve. Writing $r=-A^2$, *the asymptotic curves of the scroll* $x^2z=y^2w$ *are given by its intersections with the system of quadrics* $xy=A^2z\,w$. The equations of the asymptotic curves can be written as

$$x:y:z:w=At:At^3:t^4:1.$$

The correctness of the result is at once verified by showing that the osculating plane of the curve coincides with the tangent plane of the surface.

Wilczynski's result that any generator intersects C in two points harmonically separating the points on the directrix lines follows at once; for $y=mx$, $z=m^2w$ meets the directrices where $xw=0$, and it meets the quadric $xy=A^2zw$ where

$$x^2-A^2mw^2=0.$$

Another result can be verified. *The tangents to C belong to a linear complex.* If l, m, n, l', m', n' denote the line co-ordinates of a tangent to C, we have

$$A^2n+2\,n'=0.$$

§ 2. CAYLEY'S CUBIC SCROLL

This is the cubic scroll in which there is only one directrix line, which is a double line as well as a generator. The work of the third para of § 1 holds good for all scrolls which possess a simple directrix, or a double directrix through any point of which only one other generator can be drawn. Hence, for Cayley's cubic which has no torsal generators, we must have $(\pi-1)+k=0$. Hence $\pi=0$, $k=1$. There is therefore an algebraic (1, 1) correspondence between the double line D and an asymptotic curve C which is also rational. The curve C forms a directrix curve of the generators. The line D itself counts as a generator and must therefore meet C at the point such that the generator through that point is D itself, viz. at the unode (pinch-point) on D. This point counts as the sole united point of the correspondence. Hence, if m be the order of C, the degree of the scroll generated* by joining corresponding points$=m+1-1=3$. Hence C is a twisted cubic.

We have thus proved† that *the asymptotic curves of Cayley's cubic scroll are twisted cubics all passing through the unode of the scroll, and meeting every generator at one point.*

If the scroll be taken as $y^3=x(zx+wy)$, the plane $x=0$ can meet any asymptotic curve only at the unode, i.e. all the asymptotic curves have at the unode the same osculating plane, viz. $x=0$.

The simplest method, however, of obtaining the asymptotic curves of Cayley's cubic $y^3=x(zx+wy)$ is by using the direct method. Any generator of the scroll is given by

$$y=mx,\ z+mw=m^3x.$$

Writing these equations as

$$\frac{\frac{x}{w}}{1}=\frac{\frac{y}{w}}{m}=\frac{\frac{z}{w}+m}{m^3}=r,$$

* Edge, *loc. cit.* § 19; Baker, *loc. cit.* p. 15.

† Compare Wilczynski, *loc. cit.* p. 145; Lane, *loc. cit.* p. 58, Ex. 4.

the parametric equations of the surface can be written as

$$\frac{x}{w}=r,\ \frac{y}{w}=mr,\ \frac{z}{w}=m^3r-m.$$

The differential equation of the asymptotic lines is

$$Ldr^2+2\,Mdr\,dm+Ndm^2=0,$$

where L, M, N in terms of homogeneous co-ordinates are given by the determinants $(\xi\xi_r\xi_m\xi_{rr})$, $(\xi\xi_r\xi_m\xi_{rm})$, and $(\xi\xi_r\xi_m\xi_{mm})$, ξ being the (generalised) vector having for components x, y, z, w. In the present case, the values of L, M, N are found to be

$$L=0,\ M=1,\ N=6mr^2.$$

The differential equation becomes

$$dm(2dr+6mr^2dm)=0,$$

whence the curved asymptotics are given by

$$r(3m^2-c)=2,$$

where c is any constant.

The asymptotic curves of Cayley's cubic $y^3=x(zx+wy)$ *are therefore given by*

$$x:y:z:w=2:2m:cm-m^3:3m^2-c$$

where m is the parameter, and different values of c refer to different curves.

The correctness of the result is verified by comparing the osculating plane at any point of the curve with the tangent plane thereat.

Incidentally, it may be remarked that all the cubic curves on the surface are given by the equation

$$r(A+Bm+Cm^2+Dm^3)=1+Dm,$$

where A, B, C, D are constants. If $C=0$, the curve is a rational plane cubic. For other values of C, we get twisted cubics. There are thus altogether ∞^4 twisted cubics on the surface.

§ 3.

We have to prove that the number η of generators which touch an asymptotic curve C is equal to the number of torsal generators. In other words every torsal generator touches C, and conversely if a generator touches C it is torsal. These results will be obvious if the point where a torsal generator meets C or the point of contact of a generator with C be an ordinary point on the surface—neither a double point nor a cubical point, since the locus of parabolic points of the scroll consists of generators which are torsal.

Referring the scroll to two directrix curves, its equations may be written as

$$x=\phi_1(v)+u\psi_1(v)\ ;\ y=\phi_2(v)+u\psi_2(v)\ ;\ z=\phi_3(v)+u\psi_3(v).$$

The differential equation of the asymptotic lines is $2\,Mdu\,dv+Ndv^2=0$, the value of L being identically zero. The curved asymptotics are given by $2\,Mdu+Ndv=0$. The torsal generators are given by $M=0$*. If a generator touches an asymptotic curve, we must have at the point of contact $dv/du=0$, since the generators are given by $v=$constant. Hence $M=0$ at the point of contact, i.e. the generator is torsal. Conversely, a torsal generator being obtained as the limit when two generators through a point on the double curve tend to coincide, will always touch every asymptotic curve.

The number η is thus equal to the number of torsal generators.

§ 4. THE QUARTIC SCROLLS

Some information about the asymptotic curves of the different quartic scrolls can be had by means of the previous method.

The Quartic Scroll: Type VIII of Cayley, Type VII of Salmon, Type II B of W. L. Edge. This is formed by the chords of a twisted cubic which meet a given line l. The cubic will be the double curve of the scroll, so that through each point of it pass two generators. If AB and AC are the generators through a point A, then BC is evidently another generator and the section of the scroll by the plane $l\,A$ consists of the line l and the sides of the triangle ABC. Conversely, *there exists a definite quartic scroll of this type which passes through the sides of any two triangles not in the same plane.* This follows because six points determine uniquely a twisted cubic, and the planes of the triangles intersect in a line l which will be the directrix.

