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Page 257, last but one line, for *quartic* read *quintic*.

ON INTEGRAL FUNCTIONS OF ORDER ONE AND OF FINITE TYPE

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1. *Introduction.* Let $f(z)$ be an integral function and $M(r)$ the maximum of $|f(z)|$ on $|z|=r$. The order ρ and type k of $f(z)$ are defined by the relations

$$\rho=\overline{\lim_{r\to\infty}}\frac{\log\log M(r)}{\log r}\,;\ k=\overline{\lim_{r\to\infty}}\frac{\log M(r)}{r^{\rho}}. \tag{1}$$

If ρ is finite, the function $f(z)$ is said to be of minimal, normal or maximal type according as k vanishes, is a finite positive number, or is infinite. We shall define the number l by the relation

$$l=\underline{\lim_{r\to\infty}}\frac{\log M(r)}{r^{\rho}}, \tag{2}$$

which might be called the lower type in contrast to k which might be termed the upper type. It is evident that $l\leqslant k$. When a type is spoken of without qualification we shall mean the upper type. When there are several functions f, g, etc., we shall denote the corresponding quantities $l, k, M(r)$, etc. by $l(f), l(g)$, etc. and $M(f), M(g)$, etc. where the variable r is understood.

1.1. Polya* has suggested and Tschakaloff† and others have proved the following theorem.

THEOREM A. *Let $f(z)$ be an integral function of order one and minimal type. Let $f(n)=O(1)$, $n=0, \pm1, \pm2,\ldots$. Then $f(z)$ reduces to a constant.*

1·2. The above theorem calls for some remarks. The function $\sin \pi z$ vanishes at $z=0, \pm1, \pm2,\ldots$, and it is known that its order is one and its type π. It can be proved‡ that any function of order one vanishing at these points and having a type less than π is identically zero. The above theorem states roughly that a

* *Jahrsbericht der Deutschen Mathematiker Vereinigung*, Vol. 40 (1931), 2te Abteilung, p. 85, problem 105.

† *Ibid.* Vol. 43, 2te Abteilung, pp. 10, 11, 67.

‡ *See* Polya and Szego, *Aufgablen und Losungen*, Vol. I, III, Abschitt, p. 149.

function of order one bounded at these points cannot have a type substantially less than π unless it reduces to a constant, π being the type of $\sin \pi z$ which, being zero at these points, is certainly bounded. The object of this paper is to extend the above theorem to a general class of functions and to examine the type of a function bounded at a given set of points.

2. Let $0<\lambda_1<\lambda_2\ldots<\lambda_n\ldots$ be a sequence of numbers tending to infinity. Let $n(r)$ be the number of these $[\lambda_n]$ not exceeding r. We set

$$\phi(z) = \prod_{\nu=1}^{\infty} \left(1-\frac{z^2}{\lambda_n^2}\right) \tag{3}$$

and make the following hypothesis on $[\lambda_n]$:

(i) there exist two numbers $0<A\leqslant B$ such that

$$A = \varliminf_{r\to\infty} \frac{n(r)}{r} \leqslant \varlimsup_{r\to\infty} \frac{n(r)}{r} = B; \tag{4}$$

(ii) there is an integer $p\geqslant 0$ such that

$$\sum_{\nu=1}^{\infty} \left| \frac{1}{\phi'(\lambda_\nu)\lambda_\nu^{p+1}} \right| < \infty. \tag{5}$$

For instance, if $\lambda_n=n$, (4) is true with $A=B=1$ and (5) with $p=2$. We shall show below that if (4) holds, (3) represents an even integral function of order one and type not exceeding πB. The first part of the inequality (4) precludes the function (3) from being of minimal type and so we can use the function as a comparison function for investigating the types of other functions bounded at $z=\pm\lambda_n$. We shall refer to $\phi(z)$ as the base function with reference to the sequence $[\lambda_n]$.

2·1. Let $f(z)$ be an integral function of order one. Let $f(\lambda_n)=y_n$, $f(-\lambda_n)=y_{-n}$ and let the sequence $[y_{\pm n}]$ be bounded. The following two theorems constitute the chief results of this paper.

THEOREM 1. *Let* $[y_n]$, *in addition to being bounded, satisfy one of the following conditions:*

(α) *the real parts of* $\dfrac{y_n+y_{-n}}{\phi'(\lambda_n)}$ *and* $\dfrac{y_n-y_{-n}}{\phi'(\lambda_n)}$ *do not change sign as n varies, not all these real parts vanishing;*

(β) *the imaginary parts of* $\dfrac{y_n+y_{-n}}{\phi'(\lambda_n)}$ *and* $\dfrac{y_n-y_{-n}}{\phi'(\lambda_n)}$ *do not change sign as n varies, not all these imaginary parts vanishing.*

Under one of these conditions the type of $f(z)$ *cannot be less than* $l(\phi)$, *the lower type of the base function* $\phi(z)$.

THEOREM 2. *Let* $[\alpha_\nu]$, $\nu=1, 2, \ldots,$ *be a sequence of numbers so that* $|\alpha_\nu| \to \infty$ *as* $\nu \to \infty$. *Let* $E(\lambda, \alpha)$ *denote the set of points*

$$z = \pm \alpha_\nu \cdot \lambda_n, \quad \nu, n = 1, 2, 3, \ldots .$$

Let $f(z)$ *be a function of order one and* minimal *type. Then it reduces to a constant if it is bounded at the set of points* $E(\lambda, \alpha)$.

2·2. Theorem 2 is the direct generalisation of Theorem A which corresponds to $\lambda_n = n$, $\alpha_\nu = \nu$. Theorem 1 gives a definite lower limit for the type of functions of order one bounded in a particular manner at the zeros of the base function (3). Before proceeding to the proof of these theorems we shall establish some properties of the base function ϕ which are used in the investigation.

3. THEOREM 3. *If* (4) *holds, we have*

$$0 < \pi A \leqslant \varliminf_{r \to \infty} \frac{\log M(\phi)}{r} \leqslant \varlimsup_{r \to \infty} \frac{\log M(\phi)}{r} \leqslant \pi B, \tag{6}$$

so that $\phi(z)$ *is an integral function of order one and*

$$l(\phi) \geqslant \pi A, \quad \text{and} \quad k(\phi) \leqslant \pi B. \tag{7}$$

Moreover $\sum \frac{1}{\lambda_n^\lambda}$ *is convergent for* $\lambda > 1$ *and divergent for* $\lambda \leqslant 1$.

PROOF: It is easy to see that for a $\lambda > 0$, the series $\sum^{\infty} \frac{1}{\lambda_n^\lambda}$ and $\int^{\infty} \frac{n(u)}{u^{1+\lambda}}\, du$ converge or diverge at the same time. Now using (4) we find that the integral $\int^{\infty} \frac{n(u)}{u^{1+\lambda}}\, du$ converges for $\lambda > 1$ and diverges for $\lambda \leqslant 1$. Hence $\phi(z)$ given by (3) is an integral function. Moreover,

$$\begin{aligned} \log M(\phi) &= \sum_{\nu=1}^{\infty} \log\left(1 + \frac{r^2}{\lambda_\nu^2}\right) \\ &= \int_1^{\infty} \log\left(1 + \frac{r^2}{t^2}\right) dn(t) \\ &= 2r^2 \int_1^{\infty} \frac{n(t)\, dt}{t(t^2 + r^2)} - n(1) \log(r^2 + 1) \end{aligned} \tag{8}$$

since $\frac{n(u)}{u^2} \to 0$ by (4). Again using (4) we get from (8)

$$2Ar^2\int_1^\infty \frac{dt}{t^2+r^2}+O(\log r)\leqslant \log M(\phi)\leqslant 2Br^2\int_1^\infty \frac{dt}{t^2+r^2}+O(\log r)$$

which gives (6) and consequently (7). So the theorem is proved.

3·1. The following result is known* and so we state it in the form of a lemma.

LEMMA 1. *Let* $\varepsilon>0$ *and* $0<\eta<\varepsilon$. *Then there is a finite positive number* $Q=Q(\varepsilon)$ *so that*

$$|\phi(z)|>\exp(-r^{1+\varepsilon}),\ |z|=r, \tag{9}$$

except for a set of values of r *over which the variation of* r^η *does not exceed* Q.

3·2. THEOREM 4. *Let* $0<\xi<1$. *Then on the lines* $\theta=\theta_0$ *where* $\cos 2\theta_0=-(1-\xi)^2$,

$$|\phi(z)|>\exp\left\{(1-\xi)[l(\phi)-\varepsilon]r\right\} \tag{10}$$

where $\varepsilon>0$ *is as small as we please and* $|z|=r\geqslant r_0=r_0(\varepsilon)$.

PROOF: On the lines $\cos 2\theta_0=-(1-\xi)^2$, we find

$$|\phi(z)|^2=\prod_{\nu=1}^{\infty}\left\{1+2(1-\xi)^2\frac{r^2}{\lambda_\nu^2}+\frac{r^4}{\lambda_\nu^4}\right\}$$
$$\geqslant\prod_{\nu=1}^{\infty}\left\{1+(1-\xi)^2\frac{r^2}{\lambda_\nu^2}\right\}^2$$
$$\geqslant\left\{M[r(1-\xi)]\right\}^2$$

where $M(r)=\max_{|z|=r}|\phi(z)|$. The result now follows from the definition of the lower type.

4. We now proceed to establish the fundamental formula on which the proof of the two theorems is based. It is suggested by the method of proof given by Tschakoloff for Theorem A. It is based on the general formula for the representation of a meromorphic function in terms of its principal parts.

THEOREM 5. *Let* $[y_n]$, *the values of* $f(z)$ *at* $z=\pm\lambda_n$ *be bounded. Let the type of* $f(z)$, *which is of order one, be less than* $l(\phi)$, *the lower type of the base function. Let*

$$g(z)=\sum_{\nu=1}^{\infty}\frac{1}{\phi'(\lambda_\nu)}\left\{\frac{y_\nu}{z-\lambda_\nu}+\frac{(-1)^{p+1}y_{-\nu}}{z+\lambda_\nu}\right\}\left(\frac{z}{\lambda_\nu}\right)^p \tag{11}$$

* For a proof see R. Nevanlinna, *Fonctions Meromerphes*, Paris, pp. 32, 41.

which, owing to (5), *represents a meromorphic function with poles at* $z=\pm\lambda_\nu$ *and residues* $y_{\pm\nu}/\phi'(\pm\lambda_\nu)$. *Then*

$$\frac{f(z)}{\phi(z)}=g(z)+C_{p-1}(z) \tag{12}$$

where $C_{p-1}(z)$ *is a polynomial of degree* $p-1$ *at most. Here* p *is the integer in* (5).

PROOF: We set

$$G(z)=\frac{f(z)}{\phi(z)}-g(z).$$

Then from (10), we find that $G(z)$ is an integral function. We shall first require an expression for $M(G)$ for which we need the following

4·1. LEMMA 2. *Let* $h>0$ *be given. Let* z, $|z|=r$, *lie outside the circles with centres* $z=\pm\lambda_\nu$ *and radii* λ_ν^{-h}. *Then there is a constant* H *so that*

$$|g(z)|\leqslant Hr^{p+h+1}. \tag{13}$$

PROOF: We have, if $|y_n|\leqslant D$,

$$|g(z)|\leqslant 2Dr^p\sum_{\nu=1}^{\infty}\frac{1}{\lambda_\nu^p|\phi'(\lambda_\nu)|}\,\frac{1}{|\lambda_\nu-z|}$$

$$\leqslant 2Dr^p\left\{\sum_{\nu=1}^{\nu_0}+\sum_{\nu=\nu_0+1}^{\infty}\right\}$$

where ν_0 is such that $\lambda_{\nu_0}\leqslant\beta r<\lambda_{\nu_0+1}$ where $\beta>1$.

Then we have

$$|g(z)|\leqslant 2Dr^p\left\{\sum_{\nu=1}^{\nu_0}\frac{\lambda_\nu^{1+h}}{\lambda_\nu^{p+1}|\phi'(\lambda_\nu)|}+\frac{\beta}{\beta-1}\sum_{\nu=\nu_0+1}^{\infty}\frac{1}{\lambda_\nu^{p+1}\phi'(\lambda_\nu)}\right\}$$

$$\leqslant Hr^{p+h+1}$$

in virtue of (5) and the fact $\lambda_{\nu_0}\leqslant\beta r$. So the lemma is proved.

4·2. We also need the following lemma.

LEMMA 3.* *Let* $\beta>1$. *Then there is a sequence* $[r_n]$, $r_n\to\infty$, $r_{n+1}/r_n<\beta$ *so that for* $|z|=r_n$, *the results* (9) *and* (13) *of Lemmas* 1 *and* 2 *hold simultaneously.*

PROOF: Consider the interval $(r\leqslant t\leqslant r\sqrt{\beta})$. The total length of the exceptional intervals of Lemma 2 lying in $(r, r\sqrt{\beta})$

is $4\sum_{\nu=n(r)}^{n(\sqrt{\beta}r)}\lambda_\nu^{-h}$ which is of the form $O(r^{1-h})$. In the remaining intervals the total variation of r^η is greater than

$$(\beta^{\eta/2}-1)\,r^\eta-2\sum_{\nu=n(r)}^{n(\sqrt{\beta}r)}[(\lambda_\nu+\lambda_\nu^{-h})^\eta-(\lambda_\nu-\lambda_\nu^{-h})^\eta]$$

which is of the form $(\beta^{\eta/2}-1)\,r^\eta+O(r^{\eta-h})$ and so tends to infinity. Hence by Lemma 1 there is at least one r_n in $(\beta^{n/2},\sqrt{\beta}\beta^{n/2})$ for which (9) and (13) hold at the same time and obviously $r_{n+1}/r_n<\beta$, if n is sufficiently large. So the lemma is proved.

4·3. We can now find an expression for $M(G)$. We have by Lemmas 1, 2 and 3, for $|z|=r_n$,

$$|M(G)|\leqslant\exp\left\{[k(f)+\varepsilon]r_n+r_n^{1+\varepsilon}\right\}+Hr_n^{p+h+1}$$
$$\leqslant\exp[r_n^{1+2\varepsilon}].$$

But G is an integral function and so $M(G)$ is steadily increasing while $r_{n+1}/r_n<\beta$. Hence for *all* $r\geqslant r_0$ we have

$$M(G)\leqslant\exp[\beta^{1+2\varepsilon}r^{1+2\varepsilon}]<\exp[r^{1+3\varepsilon}], \tag{14}$$

which is the required expression. Our next step is to find an upper bound for $|G|$ on the lines $\cos 2\theta_0=-(1-\xi)^2$ of Theorem 4. We already know one for $|\phi(z)|^{-1}$ given by (10). We shall find an upper bound for $g(z)$ in

4·4. LEMMA 4. *On the lines* $\cos 2\theta_0=-(1-\xi)^2$,

$$|g(z)|=o(|z|^p). \tag{15}$$

PROOF: We have, if $|z|=r$,

$$|g(z)|\leqslant 2Mr^p\sum_{\nu=1}^{\infty}\frac{1}{\lambda_\nu^p|\phi'(\lambda_\nu)|\,|z-\lambda_\nu|}. \tag{16}$$

But $|z-\lambda_\nu|^2=r^2-2r\lambda_\nu\cos\theta_0+\lambda_\nu^2$ where $2\cos^2\theta_0=1-(1-\xi)^2$, so that $|\cos\theta_0|=\sqrt{\dfrac{1-(1-\xi)^2}{2}}=\mu$, say where $\mu\to 0$ as $\xi\to 0$. Hence $|z-\lambda_\nu|^2\geqslant r^2+\lambda_\nu^2-2\mu r\lambda_\nu\geqslant(1-\mu)(r^2+\lambda_\nu^2)$. So (16) gives

$$|g(z)|\leqslant\frac{2Mr^p}{\sqrt{1-\mu}}\sum_{\nu=1}^{\infty}\frac{1}{\lambda_\nu^{p+1}|\phi'(\lambda_\nu)|}\,\frac{1}{\sqrt{1+r^2/\lambda_\nu^2}},$$

from which it is easy to see that $\frac{|g(z)|}{r^p} \to 0$ which is equivalent to (15).

4·5. Resuming the proof of Theorem 5 we find that on the lines $\cos 2\theta_0 = -(1-\xi)^2$,

$$|G(z)| \leqslant \exp\left\{ [k(f)+\varepsilon]\, r - [l(\phi)-\varepsilon](1-\xi) r \right\} + o(r^p) \quad (17)$$

in virtue of (10) and (15). Since $k(f)-l(\phi)<0$, by hopothesis, we can find a $\rho>0$ so that on $\cos 2\theta_0 = -(1-\xi)^2$,

$$|G(z)| \leqslant e^{-\rho r} + o(r^p) \quad (18)$$

in virtue of (17). Now we apply to the function $G(z)$ the following classical theorem of Phragman and Lindelof* which we quote in the form of a lemma.

4·6. Lemma 5. *Let $f(z)$ be a function regular at infinity and let it be bounded as $|z| \to \infty$ on the sides of an angle of magnitude π/γ. If the order of $f(z)$ is less than γ in the angle then $f(z)$ is bounded uniformly as $|z| \to \infty$ in the whole angle.*

4·7. Now the order of $G(z)$ in any angle in the z—plane cannot exceed $1+3\varepsilon$ by (14), ε being as small as we please. Now if ξ be chosen so that on $\cos 2\theta_0 = -(1-\xi)^2$, (18) holds and we denote by π/γ the greatest of the angles between the consecutive lines of the set $\cos 2\theta_0 = -(1-\xi)^2$, it is evident that $\lambda>1$. So if we choose ε so that $1+3\varepsilon<\lambda$ we can apply Lemma 5 to the function $G(z)/z^p$ which by (18) is bounded on the said lines. Hence we conclude that $G(z)/z^p$ is bounded as $|z| \to \infty$ which is possible only if the integral function $G(z)$ reduces to a polynomial of degree p at most. But on the lines $\cos 2\theta_0 = -(1-\xi)^2$ we find that $G(z) = o(r^p)$ and so the degree of $G(z)$ cannot exceed $p-1$. This completes the proof of Theorem 5.

5. Now we can proceed to the proof of Theorem 1.

Proof of Theorem 1: Suppose the theorem is not true. Then we can suppose $k(f)<l(\phi)$. Under this assumption the expression (12) is valid since by hypothesis $[y_{\pm n}]$ is a bounded sequence. We shall first show that it is sufficient to consider the case when $f(z)$ is an even function, real at $z=\pm\lambda_n$ and satisfying the condition (α). For if $f(z)$ were not real, we can take $\sigma(z) = \frac{f(z)+\overline{f(\bar{z})}}{2}$ which is an integral function whose

* *See* G. Valiron, *Lectures on Integral Functions*, p. 125, Theorem 34.

type satisfies the condition $h(\sigma)\leqslant k(f)<l(\phi)$ and which is real at $z=\pm\lambda_n$. So we can suppose $f(z)$ is real. If it were not even we take $\psi(z)=\frac{f(z)+f(-z)}{2}$ which is even and satisfies the condition $k(\psi)\leqslant k(f)<l(\phi)$. Again it is sufficient to suppose (α) is satisfied for if not we can deal with $if(z)$. It is easy to see that if $f(z)$ satisfies (α) so do the functions $\sigma(z)$ and $\psi(z)$. Hence it is sufficient to treat the case when $f(z)$ is even and real at $z=\pm\lambda_n$, so that $y_n=y_{-n}$ and the condition (α) is true. Turning to formula (12) we can suppose without loss of generality that p is odd equal to $2\lambda+1$. Putting $z=ir$ in (12) we get, using (α),

$$M(f)\geqslant|f(ir)|$$

$$\geqslant M(\phi)\left|C_{2\lambda}(ir)+2(ir)^{2\lambda+2}\sum_{\nu=1}^{\infty}\frac{y_\nu}{\lambda_\nu^{2\lambda+1}\phi'(\lambda_\nu)}\,\frac{-1}{\lambda_\nu^2+r^2}\right|$$

$$\geqslant M(\phi)\left|r^{2\lambda+2}\sum_{\nu=1}^{\infty}\left|\frac{2y_\nu}{\phi'(\lambda_\nu)}\right|\frac{1}{\lambda_\nu^{2\lambda+1}(\lambda_\nu^2+r^2)}+h_{2\lambda}(r)\right|$$

taking the real part alone; here $h_{2\lambda}(r)$ is a polynomial with real coefficients of degree 2λ at most. We now require a lower bound for the expression multiplying $M(\phi)$ and for this purpose we prove the following lemma.

5·1 LEMMA 6. *Let Σk_ν be a convergent series of non-negative terms such that $\Sigma k_\nu\neq 0$. Let $H_{2\lambda}(r)$ be an even polynomial with real coefficients and h any integer so that $\Sigma k_\nu\lambda_\nu^{h-2}$ converges. Then there is an integer P positive or negative so that*

$$\varliminf_{r\to\infty} r^P\left|r^{2\lambda+2}\sum_{\nu=1}^{\infty}\frac{k_\nu\lambda_\nu^h}{\lambda_\nu^2+r^2}+H_{2\lambda}(r)\right|>0.$$

PROOF: Let $H_{2\lambda}(r)=a_{2\lambda}r^{2\lambda}+a_{2\lambda-2}r^{2\lambda-2}+\ldots+a_0$ and

$$v(r)=r^2\sum_{\nu=1}^{\infty}\frac{k_\nu\lambda_\nu^h}{\lambda_\nu^2+r^2}+a_{2\lambda}+\frac{a_{2\lambda-2}}{r}+\ldots+\frac{a_0}{r^{2\lambda}}. \tag{19}$$

If $\Sigma k_\nu\lambda_\nu^h$ diverges we have

$$r^2\sum_{\nu=1}^{\infty}\frac{k_\nu\lambda_\nu^h}{\lambda_\nu^2+r^2}>\tfrac{1}{2}\sum_{\nu=1}^{n(r)}k_\nu\lambda_\nu^h\to\infty \text{ as } r\to\infty.$$

So the lemma is true in this case. If $\Sigma k_\nu \lambda_\nu^h$ converges and is not equal to $-a_{2\lambda}$ then again the lemma is true. If $\Sigma k_\nu \lambda_\nu^h + a_{2\lambda} = 0$ we get by (19)

$$-r^2 v(r) = r^2 \sum_{\nu=1}^{\infty} \frac{k_\nu \lambda_\nu^{h+2}}{\lambda_\nu^2 + r^2} - a_{2\lambda-2} - \frac{a_{2\lambda-4}}{r^2} - \ldots$$

which is of the same form as (19) with $h+2$ for h and $2\lambda-2$ for 2λ. Since $\Sigma k_\nu \lambda^{h+2-2}$ converges, we can repeat the argument and in a finite number of steps arrive at the result stated.

5·2. Returning to the proof of Theorem 1 we find from (12) that $C_{2\lambda}(z)$ is an even polynomial since $f(z)$ is an even function. Therefore $h_{2\lambda}(r)$ occurring in the inequality for $M(f)$ is even. Also under (α) all y_n do not vanish. Hence we can apply Lemma 6 with $h=1$, $k_\nu = \left| \frac{y_\nu}{\phi'(\lambda_\nu)\, \lambda^{2\lambda+2}} \right|$. Thus we get that

$$M(f) \geqslant D \frac{M(\phi)}{r^p}$$

where $D > 0$ is a constant. We deduce at once that

$$k(f) \geqslant l(\phi)$$

which contradicts the hypothesis. So the theorem is true.

6. We now proceed to the proof of Theorem 2. As before we can suppose $f(z)$ to be even, since otherwise we can consider the two functions $\frac{f(z)+f(-z)}{2}$ and $\frac{f(z)-f(-z)}{2z}$ both of which are even and of minimal type and also bounded at $E(\lambda, \alpha)$. If the theorem be proved for these functions we should have

$$f(z)+f(-z) = C, \text{ a constant}$$

and

$$f(z)-f(-z) = Dz,$$

where, since $f(z)-f(-z)$ is bounded at a set of points tending to infinity, we must have $D=0$. Adding up we find that $f(z)$ itself is a constant. So we shall suppose $f(z)$ to be even and take $p = 2\lambda+1$ as in Theorem 1.

Let $a \neq 0$ and consider $\phi(z/a)$. Its lower type is $l(\phi)/|a|$ and so is greater than the type of $f(z)$ which is minimal. Hence we can use $\phi(z/a)$ as the base function. If

$$\sigma(z) = \phi(z/a), \quad \sigma'(a\lambda_n) = \frac{1}{a}\phi'(\lambda_n)$$

we get by (12),

$$\frac{f(z)}{\phi\left(\frac{z}{a}\right)}=C_{2\lambda}(z,a)+\sum_{\nu=1}^{\infty}\frac{z^{2\lambda+1}f(a\lambda_n)}{a^{2\lambda}\phi'(\lambda_n)\lambda^{2\lambda+1}}\left\{\frac{1}{z-a\lambda_\nu}+\frac{1}{z+a\lambda_\nu}\right\} \tag{20}$$

where $C_{2\lambda}(z,a)$ is a polynomial of degree 2λ whose coefficients depend on a. Putting $a=\alpha_\nu$ in (20) and letting $\nu\to\infty$ we find, since $f(z)$ is bounded at $E(\lambda,\alpha)$,

$$f(z)=\lim_{\alpha_\nu\to\infty} C_{2\lambda}(z,\alpha_\nu)$$

$$=\lim_{\nu\to\infty}[a_{2\lambda,\nu}z^{2\lambda}+a_{2\lambda-1,\nu}z^{2\lambda-1}+\ldots],$$

which can happen only if $a_{k,\nu}\to a_k$ as $\nu\to\infty$. Hence $f(z)$ is a polynomial of degree 2λ at most which being bounded at $E(\lambda,\alpha)$ must reduce to a constant.

6·1. We shall consider some particular cases of Theorem 2. First let $\lambda_n=n$. Then

$$\phi(z)=\frac{\sin \pi z}{\pi z},$$

$$l(\phi)=k(\phi)=\pi.$$

Here

$$\phi'(\lambda_n)=\frac{(-1)^n}{n}$$

and so we can take $p\geqslant 2$ in (5). So Theorem 2 holds in this case and reduces to Theorem A when $\alpha_\nu=\nu$ as has already been remarked.

6·11. Next let $\lambda_n=a+nb$, $a\geqslant 0$, $b>0$. Here we have

$$\phi(z)=\frac{\left\{\Gamma\left(\frac{a}{b}\right)\right\}^2}{\left(1-\frac{z^2}{a^2}\right)\Gamma\left(\frac{a}{b}+\frac{z}{b}\right)\Gamma\left(\frac{a}{b}-\frac{z}{b}\right)}$$

and

$$l(\phi)=k(\phi)=\frac{\pi}{b}.$$

We find

$$\phi'(\lambda_n)=(-1)^n n^{-(2a/b+1)}[1+o(1)]$$

and so we can take p as the integer next to $\frac{2a}{b}+2$. Taking $\alpha_\nu=\nu$ we get the following

THEOREM 2-A. *Let $f(z)$ be a function of order one and minimal type. Let it be bounded at the points $z=am+bn$ where $m, n=\pm 1, \pm 2, \ldots$. Then $f(z)$ is a constant.*

7. REMARKS ON THEOREM 1. It is easy to verify that $(-1)^n\phi'(\lambda_n)$ is positive so that the hypothesis (α) and (β) could be replaced by

(α') the real parts of $(-1)^n(y_n \pm y_{-n})$ do not change sign and all these real parts do not vanish;

(β') the same is true of the imaginary parts of $(-1)^n(y_n \pm y_{-n})$.

It can be proved that if the real as well as the imaginary parts of $(-1)^n(y_n \pm y_{-n})$ keep constant sign we can relinquish the condition that not all these are zero if in the conclusion we add that $f(z) \equiv 0$ if $k(f) < l(\phi)$.

7·1. It may be noted that the second part of the inequality (4) has been made use of to prove that $\phi(z)$ is of order one and that $n(u)/u^2 \to 0$ in (8). Both these will be true if it is merely assumed that $\phi(z)$ is of order one. So we can replace the second part of the inequality (4) by the hypothesis that $\phi(z)$ is of order one and all the results will be true.

7·2. We conclude with the statement of a result allied to problems on Diophantine approximations derivable from Theorem 1.

Let $[\lambda_n]$ be the sequence satisfying the first part of inequality (4) that is,

$$\varliminf_{r \to \infty} \frac{n(r)}{r} = A > 0.$$

Let ρ be any real number such that $0 < \rho < A\pi$. Consider the function

$$f(z) = e^{i\rho z}$$

which is bounded on the whole of the real axis and therefore also at $z = \pm\lambda_n$. Since

$$k(f) \leqslant A\pi < l(\phi)$$

we must have from Theorem 1 that

(i) $(-1)^n \cos \rho\lambda_n$ cannot retain the same sign as n varies unless all these are zero;

(ii) $(-1)^n \sin \rho\lambda_n$ cannot retain the same sign as n varies unless all these are zero.

It is easy to see that both these sets cannot vanish at the same time. We can verify that these statements are equivalent to the following

THEOREM 1-A. *Let* $[\lambda_n]$ *be such that*

$$\underline{lim} \frac{n(r)}{r} = A > 0;$$

let ρ *be a real number such that* $0 < \rho < \pi A$;
let the interval $(0, 1)$ *be divided into four closed intervals* $(0, \frac{1}{4})$, $(\frac{1}{4}, \frac{1}{2})$, $(\frac{1}{2}, \frac{3}{4})$, $(\frac{3}{4}, 1)$, *the dividing points being supposed to belong to both the intervals adjacent to them,* 0 *and* 1 *both being regarded as belonging to the first and last of these intervals;*

and let σ_n *denote the fractional part of* $\frac{\rho\lambda_n}{2\pi}$; *then the two sets* $[\sigma_{2n}]$ *and* $[\sigma_{2n+1}]$ *cannot lie entirely in two different intervals of the set considered above.*

7.3. Lastly we may note that taking $f(z) \equiv 1$, the expression (12) gives a formula for the function $1/\phi(z)$ in terms of its principal parts. The polynomial $C_{p-1}(z)$ could be evaluated by expanding both sides of (12) in the neighbourhood of $z=0$.

TWO THEOREMS ON THE CONVEX OVAL

By R. C. BOSE, Calcutta

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1. It has been shown by Ganapati† that all the cyclic points of a convex oval cannot lie on the same circle. The object of this note is to establish the following similar theorems:

THEOREM (A). *The tangents to the oval at the cyclic points cannot all touch the same circle.*

This theorem may in a sense be regarded as the dual of Ganapati's theorem, but is not derivable from his theorem by polar reciprocation.

THEOREM (B). *All the sextactic points on a convex oval cannot lie on the same conic.*

This theorem may be regarded as the affine analogue of Ganapati's theorem. The affine analogue of Theorem (A) given above follows at once, from Theorem (B) by reciprocation, viz.

THEOREM (B*). *The tangents to the oval at the sextactic points cannot all touch the same conic.*

2. Let ψ be the angle which the positive tangent to the oval, makes with the positive direction of the x—axis, and p the length of the perpendicular from the origin on the tangent, p being taken as positive when the origin lies to the left of the tangent. Then we shall take the tangential polar equation of the oval to be

$$p=f(\psi).$$

If ρ is the radius of curvature, then

$$\text{I} \qquad \int_0^{2\pi} p\frac{d\rho}{d\psi}d\psi=\int_0^{2\pi} p(p'+p''')d\psi$$

$$=\Big[\tfrac{1}{2}p^2\Big]_0^{2\pi}+\Big[pp''\Big]_0^{2\pi}-\int p'p''d\psi$$

$$=\Big[-\tfrac{1}{2}p'^2\Big]_0^{2\pi}$$

$$=0,$$

where the dashes denote differentiation with respect to ψ.

† P. Ganapati, 'On certain ovaloids', *Jour. Ind. Math. Soc.* Vol. XIX, 225-32.

II $$\int_0^{2\pi} \sin\psi \frac{d\rho}{d\psi} d\psi = \Big[\rho \sin\psi\Big]_0^{2\pi} - \int_0^{2\pi} \rho \cos\psi d\psi$$
$$=0.*$$

III Similarly $$\int_0^{2\pi} \cos\psi \frac{d\rho}{d\psi} d\psi = 0.$$

From I, II, III, it however follows, that for all values of the constants a, b, c,

$$\int_0^{2\pi} (p - a \sin\psi - b \cos\psi - c) \frac{d\rho}{d\psi} d\psi = 0. \qquad (1)$$

Now if possible suppose the tangents to the oval at the cyclic points, all touch the same circle. Let the tangential polar equation of this circle be

$$p = a \sin\psi + b \cos\psi + c.$$

Suppose that the oval has exactly $2n$ cyclic points. Let $\psi_1, \psi_2, \ldots, \psi_{2n}$ be the angles which the tangents at the $2n$ cyclic points, make with the x—axis. The circle in question cannot have any other common tangents with the oval, for otherwise it would have more than $2n$ intersections with the oval, and hence from a theorem of Blaschke† the oval would have more than $2n$ cyclic points. When p refers to the oval, the expression

$$p - a \sin\psi - b \cos\psi - c$$

is alternately positive and negative, or negative and positive, in the intervals (ψ_1, ψ_2), (ψ_2, ψ_3), . . . , (ψ_{2n}, ψ_1), and a similar statement is true for $\frac{d\rho}{d\psi}$. Hence the integrand in the left hand side of (1) maintains an invariable sign. This however makes the equation (1) an impossibility. Hence we have the following

THEOREM (A). *The tangents to be the oval at the cyclic points cannot all touch the same circle.*

COROLLARY 1. Ovals with only four cyclic points have been studied by C. Juel‡. It follows from our theorem that *if $PQRS$ is the quadrilateral formed by the tangents at the cyclic points to such an oval, then a circle cannot be inscribed in $PQRS$. Consequently*

$$PQ + RS \neq QR + SP.$$

* *Cf.* W. Blaschke, *Kreis und Kugel*, p. 116.

† W. Blaschke, *Circolo di Palermo*, 36 (1913) 220-222. H. Mohrmann, *Circolo di Palermo*, 37 (1914) 267-268.

‡ C. Juel, "On simple cykliske kurver" *Danske Vidensk. Selsk. Skrifter*, Vol. 8, (1911).

COROLLARY 2. If we invert the oval with respect to any point which lies outside or inside all the circles of curvature at the cyclic points, then the oval goes over into an oval, cyclic points into cyclic points, and the tangents to the oval at the cyclic points into circles touching the oval at the cyclic points and passing through the centre of inversion. Hence we get the following result.

If a point O lies inside or outside all the circles of curvature at the cyclic points, and circles be drawn passing through O and touching the oval at the cyclic points, then these circles cannot all be touched by the same circle.

3. If s denotes the affine length along the oval, and k denotes the affine curvature, it is known† that the integral

$$\int (ax^2+2hxy+by^2+2gx+2fy+c)\,\frac{dk}{ds}\,ds \tag{2}$$

when taken round the whole oval, always vanishes whatever be the value of the constants a, b, c, f, g, h.

Let the oval possess exactly $2n$ ($n \geqslant 3$) sextactic points, and if possible suppose that all these points lie on the same conic. Then this conic cannot cut the oval in any other point, for otherwise by a theorem of Mukhopadhyaya‡ the oval would possess more sextactic points. If then we take the equation of this conic to be

$$ax^2+2hxy+by^2+2gx+2fy+c=0$$

then the expression within the brackets in the integrand of (2) is alternately positive and negative on the successive arcs into which the oval is divided by the sextactic points. The same holds for $\frac{dk}{ds}$. Hence the integrand in (2) always maintains its sign, and the integral cannot vanish when taken round the whole oval. We thus get

THEOREM (B). *All the sextactic points on a convex oval cannot lie on the same conic.*

Reciprocating we get

THEOREM (B*). *The tangents to the oval at the sextactic points cannot all touch the same conic.*

† W. Blaschke, *Vorlesungen uber Differential Geometrie*, Vol. II, p. 43.

‡ S. Mukhopadhyaya, 'Extended Minimum—Number Theorems of Cyclic and Sextactic points on a Plane Convex Oval', *Math. Zeitschrift*, Bd. 33, (1931) 648-62, Theorem X.

ON WARING'S PROBLEM

By S. S. PILLAI, D.Sc., Annamalai University

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§ 1. *Introduction*

1. Let $g(n)$ denote the least value of s required to represent every positive integer as the sum of s non-negative nth powers. Further let $l=[(3/2)^n]$ and $j=[(4/3)^n]$, where $[x]$ denotes the integral part of x. The object of this paper is to prove the following results:

THEOREM I. *If* $n \geqslant 30$ *and* $\{(3/2)^n\} \leqslant 1-(l+3)/2^n$

then
$$g(n)=2^n+l-2, \tag{1}$$
where $\{x\}$ *denotes the fractional part of* x.

THEOREM II. *If* $\{(3/2)^n\} \geqslant 1-l/2^n$

then
$$g(n) \geqslant 2^n+l+j-3.$$

THEOREM III.
$$g(n)=2^n+l+O(4/3)^n.$$

THEOREM IV. *When* $8 \leqslant n \leqslant 100$,
$$g(n)=2^n+l-2.$$

THEOREM V. *If* $K(x)$ *denotes the number of n's less than* x *for which* (1) *is true, then*
$$K(x) \geqslant \frac{\log (4/3)}{\log 3} x + O(1).$$

Theorem I when $n > n_0$ and Theorems II and III were proved by the author in *The Journal of the Annamalai University**.

Vinogradow proved recently that
$$G(n)=O(n \log n).$$
The proof given below is based on his method, and is slightly different from the original proof given in *The Annamalai University Journal.*

* S. S. Pillai, (R1).

§ 2. *Notation and lemmas*

2. n is a positive integer $\geqslant 8$, $\nu=\frac{1}{n}$.

$k=[4n\log n+1]$; $t=n^2+1$; $f=4/t$; $\sigma=n(1-\nu)^k$;

N_0 is a positive integer,

$$P=[3^{-1}N_0^{\nu}+1];\ P_1=[n^{\nu}P^{1-\nu}];\ R=[P^{1-\nu/2}];$$
$$R_1=[n^{\nu}R^{1-\nu}];\ Y=[P^{(\nu-\nu^2)/2}];\ \tau=2n3^{n-1}P^{n-\frac{1}{2}};$$

N is an integer lying in the interval

$$N_0-N_0P^{-\frac{1}{4}}<N\leqslant N_0$$
$$N_0\geqslant\beta,\ \text{where}\ \log\beta=\frac{25n^{5\frac{1}{4}}}{2}. \qquad (2)$$

Then we can easily verify that

$$\left.\begin{aligned}&2^nP^n-N_0P^{-1/4}>P^n;\ N_0>80\times3^{n-4}P^n;\\&Y^n\leqslant 2^{-1}\sqrt{P};\ \tau\leqslant P^n.\end{aligned}\right\} \qquad (3)$$

p is a prime, $E\{x\}$ denotes e^x; $\eta=E\left\{\frac{2\pi i}{q}\right\}$.

If $q>0$, and $(a,q)=1$, we put

$$S_{a,q}=\sum_{r=0}^{q-1}E\left\{\frac{2\pi i a r^n}{q}\right\};\ B_{a,q}=(1/q)S_{a,q}.$$

$$S'_{a,q}=\sum_{h}E\left\{\frac{2\pi i a h^n}{q}\right\},\ \text{where}\ (h,q)=1,\ 0<h\leqslant q.$$

$$A_q(N)=\sum_a B_{a,q}^t E\left\{\frac{-2\pi i a N}{q}\right\},$$

where a runs through all integers less than and prime to q.

$\mathfrak{S}=\sum_{q=1}^{\infty}A_q(N)$, $\Theta=$ the highest power of a prime p which divides n.

$$\gamma=\begin{cases}\Theta+2, & \text{if } p=2\\ \Theta+1, & \text{if } p>2.\end{cases}$$

$$\alpha=\frac{(p^{\gamma}-1)(n,p-1)}{(p-1)},\ \chi_p=\sum_{r=0}^{\infty}A_{p^r}(N).$$

$$\vartheta(x) = \sum_{p \leqslant x} \log p.$$

$A = \theta[B]$ denotes that $|A| \leqslant |B|$.

(x) denotes the distance of x to the nearest integer.

$d = (4/3)^n + 5(5/4)^n + n\,(5\tfrac{1}{6}n + 3.2) - 8.$

$\delta = l + d.$

$g = 2^n + l - 2.$

LEMMA 1. *If* $x \geqslant 16$, *then* $\vartheta(x) \leqslant 1{\cdot}5x$.

This can be easily proved starting from $\binom{2n}{n}$ and applying Stirling's theorem.

LEMMA 2. $|B_{a,q}| \leqslant n^5/q^f$.

(a) Let $p|n,\ \lambda = \alpha n + \mu,\ 0 < \mu \leqslant n$.

Then from 3·47 in R.2.,*

$$|B_{a,q}| = q^{-\nu}\,\Pi\,|B_{a,p^\mu}|\,p^\mu \leqslant q^{-\nu} \prod_{p|n} p \leqslant nq^{-\nu}; \tag{4}$$

(b) If p does *not* divide n, then from 3·44, and 3·45 in R.2., we get

$$|B_{a,q}| \leqslant (q_2^n Q)^{-\nu} B_{a,q_1}$$

where q_1 is a product of different primes, i.e.

$$|B_{a,q}| \leqslant (q_2^n Q)^{-\nu} \prod_p |B_{a,p}|.$$

But $|B_{a,p}| \leqslant n/\sqrt{p} \leqslant 1/p^f$,

if $p \geqslant n^{2+8(n^2-4)} = \lambda$, say.

Then $$|B_{a,q}| \leqslant (q_2^n Q)^{-\nu} \prod_{p \geqslant \lambda} p^{-\nu}$$

$$\leqslant (q_2^n Q)^{-\nu}\, e^8/q_1^f \leqslant e^8/q^f; \tag{5}$$

for from Lemma 1 we get $\prod_{p \leqslant \lambda} p^f \leqslant e^8$.

Since when

$$n \geqslant 8,\ n^4 > e^8 \text{ and } f \leqslant \nu,$$

the lemma follows from (4) and (5).

LEMMA 3. $A_q(N) = \theta[n^{5t}/q^3]$.

* The references relate to the bibliography given at the end of the paper.

Now
$$A_q(N)=\theta\left[\sum_a |B^t_{a,q}|\right]$$
$$=\theta\left[\sum_a n^{5t}/q^4\right] \text{ from Lemma 2}$$
$$=\theta[n^{5t}/q^3].$$

LEMMA 4. $\infty=\prod_{p\leqslant b} b(p) \prod_{p>b}(1-p^{-3/2})=\Pi_1\Pi_2,$

where $b(p)=\chi_p\geqslant p^{-\gamma\alpha}$

and $b=(1+k^t)^{2/(t-5)}.$

Theorems 25 and 10 in R.3. give this result.

LEMMA 5. $\operatorname{Log}\infty\geqslant-2n^{3\frac{1}{6}}.$

(a) Let us suppose that p does not divide n and $p>2$. Then
$$\gamma\alpha=\alpha=\frac{p-1}{p-1}(n,p-1)\leqslant n.$$

So if $F_1=\Sigma\gamma\alpha\log p$, where $p\leqslant b$ and p does not divide n, then from Lemma 1 $F_1\leqslant n\sum_{p\leqslant b}\log p\leqslant 1{\cdot}5nb,$
$$\leqslant 1{\cdot}5n(1+n^{-65})^{13/6}n^{13/6}, \text{ for } t\geqslant 65$$
$$\leqslant 1{\cdot}51n^{19/6}. \tag{6}$$

(b) Let $p>2$ and $p|n$. Then
$$\gamma\alpha\leqslant\frac{p}{p-1}(\Theta+1)\leqslant\frac{p}{p-1}n\left(\frac{\log n}{\log p}+1\right)$$
$$\leqslant\frac{2np\log n}{(p-1)\log p}.$$

So
$$F_2=\sum_{p|n}\gamma\alpha\log p\leqslant 2n\log n\sum_{p|n}p/(p-1)$$
$$\leqslant 3n\log n\sum_{p|n}1\leqslant 3n(\log n)^2/\log 2. \tag{7}$$

(c) Let $p=2$. Then
$$\gamma\alpha\log 2\leqslant(\Theta+2)4n\log 2\leqslant 8n\log n. \tag{8}$$

Hence from (6), (7) and (8) it follows that
$$-\log\Pi_1+3\leqslant n^{19/6}\left[1{\cdot}51+\frac{3(\log n)^2}{n^{13/6}\log 2}+\frac{8\log n}{n^{13/6}}+\frac{3}{n^{19/6}}\right]$$
$$\leqslant 2n^{19/6}. \tag{9}$$

and $$-\log \Pi_2 \leqslant 3. \tag{10}$$

From (9) and (10), the lemma follows.

LEMMA 6. *If every integer in* $[g, h]$ *is a sum of* $s-1$, *nth powers, and if m is the greatest integer such that*

$$(m+1)^n - m^n < h - g,$$

then every integer in $[g, h+(m+1)^n]$ *is a sum of s, nth powers.*

This is proved in **R.1**.

LEMMA 7. *If every integer in* $[g, h]$ *is a sum of s, nth powers and if* $h-g=L>n^n$, *then every integer in* $[g, C]$ *is a sum of*

$$s+\left[\frac{\log\log C-\log(\log L-n\log n)}{\log n-\log(n-1)}\right]+1,$$

nth powers.

This is proved in **R.3.** with $g=0$. A trivial modification of that proof will give the above result.

LEMMA 8. *If every integer in* $[F, F+2^n]$ *is the sum of s, nth powers, then every integer in* $[F, \beta]$ *is the sum of* $s+\delta$, *nth powers.*

Starting from $[F, F+2^n]$, by the repeated application of Lemma 6, we get that every integer in $[F, F+(2n)^n]$ is the sum of $s+u$, nth powers, where

$$u=(3/2)^n+(4/3)^n+\ldots+\left(\frac{2n}{2n-1}\right)^n.$$

Now

$$\left(\frac{6}{5}\right)^n+\left(\frac{7}{6}\right)^n+\ldots+\left(\frac{2n}{2n-1}\right)^n<\int_4^{2n-1}\left(1+\frac{1}{x}\right)^n dx$$

$$=\left[x+n\log x-\binom{n}{2}\frac{1}{x}-\binom{n}{3}\frac{1}{2}\cdot\frac{1}{x}-\ldots\right]_4^{2n-1}$$

$$\leqslant 2n-1-4+n\log(2n-1)-n\log 4+\binom{n}{2}\frac{1}{4}+\binom{n}{3}\frac{1}{2}\cdot\frac{1}{4^2}+\ldots$$

$$\leqslant 2n-5+n\log 2n-n\log 4+4(1+1/4)^n-4-n$$

$$=4(5/4)^n+n\log n+n(1-\log 2)-9.$$

So

$$u\leqslant\left(\frac{3}{2}\right)^n+\left(\frac{4}{3}\right)^n+5\left(\frac{5}{4}\right)^n+n\log n+n(1-\log 2)-9. \tag{11}$$

Again by Lemma 7, every integer in $(F, F+\beta)$ is the sum $s+u+v$, nth powers, where

$$v=\left[\frac{\log\log\beta-\log\{\log(2n)^n-n\log n\}}{\log n-\log(n-1)}\right]+1$$

$$\leqslant n[5\tfrac{1}{6}\log n+\log 12{\cdot}5-\log n-\log\log 2]+1, \tag{12}$$

Hence from (11) and (12)

$$u+v \leqslant \left(\frac{3}{2}\right)^n+\left(\frac{4}{3}\right)^n+5\left(\frac{5}{4}\right)^n+5\tfrac{1}{6}n\log n$$
$$+n[1-\log 2+\log 12{\cdot}5-\log\log 2]-9+1$$
$$=\delta.$$

So the lemma follows.

LEMMA 9. *Let* $3^n=l2^n+r.$

If $d+3 \leqslant r \leqslant 2^n-\delta-1,$

then every positive integer $\leqslant \beta$, *is the sum of* g, *nth powers.*

(a) Let $l2^n \leqslant a < 3^n.$

Then $a=l2^n+(a-l2^n).$

So a is the sum of u, nth powers, where

$$u=l+a-l2^n \leqslant l+3^n-1-l2^n$$
$$=l+r-1 \leqslant 2^n-d-2. \qquad (13)$$

(b) Let $3^n \leqslant b < (l+1)\,2^n.$

Then $b=3^n+(b-3^n).$

So b is the sum of v, nth powers, where

$$v=1+b-3^n \leqslant 1+(l+1)\,2^n-1-3^n$$
$$=2^n-r \leqslant 2^n-d-3. \qquad (14)$$

From (13) and (14), it follows that every integer in

$$[l2^n, (l+1)2^n]$$

is the sum of 2^n-d-2, nth powers.

Hence by Lemma 9, every integer in $[l2^n, \beta]$ is the sum of $2^n-d-2+\delta=2^n+l-2$, nth powers. (15)

But every positive integer $\leqslant l2^n$, is the sum of g, nth powers. (16)

From (15) and (16), the lemma follows.

LEMMA 10. *If* $n \geqslant 30$ *and* $r \leqslant d+3,$

then every positive integer $\leqslant \beta$, *is the sum of* g, *nth powers.*

Let $ul2^n < u3^n < (ul+1)\,2^n.$

(a) Let $ul2^n \leqslant a < u3^n.$

Then $a=ul2^n+a-ul2^n.$

Hence a is the sum of v, nth powers, where

$$v=ul+a-ul2^n \leqslant ul+u3^n-1-ul2^n$$
$$=ul+ur-1=u(l+r)-1 \leqslant 2^n-d-2,$$

if $u \leqslant (2^n-d-1)/(l+r)$

or if $u \leqslant (2^n-d-1)/(\delta+3)$; for $r \leqslant d+3.$ (17)

(b) Let $u3^n \leqslant b < (ul+1)\,2^n.$

Then $$b=u3^n+b-u3^n.$$

So b is the sum of v_1, nth powers, where

$$v_1=u+b-u3^n\leqslant u+(ul+1)\,2^n-1-u3^n$$
$$=u+2^n-1-ur=2^n-u(r-1)-1\leqslant 2^n-d-2,$$

if $$u\geqslant\frac{d+1}{r-1}. \tag{18}$$

If $n\geqslant 30$ and $r\geqslant 3$, it can be easily verified that

$$\frac{2^n-d-1}{\delta+3}\geqslant\frac{d+1}{r-1}+1.$$

Hence u has an integral value and further $ul2^n<u3^n<(ul+1)2^n$.

Therefore every integer between $ul2^n$ and $(ul+1)2^n$ is the sum of 2^n-d-2, nth powers.

Consequently, by Lemma (8), every integer in $[ul2^n, \beta]$, is the sum of g, nth powers. (19)

Let m be an integer such that $1\leqslant m\leqslant u-1$. Then from (a) and (b), every integer in

$$[ml2^n, (ml+1)2^n]$$

is the sum of v_2, nth powers, where

$$v_2=\text{Max}\,[m(l+r)-1, 2^n-u(r-1)-1].$$

Now

$$m(l+r-1)\leqslant(u-1)(l+r)\leqslant\left(\frac{2^n-d-1}{l+r}-1\right)(l+r)\leqslant 2^n-3,$$

and $$2^n-u(r-1)-1\leqslant 2^n-3, \text{ when } r\geqslant 3.$$

Hence $$v_2\leqslant 2^n-3.$$

Therefore, by Lemma (6) every integer in

$$[ml2^n, (m+1)\,l2^n]$$

is the sum of g, nth powers. (20)

Further every positive integer $\leqslant l2^n$, is the sum of g, nth powers. (21)

Moreover when $n\geqslant 5$,

$$r\neq 1 \text{ or } 2. \tag{22}$$

From (19)—(22), the lemma follows.

LEMMA 11. *If* $$n\geqslant 30$$
and $$2^n-\delta-1\leqslant r\leqslant 2^n-(l+3), \tag{23}$$
then every positive integer $\leqslant\beta$, *is the sum of* g, *nth powers.*

Let $$(ul+u-1)\,2^n<u3^n<(ul+u)2^n,$$

(a) Let $(ul+u-1)2^n \leqslant a < u3^n$.

Then $a=(ul+u-1)\,2^n+a-(ul+u-1)2^n$.

Hence a is the sum of v, nth powers, where

$$\begin{aligned} v &= ul+u-1+a-(ul+u-1)2^n \\ &\leqslant ul+u-1+u3^n-1-(ul+u-1)2^n \\ &= ul+u-2+ur-(u-1)\,2^n \\ &\leqslant ul+u-2+2^n-u\,(l+3), \text{ from } (23) \\ &= 2^n-2u-2. \end{aligned}$$

Hence $v \leqslant 2^n-d-2$,

if $2^n-2u-2 \leqslant 2^n-d-2$

or if $u \geqslant \frac{d}{2}$. (24)

(b) Let $u3^n \leqslant b < (ul+u)2^n$.

So $b=u3^n+b-u3^n$.

Hence b is the sum of v_1 nth powers where

$$\begin{aligned} v_1 = u+b-u3^n &\leqslant u+(ul+u)2^n-1-u3^n \\ &= u+u2^n-1-ur \\ &\leqslant u-1+u(\delta+1)=u(\delta+2)-1. \end{aligned}$$

Therefore, $v_1 \leqslant 2^n-d-2$,

if $u(\delta+2)-1 \leqslant 2^n-d-2$,

or if $u \leqslant \dfrac{2^n-d-1}{\delta+2}$. (25)

Now since $n \geqslant 30$, it can be verified that

$$\frac{2^n-d-1}{\delta+2} \geqslant \frac{d}{2}+1.$$

Hence u has an integral value and further

$$(ul+u-1)2^n < u3^n < (ul+u)2^n.$$

Therefore, by (24) and (25), every integer in

$$[(ul+u-1)2^n,\ (ul+u)2^n],$$

is the sum of 2^n-d-2, nth powers.

Consequently, by Lemma (8), every integer in

$$[(ul+u-1)2^n,\ \beta]$$

is the sum of g, nth powers.

As in the previous lemma it can be proved that every positive integer

$$\leqslant (ul+u-1)2^n,$$

is the sum of g, nth powers.

Thus the lemma is completely proved.

LEMMA 12. *If*

$$n \geqslant 30, \{(3/2)^n\} \leqslant 1-(l+3)/2^n,$$

then every positive integer $\leqslant \beta$, *is the sum of* 2^n+l-1, *nth powers.*

Combing the results of Lemmas (9), (10) and (11), we get the present lemma.

LEMMA 13. *If* $0 \leqslant f'(t) \leqslant \frac{1}{2}$, $f''(t) \geqslant 0$, $g \leqslant t \leqslant h$,

then $$\sum_{t>g}^{h} E[\pm 2\pi i f(t)] = \int_g^h E[\pm 2\pi i f(t)]\, dt + \theta(5).$$

LEMMA 14. *Let* λ *be real but not an integer, and* $G<H$.

Then $$\left| \sum_{x>G}^{H} E[2\pi i \lambda x] \right| < \frac{1}{2(\lambda)}.$$

These are Lemmas C, D, in Vinogradow's paper. Hereafter I closely follow his paper.

§ 3.

3·1 *The numbers u.* Let ξ denote a number which does not exceed P_1, and which can be expressed as the sum of $k-1$, nth powers; and X the number of different ξ's and

$$u = \xi + v^n,$$
$$v = P, P+1, \ldots, 2P-1.$$

Then, it is proved in R.4., that

$$X > P^{n-1-\sigma}/n^{n+1}. \tag{26}$$

Further the numbers u are all different and

$$P^n < u < 2^n P^n.$$

Similarly let ξ_1, u_1 and X_1 be defined with respect to R_1 and R.

Then $$X_1 > P^{(1-\nu/2)(n-1-\sigma)}/n^{n+1}. \tag{27}$$

3·2. *An integral.*

Let y be an integer $0 < y \leqslant Y$.

We introduce the sums depending on α;

$$T=\sum_{x=1}^{P} E[2\pi i\alpha x^n]\,; \quad T_1=\sum_{x=1}^{3P} E[2\pi i\alpha x^n]\,;$$

$$V=\sum_{v} E[2\pi i\alpha x^n]\,; \quad V_y=\sum_{v_1} E[2\pi i\alpha y^n v_1^n]\,;$$

$$S=\sum_{\xi} E[2\pi i\alpha\xi]\,; \quad S_y=\sum_{\xi_1} E[2\pi i\alpha y^n\xi_1]\,;$$

and the integral

$$I_{y,N}=\int_0^1 T^{t-3}T_1V^2S^2V_y\,S_yE[-2\pi i\alpha N]\,d\alpha,$$

where for the interval of integration, we can substitute

$$-\tau^{-1}\leqslant\alpha\leqslant 1-\tau^{-1}.$$

For every α in the last integral, we have

$$\alpha=(a/q)+z\,;\ (a,q)=1\,;\ 0<q<\tau\,;\ |z|\leqslant 1/q\tau. \qquad (28)$$

We divide the interval of integration into two classes. In the first class, we put the intervals with

$$\alpha=(a/q)+z\,;\ (a,q)=1\,;\ 0<q\leqslant\sqrt{P}\,;\ -\tau^{-1}\leqslant z\leqslant\tau^{-1},$$

and in the second class the remaining intervals. The intervals in the first class do not overlap.

Corresponding to this division of the interval of integration, the integral is represented as the sum of two terms

$$I_{y,N}=H_{y,1}+H_{y,2}.$$

3·3. *Asymptotic formula for* $H_{y,1}$.

Now we introduce the integrals:

$$\phi=\int_0^P E[2\pi izx^n]\,dx\,; \quad \phi_1=\int_0^{3P} E[2\pi izx^n]\,dx\,;$$

$$\psi=\int_P^{2P} E[2\pi izx^n]\,dx\,; \quad \psi_y=\int_R^{2R} E[2\pi izy^nx^n]\,dx.$$

In T, for x, we put

$$x=tq+r,\ r=0,1,\ldots,q-1,$$

where for every r, t takes all integral values satisfying the condition

$$0<qt+r\leqslant P.$$

So
$$T=\sum_{r=0}^{q-1}\sum_{t} E\left[2\pi i\left(\frac{a}{q}+z\right)(qt+r)^n\right]$$
$$=\sum_{r=0}^{q-1} E\left[\frac{2\pi i a r^n}{q}\right] K_r,$$

where
$$K_r=\sum_{t} E[2\pi i z(qt+r)^n].$$

The function $f(t)=|z|(qt+r)^n$ satisfies in the interval of §3·2 the conditions of Lemma 13.

