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'*On skew fields of a given degree*' *by F. W. Levi, Vol. XI.* (1947).

p. 86 formula (4) for $=$ read $\leqslant$

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p. ,, ,, (12) for $= [(c, b^{-1}a)]$ read $- [(c, b^{-1}a)]$.

ON A THEOREM OF POLYA

BY

M. H. STONE, (*Chicago, U.S.A.*).

[Received 24 April, 1948].

In a well-known note, G. Pólya has given an interesting criterion that an integral function may reduce to a polynomial.* Certain questions in the theory of normed rings have been settled by E. Hille through an appeal to Pólya's result, which consequently attracted renewed attention.† The purpose of the present note is to give a new and simplified proof of Pólya's criterion, generalized in a manner suitable for immediate application to the situation considered by Hille. For the sake of completeness this application will be carried out, only a few lines being needed for the purpose.

The generalization we need consists in allowing the functions discussed to assume values in a complex Banach space rather than in the complex field. A Banach space B is a system of vectors $a, b, c, \ldots, x, y, z, \ldots$, admitting the operations of vector addition and multiplication by complex scalars (denoted here by Greek letters), each vector x having a real-valued length or norm $|x|$ such that $|x| \geqslant 0$ (equality holding only for the null vector 0), while $|\lambda x| = |\lambda||x|$ and $|x+y| \leqslant |x| + |y|$—and the whole system being complete in the sense that every

* *Jahresbericht der Deutschen Mathematiker Vereinigung*, 40 (1931), Problem 106, p. 81 of the Problem Section. Various solutions have appeared in Volume 43 (1931), 13-7.

† *Proceedings of the National Academy of Sciences*, U.S.A., 30 (1944), pp. 58-60.

Cauchy sequence of vectors has a limit [$\lim_{m,n\to\infty} |x_m - x_n| = 0$ implies the existence of a vector x such that $\lim_{n\to\infty} |x - x_n| = 0$]. The complex field furnishes the simplest example of a complex Banach space. Since a fairly routine development suffices to provide the analytical techniques for handling B-valued functions of a real or complex argument, we shall use those techniques without further justification.*

We now state the basic result.

THEOREM. *Let f be a function of the complex variable λ with values in a complex Banach space B; let f be an integral function in the sense that it has a power series development $f(\lambda) = \sum_{n=0}^{\infty} \lambda^n a_n$, $a_n \in B$, convergent for all λ [and hence necessarily uniformly convergent on any bounded part of the λ-plane]; and for large r let $|f(\lambda)| \leqslant c\, r^N |\sec\theta|^N$, where $\lambda = r e^{i\theta}$, c is a positive real constant, and N is a non-negative integer. Then f is a polynomial of degree not exceeding N, in the sense that $a_k = 0$ for $k \geqslant N+1$.*

PROOF:

Since f is a continuous function, we can form the integral expression

$$I(r,k) = \frac{1}{2\pi i} \int_{|\lambda|=r} \lambda^{-k-1} \left(1 + \frac{\lambda^2}{r^2}\right)^N f(\lambda)\, d\lambda.$$

Since $|1 + \lambda^2/r^2| = 2|\cos\theta|$ we have for $k \geqslant N+1$

$$|I(r,k)| \leqslant 2^N c\, r^{N-k}, \quad \lim_{r\to\infty} I(r,k) = 0.$$

* The reader who wishes details will find them in a forthcoming book of E. Hille, *Functional Analysis and semi-groups*, Colloquim Series of the American Mathematical Society pp. 1-64, especially pp. 52-60.

On the other hand, a simple calculation expresses the integrand as a series in powers of λ, namely

$$\lambda^{-k}\left(1+\frac{\lambda^2}{r^2}\right)^N f(\lambda)$$
$$=\sum_{p=0}^{\infty}\lambda^{p-k-1}\Big(\sum_{2m+n=p}\frac{N!}{m!\,N-m!}\frac{1}{r^{2m}}a_n\Big).$$

Cauchy's integral theorem thus allows us to make the evaluation

$$I\ (r,k)=\sum_{2m+n=k}\frac{N!}{m!\ N-m!}\frac{1}{r^{2m}}a^n.$$

It follows that $\lim_{r\to\infty} I\ (r,k)=a_k$. The theorem then follows directly from the earlier result.

A simple application now yields

PÓLYA'S THEOREM. *If f is an integral function as before, and if $g(\mu)=f\left(\frac{1+\mu}{1-\mu}\right)$ has the power series developments*

$$g(\mu)=\sum_{n=0}^{\infty}\mu^n b_n,\ |b_n|=O(n^M),\ |\mu|<1,$$
$$g(\mu)=\sum_{n=0}^{\infty}\mu^{-n}c_n,\ |c_n|=O(n^M),\ |\mu|>1,$$

where b_n and c_n are in B and M is a non-negative integer, then f is a polynomial of degree not exceeding $N=M+1$.

PROOF. Elementary direct calculations given by Pólya in the case of complex-valued functions show that f satisfies the inequality of the preceding theorem, with $N=M+1$; and the result then follows. It is evident that the assumptions concerning the magnitudes of $|b_n|$ and $|c_n|$ require the existence of a positive real constant

c such that $|b_n| \leqslant c \frac{(n+M)!}{n!\,M!}$, $|c_n| \leqslant c \frac{(n+M)!}{n!\,M!}$. We therefore have

$$|g(\mu)| \leqslant c \sum_{n=0}^{\infty} \frac{(n+M)!}{n!\,M!} |\mu|^n = c(1-|\mu|)^{-M-1}, \; |\mu| \leqslant 1,$$

$$|g(\mu)| \leqslant c \sum_{n=0}^{\infty} \frac{(n+M)!}{n!\,M!} |\mu|^{-n} = c(1-|\mu|^{-1})^{-M-1}, \; |\mu| \geqslant 1,$$

and hence

$$|g(\mu)| \leqslant c(1-\mu^*)^{-M-1}, \; \mu^* = \min\,(|\mu|, |\mu|^{-1}).$$

Since

$$\mu = -\frac{1-\lambda}{1+\lambda}, \; \lambda = re^{i\theta},$$

we obtain

$$|\mu| = \left(\frac{1+r^2-2\,r\cos\theta}{1+r^2+2r\,\text{ros}\,\theta}\right)^{1/2}; \mu^* = \left(\frac{1+r^2-2r\,|\cos\theta|}{1+r^2+2r\,|\cos\theta|}\right)^{1/2};$$

$$(1-\mu^*)^{-1} = \frac{1+\mu^*}{1-(\mu^*)^2} \leqslant \frac{2}{1-(\mu^*)^2} = \frac{1+r^2+2\,r\,|\cos\theta|}{2\,r\,|\cos\theta|}$$

$$\leqslant \frac{(1+r)^2}{2}\,|\sec\theta| \leqslant r\,|\sec\theta| \text{ for } r \geqslant 1;$$

$$|f(\lambda)| = |g(\mu)| \leqslant c\,r^N |\sec\theta|^N \text{ for } N = M+1, \; r \geqslant 1.$$

This completes the proof.

The applications to be made of these two theorems concern the case in which a product xy is defined for all x and y in B with the following properties:

(1) xy is linear in each factor*;

(2) the resulting algebra is power-associative—that is, if x^n is defined recursively for $n \geqslant 1$ by the equation $x^{n+1} = xx^n$, then $x^m x^n = x^{m+n}$;

* By linearity we mean that $(\alpha x+\beta y)\,z = \alpha xz+\beta yz$, $x(\alpha y+\beta z) = \alpha xy+\beta xz$; these relations can be extended to infinite sums by using (4) to justify the required passages to the limit.

(3) there is a unit element e—that is, an element e such that $ex = xe = x$;

(4) $|xy| \leqslant |x|\,|y|$.

It is easily verified that the polynomials $\alpha_0 e + \alpha_1 x + \ldots + \alpha_n x^n$ and their limits in B constitute a subalgebra $B(e, x)$ which has all the properties prescribed for B and in addition is subject to the commutative and associative laws for multiplication. An element x is said to be nilpotent if $x^n = 0$ for some n. An element x is said to be a generalized nilpotent if $\lim_{n \to \infty} |x^n|^{1/n} = 0$. The theorem which we intend to prove offers a criterion for a generalized nilpotent element to be nilpotent. As a preliminary to the proof of this theorem we note the following

LEMMA. *If x is a generalized nilpotent and y any element in $B(e, x)$, then xy is also a generalized nilpotent. If x is a generalized nilpotent, then the element $y = e - x$ is in $B(e, x)$ and has in $B(e, x)$ a unique inverse y^{-1} given by the power series $y^{-1} = e + x + x^2 + \ldots + x^n + \ldots$.*

PROOF. When x is a generalized nilpotent and y is in $B(e, x)$, we can use the commutative and associative laws to infer that $|(xy)^n| = |x^n y^n| \leqslant |x^n|\,|y^n| \leqslant |x^n|\,|y|^n$, $|(xy)^n|^{1/n} \leqslant |x^n|^{1/n}\,|y| \to 0$. Hence xy is also a generalized nilpotent. When x is a generalized nilpotent the series $e + x + x^2 + \ldots + x^n + \ldots$ obviously converges, since for large n we have $|x^n| \leqslant r^n$, where $r < 1$. Its sum is in $B(e, x)$ and direct calculation shows that $y(e + x + x^2 + \ldots + x^n + \ldots) = (e + x + x^2 + \ldots + x^n + \ldots)\,y = e$. Hence the sum is an inverse of y. As usual in a commutative and associative algebra, this inverse is unique and may be denoted as y^{-1}. It should be observed, of course, that we have *not* asserted the absence of other inverses outside of $B(e, x)$.

We are now prepared to give the

THEOREM OF GELFAND AND HILLE. *If x is a generalized nilpotent element in an algebra B of the type described above, then a necessary and sufficient condition that $x^{N+1} = 0$ for $N \geqslant 0$ (respectively, that $x^N = 0$ for $N \geqslant 1$) is that the element $y = e - x$ and its inverse y^{-1} in $B(e, x)$ have the property $|y^n| = O(|n|^N)$ for $n = 0, \pm 1, \pm 2, \ldots$ (respectively, the property $|y^n| = o(|n|^N)$ for $n = 0, \pm 1, \pm 2, \ldots$).**

PROOF. Since all our calculations are made in the subalgebra $B(e, x)$ we may rely on the commutative and associative laws and their standard consequences, particularly those relating to the treatment of inverses. Since x is a generalized nilpotent the element $\lambda z = -\frac{\lambda}{2} x(e - 1/2\, x)^{-1}$ exists and is a generalized nilpotent in accordance with the lemma. (Note that $1/2\, x = x(1/2\, e)$ is a generalized nilpotent, hence that $(e - 1/2\, x)^{-1}$ exists in $B(e, 1/2\, x) = B(e, x)$, and hence that λz is a generalized nilpotent). The power series $e + \lambda z + \lambda^2 z^2 + \ldots + \lambda^n z^n + \ldots$ is therefore convergent for all λ to the inverse $(e - \lambda z)^{-1}$ in $B(e, \lambda z)$, which is part of $B(e, x)$. Hence the function

$$f(\lambda) = \frac{1}{2}(1+\lambda)\ (e - \frac{1}{2}x)^{-1}\ (e - \lambda z)^{-1},\ z = -\frac{1}{2} x(e - \frac{1}{2}x)^{-1}$$

is an integral function. We now compute the function $g(\mu) = f\left(\frac{1+\mu}{1-\mu}\right)$. We have on putting $\lambda = \frac{1+\mu}{1-\mu}$ at the appropriate stage

$$f(\lambda) = \frac{1}{2}(1+\lambda)\left\{(e - \frac{1}{2}x) + \frac{1}{2}\lambda x\right\}^{-1} = (1+\lambda)\left\{(2e + (\lambda - 1)x\right\}^{-1}$$

* The cases $N = 0, 1$ were treated (in part) by Gelfand, *Matematicheskii Sbornik*, 9 (51) (1941), pp. 49-50. Hille gave the general case in *Proceedings of the National Academy of Sciences*, loc. cit; it will also appear as Theorem 22.17.3 of his forthcoming book.

$$= \frac{2}{1-\mu} - (2e + \frac{2}{1-\mu} x)^{-1} = \{ (1-\mu) e + \mu x \}^{-1}$$

$$= (e - \mu y)^{-1}$$

Thus $g(\mu) = (e - \mu y)^{-1}$. Elementary calculation shows that when $|y^n| = O(|n|^N)$ the series

$$e + \mu y + \mu^2 y^2 + \ldots + \mu^n y^n + \ldots, \; |\mu| < 1$$
$$\mu^{-1} y^{-1} + \mu^{-2} y^{-2} + \ldots + \mu^{-n} y^{-n} \ldots, \; |\mu| > 1$$

converge to $g(\mu)$ and that this function satisfies the requirements of Pólya's theorem. Hence $f(\lambda)$ is a polynomial of degree at most $N+1$. It follows that $(e - \lambda z)^{-1}$ is a polynomial of degree at most N. Thus we must have $z^{N+1} = 0$ and hence $x^{N+1} = 0$. In case the stronger relation $|y^n| = o(|n|^N)$ holds, we have

$$y^n = (e-x)^n = \sum_{k=0}^{N} (-1)^k \frac{n!}{k!\,(n-k)!} x^k \text{ for } m \geqslant N$$

and hence

$$\frac{1}{N!} x^N = \lim_{n \to \infty} (-n)^{-N} \sum_{k=0}^{N} (-1)^k \frac{n!}{k!\,(n-k)!} x^k$$
$$= \lim_{n \to \infty} (-n)^{-N} y^n = 0.$$

Thus the conditions stated in the theorem are found to be sufficient. On the other hand if $x^{N+1} = 0$ we see that for $n \geqslant 1$ we must have

$$|y^n| = \sum_{k=0}^{N} \frac{n!}{k!\,(n-k)!} |x|^k = O(n^N).$$

Similarly $y^{-1} = e + x + x^2 + \ldots + x^N = e + z$, where $z^{N+1} = 0$, and

$$|y^{-n}| \leqslant \sum_{k=0}^{N} \frac{n!}{k!\,(n-k)!} |z|^k = O(n^N).$$

Thus the conditions stated in the theorem are also necessary.

ON THE STRONG SUMMABILITY OF FOURIER SERIES.

BY

PAUL TURÀN, *The Institute for Advanced Study, Princeton, N. J.*

[Received 1 May, 1948.]

(In memory of my late friend, Dr. Gèza Grünwald.)

1. Let $f(x)$ be a function belonging to the class L^p, $p > 1$ in the interval $[0, 2\pi]$ and

$$f(x) \sim \sum_{\nu=0}^{\infty} (a_\nu \cos \nu x + b_\nu \sin \nu x). \tag{1.1}$$

Let

$$S_n(x, f) = \sum_{\nu \leqslant n} (a_\nu \cos \nu x + b_\nu \sin \nu x). \tag{1.2}$$

It is well known* that if x_0 is an arbitrary continuity-point of $f(x)$, and k an arbitrary large positive number, then the Fourier series (1.1) is at $x = x_0$ strongly summable with the exponent k to the value $f(x_0)$, that is

$$\lim_{n \to \infty} \frac{1}{n} \sum_{\nu \leqslant n} | S_\nu(x_0, f) - f(x_0) |^k = 0 \tag{1.3}$$

2. The larger the value of k the stronger is the information that can be derived for the behaviour of the partial-sums $S_\nu(x_0, f)$. Gèza Grünwald raised the question whether or not the constant k can be replaced by a

* For $k = 2$ this was discovered essentially by Hardy and Littlewood in their paper, "Sur la serie de Fourier d'une function a carre sommable," *C.R.* 156 (1913), 1307-9. In the case of general k see T. Carleman: "A theorem concerning Fourier series", *Proc. Lond. Math. Soc.*, 21 (1923), 483-92 and the paper of Sutton which is quoted but not given by title either in the text or in the bibliography of Zygmund's book *Trigonometrical series* p. 238.

$k=k(n)$ which tends to infinity.* The aim of this short note is prove that there is no such strong-summability theorem for the class L_p. More generally, I shall prove the following

THEOREM. *Given a positive $k(n)$ tending monotonically, arbitrarily slowly to $+\infty$, we can construct a function $f(x)$ continuous everywhere such that for an appropriate $x=x_0$ the relation*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{\nu\leqslant n} |S_\nu(x_0,f)-f(x_0)|^{k(n)}=0 \qquad (2.1)$$

is not true.

3. It is easy to modify the construction so that the relation (2.1) does not hold for an enumerable set. This fact does not exclude of course the possibility that (2.1) should be true for almost all values of x_0, which is certainly true in the case of strong-summability in the sense (1.3) if $p>1$ and k is a positive constant. I cannot assure the strong-summability in the sense (2.1) almost everywhere even in the case when $f(x)$ is everywhere continuous.

4. To prove our theorem we have two cases.

CASE I. There is an infinite sequence of positive integers

$$2^{27}<n_1<n_2<\ldots<n_\nu<\ldots \qquad (4.1)$$

such that

$$k(n_\nu)>\frac{\log n_\nu}{\sqrt{\log\log n_\nu}}. \qquad (4.2)$$

* This generalization of the usual notion of strong-summability is analogous to the generalization of the notion "convergence in mean" given by A. Haar in his paper "Über die Konvergenz von Function folgen", *Acta Litt. ac Scient. Szeged*, Bd. I. p. 167-79.

We take simply Fejer's* example of a continuous function

$$f_0(x) = \sum_{j=1}^{\infty} \frac{1}{j^2} Q(x, 2^{j^3}, 2^{j^3}) \tag{4.3}$$

if $Q(x, \mu, n)$ denotes throughout this paper the Fejer-polynomials

$$Q(x, \mu, n) = \frac{\cos \mu x}{n} + \frac{\cos (\mu+1) x}{n-1} + \ldots + \frac{\cos (\mu+n-1) x}{1}$$
$$- \frac{\cos (\mu+n+1) x}{1} - \ldots - \frac{\cos (\mu+2n) x}{n}. \tag{4.4}$$

For an n_ν first we determine r by

$$2^{r^3} \leqslant n_\nu < 2^{(r+1)^3}, \; r \text{ being an integer;} \tag{4.5}$$

then $r \geqslant 3$. Taking into account that

$$S_{2.2^{(r-1)^3}}(0, f_0) > (r-1) \log 2 > \frac{r+1}{2} \log 2 > \frac{\log^{\frac{2}{3}} 2}{2} \log^{\frac{1}{3}} n_\nu$$

and $f_0(0) = 0$ we obtain

$$\sum_{j \leqslant n_\nu} | S_j(0, f_0) - f_0(0) |^{k(n_\nu)} > \left(\frac{\log^{\frac{1}{3}} 2}{2} \log^{\frac{1}{3}} n_\nu \right)^{k(n_\nu)},$$

i.e. for sufficiently large ν this sum is

$$> e^{k(n_\nu)\frac{1}{4} \log \log n_\nu} > e^{\frac{1}{4} \log n_\nu \sqrt{\log \log n_\nu}}$$

which proves our assertion in the first case.

5. Case II. For $n > N_0$ we have

$$k(n) \leqslant \frac{\log n}{\sqrt{\log \log n}}. \tag{5.1}$$

In this case we use again Fejer's construction-principle. Let

$$f_1(x) = \sum_{\nu=3}^{\infty} \frac{1}{\nu \log^2 \nu} Q(x, m_\nu, m_\nu), \tag{5.2}$$

* L. Fejer: "Sur les singularites des series de Fourier de fonctions continues", *Ann. de l'Ecole Norm. Sup.*, (28) (1911), 63-103.

where the positive integers (m_ν) will be determined later; at this moment we require only

$$m_\nu > 3m_{\nu-1} \tag{5.3}$$

in order to avoid overlappings. $f_1(x)$ is obviously continuous everywhere and

$$f_0(0) = 0.$$

Since

$$U_{2m_\nu} = \sum_{j \leqslant 2m_\nu} |S_j(0, f_1) - f_1(0)|^{k(2m_\nu)}$$

$$> \sum_{m_\nu \leqslant j \leqslant 2m_\nu - 1} |S_j(0, f_1)|^{k(m_\nu)}$$

and for the remaining values of j we have

$$S_j(0, f_1) = \frac{1}{\nu \log^2 \nu}\left(\frac{1}{m_\nu} + \frac{1}{m_\nu - 1} + .. + \frac{1}{2m_\nu - j}\right)$$

$$> \frac{1}{\nu \log^2 \nu} \log \frac{m_\nu}{2m_\nu - j}$$

we obtain

$$U_{2m_\nu} > \left(\frac{1}{\nu \log^2 \nu}\right)^{k(m_\nu)} \sum_{m_\nu \leqslant j \leqslant 2m_\nu - 1} \log^{k(m_\nu)} \frac{m_\nu}{2m_\nu - j}$$

$$> \left(\frac{1}{\nu \log^2 \nu}\right)^{k(m_\nu)} \int_{m_\nu}^{2m_\nu - 1} \log^{k(m_\nu)} \frac{m_\nu}{2m_\nu - x} dx$$

$$= \left(\frac{1}{\nu \log^2 \nu}\right)^{k(m_\nu)} \int_1^{m_\nu} \log^{k(m_\nu)} \frac{m_\nu}{y} dy$$

$$= \left(\frac{1}{\nu \log^2 \nu}\right)^{k(m_\nu)} m_\nu \int_0^{\log m_\nu} e^{-t} t^{k(m_\nu)} dt.$$

Since for all sufficiently large ν's from (5.1)

$$k(m_\nu) < \tfrac{1}{2} \log m_\nu$$

the integral

$$> \tfrac{1}{2}\Gamma(1 + k(m_\nu)) > \tfrac{1}{3}\left(\frac{k(m_\nu)}{e}\right)^{k(m_\nu)}$$

and hence

$$\frac{1}{2m_\nu} U_{2m_\nu} > \frac{1}{6}\left(\frac{k(m_\nu)}{e\nu \log^2\nu}\right)^{k(m_\nu)}$$

Thus if m_ν is chosen so that for all ν's

$$k(m_\nu) > 3\nu \log^2\nu, \tag{5.4}$$

then

$$\frac{1}{2m_\nu} U_{2m_\nu} \to +\infty$$

i.e. (2.1) is not true. But the choice (5.4) is possible; starting with $m_1 = 1$ and if $m_{\nu-1}$ is defined already, then defining m_ν as the least integer satisfying (5.3) as well as (5.4) case II is also settled.

ON THE SPACE OF INTEGRAL FUNCTIONS—I.*

BY

V. GANAPATHY IYER, (*Annamalai University*).

[Received in revised form 6 May, 1948.]

1. INTRODUCTION. The object of this paper is to make a preliminary study of the space of all Integral Functions suitably topologized into a complete metric space. Except in Theorems 3 and 10 of this paper, the idea of an integral function as such is not essential. It is possible to study the space as a special class of sequences of complex numbers. But in view of the applications to be made later, the notations and terminology are so chosen as to emphasize the fact that the space in question is the space of integral functions.

1.1. DEFINITIONS AND NOTATIONS. Throughout this paper z will stand for the complex variable. The small Greek letters α, β and γ (with or without suffixes) will stand for integral functions. We use the suggestive notation

$$\alpha = \alpha(z) \doteq \sum_{0}^{\infty} a_n z^n \tag{1}$$

to denote an integral function. It is well known that (1) is an integral function if and only if

$$|a_n|^{1/n} \to 0 \text{ as } n \to \infty. \tag{2}$$

The number $|\alpha|$ is defined† by

$$|\alpha| = \text{upper bound } [|a_0|, |a_n|^{1/n}, n \geqslant 1]. \tag{3}$$

* An abstract of this paper was communicated to the 15th conference of the Indian Mathematical Society held at Waltair in December, 1947.

† The symbol $||$ is used for the modulus of a complex number as well as for the norms in a normed space (see § 3 below). The same notation is here used, for convenience, to denote (3). The meaning of $||$ where it occurs will be evident from the context.

It is easily verified that $|\alpha|$ satisfies the following conditions:—

(*a*) $|\alpha| \geqslant 0$ and $|\alpha| = 0$ if and only if $\alpha = \theta$, the identically zero function,

(*b*) $|\alpha + \beta| \leqslant |\alpha| + |\beta|$,

(*c*) $|t\alpha| \leqslant A(t) . |\alpha|$, where $A(t) = \max (1, |t|)$, t being any complex number. (4)

1.2. If $\beta = \beta(z) = \sum_{0}^{\infty} b_n z^n$ be another integral function, it follows from (*a*) and (*b*) of (4) that $|\alpha - \beta|$ defines a distance in the class of all integral functions. We shall denote by Γ the space of all Integral Functions topologized by means of the distance just mentioned.

1.3. A complex-valued function $f(\alpha)$ defined for $\alpha \varepsilon \Gamma$ will be called a functional. A functional $f(\alpha)$ is said to be linear if $f(a\alpha + b\beta) = a f(\alpha) + b f(\beta)$. We shall denote by $\bar{\Gamma}$ the set of all *continuous linear functionals* defined on Γ. We shall call $\bar{\Gamma}$ the *adjoint* of the space Γ. We shall prove that each $f \varepsilon \bar{\Gamma}$ is of the form

$$f(\alpha) = \sum_{0}^{\infty} c_n a_n, \quad \alpha = \sum_{0}^{\infty} a_n z^n, \tag{5}$$

where the sequence

$$\{ |c_n|^{1/n} \} \text{ is bounded.} \tag{6}$$

Now each such sequence (6) determines uniquely a power series

$$\sum_{0}^{\infty} c_n z^n \tag{7}$$

with positive radius of convergence at $z = 0$ and conversely. Hence we may regard $\bar{\Gamma}$ *as the space of all power series with* *positive radius of convergence at* $z=0$*. Throughout this paper the usual function-symbols $f, g, \ldots$, will stand for elements of $\bar{\Gamma}$. As in the case of elements of Γ, we shall use the suggestive notation

$$f=f(z)=\sum_{0}^{\infty} c_n z^n$$

to denote an element of $\bar{\Gamma}$. If $f \in \bar{\Gamma}$ is defined as above, $f(\alpha)$, the value of f at $\alpha \in \Gamma$ will be given by (5). If we define

$$|f| = \text{upper bound } [|c_0|, |c_n|^{1/n}], n \geqslant 1$$

for $f=\sum_{0}^{\infty} c_n z^n \in \bar{\Gamma}$, then $|f-g|, f, g \in \bar{\Gamma}$ defines a distance in $\bar{\Gamma}$. Unless otherwise stated, we suppose $\bar{\Gamma}$ *is topologized by this distance.* So Γ is a sub-space of $\bar{\Gamma}$ in the sense that Γ is an isometric subset of $\bar{\Gamma}$.

1.4. A class L of elements $\alpha, \beta, \ldots$, is said to be a *linear space* if $a\alpha+b\beta$, a, b complex members, is defined and belongs to L. If L is also a topological space and $a\alpha+b\beta$ is continuous in the topology of L we say that L is a *linear topological space.* If the topology is, moreover, induced by a metric or distance we shall call L a linear metric space. Obviously Γ and $\bar{\Gamma}$ are linear spaces with the usual definitions of addition and scalar multiplication. We shall prove *that* Γ *is a linear metric space* but $\bar{\Gamma}$ *is not.*

* I am thankful to the referee for pointing out that it is not correct to identify $\bar{\Gamma}$ with the class of all functions regular at $z=0$ as was done in the original version. As a matter of fact, uniform functions regular at $z=0$ determine a unique element of $\bar{\Gamma}$ but each multiform function determines as many elements of $\bar{\Gamma}$ as there are branches of the function regular at $z=0$. The referee also suggests that $\bar{\Gamma}$ should be regarded as the class of sequences satisfying (6) and then to regard Γ as its subspace and (5) as a scalar product. The author however prefers the viewpoint adopted in the paper.

2. THEOREM 1. *Γ is a complete, separable linear metric space.*

PROOF. (i) *Γ is complete.* To prove this, let (α_p),

$$\alpha_p = \sum_{n=0}^{\infty} a_{pn} z^n$$

be a sequence in Γ such that $|\alpha_p - \alpha_q| \to 0$ as $p, q \to \infty$. Then, by (3), given $\varepsilon > 0$, we can find p_0 so that

$$|a_{p0} - a_{q0}| \leqslant \varepsilon, \; |a_{pn} - a_{qn}|^{1/n} \leqslant \varepsilon, \; n \geqslant 1 \text{ for } p, q \geqslant p_0. \quad (8)$$

This relation shows that for each $n \geqslant 0$, $a_{pn} \to a_n$ (say) as $p \to \infty$. Using this in (8) we see that

$$|a_{p0} - a_0| \leqslant \varepsilon, \; |a_{pn} - a_n|^{1/n} \leqslant \varepsilon, \; n \geqslant 1 \text{ for } p \geqslant p_0. \quad (9)$$

Since for each fixed p, $|a_{pn}|^{1/n} \to 0$ as $n \to \infty$, it follows, by taking $p = p_0$ in (9), that $|a_n|^{1/n} \to 0$ as $n \to \infty$. Hence $\alpha = \sum_{0}^{\infty} a_n z^n$ belongs to Γ. Also by (9), $|\alpha_p - \alpha| \leqslant \varepsilon$ for $p \geqslant p_0$. Hence $\alpha_p \to \alpha \,\varepsilon\, \Gamma$. So Γ is complete.

(ii) *Γ is a linear metric space.* We have to prove that $\alpha + \beta$ and $c\alpha$, where $\alpha, \beta \,\varepsilon\, \Gamma$ and c is a complex number, are continuous in Γ. That $\alpha + \beta$ is continuous follows from the property (*b*) of (4) of $|\alpha|$. Since Γ is a metric space, to prove that $c\alpha$ is continuous it is enough to prove that $\alpha_p \to \alpha$ in Γ implies $c\alpha_p \to c\alpha$ *and* that $c_p \to c$ implies $c_p\alpha \to c\alpha$ for each $\alpha \,\varepsilon\, \Gamma$. The first statement is an immediate consequence of (*c*) of (4). To prove the second result we may suppose that $c = 0$. Let $\alpha = \sum_{0}^{\infty} a_n z^n$. Given $\varepsilon > 0$, we can, by (2), find n_0 so that

$$|a_n|^{1/n} \leqslant \varepsilon \text{ for } n \geqslant n_0. \quad (10)$$

Since $c_p \to 0$ we may suppose that $|c_p| \leqslant 1$ for all p and that

$$|c_p a_0| \leqslant \varepsilon, \; |c_p a_n|^{1/n} \leqslant \varepsilon, \; n = 1, 2, \ldots, n_0 - 1$$

for $p \geqslant p_0$. By (*c*) of (4) and (10) we get

$$|c_p a_n|^{1/n} \leqslant \varepsilon \text{ for all } p \geqslant 1 \text{ and } n \geqslant n_0. \qquad (11)$$

Combining these results we see that

$$|c_p a_0| \leqslant \varepsilon, \ |c_p a_n|^{1/n} \leqslant \varepsilon, \ n \geqslant 1 \text{ for } p \geqslant p_0.$$

This shows that $|c_p \alpha| \leqslant \varepsilon$ for $p \geqslant p_0$. That is $c_p\alpha \to 0$ as $p \to \infty$. This proves that Γ is a linear metric space.

(iii) Γ *is separable.* Using (2) it is easily seen that the set of all polynomials with complex rational coefficients is dense in Γ. This set is enumerable. So Γ is separable.

3. Let L be a linear topological space. A set $S \subset L$ is said to be *bounded* if given any neighbourhood U of the zero element, we can find a number c such that $S \subset cU$, where cU denotes the set of all elements $c\alpha$, $\alpha \varepsilon U$. A non-negative quantity $\|\alpha\|$ defined for $\alpha \varepsilon L$ is said to be a norm* on L if (i) $\|\alpha\| = 0$ is equivalent to $\alpha =$ the zero element; (ii) $\|\alpha+\beta\| \leqslant \|\alpha\| + \|\beta\|$; and (iii) $\|c\alpha\| = |c| \, \|\alpha\|$, where c is any complex number. The quantity $\|\alpha-\beta\|$ is a distance in L and if the topology of L can be induced by this distance we say that L *can be normed* and that L is a normed linear space. We shall now prove that Γ *cannot* be normed.