The chords of a twisted cubic which belong to a linear complex determine a symmetrical (2, 2) correspondence of points on the cubic, of the form

$$ap^2q^2+bpq(p+q)+c(p^2+q^2)+a'pq+b'(p+q)+c'=0.$$

Conversely, it may be proved that any such correspondence determines chords belonging to a linear complex given by

$$cl+b'm+c'n+(a'-c)l'-bm'+an'=0.$$

*Vide *Journal Indian Mathematical Society*, Vol. XIX, Part II, p. 219 and p. 242.

The condition that the correspondence is involutary is the same as the condition that this complex is a special complex, viz.

$$c^2 = ac' + a'c - bb'.$$

The surface is rational and has four torsal generators. Hence, using § 1, we have

$$2(\pi - 1) + 2k = 4, \text{ or } k = 3 - \pi.$$

We cannot have $\pi = 2$, for this would set up a (1, 1) correspondence between an asymptotic curve C and the simple directrix line l, which would mean that $\pi = 0$. If $\pi = 1$, then $k = 2$, and a (1, 1) involution would be formed on C by the generators. The order of the scroll is then equal to $m - 1$ where m is the order of C, i.e. $m = 5$. But any plane through l has three generators and hence would have six points of C. We cannot therefore have $\pi = 1$. Hence $\pi = 0$, $k = 3$.

The argument can be slightly altered so as to refer to any rational quartic scroll which has four torsal generators. As before $\pi \neq 2$, since instead of the line l, we could take a conic or a twisted cubic as the directrix curve in every case. If $\pi = 1$, a (1, 1) involution is formed on C, and this will have two and only two coincidences. But every torsal generator touches C and hence must give a coincidence. The value of π cannot therefore be 1. Hence, whenever $\eta = 4$ for a rational quartic scroll, we must have $\pi = 0$, $k = 3$.

The asymptotic curves on each of the following types of quartic scrolls are rational and meet every generator in three points.

Type I, Type II-B, Type II-A, Type IV-A. The numbers assigned to the types are those of Edge.

A passing remark may be made regarding Type IV-A. The surface has one triple line and one simple directrix line. *This type can be regarded as a limiting case of Type II-B (Edge)*, by supposing that the two triangles considered at the beginning of this section have dwindled into points. In other words, three concurrent lines in one plane and three concurrent lines in another plane determine a definite quartic scroll of this type.

For rational quartic scrolls for which $\eta = 2$, we have $k = 2 - \pi$. As before, we cannot have $\pi = 1$. Hence $\pi = 0$, and $k = 2$. The generators by their intersections with C form a (1, 1) involution on C. If m is the degree of C, the degree of the scroll is $m - 1$, whence $m = 5$.

The asymptotic curves on the following types of quartic scrolls are rational quartic curves which are met by any generator in two points.

Type III-(A), Type II-(C), Type IV-(B), Type V-(A). There are two rational quartic scrolls for which $\eta=0$, viz. the surfaces Type III-B and Type V-B. For these surfaces, $\pi+k-1=0$, so that $\pi=0$, $k=1$. *The asymptotic curves are rational and are met by any generator in one point.*

There are two types of *elliptic* quartic scrolls, one having two directrices, the other having only one. It is easy to prove that these have respectively four and two torsal generators. Since $p=1$, we now get $\pi=3$ or 2. *The asymptotic curves of the elliptic quartic scrolls are curves of genus 3 or 2 according to the type of scroll.*

ON THE AFFINE CLASSIFICATION OF QUADRIC LOCI

By R. VAIDYANATHASWAMY, M.A., D.Sc.

The classification of plane conics into hyperbolas, parabolas and ellipses can, it is well known, be extended to Euclidean-affine space of any number of dimensions. It is the purpose of this note to shew that the classification can be completely described by four numbers which are respectively the rank and signature of the quadric itself, and the rank and signature of its section by the prime at infinity. These four numbers are independent in the sense that no one of them can be unambiguously determined from the others. The description of quadric-types by the values of these four numbers is very convenient, and does not appear to have been stated before. The usual terms, *ellipse, hyperbola, parabola, cylinder* are used in an extended sense, and serve to characterise affine types.

I. The projective types of quadric loci

Let S_n be a real projective space of n dimensions, $X_0, X_1, \ldots, X_n$ real projective co-ordinates in S_n. By a 'quadric locus' will be meant in this paper, the locus obtained by equating to zero a *real* quadratic form $f(X_0, X_1, \ldots, X_n)$. If f is of rank r, it is known that it can be transformed by a real projective transformation into the shape

$$\varepsilon_1 Y_1^2 + \varepsilon_2 Y_2^2 + \ldots + \varepsilon_r Y_r^2,$$

where the Y's are independent linear functions of the X's, and each $\varepsilon = \pm 1$. If s of the ε's are -1 and the remaining $r-s$ equal to $+1$, we define the signature of the quadric locus to be the smaller of the two numbers s, $r-s$; thus the signature of a quadric locus of rank $r \leqslant \frac{1}{2}r$. (As the quadric locus $f=0$ remains the same when f is multiplied by a negative number, we cannot identify the signature of the locus with the signature of the form f, but must adopt this definition).

A quadric locus in S_n, of rank r and signature s contains real flat spaces of $n-(r-s)$ dimensions (and no flat spaces of higher dimension).

For, such a locus can be put into the form:

$$Y_1^2 - Y_2^2 + Y_3^2 - Y_4^2 + \ldots + Y_{2s-1}^2 - Y_{2s}^2 + Y_{2s+1}^2 + \ldots + Y_r^2 = 0,$$

The locus therefore contains the real space of $n-(r-s)$ dimensions, determined by the $r-s$ equations

$$Y_1-Y_2=Y_3-Y_4=\ldots=Y_{2s-1}-Y_{2s}=Y_{2s+1}=\ldots=Y_r=0.$$

THEOREM I. *The total number of projective types of quadric loci in* S_n *is equal to* $[(n+3)/2]\times[(n+4)/2]$, *where* $[x]$ *is the greatest integer in* x.