So
$$K_r=\int_{r/q}^{(P-r)/q} E[2\pi i z(qt+r)^n]dt+\theta(5)$$
$$=\phi/q+\theta(5).$$

Hence
$$T=\phi B_{a,q}+\theta(5q). \tag{29}$$

Similarly we get
$$T_1=\phi_1 B_{a,q}+\theta(5q) \tag{30}$$
$$V=\psi B_{a,q}+\theta(5q) \tag{31}$$

and
$$V_y=(\psi_y/q)\sum_{r=0}^{q-1} E[2\pi i a y^n r^n/q]+\theta(5q). \tag{32}$$

$$\sum_{v_1} E[2\pi i a y^n v_1^n/q]=(R/q)\sum_{r} E[2\pi i a y^n r^n/q]+\theta(q).$$

So
$$V_y=(\psi_y/R)\sum E[2\pi i a y^n v_1^n/q]+\theta(5q)+\theta(\psi_y q/R)$$
$$=(\psi_y/R)\sum_{v_1} E[2\pi i a y^n v_1^n/q]+\theta(6q). \tag{33}$$

Now
$$|\phi|=\left|\int_0^P E[2\pi i z x^n]dx\right|$$
$$=\left|\frac{1}{nz^\nu}\int_0^{zP^n} E[2\pi i y]dy/y^{1-\nu}\right|$$

$$\leqslant \frac{1}{n|z|^\nu}\int_0^1 \frac{dy}{y^{1-\nu}} + \left| \frac{1}{nz^\nu}\int_1^{zP^n} E[2\pi iy]\,dy/y^{1-\nu} \right|$$

$$\leqslant 1/|z|^\nu + (2+\sqrt{2})/(n|z|^\nu)$$

$$\leqslant 2/|z|^\nu, \text{ when } n \geqslant 4. \tag{34}$$

Further, let

$$Z = P, \text{ if } |z| \leqslant P^{-n}$$

and

$$Z = |z|^{-\nu}, \text{ if } |z| \geqslant P^{-n}. \tag{35}$$

Then from (34) and Lemma 2, we get

$$T' = \phi\, B_{a,q} = \theta(A), \tag{36}$$

where

$$A = 2n^5 Z q^{-f} \tag{37}$$

$$T'_1 = \phi_1\, B_{a,q} = \theta(3A).$$

Further let $r_1 = 5Z^{-1}\, q^{1+f}/(2n^5)$.

Now, since $q \leqslant \sqrt{P}$, we have

$$tr_1 \leqslant \frac{5t}{2n^5}\,\frac{P^{(1+f)/2}}{(2n\,3^{n-1}\,P^{n-1/2})^\nu} < 1/P^{-1/4}. \tag{38}$$

Therefore

$$T^{t-3} = [T' + \theta(5q)]^{t-3} = [T' + \theta(Ar_1)]^{t-3}$$

$$\leqslant T'^{t-3} + \theta(A^{t-3})\left[tr_1 + \frac{t^2 r_1^2}{2!} + \dots\right] \tag{39}$$

$$= T'^{t-3} + \theta(2A^{t-3} tr_1).$$

From (30) and (37) we get

$$T_1 = T_1' + \theta(Ar_1). \tag{40}$$

From (39) and (40) we have

$$T^{t-3}T_1 = T'^{t-3}T_1' + \theta(A^{t-2}tr_1)\left[6 + \frac{1}{t} + 2r_1\right]$$

$$= T'^{t-3}T_1' + \theta(7A^{t-2}tr_1). \tag{41}$$

As in (34), we can verify that

$$|\psi| \leqslant 2|z|^{-\nu}.$$

So as in (36), we have

$$V' = \psi\, B_{a,q} = \theta(A). \tag{42}$$

From (31), we get

$$V = V' + \theta(Ar_1)$$

So $V^2 = V'^2 + \theta(A^2 r_1)(2 + r_1) = V'^2 + \theta(3A^2 r_1)$. (43)

From (41) and (43), we get

$$T^{t-3}T_1V^2 = T'^{t-3}T_1'V'^2 + \theta(A^t r t_1)\left[7+\frac{9}{t}+21r_1\right]$$
$$= T'^{t-3}T_1' + \theta(8A^t r_1 t)$$
$$= T'^{t-3}T_1' + \theta(8A^t P^{-1/4}), \text{ (from 38).} \qquad (44)$$

Let V_y' denote the main term in (33). Then $V_y' = \theta(R)$, and

$$V_y = V_y' + \theta(RP^{-1/4}).$$

So $T^{t-3}T_1V^2V_y = T'^{t-3}T_1'V'^2V_y' + \theta(A^tRP^{-1/4})[8+1+P^{-1/4}]$

$$= T'^{t-3}T_1'V'^2V_y' + \theta(10A^tRP^{-1/4}). \qquad (45)$$

$$E[-2\pi i\alpha N] = E[-2\pi iaN/q - 2\pi izN_0]\, E[-2\pi iz(N-N_0)]$$
$$= E' + \theta[2\pi|z|3^nP^{n-1/4}], \qquad (46)$$

where $E' = E[-2\pi iaN/q - 2\pi izN_0] = \theta(1)$.

Now
$$S_y = \sum_{\xi_1} E[2\pi iay^n\xi_1/q] + \theta(2\pi|z|X_1Y^nR_1^n)$$
$$= S_y' + \theta(2\pi|z|X_1Y^nR_1^n) \qquad (47)$$
$$= S_y' + \theta(2\pi n|z|X_1P^{n-1})$$

where $S_y' = \theta(X_1)$.

From (46) and (47) we get

$$S_yE[-2\pi i\alpha N] = S_y'E' + \theta(2\pi|z|X_1P^{n-1/4})[3^n + n/P^{3/4} + 2\pi n|z|3^nP^{n-1}]$$

and
$$2\pi n3^nP^{n-1}|z| \leqslant 3\pi/P^{\frac{1}{4}} < \tfrac{1}{2}.$$

Hence

$$S_yE[-2\pi i\alpha N] = S_y'E' + \theta[2\pi(3^n+1)|z|X_1P^{n-1/4}]. \qquad (48)$$

Further

$$S = \sum_{\xi_1} E[2\pi ia\xi/q] + \theta[2\pi n|z|P^{n-1}]$$
$$= S' + \theta(2\pi n|z|P^{n-1}),$$

where $S' = \theta(X)$.

So
$$S^2 = S'^2 + \theta(2\pi n|z|X^2P^{n-1})(2+2\pi|z|nP^{n-1})$$
$$= S'^2 + \theta(6\pi n|z|X^2P^{n-1}). \qquad (49)$$

Hence from (48) and (49), it follows that

$$S^2S_yE[-2\pi i\alpha N] = S'^2S'_yE'$$
$$+\theta(2\pi|z|X^2X_1P^{n-1/4})[3^n+1+3nP^{-3/4}+(3^n+1)3n2\pi|z|P^{n-1}]$$
$$= S'^2S_y'E' + \theta(7.3^n|z|X^2X_1P^{n-1/4}). \qquad (50)$$

Hence from (45) and (50), we have

$$T^{t-3}T_1V^2V_yS^2S_yE[-2\pi i\alpha N]=T'^{t-3}T_1'V'^2V_y'S'^2S_y'E'$$
$$+\theta(X^2X_1RA^tP^{-1/4})[10+21P^n3^n|z|+70|z|3^nP^{n-1/4}]$$
$$=T'^{t-3}T_1'V'^2V_y'S'^2S_y'E'+\theta(X^2X_1RA^tP^{-1/4})[10+22|z|3^nP^n].$$
$$=F_yE_y+\theta(X^2X_1RA^tP^{-1/4})(10+22|z|3^nP^n), \tag{51}$$

where F_y depends only on z, y, n and N_0, and

$$E_y=B^t_{a,q}\Big(\sum_{\xi}e^{2\pi ia\xi/q}\Big)^2\sum_{u_1}e^{2\pi iay^nu_1/q}\,e^{-2\pi iaN/q}, \tag{52}$$

and hence we get

$$E_y=\theta[n^{5t}X^2X_1Rq^{-4}]. \tag{53}$$

Now integrating (51) with respect to z between the limits $z=-\tau^{-1}$ and $z=\tau^{-1}$, we get that the part of $H_{y,1}$ corresponding to any fixed a and q is

$$Q_yE_y+L,$$

where Q_y depends only on y, n and N_0.

Now
$$\int_0^{\tau^{-1}}Z^tdz=\int_0^{P^{-n}}P^tdz+\int_{P^{-n}}^{\tau^{-1}}|z|^{-\nu t}\,dz$$
$$=\theta[P^{t-n}+P^{t-n}/(\nu t-1)]=\theta(4P^{t-n}/3) \tag{54}$$

and
$$\int_0^{\tau^{-1}}Z^t|z|dz=\theta(P^{t-2n}). \tag{55}$$

Hence from (51), (54), (55) and (37) we have

$$L=\theta[X^2X_1RP^{-1/4}2^tq^{-4}]\Big[20\int_0^{\tau^{-1}}Z^tdz+44P^n3^n\int_0^{\tau^{-1}}Z^t|z|dz\Big]$$
$$=\theta[45.3^n2^tn^{5t}X^2X_1RP^{t-n-1/4}]. \tag{56}$$

Now
$$\sum_{q<\sqrt{P}}\sum_a q^{-4}=\theta\Big[\sum q^{-3}\Big]=\theta(5/4) \tag{57}$$

and since $n\geqslant 8$, we see that

$$45\times\tfrac{5}{4}3^n2^t=\theta(n^t). \tag{58}$$

So from (56), (57) and (58), we get

$$\sum_{q\leqslant\sqrt{P}}\sum_a L=\theta(n^{6t}X^2X_1RP^{t-n-1/4}). \tag{59}$$

Therefore

$$H_{y,1} = \sum_{q \leqslant \sqrt{P}} \sum_{a} \Big[Q_y E_y + L \Big]$$

$$= Q_y D_y + \theta (n^{6t} X^2 X_1 R P^{t-n-1/4}), \qquad (60)$$

where

$$D_y = \sum_{q=1}^{\sqrt{P}} \sum_{a} E_y. \qquad (61)$$

3·4. *Estimation of Q_y.*

We sum up the equality

$$I_{y,N} = H_{y,1} + H_{y,2} \qquad (62)$$

of § 3·2, for all values of N satisfying the condition

$$N_0 - N_0 P^{-1/4} < N \leqslant N_0.$$

The result of summation of the left side is equal to the number of solutions for x and N_1 of the system of inequalities

$$N_1 - N_0 P^{-1/4} < x^n \leqslant N_1, \qquad (63)$$

where $\qquad N_1 = N_0 - x_1^n - \ldots - x_{t-3}^n - u - u' - y^n u_1,$

and x runs over the values $1, 2, \ldots, 3P$; every x_i runs over the set $1, 2, \ldots, P$; u, u_1 are the values described in § 3·1, and u' takes on the same values as u but independent of u.

Now with the help of (2) and (3), we can easily see that the number of solutions of (63) for every given N_1 is

$$N_1^{\nu} - (N_1 - N_0 P^{-1/4})^{\nu} + \theta(1) \geqslant (2/n) P^{1-1/4}. \qquad (63\text{-a})$$

Now the whole number of solutions of (63) is

$$\sum_{N} I_{y,N}.$$

Therefore from (63) and (63-a), we get

$$\sum_{N} I_{y,N} \geqslant (2/n) X^2 X_1 R P^{t-1/4}. \qquad (64)$$

When $q=1$, from (52), we get

$$E_y = X^2 X_1 R. \qquad (65)$$

By Lemma (12), we have

$$\Big| \sum_{N} E[-2\pi i a N/q] \Big| < 1/2\,(a/q) = q/(2a). \qquad (66)$$

Now

$$\sum_N Q_y D_y = \sum_N Q_y \sum_{q=1}^{\sqrt{P}} \sum_a E_y$$

$$= \sum_N Q_y \sum_a E_y, \text{ when } q=1$$

$$+ \sum_N Q_y \sum_{q=2}^{\sqrt{P}} \sum_a E_y = S_1 + S_2 \text{ (say).} \quad (67)$$

From (65) we get

$$S_1 = \sum_N Q_y X^2 X_1 R = Q_y X^2 X_1 R [N_0 P^{-1/4} + \theta(1)]. \quad (68)$$

$$|S_2| = \left| \sum_{q=2}^{\sqrt{P}} \sum_a \sum_N Q_y E_y \right|$$

$$\leqslant |Q_y| X^2 X_1 R \sum_{q=2}^{\sqrt{P}} n^{5t} q^{-4} \sum_q \left| \sum_N E[-2\pi i a N/q] \right| \text{ from (52), (53)}$$

$$\leqslant \tfrac{1}{2} n^{5t} |Q_y| X^2 X_1 R \sum_{q=2}^{\sqrt{P}} q^{-4} \sum_a (q/a)$$

$$< (1/4) n^{5t} |Q_y| X^2 X_1 R. \quad (69)$$

Hence from (67), (68) and (69) it follows that

$$\sum Q_y D_y \leqslant Q_y X^2 X_1 R [N_0 P^{-1/4} + \theta(1) + \theta(n^{5t}/4)]$$

$$\leqslant 2.3^n Q_y X^2 X_1 R P^{n-1/4}. \quad (70)$$

So from (60) and (70) we get

$$\sum_N H_{y,1} \leqslant 2.3^n Q_y X^2 X_1 R P^{n-1/4} + \theta(n^{6t} X^2 X_1 R P^{t-n-1/4}), \quad (71)$$

and

$$\left| \sum_N H_{y,2} \right| \leqslant \int_0^1 |T^{t-3} T_1 V_y S_y| V^2 S^2 \left| \sum_N E[-2\pi i \alpha N] \right| d\alpha$$

$$\leqslant 3P^{t-2}X_1R(1/2(\alpha))\int_0^1 V^2S^2\,d\alpha, \qquad \text{from Lemma (13)}$$

$$\leqslant n3^nP^{t+n-2\frac{1}{2}}X_1R\int_0^1\sum_u\sum_{u'}E[-2\pi\alpha(u-u')]\,d\alpha$$

$$=n3^nXX_1RP^{t+n-1}. \tag{72}$$

From (71), (62), (64) and (72) we have

$$2.3^nQ_yX^2X_1RP^{n-1/4}+\theta(n^{6t}X^2X_1RP^{t-n-1/4})$$

$$\geqslant\sum_N H_{y,1}=\sum_N I_{y,N}-\sum_N H_{y,2}$$

$$\geqslant(2/n)X^2X_1RP^{t-1/4}-n.\,3^nXX_1RP^{t+n-1\frac{1}{2}}.$$

Since

$$k=[4n\log n+1],$$
$$\sigma\leqslant 1/n^3. \tag{73}$$

Hence from (26) it follows that

$$X>P^{n-1-1/n^3}/n^{n+1}. \tag{74}$$

So from (73) and (74), we get

$$2.3^nQ_yX^2X_1RP^{n-1/4}\geqslant(2/n)X^2X_1RP^{t-1/4}$$
$$-n^{n+1}3^nX^2X_1RP^{t-\frac{1}{2}+1/n^3}-\theta(n^{6t}X^2X_1RP^{t-n-1/4})$$

$$\geqslant(1/n)X^2X_1RP^{t-1/4}[2-n^{n+2}3^nP^{-1/4+1/n^3}-n^{6t+1}P^{-n}]$$

$$\geqslant(1/n)X^2X_1RP^{t-1/4}, \text{ for } N_0\geqslant\beta.$$

Hence

$$Q_y\geqslant P^{t-n}/(2n\,3^n). \tag{75}$$

3·5. *Estimation of D_y.*

It follows from (52), that

$$E_y=\sum_{N_2}E[-2\pi iaN_2/q]\,B^t_{a,q},$$

where

$$N_2=N-\xi-\xi_1-y^nu_1. \tag{76}$$

Hence

$$D_y=\sum_{N_2}\sum_{q=1}^{\sqrt{P}}A_q(N_2)=\sum_{N_2}\infty(N_2)-\sum_{N_2}\sum_{q>\sqrt{P}}A_q(N_2)$$

$$\geqslant X^2X_1RE[-2n^{3\frac{1}{8}}]-X^2X_1Rn^{5t}\sum_{q>\sqrt{P}}1/q^3,$$

from Lemmas (5) and (2)

$$\geqslant X^2X_1RE[-2n^{3\frac{1}{6}}]-X^2X_1Rn^{5t}P^{-1}$$
$$>\tfrac{1}{2}E[-2n^{3\frac{1}{6}}]X^2X_1R, \text{ for } N_0\geqslant\beta. \quad (77)$$

Therefore, from (75) and (77), we get

$$Q_yD_y\geqslant\frac{X^2X_1RP^{t-n}}{2n3^nE[2n^{3\frac{1}{6}}]}=M, \text{ say.} \quad (78)$$

It follows from (60) and (78), that since

$$n^{6t}X^2X_1RP^{t-n-1/4}=\theta(M/2),$$
$$H_{y,1}\geqslant\frac{X^2X_1RP^{t-n}}{4n3^nE[2n^{3\frac{1}{6}}]}. \quad (79)$$

3·6. *The fundamental Integral.*

Now we put $N=N_0$ and consider the integral

$$I_{N_0}=\sum_{y=1}^{Y}I_{y,N_0}=\int_0^1 T^{t-3}T_1V^2S^2\sum_y V_yS_yE[-2\pi i\alpha N_0]d\alpha. \quad (80)$$

By dividing the interval of integration into two classes as in § 3·2, we write

$$I_{N_0}=H_1+H_2. \quad (81)$$

From (79), we have

$$H_1=\sum_y H_{y,1}\geqslant\frac{YX^2X_1RP^{t-n}}{4n3^nE[2n^{3\frac{1}{6}}]}. \quad (82)$$

In the intervals of the second class,

$$\sqrt{P}<q\leqslant\tau=2n3^{n-1}P^{n-\frac{1}{2}};\quad Y<(q/2)^{\nu}.$$

So

$$\sum_y V_yS_y=\sum_y\sum_{u_1}E[2\pi i\alpha y^nu_1]$$
$$\leqslant\sum_{u_1}\left|\sum_y E[2\pi i\alpha y^nu_1]\right|$$
$$\leqslant\sqrt{\left[X_1R\sum_x\left|\sum_y E[2\pi i\alpha y^nx]\right|^2\right]},\ x\leqslant 2^nR^n$$
$$\leqslant\sqrt{\left[X_1R\sum_{y=1}^{Y}\sum_{y_1=1}^{Y}\left|\sum E[2\pi i\alpha x(y^n-y_1^n)]\right|\right]},\ x\leqslant 2^nR^n$$
$$=\sqrt{[X_1R(S_1+S_2)]}, \quad (83)$$

where S_1 represents the coefficient of X_1R when $y \equiv y_1^n \pmod q$ and S_2 the other coefficient.

Now obviously

$$S_1 = 2^n R^n Y.$$

If $$c = y^n - y_1^n \neq 0,$$

then $$|c| < q/2, \ (a, q) = 1.$$

Since $$y \leqslant Y \text{ and } y_1 \leqslant Y,$$

when c is fixed, c cannot be represented in the form $(y^n - y_1^n)$ more than Y ways. Further, since

$$|c| < q/2, \ (a, q) = 1,$$

if $ac \equiv s \pmod q$, where

$$1 \leqslant s \leqslant q - 1,$$

then s is different for different values of c. So from Lemma 13

$$S_2 \leqslant \sum_{y} \sum_{y_1} \frac{1}{2(\alpha c)^{-1}}, \ c = y^n - y_1^n \neq 0,$$

$$\leqslant \left(\frac{Y}{2}\right) \sum_{c \leqslant Y^n} \left(\frac{ac}{q} + \frac{\theta c}{q\tau}\right)^{-1}, \ |\theta| < 1,$$

$$\leqslant \left(\frac{Y}{2}\right) \sum \left(\frac{s}{q} + \frac{\theta}{2q}\right)^{-1}, \text{ at most } Y^2 \text{ terms}$$

$$\leqslant Yq \sum \frac{1}{s} \qquad \text{"} \qquad \text{"}$$

$$\leqslant Yq(2 \log Y + 1).$$

Since $$N_0 > \beta, 2 \log Y + 1 \leqslant P^{1/(2n^2)}$$

Further $$q \leqslant \tau.$$

So $$S_2 \leqslant n 3^n Y P^{n - \frac{1}{2} + 1/(2n^2)}.$$

Therefore,

$$S_1 + S_2 \leqslant Y^2 R [2^n R^{n-1} Y^{-2} + n 3^n P^{n - \frac{1}{2} + 1/(2n^2)} Y^{-1} R^{-1}]$$

$$\leqslant Y^2 R \, 3^{2n} P^{n - (3/2) + 3/(4n^2)}. \tag{84}$$

Hence from (83) and (84), we get

$$\left| \sum V_y S_y \right| \leqslant YR\sqrt{X_1}\, 3^n P^{(n/2) - (3/4) + 3/(8n^2)}.$$

Therefore

$$|H_2| \leqslant 3^{n+1} YR\sqrt{X_1} P^{(n/2) - (3/4) + 3/(8n^2) + t - 2} \int_0^1 \sum_u \sum_{u'} e^{2\pi i \alpha(u - u')} d\alpha$$

$$\leqslant 3^{n+1} YRX_1^{\frac{1}{2}} X P^{t - (4/7) + (n/2) + 3/(8n^2)}$$

$$\leqslant \frac{YX^2X_1RP^{t-n}}{8n3nE[2n^{3\frac{1}{3}}]}; \text{ (from (2), (26), (27) and (73)).} \tag{85}$$

Hence from (81), (82) and (85), we see that

$$I_{N_0} \geqslant \frac{YX^2X_1RP^{t-n}}{8n3^nE[2n^{3\frac{1}{8}}]} > 0. \tag{86}$$

3·7. *The fundamental Lemma.*

I_{N_0} is the number of representations of N_0 in the form

$$N_0 = x^n + x_1^n + \dots + x_{t-3}^n + u + u' + y^n u_1,$$

where $x, x_1, \dots, x_{t-3}, u, u', u_1, y$ run over the set of numbers we have already considered.

The right hand side of the above equality is a sum of

$$t-2+3k \leqslant n^2+1-2+3(4n\log n+1) < 2^n+l-2.$$

Further from (86), we get

$$I_{N_0} > 0.$$

Hence we get

LEMMA 15. *When* $n \geqslant 8$, *every integer* $\geqslant \beta$, *is the sum of* 2^n+l-2, *nth powers.*

§ 4. *Proofs of Theorems*

4·1. *Proof of Theorem I.*

From Lemmas (12) and (15), Theorem I follows.

4·2. *Proof of Theorem II.*

Under the hypothesis of Theorem II, it can be easily verified that

$$N = (j-1)3^n + l2^n - 1,$$

requires exactly $2^n+j+l-3$, nth powers. So Theorem II is proved.

4·3. *Proof of Theorem III.*

Every integer in $[0, 2^n]$ is the sum of 2^n, nth powers. Hence by Lemma (8), every integer in $[0, \beta]$ is the sum of $2^n+\delta$, nth powers. (87)

But

$$2^n - \delta = 2^n + l + d$$
$$= 2^n + l + O(4/3)^n.$$

Hence from (87) and Lemma (15), we get

$$g(n) \leqslant 2^n + l + O(4/3)^n. \tag{88}$$

But $N = l2^n - 1$, requires 2^n+l-2, nth powers. So

$$g(n) \geqslant 2^n + l - 2. \tag{89}$$

From (88) and (89), we have

$$g(n) = 2^n + l + O(4/3)^n.$$

4·4. *Proof of Theorem IV.*

(a) In **R.3**, R. D. James has proved that every positive integer $\leqslant 10^{3920000}$ is the sum of 279, nth powers. But when $n=8$,

$$\beta \leqslant 10^{3920000}$$

Hence from Lemma (15), it follows that

$$g(8)=279. \tag{90}$$

(b) When $n \geqslant 9$, it can be easily proved that

$$\delta \leqslant 3^{3/4} l.$$

With the help of this, and from the table in the appendix, it can be verified that for $9 \leqslant n \leqslant 29$, the hypothesis of Lemma (9) is satisfied. Hence from Lemmas (15) and (9) it follows that when $9 \leqslant n \leqslant 29$,

$$g(n)=2^n+l-2. \tag{91}$$

(c) When $30 \leqslant n \leqslant 100$, from the table it can be easily seen that the hypothesis of Lemma (12) is satisfied. Hence from Lemmas (15) and (12), it follows that when $30 \leqslant n \leqslant 100$,

$$g(n)=2^n+l-2. \tag{92}$$

From (90), (91) and (92), Theorem IV follows.

4·5. *Proof of Theorem V.*

If $\left\{ (3/2)^s \right\} < 1-(3/4)^s$, let us call it hypothesis A. So

$$A: \left\{ (3/2)^s \right\} < 1-(3/4)^s.$$

Let

$$3^n = l2^n + 2^n - a. \tag{93}$$

(a) Let $a \leqslant (3/2)^n$.

Then $$(3/2)^{n+r} = (l+1)/(3/2)^r - a3^r/2^{n+r}.$$

Now in order that A may be false for all s such that $n \leqslant s \leqslant n+r$, we should have

(α) $$a3^r \leqslant (3/2)^{n+r}$$

and (β) $$2^r | (l+1).$$

From (β) $$r \leqslant \frac{n \log(3/2)}{\log 2} + O(1).$$

Hence for more than $\dfrac{n \log(3/2)}{\log 2} + O(1)$

consecutive values, A cannot be false. (94)

(b) Let $a \leqslant (3/2)^n$ and let $n+k$ be the first number from n for which A is true. Then either

(α) $a3^k > (3/2)^{n+k}$, but $a3^{k-1} < (3/2)^{n+k-1}$, or (β) 2^k does not divide $(l+1)$, but $2^{k-1} \mid (l+1)$.

Consider

$$\left(\frac{3}{2}\right)^{n+k+s} = \frac{(l+1)3^{k+s}/2^{k-1}}{2^{s+1}} - \frac{a3^{k+s}}{2^{n+k+s}}.$$

The fractional part of $\frac{(l+1)\, 3^{k+s}/2^{k-1}}{2^{s+1}}$ is either

(γ) 0

or (δ) $\geqslant 1/2^{s+1}$.

Now,

if $$t=(n+k)\log(4/3)/(\log 3)+O(1), \tag{95}$$

then $$a3^{k+t}/2^{n+k+t} \leqslant 2^{t+1}.$$

Therefore if $s \leqslant t$, A is true.

Further $$k+t \leqslant n+O(1)$$
from (94) and (95).

Again $t/(k+t) \geqslant \log(4/3)/(\log 2)+O(1)$.

Hence in the interval $[n, 2n]$, there are at least t numbers for which A is true. Therefore, if n is a number for which A is false. then between n and $2n$, there are more than

$$\log(4/3)/\log 2+O(1)$$

numbers for which A is true.

Let $K_1(x)$ denote the number of $s \leqslant x$, for which A is true.

Further let m be a number at which an interval for which A is true ends. Then from the above

$$K_1(m) \geqslant [\log(4/3)/\log 2]\, m+O(1).$$

Let n be the number following m at which the interval for which A is false ends. Then

$$n-m \leqslant m\log(3/2)/\log 2+O(1).$$

In this case

$$K_1(n)=K_1(m) \geqslant n\log(4/3)/\log 3+O(1).$$

This is the most unfavourable case. Hence

$$K_1(n) \geqslant n\log(4/3)\log n+O(1). \tag{96}$$

Let $3^n = l2^n+2^n-l-1$.

Then $3^n=(l+1)(2^n-1)$.

So $3^r=2^n-1$.

Hence $$n \leqslant 2. \tag{97}$$

So $$3^n \neq l2^n + 2^n - l - 1,$$
when $$n \geqslant 3.$$

Let $$3^n = l2^n + 2^n - l - 2.$$

Then $$\left(\tfrac{3}{2}\right)^n = l + 1 - \frac{l+2}{2^n}.$$

So $$\left(\tfrac{3}{2}\right)^{n+r} = \frac{3^r(l+1)}{2^r} - \frac{3^r(l+2)}{2^{n+r}}.$$

As before, if $$r < n \log\left(\tfrac{4}{3}\right) + O(1),$$
it can be verified that
$$\left\{\left(\tfrac{3}{2}\right)^{n+r}\right\} < 1 - \frac{(3/2)^{n+r} + 2}{2^{n+r}}.$$

Hence if
$$3^n = l2^n + 2^n - l - 2,$$
then for more than
$$n \log(4/3) + O(1)$$
succeeding values we have
$$3^s \neq [(3/2)^s](2^n - 1) + 2^n - 2. \tag{98}$$

From (96), (97) and (98), it follows that the number of n's $\leqslant x$ for which
$$\{(3/2)^n\} \leqslant 1 - (l+3)/2^n$$
is asymptotically equal to $K_1(x)$.

Hence from Theorem I and (96), we get
$$K(x) \sim K_1(x) \geqslant \frac{x \log(4/3)}{\log 3} + O(1).$$

§ 5. *Conclusion.*

When n is even, it is obvious that
$$3^n \neq l2^n + 2^n - l - 2.$$

Hence, from (97), when n is even and greater than 2, Theorems I and II are completely complementary. That is, when n is even and greater than 2,

(1) if $$\{(3/2)^n\} \leqslant 1 - (l+1)/2^n,$$
then $$g(n) = 2^n + l - 2;$$

(2) if the above hypothesis is not true, then
$$g(n) > 2^n + l - 2.$$

When n is odd, the case

$$3^n = l2^n + 2^n - l - 2,$$

offers difficulty.

In this case, we have only

$$g(n) = 2^n + l + O(5/4)^n.$$

Further, if $$\left[\left(1+\frac{1}{r}\right)^n\right] = l_r$$

and if $$\left\{\left(1+\frac{1}{r}\right)^n\right\} \leqslant 1 - l_r/r^n, \quad r = 1, 2, 3, \ldots, m,$$

then it can be easily shown that

$$g(n) \geqslant l_1 + l_2 + \ldots + l_m - m.$$

But if $$\left\{(4/3)^n\right\} \geqslant 1 - l_3/3^n,$$

then it can be proved that

$$\left\{(3/2)^n\right\} < 1 - (l_2 + 3)/2^n.$$

But it is highly probable that

$$\left\{(3/2)^n\right\} \leqslant 1 - (l + 3)/2^n, \quad \text{for all } n \geqslant 3.$$

If this is true, Theorem II loses its interest and there is no point in trying to improve it.

In conclusion, it may not be too much to expect to get a proof for $g(n)$ from the consideration of fractional parts of the numbers $(a/b)^n$ alone.

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APPENDIX: TABLE

To verify from the table that

$$\{(3/2)^n\} \leqslant 1-(l+3)/2,$$

one has merely to see that

$$2^n \geqslant l+r+3.$$

The table was calculated by P. Jagannathan and the author. Then $(3/2)^{100}$ was directly calculated and the results tallied. Hence the table is correct.

TABLE

$3^n = l2^n + r$

n	2^n	l	r
1	2	1	1
2	4	2	1
3	8	3	3
4	16	5	1
5	32	7	19
6	64	11	25
7	128	17	11
8	256	25	161
9	512	38	227
10	1024	57	681
11	2048	86	1019
12	4096	129	3057
13	8192	194	5075
14	16384	291	15225
15	32768	437	29291
16	65536	656	55105
17	131072	985	34243
18	262144	1477	233801

n	2^n	l	r
19	524288	2216	439259
20	1,048,576	3325	269201
21	2,097,152	4987	1856179
22	4,194,304	7481	3471385
23	8,388,608	11222	6219851
24	16,777,216	16834	1,882,337
25	33,554,432	25251	5,647,011
26	67,108,864	37876	50,495,465
27	34,217,728	56815	17,268,667
28	268,435,456	85222	186,023,729
29	536,870,912	127,834	21,200,275
30	1,073,741,824	191,751	63,600,825
31	2,147,483,648	287,626	1,264,544,299
32	4,294,967,296	431,439	3,793,632,897
33	8,589,934,592	647,159	7,085,931,395
34	17,179,869,184	970,739	12,667,859,593

n	2^n	l	r
35	34,359,738,368	1,456,109	20,823,709,595
36	68,719,476,736	2,184,164	28,111,390,417
37	137,438,953,472	3,276,246	84,334,171,251
38	274,877,906,944	4,914,369	253,002,513,753
39	549,755,813,888	7,371,554	484,129,634,315
40	1,099,511,627,776	11,057,332	352,877,275,169
41	2,199,023,255,552	16,585,998	1,058,631,825,507
42	4,398,046,511,104	24,878,997	3,175,895,476,521
43	8.796,093,022,208	37,318,496	5,129,639,918,459
44	17,592,186,044,416	55,977,744	15,388,919,755,377
45	35,184,372,088,832	83,966,617	10,982,387,177,299
46	70,368,744,177,664	125,949,925	68,131,533,620,729
47	140,737,488,355,328	188,924,888	134,025,856,684,523
48	281,474,976,710,656	283,387,333	120,602,593,342,913
49	562,949,953,421,312	425,081,000	80,332,803,318,083
50	1,125,899,906,842,624	637,621,500	240,998,409,954,249

n	2^n	l	r
51	2,251,799,813,685,248	956,432,250	722,995,229,862,747
52	4,503,599,627,370,496	1,434,648,375	2,168,985,689,588.241
53	9,007,199,254,740,992	2,151,972,563	2,003,357,441,394,227
54	18,014,398,509,481,984	3,227,958,844	15,017,271,578,923,673
55	36,028,797,018,963,968	4,841,938,267	9,023,017,717,807,051
56	72,057,594,037,927,936	7,262,907,400	63,097,850,172,385,121
57	144,115,188,075,855,872	10,894,361,101	45,178,362,441,299,491
58	288,230,376,151,711,744	16,341,541,651	279,650,275,399,754,345
59	576,460,752,303,423,488	24,512,312,477	550,720,450,047,551,291
60	1,152,921,504,606,846,976	36,768,468,716	1,075,700,597,839,230,385
61	2,305,843,009,213,693,952	55,152,703,075	921,258,784,303,997,203
62	4,611,686,018,427,387,904	82,729,054,613	457,933,343,698,297,657
63	9,223,372,036,854,775,808	124,093,581,919	5,985,486,049,522,280,875
64	18,446,744,073,709,551,616	186,140,372,879	8,733,086,111,712,066,817
65	36,893,488,147,419,103,232	279,210,559,319	7,752,514,261,426,648,835
66	73,786,976,294,838,206,464	418,815,838,978	60,151,030,931,699,049.737
67	147,573,952,589,676,412,928	628,223,758,468	32,879,140,205,420,736,283

68	295,147,905,179,352,825,856	942,335,637,702	98,637,420,616,262,208,849
69	590,295,810,358,705,651,712	1,413,503,456,553	295,912,261,848,786,626,547
70	1,180,591,620,717,411,303,424	2,120,255,184,830	297,440,975,187,654,227,929
71	2,361,183,241,434,822,606.848	3,180,382,777,245	892,322,925,562,962,683,787
72	4,722,366,482,869,645,213,696	4,770,574,165,868	315,785,535,254,065,444,513
73	9,444,732,965,739,290,427,392	7,155,861,248,802	947,356,605,762,196,333,539
74	18,889,465,931,478,580,854,784	10,733,791,873,203	2,842,069,817,286,589,000,617
75	37,778,931,862,957,161,709,568	16,100,687,809,804	27,415,675,383,338,347,856,635
76	75,557,863,725,914,323,419,136	24,151,031,714,707	6,689,162,424,100,720,150,769
77	151,115,727,451,828,646,838,272	36,226,547,572,060	95,625,350,998,216,483.871,443
78	302,231,454,903,657,293,676,544	54,339,821,358,090	286,876,052,994,649,451,614,329
79	604,462,909,807,314,587,353,088	81,509,732,037,136	256,165,249,176,633,767,489,899
80	1,208,925,819,614,629,174,706,176	122,264,598,055,704	768,495,747,529,901,302,469.697
81	2,417,851,639,229,258,349,412,352	183,396,897,083,556	2,305,487,242,589,703,907,409,091
82	4,835,703,278,458,516.698,824,704	275,095,345,625,335	2,080,758,449,310,595,023,402,569
83	9,671,406,556,917,033,397,649,408	412,643,018,438,003	1,406,572,069,473,268,371,383,003
84	19,342,813,113,834,066,795,298,816	618,964,527,657,004	13,891,122,765,336,838,511,798,417
85	38,685,626,227,668,133,590,597,632	928,446,791,485,507	2,987,742,068,342,381,944,797,619
86	77,371,252,455,336,267,181,195,264	1,392,670,187,228,260	47,648,852,432,695,279,424,990,489

n	2^n	l	r
87	154,742,504,910,672,534,362,390,528	2,089,005,280,842,390	142,946,557,298,085,838,274,971,467
88	309,485,009,821,345,068,724,781,056	3,133,507,921,263,586	119,354,662,072,912,446,100,133,345
89	618,970,019,642,690,137,449,562,112	4,700,261,881,895,379	358,063,986,218,737,338,300,400,035
90	1,237,940,039,285,380,274,899,124,224	7,050,392,822,843,069	455,221,939,013,521,877,451,637,993
91	2,475,880,078,570,760,549,798,248,448	10,575,589,234,264,604	127,725,777,755,185,357,455,789,755
92	4,951,760,157,141,521,099,596,496,896	15,863,383,851,396,906	383,177,333,265,556,072,367,369,265
93	9,903,520,314,283,042,199,192,993,792	23,795,075,777,095,359	1,149,531,999,796,668,217,102,107,795
94	19,807,040,628,566,084,398,385,987,584	35,692,613,665,643,038	13,352,116,313,673,046,850,499,317,177
95	39,614,081,257,132,168,796,771,975,168	53,538,920,498,464,558	442,267,683,886,971,754,725,976,363
96	79,228,162,514,264,337,593,543,950,336	80,308,380,747,696,837	1,326,803,051,660,915,264,177,929,089
97	158,456,325,028,528,675,187,087,900,672	120,462,571,121,545,255	83,208,571,669,247,083,386,077,737,603
98	316,912,650,057,057,350,374,175,801,344	180,693,856,682,317,883	91,169,389,979,212,574,971,145,312,137
99	633,825,300,114,114,700,748,351,602,688	271,040,785,023,476,824	590,420,819,994,695,075,287,611,737,755
100	1,267,650,600,228,229,401,496,703,205,376	406,561,177,535,215,237	503,611,859,755,855,824,366,132,007,889

GENERALISATIONS OF THE THEOREMS OF MALUS-DUPIN, BELTRAMI AND RIBAUCOUR IN RECTILINEAR CONGRUENCES

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1. Consider a thin pencil formed by rays adjacent to a ray l of the rectilinear congruence given by

$$\xi = x + tX,\ \eta = y + tY,\ \zeta = z + tZ$$

where x, y, z, and X, Y, Z are functions of two parameters u and v. Let C be the closed curve on the surface of reference which forms the boundary of the area dS on it cut off by the pencil. Let (x, y, z) be the point where the ray l meets C. The equation of the orthogonal trajectories of the generators is

$$u = \text{const.} - \int (Xdx + Ydy + Zdz).$$

The orthogonal trajectories are closed curves if

$$\int_C (Xdx + Ydy + Zdz) = 0$$

where the line integral is taken over the boundary of the closed curve C. Hence the distance between the two points where an orthogonal trajectory cuts l is given by

$$\int_C (Xdx + Ydy + Zdz).$$

We call this distance, for convenience, the pitch (p) of the pencil at l. Various properties of the 'pitch' have been obtained in my paper* on "A Significant Integral invariant in the Theory of Rectilinear Congruences".

* *Jour. Ind. Math. Soc.* (New Series) Vol. I, No. 4, (1934), 135-42.

The object of this paper is to obtain generalisations of the theorems of Malus-Dupin†, Beltrami‡ and Ribaucour§ in terms of the 'pitch'.

2. Let the direction cosines of the incident ray PO meeting a reflecting surface at O be X, Y, Z and let those of the reflected ray OQ, and the normal to the surface at O be respectively X', Y', Z'; and λ, μ, ν. Then equating the sums of the projections of OP and OQ to twice the projection of OM (where $OP=OQ=1$ and M is the point where the line PQ cuts the normal at O) on the co-ordinate axes, we have

$$(X'-X)=2\lambda \cos\phi,\ (Y'-Y)=2\mu \cos\phi,\ (Z'-Z)=2\nu \cos\phi,$$

where

$$\cos\phi=X'\lambda+Y'\mu+Z'\nu=-(X\lambda+Y\mu+Z\nu).$$

Pitch of a pencil of the congruence formed by the incident rays

$$=\int_C \Sigma X dx. \qquad (1)$$

Pitch of a pencil of the congruence formed by these very rays after reflection by the surface

$$=\int_C \Sigma X' dx=\int_C \Sigma(X+2\lambda \cos\phi) dx$$

$$=\int_C \Sigma X dx+2\cos\phi\int_C \Sigma \lambda dx$$

$$=\int_C \Sigma X dx,$$

since $\Sigma \lambda dx=0$.

The same process can be repeated for any number of reflections, hence we get the result:

The pitch of a pencil of a congruence formed by the incident rays remains unaltered by reflection.

This is an extension of the Malus and Dupin's theorem, viz. that "if a bundle of rays of light forming a normal congruence be reflected any number of times by a series of surfaces, the rays continue to constitute a normal congruence".

† See Eisenhart, *Differential Geometry*. p. 403; Darboux, *Lecons*, Vol. 2, pp. 280-81; Bianchi, *Lezioni di geometria differenziale*, Vol. 1, p. 476.

‡ *Giornalle di matematiche*, Vol. II (1864), p. 281; Darboux, *Lecons*, Vol. 3, p. 348; Bianchi, *Lezioni*, Vol. I, p. 476.

§ Salmon, *Analytic Geometry of three dimensions*, Vol. II (1915), p. 70.

Here C has been supposed to be the closed curve bounding a small closed area on the reflecting surface at O.

3. Again let X, Y, Z be the direction cosines referred to any rectangular axes, of the incident ray PO; X', Y', Z' those of the refracted ray OQ, and let p, q, r be the direction cosines of the normal MN to the refracted surface at O, taken in the direction from the medium in which the light is incident into the medium in which it is refracted.

Then equating the sum of the projections of PO, OM to the sum of the projections of NO and OQ on the axes of reference we have

$$\mu(X-\cos\phi .\ p)=\mu'(-\cos\phi' .\ p+X'),$$

with two similar equations, where μ and μ' are the refracting coefficients of the two media.

Hence

$$\frac{(\mu'X'-\mu X)}{p}=\frac{(\mu'Y'-\mu Y)}{q}=\frac{(\mu'Z'-\mu Z)}{r}=\mu'\cos\phi'-\mu\cos\phi. \quad (2)$$

Also $$\cos\phi=Xp+Yq+Zr,$$

and ϕ' is given by the equation

$$\mu\sin\phi=\mu'\sin\phi'.$$

Now pitch of a pencil at a ray of the congruence formed by the incident rays

$$=\int_C \Sigma X dx. \quad (3)$$

And pitch of a pencil at a ray of the congruence formed by these very rays after refraction

$$=\int_C \Sigma X' dx$$

$$=\int_C \sum\left[\frac{\mu X+p(\mu'\cos\phi'-\mu\cos\phi)}{\mu'}\right] dx \text{ from (2)},$$

$$=\frac{\mu}{\mu'}\int_C \Sigma X dx,$$

since $$\int_C \Sigma p dx=0,$$

which is the same as (3) except for the factor $\frac{\mu}{\mu'}$.

Hence the pitch of a pencil at a ray of the congruence formed by the incident rays remains unaltered by refraction except for a *factor which is equal to the ratio of the refracting indices of the two media.*

By putting $\frac{\mu}{\mu'}=-1$, we get the case (1), viz. of reflection.

4. The pitch of a pencil of a congruence at a ray is given by

$$p=\int_C \Sigma X dx=\int_C (\Sigma X x_1 du+\Sigma X x_2 dv).$$

Let $v=$const. and $u=$const. be an orthogonal system of parametric curves on the surface of reference, and suppose that a ray of the congruence meets the curve $v=$const. making an angle θ with the tangent to it at the point of intersection, then

$$\cos\theta=\frac{\Sigma X x_1}{\sqrt{E}}.$$

Hence

$$\Sigma X x_1=\sqrt{E}\cos\theta.$$

Similarly

$$\Sigma X x_2=\sqrt{G}\cos\phi$$

where ϕ is the angle which the ray makes with $u=$const. Hence

$$\text{pitch}=\int_C (\sqrt{E}\cos\theta du+\sqrt{G}\cos\phi dv).$$

Hence if the surface of reference be deformed and with it the congruence as in Beltrami's theorem, then since E and G remain unaltered, it follows that the pitch remains unaltered.

This gives an extension of Beltrami's theorem which says that: "If a surface of reference of a normal congruence be deformed in such a way that the directions of the lines of the congruence with respect to the surface be unaltered, the congruence continues to be normal"*.

5. Let a line l be drawn in the tangent plane at each point M of a surface of reference S so as to give a congruence. Let S be referred to parametric curves $u=$const. and $v=$const. Suppose l makes an angle θ with the tangent to the curve $v=$const. and an angle ϕ with the tangent to the curve $u=$const.

* For other proofs of Beltrami's theorem, see C. H. Rowe, 'A kinematical treatment of some theorems on Normal Rectilinear Congruences, *Transactions of the American Mathematical Society*, Vol. 31, p. 922.

Darboux, *Lecons*, Vol. 3, p. 348.

Suppose the direction cosines of the normal to the surface of reference at M are λ, μ, ν then

$$\Sigma\lambda dx=0,$$

i.e.
$$\Sigma\lambda x_1 du+\Sigma\lambda x_2 dv=0.$$

Hence

$$\Sigma\lambda x_1=0,\ \Sigma\lambda x_2=0:$$

also if the direction cosines of the line are $X, Y, Z,$ then since l lies in the tangent plane,

$$\Sigma\lambda X=0.$$

Eliminating λ, μ, ν from these three equations we get

$$\begin{vmatrix} X & Y & Z \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix}=0.$$

Hence

$$X=ax_1+bx_2,\ Y=ay_1+by_2,\ Z=az_1+bz_2,$$

where a and b are constants.

Let the ray (X, Y, Z) of the congruence meet the tangent to the curve $v=$const. at P, then the co-ordinates of P are given by

$$\xi=x+r(u, v)x_1,\ \ldots\ldots,\ \ldots\ldots,$$

where
$$x_1\equiv\frac{\partial x}{\partial u},\ \ldots,\ \ldots\ .$$

The pitch of a pencil of the congruence at the ray (X, Y, Z) is given by

$$\begin{aligned} p&=\int_C \Sigma X d\xi \\ &=\int_C \Sigma X(dx+rdx_1+x_1 dr) \\ &=\int_C \left\{ \Sigma Xx_1 du+\Sigma Xx_2 dv \right\} \\ &\qquad\qquad +\Sigma X\left\{ r(x_{11}du+x_{12}dv)+x_1(r_1 du+r_2 dv) \right\} \\ &=\int_C \Sigma Xx_1(1+r_1)du+\Sigma Xx_2 dv \\ &\qquad\qquad +\Sigma Xx_1 r_2 dv+r(\Sigma Xx_{11}du+\Sigma Xx_{12}dv). \end{aligned}$$

But
$$\Sigma Xx_1=\sqrt{E}\cos\theta,$$

and
$$\Sigma Xx_2=\sqrt{G}\cos\phi,$$

where θ and ϕ are the angles which the ray makes with $v=$const. and $u=$const.

Also

$$\Sigma X x_{11} = \Sigma x_{11}(a x_1 + b x_2) = a\Sigma x_1 x_{11} + b\Sigma x_2 x_{11} = a\tfrac{1}{2}E_1 + b(F_1 - \tfrac{1}{2}E_2),$$

and $\Sigma X x_{12} = \Sigma x_{12}(a x_1 + b x_2) = a\Sigma x_1 x_{12} + b\Sigma x_2 x_{12} = a\tfrac{1}{2}E_2 + b\tfrac{1}{2}G_1$*.

Thus the pitch depends upon E, F, G and their differentials, and upon θ and ϕ.

Hence if the surface of reference is deformed and with it the congruence as in Ribaucour's theorem, the pitch of any pencil of the congruence remains unaltered by deformation.

This gives an extension of Ribaucour's theorem in Rectilinear Congruences.

* See Forsyth, *Differential Geometry*, p. 44.

A NOTE ON THE ZEROS OF PARABOLIC CYLINDER FUNCTIONS OF THE SECOND KIND

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1. The cylinder function $D_n(z)$ of order n is the standard solution of Weber's equation

$$\frac{d^2y}{dz^2}+(n+\tfrac{1}{2}-\tfrac{1}{4}z^2)y=0. \tag{1}$$

Here n is any real or complex number and $D_n(z)$ is given by

$$D_n(z)=e^{-\frac{1}{4}z^2}\sum_{m=0}^{\infty}\frac{\Gamma(\frac{1}{2}m-\frac{1}{2}n)}{\Gamma(m+1)\Gamma(-n)}(\sqrt{2})^{m-n-2}(-z)^m \tag{2}$$

when n is not a positive integer. When n is a positive integer $D_n(z)$ is known to be the product of $e^{-\frac{1}{4}z^2}$ and a polynomial and is explicitly given by

$$D_n(z)=(-1)^n e^{\frac{1}{4}z^2}\frac{d^n}{dz^n}(e^{-\frac{1}{2}z^2}). \tag{3}$$

Hence in all cases $D_n(z)$ is an integral function of z.

1·1. We shall designate the function $E_n(x)$ given by

$$E_n(x)=\pm e^{\mp n\pi i}i\sqrt{2\pi}\,\Gamma(n+1)D_{-n-1}(\mp ix)e^{-\frac{1}{4}x^2}, \tag{4}$$

where the upper or the lower sign is to be taken according as $I(x)>0$ or $I(x)<0$, a parabolic cylinder function of the second kind. Dr. Watson* proves that (4) is the value of the definite integral

$$\int_{-\infty}^{\infty}\frac{e^{-\frac{1}{4}z^2}D_n(z)}{z-x}dz \tag{5}$$

when x is not real and n is a positive integer. From (4) we can define $E_n(x)$ for all x not real and all n such that $(n+1)$ is not a negative integer. The object of this note is to prove the following theorem:

THEOREM: *The functions $E_n(x)$ and $E_{n+m}(x)$ have no common zeros, $n+1$ being not a negative integer, and m being a positive integer.*

* Watson, *Proc. Lond. Math. Soc.* 2nd Series, Vol. 8, p. 417.

NOTE: By (4), the real axis is a cut for $E_n(x)$ so that when we speak of the zeros of $E_n(x)$ we mean only those lying in $I(x)>0$ and $I(x)<0$.

1·2. PROOF: From (1) and (4) it is easy to show that the differential equation satisfied by $E_n(x)$ is

$$E''_n - xE'_n + (n+1)E_n = 0. \tag{6}$$

Also using the recurrence formula† for $D_n(z)$ we get the two equations

$$\left.\begin{aligned} E_{n+1}(x) - xE_n(x) + nE_{n-1}(x) &= 0 \\ E'_n(x) + xE_n(x) - nE_{n-1}(x) &= 0. \end{aligned}\right\} \tag{7}$$

From (6) we deduce that $E'_n(x)$ and $E_n(x)$ cannot vanish simultaneously, since then $E_n^{(p)}(x)=0$ for all p and so $E_n(x)$ would vanish identically. So from (7) we deduce that E_n and E_{n-1} do not become zero at the same time. Next using the recurrence formula in (7) we get

$$E_{n+m}(x) = \begin{vmatrix} x & n+m-1 & 0 & \cdots & 0 & 0 \\ 1 & x & n+m-2 & \cdots & 0 & 0 \\ 0 & 1 & x & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & E_{n-1}(x) & E_n(x) \end{vmatrix}$$

$$= L(x)E_n(x) - M(x)E_{n-1}(x), \tag{8}$$

where $L(x)$ and $M(x)$ are polynomials, the latter being of degree $m-1$. Using the method followed in a previous paper‡ by the author, it can be deduced that

$$M(x) = \begin{vmatrix} x & n+m+1 & 0 & \cdots & \cdots & 0 \\ 1 & x & 0 & \cdots & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & x \end{vmatrix}$$

and the functions $M_r(x)$ obtained from $M(x)$ by suppressing the 1st r rows and columns $(r=0,..m-1)$ form a Sturm chain of polynomials, and therefore that $M(x)$ has $m-1$ real roots, and being of degree $m-1$ has no other roots. It follows from (8) and the fact that we consider $E_n(x)$ only in the two half-planes $I(x)>0$ and $I(x)<0$, that if E_{n+m} and E_n have a common zero that zero will also be a root of $E_{n-1}(x)=0$, that is, $E_n(x)$ and $E_{n-1}(x)$ have a common zero which has been proved to be untrue. So the theorem is proved.

† *l. c.* 418.

‡ D. P. Banerjee, "On the Zeros of Bessel Functions", *Journal of the Ind. Math. Soc.* Vol. I (New Series), No. 8, (1935) pp. 266-68.

ON INTEGRAL FUNCTIONS OF FINITE ORDER BOUNDED AT A SEQUENCE OF POINTS

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I. Introduction

1. Let $f(z)$ be an integral function and $M(r)=\max_{|z|\leqslant r} |f(z)|$. The order ρ of the function is defined by

$$\rho = \overline{\lim_{r\to\infty}} \frac{\log \log M(r)}{\log r}.$$

If ρ is finite, the upper type k and the lower type l are defined by

$$l= \underline{\lim_{r\to\infty}} \frac{\log M(r)}{r^\rho} \leqslant \overline{\lim_{r\to\infty}} \frac{\log M(r)}{r^\rho}=k.$$

The function is said to be of maximal, normal or minimal type according as $k=\infty$, a finite positive number or zero. Let $z=re^{i\theta}$. We define the upper and lower types, $k(\theta)$ and $l(\theta)$, respectively on the line $\operatorname{amp}(z)=\theta$ by the relations

$$l(\theta) = \underline{\lim_{r\to\infty}} \frac{\log|f(re^{i\theta})|}{r^\rho} \leqslant \overline{\lim_{r\to\infty}} \frac{\log|f(re^{i\theta})|}{r^\rho}=k(\theta).$$

When more than one function enters into a discussion we distinguish the above quantities for the respective functions by writing $k(f)$, $l(f)$, $k(\theta,f)$, etc., and $M(f)$, etc., where, in the last symbol, the variable r is understood.

1.1. In a recent paper* I have investigated the properties of functions of order *one* bounded at a sequence of points $z=\pm\lambda_n$, where

$$0<\lambda_1<\lambda_2 \ldots <\lambda_n\to\infty$$

and the function

$$\phi(z)=\prod_{\nu=1}^{\infty}\left(1-\frac{z^2}{\lambda^2}\right) \tag{1}$$

is of order one and positive lower type with an additional hypothesis on the order of increase of $|\phi'(\lambda_n)|$. In the present

*V. Ganapati Iyer, 'On Integral Functions of order one and finite type,' *Jour. Ind. Math. Soc.* Vol. II, No. 1, New Series, pp. 1—12.

paper it is proposed to extend the results there obtained to functions of any finite order. Section II contains the treatment of the case $0<\rho<1$ and section III, the case $\rho>1$. The last section IV contains some additional remarks and particular cases of the general theorems in II and III.

II. FUNCTIONS FOR WHICH $0<\rho<1$

2. By an analysis of the method used to prove the results contained in the author's paper quoted above which we shall hereafter refer as (A), for the case $\rho=1$, it can be shown that, if $f(z)$ is of order ρ, $0<\rho<1$ and bounded at $z=\pm\lambda_n$ where the 'base function'(1) for this $[\lambda_n]$ is of order ρ and positive lower type, all the results in (A) remain true provided the corresponding hypothesis on the order of increase of $|\phi'(\lambda_n)|$ is satisfied. Therefore we consider here a more general case where the function $f(z)$ is bounded at a sequence of points on a line tending to infinity only in one direction on the line. We can suppose, without loss of generality, that this direction is that of the negative real axis and the points in question lie to the left of the origin.

Let, therefore, $0<\lambda_1<\lambda_2 \ldots$ be a sequence tending to infinity and satisfying the condition

$$0<A=\varliminf_{r\to\infty}\frac{n(r)}{r^{\rho}}\leqslant\varlimsup_{r\to\infty}\frac{n(r)}{r^{\rho}}=B, \tag{2}$$

where $n(r)$ is the number of these $[\lambda_n]$ not exceeding r. It is seen from (2) that $\sum\frac{1}{\lambda_n}$ converges. We set

$$\sigma(z)=\prod_{\nu=1}^{\infty}\left(1+\frac{z}{\lambda_n}\right) \tag{3}$$

and make the following additional hypothesis, that there is an integer $p\geqslant 0$ so that

$$\sum_{\nu=1}^{\infty}\left|\frac{1}{\sigma'(-\lambda_\nu)}\frac{1}{\lambda_\nu^{p+1}}\right|<\infty. \tag{4}$$

We shall show below that $\sigma(z)$ is an integral function of order ρ and positive lower type, We shall use this as the base—, or the comparison-function to study the properties of functions of order ρ bounded at $z=-\lambda_n$.

2·1. Let $f(z)$ be a function of order ρ. Let $f(-\lambda_n)=y_n$. We assume that $[y_n]$ is bounded and prove the following two theorems corresponding to Theorems 1 and 2 of (A).

THEOREM 1. *Let* $[y_n]$, *in addition to being bounded satisfy one of the following conditions*:

(α) *the real part of* $\frac{y_n}{\sigma'(-\lambda_n)}$ *does not change sign as n varies, not all these real parts vanishing*;

(β) *a similar hypothesis on the imaginary part*;

(γ) *all* $y_n=0$.

Under (α) *or* (β) *we must have*

$$k(f)\geqslant l(\theta_\rho, \sigma), \text{ where } \theta_\rho=\max\left(\pi-\frac{\pi}{2\rho}, 0\right).$$

Under (γ), $k(f)\geqslant l(\theta_\rho, \sigma)$ *or else* $f(z)\equiv 0$.

THEOREM 2. *Let* $[\alpha_\nu]$, $|\alpha_\nu|\to\infty$ *as* $\nu\to\infty$, *be any sequence of numbers. Let* $E(\lambda, \alpha)$ *denote the set*

$$z=-\alpha_\nu\lambda_n,\ \nu,\ n=1, 2, \ldots.$$

If $f(z)$ *is of order* ρ *and* minimal *type and bounded at* $E(\lambda, \alpha)$, $f(z)$ *reduces to a constant.*

2·11. As in (A), we first establish some properties of $\sigma(z)$ and next establish the fundamental interpolation formula which is used to prove Theorems 1 and 2.