3.1. THEOREM 2. *Γ contains no bounded open set and so Γ cannot be normed.*

PROOF (i) *Γ contains no bounded open set.* Let $U(\varepsilon)$ denote the set of all $\alpha \varepsilon \Gamma$ such that $|\alpha| < \varepsilon$. The prove the result stated it is enough to show that no $U(\varepsilon)$ is bounded. That is, given $U(\varepsilon)$, we have to prove that there exists a $U(\eta)$ for which there is no c such that $U(\varepsilon) \subset cU(\eta)$. For this purpose, take $\eta = \varepsilon/4$. Given c, we can find the positive integer m so great that $|c|^{1/m} < 2$. Let $\alpha = (\varepsilon/2)^m z^m$. Then $|\alpha| = \varepsilon/2$; so $\alpha \varepsilon U(\varepsilon)$. But $|\alpha/c| = \varepsilon/2 \, |c|^{-1/m} > \varepsilon/4 = \eta$. So $\frac{\alpha}{c}$ does not belong to $U(\eta)$ that is, α is not in $cU(\eta)$. This shows that $U(\varepsilon)$ is not bounded.

* See S. Banach, *Operations Lineaires*, (Warsaw, 1932), p. 53.

(ii) Γ *cannot be normed.** Suppose that Γ can be normed. We shall denote by $U(\varepsilon)$ the neighbourhoods of $\alpha=\theta$ mentioned in the previous paragraph and by $V(\varepsilon)$ the neighbourhood $\|\alpha\|<\varepsilon$ where $\|\alpha\|$ is the norm in question. The topology of Γ will be induced by the norm if and only if every $U(\varepsilon)$ contains a $V(\eta)$ and every $V(\varepsilon)$ contains some $U(\eta)$. Given $V(\varepsilon)$ we can find $U(\varepsilon_1)\subset V(\varepsilon)$. Also given $U(\eta)$ we can find $V(\eta_1)\subset U(\eta)$. But by the property (iii) of the norm defined in §3, we can find c so that $V(\varepsilon)\subset cV(\eta_1)$. Hence $U(\varepsilon_1)\subset V(\varepsilon)\subset cV(\eta_1)\subset cU(\eta)$. So $U(\varepsilon_1)$ is bounded contrary to the result proved in (i). Hence Γ cannot be normed.

4. The following theorem gives the relation between convergence of elements in Γ and the classical notion of uniform convergence of sequences of integral functions.

THEOREM 3. *The statement that $\alpha_p\to\alpha$ as $p\to\infty$, in the space Γ is equivalent to the statement that $\alpha_p(z)\to\alpha(z)$ uniformly in any finite circle.*

PROOF. Let $\alpha_p=\sum_0^\infty a_{pn}z^n$ and $\alpha=\sum_0^\infty a_n z^n$. Given R and ε, choose η so that $\eta R<1$ and $\eta\left(1+\frac{R}{1-\eta R}\right)\leqslant\varepsilon$. Let $\alpha_p\to\alpha$ in Γ as $p\to\infty$. We can find p_0 so that $|\alpha_p-\alpha|\leqslant\eta$ for $p\geqslant p_0$. This implies that $|a_{p0}-a_0|\leqslant\eta$ and $|a_{pn}-a_n|\leqslant\eta^n$, $n\geqslant 1$. Hence

$$|\alpha_p(z)-\alpha(z)|\leqslant\eta+\sum_1^\infty\eta^n R^n=\eta\left(1+\frac{R}{1-\eta R}\right)\leqslant\varepsilon,$$

for $p\geqslant p_0$. This proves that $\alpha_p(z)\to\alpha(z)$ uniformly in $|z|\leqslant R$.

Conversely suppose that $\alpha_p(z)\to\alpha(z)$ uniformly in any finite circle. In particular it follows that $a_{p0}\to a_0$.

* It is known that (i) implies (ii). (See J. V. Wehausen, *Duke Math. Jour.*, 4 (1938), 167-8). A proof is added here for the sake of completeness.

Given ϵ, choose $R \geqslant 1/\epsilon$. By hypothesis we can find p_0 such that

$$\left.\begin{array}{l} |a_{p_0} - a_0| \leqslant \varepsilon \\ |\alpha_p(z) - \alpha(z)| \leqslant 1 \end{array}\right\} \text{for } p \geqslant p_0,\ |z| \leqslant R. \qquad (12)$$

By Cauchy's inequalities it follows from (11) that

$$|a_{pn} - a_n|\ R^n \leqslant 1,\ n \geqslant 1,\ p \geqslant p_0.$$

So $|a_{pn} - a_n|^{1/n} \leqslant 1/R \leqslant \varepsilon$, $n \geqslant 1$, $p \geqslant p_0$. Hence $|\alpha_p - \alpha| \leqslant \varepsilon$ for $p \geqslant p_0$. So $\alpha_p \rightarrow \alpha$ in Γ.

4.1. Remark 1. This result is interesting since the corresponding result for $\bar{\Gamma}$ is not true.

Remark 2. Since the product of two integral functions is an integral function we can define multiplication among the elements of Γ. So we may regard Γ as a linear ring. From Theorem 3 (or directly as in Theorem 1) it follows that $\alpha\beta$, $\alpha, \beta \varepsilon \Gamma$, is continuous in the topology of Γ. So we may regard Γ as a *linear topological ring*.

5. The space of continuous linear functionals $\bar{\Gamma}$ on Γ.

We shall now prove

Theorem 4. *Every continuous linear functional $f(\alpha)$ defined for $\alpha \varepsilon \Gamma$ is of the form*

$$f(\alpha) = \sum_0^\infty c_n a_n,\ \alpha = \sum_0^\infty a_n z^n,$$

where $\{|c_n|^{1/n}\}$ is a bounded sequence.

5.1. To prove this we require the following

Lemma. *A necessary and sufficient condition that $\sum_0^\infty c_n a_n$ should be convergent for every sequence (a_n) satisfying (2) is that $\{|c_n|^{1/n}\}$ should be bounded.*

Proof. Suppose (6) holds. Then we can find M so that $|c_0| \leqslant M$, $|c_n|^{1/n} \leqslant M$ for $n \geqslant 1$. By (2) we can find n_0 so that $|a_n| \leqslant \left(\frac{1}{2M}\right)^n$, $n \geqslant n_0$. Hence $|c_n a_n|$ $\leqslant \frac{1}{2^n}$, $n \geqslant n_0$. So $\Sigma\, c_n a_n$ converges.

Suppose now that (6) does not hold. Then we can find an increasing sequence (n_p) of integers such that

$$|c_{n_p}| \geqslant p^{n_p},\ p = 1, 2, \ldots \tag{13}$$

Take $a_n = 0$ if $n \neq n_p$ and $a_n = p^{-n}$ if $n = n_p$, $p = 1, 2, \ldots$. Then $|a_n|^{1/n} = 0$ or $1/p$ according as $n \neq$ or $= n_p$. Hence (a_n) satisfies (2) but $|c_n a_n| \geqslant 1$ for $n = n_p$ so that $\Sigma\, c_n a_n$ does not converge.

5.2. Proof of Theorem 4. Let $\alpha = \sum_0^\infty a_n z^n$ and $f(\alpha)$ a continuous linear functional on Γ. Let $\alpha_n = z^n$ and $f(\alpha_n) = c_n$. Then

$$\begin{aligned} f(\alpha) &= \lim_{n\to\infty} f(a_0\alpha_0 + a_1\alpha_1 + \ldots + a_n\alpha_n) \\ &= \lim_{n\to\infty} (c_0 a_0 + c_1 a_1 + \ldots + c_n a_n). \end{aligned}$$

So for every $\alpha \in \Gamma$, $\sum_0^\infty c_n a_n$ converges and $f(\alpha) = \sum_0^\infty c_n a_n$. So by the lemma, $\{|c_n|^{1/n}\}$ is bounded.

Conversely suppose that (6) holds. Then by the lemma $\sum_0^\infty c_n a_n$ converges for every $\alpha \in \Gamma$. The functional $f(\alpha) = \sum_0^\infty c_n a_n$ is obviously linear on Γ. We shall show that it is continuous. For this purpose it is enough to show that if $|\alpha_p| \to 0$ as $p \to \infty$ then $f(\alpha_p) \to 0$. Let $\alpha_p = \sum_0^\infty a_{pn} z^n$. By (6) we can find M such that $|c_0| \leqslant M$, $|c_n| \leqslant M^n$, $n \geqslant 1$.

Given ϵ, choose η so that $\eta < \frac{1}{M}$ and $\eta M \left(1+\frac{1}{1-\eta M}\right) \leqslant \epsilon$. Since $|\alpha_p| \to 0$, we can find p_0 so that $|\alpha_p| \leqslant \eta$ for $p \geqslant p_0$. So

$$|f(\alpha_p)| \leqslant \eta M + \sum_{1}^{\infty} (\eta M)^n = \eta M\left(1+\frac{1}{1-\eta M}\right) \leqslant \epsilon$$

for $p \geqslant p_0$. So $f(\alpha_p) \to 0$ as $p \to \infty$. This completes the proof of Theorem 4.

6. As explained in §1.3, we can regard $\bar{\Gamma}$ as *the space of all power series with positive radius of convergence at $z=0$* and that it can be topologized by a metric which coincides with that of Γ for integral functions. We shall now prove

THEOREM 5. *$\bar{\Gamma}$ is a complete metric space. It is not a linear metric space. More precisely, if $f, g \in \bar{\Gamma}$. then (i) $f+g$ is continuous; (ii) if $f_n \to f$, $cf_n \to cf$; (iii) if (c_n) is a sequence of complex numbers $\to c$ then, in general, $(c_n f)$ does not converge to cf.*

PROOF. That $\bar{\Gamma}$ is complete follows as for Γ. The relation (*i*) and (*ii*) are consequences of the propertie (4) of $|f|$. To prove (*iii*), it is enough to find a sequence (c_p) tending to zero and one $f \in \bar{\Gamma}$ such that $|c_p f|$ does not tend to zero as $p \to \infty$. Let $c_p = 2^{-p}$ so that $c_p \to 0$. Let $f = \sum_{0}^{\infty} z^n$. Then $|c_p f| \geqslant \frac{1}{2}, p \geqslant 1$. This proves (*iii*).

6.1. $\bar{\Gamma}$ is a group with respect to addition. The result (*i*) of Theorem 5 shows that $\bar{\Gamma}$ is also a topological group since $f \pm g$ is continuous in its topology, But it is not a linear topological space like Γ since cf is continuous in f but is not simultaneously continuous in c and f by (*ii*) and (*iii*) of the previous theorem. The following theorem throws more light on this point and at the same time characterizes Γ as a sub-space of $\bar{\Gamma}$.

THEOREM 6. *Γ is the greatest sub-set of $\bar{\Gamma}$ which is at the same time a linear metric space, that is, if S is a linear sub-set of*

$\bar{\Gamma}$ and S regarded as a metric sub-space of $\bar{\Gamma}$ is also a linear metric space, then $S \subset \Gamma$.

PROOF. $\Gamma \subset \bar{\Gamma}$ and Γ is a linear metric sub-space by Theorem 1. Let $S \subset \bar{\Gamma}$ be a linear metric sub-space. We have to prove that $S \subset \Gamma$. If this is not so there is an element $f \in \bar{\Gamma}$ such that $f \in S$ but f does not belong to Γ. Then the same is true of cf, where c is any complex number. Let $f = \sum_0^\infty c_n z^n$. Then, by (2), $\lim |c_n|^{1/n} = \rho > 0$. So we can find a divergent increasing sequence (n_p) of integers such that

$$|c_n| \geqslant \left(\frac{\rho}{2}\right)^n \text{ for } n = n_p. \tag{14}$$

Take $a_n = \frac{1}{(1+\rho)^n}$ so that $a_n \to 0$ as $n \to \infty$.

By (14), $|a_{n_p} f| \geqslant \frac{\rho}{2(1+\rho)} > 0$ for $p = 1, 2, \ldots$ Hence the sequence $(a_n f)$ does not tend to zero while $a_n \to 0$. So S is not a linear metric sub-space of $\bar{\Gamma}$ contrary to hypothesis. So $S \subset \Gamma$.

6.2. THEOREM 7. *If $f_n \to f$ in $\bar{\Gamma}$ then $f_n(\alpha) \to f(\alpha)$ for each $\alpha \in \Gamma$.*

PROOF. Let $\alpha = \sum_0^\infty a_n z^n$. Then

$$|f_n(\alpha) - f(\alpha)| \leqslant |a_0|\, |f_n - f| + \sum_{p=1}^\infty |a_p|\, |f_n - f|^p.$$

Since $|f_n - f| \to 0$ as $n \to \infty$ it follows that $f_n(\alpha) \to f(\alpha)$.

7. WEAK AND STRONG CONVERGENCE IN Γ AND $\bar{\Gamma}$.

A sequence (α_p) of elements of Γ is said to converge weakly to the element $\alpha \in \Gamma$ if $f(\alpha_p) \to f(\alpha)$ as $p \to \infty$ for every $f \in \bar{\Gamma}$. By definition, if $|\alpha_p - \alpha| \to 0$, then $f(\alpha_p) \to f(\alpha)$ for every $f \in \bar{\Gamma}$. We shall therefore call convergence in the topology of Γ as strong convergence in contrast to the notion of weak convergence in Γ.

A sequence (f_p) *of elements of* $\bar{\Gamma}$ *is said to converge weakly to* $f \epsilon \bar{\Gamma}$ *if* $f_p(\alpha) \to f(\alpha)$ *for every* $\alpha \epsilon \Gamma$. By Theorem 7, $|f_p - f| \to 0$ in $\bar{\Gamma}$ implies that $f_p(\alpha) \to f(\alpha)$ for eaeh $\alpha \epsilon \Gamma$. As before we shall distinguish the convergence in $\bar{\Gamma}$ as strong convergence. In both these notions of weak convergenee, strong convergence, as indicated above, implies weak convergence. We shall now prove that *in the case of* Γ, *weak and strong convergences are equivalent but that this is not so in* $\bar{\Gamma}$.

7.1. THEOREM 8. *The notions of strong and weak convergence in* Γ *are equivalent.*

PROOF. As already pointed out strong convergence implies weak convergence. To prove the converse it is enough to prove that, if (α_p) is a sequence in Γ such that $f(\alpha_p) \to 0$ as $p \to \infty$ for every $f \epsilon \bar{\Gamma}$, then $|\alpha_p| \to 0$. Let

$$\alpha_p = \sum_{0}^{\infty} a_{pn} z. \tag{15}$$

Suppose if possible that $\overline{\lim_{p \to \infty}} \; |\alpha_p| > 0$. Then, by selecting a sub-sequence, if necessary, we may suppose that

$$|\alpha_p| \geqslant l > 0 \text{ for } p \geqslant 1. \tag{16}$$

Let $g_i \epsilon \bar{\Gamma}$ be defined by $g_i(\alpha) = a_i$, $\alpha = \sum_{0}^{\infty} a_n z^n$ (i.e. in the notation of § 1.3, $g_i = z^i$). Since $g_i(\alpha_p) = a_{pi}$ it follows by hypothesis, that

$$a_{pn} \to 0 \text{ as } p \to \infty \text{ for each } n \geqslant 0. \tag{17}$$

By using (2), (3), (16) and (17) we now construct by induction triplets of integers $(p_i\; n_i,\; n'_i)$ as follows :—

(i) $p_1 = 1$; n_1 is the least integer such that

$$|a_{p_1 n_1}| \geqslant \left(\frac{l}{2}\right)^{n_1};$$

and $n'_1 > n_1$ the least integer such that

$$\sum_{n=n'_1+1}^{\infty} |a_{p_1 n}| \left(\frac{2}{l}\right)^n \leqslant \tfrac{1}{4};$$

(ii) having constructed (p_i, n_i, n'_i) for $i = 1, 2, \ldots, k$ we choose the least integer $p_{k+1} > p_k$ such that

$$\left|a_{p_{k+1}0}\right| \left(\frac{2}{l}\right) + \sum_{n=1}^{n_k} \left|a_{p_{k+1}n}\right| \left(\frac{2}{l}\right)^n \leqslant \tfrac{1}{4}; \qquad (18)$$

and such that the least integer n_{k+1} for which

$$|a_{p_{k+1}\, n_{k+1}}| \geqslant \left(\frac{l}{2}\right)^{n_{k+1}} \qquad (19)$$

satisfies the condition $n_{k+1} > n'_k$; this is possible by (16) and (17). Then choose $n'_{k+1} > n_{k+1}$ to be the least integer such that

$$\sum_{n=n'_{k+1}+1}^{\infty} |a_{p_{k+1}n}| \left(\frac{2}{l}\right)^n \leqslant \tfrac{1}{4}, \qquad (20)$$

this being possible by (2).

Now we define $f = \sum_{0}^{\infty} c_n z^n \epsilon \bar{\Gamma}$ as follows:

$$\left.\begin{aligned} c_n &= 0 \text{ if } n \neq n_i \\ c_n &= \left(\frac{2}{l}\right)^{n_i} \operatorname{sgn}\ (a_{p_i n_i}) \text{ if } n = n_i \end{aligned}\right\}. \qquad (21)$$

Now

$$\begin{aligned} f(\alpha_{p_i}) &= \sum_{0}^{\infty} c_n a_{p_i n} \\ &= S_1 + S_2 + S_3, \text{ say,} \end{aligned}$$

where

$$\begin{aligned} |S_1| &= \left| \sum_{0}^{n_i - 1} c_n a_{p_i n} \right| \\ &= \left| \sum_{0}^{n_i - 1} c_n a_{p_i n} \right|, \text{ by (21)} \\ &\leqslant \tfrac{1}{4}, \text{ by (18)}; \end{aligned}$$

also

$$|S_2| = |c_{n_i}\, a_{p_i n_i}| \geqslant 1, \text{ by (19) and (21)};$$

and

$$|S_3| = \left|\sum_{n=n_i+1}^{\infty} c_n\, a_{p_i n}\right|$$
$$= \left|\sum_{n=n'_i+1}^{\infty} c_n\, a_{p_i n}\right|, \text{ by (21)}$$
$$\leqslant \tfrac{1}{4}, \text{ by (20)}.$$

So we get $|f(\alpha_p)| \geqslant 1-\frac{1}{4}-\frac{1}{4}=\frac{1}{2}$. So $f(\alpha_p)$ does not tend to zero contrary to hypothesis. Hence $|\alpha_p| \to 0$. This proves Theorem 8.

7.2. It is easy to show that in $\bar{\Gamma}$ *weak and strong convergence are not equivalent.* For instance, take $f_p = z^p$. If $\alpha = \sum_0^\infty a_n z^n$, then $f_p(\alpha) = a_p \to 0$ as $p \to \infty$ by (2). This is true for each $\alpha \in \Gamma$. Hence f_p converges weakly to the zero functional in $\bar{\Gamma}$. But $|f_p| = 1$ and so f_p does not converge strongly in $\bar{\Gamma}$. We shall now prove

THEOREM 9. *A set of necessary and sufficient conditions that a sequence* (f_p) *of* $\bar{\Gamma}$, $f_p = \sum_0^\infty c_{pn} z^n$, *should converge weakly to* $f = \sum_0^\infty c_n z^n \in \bar{\Gamma}$ *is*

(*i*) $\{|f_p|\}$ *is a bounded sequence;*

(*ii*) $c_{pn} \to c_n$ *as* $p \to \infty$ *for each* $n \geqslant 0$.

PROOF. *The conditions are sufficient.* To see this, let $\alpha = \sum_0^\infty a_n z^n \in \Gamma$. We have to prove that $f_p(\alpha) \to f(\alpha)$ as $p \to \infty$. Now, by (i) and (ii), it follows that $\{|c_n|^{1/n}\}$ is bounded so that $f = \sum_0^\infty c_n z^n \epsilon \bar{\Gamma}$. Also we can find M so that $|c_{pn}| \leqslant M^n$, $|c_n| \leqslant M^n$, $n \geqslant 1$, by (i) and (ii). So by (α) we find n_0 so that

$$\sum_{n_0+1}^{\infty} |c_{pn}-c_n|\, |a_n| \leqslant 2 \sum_{n_0+1}^{\infty} |a_n|\, M^n \leqslant \epsilon.$$

Also, by (ii), we can find p_0 such that

$$\left| \sum_0^{n_0} (c_{pn}-c_n)\, a_n \right| \leqslant \epsilon \text{ if } p \geqslant p_0.$$

Combining these two we get

$$|f_p(\alpha) - f(\alpha)| = \left| \sum_0^{\infty} (c_{pn}-c_n)\, a_n \right| \leqslant 2\epsilon$$

if $p \geqslant p_0$. So the conditions are sufficient.

7·3. To prove the necessity of the conditions we require the following known result.*

Lemma. *Let $F_n(x)$ be a sequence of continuous functional defined on a complete metric space E. Let $\overline{\lim}_{n\to\infty} |F_n(x)| < \infty$ for each $x \epsilon E$. Then there exists a fixed number M and a closed sphere $S \subset E$ such that $|F_n(x)| \leqslant M$ for $x \epsilon S$ and for all $n \geqslant 1$.*

7·4. *The conditions are necessary.* That (ii) is necessary follows by taking $\alpha = z^n$, $n = 1, 2, \ldots$, so that $f_p(\alpha) = c_{pn} \to f(\alpha) = c_n$. To prove (i), we see, by hypothesis, that

$$|f_p(\alpha)| \to |f(\alpha)| < \infty \text{ for each } \alpha \epsilon \Gamma.$$

Also Γ is a complete metric space by Theorem 1. So by the lemma just quoted, there is a closed sphere $S \subset \Gamma$ and a fixed number M such that

$$|f_p(\alpha)| \leqslant M, \ \alpha \epsilon S, p \geqslant 1. \tag{22}$$

Since the functionals are linear and the space Γ is also a linear metric space, we may suppose that S is the

* For the notions of weak convergence and the result of the lemma, see S. Banach, *l. c.*, pp. 126-32, 137, 19, and Theorem 11 respectively.

sphere $|\alpha| \leqslant d$. Take $\alpha = (d/2)^n z^n$ so that $|\alpha| = d/2$ and therefore $\alpha \varepsilon S$. Then by (22),

$$|f_p(\alpha)| = \left| c_{pn} \left(\frac{d}{2}\right)^n \right| \leqslant M,$$

that is

$$|c_{pn}|^{1/n} \leqslant M^{1/n} \frac{2}{d} \leqslant \frac{2\ A(M)}{2},$$

where $A(M) = \max\ (1, M)$. So $|f_p| \leqslant \frac{2\ A(M)}{d}$, for all $p \geqslant 1$. This proves that (i) is necessary.

7.5. Remark 1. Theorem 1 shows that Γ is what is called an F-space by Banach.* Bur $\bar{\Gamma}$ while it is complete and is a topological group (for addition) is not an F-space. Γ is separable but it is easy to show that $\bar{\Gamma}$ is not so. It may be noted that Γ is a *closed* subspace of $\bar{\Gamma}$.

Remark 2. If $f \varepsilon \bar{\Gamma}$ be represented by $f = f(z) = \sum_0^\infty c_n z^n$, then we can write

$$f(\alpha) = \frac{1}{2\pi i} \int_{|z|=\rho} f(z)\, \alpha \left(\frac{1}{z}\right) \frac{dz}{z},$$

where $\alpha = \alpha(z) = \sum_0^\infty a_n z^n \varepsilon \Gamma$ and ρ is any number less then the radius of convergence of the power series $f(z)$. So every functional of $\bar{\Gamma}$ can be represented as a contour integral.

Remark 3. The result of Theorem 8 is noteworthy. The only example known so far of a linear topological space for which such a result is true is the normed space of absolutely convergent series. It may also be noted that Theorem 9 closely resembles the corresponding results known for many normed spaces.†

* *See* S. Banach, *l.c.*, p. 35.

† *Ibid.* pp. 126-32, 137.

8. It is well known that a *complete metric space is of the second category.* So Γ is of the second category. If we regard the set of elements in a set of the *first category* in a space of the second category as '*rare*', the following theorem shows that many of the well-known classes of integral functions are rare compared to the class of all integral functions.

THEOREM 10. *The sapce Γ is of the second category. The following classes of functions regarded as subsets of Γ are of the first category:*

(1) *functions of finite order;*

(2) *functions vanishing at a finite or an enumerably infinite set of given points;*

(3) *functions having some Picard exceptional value; in particular functions without zeros;*

(4) *the set of all periodic functions.*

Of these, (2) and functions without zeros being closed sets will be non-dense subsets of Γ.

PROOF. Γ is complete by Theorem 1 and so is of the second category. We take the others in order.

(1) Let $\Gamma(\lambda, \mu)$ denote the set of all integral functions $\alpha = \sum_{0}^{\infty} a_n z^n$ such that

$$\left.\begin{array}{l} |a_n| \leqslant \mu^n n^{-n/\lambda},\ n \geqslant 1 \\ |a_0| \leqslant \mu \end{array}\right\}. \tag{23}$$

By Theorem 3 and classical results it follows that $\Gamma(\lambda, \mu)$ is a closed set and consists only of functions of finite order. If $\Gamma(\lambda, \mu)$ is not non-dense then it will contain a sphere S since $\Gamma(\lambda \ \mu)$ is closed. Let α_0 be the centre of S. It belongs to S and so is of finite order. Let α be a function of infinite order; then $\alpha_0 + \eta\alpha$ is also of infinite order for $\eta \neq 0$. But, by Theorem 1, $|\eta\alpha| \to 0$ as $|\eta| \to 0$. Hence for $|\eta|$ sufficiently small $\alpha_0 + \eta\alpha \,\varepsilon\, S$ which is a contradiction since $S \subset \Gamma(\lambda, \mu)$. Hence $\Gamma(\lambda, \mu)$ is non-dense. Now the set of functions of finite order is precisely $\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \Gamma(m, n)$ and so is of the first category.

(2) It is enough to prove that the set of functions vanishing at a given point $z = z_0$ is non-dense. By Theorem 3 and classical results, this set is closed. So if it is not non-dense, it will contain a sphere S whose centre is α_0, say. If $\alpha \varepsilon \Gamma$ be such that $\alpha(z_0) \neq 0$, then $\alpha_0 + \lambda \alpha \neq 0$ at $z = 0$ for $\lambda \neq 0$ and if $|\lambda|$ is sufficiently small, $\alpha_0 + \lambda \alpha \varepsilon S$. This contradiction proves the result.

(3) *We say that x is a Picard exceptional value for a function $\alpha \varepsilon \Gamma$ if $\alpha(z) - x = 0$ has at most a finite number of roots.* Let $\Gamma(N, R)$ denote the set of all integral functions $\alpha(z)$ such that for some x with $|x| \leqslant R$, the equation $\alpha(z) - x = 0$ has at most N roots. The set $\Gamma(N, R)$ is closed. To see this, let $\alpha_p \to \alpha$ as $p \to \infty$, $\alpha_p \varepsilon \Gamma(N, R)$. By hypothesis there is an x_p, $|x_p| \leqslant R$, such that $\alpha_p(z) - x_p$ does not have more than N zeros. The set (x_p) has at least one limit point x, $|x| \leqslant R$. So by Theorem 3 and classical results $\alpha(z) - x$ does not have more than N zeros; so $\alpha \varepsilon \Gamma(N, R)$, that is, $\Gamma(N, R)$ is closed. Now, if $\Gamma(N, R)$ is not non-dense, it will contain a sphere S whose centre $\alpha \varepsilon \Gamma(N, R)$. Now let α be an integral function such that α/α_0 is not a rational function. Then $\beta = \alpha_0 + \lambda \alpha$ cannot have any Picard exceptional value for more than one value of $\lambda \neq 0$. To see this, suppose there are two such values λ_1 and λ_2, $\lambda_1 \neq \lambda_2$. Then, supposing as we may, that 0 is the Picard value of α_0, we get the relations

$$\alpha = A_0 e^{B_0}, \ \alpha_0 + \lambda_1 \alpha = A_1 e^{B_1}, \ \alpha_0 + \lambda_2 \alpha = A_2 e^{B_2},$$

where A_0, A_1, A_2 are polynomials and B_0, B_1, B_2 are integral functions. These give

$$\lambda_2 A_1 e^{B_1} - \lambda_1 A_2 {}^{B_2} = (\lambda_2 - \lambda_1) A_0 e^{B_0},$$

where $\lambda_2 - \lambda_1 \neq 0$. By a well-known result* the above equation is impossible unless $B_1 - B_0$, and $B_2 - B_0$ are constants. But then α/α_0 will be a rational function con-

* See Borel, *Acta Mathematica*, 20 (1896-97), p. 357-96.

trary to hypothesis. From this it follows that if $|\lambda|$ is sufficiently small $\alpha_0+\lambda\alpha\epsilon\Gamma(N, R)$ while $\alpha_0+\lambda\alpha\,\epsilon\, S\subset\Gamma(N, R)$. So $\Gamma(N, R)$ is non-dense. Now the set in (3) is precisely $\sum_{N=1}^{\infty}\sum_{n=1}^{\infty}\Gamma(N, n)$ and so is of the first category.

(4) Let $\Gamma(R)$ denote the set of all functions of Γ having a period λ with $|\lambda|\leqslant R$. The set $\Gamma(R)$ is closed. To see this, let $\alpha_p\rightarrow\alpha$; $\alpha_p\,\epsilon\,\Gamma(R)$, let λ_p be a period of α_p with $|\lambda_p|\leqslant R$. If λ is a limit point of the set λ_p then $|\lambda|\leqslant R$ while the equation $\alpha_p(z+\lambda_p)=\alpha_p(z)$ and $\alpha_p\rightarrow\alpha$ gives $\alpha(z+\lambda)=\alpha(z)$ by Theorem 3. So $\alpha\,\epsilon\,\Gamma(R)$ and therefore is closed. Now if $\Gamma(R)$ is not non-dense it will contain a sphere S with centre $\alpha_0\,\epsilon\,\Gamma(R)$. Taking $\alpha\equiv z$, the function $\alpha_0+\eta\alpha$ cannot have any period for $\eta\neq 0$ and if $|\eta|$ is small enough, $\alpha_0+\eta\alpha\,\epsilon\, S$. This contradiction proves that $\Gamma(R)$ is non-dense. The set in (4) is precisely $\sum_{n=1}^{\infty}\Gamma(n)$ and so is of the first category.

A NOTE ON LOWER PROXIMATE ORDERS

BY

S. M. SHAH, *Aligarh.*

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1. Let $f(z)$ be an integral function of order ρ and let $\lambda = \varliminf_{r\to\infty} \frac{\log\log M(r)}{\log r}$ $(0 \leqslant \lambda < \infty)$. We consider the class of functions $\lambda(r)$ having the following properties :—

(1) $\lambda(r)$ is a non-negative continuous function of r for $r > r_0$,

(2) $\lambda(r)$ is differentiable for $r > r_0$ except at isolated points at which $\lambda'(r-0)$ and $\lambda'(r+0)$ exist,

(3) $\lim_{r\to\infty} r\lambda'(r) \log r = 0$,

(4) $\lim_{r\to\infty} \lambda(r) = \lambda$,

(5) $\varliminf_{r\to\infty} \frac{\log M(r)}{r^{\lambda(r)}} = 1$.

These functions $\lambda(r)$ are thus defined in the same way, except for (4) and (5), as the proximate orders $\rho(r)$[1] and it is natural to call $\lambda(r)$ as a lower proximate order for the function $f(z)$. It seems worth while to put on record that such functions exist, but I omit the proof of the existence of such functions since it is so very similar to that given in [2] for the construction of proximate orders $\rho(r)$.