For, since the signature s of a quadric of rank r is such that $0\leqslant s\leqslant\frac{1}{2}r$, the number of projective types for a given rank r is $[(r+2)/2]$. Since r can vary from 0 to $n+1$, the total number of projective types is

$$\left[\frac{2}{2}\right]+\left[\frac{3}{2}\right]+\ldots+\left[\frac{n+3}{2}\right].$$

To sum this series, we note that

$$\left[\frac{p}{2}\right]+\left[\frac{q}{2}\right]=\left[\frac{p+q}{2}\right],$$

unless p and q are both odd.

Hence, adding the first term of the series to the last and so on, and dividing by 2, we see that if n is even, the sum of the series is

$$\frac{n+2}{2}\left[\frac{n+5}{2}\right]=\left[\frac{n+3}{2}\right]\times\left[\frac{n+4}{2}\right].$$

When n is odd, we add $[\frac{1}{2}]$ as the initial term and sum as before, and obtain

$$\frac{n+3}{2}\left[\frac{n+4}{2}\right]=\left[\frac{n+3}{2}\right]\times\left[\frac{n+4}{2}\right].$$

This proves the theorem. In the enumeration of this theorem, the identically vanishing quadric is also included.

II. ELLIPTIC AND HYPERBOLIC AFFINE TYPES

Let Q be a quadric locus of rank-signature (r, s) in S_n. Let S_{n-1} be a real prime ($=$flat subspace of $n-1$ dimensions) in S_n. The intersection Q' of S_{n-1} with Q is determined by equating to zero a real quadratic form in the n projective co-ordinates in S_{n-1}, and is therefore a 'quadric locus' in the sense defined. Let (r_1, s_1) be the rank-signature of Q'. We have then the following relation between the four numbers r, s, r_1, s_1.

THEOREM II. r_1-s_1 *is equal to either* $r-s$ *or* $r-s-1$.

For, from the geometrical significance of signature, Q' contains real flat regions of maximum dimensions $n-1-(r_1-s_1)$, and Q contains real flat regions of maximum dimensions $n-(r-s)$.

Since the intersection of S_{n-1} with a real flat region in Q is a real flat region in Q', we have

$$n-1-(r_1-s_1) \geqslant n-1-(r-s)$$
$$n-1-(r_1-s_1) \leqslant n-(r-s).$$

Hence $(n-1)-(r_1-s_1)$ is equal either to $n-1-(r-s)$ or to $n-(r-s)$; in other words r_1-s_1 is either equal to $r-s$ or to $r-s-1$.

Supposing S_n to be an affine space, and S_{n-1} the prime at infinity, we see that the affine types of quadric loci in S_n fall into two classes, which we may call the *elliptic class* ($r_1-s_1=r-s$) and the *hyperbolic class* ($r_1-s_1=r-s-1$) respectively. For a hyperbolic quadric certain flat regions of maximum dimension lying on the quadric, are situated entirely at infinity; for an elliptic quadric this is no longer the case.

The real ellipse in the plane belongs to the elliptic type; for here, we have

$$n=2,\ r=3,\ s=1,\ r_1=2,\ s_1=0;\ r-s=r_1-s_1.$$

Similarly, *the plane hyperbola belongs to the hyperbolic type*; for

$$n=2,\ r=3,\ s=1,\ r_1=2,\ s_1=1;\ r_1-s_1=r-s-1.$$

Lastly, *the plane parabola and the imaginary ellipse both belong to the hyperbolic type*; for, we have respectively for these:

$$r=3,\ s=1,\ r_1=1,\ s_1=0;\ r_1-s_1=r-s-1$$
$$r=3,\ s=0,\ r_1=2,\ s_1=0;\ r_1-s_1=r-s-1.$$

III. Rank of quadric and of a prime section

To find the relation between the rank of a quadric and a prime section of it, suppose first that the quadric is non-singular, so that its rank $r=n+1$. Then the section by a general prime is also non-singular, so that its rank $r_1=n=r-1$. The section by a tangent prime however has one singular point, namely, the unique point of contact; hence for this case $r_1=n-1=r-2$. If the quadric is singular, and of rank r, the section by a general prime has a singular region of one dimension lower, and is therefore of the same rank r; if the section contains the vertical (or singular) region of the quadric, then $r_1=r-1$; lastly if the section is tangent to the quadric (at a non-singular point), then $r_1=r-2$. These exhaust all possible cases.

Consider an affine space and let the ranks of the quadric and its section by the prime at infinity be r, r_1; then we have seen that $r_1=r$ or $r-1$ or $r-2$. If $r_1=r-2$, we shall say that *the quadric is parabolic*; it is clear that in this case the prime at infinity is a tangent prime to the quadric at a non-singular point. A singular quadric may be called *conical,* and its singular region the *vertical region.* A conical quadric whose vertical region lies at infinity will be called *cylindrical.* It follows that if $r_1=r-1$, the quadric is either cylindrical or non-singular. Finally if $r_1=r$, the quadric must be conical (without being cylindrical).

THEOREM III. *A parabolic quadric is necessarily of the hyperbolic type; a conical (non-cylindrical) quadric is necessarily of the elliptic type.*

For, let r, s, r_1, s_1 represent the ranks and signatures of the quadric and its section by the prime at infinity. For a parabolic quadric $r=r_1+2$; if it is also elliptical $s=s_1+2$. This is impossible as the prime at infinity being a tangent contains a flat region of maximum dimensions lying on the quadric; the quadric is therefore hyperbolic so that $s=s_1+1$.

For a conical quadric, a flat region of maximum dimensions lying on the quadric necessarily contains the vertical region; if the quadric is not a cylinder, the prime at infinity does not contain the vertical region and therefore does not contain any region of maximum dimension lying on the quadric. The quadric is therefore elliptical.

This theorem shews the possible projective types of the quadric (i.e. the values of r, s), when its intersection with the prime at infinity is of given projective type (i.e. when r_1, s_1 are known); namely the possible values of r, s are

$$(r_1, s_1), (r_1+1, s_1+1), (r_1+1, s_1), (r_1+2, s_1+1).$$

We are now in a position to shew that the four numbers r, s, r_1, s_1 completely characterise the affine type of the quadric locus—that is, that any two quadrics with the same four characteristic numbers can be transformed into each other by an *affine* transformation.