2·2. THEOREM 3. *For the function* $\sigma(z)$ *the following relations are valid*:

$$(i)\quad \frac{\pi A}{\sin \pi\rho}\leqslant l(\sigma)\leqslant k(\sigma)\leqslant\frac{\pi B}{\sin \pi\rho}; \tag{5}$$

(*ii*) *the quantities* $l(\theta, \sigma)$ *and* $k(\theta, \sigma)$ *are non-increasing continuous functions of* θ *in* $0\leqslant\theta\leqslant\frac{\pi}{2}$ *and*

$$l(\theta)\geqslant l\left(\frac{\pi}{2}\right)\geqslant\frac{\pi A}{2\sin \pi\rho/2}. \tag{6}$$

Also $\qquad l(\theta)=l(-\theta),\ k(\theta)=k(-\theta).$

PROOF: It is evident that $M(\sigma)$ is attained on the positive real axis so that

$$l(0, \sigma)=l(\sigma),\ k(0, \sigma)=k(\sigma).$$

It is also obvious from (3) that

$$l(\theta)=l(-\theta),\ k(\theta)=k(-\theta)$$

and that $l(\theta)$ is a non-increasing function for $0\leqslant\theta\leqslant\frac{\pi}{2}$. Putting $\cos\theta=\xi$, $0\leqslant\theta\leqslant\frac{\pi}{2}$ so that $0\leqslant\xi\leqslant 1$, we get from (3),

$$2\log|\sigma(re^{i\theta})| = \sum_{\nu=1}^{\infty}\log\left(1+\frac{2r\xi}{\lambda_\nu}+\frac{r^2}{\lambda_\nu^2}\right)$$

$$=\int_0^\infty \log\left(1+\frac{2r\xi}{t}+\frac{r^2}{t^2}\right)dn(t)$$

$$=O(\log r)+2r\int_0^\infty \frac{n(t)\,(r+t\xi)}{(r^2+2rt\xi+t^2)}\frac{dt}{t},$$

in which, putting

$$N(t)=\frac{n(t)}{t^\rho},\ t=vr,$$

we get

$$2\log|\sigma(re^{i\theta})| = O(\log r)+2r^\rho\int_0^\infty \frac{v^{\rho-1}(1+v\xi)N(vr)}{v^2+2v\xi+1}dv, \qquad (7)$$

where it is easily seen that the constant in $O(\log r)$ can be chosen to be independent of ξ, $0\leqslant\xi\leqslant1$. Let $\cos\theta_1=\xi_1$, $\cos\theta_2=\xi_2$ and $\chi(v,\xi)$ be the factor multiplying $N(vr)$ in (7). Then we get by (2),

$$N(vr)\leqslant C, \text{ a constant,}$$

and

$$\overline{\lim_{r\to\infty}}\left|\frac{\log|\sigma(re^{i\theta_1})|}{r^\rho}-\frac{\log|\sigma(re^{i\theta_2})|}{r^\rho}\right| \leqslant C\int_0^\infty |\chi(v,\xi_1)-\chi(v,\xi_2)|\,dv,$$

from which, since $\int_0^\infty \chi(v,\xi)dv$ is uniformly convergent in ξ, we deduce that $l(\theta,\sigma)$, $k(\theta,\sigma)$ are continuous functions of θ in the range considered. To prove (5), we put $\xi=1$ corresponding to $\theta=0$, in (7). We get in virtue of (2),

$$l(\sigma)\geqslant A\int_0^\infty \frac{v^{\rho-1}}{1+v}dv=\frac{\pi A}{\sin\pi\rho},$$

and

$$k(\sigma)\leqslant B\int_0^\infty \frac{v^{\rho-1}}{1+v}dv=\frac{\pi B}{\sin\pi\rho}.$$

Putting $\xi=0$ in (7) corresponding to $\theta=\frac{\pi}{2}$, we get

$$l\left(\frac{\pi}{2}\right)\geqslant A\int_0^\infty \frac{v^{\rho-1}}{1+v^2}dv=\frac{\pi A}{2\sin\pi\rho/2}.$$

This completes the proof of the theorem.

2·21. The following result, which is true for functions of any finite order λ, is known† and so we quote it as a lemma.

† See R. Nevanlinna, *Fonctions Meromorphes*, Paris, pp. 31, 34.

LEMMA 1. *Let $f(z)$ be of order λ. Let $|z|=r$. Let $\varepsilon>0$ be given and $0<\eta<\varepsilon$. Then there is a finite number $P=P(\varepsilon)$ so that*

$$|f(z)|\geqslant\exp(-r^{\lambda+\varepsilon}) \tag{8}$$

except for a set of values of r over which the total variation of r^{η} does not exceed $P(\varepsilon)$.

2·3. We now establish the fundamental interpolation formula.

THEOREM 4. *Let $f(z)$ be a function of order ρ and let $f(-\lambda_n)=y_n$ be bounded. Let $k(f)<l(\theta_\rho, \sigma)$, where θ_ρ has the value given in Theorem 1. Let*

$$g(z)=(-1)^{p}\sum_{\nu=1}^{\infty}\frac{z^{p}}{\sigma'(-\lambda_\nu)\lambda_\nu^{p}}\frac{y_\nu}{z+\lambda_\nu}, \tag{9}$$

so that $g(z)$, in virtue of (4) represents a meromorphic function with simple poles at $z=-\lambda_\nu$ and residues $\dfrac{y_\nu}{\sigma'(-\lambda_\nu)}$. Then,

$$\frac{f(z)}{\sigma(z)}=C_{p-1}(z)+g(z), \tag{10}$$

where $C_{p-1}(z)$ is a polynomial of degree $p-1$ at most.

PROOF: Let

$$G(z)=\frac{f(z)}{\sigma(z)}-g(z). \tag{11}$$

Then $G(z)$ is an integral function. To find the order of $M(G)$ we proceed as in (A) and use the following two lemmas which could be proved exactly as the corresponding lemmas in (A).

2·31. LEMMA 2. *Let $h>0$ be given. Let z, $|z|=r$, lie outside the circles with centres $z=-\lambda_n$ and radii λ_n^{-h}. Then there is a constant H so that*

$$|g(z)|\leqslant Hr^{p+h+1} \tag{12}$$

2·311. LEMMA 3. *Let $\beta>1$. Then there is a sequence $[r_n]$, $r_n\to\infty$, $r_{n+1}/r_n<\beta$, so that on $|z|=r_n$, (8) and (12) hold simultaneously.*

2·32. Resuming the proof, we find, as in (A), using (8), (12) and Lemma 3, along with the fact that $M(G)$ is an increasing function of r, that there is an $r_0=r_0(\varepsilon)$ so that for $r\geqslant r_0$,

$$M(G)\leqslant\exp[r^{\rho+2\varepsilon}]. \tag{13}$$

Next we find an upper bound for $M(G)$ on the lines $\theta=\theta_\rho+\eta$ where $\eta>0$ is to be properly chosen. We need the following lemma.

2·321. LEMMA 4. *On*

$$\theta=\theta_\rho+\eta,\ \theta_\rho+\eta<\frac{\pi}{2},$$

$$g(z)=o(|z|^p). \tag{14}$$

PROOF: We have, if $|z|=r$,

$$|g(z)|\leqslant Mr^p\sum_{\nu=1}^{\infty}\frac{k_\nu\lambda_\nu}{|z+\lambda_\nu|}, \tag{15}$$

where

$$k_\nu=\left|\frac{1}{\lambda_\nu^{p+1}\sigma'(-\lambda_\nu)}\right|$$

so that Σk_ν converges. Now,

$$|z+\lambda_\nu|^2=r^2+2r\cos(\theta_\rho+\eta)+\lambda_\nu^2\geqslant r^2+\lambda_\nu^2.$$

So, (15) gives

$$|g(z)|\leqslant Mr^p\sum_{\nu=1}^{\infty}\frac{k_\nu}{\sqrt{1+r^2/\lambda_\nu^2}},$$

which gives (14).

2·33. On $\theta=\theta_\rho+\eta,$

we get

$$|G(z)|\leqslant o(r^p)+\exp\left[k(f)-l(\theta_\rho+\eta,\sigma)-2\varepsilon\right]r^\rho.$$

Now, $k(f)<l(\theta_\rho)$, by hypothesis. Also $l(\theta)$ is a continuous function by Theorem 3. Therefore, if

$$\delta=\tfrac{1}{2}[l(\theta_\rho)-k(f)],$$

we can find an $\eta>0$ so that, $\theta_\rho+\eta<\frac{\pi}{2}$ and

$$|G(z)|\leqslant o(r^p)+\exp(-\delta r^\rho). \tag{16}$$

Moreover (16) is true on $\theta=-(\theta_\rho+\eta)$. Let π/γ be the greater of the angles between the lines $\theta=\pm(\theta_\rho+\eta)$. Then obviously $\rho<\gamma$. Now choose ε in (13) so that $\rho+2\varepsilon<\gamma$. Now we apply the following classical theorem due to Phragman and Lindelof‡.

‡ See G. Valiron, *Integral Functions*, p. 125.

2·331. LEMMA 5. *Let* $|f(z)|=O(1)$ *as* $|z|\to\infty$ *along the sides of an angle of magnitude* π/γ *and let the order of* $f(z)$ *in the angle be less than* γ. *Then*

$$f(z)=O(1)$$

as $|z|\to\infty$ *uniformly in the angle.*

2·34. By (16) we see that $\left|\dfrac{G(z)}{z^p}\right|$ is bounded on the lines $\theta=\pm(\theta_\rho+\eta)$ and the order in any one of the angles between these lines is less than γ where π/γ is the magnitude of the greater of these angles. Hence by Lemma 5

$$|G(z)|=O(|z|^p) \text{ as } |z|\to\infty$$

so that $G(z)$ is a polynomial of degree p at most. But $G(z)=o(r^p)$ on $\theta=\theta_\rho+\eta$. Hence $G(z)$ is a polynomial of degree $p-1$ at most. Therefore the theorem is proved.

2·4. PROOF OF THEOREM 1. We can now prove Theorem 1. First, we shall dispose of (γ). Let all $y_n=0$ and $k(f)<l(\theta_\rho, \sigma)$. Then (11) is valid and we get

$$f(z)\equiv C_{p-1}(z)\sigma(z),$$

from which we deduce that $C_{p-1}(z)\equiv 0$ since otherwise $k(f)\geqslant l(\theta_\rho, \sigma)$. Hence $f(z)\equiv 0$.

Next consider (α) or (β). It is sufficient to consider (α) since if (β) is true, $if(z)$ satisfies (α). Suppose that, under (α) the theorem is not true. Then $k(f)<l(\theta_\rho, \sigma)$. So we can make use of (10). Putting $z=r$ in (10), we get

$$M(f)\geqslant M(\sigma)\left|(-1)^p\sum_{\nu=1}^{\infty}\frac{r^p}{r+\lambda_\nu}\frac{y_\nu}{\lambda_\nu^p\sigma'(-\lambda_\nu)}+C_{p-1}(r)\right|$$

$$\geqslant M(\sigma)\left|r^p\sum_{\nu=1}^{\infty}\frac{k_\nu\lambda_\nu}{r+\lambda_\nu}+D_{p-1}(r)\right|, \tag{17}$$

considering the real part alone, where

$$k_\nu=\left|\frac{R(y_\nu)}{\sigma'(-\lambda_\nu)}\frac{1}{\lambda_\nu{}^{p+1}}\right|,$$

$R(z)$ being the real part of z and $D_{p-1}(r)$ is a polynomial with real coefficients and of degree $p-1$ at most. We now require a lower bound for the factor multiplying $M(\sigma)$ in (17). We prove the following lemma.

2·41. LEMMA 6. *Let Σk_ν be a convergent series of non-negative terms such that $\Sigma k_\nu \neq 0$. Let $h \geqslant 1$ be such that $\Sigma k_\nu \lambda_\nu^{h-1}$ converges. Then there is an integer P, positive or negative so that*

$$\varliminf_{r\to\infty} r^P \left| r^p \sum_{\nu=1}^{\infty} \frac{k_\nu \lambda_\nu^h}{r+\lambda_\nu} + D_{p-1}(r) \right| > 0,$$

where $D_{p-1}(r)$ is a real polynomial of degree $p-1$ at most.

PROOF: Let

$$D_{p-1}(r) = a_{p-1} r^{p-1} + a_{p-2} r^{p-2} + \ldots + a_0$$

and

$$V(r) = r \sum \frac{k_\nu \lambda_\nu^h}{r+\lambda_\nu} + a_{p-1} + \frac{a_{p-2}}{r} + \ldots + \frac{a_0}{r^{p-1}}. \tag{18}$$

If $\Sigma k_\nu \lambda_\nu^h$ diverges, we find

$$r \sum \frac{k_\nu \lambda_\nu^h}{r+\lambda_\nu} \geqslant \frac{1}{2} \sum_{1}^{n(r)} k_\nu \lambda_\nu^h \to \infty \text{ as } r \to \infty,$$

so that the lemma is true in this case.

If $\Sigma k_\nu \lambda_\nu^h$ converges and is not equal to $-a_{p-1}$, the lemma is true. If

$$\Sigma k_\nu \lambda_\nu^h + a_{p-1} = 0,$$

we can write (18) as

$$-rV(r) = r \sum \frac{k_\nu \lambda_\nu^{h+1}}{r+\lambda_\nu} - a_{p-2} - \frac{a_{p-3}}{r} - \ldots - \frac{a_0}{r^{p-2}}$$

which is of the same form as (18) with $h+1$ for h and $p-1$ for p since $\Sigma k_\nu \lambda_\nu^{h+1-1}$ converges. So we can repeat the argument and prove the lemma in a finite number of steps.

2·42. We can now complete the proof of Theorem 1. By (α), all k_ν do not vanish so that $\Sigma k_\nu \neq 0$. So applying Lemma 6 with $h=1$, we get by (17)

$$M(f) \geqslant \frac{D}{r^P} M(\sigma)$$

where $D>0$ is a constant. This gives $k(f) \geqslant l(\sigma)$ and *a fortiori* $k(f) \geqslant l(\theta_\rho, \sigma)$ which leads to a contradiction. So the theorem is true.

2·5. PROOF OF THEOREM 2: Let a be such that $|a| \neq 0$. Consider the function $\sigma(z/a) = \sigma_a(z)$. We find $l(\theta_\rho, \sigma_a)$ is $\dfrac{(\theta_\rho, \sigma)}{|a^\rho|}$ and so is greater than $k(f)=0$. Hence we can apply Theorem 4 using $\sigma(z/a)$ instead of $\sigma(z)$.

Also

$$\sigma'_a(-a\lambda_\nu)=\frac{1}{a}\sigma'(-\lambda_\nu).$$

Taking $p>1$ in (4) we get by (11),

$$\frac{f(z)}{\sigma(z/a)}=C_{p-1}(z,a)+(-1)^p\sum_{\nu=1}^{\infty}\frac{y_\nu(a)}{z+a\lambda_\nu}\left(\frac{z}{a\lambda_\nu}\right)^p\frac{a}{\sigma'(-\lambda_\nu)},$$

where $$y_\nu(a)=f(-a\lambda_\nu).$$

Putting $a=\alpha_\nu$ and letting $\nu\to\infty$, we get

$$f(z)=\lim_{\nu\to\infty}C_{p-1}(z,\alpha_\nu)$$
$$=\lim_{\nu\to\infty}[a_{p-1,\nu}\,z^{p-1}+a_{p-2,\nu}\,z^{p-2}+\ldots+a_{0,\nu}],$$

which being true for an infinity of values of z, we must have

$$a_{k\nu}\to a_k \text{ as } \nu\to\infty,\ k=0,\ldots,p-1,$$

and therefore $f(z)$ is a polynomial of degree $p-1$ at most. But $f(z)$ is bounded at $E(\lambda,\alpha)$ and hence $f(z)$ is a constant.

2·51. NOTE. In § 2·5 we have argued as if $a>0$. But this is not necessary since if $a=|a|e^{i\omega}$, we can apply Theorem 4 to $\dfrac{f(ze^{+i\omega})}{\sigma(z/|a|)}$ and replacing z by $ze^{-i\omega}$ we can arrive at the formula required.

III. FUNCTIONS FOR WHICH $\rho>1$

3. Let, as before,

$$0<\lambda_1<\lambda_2\ldots<\lambda_n\to\infty$$

and let $[\lambda_n]$ satisfy the condition

$$0<A=\underline{\lim}_{r\to\infty}\frac{n(r)}{r^\rho}\leqslant\overline{\lim}_{r\to\infty}\frac{n(r)}{r^\rho}=B. \tag{19}$$

Let q be an integer so that

$$\rho\leqslant q<\rho+1.$$

We use

$$\chi(z)=\prod_{\nu=1}^{\infty}\left(1-\frac{z^{2q}}{\lambda_\nu^{2q}}\right) \tag{20}$$

as the base-function and make the additional hypothesis that there is an integer $p\geqslant0$ so that

$$\sum_{\nu=1}^{\infty}\left|\frac{1}{\chi'(\lambda_\nu)\lambda_\nu^{p+1}}\right|<\infty, \tag{21}$$

similar to (4).

3·1. Let ω be a primitive $2q$th root of unity. Let $f(z)$ be a function of order ρ and

$$f(\lambda_\nu \omega^\mu) = y_{\nu\mu}, \ \mu = 0, 1, 2, \ldots, 2q-1.$$

We prove the following two theorems corresponding to Theorems 1 and 2 above.

THEOREM 5. *Let* $[y_{\nu\mu}]$ *be bounded. Let*

$$\alpha_\nu(t) = \sum_{\mu=0}^{2q-1} y_{\nu\mu}\,\omega^{\mu t}, \ t = 0, 1, \ldots, 2q-1.$$

Let $[y_{\nu\mu}]$ *satisfy one of the following conditions:*

(*a*) *the real parts of* $\dfrac{\alpha_\nu(t)}{\chi'(\lambda_\nu)}$ *do not change sign as* ν *varies for each* $t = 0, 1, \ldots, 2q-1$, *not all these real parts vanishing;*

(*b*) *a similar hypothesis on the imaginary part;*

(*c*) *all* $y_{\nu\mu} = 0$.

Under (*a*) *or* (*b*) *we must have* $k(f) \geqslant l(\chi)$. *Under* (*c*), $k(f) \geqslant l(\chi)$ *or else* $f(z) \equiv 0$.

THEOREM 6. *Let* $[\alpha_\nu]$, $|\alpha_\nu| \to 0$ *as* $\nu \to \infty$, *be any sequence of numbers. Let* $E(\lambda, \alpha)$ *denote the set*

$$z = \alpha_\nu \lambda_n \omega^\mu, \ \nu, n = 1, 2, \ldots; \ \mu = 0, 1, \ldots, 2q-1.$$

Let $f(z)$ *be of order* ρ *and minimal type. Let it be bounded at* $E(\lambda, \alpha)$. *Then* $f(z)$ *reduces to a constant.*

3·2. PROOF OF THEOREMS 5 AND 6. We shall first prove these theorems for a function of order ρ of the special form $\psi(z^{2q})$ where $\psi(u)$ is an integral function in u. Putting $z^q = u$, we get

$$\psi(u^2) = f(z)$$
$$\chi_1(u^2) = \chi(z).$$

It is easy, now, to verify the following statements:

(i) $\psi(u^2), \chi_1(u^2)$ are even integral functions in u of order $\rho/q \leqslant 1$;

(ii) $\chi_1(u^2) = \phi(u)$ is a base-function of the form (1) of order not exceeding one and the hypothesis (19) and (21) are equivalent to similar hypothesis on $\psi(u)$;

(iii) the lower and upper types are unchanged by the substitution $z^q = u$.

As regards the hypothesis (*a*), (*b*) or (*c*), we find that, for the function $\psi(z^{2q})$, $y_{\nu,\mu}$ does not depend on μ. Let y_ν denote the common value.

Then

$$\alpha_\nu(0)=2qy_\nu \text{ and } \alpha_\nu(t)=0,\ t=1,2,\ldots,2q-1.$$

Hence the hypothesis (a), (b) or (c) reduces to the corresponding hypothesis on $\psi(u^2)$ for the case $\rho\leqslant 1$. Now we can make use of the theorems of (A) and the remark made at the beginning of § 2 and arrive at the conclusion that Theorem 5 is true for the special type of functions considered. A similar proof holds for Theorem 6 in this special case.

3·3. We now turn to the general case. Let

$$\sigma_t(z)=z^{-t}\sum_{\mu=0}^{2q-1}\omega^{-\mu t}f(z\omega^{\mu}),\ \ t=0,\ldots,2q-1. \tag{22}$$

Each of these $2q$ functions σ_t is of the special form considered in § 3·2. It is easy to see that

$$k(\sigma_t)\leqslant k(f),\ t=0,\ldots,2q-1.$$

Since the matrix $||\omega^{-\mu t}||$ has a non-vanishing determinant, it follows that $k(f)$ cannot exceed the greatest of

$$k(\sigma_t),\ t=0,\ldots,2q-1.$$

Hence $k(f)$ is equal to the greatest of the types of $\sigma_t(z)$. If the hypothesis (b) is true, (a) would be true for $if(z)$. So we shall suppose (a) is satisfied. We find that all σ_t satisfy (a) or (c) while at least one σ_t satisfies (a). Hence from § 3·2, it follows that either $\sigma_t(z)\equiv 0$ or $k(\sigma_t)\geqslant l(\chi)$ and the latter alternative must hold for one t at least. Hence $k(f)\geqslant l(\chi)$ since $k(\rho)$ is equal to the greatest of the types of $\sigma_t(z)$. Next if (c) is true, all $\sigma_t(z)$ verify (c) so that $\sigma_t(z)\equiv 0$ for all t and consequently from (22), $f(z)\equiv 0$. So Theorem 5 is proved completely.

Regarding Theorem 6, if $f(z)$ is of minimal type so is each $\sigma_t(z)$ which are also bounded at $E(\lambda,\alpha)$ if $f(z)$ satisfies this condition. Hence $\sigma_t(z)=c_t$, a constant, for each t. From (22), we find that $f(z)$ is a polynomial of degree $2q-1$ at most; but $f(z)$ is bounded at $E(\lambda,\alpha)$. Hence $f(z)$ reduces to a constant. So Theorem 6 is completely proved.

3·4. We can also derive the following result by using the methods of § 3·1 and § 3·2.

THEOREM 7. *Let*

$$g(z)=\sum_{\nu=1}^{\infty}\frac{z^p}{\lambda_\nu^p\chi'(\lambda_\nu)}\left(\sum_{\mu=0}^{2q-1}\frac{y_{\nu,\mu}\,\omega^{\mu(1-p)}}{z-\lambda_\nu\omega^{\mu}}\right), \tag{23}$$

and let $$k(f) < l(\chi),$$
then

$$\frac{f(z)}{\chi(z)} = g(z) + C_{p-1}(z),$$

where $C_{p-1}(z)$ *is a polynomial of degree* $p-1$ *at most.*

3·5. Similarly we can prove the following theorem.

THEOREM 8. *Under the condition* (19), $\chi(z)$ *is a function of order* ρ *and*

$$0 < \frac{\pi A}{\sin \pi\rho/2q} \leqslant l(\chi) \leqslant k(\chi) \leqslant \frac{\pi B}{\sin \pi\rho/2q}. \qquad (24)$$

IV. REMARKS AND ILLUSTRATIONS

4. REMARKS ON THEOREM 1. As in (A), we can easily see that

$$(-1)^{n-1}\sigma(-\lambda_n)$$

is positive so that in the conditions (α), (β) and (γ) we can replace $\frac{y_n}{\sigma'(-\lambda_n)}$ by $(-1)^{n-1}y_n$. The second part of the inequality (2) is used to prove that $\sigma(z)$ is of order ρ and the functions $l(\theta, \sigma)$, $k(\theta, \sigma)$ are continuous in $0 \leqslant \theta \leqslant \pi/2$. Therefore we can replace this second part by the above two conditions.

4·1. REMARKS ON THEOREM 2. By an analysis of the proof of Theorem 2 given in § 2·5, we can show that the condition that $f(z)$ should be bounded at $E(\lambda, \alpha)$ could be replaced by the condition that $\frac{y_\nu(\alpha_n)}{|\alpha_n|^\lambda}$ is bounded where λ is any fixed number, positive or negative and $f(-\alpha_n\lambda_\nu) = y_\nu(\alpha_n)$.

4·2. ILLUSTRATION. Let

$$\lambda_n = n^{1/\rho},\ 0 < \rho < 1.$$

Then (2) is true with $A = B = 1$. $\sigma(z)$ is of order ρ and

$$l(\sigma) = k(\sigma) = \frac{\pi}{\sin \pi\rho}.$$

We have to find out when (4) is true. Let

$$X_n = |\sigma'(-n^{1/\rho})| \text{ and } \alpha = 1/\rho \text{ so that } \alpha > 1.$$

Differentiating (3), we get

$$X_n = \frac{1}{n^\alpha} \prod_{\nu=1}^{n-1} \left[\left(\frac{n}{\nu}\right)^\alpha - 1\right] \prod_{\nu=n+1}^{\infty} \left[1 - \left(\frac{n}{\nu}\right)^\alpha\right].$$

Therefore,

$$\log X_n = -\alpha \log n + (n-1)\alpha \log n - \alpha \log \Gamma(n) + \sum_{\nu=1}^{n-1} \log\left[1-\left(\frac{\nu}{n}\right)^{\alpha}\right] + \sum_{\nu=n+1}^{\infty} \log\left[1-\left(\frac{n}{\nu}\right)^{\alpha}\right]$$

$$= -\alpha \log n + (n-1)\alpha \log n - \alpha[(n-\tfrac{1}{2})\log n - n + O(1)]$$

$$+\sum_{p=1}^{\infty} \frac{1}{pn^{p\alpha}}[1^{p\alpha}+\ldots+(n-1)^{p\alpha}] - \sum_{p=1}^{\infty} \frac{n^{p\alpha}}{p}\left[\frac{1}{(n+1)^{p\alpha}}+\frac{1}{(n+2)^{p\alpha}}+\ldots\right]$$

$$= O(\log n) + \alpha n - \sum_{p=1}^{\infty} \frac{1}{pn^{p\alpha}}\left[\frac{(n-1)^{p\alpha+1}}{p\alpha+1}\right] + O\left\{\log\left[1-\left(\frac{n-1}{n}\right)^{\alpha}\right]\right\}$$

$$-\sum_{p=1}^{\infty} \frac{n^{p\alpha}}{p}\left[\frac{1}{(p\alpha-1)(n+1)^{p\alpha-1}}\right] + O\left\{\log\left[1-\left(\frac{n}{n+1}\right)^{\alpha}\right]\right\}$$

$$= O(\log n) + \alpha n - (n-1)\sum_{p=1}^{\infty} \frac{1}{p(p\alpha+1)}\left(1-\frac{1}{n}\right)^{p\alpha}$$

$$-n\sum_{p=1}^{\infty} \frac{1}{p(p\alpha-1)}\left(1-\frac{1}{n+1}\right)^{p\alpha-1}$$

$$= O(\log n) + \alpha n\left[\left\{1-\sum_{p=1}^{\infty} \frac{2}{p^2\alpha^2-1}\right\} + O\left(\frac{\log n}{n}\right)\right]$$

$$= O(\log n) + \alpha n\left\{1-\sum_{p=1}^{\infty} \frac{2}{p^2\alpha^2-1}\right\}. \tag{25}$$

Let

$$T(\alpha) = 1 - \sum_{p=1}^{\infty} \frac{2}{p^2\alpha^2-1}.$$

Then

$$T(2)=0,\ T(\alpha)>0 \text{ for } \alpha>2 \text{ and } T(\alpha)<0 \text{ for } \alpha<2.$$

Hence by (25) we deduce that (4) is true for $\alpha \geqslant 2$ but not true for $\alpha<2$. Therefore we can state the following theorems as particular cases of Theorems 1 and 2.

THEOREM 1 A. *Let $f(z)$ be a function of order ρ, $0<\rho\leqslant\frac{1}{2}$. Let*

$$f(-n^{1/\rho})=y_n.$$

Let $[y_n]$ be bounded, and let the real or the imaginary part of $(-1)^n y_n$ not change sign as n varies not all these vanishing. Then

$$k(f)\geqslant\frac{\pi}{\sin \pi\rho}.$$

If all $y_n=0$, then $f(z)\equiv 0$, if $k(f)<\frac{\pi}{\sin \pi\rho}$.

THEOREM 2 A. *Let $f(z)$ be of order ρ, $0<\rho\leqslant\frac{1}{2}$ and minimal type. Let*

$$f(-n^{1/\rho})=O(1).$$

Then $f(z)$ reduces to a constant.

NOTE. Here we take $\alpha_\nu=\nu^{1/\rho}$ when applying Theorem 2.

4·3. When $\frac{1}{2}<\rho<1$, we can use the remark at the beginning of § 2 and state corresponding theorems in this case also by using the results of (A). But here we have to consider the values of $f(z)$ at $z=\pm n^{1/\rho}$ instead of at $z=-n^{1/\rho}$ alone. With this understanding Theorems 1 A and 2 A remain true under similar hypothesis.

5. Remarks similar to those in § 4 apply to Theorem 5. As a particular case consider $\lambda_n=n^{1/\rho}$, $\rho>1$. The function $\chi(z)$ given by (20) is of order ρ and $A=B=1$ in (19). By putting $\eta_n=n^{2q/\rho}$ and $\alpha=2q/\rho\geqslant 2$ in (25) we see that the hypothesis (21) holds. Thus we get

THEOREM 5 A. *Let $f(z)$ be a function of order $\rho\geqslant 1$. Let q be the integer $\rho\leqslant q<\rho+1$. Let*

$$f(n^{1/\rho}\omega^\mu)=y_{n\mu}$$

and let the hypothesis of Theorem 5 be true for the quantities $[y_{n\mu}]$. Then

$$k(f)\geqslant\frac{\pi}{\sin \pi\rho/2q}$$

or else $f(z)\equiv 0$.

THEOREM 6 A. *Let $f(z)$ be of order ρ and minimal type. Let* $f(n^{1/\rho}\omega^\mu)=O(1)$, $\mu=0, 1\ldots, 2q-1$.

Then $f(z)$ is a constant.

ON EULER'S Φ-FUNCTION AND ITS EXTENSIONS

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I. Introduction

The type of argument given in text-books to derive the form of $\phi(n)$—the number of numbers less than n and prime to n, is as follows. If n be the given number and $p_1, p_2, \ldots, p_q$ be the different prime factors of n, then there are $\frac{n}{p_i}$ numbers divisible by p_i, $\frac{n}{p_j}$ numbers divisible by p_j, and so on. Generally there are $\frac{n}{p_ip_jp_k\cdots}$ numbers divisible by $p_i, p_j, p_k, \ldots,$ simultaneously. Then the number of numbers less than n and not prime to n is

$$\sum\frac{n}{p_1}-\sum\frac{n}{p_1p_2}+\sum\frac{n}{p_1p_2p_3}-\ldots.$$

Hence

$$\phi(n)=n-\sum\frac{n}{p_1}+\sum\frac{n}{p_1p_2}-\ldots$$
$$=n\left(1-\frac{1}{p_1}\right)\left(1-\frac{1}{p_2}\right)\ldots\left(1-\frac{1}{p_q}\right)$$

The kind of logical argument used here has been studied to some extent by Silva and Betti*. The object of this paper is firstly to formulate in abstract terms the general type of argument, and use it systematically to derive the form of $\phi(n)$ and its generalizations by Jordan, Schemmel and Lucas†; and secondly by using an extended conception of *g. c. d.*—greatest common divisor, of a set of numbers, to reach a generalization of Jordan's function, which I belive to be new.

The logical argument used in all these cases can be stated thus in general terms. Let a finite universe of discourse consisting of n elements be given, and let $A_1, A_2, \ldots, A_k$ be certain given

* L. E. Dickson, *History of the Theory of Numbers*, Vol. I, pp. 119, 120.

† *Loc. cit.*, p. 147.

attributes. If we know the number of elements of the universe which possess any assigned set of attributes chosen from these, we can find an expression for the number of elements possessing one at least or none at all of the attributes $A_1, A_2, \ldots, A_k$. For let N_i be the number of elements of the universe which possess A_i. Similarly let $N_{ijk\ldots}$ be the number possessing $A_i, A_j, A_k \ldots$, simultaneously, and so on. Then the number of elements of the universe which possess at least one of these attributes is*

$$\Sigma N_1 - \Sigma N_{12} + \Sigma N_{123} - \ldots \quad (1)$$

For, an element which possesses exactly k (>0) of these attributes, is enumerated in (1) a number of times equal to

$$_kC_1 - {_kC_2} + {_kC_3} - \ldots \quad = 1.$$

II. EULER'S FUNCTION—$\phi(n)$

In the case of Euler's function, the universe of discourse consists of the n numbers $\leqslant n$. An element of this universe may be said to possess the attribute A_i, when it is divisible by p_i. Thus

$$N_{ijk\ldots} = \frac{n}{p_i p_j p_k \ldots}.$$

Hence from (1), we get the number of numbers less than n and not prime to n to be

$$\sum \frac{n}{p_1} - \sum \frac{n}{p_1 p_2} + \sum \frac{n}{p_1 p_2 p_3} - \ldots.$$

Hence the number of numbers less than n and prime to n is given by

$$\phi(n) = n - \sum \frac{n}{p_1} + \sum \frac{n}{p_1 p_2} - \ldots$$

$$= n\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right)\ldots\left(1 - \frac{1}{p_q}\right).$$

III. JORDAN'S EXTENSION OF $\phi(n) - J_r(n)$

Precisely the same argument can be used to determine $J_r(n)$, which represents the number of sets of r numbers ($\leqslant n$) whose $g.c.d.$ is prime to n. To find its form, we take as the universe of discourse the n^r elements, where any element is an ordered set of

* This is substantially identical with Theorem I of the paper by R. Vaidyanathaswamy, 'On the Arithmetico-Logical Symmetric Functions of n attributes', *Pro. Indian Academy of Sc.* Vol. II, No. I (1935).

integers $(C_1, C_2, \ldots, C_r)$ each $\leqslant n$. The element $(C_1, C_2, \ldots, C_r)$ is said to possess the attribute A_i when each C is divisible by p_i. Now since the number of integers $(\leqslant n)$ divisible by $p_i, p_j, p_k, \ldots$, simultaneously is $\frac{n}{p_i p_j p_k \ldots}$ it follows that

$$N_{ijk\ldots} = \left(\frac{n}{p_i p_j p_k \ldots}\right)^r.$$

Hence $J_r(n)$, the number of elements possessing none of the attributes $A_i, A_j, A_k \ldots$, is given by

$$J_r(n) = n^r - \sum\left(\frac{n}{p_1}\right)^r + \sum\left(\frac{n}{p_1 p_2}\right)^r - \ldots$$

$$= n^r\left(1-\frac{1}{p_1^r}\right)\left(1-\frac{1}{p_2^r}\right)\cdots\left(1-\frac{1}{p_q^r}\right).$$

IV. Schemmel's extension of $\phi(n) - \Phi_m(n)$

We can apply the same argument to derive the form of Schemmel's extension of $\phi(n)$, namely $\Phi_m(n)$, which is defined as the number of sets of m consecutive numbers each less than n, and relatively prime to n, where $n = p_1^{\alpha_1} p_2^{\alpha_2} \ldots$. Now if none of a set of m consecutive numbers beginning with A is divisible by p, it amounts to saying that A does not leave any of the remainders $-1, -2, \ldots, -m$ when divided by p. Hence $\Phi_m(n)$ is simply the number of numbers among $1, 2, \ldots, n$, which when divided by any prime factor of n leave none of the remainders $-1, -2, \ldots, -m$.

As before take the universe of discourse to consist of the numbers $1, 2, \ldots, n$. An element of this universe is said to possess the attribute A_i when it leaves one of the remainders $-1, -2, \ldots, -m$ on division by p_i. So that $N_{ijk\ldots}$ is equal to the number of numbers $(\leqslant n)$ which when divided by $p_i, p_j, p_k, \ldots$, leave in each case a remainder from the series $-1, -2, \ldots, -m$.

To evaluate N_i, we note that the number of numbers in the series $1, 2, \ldots, n$, which when divided by a given factor d of n leaves an assigned remainder a is n/d. For the numbers in question are precisely

$$a,\ a \pm d,\ a \pm 2d, \ldots$$

and exactly n/d of these fall within the range $1, 2, \ldots, n$. Now supposing that d is a prime factor p_i of n and that we are seeking the number of numbers in the series $1, 2, \ldots, n$ which leave some one of the m remainders $-1, -2, \ldots, -m$ when divided by p_i, it follows that the number of such numbers is

$$\frac{n}{p_i} \times m.$$

Hence
$$N_i = \frac{n}{p_i} \times m.$$

Next to evaluate $N_{ijk}\ldots$ we make use of the Chinese Remainder Theorem* and its Corollary, which run as follows.

THEOREM 1. *If $M_1, M_2, \ldots, M_r$ are relatively prime in pairs there exists an integer x which when divided by them respectively leave given remainders $a_1, a_2, \ldots, a_r$. All such integers x differ from one another by multiples of $M_1 M_2 \ldots M_r$.*

COROLLARY. *There exists one and only one solution x of the problem which falls within the interval 1 to $M_1 M_2 \ldots M_r$ and hence within any interval of the same length.*

The corollary is obvious, for since by Theorem 1 the possible values of x differ by multiples of $M_1 M_2 \ldots M_r$, only one solution of x can fall between consecutive numbers of any arithmetic progression whose common difference is $M_1 M_2 \ldots M_r$.

We are now in a position to evaluate $N_{ijk}\ldots$. Let $M_1 M_2 \ldots M_r$ be a set of q prime factors $p_i, p_j, p_k, \ldots$ of n. Then the number of numbers chosen from $1, 2, \ldots, n$ which leave assigned remainders on division by $p_i, p_j, p_k, \ldots$, is $\frac{n}{p_i p_j p_k \ldots}$. For from the corollary stated above we see that there is only one such integer in any interval of length $p_i p_j p_k \ldots$; since the interval $1 \ldots n$ is composed of $\frac{n}{p_i p_j p_k \ldots}$ such intervals the statement follows.

Now $N_{ijk}\ldots$ is the number of numbers which when divided by $p_i, p_j, p_k, \ldots$, leave in each case a remainder from $-1, -2, \ldots, -m$. Hence there are m^q sets of admissible remainders to choose from. Therefore there are

$$\frac{n}{p_i p_j p_k \ldots} \times m^q$$

numbers enumerated in $N_{ijk}\ldots$. Thus

$$N_{ijk\ldots} = \frac{n}{p_i p_j p_k \ldots} \times m^q$$

* For proof see L. E. Dickson, *Introduction to the Theory of Numbers.*

where q is the number of subscripts $i, j, k, \ldots$. Therefore from (1)

$$\Phi_m(n) = n - m\sum\frac{n}{p_1} + m^2\sum\frac{n}{p_1p_2} - \ldots + (-)^q m^q \sum\frac{n}{p_ip_j\ldots p_q}$$
$$= n\left(1-\frac{m}{p_1}\right)\left(1-\frac{m}{p^2}\right)\ldots\left(1-\frac{m}{p_q}\right).$$

V. Lucas' extension of $\Phi_m(n)$

The above argument tacitly assumes that the remainders $-1, -2, \ldots, -m$ are all distinct from one another for division by each p_i. In other words it is assumed that none of the prime factors p_i is smaller than m. If some of the p's be less than m the argument must be slightly modified. Lucas generalizes this case and studies the function $\psi(n; e_1, e_2, \ldots, e_q)$ defined as the number of integers x chosen from $1, 2, \ldots, n$ such that $h-e_1, h-e_2 \ldots$, are prime to n, $e_1, e_2, \ldots, e_q$ being given integers. If $\lambda_1, \lambda_2, \ldots$ be the number of distinct residues of $e_1, e_2 \ldots, e_q$, mod p_1, mod $p_2, \ldots$ respectively, Lucas proves that

$$\psi(n, e_1, e_2, \ldots, e_q) = n\left(1-\frac{\lambda_1}{p_1}\right)\left(1-\frac{\lambda_2}{p_2}\right)\ldots\left(1-\frac{\lambda_q}{p_q}\right).$$

This formula can be derived by the same procedure as was adopted in deriving Schemmel's function, the only difference being that here we consider in each case the set of distinct residues instead of the set of numbers $-1, -2, \ldots, -m$.

VI. Extension of Jordan's $J_r(n) - J_{rs}(n)$

Lastly Jordan's $J_r(n)$ can be generalized by using an extension* of the concept of the *g.c.d.* of any set of integers.

Let $b_1, b_2, \ldots b_r$ be a set of integers and p any prime factor occurring $k_1, k_2, \ldots$ times in $b_1, b_2, \ldots b_r$ respectively, and let α be the sth term in ascending order of magnitude in the series $k_1, k_2, \ldots$. Then the sth *g.c.d.* of the set of integers $(b_1, b_2, \ldots, b_r)$ is defined as the product of all the prime powers of the form p^α.

The problem now is to find an expression for the number of sets of r integers ($\leqslant n$) whose sth *g.c.d.* is prime to n. This will be denoted by $J_{rs}(n)$. Since the first *g.c.d.* of r numbers is their usual *g.c.d.* $J_{rs}(n)$ for $s=1$, reduces to $J_r(n)$. As in Jordan's

* For the definition of the 1st, 2nd, ...sth, ...rth *g.c.d*'s of a set of r numbers see § 5 of the paper '*On the Arithmetico—Logical etc.*' by R. Vaidyanathaswamy, *loc. cit.*

function, the universe of discourse consists of n^r elements, each an ordered set of r integers $C_1, C_2, \ldots, C_r (\leqslant n)$. We shall say that the element $(C_1, C_2, \ldots, C_r)$ possesses the attribute A_i when the sth $g.c.d.$ of $C_1, C_2, \ldots, C_r$ is divisibly by p_i. As in the previous cases we evaluate $N_i, N_{ijk} \ldots$, etc.

The given set of integers being $C_1, C_2 \ldots, C_r$ let the highest power of p in them be $n_1, n_2, \ldots, n_r$ respectively. If p is to divide the sth $g.c.d.$ then from the definition of the sth $g.c.d.$ the sth term in ascending order of magnitude in $n_1, n_2, \ldots n_r$ is greater than zero, i.e. at most $s-1$ of the numbers $C_1, C_2, \ldots, C_r$ are prime to p. Hence to evaluate N_i, the following are the possible cases to be considered.

(1) All the numbers $C_1, C_2, \ldots, C_r$ are divisible by p_i. Then each of the numbers C can be chosen in n/p_i ways to satisfy this condition. Hence the total number of elements of this kind is $(n/p_i)^r$.

(2) $r-1$ of the numbers $C_1, C_2 \ldots, C_r$ are divisible by p_i. and the remaining one is not divisible by p_i. The total number of elements of this kind is

$$\binom{r}{1}\left(\frac{n}{p_i}\right)^{r-1}\left(n-\frac{n}{p_i}\right).$$

And so on up to the sth case in which $s-1$ of the numbers are not divisible by p_i while the remaining $r-s+1$ are divisible by p_i. The number of elements in this case is

$$\binom{r}{s-1}\left(\frac{n}{p_i}\right)^{r-s+1}\left(n-\frac{n}{p_i}\right)^{s-1}.$$

Hence

$$N_i=\left(\frac{n}{p_i}\right)^r+\binom{r}{1}\left(\frac{n}{p_i}\right)^{r-1}\left(n-\frac{n}{p_i}\right)+\ldots+\binom{r}{s-1}\left(\frac{n}{p_i}\right)^{r-s+1} \times\left(n-\frac{n}{p_i}\right)^{s-1}$$

$$=n^r\left[\frac{1}{p_i^r}+\binom{r}{1}\frac{1}{p_i^{r-1}}\left(1-\frac{1}{p_i}\right)+\ldots\binom{r}{s-1}\frac{1}{p_i^{r-s+1}}\left(1-\frac{1}{p_i}\right)^{s-1}\right]$$

$$=n^r f_{rs}\left(\frac{1}{p_i}\right),$$

where $f_{rs}(x)$ is the polynomial of the rth degree given by

$$f_{rs}(x)=x^r+\binom{r}{1}x^{r-1}(1-x)+\ldots+\binom{r}{s-1}x^{r-s+1}(1-x)^{s-1}.$$

We may note that if $s=r$

$$f_{rr}(x)=x^r+\binom{r}{1}x^{r-1}(1-x)+\ldots+\binom{r}{r-1}x(1-x)^{r-1}$$
$$=1-(1-x)^r.$$

Next to evaluate $N_{ijk\ldots}$ we make use of a consequence of of Theorem 1 which asserts that

If $M_1, M_2, \ldots, M_r$ be relatively prime then given a set of integers $K_1 \pmod{M_1}$, $K_2 \pmod{M_2}, \ldots K_r \pmod{M_r}$ there exists a unique number

$$x \pmod{M_1 M_2 \ldots M_r}$$

such that

$$x=K_1 \pmod{M_1},\ x=K_2 \pmod{M_2}, \ldots,\ x=K_r \pmod{M_r};$$

and conversely, given $x \pmod{M_1 M_2 \ldots M_r}$ from these very equations we can determine

$$K_1, K_2, \ldots, K_r, \bmod M_1, \bmod M_2, \ldots, \bmod M_r$$

respectively.

Thus the theorem establishes a one-to-one correspondence between the residue classes $x \pmod{M_1 M_2 \ldots M_r}$ and the sets of residue classes $K_1, K_2, \ldots, K_r$, mod M_1, mod $M_2, \ldots$, mod M_r respectively.

A very important property of this correspondence is the following:

If x has the g.c.d. δ_i with M_i, then K_i has the same g.c.d. δ_i with M_i and conversely.

For the equation $x=K_i \pmod{M_i}$ or $x=K_i+\lambda M_i$ where λ is some integer, shows that any common factor of x and M_i is also a common factor of K_i and M_i and vice versa.

The following theorem is a consequence of this property.

THEOREM 2. *The sth g.c.d. of r values $x_1, x_2, \ldots x_r$ of x is or is not divisible by m (a prime factor of M_i) according as the sth g.c.d. of the corresponding values $K_{i_1}, K_{i_2}, \ldots, K_{i_r}$ of K is or is not divisible by m; and vice versa.*

To prove this we recall that if m divides the sth *g.c.d.* $x_1, x_2, \ldots, x_r$ at most $s-1$ of them are prime to n, and since by the *g.c.d.* property proved above the *g.c.d.* of x_t and M_i is the same as the *g.c.d.* of K_{it} and M_i, it follows at once that not more than $s-1$ of $K_{i_1}, K_{i_2}, \ldots$, are prime to m. Hence the sth *g.c.d.* of $K_{i_1}, K_{i_2}, \ldots$ is divisible by m and vice versa.

In the present case we take for $M_1, M_2, \ldots, M_r$ the q elementary block-factors* $p_1^{\alpha_1}, p_2^{\alpha_2}, \ldots, p_q^{\alpha_q}$ of $n = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_q^{\alpha_q}$. The element we are concerned with is a set of r integers $C_1, C_2, \ldots, C_r$ ($\leqslant n$), i.e. a set of r residue classes $C_1, C_2, \ldots, C_r \pmod{n}$. By Theorem 1 there corresponds to each C_i a set of q integers (or residue classes) $C_{i_1}, C_{i_2}, \ldots, C_{i_q}$, mod $p_1^{\alpha_1}$, mod $p_2^{\alpha_2} \ldots$, mod $p_q^{\alpha_q}$, respectively. Thus to our original element $(C_1, C_2, \ldots, C_r)$ there corresponds a set of q partial elements which we shall call the q partial components of the given element. Thus the tth partial component is $C_{1t}, C_{2t}, \ldots, C_{rt} \pmod{p_t^{\alpha t}}$.

Since the correspondence established by Theorem 1 is one-to-one it follows that if the q partial components ($C_{1t}, C_{2t}, \ldots, C_{rt}$ mod $p_t^{\alpha t}$, where $t = 1, \ldots, q$) be given we can determine the original element $(C_1, C_2, \ldots, C_r)$. From Theorem 2 we see that the sth *g.c.d.* of the integers in the tth partial component is or is not divisible by p_t according as the sth *g.c.d.* of the original element $(C_1, C_2, \ldots, C_r)$ is or is not divisible by p_t and conversely. Thus if the sth *g.c.d.* of the original element is divisible by $p_i, p_j, p_k, \ldots$, then the sth *g, c.d*'s of the ith, jth, kth... partial components are divisible by $p_i, p_j, p_k, \ldots$ respectively; and conversely.

Now P_i, the total possible number of the ith partial components is given by

$$P_i = (p_i^{\alpha_i})^r.$$

By the formulae established for N_i, the number $\overline{N}_i$ of the ith partial components which satisfy the condition that the sth *g. c. d.* is divisible by P_i is given by

$$\overline{N}_i = (p_i^{\alpha_i})^r f_{rs}\left(\frac{1}{p_i}\right).$$

Hence $N_{ijk\ldots}$, the number of sets of numbers whose sth *g c.d.* is divisible by $p_i, p_j, p_k, \ldots$, is given by

$$N_{ijk\ldots} = \Pi P_y \times \Pi \overline{N}_z,$$

where in the first product y takes all values from 1 to q other

* The term Block-factor is used by R. Vaidyanathaswamy [See *Transactions of the American Math. Soc.*, Vol. 33, No. 2] to denote a factor δ in which each prime factor has the same exponent as in n, *i e.* a factor δ which is relatively prime to n/δ.

An elementary block-factor is the highest power of any single prime factor which occurs in the given number. Thus $p_i^{\alpha_i}$ is an elementary block-factor of n.

than $i, j, k, \ldots$, and in the second product z takes all the values $i, j, k, \ldots$. Thus

$$N_{ijk\ldots}=\left[\frac{n}{p_i^{\alpha_i}p_j^{\alpha_j}p_k^{\alpha_k}\ldots}\right]^r\times\left[(p_i^{\alpha_i})^r f_{rs}\left(\frac{1}{p_i}\right)\times(p_j^{\alpha_j})^r f_{rs}\left(\frac{1}{p_j}\right)\times\ldots\right]$$

$$=n^r\times f_{rs}\left(\frac{1}{p_i}\right)f_{rs}\left(\frac{1}{p_j}\right)f_{rs}\left(\frac{1}{p_k}\right)\ldots.$$

Hence from (1),

$$J_{rs}(n)=N-\Sigma N_1+\Sigma N_{12}-\ldots$$

$$=n^r-n^r\sum f_{rs}\left(\frac{1}{p_1}\right)+n^r\sum f_{rs}\left(\frac{1}{p_1}\right)f_{rs}\left(\frac{1}{p_2}\right)-\ldots$$

$$=n^r\left[1-f_{rs}\left(\frac{1}{p_1}\right)\right]\left[1-f_{rs}\left(\frac{1}{p_2}\right)\right]\ldots\left[1-f_{rs}\left(\frac{1}{p_q}\right)\right].$$

An interesting special case of the above arises when $s=r$. Here the general formula gives

$$J_{rr}(n)=n^r\left[1-f_{rr}\left(\frac{1}{p_1}\right)\right]\left[1-f_{rr}\left(\frac{1}{p_2}\right)\right]\ldots$$

$$=n^r\left(1-\frac{1}{p_1}\right)^r\left(1-\frac{1}{p_2}\right)^r\ldots\left(1-\frac{1}{p_q}\right)^r$$

$$=[\phi(n)]^r.$$

It is easy to verity this result directly. For by the definition* the rth *g.c.d.* of a set of r numbers is their *l.c.m.* If this be prime to n, each of the numbers is prime to n and can therefore be chosen in $\phi(n)$ ways. Hence

$J_{rr}(n)$ = the number of sets of numbers whose rth *g.c.d.* that is to say whose *l.c.m.* is prime to n

$$=[\phi(n)]^r.$$

I am indebted to Dr. R. Vaidyanathaswamy for his kind guidance in the preparation of this paper.

* *Cf.* R. Vaidyanathaswamy '*On the Arithmetico-Logical etc.*' *loc. cit.*

ON THE MOST GENERAL STATIC FIELD IN THE RELATIVITY THEORY

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1. The ten gravitational equations of Einstein's Relativity theory are reducible to a system of only seven equations of a more tractable type when the metric can be written in the form

$$ds^2 = V^2(x^1, x^2, x^3)dt^2 - \sum_{i,j}^{1,2,3} g_{ij}(x^1, x^2, x^3)dx^i dx^j. \tag{1,1}$$

This case has been studied and called the static case by Prof. Levi-Civita*. The space-time then admits a group of isometry defined by the transformation $t' = t + \text{constant}$.

The equations of Einstein, if there is no electro-magnetic field, and outside matter are

$$\begin{aligned} &\bar{R}^i_j + \frac{V_{,j}^{\ i}}{V} = 0 \\ &\bar{\Delta} V = \sum_{i=1}^{3} V_{,i}^{\ i} = 0, \end{aligned} \qquad (i,j=1,2,3) \tag{1,2}$$

where the $\bar{R}^i_j$'s are the components of the contracted tensor of Riemann in the metric

$$d\sigma^2 = \sum_1^3 g_{ij} dx^i dx^j$$

and the $V_{,j}^{\ i}$'s the covariant derivatives of the vector

$$V^i = \sum_{\alpha=1}^{3} g^{i\alpha} \frac{\partial V}{\partial x_\alpha}$$

in the same metric.

* *Cf.* Levi-Civita, 'ds^2 einsteiniani in Campi Newtoniani', *Rendiconti Ac. Lincei*, 1917-1918.

In spite of this simplification, the static case is still a very difficult one and it has been integrated only in two particular instances, that of the famous ds^2 of Schwarzschild and that of the axial symmetry studied by Palatini, Bach and Chazy. The solution of Chazy is the most general and is obtained by very elegant methods*.

I have studied the most general static case, namely the case of a space-time whose metric is reducible to the following form

$$(1.3)\qquad ds^2=V^2(x^1,x^2,x^3)dt^2+\sum_{i=1}^{3} g_{io}(x^1,x^2,x^3)dx^i dt + \sum_{i,j}^{1,2,3} g_{ij}(x^1,x^2,x^3)dx^i dx^j.$$

For such a metric I shall prove

(*a*) that when the space-time is quasi euclidean† and asymptotic to the euclidean space-time it cannot be regular everywhere without being identically the euclidean space-time;

(*b*) that the case of axial symmetry can be generalized. Then the gravitational equations reduce to a system of two equations and a quadrature.

Moreover I have written the gravitational equations of Einstein in what I think is the most natural form.

2. I shall exclusively apply, in what follows, the method of the "repere mobile" developed by Prof. E. Cartan‡. It leads most naturally to the results. I give here a brief summary of this method.

An n-dimensional space being defined by the real fundamental quadratic form (which we shall first suppose definite and positive)

$$ds^2=\sum_{i,j}^{1\ldots n} g_{ij}dx^i dx^j$$

* *Cf.* J. Chazy, *Bull. Soc. Math. France* (1924), pp. 17-38.

G. Darmois, *Memorial des Sciences Mathematiques*, fasc. XXV. Gauthier-Villars, (1927) p. 35.

† For an accurate definition of spaces which are quasi-euclidean and asymptotic to the euclidean one, see C. Racine, *Le probleme des n corps dans la theorie de la Relativite*, these, Gauthier-Villars, (1934) Chapter I and C. Racine *C. R. Ac. Sc.* Paris, t. 197 (1933), p. 302.

‡ *Cf. Journ. de Math.* t. I, (1922), pp. 141-203.

this form may be represented by the sum of n squares

$$ds^2=\sum_{i=1}^{n} \omega_i^2$$

where the ω_i's are n Pfaffian forms

$$\omega_i=\sum_{h=1}^{n} a_{ih}dx^h.$$

The bilinear covariants of these forms are

$$\omega'_i=\sum_{(\alpha,\beta)}^{1\ldots n} c_{\alpha\beta i}[\omega_\alpha\omega_\beta],$$

where the brackets denote outer products. The ω'_i's define $\frac{n(n-1)}{2}$ new Pfaffian forms ω_{ij} such that

$$\omega'_i=\sum_{h=1}^{n} [\omega_h\omega_{hi}], \quad \text{where } \omega_{hi}=\sum_{k=1}^{n} \gamma_{hik}\omega_k$$

so that

$$\gamma_{hik}=\tfrac{1}{2}(c_{hik}-c_{ikh}-c_{khi}).$$

The bilinear covariants of the ω_{ij}'s define the components of the Riemannian tensor by means of the relations*

$$\omega'_{ij}=\sum_{h=1}^{n} [\omega_{ih}\omega_{hj}]+\Omega_{ij}, \quad \Omega_{ij}=\sum_{(h,k)}^{1\ldots n} R_{ij\,hk}[\omega_h\omega_k].$$

Hence the contracted tensor

$$R_{ij}=\sum_{\alpha=1}^{n} R_{i\alpha j\alpha}$$

and the invariant curvature (or total curvature)

$$2\bar{R}=\sum_{i=1}^{n} R_{ii}.$$

* With Cartan's method there is no need of any distinction between subscript and superscript letters. For the sake of more precision I shall make use of Σ for the summations.