2. It is possible to have a (smaller) class of functions $\lambda(r)$ satisfying (1) to (5) and the relation

$$\lim_{r\to\infty} r\lambda'(r) l_1 r l_2 r \ldots l_k r = 0.$$

In fact we have to take curves of the form

$$y = A \pm l_{k+1}(r) \quad (A \text{ a is constant, } l_1 r = \log r, \text{ etc.})$$

instead of $y = A \pm l_3 r$ in our construction for $\lambda(r)$. A similar remark applies for the functions $\rho(r)$.

3. These functions $\rho(r)$ and $\lambda(r)$ satisfy also the relations [2]

$$r^{\lambda(r)} \leqslant \log M(r) \leqslant r^{\rho(r)} \quad (r > r_0 = r_0(f))$$

$\log M(r) = r^{\lambda(r)}$ for sequence of values of $r \to \infty$

$$\log M(r) = r^{\rho(r)}$$

for another sequence of values of $r \to \infty$.

Hence these orders become very helpful for a detailed study of $M(r)$ and of the relations between $M(r)$ and the associated functions such as $n(r)$. It is known for instance[3] that for functions of finite order ρ

$$n(r) \leqslant K r^{\rho(r)}$$

for all $r > r_0$.

We can prove that for functions of finite lower order,

$$n(r) \leqslant K r^{\lambda(r)}$$

for sequence of values of $r \to \infty$.

If $\nu(r)$ denote the rank of the maximum term of $f(z) = \sum_{0}^{\infty} a_n z^n$ for $|z| = r$, then we can show that

$$\nu(r) \leqslant K r^{\lambda(r)}$$

for a sequence of values of $r \to \infty$.

If $f(z)$ be of finite order, then

$$\nu(r) \geqslant \frac{K r^{\rho(r)}}{\log r}$$

for sequence of values of r tending to infinity.

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2. *See* S. M. SHAH, *Loc. Cit.*

3. *See* G. VALIRON, *Loc. Cit.* p. 68.

ON THE FRACTIONAL PARTS OF POWERS OF A NUMBER, IV.

BY

T. VIJAYARAGHAVAN, *Andhra University, Waltair.*

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Suppose that θ is a real positive number, and that $G(\theta)$ denotes the limit points of the set of fractional parts of the numbers $\theta, \theta^2, \theta^3, \ldots$. If $0 \leqslant \theta \leqslant 1$ then $G(\theta)$ merely consists of the point zero only; in what follows we suppose that $\theta > 1$. It has been proved by the author that the set of numbers θ for which $G(\theta)$ consists of finite number of points is an enumerable set.* It is also known that $G(\theta)$ consists of the entire unit interval except for a set E of θ's of Lebesgue measure zero. An obvious corollary of of the theorem proved in this note is that E has the power of the continuum. Suppose that θ_n $(n = 1, 2, 3, \ldots,)$ denotes the fractional part of θ^n; in this note, to put it roughly, we show that if M is a mode of distribution of of $\theta_1, \theta_2, \ldots, \theta_n, \ldots,$ where M is arbitrary within limits, then there corresponds to it a set S_M of numbers θ, where S_M has the power of the continuum and is such that to every θ of S_M the pattern of distribution of $\theta_1, \theta_2, \ldots, \theta_n, \ldots,$ is the prescribed mode M. The meaning of the above sentence is made precise and clearer by the following. Let $i_1, i_2, \ldots, i_n \ldots,$ be an arbitrarily given sequence of intervals (α_n, β_n), $n = 1, 2, 3, \ldots,$ all of which are in the unit interval, and are subject to the single restriction that if l_n denotes the length of the interval i_n, then the largest

* *The Journal of the Lond. Math. Soc.* 17 (1942), 137-8.

lower bound of the interval l_n $(n = 1, 2, 3, \ldots,)$ is positive*. We define the set S_M of numbers θ by stating that θ belongs to S_M if and only if for every positive integral n the fractional part θ_n of θ^n lies in i_n. We can now state the

THEOREM 1. *The set S_M has the power of the continuum.*

We prove in the first place that the set S_M contains at least one number and deduce from it that S_M has the power of the continuum. Let δ be the greatest lower bound of the numbers $l_1, l_2, l_3, \ldots,$ and $H = [2/\delta]+1$, i.e. H is the smallest integer that exceeds $2/\delta$. We denote by I_1 the interval that lies in $(H, H+1)$ and is congruent to i_1 (mod 1); in other words $I_1 = (H+\alpha_1, H+\beta_1)$. Let T_2 be the interval $\{(H+\alpha_1)^2, (H+\beta_1)^2\}$; it is of length greater than 2 since

$$(H+\beta_1)^2-(H+\alpha_1)^2 > \{(H+\beta_1)-(H+\alpha_1)\}\ (H+\alpha_1)$$
$$= (\beta_1-\alpha_1)\ (H+\alpha_1) \geqslant \delta H > 2.$$

Let H_2 be the smallest integer that exceeds $(H+\alpha_1)^2$, and let c_2 be the interval $(H_2+\alpha_2, H_2+\beta_2)$ which is evidently congruent to i_2 (mod 1). We denote by I_2 the interval $\{\sqrt{(H_2+\alpha_2)}, \sqrt{(H_2+\beta_2)}\}$. Plainly c_2 is contained in T_2 and therefore I_2 is contained in I_1. It is clear that if θ is in I_2 then θ is in I_1 too, and hence θ_1 is in i_1. Next consider T_3 which is the interval $\{H_2+\alpha_2)^{3/2}, (H_2+\beta_2)^{3/2}\}$; its length exceeds 2 since

$$(H_2+\beta_2)^{3/2}-(H_2+\alpha_2)^{3/2} > \{(H_2+\beta_2)-(H_2+\alpha_2)\}\ (H_2+\alpha_2)^{\frac{1}{2}}$$
$$= (\beta_2-\alpha_2)\ \sqrt{(H_2+\alpha_2)} > \delta H > 2.$$

* Plainly the intervals cannot be non-overlapping. We can, for instance, take i_n $(n=1, 2, 3, \ldots,)$ to be the interval (α_n, β_n), where α_n is half the fractional part of $n^2\sqrt{5}$, $\beta_n = \alpha_n + \frac{1+r_n}{1000}$, r_n being the nth digit in the decimal representation of $\sqrt{3}$. In so far as the theorem of this note is concerned there will be no loss of generality if we suppose that all the intervals i_n $(n=1, 2, 3, \ldots,)$ are of equal length.

Let H_3 be the smallest integer that exceeds $(H_2+\alpha_2)^{3/2}$, c_3 the interval $(H_3+\alpha_3, \ H_3+\beta_3)$ and I_3 the interval $\{(H_3+\alpha_3)^{1/3}, (H_3+\beta_3)^{1/3}\}$. Plainly I_3 is contained in I_2 and if θ is in I_3 then $\theta_1, \theta_2, \theta_3$ are respectively in i_1, i_2, i_3. The argument can be repeated successively and we get that if θ is the number that is common to the closed intervals $I_1, I_2, I_3, \ldots$, then θ belongs to S_M. Thus we see that S_M contains at least one number.

It is shown below that S_M has the power of the continuum. We divide each of the intervals i_n $(n=1, 2, 3, \ldots,)$ into three equal parts and denote the first part, viz. $\left(\alpha_n, \frac{2\alpha_n+\beta_n}{3}\right)$ by i_{0n} and the last part $\left(\frac{\alpha_n+2\beta_n}{3}, \beta_n\right)$ by i_{1n}. Let $0 \leqslant x \leqslant 1$ and let the representation* of x in the dyadic scale be $.a_1 a_2 a_3 \ldots$, where $a_n = 0$ or 1 for $n=1, 2, 3, \ldots$. And with x we associate the sequence of intervals $b_1, b_2, b_3, \ldots$, where $b_n = i_{a_n n}$ $(n=1, 2, 3, \ldots)$. Since the lower bound of the lengths of the intervals b_n is $\frac{1}{3}\delta > 0$ it follows from what has been proved already that there exists a number $\theta = \theta(x)$ such that θ_n is in b_n for $n=1, 2, 3, \ldots$ Clearly, θ belongs to S_M since b_n is in i_n $(n=1, 2, 3, \ldots)$. Also, if $x \neq x'$, $x = .a_1 \ a_2 \ a_3, \ldots$, $x' = .a'_1 \ a'_2 \ a'_3 \ldots$ then $a_n \neq a'_n$ for at least one value of n, and hence the corresponding intervals b_n and b'_n are two disjoint intervals (namely the first and the last of the three equal intervals into which we divide i_n) and hence $(\theta(x))^n$ and $(\theta(x'))^n$ are unequal, as they are in b_n and b'_n respectively. Hence $\theta(x) \neq \theta(x')$. Thus we see that $\theta(x)$ exists, for all values of x for which $0 \leqslant x \leqslant 1$, $\theta(x) \neq \theta(x')$ whenever $x \neq x'$, and that $\theta(x)$ belongs to S_M whenever $0 \leqslant x \leqslant 1$. Hence we see that S_M has the power of the continuum.

* If $x = p/2^m$, where p is odd then there will be two representations of x in the dyadic scale, but for the purposes of our argument it will not matter which representation we take ; only the value of a_n should be specified for all n even if $a_n = 0$ for all large values of n.

By a similar reasoning but with slight changes we can prove that if $E(a, b)$ denotes the set of numbers that belong to both E and the interval (a, b), then $E(a, b)$ has the power of the continuum in every interval (a, b) whenever $3 < a < b$. Plainly in proving the above result we can suppose that $b-a$ is small. We take $\delta = 1/2\,(a+1)$, $b = a+\delta$, $\eta = \frac{1}{2}(a-3)\,\delta$, and I to be a sub-interval of the unit interval, with length η and such that no point of (a, b) is congruent (mod 1) to any point of I. The choice of such an interval I is possible since $\eta < \frac{1}{2}(1-\delta)$. Let S stand for the set of numbers θ for which $a \leqslant \theta \leqslant b$ and are such that none of the fractional parts of $\theta, \theta^2, \theta^3, \ldots$, is in the interval I. Plainly if θ belongs to S then $G(\theta)$ contains no point of the open interval I and therefore $G(\theta)$ does not consist of the whole of the unit interval and hence θ belongs to E.

We shall now prove that S has the power of the continuum. A closed interval (x, y) is said to be of type k if

$$a \leqslant x < y \leqslant b,\, y^k - x^k = \delta$$

and none of the intervals $(x, y), (x^2, y^2), \ldots, (x^k, y^k)$ contains a point congruent (mod 1) to any point in I. We shall now show that every interval of type k contains two disjoint (closed) intervals of type $k+1$. We have

$$a\,\delta \leqslant x\,(y^k - x^k) < y^{k+1} - x^{k+1} < y\,(y^k - x^k) < b.\delta,$$

since $a \leqslant x < y \leqslant b,\, y^k - x^k = \delta$. Hence we have

$$y^{k+1} - x^{k+1} + \eta < b\,\delta + \eta < (a+1)\,\delta + \eta = \frac{3a-1}{4a+4} < 1,$$ i.e. $y^{k+1} - x^{k+1} < 1 - \eta$. Therefore, if we remove from (x^{k+1}, y^{k+1}) those points, if any, that are congruent to points in the interval I then what remains, call it R, is either one (connected) interval or two intervals. The length of the remaining interval or two intervals as the case may be is not less than

$$y^{k+1} - x^{k+1} - \eta > a\,\delta - \eta = (\tfrac{1}{2}a + \tfrac{3}{2})\,\delta > 3\,\delta.$$

Hence we can cut out two disjoint closed intervals each of length δ from R. This completes the proof of the assertion that every interval of type k contains (at least) two disjoint intervals of type $k+1$. Now (a, b) is of type 1, and therefore it contains two disjoint intervals of type 2, each of which intervals of type 2 contains two disjoint intervals of type 3, and so on. We can now choose a sequence of closed intervals I_1 $(=(a, b))$, I_2, I_3, ... where I_k $(k=1, 2, 3, \ldots)$ is of type k and contains I_{k+1}. At each stage after the first there are two choices and hence the number of choices is as numerous as the set of real numbers. Moreover to each sequence of intervals $I_1, I_2, I_3, \ldots$, there corresponds one and only one number θ which is common to all of them and unless two sequences are identical the corresponding θ's are distinct. This establishes that S has the power of the continuum.

If the intervals $i_1, i_2, i_3, \ldots$, cover only a part of the unit interval as is the case when, for instance, we take each i_r to denote the intervals $(0, 1/10)$ then plainly $G(\theta)$ is contained in $(0, 1/10)$ and therefore θ belongs to E. Hence it follows from the theorem proved above that E has the power of the continuum.

The above reasoning, with some modifications, will yield

THEOREM 2. *If* $1 < a < b$ *then the meet of* E *and* (a, b) *has the power of the continuum.*

PROOF. We take arbitrarily the number $a > 1$; it is sufficient to prove the theorem for sufficiently small intervals (a, b). Let

(1) h be the smallest positive integer for which $a^h > h+3$,

(2) $b = a+\delta$, where $0 < \delta < 1/(a+1)^h$,

(3) $\eta = 1/(1+b+b^2+\ldots+b^{h-1})$,

(4) I be an open subinterval of the unit interval, of length η and containing no point congruent (mod 1) to any point of (a, b).

We can easily verify that $\eta < \frac{1}{2}(1-\delta)$ and hence the choice of I is possible. Let d_{kn} $(k, n = 1, 2, 3, \ldots,)$ denote the set of points θ where θ belongs to d_{kn} if and only if $\theta_k - n$ is a point of I; and let*

$$D_k = d_{k1} + d_{k2} + d_{k3} + \ldots \quad (k = 1, 2, 3, \ldots).$$

It is implicit in what we prove below that if O_k is the set that is complementary to $D_1 + D_2 + \ldots + D_k$ in the interval $(1, \infty)$ then O_k is not the null set. Plainly O_k is a closed set and if θ belongs to O_k then $\theta \geqslant 1$ and the fractional parts of the numbers $\theta, \theta^2, \ldots, \theta^k$ all lie outside I. If (x, y) is a closed interval which is contained in the meet of (a, b) and O_k, and is such that $y^k - x^k = \delta$ $(= b-a)$, then we say that (x, y) is of type k. For instance (a, b) is of type 1 we shall now prove that†

(*A*) *If (x, y) is of type k then it contains (at least) two disjoint intervals of type $k+h$, where h is the number defined by* (1) *above.*

From (A) it will follow, as before, that the set of θ's that belong to O, where O is the meet of $O_1, O_2, O_3, \ldots,$ has the power of the continuum.‡ Now if θ belongs to O then the fractional parts of $\theta, \theta^2, \theta^3, \ldots,$ all lie outside I, and hence θ belongs to E. It only remains to complete the proof of (A). If (c, d) is any interval i then we refer to (c^n, d^n) as the n-transform of (c, d) and write $(i)_n = (c, d)_n = d^n - c^n$ which is the length of the transform of (c, d). If $1 \leqslant r \leqslant b$, and (x, y) is of type k then $a^r\delta < (x, y)_{k+r} < b^r\delta < \frac{1}{2} < 1-\eta$, since $a^r\delta \leqslant x^r(y^k - x^k) < y^{k+r} - x^{k+r} < y^r(y^k - x^k) \leqslant b^r\delta \leqslant b^h\delta < (a+1)^h\delta < 1/a+1 < \frac{1}{2}$. Therefore if $1 \leqslant r \leqslant h$ and m_r denotes the meet of (x, y) and D_{k+r} then m_r is either the null set or a part or whole of one and

* A point θ belongs to D_k if and only if $\theta > 1$ and θk is congruent (mod 1) to a point in I.

† Since (a, b) is of type 1 we have that (A) implies that intervals of type $1+h, 1+2h, 1+3h, \ldots,$ all exist and therefore the sets O_{1+h} $O_{1+2h}, \ldots,$ all exist; since O_k contains O_{k+1} $(k = 1, 2, 3, \ldots)$ it follows that O_k is non-null for every k.

‡ As k becomes large the intervals $d_{k1}, d_{k2}, \ldots,$ come near each other and it is easy to see that O is a nowhere dense closed set.

only one of the intervals d_{k+r1}, d_{k+r2}, . ., since $(x, y)_{k+r} < 1-\eta$ and the $(k+r)$-transforms of d_{k+r1}, d_{k+r2}, d_{k+r3}, . . ., are intervals of length η and are such that the distance between the left end points of neighbouring (transformed) intervals is equal to unity. We see then without difficulty that the removal of $m_1, m_2, \ldots, m_k$ from (x, y) leaves behind (besides possibly isolated end points, not more than h in number) a set of disjoint closed intervals, not more than $h+1$ in number. Also we see easily that $(m_r)_{k+h} \leqslant \eta(b^{h-r})$, and

$$\{(x, y)-m_1-m_2-\ldots-m_h\}_{k+h} > a^h\delta-\eta(b^{h-1}+b^{h-2}+\ldots+b+1)$$
$$> (h+3)\delta-\delta = (h+2)\delta.$$

Hence the $(k+h)$-transform of $\{(x, y)-m_1-m_2-\ldots-m_h\}$ either contains (at least) two intervals each of length $\geqslant \delta$ or one interval of length $> 2\delta$. Hence in any case $\{(x, y)-m_1-m_2-\ldots-m_h\}_{k+h}$, which is contained in O_{k+h}, contains two disjoint closed intervals of length δ. This proves (A), and completes the proof of Theorem 2.

MINIMAL-BICOMPACT SPACES*

BY

A. RAMANATHAN, *Pachaiyappa's College, Madras.*

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1. INTRODUCTION. Of the two questions relating to bicompact spaces, viz. whether there exist (i) maximal-Hausdorff non-bicompact spaces, and (ii) minimal-bicompact non-Hausdorff spaces, I have solved the former by proving that the maximal-Hausdorff spaces are identical with the *H*-closed semi-regular spaces.† The purpose of the present paper is to produce a minimal-bicompact non-Hausdorff space. The space has been obtained from the one constructed by Urysohn for a different purpose.‡

2.1. DEFINITIONS. A topological space is said to be *bicompact* if, from every open covering of the space, we can select a finite covering of the space. A bicompact space $(R; f)$ is said to be *minimal-bicompact*, if any topology on R, which is weaker than f, renders the space non-bicompact.

2.2. We know that *any closed subset of a bicompact space, considered as relative space, is necessarily bicompact*; and that *any bicompact subset of a bicompact space* $(R; f)$ *is necessarily a closed set, either of the topology* f, *or of some bicompact topology* ϕ *weaker than* f§. Thus *a bicompact space is*

* I wish to thank Dr. R. Vaidyanathaswamy for his help in the preparation of this paper.

† *See* my Papers (*a*) Maximal-Hausdorff Spaces, *Proc. Ind. Ac. Sc.*, 26 (1947); and (*b*) A characterization of Maximal-Hausdorff Spaces, *Jour. Ind. Math. Soc.*, (2) 11 (1947).

‡ Über die Mächtigkeit Zusammenhängender Mengen, *Mathematische Annalen*, Band 94.

§ R. Vaidyanathaswamy, *Set Topology*, Part I, pp. 101-22.

minimal-bicompact if and only if it contains no non-closed bicompact subsets.

3. The space R is defined as follows:

$$R = a + b + \{a_{ij}\} + \{b_{ij}\} + c_i, \quad (i, j = 1, 2, \ldots).$$

R is defined as a topological space by assigning the following neighbourhood system $\{U\}$:

$$U(a_{ij}) = a_{ij}; \; U(b_{ij}) = b_{ij},$$

(i.e. the points a_{ij}, b_{ij} are all isolated points in R);

$$U^n(c_i) = c_i + \sum_{j=n}^{\infty} (a_{ij} + b_{ij}), \quad U^n(b) = b + \sum_{j=1}^{\infty} \sum_{i=n}^{\infty} b_{ij}$$

$$n = 1, 2, \ldots.$$

The neighbourhoods of the point a on the other hand are defined as follows in terms of an arbitrary integer n and an arbitrary integer-valued function $f = f(i)$ $(i = 1, 2, \ldots)$ such that $f(i)$ tends to infinity with i:

$$U_f^n(a) = a + \sum_{j=1}^{\infty} \sum_{i=n}^{\infty} a_{ij} + \sum_{i=n}^{\infty} U^{m_i}(c_i),$$

where $n = 1, 2 \ldots$; $m_i = f(i)$, $(i = 1, 2 \ldots)$, is any sequence of positive integers tending to infinity with i. It will be convenient to call the set of points $(a_{i1}, a_{i2} \ldots)$ or $(b_{i1}, b_{i2} \ldots)$ a *row* of the space R and the set of points $(a_{1j}, a_{2j}, \ldots)$ or $(b_{1j}, b_{2j}, \ldots)$ a *column* of the space R. It is clear that no neighbourhood of a will contain a complete column of b_{ij}'s.

The neighbourhood system thus defined satisfies Hausdorff's postulates H_1, H_2, H_3. The space R thus defined is a T_1-space. It is not, however, Hausdorff, since the points a and b do not possess disjoint neighbourhoods. We first of all show that

The topological space R defined by this neighbourhood system is bicompact.

For this purpose, let us consider an arbitrary open covering $\{G\}$ of R. Let us denote by $G(a)$, $G(b)$ a pair of elements of $\{G\}$ which contain the points a and b

respectively. Then $G(a)$, being an open set of R, will contain a neighbourhood of a and will therefore cover all but a finite number of the c's, say $c_{n+1}, c_{n+2}, \ldots$ and all but a finite number of rows of a_{ij}'s. Similarly $G(b)$ will cover all but a finite number of rows of b_{ij}'s. The remaining c's, viz. $c_1, c_2, \ldots, c_n$ will be covered by a finite number of G's, say $G_1, G_2, \ldots, G_k$. It follows that

$$R-[G(a)+G(b)+G_1+G_2+\ldots\ldots+G_k]$$

is a finite set of a_{ij}'s and b_{ij}'s, which will evidently be covered by a further finite number of G's. Hence R can be covered by a finite number of G's. Thus the given open covering of R contains a finite covering of R. Hence R is bicompact.

In order to prove that R is minimal-bicompact, we have to show that R contains no non-closed bicompact subset. Let, if possible, S be a non-closed subset of R, which is bicompact. Then, clearly, S should have either a or b or one of the c's as contact point not belonging to it.

Case I. Let S have c_i as contact point not belonging to it. Then we can select from S a sequence of distinct points of the type

$$A=\{a_{ij_1}, a_{ij_2}, \ldots, a_{ij_n}, \ldots\}$$

or a sequence of distinct points of the type

$$B=\{b_{ij_1}, b_{ij_2}, \ldots, b_{ij_n}, \ldots\},$$

(where $j_1<j_2<\cdot\cdot<j_n<\ldots.$) which converges* to c_i. The set A (or B) is relatively closed in S, since S does not contain c_i, by hypothesis. But a set of type A (or B) cannot be bicompact, since it is a T_1-space consisting of an infinity of isolated points. Hence S contains a relatively closed non-bicompact subset, and so, by §2.2, S cannot be bicompact.

Case II. Let S have b as contact point not belonging to it. Since every neighbourhood of b intersects S,

* In a T_1-space, a sequence of points $\{x_1, x_2 \ldots . x_n \ldots\}$ is said to *converge* to a point x, if every neighbourhood of x contains all but a finite number of points of the sequence.

we can select from S, a sequence of distinct points of the type

$$\beta = \{ b_{i_1 j_1}, b_{i_2 j_2}, \ldots b_{i_n j_n}, \ldots \}$$

(where $i_1, i_2, \ldots i_n \ldots$ is an increasing sequence of positive integers and $j_1, j_2 \ldots j_n \ldots$ is some sequence of positive integers) which converges to the unique limit b, (since it is clear that β cannot have a as accumulation point, as we can always construct a neighbourhood of a disjoint with β by choosing $f(i)$ so that $f(i_k) > j_k$). Hence β is relatively closed in S. Since β is a T_1-space consisting of an infinity of isolated points, it is not bicompact. Hence by §2.2, S cannot be bicompact.

Case III. Let S have a as contact point not belonging to it.

Firstly, if S contains points a_{ij} from an infinity of rows, we can select a sequence of distinct points of the type

$$\alpha = \{ a_{i_1 j_1}, a_{i_2 j_2}, \ldots a_{i_n j_n} \ldots \}$$

(where $i_1, i_2 \ldots . i_n \ldots$ is an increasing sequence of positive integers and $j_1, j_2 \ldots j_n \ldots$ is some sequence of positive integers) which converges to a. As before α is a non-bicompact relatively closed subset of S. Hence, by §2.2, S cannot be bicompact.

Secondly, if S does not contain a_{ij}'s from an infinity of rows but contains an infinity of c's say $c_{i_1}, c_{i_2} \ldots c_{i_n} \ldots$ then the sequence of distinct points

$$\gamma = \{ c_{i_1}, c_{i_2}, \ldots c_{i_n} \ldots \}$$

converges to a. Hence it forms a non-bicompact relatively closed subset of S. Hence again, by §2.2, S cannot be bicompact.

If neither of these two sub-cases happens, i.e. if S contains only a finite number of c's and contains a_{ij}'s only from a finite number of rows, then, since S has a as contact point outside it, it must contain an infinity of b_{ij}'s from each of an infinity of rows (for, otherwise, we can construct a neighbourhood of a disjoint with S). This means that S will have some c's outside it as contact points, since by hypothesis, it contains only a finite number of c's. Thus this leads to Case I discussed above, so that S cannot be bicompact under any circumstances in Case III.

Thus, finally, in no case can a non-closed subset be bicompact in R, so that the space R constructed as above is a non-Hausdorff minimal-bicompact space.

4. Character of the point a in R. The neighbourhood system given for the point a in R is connected with some general questions of topological interest which we proceed to mention. In the first place, there exists a sequence of distinct points which converges to a, for example, the sequence α. In the second place, the set S of all b_{ij}'s is such that $\bar{S}=S+a$, but there exists no subsequence of b_{ij}'s which converges to a (see Case II above). It follows that a cannot have an equivalent system of countable neighbourhoods. For, if $\{V^n(a)\}$ is a system of countable neighbourhoods equivalent to $\{U^n_f(a)\}$, then choosing a b_{ij} from each of these neighbourhoods $V^n(a)$, we will get a sequence of b_{ij}'s which will converge to a; but this has been shown to be impossible.

Alexandroff and Urysohn* have defined the following *local properties* of a point of a Hausdorff space which may also be considered for a general T_1-space. A point ξ of a general T_1-space is said to have (1) the property (θ), if there exists a sequence of distinct points $\{\xi_1, \xi_2, \ldots\}$ of the space which converges to ξ; (2) the property (ι) if there exists an equivalent system of countable neighbourhoods for the point ξ; (3) the property (δ) if the point ξ is the intersection of a countable family of open sets. Now the point a of our bicompact space R possesses the properties (θ) and (δ) but not (ι). This verifies the

* Zur Theorie der Topologischen Räume, *Mathematische Annalen*, Band 92.

remark of Alexandroff and Urysohn that the properties (θ) and (δ) do not imply (ι) separately or together, although, obviously, the property (ι) implies (θ) and (δ). We further see that the property (δ) is not sufficient in order that every set having ξ as accumulation point may contain a sequence converging to ξ.*

A more general classification of points given by Alexandroff† for points of a Hausdorff space, may also be considered for a general T_1-space. If M be a subset of a general T_1-space, R, then the cardinal numbers of the set M and of the whole space R may be denoted by $|M|$ and $|R|$ respectively. Alexandroff's generalized notion of convergence is as follows.

Definition 1. A subset M of a T_1-space R is said to *flow* to a point ξ, not contained in M, if, for every arbitrary neighbourhood $U(\xi)$ of ξ we have

$$|U(\xi).\ M| > |[R-U(\xi)].\ M|.$$

In particular, a countable subset M of R can flow to a point ξ, not contained in M, if and only if every neighbourhood of the point ξ contains all but a finite number of the points of M; in other words, if and only if M can be arranged as a sequence converging to ξ. Clearly any subset of a Hausdorff space cannot flow to more than one point; this need not be true for a general T_1-space.

Definition 2. The smallest cardinal of subsets of R, not containing an assigned point $x \epsilon R$, which flow to x is defined to be the *convergence character* of the point x and is denoted by $\phi_R(x)$. We say that the convergence character of an isolated point is 1.

Definition 3. The smallest cardinal of a family of open sets of which the point $x \epsilon R$ is the sole intersection is called the *intersection character* of the point x and is denoted by $\psi_R(x)$.

* Compare the question raised in Ex. 27, page 276 in R. Vaidyanathaswamy, *Set Topology*, Part I.

† Über die Struktur der Bikompakten Topologischen Räume, *Mathematische Annalen*, Band 92.

DEFINITION 4. The minimum cardinal of the neighbourhood system of the point $x \epsilon R$, equivalent to the given neighbourhood system of the point x is called simply the *character* of the point x and is denoted by $\chi_R(x)$.

It is quite possible that there exists no subset M (not containing x) which flows to x, so that the convergence character may or may not exist for a point $x \epsilon R$. (This is the case, for example, for the subspace $S+a$, where S is the set of all b_{ij}'s; see below.) It is of course true that a point x may be contact point of a subset M of R without, however, being 'flow point' of M. From the definitions, it is clear that for an arbitrary point $x \epsilon R$, the intersection character $\psi_R(x)$ and character $\chi_R(x)$ are well defined cardinals and that $\psi_R(x) \leqslant \chi_R(x)$. All the three cardinals may also exist and be different.

Let us now take the bicompact space R constructed above. Since the family of functions $\{f(i)\}$ has the cardinal $\mathcal{N}$ of the continuum we see that the cardinal of the given neighbourhood system of the point a is $\mathcal{N}.\mathcal{N}_0 = \mathcal{N}$. As has already been shown, the point a does not possess a countable system of neighbourhoods equivalent to the given neighbourhood system. Hence assuming the Continuum Hypothesis, viz. that there exists no cardinal between $\mathcal{N}_0$ and $\mathcal{N}$, we see that character $\chi(a) = \mathcal{N}$. Further, since there exists a sequence $\{a_{i_n j_n}\}$, which converges to a, we see that convergence character exists and is equal to $\mathcal{N}_0$. Next, since, for a given f, the intersection of all the neighbourhoods $U_f^n(a)$, i.e.

$$U_f^1(a),\ U_f^2(a), \ldots.$$

of the point a is the point a itself, we see that intersection character $\psi(a) = \mathcal{N}_0$. Hence, we see that $\phi(a) = \psi(a) < \chi(a)$.

Lastly let S denote the subset of all b_{ij}'s. Then, in the subspace $S+a$, the point a is contact point of S but a flow point neither of S nor of any subset of S; hence the convergence character of a does not exist in the subspace $S+a$.

MINIMAL BICOMPACT SPACE.*

BY

V. K. BALACHANDRAN, *Madras University.*

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A Ramanathan§ has given an example of a non-Hausdorff minimal bicompact space. A simpler example of the same space was constructed by Hing Tong.† The following space R is a still simpler example of a non-Hausdorff minimal bicompact space.

The points of R consist of elements of the double array (x_{ij}), $i, j = 1, 2, \ldots$, together with two other elements x, y. The neighbourhood system of R is defined as follows :—

Each point (x_{ij}) of the array is its own sole neighbourhood; the neighbourhoods of x are sets of the form (x+all terms of the array except a *finite* number from each row); the neighbourhoods of y are of the form (y+all terms of the array except a *finite* number of rows).

The verification of the Hausdorff postulates H_1, H_2, H_3, is immediate. The resulting topological space R has the points (x_{ij}), $i, j = 1, 2, \ldots$, as isolated points. That R is T_1 follows at once, since every neighbourhood of x is disjoint with y, and vice-versa, while all the other points (x_{ij}) are isolated. Since every neighbourhood of x intersects every neighbourhood of y, x and y are T_2—inseparable, and hence R is non-Hausdorff. We shall now prove that (1) R is minimal bicompact by showing that R is bicompact, and (2) that all bicompact subsets of R are closed.‡

To prove (1), let us consider any open covering $\{g_i\}$ of R. Let g_x, g_y be the open sets of the covering con-

* I wish to thank Dr. R. Vaidyanathaswami for his help in the preparation of this paper.