We may choose n of the points of reference $X_1, X_2, \ldots, X_n$ (say) at infinity, and X_0 in the finite part of S_n. We may suppose the reference points at infinity to have been so chosen that the section by X_0 (i.e. the prime at infinity) has the form:

$$Q_1\equiv -X_1^2-X_2^2-\ldots-X_{s_1}^2+X_{s_1+1}^2+\ldots+X_{r_1}^2=0 \quad (s_1 \leqslant \tfrac{1}{2}r_1).$$

It follows then that the quadric Q in S_n has the form

$$Q\equiv Q_1+X_0X=0,$$

where X represents some prime. The linear form X may be (1) linearly independent of $X_0X_1\ldots X_{r_1}$ (*this can happen only if* $r_1<n$), (2) linearly dependent on $X_0X_1X_2\ldots X_{r_1}$.

In case (1) X can be taken in the place of one of the missing co-ordinates. The product X_0X can be expressed as the difference of two real squares and therefore $r=r_1+2$, $s=s_1+1$. The expression Q_1+X_0X is clearly an affine canonical form for this case, since any two expressions of this type can be transformed into each other by a transformation which leaves X_0 (i.e. the prime at infinity) invariant.

In case (2), we can write

$$2a_1X_0X_1-X_1^2=-(X_1-a_1X_0)^2+a_1^2X_0^2.$$

Thus by a linear transformation which leaves the prime at infinity invariant, the expression for Q can be reduced to

$$aX_0^2-X_1^2-\ldots-X_{s_1}^2+\ldots+X_{r_1}^2,$$

where a may be positive, zero or negative. This gives three canonical reduced forms corresponding respectively to $(r=r_1+1, s=s_1)$, $(r=r_1, s=s_1)$, $(r=r_1+1, s=s_1+1)$. This shews in a general manner, that there is effectively only one affine type corresponding to each set of values of (r, s, r_1, s_1).

IV. The elliptic and hyperbolic affine types

Let us denote by $P(n)$ the number of projective types of quadric loci in S_n, so that

$$P(n)=\left[\frac{n+3}{2}\right]\times\left[\frac{n+4}{2}\right],$$

as already shewn; let $A(n)$ denote the number of affine types in S_n and $EA(n)$, $HA(n)$ the number of elliptic and hyperbolic affine types respectively. We have seen that for given (r_1, s_1) there are in general two elliptic types in S_n given by $(r=r_1, s=s_1)$ and $(r=r_1+1, s=s_1+1)$ respectively, and two hyperbolic types given by $(r=r_1+1, s=s_1)$ and $(r=r_1+2, s=s_1+1)$. If, however, r_1 is even and $s_1=\frac{1}{2}r_1$, then in the former case s cannot take the value s_1+1, since $s_1+1>\frac{1}{2}(r_1+1)$. Thus $EA(n)$, the number of elliptic affine types in S_n is equal to $2P(n-1)$ diminished by the number of projective types in S_{n-1} in which $s_1=\frac{1}{2}r_1$. Thus

$$EA(n)=2P(n-1)-\left\{1+[n/2]\right\}.$$

In the same way, the value $r=r_1+2$ occurring in the enumeration of hyperbolic types becomes inadmissible when $r_1=n$; the number of projective types in S_{n-1} for which $r_1=n$ is

$$1+[n/2].$$

Hence

$$HA(n)=2P(n-1)-\{1+[n/2]\}=EA(n).$$

Hence

$$A(n)=HA(n)+EA(n)$$
$$=4P(n-1)-2-2[n/2].$$

If $\varepsilon(n)=0$ or 1 according as n is even or odd, we have,

$$4P(n-1)=\{n+2-\varepsilon(n)\}\{n+2+\varepsilon(n)\}$$
$$=(n+2)^2-\varepsilon(n).$$

Hence

$$A(n)=(n+2)^2-\varepsilon(n)-2-\{n-\varepsilon(n)\}$$
$$=(n+1)(n+2).$$

We have therefore

THEOREM IV. *The total number of affine types in S_n is $(n+1)(n+2)$; one half of these are elliptic, the other half hyperbolic.*

V. TABLE OF THE TWELVE AFFINE TYPES IN THE PLANE

r_1	s_1	r	s	Types	
0	0	0	0	Identically vanishing conic	E
		1	0	Squared line at infinity	H
		2	1	Line at infinity and finite line	H
1	0	1	0	Squared line	E
		2	1	Real parallel lines	E
		2	0	Imaginary parallel lines	H
		3	1	Parabola	H
2	0	2	0	Imaginary lines with real vertex	E
		3	0	Imaginary conic	H
		3	1	Ellipse	E
2	1	2	1	Real lines	E
		3	1	Hyperbola	H

In the above the elliptic types are marked E and the hyperbolic types marked H.

VI. Table of the 20 affine types in space

r_1	s_1	r	s	Types	
0	0	0	0	Identically vanishing quadric	E
		1	0	Squared plane at ∞	H
		2	1	Plane at ∞ and another plane	H
1	0	1	0	Squared plane	E
		2	1	Real parallel planes	E
		2	0	Imaginary parallel planes	H
		3	1	Parabolic cylinder	H
2	0	2	0	Conjugate imaginary planes through a real line	E
		3	1	Elliptic cylinder	E
		3	0	Imaginary cylinder	H
		4	1	Elliptic paraboloid	H
2	1	2	1	Real planes	E
		3	1	Hyperbolic cylinder	H
		4	2	Hyperbolic paraboloid	H
3	0	3	0	Imaginary cone with finite vertex	E
		4	1	Ellipsoid	E
		4	0	Imaginary quadric	H
3	1	3	1	Real cone	E
		4	2	Hyperboloid of one sheet	E
		4	1	Hyperboloid of two sheets	H

ON THE ZEROS OF BESSEL FUNCTIONS

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1. INTRODUCTION. Bourget* assumed that $J_n(x)$ and $J_{n+m}(x)$ have no zeros in common except may be $x=0$ if n and m are positive integers including zero. But it seems possible to prove corresponding theorems for some of the associated Bessel Functions. This paper contains three theorems relating to the zeros of such functions.