Let us now consider a function U of the n co-ordinates. Then

$$dU=\sum_{1}^{n} U_i\omega_i;$$

U_i may be called the Pfaffian derivative of U with regard to ω_i. The U_i's are components of a vector. Likewise

$$dU_i=\sum_{\alpha=1}^{n} U_{i.\alpha}\omega_\alpha.$$

Henceforth a point followed by a letter will define a Pfaffian derivative with regard to the ω_i of same subscript. A comma will mean a covariant derivation, so that, for instance,

$$U_{i,j}=U_{i.j}+\sum_{\alpha=1}^{n} U_\alpha\gamma_{\alpha ij}.$$

3. Let us now consider a four dimensional Riemannian space whose metric is

$$ds^2=\omega_0^2-\omega_1^2-\omega_2^2-\omega_3^2, \tag{3.1}$$

where

$$\omega_0=V(x_1,x_2,x_3)dt, \quad \omega_i=\bar{\omega}_i+\lambda_i(x_1,x_2,x_3)dt$$

$$\bar{\omega}_i=\sum_{h=1}^{3} a_{ih}(x_1,x_2,x_3)dx_h \quad (i=1,2,3), \tag{3.2}$$

with $\quad V^2-\lambda_1^2-\lambda_2^2-\lambda_3^2>0.$

We have

$$\bar{\omega}'_i=\sum_{(\alpha,\beta)}^{1,2,3} c_{\alpha\beta i}[\bar{\omega}_\alpha\bar{\omega}_\beta]. \tag{3.3}$$

If U is a function of x_1,x_2,x_3 only, then

$$dU=U_0\omega_0+\sum_{1}^{3} U_i\omega_i=\left(VU_0+\sum_{1}^{3}\lambda_iU_i\right)dt+\sum_{1}^{3}\bar{U}_i\omega_i \tag{3.4}$$

hence

$$U_0=-\sum_{1}^{3}\frac{\lambda_iU_i}{V}. \tag{3.5}$$

Moreover

$$\omega'_0=[dVdt]=\sum_{1}^{3}\frac{V_i}{V}[\omega_i\omega_0]$$

$$\omega'_i=\bar{\omega}'_i+[d\lambda_i dt]$$

$$(3.6)\qquad =\sum_{(\alpha,\beta)}^{1,2,3} c_{\alpha\beta i}[\bar{\omega}_\alpha\bar{\omega}_\beta]+\sum_{\rho=1}^{3}\frac{\lambda_{i.\rho}}{V}[\omega_\rho\omega_0]\quad (i=1,2,3)$$

$$=\sum_{(\alpha,\beta)}^{1,2,3} c_{\alpha\beta i}[\omega_\alpha\omega_\beta]+\sum_{\rho=1}^{3}A_{i\rho}[\omega_\rho\omega_0],$$

where

$$A_{i\rho}=\frac{\lambda_{i.\rho}+\sum_{\alpha=1}^{3}\lambda_\alpha c_{\alpha\rho i}}{V}\qquad (i,\rho=1,2,3).$$

Now if we write

$$(3{\cdot}7)\qquad \omega_{i0}=\frac{V_i}{V}\omega_0+\sum_{\alpha=1}^{3}P_{i\alpha}\omega_\alpha\quad (i=1,2,3)$$

$$\omega_{ij}=\sum_{\alpha=1}^{3}\gamma_{ij\alpha}\omega_\alpha+\bar{Q}_{ij}\omega_0\quad (i,j=1,2,3)$$

it follows that

$$(3.8)\qquad \begin{aligned} &P_{ij}=\gamma_{i0j}=\tfrac{1}{2}(A_{ji}+A_{ij})\\ &P_{ii}=A_{ii}\\ &\bar{Q}_{ij}=\gamma_{ij0}=\tfrac{1}{2}(A_{ji}-A_{ij})\\ &A_{ji}=P_{ij}+\bar{Q}_{ij}. \end{aligned}$$

Let us now call $\bar{\lambda}_{i,j}(i,j=1,2,3)$ the tensor obtained from the vector λ_i by covariant derivation in the three-dimensional space

$$(3.9)\qquad d\sigma^2=\bar{\omega}_1^2+\bar{\omega}_2^2+\bar{\omega}_3^2$$

which we shall call a space-section of the space-time. Then

$$(3.10)\qquad P_{ij}=\frac{\bar{\lambda}_{i,j}+\bar{\lambda}_{j,i}}{2V}\quad (i,j=1,2,3).$$

This result can be obtained very easily. It is evident that the vector $\lambda_0=V,\lambda_1,\lambda_2,\lambda_3$ defines in space-time the infinitesimal displacement of a group of isometry. Hence from Killing's theorem

$$\lambda_{i,j}+\lambda_{j,i}=0 \quad (i,j=0,1,2,3).$$

If we consider the equations for which $i,j=1,2,3$ we have

$$0=\lambda_{i,j}+\lambda_{j,i}=\bar{\lambda}_{i,j}+\lambda_{j,i}+\lambda_0(\gamma_{0ij}+\gamma_{0ji})$$
$$=\bar{\lambda}_{i,j}+\lambda_{j,i}-2VP_{ij}.$$

It follows that

$$P_{ij}=\frac{\bar{\lambda}_{i,j}+\bar{\lambda}_{j,i}}{2V}.$$

Now we can write $\bar{Q}_{ij}$ in the form

$$\bar{Q}_{ij}=\frac{1}{2V}\left[\lambda_{j.i}-\lambda_{i.j}+\sum_{\alpha=1}^{3}\lambda_\alpha(c_{\alpha ij}-c_{\alpha ji})\right].$$

The $\bar{Q}_{ij}$'s are not the components of a tensor. Let Q_{ij} be defined by

(3.11) $$Q_{ij}=\frac{\bar{\lambda}_{j,i}-\bar{\lambda}_{i,j}}{2V} \quad (i,j=1,2,3).$$

The Q_{ij}'s are now components of a skew-symmetric tensor and

(3.12) $$Q_{ij}=\bar{Q}_{ij}+\sum_{\alpha=1}^{3}\frac{\lambda_\alpha\gamma_{ij\alpha}}{V},$$

thus we have

(3.13) $$\omega_{ij}=\bar{\omega}_{ij}+Q_{ij}\omega_0,\ \bar{\omega}_{ij}=\sum_{\alpha=1}^{3}\gamma_{ij\alpha}\bar{\omega}_\alpha \quad (i,j=1,2,3).$$

The expression of the Q_{ij}'s can be easily deduced from the equations of Killing

$$\lambda_{i,0}+\lambda_{0,i}=0 \quad (i=1,2,3).$$

It is now possible to calculate the components of the Riemannian tensor of our space-time (3.1) and to write the gravitational of equations Einstein. Let the $\bar{R}_{ijhk}$'s be the components of the Riemannian tensor attached to the metric (3.9). Then

$$R_{ijik}=-\bar{R}_{ijik}+(P_{ii}P_{jk}-P_{ik}P_{ji}) \quad (i,j,k=1,2,3)$$
$$R_{i0ij}=P_{ij,i}-P_{ii,j}\ (*) \quad (i,j=1,2,3)$$

* Henceforth the comma will only mean a covariant derivation in the metric (3.9).

$$R_{i0j0} = -\frac{V_{i,j}}{V} + \sum_{h=1}^{3}\left[P_{ih}P_{hj} - (P_{ih}\bar{Q}_{hj} - \bar{Q}_{ih}P_{hj}) + \frac{\lambda_h}{V}P_{ij.h}\right].$$

After a very simple transformation we have

$$R_{i0j0} = -\frac{V_{i,j}}{V} + \sum_{h=1}^{3}\left[P_{ih}P_{hj} - (P_{ih}Q_{hj} - Q_{ih}P_{hj}) + \frac{\lambda_h}{V}P_{ij,h}\right] (i,j=1,2,3).$$

Hence

$$R_{ij} = -\left(\bar{R}_{ij} + \frac{V_{i,j}}{V}\right) + PP_{ij} - \sum_{h=1}^{3}\left[(P_{ih}Q_{hj} - Q_{ih}P_{hj}) - \frac{\lambda_h}{V}P_{ij,h}\right]$$

$$(i,j=1,2,3)$$

$$R_{0i} = \sum_{h=1}^{3} P_{ih,h} - P_i, \text{ where } P = \sum_{1}^{3} P_{hh} \quad (i=1,2,3)$$

$$\sum_{1}^{3} R_{ii} - R_{00} = -2\bar{R} - H^2, \text{ where } H^2 = \sum_{\alpha,\beta}^{1,2,3} P^2_{\alpha\beta}.$$

So the gravitational equations—outside matter and when there is no electro-magnetic field—are

$$R_{ij} + \frac{V_{i,j}}{V} - PP_{ij} + \sum_{h=1}^{3}\left[(P_{ih}Q_{hj} - Q_{ih}P_{hj}) - \frac{\lambda_h}{V}P_{ij,h}\right] = 0 \quad (i,j=1,2,3).$$

(3.14)
$$\sum_{h=1}^{3} P_{ih,h} - P_i = 0;$$

$$2\bar{R} + H^2 = 0.$$

These are the gravitational equations in the most general case of a static field in the Relativity theory. It is manifest that they reduce to those of Levi-Civita:

(1) when $P_{ij}=0 \quad (i,j=1,2,3)$;

then the vector $\lambda_1, \lambda_2, \lambda_3$ generates a group of motion in (3.9), the above equations being the Killing equations,

(2) when the vector $\lambda_0 = V, \lambda_1, \lambda_2, \lambda_3$ is integrable in space-time. Then it is orthogonal to a ∞^1 family of three-dimensional hypersurfaces. The Pfaffian form

$$(V^2 - \lambda_1^2 - \lambda_2^2 - \lambda_3^2)dt - 2\lambda_1\bar{\omega}_1 - 2\lambda_2\bar{\omega}_2 - 2\lambda_3\bar{\omega}_3$$

is reducible to

$$A(x_1,x_2,x_3)\ [dt+dB(x_1,x_2,x_3)]$$

and the quantities V, $\lambda_1,\lambda_2,\lambda_3$ satisfy four differential equations, one of them being

$$\lambda_1 Q_{23}+\lambda_2 Q_{31}+\lambda_3 Q_{12}=0.$$

Thus *if* $\lambda_0,\lambda_1,\lambda_2,\lambda_3$ *is integrable in space-time,* $\lambda_1,\lambda_2,\lambda_3$ *is integrable in the space-sections.*

The case $\qquad Q_{ij}=0 \quad (i,j=1,2,3)$

would deserve a special development but not being connected with any problem of Celestial Dynamics I shall not deal with it here.

4. *Particular systems of co-ordinates.*

We shall consider the simplifications introduced by the following change of co-ordinates:

$$t'=t+X(x_1,x_2,x_3).$$

This transformation does not change the canonical form (1.3) of the ds^2.

It is easy to show that the function X can be chosen so that

$$\lambda_1 Q_{23}+\lambda_2 Q_{31}+\lambda_3 Q_{12}=0.$$

Then $\lambda_1,\lambda_2,\lambda_3$ is integrable in (3.9)—which does not mean that $\lambda_0,\lambda_1,\lambda_2,\lambda_3$ is integrable in space-time.

With these co-ordinates we can determine the 'repere mobile,' so that the metric of (3.9) becomes

$$\bar{\omega}_1=\omega_1 \qquad \bar{\omega}_2=\omega_2$$

$$\omega_3=a(x_1,x_2,x_3)\,dx_3+\lambda(x_1,x_2,x_3)\,dt.$$

It is also possible to determine X so that

$$P=\sum_1^3 P_{ii}=0.$$

This means that the hypersurfaces $t=$const. are extremal in space-time.

Unfortunately the theory of the partial derivative equations of the second order is not yet sufficiently developed to justify such a change of co-ordinates outside a very limited domain. A thorough study of extremal hypersurfaces in space-time would constitute a considerable improvement of the Relativity theory. The system of reference defined by a family of extremal three-dimensional hypersurfaces as space-sections introduces, in the general case the greatest simplifications.

5. *Space-times regular everywhere.*

I have proved in my thesis that the space-time representing the static field of Levi-Civita cannot be regular everywhere without being everywhere locally euclidean. In the case of the most general static fields which I am studying here, it is doubtful whether such a proposition still holds good. I have proved in my thesis that in the case of the general field of the Relativity theory, this is certainly not true and that there exist space-times in which there is gravitation without there being any matter. This is a curious paradox of the the theory and shows that its axiomatic is not yet perfect.

As to the general static case I deal with here, it is possible to prove the following theorem:

If V, the λ_i's, and the a_{ih}'s depend at least on one parameter m, (mass at rest of the matter generating the field), so that the ds^2 tends to the euclidean one, namely

$$dt^2 - dx_1^2 - dx_2^2 - dx_3^2$$

when m tends to zero, and moreover is asymptotically euclidean when m is smaller than a given number, then the space-time cannot be regular everywhere without being identically euclidean.

The demonstration of this proposition is based on the following lemma which I have proved in my thesis:

If the space-time is regular everywhere and if the terms of the first order of smallness of its metric, supposed quasi-euclidean, satisfy the equations (1 2) *of Levi-Civita, then this metric is reducible to*

$$dt^2 - dx_1^2 - dx_2^2 - dx_3^2$$

to the second order of smallness.

Now the above theorem is an immediate consequence of this lemma. First, the equations of gravitation (3.14) reduce to the equations of Levi-Civita (1.2) and to the following

$$\frac{\partial^2 \lambda_i}{\partial x_1^2} + \frac{\partial^2 \lambda_i}{\partial x_2^2} + \frac{\partial^2 \lambda_i}{\partial x_3^2} = 0 \qquad (i = 1, 2, 3) \tag{5.1}$$

when we neglect the second order of smallness, The equations (5.1) if the space-time is regular everywhere cannot admit any solution regular everywhere and zero at infinity other than zero. So, to the second order of smallness

(5.2) $$\lambda_i=0 \qquad (i=1, 2, 3).$$

Moreover the equations of Levi-Civita according to our lemma permit us to write the metric of the space-time, to the second order of smallness, in the form

$$dt^2-dx_1^2-dx_2^2-dx_3^2.$$

Proceeding in the same way, we successively prove that the metric can be written as above to the second, then to the third etc...order of smallness. So it is identical with the euclidean metric and the theorem is proved.

In other words, if the metric is euclidean to the second order of smallness as a consequence of the gravitational equations it is identically euclidean.

6. *The case of axial symmetry.*

The axial symmetry is defined by the following equations:

(6.1) $$\omega_0=V(x,y)dt, \quad \omega_1=A(x,y)dx, \quad \omega_2=A(x,y)dy, \quad \omega_3=B(x,y)d\theta+\lambda_3(x,y)dt,$$

where $$0\leqslant\theta\leqslant 2\pi.$$

Let β and ϕ be defined by

$$\beta=\log B, \ \phi=\log V.$$

Moreover let us write

$$\Delta=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}, \quad \Delta_1(u,v)=\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\frac{\partial v}{\partial y}, \quad \Delta_1(u)=\Delta_1(u,u).$$

Then we obtain as a result of a very easy calculation

(6.2) $$P_{11}=P_{22}=P_{33} = P_{12}=0$$
$$P_{13}=\frac{B}{2AV}\frac{\partial I}{\partial x}, \qquad P_{23}=\frac{B}{2AV}\frac{\partial I}{\partial z},$$
$$Q_{13}=P_{13}+\frac{I}{AV}\frac{\partial B}{\partial x}, \quad Q_{23}=P_{23}+\frac{I}{AV}\frac{\partial B}{\partial z},$$

where

$$I=\frac{\lambda_3}{B}.$$

Let H^2 be defined by

$$H^2=2(P_{13}^2+P_{23}^2),$$

then the gravitational equations

$$R_{00}=0; \quad R_{33}=0$$

become

$$\Delta\beta+\Delta_1(\beta,\beta+\phi)+a^2H^2=0$$

$$\Delta\phi+\Delta_1(\phi,\phi+\beta)-a^2H^2=0. \tag{6.3}$$

Let h be a function of x and y only such as to have

$$\log h=\log BV.$$

Then from (6.3) we obtain

$$\Delta(\log h)+\Delta_1(\log h)=0.$$

Hence

$$\Delta h=0.$$

As in the case of the ds^2 of Chazy we can use a transformation of co-ordinates, the new ones being

$$r=f_1(x,y) \quad z=f_2(x,y) \quad \theta=\theta,$$

so as to have

$$r+iz=F(x+iy)$$

$$h(x,y)=r.$$

Then the metric of the space-time is such that the Pfaffian forms (6.1) are transformed into

$$\omega_0=V(r.z)dt, \quad \omega_1=a(r,z)dt,$$

$$\omega_2=a(r,z)dz, \quad \omega_3=\frac{r}{V}d\theta+\lambda_3(r,z)dt. \tag{6.4}$$

Now we have with these new co-ordinates

$$2P_{13}Q_{31}-\frac{\lambda_3}{V}P_{11,3}=-2P_{13}^2, \quad P_{13}=\frac{r}{2a^2V}\frac{\partial I}{\partial r}$$

$$2P_{23}Q_{32}-\frac{\lambda_3}{V}P_{22,3}=-2P_{23}^2, \quad P_{23}=\frac{r}{2a^2V}\frac{\partial I}{\partial z} \tag{6.5}$$

$$2(P_{31}Q_{13}+P_{32}Q_{23})-\frac{\lambda_3}{V}P_{33,3}=H^2 \qquad I=\frac{\lambda_3 V}{r}$$

$$P_{13}Q_{32}-Q_{13}P_{32}-\frac{\lambda_3}{V}P_{12,3}=-2P_{13}P_{23}.$$

Hence the gravitational equations. If we we put $\alpha=\log a$, the equations $R_{11}=0$ and $R_{22}=0$ give

$$\Delta\alpha-\frac{1}{r}\frac{\partial\alpha}{\partial r}-\frac{2}{r}\frac{\partial\phi}{\partial r}+2\left(\frac{\partial\phi}{\partial r}\right)^2-\frac{r^2}{2V^4}\left(\frac{\partial I}{\partial r}\right)^2=0$$

$$\Delta\alpha+\frac{1}{r}\frac{\partial\alpha}{\partial r} \qquad +2\left(\frac{\partial\phi}{\partial z}\right)^2-\frac{r^2}{2V^4}\left(\frac{\partial I}{\partial z}\right)^2=0,$$

which by addition and subtraction become, with $\psi=\alpha+\phi$

$$\frac{\partial\psi}{\partial r}-r\left[\left(\frac{\partial\phi}{\partial r}\right)^2-\left(\frac{\partial\phi}{\partial z}\right)^2\right]+\frac{r^3}{4V^4}\left[\left(\frac{\partial I}{\partial r}\right)^2-\left(\frac{\partial I}{\partial z}\right)^2\right]=0$$

$$(6.6)\qquad \Delta\psi-\Delta\phi-\frac{1}{r}\frac{\partial\phi}{\partial r}+\Delta_1\phi-\frac{r^2}{4V^4}\Delta_1 I=0.$$

Then $R_{00}=0$ and $R_{03}=0$ give

$$\Delta\phi+\frac{1}{r}\frac{\partial\phi}{\partial r}-\frac{r^2}{2V^4}\Delta_1 I=0$$

$$(6.7)\qquad \Delta I+\frac{3}{r}\frac{\partial I}{\partial r}-4\Delta_1(I,\phi)=0.$$

Then $R_{12}=0$ is

$$\frac{\partial\psi}{\partial z}-2r\left(\frac{\partial\phi}{\partial r}\frac{\partial\phi}{\partial z}-\frac{r^2}{4V^4}\frac{\partial I}{\partial r}\frac{\partial I}{\partial z}\right)=0.$$

The gravitational problem therefore depends only on the two equations (6.7). ϕ and I being calculated by means of this system, ψ is obtained by a quadrature, for

$$\frac{\partial\psi}{\partial r}=r\left(\nabla_1\phi-\frac{r^2}{4V^4}\nabla_1 I\right),\quad \nabla_1=\left(\frac{\partial}{\partial r}\right)^2-\left(\frac{\partial}{\partial z}\right)^2$$

$$(6.8)\qquad \frac{\partial\psi}{\partial z}=2r\left(\frac{\partial\phi}{\partial r}\frac{\partial\phi}{\partial z}-\frac{r^2}{4V^4}\frac{\partial I}{\partial r}\frac{\partial I}{\partial z}\right).$$

There remains to write the condition of integrability of (6.8). By means of the first equation of (6.7) this condition becomes

$$-\frac{r^3}{2V^4}\frac{\partial I}{\partial z}\left[\Delta I+\frac{3}{r}\frac{\partial I}{\partial r}-4\Delta_1(I,\phi)\right]=0.$$

So it is identically verified because of the second equation of (6.7).

Finally we must verify that

$$(6.9)\qquad \Delta\psi-\Delta\phi-\frac{1}{r}\frac{\partial\phi}{\partial r}+\Delta_1\phi-\frac{r^2}{4V^4}\Delta_1 I=0$$

is identically a consequence of (6.7) and (6.8). As a matter of fact if we calculate $\Delta\psi$ by means of (6.8) and simplify the result by means of the first equation of (6.7), this last expression becomes

$$-\frac{r^3}{2V^4}\frac{\partial I}{\partial x}\left[\Delta I+\frac{3}{r}\frac{\partial I}{\partial r}-4\Delta_1(I,\phi)\right]$$

and it vanishes in virtue of the second equation of (6.7).

Then the metric is

$$(6.10)\quad ds^2=e^{2\phi}dt^2-\frac{2r}{V}\lambda_3 dt\,d\theta-e^{-2\phi}\left[e^{2\psi}(dr^2+dz^2)+r^2d\theta^2\right]$$

λ_3 must be equal to zero on the axis of symmetry ($r=0$), and must tend to zero when $\sqrt{r^2+z^2}$ tends to infinity.

Therefore we shall determine the solutions I and ϕ of the system (6.7) by the conditions that these functions vanish at infinity and that on a segment of the axis of symmetry, say $r=0$, $-l\leqslant z\leqslant l$, they become equal to $-\infty$.

One knows how to transform afterwards the space-sections so that the singularities of the metric become the points of a sphere of radius l. The transformation is defined by

$$z+ir=Z,\quad x+iy=l\zeta,$$

$$Z=\frac{l}{2}\left(\zeta+\frac{1}{\zeta}\right).$$

7. I propose to study the system (6.7) in another paper. I shall only deal here with the case of a quasi-euclidean solution. Then the quantities ϕ, ψ, and I depend on a parameter m and the space-time tends to the euclidean one when m tends to zero. It is easy to get the first two approximations of the metric.

Let, first, r_1 and r_2 be defined by

$$r_1^2=r^2+(z+l)^2,\quad r_2^2=r^2+(z-l)^2.$$

The first approximation of ϕ will be the solution of

$$\Delta\phi_1+\frac{1}{r}\frac{\partial\phi_1}{\partial r}=0.$$

So it is the expression given by Chazy for his ds^2*

$$\phi_1=\frac{k}{2l}\log\frac{r_1+r_2-2l}{r_1+r_2+2l},\quad k=\text{constant}.$$

Then the first approximation of I will be the solution of

$$\Delta I_1+\frac{3}{r}\frac{\partial I_1}{\partial r}=0$$

with the above mentioned boundary conditions.

* *Cf.* G. Darmois, *loc. cit.*

Let us put $\mu=r_1+r_2$. Then we have

$$\frac{\partial\mu}{\partial r}=\frac{r}{r_1}+\frac{r}{r_2}, \qquad \frac{\partial\mu}{\partial z}=\frac{z+l}{r_1}+\frac{z-l}{r_2}$$

$$\frac{\partial^2\mu}{\partial r^2}=\frac{1}{r_1}+\frac{1}{r_2}-\frac{r^2}{r_1^3}-\frac{r^2}{r_2^3}, \quad \frac{\partial^2\mu}{\partial z^2}=\frac{1}{r_1}+\frac{1}{r_2}-\frac{(z+l)^2}{r_1^3}-\frac{(z-l)^2}{r_2^3}$$

$$\Delta_1\mu=\frac{\mu^2-4l^2}{r_1r_2}, \qquad \Delta\mu=\frac{\mu}{r_1r_2}.$$

So for any function U of μ only we have

$$\Delta U+\frac{n}{r}\frac{\partial U}{\partial r}=\frac{1}{r_1r_2}\left[(\mu^2-4l^2)\frac{d^2U}{d\mu^2}+(1+n)\mu\frac{dU}{d\mu}\right].$$

Now let us determine I as a function of μ only. We obtain, h being a constant,

$$I_1=h\left[\frac{1}{r_1+r_2-2l}+\frac{1}{r_1+r_2+2l}+\frac{1}{2l}\log\frac{r_1+r_2-2l}{r_1+r_2+2l}\right].$$

The second approximation of I will be the solution of

$$\Delta I_2+\frac{3}{r}\frac{\partial I_2}{\partial r}-4\Delta_1(I_1,\phi_1)=0,$$

and it is still possible to determine it as a function of μ only. The equation then becomes

$$(\mu^2-4l^2)\frac{d^2I}{d\mu^2}+4\mu\frac{dI}{d\mu}+\frac{128hkl^2}{(\mu^2-4l^2)^2}=0.$$

Finally the second approximation of ϕ will be the solution of

$$\Delta\phi_2+\frac{1}{r}\frac{\partial\phi_2}{\partial r}=\frac{r^2}{2V^4}\Delta_1 I_1.$$

It is no longer possible to determine it as a function of μ only. But if we remark that

$$\Delta+\frac{1}{r}\frac{\partial}{\partial r}$$

is the ordinary Laplacian, in the euclidean space, for cylindric potentials, then we may obtain ϕ_2 by means of the integral

$$\phi_2(\xi,\eta,\zeta)=\iiint_\delta\frac{\rho(x,y,z)\ dxdydz}{\sqrt{(x-\xi)^2+(y-\eta)^2+(z-\zeta)^2}},$$

where

$$\rho(x,y,z)=\frac{r^2}{2V^4}\Delta_1 I,$$

with cartesian co-ordinates; the domain δ of integration being the whole space with the exclusion of a finite domain containing the singularities of the metric, where I_1 and its derivatives become infinite.

Of course these approximations will prove to be quite sufficient for the applications of the Relativity theory to problems of Celestial Dynamics. But there remains to state an existence theorem for the solutions of (6,7) defined by the above mentioned conditions.

Another method of approximation is known. It is the method of de Sitter*. But it makes use of a system of reference less natural and I think that it is less satisfactory than the method developed in this paper.

* The gravitational field due to a spherical mass rotating round one of its diameters has been studied by some mathematicians. *Cf.* De Sitter, *Monthly Notices*, Vol. 76 (1916) pp. 720-7; Lense and Thirring, *Physikalische Zeitschrift*, Band XIX (1918) pp. 156-163; Lense, *Astronomische Nachrichten*, Band 206 (1918) p. 117; Kramers, *Proceedings*, Amsterdam, Vol. XXIII (1920) p. 1059; Bach, *Mathematische Zeitschrift*, Band 13 (1922) p. 123.

A summary of their methods and results is given by Chazy, *La Theorie de ta Relativite et la Mecanique Celeste*, Gauthier-Villars, (1930), tome II, pp. 171-175. The ds^2 thus obtained is such that

$$I=\frac{A}{\rho^3},\ A=\frac{6fM\omega R^2}{5},$$

where ρ is the radial distance of a point, f a constant which depends on the system of units, M the mass of the sphere, R its radius and ω its angular velocity.

This approximation gives only the first term of the expansion of the above I_1 in a series of the powers of $\frac{1}{\rho}=\frac{1}{r_1+r_2}$. We have

$$I_1=\frac{16}{3}\frac{hl^2}{\rho^3}+\dots;\ \text{hence}\ \frac{4h}{3}=\frac{f}{5}M\omega.$$

The method developed in this paper not only gives a better approximation in the case of the spherical mass, but applies to the most general case of the axial symmetry.

ERRATA: Vol. XX (JUBILEE VOLUME)

Page 215, lines 2 and 4, *for* 'right-helicoid' *read* 'right-conoid'.

RULED SURFACES THROUGH A RAY OF A RECTILINEAR CONGRUENCE

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1. Let a rectilinear congruence be defined by the co-ordinates (x, y, z) of a point M on the surface of reference S and by the direction cosines (X, Y, Z) of the line l passing through M, where $x, y, z; X, Y, Z$ are functions of the two parameters u and v. The co-ordinates of any point P of the ray are given by

$$\xi = x + tX,\ \eta = y + tY,\ \zeta = z + tZ,$$

where t is the distance of P measured from the point M.

Corresponding to any curve on the surface of reference there is a ruled surface formed by the lines of the congruence which meet the curve. Hence a curve $\phi(u,v)=0$, on the surface of reference defines the corresponding ruled surface of the line congruence, and a differential equation of the form

$$\frac{dv}{du} = F(u,v)$$

defines a family of ruled surfaces of the line congruence.

2. Delgleize* has proved that through a line l of the congruence, two ruled surfaces pass which have the same central point and that the sum of their parameters of distribution is equal to the mean parameter of the congruence.

In this paper I have first obtained Delgleize's result in a much simpler way and have then obtained some properties of the following types of ruled surfaces through a line of a rectilinear congruence:

(i) which have the same central point, (ii) whose lines of striction lie on the focal sheets, and (iii) whose parameters of distribution are equal to a given constant.

*See 'Contribution a la theorie des congruences rectilignes', *Memoires de la Societe Royale des Sciences de Liege*, Vol. XVI, (1931), 1-9.

The distance t to the central point of a ruled surface of the congruence is given by

$$t = -\frac{edu^2 + (f+f')dudv + gdv^2}{Edu^2 + 2Fdudv + Gdv^2}, \tag{1}$$

where E, F, G ; e, f, f', g are the elements of Kummer's* quadratic differential forms.

Putting $t = \text{const.} \equiv A$, say, we have

$$(AE+e)du^2 + (2FA+f+f')dudv + (AG+g)dv^2 = 0,$$

which is a quadratic equation in dv/du and hence shows that through a line $l(u,v)$ two ruled surfaces pass which have the same central point. Call these ruled surfaces R_1 and R_2.

Taking the curves on the unit sphere which represent R_1 and R_2 as the parametric curves we have

$$AE+e=0, \quad AG+g=0,$$

or

$$\frac{e}{E} = \frac{g}{G} = -A. \tag{2}$$

Also, the parameter of distribution of a ruled surface through l is given by

$$P = \frac{\begin{vmatrix} Edu+Fdv & Fdu+Gdv \\ edu+fdv & f'du+gdv \end{vmatrix}}{(Edu^2+2Fdudv+Gdv^2)\sqrt{EG-F^2}} \dagger.$$

Hence the parameters of distribution of R_1 and R_2 which correspond to $v = \text{const.}$, and $u = \text{const.}$, are given by

$$P_1 = \frac{Ef' - Fe}{E\sqrt{EG-F^2}},$$

and

$$P_2 = \frac{Fg - Gf}{G\sqrt{EG-F^2}}.$$

Hence

$$P_1 + P_2 = \frac{1}{\sqrt{EG-F^2}}\left(f' - \frac{F}{E}e + F\frac{g}{G} - f\right)$$

$$= \frac{f'-f}{\sqrt{EG-F^2}} \text{ from (2)}$$

$$\equiv \frac{E\delta'' + G\delta - 2F\delta'}{EG-F^2},$$

* See 'Allgemeine Theorie der geradlinigen strahlensysteme' *Journal fur Mathematik*, (1860), 189-230.

† See Bianchi, *Lezioni*, Vol. I, 459.

where δ, δ', δ'' are the elements of Sannia's second quadratic differential form, which is the 'mean parameter' of the congruence by definition*.

But the mean parameter of the congruence is equal to $\frac{dp}{d\sigma}$ where p denotes the pitch† of the pencil of the congruence at l.

Hence we get the result:

The pitch of a pencil of the congruence at l is such that the value of $\frac{dp}{d\sigma}$ at l is equal to the sum of the parameters of distribution of the two ruled surfaces through l which have the same central point.

Also, if the sum of the parameters of distribution of the two ruled surfaces through l which have the same central point is zero, the congruence is normal and the pitch vanishes.

3. We know that through any line l of the congruence two ruled surfaces (non-developable) pass whose lines of striction lie on the focal sheets and that the parameter of distribution of each of these two is equal to the mean parameter of the congruence‡.

Hence we get another expression for the value of $\frac{dp}{d\sigma}$ namely:

The value of $\frac{dp}{d\sigma}$ at the ray l is equal to the parameter of distribution of each of the two ruled surfaces through l whose lines of striction lie on the focal sheets.

Also, if the parameter of distribution of any one of the two ruled surfaces through l whose lines of striction lie on the focal sheets vanishes then the congruence is normal and the pitch is zero.

4. The parameter of distribution of a ruled surface through l is given by

$$P = \frac{\delta du^2 + 2\delta' du dv + \delta'' dv^2}{E du^2 + 2F du dv + G dv^2} \|.$$

* See Sannia, *Annali di Matematica,* (1908), 152; or Sannia *Mathematische Annalen,* (1910), 411.

† See my paper on 'A Significant integral invariant in the Theory of Rectilinear Congruences', *Jour. Ind. Math. Soc.*, (New Series), Vol. I, No. 4 (1934), 136.

‡ See Delgleize, *loc. cit.* pp. 1-9.

|| See Bianchi, *Lezioni*, Vol. I, 459.

Putting $P=\text{const.}=B$, say, we have

$$(BE-\delta)du^2+(2FB-2\delta')dudv+(BG-\delta'')dv^2=0,$$

which is a quadratic in $\frac{dv}{du}$ and hence shows that through a line l two ruled surfaces pass which have a given parameter of distribution.

If the curves which represent these ruled surfaces say, S_1 and S_2, on the unit sphere be taken as the parametric curves $v=\text{const.}$, and $u=\text{const.}$, we have

$$BE-\delta=0,\ BG-\delta''=0,$$

or

$$\frac{\delta}{E}=\frac{\delta''}{G}=B. \tag{3}$$

We know that the distance to the central point of a ruled surface through l is given by

$$t=-\frac{edu^2+(f+f')dudv+gdv^2}{Edu^2+2Fdudv+Gdv^2}.$$

Hence the distances of the central points of the ruled surfaces S_1 and S_2 are given by

$$t_1=-\frac{e}{E}, \text{ and } t_2=-\frac{g}{G}. \tag{4}$$

Hence

$$\frac{1}{t_1}+\frac{1}{t_2}=-\left(\frac{E}{e}+\frac{G}{g}\right)$$

$$=-\left(\frac{\delta}{e}+\frac{\delta''}{g}\right).\frac{1}{B} \text{ from (3)}$$

$$=-\frac{1}{B}\left(\frac{Ef'-Fe}{e\sqrt{EG-F^2}}+\frac{Fg-Gf}{g\sqrt{EG-F^2}}\right)$$

$$=-\frac{1}{B\sqrt{EG-F^2}}\left(\frac{E}{e}f'-F+F-\frac{G}{g}f\right).$$

Hence

$$\frac{t_1+t_2}{t_1t_2}=-\frac{1}{B\sqrt{EG-F^2}}\left(-\frac{f'}{t_1}+\frac{f}{t_2}\right)$$

$$=-\frac{ft_1-f't_2}{t_1t_2B\sqrt{EG-F^2}},$$

or

$$t_1\left(1+\frac{f}{B\sqrt{EG-F^2}}\right)=t_2\left(\frac{f'}{B\sqrt{EG-F^2}}-1\right),$$

or

$$\frac{t_1}{t_2}=\frac{f'-B\sqrt{EG-F^2}}{f+B\sqrt{EG-F^2}}.$$

Hence the ratio of the distances to the central points of the two ruled surfaces through l which have their parameters of distribution equal to a given constant B is

$$(f'-B\sqrt{EG-F^2})/(f+B\sqrt{EG-F^2}).$$

In particular if $t_1=t_2$ we get

$$2B=\frac{f'-f}{\sqrt{EG-F^2}}$$

=mean parameter of the congruence.

Hence we have the following results:

(*i*) *The two ruled surfaces through l which have their parameters of distribution equal to the semi mean parameter of the congruence have the same central point.*

(*ii*) *The value of* $\frac{dp}{d\sigma}$ *at l is equal to twice the parameter of distribution of any one of the two ruled surfaces through l whose parameters of distribution are equal and whose central points coincide.*

A THEOREM ON EQUIANGULAR CONVEX POLYGONS CIRCUMSCRIBING A CONVEX CURVE

By R. C. BOSE, Calcutta

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1. The object of this note is to prove the following theorem, connecting the perimeter of a convex curve, with the perimeter of convex equiangular polygons circumscribing it.

THEOREM. *Given a closed convex curve V, with perimeter L, and an integer $n \geqslant 3$ we can find at least two n-sided convex equiangular polygons, circumscribing V, and having the perimeter*

$$\frac{nL}{\pi} \tan \frac{\pi}{n} \tag{1}$$

i.e. a perimeter equal to the perimeter of a regular n-gon circumscribing a circle of perimeter L.

It follows as a corollary that, *if $\bar{S}$ and $\underline{S}$ denote the maximum and minimum perimeters of n-sided convex equiangular polygons circumscribing V, then*

$$\bar{S} \geqslant \frac{nL}{\pi} \tan \frac{\pi}{n} \geqslant \underline{S}. \tag{2}$$

The proof depends upon Hurwitz' theorem on the zeros of Fourier series lacking first terms, and the trigonometrical identities

$$\sum_{k=0}^{n-1} \sin m\left(\phi+\frac{2k\pi}{n}\right)=0, \quad \sum_{k=0}^{n-1} \cos m\left(\phi+\frac{2k\pi}{n}\right)=0, \tag{3}$$

where m and n are integers and m is not divisible by n.

2. A convex polygon with all its angles equal may be called an *equiangular convex polygon.* The following property of the equiangular convex polygon may be easily proved from elementary geometry.

If S denotes the perimeter of an n-sided equiangular convex polygon, and U denotes the sum of the perpendiculars drawn from any interior point to the sides of the polygon, then

$$S=2U \tan \frac{\pi}{n}. \tag{4}$$

It follows as a corollary that *the sum of the perpendiculars drawn from any interior point, to the sides of a convex equiangular polygon is constant.*

3. A plane set of points which is (*a*) bounded (*b*) closed and (*c*) has the convexity property that all the points of the segment joining any two points of the set, are members of the set, may be called a *plane convex domain.*

The boundary points of a plane convex domain form a *closed convex curve.*

A line t may be called a *stutz-line* of a closed curve V if t contains at least one point of V, while one of the two half-planes in which t divides the plane is free from points of V. If we so orient t, that the half-plane to the right of t, is free from points of V, then t may be called an *oriented stutz-line* of V.

Corresponding to any angle $\phi (0 \leqslant \phi \leqslant 2\pi)$ there exists just one oriented stutz-line of V, which makes an angle ϕ with the positive direction of the x-axis. If p is the length of the perpendicular from the origin on this line (reckoned positively when the origin is to the left of the oriented stutz-line), then p is a function of ϕ, say $p = p(\phi)$. The function $p(\phi)$ has been called by Minkowski* the *Stutz-function* of V, and he has studied the most important properties of this function.

If L denote the perimeter of the curve V, then

$$L = \int_0^{2\pi} p(\phi) d\phi. \tag{5}$$

4. A convex polygon Σ may be said to *circumscribe* the closed convex curve V, if all the sides of Σ are stutz-lines of V, and if all points of V which do not lie on the sides of Σ, are interior points of V.

Consider now an n-sided convex equiangular polygon Σ, circumscribing a closed convex curve V with perimeter L. Let $p(\phi)$ be the stutz-function of V. Let us orient the sides of Σ, so that they become oriented stutz-lines according to the definition of the previous paragraph. Then if ϕ be the angle which one of the oriented sides makes with the positive direction of the x-axis, then the angles which the oriented sides successively make with the positive direction of the x-axis may be taken to be

$$\phi, \phi + \frac{2\pi}{n}, \phi + \frac{4\pi}{n}, \ldots, \phi + \frac{2(n-1)\pi}{n}. \tag{6}$$

* Minkowski, 1. 'Theorie der knovexen körper, insbesondre Begrundung ihrer Oberflächenbegriffs', *Ges. Abh.* Bd. 2. 131-229. 2. Volumen und Oberfläche', *Math. Ann.* Bd. 57 (1903), 447-495.

We may suppose the origin to be an interior point of the convex domain bounded by V. If $U(\phi)$ denotes the sum of the perpendiculars from the origin to the sides of Σ, then

$$U(\phi)=\sum_{k=0}^{n-1} p\left(\phi+\frac{2k\pi}{n}\right). \tag{7}$$

Suppose now $p(\phi)$ to be expanded in Fourier series so that

$$p(\phi)=\frac{a_0}{2}+(a_1\cos\phi+b_1\sin\phi)+(a_2\cos 2\phi+b_2\sin 2\phi)+\ldots, \tag{8}$$

then
$$a_0=\frac{1}{\pi}\int_0^{2\pi} p(\phi)\,d\phi=\frac{L}{\pi}. \tag{9}$$

Hence in virtue of the trigonometrical identities (3), we get from (7), (8) and (9)

$$U(\phi)=\frac{nL}{2\pi}+n\sum_{k=1}^{\infty}(a_{nk}\cos nk\phi+b_{nk}\sin nk\phi). \tag{10}$$

Thus the Fourier expansion of

$$U(\phi)-\frac{nL}{2\pi} \tag{11}$$

begins from terms of the type sin $n\phi$, cos $n\phi$. It follows from Hurwitz theorem* on the zeros of Fourier series lacking first terms, that $U(\phi)$ attains the value $nL/(2\pi)$, at least $2n$ times in the interval $(0,2\pi-0)$. But if $S(\phi)$ denotes the perimeter of Σ, then from (4)

$$S(\phi)=2U(\phi)\tan\frac{\pi}{n}. \tag{12}$$

Hence $S(\phi)$ attains the value $(nL/\pi)\tan(\pi/n)$, at least $2n$ times in the interval $(0,2\pi-0)$. But to every n-sided equiangular polygon circumscribing V, there correspond n values of ϕ, in the interval $(0,2\pi-0)$ since any side of Σ can be taken as the first side. Hence we get the Theorem and the Corollary stated above.

In conclusion my best thanks are due to Prof. F. Levi, for the keen interest taken by him in the investigation.

* A. Hurwitz, 'Über die Fourierschem Konstanten integrierbarer Funktionen', *Math. Ann.* Bd. 57 (1903), 444. W. Suss, 'Einige mit dem Vierscheitelsatze fur Eilinien Zuzammenhängende Satze', *Tohoku Math. J.* Vol. 28 (1927), 219.

ON THE SIGNIFICANCE AND THE EXTENSION OF THE CHINESE REMAINDER THEOREM*

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Introduction

A set of elements is said to form a *group* under a composition rule R or simply an R-group if

(i) for every two elements a and b of the set aRb is also an element of the set, i.e. the set is closed under R;

(ii) R is associative, i.e. for any three elements a, b, c, of the set

$$(aRb)Rc = aR(bRc);$$

(iii) there exists an element e, called the *identity element*, such that for every element a of the set

$$aRe = eRa = a;$$

(iv) to every element a, there exists an element $x = a^{-1}$ called the *inverse* of a, such that

$$aRx = e.$$

The group is called Abelian if R is commutative.

The composition process R is often called addition or multiplication according as it is or is not commutative. Then aRb is called the sum or the product of a and b and written $a+b$ or ab according as R is addition or multiplication. Naturally the identity element in these cases would be written 0 or 1 respectively.

An additive group is also called a *module*.

An *algebra* is a module which is closed under multiplication and in which multiplication is associative and distributive both ways with respect to addition.

Let $B_1, B_2, \ldots, B_n$ be a set of n groups under the composition rules $R_1, R_2, \ldots, R_n$ respectively. Consider the vectors

* I wish to express my thanks to Dr. R. Vaidyanathaswamy, for his valuable suggestions and helpful criticisms in the course of writing out this paper.

$(b_1, b_2, \ldots, b_n)$ where b_r is an element of $B_r (r=1, 2, \ldots, n)$. If we define

$(b_1, b_2, \ldots, b_n) R (b_1', b'_2, \ldots, b'_n)$ as $(b_1 R_1 b'_1, b_2 R_2 b'_2, \ldots, b_n R_n b'_n)$ the set of vectors $(b_1, b_2, \ldots, b_n)$ evidently forms an R-group which we call the *vector compound* of the n groups $B_1, B_2, \ldots, B_n$.

Similarly, if $A_1, A_2, \ldots, A_n$ is a set of n algebras we consider all the vectors $(a_1, a_2, \ldots, a_n)$ where a_r is an element of A_r $(r=1, 2, \ldots, n)$. Let us denote addition and multiplication in A_i by S_i and R_i respectively. If we define the sum and the product of two vectors $(a_1, a_2, \ldots, a_n)$, $(a'_1, a'_2, \ldots, a'_n)$ as $(a_1 S_1 a'_1, a_2 S_2 a'_2, \ldots, a_n S_n a'_n)$ and $(a_1 R_1 a'_1, a_2 R_2 a'_2, \ldots, a_n R_n a'_n)$ respectively, the set of vectors $(a_1, a_2, \ldots, a_n)$ evidently forms an algebra which we call the *vector compound* of the n algebras $A_1, A_2, \ldots, A_n$.

As a simple illustration of this definition, let $M_1, M_2, \ldots, M_n$ be a set of n integers. Then every M_r defines a residue class commutative algebra $\bmod (M_r)$ $(r=1, 2, \ldots, n)$. The vector compound of the n residue class algebras consists of the totality of the vectors $(m_1 \bmod M_1, m_2 \bmod M_2, \ldots, m_n \bmod M_n)$ which we may write for simplicity as $(m_1, m_2, \ldots, m_n) \bmod (M_1, M_2, \ldots, M_n)$. We call the vector compound of the n residue class algebras $\bmod (M_1), \bmod (M_2), \ldots, \bmod (M_n)$ as the *vector algebra* $\bmod (M_1, M_2, \ldots, M_n)$. Per contra, a residue class algebra $\bmod (M)$ is a *scalar algebra* $\bmod (M)$.

From the point of view developed in this paper, the Chinese remainder theorem is regarded as establishing a definite isomorphism between a vector modular algebra and a scalar modular algebra. This is utilised in II and III to show that this itself is a special case of an isomorphism between two vector modular algebras. We determine in IV, the linear transformations by which we obtain the elements of one vector modular algebra from the corresponding elements of the other. The equations of the isomorphism so obtained constitute the extension of the Chinese remainder theorem. In V we find the necessary and sufficient conditions for the solvability of the congruences

$$x \equiv m_1 \bmod M_1,\ x \equiv m_2 \bmod M_2, \ldots, x \equiv m_n \bmod M_n. \quad (1)$$

In the case in which integers x exist for which (1) is true, an expression for x is given which can be easily identified with that given by Matthiesen*. Finally we show, by means of the

* Matthiesen (1).

extended Chinese remainder theorem, how to determine a set of normal basis* elements from a given set of basis elements of an Abelian group.

I. Chinese remainder theorem and its significance

Theorem 1. (*Chinese remainder theorem*). *If $M_1, M_2,\ldots,M_n$ is a set of n integers relatively prime in pairs, then integers x exist for which simultaneously*

$$x\equiv m_1 \bmod M_1,\ x\equiv m_2 \bmod M_2,\ldots,\ x\equiv m_n \bmod M_n \qquad (1)$$

and all such integers are congruent $\bmod(M_1.M_2\ldots M_n=M)$.

Proof: Since $\left(M_r, \frac{M}{M_r}\right)=1,\ r=1, 2,\ldots, n$

we can find integers $A_1, A_2,\ldots, A_n$ such that

$$\left.\begin{aligned} A_r &\equiv 1 \bmod M_r \\ &\equiv 0 \bmod \frac{M}{M_r} \end{aligned}\right\} r=1, 2,\ldots, n.$$

$x=A_1m_1+A_2m_2+\ \ldots\ +A_nm_n$ evidently satisfies (1). The difference between any two solutions of (1) is divisible by $M_1, M_2,\ldots, M_n$ and hence by their product, as they are relatively prime in pairs.

It therefore follows that given a set of n residue classes $m_1 \bmod M_1$, $m_2 \bmod M_2,\ldots, m_n \bmod M_n$, the integers that satisfy the congruences (1) form a definite residue class $g \bmod(M)$. In other words, given a vector$(m_1, m_2,\ldots, m_n) \bmod (M_1, M_2,\ldots, M_n)$ Theorem 1 uniquely defines the scalar $(g) \bmod (M)$. And the scalar $(g) \bmod (M)$ by (1) defines the vector $(m_1, m_2,\ldots, m_n) \bmod (M_1, M_2,\ldots, M_n)$ uniquely. This (1, 1) correspondence may be denoted by

$$(m_1, m_2,\ldots, m_n) \bmod (M_1, M_2,\ldots, M_n)\rightarrow(g) \bmod (M)$$

or simply $$(m_1, m_2,\ldots, m_n)\rightarrow(g).$$

If now $$(m_1, m_2,\ldots, m_n)\rightarrow(g)$$

and $$(m'_1, m'_2,\ldots, m'_n)\rightarrow(g')$$

then $$(m_1, m_2,\ldots, m_n)+(m'_1, m'_2,\ldots, m'_n)$$
$$=(m_1+m'_1, m_2+m'_2,\ldots, m_n+m'_n)\rightarrow(g+g')=(g)+(g'),$$

and $$(m_1, m_2,\ldots, m_n)\ .\ (m'_1, m'_2,\ldots, m'_n)$$
$$=(m_1m'_1, m_2m'_2,\ldots, m_nm'_n)\rightarrow(g.g')=(g).(g').$$

* Frobenius (2).

This is obvious if we write (1) in the form

$$(g, g, \ldots, g) \rightarrow (g).$$

Hence the (1,1) correspondence is an isomorphism and the vector algebra mod $(M_1, M_2, \ldots, M_n) \cong$* the scalar algebra mod (M).

The question then arises whether a similar isomorphism can be established when $M_1, M_2, \ldots, M_n$ are not relatively prime in pairs. We begin with the case of two moduli say M_1, M_2. Let G_1 be the *g.c.d.* and G_2 be the *l.c.m.* of M_1 and M_2. The number of distinct vectors in the vector algebra mod (M_1, M_2) is $M_1.M_2$ and in the vector algebra mod (G_1, G_2) is $G_1.G_2$. Since

$$M_1.M_2 = G_1.G_2 \tag{1·1}$$

the number of distinct vectors is the same in the two algebras. This suggests that the two vector modular algebras may be simply isomorphic. Such isomorphism would be quite consistent with Theorem 1 for, when M_1, M_2 are relatively prime $G_1 = 1$ and $M_1.M_2 = G_2$; thus $g_1 = 0$ in all the vectors (g_1, g_2) and hence the vector algebra mod(G_1, G_2) reduces to the scalar algebra mod (G_2). We shall now show by Theorem 1 that the two vector algebras mod (M_1, M_2) and mod (G_1, G_2) are in fact simply isomorphic.

II. THE ISOMORPHISM FOR TWO MODULI

DEFINITION. A factor A of B is called a block factor†, if A is prime to B/A. An elementary block factor is a block factor which is a prime power.

We begin by showing that M_1, M_2, G_1, G_2 can be expressed as products of relatively prime factors in the form

$$\left.\begin{aligned} M_1 = M_{11}.M_{12}. \quad G_1 = M_{11}.M_{21} \\ M_2 = M_{21}.M_{22}. \quad G_2 = M_{12}.M_{22} \end{aligned}\right\}. \tag{2·1}$$

We show how to determine $M_{rs} (r, s = 1, 2)$.

If p^a is an elementary block factor in only one of the M's say in M_r, it will be an elementary block factor in only one of the G's say in G_s. We take in this case, p^a as an elementary block factor in M_{rs}. If p^a is an elementary block factor in M_1 and M_2, it will be an elementary block factor in G_1 and G_2. We then take

* The symbol $\cong$ denotes simple isomorphism.

† R. Vaidyanathaswamy (3).

p^a as an elementary block factor either in M_{11} and M_{22} or in M_{12} and M_{21}. By taking all the elementary block factors in M_1 and M_2 we determine as above all the elementary block factors in $M_{rs}(r, s=1, 2)$.

By our construction, M_{r1}, M_{r2} are block factors in M_r and every elementary block factor of M_r is taken either in M_{r1} or in M_{r2} and therefore once only in $M_{r1}.M_{r2}$. Hence $M_r = M_{r1}.M_{r2}$. Since M_{r1}, M_{r2} are block factors in M_r they are therefore relatively prime. Similarly M_{1r}, M_{2r} are block factors in G_r and every elementary block factor of G_r is taken either in M_{1r} or in M_{2r} and therefore once only in $M_{1r}.M_{2r}$ so that $G_r = M_{1r}.M_{2r}$. Since M_{1r}, M_{2r} are block factors in G_r they are therefore relatively prime. Hence the representation (2·1) is established.

NOTE. If M_1 and M_2 have no common block factors, M_{rs} must be clearly taken as the greatest common block factor in M_r and G_s and the representation (2·1) is unique. If M_1, M_2 have common block factors, every common elementary block factor gives rise to two representations of the form (2·1). Therefore if there are r common elementary block factors in M_1 and M_2 there will be 2^r distinct representations of the form (2·1).

Since M_{r1}, M_{r2} are relatively prime by Theorem 1 $(r=1, 2)$
the vector algebra mod $(M_{11}, M_{12}) \cong$ the scalar algebra mod (M_1)
and
the vector algebra mod $(M_{21}, M_{22}) \cong$ the scalar algebra mod (M_2),
so that

the vector algebra mod $(M_{11}, M_{12}, M_{21}, M_{22})$
$\cong$ the vector algebra mod (M_1, M_2).

Similarly,

the vector algebra mod $(M_{11}, M_{21}, M_{12}, M_{22})$
$\cong$ the vector algebra mod (G_1, G_2).

The two vector algebras mod $(M_{11}, M_{12}, M_{21}, M_{22})$ and mod $(M_{11}, M_{21}, M_{12}, M_{22})$ differ only in the order of taking the component residue class algebras. Therefore they are simply isomorphic.

Hence
the vector algebra mod $(M_1, M_2) \cong$ the vector algebra mod (G_1, G_2).

III. THE CASE OF n MODULI

By an extension of the notion of the *g.c.d.* and the *l.c.m.* a relation of the type (2) can be established when there are n

integers $M_1, M_2, \ldots, M_n$. The rth $g.c.d$,* of a set of n integers $M_1, M_2, \ldots, M_n$ is defined as the $g.c.d.$ of the $l.c.m.$'s taken r at a time of $M_1, M_2, \ldots, M_n$. Let $G_1, G_2, \ldots, G_n$ be the successive $g.c.d$'s. Let $p^{a_1}, p^{a_2}, \ldots, p^{a_n}$ be elementary block factors in $M_1, M_2, \ldots, M_n$ respectively for a prime p. Then $p^{a_1+a_2+\cdots+a_n}$ is an elementary block factor in $M_1.M_2\ldots M_n$. If $p^{b_1}, p^{b_2}, \ldots, p^{b_n}$ are elementary block factors in $G_1, G_2, \ldots, G_n$ respectively $b_1, b_2, \ldots, b_n$ will be the 1st, 2nd,..., nth in the ascending order of magnitude of $a_1, a_2, \ldots, a_n$ so that $p^{b_1+b_2+\cdots+b_n}=p^{a_1+a_2+\cdots+a_n}$ is an elementary block factor in $G_1.G_2\ldots G_n$ also. This is true for any prime. Therefore

$$M_1.M_2\ldots M_n=G_1.G_2\ldots G_n.$$

We now show that the vector algebras mod $(M_1, M_2, \ldots, M_n)$ and mod $(G_1, G_2, \ldots, G_n)$ are simply isomorphic. The proof follows as in II.

LEMMA. 1. *$M_r, G_r (r=1, 2, \ldots, n)$ can be expressed as products of mutually relatively prime factors in the form*

$$\left.\begin{aligned} M_r&=M_{r1}.M_{r2}\ldots M_{rn}.\\ G_r&=M_{1r}.M_{2r}\ldots M_{nr}.\end{aligned}\right\} \qquad (3{\cdot}1)$$

We show how to determine $M_{rs}(r, s=1, 2, \ldots, n)$. If p^a is an elementary block factor in exactly r of the M's, say in $M_p, M_q, \ldots$ then it is clear that it will be an elementary block factor in r consecutive G's, say in $G_k, G_{k+1}, \ldots, G_{k+r-1}$. We choose p^a as an elementary block factor in r M's each having the first suffix as one of the r numbers $p, q, \ldots$and the second suffix as one of the r numbers $k, k+1, \ldots, k+r-1$, no two of the r M's having the same first or second suffix. Evidently the number of ways of choosing such r M's is equal to the number of terms in the expansion of a determinant of r rows and r columns, i.e. $r!$ We take all the elementary block factors in $M_1, M_2, \ldots, M_n$ and determine as above all the elementary block factors of $M_{rs}(r, s=1, 2, \ldots, n)$.

By our construction $M_{r1}, M_{r2}, \ldots, M_{rn}$ are block factors in M_r and every elementary block factor in M_r is counted once and only once in the product $M_{r1}M_{r2}\ldots M_{rn}$. Therefore $M_r=M_{r1}M_{r2}\ldots M_{rn}(r=1, 2, \ldots n)$. Since $M_{r1}, M_{r2}, \ldots, M_{rn}$ are block factors in M_r they are relatively prime in pairs. Similarly $G_r=M_{1r}M_{2r}\ldots M_{nr}(r=1, 2, \ldots, n)$ and $M_{1r}, M_{2r}, \ldots, M_{nr}$ are relatively prime in pairs. Hence the representation (3·1) is established.

* R. Vaidyanathaswamy (4).

NOTE. If p^a is an elementary block factor in exactly r of the M's the number of representations of the form (3·1) is the product of a number of factors of the form $r!$

Since $M_{r1}, M_{r2}, \ldots, M_{rn}$ are relatively prime in pairs by Theorem 1

the vector algebra mod $(M_{r1}, M_{r2}, \ldots M_{rn})$

$$\cong \text{the scalar algebra mod } (M_r) \quad (r=1, 2, \ldots, n).$$

Therefore the vector algebra mod $(M_{11}, M_{12} \ldots, M_{1n}, M_{21}, M_{22}, \ldots, M_{2n}, \ldots, M_{n1}, M_{n2}, \ldots, M_{nn})$

$$\cong \text{the vector algebra mod } (M_1, M_2, \ldots, M_n).$$

Similarly the vector algebra mod $(M_{11}, M_{21}, \ldots, M_{n1}, M_{12}, M_{22}, \ldots, M_{n2}, \ldots, M_{1n}, M_{2n}, \ldots, M_{nn})$

$$\cong \text{the vector algebra mod } (G_1, G_2, \ldots, G_n).$$

The vector algebras mod $(M_{11}, M_{12}, \ldots, M_{1n}, M_{21}, M_{22}, \ldots, M_{2n}, \ldots M_{n1}, M_{n2}, \ldots, M_{nn})$ and mod $(M_{11}, M_{21}, \ldots, M_{n1}, M_{12}, M_{22}, \ldots M_{n2}, \ldots, M_{1n}, M_{2n}, \ldots, M_{nn})$ differ only in the order of taking the component residue class algebras and therefore they are simply isomorphic. Hence

the vector algebra mod $(M_1, M_2 \ldots M_n)$

$$\cong \text{the vector algebra mod } (G_1, G_2, \ldots G_n).$$

IV. EQUATIONS OF THE ISOMORPHISM

We now proceed to determine the linear transformation A by which we obtain $(g_1, g_2, \ldots, g_n)$ mod $(G_1, G_2, \ldots, G_n)$ from the corresponding vector $(m_1, m_2, \ldots, m_n)$ mod $(M_1, M_2, \ldots, M_n)$.

Let the matrix of A be

$$\begin{vmatrix} \alpha_{11} & \alpha_{12} & \ldots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \ldots & \alpha_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ \alpha_{n1} & \alpha_{n2} & \ldots & \alpha_{nn} \end{vmatrix}$$

so that

$$g_r \equiv \alpha_{r1} m_1 + \alpha_{r2} m_2 + \ldots + \alpha_{rn} m_n \text{ mod } (G_r) \quad (r=1, 2, \ldots, n).$$

Since $\quad G_r = M_{1r} M_{2r} \ldots M_{nr}$

by Theorem 1 $\quad g_r \equiv m_{pr} \text{ mod } M_{pr} \quad (p=1, 2, \ldots, n).$

Since $\quad M_p = M_{p1} M_{p2} \ldots M_{pn}$

by Theorem 1 $\quad m_p \equiv m_{pr} \text{ mod } (M_{pr}) \quad (r=1, 2, \ldots, n).$

Therefore $\quad g_r \equiv m_p \text{ mod } (M_{pr})$

and $\quad m_p \equiv \alpha_{r1} m_1 + \alpha_{r2} m_2 + \ldots + \alpha_{rn} m_n \text{ mod } (M_{pr}).$

Since the above congruence is true for all values of $m_1, m_2, \ldots, m_n$

therefore $\alpha_{rs} \equiv 0 \bmod M_{pr}$ for $s \neq p$

$$\alpha_{rp} \equiv 1 \bmod M_{pr}.$$

Therefore

$$\alpha_{rs} \equiv 0 \bmod M_{1r} M_{2r} \ldots M_{(s-1)r} M_{(s+1)r} \ldots M_{nr}.$$

$$\equiv 1 \bmod M_{sr}.$$

.e. $$\alpha_{rs} \equiv \frac{G_r}{M_{sr}} \text{ and} \equiv 1 \bmod M_{sr}.$$

Giving all values to $r, s=1, 2, \ldots, n$ we get the required transformation.