§ *Jour. Ind. Math. Soc.*, (2) 12 (1948), 40-7.

† Minimal Bicompact spaces, *Bull. Am. Math. Soc.*, 54 (1948), 478.

‡ R. Vaidyanathaswamy, *Treatise on Set Topology*, p. 102, Art. 31.6.

taining x, y respectively. Then $g_x \cup g_y$ being an open set containing x and y must contain an x-neighbourhood as well as a y-neighbourhood. Now it is clear that the union of any x-neighbourhood with any y-neighbourhood can exclude at most a finite number of points of R. Therefore, it follows that $g_x \cup g_y$ (in fact, any open set containing x and y) must include all but a finite number of points of R. Thus R can be covered by a finite number of the g's, that is, R is bicompact.

To prove (2) let S be a non-closed subset of R. Since all the points of R other than x and y are isolated, S must have one of x, y as accumulation point without containing it.

CASE 1 :—*x is an accumulatton point of S, not contained in S.* Here S must contain an *infinity* of terms $\{x_{ij_1}, x_{ij_2}, \ldots\}$ from at least one row, say the ith. For, if S contains only a *finite* number from each row, the set consisting of x and all the terms of the array (x_{ij}) except those belonging to S will be a neighbourhood of x disjoint with S, which contradicts the fact that x is an accumulation point of S. Let g_y be the open neighbourhood of y, consisting of all the x_{ij}'s other than those in the ith row. Now consider the open covering of S consisting of $(g_y \cap S, (x_{ij_1}), (x_{ij_2}), \ldots)$. It is clear that this cannot include a finite covering. Thus S is not bicompact.

CASE 2 :—*y is an accumulation point of S, not belonging to S.* In this case, S must contain points from an *infinity* of *distinct* rows, say the i_1th, i_2th, ... Let $x_{i_n j_n}$ be the first element of the i_nth row, $n = 1, 2, \ldots$, belonging to S. Let g_x be the open neighbourhood of x containing all terms of the i_nth row after the j_nth, $n = 1, 2, \ldots$. Then as before the covering of S consisting of $g_x \cap S$ and $(x_{i_n j_n})$ $n = 1, 2, \ldots$ does not include a finite covering. Thus in this case also S is not bicompact.

In other words, we have proved that no *non-closed* subset of R can be bicompact. Hence R is minimal bicompact.

IDEALS OF THE DISTRIBUTION LATTICE.*

BY

V. K. BALACHANDRAN, *University of Madras.*

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In a distributive lattice L with units, the set π_α of elements whose product-complement is 0 constitute an α-ideal.(1) In this paper we are primarily concerned with distributive lattices closed for product-complements. For this case, it is known that the normal elements form a Boolean-algebra N, for which the boolean sum and product are respectively the normalized sum and lattice product in L(5), and that the quotient lattice L/π_α is isomorphic to N.(2) From V. S. Krishnan's investigation(3) of the lattice homomorphism from L to N, it follows that each residue-class of π_α in N contains a unique normal element and every element in the same residue-class has the same product-complement. I have also obtained this result below directly from the theory of product-complements. From this property, I deduce a direct characterization of all elements having a given normal element as product-complement (Section 1).

A similar, but more difficult problem, which is solved (Section 2) is the characterization of all μ-ideals in L which have a given normal ideal as product-complement. (I believe that this has not been solved so far in the existing literature). The solution will of course depend on the properties of the ideal $\pi_{\alpha\mu}$ (consisting of all μ-ideals whose product-complement is 0) and of the residue-classes of $\pi_{\alpha\mu}$ in L_μ (the lattice of μ-ideals) of L. I, therefore, proceed first to characterize the ideal $\pi_{\alpha\mu}$ in terms of π_α and then, by considering the residue-classes of $\pi_{\alpha\mu}$ in L_μ deduce a solution of the problem.

* I wish to thank Dr. R. Vaidyanathaswami of the University of Madras for his valuable guidance.

1. Throughout the investigation, L is a distributive lattice with units closed for product complements unless otherwise stated.

THEOREM 1 :—*The elements in any residue-class of* Π_α *in* L *have the same product-complement. If* a' *denote the product-complement of* a, *then* a'' *is the highest element of the residue-class containing* a.

PROOF : We say that, $a \equiv b \pmod{\Pi_\alpha}$ in L if there exist elements x and y in Π_α such that $ax = by$. Then $axy = by.y = by$ and $bxy = byx = ax.x = ax$. Therefore $axy = bxy$, and xy is in Π_α since Π_α is an α-ideal and x, y are in it. So $a \equiv b \pmod{\Pi_\alpha}$ if and only if there exists an element t in Π_α such that $at = bt$.* $(a+a')' = a'.a'' = 0'$, and therefore $a+a' \in \Pi_\alpha$; since $a(a+a') = a = a''(a+a')$ it follows that $a'' = a \pmod{\Pi_\alpha}$. Further if t be in Π_α and $a.t = b.t$, then $(a.t)' = (b.t)'$ or $(a'+t')'' = (b'+t')''$. But $t' = 0$ since $t \in \Pi_\alpha$. Therefore we have $a''' = b'''$ or $a' = b'$. Thus, if $c \equiv a \pmod{\Pi_\alpha}$ $c' = a'$ and so $c < c'' = a''$ or a'' is the highest element of the residue class containing a. Thus we see that every residue-class contains a unique normal element which is equal to the double-product complement of every element of the residue-class. This proves the theorem.

THEOREM 2 :—*Every element* b *in the residue-class of* Π_α *in* L *containing* a *is of the form* $b = a''x$, *where* x *is in* Π_α.

For, by Theorem 1, if $b \equiv a \pmod{\Pi_\alpha}$, $b'' = a''$ and therefore $b = b.b'' = b.b''+b'.b'' = (b+b')b'' = (b+b')\ a''$; but $x = (b+b')$ is in Π_α since $(b+b)' = b'.b'' = 0$†.

Thus Theorems 1 and 2 give

THEOREM 3 :—*The set of elements having a given normal element* n *as product-complement is given by* $n'x$, *where* x *varies in* Π_α.

* See p. 45 of (5).

† For various properties of product-complements, see (5) pp. 42-44.

As an example to Theorem 3, consider the lattice Γ of open sets of a topological space R. It is known that Γ is a complete lattice closed for product-elements.(5) The product-complement of any open set is its exterior. The normal elements of Γ are called the open domains. The dense open sets of Γ constitute the ideal Π_α. Theorem 3 gives in this case the result:—The open sets of Γ whose exterior is a given open domain d are the intersections of Ext d with a dense open set g. But $(\text{Ext } d) \cap g = (\text{Ext } d) - (\text{Ext } d) \cap g'$ (g' being the set-complement of g). Further $(\text{Ext } d) \cap g'$ is a non-dense, relatively closed subset of Ext d. Hence any open set, whose exterior is the open domain d, is obtained by the removal of a non-dense relatively closed subset from Ext d.(5)

2. Thus the problem of specifying all elements whose product-complement is a given normal element in terms of the ideal Π_α is completely solved by Theorem 3. To obtain a similar characterization of ideals having a given normal ideal as product-complement, we must first determine the ideal $\Pi_{\alpha\mu}$ (that is, the class of ideals in L whose product-complement is 0). For this purpose we have

THEOREM 4:—*In a distributive lattice with units (not necessarily closed for product-complements) every normal ideal is the ideal sum of double-product-complements of all principal ideals contained in it.*

PROOF:—First observe that if a is any element of the normal ideal P_μ, $P_\mu''(a) \subset P_\mu'' = P_\mu$ (since P_μ is normal). Therefore, $P_\mu = \Sigma P_\mu(a) \subset \Sigma P_\mu''(a) \subset P_\mu$,(5) whence $P_\mu = \Sigma P_\mu''(a)$ $(a \varepsilon P_\mu)$.

COR.:—*If L be closed for product-complements, every normal ideal is the ideal sum of principal normal ideals contained in it. Hence, a normal ideal is completely determined by the set of normal elements it contains.*

PROOF:—For then $P_\mu = \Sigma P_\mu''(a) = \Sigma P_\mu(a'')$ since L is closed for product-complements.

We can also arrive at this result from V. S. Krishnan's extension, theory[1] according to which a normal μ-ideal of L (the given lattice) is the extension of an ideal of $\mathcal{N}$ (the boolean-algebra of normal elements of L). And since, every ideal of $\mathcal{N}$ has a unique extension, the result follows.

THEOREM 5:—*If P_μ and Q_μ are normal, then the conditions $Q_\mu \subset P_\mu$, and $Q_\mu' \supset P'_\mu$ imply one another.*

If $P_\mu \supset Q_\mu$, then, of course, $Q'_\mu \supset P'_\mu$. Conversely. $Q'_\mu \supset P'_\mu$ together with the fact that that P_μ, Q_μ are normal give $Q_\mu = Q''_\mu \subset P''_\mu = P_\mu$.

If L is closed for product-complements, we have the deeper

THEOREM 6:—*The normal elements (n) of L contained in a normal ideal P'_μ are precisely those with the property $P_\mu(n') \supset P_\mu$.*

The proof is on the same lines as that in Theorem 5. If n is a normal element of P'_μ, $P_\mu = P_\mu(n)^* \subset P'_\mu$ and so $P_\mu(n') \supset P''_\mu \supset P_\mu$. Again if n is such that $P_\mu(n) \supset P_\mu$, $P_\mu(n') = P'_\mu(n) \subset P'_\mu$, that is, $n' \varepsilon P'_\mu$, this together with the relation $n = (n')'$ since n is normal, proves the theorem.

From Theorem 6, and Cor. Theorem 4, we can deduce

THEOREM 7. *The product-complement of a μ-ideal is identical with that of its comprincipal envelope, when L is closed for product-complements.*†

PROOF. The comprincipal envelope of an ideal is the intersection of all principal ideals containing it. The cut-complement $(P_\mu)_c$ of a μ-ideal P_μ is the set of all elements x greater than every element of P_μ. It is known that $(P_\mu)_c$ is an α-ideal and that $(P_\mu)_{cc}$—that is, the cut-complement of its cut-complement—is the same as the comprincipal

* The set of elements of a distributive lattice $L < a$ given element a is a μ-ideal called the principal ideal, it is denoted by $P_\mu(a)$.

† See example under Theorem 8.

envelope of P_μ. Hence the comprincipal envelope of P_μ may be denoted by $(P_\mu)_{cc}$. Dual definitions and results hold for α-ideals.

For the proof of the theorem, we observe that the principal ideals containing P_μ and those containing $(P_\mu)_{cc}$ are the same, Therefore, the normal elements in P'_μ and $(P_\mu)'_{cc}$ are the same (Theorem 6). Hence $(P_\mu)' = (P_\mu)'_{cc}$ since these contain the same normal elements.

A more direct proof is as follows: *When L is closed for product-complements every normal ideal in L is comprincipal.* For every normal ideal is of the form P'_μ. Since $P_\mu = \Sigma P_\mu(a), a \in P_\mu, P'_\mu = [\Sigma P_\mu(a)]' = \Pi P_\mu(a')$ showing that P_μ' is comprincipal. Since the comprincipal envelope of an ideal is the smallest comprincipal ideal containing it, we have $P_\mu \subset (P_\mu)_{cc} \subset P''_\mu$; hence $P'_\mu \supset (P_\mu)''_c \supset P'_\mu$ that is, $(P_\mu)'_{cc} = P'_\mu$.

Cor. *In a complete distributive lattice closed for product-complements, every normal ideal is principal.*

For, if t be the sum of the elements of P_μ (which exists since L is complete)* $P_\mu(t)$ is the comprincipal envelope of P_μ. Therefore $P_\mu' = P_\mu'(t) = P_\mu(t')$. Hence, since every normal ideal is of the form P_μ' the result follows.

Theorem 8. *The ideals P_μ such that $P_\mu' = 0$ are precisely those whose cut-complement is contained in* Π_α.

Proof. Suppose $P'_\mu = 0$. Then if $x \in (P_\mu)_c$ (the cut-complement of P_μ), $P_\mu(x)$ contains P_μ. So $P'_\mu(x) = P_\mu(x')^*$ $\subset P'_\mu = 0$ or $x' = 0$; that is x is in Π_α. Conversely, if $(P_\mu)_c$ is contained in Π_α, the product-complement of every element a in $(P_\mu)_c$ (that is every element a such that $P_\mu(a)$ contains P_μ) is 0. But these elements a are precisely those normal elements of L contained in P'_μ (Theorem 6). In other words, 0 is the only normal element in P'_μ; or $P'_\mu = 0$ (Cor. Theorem 4).

* See (5) p. 51, §12.71.

The following example constructed by V. S. Krishnan* for a different purpose, shows that Theorem 8 no longer holds if the restriction that L is closed for product-complements is removed.

Consider the infinite double chain L with the following ordering relations: $0 < a_0 < a_1 \ldots . < a_n \ldots < 1$; $0 < b_1 < b_2 \ldots < b_n \ldots < 1$; $b_n < a_n$, $n = 1, 2, \ldots$; L is easily verified to be a distributive lattice. L is not closed for product-complements, since a'_0 does not exist. Here Π_α is the set $\{1, a_n\}$ $n = 1, 2, \ldots$. The set $\{0, b_n\}$ $n = 1, 2, \ldots$, is evidently a μ-ideal, say P_μ ; also $(P_\mu)_c = 1 \subset \Pi_\alpha$; but $P'_\mu =$ the set $\{a_0, 0\} \neq 0$, whence the part of Theorem 8—$(P_\mu)_c \subset \Pi_\alpha$ implies $P'_\mu = 0$ fails. It is easy to see that the part—$P'_\mu = 0$ implies $(P_\mu)_c \subset \Pi_\alpha$—holds irrespective of the fact that L is closed for product-complements. For if $P'_\mu = 0$ and $x \varepsilon (P_\mu)_c$, $P_\mu(x) \supset P_\mu$. So $P'_\mu(x) \subset P'_\mu = 0$, that is $P'_\mu(x) = 0$. If t be such that $x t = 0$, then $P_\mu(x)$. $P_\mu(t) = P_\mu(x t) = P_\mu(0) = 0$, and therefore $P_\mu(t) \subset P'_\mu = 0$ or $t = 0$. This means that x' exists and $= 0$. Thus we have proved that any arbitrary element of $(P_\mu)_c$ is contained in Π_α, whence the result.

Again it may be observed that $(P_\mu)'_{cc} = 0 \neq P'_\mu$. So we find that Theorem 7 also does not always hold good in the general distributive lattice.

THEOREM 9. *If L be complete, the μ-ideals whose product-complement is 0, are precisely those whose comprincipal envelope is a principal ideal of the form $P_\mu(t)$, where t is in Π_α.*

This follows immediately from the equation $P'_\mu(t) = P_\mu(t')$, where $P_\mu(t)$ is the comprincipal envelope of P_μ.

An example to this theorem is provided by the lattice Γ of open sets of a topological space R. Since Γ is

* V. S. Krishnan: Thesis for the M. Sc. degree.

complete, every normal ideal is a principal ideal defined by an open domain. Therefore, the ideals whose product-complement is 0, are precisely those whose comprincipal-envelope is a principal ideal defined by a dense open-set. Alternatively, we may state this as follows :—The μ-ideals of Γ whose product-complement is 0, are precisely those which are generated from an open covering* of a dense open set (in particular of δ space).

THEOREM 10. *If in a distributive lattice L closed for product-complements* $\Pi_\alpha = 0$, *L is a boolean-algebra; in this case* $\Pi_{\alpha\mu}$ *consists of all* μ*-ideals whose comprincipal-envelope is* 1 *(that is the whole lattice).*

PROOF. We have $(x+x')' = x'x'' = 0$, for every x in L. Therefore $x+x'$ is in Π_α and since $\Pi_\alpha = 0$, $x+x' = 1$ or every element of L is simple. That is, L is a boolean-algebra. Also since Π_α consists of the element 1 alone "an ideal whose cut-complement is contained in Π_α" is equivalent to "an ideal whose comprincipal envelope is is the whose lattice". Hence the result follows from Theorem 9.

Having thus determined the ideal $\Pi_{\alpha\mu}$ in L_μ, we proceed to specify all ideals with a given normal ideal as as product-complement. By Theorems 3 and 8 we get

THEOREM 11. *Every* μ*-ideal, of a distributive lattice (closed for product-complements) whose product-complement is a given normal ideal* P_μ, *is the intersection of* P'_μ *with an ideal whose cut-complement is contained in* Π_α.

In case $\Pi_\alpha = 0$, that is, L is a boolean-algebra the term "an ideal whose cut-complement is contained in Π_α" may be replaced by the equivalent term "an ideal whose comprincipal envelope is the whole lattice."

As an example, we may consider the μ-ideals of the lattice Γ of open sets of a topological space R. As stated

* See Art. 21.5 of (5).

above every normal ideal in Γ is a principal ideal defined by an open domain. Therefore, Theorem 11 gives:— The most general ideal whose product-complement is a given normal ideal (principal ideal defined by an open domain g) is generated from an open covering of any open set whose exterior is g.

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NOTICE

The Sixteenth Conference of the Indian Mathematical Society will be held in Madras in the last week of December 1949 under the auspices of the University of Madras.

Papers intended for the Conference are to be sent to Dr. A. Narasinga Rao, Andhra University, Waltair with a brief abstract before the end of September 1949.

REGULAR BANACH ALGEBRAS

BY

IRVING KAPLANSKY, *University of Chicago.*

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1. INTRODUCTION. Von Neumann(6) defines a ring to be regular if for any a there exists an x such that $axa = a$. For algebras of finite order, or more generally for rings with the descending chain condition, regularity coincides with the classical concept of semi-simplicity. For rings without a finiteness assumption, regularity thus presents itself as a possible candidate for the generalization of semi-simplicity. In connection with von Neumann's continuous geometries and their associated rings, it has in fact proved to be a useful generalization. However in the case of Banach algebras, evidence has slowly accumulated that regularity is too severe an assumption. We may remark parenthetically that Jacobson(4) has proposed a weaker version of semi-simplicity, which appears to be the most useful one to use for Banach algebras.

There are four results in the literature concerning regular Banach algebras.

(1) Von Neumann* showed that a regular ring of operators on Hilbert space is finite-dimensional.

(2) Segal† showed that the group algebra of a locally compact group is regular only if the group is finite.

(3) Rickart(8) [Cor. 3.7] proved (among other things) that in certain regular Banach algebras all principal ideals are closed.

(4) Arens and the author(2) [Th. 3.5] proved that a commutative regular Banach algebra is finite-dimensional.

* See (7) pp. 22-24.

† In his thesis (Yale 1940). In the published abstract (9) of this thesis, the word "compact" should be replaced by "finite" on p. 351, l. 25.

Actually the following somewhat more general result is given there: a strongly regular Banach algebra is finite-dimensional. (A ring is strongly regular if for any a there exists an x such that $a^2x = a$. The x in question can be shown to commute with a, and so, as the name implies, strong regularity implies regularity. Of course for commutative rings the two concepts coincide.)

In this paper we shall subsume these four results in the following theorem.

THEOREM. *Any regular Banach algebra is finite-dimensional.*

2. DEFINITIONS. We shall use the term regular as defined above; we remark that, unlike von Neumann, we do not require a unit element.

A Banach algebra is a Banach space over the real numbers, and an algebra, with the norm satisfying the condition

$$\|xy\| \leqslant \|x\| \, \|y\|.$$

In comparing this with Gelfand's normed rings (3), it should be observed that we assume only real scalars, that we do not assume a unit element, and that finally commutativity is not assumed.

3. PROOF OF THE THEOREM. The proof follows the same fundamental idea as von Neumann's, in that we first prove the following lemma.

LEMMA 1. *A regular Banach algebra cannot have an infinite set of orthogonal idempotents.*

PROOF. Suppose on the contrary that $e_1, e_2, \ldots, e_i, \ldots$ are orthogonal non-zero idempotents: that is, $e_i^2 = e_i$, $e_ie_j = e_je_i = 0 \, (i \neq j)$. Write $\|e_i\| = M_i$, and set $N_i = 1/2^iM_i$. Then $\|N_ie_i\| = 2^{-i}$ and hence ΣN_ie_i converges, say to a. We have $e_ia = ae_i = N_ie_i$, because of the orthogonality of the e's. Find the x with $axa = a$. Then

$$N_i^2e_ixe_i = e_iaxae_i = e_iae_i = N_ie_i.$$

Take norms of the two end members of this equation, and cancel $N_i M_i$; we get $2^{-i} \|x\| \geqslant 1$, which cannot be true for all i.

We supplement Lemma 1 with the following purely algebraic lemma.

LEMMA 2. *If a regular ring A does not have an infinite set of orthogonal idempotents, then it satisfies the descending chain condition on right ideals (and so is the direct sum of a finite number of matrix rings over division rings).*

PROOF. Although it is the descending chain condition we are after, let us begin by proving the *ascending* chain condition, and for the moment we will prove it only for principal right ideals. Now any principal right ideal can be generated by an idempotent[(6)] [Lemma 5]. So the denial of the ascending chain condition leads us to a sequence $\{f_i\}$ of idempotents with $f_i A$ properly contained in $f_{i+1}A$. In particular $f_i f_j = f_j$ for $i \geqslant j$. Set

$$e_i = (1-f_1)(1-f_2)\dots(1-f_{i-1})f_i.$$

(This product is well defined even if A has no unit element). We verify that the e_i's are orthogonal idempotents—this can be done quickly by noting that $(1-f_{i-1})f_i f_j = 0$ for $i > j$. Our hypothesis is contradicted unless from some point on all the e's are 0. But $e_i = 0$ tells us that f_i is in $f_{i-1}A$, contradicting the assumption that the latter is a proper part of $f_i A$.

Thus we have proved the ascending chain condition on principal right ideals. But every finitely generated right ideal is principal*; hence we even have the ascending chain condition on finitely generated right ideals. From this the ascending chain condition on all right ideals follows (this is a remark valid in any ring). In short we have proved that every right ideal is principal. Of course similarly every left ideal is principal. But [6, Th. 1] asserts that there is a one to one order-inverting correspondence between principal left ideals and principal

* See (6) Lemma 15.

right ideals. We are thus able to translate the ascending chain condition into the descending chain condition, and this concludes the proof of Lemma 2.

To complete the proof of our main theorem we need only to supplement Lemmas 1 and 2 with the theorem of Mazur*: the only Banach division algebras are the reals, complexes, and quaternions. Since these are finite-dimensional, it follows that every regular Banach algebra is finite-dimensional.

REMARK 1. Inspection of the above proof shows that it is not of vital importance to have an algebra over the reals. In fact, if we start with a regular complete normed algebra over a field with a valuation, we may, with slight modifications, carry through Lemma 1 and thus arrive at the descending chain condition. Of course we cannot expect to push on to finite-dimensionality, since infinite-dimensional fields will in general exist over the given field.

REMARK 2. On the other hand, the completeness (inherent in the definition of a Banach algebra) is indispensable. For a counter-example, take the set of all sequences $\{a_i\}$ of real numbers, with only a finite number of non-zero entries A suitable norm is Max $|a_i|$. This is a regular ring and a Banach algebra except for its failure to be complete.

4. ANOTHER KIND OF REGULARITY. Segal† has suggested the following "inversion" of von Neumann's regularity: for any $x \neq 0$ there exists an $a \neq 0$ such that $axa = a$. In anticipation of the remarks below, we shall call this *weak regularity*.

Weak regularity is equivalent to the following condition: every non-zero right ideal contains a non-zero idempotent. For if a right ideal I contains $x \neq 0$, choose $a \neq 0$ with $axa = a$; then I contains the non-zero idempotent

* See (5) and (1).

† In conversation with the author.

xa. Conversely, given x, let $e = xy$ be a non-zero idempotent in the right ideal generated by x. Set $a = yxy$; then $axa = ye^3 = ye = a$, and $a \neq 0$ since $e = xa$.

Now in a regular ring every principal right ideal is generated by an idempotent; since it is clear that any non-zero right ideal contains a non-zero principal right ideal, it follows that regularity implies weak regularity.

For rings with the descending chain condition, weak regularity is easily seen to coincide with semi-simplicity and hence with regularity. But in general the two concepts are distinct, as is borne out sharply by the existence of infinite-dimensional weakly regular Banach algebras.

THEOREM. *Let X be a compact Hausdorff space and $C(X)$ the ring of continuous real functions on X. Then $C(X)$ is weakly regular if and only if every open set in X contains an open and closed set.*

PROOF. Let U be an open set in X and I the ideal of functions vanishing in the complement of U. Then a non-zero idempotent in I is necessarily the characteristic function of an open and closed set contained in U. Conversely, given $x \neq 0$ in $C(X)$, let U be the subset of X where x does not vanish, and V an open and closed set contained in U. Define $a = x^{-1}$ on V, $a = 0$ on the complement of V. Then $axa = a$.

This condition on X is satisfied if X is totally disconnected, but on the other hand X does not have to be totally disconnected. For example, the subset of the plane consisting of the points

$$\begin{array}{ll} (t, 0) & (0 \leqslant t \leqslant 1) \\ (i/n, 1/n) & (n = 1, 2, \ldots, i = 1, 2, \ldots, n) \end{array}$$

contains an arc, but satisfies the above condition since its isolated points are dense.

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ON ARITHMETICAL PROPERTIES OF LAMBERT SERIES

BY

P. ERDÖS, *University of Syracuse.*

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Let

$$f(x)=\sum_{n=1}^{\infty}\frac{x^n}{1-x^n} \text{ and } g(x)=\sum_{n=1}^{\infty}\frac{x^n}{1-x^n}\sin\frac{n\pi}{2}.$$

Chowla* has proved that if t is an integer $\geqslant 5$, then $g(1/t)$ is irrational. He also conjectures that for rational $|x|<1$ both $f(x)$ and $g(x)$ are irrational.

In the present note we prove the following

THEOREM. *Let $|t|>1$ be any integer. Then both $f(1/t)$ and $g(1/t)$ are irrational.*

We only give the details for $f(1/t)$; the proof for $g(1/t)$ follows by the method of this note and that of Chowla.

Let us first assume that t is positive and that n is large. Put $k=[(\log n)^{1/10}]$ and let $p_1, p_2, \ldots,$ be the sequence of consecutive primes greater than $(\log n)^2$. Put

$$A=\Big\{\prod_{1\leqslant i\leqslant \frac{k(k+1)}{2}} p_i\Big\}^t.$$

From elementary results about the distribution of primes it follows that $p_i<2\ (\log n)^2$ for $i\leqslant\dfrac{k(k+1)}{2}$. Thus by a simple computation we obtain

$$A<\{2\ (\log n)\}^{tk^2}<e^{(\log n)^{1/4}}. \qquad (1)$$

Consider now the following congruences:

$$x\equiv p_1^{t-1} \pmod{p_1^t}$$

$$x+1\equiv (p_2\, p_3)^{t-1}\,\{\bmod\,(p_2\, p_3)^t\}$$

$$\cdots$$

$$x+k-1\equiv (p_u\, p_{u+1}\cdots p_{u+k-1})^{t-1}\,\{\bmod\,(p_u\cdots p_{u+k-1})^t\}, \qquad (2)$$

* *Proc. Nat. Inst. of Sciences of India*, 13 (1947).

where $u=\frac{k(k-1)}{2}+1$. The integers less than n satisfying the congruences (2) are clearly of the form

$$x+y.A,\ 0<x<A,\ 0\leqslant y<[n/A].$$

We evidently have from (2) that for $0\leqslant j<k$

$$d(x+j+y.A)\equiv 0 \pmod{t^{j+1}},$$

where $d(m)$ denotes, as usual, the number of divisors of m. Thus if we rewrite

$$\sum_{r<x+k+yA} \frac{d(r)}{t^r}$$

in the scale of t, then $t^{-x-yA+1}$ will be the lowest power of t which will occur.

Now if we proceed to determine y in such a way that

$$\sum_{r\geqslant x+k+yA} \frac{d(r)}{t^r}<\frac{1}{t^{x+k/2+yA}}, \tag{3}$$

then the representation of $\sum_{r=1}^{\infty} d(r)/t^r$ in the scale of t will contain at least $\frac{1}{2}k$ consecutive zeros. Thus since $k=[(\log n)^{1/10}]$ can be made arbitrarily large, our number is irrational. [It is clear that the representation of $\sum_{r=1}^{\infty} d(r)/t^r$ in the scale of t is not finite, since

$$\sum_{r\geqslant x+k+yA} d(r)/t^r>0.]$$

To complete our proof we will determine a value $y_0<[n/A]$ satisfying (3). First of all we show that

$$\sum_{r>x+k+10\log n+yA} \frac{d(r)}{t^r}<\frac{1}{t^{x+k+yA}}. \tag{4}$$

Now (4) follows by a simple computation by remarking that $d(r)<r$ and $k=(\log n)^{1/10}$ Thus it will suffice to find a $y_0<[n/A]$ for which

$$\sum{}' \frac{d(r)}{t^r} < \tfrac{1}{2}\frac{1}{t^{x+k/2+y_0A}}, \tag{5}$$

where the dash indicates that

$$x+k+y_0A \leqslant r \leqslant x+k+y_0A+10 \log n;$$

clearly if y_0 satisfies (5) it also satisfies (3). Thus r lies in one of the $[10 \log n]$ arithmetic progressions

$$x+k+s+yA,\ y < [n/A],\ 0 \leqslant s < 10 \log n.$$

First we prove that there exists a $y_0 < [n/A]$ so that

$$d(x+k+s+y_0.A) < 2^{k/4}, \text{ for all } 0 \leqslant s < 10 \log n. \tag{6}$$

It is easy to see that

$$(x+k+s, A) = 1 \text{ for all } 0 \leqslant s < 10 \log n.$$

For, if not, then there exists an s so that

$$x+k+s \equiv 0 \pmod{p_j}, \text{ where } j \leqslant \frac{k(k+1)}{2}.$$

But from (2) we have

$$x+i \equiv 0 \pmod{p_j} \text{ for some } i < k.$$

Thus $k+s-i \equiv 0 \pmod{p_j}$, which is impossible since

$$0 < k+s-i < 11 \log n \text{ and } p_j > (\log n)^2.$$

This completes the proof.

Put $x+k+s = \vartheta$. We have from $(\vartheta, A) = 1$,

$$\sum_{y \leqslant [n/A]} d(\vartheta+yA) < 2 \sum_{l=1}^{\sqrt{n}} \left(\frac{n}{Al}+1\right) \quad c\frac{n \log n}{A},$$

since $A < n^{\epsilon}$. Thus the number of y's for which

$$d(\vartheta+yA) > 2^{k/4} \text{ is less then } c\frac{n \log n}{A.2^{k/4}},$$

and the number of y's for which for some s

$$d(x+k+s+y.A) > 2^{k/4} \text{ is less than}$$

$$10\,c\frac{n\,(\log n)^2}{A.2^{k/4}} < \tfrac{1}{2}\frac{n}{A}.$$

Thus there clearly exists a $y_0 < [n/A]$ satisfying (6). Now clearly

$$\sum{}' \frac{d(r)}{t^r} < 2^{k/4} \sum{}' \frac{1}{t^r} < \tfrac{1}{2}\frac{1}{t^{x+k/2+y_0A}},$$

which proves (5) and completes the proof of the theorem for $t > 1$.

If t is negative the proof is similar to the one just given except that we have to make sure that the expansion of $\sum_{r=1}^{\infty} d(r)/t^r$ in the scale of t is not finite. This is certainly the case if we can prove the existence of a $y_0 < [n/A]$ satisfying (6), for which

$$\sum_{r > x+k+y_0 A} d(r)/t^r \neq 0.$$

This can be done by methods similar to those used above. We do not give the details.

The analogous problems about

$$\sum_{n=1}^{\infty} \frac{\phi(n)}{t^n}, \sum_{n=1}^{\infty} \frac{\phi'(n)}{t^n}, \sum_{n=1}^{\infty} \frac{\vartheta(n)}{t^n},$$

where $\phi(n)$ denotes Euler's ϕ-function, $\phi'(n)$ denotes the sum of the divisors of n, and $\vartheta(n)$ denotes the number of prime factors of n, seem to present difficulties.