2. THEOREM I. *$G_n(x)$ and $G_{n+m}(x)$ have no common root except may be $x=0$ when n is real and $\geqslant -1$ and $m \geqslant 1$ is a positive integer.*

PROOF: The following relations† are known

$$G_{n+1}(x)+G_{n-1}(x)=\frac{2n}{x}G_n(x) \tag{1}$$

$$G'_n(x)=\frac{n}{x}G_n(x)-G_{n+1}(x) \tag{2}$$

and
$$x^2G''_n(x)+xG'_n(x)+(x^2-n^2)G_n(x)=0. \tag{3}$$

By (2), G_n and G_{n+1} have no common root except may be $x=0$ for any such root will make $G_n=0$ and then (3) gives $G''_n = G'''_n = \ldots = 0$ and so $G_n(x)=0$ identically. Next, using (1) successively, we get

$$G_{n+m}(x)=\begin{vmatrix} \frac{2}{x}(n+m-1) & 1 & 0 & 0 & .. & 0 & 0 \\ 1 & \frac{2}{x}(n+m-2) & 1 & 0 & .. & 0 & 0 \\ 0 & 1 & .. & .. & .. & .. & .. \\ .. & .. & .. & .. & .. & .. & .. \\ 0 & .. & .. & .. & .. & G_n(x) & G_{n+1}(x) \end{vmatrix}$$

$$=\Delta_m^{(x)}G_{n+1}(x)-A_m^{(x)}G_n(x) \tag{4}$$

* G. N. Watson, *Theory of Bessel Functions*, (1922) p. 484.
† Andrew Gray, *Bessel Functions*, (1922) p. 23.

where

$$\Delta_m^{(x)} = (n+1)\ldots(n+m-1)\begin{vmatrix} \frac{2}{x} & \frac{1}{n+1} & 0 & \ldots & \ldots & 0 \\ \frac{1}{n+2} & \frac{2}{x} & \frac{1}{n+2} & \ldots & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & \ldots & \ldots & \ldots & \frac{2}{x} \end{vmatrix}$$

$$=(n+1)\ldots(n+m-1)\; f_0(\beta), \text{ say where } \beta=\frac{2}{x}. \quad (5)$$

Now by (4) any root $x=x_0$ common to G_{n+m} and G_n must make $\Delta_m^{(x)}$ zero since x_0 cannot be a root of G_{n+1} as well. Hence $\beta_0=2/x_0$ will be a root of $f_0(\beta)$. Now it is known that G_n as well as G_{n+m} has no real root. Hence if we prove that all the roots of $f_0(\beta)$ are real, G_n and G_{n+m} can have no common zero. Since $f_0(\beta)$ is a polynomial of degree m in β it is sufficient to prove that $f_0(\beta)$ has m real roots. Let $f_r(\beta)$ denote the determinant obtained from $f_0(\beta)$ by suppressing the first r rows and columns. Consider the sequence of functions

$$f_0(\beta), f_1(\beta), \ldots, f_r(\beta), \ldots, f_m(\beta)=1.$$

These satisfy the relations

$$f_r(\beta)=\beta f_{r+1}(\beta)-\frac{f_{r+2}(\beta)}{(n+r+1)\,(n+r+2)},\; r=0,\ldots,\; m-1. \quad (6)$$

Now if $f_0(\alpha)=0$, we have

$$\frac{f_0(\alpha+h)}{f_1(\alpha+h)}=h\left[1-\frac{d}{d\alpha}\frac{f_2(\alpha)}{f_1(\alpha)}\right]+\text{higher powers of } h;$$

or,

$$f'(\alpha)=f_1(\alpha)\left[1-\frac{d}{d\alpha}\frac{f_2(\alpha)}{f_1(\alpha)}\right]+h[\ \ldots]. \quad (7)$$

Now,

$$\frac{d}{d\alpha}\frac{f_2(\alpha)}{f_1(\alpha)}=\frac{1}{f_1^2}\left[-f_2^2+\frac{f_2^2}{(n+2)\,(n+3)}\frac{d}{d\alpha}\frac{f_3}{f_2}\right]$$

$$=-\frac{1}{f_1^2}\left[f_2^2+\frac{f_3^2}{(n+2)\,(n+3)}+\ \ldots\ \right] \quad (8)$$

by using (6).

So by (7) and (8),

$$f'_0(\alpha)=f_1(\alpha)\ [\text{a positive quantity}]. \quad (9)$$

Therefore the sequence of polynomials $f_0(\beta), \ldots, f_m(\beta)$ possesses the following properties:

(*i*) no two consecutive polynomials can vanish for the same value of β,

(ii) if $\beta=\alpha$ is a root of $f_r(\beta)$, $f_{r-1}(\alpha)$ and $f_{r+1}(\alpha)$ have opposite signs, $r=1, 2, \ldots$,

(iii) if $f(\alpha)=0$, $f_1(\alpha)$ and $f'(\alpha)$ have the same sign,

(iv) the last function $f_m(\beta)=1$ is of constant sign. Hence these form a Sturm* series of polynomials and therefore if $f(a)\neq 0$, $f(b)\neq 0$, the number of real roots of $f(\beta)$ lying in (a, b) is the difference between the number of changes of sign in the two sequences $[f_r(a)]$ and $[f_r(b)]$, $r=0, 1, \ldots, m$. Considering the interval $(-\infty, \infty)$ we find that $[f_r(\infty)]$ has the sequence of signs $(+, +, +, \ldots, +)$ and $[f_r(-\infty)]$ the sequence $[(-1)^m, (-1)^{m-1}, \ldots, -1, +1]$ and so $f(\beta)$ has exactly m roots in $(-\infty, \infty)$, i.e. all the roots of $f(\beta)$ are real. Hence the proof of the theorem is complete.