In a similar manner we find the linear transformation B by which we obtain $(m_1, m_2, \ldots, m_n) \bmod (M_1, M_2, \ldots, M_n)$ from $(g_1, g_2, \ldots, g_n) \bmod (G_1, G_2, \ldots, G_n)$.

If the matrix of B is

$$\begin{vmatrix} \beta_{11} & \beta_{12} & \ldots & \beta_{1n} \\ \beta_{21} & \beta_{22} & \ldots & \beta_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ \beta_{n1} & \beta_{n2} & \ldots & \beta_{nn} \end{vmatrix}$$

$$m_r \equiv \beta_{r1} g_1 + \beta_{r2} g_2 + \ldots + \beta_{rn} g_n \bmod (M_r) \quad (r=1, 2, \ldots, n).$$

By Theorem 1 $\quad m_r \equiv m_{rp} \bmod (M_{rp}) \quad (p=1, 2, \ldots, n)$

$$\equiv g_p \bmod (M_{rp}).$$

Therefore $\quad g_r \equiv \beta_{r1} g_1 + \beta_{r2} g_2 + \ldots + \beta_{rn} g_n \bmod (M_{rp}).$

Since the above congruence is true for all values of $g_1, g_2, \ldots, g_n$

$$\beta_{rs} \equiv 0 \bmod (M_{rp}) \quad s \neq p$$

$$\beta_{rp} \equiv 1 \bmod (M_{rp}).$$

Therefore $\quad \beta_{rs} \equiv 0 \bmod M_{r1} M_{r2} \ldots M_{r, s-1} M_{r, s+1} \ldots M_{rn}.$

$$\equiv 1 \bmod M_{rs}.$$

i.e. $$\beta_{rs} \equiv 0 \bmod \frac{M_r}{M_{rs}} \text{ and} \equiv 1 \bmod M_{rs}.$$

Giving all values to $r, s=1, 2, \ldots, n$ we get the transformation B.

V. THE CONGRUENCES OF THE CHINESE REMAINDER THEOREM IN THE GENERAL CASE

LEMMA 2. *The necessary and sufficient condition that integers x exist for which*

$$x \equiv m_1 \bmod M_1, \quad x \equiv m_2 \bmod M_2 \tag{5·1}$$

is that $$m_1 \equiv m_2 \bmod \delta(M_1, M_2),$$

where $\delta(M_1, M_2)$ denotes the g.c.d. of M_1 and M_2.

PROOF: The condition is necessary for if

$$x=m_1+k_1M_1=m_2+k_2M_2$$
$$m_1-m_2=k_2M_2-k_1M_1\equiv 0 \bmod \delta\ (M_1, M_2).$$

The condition is also sufficient for if

$$AM_1+BM_2=\delta(M_1, M_2)$$

writing δ for $\delta(M_1, M_2)$ we have

$$A.M_1.\frac{m_2-m_1}{\delta}+B.M_2.\frac{m_2-m_1}{\delta}=m_2-m_1,$$

or $$A.M_1.\frac{m_2-m_1}{\delta}+m_1=B.M_2.\frac{m_1-m_2}{\delta}+m_2=l_1, \text{ say.}$$

$x=l_1$ satisfies (5·1). If L_1 is the *l.c.m.* of M_1, M_2 all other solutions of (5·1) differ from l_1 by a multiple of L_1.

LEMMA 3. *The necessary and sufficient conditions for the solvability of the congruences*

$$x\equiv m_1 \bmod M_1,\ x\equiv m_2 \bmod M_2, \ldots, x\equiv m_n \bmod M_n \qquad (1)$$

are $$m_r\equiv m_s \bmod \delta(M_r, M_s) \quad r, s=1, 2, \ldots, n. \qquad (5·2)$$

PROOF: Taking the congruences (1) in pairs we find from Lemma 2 that the conditions are necessary.

The conditions are also sufficient for by Lemma 2 integers x exist for which

$$x\equiv m_1 \bmod M_1,\ x\equiv m_2 \bmod M_2$$

and they form a definite residue class l_1 mod L_1. Since

$$m_1\equiv m_3 \bmod \delta(M_1, M_3),\ m_2\equiv m_3 \bmod \delta(M_2, M_3)$$
$$l_1\equiv m_3 \bmod \delta(M_1, M_3),\ l_1\equiv m_3 \bmod \delta(M_2, M_3)$$

hence $$l_1\equiv m_3 \bmod \delta(L_1, M_3).$$

By Lemma 2 again integers x exist for which

$$x\equiv l_1 \bmod L_1,\ x\equiv m_3 \bmod M_3,$$

and they form a definite residue class l_2 mod L_2 where L_2 is the *l.c.m.* of L_1 and M_3, i.e. the *l.c.m.* of M_1, M_2, M_3.

By repeating the same process we find that the conditions are sufficient and that the integers that satisfy the congruences (1) form a definite residue class mod (the *l.c.m.* of $M_1, M_2, \ldots, M_n$ namely G_n).

Conversely we show that all the elements of the residue class g_n mod G_n satisfy the congruences (1) when the conditions (5·2) are satisfied. Now

$$g_n\equiv \alpha_{n1}m_1+\alpha_{n2}m_2+\ldots+\alpha_{nn}m_n \bmod G_n$$
$$\equiv(\alpha_{n1}+\alpha_{n2}+\ldots+\alpha_{nn})\ x \bmod G_n.$$

$$\alpha_{n1}+\alpha_{n2}+\ldots+\alpha_{nn}\equiv 1 \bmod (M_{1n}, M_{2n},\ldots M_{nn})$$
$$\equiv 1 \bmod G_n.$$

Therefore $$x\equiv g_n \bmod G_n. \quad (5\cdot3)$$

Since G_n is a multiple of $M_1, M_2\ldots, M_n$ it follows from (1) and (5·3) that g_n satisfies (1). All other solutions differ from g_n by a multiple of G_n.

VI. CONNECTION WITH THE THEORY OF ABELIAN GROUPS

LEMMA 4. *The invariant factors (or the elementary divisors) of a square matrix A, whose elements are all zero except those in the leading diagonal, are the successive g.c.d.'s of the diagona elements of A.*

PROOF: Let

$$A=\begin{vmatrix} M_1 & 0 & \ldots & 0 \\ 0 & M_2 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & M_n \end{vmatrix}.$$

If δ_k is the *g.c.d.* of the k rowed sub-determinants of the determinant of A ($k=1, 2,\ldots, n$), δ_k is called the kth determinant divisor of A, and the invariant factors can be obtained from the relation

$$\delta_k=\delta_{k-1}.\varepsilon_k{}^{*} \quad (k=1, 2,\ldots, n), \quad (6\cdot1)$$

where ε_k is the kth invariant factor of A.

If $p^{a_1}, p^{a_2},\ldots p^{a_n}$ are elementary block factors in $M_1, M_2,\ldots, M_n$ for a prime p and if $b_1, b_2,\ldots, b_n$ are the 1st, 2nd,..., nth in the ascending order of magnitude of $a_1, a_2,\ldots, a_n$ then it is clear that $p^{b_1+b_2+\ldots+b_k}$ is an elementary block factor in δ_k. Therefore by (6·1) p^{b_k} is an elementary block factor in ε_k. p^{b_k} is also an elementary block factor in G_k, i.e. p^{b_k} is a common elementary block factor in ε_k and G_k. This is true for any prime. Hence $\varepsilon_k=G_k$, $k=1, 2,\ldots, n$.

Now let A be a finite Abelian group under any composition rule R, for instance multiplication. We write $H.H\ldots p$ times for brevity as H^p. (If R is addition we naturally write $H+H+\ldots p$ times as $p.H$). The element H is said to be of order p, if p is the least positive integer $\neq 0$ such that $H^p=1$.

* Van der Waerden (5).

Given a finite Abelian group A, a set of elements $A_1, A_2, \ldots, A_n$ of orders $M_1, M_2, \ldots, M_n$ is said to form a basis if every element of A can be put in one and only one way in the form

$$A_1^{m_1} A_2^{m_2} \ldots A_n^{m_n};\ m_r = 1, 2, \ldots M_r:\ r = 1, 2, \ldots n.$$

A generating relation of the basis elements is a relation of the form

$$A_1^{\alpha_1} A_2^{\alpha_2} \ldots A_n^{\alpha_n} = 1.$$

A set of generating relations may be said to form a basis generating relations of A, if every generating relation of A can be uniquely expressed in terms of the generating relations of the set. Evidently

$$A_1^{M_1} = 1,\ A_2^{M_2} = 1, \ldots, A_n^{M_n} = 1$$

form a set of basis generating relations of A.

A normal basis of the Abelian group is one, in which among the orders of any two elements of the basis, one is a divisor of the other. The orders of the elements of a normal basis are known to be invariants, called the normal or the Schering invariants*. The normal invariants can be found as the invariant factors of the matrix of exponents in the set of generating relations for the basis elements $A_1, A_2, \ldots, A_n$ of A*. This matrix is equivalent† to the matrix of exponents in a set of basis generating relations and hence equivalent to

$$\begin{vmatrix} M_1 & 0 & \ldots & 0 \\ 0 & M_2 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & M_n \end{vmatrix}.$$

By Lemma 4 the invariant factors of this matrix are $G_1, G_2 \ldots, G_n$. Hence $G_1, G_2, \ldots, G_n$ are the normal invariants of A.

Every element H of A can be put uniquely in the form

$$H = A_1^{m_1} A_2^{m_2} \ldots A_n^{m_n};\quad m_r = 1, 2, \ldots, M_r;\ r = 1, 2, \ldots, n.$$

By means of the extended Chinese remainder theorem we find the vector $(g_1, g_2, \ldots, g_n)$ mod $(G_1, G_2, \ldots, G_n)$ corresponding to the vector $(m_1, m_2, \ldots, m_n)$ mod $(M_1, M_2, \ldots, M_n)$. Every element H of A can be associated with a vector $(m_1, m_2, \ldots, m_n)$ mod

* Brown (6).

† Two matrices are said to be equivalent when one can be obtained from the other by means of unimodular transformations only.

$(M_1, M_2, \ldots, M_n)$ and thereby with the corresponding vector $(g_1, g_2, \ldots, g_n)$ mod $(G_1, G_2, \ldots, G_n)$. Evidently the correspondence between the elements H and the vectors $(g_1, g_2 \ldots, g_n)$ mod $(G_1, G_2, \ldots, G_n)$ is $(1, 1)$. If e_k is the vector $(g_1, g_2, \ldots, g_n)$ mod $(G_1, G_2, \ldots, G_n)$ where $g_r = 0$ for $r \neq k$ and $g_k = 1$, we take B_k as the element of A which corresponds to the vector e_k $(k = 1, 2, \ldots, n)$.

Now every element H of A can be uniquely put in the form

$$H = A_1^{m_1} A_2^{m_2} \ldots A_n^{m_n} = B_1^{g_1} B_2^{g_2} \ldots B_n^{g_n}$$

$$(m_r = 1, 2, \ldots M_r;\ g_r = 1, 2, \ldots G_r;\ r = 1, 2, \ldots n),$$

and B_k is of order $G_k (k = 1, 2, \ldots, n)$. Hence $B_1, B_2, \ldots, B_n$ form a set of normal basis elements of A.

It should be noted that a set of normal basis elements is not unique whereas the orders of the elements of a normal basis are unique.

I reserve for a later communication the discussion of the automorphisms of the vector algebra mod $(M_1, M_2, \ldots, M_n)$.

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TWO REMARKS ON HILBERT'S DOUBLE SERIES THEOREM

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We assume throughout that a_m and b_n are real non-negative numbers such that neither all $a_m=0$ nor all $b_n=0$, and that $f(x)$ and $g(y)$ are real non-negative functions, defined for almost all real positive x and y, such that $f(x)\neq 0, g(y)\neq 0$.

1. The so called " sharpest " form of Hilbert's double series theorem is*

$$(1\cdot 1) \qquad \sum_0^\infty\sum_0^\infty \frac{a_m b_n}{m+n+1} < \pi\Big(\sum_0^\infty a_m^2\Big)^{\frac{1}{2}}\Big(\sum_0^\infty b_n^2\Big)^{\frac{1}{2}}.$$

It is of some interest that a still "sharper" form exists, in the sense that the coefficient $1/(m+n+1)$ of the general term on the left hand side of (1·1) can be increased, the inequality still remaining true. *In fact we prove*

(1·2)

$$\sum_0^\infty\sum_0^\infty \log\frac{(m+n)^{m+n}(m+n+2)^{m+n+2}}{(m+n+1)^{2(m+n+1)}}\cdot a_m b_n < \pi\Big(\sum_0^\infty a_m^2\Big)^{\frac{1}{2}}\Big(\sum_0^\infty b_n^2\Big)^{\frac{1}{2}}.\S$$

That

$$(1\cdot 3) \qquad k(m+n)=k(\nu)=\log\frac{\nu^\nu(\nu+2)^{\nu+2}}{(\nu+1)^{2(\nu+1)}}>\frac{1}{\nu+1}$$

for all $\nu\geqq 0$ will be shown in due course†.

It is plainly sufficient to prove (1·2) in the case $b_n=a_n$, since a symmetric bilinear form in [2, 2] has a bound equal to that of

* The original left hand side of Hilbert's theorem was of course $\sum_1^\infty\sum_1^\infty \frac{a_m b_n}{m+n}$. A detailed discussion of Hilbert's double series theorem may be found in "*Inequalities*" by G. H. Hardy, J. E. Littlewood and G. Polya, (Cambridge Univ. Press, 1934), Ch. IX.

§ That π is the best possible constant in (1·2) follows from the fact that it is the best constant in (1·1).

† $k(0)$ is of course $2\log 2$.

the corresponding quadratic form. Now, according to a result of Hardy‡

$$(1\cdot4)\qquad \int_0^\infty\left\{\int_0^\infty e^{-xy}f(y)\,dy\right\}^2dx<\pi\int_0^\infty f^2(x)\,dx \quad (f\varepsilon L^2(0,\infty)).$$

For the sake of completeness we insert Hardy's proof of (1·4). By Hoelder's inequality

$$(1\cdot5)\quad J=\int_0^\infty\int_0^\infty e^{-xy}f(x)g(y)\,dx\,dy$$

$$=\int_0^\infty\int_0^\infty e^{-\frac{1}{2}xy}\left(\frac{x}{y}\right)^{\frac{1}{4}}f(x)\,.\,e^{-\frac{1}{2}xy}\left(\frac{y}{x}\right)^{\frac{1}{4}}g(y)\,dx\,dy\leqq P^{\frac{1}{2}}Q^{\frac{1}{2}},$$

say, where

$$P=\int_0^\infty f^2(x)\,dx\int_0^\infty e^{-xy}\sqrt{\frac{x}{y}}\,dy=\int_0^\infty f^2(x)\,dx\int_0^\infty e^{-z^2}2dz=\sqrt{\pi}\int_0^\infty f^2(x)\,dx,$$

and similarly

$$Q=\int_0^\infty g^2(y)\,dy\int_0^\infty e^{-xy}\sqrt{\frac{y}{x}}\,dx=\sqrt{\pi}\int_0^\infty g^2(y)\,dy.$$

Hence

$$J\leqq\sqrt{\pi}\left\{\int_0^\infty f^2(x)\,dx\right\}^{\frac{1}{2}}\left\{\int_0^\infty g^2(y)\,dy\right\}^{\frac{1}{2}}$$

for every $g\varepsilon L^2(0,\infty)$, and (1·4) follows by the converse of Hoelder's inequality with $\leqq$ instead of $<$. To eliminate the sign of equality observe that for (1·5) to be an equality it is necessary that

$$\left(\frac{x}{y}\right)^{\frac{1}{4}}f(x)\equiv\text{const.}\left(\frac{y}{x}\right)^{\frac{1}{4}}g(y),$$

which leads to $f(x)$ and $g(y)$ not belonging to $L^2(0,\infty)$.

To deduce the quadratic form of (1·2) from (1·4) put $f(x)=a_\nu\,(\nu\leqq x<\nu+1,\ \nu=0,1,2,\ldots)$ and $x=\log\frac{1}{\xi}$. Then

$$\int_0^\infty e^{-xy}f(y)\,dy=\sum_0^\infty a_\nu\int_\nu^{\nu+1}e^{-xy}\,dy=\frac{1-e^{-x}}{x}\sum_0^\infty a_\nu e^{-\nu x}=\frac{1-\xi}{\log\frac{1}{\xi}}\sum_0^\infty a_\nu\xi^\nu,$$

$$\int_0^\infty\left\{\int_0^\infty e^{-xy}f(y)\,dy\right\}^2dx=\int_0^1\left(\frac{1-\xi}{\log\frac{1}{\xi}}\right)^2\left(\sum_0^\infty a_\nu\xi^\nu\right)^2\frac{d\xi}{\xi}$$

‡ "The constants of certain inequalities", *Journal L.M.S.*, 8 (1933), 118.

$$=\sum_{\nu=0}^{\infty}k(\nu)\sum_{n=0}^{\nu}a_n a_{\nu-n},$$

where

(1·6) $$k(\nu)=\int_0^1\left(\frac{1-\xi}{\log\frac{1}{\xi}}\right)^2\xi^{\nu-1}d\xi.$$

Since $$\int_0^{\infty}f^2(x)dx=\sum_0^{\infty}a_\nu^2,$$

we have, by (1·4),

$$\sum_{\nu=0}^{\infty}k(\nu)\sum_{n=0}^{\nu}a_n a_{\nu-n}<\pi\sum_0^{\infty}a_\nu^2$$

with $k(\nu)$ from (1·6), and (1·2) will be proved if we establish the identity of this $k(\nu)$ with that of (1·3). This is easily done by putting

$$\phi_\nu(z)=\int_0^{\infty}\left(\frac{1-e^{-x}}{x}\right)^2 e^{-(z+\nu)x}dx \quad (z\geqq 0)$$

and observing that $\phi_\nu(0)=k(\nu)$ of (1·6),

$$\phi_\nu''(z)=\frac{1}{\nu+z}-\frac{2}{\nu+z+1}+\frac{1}{\nu+z+2}$$

and that $$\lim_{z\to\infty}\phi_\nu(z)=\lim_{z\to\infty}\phi_\nu'(z)=0.$$

Finally we verify the inequality (1·3). Since

$2x<e^x-e^{-x}\,(0<x<\infty)$, or $2\log\frac{1}{u}<\frac{1-u^2}{u}\,(0<u<1)$, $\frac{\log\frac{1}{u^2}}{1-u^2}<\frac{1}{u}$,

$\left(\frac{\log\frac{1}{\xi}}{1-\xi}\right)^2<\frac{1}{\sqrt{\xi}}\,(0<\xi<1)$, $\left(\frac{1-\xi}{\log\frac{1}{\xi}}\right)^2>\xi$, we have, by (1·6),

$$k(\nu)=\int_0^1\left(\frac{1-\xi}{\log\frac{1}{\xi}}\right)^2\xi^{\nu-1}d\xi>\int_0^1\xi.\xi^{\nu-1}d\xi=\frac{1}{\nu+1}.$$

2. The second remark concerns the two-parameter extension of Hilbert's double series theorem. The following result is known:*

* "*Inequalities*", loc. cit., 253.

If $p>1, q>1,\ 1/p+1/q \geqq 1,\ p'=p/(p-1),\ q'=q/(q-1)$, $\lambda=1/p'+1/q'(=2-1/p-1/q)$, *so that* $0<\lambda\leqq 1$, *then*

$$(2\cdot1)\qquad \sum_1^\infty\sum_1^\infty \frac{a_m b_n}{(m+n)^\lambda} \leqslant K\left(\sum_1^\infty a_m^p\right)^{\frac{1}{p}}\left(\sum_1^\infty b_n^q\right)^{\frac{1}{q}},$$

where $K=K(p,q)$ *is a function of* p *and* q *only*†.

The best value of $K(p,q)$ is known only in the case $q=p'$; it is π cosec $(\pi/p)=\pi$ cosec (π/p'). The best value of $K(p,q)$ for general p and q is not known. "*Inequalities*" (loc. cit.) gives only $K(p,q)\leqq p'^p+q'^q$. *Our object is to prove that* (2·1) *holds with*

$$(2\cdot2)\qquad K=\bar{K}(p,q)=\left(\pi \operatorname{cosec}\frac{\pi}{\lambda p'}\right)^\lambda=\left(\pi \operatorname{cosec}\frac{\pi}{\lambda q'}\right)^\lambda.$$

This is most probably also not the best possible value for K, but it reduces to the best possible one in the case $q=p'$, i.e. $\lambda=1$.

We prove (2·1) with (2·2) in its integral form

$$(2\cdot3)\qquad \int_0^\infty\int_0^\infty \frac{f(x)g(y)}{(x+y)^\lambda}dxdy \leqslant \bar{K}(p,q)\left\{\int_0^\infty f^p(x)dx\right\}^{\frac{1}{p}}\left\{\int_0^\infty g^q(y)dy\right\}^{\frac{1}{q}},$$

$$f\ \varepsilon\ L^p(0,\infty),\ g\ \varepsilon\ L^q(0,\infty),$$

from which (2·1) follows immediately by putting $f(x)=a_m$ $(m-1\leqslant x<m)$, $g(y)=b_n(n-1\leqslant x<n)$, $m, n=1, 2, \ldots$, for

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{(x+y)^\lambda}dxdy$$

$$=\sum_1^\infty\sum_1^\infty \int_{x=m-1}^{m}\int_{y=n-1}^{n} \frac{a_m b_n}{(x+y)^\lambda}dxdy > \sum_1^\infty\sum_1^\infty \frac{a_m b_n}{(m+n)^\lambda}.$$

We prove (2·3) in the following way. Let

$$r=\lambda q',\ r'=\lambda p', \text{ so that } r>1 \text{ and } 1/r+1/r'=1,$$

$$\alpha=p/r,\ \beta=q/r', \text{ so that } 0<\alpha,\ \beta\leqslant 1,$$

and the following identities hold:

$$(1-\beta)r=(1-\lambda)q,\ (1-\alpha)r'=(1-\lambda)p,$$

$$\frac{r}{p'+q'}=\frac{1}{p'},\ \frac{r'}{p'+q'}=\frac{1}{q'}.$$

† A different two-parameter extension of Hilbert's double series theorem is given in my paper "On the two-parameter extension and analogue of Hilbert's inequality", *Journal L.M.S.* 11 (1936), 119-124, where also the result to be established below is announced.

Then, by Hoelder's inequality,

$$(2\cdot4)\qquad \int_0^\infty\int_0^\infty \frac{f(x)g(y)}{(x+y)^\lambda}dxdy$$

$$=\int_0^\infty\int_0^\infty \frac{f^\alpha g^{1-\beta}}{(x+y)^{\lambda/r}}\left(\frac{y}{x}\right)^{-\frac{1}{p'+q'}}\cdot\frac{f^{1-\alpha}g^\beta}{(x+y)^{\lambda/r'}}\left(\frac{x}{y}\right)^{-\frac{1}{p'+q'}}dxdy$$

$$\leqslant S^{1/r}T^{1/r'},$$

say, where again by Hoelder's inequality,

$$(2\cdot5)\quad S=\int_{x=0}^\infty f^{\alpha r}(x)dx\int_{y=0}^\infty \frac{g^{(1-\beta)r}(y)}{(x+y)^\lambda}\left(\frac{y}{x}\right)^{-\frac{r}{p'+q'}}dy$$

$$=\int_{x=0}^\infty f^p(x)dx\int_{y=0}^\infty g^{(1-\lambda)q}(y)\frac{(y/x)^{-1/p'}}{(x+y)^\lambda}dy$$

$$\leqslant\int_{x=0}^\infty f^p(x)dx\left\{\int_0^\infty g^q(y)dy\right\}^{1-\lambda}\left\{\int_{y=0}^\infty\frac{(y/x)^{-1/(\lambda p')}}{x+y}dy\right\}^\lambda$$

$$=\int_0^\infty f^p(x)dx\left\{\int_0^\infty g^q(y)dy\right\}^{1-\lambda}\left\{\int_0^\infty\frac{v^{-1/(\lambda p')}}{1+v}dv\right\}^\lambda,$$

and similarly

$$(2\cdot6)\qquad T=\int_{y=0}^\infty g^{\beta r'}(y)dy\int_{x=0}^\infty\frac{f^{(1-\alpha)r'}(x)}{(x+y)^\lambda}\left(\frac{x}{y}\right)^{-\frac{r'}{p'+q'}}dx$$

$$\leqslant\int_0^\infty g^q(y)dy\left\{\int_0^\infty f^p(x)dx\right\}^{1-\lambda}\left\{\int_0^\infty\frac{u^{-1/(\lambda q')}}{u+1}du\right\}^\lambda.$$

Since

$$\int_0^\infty\frac{u^{-1/(\lambda q')}}{u+1}du=\int_0^\infty\frac{v^{-1/(\lambda p')}}{1+v}dv=\pi\text{ cosec}\frac{\pi}{\lambda p'}=\pi\text{ cosec}\frac{\pi}{\lambda q'},$$

(2·4) in combination with (2·5) and (2·6) gives

$$\int_0^\infty\int_0^\infty\frac{f(x)g(y)}{(x+y)^\lambda}dxdy$$

$$\leqslant\left(\pi\text{ cosec}\frac{\pi}{\lambda p'}\right)^{\frac{\lambda}{r}+\frac{\lambda}{r'}}\left\{\int_0^\infty f^p(x)dx\right\}^{\frac{1}{r}+\frac{1-\lambda}{r'}}\left\{\int_0^\infty g^q(y)dy\right\}^{\frac{1}{r'}+\frac{1-\lambda}{r}}$$

$$=\bar{K}(p,q)\left\{\int_0^\infty f^p(x)dx\right\}^{\frac{1}{p}}\left\{\int_0^\infty g^q(y)dy\right\}^{\frac{1}{q}},$$

and our result is established.

ON SETS OF SQUARE-FREE NUMBERS

By S. S. PILLAI, D.Sc., Annamalai University

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1. Let $N(x)=N(x, d_1, d_2, \ldots, d_{r-1})$ denote the number of groups of square-free numbers $q_1, q_2, \ldots, q_r$ not exceeding x, such that $q_m-q_1=d_{m-1}$, $(m=2, 3, \ldots, r)$. Further let $f(p)$ be the number of different residue classes modulus p^2 and not congruent to 0 contained in the set $d_1, d_2, \ldots, d_{r-1}$. The object of this note is to prove

THEOREM. $$N(x)=Ax+O\left(\frac{x}{\log x}\right),$$

where $$A=\prod_p\left\{1-\frac{1+f(p)}{p^2}\right\}.$$

2. Consider the set

$$a, a+d_1, a+d_2, \ldots, a+d_{r-1}.$$

Let a run from 1 to p^2. Then every $a+d_m$ will become divisible by p^2 once and only once. So the number of sets which contain at least one number which is divisible by p^2 is $1+f(p)$. Since a takes the value p^2, we get $1+f(p)$. Hence the number of groups which contain no number divisible by p^2 is

$$p^2\left\{1-\frac{1+f(p)}{p^2}\right\}.$$

Again consider the set

$$a, a+d_1, \ldots, a+d_{r-1},$$

where $a=1, 2, 3, \ldots, p_1^2 p_2^2$.

Then it is easy to see that

(1) the number of sets which contain at least one number which is divisible by p_1^2 is

$$p_2^2\{1+f(p_1)\},$$

(2) the number of sets which contain at least one number which is divisible by p_2^2 is

$$p_1^2\{1+f(p_2)\},$$

(3) the number of sets which contain at least one number which is divisible by p_1^2 and at least one number which is divisible by p_2^2 (the two need not be different) is

$$\{1+f(p_1)\}\{1+f(p_2)\}.$$

Hence the number of groups which contain no number divisible either by p_1^2 or p_2^2, is

$$p_1^2 p_2^2\left\{1-\frac{1+f(p_1)}{p_1^2}\right\}\left\{1-\frac{1+f(p_2)}{p_2^2}\right\}.$$

Similarly proceeding, we can easily see that the number of groups $a, a+d_1, \ldots, a+d_{r-1}$, not exceeding $p_1^2 p_2^2 \ldots p_n^2$, which contain no number divisible by any of $p_1^2, p_2^2, \ldots, p_n^2$, is

$$p_1^2 p_2^2 \ldots p_n^2\left\{1-\frac{1+f(p_1)}{p_1^2}\right\}\ldots\left\{1-\frac{1+f(p_n)}{p_n^2}\right\}.$$

Hence the number of groups $a, a+d_1, \ldots, a+d_{r-1}$, not exceeding x, which contain no number divisible by any of $p_1^2, p_2^2, \ldots, p_n^2$ is

$$x \prod_{m=1}^{n}\left\{1-\frac{1+f(p_m)}{p_m^2}\right\}+O(p_1 \ldots p_n)^2. \tag{1}$$

Again, since $f(p) \leqslant r-1$, the number of groups $a, a+d_1, \ldots, a+d_{r-1}$, not exceeding x, which contain at least one number divisible by any one of $p_{n+1}^2, p_{n+2}^2, \ldots, p_l^2$, where $p_l \leqslant \sqrt{x} < p_{l+1}$, is

$$O(x)\left\{\frac{1}{p_{n+1}^2}+\frac{1}{p_{n+2}^2}+\ldots\right\}=O\left(\frac{x}{p_n}\right). \tag{2}$$

Further,

$$\left|A-\prod_{m+1}^{n}\left\{1-\frac{1+f(p_m)}{p_m^2}\right\}\right|$$
$$\leqslant O\left\{\sum_{p \geqslant p_{n+1}} \frac{1}{p^2}\right\}=O\left(\frac{1}{p_n}\right). \tag{3}$$

Therefore from (1), (2) and (3)

$$N(x)=A(x)+O\left(\frac{x}{p_n}\right)+O(p_1^2 \ldots p_n^2).$$

Choose, p_n so that $$p_n \leqslant \frac{\log x}{8} < p_{n+1}.$$

Then $$N(x)=A(x)+O\left(\frac{x}{\log x}\right).$$

3. In particular, let $N(x, k)$ be the number of pairs of square-free numbers not exceeding x, whose common difference is k, Then from the above it follows that

$$N(x, k) \sim Ax,$$

where $$A=\prod_{p^2 | k}(1-1/p^2)\prod_{p^2 \nmid k}(1-2/p^2).$$

Again let $N(x, \overline{k, l})$ be the number of pairs of square-free numbers $\leqslant x$, whose common difference is k, in the arithmetic progression $l+mk$, $(m=0, 1, 2, \ldots)$. Then it can be proved that

$$N(x, \overline{k, l}) \sim \frac{Ax}{k},$$

where if $p^2 | (l, k)$, then $A=0$, otherwise

$$A=\prod_{p | (l,k)}(1-2/p)\prod_{p \nmid (l,k)}(1-2/p^2).$$

Lastly, let $N_k(x)$ denote the groups of kth power free numbers a, $a+d_1, \ldots, a+d_{r-1}$ not exceeding x. Then arguing as before it can be proved that

$$N_k(x)=Ax+O(x/\log^{k-1}x),$$

where $$A=\prod_p\left\{1-\frac{1+f_k(p)}{p^k}\right\}$$

and $f_k(p)$ denotes the number of different residue classes different from 0 modulus p^k, in the set $d_1, d_2, \ldots, d_{r-1}$.

ON $a^x - b^y = c$

By S. S. PILLAI, D.Sc., Annamalai University

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Aaron Herschefeld* has recently proved the very interesting result that

$$2^x - 3^y = c \tag{1}$$

has got at most one solution when c is large. Further, by applying the following theorem of Siegel, namely,

$$ax^n - by^n = k \text{ (fixed } n \geqslant 3)$$

has at most one solution if $|ab|$ is sufficiently large, he proves that

$$a^x - b^y = c \tag{2}$$

has at most 9 solutions.

In Siegel's theorem, it may be that $|ab|$ should be large as a function of k. If it is so, an easy examination of his proof will convince any one that Herschefeld's deduction is not valid and that his proof does not lead to any new result. However, his method of proof for (1) can be applied to (2) also. The object of this note is to show that his method can be extended to prove

THEOREM I. *When c is large, the equation*

$$a^x - b^y = c$$

has at most one solution.

In a paper published in 1931†, the author has proved that the number of solutions of

$$0 < a^x - b^y \leqslant c$$

is asymptotically equal to

$$\frac{(\log c)^2}{2 \log a \log b}.$$

* 'The equation $2^x - 3^y = d$', *Bulletin of the American Mathematical Society*, Vol. XLII No. 4. (1936), 231-4.

† 'On the inequality $0 < a^x - b^y \leqslant n$', *The Journal of the Indian Mathematical Society*, Vol. XIX (1931).

From this and Theorem I we get

THEOREM II. *If $N(n)$ is the number of distinct numbers not exceeding n, which can be expressed in the form*

$$a^x - b^y,$$

then
$$N(n) \sim \frac{(\log n)^2}{2 \log a \log b}.$$

2. To prove our result, we require the following lemmas.

LEMMA 1. *If $(a, b)=1$, there are values x and y, such that*

(1) $$x \geqslant 2$$

(2) $$b^y = la^x,$$

where $(l, a)=1$.

Let
$$a = p_1^{\alpha_1} \dots p_r^{\alpha_r}, \tag{3}$$
where p's are different prime factors of a. Since $(a, b)=1$, there is an y such that

$$b^y \equiv 1 \pmod{a^2}.$$

Let y_1 be the smallest value of $y \geqslant 1$ satisfying the above congruence. Then we can write b^{y_1} in the form

$$b^{y_1} = 1 + M_1 p_1^{\beta_1} \dots p_r^{\beta_r} a^{x_1}, \tag{4}$$

where $x_1 \geqslant 2$, $(M_1, a)=1$ and $\beta_s \leqslant \alpha_s - 1$ at least for one value of s.

Let $t_1 = \Pi\, p_s^{\alpha_s - \beta_s}$, where s runs through all values for which $\beta_s \leqslant \alpha_s - 1$. Then, from (4)

$$b^{y_1 t_1} \equiv 1 + M_1 t_1 p_1^{\beta_1} \dots p_r^{\beta_r} a^{x_1} \pmod{a^{2x_1}}$$
$$\equiv 1 + M_1 p_1^{\gamma_1} \dots p_r^{\gamma_r} a^{x_1+1} \pmod{a^{2x_1}}, \tag{5}$$

where at least one γ_s is zero and $\gamma_s \leqslant \beta_s - 1$ for all s for which $\beta_s \geqslant 1$.

Since
$$x_1 \geqslant 2,\ 2x_1 \geqslant x_1 + 2.$$
So from (5), we have

$$b^{y_1 t_1} = 1 + M_2 p_1^{\gamma_1} \dots p_r^{\gamma_r} a^{x_1+1}, \tag{6}$$

where
$$(a, M_2)=1,\ \gamma_s \leqslant \beta_s - 1, \text{ when } \beta_s \neq 1.$$

Let $t_2 = \Pi\, p_s^{\alpha_s - \gamma_s}$, where s runs through all values for which $\gamma_s \leqslant \alpha_s - 1$. Then from (6)

$$b^{y_1 t_1 t_2} \equiv 1 + M_2 t_2 p_1^{\gamma_1} \dots p_r^{\gamma_r} a^{x_1+1} \pmod{a^{2x_1+2}}$$
$$\equiv 1 + M_3 p_1^{\theta_1} \dots p_r^{\theta_r} a^{x_1+1} \pmod{a^{2x_1+2}}.$$

Hence
$$b^{y_1 t_1 t_2} = 1 + M_3 p_1^{\theta_1} \dots p_r^{\theta_r} a^{x_1+2},$$

where $(a, M_3)=1$ and $\theta_s \leqslant \gamma_s - 1$, when $\gamma_s \neq 0$ and $\theta_s = 0$, when $\gamma_s = 0$, Repeating this process, after a finite number of steps, we arrive at the lemma.

LEMMA 2. *Let m and n be the smallest values of $x \geqslant 2$ and $y \geqslant 1$, satisfying*

$$b^y = la^x + 1,$$

where $(a, b) = 1$ and $(l, a) = 1$ and a, b are given. Then if

$$b^N \equiv 1 \pmod{a^M},$$

N is a multiple of na^{M-m}.

This can easily be proved by induction.

LEMMA 3. *If $a^x - b^y = a^X - b^Y$, where $X > x$ and $(a, b) = 1$ then*

$$Y \geqslant a^{x-m}.$$

From the above equation

$$a^x(a^{X-x} - 1) = b^y(b^{Y-y} - 1).$$

Then
$$b^{Y-y} \equiv 1 \pmod{a^x}.$$

So from Lemma 2,

$$Y - y \geqslant n\, a^{x-m}.$$

Hence
$$Y \geqslant a^{x-m}.$$

LEMMA 4. *When $a^x > b^y$, there is an x_0 such that for all $x > x_0$,*

$$a^x - b^y \geqslant a^{x/2}.$$

This is a particular case (with $\delta = \frac{1}{2}$) of Theorem I in my paper referred to above.

3. If $(a, b) \neq 1$, obviously (2) can have at most one solution. So we can assume that

$$(a, b) = 1 \text{ and } a \geqslant 2,\ b \geqslant 2. \tag{7}$$

Let
$$x_1 = \text{Max}\left\{ x_0, 3(m+1) \right\}. \tag{8}$$

We assume that
$$c > a^{x_1} \text{ and so } x > x_1. \tag{9}$$

If possible let (2) have the two solutions

$$c = a^x - b^y = a^X - b^Y, \text{ where } X > x. \tag{10}$$

Then from Lemma 4, (for, $x > x_0$)

$$a^x = c + b^y > c = a^X - b^Y \geqslant a^{X/2}. \tag{11}$$

So
$$X < 2x.$$

Again $a^X = c + b^Y > b^Y$

$$> b^{a^{x-m}}, \text{ from Lemma 3,}$$

$$\geqslant 2^{a^{x-m}}, \text{ from (7)}$$

$$> 1 + a^{x-m} + \frac{a^{x-m}(a^{x-m}-1)}{2}$$

$$+ \frac{a^{x-m}(a^{x-m}-1)(a^{x-m}-2)}{6}$$

$$> \frac{a^{3(x-m)}}{6}$$

$$> a^{3(x-m-1)}, \text{ from (7).}$$

So

$$X > 3(x-m-1). \qquad (12)$$

But from (8) and (9)

$$x \geqslant 3(m+1). \qquad (13)$$

Hence from (12) and (13), we have

$$X > 2x.$$

Now (11) and (14) are contradictory. Hence (10) is impossible.

So Theorem I is proved.

4. Proceeding similarly we can prove

THEOREM III. *If a, A, b, B, c are given positive integers and c is sufficiently large, then the equation*

$$Aa^x - Bb^y = c$$

can have at most one solution.

In conclusion, it may be that when c alone is given and X, x, Y, y are unknown, the equation

$$X^x - Y^y = c$$

will have only a finite number of solutions, provided $x \geqslant 2$ and $y \geqslant 2$.

AN APPLICATION OF A THEOREM OF LIE AND KOENIGS TO THE EQUATIONS OF MOTION

By K. NAGABUSHANAM, M.A., Andhra University, Waltair

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When the Hamiltonian is independent of time, the solution of the dynamical problem can be reduced* to depend on that of another system whose number of degrees of freedom is one less than that of the original system. The same result is here derived as an application of a theorem† of Lie and Koenigs, which may be stated as follows:

A given set of differential equations

$$\frac{dx^i}{dt}=\xi^i, \quad (i=1,2,\ldots,m), \tag{1}$$

can be expressed as $2l$ equations in the Hamiltonian form, together with $m-2l$ equations of the type

$$\frac{du_s}{dt}=U_s, \quad (s=1,2,\ldots,m-2l).$$

Either we can look upon t as a parameter or regard it as the $m+1$th variable of the $m+1$ dimensional space S_{m+1} of (x^i, t). Here the latter point of view is adopted.

In S_{m+1} (1) can be written as

$$\frac{dx^1}{\xi^1}=\frac{dx^2}{\xi^2}=\ldots=\frac{dx^m}{\xi^m}=dt. \tag{2}$$

These can be taken as the equations of a congruence of curves in S_{m+1}. Following the method of integral invariants, used in the proof of the theorem of Lie and Koenigs, we can make $2l$ attain the maximum value, which is m if m is even, and $m-1$ if m is odd. Hence we can state the following result‡:

* E. T. Whittaker, *Analytical Dynamics*, p. 265.

† E. T. Whittaker, *Ibid*. p. 275.
I have not been able to see the original papers of Lie or Koenigs.

‡ The same result is proved without the use of integral invariants by the author in his note 'On expressing the equations of a congruence of curves in the Hamiltonian form', *Mathematics Student*, Vol. 2. 129-131.

The differential equations of a congruence of curves in S_{m+1} can be expressed as m Hamiltonian equations when m is even, and as $m-1$ Hamiltonian equations together with an eqution of the type

$$\frac{du}{dt}=U, \tag{3}$$

when m is odd.

Let us consider the equations of motion of a dynamical system with n degrees of freedom, the Hamiltonian being independent of time. Then the equation of the integral of energy can be written as

$$H=h, \text{ a constant.} \tag{4}$$

For nontriviality of the equations of motion, we assume that not all the q's and p's are integrals of motion. Hence one of the variables, say q^α, can be taken as varying along the trajectories. Then the equations of motion may be put into the form

$$\frac{dq^s}{\frac{\partial H}{\partial p^s}\Big/\frac{\partial H}{\partial p^\alpha}}=\frac{dp^r}{-\frac{\partial H}{\partial q^r}\Big/\frac{\partial H}{\partial p^\alpha}}=dq^\alpha, \quad \begin{pmatrix} r=1, 2, \ldots, n \\ s=1, 2, \ldots, \alpha-1, \alpha+1, \ldots, n \end{pmatrix}, \tag{5}$$

which are similar to (2). These being the equations of a congruence of curves in S_{2n} of $(q^1, \ldots, q^n, p^1, \ldots, p^n)$, it is clear by the above result that (5) can be expressed as $2n-2$ Hamiltonian equations together with one equation of the form (3). Separating the equations

$$H=h, \text{ a constant,}$$

and

$$\frac{du}{dq^\alpha}=U$$

from the rest, the $2n-2$ Hamiltonian equations can be regarded as defining the motion of a dynamical system with $n-1$ degrees of freedom.

ON INTEGRALS INVOLVING LAME′ FUNCTIONS

By J. L. SHARMA, M.A., Christ Church College, Cawnpore

[Received 25th April, 1936]

1. Introduction

Lame″s equation in its Weierstrassian form, viz.

$$\frac{d^2y}{du^2}=[n(n+1)\wp(u)+B]y, \tag{1·1}$$

where n is an integer and B is arbitrary, admits solutions of the type*

$$y_1=F_n(u)=\prod_1^n\left[\frac{\sigma(u+a_r)}{\sigma(u)\sigma(a_r)}e^{-u\zeta(a_r)}\right], \tag{1·2}$$

$$y_2=F_n(-u)=\prod_1^n\left[\frac{\sigma(a_r-u)}{\sigma(-u)\sigma(a_r)}e^{u\zeta(a_r)}\right].$$

These functions will be referred to hereafter as generalized Lame′ functions. One of these solutions reduces to a polynomial in $\wp(u)$ and $\wp'(u)$ for $2n+1$ values of B, which are given by the equations†

$$\text{(i) } P^n(B)=0\,^{\S} \qquad \text{(ii) } Q_1^n(B)=0 \tag{1·3}$$

$$\text{(iii) } Q_2^n(B)=0 \qquad \text{(iv) } Q_3^n(B)=0.$$

These sets of values of B give rise to four species of functions represented by K_n^p, L_n^q, M_n^r and N_n^s, or collectively by E_n^m, which are called Lame′ functions of the first kind. While there exist various types of integrals involving almost all the functions occurring in harmonic analysis, we hardly come across any integrals involving Lame′ functions. The object of the present paper is to evaluate certain definite integrals involving the above functions and to investigate all possible non-orthogonal systems of solutions of Lame′ equation. So far as is known this property

*Halphen, *Fonctions Elliptiques*, t, II. 494-502.

†Halphen has represented these equations as $P(B)Q(B)=0$ but the above form is preferable, for in this $P(B)=0$ gives the values of B, for which there exist Lame′ functions of the first species and $Q_r(B)=0$ gives those values which give rise to functions of $(r+1)$th species, $(r=1, 2, 3)$.

§They are different from Legendre functions $P_n(\mu)$ and $Q_n(\mu)$.

was not observed in the case of any harmonic function except that of conal functions by Neumann‡ and that of Legendre functions by G. Prasad‖.

2. INTEGRAL OF THE PRODUCT OF TWO GENERALIZED LAME' FUNCTIONS

Let $F_p^t(u)$ and $F_p^s(u)$ be the solutions of the differential equations

$$\frac{d^2}{du^2}\left\{ F_p^t(u) \right\} = [p(p+1)\wp(u)+B_t]\ F_p^t(u), \tag{2·1}$$

$$\frac{d^2}{du^2}\left\{ F_p^s(u) \right\} = [p(p+1)\wp(u)+B_s]\ F_p^s(u), \tag{2·2}$$

where B_t and B_s are two values of B, which are different from the characteristic values for which F_p reduces to E_p, i.e. B_t and B_s are different from the roots of the equation

$$P^p(B)Q_1^p(B)Q_2^p(B)Q_3^p(B)=0. \tag{2·3}$$

Multiply (2·1) by $F_p^s(u)$ and (2·2) by $F_p^t(u)$, subtract and integrate; then,

$$(B_t-B_s)\int_{\omega_1}^{\omega_2} F_p^s(u)F_p^t(u)\,du$$

$$=\left\{F_p^s(u)\frac{d}{du}[F_p^t(u)]-F_p^t(u)\frac{d}{du}[F_p^s(u)]\right\}_{\omega_1}^{\omega_2}$$

$$=\phi(\omega_2)-\phi(\omega_1), \tag{2·4}$$

putting $\phi(u)$ for the function within the brackets.

3. EVALUATION OF $\phi(\omega_1)$, $\phi(\omega_2)$ AND $\phi(\omega_3)$

From (1·2) $\quad F_p^s(u)=\prod_1^n\left[\dfrac{\sigma(u+a_r^{ps})}{\sigma(u)\sigma(a_r^{ps})}e^{-u\zeta(a_r^{ps})}\right],$

therefore,

$$\frac{d}{du}\left\{ F_p^s(u) \right\} = \tfrac{1}{2}F_p^s(u)\left[\sum\frac{\wp'(u)-\wp'(a_r^{ps})}{\wp(u)-\wp(a_r^{ps})}\right].$$

Hence

$$\phi(u)=\tfrac{1}{2}F_p^s(u)F_p^t(u)\left[\sum\frac{\wp'(u)-\wp'(a_r^{pt})}{\wp(u)-\wp(a_r^{pt})}-\sum\frac{\wp'(u)-\wp'(a_r^{ps})}{\wp(u)-\wp(a_r^{ps})}\right]. \tag{3·1}$$

‡ Neumann, 'Ueber die Mehlerschen Kegelfunctionen', *Math. Ann.* Vol. 18. 206.

‖ G. Prasad, 'On non-orthogonal systems of Legendre functions', *Proc. Ben. Math. Soc.* Vol. XII. 1-10.

Since $F_p^t(u)$ is a doubly periodic function of the second kind, so

$$F_p^t(u+2\omega_1)=\mu_1^t F_p^t(u), \tag{3·2}$$

where $\mu_1^t=\exp(-2\omega_1\Sigma\zeta(a_r^{pt})+2\eta_1\Sigma a_r^{pt})$.

Putting $u=-\omega_1$ in (3·2), we get,

$$F_p^t(\omega_1)=\mu_1^t F_p^t(-\omega_1). \tag{3·3}$$

Therefore, the product of two linearly independent solutions of (2·1) at the point $u=\omega_1$, viz.

$$Y_p^t(\omega_1)=F_p^t(\omega_1)F_p^t(-\omega_1)$$
$$=\frac{1}{\mu_1^t}[F_p^t(\omega_1)]^2.$$

Hence

$$F_p^t(\omega_1)=\sqrt{\mu_1^t Y_p^t(\omega_1)}*=\sqrt{\mu_1^t c_p^2 Q_2^p(B_t)Q_3^p(B_t)}, \tag{3·4}$$

where
$$c_p=\frac{1}{1.3.5.7\ldots(2p-1)}. \tag{3·5}$$

Further†
$$\sum_{r=1}^{n}\frac{\wp'(a_r^{pt})}{\wp(\omega_1)-\wp(a_r^{pt})}=\frac{2C_{pt}}{Y_p^t(\omega_1)}, \tag{3·6}$$

where*

$$C_{pt}=c_p^2\sqrt{P^p(B_t)Q_1^p(B_t)Q_2^p(B_t)Q_3^p(B_t)}. \tag{3·7}$$

Using these relations in (3·1), we get,

$$\phi(\omega_1)=\tfrac{1}{2}F_p^t(\omega_1)F_p^s(\omega_1)\left[\sum\frac{\wp'(a_r^{ps})}{\wp(\omega_1)-\wp(a_r^{ps})}-\sum\frac{\wp'(a_r^{pt})}{\wp(\omega_1)-\wp(a_r^{pt})}\right]$$
$$=c_p^2\sqrt{\mu_1^t\mu_1^s}\left\{\sqrt{P^{ps}Q_1^{ps}Q_2^{pt}Q_3^{pt}}-\sqrt{P^{pt}Q_1^{pt}Q_2^{ps}Q_3^{ps}}\right\}, \tag{3·8}$$

where Q_r^{ps} stands for $Q_r^p(B_s)$, and P^{ps} for $P^p(B_s)$.

Similarly

$$\phi(\omega_2)=c_p^2\sqrt{\mu_2^t\mu_2^s}\left\{\sqrt{P^{ps}Q_2^{ps}Q_1^{pt}Q_3^{pt}}-\sqrt{P^{pt}Q_2^{pt}Q_1^{ps}Q_3^{ps}}\right\}$$
$$\phi(\omega_3)=c_p^2\sqrt{\mu_3^t\mu_3^s}\left\{\sqrt{P^{ps}Q_3^{ps}Q_1^{pt}Q_2^{pt}}-\sqrt{P^{pt}Q_3^{pt}Q_1^{ps}Q_2^{ps}}\right\}.$$

Hence

$$\int_{\omega_1}^{\omega_2}F_p^t(u)F_p^s(u)\,du=\phi(\omega_2)-\phi(\omega_1) \tag{3·9}$$
$$=\frac{c_p^2}{B_t-B_s}\Big[\sqrt{\mu_2^s\mu_2^t}\left\{\sqrt{P^{ps}Q_2^{ps}Q_1^{pt}Q_3^{pt}}-\sqrt{P^{pt}Q_2^{pt}Q_1^{ps}Q_3^{ps}}\right\}$$
$$-\sqrt{\mu_1^s\mu_1^t}\left\{\sqrt{P^{ps}Q_1^{ps}Q_2^{pt}Q_3^{pt}}-\sqrt{P^{pt}Q_1^{pt}Q_2^{ps}Q_3^{ps}}\right\}\Big].$$

* Halphen, *loc. cit.*

† Whittaker and Watson, *Modern Analysis*, p. 572,

4. Integral of FE

If B_s is a root of $P^p(B)=0$, then $P^{ps}=0$. Hence substituting this in (3·9) and remembering in this case that $\mu_1^s=1=\mu_2^s=\mu_3^s$, we get,

$$\int_{\omega_1}^{\omega_2} F_p^t(u)K_p^s(u)\,du=\frac{c_p^2}{(B_t-B_s)}\Big\{\sqrt{\mu_1^t}\sqrt{P^{pt}Q_2^{pt}Q_1^{ps}Q_3^{ps}} - \sqrt{\mu_2^t}\sqrt{P^{pt}Q_2^{pt}Q_1^{ps}Q_3^{ps}}\Big\}.$$

If B_s is a root of (ii) of (1·3), then $Q_1^{ps}=0$, hence

$$\int_{\omega_1}^{\omega_2} F_p^t(u)L_p^s(u)\,du=\frac{c_p^2}{(B_t-B_s)}\Big\{\sqrt{\mu_2^t}\sqrt{P^{ps}Q_2^{ps}Q_1^{pt}Q_3^{pt}} + \sqrt{\mu_1^t}\sqrt{P^{pt}Q_1^{pt}Q_2^{ps}Q_3^{ps}}\Big\}.$$

Putting $Q_2^{ps}=0$ and $Q_3^{ps}=0$ respectively, we get,

$$\int_{\omega_1}^{\omega_2} F_p^t(u)M_p^s(u)\,du=\frac{c_p^2}{B_s-B_t}\Big\{\sqrt{\mu_2^t}\sqrt{P^{pt}Q_2^{pt}Q_1^{ps}Q_3^{ps}} + \sqrt{\mu_1^t}\sqrt{P^{ps}Q_1^{ps}Q_2^{pt}Q_3^{pt}}\Big\}.$$

$$\int_{\omega_1}^{\omega_2} F_p^t(u)N_p^s(u)\,du=\frac{c_p^2}{B_t-B_s}\Big\{\sqrt{\mu_2^t}\sqrt{P^{ps}Q_2^{ps}Q_1^{pt}Q_3^{pt}} - \sqrt{\mu_1^t}\sqrt{P^{ps}Q_1^{ps}Q_2^{pt}Q_3^{pt}}\Big\}.$$

5. Integral of the product of two functions of different species

If B_s is a root of (i) of (1·3) and B_t of (ii) of (1·3), then $P^{ps}=0$ and $Q_1^{pt}=0$ and we get from (3·9)

$$\int_{\omega_1}^{\omega_2} K_p^s(u)L_p^t(u)\,du=\frac{-c_p^2}{B_t-B_s}\sqrt{P^{pt}Q_2^{pt}Q_1^{ps}Q_3^{ps}}.$$

Similarly putting $P^{ps}=0$ and $Q_2^{pt}=0$, we get,

$$\int_{\omega_1}^{\omega_2} K_p^s(u)M_p^t(u)\,du=\frac{-c_p^2}{B_t-B_s}\sqrt{P^{pt}Q_1^{pt}Q_2^{ps}Q_3^{ps}}$$

putting $P^{ps}=0$ and $Q_3^{pt}=0$, we get,

$$\int_{\omega_1}^{\omega_2} K_p^s(u)N_p^t(u)\,du=\frac{-c_p^2}{B_t-B_s}\Big\{\sqrt{Q_2^{pt}Q_1^{ps}}-\sqrt{Q_1^{pt}Q_2^{ps}}\Big\}\sqrt{P^{pt}Q_3^{ps}}.$$

Thus we can obtain integrals of the product of two functions of the first kind but not of the same species. Moreover if the interval is taken as (ω_2, ω_3), or (ω_3, ω_1) the value of the integral can be found by making the corresponding change in the functions used.

6. Integral of F^2

Now suppose $B_t \to B_s$, then the right hand side of (3·9) becomes indeterminate but tends to a definite limit. Hence it can be deduced from (3·9) that

$$\int_{\omega_1}^{\omega_2} [F_p^s(u)]^2 du = \frac{c_p^2}{2\sqrt{P^{ps} Q_1^{ps} Q_2^{ps} Q_3^{ps}}} \times \left\{ \mu_2^s (P^{ps} Q_2^{ps} Q_1'^{ps} Q_3'^{ps} - Q_1^{ps} Q_3^{ps} P'^{ps} Q_2'^{ps}) - \mu_1^s (P^{ps} Q_1^{ps} Q_2'^{ps} Q_3'^{ps} - Q_2^{ps} Q_3^{ps} P'^{ps} Q_1'^{ps}) \right\}.$$

7. Orthogonality of functions of the same species

If B_s and B_t are roots of the same equation of (1·3) say, of $P^p(B)=0$ then putting $P^{ps}=0=P^{pt}$ we get,

$$\int_{\omega_1}^{\omega_2} K_p^s(u) K_p^t(u) du = 0, \quad s \neq t.$$

Similar results will be obtained for functions of other species. Hence we obtain a new proof of the theorem, viz. *the functions of the same species form an orthogonal set of functions.* We cannot deduce as we did in § 6, the value of

$$\int_{\omega_1}^{\omega_2} [E_p^t(u)]^2 du,$$

for $B_t \to B_s$ will imply that two roots of $P(B)Q(B)=0$ are coincident which is not true.

8.

Theorem:—*Generalized Lame' functions form a non-orthogonal set of functions.*

Proof: If they do not, then the integral (3·9) is zero. Since all the roots of (2·3) are simple, hence all the functions under the radical signs occurring above are algebraically irrational and so the expression (3·9) can be zero only if

(i) all of them are separately zero,

or (ii) $$\sqrt{P^{pt} Q_1^{pt} Q_2^{ps} Q_3^{ps}} = \sqrt{P^{ps} Q_1^{ps} Q_2^{pt} Q_3^{pt}}$$

and $$\sqrt{P^{pt}Q_2^{pt}Q_1^{ps}Q_3^{ps}}=\sqrt{P^{ps}Q_2^{ps}Q_1^{pt}Q_3^{pt}}$$

or (iii) $$\sqrt{P^{ps}Q_1^{ps}Q_2^{pt}Q_3^{pt}}=\frac{\sqrt{\mu_2^t\mu_2^s}}{\sqrt{\mu_1^t\mu_1^s}}\sqrt{P^{ps}Q_2^{ps}Q_1^{pt}Q_3^{pt}}$$

and $$\sqrt{P^{pt}Q_1^{pt}Q_2^{ps}Q_3^{ps}}=\frac{\sqrt{\mu_2^t\mu_2^s}}{\sqrt{\mu_1^t\mu_1^s}}\sqrt{P^{pt}Q_2^{pt}Q_1^{ps}Q_3^{ps}}.$$

Since $F_p^t(u)$ and $F_p^s(u)$ are symmetric in ω_1, ω_2, ω_3, so similar two sets of such conditions will be obtained for the intervals (ω_2, ω_3) and (ω_3, ω_1). Now (i) cannot hold because B_s and B_t are different from the roots of $P(B)Q(B)=0$; (ii) and (iii) along with two other similar sets give rise to

$$\frac{P^p(B_s)}{Q_1^p(B_s)}=\frac{P^p(B_t)}{Q_1^p(B_t)}\ ;\ \frac{P^p(B_s)}{Q_2^p(B_s)}=\frac{P^p(B_t)}{Q_2^p(B_t)}\ ;\ \frac{P^p(B_s)}{Q_3^p(B_s)}=\frac{P^p(B_t)}{Q_3^p(B_t)};$$

and $$\mu_1^t=\mu_2^t=\mu_3^t\ ;\ \mu_1^s=\mu_2^s=\mu_3^s.$$

The first three are impossible unless $B_s=B_t$, which is not; and others cannot hold because $F_p^t(u)$ is a doubly periodic function of the second kind. Therefore the supposition is wrong, i.e. the integral (3·9) is definitely different from zero and therefore the functions are non-orthogonal.