SOME REMARKS ON DIOPHANTINE APPROXIMATIONS

BY

P. ERDÖS, *University of Syracuse.*

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1. The present note contains some disconnected remarks on diophatine approximations.

First we collect a few well-known results about continued fractions, which we shall use later[1]. Let α be an irrational number, $q_1 < q_2 < \ldots$ be the sequence of the denominators of its convergents. For almost all α we have for $k > k_0(\alpha)$, $q_{k+1} < q_k (\log q_k)^{1+\varepsilon}$. Thus if n is large and $q_r \leqslant n < q_{r+1}$ we have $q_r > \frac{n}{(\log n)^{1+\varepsilon}}$. Further for almost all α

$$\frac{1}{q_k^2 (\log q_k)^{1+\varepsilon}} < \left| \alpha - \frac{p_k}{q_k} \right| < \frac{1}{q_k^2}, \tag{1}$$

the second inequality is true for all α.

Also if $|\alpha - a/b| < \frac{1}{2} b^2$ and $q_k \leqslant b < q_{k+1}$, then $b \equiv 0 \pmod{q_k}$. Hence if

$$\frac{1}{\{m\alpha\}} > 2n, m < n \text{ then } m \equiv 0 \pmod{q_r}, \tag{2}$$

where $q_r \leqslant n < q_{r+1}$, and we denote by $\{u\}$ the distance of u from the nearest integer. It is easy to obtain from (1) that for almost all α and $m \geqslant m_0(\alpha)$

$$\frac{1}{\{m\alpha\}} < m (\log m)^{1+\varepsilon}. \tag{3}$$

A theorem of Behnke[2] states that for almost all α $(q_r \leqslant n < q_{r+1})$

1. The results in question can all be found in Koksma, Diophantische Approximation, *Ergebnisse der Math.* 4 (4).

2. *Hamburgische Abhandlengen*, 3 (1924), p. 289.

$$\sum_{\substack{m=1 \\ q_r \neq m}}^{n} \frac{1}{\{ m\alpha \}} < c_1 n \log n. \qquad (4)$$

Denote by $N_n(a, b)$ the number of integers $m \leqslant n$ for which $a \leqslant n\alpha - [n\alpha] \leqslant b$. A theorem of Khintchine-Ostrowsky[1] states that

$$(b-a)n - c_2 (\log n)^{1+\varepsilon} < N_n (a, b) < (b-a)n + c_3 (\log n)^{1+\varepsilon}, \qquad (5)$$

where c_2 and c_3 are independent of a, b and n and depend only on α and ϵ.

2. Denote by $d(n)$ the number of divisors of n, by $r_2(n)$ the number of representations of n as the sum of two squares and by $r_4(n)$ the number of representations of n as the sum of four squares. Walfisz[2] proved, sharpening previous results of Chowla[3], that for almost all α

$$\sum_{m=1}^{n} d(m)\, e^{2\pi i m\alpha} = O(n^{1/2} (\log n)^{1+\varepsilon}) \qquad (6)$$

$$\sum_{m=1}^{n} r_2(m)\, e^{2\pi i m\alpha} = O(n^{1/2} (\log n)^{1+\varepsilon}) \qquad (7)$$

$$\sum_{m=1}^{n} r_4(m)\, e^{2\pi i m\alpha} = O(n^{\frac{1}{2}} (\log n)^{2+\varepsilon}). \qquad (8)$$

By a slight modification of their argument we obtain that for almost all α

$$\sum_{m=1}^{n} d(m)\, e^{2\pi i m\alpha} = O(n^{\frac{1}{2}} \log n) \qquad (9)$$

1. Khintchine, *Math. Zeitschrift*, 18 (1923), p. 297-300. See also Ostrowsky, *Hamburgische Abhandlungen*, 1 (1922), p. 95.

2. *Math. Zeitschrift*, 35 (1935), p. 774-778.

3. *Ibid.*, 33 (1935), p. 544-563.

$$\sum_{m=1}^{n} r_2(m)\, e^{2\pi i m\alpha} = O(n^{\frac{1}{2}} \log n) \tag{10}$$

$$\sum_{m=1}^{n} r_4(m)\, e^{2\pi i m\alpha} = O(n^{\frac{1}{2}} (\log n)^2). \tag{11}$$

(9), (10) and (11) were proved by Chowla[1] in case α has bounded partial fractions in its continued fraction development. But it is well known that these α's have measure 0.

It will suffice to prove (9), the proof of (10) and (11) follows the same pattern.

$$\sum_{m=1}^{n} d(m)\, e^{2\pi i m\alpha} = \sum_{ab \leqslant n} e^{2\pi i ab\alpha}$$

$$= 2 \sum_{a=1}^{n^{\frac{1}{2}}} \sum_{a \leqslant b \leqslant n/a} e^{2\pi i ab\alpha} - \sum_{a=1}^{n^{\frac{1}{2}}} e^{2\pi i a^2\alpha}. \tag{12}$$

Now clearly for every irrational number α

$$\left| \sum_{a \leqslant b \leqslant m/a} e^{2\pi i ab\alpha} \right| < \frac{c_4}{\sin a\pi\alpha} < \frac{c_5}{\{a\alpha\}}. \tag{13}$$

Also trivially

$$\left| \sum_{a \leqslant b \leqslant n/a} e^{2\pi i ab\alpha} \right| < \frac{n}{a}. \tag{14}$$

Put $q_r \leqslant n^{1/2} < q_{r+1}$. We have from (12), (13), (14) and (3)

$$\left| \sum_{m=1}^{n} d(m)\, e^{2\pi i m\alpha} \right| < \sum_{\substack{a=1 \\ q_r \nmid a}}^{n^{\frac{1}{2}}} \frac{1}{\{a\alpha\}} + \sum{}' \min\left(\frac{r_5}{\{a\alpha\}}, \frac{n}{a} \right) + O(n^{\frac{1}{2}})$$

$$< c_6 n^{\frac{1}{2}} \log n + \sum{}'. \tag{15}$$

The dash indicates that the summation is extended over the $a \equiv 0 \pmod{q_r}$.

1. *Ibid.*, 33 (1935), p. 544-563.

Now we estimate Σ'. As stated in the introduction $q_r > n^{\frac{1}{2}}/(\log n)^{1+\varepsilon}$. We distinguish two cases. In case I we have

$$n^{\frac{1}{2}}/(\log n)^{1+\varepsilon} < q_r < (n/\log n)^{\frac{1}{2}}. \tag{16}$$

From (1) we evidently have that for $k < (\log n)^2$, $\{k\, q_r\alpha\} = k\{q_r\alpha.\}$ Thus from (15), (16) and (2)

$$\Sigma' < \sum_{k<(\log n)^2} \frac{1}{\{kq_r\alpha\}} = \sum_{k<(\log n)^2} \frac{1}{k\{q_r\alpha\}} < q_r(\log q_r)^{1+\varepsilon}$$
$$\times \sum_{k<(\log n)^2} \frac{1}{k} < n^{\frac{1}{2}}(\log n)^{\frac{1}{2}+\varepsilon} \sum_{k<(\log n)^2} \frac{1}{k} = o(n^{\frac{1}{2}}\log n). \tag{17}$$

In case II, $q_r > \left(\frac{n}{\log n}\right)^{\frac{1}{2}}$. We evidently have from (14)

$$\Sigma' < \sum_{k \leqslant (\log n)^{\frac{1}{2}}} \frac{n}{kq_r} < (n\log n)^{1/2} \sum_{k<(\log n)} \frac{1}{k} = o(n^{1/2}\log n). \tag{18}$$

(9) clearly follows from (15), (17) and (18).

3. Spencer[1] proved that for almost all α

$$\sum_{m=1}^{n} \frac{1}{m\{m\alpha\}} = O\big((\log n)^2\big). \tag{19}$$

He remarks that (19) is in a sense best possible since by a theorem of Hardy-Littlewood[2] we have for all irrational α

1. *Proc. Cambridge Phil. Soc.*, 35 (1939), p. 521-547. In fact Spencer considers $\sum_{m=1}^{n} \frac{\operatorname{cosec} m\pi\alpha}{m}$ but it is easy to see that asymptotically this is the same as $\sum_{m=1}^{n} \frac{1}{m\{m\alpha\}}$.

2. *Bull. Calcutta Math. Soc.*, 20 (1930), p. 251-266.

$$\sum_{m=1}^{n} \frac{1}{m\{m\alpha\}} > c_7 (\log n)^2.$$

Spencer conjectured[1] that for almost all α

$$\sum_{m=1}^{n} \frac{1}{m\{m\alpha\}} = (1+o(1)) (\log n)^2. \tag{20}$$

We shall prove (20) and a few related results.

First we prove the following

LEMMA. *For almost all α we have*

$$\sum{}' \frac{1}{\{m\alpha\}} = (1+o(1))\; 2\,n \log n, \tag{21}$$

where in Σ' the summation is extended over the m for which $m \leqslant n$ and $\frac{1}{\{m\alpha\}} \leqslant 2n$.

We write

$$\sum{}' \frac{1}{\{m\alpha\}} = \sum_1 + \sum_2 \tag{22}$$

where in $\sum_1$ the summation is over all such m for which

$$\frac{1}{\{m\alpha\}} \leqslant \frac{n}{(\log n)^{10/9}}$$

and in $\sum_2$,

$$2n \geqslant \frac{1}{\{m\alpha\}} > \frac{n}{(\log n)^{10/9}}.$$

We obtain by (5) by a simple argument (re-ordering the terms in the summation) that

$$\sum_1 = (1+o(1)) \sum_{k < n/(\log n)^{10/9}} \left(N_n\left(0, \frac{1}{k}\right) + N_n\left(1-\frac{1}{k}, 1\right)\right) =$$

$$(1+o(1)) \quad 2 \sum_{k < n/(\log n)^{10/9}} \frac{n}{k} = (1+o(1))\, n \log n. \tag{23}$$

1. Oral communication.

Next we estimate $\sum_2$. Put $A = \frac{(\log n)^{1/8}}{n}$. We evidently have from (5) and the fact that each summand in Σ_2 is less than $2n$

$$\sum_2 < 2n\Big(\mathcal{N}_n(0, A) + \mathcal{N}_n(1-A, 1)\Big) + 3(\log n)^{10/9}\frac{n}{(\log n)^{1/8}} \tag{24}$$

(by (5) the number of terms in Σ_2 is less then $3(\log n)^{10/9}$).

Now we have to estimate $\mathcal{N}_n(0, A) + \mathcal{N}_n(1-A, 1)$. Let $0 < x < 1$ be arbitrary. Denote by $v_1 < v_2 < \ldots < v_k$ the integers $\leqslant n$ for which $x \leqslant v_i\alpha - [v_i\alpha] \leqslant x + 1/2.n$. Clearly the numbers $(v_i - v_1)\alpha - [(v_i - v_1)\alpha]$ all are either in $(0, 1/2.n)$ or in $(1-1/2.n, 1)$. Thus

$$\mathcal{N}_n(x, x+1/2.n) < \mathcal{N}_n(0, 1/2.n) + \mathcal{N}_n(1-1/2.n, 1) + 1,$$

or splitting $(0, A)$ and $(1-A, 1)$ into intervals of length $\frac{1}{2n}$ we have $\mathcal{N}_n(0, A) + \mathcal{N}_n(1-A, 1) <$

$$2(\log n)^{1/8}[\mathcal{N}_n(0, 1/2.n) + \mathcal{N}_n(1-1/2.n, 1)] + 2(\log n)^{1/8}. \tag{25}$$

By what has been said in the introduction all the integers m, for which $\frac{1}{\{m\alpha\}} \geqslant 2n$ satisfy $m \equiv 0 (\mathrm{mod}\ q_r)$, where $q_r \leqslant n < q_{r+1}$. We distinguish two cases.

Case I. $q_r \geqslant n/(\log n)^{1/2}$.

Then clearly

$$\mathcal{N}_n(0, 1/2.n) + \mathcal{N}_n(1-1/2.n, 1) < (\log n)^{1/2}. \tag{26}$$

Case II. $q_r < n/(\log n)^{1/2}$.

But then by (3)

$$\frac{1}{\{q_r\alpha\}} < q_r(\log q_r)^{1+\varepsilon} < n(\log n)^{1/2+\varepsilon}$$

Thus if $k.q_r.\alpha - [k.q_r.\alpha]$ is in $(0, 1/2.n)$ or in $(1-1/2.n, 1)$ we have $k < (\log n)^{1/2+\varepsilon}$ Thus in case II

$$\mathcal{N}_n(0,\ 1/2.n)+\mathcal{N}_n(1-1/2.n,\ 1) < (\log n)^{1/2+\varepsilon} \qquad (27)$$

Hence from (26), (27) and (24) we obtain

$$\Sigma_2 = o(n \log n). \qquad (28)$$

The lemma now follows from (23) and (28).

Now we prove (20). We have

$$\sum_{m=1}^{n} \frac{1}{m\{m\alpha\}} = \sum_3 + \sum_4, \qquad (29)$$

where in $\sum_3$, $\qquad \frac{1}{(m\alpha)} \leqslant 2.n$

and in $\sum_4$, $\qquad \frac{1}{(m\alpha)} > 2.n.$

We obtain from (21) by partial summation that

$$\sum_3 = (1+o(1)) \sum_{m \leqslant n} \frac{2 \log m}{m} = (1+o(1))\,(\log n)^2. \qquad (30)$$

For the m in Σ_4 we have as before that $m \equiv 0 (\mathrm{mod}\ q_r)$, hence from $q_r > n/(\log n)^{1+\varepsilon}$ we have

$$\sum_4 \leqslant \sum_{k \leqslant n/q_r} \frac{1}{kq_r\{kq_r\alpha\}} \leqslant \sum_{k < (\log n)^2} \frac{1}{k^2 q_r \{q_r\alpha\}} <$$

$$(\log n)^{1+\varepsilon} \sum_{k=1}^{\infty} \frac{1}{k^2}\, o(\log n)^2. \qquad (31)$$

(20) follows from (30) and (31).

Similarly we can prove that for almost all α and $0 < a < 1$

$$\sum_{m=1}^{n} \frac{1}{m^a\{m\alpha\}} = (1+o(1))\,\frac{2n^{1-a}\log n}{a}.$$

Before concluding the paper we state a few results without proof:

I. For almost all α

$$\sum_{n=1}^{x} \frac{1}{\sum_{m=1}^{n} \{m\alpha\}^{-1}} = (1+o(1))\,\frac{\log \log x}{2}. \qquad (32)$$

Thus in particular for almost all α,

$$\sum_{n=1}^{\infty} \frac{1}{\sum_{n=1}^{n} \{m\alpha\}^{-1}}$$

diverges.

The proof of (32) is not difficult, it follows from (21) without much difficulty.

II. Let $f(n)$ be an increasing function of n for which $f(n) > (2+c).n.\log n$ and $\sum_{n=1}^{\infty} \frac{1}{f(n)}$ converges. Then for almost all α and $n > n_0(\alpha)$

$$\sum_{m=1}^{\infty} \frac{1}{\{m\alpha\}} < f(n).$$

The proof of (II) is not quite simple and is not given here. (I) and (II) were suggested to me by the beautiful work of Khintchine[1] and Paul Levy[2] on continued fractions.

1. *Compositio Math.*, 1 (1935), p. 381.
2. *Ibid.*, 3 (1936), p. 302.

SOME ASYMPTOTIC FORMULAS IN NUMBER THEORY

BY

P. ERDÖS, *University of Syracuse.*

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Szele[1] recently proved that the necessary and sufficient condition that there should be only one abstract group of order m is that $(m, \phi(m)) = 1$. In the present note we are going to investigate how many such integers there are up to n. In fact we prove the following

THEOREM. *Denote by $A(n)$ the number of integers $m < n$ for which $(m, \phi(m)) = 1$. Then*

$$A(n) = (1+o(1))\frac{ne^{-\gamma}}{\log\log\log n},$$

where γ is Euler's constant.

Throughout this paper p, q, r and s will denote primes, the c's denote absolute constants, $\epsilon > 0$ is a number which can be chosen arbitrarily small.

Clearly $(m, \phi(m)) = 1$ if and only if m is squarefree and m is not divisible by $p.\,q$, where $q \equiv 1 \pmod{p}$.

Denote by $A_p(n)$ the number of integers $m \leqslant n$ for which $(m, \phi(m)) = 1$ and the smallest prime factor of m is p. Clearly

$$A(n) = \sum_{p \leqslant n} A_p(n) = \Sigma_1 + \Sigma_2 + \Sigma_3, \tag{1}$$

where in Σ_1, $\quad p < (\log\log n)^{1-\epsilon}$,

in Σ_2, $\quad (\log\log n)^{1-} \leqslant p \leqslant (\log\log n)^{1+\epsilon}$

and in Σ_3, $\quad (\log\log n)^{1+\epsilon} < p$.

First we prove three lemmas.

LEMMA I. *Let $p < (\log\log n)^{1-\epsilon}$. Then*

$$\sum{}' \frac{1}{q} > c_1 \frac{\log\log n}{p} > (\log\log n)^{\epsilon/2},$$

where the dash indicates that the summation is extended over the $q \equiv 1 \pmod{p}$ which satisfy $q < n^{1/(\log\log n)^2}$.

1. *Comment. Math. Helv.*, 20 (1947), p. 265-7.

A result of Page[1] states that if $\pi(x, 1, k)$ denotes the number of primes $q \equiv 1 \pmod{k}$, then

$$\pi(x, 1, k) = (1+o(1)) \frac{x}{\phi(k) \log x}$$

uniformly for $k < \log x$. Thus if $x > \log n > e^p$, we have

$$\pi(x, 1, p) > \tfrac{1}{2} \frac{x}{p \log x}. \tag{2}$$

From (2) we obtain

$$\sum{}' \frac{1}{q} > \sum \frac{1}{4\, pl \log l} > c_1 \frac{\log \log n}{p},$$

where $\log n < l < n^{1/(\log \log n)^3}$ which proves the lemma.

LEMMA II. *Let p be any prime. Then*

$$\sum{}' \frac{1}{q} < c_2 \left(\frac{\log p + \log \log n}{p} \right),$$

where the dash indicates that $q \equiv 1 \pmod{p}$, $q \leqslant n$.

We have

$$\sum{}' \frac{1}{q} < \sum_{a=1}^{p} \frac{1}{1+ap} + \sum{}'' \frac{1}{q} < c_2 \frac{\log p}{p} + \sum{}'' \frac{1}{q}, \tag{3}$$

where in Σ'', $q \equiv 1 \pmod{p}$, $p^2 < q \leqslant n$. By a result of Titchmarsh[2] the number of primes $q \equiv l \pmod{p}$, $q \leqslant x$ is for $x > p^2$ less than

$$\frac{c_3\, x}{p \log x}.$$

Thus a simple argument shows that

$$\sum{}'' \frac{1}{q} < \frac{c_3}{p} \sum \frac{1}{x \log x} < \frac{c_2}{p} \log \log n. \tag{4}$$

Lemma II follows from (3) and (4).

LEMMA III. *Let* $x \leqslant (\log \log n)^{1+\varepsilon}$ $(x \to \infty)$. *Denote by* $B_x(n)$ *the number of integers* $m \leqslant n$ *not divisible by any prime* $p \leqslant x$. *Then uniformly in* x

1. *Proc. London Math. Soc.*, (2) (39) (1935), p. 136 equation (36).

2. *Rend. di Palermo*, 57 (1933), p. 478-9.

$$B_x(n) = (1+o(1))\, e^{-\gamma} \frac{n}{\log\log x}.$$

By the sieve of Eratosthenes we have

$$B_x(n) = n - \sum_{p \leqslant x} \left[\frac{n}{p}\right] + \sum \left[\frac{n}{p_1 p_2}\right] - \cdots$$

$$= \prod_{p \leqslant x} \left(1 - \frac{1}{p}\right) + O(2^x) = (1+o(1)) \frac{n e^{\gamma}}{\log\log x}.$$

From Lemma III we immediately obtain the following

COR. *Let $p \leqslant (\log\log n)^{1+\varepsilon}$. Denote by $C_p(n)$ the number of integers $m \leqslant n$ for which the least prime factor of m is p. Then*

$$C_p(n) = B_p\left(\frac{n}{p}\right) < c_3 \frac{n e^{-\gamma}}{p \log\log p}.$$

Now we can prove our theorem. First we estimate Σ_1. Let $p < (\log\log n)^{1-\varepsilon}$. $A_p(n)$ is clearly greater than the number of integers $m \leqslant n$ not divisible by any $q \equiv 1 \pmod p$ satisfying $q < n^{1/(\log\log n)^2}$. By Brun's method[1] we thus obtain from Lemma I that

$$A_p(n) < c_4 n \prod{}' \left(1 - \frac{1}{q}\right) < c_5 n e^{-(\log\log)^{\varepsilon/2}} = o\left(\frac{n}{(\log\log n)^2}\right),$$

where the dash indicates $q \equiv 1 \pmod p$, $q < n^{1/(\log\log n)^2}$. Thus

$$\sum\nolimits_1 < \log\log n \max_{p \leqslant (\log\log n)^{1-\varepsilon}} A_p(n) = o\left(\frac{n}{\log\log n}\right) \tag{5}$$

Now we estimate Σ_2. We have by the corollary to Lemma III that

$$\sum\nolimits_2 < \sum{}' c_p(n) < c_6 \frac{n e^{-\gamma}}{\log\log\log n} \sum{}' \frac{1}{p} < c_7 \frac{\varepsilon n}{\log\log\log n}, \tag{6}$$

where the dash indicates that

$$(\log\log n)^{1-\varepsilon} \leqslant p \leqslant (\log\log n)^{1+\varepsilon}.$$

1. P. Erdös, *Proc. Cambridge Phil. Soc.*, 33 (1937), p. 8 Lemma 2. In this case one does not need the full strength of the method and the simpler arguments in Landau, *Zahlentheorie*, Vol. 1, will suffice.

Finally we estimate Σ_3. Put $x = (\log\log n)^{1+\varepsilon}$. Clearly by our remark at the beginning of the proof, i.e. $(m, \phi(m)) = 1$ if and only if m is squarefree, and is not divisible by any $p.q$ with $q \equiv 1 \pmod p$ we have

$$B_x(n) > \Sigma_3 > B_x(n) - \sum_{r > x} \frac{n}{r^2} - \sum{}' \frac{n}{s_1 s_2},$$

where the dash indicates that $s_1 > x$ and $s_2 \equiv 1 \pmod{s_1}$. By Lemmas II and III

$$(1+o(1)) \frac{e^{-\gamma} n}{(1+\varepsilon)\log\log\log n} >$$

$$\Sigma_3 > (1+o(1)) \frac{e^{-\gamma} n}{(1+\varepsilon)\log\log\log n)} - \frac{n}{x} - \sum_{s > x} \frac{\log s + \log\log n}{s^2}$$

$$> (1+o(1)) \frac{e^{-\gamma} n}{(1+\varepsilon)\log\log\log n} - \frac{n}{x} - c_8 \frac{\log x}{x} - \frac{\log\log n}{x}$$

$$= (1+o(1)) \frac{e^{-\gamma} n}{(1+\varepsilon)\log\log\log n}. \quad (7)$$

Since ϵ can be chosen arbitrarily small, we obtain the theorem from (5), (6) and (7).

By more complicated methods we can prove the following result: Denote by $v(x)$ the number of prime factors of x. Then the number of integers $m \leqslant n$ for which $v\{m, \phi(m)\}$ does not satisfy

$$(1-\varepsilon)\log\log\log\log m < v\{(m, \phi(m))\} < (1+\varepsilon)\log\log\log\log m \text{ is } o(n).$$

An analogous but much harder problem was raised by Pillai[1]: Find an asymptotic formula for the number of integers $m \leqslant n$ which have no factor of the form $p(a.p+1)$. I can prove by much more complicated methods that the asymptotic formula for the number of these integers is

$$\frac{e^{-\gamma}}{\log 2} \frac{n}{\log\log n}.$$

I hope to return to this at another occasion.

1. *The Journal of Indian Math. Soc.*, 18 (1929-1930), p. 51-9.

ON THE EQUIVALENCE OF CERTAIN INFINITE SERIES AND THE CORRESPONDING INTEGRALS

BY

K. S. KRISHNAN, *National Physical Laboratory of India, Delhi.*

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1. In calculating the intensity of light scattered in a homogeneous medium, one comes across the infinite series.

$$\alpha \sum_{n=-\infty}^{+\infty} \frac{\sin^2 (n\alpha+\theta)}{(n\alpha+\theta)^2},$$

where θ is a constant, and n an integer. α is a positive number which under the conditions under which light-scattering is generally studied, can be made arbitrarily small, and hence the sum is usually replaced by the corresponding integral*

$$\alpha \sum_{n=-\infty}^{+\infty} \frac{\sin^2 (n\alpha+\theta)}{(n\alpha+\theta)^2} = \int_{-\infty}^{\infty} \frac{\sin^2 x}{x^2}\, dx = \pi. \tag{1}$$

It is easy to show†, however, that (1) holds not only in the limit when α tends to zero, but for any value of α in the range $0 < \alpha \leqslant \pi$. The proof is as follows.

Consider the series‡

$$\sum_{n=-\infty}^{\infty} \frac{\sin (n+\beta) z}{n+\beta} = \pi, \tag{2}$$

where β is a constant, n an integer, and $0 < z < 2\pi$. The series can be integrated term by term with respect to z in any closed interval (γ, δ), where $0 < \gamma < \delta < 2\pi$, since it is uniformly convergent in this interval. We thus obtain

* Einstein. *Ann. der Physik*, 33 (1910), 1294.

† Bhatia and Krishnan, *Proc. Royal Soc.* A., 192 (1947), 184.

‡ Bromwich, *An introduction to the theory of infinite series*, (1931), 371, Ex. 5.

$$\sum_{n=-\infty}^{\infty} \frac{\cos(n+\beta)\gamma}{(n+\beta)^2} - \sum_{n=-\infty}^{\infty} \frac{\cos(n+\beta)\delta}{(n+\beta)^2} = \pi(\delta-\gamma). \qquad (3)$$

Keeping δ constant and making $\gamma \to 0$, (3) reduces to

$$\sum_{n=-\infty}^{\infty} \frac{1-\cos(n+\beta)\delta}{(n+\beta)^2} = \pi\delta,$$

since the first series on the left side of (3) is uniformly convergent and therefore represents a continuous function of γ. Putting now $\delta = 2\alpha$, $\alpha\beta = \theta$, and dividing both sides by 2α ($\neq 0$), we obtain for $0 < \alpha < \pi$,

$$\alpha \sum_{n=-\infty}^{\infty} \frac{\sin^2(n\alpha+\theta)}{(n\alpha+\theta)^2} = \pi. \qquad (4)$$

This can be seen to be true for $\alpha = \pi$ also, and hence (4) holds over the interval $0 < \alpha \leqslant \pi$.

The above result implies that the area subtended between the curve $y = \sin^2 x/x^2$ and the x-axis may be obtained just as well by adding up the ordinates at equal intervals α, and multiplying by α (i.e. by simple rectangulation), as by integration, provided $0 < \alpha \leqslant \pi$. It may be noted here that the sum in (4) is independent of θ, which shows that we may start the division of the x-axis into equal steps α from any value of x. In particular, $x = 0$ need not be one of the points of division.

2. As we shall show presently, the same property, viz.

$$\alpha \sum_{n=-\infty}^{\infty} f(n\alpha+\theta) = \alpha \sum_{n=-\infty}^{\infty} f(n\alpha) = \int_{-\infty}^{\infty} f(x)dx, \qquad (5)$$

for a suitable range of values of α, $0 < \alpha \leqslant l$, say, holds for several other functions too. Before considering such functions, we shall refer here to an alternative proof of (4), given by Prof. Norbert Wiener.* The proof is very

* See Bhatia and Krishnan, *loc. cit.*, foot-notes on pp. 184 and 185.

suggestive, and enables us to determine the conditions for the validity of the equations (5).*

Consider an even function $f(x)=f(-x)$, and its Fourier transform defined by

$$g(v)=\frac{1}{(2\pi)^{\frac{1}{2}}}\int_{-\infty}^{\infty}f(x)e^{ivx}\,dx. \tag{6}$$

Let $f(x)$ be such that $g(v)$ has non-zero values in the range $-a<v<a$, where a is a positive number. Obviously $f(x)=\sin^2 x/x^2$ satisfies this condition, since $g(v)=(\pi/2)^{\frac{1}{2}}(2-|v|)$, if $0\leqslant|v|\leqslant 2$, and $=0$ if $|v|\geqslant 2$.

According to Poisson's summation formula,†

$$\sum_{n=-\infty}^{\infty}f(n\alpha)=\frac{(2\pi)^{\frac{1}{2}}}{\alpha}\sum_{N=-\infty}^{\infty}g\left(\frac{2\pi N}{\alpha}\right), \tag{7}$$

where N is an integer. If now $0<\alpha\leqslant 2\pi/a$, there is only one value of N, viz. $N=0$, for which $g(2\pi N/\alpha)$ differs from 0.

Hence

$$\sum_{n=-\infty}^{\infty}f(n\alpha)=\frac{(2\pi)^{\frac{1}{2}}}{\alpha}g(0), \tag{8}$$

whence substituting for $g(0)$ from (6) we obtain

$$\alpha\sum_{n=-\infty}^{\infty}f(n\alpha)=\int_{-\infty}^{\infty}f(x)\,dx. \tag{9}$$

Further, for a fixed θ, it can be seen that

$$\int_{-\infty}^{\infty}f(x+\theta)e^{ivx}\,dx=e^{-iv\theta}\int_{-\infty}^{\infty}f(x)e^{ivx}\,dx, \tag{10}$$

which shows that for $v=0$, the Fourier transform of $f(x+\theta)$ is the same as that of $f(x)$. Hence it follows that

* $f(x)=$ constant is a trivial example that satisfies (5).

† See Titchmarsh, *Introduction to the Theory of Fourier Integrals*, p. 60, where Poisson's summation formula is proved with reference to the Fourier cosine transform,

$$g_c(v)=\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\int_0^{\infty}f(x)\cos vx\,dx;$$

but, for the *even* functions that we are considering, the exponential and the cosine transforms become identical.

if $f(x)$ be such that its Fourier transform $g(v)$ has non zero values when $|v| < a$, and zero value otherwise, then

$$\alpha \sum_{n=-\infty}^{\infty} f(n\alpha+\theta) = \alpha \sum_{n=-\infty}^{\infty} f(n\alpha) = \int_{-\infty}^{\infty} f(x)dx, \qquad (11)$$

provided that $0 < \alpha \leqslant 2\pi/a$.

We have seen that $\sin^2 x/x^2$ is such a function, the range of α over which (11) holds being $0 < \alpha \leqslant \pi$. *More generally*, the functions $f_{m,n}(x) = \sin^m x/x^n$, where m and n are positive integers, both odd or both even, and $n \leqslant m$, are examples of such functions; their Fourier transforms $g_{m,n}(v)$ can be seen to have zero value if $|v| \geqslant n$ ($|v| > n$ when $n = 1$), and non-zero values otherwise, and hence for these functions, relations (11) will be valid if

$$0 < \alpha \leqslant 2\pi/n \quad (0 < \alpha < 2\pi/n \text{ when } n = 1).$$

The Fourier transforms of $(\sin x/x)^m$, $m = 1, 2, 3$, are plotted in Fig. 1, *a*, *b*, *c*.