THEOREM II. *$K_n(x)$ and $K_{n+m}(x)$ have no common zero except may be $x=0$, where n is real and >-1 and $m \geqslant 1$ is a positive integer.*

By definition,

$$K_n(x)=e^{\frac{\pi i n}{2}} G_n(ix)$$

$$K_{n+m}(x)=e^{\frac{\pi i(n+m)}{2}} G_{n+m}(ix).$$

Therefore if $x=\alpha$ were a common root of K_n and K_{n+m} we shall have $x=i\alpha$ a common root of $G_n(x)$ and $G_{n+m}(x)$ which is impossible by theorem (I).

THEOREM III. *$I_n(x)$ has no complex zeros, that is no zero of the form $p+iq$ where $p\neq 0$, $q\neq 0$.*

PROOF: If λ and μ are two roots of $I_n(x)=0$, then it is known that†

$$(\lambda^2-\mu^2)\int_0^1 x\, I_n(\lambda x)\, I_n(\mu x)=0. \tag{10}$$

Now if $p+iq$ were a root of I_n so is $p-iq$ since $I_n(x)$ is real for real x. Hence taking $\lambda=p+iq$, $\mu=p-iq$ we have

$$I_n(\overline{p+iq}\,x)\, I_n(\overline{p-iq}\,x)>0, \tag{11}$$

$$\lambda^2-\mu^2=(\lambda-\mu)(\lambda+\mu)=4ipq\neq 0, \tag{12}$$

while (10) gives owing to (12),

$$\int_0^1 x\, I_n(\lambda x)\, I_n(\mu x)=0$$

which contradicts (11). Hence the theorem is true.

* Burnside and Panton, *The Theory of Equations*, Vol. 1 (1918) Art. 96.
† *See* Andrew Gray, *loc. cit.* p. 82.

ON THE ARITHMETICO-LOGICAL PRINCIPLE OF DUALITY*

BY MISS S. PANKAJAM, M.A., L.T.

INTRODUCTION

The Logical Calculus contains two operations $\oplus$ and $\otimes$, having respectively the sense of 'or' and 'and'. They are connected by the following laws:

(α) Commutative $\qquad x\oplus y=y\oplus x.$
$\qquad x\otimes y=y\otimes x.$

(β) Distributive $\qquad x\otimes(y\oplus z)=(x\otimes y)\oplus(x\otimes z).$
$\qquad x\oplus(y\otimes z)=(x\oplus y)\otimes(x\oplus z).$

(γ) Associative $\qquad x\oplus(y\oplus z)=(x\oplus y)\oplus z.$
$\qquad x\otimes(y\otimes z)=(x\otimes y)\otimes z.$

We see that in these axioms, $\oplus$ and $\otimes$ occur symmetrically, i.e. the form of the axioms remains unchanged (or invariant) for an interchange of $\oplus$ and $\otimes$. The universe U and the Null element 0, satisfy the following relations

$$\left.\begin{array}{l} U\otimes A=0\oplus A=A \\ U\oplus A=U;\ A\otimes 0=0. \end{array}\right\} \qquad \text{(i)}$$

Also, the converse operation $\bar{A}$ has the sense of 'not A'. We have for this the two axioms,

$$\left.\begin{array}{l} A\oplus\bar{A}=U \\ A\otimes\bar{A}=0. \end{array}\right\} \qquad \text{(ii)}$$

From the fact, that the form of the axioms α, β, γ remains invariant for an interchange of $\oplus$ and $\otimes$, we can infer† a Logical Principle of Duality, namely:

If $f(A, B, C, \ldots, K)\equiv\phi(A, B, C, \ldots, K)$ is identically true in $A, B, C, \ldots$ then, the relation continues to be valid, after $\oplus$ and $\otimes$, are interchanged on both sides.

* I am indebted to Dr. Vaidyanathaswamy for his kind interest and help in the preparation of this paper.

† This is the method followed by Whitehead, in his *Universal Algebra* for proving the principle of Duality. Another method of proof is given in § 1 of this paper. This Reciprocity was first discovered by Pierce—*Proceedings of the American Academy of Arts and Sciences* (1867), later, by Schröder—*Der Operationkras Logikkalkuls.* Both these works have not been available to the present writer.

The symbols of the Logical Calculus can be interpreted in two ways either (*a*) as propositions or (*b*) as classes (or attributes which define classes). The two interpretations are instances of the same abstract calculus. For our present purpose, we are concerned only with the Calculus of Classes. The scope of the Calculus of Classes, can still be further extended if we let the class symbols A, B, C, . . . also represent the Cardinal Numbers of the corresponding classes. (We assume of course, that the Cardinal Numbers of all the classes which occur, as well as of the Universe of Discourse to be finite). In the Symbolic Calculus thus extended, we will have to deal in addition to logical operations $\oplus$ and $\otimes$, with the arithmetical operations of $+$ and $-$ and multiplication by scalars, i.e. with expressions of the form— $c_1 f_1 \pm c_2 f_2 \pm \ldots$ where $c_1, c_2, \ldots$ are Arithmetical Scalars and $f_1, f_2 \ldots$ are functions of arbitrary variables or classes, formed by purely logical operations. The new axiom which connects the Arithmetical with the Logical Operations is

$$A+B=(A\oplus B)+A\otimes B \qquad \text{(iii)}$$

which may be called the *Arithmetico-Logical Axiom*. This Axiom is put in evidence, by the usual geometrical representation of A and B, as intersecting regions. Any identity which contains Logical and Arithmetical Operations, and which can be derived purely from axioms α, β, γ and (iii), may be called an *Arithmetico-Logical Identity*.

In Dr. Vaidyanathaswamy's paper* "On the Arithmetico-Logical Symmetric Functions of n attributes", three pairs of identities, (quoted below), are derived. These are related to each other, by a dual symmetry, i.e. each identity of a pair is derived from the other, by interchanging $\oplus$ and $\otimes$ without interference with the Arithmetical Operations.

For instance, in the following theorems,*

THEOREM I.

$A_1 \oplus A_2 \oplus \ldots \oplus A_n = \Sigma A_1 - \Sigma (A_1 A_2) + \ldots$ where, Σ's represent always arithmetical sums.

THEOREM II.