9.

Similarly we can show that the integrals obtained in §§ 4-5 are different from zero. Hence we obtain the following

THEOREM. *The integral of the product of any two solutions of Lame' equation for a fixed value of n, is different from zero unless both of them are of the first kind and of the same species.*

ERRATA

For the Paper "On Waring's Problem",
Vol. II (New Series) No. 1. pp. 16-44.

Page	*Line*	*For*	*Read*
16	Footnote	$\mathbf{R}(1)$	$\mathbf{R}(4)$
18	4	$5\frac{1}{6}n$	$5\frac{1}{6}\log n$
,,	13	$p^{\mu}\leqslant$	$p^{\mu\nu}\leqslant$
,,	19	n	$(n-1)$
,,	20	$n^{2+8(n^2-4)}$	$n^{2+16/(t-8)}$
19	6	k	n
20	24	$\binom{n}{3}\frac{1}{2}\cdot\frac{1}{x}$	$\binom{n}{3}\frac{1}{2}\cdot\frac{1}{x^2}$
21	22	9	8
24	8	2^n+l-1	$2^n+l-2.$
,,	18	P_1	P_1^n
33	2	$\frac{1}{2}$	$1/\sqrt{2}$
,,	4	$2n3^n$	$2\sqrt{2}n3^n$
,,	6	$M/2$	$M(1-1/\sqrt{2})$
34	14	$2(ac)^{-1}$	$2(ac)$
,,	19, 21, 23	$2n^2$	$4n^2$
35	28	$2^n-\delta$	$2^n+\delta$
36	3	nth	9th
,,	9	$3^{3/4}l$	$3\frac{3}{4}\,l$
37	17	$\log(4/3)$	$n\log(4/3)$
,,	29	$\log(4/3)\log n$	$\log(4/3)/\log 3$
39	7	$\leqslant$	$\geqslant$
,,	14	$n\geqslant 3$	$n\geqslant 4.$

ON INTEGRAL FUNCTIONS OF FINITE ORDER AND MINIMAL TYPE

By V. GANAPATHY IYER, Madras University

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1. INTRODUCTION. Let $[z_n]$ be a sequence of distinct complex numbers such that $|z_n| \to \infty$ as $n \to \infty$. It is supposed that $[z_n]$ are arranged according to non-decreasing moduli, the numbers with the same modulus being arranged according to their amplitudes. We shall refer to the exponent of convergence ρ of $[z_n]$ as its order. We suppose $0 < \rho < \infty$. Let $\sigma(z)$ be the canonical product with simple zeros at z_n. It is well known that the order of the integral function $\sigma(z)$ is ρ. We define the index of distribution* (I.D.) of $[z_n]$ as the greatest lower bound of numbers h such that

$$\sum \left| \frac{1}{\sigma'(z_n) z_n^{h+1}} \right|$$

converges. The I.D. may vary from $-\infty$ to ∞. When it is not $+\infty$ we shall say that it is finite. The author has discussed elsewhere† the properties of integral functions bounded at a sequence of points with finite I.D. The object of the present paper is to discuss certain cases where the I.D. may be infinite while the value of the function at $[z_n]$ is subjected to more stringent hypothesis than mere boundedness. The results of this paper include cases where the I.D. may be finite though in many cases more precise results can be proved when the I.D. is supposed to be finite. In some cases‡, an exact knowledge of the behaviour of $[\sigma'(z_n)]$ as $|z_n| \to \infty$ would give very precise results as to the type of $f(z)$ on the supposition $f(z_n) = O(1)$.

* This is not the same as the index of condensation of a sequence $[\lambda_n]$ of real positive numbers (see Bernstien, *Series de Derichlet*, Borel Tracts, (1933) and § 5·3 below).

† "On Integral Functions of order one and finite type", *Jour. Ind. Math. Soc.* II, 1 (New Series) 1-12; "On Integral Functions of finite order bounded at a sequence of points", Ibid. No. 2, pp. 53-66. These will be referred to as I and II in the sequel.

‡ See, for instance, the author's note "On Integral Functions of order two bounded at the lattice points", *Jour. Lond. Math. Soc.* Vol. II, 1936.

1·1. We treat the three cases $0<\rho<\frac{1}{2}$, $0<\rho<1$ and $\rho\geqslant 1$ in §§ 2, 3 and 4 respectively. In § 5, we consider some illustrations of the general cases discussed in §§ 2-4. In § 6 we prove some results allied to a theorem due to Faber that, for a function $f(z)$ of order one and minimal type, $|f(n)|>e^{-\alpha n}$ for any given $\alpha>0$, the inequality holding for almost all positive integers. This result itself does not seem capable of proof by the methods of this paper.

1·2. NOTATION. We denote by $n(r)$ the number of $[z_n]$ lying in $|z|\leqslant r$. Let $f(z)$ be an integral function. We write $M(r,f)=\max_{|z|\leqslant r}|f(z)|$. The order ρ, the upper type $k(f)$ and the lower type $l(f)$ are defined by the relations:

$$\rho=\overline{\lim_{r\to\infty}}\frac{\log\log M(r,f)}{\log r};$$

$$l(f)=\underline{\lim_{r\to\infty}}\frac{\log M(r,f)}{r^{\rho}}\leqslant\overline{\lim_{r\to\infty}}\frac{\log M(r,f)}{r^{\rho}}=k(f).$$

When $k(f)=0$, the function $f(z)$ is said to be of minimal type.

2. For $0<\rho<\frac{1}{2}$ we can prove the following

THEOREM 1. *Let* $[z_n]$ *be a sequence of order* ρ, $0<\rho<\frac{1}{2}$ *such that* $\underline{\lim_{r\to\infty}}\frac{n(r)}{r^{\rho}}=A>0$. *Let* $f(z)$ *be a function of order* ρ *and minimal type. Let* $f(z_n)=y_n$ *and*

$$\chi_n=\max\left[|y_n|,\frac{|y_n|}{|\sigma'(z_n)|}\right].$$

If $\chi_n=O(1)$, *then* $f(z)$ *will reduce to a constant.*

PROOF: Since $0<\rho<\frac{1}{2}$, $\sum\frac{1}{|z_n|}$ will converge. Also it is easy to see, by using Jensen's formula, that $l(\sigma)\geqslant A/\rho>0$. If $m(r,\sigma)=\min_{|z|=r}|\sigma(z)|$, a theorem due to Wiman* states that

$$\overline{\lim_{r\to\infty}}\frac{\log m(r,\sigma)}{\log M(r,\sigma)}\geqslant\cos\pi\rho.$$

Hence using the fact that $l(\sigma)>0$, $\cos\pi\rho>0$, we can find a $d>0$ and a sequence $[R_n]$, $R_1<R_2\ldots<R_n\to\infty$ such that on $|z|=R_n$

$$\left|\frac{[f(z)]^{p}}{\sigma(z)}\right|\leqslant\exp(p\log M(R_n,f)-dR_n^{\rho}),\qquad(1)$$

* A. Wiman, *Math. Ann.* 76 (1915), 197-211.

where p is any positive integer. Let C_{R_n} be the circle $|z|=R_n$. Consider the integral

$$\frac{1}{2\pi i}\int_{C_{R_n}} \frac{[f(z)]^p}{\sigma(z)\,(z-x)}\,dz. \tag{2}$$

In virtue of (1) and the fact that $k(f)=0$, we find that (2) tends to zero uniformly in any circle $|x|\leqslant R$. But (2) is equal to

$$\frac{[f(x)]^p}{\sigma(x)} - \sum_{|z_\nu|<R_n} \frac{y_\nu^p}{\sigma'(z_\nu)\,(x-z_\nu)}. \tag{3}$$

But by hypothesis,

$$\left|\frac{y_\nu^p}{\sigma'(z_\nu)}\right| \leqslant \chi_\nu^p = O(1)$$

for any fixed p, so that

$$\sum \left|\frac{y_\nu^p}{\sigma'(z_\nu)\,(x-z_\nu)}\right|$$

converges. Combining these results we get, from (3)

$$\frac{[f(x)]^p}{\sigma(x)} = \sum_{\nu=1}^{\infty} \frac{y_\nu^p}{\sigma'(z_\nu)\,(x-z_\nu)} \tag{4}$$

for $p\geqslant 1$*. Now, the double series

$$\sum_{\nu,p=1}^{\infty} \left|\frac{y_\nu^p}{(p-1)!\;\sigma'(z_\nu)\,(x-z_\nu)}\right|$$

converges because, a summation with respect to p, gives

$$\sum_{\nu=1}^{\infty} \left|\frac{y_\nu e^{|y_\nu|}}{\sigma'(z_\nu)\,(x-z_\nu)}\right|$$

which converges since $|y_\nu|=O(1)$ and $\left|\dfrac{y_\nu}{\sigma'(z_\nu)}\right| = O(1)$.

Therefore we get from (4),

$$\frac{f(x)\,e^{f(x)}}{\sigma(x)} = \sum_{\nu=1}^{\infty} \frac{y_\nu e^{y_\nu}}{\sigma'(z_\nu)}\,\frac{1}{x-z_\nu}. \tag{5}$$

Using the hypothesis $\chi_n=O(1)$ and the methods of § 2·3, II, we can conclude that $f(x)e^{f(x)}$, given by (5), is a function of order ρ.

* When $p=0$, we cannot assert that $\sum \frac{1}{\sigma'(z_\nu)(x-z_\nu)}$ converges absolutely though a proper grouping of terms would render it absolutely convergent.

This would require that $e^{f(x)}$ is of finite order which is equivalent to the assertion that $f(x)$ is a polynomial. But $|f(z_n)| \leqslant \chi_n = O(1)$. Hence $f(x)$ reduces to a constant.

3. Let $0<\rho<1$. Here we prove

THEOREM 2. *Let $z_n = \lambda_n$ be real and positive. Let $[\lambda_n]$ be of order ρ, $0<\rho<1$ and $\lim_{r\to\infty} \frac{n(r)}{r^\rho} = A > 0$. Let $f(z)$ be a function of order ρ and* minimal *type. Let $f(z_n) = y_n$ and*

$$\chi_n = \max\left[|y_n|, \left|\frac{y_n}{\sigma'(\lambda_n)}\right|\right].$$

If $\chi_n = O(1)$, then $f(z)$ will reduce to a constant.

PROOF: As in § 2, the series

$$\sum \frac{|y_\nu|^p}{|\sigma'(\lambda_\nu)(z-\lambda_\nu)|}$$

converges for $p \geqslant 1$ since $\chi_n = O(1)$ and $\rho < 1$. Let

$$H_p(z) = \frac{[f(z)]^p}{\sigma(z)} - \sum_{\nu=1}^{\infty} \frac{y_\nu^p}{\sigma'(\lambda_\nu)(z-\lambda_\nu)}.$$

By using the method of § 2·32, II, we show that $H_p(z)$ is an integral function of order ρ. Next, combining the facts that $k(f)=0$ and $l(\sigma)>0$ we can show, by the method of § 2·321, II, that $H_p(z)\to 0$ along two lines through the origin, the greater of the angles between them being π/γ where $\rho<\gamma$. Hence we conclude as in § 2·34, II, that $H_p(z)\equiv 0$, so that we get

$$\frac{[f(z)]^p}{\sigma(z)} = \sum_{\nu=1}^{\infty} \frac{y_\nu^p}{\sigma'(\lambda_\nu)(z-\lambda_\nu)}. \tag{6}$$

From (6) we conclude, as in § 2 above, that $f(z)$ reduces to a constant.

4. For $\rho \geqslant 1$, we choose an integer q such that $\rho \leqslant q < \rho+1$. We can now state

THEOREM 3. *Let $[\lambda_n]$, $0<\lambda_1<\lambda_2 \ldots$, be a sequence of order ρ, $\rho \geqslant 1$ such that $\lim_{r\to\infty} \frac{n(r)}{r^\rho} = A > 0$. Let $[z_n]$ denote the zeros of*

$$\sigma(z) = \prod_{\nu=1}^{\infty}\left(1 - \frac{z^{2q}}{\lambda_\nu^{2q}}\right).$$

Let $f(z)$ be of order ρ and minimal *type. Let $f(z_n)=y_n$ and*

$$\chi_n=\max\left[|y_n|,\ \left|\frac{y_n}{\sigma'(z_n)}\right|\right].$$

If $\chi_n=O(1)$, then $f(z)$ reduces to a constant.

PROOF: Here

$$\sum_{\nu=1}^{\infty}\left|\frac{y_\nu^p}{\sigma'(z_\nu)}\frac{1}{z-z_\nu}\left(\frac{z}{z_\nu}\right)^q\right|$$

converges. Put

$$H_p(z)=\frac{[f(z)]^p}{\sigma(z)}-\sum_{\nu=1}^{\infty}\frac{y_\nu^p}{\sigma'(z_\nu)}\frac{1}{z-z_\nu}\left(\frac{z}{z_\nu}\right)^q. \tag{7}$$

Following the line of argument indicated in § 3 above, we can conclude that $H_p(z)$ is polynomial of degree $q-1$ at most whose coefficients might depend on p. From (7) we get, as in § 2,

$$\frac{f(z)e^{f(z)}}{\sigma(z)}=\lim_{\lambda\to\infty}\sum_{p=1}^{\lambda}\frac{H_p(z)}{(p-1)!}+\sum_{\nu=1}^{\infty}\frac{y_\nu e^{y_\nu}}{\sigma'(z_\nu)}\frac{1}{z-z_\nu}\left(\frac{z}{z_\nu}\right)^p. \tag{8}$$

Since $H_p(z)$ is a polynomial of degree $q-1$ at most we can write

$$\sum_{p=1}^{\lambda}\frac{H_p(z)}{(p-1)!}=a_{0,\lambda}z^{q-1}+a_{1,\lambda}z^{q-2}+\ldots+a_{q-1,\lambda}$$

from which we conclude that the limit on the right side of (8) exists and is a polynomial of degree $q-1$ at most. We can now repeat the argument of § 2 and conclude that $f(z)$ is a constant.

5. ILLUSTRATIONS. As a special case of Theorems (1)—(3) we state

THEOREM 4. *Let $[z_n]$ denote the zeros of the function $\sigma(z)$ occurring in Theorems* 1—3 *and let the appropriate conditions imposed on $[z_n]$ in the respective cases be satisfied. Let $f(z)$ be a function of order $\rho>0$ and minimal type. Let $[k_n]$ be any sequence of positive numbers tending to infinity. If the two conditions*

$$\left.\begin{aligned}&\overline{\lim_{n\to\infty}}\ \frac{1}{k_n}\log\frac{1}{|\sigma'(z_n)|}<+\infty\\&\overline{\lim_{n\to\infty}}\ \frac{1}{k_n}\log|y_n|\qquad<0\end{aligned}\right\} \tag{9}$$

be simultaneously satisfied, then $f(z)$ will be identically zero.

PROOF: Let α and $-\beta$ respectively denote the limits occurring on the right side of (9). Let p_0 be an integer such that

$p_0\beta > \alpha$, this being possible because $\alpha < \infty$ and $\beta > 0$ by hypothesis. Let $F(z) = [f(z)]^{p_0}$. Then

$$\overline{\lim_{n\to\infty}} \frac{1}{k_n} \log |F(z_n)| = -p_0\beta$$

while

$$\overline{\lim_{n\to\infty}} \frac{1}{k_n} \log \frac{|F(z_n)|}{|\sigma'(z_n)|} \leqslant -p_0\beta + \alpha < 0.$$

Hence $F(z)$ satisfies the conditions of Theorems 1—3 so that $F(z)$ reduces to a constant which will be zero since $y_n \to 0$ as $n \to \infty$. Therefore, also, $f(z) \equiv 0$.

5·1. REMARK. If $y_n = O(1)$, and $\alpha < 0$ in Theorem 4, $f(z)$ will reduce to a constant. Now, $\alpha < 0$ is, in general, a more stringent condition than the supposition that I.D. is finite. As a matter of fact, $f(z)$ will be a constant if $y_n = O(1)$ and the I.D. is finite*.

5·2. Let $\lambda_n = n^\alpha$, $\alpha > 0$. The order $\rho = 1/\alpha$. The cases $0 < \rho \leqslant \frac{1}{2}$ and $\rho \geqslant 1$ have been considered in I and II. We take the case $\frac{1}{2} < \rho < 1$. By § 4·2, II, we have

$$\log |\sigma'(n^{1/\rho})| = n\pi \cot \pi\rho + O(\log n)$$

so that

$$\lim_{n\to\infty} \frac{1}{n} \log \frac{1}{|\sigma'(n^{1/\rho})|} = -\pi \cot \pi\rho < +\infty$$

though $\pi \cot \pi\rho < 0$. Hence we get

THEOREM 5. *Let $f(z)$ be a function of order ρ, $\frac{1}{2} < \rho < 1$ and minimal type. Let*

$$\overline{\lim_{n\to\infty}} \frac{1}{n} \log |f(n^{1/\rho})| < 0.$$

Then $f(z)$ is identically zero.

5·3 When $\rho = 1$, we get the following

THEOREM 6. *Let $[\lambda_n]$, $0 < \lambda_1 < \lambda_2 \ldots < \lambda_n \to \infty$ be a sequence of order one and finite index of condensation*†. *Let*

* The author has proved this result in a paper not yet published.

† For the definitions of the index of condensation and maximum density see Bernstien, l.c. When $[\lambda_n]$ is measurable, it is proved in the reference quoted (p. 289) that the index of condensation is actually equal to $\overline{\lim_{n\to\infty}} \frac{1}{\lambda_n} \log \frac{1}{|\sigma'(\lambda_n)|}$. When $[\lambda_n]$ is not measurable, we can find a measurable sequence $[\Lambda_n]$ of density D containing $[\lambda_n]$ and having the same index of condensation δ. If $[\mu_n]$ is the complimentary sequence of $[\lambda_n]$ in $[\Lambda_n]$ and

$\lim_{r\to\infty} \frac{n(r)}{r} = A > 0$. *Let $f(z)$ be a function of order one and minimal type. If*

$$\overline{\lim_{n\to\infty}} \frac{1}{\lambda_n} \log |f(\pm\lambda_n)| < 0,$$

then $f(z)$ will be identically zero.

PROOF: Let

$$\sigma(z) = \prod_{\nu=1}^{\infty} \left(1 - \frac{z^2}{\lambda^2}\right).$$

Let δ be the index of condensation and D the maximum density of $[\lambda_n]$. Then it is known that

$$\overline{\lim} \frac{1}{\lambda_n} \log \frac{1}{|\sigma'(\lambda_n)|} \leqslant \delta + \pi D.$$

The result required is now a consequence of Theorem 4 since $\sigma'(-\lambda_n) = -\sigma(\lambda_n)$.

6. We proceed to prove some theorems allied to Faber's* result mentioned in § 1·1 when $\rho = 1$.

THEOREM 7. *Let $[\lambda_n]$, $0 < \lambda_1 < \lambda_2 \ldots \lambda_n \to \infty$, be a sequence of order one and finite index of condensation δ. Let*

$$\sigma(z) = \prod_{\nu=1}^{\infty} \left(1 - \frac{z^2}{\lambda_\nu^2}\right)$$

and $l(\sigma) > 0$. Let $f(z)$ be a function of minimal type and order one. If $|f(\lambda_n)| \leqslant e^{-\alpha\lambda_n}$ for some $\alpha > 0$, then $f(z) \equiv 0$.

REMARK. In Theorem 6 we have supposed that $|f(\pm\lambda_n)| \leqslant e^{-\alpha\lambda_n}$ for some $\alpha > 0$. Theorem 7 is therefore sharper than and includes Theorem 6.

PROOF: Let D be the maximum density of the sequence. Let p_0 be an integer so that $p_0\alpha > 2(\delta + \pi D)$. Let

$$F(z) = \left[e^{\frac{\alpha z}{2}} f(z)\right]^{p_0}, \quad \lambda_{-n} = -\lambda_n, \text{ and } F(\lambda_n) = y_n, -\infty < n < \infty.$$

$\sigma_1(z)$ the corresponding product, we shall have

$$\overline{\lim_{n\to\infty}} \frac{1}{\lambda_n} \log \frac{1}{|\sigma'(\lambda_n)|} \leqslant \delta + \overline{\lim_{n\to\infty}} \frac{1}{\lambda_n} \log |\sigma_1(\lambda_n)| \leqslant \delta + \pi D$$

and D is known to be finite when δ is finite (see pp. 264, 267-293, of the reference quoted).

* G. Faber, *Jahr. der. Deut. Math. Ver.* 16 (1907), 285-298.

Put

$$H_p(z)=\frac{[F(z)]^p}{\sigma(z)}-\sum_{\nu=-\infty}^{\infty}\frac{y_\nu^p}{\sigma'(\lambda_\nu)}\frac{1}{z-\lambda_\nu}, \tag{10}$$

where p is a positive integer $\geqslant 1$. We have

$$\overline{\lim_{|\nu|\to\infty}}\frac{1}{|\lambda_\nu|}\log\left|\frac{y_\nu^p}{\sigma'(\lambda_\nu)}\right|\leqslant\delta+\pi D-\frac{\alpha}{2}pp_0$$

since $k(f)=0$, $p_0\alpha>2(\delta+\pi D)$ and $p\geqslant 1$. Hence the series on the right side of (10) represents a meromorphic function bounded outside small circles round $z=\lambda_\nu$. On $\theta=\frac{\pi}{2}\pm\eta$, $\eta>0$ being sufficiently small, we have

$$\overline{\lim_{|z|\to\infty}}\frac{1}{|z|}\log\left|\frac{[F(z)]^p}{\sigma(z)}\right|\leqslant pp_0\frac{\alpha}{2}\sin\eta-l(\sigma)[1-\xi(\eta)],$$

where $\xi(\eta)\to 0$ as $\eta\to 0$. Since $l(\sigma)>0$, we conclude that

$$\frac{[F(z)]^p}{\sigma(z)}\to 0 \text{ as } |z|\to\infty$$

along these lines. The same is true of the series in (10). Hence we can conclude as in § 4·7, I, that $H_p(z)\equiv 0$ so that

$$\frac{[F(z)]^p}{\sigma(z)}=\sum_{-\infty}^{\infty}\frac{y_\nu^p}{\sigma'(\lambda_\nu)}\frac{1}{z-\lambda_\nu}. \tag{11}$$

Also, $|y_\nu|<e^{-\frac{p_0\alpha}{2}|\lambda_\nu|}$ and $\left|\frac{y_\nu}{\sigma'(\lambda_\nu)}\right|\leqslant\exp\left[\left(\delta+\pi D-\frac{p_0\alpha}{2}\right)|\lambda_\nu|\right]$, where $p_0\alpha>2(\delta+\pi D)$. We can now repeat the argument of § 2 and conclude that $F(z)\equiv 0$. Therefore, also, $f(z)\equiv 0$.

6·1. Another theorem of the same type is

THEOREM 8. *Let* $[\lambda_n]$ *and* $\sigma(z)$ *satisfy the conditions of Theorem 7. Let* $f(z)$ *be a function of order one such that* $\lim_{n\to\infty}\frac{1}{\lambda_n}\log|f(\lambda_n)|=-\infty$. *If* $k(f)<l(\sigma)$, *then* $f(z)\equiv 0$.

PROOF: Let $\beta>\delta+\pi D+k(f)$. Let $F_m(z)=e^{\beta z}f(z)\,z^m$, m being a non-negative integer. We have, in the notation of the previous paragraph,

$$\overline{\lim_{\nu\to\infty}}\frac{1}{|\lambda_\nu|}\log\left|\frac{F_m(\lambda_\nu)}{\sigma'(\lambda_\nu)}\right|\leqslant-\beta+k+\delta+\pi D<0,$$

where $k=k(f)$. Hence as in § 6, we can conclude in virtue of the fact $k(f)<l(\sigma)$, that

$$\frac{z^m e^{\beta z} f(z)}{\sigma(z)}=\sum_{\nu=-\infty}^{\infty}\frac{e^{\beta\lambda_\nu}f(\lambda_\nu)\lambda_\nu^m}{\sigma'(\lambda_\nu)\,(z-\lambda_\nu)}. \tag{12}$$

Now, let $\chi(z)=c_0+c_1z+c_2z^2+\dots$ be any function of order one and type $k(\chi)<\beta-(k+\delta+\pi D)$. Then the double series

$$\sum_{(\nu,\,m)}\left|\frac{c_m\lambda_\nu^m e^{\beta\lambda_\nu}f(\lambda_\nu)}{\sigma'(\lambda_\nu)\,(z-\lambda_\nu)}\right|$$

converges. Hence, we get, from (12), that

$$\frac{e^{\beta z}f(z)\chi(z)}{\sigma(z)}=\sum_{-\infty}^{\infty}\frac{e^{\beta\lambda_\nu}f(\lambda_\nu)\chi(\lambda_\nu)}{\sigma'(\lambda_\nu)\,(z-\lambda_\nu)}. \tag{13}$$

Now let $\xi>0$ and $\beta=(k+\xi)p+\delta+\pi D$. Then we can take $\chi(z)=[f(z)]^{p-1}$ in (13) and arrive at the formula

$$\frac{e^{(\delta+\pi D)z}[e^{(k+\xi)z}f(z)]^p}{\sigma(z)}=\sum_{\nu=-\infty}^{\infty}\frac{e^{(\delta+\pi D)\lambda_\nu}[e^{(k+\xi)\lambda_\nu}f(\lambda_\nu)]^p}{\sigma'(\lambda_\nu)\,(z-\lambda_\nu)}. \tag{14}$$

Let y_ν^p denote the coefficient of $\dfrac{e^{(\delta+\pi D)\lambda_\nu}}{(z-\lambda_\nu)\,\sigma'(\lambda_\nu)}$ on the right side of (14). Then for $\nu>0$, we have, by hypothesis,

$$\lim_{\nu\to\infty}\frac{1}{\lambda_\nu}\log|y_\nu|=-\infty,$$

while for $\nu<0$, we have $\quad \overline{\lim}_{|\nu|\to\infty}\dfrac{1}{|\lambda_\nu|}\log|y_\nu|\leqslant-\xi<0.$

These together with the fact that $\dfrac{e^{(\delta+\pi D)\lambda_\nu}y_\nu}{\sigma'(\lambda_\nu)}$ satisfies the same inequalities, enable us to repeat the argument of § 2 and conclude that $f(z)\equiv 0$.

6·2. As special cases of Theorems 7 and 8, we get

THEOREM 9. *Let $[\lambda_n]$ be a measurable sequence of positive density and finite index of condensation. Then, a function of order one and minimal type satisfying the condition $|f(\lambda_n)|\leqslant e^{-\alpha\lambda_n}$ for some $\alpha>0$, is identically zero.*

THEOREM 10. *Let $[\lambda_n]$ be a measurable sequence of density D and finite index of condensation. Let $\lim_{n\to\infty}\dfrac{1}{\lambda_n}\log|f(\lambda_n)|=-\infty$, where $f(z)$ is a function of order one. Then $\overline{\lim}_{r\to\infty}\dfrac{\log M(r,f)}{r}\geqslant\pi D$ or else $f(z)\equiv 0$.*

These results follow from the fact that when $[\lambda_n]$ is measurable and has density D, $l(\sigma)=k(\sigma)=\pi D$.

6·3. When $[\lambda_n]$ are positive integers, the index of condensation is always zero. So we get

THEOREM 11. *Let $[\lambda_n]$ be any sequence of positive integers with $\varliminf_{r\to\infty}\frac{n(r)}{r}>0$. Then a function of order one and minima type such that $|f(\lambda_n)|\leqslant e^{-\alpha\lambda_n}$ for some $\alpha>0$ is identically zero.*

NOTE. Faber's result is that even if $\overline{\lim}\, n(r)/r>0$, the conclusion of the above theorem is true. The assumption $\varliminf n(r)/r>0$ enables us to show that $l(\sigma)>0$. Hence, if it could be shown that, when $[\lambda_n]$ are integers, $\overline{\lim}\, n(r)/r>0$ involves $l(\sigma)>0$, Faber's result would follow from Theorem 7 but the last mentioned proposition is not true in general, as may be seen by taking for $[\lambda_n]$ the set of integers lying in $[2^{(2k-1)^2}, 2^{(2k)^2}]$, $k=1, 2, \ldots$. It can be shown that when $r_k=2^{(2k)^2+\frac{4k-1}{2}}$, $\lim_{k\to\infty}\frac{\log M(r_k, \sigma)}{r_k}=0$ so that $l(\sigma)=0$.

6·4. THEOREM 12*. *Let $[\lambda_n]$ be a measurable sequence of integers with density D. Let $f(z)$ be a function of order one such that $\lim_{n\to\infty}\frac{1}{\lambda_n}\log|f(\lambda_n)|=-\infty$. Then, either $k(f)\geqslant\pi D$ or else $f(z)\equiv 0$.*

6·5. REMARKS. In a sense, Theorem 12 is a best possible result. The function $f(z)=\sin\pi z$ vanishes at $\lambda_n=n$, which is measurable and has density one. Therefore $f(z)$ satisfies the condition of Theorem 12 while its type is precisely π. Again, the function $f(z)=e^{-\delta z}$ has type $\delta>0$ and over any set $[\lambda_n]$ satisfies the condition $|f(\lambda_n)|=e^{-\delta\lambda_n}$. Hence, the condition that $f(z)$ is of minimal type is the best possible in Theorems 7, 9 and 11. In Theorem 12, we can suppose $[\lambda_n]$ to be any measurable sequence of numbers with $\varliminf_{n\to\infty}(\lambda_{n+1}-\lambda_n)>0$ for, in this case, the index of condensation is zero; the density D of $[\lambda_n]$ is assumed to be positive.

* Results of the type contained in Theorem 12 are proved by M.L. Cartwright under broader conditions on $[\lambda_n]$ with additional restrictions on $f(z)$. Cf. 'Some uniqueness theorems', *Proc. Lon. Math. Soc.* 2nd Series Vol. 41, 33-47.

DIFFERENTIAL GEOMETRY OF THE LAPLACE EQUATION

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Given any n—dimensional Euclidean space in general co-ordinates, with a metric $ds^2 = g_{ij} dx^i dx^j$, Laplace's equation for the given system of co-ordinates becomes

$$\text{div. grad. } u \equiv g^{ij} u_{|i|j} \equiv g^{ij} u_{,i,j} - g^{ij} u_{,r} \Gamma_{ij}^{r} = 0. \tag{1}$$

For a general Riemannian space with a non-vanishing curvature tensor, (1) may still be taken as the generalized Laplace equation. I propose to deal with the simple inverse problem:

Given a linear partial differential equation of the second order

$$a^{ij} u_{,i,j} + b^i u_{,i} = 0; \tag{2}$$

under what conditions may it be regarded as the Laplace equation associated with some Riemann space?

In this connection, the term Laplace equation includes such types as the classical wave equation $\frac{\partial^2 u}{c^2 \partial t^2} - \Delta_2 u = 0$. The space here will obviously be that of special relativity:

$$ds^2 = c^2 dt^2 - \Sigma (dx^i)^2. \tag{3}$$

It is clear that the equation (2) must be tensor invariant, and if associated with a Riemann space can differ at most from (1) by a factor λ. The space is, therefore, conformal to that with a fundamental tensor a_{ij} obtained from the equations $a^{ir} a_{rj} = \delta^i_j$, or, what is the same thing, the tensor obtained by dividing the cofactor of a^{ij} in $|a^{ij}|$ by the determinant itself. To this end, a first condition is that $|a^{ij}| \neq 0$. That a^{ij} must be a contravariant tensor is seen from the law of transformation of $u_{,i,j}$,

$$\bar{u}_{,i,j} = u_{,r\,s} \frac{\partial x^r}{\partial \bar{x}^i} \frac{\partial x^s}{\partial \bar{x}^j} + u_{,r} \frac{\partial^2 x^r}{\partial \bar{x}^i \partial \bar{x}^j}, \tag{4}$$

and from the tensor invariance postulated for (2). In fact, we must have

$$\bar{a}^{ij}=Wa^{rs}\frac{\partial \bar{x}^i}{\partial x^r}\frac{\partial \bar{x}^s}{\partial x^j}. \tag{5}$$

The weighting factor W may be assimilated to the transformed λ, or, assumed to be unity.

Let Γ^i_{jk} be the usual Christoffel symbols calculated for a tensor g_{ij} and $\left\{ {i \atop jk} \right\}$ the corresponding expressions for the tensor a_{ij}. If we put λa^{ij} for g^{ij} in (1), the equation (1) now has the form

$$\lambda a^{ij}u_{,i,j}-\lambda a^{jk}\left(\left\{ {i \atop jk} \right\} - \frac{a^{ir}}{2\lambda}[a_{jr}\lambda_{,k}+a_{kr}\lambda_{,j}-a_{jk}\lambda_{,r}] \right)u_{,i}=0, \tag{6}$$

since

$$g_{ij}=\frac{1}{\lambda}a_{ij}.$$

As we require this to be of type (2) but for the factor λ we have the condition:

$$b^i+a^{jk}\left\{ {i \atop jk} \right\}+\frac{n-2}{2\lambda}a^{ir}\lambda_{,r}=0. \tag{7}$$

Putting $\mu=\log\lambda$, and $\beta_i=a_{ir}\left(b^r+a^{jk}\left\{ {i \atop jk} \right\} \right)$ this is equivalent to

$$\frac{n-2}{2}\mu_{,i}+\beta_i=0. \tag{8}$$

Though indeterminate for $n=2$, this system of partial differential equations for the unknown μ has a solution provided the conditions $\beta_{i,j}-\beta_{j,i}=0$ are satisfied; the metric of the space is then of the form

$$ds^2=e^{-\mu}a_{ij}dx^idx^j.$$

All these results may be summed up in the form of

THEOREM I. *The equation $a^{ij}u_{,i,j}+b^iu_{,i}=0$ may be regarded as the Laplace equation for a Riemann space if and only if: (a) a^{ij} is a symmetric contravariant tensor with non-vanishing determinant, $|a^{ij}|\neq 0$, (b) $n=2$ and $\beta_i=0$, or (b′) $n>2$, $\beta_{i,j}-\beta_{j,i}=0$. Under these conditions, the space for $n=2$ is any conformal to that of the fundamental tensor a_{ij}; for $n>2$, the metric of the space is given by*

$$ds^2=e^{-\mu}a_{ij}dx^idx^j$$

and determined to a constant factor by the partial differential equation $\frac{n-2}{2}\mu_{,i}+\beta_i=0.$

It is clear that for $n=1$, the problem is trivial. The theory of functions of a complex variable and the fact that any surface with a regular positive-definite groundform may be represented conformally on the Euclidean plane lead us to expect the result given for $n=2$.

Equations of a more general type than (2) cannot be treated without recourse to some kind of a transformation. For instance, the type

$$a^{ij}u_{,i,j}+b^{i}u_{,i}+cu=R \tag{9}$$

must be reduced to the homogeneous form ($R=0$) by the usual methods, and the substitution $u=fv$ must be employed to get rid of the last term. For this, f must be a known solution of (9) with $R=0$, and we have a reduced form:

$$a^{ij}v_{,i,j}+(2a^{ij}\frac{f_{,j}}{f}+b^{i})v_{,i}=0. \tag{10}$$

This is much the same equation as (2), but with $a^{ij}\phi_{,j}$ added to b^{i}, or, a term $\phi_{,i}$ added to β_i, where $\phi=2\log f$ and f is any particular non-trivial solution of (9) with $R=0$. This allows us to generalize the theorem stated, as (2) is a special case of (9); we obtain a relaxation of condition (b) for $n=2$, and also a new space for every solution of the given equation, though a space conformal to that of the tensor a_{ij}. We sum these up in

Theorem II. *The conditions and conclusions of Theorem I are applicable to the equation*

$$a^{ij}u_{,i,j}+b^{i}u_{,i}+cu=0$$

when the transformation $v=fu$ *is allowed; but condition* (b) *becomes:*

$$n=2,\ \beta_{i,j}-\beta_{j,i}=0;\ a^{ij}(\beta_i\beta_j-2\beta_{i,j})-2b^{i}\beta_i+4c=0.$$

And for $n>2$, *the metric is given by*

$$ds^2=f^{4/(n-2)}e^{-\mu}a_{ij}dx^{i}dx^{j},$$

where f *is a solution of the given equation,* μ *being, as before, a solution if any, of*

$$\frac{n-2}{2}\mu_{,i}+\beta_i=0.$$

This says nothing about equations of a more general type, and transformations which are less simple. It would seem, however, that the entire problem is better approached from a different point of view. One should discuss necessary and sufficient conditions for the given equations to be deducible from a variational principle (self-adjoincy) and then see whether some sort of geometry may be associated with the integrand of the variational principle so obtained.

A NOTE ON THE MORLEY-PETERSON THEOREM

By Dr. R. VAIDYANATHASWAMY

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The Peterson-Morley theorem is as follows:

Theorem. *Let a, b, c be three skew-lines, a', b', c' the lines along which the shortest distances between (b, c), (c, a), (a, b) lie. If α, β, γ be the shortest distances between (a, a'), (b, b'), (c, c'), there is a common transversal of α, β, γ which meets them perpendicularly.*

Various proofs of this theorem have been given, the most recent being from Prof. H. F. Baker*. I am concerned here with some simple considerations which throw light on the nature of this theorem, and help incidentally to prove it.

1. Given a plane conic S, regarded as the carrier of a binary variable x, any line in the plane corresponds to the quadratic in x which determines its intersections with S. A point in the plane may be considered to correspond to the same quadratic in x as its polar line with respect to S. It is clear that if two quadratics a_x^2, b_x^2 are apolar, then (1) the points (or lines) which correspond to them are *conjugate* with respect to S, and (2) the *point* which corresponds to either is incident with the *line* which corresponds to the other.

Suppose that ABC is a triangle in the plane of S, and $A'B'C'$ is the polar triangle for S. The conics through ABC outpolar to S must form a pencil and have therefore a fourth common point O. The line-pair AA', BC being a conjugate line-pair of S must be included among these conics; hence AA' (and similarly BB', CC') passes through O. Thus the polar triangles ABC, $A'B'C'$ are in perspective with O as centre of perspective. We call $ABCO$ an apolar quadrangle of S.

Further, the conic envelope S and the four squared points A, B, C, O must be linearly related, since they have a pencil of common outpolar conics. Hence there is a linear relation mod S between the squared points A, B, C, O or in other words there is a

* *Journal of the London Mathematical Society.* (Jan. 1936).

linear relation between the squares of the quadratics corresponding to $ABCO$. It is known that a net of binary quartics has exactly four perfect squares; hence this condition suffices to determine uniquely any one of the points A, B, C, O when the other three are known.

2. We can develop parallel ideas in the case of the quadric surface S in three dimensions. The quadric S contains two families of generators, which may be represented by two binary parameters λ, μ respectively. Two quadratics $f(\lambda)$, $\phi(\mu)$ correspond to four generators which make up the sides of a skew quadrilateral; the diagonals of this quadrilateral are a pair of polar lines of S. Thus each polar line-pair of S corresponds to a pair of quadratics $f(\lambda)$, $\phi(\mu)$. In the two dimensional case we had a dual representation of binary quadratics as either *points* or *lines* of the plane; in three dimensions we have a representation of pairs of binary quadratics $f(\lambda)$, $\phi(\mu)$ by polar line-pairs of the fundamental quadric S.

Now two mutually polar lines of S are fixed lines of an involutoric collineation which transforms the quadric into itself and each system of generators into itself. Hence any transversal of the polar line-pair corresponding to the quadratics $f(\lambda)$, $\phi(\mu)$ cuts S in two points the λ and μ generators through which are given by quadratics apolar to $f(\lambda)$, $\phi(\mu)$ respectively. Hence we have:

THEOREM 1. *If the quadratics $f(\lambda)$, $f'(\lambda)$ are apolar, and the quadratics $\phi(\mu)$, $\phi'(\mu)$ are also apolar, the polar line-pairs corresponding to $\{f(\lambda), \phi(\mu)\}$ and $\{f'(\lambda), \phi'(\mu)\}$ are transversal pairs of each other; and conversely.*

In the plane we had, on account of the operation of duality, a two-fold representation of the apolarity of quadratics; namely, as the *conjugacy* of the points (or lines) corresponding to the two quadratics, and as the *incidence* of the point corresponding to the one and the line corresponding to the other. Analogously, the simultaneous apolarity of the two pairs of quadratics corresponds here to the mutual *transversality* of the two corresponding polar line-pairs.

3. Since we have obtained the geometrical significance of apolarity, we may follow out for three dimensions the same order of ideas that gave us the *apolar* quadrangle in two dimensions. Let a, b, c be three polar line-pairs of S, corresponding to the points A, B, C of the two-dimensional case. The line BC or its pole

A' for the fundamental conic would here correspond to the transversal pair a' of the four lines b, c; it is clear that in the general case a' will consist of a pair of polar lines of S. The line AA' would correspond to the transversal pair α of the four lines a, a'; α would also be a pair of polar lines of S. The concurrency of AA', BB', CC' in O, corresponds to the statement that the three line-pairs α, β, γ have a transversal-pair O which is also a polar line-pair of S. We have thus

THEOREM 2. *Let a, b, c be three polar line-pairs of a quadric S; a', b', c' the transversal-pairs of (b, c), (c, a), (a, b); α, β, γ the transversal-pairs of (a, a'), (b, b'), (c, c'). Then α, β, γ are all met by a pair O of polar lines of S.*

The symmetric property of the apolar quadrangle $ABCO$ in plane is parallelled by

THEOREM 3. *The four line-pairs a, b, c, o have the symmetrical property that the transversal-pair of any two of them meet the transversal-pair of the remaining two.*

When the fundamental quadric degenerates into the imaginary circle at infinity, we get the Morley-Peterson theorem.

TAUBERIAN THEOREMS ON DIRICHLET'S SERIES

By S. MINAKSHISUNDARAM, M.A., Madras

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1. Let $\sum_{\nu=1}^{\infty} a_\nu$ be a given infinite series, $1 \leqslant l_1 < l_2 < \ldots$, a sequence tending to infinity and

$$A^r(\omega) = \sum_{\rho_n \leqslant \omega} (\omega - l_n)^r a_n$$

$$= r \int_0^\omega (\omega - t)^{r-1} A(t)\, dt,$$

where we write $A(t)$ for $A^0(t)$. With this notation the following theorem was stated and proved by Ananda Rau*.

THEOREM A. *Let* $\dfrac{a_n}{l_n - l_{n-1}} = O(l_n^\alpha)$, $\alpha + 1 \geqslant 0$ *and let* $A^r(\omega) = S\omega^r + o(\omega^\beta)$, *where* S *is a constant,* $0 < r \leqslant 1$ *and* $\beta > 0$. *Then*

$$A(\omega) - S = o\left(\omega^{\frac{\beta + \alpha r}{1+r}}\right).$$

1·1. A theorem more general in form can be proved where (a_n) satisfies the condition

$$\sum_{\nu=1}^{n} |a_\nu|^p l_\nu^p (l_\nu - l_{\nu-1})^{1-p} = O(l_n^{p(\alpha+1)+1})\dagger,$$

where $\alpha + 1 \geqslant 0$, $p > 1$.

The object of this paper is to prove that these theorems are true even when $r > 1$. The argument used is similar to that used in Theorem 22 of the Cambridge Tract on *The General Theory of Dirichlet's Series* by Hardy and Riesz‡. Some remarks are also made in the concluding section about the applications of these

* K. Ananda Rau (1) Theorem 4.

† V. Ganapati Iyer (2) Theorem 4.

‡ This will be referred to in future as the *Tract.*

Tauberian theorems to obtain certain precise results on the abcissae of summability of Dirichlet's series. These results have been anticipated by Ananda Rau*.

I wish to express my thanks to Prof. K. Ananda Rau who helped me in preparing this paper.

2. We require the following lemmas†.

LEMMA 1. *If* $k>0$, $\mu>0$, *then*

$$A^{k+\mu}(\omega)=\frac{\Gamma(k+\mu+1)}{\Gamma(k+1)\Gamma(\mu)}\int_0^\omega(\omega-t)^{\mu-1}A^k(t)dt.$$

LEMMA 2. If $0\leqslant\xi\leqslant\omega$, $k>0$ *and* $0<\mu\leqslant1$, *then*

$$\left|\frac{\Gamma(k+\mu+1)}{\Gamma(k+1)\Gamma(\mu)}\int_0^\xi A^k(t)(\omega-t)^{\mu-1}dt\right|\leqslant\max_{0\leqslant\tau<\xi}\left|A^{k+u}(\tau)\right|.$$

LEMMA 3. *Let* $r>k>0$ *and* $V(x)$ *and* $W(x)$ *be two positive non-decreasing functions of* x, $x\geqslant0$. *Suppose*

$$\phi(x)=O(V)\quad and\quad \phi_r(x)=o(W),$$

where
$$\phi_r(x)=\frac{1}{\Gamma(r)}\int_0^x\phi(t)(x-t)^{r-1}dt,$$

then
$$\phi_k(x)=o(V^{1-k/r}W^{k/r}).$$

LEMMA 4. *If* $\alpha_\nu\geqslant0$, $\beta_\nu\geqslant0$, $1/p+1/q=1$

then
$$\Sigma\alpha_\nu\beta_\nu\leqslant(\Sigma\alpha_\nu^p)^{1/p}(\Sigma\beta_\nu^q)^{1/q}.$$

2·1. We shall now deduce a formula that will be used in this section. Let n be a positive integer and $y>x>0$. Then

$$A^n(y)-A^n(x)=n\int_x^y A^{n-1}(t)dt$$

$$A^n(y)-A^n(x)-n(y-x)A^{n-1}(x)$$
$$=n\int_x^y A^{n-1}(t)dt-n\int_x^y A^{n-1}(x)dt$$
$$=n\int_x^y\left\{A^{n-1}(t)-A^{n-1}(x)\right\}dt$$

* K. Ananda Rau (1), first foot note on p. 432.

† For the proofs of Lemmas 1 and 2 see the *Tract* pp. 27-29 (Lemmas 6 and 8); for the proof of Lemma 3 see (3). Lemma 4 is Hölder's Inequality.

$$=n(n-1)\int_x^y dt\int_x^t A^{n-2}(u)du$$

$$=n(n-1)\int_x^y A^{n-2}(u)du\int_u^y dt$$

$$=n(n-1)\int_x^y (y-u)A^{n-2}(u)du.$$

More generally it can be seen by induction that if μ is an integer $<n$

$$A^n(y)-A^n(x)-n(y-x)A^{n-1}(x)-\ldots$$
$$-\frac{n(n-1)\ldots(n-\mu+1)}{1.2\ldots\mu}A^{n-\mu}(x)$$

$$=\frac{n(n-1)\ldots(n-\mu+1)}{1.2\ldots\mu}\int_x^y (y-t)^\mu A^{n-\mu-1}(t)dt.$$

Putting $\mu=n-1$ we have the important formula

$$A^n(y)=A^n(x)+\binom{n}{1}(y-x)A^{n-1}(x)+\ldots$$

$$+\binom{n}{n-1}(y-x)^{n-1}A^1(x)\ +n\int_x^y (y-t)^{n-1}A(t)dt. \qquad (1)$$

2·2. We now proceed to the proof of the theorem in view.

THEOREM 1. *Let* $\sum_{\nu=1}^{\infty} a_\nu$ *be a given infinite series and* $1\leqslant l_1<l_2<\ldots$ *a sequence tending to infinity. Also let*

$$\frac{a_n}{l_n-l_{n-1}}=O(l_n^\alpha),\ \alpha+1\geqslant 0$$

and
$$A^r(\omega)=S\omega^r+o(\omega^\beta),$$

where $r>0$, $\beta>0$ *and* S *is a constant. Then*

$$A^k(\omega)=S\omega^k+o(\omega^{\rho_k}),$$

where $0\leqslant k\leqslant r$ *and* $\rho_k=\beta-\dfrac{\beta-\alpha}{r+1}(r-k)$.

PROOF: Without loss of generality we can assume that $S=0$. Also let

$$\rho=\rho_0=\beta-\frac{\beta-\alpha}{r+1}r=\frac{\alpha r+\beta}{r+1}.$$

The two cases where (i) $\beta>r+\alpha+1$ and (ii) $\beta\leqslant r+\alpha+1$, $0<r\leqslant 1$ have been considered in the proof of Theorem A.

So it is enough if we consider the case (iii) $\beta\leqslant r+\alpha+1$, $r>1$.

Put $\lambda = \frac{\beta-\alpha}{r+1}$ so that $\lambda \leqslant 1$ and let n be the greatest integer just less than r. Also let $l_m \leqslant \omega < l_{m+1}$.

By hypothesis and by Lemma 1

$$A^r(l_m+H\omega^\lambda) = \frac{\Gamma(r+1)}{\Gamma(n+1)\Gamma(r-n)} \int_0^{l_m+H\omega^\lambda} (l_m+H\omega^\lambda-t)^{r-n-1} A^n(t)\,dt$$
$$= o(\omega^\beta),$$

where H is a constant positive and <1. Also by Lemma 2

$$\left| \frac{\Gamma(r+1)}{\Gamma(n+1)\Gamma(r-n)} \int_0^{l_m} (l_m+H\omega^\lambda-t)^{r-n-1} A^n(t)\,dt \right|$$
$$\leqslant \max_{0\leqslant\tau\leqslant\omega} A^r(\tau) = o(\omega^\beta).$$

Hence, we have

$$\frac{\Gamma(r+1)}{\Gamma(n+1)\Gamma(r-n)} \int_{l_m}^{l_m+H\omega^\lambda} (l_m+H\omega^\lambda-t)^{r-n-1} A^n(t)\,dt = o(\omega^\beta).$$

If we use (1), with $x=l_m$ and $y=t$ the above relation becomes

$$\frac{\Gamma(r+1)}{\Gamma(n+1)\Gamma(r-n)} \sum_{\mu=0}^{n-1} \binom{n}{\mu} A^{n-\mu}(l_m) \int_{l_m}^{l_m+H\omega^\lambda} \Big\{(t-l_m)^\mu$$
$$\times (l_m+H\omega^\lambda-t)^{r-n-1}\Big\}dt$$
$$+\frac{n\Gamma(r+1)}{\Gamma(n+1)\Gamma(r-n)} \int_{l_m}^{l_m+H\omega^\lambda} (l_m+H\omega^\lambda-t)^{r-n-1}$$
$$\times \left\{\int_{l_m}^t (t-u)^{n-1} A(u)\,du\right\}dt$$
$$= o(\omega^\beta),$$

which, after simplification* becomes

* The first n terms of (2) are obtained by the application of the result

$$\int_a^b (t-a)^\mu (b-1)^{r-n-1} dt = \frac{\Gamma(\mu+1)\Gamma(r-n)}{\Gamma(r-n+\mu+1)} (b-a)^{r-n+\mu};$$

while the $(n+1)$th term follows from

$$\int_{l_m}^{l_m+H\omega^\lambda} (l_m+H\omega^\lambda-t)^{r-n-1} dt \int_{l_m}^t (t-u)^{n-1} A(u)\,du$$
$$= \int_{l_m}^{l_m+H\omega^\lambda} A(u)\,du \int_u^{l_m+H\omega^\lambda} (t-u)^{n-1} (l_m+H\omega^\lambda-t)^{r-n-1} dt.$$

$$\sum_{\mu=0}^{n-1}\frac{\Gamma(r+1)}{\Gamma(n-\mu+1)\Gamma(r-n+\mu+1)}(H\omega^{\lambda})^{r-n+\mu}A^{n-\mu}(l_m)$$
$$+r\int_{l_m}^{l_m+H\omega^{\lambda}}(l_m+H\omega^{\lambda}-t)^{r-1}A(t)\,dt=o(\omega^{\beta}). \qquad (2)$$

This may also be written as

$$\sum_{\mu=0}^{n-1}\frac{\Gamma(r+1)\quad H^{r-n+\mu}}{\Gamma(n-\mu+1)\Gamma(r-n+\mu+1)}\;\omega^{\lambda(r-n+\mu)}A^{n-\mu}(l_m)+H^{r}\omega^{\lambda r}A(\omega)$$
$$=o(\omega^{\beta})-r\int_{l_m}^{l_m+H\omega^{\lambda}}[A(t)-A(\omega)](l_m+H\omega^{\lambda}-t)^{r-1}dt. \qquad (2')$$

But by hypothesis, it can be observed, that

$$A(t)-A(\omega)=O[\omega^{\alpha}(t-\omega)].$$

For, if $\quad l_{m+h}\leqslant t<l_{m+h}+1$

$$|A(t)-A(\omega)|\leqslant\sum_{\nu=1}^{h}|a_{m+\nu}|$$
$$\leqslant K\sum_{\nu=\alpha}^{h}l^{\alpha}_{m+\nu}(l_{m+\nu}-l_{m+\nu-1})$$
$$\leqslant K\,\omega^{\alpha}(t-l_m)^{*}.$$

Using this the right side of (2′) becomes

$$=o(\omega^{\beta})+O\left\{\omega^{\alpha}\int_{l_m}^{l_m+H\omega^{\lambda}}(t-l_m)(l_m+H\omega^{\lambda}-t)^{r-1}dt\right\}$$
$$=o(\omega^{\beta})+O(\omega^{\alpha+\lambda(r+1)})$$
$$=O(\omega^{\beta}).$$

Thus we have the order equation

$$\sum_{\mu=0}^{n-1}\frac{\Gamma(r+1)\quad H^{r-n+\mu}}{\Gamma(n-\mu+1)\Gamma(r-n+\mu+1)}\omega^{\lambda(r-n+\mu)}A^{n-\mu}(l_m)$$
$$+H^{r}\omega^{\lambda r}A(\omega)=O(\omega^{\beta}). \qquad (3)$$

* Clearly $\omega\leqslant t\leqslant(1+H)\omega\leqslant 2\omega$, if $H<1$.

If $\alpha<0$, $2^{\alpha}\omega^{\alpha}\leqslant l^{\alpha}_{m+\nu}\leqslant\omega^{\alpha}$, $\nu=1,2,\dots h$ since $\omega\leqslant l_{m+r}\leqslant 2\omega$.

If $\alpha>0$, $\omega^{\alpha}\leqslant l^{\alpha}_{m+\nu}\leqslant 2^{\alpha}\omega^{\alpha}$.

In any case $l^{\alpha}_{m+\nu}=O(\omega^{\alpha})$.

K is used to denote a constant independent of H and ω.

Supposing we now give $n+1$ distinct values to H, viz.— $H_0, H_1, \ldots, H_n$, so that $H_\mu \neq H_\nu$ if $\mu \neq \nu$, we get $n+1$ relations of the form (3). The $n+1$ linear expressions of the form on the left side of (3)—each of which is $O(\omega^\beta)$—can be easily seen to be linearly independent; for the determinant of the matrix formed by the coefficients of the functions of ω (since l_m is defined by ω, $A^{n-\mu}(l_m)$ is regarded as a function of ω) is

$$\begin{vmatrix} \frac{\Gamma(r+1)}{\Gamma(n+1)\Gamma(r-n+1)} H_0^{r-n} & \cdots & \frac{\Gamma(r+1)\, H_0^{r-n+\mu}}{\Gamma(n-\mu+1)\Gamma(r-n+\mu+1)} & \cdots & H_0^r \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{\Gamma(r+1)}{\Gamma(n+1)\Gamma(r-n+1)} H_\nu^{r-n} & \cdots & \frac{\Gamma(r+1)\, H_\nu^{r-n+\mu}}{\Gamma(n-\mu+1)\Gamma(r-n+\mu+1)} & \cdots & H_\nu^r \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{\Gamma(r+1)}{\Gamma(n+1)\Gamma(r-n+1)} H_n^{r-n} & \cdots & \frac{\Gamma(r+1)\, H_n^{r-n+\mu}}{\Gamma(n-\mu+1)\Gamma(r-n+\mu+1)} & \cdots & H_n^r \end{vmatrix}$$

$$= \frac{[\Gamma(r+1)]^{n+1} \prod_{\nu=0}^{n} H_\nu^{r-n}}{\prod_{\mu=0}^{n} \Gamma(n-\mu+1)\Gamma(r-n+\mu+1)} \begin{vmatrix} 1 & \cdots & H_0^\mu & \cdots & H_0^n \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & \cdots & H_\nu^\mu & \cdots & H_\nu^n \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & \cdots & H_n^\mu & \cdots & H_n^n \end{vmatrix}$$

$$\neq 0,$$

since $H_\mu \neq H_\nu$, when $\mu \neq \nu$.

Hence every term in the linear combination of (3) is $O(\omega^\beta)$. In particular

$$A(\omega) = O(\omega^{\beta - \lambda r}) = O(\omega^\rho).$$

Hence, by Lemma 3,

$$A^k(\omega) = o(\omega^{\rho k}), \quad 0 < k \leqslant r.$$

But we have yet to prove that $\quad A(\omega) = o(\omega^\rho)$.

For this, take a $k < 1$, and we have since $A^k(\omega) = o(\omega^{\rho k})$, by the application of Theorem A

$$A(\omega) = o\left(\omega^{\frac{\alpha k + \rho k}{k+1}}\right) = o(\omega^\rho)^*.$$

* $\rho k = \beta - \frac{\beta - \alpha}{r+1}(r-u) = \rho + k\lambda$. So

$$\frac{\rho k + \alpha k}{k+1} = \frac{\rho + k\lambda + \alpha k}{k+1} = \frac{\rho + k\rho}{k+1} = \rho.$$

Thus the proof of the theorem is complete.

2·3 THEOREM 2. *If*

$$\sum_{\nu=1}^{n} |a_\nu|^p l_\nu^p (l_\nu - l_{\nu-1})^{1-p} = O(l_n^{p(\alpha+1)+1})$$

and $A^r(\omega) = S\omega^r + o(\omega^\beta)$ *where* $\alpha+1 \geqslant 0$, $p>1$, $r>0$ *and* $l_0=0$, *then*

$$A^k(\omega) = S\omega^k + o(\omega^{\rho_k}),$$

where $\rho_k = \beta - \dfrac{\beta-\alpha-1/p}{r+1-1/p}(r-k)$ *and* $0 \leqslant k \leqslant r$.