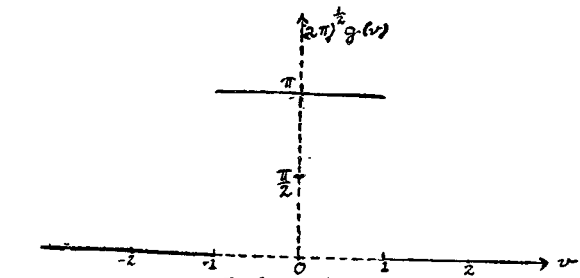


Fig 1 a; m = 1

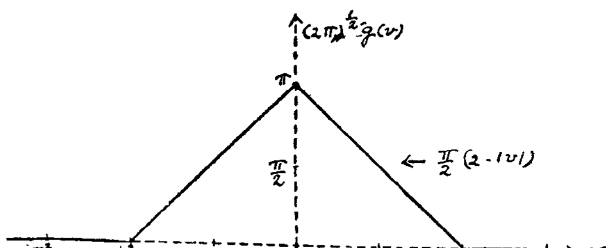
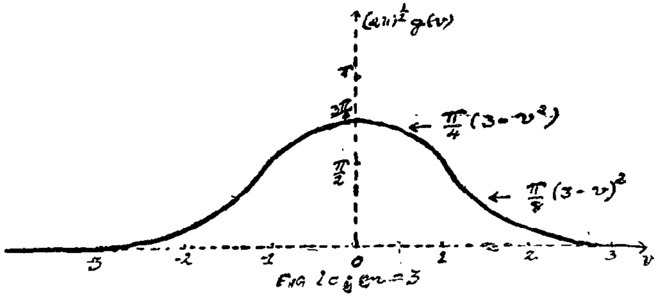


Fig 1 b; m = 2



3. Since the Fourier transforms are known for a large number of functions, and many of them have been conveniently tabulated, it is easy to select other examples of functions that satisfy the criterion stated above, and for which therefore relations (10) are valid. We give in the following tables a few. Those given in Table I are taken from a paper by Ramanujan,* entitled *A class of definite integrals*, in which among others are given the values of several Fourier integrals, some of which are found to satisfy the above criterion. Those given in Table II are taken from a monograph† by Campbell and Foster.

TABLE I.

$f(x)$	$2\pi)^{\frac{1}{2}} g(v)$, when $\lvert v\rvert < \pi$ [$g(v)=0$ otherwise.]
$\dfrac{1}{\Gamma(\alpha+x)\,\Gamma(\beta-x)}$	$\dfrac{(2\cos v/2)^{\alpha+\beta-2}}{\Gamma(\alpha+\beta-1)} e^{\frac{1}{2}iv(\beta-\alpha)}$, convergence condition $R(\alpha+\beta)>1$
$\dfrac{J_{\alpha+x}(\lambda)\,J_{\beta-x}(\mu)}{\lambda^{\alpha+x}\,\mu^{\beta-x}}$	$\left(\dfrac{2\cos v/2}{\Omega}\right)^{\frac{1}{2}(\alpha+\beta)} e^{\frac{1}{2}v(\beta-\alpha)i}$ $+ J_{\alpha+\beta}\,[\sqrt{(2\,\Omega\cos v/2)}]$, where $\Omega=\lambda^2\,e^{-\frac{1}{2}vi}+\mu^2\,e^{\frac{1}{2}vi}$, converge condition $R(\alpha+\beta)>-1$.

* *Quarterly Jour. Math.* 48 (1920), 294; *Collected Papers*, Cambridge (1927), 216.

† *Fourier Integrals for Practical Applications*, Bell Telephone Publications, Monograph B—584 (1931).

Table II.

$f(x)$	$(2\pi)^{\frac{1}{2}}\, g(v)$, when $\lvert v\rvert \leqslant a$ [$g(v)=0$ otherwise.]
$\dfrac{\sin a(x^2+\lambda^2)^{\frac{1}{2}}}{(x^2+\lambda^2)^{\frac{1}{2}}}$	$\pi J_0[\lambda(a^2-v^2)^{\frac{1}{2}}]$
$\dfrac{\sin a(x^2-\lambda^2)^{\frac{1}{2}}}{(x^2-\lambda^2)^{\frac{1}{2}}}$	$\pi I_0[\lambda(a^2-v^2)^{\frac{1}{2}}]$
$\cos a(x^2+\lambda^2)^{\frac{1}{2}} - \cos ax$	$-\dfrac{\pi a\lambda J_1[\lambda(a^2-v^2)^{\frac{1}{2}}]}{(a^2-v^2)^{\frac{1}{2}}}$
$\dfrac{\sin a(1-x)}{1-x}+\dfrac{\sin a(1+x)}{1+x}$	$2\pi \cos v$
$\dfrac{\sin^2 a(1-x)/2}{1-x}+\dfrac{\sin^2 a(1+x)/2}{1+x}$	$\pi \sin v$
$\dfrac{\sin ax}{x^2}-\dfrac{a\cos ax}{x}$	$i\pi v.$

In the above, a is a positive real finite number, and λ is a complex number, not infinite.

Relations (11) will be valid for all the functions in Tables I and II. Taking for example the first function entered in Table II, (11) will read as follows:

If $0<\alpha\leqslant 2\pi/a$,

$$\alpha \sum_{n=-\infty}^{\infty} \frac{\sin a\{(n\alpha+\theta)^2+\lambda^2\}^{\frac{1}{2}}}{\{(n\alpha+\theta)^2+\lambda^2\}^{\frac{1}{2}}} = \alpha \sum_{n=-\infty}^{\infty} \frac{\sin a(n^2\alpha^2+\lambda^2)^{\frac{1}{2}}}{(n^2\alpha^2+\lambda^2)^{\frac{1}{2}}}$$

$$= \int_{-\infty}^{\infty} \frac{\sin a(x^2+\lambda^2)^{\frac{1}{2}}}{(x^2+\lambda^2)^{\frac{1}{2}}}\, dx = \pi J_0(\lambda a). \quad (12)$$

When $\lambda=0$ and $a=1$, this reduces to the case $f(x)=\sin x/x$.

The last two functions given in Table II however differ from the rest in that $g(v)$, besides being zero when $\lvert v\rvert > a$, is zero at $v=0$ also. Considering the last function, we obtain, if $0<\alpha<\pi$,

$$\alpha \sum_{n=-\infty}^{\infty} \left[\frac{\sin(n\alpha+\theta)}{(n\alpha+\theta)^2} - \frac{\cos(n\alpha+\theta)}{n\alpha+\theta}\right]$$

$$= \alpha \sum_{n=-\infty}^{\infty} \left[\frac{\sin(n\alpha)}{n^2\alpha^2} - \frac{\cos(n\alpha)}{n\alpha}\right]$$

$$= \int_{-\infty}^{\infty} \left(\frac{\sin x}{x^2} - \frac{\cos x}{x}\right) dx \tag{13}$$

$$= g(0) = 0.$$

That the value of the integral in (13) is zero is otherwise obvious. We thus obtain when $0 < \alpha < \pi$ and $n\alpha+\theta$ is not equal to θ for any value of n,

$$\sum_{n=-\infty}^{\infty} \frac{\sin(n\alpha+\theta)}{(n\alpha+\theta)^2} = \sum_{n=-\infty}^{\infty} \frac{\cos(n\alpha+\theta)}{n\alpha+\theta} \tag{14}$$

The corresponding integrals $\int_{-\infty}^{\infty} \frac{\sin x}{x^2}\, dx$ and $\int_{-\infty}^{\infty} \frac{\cos x}{x}\, dx$ are however infinite.

The last but one function entered in Table II, for which too $g(a) = 0$ at $v = 0$, as also when $|v| > a$, yields similarly, if $0 < \alpha < \pi/a$,

$$\sum_{n=-\infty}^{\infty} \frac{\sin^2 a(n\alpha+\theta)}{n\alpha+\theta} = \sum_{n=-\infty}^{\infty} \frac{\sin^2 a(n\alpha+\theta+2)}{n\alpha+\theta+2}, \tag{15}$$

though

$$\int_{-\infty}^{\infty} \frac{\sin^2 ax}{x}\, dx = 0.$$

4. The class of functions that we have been considering here, which is characterized by the Fourier transforms being zero when $|v|$ is greater then a certain positive number a, has other interesting properties. Analogous to Poisson's summation formula, which we have used, and which we may write in the more familiar form

$$\sqrt{\alpha}\left\{\tfrac{1}{2}f(0)+f(\alpha)+f(2\alpha)+\dots\right\}$$
$$= \sqrt{\beta}\left\{\tfrac{1}{2}g_c(0)+g_c(\beta)+g_c(2\beta)+\dots\right\}, \tag{16}$$

where $\alpha\beta = 2\pi$, there are others of the same type due to Ramanujan*. Two of the typical formulae are given below.

$$\sqrt{\alpha}\{f(\alpha)-f(3\alpha)-f(5\alpha)+f(7\alpha)+f(9\alpha)-\ldots\}$$
$$=\sqrt{\beta}\{g_c(\beta)-g_c(3\beta)-g_c(5\beta)+g_c7(\beta)+g_c(9\beta)-\ldots\}, \qquad (17)$$

where $\alpha\beta = \pi/4$;

$$\sqrt{\alpha}\{f(\alpha)-f(5\alpha)-f(7\alpha)+f(11\alpha)+f(13\alpha)-\ldots\}$$
$$=\sqrt{\beta}\{g_c(\beta)-g_c(5\beta)-g_c(7\beta)+g_c(11\beta)+g_c(13\beta)-\ldots\}, \qquad (18)$$

where $\alpha\beta = \pi/6$, and 1, 5, 7, 11, 13, ..., are the numbers prime to 6.

Unlike in Poisson's formula (16), in which the first term on the right side is $\frac{1}{2}g_c(0)$, the first term in (17), (18) and similar formulae, is $g_c(\beta)$. If now $\beta > a$, i.e. if α is chosen small enough to make $\beta > a$, then all the terms on the right side of (17) and (18) vanish, and we get the following interesting results. From (17), for example, we obtain, if $0 < \alpha < \pi/4a$,

$$f(\alpha)+f(7\alpha)+f(9\alpha)+f(15\alpha)+f(17\alpha)+\ldots$$
$$=f(3\alpha)+f(5\alpha)+f(11\alpha)+f(13\alpha)+\ldots \qquad (19)$$

Similarly, from (18), if $0 < \alpha < \pi/6a$,

$$f(\alpha)+f(11\alpha)+f(13\alpha)+f(23\alpha)+f(25\alpha)+\ldots$$
$$=f(5\alpha)+f(7\alpha)+f(17\alpha)+f(19\alpha)+\ldots \qquad (20)$$

Taking $\sin x/x$ as an example of such a function, we obtain from (19), if $|\alpha| < \pi/4$,

$$\frac{\sin\alpha}{\alpha}+\frac{\sin 7\alpha}{7\alpha}+\frac{\sin 9\alpha}{9\alpha}+\ldots$$
$$=\frac{\sin 3\alpha}{3\alpha}+\frac{\sin 5\alpha}{5\alpha}+\frac{\sin 11\alpha}{11\alpha}+\ldots$$
$$=\frac{1}{8\alpha}\int_{-\infty}^{\infty}\frac{\sin x}{x}\,dx=\frac{\pi}{8\alpha}. \qquad (21$$

More generally,

Collected papers, p. 63; *Messenger of Maths.*, 44 (1915), 75.

$$\frac{\sin^m \alpha}{\alpha^n} + \frac{\sin^m 7\alpha}{(4\alpha)^n} + \frac{\sin^m 9\alpha}{(9\alpha)^n} + \dots$$
$$= \frac{\sin^m 3\alpha}{(3\alpha)^n} + \frac{\sin^m 5\alpha}{(5\alpha)^n} + \frac{\sin^m 11\alpha}{(11\alpha)^n} + \dots$$
$$= \frac{1}{8\alpha}\int_{-\infty}^{\infty} \frac{\sin^m x}{x^n}\, dx, \qquad (22)$$

where $|\alpha| \leqslant \frac{\pi}{4n}$, m and n are positive integers both odd or both even, and $0 < n \leqslant m$.

Now the Fourier transform of $f(x+\theta)$ differs from that of $f(x)$ by a multiplying factor $e^{-iv\theta}$, or $\cos v\theta$ in the case of cosine transforms, and hence the g's on the right side of (17) and (18) will remain zero even when $f(x)$ is changed to $f(x+\theta)$. Hence, we obtain from (22), even more generally,

$$\frac{\sin^m(\alpha+\theta)}{(\alpha+\theta)^n} + \frac{\sin^m(\alpha-\theta)}{(\alpha-\theta)^n} + \frac{\sin^m(7\alpha+\theta)}{(7\alpha+\theta)^n} + \frac{\sin^m(7\alpha-\theta)}{(7\alpha-\theta)^n}$$
$$+ \frac{\sin^m(9\alpha+\theta)}{(9\alpha+\theta)^n} + \frac{\sin^m(9\alpha-\theta)}{(9\alpha-\theta)^n} + \dots$$
$$= \frac{\sin^m(3\alpha+\theta)}{(3\alpha+\theta)^n} + \frac{\sin^m(3\alpha-\theta)}{(3\alpha-\theta)^n} + \frac{\sin^m(5\alpha+\theta)}{(5\alpha+\theta)^n} + \frac{\sin^m(5\alpha-\theta)}{(5\alpha-\theta)^n} + \dots \qquad (23)$$

under the same conditions as before.

Similar series can be constructed from (20) and the other formulae analogous to Poisson's, and for all the functions in Tables I and II.

Indeed, equations (19), (21), (22) and (23) can be seen to be special cases of

$$\sum_{n=-\infty}^{\infty} f(nA+\Omega) = \frac{1}{A}\int_{-\infty}^{\infty} f(x)\, dx, \qquad (24)$$

and therefore independent of Ω, when $0 < A < 2\pi/a$, and $g(v) = 0$ when $|v| \geqslant a$. By putting $A = 8\alpha$, it can be seen that the left sides of (19), (21) and (22) correspond to $\Omega = \alpha$, and the right sides to $\Omega = 3\alpha$; the left side of (23) corresponds to $\Omega=\alpha\pm\theta$ while the right side corresponds to $\Omega=3\alpha\pm\theta$, if we remember that f is an even function, so that $f(-nA+\Omega)=f(nA-\Omega)$.

Equation (24) moreover enables us to evaluate the series in all these equations.

Similarly, (20) corresponds to $A=12\alpha$ and $\Omega=\alpha$ and 5α respectively.

5. Till now, we have confined ourselves to the Fourier cosine or exponential transforms. There are formulae analogous to Poisson's, applicable to Fourier sine transforms, again due to Ramanujan*, of which we shall quote here just one.

If $g_s(v)=\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\int_0^\infty f(x)\sin vx\,dx$, then

$$\alpha\{f(\alpha)-f(3\alpha)+f(5\alpha)-\ldots\}=g_s(\beta)-g_s(3\beta)+g_s(5\beta)-\ldots$$

where $\alpha\beta=\pi/2$. (25)

If $f(x)$ be such that its Fourier sine transform $g_s(v)$ is zero for $|v|>a$, and if $\beta>a$, i.e. if $0<\alpha<\pi/2a$, all the terms on the right side of (25) vanish, and we obtain

$$\sum_{n=0}^{\infty} f\Big[(4n+1)\alpha\Big]=\sum_{n=0}^{\infty} f\Big[(4n+3)\alpha\Big]. \qquad (26)$$

As examples of such functions, we may mention†

$$(1)\begin{cases} f(x)=2^{\nu-\frac{3}{2}}\,\Gamma(\nu-\tfrac{1}{2})\;x^{1-\nu}\,J_\nu(x)\\ g_s(v)=v(1-v^2)^{\nu-\frac{3}{2}} \text{ if } 0<v<1\\ \quad =0 \qquad\qquad \text{ if } v>1\;; \end{cases} \qquad (27)$$

$$(2)\begin{cases} f(x)=2^{\nu-\frac{1}{2}}\,\Gamma(\nu+\tfrac{1}{2})\;x^{-\nu}\,H_\nu(x)\\ g_s(v)=(1-v^2)^{\nu-\frac{1}{2}} \text{ if } 0<v<1\\ \quad =0 \qquad\qquad \text{ if } v>1\;, \end{cases} \qquad (28)$$

where $H_\nu(x)$ is Struve's function of order ν.

* *Collected Papers*, p. 64.

† Titchmarsh : *loc. cit.* pp. 179 and 180.

EXTENSIONS OF PARTIALLY ORDERED SETS-II: CONSTRUCTIONS

BY

V. S. KRISHNAN, *Presidency College, Madras.*

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After the discussions in Part I*, on the nature and characterestic properties of canonical extensions, we now proceed to consider the question of the existence of such extensions for specific cases. Given an ordered set (K) of a particular type, and a set (θ) of order operations, is there a canonical extension of K to the θ, or can the ordered set fail to have any such canonical extension? This question is answered definitively here for the gen ral, arbitrary, ordered sets, and the ordered sets with units, when (θ) is *any* selection of operations from the following twelve:†

> adjunction of either unit, O I;
> four types of addition, $+$, Σ, $\overline{+}$, $\bar{\Sigma}$;
> four types of multiplication, $\times$, π $\overline{\times}$, $\bar{\pi}$;
> sum$-$, or product$-$, complementation, S, P.

REMARKS.

$O(A)$ is defined for any subset of an ordered set K as an element $<$ every element of K; this *operation* O is to be distinguished from the *element*, o, of an ordered set K; similarly for I and 1; the operations $+$, $\overline{+}$, $\times$, $\overline{\times}$ are defined only for non-null finite subsets of an ordered set; while Σ, $\bar{\Sigma}$, π, $\bar{\pi}$ are defined for all non-null subsets; the bars on the top for $\overline{+}$, $\bar{\Sigma}$, $\overline{\times}$, $\bar{\pi}$ signify that the sum or product defined should be distributive, (according to the general definition given by MacNeille); finally P, S are defined only for one element sets; $P(a) = a'$ if a' is the maximal element y such that $x < a$, $x < y$ imply $x <$ every element of K.

* For Part I of this paper see Vol. XI, (1947) of this Journal.

† In the sequel whenever 'operations' are mentioned only those from this list are meant.

Only three canonical extensions are constructed fully; two of these are due to MacNeille, and the third is my own. From these constructed extensions, others are obtained by using results from Part I, like the modified combination theorem, resolution theorem and the duality theorem. Then for all other cases examples of ordered sets are given to show that no canonical extension need exist; in fact, in all such cases even more is asserted; namely that either the chosen ordered set has no extension at all to the set of operations in question, or has no minimal extension at all.

A corresponding complete analysis for all the possible selections of operations is not yet ready for the cases of special classes of ordered sets, like multiplicative systems, modular or distributive lattices, etc. I expect to deal with these in subsequent papers.

The possible selections of operations are covered up in the following sections, according to the following scheme, under which it may be observed that all subsets of the set of twelve operations are dealt with:

§5. Subsets of $(O, I, +, \Sigma, \times, \pi)$;

§6. Subsets of $(O, I, \pi, \times, P)$ containing (P); and their duals:

§7. Sets containing one operation from each of the sets: $(+, \Sigma)$; $(\times, \pi)$; and $(P, S, \mp; \bar{\Sigma}, \overline{\times}, \bar{\pi})$;

§8. Sets not containing either of $(+, \Sigma)$; and their duals.

5. *Extensions to subsets of* $(O, I, +, \Sigma, \times, \pi)$

5.1 THEOREM. *For any ordered set* (K) *there exist strong canonical extensions* $K(O)$, $K(I)$ *respectively to* O *and to* I; *these extensions are further supersystems of the images of* K *in them for all* operations.*

PROOF. We take $K(O)$ to be K itself if K has a o, ($<$ every element of K). Otherwise $K(O)$ contains all elements of K together with an extra element o; the

* The 'all' here includes only the twelve operations cited earlier.

relation of order between elements of K remain unaffected, while o is defined to be $<$ every other element in $K(O)$. It is easily verified that $K(O)$ is an extension of K to (O) and a supersystem of K [which is itself the image of K in $K(O)$] to all operations. The relations between the elements in $K(O)$ are strongly required, since if K' is the image of K in any superset L of K to (O), in L also $O(K')$ $<$ every element in K'. Finally, given two distinct elements a, b of $K(O)$ at least one of them is in K, and so is an element of K separating a, b. This completes the proof that $K(O)$ is a strong canonical extension of K to O.

The extension $K(I)$ is similarly constructed, and can be shown to be a strong canonical extension by dual reasoning. [$K(I)$ is either the same as K, when K has a 1, or is K together with a new element 1 which is $>$ every element of K.]

Now we use the combination theorem (of §4.32) by which, a strong canonical extension M to (ϕ) of a (strong) canonical extension L to (ϕ') of K, where $(\phi') \subset (\phi)$, is also a (strong) canonical extension of K to (ϕ) provided L is a supersystem of the image of K for (ϕ) and the elements of M are separated by elements in the image of K in L. [It was proved in §4.32 that L is a minimal extension to (ϕ) of K, and is also isomorphic to the canonical extension if one exists; it is then a corollary that the condition for M to be a (strong) canonical extension is that the separation condition should be true in M (considered as an extension of K].

In the present case we have a (strong) canonical extension $K(O)$ of K to (O); and the strong canonical extension $[K(O)]$ (I) of this to (I) being evidently an extension to (O, I) is, by the theorem of § 4.342, a strong canonical extension of $K(O)$ to (O, I) also. And the separation of any two distinct elements a, b of this extension by an element of K, [not merely by an element of $K(O)$], can be proved also; for if either a or b is in K, then it is itself an element of K separating a, b;

while if a, b are both outside K, it means that K has neither units 0 or 1 and a,b are the units of $[K(O)]\ (I)$. Then K must contain elements c, d with $c \not< d$; and as $c >$ the 0 but $\not>$ the 1 of $[K(O)]\ (I)$, it separates (0, 1), i.e. (a, b). Hence we have proved the

THEOREM. *Any ordered set K has a (strong) canonical extension to (O, I).*

5·2 We consider now the extension by cuts of Mac Neille, which is the second fundamental of the three extensions.

THEOREM. *For any ordered set K with units, there is a strong canonical extension L to (π), to (Σ), and to all subsets of $(O, I, +, \Sigma, \times, \pi)$ containing either π or Σ.*

PROOF. In view of the theorem of (§4·342) it is enough to prove that there is an extension L to $(O, I, +, \Sigma, \times, \pi)$ which is a strong canonical extension to π, as also one to Σ, with the same image of K.

The extension L consists of the cuts $[A, B]$ of K, as defined by Mac Neille, with a ordering relation $[A, B] \subset [C, D]$ which is equivalent to $A \subset C$, and also to $B \supset D$. The image, K^*, of K consists of cuts of the form $[A, \bar{A}]$, corresponding to single elements a of K and defined by the conditions: A is the set of elements of $K < a$, $\bar{A}$ the set of elements of $K > a$.

We refer the reader to Mac Neille's paper for the proofs of the results that

$$[A, B] = \Sigma\, [A_i, \bar{A}_i].\ a_i \epsilon A\dagger$$
$$= \pi\, [B_j, \bar{B}_j],\ b_j \epsilon B\ ;$$

and the relation $\subset$ holds between two cuts $[A, B]$, $[C, D]$ in L if and only if $\bar{A} < \bar{B}$ holds for elements with similar representations (as sums or as products, as the case may be) in *any* superset $\bar{L}$ of K to Σ, or π; also it is proved by Mac Neille that L is closed for, and a supersystem of K^* for, $(O, I; +, \Sigma, \times, \pi)$.

† Refer M: §§ 11.10, 11.12.

We have then only the separation condition left to be proved. Let $[A, B]$, $[C, D]$ be distinct cuts of K; then either $A \not\leq C$ or $C \not\leq A$, and so an element x of K can be found which does not lie in both A, C but only in one of them. Then the corresponding element $(X, \bar{X})$ in κ is $\subset$ one of $[A; B]$, $[C, D]$ but not $\subset$ both; this $[X, \bar{X}]$ is then an element of K separating the given distinct elements of L. This completes the proof that L is a strong canonical extension to Σ, as also to π. The later statements follow, as mentioned before, by the use of theorem (4.342), since L is an extension to $(O, I, +, \Sigma, \times, \pi)$.

5.3. The last theorem gives extensions for ordered sets with units only. For the general ordered set, we combine with this the results of §(5.1), and using the composition theorem (§4.32), obtain the

THEOREM. *For any ordered set K, there is a canonical extension L to subsets of $(O, I, +, \Sigma, \times, \pi)$ containing either (O, I, Σ) or (O, I, π).*

PROOF. For L we take the extension by cuts of the extension $K(O, I)$ of K^* to units. For the proof we have only to establish the separation of any two distinct elements $[A, B]$, $[C, D]$ of L by some element of K^*, the image of K in L, consisting of cuts of the form $[X, \bar{X}]$, for x in K. The image $K(O, I)^*$ of $K(O, I)$ in L consists of cuts $[X, \bar{X}]$, $x \in K(O, I)$. Thus the only extra elements in $K(O, I)^*$ which may not be in K^* are the 0 and 1 of L (which are not in K if K^* has no 0 or 1, as the case may be). By the last theorem there is a $[X, \bar{X}]$, $x \in K(O, I)$ separating the distinct elements $[A, B]$, $[C, D]$ of L; in fact, $[X, \bar{X}]$ is $\subset$ one, say, $[A, B]$ but not $\subset$ the other $[C, D]$. If $[X, \bar{X}]$ is in K^*, nothing remains to be proved. If it is not in K^*, it cannot be the 0 of L, as it is not $\subset [C, D]$; so it must be the 1 of L, and then, as $[X, \bar{X}] \subset [A, B]$, $[A, B]$ is also the 1 of L and $[C, D]$ is $\not\supset [A, B]$. Now C cannot cannot be $\supset K$; for K now has no 1, and so $1 = \Sigma x, x \in K$, in $K(O, I)$; whence in L also the $1 = \Sigma[x, \bar{x}], x \in K =$

$[K, \pi \bar{X}]$; thus the 1 of L alone has the form $[K, B]$. Thus $C \not\supset K$ and an *element* x of K can be found which is not in C. Then $[X, \bar{X}]$ is in K^*, is $\subset [A, B] =$ the 1 of L, but $[X, \bar{X}] \not\subset [C, D]$. This $[X, \bar{X}]$ separates $[A, B], [C, D]$. The proof of the theorem now follows from the theorems of §§4.32 and 5.2.

5.4. From the above, by using the resolution theorem, (§4.33), we obtain further extensions.

THEOREM. *For any ordered set K, there exists*

(*a*) *a canonical extension, L, to subsets of $(O, I, +, \Sigma, \times, \pi)$ containing (O, Σ) or (I, π) or (Σ, π);*

(*b*) *a canonical extension, L', to subsets of $(1, \Sigma, +)$ containing Σ, and*

(*c*) *a canonical extension, L^*, to subsets of $(O, \pi, \times)$ eonta.ning π.*

PROOF. (*a*) From the canonical extension, L, given in the last section, of K to $(O, I, +, \Sigma, \times, \pi)$ if we select only the elements required by K and (O, Σ) or (I, π) or (Σ, π), we get all the elements of L. Also set L is, trivially a subsystem of L for all operations. So we can apply the conclusions of the resolution theorem if we can represent the elements in L in terms of K and (O, Σ) or (I, π) or (Σ, π) so that the relations between them are required by these representations in all extensions of a similar sort.

Let us first consider the case when the operations are (O, Σ). Then any $[A, B]$ of L other than the o element may be represented as $[A, B] = \Sigma [A_i, \bar{A}_i]$, $a_i \in A \cap K$ ($A \cap K$ is not null) while the element o is represented as $O(K^*)$. The above representation of $[A, B]$ as a sum differs, if at all, from the earlier one in terms of Σ and elements of $K(O, I)^*$ by the omission of $(A_i, \bar{A}_i)$ from $K(O, I)^* - K^*$, when such a term is the o of L its omission does not affect the sum, as some summands are still left over; the omitted term may be the I of L, if K does not have a 1. Then

$[A, B]$ must be the I of L, and $1 = \Sigma a_i, a_i \varepsilon K$ in $K(O, I)$, so that $[A, B] =$ the I of $L = \Sigma(A_i \bar{A}_i)$, $a_i \varepsilon A \cap K$, as L is a supersystem of $K(O, I)$ for Σ. Thus the representation is always valid.

Let $[A, B]$, $[C, D]$ be two elements of L with representations as given above, and α, γ be elements with similar representations in any extension $\bar{L}$ of K to (O, Σ) with $\bar{K}$ as image of K. Then we can prove that $[A, B] \subset [C, D]$ if, and only if, $\alpha < \gamma$. For if $[A, B] \subset [C, D]$ $\alpha < \gamma$ follows when $[A, B]$ or $[C, D]$ equals a unit 0 or 1 of L. Then either $[A, B] = 0 = O(K)$ or $[C, D] = 1 = \Sigma[A_i A_i]$, $a_i \varepsilon k$; and then $\alpha = O(\bar{K})$ or $\gamma = \Sigma \bar{a}_i, \bar{a}_i, \varepsilon \bar{K}$, and in both cases $\alpha < \gamma$. If we assume that neither $[A, B]$ nor $[C, D]$ equal either unit of L then $A-(A \cap K)$, $C-(C \cap K)$ are both null, or consist of 0 only; $B-(B \cap K)$, $D-(D \cap K$ are both null, or consist of 1 only. So $[A, B] \subset [C, D]$ if, and only if, $(A \cap K) \subset (C \cap K)$, or equivalently, $(B \cap K) \supset (D \cap K)$. But $(A \cap K) \subset (C \cap K)$ implies $\alpha = \Sigma \bar{x}$, $(x \varepsilon A \cap K)$ is $< \Sigma \bar{x}$, $(x \varepsilon C \cap K) = \gamma$; while $\alpha < \gamma$ implies that any d in $(D \cap K)$ is in $(B \cap K)$, as $\bar{d} > \Sigma \bar{x}$, $(x \varepsilon C \cap K) = \gamma > \alpha = \Sigma \bar{x}$, $(x \varepsilon A \cap K)$ and so $d >$ each x in $(A \cap K)$ and $x > 0$, so that $x \varepsilon B$ and in K. Hence in all cases $[A, B] \subset [C, D]$ is equivalent to $\alpha < \gamma$. This proves the result for the case when the operations are (O, Σ). A dual reasoning applies for the case when the operations are (I, π). When they are (Σ, π), the above representation is changed only for 0 which is now written as $\pi[X, \bar{X}]$, $x \varepsilon X$.

These prove that L is a canonical extension of K to (O, Σ), to (I, π) and to (Σ, π). As it is also an extension to $(O, I, +, \Sigma, \times, \pi)$, it is a canonical extension to subsets of $(O, I, +, \Sigma, \times \pi)$ containing any one of the three sets given above; (by §4.342).

(*b*) For L' we take the elements of L required by K and (Σ); i.e. the cuts $[A, B]$ for which $A \cap K$ is non-null. Then representing such a $[A, B]$ as $\Sigma[A_i, \bar{A}_i]$, $a_i \varepsilon A \cap K$, we can prove, as in (a), that the relation $\lessdot$ holds between two of these elements when, and only when, the relation $<$ holds between elements with similar representations in any extension $\bar{L}$ of K to (Σ).

(c) For this we take L^* to consist of $[C, D]$ for which $D \cap K$ is non-null and argue as in (b).

Thus canonical extensions L', L^* of K to $[\Sigma]$ and to (π) exist; and as they are also evidently extensions to $(\Sigma, +, I)$ and $(\pi, \times, O)$ respectively the results (b), (c) follow by the theorem of 4.342.

5.5 We construct examples now to show that an ordered set, or one with units, need not have even a minimal extension to those subsets of $(O, I, +, \Sigma, \times, \pi)$ for which a canonical extension has not been shown to exist in the previous sections.

5.51. Example of an ordered set with units having no minimal extension to subsets of $(O, I, +, \times)$ containing $+$ or $\times$.