$(A_1 A_2 A_3 \ldots A_n) = \Sigma A_1 - \Sigma S_{12} + \Sigma S_{123} - \Sigma S_{1234} + \ldots$ where $S_{pqr} \ldots$ denotes $A_p \oplus A_q \oplus A_r \oplus \ldots$

THEOREM IV.

$$\alpha_r = A_r^r - \binom{r}{1} A_{r+1}^{r+1} + \binom{r+1}{2} A_{r+2}^{r+2} - \ldots$$

* *Proceedings of the Indian Academy of Sciences* Vol. II, No. I, July 1935.

THEOREM V.

$$\alpha_{n-r+1}=A^1_r-\binom{r}{1}A^1_{r+1}+\binom{r+1}{2}A^1_{r+2}-$$

THEOREM VI.

$$\binom{n\quad p}{r-p}\alpha_p=A^p_r-\binom{p}{1}A^{p+1}_{r+1}+\binom{p+1}{2}A^{p+2}_{r+2}-\ \ldots\ .$$

THEOREM VII.

$$\binom{n-r+p-1}{p-1}\alpha_{n-r+p}=A^p_r-\binom{r-p+1}{1}A^p_{r+1}$$
$$+\binom{r-p+2}{2}A^p_{r+2}-\ \ldots$$

it is clear that, II, V, VII can be obtained from I, IV, VI, respectively by the interchange of $\oplus$ and $\otimes$.

It is clear that Theorem I is an extension of axiom (iii), to which it reduces, if we put $n=2$. It may be instructive to derive this theorem, purely from the axioms of the calculus, by straightforward induction.

Assume the theorem to be true for n attributes, i.e.
$A_1\oplus A_2\oplus A_3\oplus\ \ldots\ \oplus A_n=\Sigma A_1-\Sigma A_1A_2+\ \ldots\ +A_1A_2\ \ldots\ A_n$.
By changing A_n into $A_n\oplus A_{n+1}=A_n+A_{n+1}-(A_n\otimes A_{n+1})$ (from axiom iii),

L. H. S. of Theorem I, becomes

$$A_1\oplus A_2\oplus\ \ldots\ \oplus A_n\oplus A_{n+1}.$$

R. H. S. becomes

$$A_1+A_2+\ \ldots\ +A_n-A_n\otimes A_{n+1},$$

(from axiom iii)

$$=\Sigma A_1-\Sigma A_1A_2+\ \ldots$$

where the number of attributes is now $n+1$. This is of the same form as the expression assumed for n attributes.

And, since for $n=2$, Theorem I coincides with the axiom, the theorem is thus proved.

The extension of the Logical Principle of Duality to the case in which, (as here), the Arithmetical Operations occur, in addition to Logical Operations, may be called, the *Arithmetico-Logical Principle of Duality*.

It is the object of this paper, to give a precise formulation and proof of this Principle of Duality.

§ 1. LOGICAL PRINCIPLE OF DUALITY

If we denote by $\bar{A}$, (as stated in the Introduction), the 'not' of A, we can see from the following, that there is a close relationship between the operation 'not' and the interchange of $\oplus$ and $\otimes$:—

$$\left.\begin{aligned}\bar{A}\oplus\bar{B}&=\overline{A\otimes B}\\ \bar{A}\otimes\bar{B}&=\overline{A\oplus B}.\end{aligned}\right\} \qquad (iv)$$

These can at once be proved if we represent A and B, geometrically, as intersecting regions.

Now, if there is a function $f\,(A, B, C, \ldots)$ formed from the symbols $A, B, C, \ldots$ by means of the logical operations, and if the identity

$$f(A, B, C, \ldots) \equiv \phi(A, B, C, \ldots)$$

holds good for all values of $A, B, C, \ldots$ then, the Principle of Duality, expresses the fact, that the identity continues to be valid after the interchange of $\oplus$ and $\otimes$ on both sides.

To prove this, we require the following lemma.

LEMMA 1.—*If Df represents the operation of interchanging $\oplus$ and $\otimes$ in f, then $\overline{Df}=f(\bar{A}, \bar{B}, \bar{C}, \ldots)$.*

We shall prove this by Induction, with the help of the 'Grade' of a logical function which is defined as the number of times the logical operations are performed to arrive at the function. For example a function of the first grade, can be of the types $A\oplus B$ and $A\otimes B$ only.

PROOF.

This lemma is obviously true for a function of the first grade. For, we have in these cases,

$$f(\bar{A}, \bar{B}) \equiv \bar{A}\oplus\bar{B} = (\overline{A\otimes B}) \text{ (by axiom iv)}$$
$$= \left\{ \overline{Df(A, B)} \right\};$$

and $$f(\bar{A}, \bar{B}) = \bar{A}\otimes\bar{B} = \overline{Df(A, B)}.$$

Thus, the lemma is true for functions whose grade $n=1$.

Assume the lemma to hold good for any function F_n of nth grade.

Then, functions of the $(n+1)$th grade, can be only of the forms $F_n\oplus T$ and $F_n\otimes T$. Now let

$$F_{n+1}(A, B, \ldots, K, T) \equiv F_n\oplus T.$$

Applying the lemma to this, we have

$$\overline{DF_{n+1}(A, B, \ldots, K, T)} = \overline{DF_n(A, B, \ldots, K) \otimes T}$$
$$= \overline{DF_n(A, B, \ldots K)} \oplus \overline{T}$$
(from axiom iv)
$$= F_n(\overline{A}, \overline{B}, \ldots, \overline{K}) \oplus \overline{T}$$
$$= F_{n+1}(\overline{A}, \overline{B}, \ldots, \overline{K}, \overline{T}).$$

A similar proof, holds good for the case $F_{n+1} = F_n \otimes T$. We thus see, that the lemma is true for a function of $(n+1)$th grade also.

But, the lemma is true for a function of the first grade. Hence, it is true for a function of any grade whatever.