This theorem can be proved arguing substantially in the same manner as in Theorem 1. Only we have to observe that

$$\lambda = \frac{\beta-\alpha-1/p}{r+1/q}, \quad 1/p+1/q=1$$

and

$$A(t) - A(\omega) = O[\omega^{\alpha+1/p}(t-l_m)^{1/q}]. \tag{4}$$

In fact, if $l_m \leqslant \omega < l_{m+1}$ and $l_{m+h} \leqslant t < l_{m+h+1}$

$$|A(t) - A(\omega)| \leqslant \sum_{\nu=1}^{h} |a_{m+\nu}|$$

$$= \sum_{\nu=1}^{h} |a_{m+\nu}| \, l_{m+\nu}(l_{m+\nu} - l_{m+\nu-1})^{(1-p)/p} \, \frac{(l_{m+\nu} - l_{m+\nu-1})^{1/q}}{l_{m+\nu}}$$

$$\leqslant \left\{ \sum_{\nu=1}^{h} |a_{m+\nu}|^p l_{m+\nu}^p (l_{m+\nu} - l_{m+\nu-1})^{1-p} \right\}^{1/p}$$

$$\times \left\{ \sum_{\nu=1}^{h} \frac{(l_{m+\nu} - l_{m+\nu-1})}{l_{m+\nu}^q} \right\}^{1/q}$$

$$\leqslant K \frac{l_{m+h}^{\alpha+1+1/p}}{l_{m+1}} (t-l_m)^{1/q}$$

$$\leqslant K\omega^{\alpha+1/p}(t-l_m) \quad \text{since } t \leqslant 2\omega.$$

NOTE. If $p=1$, $1/q=0$, (4) will still hold good, so that from (3) we shall have

$$A(\omega) = O(\omega^\rho)$$

and

$$A^k(\omega) = o(\omega^{\rho_k}) \quad 0 < k \leqslant r.$$

But we cannot assert that $A(\omega)=o(\omega^{\rho})$ if $p=1$, $1/q=0$*. If the 'O' condition in the hypothesis of Theorem 2 is replaced by 'o' condition, we can assert $A(\omega)=o(\omega^{\rho})$ otherwise

$$A(\omega)=O(\omega^{\rho}).$$

3. THEOREM 3. *Let* $\sum_{n=1}^{\infty}\frac{a_n}{l_n^s}$ *be a Dirichlet's series,* $s=\sigma+it$ *and* $\frac{a_\alpha}{l_n-l_{n-1}}=O(l_n^\alpha)$ *and let* $\sum_{n=1}^{\infty} a_n l_n^{-\sigma_r}$ *be summable* (l, r) *then the Dirichlet's series is summable* (l, k) $0\leqslant k\leqslant r$ *for*

$$\sigma>\frac{(\alpha+1)(r-k)+\sigma_r(k+1)}{r+1}$$

and for†

$$s=\frac{(\alpha+1)(r-k)+\sigma_r(k+r)}{r+1}.$$

The first part of this theorem has been proved by Ananda Rau‡. Using the same argument as in Theorem 8 of his paper and applying Theorem 1 of this paper it can be proved that the Dirichlet's series is summable at the real point on the line

$$\sigma=\frac{(\alpha+1)(r-k)+\sigma_r(k+1)}{r+1}.$$

NOTE. It follows from the above that if σ_r is the abcissa of summability (l, r) of the Dirichlet's series

$$\sigma_k\leqslant\frac{(\alpha+1)(r-k)+\sigma_r(k+1)}{r+1},\quad 0\leqslant k\leqslant r.$$

That this inequality is the best possible one can be seen from the following example. It is known that the abcissa of summability (n, r) of the series

$$1-\frac{1}{2^s}+\frac{1}{3^s}-\frac{1}{4^s}+\dots$$

is $-r$. That is $\sigma_r=-r$. Also $l_n=n$, $a_n=(-1)^{n-1}$ so that $\alpha=0$. Now applying Theorem 3, we observe that

$$\sigma_k\leqslant\frac{(r-k)-r(k+1)}{r+1}=-k.$$

* In para 7 of his paper (2) Ganapathy Iyer points that the proof of his Theorem 4 and therefore Theorem 5, breaks down if $p=1$. From what has been said above it follows that Theorem 5 is true if $k>0$ and if $k=0$ 'O' takes the place of 'o'.

† The series is in fact summable (l, k) at every point on the line

$$\sigma=\frac{(\alpha+1)(r-k)+\sigma_r(k+1)}{r+1}$$

if $\alpha+1-\sigma_r\neq0$.

‡ K. Ananda Rau (1) Theorems 7 and 8.

But, in fact,

$$\sigma_k = -k.$$

Hence the inequality is the best possible one.

Similarly we may prove

THEOREM 4. *Let* $\sum_{\nu=1}^{\infty} a_\nu l_\nu^{-s}$ *be a Dirichlet's series,* $s=\sigma+it$, *summable* (l, r) *at the real point* σ_r, *and let*

$$\sum_{\nu=1}^{n} |a_\nu|^p l_\nu^p (l_\nu - l_{\nu-1})^{1-p} = O(l_n^{p(\alpha+1)+1})$$

$p>1$, $\alpha+1+1/p \geqslant 0$. *Then, the series is summable* (l, k), $0 \leqslant k \leqslant r$ *for*

$$\sigma > \frac{(\alpha+1)(r-k)+\sigma_r(k+1/q)}{r+1/q}, \quad 1/p+1/q=1$$

and for $\quad s = \dfrac{(\alpha+1)(r-k)+\sigma_r(k+1/q)}{r+1/q}.$

NOTE. If $\alpha+1-\sigma_r \neq 0$, the Dirichlet's series is summable at every point on the line

$$\sigma = \frac{(\alpha+1)(r-k)+\sigma_r(k+1/q)}{r+1/q}.$$

In any case, if σ_r is the abcissa of summability (l, r), then

$$\sigma_k \leqslant \frac{(\alpha+1)(r-k)+\sigma_r(k+1/q)}{r+1/q}, \quad 0 \leqslant k \leqslant r.$$

There seems to be little doubt that the above inequality is the best possible one.

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PROJECTIVE TRANSFORMATIONS AND THE HAMILTONIAN

By K. NAGABUSHANAM, M.A., Andhra University, Waltair

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. In a previous paper* I have defined the notions of time, the Hamiltonian and the Lagrangian for a given time measure in terms of a Pfaffian $X_i dx^i$ of class $2n+1$ in the $2n+1$ variables of the manifold of states and time, representing the action of a dynamical system in an infinitesimal displacement. Adopting the the same point of view, I here derive invariant expressions for the kinetic and the potential energies of the system by restricting the admitted transformations to the projective subgroup of the general analytic transformations.

A repeated index stands for summation; the ranges of variation of the indices are as follows:

$$i: 1 \text{ to } 2n+1, \quad j, j': 1 \text{ to } 2n, \quad r: 1 \text{ to } n.$$

I briefly recall the notions of the Hamiltonian and the Lagrangian. The Hamiltonian H for any given time measure t is defined by the condition that the class of $X_i dx^i + Hdt$ shall be $2n$. The Lagrangian is given invariantly by the expression

$$L = X_i \xi^i \Big/ \frac{\partial t}{\partial x^k} \xi^k,$$

where (ξ^i) denotes a contravariant vector everywhere co-directional with the trajectories.

This definition of the Hamiltonian does not lead to a unique function; the extent of its arbitrariness under the general analytic transformations is very great indeed. For example, if we consider the transformations induced in the $2n+1$ manifold by

$$q^r = q^r(\bar{q}^r, \bar{t})$$
$$t = t(\bar{q}^r, \bar{t})$$

the action form $p_r dq^r - Hdt$, to which the form $X_i dx^i$ can be reduced, is transformed invariantly into $\bar{p}_r d\bar{q}^r - \bar{H} d\bar{t}$. It is easy to

* 'On the form $p_r dq_r - Hdt$', *Proc. Indian Ac. Sci.* A. 1 (1935), 555-561.

see that any such $\bar{H}$ is a Hamiltonian, if $\bar{t}$ is a measure of time. But if we confine ourselves to the projective subgroup in which the 'gauge variable'* is always a time measure

$$\begin{aligned} \bar{t} &= t+\rho(x^1, x^2, \ldots, x^{2n}) \\ \bar{x}^j &= \bar{x}^j(x^1, x^2, \ldots, x^{2n}), \end{aligned} \tag{1}$$

the definition of the Hamiltonian yields an invariant when we make reasonable assumptions.

For a given set of t and H the form $X_i dx^i + H dt$ of class $2n$ can be expressed in terms of $2n$ independent variables (x^j). The variables (x^j, t) will be referred to as constituting a specialised co-ordinate system. We assume that the scheme of projective co-ordinate systems to be employed here includes at least one specialised system. By this we make provision for the system $(q^1, q^2, \ldots, q^n, p_1, p_2, \ldots, p_n, t)$ in which the action has the form $p_r dq^r - H dt$ as an allowed system of co-ordinates. Thus the variables of any co-ordinate system can be supposed to be derived from those of a specialised one by the projective transformations (1). In a specialised system of variables (x^j, t) the action form can be written as

$$X_j dx^j - H dt.$$

Let $(\bar{x}^j, \bar{t})$ be the variables of any other co-ordinate system related to those of the specialised system by (1). If the action form becomes $\bar{X}_j\, d\bar{x}^j + \bar{X}_{2n+1} d\bar{t}$, we have

$$\bar{X}_j = X_{j'} \frac{\partial x^{j'}}{\partial \bar{x}^j} - H \frac{\partial t}{\partial \bar{x}^j} \tag{2}$$

$$\bar{X}_{2n+1} = X_{j'} \frac{\partial x^{j'}}{\partial \bar{t}} - H \frac{\partial t}{\partial \bar{t}} = -H. \tag{3}$$

As our present object is to seek an invariant function for H so that it represents the total energy of the system, it is natural to suppose that H is independent of t, for then only does H represent the energy. Then the equations (2) show that

$$\bar{X}_j = \bar{X}_j(x^1, x^2, \ldots, x^{2n}). \tag{4}$$

By the definition of a time measure, the substitution $\bar{t}=$constant, $d\bar{t}=0$, in $\bar{X}_i d\bar{x}^i$ lowers the class by one unit exactly; and therefore $\bar{X}_j d\bar{x}^j$ is of class $2n$. It then becomes clear that $-\bar{X}_{2n+1}$ is a

* This term is used as the equivalent of the German *Eichvariable*. See Veblen, *Projective Relativitats-theorie*, p. 10.

Hamiltonian for the time measure $\bar{t}$. Denoting this by $\bar{H}$, we have by (3)

$$\bar{H} = -\bar{X}_{2n+1} = H, \tag{5}$$

which shows that the Hamiltonian is an invariant.

The assumptions made in this connection may be recounted.

1. The transformations are restricted to the projective subgroup, with time as the gauge variable.

2. The co-ordinate systems employed include at least one specialised system.

3. The Hamiltonian in the specialised variables is independent of time.

The invariance of H can be exhibited in a more vivid form. In the projective co-ordinate systems there exists a unique contravariant vector field $\lambda^i = (0, 0, \ldots, 0, 1)$, having the same components in all systems. The vector field constitutes the transversals of the hypersurfaces t=constant, and is everywhere tangential to the parametric lines of t. The invariant $X_i\lambda^i$ always gives X_{2n+1}. Thus we may write

$$H = -X_{2n+1} = -X_i\lambda^i.$$

This equation can be interpreted as the invariance of the time-component of action.

In the specialised system of variables H is assumed to be independent of t, and hence the covariant vector (X_i) is independent of t. It can be proved* that in this case H is an integral of motion. Under the transformations allowed here the Hamiltonian is an invariant; and it will therefore continue to be an integral of motion in all projective co-ordinate systems employed.

If we put

$$H = T + V, \text{ and } L = T - V,$$

where T and V are the kinetic and the potential energies, we get the two kinds of energy uniquely defined for a given time measure. It may be mentioned that L changes from one projective co-ordinate system to another, for t changes. Therefore the kinetic and potential energies vary in the different co-ordinate systems, but their sum remains the same.

Out of the great variety of time measures conceived of, those connected with one another as the gauge variables of projective co-ordinate systems seem to be specially significant from the physial point of view, because these help to define the two energies of the system uniquely.

* See § 6, p. 559, of the paper of the author cited before.

ON THE PEDAL QUARTICS OF A QUADRIC

By K. RANGASWAMI, M.Sc.

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1. In a previous paper* the author has discussed at length the classical theory of normals to a system of confocal quadrics and has studied in detail certain loci of points and lines which arise in considering the relations of a linear complex to a quadric Q of the system singled out. To establish continuity with the above paper it is found desirable to retain the same terminology. Further, the results apply equally to Euclidean or Non-Euclidean space so long as a particular, though unspecified member of the quadric system, is taken as the Absolute of the the metric of space.

2. The tetrahedron Δ apolar† to the quadrics of the confocal system has been termed the *fundamental tetrahedron*; further the quadrics of the net which are apolar to Δ and outpolar to Q will be termed the *fundamental quadrics* associated with Q, and the net of quartics cut out by them on Q the *fundamental quartics* of Q. It is shown in the previous paper that the locus of points whose polar planes with respect to Q and the nul-planes in a linear complex C are orthogonal is the Apollonian quadric H associated with C. Such a quadric circumscribes the fundamental tetrahedron and cuts out on Q a *pedal quartic* which is the locus of points on Q the normals at which belong to the linear complex C. It is the purpose of this paper to study the pedal quartics from the point of view of their double binary apolarity specification.

3. Since the Apollonian quadrics pass through the four fundamental points, they are outpolar to every quadric of the confocal system and hence they constitute a linear ∞^5—system of equilateral‡ quadrics. Further since every Apollonian quadric cuts out on Q a pedal quartic we have a linear ∞^5—system of pedal quartics on Q. We thus see that

The pedal quartics on Q are cut out by the linear ∞^5—system of Apollonian quadrics, namely, the equilateral quadrics which pass through the fundamental points. (3·1)

* The Theory of Normals to a Quadric in Hyperspace, *Proc. Ind. Acad. Sci.* I. 931-51.

† To mean having Δ for a self polar tetrahedron.

‡ An equilateral quadric is outpolar to the Absolute quadric.

Now the pedal quartics on Q form a linear ∞^5—system. The system of quartics apolar to this must be a linear ∞^2—system. Let H be an Apollonian quadric and Q' any quadric apolar to Δ. Then h, the quartic of intersection of Q and H is the contact quartic of Q and the polar reciprocal, S, of H with respect to Q. If Q'' is the quadric in the pencil determined by Q and Q' outpolar to Q, it follows that Q'' is outpolar to S. Hence by the known result*, viz.

If a quadric Q_1 is outpolar to the quadrics Q and Q_2, the intersection quartic of Q and Q_1 is apolar to the contact quartic of Q and Q_2 (3·2)

the quartic of intersection of Q'' and, therefore, of Q' with Q is apolar to the pedal quartic h. Thus a quadric Q' apolar to Δ interesects Q in a quartic apolar to the linear ∞^5—system of pedal quartics. Since we may choose Q' outpolar to Q without prejudice to the intersection with Q, the apolar system is the net of fundamental quartics. Thus we have the following apolarity specification of pedal quartics on Q, viz.

The linear ∞^5—system of pedal quartics on Q are apolar to the net of fundamental quartics. (3·3)

4. *Pedal Cubics on a Quadric.*

Let an Apollonian quadric H cut Q in a generator g and a twisted cubic G. Then G is a *pedal cubic* and in this case the pedal quartic h splits up into the pedal cubic G and the generator g. The three linearly independent class quadrics inscribed in G can be taken to be Q, the Steiner† quadric S and another quadric T which is not contained in the tangential pencil determined by Q and S. Now, among the net of fundamental quadrics of Q there is only a pencil γ of quadrics also outpolar to T. Since the quadrics of this pencil are also outpolar to Q and S, we have by (3·2) the result, that *the pedal cubic G is apolar to the pencil γ of fundamental quartics cut out on Q by the fundamental quadrics of the pencil γ.*

Conversely suppose that some cubic G on Q is apolar to a pencil γ of fundamental quartics. Since the six linearly independent Steiner quadrics are inpolar to the net of fundamental quadrics they are, therefore, inpolar to the quadrics of the pencil γ. Further since, by hypothesis, there are three linearly independent

* *Cubic Transformations associated with a desmic system* by Dr. R. Vaidyanathaswamy, p. 79.

† The reciprocal of an Apollonian quadric H with respect to Q is a Steiner quadric.

class quadrics say Q, T, T' inscribed in the cubic and also inpolar to the quadrics of the pencil γ, we have, therefore, nine linearly independent class quadrics inpolar to the quadrics of the pencil γ. This is impossible, so that there is a Steiner quadric S inscribed in the cubic. Consequently there is an Apollonian quadric H through G and thus G is a pedal cubic. We have, therefore, the result:

Any twisted cubic on Q is a pedal cubic if it is apolar to a pencil of fundamental quartics of Q. (4·1)

Through every generator of Q there pass ∞^2 Apollonian quadrics and therefore, cut out on Q ∞^2 different pedal cubics. Thus corresponding to the two systems of generators on Q we have two systems of ∞^3 pedal cubics of which the cubics of one system meet the generators of one system on Q in two points while those of the other system meet these generators in only one point. Further, as there are only ∞^2 pencils of fundamental quartics of Q it follows from (4·1) that there are two sets of ∞^1 pedal cubics which are apolar to the same pencil of fundamental quartics.

Now, it is easily seen that a degenerate pedal quartic consisting of a twisted cubic and a generator is also subrational*; for such a quartic posseses the characteristic property of admitting ∞^1 inscribed generator quadrangles of Q. Also it is shown§ in the paper refered to at the begining that in every quadric ϕ apolar to Δ there is a unique pair of inscribed tetrahedra Δ_1, Δ_2 forming with Δ a desmic system; further, when ϕ is one of the fundamental quadrics of Q, the vertices of the tetrahedra Δ_1, Δ_2 also lie on Q. Since any two quadrics† of different desmic nets are doubly apolar, the Apollonian quadrics through (Δ, Δ_1), (Δ, Δ_2) are doubly apolar to Q and hence cut out on Q subrational pedal quartics. Further, since the Apollonian quadrics through (Δ, Δ_1), (Δ, Δ_2) are invariant for the transformations‡ Γ_1, Γ_2 and Q is circumscribed to Δ_1, Δ_2, it follows that:

The cubics which are the transforms of the generators of one system on Q in Γ_1, and those of the second system in Γ_2 are pedal cubics. (4·2)

* Subrational pedal quartics on Q are those which are cut out on Q by Apollonian quadrics doubly apolar to Q.

§ *Pro. Ind. Acad. Sc.* loc. cit. p. 946.

† *Cubic Trans.* etc. . . . loc. cit. p. 51.

‡ The transformation Γ_1 has the vertices of Δ_1 for singular points and the eight vertices of two tetrahedra Δ, Δ_2 for fixed points. Similarly for Γ_2.

Let G be a pedal cubic and H the unique Apollonian quadric through it. It is known from the theory of triads of desmic tetrahedra that the net of quadrics through the vertices of Δ_1, Δ_2 are apolar to Δ. Let γ be the pencil of quadrics in this net also outpolar to Q. As there are two class quadrics Q and S inscribed in G and also inpolar to the quadrics of the pencil γ, every class quadric inscribed in G is inpolar to the quadrics of the pencil γ. In other words G is apolar to the fundamental quartics of the pencil γ. Further, the cubics which are the transforms of one system of generators on Q, say, in Γ_1 all pass through the vertices of Δ_1 and hence belong to a pencil. Similar results hold good with regard to the cubics through Δ_2. We have thus the result:

The two families of ∞^1 pedal cubics which are apolar to the same pencil of fundamental quartics on Q belong to two pencils and are the transformations in Γ_1, Γ_2 of one or other system of generators on Q. (4·3)

A NOTE ON STRAZZERI'S FORMULA IN RECTILINEAR CONGRUENCES

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1. Consider a rectilinear congruence defined by the co-ordinates (x, y, z) of a point M on the surface of reference S and by the direction cosines (X, Y, Z) of the line l passing through M, where x, y, z; X, Y, Z are functions of the two parameters u and v.

The co-ordinates of any point P of the line l are given by $\xi = x + tX$, $\eta = y + tY$, $\zeta = z + tZ$, where t is the distance of P measured from the point M.

The object of this note is to give a simple proof of Strazzeri's formula $\cos\theta . dS/d\sigma = \rho_1\rho_2$, where ρ_1 and ρ_2 are the focal distances of l, θ is the angle which l makes with the normal to the surface of reference at M, dS is the element of area of the surface of reference at M, and $d\sigma$ is the element of area of the spherical representation.

2. If ρ_1 and ρ_2 are the focal distances of l we have

$$\rho_1\rho_2 = \frac{eg - ff'}{\mathbf{EG} - \mathbf{F}^2}, *$$

where

$$\mathbf{E} = \Sigma X_1^2,\ \mathbf{F} = \Sigma X_1 X_2,\ \mathbf{G} = \Sigma X_2^2;$$

$$e = \Sigma x_1 X_1,\ f = \Sigma x_2 X_1,\ f' = \Sigma x_1 X_2,\ g = \Sigma x_2 X_2,$$

the subscripts 1 and 2 denoting differentiation with regard to u and v respectively. Hence $\rho_1\rho_2$

$$= \frac{1}{\mathbf{EG} - \mathbf{F}^2} \begin{vmatrix} e & f \\ f' & g \end{vmatrix} = \frac{1}{\mathbf{EG} - \mathbf{F}^2} \begin{vmatrix} \Sigma X_1 x_1 & \Sigma X_1 x_2 \\ \Sigma X_2 x_1 & \Sigma X_2 x_2 \end{vmatrix}$$

$$= \frac{1}{\mathbf{EG} - \mathbf{F}^2} \left\| \begin{matrix} x_1 y_1 z_1 \\ x_2 y_2 z_2 \end{matrix} \right\| . \left\| \begin{matrix} X_1 Y_1 Z_1 \\ X_2 Y_2 Z_2 \end{matrix} \right\|$$

$$= \frac{1}{\sqrt{\mathbf{EG} - \mathbf{F}^2}} \left\| \begin{matrix} X_1 Y_1 Z_1 \\ X_2 Y_2 Z_2 \end{matrix} \right\| \times \frac{1}{\sqrt{EG - F^2}} \left\| \begin{matrix} x_1 y_1 z_1 \\ x_2 y_2 z_2 \end{matrix} \right\| \times \frac{\sqrt{EG - F^2}}{\sqrt{\mathbf{EG} - \mathbf{F}^2}},$$

* See Eisenhart, *Differential Geometry*, p. 399.

where $E=\Sigma x_1^2,\ F=\Sigma x_1x_2,\ G=\Sigma x_2^2.$

Hence $$\rho_1\rho_2=\cos\theta\,\frac{\sqrt{EG-F^2}}{\sqrt{\mathbf{EG}-\mathbf{F}^2}},$$

where θ is the angle which l makes with the normal to the surface of reference at M, $=\cos\theta.\ dS/d\sigma$.*

If the congruence is normal, we have $\theta=0$, $\cos\theta=1$, and hence $\rho_1\rho_2=dS/d\sigma$ which corresponds to the well-known theorem of Gauss†.

3. We give below a simple proof of Strazzeri's formula.

It is known that if a variable plane remains parallel to a fixed plane and cuts a tubular ruled surface, the area of the section is a quadratic function of the distance of the variable plane from the fixed plane. Take a line l of a congruence and a thin pencil formed by lines adjacent to l. Let P be any point on l distant t from some fixed point M and let $d\Sigma$ be the area of the normal cross-section of the pencil at P.

Then $d\Sigma=K(t^2+2at+b)$. But $d\Sigma=0$ when P is at a focus. Hence if $t=t_1$, $t=t_2$ give the foci, we have $d\Sigma=K(t-t_1)(t-t_2)$, where K depends only on the choice of the pencil.

Let $t\to\infty$ so that $d\Sigma\sim Kt^2$. But if $d\sigma$ is the area of the spherical representation of the pencil, $d\Sigma\sim t^2 d\sigma$. Hence $K=d\sigma$ and $d\Sigma=(t-t_1)(t-t_2)d\sigma$.

Now if dS is the area of the section of the pencil by the surface of reference at M, then

$$d\Sigma=dS\cos\theta.$$

Hence $$dS\cos\theta=(t-t_1)(t-t_2)d\sigma,$$

or $$\frac{dS}{d\sigma}\cos\theta=(t-t_1)(t-t_2)$$

or $$\frac{dS}{d\sigma}\cos\theta=\rho_1\rho_2.$$

* Cf. Strazzeri, 'Sulle congruenze di rette', *Rendiconti del Circolo Matematico di Palermo* (1927), p. 138.

† See Eisenhart, *Differential Geometry*, p. 145; Kommerell, *Theorie der Raumkurven und Krummen flächen*, Vol. I, p. 126.

ANALOGUE OF A THEOREM OF BLASCHKE

By R. C. BOSE, Calcutta

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1. If $\mathbf{r}=\mathbf{r}(s)$ is the vector equation of a plane curve V, s denoting the affine length, then $\mathbf{r}_1=\mathbf{r}'(s)$ and $\mathbf{r}_2=\mathbf{r}''(s)$, are the vector equations of what have been called the tangent and curvature forms of V, the dashes denoting differentiation with respect to s. Let V' and V'' denote the tangent and curvature forms of V. If ρ denotes the radius of curvature at any point of V, r_1, r_2 denote the lengths of the radii vectors to the corresponding points on V' and V'', and p_1, p_2 denote the lengths of the perpendiculars from the origin to the tangents at these points then it is known that*

$$r_1=\rho^{1/3}, \tag{1}$$

$$p_2=\rho^{-1/3}. \tag{2}$$

We shall first prove the analogous formulae

$$r_2=l^{-1/3}, \quad p_1=l^{1/3},$$

where l is the semi-latus rectum of the osculating parabola to V.

Let ψ denote the angle of aberrency for V, i.e. the angle between the normal and the affine normal. Then it can be easily proved that

$$l=\rho \cos^3\psi. \tag{3}$$

The relation has been noticed by Mukhopadhyaya†.

Now the radii vectors of V'' are parallel to the affine normals of V, whereas the tangents are parallel to tangents. Hence the angle between the tangent and radius vector of V'' is the complement of ψ. Consequently

$$p_2=r_2 \sin\left(\frac{\pi}{2}-\psi\right). \tag{4}$$

From (2), (3) and (4), it follows at once, that

$$r_2=l^{-1/3}. \tag{5}$$

* W. Blaschke, *Vorlesungen über Differential Geometrie*, II, p. 32.

† S. Mukhopadhyaya, *Collected works*, Part I, pp 125 and 145.

Again from the reciprocal relation between V' and V'' we have*

$$p_1 = 1/r_2 = l^{1/3}. \tag{6}$$

2. A convex oval for which the osculating conic at every point is an ellipse, is called an elliptic convex oval. It is well known that for such an oval, the conic through any five points is an ellipse†. We shall prove the following

THEOREM. *On an elliptic convex oval we can find at least three pairs of points, such that the latera recta of the osculating parabolas are equal, and the affine normals are parallel.*

Let us identify the plane curve V, considered in the previous paragraph, with an elliptic convex oval. Then V' will be a convex oval, and the origin O will be the centre of gravity of its area‡. Hence from a theorm of mine‖ there will exist at least three pairs of parallel tangents to V', such that the members of any pair are equidistant from O. Since tangents to V' are parallel to the affine normals of V the required theorem follows from relation (6).

The theorem given here may be regarded as an analogue of Blaschke's theorem§, viz.:

On a convex oval there exist at least three pairs of points, such that the tangents are parallel and the radii of curvature equal.

* W. Blaschke, *loc. cit.* p. 30.

† P. Bohmer, *Mathematische Annalen*, 60 (1905); 256-62.
S. Mukhopadhyaya, *Math. Zeitschrift*, 30 (1929), 560-71.

‡ W. Blaschke, *loc. cit.* p 31 and p. 21.

‖ R. C. Bose, *Bull. Cal. Math. Soc.* 27 (1935), 60.

§ W. Blaschke, *Aufgabe* 540, Archiv. Math. Phys. 26 (1917), 65.
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ON SOME RESULTS INVOLVING CONFLUENT HYPERGEOMETRIC FUNCTIONS

By N. G. SHABDE, D.Sc., (Edin.), College of Science, Nagpur

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Introduction. The object of this paper is to collect a number of results involving the confluent hypergeometric functions such as the k-functions, D_n functions, Laguerre functions and the Bessel functions. The results have been obtained either by the methods of operational calculus or otherwise and are believed to be new.

The fundamental rule governing the operational representation of a function is given by

$$\phi(u)=u\int_0^\infty e^{-ut}f(t)\,dt,$$

where, the integral on the right being convergent, $\phi(p)$ is the operational representation of $f(z)$. The second rule is, if $\phi_1(p)$, $\phi_2(p)$ are the operational representations of $f_1(z)$ and $f_2(z)$, $\dfrac{\phi_1(p)\phi_2(p)}{p}$ represents $\chi(z)=\int_0^z f_1(\xi-z)f_2(\xi)\,d\xi$.

1. P. Humbert* gives the operational representation of ber x and bei x. We obtain here the operational representations of ker(x) and kei(x) functions.

We have

$$\int_0^\infty e^{-pt}K_0(t)\,dt=\frac{\cos^{-1}p}{\sqrt{1-p^2}}, \tag{1·1}$$

and

$$K_0(t\sqrt{i})=\mathrm{ker}(t)+i\,\mathrm{kei}(t). \tag{1·2}$$

This gives

$$p\sqrt{i}\int_0^\infty e^{-pt\sqrt{i}}K_0(t\sqrt{i})\,dt=\frac{p\cos^{-1}p}{\sqrt{1-p^2}}. \tag{1·3}$$

Separating the real and imaginary parts we find that

$$\mathrm{ker}(x)\doteqdot\frac{p}{\sqrt{2(1+p^4)}}\Big[\left\{\sqrt{1+p^4}-p^2\right\}^{\frac{1}{2}}\cos^{-1}\left\{\tfrac{1}{2}\sqrt{1+p^2+p\sqrt{2}}-\tfrac{1}{2}\sqrt{1+p^2-p\sqrt{2}}\right\}$$
$$-\left\{\sqrt{1+p^4}+p^2\right\}^{\frac{1}{2}}\log\left\{u+\sqrt{u^2-1}\right\}\Big] \tag{1·4}$$

* Some new operational representations, *Proc. Edin. Math. Soc.* (2) 4 (1933), 232.

and

$$\operatorname{kei}(x) \doteqdot \frac{-p}{\sqrt{2(1+p^4)}}\Big[\{\sqrt{1+p^4}+p^2\}^{\frac{1}{2}}\cos^{-1}\Big\{\tfrac{1}{2}\sqrt{1+p^2+p\sqrt{2}} - \tfrac{1}{2}\sqrt{1+p^2-p\sqrt{2}}\Big\} + \{\sqrt{1+p^4}-p^2\}^{\frac{1}{2}}\log\{u+\sqrt{u^2-1}\}\Big], \tag{1·5}$$

where $u=\frac{1}{2}\sqrt{1+p^2+p\sqrt{2}}+\frac{1}{2}\sqrt{1+p^2-p\sqrt{2}}$.

This leads us to $\int_0^x \ker(x-y)\operatorname{kei} y\,dy$

$$\doteqdot \frac{-p}{2(1+p^4)}\Big[(\cos^{-1}v)^2-\{\log(u+\sqrt{u^2-1})\}^2 - 2p^2\cos^{-1}v\log(u+\sqrt{u^2-1})\Big], \tag{1·6}$$

where $v=\frac{1}{2}\sqrt{1+p^2+p\sqrt{2}}-\frac{1}{2}\sqrt{1+p^2-p\sqrt{2}}$;

and

$$\frac{1}{\sqrt{\pi x}}\int_0^\infty \exp\left(-\frac{s^2}{4x}\right)\ker(s)\,ds$$

$$\doteqdot \sqrt{\frac{p}{2}}\,\frac{1}{\sqrt{1+p^2}}\Big[\{\sqrt{1+p^2}-p\}^{\frac{1}{2}}\cos^{-1}\Big\{\tfrac{1}{2}\sqrt{1+p+\sqrt{2p}} - \tfrac{1}{2}\sqrt{1+p-\sqrt{2p}}\Big\} - \{\sqrt{1+p^2}+p\}^{\frac{1}{2}}\log\{U+\sqrt{U^2-1}\}\Big], \tag{1·7}$$

where $U=\frac{1}{2}\sqrt{1+p+\sqrt{2p}}+\frac{1}{2}\sqrt{1+p-\sqrt{2p}}$.

2. We have

$$\left.\begin{aligned} k_{2n}(x)&=(-1)^{n-1}2x\,e^{-x}\,{}_1F_1(1-n;2;2x)\\ k_{2n}(y)&=(-1)^{n-1}2y\,e^{-y}\,{}_1F_1(1-n;2;2y).\end{aligned}\right\} \tag{2·1}$$

Hence

$$k_{2n}(x)\,k_{2n}(y)=4xy\,e^{-(x+y)}\,{}_1F_1(1-n;2;2y)\,{}_1F_1(1-n;2;2x). \tag{2·2}$$

Now ${}_1F_1(\alpha;\gamma;z)\;\;{}_1F_1(\alpha;\gamma;Z)$

$$=\sum_{r=0}^{\infty}\frac{(-1)^r(\alpha)_r(\gamma-\alpha)_r}{r!\,(\gamma)_r(\gamma)_{2r}}(zZ)^r\,{}_1F_1[\alpha+r;\gamma+2r;z+Z] \tag{2·3}$$

and

$$L_n^{(\alpha)}(x)=\frac{(1+\alpha)_n}{n!}\,{}_1F_1(-n;1+\alpha;x)^*. \tag{2·4}$$

* Bailey, "On the product of two Legendre Polynomials with different arguments", *Proc. Lond. Math. Soc.* 41 (1936), 218.

Therefore, (2·2) gives

$$k_{2n}(x)\,k_{2n}(y)$$

$$=4xy\,e^{-(x+y)}\sum_{r=0}^{\infty}\frac{(-1)^r(1-n)_r(1+n)_r}{r!\,(2)_r(2)_{2r}}(4xy)^r \times {}_1F_1[1-n+r;2+2r;2(x+y)]$$

$$=4xy\,e^{-(x+y)}\sum_{r=0}^{n-1}(4xy)^r\frac{(n-r)!\,(n-r-1)!}{(n!)^2 r!\,(r+1)!}L_{n-r-1}^{(1+2r)}(2x+2y). \tag{2·5}$$

Putting $x=y$ in this result we get

$$k_{2n}^2(x)=\sum_{r=0}^{n-1}\frac{(n-r)!\,(n-r-1)!}{(n!)^2 r!\,(r+1)!}e^{-2x}(4x^2)^{r+1}L_{n-r-1}^{(1+2r)}(4x). \tag{2·6}$$

3. We have the known results

$$k_{2n}(x)\doteqdot\frac{2p(1-p)^{n-1}}{(1+p)^{n+1}}* \tag{3·1}$$

and

$$x^{-\frac{1}{2}}D_{2m}(2\sqrt{x})\doteqdot\frac{\sqrt{\pi}(2m)!\,p(1-p)^m}{2^m\,m!\,(1+p)^{m+\frac{1}{2}}}. \tag{3·2}$$

Hence by a result in operational calculus

$$\int_0^x\frac{k_{2n}(\xi)\,D_{2m}(2\sqrt{x-\xi})\,d\xi}{(x-\xi)^{\frac{1}{2}}}$$

$$\doteqdot\left[\frac{(2m)!\sqrt{2\pi}}{2^m\,m!}\sum_{r=1}^{\infty}\frac{(-1)^{r-1}1.3\ldots(2r-3)}{2^r\,r!}\,\frac{(1-p)^{m+n+r-1}}{(1+p)^{m+n+r+1}}+\frac{(2m)!\sqrt{2\pi}}{2^m\,m!}\,\frac{(1-p)^{m+n-1}}{(1+p)^{m+n+1}}\right]. \tag{3·3}$$

Interpreting the right hand side term by term we get

$$\int_0^x\frac{k_{2n}(\xi)\,D_{2m}(2\sqrt{x-\xi})\,d\xi}{(x-\xi)^{\frac{1}{2}}}=\sqrt{2\pi}\frac{(2m)!}{2^m\,m!}k_{2(m+n)}(x)$$

$$+\sqrt{2\pi}\frac{(2m)!}{m!\,2^m}\sum_{r=1}^{\infty}\frac{(-1)^{r-1}1.3\ldots(2r-3)}{2^r\,r!}k_{2(m+n+r)}(x). \tag{3·4}$$

4. We have the known operational representation

$$D_{2m+1}(2\sqrt{x})\doteqdot\frac{\sqrt{\pi}(2m+1)!}{2^m\,m!}\,\frac{p(1-p)^m}{(1+p)^{m+3/2}}. \tag{4·1}$$

*Bateman, "The k-function—a particular case of confluent hypergeometric function", *Transactions American Math. Soc.* 33 (1931), 828.

This gives

$$D_{2m+1}(2\sqrt{x}) \doteqdot \frac{\sqrt{\pi}\,(2m+1)!}{2^m\, m!\,\sqrt{2}}\;\frac{p(1-p)^m}{(1+p)^{m+1}}$$

$$\times\left[1+\sum_{r=1}^{\infty}\frac{\{1.3\ldots(2r-3)\}(-1)^{r-1}}{2^r\, r!}\left(\frac{1-p}{1+p}\right)^r\right].$$

Interpreting the right hand side we have

$$D_{2m+1}(2\sqrt{x}) = \sqrt{\frac{\pi}{2}}\;\frac{(2m+1)!\,(-1)^{m+1}}{2^m\, m!}\Big[e^{-x}L_m(2x)$$

$$+\sum_{r=1}^{\infty}\frac{1.3\ldots(2r-3)}{2^r\, r!}e^{-x}L_{m+r}(2x)\Big]. \quad (4{\cdot}2)$$

Similarly from the operational representation of $x^{-\frac{1}{2}}D_{2m}(2\sqrt{x})$ we get

$$x^{-\frac{1}{2}}D_{2m}(2\sqrt{x}) = \frac{\sqrt{2\pi}\,(2m)!\,(-1)^m e^x}{2^m\, m!}\Big[L_m(2x)+\sum_{r=1}^{\infty}\frac{1.3\ldots(2r-1)}{2^r\, r!}$$

$$\times L_{m+r}(2x)\Big]. \quad (4{\cdot}3)$$

Using these with the orthogonal relations for $L_n(x)$ functions we have

$$\int^{\infty} e^{-x}\, x^{-\frac{1}{2}}D_{2m}(2\sqrt{x})\,L_n(2x)\,dx$$

$$\left.\begin{aligned} &=0 \quad \text{if } n<m,\\ &=\tfrac{1}{2}\,\frac{\sqrt{2\pi}(2m)!}{2^m m!}(-1)^m \quad \text{if } n=m\\ \text{and}\quad &=\tfrac{1}{2}\,\frac{\sqrt{2\pi}(2m)!\,(-1)^m}{2^m m!}\;\frac{1.3\ldots(2n-2m-1)}{2^{n-n}\,(n-m)!}\text{if } n>m.\end{aligned}\right\}\quad (4{\cdot}4)$$

Similarly

$$\left.\begin{aligned} &\int_0^{\infty} e^{-x}\, D_{2m+1}(2\sqrt{x})\,L_n(2x)\,dx\\ &=0 \quad \text{if } n<m\\ &=\tfrac{1}{2}\sqrt{\frac{\pi}{2}}\,\frac{(2m+1)!\,(-1)^{m+1}}{2^m\, m!} \quad \text{if } n=m\\ \text{and}\quad &=\tfrac{1}{2}\sqrt{\frac{\pi}{2}}\,\frac{(2m+1)!\,(-1)^{m+1}}{2^m\, m!}\;\frac{1.3.5\ldots 2n-2m-3}{2^{n-m}(n-m)!}\\ &\qquad\qquad \text{if } n>m.\end{aligned}\right\}\quad (4{\cdot}5)$$

5. For all values of n and x we have the known result

$$k_{2n}(x) \doteqdot \frac{1}{\pi \sin n\pi} \sum_{m=0}^{\infty} \frac{(-1)^{m+1} p(1-p)^m}{(m-n)(m+1-n)(1+p)^{m+1}}. \quad (5{\cdot}1)$$

Again,
$$\sqrt{\pi} p / \sqrt{1+p} \doteqdot x^{-\frac{1}{2}} e^{-x}. \quad (5{\cdot}2)$$

Hence by a formula in operational calculus

$$\int_0^x e^{-\xi} \xi^{-\frac{1}{2}} k_{2n}(x-\xi)\, d\xi$$

$$\doteqdot \frac{1}{\sqrt{\pi} \sin n\pi} \sum_{m=0}^{\infty} \frac{(-1)^{m+1} p(1-p)^m}{(m-n)(m+1-n)(1+p)^{m+3/2}}.$$

Interpreting the right hand side we get

$$\int_0^x e^{-\xi} \xi^{-\frac{1}{2}} k_{2n}(x-\xi)\, d\xi$$

$$= \frac{1}{\pi \sin n\pi} \sum_{m=0}^{\infty} \frac{(-1)^{m+1} 2^m m!\, D_{2m+1}(2\sqrt{x})}{(2m+1)!\,(m-n)\,(m+1-n)}. \quad (5{\cdot}3)$$

Goldstein* gives

$$\Gamma(k-\lambda) x^{k-1} e^{-\frac{1}{2}x} W_{\lambda,m}(x) = \int_x^{\infty} u^{\lambda-1} (u-x)^{k-\lambda-1} e^{-\frac{1}{2}u} W_{k,m}(u)\, du.$$

Converting this into k_n functions we obtain

$$\int_x^{\infty} z^{\lambda-1} (z-x)^{n-\lambda-1} e^{-z} k_{2n}(z)\, dz$$

$$= \frac{\Gamma(n-\lambda)\, \Gamma(\lambda+1)}{\Gamma(n+1)} x^{n-1} e^{-x} k_{2\lambda}(x)\,;\ n>\lambda \text{ and } x>0. \quad (5{\cdot}4)$$

* On the operational representation of Whittaker's Confluent hypergeometric functions and Weber's parabolic cylinder functions, *Proc. Lond. Math. Soc.* (2) 34 (1932), 3-125.

BIPOLAR AND TRIGEMINAL CO-ORDINATES ON A LINE

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CONTENTS.

1. *Introduction.* If the position of a variable point P on a line is to be determined by a distance from a single point A, the line is given a direction and the distance is given a sign in order that P may be distinguished from the point which is its image in A. To avert the possibility of confusing the two points without introducing direction along the line, we introduce a second base point B. As a rule, we then identify P by means of the ratio AP/PB, a ratio which needs direction to give it meaning, but is actually independent of the direction chosen and is therefore a specification of position on the undirected line. This note describes an alternative use of a pair of base points. The co-ordinates, called bipolar co-ordinates on the line, are the squares AP^2, BP^2. Direction along the line is not involved in the co-ordinates, but directed distances will be used in the proofs.

Tripolar co ordinates in a plane have been the subject of serious if clumsy investigations, but the use that can be made of the corresponding co-ordinates in studying the simplest of all configurations seems to have escaped notice. The classical correlation of circles in a plane with points in space, and the use of pentaspherical co-ordinates in the study of spheres, are immediate applications of the ideas to be introduced. The substance of § 14 was indeed determined by the chapter on pentaspherical co-ordinates in Darboux's *Principes de Géométrie Analytique*, but a new basis is suggested for the whole theory.

2. *Bipolar co-ordinates; the permanent relation and the base conic.* We write λ, μ for AP^2, BP^2 and h for the square AB^2, which is the constant of the co-ordinate system. The permanent relation between the co-ordinates is the relation between the mutual distances of three points on a line. A simple method of obtaining this relation, which is the most elementary case of the Cayley-Sylvester relation, is to remark that if O is any point of the line, then

$$BP.AO^2+PA.BO^2+AB.PO^2=-BP.PA.AB;$$

taking O at A, B, P in turn we have

$$\begin{aligned} PA.h+AB.\lambda&=-BP.PA.AB\\ BP.h\qquad+AB.\mu&=-BP.PA.AB\\ BP.\lambda+PA.\mu\qquad&=-BP.PA.AB, \end{aligned}$$

while identically

$$BP+PA+AB=0.$$

Eliminating $BP:PA:AB:BP.PA.AB$ and arranging the rows and columns in the convenient order, we have the relation required, namely

$$\begin{vmatrix} 0 & h & 1 & \lambda \\ h & 0 & 1 & \mu \\ 1 & 1 & 0 & 1 \\ \lambda & \mu & 1 & 0 \end{vmatrix}=0$$

and we recognise this as the condition for the line

$$\lambda x+\mu y+1=0$$

to touch the conic

$$hxy+x+y=0.$$

The use of bipolar co-ordinates on a line establishes a correlation between the points of the line and the tangents to a hyperbola.

We call the hyperbola the base conic, and denote it by H. The asymptotes of the hyperbola correspond to the base points A, B and the tangent at the origin corresponds to the point at infinity on the line.

3. *The representation of point pairs by points in a plane.* To two points on the line correspond in the elementary sense two tangents to the base conic. If we think of the two points not individually but as the unseparated constituents of a point pair, the correlation is with the pair of tangents, or more simply with their point of intersection.

Bipolar co-ordinates set up a correspondence between point pairs on the line and points outside the base conic.

We may look at this correlation differently. The condition which expresses that the line (λ, μ) passes through a particular point (x, y) is the linear equation $x\lambda + y\mu + 1 = 0$, and the point (x, y) in the plane corresponds to the set of points on the line whose bipolar co-ordinates satisfy this equation. If x, y are any two numbers whose sum is not zero, and if P is any point of AB, then identically

$$x.AP^2 + y.BP^2 = (x+y).GP^2 + x.AG^2 + y.BG^2,$$

where G is the mean centre of loads x at A and y at B. It follows that the equation $x\lambda + y\mu + 1 = 0$ expresses on the line AB the condition $GP^2 = \Gamma$, where G is the mean centre determined by

$$\frac{AG}{y} = \frac{GB}{x} = \frac{AB}{x+y},$$

and Γ is given by

$$\Gamma = -\frac{x.AG^2 + y.BG^2 + 1}{x+y} = -\frac{hxy + x + y}{(x+y)^2}.$$

If $hxy + x + y$ is negative, the point (x, y) is outside the hyperbola H and two tangents to the hyperbola pass through it; on the line, since Γ is positive, the condition $GP^2 = \Gamma$ is satisfied by two points, constituting a pair of which G is the midpoint and Γ may be called the radial measure. In this interpretation a point on the conic is associated with a point on the line, but the latter point must be regarded as a pair of zero radius, that is, as a pair whose components coincide; for enumerative purposes, the point on the line is double.

Since the mean centre which is the midpoint of the pair correlated with (x, y) is determined by the ratio $x : y$,

Concentric point pairs are represented by points on a line through the origin.

Since the origin is on the base conic a line u through the origin cuts the conic again at one other point U; the tangent at U is the correlate of the midpoint common to the pairs represented by points of u.

4. *Special cases of the representation.* There are two cases which have special features. If $x+y$ is zero, the relation on the line becomes

$$x.AB(AP+BP)+1=0$$

and is satisfied by one, and only one, accessible point. In the plane, (x, y) is on the tangent which corresponds to the point at infinity on the line, and there is one other tangent through (x, y). Thus while there is no geometrical peculiarity in the plane, the pair on the line is the pair which has the point at infinity for one of its components. From another point of view, omitting the fixed tangent in the plane and the point at infinity on the line, we may say that the bipolar co-ordinates set up a correspondence between points on the original line and points on the particular tangent $x\lambda+y\mu=0$.

A linear equation which is not reducible to the general form $x\lambda+y\mu+1=0$, that is, an equation of the form $x\lambda+y\mu=0$, assigns a value to the ratio $AP^2:BP^2$. Hence such a relation can be satisfied only if x and y have opposite signs, and represents then a pair of points harmonic with the pair of base points A, B. In the plane, there are two tangents with a direction determined by a given positive value of $\lambda:\mu$, and this pair of parallel tangents determines a point at infinity.

Points at infinity outside the base conic correspond to point pairs which harmonize with the pair of base points on the line.

5. *The base conic in homogeneous co-ordinates.* It is time to recognise that while our work is metrical as regards the line it is descriptive as regards the conic. If we replace the absolute co-ordinates λ, μ by $h\lambda/\nu$, $h\mu/\nu$, the permanent relation between the homogeneous co-ordinates λ, μ, ν is

$$\begin{vmatrix} 0 & 1 & 1 & \lambda \\ 1 & 0 & 1 & \mu \\ 1 & 1 & 0 & \nu \\ \lambda & \mu & \nu & 0 \end{vmatrix}=0,$$

and is the condition, relative to any triangle of reference XYZ, for the line

$$x\lambda+y\mu+z\nu=0$$

to touch the conic

$$yz+zx+xy=0.$$

The tangents to the conic at the vertices Z, X, Y of the triangle of reference correspond to the point at infinity and the two base points A, B on the line.

To a point pair on the line corresponds a point outside the conic; this point is on the tangent at Z if one constituent of the point pair is at infinity, and is on the line XY if the point pair harmonizes with the point pair A, B.

6. *The representation of point pairs by lines in a plane.* A pair of tangents to a conic is characterized as directly by the chord of contact as by the point of intersection, and in the language of pure geometry it is a matter of indifference whether the point pair is represented by a point in the plane or by the line which is the polar of that point for the base conic. But since the co-ordinates of the line are not the same as the co-ordinates of the point, analytical formulæ do depend on the choice of the representation. Or to take a more useful view of the distinction, we may expect that some formulæ in the theory of point pairs will be expressed most simply in terms of co-ordinates of the representative point, others most simply in terms of co-ordinates of the representative line.

7. *Formulae connecting a pair with the representative point and line.* The relations between the point pair (G, Γ) and the representative point and line in the plane are all implicit in the identity

(7.1) $$x.AP^2+y.BP^2+z.AB^2=(x+y)(GP^2-\Gamma),$$

which holds for an arbitrary point P on the line. Taking P at G, we have

(7.2) $$x\lambda_G+y\mu_G+z\nu_G=-(x+y)(\Gamma/h)\nu_G.$$

Taking P at A, B in turn, we have

(7.3) $$(y+z)\nu_G=(x+y)\left\{\lambda_G-(\Gamma/h)\nu_G\right\},$$

$$(z+x)\nu_G=(x+y)\left\{\mu_G-(\Gamma/h)\nu_G\right\}.$$

But the co-ordinates (λ, μ, ν) of the line which is the polar of the point (x, y, z) for the base conic are given by

$$\lambda:\mu:\nu=y+z:z+x:x+y,$$

and we have therefore

(7.4) $$\lambda:\mu:\nu=\lambda_G-(\Gamma/h)\nu_G:\mu_G-(\Gamma/h)\nu_G:\nu_G,$$

and conversely

(7.5) $$\lambda_G:\mu_G:\nu_G=\lambda+(\Gamma/h)\nu:\mu+(\Gamma/h)\nu:\nu.$$

Substituting from (7.5) in (7.2), we have

(7.6) $$\frac{\Gamma}{h}=-\frac{x\lambda+y\mu+z\nu}{2(x+y)\nu},$$

whence in terms of point co-ordinates alone,

$$\frac{\Gamma}{h} = -\frac{yz+zx+xy}{(x+y)^2}, \tag{7.7}$$

and in terms of line co-ordinates alone,

$$\frac{\Gamma}{h} = \frac{2\mu\nu+2\nu\lambda+2\lambda\mu-\lambda^2-\mu^2-\nu^2}{4\nu^2}. \tag{7.8}$$

8. *The mutual power of two point pairs.* In order to transfer to the plane the condition for two pairs on the line to harmonize, we require the condition in a form that does not introduce the individual end points. Since $G_1P_2.G_1Q_2$ is equal to $G_1G_2^2-\Gamma_2$ identically and to Γ_1 if P_2Q_2 harmonizes with P_1Q_1, a suitable form is $\pi_{12}=0$, where

$$\pi_{12} = G_1G_2^2-\Gamma_1-\Gamma_2. \tag{8.1}$$

In fact, referred to the midpoint and radial measure a point pair is the analogue on a line of a circle in a plane, of a sphere in space of three dimensions, and so on, and the harmonic relation is the relation which in higher dimensions is usually regarded as the relation of orthogonality. The measure π_{12} is the mutual power of the two point pairs, and plays precisely the same part as the mutual power in the theory of circles and spheres.

To calculate the power in terms of co-ordinates in the plane, we have from (7.1), taking P at G_2 and replacing the bipolar co-ordinates of G_2 by line co-ordinates in the plane by means of (7.5),

$$(x_1\lambda_2+y_1\mu_2+z_1\nu_2)h+(x_1+y_1)\Gamma_2\nu_2 = (x_1+y_1)(G_1G_2^2-\Gamma_1)\nu_2,$$

that is

$$\frac{\pi_{12}}{h} = \frac{x_1\lambda_2+y_1\mu_2+z_1\nu_2}{(x_1+y_1)\nu_2}. \tag{8.2}$$

From this we have immediately

$$\frac{\pi_{12}}{h} = \frac{(y_1z_2+z_1y_2)+(z_1x_2+x_1z_2)+(x_1y_2+y_1x_2)}{(x_1+y_1)(x_2+y_2)}. \tag{8.3}$$

Alternatively if we eliminate x_1, y_1, z_1 between (8.2) and the equalities

$$\frac{y_1+z_1}{\lambda_1} = \frac{z_1+x_1}{\mu_1} = \frac{x_1+y_1}{\nu_1}$$

we have

$$\frac{\nu_2\pi_{12}}{h}\begin{vmatrix} 0 & 1 & 1 & \lambda_1 \\ 1 & 0 & 1 & \mu_1 \\ 1 & 1 & 0 & \nu_1 \\ 1 & 1 & 0 & 0 \end{vmatrix} = \begin{vmatrix} 0 & 1 & 1 & \lambda_1 \\ 1 & 0 & 1 & \mu_1 \\ 1 & 1 & 0 & \nu_1 \\ \lambda_2 & \mu_2 & \nu_2 & 0 \end{vmatrix},$$

that is

$$(8.4) \qquad \frac{2\nu_1\nu_2\pi_{12}}{h} = -\begin{vmatrix} 0 & 1 & 1 & \lambda_1 \\ 1 & 0 & 1 & \mu_1 \\ 1 & 1 & 0 & \nu_1 \\ \lambda_2 & \mu_2 & \nu_2 & 0 \end{vmatrix}.$$

9. *Harmonizing point pairs.* The numerators in the formulæ for π_{12}/h are the polarised forms of the functions which describe the base conic, and

Two point pairs harmonize if the associated points in the plane are conjugate for the base conic.

We infer at once that the point pairs which harmonize with a given point pair compose a linear series, which is determined completely by any two of its members and in which a variable member is identifiable from any two fixed members by a homogeneous pair of co-ordinates. The polar of a point T outside the base conic is cut by the base conic in two points, and of the two segments bounded by these points one is outside the conic and the other inside; the points on the outside segment represent the point pairs harmonizing with the pair (T) represented by T. Conversely, if R, S are any two points on this outside segment, the line RS cuts the conic in two points, and the tangents at these points represent the two constituents of the one point pair that is harmonic with each of the pairs (R), (S).

10. *Generalization of the point pair.* In transferring the study of harmonic point pairs to the familiar ground of poles and polars, the correlation shows that there are linear series that are not definable in terms of the harmonic relation. If a line does not cut the base conic, its pole is inside the conic and does not correspond to any point pair on the original line; there is therefore no point pair with which the pairs represented by the points of the line are all harmonic, but the mutual relations which interpret the collinearity of the representative points cannot be radically altered. To put the matter differently, an account of poles and polars in the plane would be insufferably complicated by an agreement that points inside the conic were not to be mentioned or used; as soon as we find that the relation between conjugate points

in the plane is significant for the geometry of the line, it is imperative therefore that we should devise some interpretation in which points inside the conic can play a part.

The solution is simple. It is because the radial measure is derived from a point pair that it presents itself as essentially positive, and there is no reason why we should not attach measures arbitrarily to the points of a line. A point G with a positive measure Γ attached to it specifies the point pair of which G is the midpoint and $\sqrt{\Gamma}$ is the radius; a point G with a negative measure attached to it does not specify a point pair, but the correlation with points in a plane shows that the geometry of points with radial measures unrestricted in sign is necessarily smoother than the geometry of point pairs in the literal sense. It is the more general geometry that we develop, even if our concern is only with propositions in the elementary field. Some change of vocabulary is inevitable, and following the practice invariable in mathematics we abduct the primitive word to serve in the sophisticated geometry. Henceforth what we mean by a point pair on the line AB is a point G of the line associated with a radial measure Γ. If Γ is positive, the point pair (G, Γ) is said to be actual; if Γ is negative, (G, Γ) is virtual. A proposition in the theory of point pairs is always significant in terms of midpoints and radial measures; if the radial measures involved are all positive the proposition is significant also in terms of ordinary points.

Elementary geometry abounds with extensions of this kind although their nature is not always analysed. If a triangle is oblique-angled, it is self-conjugate for the polar circle, which has the orthocentre H for its centre and the common value of the three products $HA.HD$, $HB.HE$, $HC.HF$ for the square of its radius; whether the polar circle exists or not, the orthocentre exists and the three products are equal and define the 'polar measure' of the triangle. If V, T are conjugate points on a diameter of a central conic, the product $CV.CT$ is a constant for that diameter, and the sum of the values of this constant for a pair of conjugate diameters is a constant for the conic; if the conic is an ellipse or an acute hyperbola, it possesses an orthoptic circle and the constant just described is the square of the radius of this circle, but the constant, the 'orthoptic measure' of the conic, exists in any case. If a triangle is self-conjugate for a central conic, the square of the distance between the orthocentre of the triangle and the centre of the conic is the sum of the polar measure of the triangle and the orthoptic measure of the conic; this is Gaskin's theorem, that the polar circle and the orthoptic circle are mutually orthogonal, expressed in a form that cannot demand the interpretation of imaginary elements to render it intelligible.

Strictly speaking, the mutual power of two point pairs and the principle on which a correlation with points in the plane is to be set up have to be defined afresh, but there is no choice if the work is not to be stultified. The mutual power is defined always as $G_1G_2^2-\Gamma_1-\Gamma_2$. If the bipolar co-ordinates of G are λ_G, μ_G, ν_G, the point pair (G, Γ) is represented by the line whose co-ordinates are

$$\lambda_G-(\Gamma/h)\ \nu_G,\ \mu_G-(\Gamma/h)\ \nu_G,\ \nu_G$$

and by the point which is the pole of this line for the base conic. The ratios of $\lambda_G-(\Gamma/h)\nu_G$, $\mu_G-(\Gamma/h)\nu_G$ to ν_G are $(GA^2-\Gamma)/h$, $(GB^2-\Gamma)/h$, that is, measure the powers of the base points A, B for (G, Γ) in terms of the constant AB^2. The mutual power of two pairs is given in terms of co-ordinates by the formulae of § 8.