$K: O < (b_1, b_2, b_3, \dots) < (a_1, a_2, a_3, \dots) < 1;$

$\bar{L}: \bar{o} < (\bar{b}_1, \bar{b}_2, \bar{b}_3, \dots) < \bar{p} < (\bar{a}_1, \bar{a}_2, \bar{a}_3, \dots) < \bar{1};$

$\bar{K}: \bar{o};\ (\bar{b}_1, \bar{b}_2, \dots),\ (\bar{a}_1, \bar{a}_2, \dots),\ \bar{1};$

L': all finite subsets of the enumerable set $C = (c_1, c_2, \dots)$, complements of finite subsets, the null set O, and the whole set C, these being partially ordered by set-inclusion;

$K': O, [(c_1), (c_2), \dots]; [(C-c_1), (C-c_2), \dots], C.$†

The subsets $\bar{K}$, K' of $\bar{L}$, L' respectively, are subsets isomorphic to K (such an isomorphism between K, $\bar{K}$ is indicated by writing the elements in $\bar{K}$ in the same order as their correspondents in K).

† The ordered sets K, $\bar{L}$, L' will be defined as here, by giving their elements and basic relations of order between them, from which alone all other order relations are to be deduced by using the fact that $<$ is an ordering relation. Thus $\bar{b}_i < \bar{a}_j$ in $\bar{L}$.

Evidently $\bar{L}$, L' are extensions of K to $(O, I, +, \times)$ with $\bar{K}$, K' as images of K. If now L were any extension of K to a set of operations including $+$ or $\times$, and if L were $\subset \bar{L}$, then L must be isomorphic to $\bar{L}$; for if $L \subset \bar{L}$, L must be isomorphic to a subset of $\bar{L}$ containing $\bar{K}$ which is closed for $+$ or $\times$. Such a subset can be either $\bar{K}$ or $\bar{L}$ (which has just one element $\bar{p}$ besides those of $\bar{K}$); as $\bar{K}$ is not closed for $+$ or $\times$, it must be $\bar{L}$. But now $\bar{L}$ is not $\subset L'$, since any ismorphism carrying $\bar{L}$ into a subset of L' and $\bar{K}$ into K' must carry $\bar{p}$ into an element of L' which is $\subset$ every $(C - c_i)$ and also $\supset$ every (c_i); but L' contains no such element. Hence as $\bar{L} \not\subset L'$, if L is $\subset \bar{L}$ then $L \not\subset L'$. Thus K has no extension L, to any subset of $(O, 1, +, \times)$, containing $+$ or $\times$, which is at the same time $\subset$ these extensions $\bar{L}$, L' (to the same set of operations). Thus it can have no minimal extension (i.e. an extension $\subset$ every other extension), and so no canonical extension either.

5.52. Example of an ordered set K (without units) which has no minimal extensions to subsets of $(\Sigma, +, I, \times)$ containing $(\times)$.

K: $(a_1, a_2, a_3, \ldots) < 1$.

$\bar{L}$: $\bar{O} < (\bar{a}_1, \bar{a}_2, \bar{a}_3, \ldots) < \bar{1}$;

$\bar{K}$: $(\bar{a}_1, \bar{a}_2, \ldots)$, $\bar{1}$.

L': Complements of finite subsets of an enumerable set $C = (c_1, c_2, c_3, \ldots)$, together with the whole set C, ordered by set inclusion;

K': $[(C - c_1), (C - c_2), \ldots]$, C.

Again we verify that the subsets $\bar{K}$, K' of $\bar{L}$, L' are isomorphic to K; and $\bar{L}$, L' are extensions of K to $(\Sigma, +, I, \times)$ with $\bar{K}$, K' as images of K. If now L were an extension of K to a set of operations including $(\times)$, and L were $\subset \bar{L}$, L must be isomorphic to $\bar{L}$.

Now $\bar{L}$ cannot be $\subset L'$, as L' contains no elements $<$ every $C - (c_i)$ in the image of K, to correspond to the element $\bar{O}$ of $\bar{L}$ under an isomorphism of $\bar{L}$ in L'. Hence L

cannot be $\subset L'$ either. Thus no extension to $(\times)$ or subsets of $(\Sigma, +, I, \times)$ containing $\times$ can be at the same time $\subset \bar{L}$ and $\subset L'$. Thus K has no minimal extension to subsets of $(\Sigma, +, I, \times)$ containing $(\times)$.

By using the duality theorem, (§9·341), we get from the above the

COR. *There exist ordered sets (without* 1) *for which there can be no minimal extension to subsets of* $(\pi, \times, O, +)$ *containing* $+$.

With this we have completely considered the existence or otherwise of a canonical extenion of an arbitrary ordered set, as well as an ordered set with units to every subset of $(O, I, +, \Sigma, \times, \pi)$.

6. *Extensions to subsets of* $(O, I, \times, \pi, P)$ *containing* (P).

6.1. For constructing the basic extension, to (O, π, P), we briefly study a class of 'ideals' in any ordered set K with units, (0, 1).

A non-null subset A of K will be called an *M-ideal** if whenever $x \in A$ and $y < x$ $y \in A$ also.

The lower segments, A, of cuts (A, B) of K evidently form M-ideals. These will be called comprincipal M-ideals.† Each a in K gives raise to a *principal* M-ideal $M(a)$ consisting of all $x < a$; these are the lower segments of the cuts $[A, \bar{A}]$ forming the image of K in the extension by cuts described earlier. It is easily verified that all M-ideals contain the element 0, and unrestricted unions and intersections of M-ideals give M-ideals again.‡ Thus the M-ideals form a complete lattice, $K(M)$, with all sums and products distributive.

* A M-ideal, or multiplicative ideal, can also be characterised as a non-null subset containing with each x, every existing $x \times y$, $y \in K$.

† R. Vaidyanathaswamy, "The Ideal theory of the partially ordered set", *Proc. Ind. Acad. Sci.*, Vol. XIII.

‡ The complete ideals defined by Vaidyanathaswamy form a sub-family not closed for unions.

Hence, by a well-known result, $K(M)$ is closed for product-complementation and sum-complementation. While the sum-complement of every M-ideal other than $M(1)$ is $M(1)$ itself, the product-complements of M-ideals contain elements besides the units. The zero of $K(M)$ being $M(0)$, the product complement, A', of any M-ideal A is evidently the set of elements x of K for which $x \times a = 0$ for each a in A. As proved elsewhere*, these product-complements, A', of M-ideals can also be characterised as M-ideals (B) for which $B = B''$, and they form a complete Boolan algebra, $K(N)$, with the same multiplication as in $K(M)$.* They are the '*normal*' M-ideals. We saw that the comprincipal M-ideals also form a sub-family $K(C)$ of $K(M)$, and it is also a complete lattice with the same multiplication as in $K(M)$. [MacNeille shows that $[\pi A_i, \Sigma B_i]$ is a cut when $[A_i, B_i]$ is a family of cuts; or it could be inferred from the fact that the comprincipal ideals may be characterised as set-products of principal ideals]. Both $K(N)$ and $K(C)$ contain the units $K = M(1)$, and $0 = M(0)$ of $K(M)$. Denoting the family of principal M-ideals by K we can now describe the third extension as under:

6.2. THOREM. *The family* $L, = K\ (NC)$, *of subsets of an ordered set* K *with units of the form* $(B \cap C)$, *where* $B \in K(N)$ *and* $C \in K\ (C)$, *ordered by set inclusion, forms a canonical extension of* K *to subsets of* $(O, I, \pi, \times, P)$ *containing* (O, π, P), *with* K *as image of* K.

PROOF. Since $K\ (N)$, $K\ (C)$ both contain the null set 0, the whole set K, and the intersections of sub-families belonging to them, $K\ (NC)$ also contains K, 0, and set-intersections of its sub-families. Also as B or C may be taken to be K in the product $(B \cap C)$, $K\ (NC)$ contains $K\ (N)$ and $K(C)$. It is thus a subset of $K(M)$ contain-

* See, for instance, my paper on 'The last residue class in the distributive lattice', *Proc. Ind. Acad. Sci.*, Vol. XVI, 1942.

ing $K(N)$, $K(C)$ and having the same multiplication as in $K(M)$. Since $K(M)$, $K(NC)$ have the same zero, o and the same multiplication, and since $K(NC)$ contains A' whenever it contains A, $[(A' \varepsilon K(N) \subset K(NC)]$, $K(NC)$ is also closed for, and a subsystem of $K(M)$ for, product-complementation. Thus $L, = K(NC)$, is closed for $(O, I, \pi, \times, P)$, and evidently a supersystem of K for these operations. It is thus an extension for all these operations.

We next note that any element of $K(NC)$ may be characterised as an M-ideal A which satisfies the condition $A = A'' \cap \overline{A}$, where A'' is the normal envelope, and $\overline{A}$ the comprincipal envelope of A. [A'' is seen to be the least normal M-ideal containing A; while $\overline{A}$, the least comprincipal ideal containing A, is evidently $\cap \{ M(c) \}$, for the $c >$ each a in A.]. Since $A'' \varepsilon K(N)$, $\overline{A} \varepsilon K(C)$, $A'' \cap \overline{A}$ is in $K(NC)$; conversely, if $A = B \cap C$, $B \varepsilon K(N)$, $C \varepsilon K(C)$, $A \subset B$, $A \subset C$; so that $A'' \subset B'' = B$, $\overline{A} \subset \overline{C} = C$ and so $A'' \cap \overline{A} \subset B \cap C = A$ which is, evidently $\subset A'' \cap \overline{A}$ also. Thus $A = A'' \cap \overline{A}$ if $A \varepsilon K(NC)$. From this fact, we can choose the representation for elements (A), of $K(NC)$ in the form :

$$A = A'' \cap \overline{A} = (A')' \cap [\cap M(c)],\ c \varepsilon C$$
$$[C = \text{set of elements} > \text{every } a \text{ in } A\,;]$$
$$= [\cup M(b)]' \cap [\cap M(c)],\ b \varepsilon A' \text{ and } c \varepsilon C$$
$$= \cap \{ M(b) \}' \cap [\cap M(c)]$$
$$= \cap \{ P[M(b)],\ M(c) \},\ b \varepsilon (A)' \text{ and } c \varepsilon C.$$

The relations between elements A_1, A_2 of $L, = K(NC)$, are required by their representations in all extension to (O, P, π). For let $\bar{A}_1\, \bar{A}_2$ be elements with representations similar to A_1, A_2 in any extension $\bar{L}$ of K to (O, P, π) with $\bar{K}$ as image of K;

so that, for $i=1$ or 2, $A_i = \cap \{P[M(b_i)], M(c_i)\}$, $b_i \varepsilon A_i'$, $c_i \varepsilon C_i$ and $\bar{A}_i = \cap \{P(\bar{b}_i), (\bar{c}_i)\}$; where $b_i \longleftrightarrow \bar{b}_i$, $c_i \longleftrightarrow \bar{c}_i$ under the isomorphism between K, $\bar{K}$; then $A_1 \subset A_2$ in L if, and only if, $\bar{A}_1 \subset \bar{A}_2$ in $\bar{L}$. For if $A_1 \subset A_2$, $A_1' \supset A_2'$ and $C_1 \supset C_2$; thus every b_2 is a b_1 and every c_2 a c_1, so that every factor of $\bar{A}_2 = \pi\{P(\bar{b}_2), (\bar{c}_2)\}$ occurs as a factor of $\bar{A}_1 = \pi\{P(\bar{b}_1), (\bar{c}_1)\}$, and $\bar{A}_1 < \bar{A}_2$. While if, conversely $\bar{A}_1 < \bar{A}_2$, and $x \varepsilon A_1$, as $A_1 \subset A_1''$, $A_1 \subset \bar{A}_1$, if follows that $x \times$ each $b_1 = 0$, and $x <$ each c_1. So $\bar{x} \times$ each $\bar{b}_1 = \bar{O}$ in $\bar{K}$ and $\bar{x} <$ each $\bar{c}_1$, for the element $\bar{x}$ of $\bar{K}$ corresponding to x in K. But $\bar{L}$ is a supersystem of $\bar{K}$ for (O, π). So that $\bar{x} \times \bar{b} = \bar{O}$ in $\bar{L}$ also. As $\bar{L}$ is closed for (P), $\bar{x} <$ each $P(\bar{b}_1)$ follows. So $\bar{x} < \pi\{P(\bar{b}_1), \bar{c}_1\} = A_1 < A_2 = \pi\{P(\bar{b}_2), \bar{c}_2\}$. It follows that $\bar{x} <$ each $P(\bar{b}_2)$, and $<$ each $\bar{c}_2$, or $x \times \bar{b}_2 = \bar{0}$ in $\bar{L}$, $\bar{x} < \bar{c}_2$; but $\bar{K}$ contains $\bar{x}$, $\bar{b}_2$, $\bar{0}$. Hence $\bar{x} \times \bar{b}_2 = \bar{0}$ in $\bar{K}$ also $x \times b = 0$ in K and $x < c_2$. Thus $x \varepsilon A_2''$ and $x \varepsilon \overline{A}_2$ so that $x \varepsilon A_2'' \cap \overline{A}_2 = A_2$, proving $A_1 \subset A_2$, when $\bar{A}_1 < \bar{A}_2$.

Finally the separation condition is easily proved. For if A_1, A_2 are distinct elements of $K(NC)$, either $A_1 \not\subset A_2$ or $A_2 \not\subset A_1$. Hence there is an element x of K in one of A_1, A_2 but not both. Then the element $M(x)$ of K is $\subset$ one of the A_1, A_2 but not both; which shows that this element of K separates A_1, A_2. This completes the proof that $K(NC)$ is a canonical extension of K to (O, π, P).

As it has also been shown to be an extension to $(O, I, \pi, \times, P)$, it follows that it is a canonical extension to all subsets, of the above set of operations, containing (O, π, P).

6.3. Example of an ordered set (without o) having no minimal extension to subsets of $(O, I, \pi, \times, P)$ containing (P).

K: $(a, b, c) < 1$.

$\bar{L}$: $\bar{0} < \bar{d} < (\bar{a}, \bar{b}, \bar{c}) < \bar{1}$.

$\bar{K}$: $(\bar{a}, \bar{b}, \bar{c})\ \bar{1}$.

L': the Boolean algebra of subsets of the set (p, q, r), including the null set ordered by $\subset$.

K': the sets (p), (q), (r) and (p, q, r).

Clearly $\bar{L}$ is an extension of K to $(O, I, \pi, \times, P)$ of K with $\bar{K}$ as image of K, and L' another extension with K' as image of K.

If now L were any other extension of K to a set of operations including P and L were $\subset \bar{L}$ then L must be isomorphic to $\bar{L}$. For the subset of $\bar{L}$ into which L could be isomorphically mapped must contain $\bar{K}$ and both the extra elements $\bar{0}$, $\bar{d}$ if it should be closed for (P). Hence L cannot be $\subset L'$; for $\bar{L}$ cannot be $\subset L'$, as L' does not contain two distinct element $< (p)$, (q), (r) in K', to correspond to $\bar{0}$, $\bar{d}$ in $\bar{L}$. Hence L cannot be $\subset \bar{L}$ and $\subset L'$ at the same time, or K can have no minimal extension to subsets of $(O, I, \pi, \times, P)$ containing P.

6.4. Example of an ordered set, K, with units which has no minimal extension to subsets of $(I, \pi, \times, P)$ containing (P).

K: $0 < (a, b, c) < 1$.

$\bar{L}$: $\bar{0} < \bar{d} < (\bar{a}, \bar{b}, \bar{c}) < \bar{1}$.

$\bar{K}$: $\bar{d}$; $(\bar{a}, \bar{b}, \bar{c})$, $\bar{1}$.

L: subsets of (p, q, r) including the null set; ordered by $\subset$.

K': 0, (p), (q), (r), (p, q, r).

It may be verified that $\bar{L}$, L' are extension of K to $(I, \pi, \times, P)$ with $\bar{K}$, K' as images of K. [Since $\bar{L}$ is not a supersystem of $\bar{K}$ for (0), we cannot say it is an extension for $(0, I, \pi, \times, P]$. Then as in the last example, it will be found that any extension L of K to a set of operations including (P) should be isomorphic to $\bar{L}$ if it were $\subset \bar{L}$; and $\bar{L}$ is not $\subset L'$, as we saw before. So L cannot be $\subset L'$ if $L \subset \bar{L}$. So no minimal extension of K to subsets of $(I, \pi, \times, P)$ containing P, can exist.

6.5. Example of an ordered set, K, with units which has no minimal extension to subsets of $(O, I, \times, P)$ containing $\dot{\times}$.

$$K: 0 < d < (b_1, b_2, b_3, \ldots) < (a_1, a_2, a_3, \ldots) < 1.$$

$$\bar{L}: \bar{0} < \bar{d} < (\bar{b}_1\ \bar{b}_2, \bar{b}_3, \ldots) < \bar{p} < (\bar{a}_1, \bar{a}_2, \bar{a}_3, \ldots) < \bar{1}.$$

$$\bar{K}: \bar{0}, \bar{d}, (\bar{b}, \bar{b}_2, \ldots -)', (\bar{a}_1, \bar{a}, \ldots -).$$

L': *all subsets* of an enumerable set $S = (s_0, s_1, s_2, \ldots)$ *of either of the types* $f + s_0$ or $(S - f) + s_0$, where f is any finite or null subset, together with the null set 0; these being ordered by $\subset$.

$$K': 0, (a_0,) [(a_0 \cup a_1), (a_0 \cup a_2), \ldots\ldots],$$
$$[(S - a_1) \cup a_0, (S - a_2) \cup a_0, \ldots\ldots], S.$$

It will be seen that $\bar{L}$, L' are extensions of K to $(O, I, \times, P)$ with $\bar{K}$, K' as images of K. (In all three ordered sets $K, \bar{L}, L'$ it will be seen that the product-complement of each element other than, 0, is 0, while $P(0) = 1$). If L were to be an extension of K to any set of operations including $\times$, and L were $\subset \bar{L}$ then it must be isomorphic to $\bar{L}$, since the only subset of $\bar{L}$ containing $\bar{K}$ and closed for $(\times)$ is $\bar{L}$ itself. Hence L cannot be $\subset L'$, when $\bar{L}$ is not $\subset L'$; now $\bar{L} \subset L'$ is impossible as no element exists in L', to correspond to $\bar{p}$ in $\bar{L}$, which is $<$ every $[(S - a_i) + a_0]$ and $>$ every $[a_i \cup a_0]$. Hence an L cannot be $\subset \bar{L}$ and L' at the same time, so K can have no minimal among extensions to $(O, I, \times P)$ containing $(\times)$.

With this all subsets of $(O, I, \pi, \times, P)$, containing P have been considered except (P), (O, P), (I, P) and (O, I, P). That there need be no minimal extensions to these operations for an ordered set with units will be seen from an example discussed in the last section (§8.2).

6.6. Finally we observe that using the duality theorem the existence of canonical extensions of an ordered set to subsets of $(O, I, \Sigma, +, S)$, can be discussed.

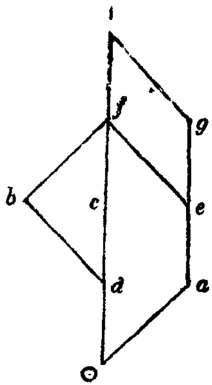
§7. *Extensions to sets of operations containing at least one from each of the sets:*

$$(+, \Sigma), (\times, \pi) \text{ and } (P, S, \bar{+}, \bar{\Sigma}, \bar{\times}, \bar{\pi}).$$

In this section we give examples of ordered sets with units which have no extension at all to the above sets of operations.

7.1. Example of an ordered set with units having no extension at all to any set of operations including (P), one of $(+, \Sigma)$ and one of $(\times, \pi)$.

$$K: \begin{Bmatrix} 0 < d < (b, c) < f < 1. \\ 0 < a < e < (f, g) < 1. \end{Bmatrix}$$



If there should be an extension L of K to a set of operations including P, one of $(+, \Sigma)$ and one of $(\times, \pi)$, replacing the image of K in L, by K itself we first show that 0 must be the zero element of L also. For as $1 = P(0)$ in K, the extension L being a supersystem of K for P, $P(0) = 1$ in L also; so that $P(0) > 0$ in L and this implies that 0 is the zero element of L. As L is an extension of K to one of $(+, \Sigma)$ and one of $(\times, \pi)$, it follows again that, even as in K,

$$a \times b = 0,$$
$$a \times c = 0,$$

and $b + c = f$. But since 0 is the zero element of L, and L is closed for (P), the first two relations imply $b < P(a)$, $c < P(a)$ and so, by the third relation, $f < P(a)$ or $f \times a = 0$ in L. But this is impossible as $a \times f = a \neq 0$ in K, implies $a \times f = a$ in L and $a \neq 0$. Hence such an extension L cannot exist.

By duality, we see there are ordered sets with units having no extension to S, one of $(+, \overline{\Sigma})$ and one of $(\times, \pi)$.

7.2. Example of an ordered set with units having no extension to a set of operations containing one of $(+, \Sigma)$, one of $(\times, \pi)$ and one of $(\mp, \overline{\Sigma}, \times, \overline{\pi})$.

$$K: 0 < (a, b, c) < 1.^*$$

* Any non-distributive lattice with units may be used.

If L were an extension to a set of operations of the sort described above, and K itself is taken as the image of K, it will be seen that L must be a distributive lattice and a supersystem of K for $+$ and $\times$. But then, as in K, so in L also we have

$$b+c = 1, \; a \times (b+c) = a,$$
$$a \times b = 0 = a \times c, \; (a \times b)+(a \times c) = 0 \text{ and } a \neq 0,$$

so that $a \times (b+c) \neq (a \times b)+(a \times c)$ in L'; i.e. L is not a distributive lattice. Hence such an extension L cannot exist.

These cases exhaust all combinations of operations containing one each from $(+, \Sigma)$, $(\times, \pi)$, and $(P, S, \bar{+}, \bar{\Sigma}, \bar{\times}, \bar{\pi})$. The sets of operations not containing any operations from the third of the above sets have been considered in § 5. Hence we have only to consider combinations not containing $(+, \Sigma)$ or not containing $(\times, \pi)$.

8. *Extension to sets of operations not including $+$ or Σ (or not including $\times$ or π).*

8.1. Example of an ordered set with units having no minimal extension to any subset of $(O, I, \times, \pi, P, S, \bar{+}, \bar{\Sigma}, \bar{\times}, \bar{\pi})$ containing at least one of the operations $S, \bar{+}, \bar{\Sigma}, \bar{\times}, \bar{\pi}$.

$$K: \; 0 < p < q < \ldots\ldots < (a, b, c) < 1;$$
$$\bar{L}: \; \bar{o} < \bar{p} < \bar{q} < \ldots\ldots < \bar{t} < (\bar{a}, \bar{b}, \bar{c}),$$
$$\bar{a} < (\bar{n}, \bar{m}) < \bar{1}, \; \bar{l} < (\bar{n}, \bar{l}) < \bar{1},$$
$$\bar{c} < (\bar{m}, \bar{l}) < \bar{1};$$
$$\bar{K}: \; \bar{0}, \bar{p}, \bar{q}, \ldots . (\bar{a}, \bar{b}, \bar{c}), \bar{1};$$
$$L': \; 0' < p' < q' \ldots . < g' < (f', e', d'),$$
$$f' < (a', b') < 1', e' < (a', c') < 1', d' < (b', c') < 1';$$
$$K': \; 0', p', q', \ldots (a' \, b', c'), 1'.$$

(The ordered set K and the extensions $\bar{L}, L'$ are as illustrated in figures 2, 3, 4, Part I; Page 60). We first note that $\bar{L}, L'$ are extensions of K to $(O, I, \bar{\times}, \bar{\pi}, \bar{+}, \bar{\Sigma}, \bar{\times}, \bar{\pi}, P, S)$. [For, the products of (a, b), (a, c), (b, c), (a, b, c) do not exist in K, nor the distributive sums or products of these subsets.] Similarly a, b, c have no product or sum complements in K. Hence $\bar{L}$, L' are supersystems of $\bar{K}$, K' for all the operations mentioned above. They are evidently closed for these operations.

We next observe that if L were any extension of K to a set of operations including at least one of $(S, \bar{+}, \bar{\Sigma}, \bar{\times}, \bar{\pi})$, which was also $\subset \bar{L}$, then it must be isomorphic to a subset of $\bar{L}$ containing $\bar{K}$ and some one of the elements $\bar{l}, \bar{m}, \bar{n}$; as subsets of $\bar{L}$ containing $\bar{K}$ and none of $\bar{l}, \bar{m}, \bar{n}$ are not closed for any one of the operations $(S, \bar{+}, \bar{\Sigma}, \bar{\times}, \bar{\pi})$. But then such a subset of $\bar{L}$ is not $\subset L'$, since in L' we have no element between l' and any of the elements a', b', c' (to correspond to $\bar{l}$, or $\bar{m}$ or $\bar{n}$). Hence if such an extension L were $\subset \bar{L}$ it cannot be $\subset L'$, so that K has no minimal extension to any subset of the operations $(O, I, \times, \pi, P, S, \bar{+}, \bar{\Sigma}, \bar{\times}, \bar{\pi})$ containing one at least of the operations $(S, \bar{+}, \bar{\Sigma}, \bar{\times}, \bar{\pi})$.

By the theorem of duality there exists an ordered set with units having no minimal extension to any subset of $(O, I, \times, \pi, S, P, \bar{+}, \bar{\Sigma}, \bar{\times}, \bar{\pi})$ containing one at least of the operations $P, \bar{+}, \bar{\Sigma}, \bar{\times}, \bar{\pi}$. In particular it has no minimal extension to subsets of (O, I, P) containing P. This completes the discussions in § 6.

8.3. With this we have exhausted all possible selections of the twelve operations. Summarising our results so far, we can say that only for the cases mentioned below, need there exist canonical extensions.

(A) *For an ordered set with units*: canonical extensions exist to (O); to (I); to (O, I); to all subsets of $(O. I, +, \Sigma, \times, \pi)$ containing Σ or π; to subsets of $(O, I, \pi, \times, P)$ containing (O, π, P); and to subsets of $(O, I, \Sigma, +, S)$ containing (I, Σ, S).

(B) *For any general ordered set* canonical extensions exist to (O); to (I); to (O, I); to subsets of $(O, I, +, \Sigma, \times, \pi)$ containing (O, Σ) or (I, π) or (Σ, π); to subsets of $(\Sigma, +, I)$ containing Σ; and to subsets of $(\pi, \times, O)$ containing π.

ON THE DUALITY OF LINEAR TENSORS IN AFFINE SPACE*

BY

N. PADMA, *University of Madras.*

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1. INTRODUCTION. The affine space A_n of n dimensions is the generalization of Riemannian space, in which a covariant differentiation is possible by means of the coefficients Γ^i_{jk} of an "affine connection," which takes the place of Christoffel symbols. Geometrically the affine connection defines an infinitesimal parallel transference whereby, the result of transferring a vector V^i at a point $x(=x_1, x_2, \ldots, x_n)$ to the point $x+\delta x$ is a vector at $x+\delta x$ with the components $V^i+\delta V^i$, where $\delta V^i = -\Gamma^i_{rs} V^r \delta x^s$ and $\Gamma^i_{rs} = \Gamma^i_{sr}$.(1) The covariant derivative of V^i is then given by $V^i_{,l} = \dfrac{\partial V^i}{\partial x^l} + V^r \Gamma^i_{rl}$. The covariant derivative of an absolute scalar F is its gradient; that of a scalar F of weight N is the vector $F_{,l}$ of weight N given by $F_{,l} = \dfrac{\partial F}{\partial x^l} - FN\Gamma^a_{la}$(2).

The covariant derivative of a mixed tensor X^k_i of weight N is of weight N and may be derived from that of the scalar $F = X^k_i \xi^i \eta_k$ of weight N (where ξ, η are absolute vectors); namely,

$$\frac{\partial F}{\partial x^l} - NF\Gamma^a_{la} = F_{,l} = X^k_{i,l} \xi^i \eta_k + X^k_r \left[\frac{\partial \xi^r}{\partial x^l} + \xi^i \Gamma^r_{il}\right]\eta_k + X^r_i \xi^i \left[\frac{\partial \eta_r}{\partial x^l} - \eta_k \Gamma^r_{rl}\right];$$

giving

$$X^k_{i,l} = \frac{\partial X^k_i}{\partial x^l} - NX^k_i \Gamma^a_{la} - X^k_a \Gamma^a_{il} + X^a_i \Gamma^k_{la}.$$

* I wish to thank Dr. R. Vaidyanathaswamy for his help and guidance in the preparation of this paper.

The same process gives the covariant derivative of any tensor of weight N and any order (= the number of indices)(3). We are concerned in this paper with *linear tensors* of any order and weight, that is, covariant and contravariant tensors, skew-symmetric in every pair of indices.

We consider the *curl* (or Stokes' tensor) to be the typical invariant process applied to covariant linear tensors, and the *divergence* as applicable only to contravariant linear tensors. The curl $X_{ij\ldots klm}$ of the covariant linear tensor $X_{ij\ldots kl}$ of order μ and weight N, being the skew-symmetric part of its covariant derivative, is of order $(\mu+1)$ and weight N; it is given by

$$X_{ij..klm} = X_{ij..kl,\,m} - X_{ij..km,\,l} + \ldots + (-1)^{\mu} X_{j..lm,\,i}$$
$$= \frac{\partial X_{ij..l}}{\partial x^m} - \frac{\partial X_{ij..km}}{\partial x^l} + \ldots + (-1)^{\mu} \frac{\partial X_{j..lm}}{\partial x^i}$$
$$-N[X_{ij..l}\,\Gamma^{\lambda}_{\lambda m} - X_{ij..km}\,\Gamma^{\lambda}_{\lambda l} + \ldots + (-1)^{\mu} X_{j..lm}\Gamma^{\lambda}_{\lambda i}].$$

The divergence of a contravariant linear tensor is the result of contracting the index of differentiation in its covariant derivative. It is thus a linear tensor of the same weight and of order less than that of the tensor by unity. The connection between these two processes is given by Theorem 2.

It does not seem to have been noticed that in the space A_n with the connection Γ^i_{jk}, we can derive from any linear tensor $X_{ij\ldots}$ or $X^{ij\ldots}$ of order μ, and arbitrary weight N, a *dual tensor* $\bar{X}^{pq\ldots}$ or $\bar{X}_{pq\ldots}$ of order $(n-\mu)$ by means of multiplication and contraction with the special ε-tensors $\varepsilon^{ij\ldots}$ and $\varepsilon_{ij\ldots}$(4) of order n, where $\varepsilon^{ij\ldots} = \varepsilon_{ij\ldots} = 1, -1$ or 0, according as the n indices $ij\ldots$ are an even or odd permutation of $1, 2, \ldots, n$, or comprise repetitions. These ε-tensors arise naturally in connection with the transformation of determinants of n covariant or contravariant vectors. For example, if $u_i, v_j, w_k\ldots$ are n

covariant vectors, their determinant is a scalar of weight 1 as is seen from the equations of transformation $u_i = \frac{\partial \bar{x}^r}{\partial x^i} \bar{u}_r$. But this determinant is equal to $\varepsilon^{ij\cdots} u_i v_j w_k \ldots\ldots$ This implies that $\varepsilon^{ijk\cdots}$ is a contravariant linear tensor of weight 1 (or, *tensor density*); similarly $\varepsilon_{ij\ldots}$ is a covariant linear tensor of weight -1. These ε-tensors are also naturally connected with linear tensors $X^{ij\cdots}$, $X_{ij\ldots}$ of order n and weight N; for, if we write $X^{ij\cdots} = X\varepsilon^{ij\cdots}$, $X_{ij\ldots} = \xi\varepsilon_{ij\ldots}$, it follows immediately that X is a scalar of weight $(N-1)$ and ξ a scalar of weight $(N+1)$.

We now proceed to define the duals, $\bar{X}_{pq..r}$, $\bar{X}^{pq..r}$ of order $(n-\mu)$ of given linear tensors $X^{ij..l}$, $X_{ij..l}$ of order μ and arbitrary weight N. These are given by:

$$D_1: \quad \bar{X}^{pq\ldots r} = \frac{1}{\mu\,!} X_{ij\ldots l}\, \varepsilon^{ij..lpq..r}$$

$$D_2: \quad \bar{X}_{pq\ldots r} = \frac{1}{\mu\,!} X^{ij..l}\, \varepsilon_{pq..rij\ldots l}.$$

It follows that the dual of a covariant (contravariant) linear tensor of order μ and weight N is a contravariant (covariant) linear tensor of order $(n-\mu)$ and weight $N+1$ $(N-1)$.