From this, we are now in a position to prove the Principle of Duality which states *that after D is applied to both sides of any logical identity in independent variables, the identity continues to be valid.*

For, since $f(A, B, C, \ldots, K) \equiv \phi(A, B, C, \ldots, K)$, is valid for all arbitrary values of $A, B, \ldots, K$, it follows that if the A's be replaced by the $\overline{A}$'s, it is still valid, i.e.

$$f(\overline{A}, \overline{B}, \ldots, \overline{K}) \equiv \phi(\overline{A}, \overline{B}, \ldots, \overline{K}).$$

By Lemma I,

$$f(\overline{A}, \overline{B}, \ldots, \overline{K}) = \overline{Df}.$$

Hence, we have,

$$\overline{Df}(A, B, \ldots, K) \equiv \overline{D\phi}(A, B, \ldots, K),$$

i.e. $$Df(A, B, \ldots, K) \equiv D\phi(A, B, \ldots, K)$$

which proves the Principle of Duality for a logical identity in independent variables.

§ 2. Arithmetico-logical principle of duality

In the Arithmetico-Logical Calculus, we have to deal with the following form

$$\lambda_1 f_1(A, B, \ldots) + \lambda_2 f_2(A, B, \ldots) + \lambda_3 f_3(A, B, \ldots) + \cdots$$

where, $\lambda_1, \lambda_2, \ldots$ are positive or negative scalars, and the f's are logical functions.

We may call an expression of this type, an *Arithmetico-Logical function*. It may also be noticed that, in our symbolism, while the logical function represents both classes and cardinal numbers, an Arithmetico-Logical Function, represents a number, but not a class.

The Arithmetico-Logical Principle of Duality, may now be formulated as follows:—*If two Arithmetico-Logical Functions are* *identically equal, (i.e. in virtue of the axioms), then, after* $\oplus$ *and* $\otimes$ *are interchanged, i.e. after D has been performed on both sides, the identity continues to be valid.*

To prove this, we require the following

LEMMA 2. *If an Arithmetico-Logical Function of independent variables is identically equal to zero, then, the sum of the scalar coefficients of the Logical Functions, is equal to zero.*

Or more generally, *if two Arithmetico-Logical Functions are identically equal then the sums of the scalar coefficients of the Logical Functions on each side, are equal.*

Symbolically: *If*

$$\lambda_1 f_1(A, B, \ldots) + \lambda_2 f_2(A, B, \ldots) + \ldots \equiv 0,$$

then, $\Sigma\lambda_1 = 0$;

and, if

$$\lambda_1 f_1(A, B, \ldots) + \lambda_2 f_2(A, B, \ldots) + \ldots \equiv \mu_1 \phi_1(A, B, \ldots) + \mu_2 \phi_2(A, B, \ldots) + \ldots$$

then $\Sigma\lambda_1 = \Sigma\mu_1$.

Now to prove that $\Sigma\lambda_1 = 0$. Let us denote the universe of Discourse by U. We can immediately see, that if we put two attributes A, B each $= U$, then,

$$A \oplus B \text{ and } A \otimes B \text{ are each } = U.$$

With this observation, if each $A, B, \ldots$ (which are the variables occurring in the logical functions $f_1, f_2, \ldots$ in the Arithmetico-Logical Identity) be put $= U$, and N be the Cardinal Number of the Universe, then the identity:

$$\lambda_1 f_1(A, B, \ldots) + \lambda_2 f_2(A, B, \ldots) + \ldots \equiv 0,$$

reduces to $N\Sigma\lambda_1 \equiv 0$.

Since $N \neq 0$, it follows that $\Sigma\lambda_1 = 0$.

The truth of this lemma, can be verified in all the three pairs of identities which Dr. Vaidyanathaswamy has derived in his paper*. For instance, Theorem IV (quoted above) is

$$\alpha_r = A'_r - \binom{r}{1} A'^{\,+1}_{r+1} + \binom{r+2}{2} A'^{\,r+2}_{r+2} - \ldots$$

Here,

$$\Sigma\lambda_1 = {}^nC_{n-r} - {}^rC_1\,{}^nC_{n-r+1} + {}^rC_2\,{}^nC_{n-r+2} - \ldots,$$

i. e. $\Sigma\lambda_1 = 1$, which is also the scalar coefficient on the left. Similarly for the other theorems.

Since the equation $\Sigma\lambda_1 f_1 = 0$, is identically true in $A, B, C, \ldots$ we can replace $A, B, C, \ldots$ by $\bar{A}, \bar{B}, \bar{C}, \ldots$. We are now in a position to prove the Arithmetico-Logical Principle of Duality.

* *loc. cit.*

If $\lambda_1 f_1 + \lambda_2 f_2 + \ldots \equiv \mu_1 \phi_1 + \mu_2 \phi_2 + \ldots$ be an Arithmetico-Logical Identity, it can be put in the form

$$\lambda_1 F_1 + \lambda_2 F_2 + \ldots \equiv 0.$$

Therefore,

$$\lambda_1 F_1 (\bar{A}, \bar{B}, \bar{C}, \ldots) + \lambda_2 F_2 (\bar{A}, \bar{B}, \bar{C}, \ldots) + \ldots \equiv 0.$$

Applying Lemma 1.

$$\lambda_1 \overline{DF_1} + \lambda_2 \overline{DF_2} + \ldots \equiv 0,$$

i. e. $$\lambda_1 (N - DF_1) + \lambda_2 (N - DF_2) + \ldots \equiv 0,$$

i. e. $$N \Sigma \lambda_1 - \Sigma \lambda_1 DF_1 \equiv 0.$$

Since $$\Sigma \lambda_1 = 0$$ by Lemma 2,

we have $$\Sigma \lambda_1 DF_1 = 0.$$

Thus, the Arithmetico-Logical Principle of Duality is proved.

In conclusion, we observe that, the Principle of Duality is closely related to the Logical Operation of 'not' (as is clear from our proofs) which indicates the Dichotomy in the calculus. Also, in Dr. Vaidyanathaswami's paper, when theorems 1, 2, 4, 5, 6 and 7 are applied to the real numbers, the same duality existing between these theorems, reappears as the duality between 'greater than' and 'less than'. And, the concepts of the rth G. C. D. and L. C. M., are defined in terms of 'greater' and 'less', and therefore, the same duality is carried forward into the formulæ on page 62 (*l.c.*) relating to rth G.C.D., and L.C.M. Therefore, the basis of all these forms of duality in the last instance is to be sought in the Dichotomy of the logical 'Not'.

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