We proceed to prove that (1) the process of forming a dual is involutary or *any tensor is the dual of its dual*, (2) the process of forming the divergence of a contravariant linear tensor and the curl of a covariant linear tensor are dual to each other in the above sense. We shall also examine in A_n, the form of the successive curls (divergences) of a covariant (contravariant) linear tensor and obtain some new results of interest. [Remark (3), Theorem 4].

2. THEOREM 1. *Any linear tensor $X_{ij\ldots l}$ or $X^{ij\ldots l}$ is the dual of its dual.*

For, the dual of $X_{ij\ldots l}$ of order μ is by D_1 the tensor of order $(n-\mu)$ given by

$$\mu\,!\ \bar{X}^{pq\ldots r} = X_{ij\ldots l}\, \varepsilon^{ij..lpq..r}.$$

If $\bar{\bar{X}}_{\alpha\beta\ldots\delta}$ of order μ denotes the dual tensor of $\bar{X}^{pq\ldots r}$ we have from D_2,

$$(n-\mu)!\,\bar{\bar{X}}_{\alpha\beta\ldots\delta} = \bar{X}^{pq\ldots r}\,\varepsilon_{\alpha\beta\ldots\delta pq\ldots r}$$

$$= \frac{1}{\mu!} X_{ij\ldots l}\,\varepsilon^{ij\ldots l\,pq\ldots r}\,\varepsilon_{\alpha\beta\ldots\delta\,pq\ldots r}$$

$$= \frac{(n-\mu)!}{\mu!}\Big[X_{\alpha\beta\ldots\delta} - X_{\beta\alpha\ldots\delta} + \ldots + (-1)^{\mu-1} X_{\beta\ldots\delta\alpha}\Big]$$

$$= \frac{(n-\mu)!}{\mu!}\,\mu!\,X_{\alpha\beta\ldots\delta} = (n-\mu)!\,X_{\alpha\beta\ldots\delta}.$$

A similar proof holds in the case of $X^{ij\ldots l}$.

REMARK 1. As observed, any linear contravariant or covariant tensor of order n and weight N is of the form $X\varepsilon^{ij\ldots}$ or $X\varepsilon_{ij\ldots}$, where X is a scalar of weight $(N-1)$ or $(N+1)$ respectively. Its dual in either case is the scalar X.

REMARK 2. Conversely, we may observe that a scalar X of weight N may be regarded for purposes of dualizing as either a contravariant or covariant tensor of order zero; in the former case its dual is $X\varepsilon_{ij\ldots}$ and in the latter case the dual is $X\varepsilon^{ij\ldots}$

In particular, in the case of the absolute scalar 1, the dual of its dual in both cases is $\frac{1}{n!}\varepsilon^{ij\ldots}\,\varepsilon_{ij\ldots} = 1$.

3. THEOREM 2. *The divergence of the dual of a covariant linear tensor $X_{ij\ldots l}$ of order μ and weight N is the dual of its curl. Hence by Theorem 1, the curl of the dual of a contravariant linear tensor $X^{ij\ldots l}$ of order μ and weight N is the dual of its divergence.*

For let the dual of $X_{ij\ldots l}$ be the tensor $\bar{X}^{pq\ldots r}$ of order $(n-\mu)$ and weight $(N+1)$.

The divergence of the tensor $\bar{X}^{pq\ldots r} = Y^{q\ldots r} = \bar{X}^{pq\ldots r}_{,p}$

$$= \frac{1}{\mu!}\left[X_{ij\ldots l}\,\varepsilon^{ij\ldots l\,pq\ldots r}\right]_{,p} \qquad \{D_1\}$$

$$= \frac{1}{\mu!} X_{ij\ldots l,p}\,\varepsilon^{ij\ldots l pq\ldots r},$$

since the derivative of $\varepsilon = 0$,

$$=\frac{1}{\mu!}\Big[\frac{\partial X_{ij..l}}{\partial x^p}-NX_{ij...l}\,\Gamma^{\lambda}_{\lambda p}$$

$$-X_{\lambda j....l}\,\Gamma^{\lambda}_{pi}-\ldots-X_{ij..\lambda}\,\Gamma^{\lambda}_{pl}\Big]\,\varepsilon^{ij..r}$$

$$=\frac{1}{\mu!}\Big[\frac{\partial X_{ij..l}}{\partial x^p}-NX_{ij...l}\,\Gamma^{\lambda}_{\lambda p}\Big]\varepsilon^{ij..lpq..r},$$

since the remaining terms vanish.

The dual tensor $\bar{\bar{X}}_{\alpha\beta\ldots\gamma\delta}$ of order $(\mu+1)$ of this divergence $Y^{q\ldots r}$ of order $(n-\mu-1)$ is given by

$$\bar{\bar{X}}_{\alpha\beta\ldots\ldots\gamma\delta}=\frac{1}{(n-\mu-1)!}\,Y^{q..r}\,\varepsilon_{\alpha\beta\ldots\delta q..r} \qquad \{D_2\}$$

$$=\frac{1}{(n-\mu-1)!}\,\frac{1}{\mu!}\Big[\frac{\partial X_{ij..l}}{\partial x^p}$$

$$-NX_{ij..l}\,\Gamma^{\lambda}_{\lambda p}\Big]\,\varepsilon^{ij..l\,pq..r}\,\varepsilon_{\alpha\beta..\gamma\delta q..r}$$

$$=\frac{1}{(n-\mu-1)!}\,\frac{1}{\mu!}(n-\mu-1)!\,\mu!$$

$$[\text{Stokes' tensor of } X_{\alpha\beta..\gamma}]$$

$=$ Stokes' tensor of $X_{\alpha\beta\ldots\gamma}$.

REMARK 1. Consider the contravariant tensor of order n and weight N, $X^{ij\ldots}=X\varepsilon^{ij\ldots}$. Its dual was already seen to be the scalar X of weight $(N-1)$. The curl of the dual should be considered to be the covariant derivative of X. This is by the above theorm equal to the dual of the divergence of the tensor.

REMARK 2. Consider the covariant tensor of order n and weight N, $X_{ij\ldots l}=X\varepsilon_{ij\ldots l}$. Its curl being a linear tensor of order $>n$ must vanish. Hence the dual of its curl also vanishes. On the other hand, the dual of the tensor is the scalar X. Hence we have to regard the divergence of any scalar to be zero. This indicates that divergence should not be applied to scalars. In other words, while for purposes of forming the dual, the scalar may be taken as a contravariant or covariant linear tensor of order zero, yet for purposes of forming the differential invariants, the curl or the divergence, the scalar must be considered as only a covariant tensor of order zero.

REMARK 3. It is a well-known general theorem that if the curl of an *absolute* tensor vanishes, the tensor is itself the curl of an absolute tensor of lower order.(5) From Theorem 2, it follows that *if the divergence of a contravariant tensor-density of order* μ *vanishes, the tensor-density must itself be the divergence of a tensor-density of order* $\mu+1$.

REMARK 4. As an instance of the general theorem of Remark 3, consider the absolute tensor $X_{ij..}$ of order n. Its curl vanishes; also, by Remark 1 of §2, $X_{ij....} = X\varepsilon_{ij...}$, where X is a scalar density. The dual of the gradient of X is then the absolute tensor of order $(n-1)$ whose curl is $X_{ij....}$

REMARK 5. An important skew-symmetric tensor connected with the structure of the space is the absolute covariant tensor $S_{mn} = \dfrac{\partial \Gamma^{\lambda}_{\lambda m}}{\partial x^n} - \dfrac{\partial \Gamma^{\lambda}_{\lambda n}}{\partial x^m}$, got by contracting the curvature tensor.

This vanishes identically in a Riemannian space.(6) We may show that, *in* A_n, *the curl of* S_{mn} *vanishes identically.* For, curl $S_{mn} = S_{mnp} = S_{mn,p} - S_{mp,n} + S_{np,m} = 0$, as follows by substitution from

$$S_{mn,p} = \frac{\partial S_{mn}}{\partial x^p} - S_{\lambda n}\,\Gamma^{\lambda}_{mp} - S_{m\lambda}\,\Gamma^{\lambda}_{np}$$

$$= \frac{\partial}{\partial x^p}\left[\frac{\partial \Gamma^{\lambda}_{\lambda m}}{\partial x^n} - \frac{\partial \Gamma^{\lambda}_{\lambda n}}{\partial x^m}\right] - S_{\lambda n}\,\Gamma^{\lambda}_{mp} - S_{m\lambda}\,\Gamma^{\lambda}_{np}.$$

Hence S_{mn} must be the curl of a vector which is determined except for an additive gradient. One such vector is the vector V_a with components Γ^{m}_{ma} in the given co-ordinate system. This does not imply that the vector V_a equals Γ^{m}_{ma} in all co-ordinate systems, because the laws of transformation of V_a, Γ^{m}_{ma} are not the same.

We can also see that the curl of $S_{mn} = 0$, by contraction of Bianchi's identity $B^{i}_{ajk,l} + B^{i}_{akl,j} + B^{i}_{alj,k} = 0$, for the covariant derivative of the curvature tensor.(7)

4. THE REPEATED CURL AND DIVERGENCE.

In a Riemannian space, it is well known that the curl of the curl of an absolute covariant linear tensor vanishes. Hence by Theorem 2, the divergence of the divergence of any contravariant linear tensor-density also vanishes.

In an affine space A_n we proceed to show

THEOREM 3. *The divergence of the divergence of a linear tensor $X^{ij..lm}$ of order μ and weight N is the contracted product of the tensor with the tensor S_{ji} already introduced, multiplied by* $\frac{N-1}{2}$.

For if $Y^{j..lm}$ denotes the divergence of $X^{ij..lm}$, we have

$$Y^{j..lm} = X^{ij..lm}_{,i} = \frac{\partial X^{ij..lm}}{\partial x^i} - N X^{ij..m}\,\Gamma^\lambda_{\lambda i} + X^{\lambda j..lm}\,\Gamma^i_{\lambda i} + \ldots + X^{ij..l\lambda}\,\Gamma^m_{\lambda i}$$

$$= \frac{\partial X^{ij..lm}}{\partial x^i} - (N-1)\, X^{ij..lm}\,\Gamma^\lambda_{\lambda i} + X^{\lambda j..lm}\,\Gamma^i_{\lambda i} + \ldots + X^{ij..l\lambda}\,\Gamma^m_{\lambda i} - X^{ij..lm}\,\Gamma^\lambda_{\lambda i}$$

$$= \frac{\partial X^{ij..lm}}{\partial x^i} - (N-1)\, X^{ij..lm}\,\Gamma^\lambda_{\lambda i},$$ since the last two terms cancel each other and the other terms vanish.

Now, the divergence of $Y^{j....lm} = Y^{j..lm}_{,j}$ of order $(\mu-2)$ is given by

$$Y^{j..lm}_{,j} = \frac{\partial Y^{j..lm}}{\partial x^j} (N-1)\, Y^{j..lm}\,\Gamma^\lambda_{\lambda j}$$

$$= \frac{\partial^2 X^{ij..lm}}{\partial x^i\,\partial x^j} - (N-1)\frac{\partial X^{ij..lm}}{\partial x^j}\,\Gamma^\lambda_{\lambda i} - (N-1)\, X^{ij..lm}\frac{\partial}{\partial x^j}\Gamma^\lambda_{\lambda i}$$

$$- (N-)\,\Gamma^\lambda_{\lambda j}\left[\frac{\partial X^{ij..lm}}{\partial x^i} - (N-1)\, X^{ij..lm}\,\Gamma^\lambda_{\lambda i}\right.$$

$$= \frac{N-1}{2} X^{ij..lm}\left[\frac{\partial}{\partial x^i}\Gamma^\lambda_{\lambda j} - \frac{\partial}{\partial x}\Gamma^\lambda_{\lambda i}\right],$$

since all the other terms cancel because of the alternating character of $X^{ij..lm}$. Thus, Div. Div. $X^{ij..lm} = \frac{N-1}{2} X^{ij..lm} S_{ji}$.

We may now use Theorem 2 to get the curl of the curl of a covariant linear tensor from Theorem 3, viz.

THEOREM 4. *The curl of the curl of a covariant linear tensor* $X_{ijk..l}$ *of order* μ *and weight* N *is* $-N/2$ *times the alternating product of the tensor with the absolute tensor* S_{mn} *or explicitly*

$$(\text{curl})^2 X_{ij...l} = -\frac{N}{2}\Big[X_{ij...l}\, S_{mn} - X_{ij..m} S_{ln} + \ldots + (-1)^{\mu} X_{j..m}\, S_{im} + X_{ij...n}\, S_{lm} + \ldots + (-1)^{\mu-1} X_{j..ln}\, S_{im} + \ldots + X_{k..lmn} S_{ij}\Big].$$

For, suppose $X_{ijk..l}$ is a linear covariant tensor of order μ and weight N and $\bar{X}^{pqrs....t}$ its dual of order $(n-\mu)$ and weight $(N+1)$.

Now, $X_{ij..lmn} = \text{curl. curl. } X_{ijk...l}$

$$= \text{curl (Dual Div. } \bar{X}^{pqrs..t}) \quad \text{(Theorem 2)}$$

$$= \text{Dual (Div. Div. } \bar{X}^{pqrs..t})$$

$$= \text{Dual}\,\frac{N}{2}\,\bar{X}^{pqrs..t}\, S_{qp} \qquad \text{(Theorem 3)}$$

$$= \frac{N}{2}\,\frac{1}{(n-\mu-2)\,!}\,\bar{X}^{pqrs..t}\, S_{qp}\,.\, \varepsilon_{ij...lmnrs...t}$$

$$= \frac{N}{2}\,\frac{1}{(n-\mu-2)\,!}\,\frac{1}{\mu!}\, X_{\alpha\beta...\delta}\, S_{qp}\, \varepsilon^{\alpha\beta..\delta\, pqrs..t}\, \varepsilon_{ij..lmnrs..t}, \qquad (D_1)$$

where $\alpha, \beta, .., \delta$ are μ in number.

$$= \frac{N}{2}\,\frac{1}{(n-\mu-2)\,!}\,\frac{1}{\mu\,!}\, X_{\alpha\beta..\delta}\, S_{qp}\, \varepsilon^{pq\alpha\beta..\delta rs..t}\, \varepsilon_{mnij..lrs..t},$$

$$= \frac{N}{2}\Big[X_{ij...l}\, S_{mn} - X_{ij..m}\, S_{ln} + \ldots + (-1)^{\mu}\, X_{j..m}\, S_{in} + X_{ij...n}\, S_{lm} + \ldots + (1-)^{\mu-1} X_{j..ln}\, S_{im} + \ldots + X_{k...lm}\, S_{ij}\Big].$$

REMARK 1. Since $S_{mn} = 0$ identically in a Riemannian space, it follows from Theorems 3 and 4, that the curl of the curl of a covariant linear tensor of any order and arbitrary weight and the divergence of the divergence of a contravariant linear tensor of any order and arbitrary weight vanish in the Riemannian space.

REMARK 2. We also see from Theorems 3 and 4 that in A_n the divergence of the divergence of a contravariant linear tensor of any order vanishes when $N = 1$ and the curl of the curl of a covariant linear tensor of any order vanishes when $N = 0$. This shows that the absolute covariant linear tensor and the contravariant linear tensor-density behave in the simplest fashion under the two differential processes.

REMARK 3. The question arises whether the successive divergences or curls of a linear tensor vanish at some stage or not.

Now, it follows by a double application of Theorem 3, that

$$\text{Div}^4 X^{ijkl..mn} = \frac{(N-1)^2}{4} X^{ijk..mn} S_{ij} S_{kl}$$
$$= \frac{1}{3}\frac{(N-1)^2}{4} X^{ijkl..mn}\Big[S_{ij} S_{kl} + S_{jk} S_{il} + S_{ki} S_{jl}\Big].$$

Similarly, $\text{Div}^{2t} X^{ijkl..mn}$ differs only by a numerical factor from the contracted product of $X^{ijkl..mn}$ with $S_{ij} S_{kl}...$ to t factors; of course only the skew-symmetric part of the product is relevant. Now the rank of the skew-symmetric tensor S_{ij} is an even number $2r$, say. It is known that the skew-symmetric part of the product of t factors $S_{ij} S_{kl}..$ vanishes if $t > r$, and does not vanish if $t \leqslant r$.(8) It follows that $\text{Div}^{2r+2} X^{ijkl..mn}$ must vanish. Since we know (Theorem 2, Remark 2) that the $(\lambda+1)$th divergence of a tensor of order λ vanishes, it follows that the smallest integer P_λ such that the P_λth divergence of every tensor of order λ vanishes, is the smaller of the two numbers $\lambda+1$, $2r+2$. Similarly, the smallest integer Q_λ such that the Q_λth curl of every tensor of order λ vanishes, is the smaller of $n-\lambda+1$, $2r+2$.

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A GENERALIZATION OF THE THEOREM OF VIRIAL.*

BY

K. NAGABHUSHANAM.

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In statistical mechanics the theorem of Virial is the time-average result of

$$\frac{d}{dt}(\Sigma\, m\, \dot{x}\, x) - x\, \frac{d}{dt}(\Sigma\, m\, \dot{x}) = \Sigma\, x\, X.$$

For a dynamical system with n degrees of freedom, there are in general no definite limits to which such averages tend over long periods of time, and the value of $\frac{d}{dt}(\Sigma p_r\, q_r)$ along the trajectory is the only analogue of the Virial theorem. When we consider the $(2n+1)$ — manifold of States and Time for such a system, then there corresponds to time-differential coefficient an infinitesimal transformation (or shortly an I.T.T.) along a trajectory. The trajectories are taken as the singular curves of the curl of a covariant vector X_i of rank † $2n+1$ in the $(2n+1)$ variables (x_i) of the manifold. In what follows the summation convention will be adopted for a repeated index.

In place of $p_r\, q_r$ it is here propesed to consider $X_i\, x_i + Ht$, where $H(x)$ is and $t\,(x)$ is not an integral of motion, (i. e. $H(x)$ admits and $t\,(x)$ does not admit the I.T.T.'s).

The object of this note is firstly to obtain in terms of the variable (x) of the manifold the value along the trajectories of $\delta(X_i\, x_i + Ht)$ or I.T.T.'s, and secondly to indicate how the familiar forms of the Virial theorm can can be obtained from it by specialising the variables and the functions.

* Read at the Indian Mathematical Conference, Waltair, Dec. 1947.

† The rank of (X_i) is the rank of the matrix $\|\text{curl } X_i ;\ X_i\|$.

Writing

$$a_{ik} = \frac{\partial X_i}{\partial x^k} - \frac{\partial X_k}{\partial x^i},$$

the differential equations of the trajectories are

$$a_{ik}\, dx^k = 0, \quad i, k = 1, 2, \ldots . 2n+1.$$

If (ξ^i) is a contravariant vector codirectional with the trajectories, and A^{ik} the cofactors of a^{ik} in the determinant $|a_{ik}|$, then

$$A^{ik} = B^i B^k, *$$

and
$$\frac{\xi^1}{B^1} = \frac{\xi^2}{B^2} = \ldots\ldots = \frac{\xi^{2n+1}}{B^{2n+1}} = \phi,$$

say.

It is easily seen that ϕ is a scalar of weight -1. Since $t(x)$ is not an integral of motion, we have for an I.T.T.

$$\delta x^i = \phi B^i \delta t.$$

The Lagrangian $L(x)$ is given invariantly* by

$$L = \frac{X_i B^i}{\frac{\partial t}{\partial x^i} B^i}.$$

It is seen that

$$\phi = L \left\{ X_i B^i \right\}^{-1}$$

We now state and prove the generalization of this paper as follows:

THEOREM. *If $H(x)$ is an integral of motion, and $t(x)$ is not, then for an* I.T.T.,

$$\delta(X_i x_i + Ht) = \left\{ h + \frac{L^2 \frac{\partial t}{\partial x^i} \left(A^{ii} \right)^{\frac{1}{2}} + L x_i \frac{\partial X_l}{\partial x^i} \left(A^{ll} \right)^{\frac{1}{2}}}{X_k (A^{kk})^{\frac{1}{2}}} \right\} \delta t,$$

which is completely determined as a function of (x), when t is given as a function of (x).

* See K. Nagabhushnam: On the form $\Sigma p_r\, dq^r - Hdt$. *Proc. Ind. Ac. Sc.* 1 556-67. See also G. Kowaleski, *Determinanten-theorie*, p. 124.

PROOF.

$$\delta(X_i\, x_i + Ht) = \frac{\partial X_i}{\partial x^k} x_i\, \phi B^k \delta t + X_i\, \delta x^i + (\delta H)\ t + H\,\delta t$$

$$= a_{ik}\, B^k\ x_i\ \phi\delta t + \frac{\partial X_k}{\partial x^i} B^k\, x_i\ \phi \delta t$$

$$+ X_i\, B^i\ \phi\delta t + (\delta H)\ \delta t + H \delta t$$

which gives the required result when it is noted that along a trajectory

(1) $a_{ik}\, B^k = 0$, (2) $B^k = (A^{kk})^{\frac{1}{2}}$, (3) $\delta H = 0$, (4) $H = h$, a constant, (5) $\phi = L \left\{ X_i\ (A^{ii})^{\frac{1}{2}} \right\}^{-1}$.

We can write the generalized result also as

$$\delta(X_i\, x_i + Ht) = \left[\left\{ \frac{L^2 \frac{\partial t}{\partial x^i} B^i}{X_k\, B^k} - h \right\} + 2h + \frac{L x_i \frac{\partial X_l}{\partial x^i} B^l}{X_k\, B^k} \right] \delta t,$$

in which form the expression in the inside brackets on the right reduces to $2U$ in particular cases, where U is the work function.

We particularize as follows :

If

$$x^r = q^r,\ x^{r+n} = p_r,\ r = 1,\ 2 \ldots n\ ;\ x^{2n+1} = t\ ;$$

$(X_i) = (p_1,\ p_2, \ldots p_n,\ 0,\ 0 \ldots 0,\ -H)$; $H = h$, the energy constant along the trajectory, then we have

(1) $B^r = \dfrac{\partial H}{\partial p_r}, B^{n+r} = -\dfrac{\partial H}{\partial q^r},\ B^{2n+1} = 1$;

(2) $X_k B^k = p_r \dfrac{\partial H}{\partial p_r} - H = L$;

(3) $\left(\dfrac{\partial t}{\partial x^i}\right) = (0 \ldots . 0,\ 0 \ldots\ 0,\ 1)$;

(4) $\dfrac{\partial X_l}{\partial x^i} x_i\ B^l = p_r \dfrac{\partial H}{\partial p_r} - q_r \dfrac{\partial H}{\partial q_r} - p_r \dfrac{\partial H}{\partial p_r} = -q_r \dfrac{\partial H}{\partial q_r}$

$= -q_r\ T_r + q_r\ U_r,$

where $T_r = \dfrac{\partial T}{\partial q^r}$, and $U_r = \dfrac{\partial U}{\partial q_r}$.

In this case

$$\delta(X_i\, x_i + Ht) = \delta(p_r\, q_r) = [2(U+h) - q_r\, T_r + q_r\, U_r]\, \delta t,$$

a result essentially the same as that derived by Aurel Wintner*. The other familiar forms of the Virial theorem can be obtained from the generalized result by making the corresponding restrictions on (x), (X_i), H, T, and U.

* *Celestial Mechanics*, pp. 114-115 §§58, 59.

A SYSTEM FOR GENERAL SET THEORY

BY

R. SHUKLA, *University of Patna.*

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In this note we construct an *enumerable model* for what we shall call Godel's general set theory (*allgemeinen Mengenlehre*). It is essentially Godel's system* Σ with the postulates of the existence of particular sets and classes being omitted. Thus this general system differs from Σ in not having an axiom of infinity as a postulate and in replacing Godel's B_1 and B_5 by slightly weaker axioms. Axioms C_2 and C_3 are strengthened. From the model it will be possible to see that the axiom of choice is independent of this general set theory. For the model, the dyadic relation ε will be interpreted in such a way that for certain classes X and Y, $X \varepsilon Y$ will be meaningless.

The following is the general set theory proposed for examination. All the notations and conventions of Godel are employed.

GROUP A.

1. cls (x)
2. $X \varepsilon Y . \supset . m(X)$.
3. $(u)[u \varepsilon X . \equiv . u \varepsilon Y] . \supset . X = Y$.
4. $(x)(y)(\exists z)[u \varepsilon z . \equiv : u = x . V . u = y]$.

GROUP B.

1′. $(y)(\exists A)(x)[<xy> \varepsilon A . \equiv . x \varepsilon y]$

2. $(A)(B)(\exists C)(u)[u \varepsilon C . \equiv : u \varepsilon A . u \varepsilon B]$

3. $(A)(\exists B)(u)[u \varepsilon B . \equiv . \sim (u \varepsilon A)]$.

4. $(A)(\exists B)(x)[x \varepsilon B . \equiv . (\exists y)(<yx> \varepsilon A)]$

5′. $(A)(\exists B)(xy)[<yx> \varepsilon B . \supset . x \varepsilon A]$

6. $(A)(\exists B)(xy)[<xy> \varepsilon B . \equiv . <yx> \varepsilon A]$

7. $(A)(\exists B)(x, y, z)[<xyz> \varepsilon B . \equiv . <yzx> \varepsilon A]$

8. $(A)(\exists B)(x, y, z)[<xyz> \varepsilon B . \equiv . <xzy> \varepsilon A]$

* Godel : *Consistency of the continuem hypothesis*, Annals of Math. Studies No. 3 Princeton.

GROUP C.

The first axiom of Σ is the axiom of infinity which is omitted.

2′. $(x)\ (\exists y)\ (u, v)\ [u \varepsilon v.\ v \varepsilon x :\equiv.\ u \varepsilon y]$

3′. $(x)\ (\exists y)\ [u \leqslant x .\equiv.\ u \varepsilon y]$

4. $(x, A)\ \{U_n(A) .\supset.\ (\exists y)\ (u)$
$[u \varepsilon y .\equiv.\ (\exists v)\ [v \varepsilon x . < uv > \varepsilon A]]\}$

AXIOM D. $\sim \text{Ex}(A) .\supset. (\exists u)\ [u \varepsilon A.\ \ \text{Ex}(uA)]$.

NOTE: cls(x) means that x is a class; $m(X)$ means that X is a set. $U_n(A)$ means that A is single-valued and Ex(uA) means that u and A have no elements in common.

Now we proceed to construct an enumerable model for the above system. If the system is consistent, the possibility of an enumerable model (or realization) is implied by the famous Löwenheim-Skolem theorem.

Our model will consist of the integers 0, 1, 2, 3.... There are no *proper classes* in the model and all the integers 0, 1, 2, 3,... will be sets. Hence all the capital and small Roman letters will denote sets.

We interpret $< x y >$ as $2^{x-1}(2y-1)$.

Similarly $< xyz > = < x < yz > >$ as $2^{x-1}(2w-1)$, where $w = 2^{y-1}(2z-1)$. Now we define ε relation between the natural numbers. For this purpose, we look upon our elements as each uniquely represented in powers of 2, the coefficients being 0's and 1's only, e.g. $1 = 1.2^0, 2 = 0.2^0+1.2^1, 3 = 1.2^0+1.2^1, 4 = 0.2^0+0.2^1+1.2^2$, $5 = 1.2^0+0.2^1+1.2^2$, and so on,

while 0 has no such representation. Each representation is looked upon as having terminated at a definite finite stage and therefore infinite representations (of the type $3 = 1.2^0+1.2^1+0.2^3+0.2^4+\ldots\ldots$) are not permitted. Now suppose that a general integer x is represented as follows

$$x = a_0 2^0+a_1 2^1+a_2 2^2+a_3 2^3+\ldots+a_k.2^k,$$

where k is the highest power of 2 used in the representation, obviously $a_k = 1$.

Now if y is any of the integers, $y \epsilon x$ is meaningful if and only if $2^y < 2^k$, i.e. if and only if $y < k$. For $y \geqslant k$ $y \epsilon x$ is meaningless. Thus for all u, $u \epsilon 0$ is not only false but meaningless. When meaningful, $y \epsilon x$ is true if the coefficient of 2^y in the representation of x is 1, and false if the coefficient is 0.

Now if x is represented as above, consider the integer y such that $y = b_0.2^1 + b_1.2^1 + b_2.2^2 + \ldots\ldots b_k.2^k$, where $b_n = 0$ or 1 according as $a_n = 1$ or 0, for all $n < k$ and $b_k = 1$. Thus y is such that (u) $[u \epsilon x . \equiv . \sim (u \epsilon y)]$ and therefore y will be called the complement of x. Obviously then x is the complement of y. It is clear that for any given integer x, we can find another integer z such that $x \varepsilon z$ and therefore every member considered is a set.

Now it is easy to verify the axioms. Since the above representation is unique, A_1, A_2, A_3, are clearly satisfied. For A_4 suppose $y > x$ then take $z = 0.2^0 + 0.2^1 + \ldots + 1.2^x + \ldots\ldots + 1.2^y$ and so A_4 is satisfied.

Group B.

B_1. For any y let $x_1, x_2, \ldots x_n$ be its members, i.e. for each r, $x_r \epsilon y$. Then take

$$A = 2^{2^{x_1 - 1}.(2y-1)} + 2^{2^{x_2 - 1}.(2y-1)} + \ldots + 2^{2^{x_n - 1}.(2y-1)}.$$

B_2 and B_3 are obvious. For B_4 let $x_1, x_2, \ldots x_n$ be the elements of A and let them each be uniquely represented in the form $2^{p-1}(2q-1)$ and let the corresponding p's be $p_1, p_2, \ldots p_n$ (all not necessarily different), then take $B = 2^{p_1} + 2^{p_2} + \ldots + 2^{p_n}$ (equal p's taken only once). For B_5, if $x \varepsilon A$ take B such that $< 1\, x > \varepsilon B$. It should be noted that B is not unique. In Godel's system Σ also, B is not considered to be unique (compare Godel's remark on p. 5. loc. cit).

B_6, B_7 and B_8 are also obvious. It is interesting to note that 0 does not need to have a complement.

GROUP C.

C_2' and C_3' (which are the strongest forms of the corresponding C_2 and C_3 of Godel) are also satisfied. For C_4 let A be represented as

$$A = 2^{2^{v_1-1}.(2u_1-1)} + 2^{2^{v_2-1}.(2u_2-1)} + \ldots + 2^{2^{v_k-1}.(2u_k-1)}.$$

Now out of $v_1, v_2, \ldots v_k$ choose those which are elements of x and call them $v_{r_1}, v_{r_2}, \ldots v_{r_p}$ and then take $y = 2^{v r_1} + 2^{v r_2} + \ldots + 2^{v r_p}$. It is interesting to note that the condition for A to be single-valued may be waived altogether.

AXIOM D. For any set A different from 0, take u to be the smallest member of A (u may be 0 possibly). Then the axiom is satisfied. That a smallest member exists is obvious.

REMARKS :

(i) Consider the axiom of choice

$$(\exists A)\{u_n(A).(x)\ [\sim \mathrm{Em}(x) .\supset. (\exists y)\ [y \epsilon x . < yx > \epsilon A]]\}.$$

Obviously there is no A in the model which satisfies the above conditions, simply because for no A, $< yx > \epsilon A$ for all non-empty x. Thus the axiom of choice is independent of the rest of the axioms. It is to be noted that even the weakest form of the axiom of choice is not satisfied.

(ii) In the Godel system $X \epsilon Y$ is meant to be meaningful for all X and Y; but in our interpretation we had to restrict the meaningfulness of $X \epsilon Y$. This savours of a theory of types.

Here I take the opportunity of expressing my thanks to Prof. Dienes for helpful criticism.