11. *Involutions.* The analysis of the linear series of point pairs is now simple. The point pairs corresponding to the points of a line are said to constitute an involution of point pairs. An involution consists of all the pairs that harmonize with one pair, the generator of the involution. Two involutions have one and only one pair which harmonizes with them both. Any line in the plane of the conic cuts the tangent $x+y=0$ in one point; an involution includes one and only one pair of which the point at infinity is one constituent, the other constituent of this pair is the centre of the involution. Like every other member of the involution, the pair which consists of the centre and the point at infinity harmonizes with the generator; hence the centre is the midpoint of the latter pair. If the centre is C and the radial measure of the generating pair is Δ, the relation between the midpoint G and the radial measure Γ of any member of the involution is given by the harmonic condition: $CG^2-\Gamma=\Delta$. A line in a plane cannot be wholly inside a conic, but it may cut the conic or be wholly outside. It follows that if the distinction between actual and virtual point pairs is to be brought into consideration, or if attention is to be confined to pairs of ordinary points, involutions are of two kinds. If the line representing the involution is wholly outside the conic, the involution consists entirely of actual pairs; the pole, which represents the generator of the involution, is inside the conic, and if P, Q are the constitutents of a point pair on the original line, the value of $CP.CQ$ is negative and therefore P and

Q are on opposite sides of C. If the line representing the involution cuts the conic, the tangents at the points of intersection represent two points E, F on the original line; the actual members of the involution correspond to point pairs harmonic with EF and satisfying therefore the condition $CP.CQ = CE^2 = CF^2$, but the midpoints of these pairs compose only the part of the line AB outside EF; that is, the midpoints of actual members of the involution are the points of AB outside EF; a member of the involution whose midpoint is between E and F is a virtual point pair whose radial measure has the negative value $CG^2 - CE^2$.

Since an involution must contain actual members, it is possible logically to use only these members, and so indeed to use only ordinary pairs of points. The correspondence between the involution and the point pair which generates it implies that if we wish to avoid virtual point pairs altogether, we have only to regard (G, Γ) as the specification of the involution $GP.GQ = \Gamma$. Formally there is no change whatever, but the concepts which are now the elements of the point pair geometry are so complicated in terms of points that it has become difficult to appreciate the significance even of the harmonic relation which must remain fundamental. Nevertheless, there is one line of development in which this alternative generalization of the point pair is appropriate: in projective geometry we can define an involution without presupposing a means of measurement on the line; an ordinary point pair does not then suggest a radial measure, and the simple process of extension which we have used is not available. The alternative, which recognises the involution as the logical equivalent of a point pair, is von Staudt's method of introducing the conjugate imaginary elements into geometry.

12. *Inversion on a line.* In the geometry of a line, inversion in a point pair (G, Γ) is the transformation which replaces a point P by the point Q such that $GP.GQ = \Gamma$, that is, such that the pair PQ harmonizes with (G, Γ). Transferred to the plane, the relation is that the tangents to the base conic which represent P and Q intersect on the polar ω of the point Ω which represents (G, Γ). We can discover the effect of this transformation on an actual point pair (U) without decomposing the pair. The tangents from U to the conic cut ω in two points R, S, and the second tangents from R, S intersect in V, the representative of the point pair inverse to (U). It follows from the elementary harmonic properties of a quadrilateral composed of four tangents to a conic that UV passes through Ω, the pole of RS, and that if UV cuts RS

in Y, the pair UV harmonizes with Y; that is to say, *inversion on the line corresponds to a homography in the plane, the centre of the homography is the point which represents the point pair with respect to which the inversion takes place, and the axis of the homography is the polar of the centre for the base conic.*

Since the centre and axis of the homography are pole and polar for the base conic, the homography transforms the base conic into itself; this is to be expected, since a point on the base conic represents a degenerate point pair whose constituent members coincide, and inversion does not separate these coincident constituents. The homography transforms a line into a line, and the pole of the one line for the base conic into the pole of the other. Hence an inversion converts the involution generated by one point pair into the involution generated by the inverse point pair.

13. *The inversion of one given pair into another.* If a homography in the plane is to transform the point U into the point V, the centre of the homography is on the line UV and the axis ω cuts UV in a point Ψ such that Ψ harmonizes with UV. We know that the pairs of ordinary points on UV which are conjugate for the base conic compose an involution, and $\Omega\Psi$ must be a pair common to this involution and the involution generated by UV. If the pair common to these involutions is actual, each of the ordinary points which belong to this pair represents an inversion which interchanges the point pairs represented by U and V. There are two of these inversions; if U and V represent actual point pairs with constituents P_u, Q_u and P_v, Q_v, one inversion changes P_u into P_v and Q_u into Q_v, the other changes P_u into Q_v and Q_u into P_v. But the question whether there are ordinary points Ω, Ψ on the line UV to represent actual inversions is entirely distinct from the question whether U and V themselves represent actual pairs or virtual pairs. If the common-member of the involution on UV generated by the pair UV and the involution of conjugate points on UV is virtual, there is no actual inversion which connects the point pairs represented by U and V, even when the meaning of inversion is extended to admit inversion in a virtual point pair. In fact the complex point, driven from the original line, reappears in the interpretation of plane geometry, and a systematic translation in terms of ordinary points though always possible threatens to become sooner or later fantastic.

Since Ω and Ψ are recognised and required as two distinct inversions, the problem presented if the pair Ω, Ψ is virtual is the interpretation of the individual constituents of a virtual point pair, and this is a problem of a different order of difficulty from that of interpreting the point pair itself. The general lines of a solution are evident enough.

14. *Trigeminal co-ordinates and inversion.* A homography in a plane is a linear transformation of the point co-ordinates (x, y, z) or of the line co-ordinates (λ, μ, ν). If a, b, c are any three numbers that are not all zero, there is a point pair represented by the point (a, b, c), and $(a\lambda+b\mu+c\nu)/\nu$ for a variable line (λ, μ, ν), is the constant multiple $(a+b)/h$ of the mutual power of this fixed point pair and the variable point pair represented by the line (λ, μ, ν). Thus if we regard the transformation in the plane not as relating one line (λ, μ, ν) to another line

$$(a_1\lambda+b_1\mu+c_1\nu,\ a_2\lambda+b_2\mu+c_2\nu,\ a_3\lambda+b_3\mu+c_3\nu),$$

but as referring the same line to another triangle of reference, we are regarding the transformation on the line not as relating two variable point pairs but as introducing for the specifications of one variable point pair a new system of homogeneous co-ordinates, namely, assigned multiples of the mutual powers $\omega_1, \omega_2, \omega_3$ of the variable pair with three fixed pairs; we call $\omega_1, \omega_2, \omega_3$ trigeminal co-ordinates of the point pair.

Since there are two independent elements in the determination of a point pair on a line, there can be no permanent relation between the ratios $\omega_1 : \omega_2 : \omega_3$. Given four point pairs it is easily shown that the four rowed determinant $|\pi_{ij}|$ is identically zero, but this relation involves not only the mutual powers of distinct pairs but also four elements of the form π_{ii}, and if Γ_i is the radial measure of a pair, π_{ii} is $-2\Gamma_i$. With three fixed pairs and one variable pair, we have therefore

$$(14.1) \qquad \begin{vmatrix} \pi_{11} & \pi_{12} & \pi_{13} & \omega_1 \\ \pi_{21} & \pi_{22} & \pi_{23} & \omega_2 \\ \pi_{31} & \pi_{32} & \pi_{33} & \omega_3 \\ \omega_1 & \omega_2 & \omega_3 & -2\,\Gamma \end{vmatrix} = 0.$$

This is not a permanent relation between the co-ordinate ratios $\omega_1 : \omega_2 : \omega_3$, but a formula for Γ when the absolute values of the various powers are known.

The operation of inversion is equivalent to a change in the triad of reference pairs, but this change is not arbitrary. As we have seen, in the plane homography the centre and the axis are pole and polar for the base conic; in other words, the homography, regarded as a transformation of co-ordinates, leaves the equation of the base conic unaltered. Inversion on the line is therefore a change which leaves a certain quadratic function of $\omega_1, \omega_2, \omega_3$ relatively invariant. This function is readily identified. The base conic is the envelope of a line which represents a point pair whose radial measure is zero. Whatever the fixed pairs, a variable pair for which Γ is zero has coordinates ω_1, ω_2, ω_3 which satisfy the homogeneous equation

$$(14.2) \qquad \begin{vmatrix} \pi_{11} & \pi_{12} & \pi_{13} & \omega_1 \\ \pi_{21} & \pi_{22} & \pi_{23} & \omega_2 \\ \pi_{31} & \pi_{32} & \pi_{33} & \omega_3 \\ \omega_1 & \omega_2 & \omega_3 & 0 \end{vmatrix} = 0.$$

Inversion does not separate the two constituents of a pair if these constituents coincide, and therefore the equation just written is the equation which is invariant in a change of trigeminal co-ordinates for the specification of point pairs, and for this purpose the system is not redundant except in the sense that it is homogeneous. Since a point *G* on the line can be identified by means of the point pair which has midpoint *G* and radial measure zero, it must be possible to use trigeminal co-ordinates for the specification of individual points. But since the individual point has only one degree of freedom, the trigeminal co-ordinates of a point form a redundant system with one permanent homogeneous equation; this is the equation (14.2).

Although it is for point pairs that trigeminal co-ordinates are most naturally introduced, their use for individual points is not an excrescence. On the contrary, it is not to be supposed that every configuration which is to be inverted is essentially a collection of pairs, and trigeminal co-ordinates are those in which inversion finds its simplest expression.

IDENTITIES BETWEEN FIELD-EQUATIONS IN THE GENERAL FIELD-THEORY OF SCHOUTEN AND VAN DANTZIG

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1. Identities between the field-equations in the general field-theory of Schouten and van Dantzig have been obtained in a recent paper* by the author by using a theorem of Emmy Noether. The object of the present paper is to verify these identities by direct substitution and to show the connexion between these identities and those found by E. T. Whittaker† between the field-equations of Einstein's General Relativity. An attempt is also made to introduce Λ, the universal constant, proportional to the square of the curvature of the world into the field equations and the identities.

2. We first collect here some known results in the projective relativity theory, which will be constantly required in this paper for reference.

The fundamental quadric is

$$\sum_{\mu\nu} G_{\mu\nu}V^{\mu}V^{\nu}=0; \quad \text{Det }(G_{\mu\nu})\neq 0, \tag{2.1}$$

where $G_{\mu\nu}$ is a projector of covariant valence 2. It is normalized by

$$\sum_{\mu\nu} G_{\mu\nu}x^{\mu}\,.\,x^{\nu}=-\omega^2, \tag{2.2}$$

where ω is a positive constant$=k\mathbf{h}/c$, k being the constant of gravitation, $\mathbf{h}=\frac{h}{2\pi}$, c the velocity of light and $\mu, \nu=0, 1, 2, 3, 4$.

$$x^{\lambda}=\omega q^{\lambda},\ x_{\lambda}=\omega q_{\lambda},\ \sum_{\mu}G_{\lambda\mu}x^{\mu}=x_{\lambda},\ \sum_{\lambda}x_{\lambda}x^{\lambda}=-\omega^2,\ \sum_{\lambda}q_{\lambda}q^{\lambda}=-1. \tag{2.3}$$

$$G_{\lambda\mu}+q_{\lambda}q_{\mu}=g_{\lambda\mu},\ \sum_{\mu}g_{\lambda\mu}g^{\mu\nu}=A_{\lambda}^{\nu},\ \sum_{j}g_{ij}g^{jk}=\delta_j^k,\ \sum_{\lambda}q^{\lambda}g_{\lambda\mu}=0. \tag{2.4}$$

* With the same title as of this paper, published in *Phil. Mag.* Nov. 1936.

† "On Hilbert's World-function", *Proc. Royal Society of London* (A) 113 (1926-27), 496-571.

$$ds^2=\sum_{\lambda\mu} g_{\lambda\mu}dx^\lambda dx^\mu,\ ds^2=\sum_{ij} g_{ij}d\xi^i d\xi^j. \tag{2.5}$$

We identify g_{ij} with the Riemann fundamental tensor of general relativity. The ξ's are the ordinary non-homogeneous co-ordinates of the four-dimensional space-time. g_{ij} specifies gravitational field and $G^{\lambda\mu}$ specifies gravitational and electromagnetic field. Covariant differentiation of a projector is defined as

$$\left.\begin{aligned}\nabla_\mu p&=\frac{\partial p}{\partial x^\mu}\ \text{if } p \text{ is a scalar,}\\ \nabla_\mu V^\nu&=\frac{\partial V^\nu}{\partial x^\mu}+\sum_\lambda \Pi^\nu_{\lambda\mu}V^\lambda\\ \nabla_\mu W_\lambda&=\frac{\partial W_\lambda}{\partial x^\mu}-\sum_\nu \Pi^\nu_{\lambda\mu}W_\nu\end{aligned}\right\} \tag{2.6}$$

$$\Pi^\nu_{\lambda\mu}=\left\{ {\nu \atop \lambda\mu} \right\}-(q-1)q^\nu q_{\lambda\mu}+(q-1)q_\lambda q^\nu_{.\mu}+(p-1)q_\mu q^\nu_{.\lambda},$$

where $\left\{ {\nu \atop \lambda\mu} \right\}$ is the Christoffel symbol formed by $G^{\lambda\mu}$; p and q are constants, which, in Schouten's 1933 theory, satisfy the relations $q=2$, p arbitrary and $q^2-2pq+2p>0$; and $q_{\lambda\mu}$ is the bivector given by

$$q_{\lambda\mu}=\tfrac{1}{2}\left(\frac{\partial q_\mu}{\partial x^\lambda}-\frac{\partial q_\lambda}{dx^\mu}\right), \tag{2.7}$$

$q_{\lambda\mu}$ is a skew affinor and is identified save for a constant with the electromagnetic bivector. In fact,

$$F_{ij}=\frac{\partial\phi_j}{\partial\xi_i}-\frac{\partial\phi_i}{\partial\xi_j}=\frac{qc}{k}\,q_{ij}, \tag{2.8}$$

where k is a constant of dimensions $M^{-\frac{1}{2}}L^{\frac{1}{2}}$ and ϕ's are components of the electromagnetic potential-vector.

$$\left.\begin{aligned}S^{\cdot\cdot\nu}_{\lambda\mu}&=\tfrac{1}{2}(\Pi^\nu_{\lambda\mu}-\Pi^\nu_{\mu\lambda})\\ P^\nu_{.\lambda}&=-p\omega q^\nu_{.\lambda}\,;\ Q^\nu_{.\lambda}=-\omega q q^\nu_{.\lambda}\\ \nabla_\mu q^\nu&=-q q^\nu_{.\mu};\ \nabla_\mu q_\nu=-q q_{\nu\mu}\\ \nabla_\mu A^\nu_\lambda&=-q(q_\lambda q^\nu_{.\mu}+q^\nu q_{\lambda\mu})\\ \nabla_\mu g_{\lambda\mu}&=-q(q_\lambda q_{\nu\mu}+q_\nu q_{\lambda\mu}).\end{aligned}\right\} \tag{2.9}$$

$\overset{R}{\nabla}_\mu$ denotes the affinor part of projective covariant derivative of an affinor.

$$\left.\begin{aligned}\overset{R}{\nabla}_\mu V^\nu &= \frac{\partial V^\nu}{\partial x^\mu} + \sum_\lambda \overset{R}{\Pi}{}^\nu_{\lambda\mu} V^\lambda, \\ \text{where } \overset{R}{\Pi}{}^\nu_{\lambda\mu} &= \left\{ {\nu \atop \lambda\mu} \right\} + q^\nu q_{\lambda\mu} - q_\lambda q^\nu_{.\mu} - q_\mu q^\nu_{.\lambda}.\end{aligned}\right\} \quad (2.10)$$

$N^{\cdot\cdot\cdot\nu}_{\omega\mu\lambda}$ is the projective curvature tensor formed from $\Pi^\nu_{\lambda\mu}$.

$K_{\lambda\mu}$ is the contracted Riemann tensor of general relativity in homogeneous co-ordinates.

$$K = \sum g^{\lambda\mu} K_{\lambda\mu} = \text{Scalar curvature.}$$

$$N_{\mu\lambda} - K_{\mu\lambda} = -qq_\lambda \sum_\rho \overset{R}{\nabla}_\rho q^\rho_{.\mu} - pq_\mu \sum_\rho \overset{R}{\nabla}_\rho q^\rho_{.\lambda} - pqq_\lambda q_\mu \sum_{\rho\sigma} q^{\rho\sigma} q_{\rho\sigma}$$
$$+ (pq - 2p - q^2) \sum_\rho q_{\rho\mu} q^\rho_{.\mu} \quad (2.11)$$

$$N - K = (2pq - 2p - q^2) \sum_{\rho\sigma} q_{\rho\sigma} q^{\rho\sigma}.$$

The identities between the field-equations obtained are

$$\sum_{\lambda\mu} \left[\nabla_\mu X^{\mu\cdot}_{.\alpha} - \left\{ (q-1) q^\lambda q_{\alpha\mu} - (q-1) q_\alpha q^\lambda_{.\mu} - (p-1) q_\mu q^\lambda_{.\alpha} \right\} X^{\mu\cdot}_{.\lambda} \right]$$
$$= 0, \ (\alpha = 0, 1, 2, 3, 4), \quad (2.12)$$

where $$X^{\mu\cdot}_{.\alpha} = \tfrac{1}{2} (P^{\mu\cdot}_{.\alpha} + P^{\cdot\mu}_{\alpha.}) ; \quad (2.13)$$

and

$$P_{\lambda\mu} \equiv K_{\lambda\mu} - \tfrac{1}{2} K g_{\lambda\mu} + (q^2 - 2pq + 2p)$$
$$\times \left\{ \tfrac{1}{2} g_{\lambda\mu} \sum_{\rho\sigma} q_{\rho\sigma} q^{\rho\sigma} + 2q_\lambda \sum_\rho \overset{R}{\nabla}_\rho q^\rho_\mu - 2 \sum_\rho q^{\cdot\rho}_\lambda q_{\mu\rho} \right\}. \quad (2.14)$$

3. *The verification of the identities.* By means of (2.13) and (2.14), we have

$$X_{\lambda\mu} = N_{\lambda\mu} - \tfrac{1}{2} N G_{\lambda\mu} + (-q^2 + 3pq - 2p) \sum_\rho q^{\cdot\rho}_\lambda q_{\mu\sigma}$$

$$-(p+q) q_\lambda \sum_\rho \overset{R}{\nabla}_\rho q^{\cdot\rho}_\mu + (q^2 - pq + 2p) \left[q_\lambda \sum_\rho \overset{R}{\nabla}_\rho q^{\cdot\rho}_\mu + q_\mu \sum_\rho \overset{R}{\nabla}_\rho q^{\cdot\rho}_\lambda \right].$$
$$(3.1)$$

This can be written as

$$X_{\lambda\mu} = Z_{\lambda\mu} + (q^2 - 2pq + 2p) Y_{\lambda\mu}, \quad (3.2)$$

where

$$Z_{\lambda\mu}=N_{\lambda\mu}-\tfrac{1}{2}NG_{\lambda\mu}+(-q^2+3pq-2p)\sum_{\rho} q^{\cdot\rho}_{\lambda}q_{\mu\rho}-(p+q)q_{\lambda}\sum_{\rho} \overset{R}{\nabla} q^{\cdot\rho}_{\mu} \tag{3.3}$$

and

$$Y_{\lambda\mu}=q_{\lambda}\sum_{\rho} \overset{R}{\nabla}_{\rho}\, q^{\cdot\rho}_{\mu}+q_{\mu}\sum_{\rho} \overset{R}{\nabla}_{\rho}\, q^{\lambda\cdot\rho}. \tag{3.4}$$

Changing α into λ, λ into σ and substituting from (3.2) the left side of (2.12) breaks up into the sum of two parts namely,

$$\sum_{\mu} \nabla_{\mu}Z^{\cdot\mu}_{\lambda}-(q-1)\sum_{\mu\sigma} Z^{\cdot\mu}_{\sigma\cdot}q^{\sigma}q_{\lambda\mu}+(q-1)q_{\lambda}\sum_{\sigma\mu} Z^{\cdot\mu}_{\sigma\cdot}q^{\sigma}_{\mu\cdot}$$
$$+(p-1)\sum_{\mu\sigma} Z^{\cdot\mu}_{\sigma\cdot}q_{\mu}q^{\sigma}_{\cdot\lambda}, \tag{3.5}$$

and

$$(q^2-2pq+2p)\Big[\sum_{\mu} \nabla_{\mu}Y^{\mu}_{\cdot\lambda}-(q-1)\sum_{\sigma\mu} Y^{\mu\cdot}_{\cdot\sigma}q^{\sigma}q_{\lambda\mu}+(q-1)q_{\lambda}\sum_{\sigma\mu} Y^{\mu\cdot}_{\cdot\sigma}\, q^{\sigma}_{\cdot\mu}$$
$$+(p-1)\sum_{\sigma\mu} Y^{\mu}_{\cdot\sigma}\, q_{\mu}q^{\sigma}_{\cdot\lambda}\Big]. \tag{3.6}$$

Making use of the following relations, (which we prove in the following sections)

$$\sum_{\mu} \nabla_{\mu}(N^{\cdot\mu}_{\lambda\cdot}-\tfrac{1}{2}N\delta^{\mu}_{\lambda})$$
$$=(3pq-q^2-2p)\sum_{\rho\sigma} q_{\lambda\sigma}\nabla_{\rho}q^{\rho\sigma}+(q^2-2pq+2p)\sum_{\mu\sigma} q^{\sigma\mu}\nabla_{\lambda}q_{\sigma\mu}$$
$$+q(p-q)\sum_{\rho\sigma} q^{\sigma\rho}\nabla^{\cdot}_{\rho}q_{\sigma\lambda}, \tag{3.7}$$

$$\sum_{\mu\rho} \nabla_{\mu}\overset{R}{\nabla}_{\rho}q^{\rho\mu}=0, \tag{3.8}$$

$$\sum_{\sigma\mu} q^{\sigma\mu}\nabla_{\lambda}q_{\sigma\mu}=2\sum_{\rho\sigma} q^{\rho\sigma}\nabla_{\rho}q_{\lambda\sigma}, \tag{3.9}$$

in evaluating (3.5), we see that it reduces to

$$2(q^2-2pq+2p)q_{\lambda\mu}\sum_{\rho} \nabla_{\rho}q^{\mu\rho}.$$

Similarly, after reduction, (3.6) comes out to be equal to

$$2(q^2-2pq+2p)\sum_{\mu\rho} q_{\mu\lambda}\nabla_\rho q^{\mu\rho}.$$

So the left side of (2.12) reduces to

$$2(q^2-2pq+2p)q_{\lambda\mu}\sum_{\rho} \nabla_\rho q^{\mu\rho}+2(q^2-2pq+2p)q_{\lambda\mu}\sum_{\rho} \nabla_\rho q^{\mu\rho},$$

and this is zero since $q_{\mu\lambda}=-q_{\lambda\mu}$. Thus the identities are verified to be true by actual substitution.

4. To prove (3.7) we make use of the following relation given by Schouten and van Dantzig*.

$$\sum_{\mu} \nabla_\mu[N_{\lambda\cdot}^{\cdot\mu}-\tfrac{1}{2}N\delta_\lambda^\mu]=-2\sum_{\mu\sigma} S_{\lambda\sigma}^{\cdot\cdot\mu}N_{\mu\cdot}^{\cdot\sigma}-\sum_{\mu\rho\sigma} S_{\rho\sigma\mu}N_{\lambda}^{\cdot\mu\rho\sigma}$$

$$=-2\sum_{\mu\sigma}\left[-(q-1)q_{\lambda\sigma}q^{\mu}+\frac{p-q}{2}q_\lambda q_{\sigma}^{\cdot\mu}-\frac{p-q}{2}q_\sigma q_{\lambda}^{\cdot\mu}\right]N_{\mu}^{\cdot\sigma}$$

$$-\sum_{\mu\rho\sigma}\left[-(q-1)q_{\rho\sigma}q_\mu+\frac{p-q}{2}q_\rho q_{\sigma\mu}-\frac{p-q}{2}q_\sigma q_{\rho\mu}\right]N_{\lambda}^{\cdot\mu\rho\sigma}, \tag{4.1}$$

remembering the values of $S_{\lambda\sigma}^{\cdot\cdot\mu}$, given by (2.9).

The first term on the right of (4.1) contains transvections of the q's with $N_{\mu\cdot}^{\cdot\sigma}$. To evaluate these we make use of the following identities, which can be easily obtained from those given by Schouten and van Dantzig in their paper in the *Annals of Mathematics*:

$$\sum_{\rho} q^\rho N_{\rho\sigma}=p\sum_{\rho} \nabla_\rho q_{\cdot\sigma}^{\rho},$$

$$\sum_{\sigma} q^\sigma N_{\rho\sigma}=q\sum_{\tau} \nabla_\tau q_{\cdot\rho}^{\tau}+q(p-q)q_\rho\sum_{\theta\tau} q^{\theta\tau}q_{\theta\tau},$$

$$\sum_{\sigma\rho} q^{\sigma\rho}N_{\rho\sigma}=0. \tag{4.2}$$

To evaluate the second term of (4.1) we observe that

$$N_{\lambda}^{\cdot\mu\rho\sigma}-K_{\lambda}^{\cdot\mu\rho\sigma}=qq^{\rho\mu\theta}\overset{R}{\nabla}_\theta q_{\cdot\lambda}^{\sigma}-qq^\sigma G^{\mu\theta}\overset{R}{\nabla}_\theta q_{\cdot\lambda}^{\rho}+pq_\lambda G^{\mu\theta}\overset{R}{\nabla}_\theta q^{\sigma\rho}$$

$$-qq^\rho\overset{R}{\nabla}_\lambda q^{\sigma\mu}+qq^\sigma\overset{R}{\nabla}_\lambda q^{\rho\mu}-pq^\mu\overset{R}{\nabla}_\lambda q^{\sigma\rho}+q^2q^{\sigma\mu}q_{\cdot\lambda}^{\rho}-2pq_{\lambda\cdot}^{\mu}q^{\sigma\rho}$$

$$+pq[-q^\sigma q_\lambda q^{\alpha\rho}q_{\alpha\cdot}^{\mu}+q^\rho q^\mu q^{\sigma\alpha}q_{\alpha\lambda}+q^\sigma q^\mu q_{\alpha\lambda}q^{\rho\alpha}-q^\rho q_\lambda q_{\cdot\alpha}q^{\alpha\mu}]. \tag{4.3}$$

* *Annals of Mathematics*, 34 (1933), 293.

Transvection of the $K^{;\mu\rho\sigma}_{\lambda}$ in (4.3) with the q's vanishes since $K^{;\mu\rho\sigma}_{\lambda...}$ is an affinor, when we substitute in the second term of (4.1) from (4.3). The transvections of the right side of (4.3) with the q's can be easily evaluated by the known formulæ of section 2. After these simplifications, (4.1) comes out as

$$\sum_{\mu} \nabla_{\mu}(N^{;\mu}_{\lambda.}-\tfrac{1}{2}N\delta^{\mu}_{\lambda})=(3pq-q^2-2p)\sum_{\rho\sigma} q_{\lambda\sigma}\nabla_{\rho}q^{\rho\sigma}$$
$$+(q^2-2pq+2p)\sum_{\mu\sigma} q^{\sigma\mu}\nabla_{\lambda}q_{\sigma\mu}+q(p-q)\sum_{\rho\sigma} q^{\sigma\rho}\nabla_{\rho}q_{\sigma\lambda},$$

which is (3.7).

5. *To prove that* $\sum_{\mu\rho} \nabla_{\mu}\overset{R}{\nabla}_{\rho}q^{\rho\mu}=0.$

We have

$$\sum_{\rho} \overset{R}{\nabla}_{\rho}q^{\mu\rho}=\sum_{\rho} \nabla_{\rho}q^{\mu\rho}+qq^{\mu}\sum_{\rho\sigma} q^{\rho\sigma}q_{\rho\sigma}. \tag{5.1}$$

This comes out by remembering that

$$\overset{R}{\nabla}_{\mu}q^{\nu}_{.\lambda}=\sum_{\pi\rho\sigma} (\delta^{\pi}_{\lambda}+q^{\pi}q_{\lambda})(\delta^{\rho}_{\mu}+q^{\rho}q_{\mu})(\delta^{\nu}_{\sigma}+q^{\nu}q_{\sigma})\nabla_{\rho}q^{\sigma}_{.\lambda}$$
$$=\nabla_{\mu}q^{\nu}_{.\lambda}+qq^{\nu}\sum_{\rho} q_{\rho\lambda}q^{\rho}_{.\mu}+qq_{\lambda}\sum_{\rho} q^{\nu\rho}q_{\rho\sigma} \tag{5.2}$$

which follows after some simplification.

In getting (5.2) we require a formula

$$\sum_{\rho} q^{\rho}\nabla_{\rho}q^{\nu}_{.\lambda}=0, \tag{5.3}$$

which can be easily seen to be true because

$$\sum_{\rho} q^{\rho}\nabla_{\rho}q^{\nu}_{.\lambda}=\omega^{-1}\sum_{\rho} x^{\rho}\frac{\partial q^{\nu}_{.\lambda}}{\partial x^{\rho}}-\sum_{\rho\sigma} q^{\rho}\Pi^{\sigma}_{\lambda\rho}q^{\nu}_{.\sigma}+\sum_{\rho\sigma} q^{\rho}\Pi^{\nu}_{\sigma\rho}q^{\sigma}_{.\lambda}.$$

The first term on the right vanishes and

$$\sum_{\rho} \Pi^{\sigma}_{\lambda\rho}q^{\rho}=\omega^{-1}(P^{\sigma}_{.\lambda}-\delta^{\sigma}_{\lambda})$$
$$\sum_{\rho} \Pi^{\nu}_{\sigma\rho}q^{\rho}=\omega^{-1}(P^{\nu}_{.\sigma}-\delta^{\nu}_{\sigma})\cdot$$

give

$$\sum_\rho q^\rho \nabla_\rho q^\nu_{.\lambda} = -\sum_\rho \omega^{-1}(P^\sigma_{.\lambda} - \delta^\sigma_\lambda) q^\nu_{.\sigma} + \sum_\sigma \omega^{-1}(P^\nu_{.\sigma} - \delta^\nu_\sigma) q^\sigma_{.\lambda} = 0.$$

We also require for (5.2)

$$\sum_\rho q^\rho \nabla_\mu q^\nu_{.\rho} = q \sum_\rho q^\nu_{\rho.} q^\rho_{.\mu} \tag{5.4}$$

$$\sum_\sigma q_\sigma \nabla_\mu q^\sigma_{.\lambda} = q \sum_\rho q_{\rho\lambda} q^\rho_{.\mu} \tag{5.5}$$

both of which follow from

$$\sum_\rho q^\rho q^\nu_{.\rho} = 0.$$

We get (5.1) from (5.2) by raising λ, changing it into $\lambda = \mu = \rho$ and putting μ for ν.

Operating on both sides of (5.1) by ∇_μ, we have

$$\sum_{\mu\rho} \nabla_\mu \overset{R}{\nabla}_\rho q^{\mu\rho} = \sum_{\mu\rho} \nabla_\mu \nabla_\rho q^{\mu\rho} + \sum_\mu q q^\mu \nabla_\mu \left\{ \sum_{\rho\sigma} q^{\rho\sigma} q_{\rho\sigma} \right\}. \tag{5.6}$$

Both the terms on the right side of (5.6) can be seen to be equal to zero, as by means of (5.3)

$$\sum_\mu q^\mu \nabla_\mu \sum_{\rho\sigma} q^{\rho\sigma} q_{\rho\sigma} = 0;$$

$$\sum_{\mu\rho} \nabla_\mu \nabla_\rho q^{\mu\rho} = \sum_{\mu\rho\tau\theta} \nabla_\mu \nabla_\rho [\tfrac{1}{2} G^{\theta\mu} G^{\tau\rho} (\nabla_\theta q_\tau - \nabla_\tau q_\theta]$$

$$= \sum_{\mu\rho\theta\tau} \tfrac{1}{2} G^{\mu\theta} \nabla_\mu \nabla_\rho \nabla_\theta q^\rho - \tfrac{1}{2} \sum_{\mu\rho\theta\tau} G^{\tau\rho} \nabla_\rho \nabla_\mu \nabla_\tau q^\mu = 0$$

by interchanging μ and ρ and changing the dummy suffix θ into τ in the first term on the right.

6. *To prove that* $\sum_{\sigma\mu} q^{\sigma\mu} \nabla_\lambda q_{\sigma\mu} = 2 \sum_{\rho\sigma} q^{\rho\sigma} \nabla_\rho q_{\lambda\sigma}$.

$$\sum_{\sigma\mu} q^{\sigma\mu} \nabla_\lambda q_{\sigma\mu} = \sum_{\sigma\mu} q^{\sigma\mu} \left[\frac{\partial q_{\sigma\mu}}{\partial x^\lambda} - \sum_\rho \Pi^\rho_{\sigma\lambda} q_{\rho\mu} - \sum_\rho \Pi^\rho_{\mu\lambda} q_{\sigma\rho} \right]$$

$$= \sum_{\sigma\mu} q^{\sigma\mu} \frac{\partial q_{\sigma\mu}}{\partial x^\lambda} - 2 \sum_{\mu\rho\sigma} q_{\rho\sigma} q^{\mu\sigma} \begin{Bmatrix} \rho \\ \lambda\mu \end{Bmatrix}, \tag{6.1}$$

by making use of relations in section 1.

$$2\sum_{\rho\sigma} q^{\rho\sigma}\nabla_\rho q_{\lambda\sigma} = 2\sum_{\rho\sigma} q^{\rho\sigma}\left[\frac{\partial q_{\lambda\sigma}}{\partial x^\rho} - \sum_\tau \Pi^\tau_{\lambda\rho} q_{\tau\sigma} - \sum_\tau \Pi^\tau_{\sigma\rho} q_{\lambda\tau}\right]$$

$$= 2\sum_{\rho\sigma} q^{\rho\sigma}\frac{\partial q_{\lambda\sigma}}{\partial x^\rho} - 2\sum_{\tau\rho\sigma} q_{\tau\sigma} q^{\rho\sigma}\begin{Bmatrix}\tau\\ \lambda\rho\end{Bmatrix}$$

$$= 2\sum_{\rho\sigma} q^{\rho\sigma}\frac{\partial q_{\lambda\sigma}}{\partial x^\rho} - 2\sum_{\mu\sigma\rho} q_{\rho\sigma} q^{\mu\sigma}\begin{Bmatrix}\rho\\ \mu\lambda\end{Bmatrix}. \quad (6.2)$$

Now

$$2\sum_{\rho\sigma} q^{\rho\sigma}\frac{\partial q_{\lambda\sigma}}{\partial x^\rho} = \sum_{\sigma\rho} q^{\rho\sigma}\frac{\partial}{\partial x^\rho}\left\{\frac{\partial q_\sigma}{\partial x^\lambda} - \frac{\partial q_\lambda}{\partial x^\sigma}\right\} = \sum_{\rho\sigma} q^{\rho\sigma}\frac{\partial^2 q_\sigma}{\partial x^\rho \partial x^\lambda}, \quad (6.3)$$

$$\sum_{\sigma\mu} q^{\sigma\mu}\frac{\partial q^{\sigma\mu}}{\partial x^\lambda} = \tfrac{1}{2}\sum_{\sigma\mu} q^{\sigma\mu}\frac{\partial^2 q_\mu}{\partial x^\lambda \partial x^\sigma} - \tfrac{1}{2}\sum_{\sigma\mu} q^{\sigma\mu}\frac{\partial^2 q_\sigma}{\partial x^\lambda \partial x^\mu}$$

$$= \tfrac{1}{2}\sum_{\rho\sigma} q^{\rho\sigma}\frac{\partial^2 q_\sigma}{\partial x^\lambda \partial x^\rho} + \tfrac{1}{2}\sum_{\rho\sigma} q^{\rho\sigma}\frac{\partial^2 q_\sigma}{\partial x^\lambda \partial x^\rho} = \sum_{\rho\sigma} q^{\rho\sigma}\frac{\partial^2 q_\sigma}{\partial x^\rho \partial x^\lambda}. \quad (6.4)$$

Hence

$$\sum_{\sigma\mu} q^{\sigma\mu}\nabla_\lambda q_{\sigma\mu} = 2\sum_{\rho\sigma} q^{\rho\sigma}\nabla_\rho q^{\lambda\sigma}$$

by means of (6.4) and (6.3).

7. *Connexion with identities between field-equations in Einstein's general relativity.*

True affinors give the tensors in four-dimensions. Hence finding the true affinor corresponding to the left side of (2.12) and putting it equal to zero we have

$$\sum_j \overset{R}{\nabla}_j Z^j_i + 2(q^2 - 2pq + 2p)\sum_{jl} q_{ji}\overset{R}{\nabla}_l q^{jl} = 0 \quad (j,i,l=1,2,3,4). \quad (7.1)$$

This must give us the identities between field-equations in Einstein's general theory of relativity.

It is easy to see that

$$Z^{\cdot j}_{i\cdot} = K^{\cdot j}_{i\cdot} - \tfrac{1}{2}K\delta^j_i + \frac{\kappa}{c^2}\left(\sum_{k=1}^{4} F^{\cdot k}_{i\cdot}F^{j}_{\cdot k} - \tfrac{1}{4}\delta^j_i \sum_{k,l=1}^{4} F_{kl}F^{kl}\right)$$

$$= A^{\cdot j}_{i\cdot} \text{ say, } (i,j=1,2,3,4). \quad (7.2)$$

Then (7.1) becomes

$$\sum_{j=1}^{4} \overset{R}{\nabla}_j A_i^j + \frac{\kappa}{c^2} \sum_{j=1}^{4} F_{ij} B^j = 0 \quad (i=1,2,3,4);$$

or

$$\sum_j \left(A_i^j \right)_j + \frac{\kappa}{c^2} \sum_{j=1}^{4} F_{ij} B^j = 0; \quad (i=1,2,3,4), \tag{7.3}$$

where

$$B^j = \sum_i \overset{R}{\nabla}_i F^{ji} = 0, \ (i,j=1,2,3,4)$$

gives Maxwell's equations and $A_i^j=0$ gives Einstein's combined gravitational and electromagnetic field-equations.

8. E. T. Whittaker at the end of his paper* on "Hilbert's world-function" gives the identities between field-equations of Einstein's general theory of relativity. He has assumed the existence of magnetic and electric currents and massive particles. When we assume that there are no currents and no massive particles his identities reduce to our identities (7.3) above.

Hence in vacuuo, the identities given by Whittaker are

$$A_p = \sum_q B^q M_{pq}, \ p,q=0,1,2,3 \tag{8.1}$$

where A_p represents the vectorial divergence of the symmetrical tensor A_{pq} which is equal to

$$\gamma(K_{pq} - \tfrac{1}{2} g_{pq} K) + \tfrac{1}{2}[\tfrac{1}{4} g_{pq} \sum_{r,s} X_{rs} X^{rs} - \sum_s X_{qs} X^{\cdot s}_{\cdot p}]$$

$$X_{rs} = M_{rs} = \frac{\partial \phi_r}{\partial x^s} - \frac{\partial \phi_s}{\partial x^r} = F_{rs} \text{ above};$$

$$B^q = \frac{1}{\sqrt{-g}} \sum_q \frac{\partial X^{pq}}{\partial X^q},$$

so that $B^q=0$ gives Maxwell's equations, and γ is a constant inversely proportional to the Newtonian constant of gravitation. Taking $\gamma = -\frac{1}{2} c^2/\kappa$, (8.1) and (7.3) can be at once seen to be the same.

* *Proc. Royal Society, London*, (A) 113 (1927), 496.

9. J. M. Whittaker* also at the end of his paper gives such identities in general relativity taking into account the wave-mechanical considerations. When we do not take these into account we shall just see that his identities also reduce to (7.3) above.

His identities, in space free from electric and magnetic currents and matter and also free from wave-mechanical restrictions become

$$\sum_{\nu} (A^{\nu}_{\mu})_{\nu} - \sum_{\nu} B^{\nu} X_{\mu\nu} = 0, \quad (\mu,\nu = 1,2,3,4), \tag{9.1}$$

where

$$\tfrac{1}{2}A^{\mu\nu} = \gamma(K^{\mu\nu} - \tfrac{1}{2}Kg^{\mu\nu}) + \tfrac{1}{2}E^{\mu\nu},$$

$$B^{\mu} = (X^{\mu\nu})_{\nu} = \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^{\nu}} (X^{\mu\nu}\sqrt{-g})$$

and

$$X_{\mu\nu} = \frac{\partial \phi_{\nu}}{\partial x^{\mu}} - \frac{\partial \phi_{\mu}}{\partial x^{\nu}},$$

ϕ_{μ} being the electromagnetic potential.

Taking $\gamma = -\frac{1}{2} c^2/\kappa$, (9.1) can at once be seen to be the same as (7.4) above.

10. W. Pauli† also in his unified field-theory gives the following identities between field equations:

$$\sum_{k} K^{\cdot k}_{i;k} - \varepsilon \sum_{k} X_{ik} K^{k}_{\cdot(0)} \equiv 0 \quad (k = 1,2,3.4), \tag{10.1}$$

where

$$K_{ij} = R_{ij} - \tfrac{1}{2}g_{ij} + \frac{\kappa}{c^2} \sum_{r} \left(F^{\cdot r}_{i.} F_{jr} - \tfrac{1}{4} g_{ij} \sum_{rs} F^{rs} F_{rs} \right)$$

$$K^{i}_{\cdot(0)} = -\frac{1}{r} \frac{\sqrt{\kappa}}{c} \sum_{k} F^{ik}_{;k}$$

$$X^{ik} = r\frac{\sqrt{\kappa}}{c} F_{ik}.$$

and $\varepsilon = \pm 1$.

Pauli's K_{ij} is A_{ij} of (7.2) and his R_{ij} is our K_{ij}. Thus (10.1) in our notation becomes

$$\sum_{k} \left(A^{k}_{i} \right)_{k} + \frac{\kappa}{c^2} \sum_{kj} F_{ik} F^{kj}_{;j} = 0$$

* 'On the Principle of least action', *Proc. Royal Society of London* (A), 121 (1928), 543-57.

† 'Über die Formulierung der Naturgesetze mit funf homogenen Koordinaten', *Ann. der Ph.* 18 (1933), 305-72.

or
$$\sum_j \left(A^i_i\right)_j + \frac{\kappa}{c^2}\sum_j F_{ij}B^j = 0, \tag{10.2}$$

where
$$B^j = \sum_k F^{jk}_{ik} = \sum_k \overset{R}{\nabla}_k F^{jk} = \sum_i \overset{R}{\nabla}_i F^{ji}.$$

Thus we see that (10.2) is the same as (7.3)

11. *Introduction of current vector Λ into the identities.*

Let us first try to introduce Λ, the universal constant proportional to the square of curvature of the world, into the field-equations and identities given in the first two sections.

If in the variational integral for obtaining the field equations we take $N-2\Lambda$ for N and proceed in the same manner we arrive at

$$K_{\lambda\mu} - \tfrac{1}{2}Kg_{\lambda\mu} + \Lambda g_{\lambda\mu} + (q^2 - 2pq + 2p)$$
$$\times\left\{\tfrac{1}{2}g_{\lambda\mu}\sum_{\rho\sigma} q_{\rho\sigma}q^{\rho\sigma} - 2\sum_{\rho} q^{\cdot\rho}_{\lambda} q_{\mu\rho}\right\} = 0 \quad (\lambda,\mu = 0,1,2,3,4). \tag{11.1}$$

$X_{\lambda\mu}$ of the identities of (2.12) now takes the form

$$\bar{X}_{\lambda\mu} = Z_{\lambda\mu} + \Lambda g_{\lambda\mu} + (q^2 - 2pq + 2p)Y_{\lambda\mu}. \tag{11.2}$$

The identities (2.12) now take the form

$$\sum_{\lambda\mu}\left[\nabla_\mu \bar{X}^{\mu}_{\cdot\alpha} - \left\{(q-1)q^{\lambda}q_{\alpha\mu} - (q-1)q_\alpha q^{\lambda}_{\cdot\mu} - (p-1)q_\mu q^{\lambda}_{\cdot\alpha}\right\}\bar{X}^{\mu\cdot}_{\cdot\lambda}\right] = 0$$
$$(\alpha = 0,1,2,3,4), \tag{11.3}$$

where $\bar{X}_{\lambda\mu}$ is given by (11.2) above.

Finding the true affinor corresponding to the left side of (11.3) we have

$$\sum_j \overset{R}{\nabla}_j \bar{Z}^j_i + 2(q^2 - 2pq + 2p)\sum_{jl} q_{ji}\overset{R}{\nabla}_l q^{jl} = 0 \quad (j,i = 1,2,3,4), \tag{11.4}$$

where

$$\bar{Z}_{ij} = K_{ij} - \tfrac{1}{2}Kg_{ij} + \Lambda g_{ij} + (q^2 - 2pq - 2p)\left\{\tfrac{1}{2}g_{ij}\sum_{k,l=1}^{4} q_{kl}q^{kl} - 2\sum_{l=1}^{4} q^{\cdot l}_{i} q_{jl}\right\}$$
$$= K_{ij} - \tfrac{1}{2}Kg_{ij} + \Lambda g_{ij} + \frac{\kappa}{c^2}\left(\sum_{k=1}^{4} F^{\cdot k}_{i\cdot}F_{jk} - \tfrac{1}{4}g_{ij}\sum_{k,l=1}^{4} F_{kl}F^{kl}\right)$$
$$= \bar{A}_{ij}, \text{ say.}$$

(11.4) can be written as

$$\sum_j \overset{R}{\nabla}_j \bar{A}_i^j + \frac{\kappa}{c^2} \sum_j F_{ij} B^j = 0$$

or

$$\sum_j \left(\bar{A}_i^j \right)_j + \frac{\kappa}{c^2} \sum_j F_{ij} B^j = 0, \tag{11.5}$$

where $\bar{A}_{ij} = 0$ gives Einstein's combined and electromagnetic field-equations with Λ introduced into them,

and $$B^j = \sum_i \overset{R}{\nabla}_i F^{ji} = 0 \quad (i,j = 1,2,3,4)$$

gives Maxwell's equations.

ON SYMMETRIC FUNCTIONS OF n ELEMENTS IN A BOOLEAN ALGEBRA*

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The elements of a Boolean Algebra may be interpreted either as classes or as attributes. We shall as a rule adopt the former interpretation in this paper. The operations of the Boolean Algebra are $+$, $\times$, and negation. The sum $A+B$ of two classes A, B is defined as the class of elements belonging either to A or to B. The product $A\,B$ is defined as the class of elements belonging to both A, B; while, the negative of A—which is denoted by A', is the class of elements not belonging to A. The Null class and the Universe of Discourse are represented by 0 and I respectively. We have clearly the relations:

$$I+A=I;\ 0\times A=0;\ I\times A=0+A=A;\ A+A'=I;\ A\times A'=0.$$

By means of these operations, we can form various types of symmetric functions from n given elements $(A_1, A_2, \ldots, A_n)$ of a Boolean Algebra. Dr. Vaidyanathaswamy† has considered a type of symmetric functions $\alpha_1, \alpha_2, \ldots, \alpha_n$ of $A_1, A_2, \ldots, A_n$, where $\alpha_r\,(A_1, A_2, \ldots, A_n)$ is defined as the class of elements belonging to at least r of the classes $A_1, A_2, \ldots, A_n$; or symbolically, α_r may be expressed as the sum of products r at a time of $A_1, A_2, \ldots, A_n$ or alternatively, as product of sums $n-r+1$ at a time of the A's. He has shown that these α's are the only set of symmetric functions which can be formed from $A_1, A_2, \ldots, A_n$ solely by the two operations $+$ and $\times$.

If however, we admit negation also, it may be expected that the α's and α''s would together form a set of symmetric functions in terms of which, every symmetric function of the elements (formed with $+$, $\times$, and negation) may be expressed by means of $+$ and $\times$ alone. The question arises whether we do really require as many as these $2n$ functions for forming a fundamental system of symmetric functions. In this paper, a more economical set

* I am grateful to Dr. R. Vaidyanathaswamy for his kind guidance during the preparation of this paper.

† R. Vaidyanathaswamy (9).

of symmetric functions $\beta_0, \beta_1, \ldots, \beta_n$ is advocated,—where $\beta_r(A_1, A_2, \ldots, A_n)$ is the class of elements belonging to *exactly r* of the classes A, (or if A's are attributes, β_r is the attribute of possessing *exactly r* of the attributes $A_1, A_2, \ldots, A_r$). Indeed, the concept *exactly r* (leading to the symmetric function β_r) is more elementary than the concept *at least r* (which gives the function α_r), in as much as the latter can be split up into the concepts *exactly r*, or *exactly* $r+1$, or... .

Apart from the operations $+$, $\times$, and negation, two other Boolean operations* $\oplus$ and $\otimes$ (which will be called Disjunction and Conjunction respectively) are current. The disjoint of two elements A and B is defined as the class of elements belonging to exactly one (i.e. to one but not to both) of the classes A, B; while the conjoint of A and B is the class of elements belonging to both or to neither of the classes A, B. These two operations can be expressed in terms of the three fundamental operations $+$, $\times$, and negation. Thus

$$A \oplus B = AB' + A'B; \quad A \otimes B = AB + A'B'.$$

The importance of these two operations is brought out by Bernstein†, who has shown that the most general operations R for which the elements of a Boolean Algebra form a group (which must be necessarily Abelian) are given by

$$ARB = a(A \oplus B) + a'(A \otimes B),$$

where a is arbitrary. We can easily verify the group properties of R; for, putting in the above expression $B = a'$,

$$ARa' = Aa + Aa' = A.$$

Hence, the identical element is a'. The inverse element is obtained as the unique solution of x in the equation

$$ARx = a(A \oplus x) + a'(A \otimes x) = a'.$$

It can be shown that x is equal to A, so that each element is its own inverse. It is obvious that R is associative and commutative.

Hence, the elements of a Boolean Algebra form an Abelian group with respect to the operations R.

* The choice of our notation $\oplus$ and $\otimes$ for disjunction and conjunction respectively, is suggested by the fact that disjunction distributes $\times$ and conjunction $+$ (Theorem II of this paper). The disjunction and conjunction occur in Whitehead (10) as Discriminants. Bernstein (4) following Peano, denotes $\oplus$ by O and $\otimes$ by Δ. He calls the former 'The operation of complete disjunction'. Stone (7) also denotes $\otimes$ by a triangle and calls it 'symmetric difference'.

† Bernstein (1), (2) and (4). He gives an erroneous version of this theorem in (1) and corrects it later in (2).

The object of this paper is to study the elementary symmetric functions $\beta_r(A_1, A_2, \ldots, A_n)$ $(r=0, 1, \ldots, n)$ and to express in terms of them various other types of symmetric functions formed from the n elements A, by the three primary Boolean operations $+, \times$, and negation, as well as the secondary operations $\oplus$ and $\otimes$. We shall also see that in a certain sense, the β's can be regarded as the extension of the disjoint of two elements of a Boolean Algebra.

I. THE BOOLEAN ALGEBRA GENERATED BY THE n ELEMENTS $A_1, A_2, \ldots, A_n$

If a Boolean Algebra B_n, be generated from n elements $A_1, A_2, \ldots, A_n$, then since $A_i + A_i' = I$, we may write, following Bernstein,*

$$I = (A_1+A_1')(A_2+A_2')\ldots(A_n+A_n')$$
$$= A_1A_2\ldots A_n + A_1'A_2\ldots A_n + \ldots + A_1'A_2'\ldots A_n'. \quad (1)$$

The expression (1) is called by Bernstein, the 'complete additive normal development of I, with respect to the generating elements† $A_1, A_2, \ldots, A_n$'. The 2^n terms of this have special properties in B_n. Thus, they are all obviously different from each other; and their sum is I by (1). Further, whatever two terms of (1) we choose, there is obviously an element A_1 occurring positively in one and negatively in the other. Hence, the product of any two terms chosen from (1) is zero. Bernstein‡ calls the terms of (1) 'minimals' and Huntington‖ names them 'irreducible elements', while Stone§ appropriately terms them 'Atomic elements'. The fundamental property of these minimals is given by

THEOREM I. *Any element of the Boolean Algebra B_n can be expressed uniquely as a sum of the terms of* (1) *i.e. as a sum of minimals; hence, the order of B_n is 2^{2^n}.*

PROOF: Since the number of minimals is 2^n, we can form 2^{2^n} sums from them, including zero also as a sum. We shall prove that the totality of these sums of minimals forms a sub-algebra (M) of B_n. For, we note that if S_1 and S_2 be any two sums of minimals, then S_1+S_2 is also a sum of minimals; namely of those minimals which occur in one at least of S_1, S_2. Again,

* Bernstein (3).

† Bernstein obviously means generating elements, when he says 'if $A_1, A_2, \ldots, A_n$ be *all* the n elements of the Boolean Algebra'. See Bernstein (3), p. 733.

‡ Bernstein (3). ‖ Huntington (6). § Stone (8).

since the product of any two minimals is zero, $S_1 \times S_2$ is the sum of those minimals which occur in both S_1 and S_2. Further, S_1' (the negative of S_1) is the sum of those minimals which do not occur in S_1. Hence, the sum, product, and the negatives of sums of minimals are also sums of minimals. Thus, the totality of sums of minimals form a subalgebra (M) of B_n. This subalgebra is identical with B_n, since the generating elements of B_n are also sums of minimals. For, we have

$$A_1 = A_1.I = A_1(A_2 + A_2')\ldots(A_n + A_n')$$
$$= \text{sums of those minimals in which } A_1 \text{ enters positively.}$$

From the identity of M and B_n, it follows that any element of B_n can be uniquely expressed as a sum of minimals of B_n. Since there are 2^{2^n} sums, it follows that B_n is of order 2^{2^n}.

We shall now consider those elements of B_n which are symmetric functions of the elements $A_1, A_2, \ldots, A_n$. Now, since the sums and products as well as the negatives of symmetric functions are also obviously symmetric functions, it is clear that the symmetric functions of the elements A form a subalgebra of the Boolean Algebra B_n. To find the order of this subalgebra we express the symmetric functions as sums of minimals by Theorem I. In such an expression, it is clear that the minimals can occur only in the following symmetric groups $n+1$ in number:

$$\beta_n = A_1\, A_2 \ldots A_n.$$
$$\beta_{n-1} = A_1\, A_2 \ldots A_{n-1}\, A'_n.$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\beta_r = A_1\, A_2 \ldots A_r\, A'_{r+1} \ldots A'_n$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\beta_0 = A'_1 A'_2 \ldots A'_n.$$

Thus, all symmetric functions are expressible as sums of these β's. Hence, $\beta_0, \beta_1, \ldots, \beta_n$ are the $n+1$ minimals of the subalgebra of the symmetric functions; therefore, the order of this subalgebra is 2^{n+1}.

We shall say a symmetric function is in the normal form, when it is expressed as a sum of β's.

The expression for β_r given above, shows that β_r is the class of elements belonging to exactly r of the classes A. If $n=2$ and $r=1$, then it is clear that $\beta_1(A_1, A_2)$ is the disjoint of A_1 and A_2. Thus, β_r can be regarded as the extension of the disjoint of two elements.

As minimals of the subalgebra of symmetric functions, the β's have the following two properties;

1. *The product of two different β's is zero.**
(This is obvious, since 'exactly r' contradicts 'exactly s', $r \neq s$).

2. *Sum of all the β's$=I$.*
The β's can be related to the functions α, α' by the following:—

(i) $\beta_r = \alpha_r \alpha'_{r+1}$

(for, 'exactly r' means 'at least r' but not 'at least $r+1$').

(ii) $\alpha_r = \beta_r + \beta_{r+1} + \ldots + \beta_n$.
$\alpha'_r = \beta_{r-1} + \beta_{r-2} + \ldots + \beta_0$.

(These follow from the meaning of the concept 'at least r').

†It is clear that the β's cannot be expressed in terms of the α's by means of $+$ and $\times$ alone.

II. THE OPERATIONS OF DISJUNCTION AND CONJUNCTION

The operations of Disjunction and Conjunction (defined above) are commutative and associative. Further we have‡

THEOREM II. *Disjunction distributes $\times$; Conjunction distributes $+$.*

PROOF: For, $A(B \oplus C)$ is the class of elements belonging to both the classes A and $B \oplus C$, i.e. belonging either to both A and B but not to C, or to both A and C, but not to B. This is the same as saying that the elements of $A(B \oplus C)$ either belong to both A, B but not to both A, C; or belong to both A, C but not to both A, B. Hence, $A(B \otimes C)$ is the class of elements belonging to exactly one of the classes AB, AC.

Hence, $A(B \oplus C) = AB \oplus AC$.

Similarly, $A+(B \otimes C)$ is the class of elements belonging to at least one of the classes A, $B \otimes C$, i.e. to A or both BC or to neither of B, C. This is the same as saying that it belongs to (i) either A or both B, C, (ii) or to A' and both B', C'. Now if an element belongs to A or BC, it belongs to both the classes $A+B$, $A+C$. If it belongs to both A', B' C', it belongs to neither $A+B$ nor $A+C$.

Hence $A+(B \otimes C)$ is the class of elements belonging either to both the classes $A+B$, $A+C$, or to none of them.

* Compare $\alpha_r \alpha_s = \alpha_s$ if $r < s$, $\Sigma \alpha_r = \alpha_1$.

† If, however, β_r be interpreted as the number of elements in the class β_r, then it is an Arithmetico-Logical Symmetric Function of the A's; and by the general theorem proved in Vaidyanathaswamy (9) can be expressed as a linear function in the α's with integral coefficients (positive or negative). The expression is in fact, $\beta_r = \alpha_r - \alpha_{r+1}$.

‡ Various identities involving these operations, will be found in Bernstein (4).

Hence $A+(B\otimes C)=(A+B)\otimes(A+C)$.

Regarding the continued disjoint and conjoint of r elements, we have the following theorem which does not appear to have been stated previously.

THEOREM III. *The continued disjoint of r classes $A_1, A_2,\ldots, A_r$ is the class of elements belonging to an odd number of the classes A; the continued conjoint of r classes $A_1, A_2,\ldots, A_r$, is the class of elements belonging to all but an even number of the classes A.*

These can be proved by induction.

By definition the theorem is obviously true for disjoint and conjoint of two classes A_1 and A_2. Assume it to be true for r classes $A_1, A_2,\ldots A_r$; and write

$$V_r=A_1\oplus A_2\oplus\ldots\oplus A_r;\ V_{r+1}=A_1\oplus A_2\ldots\oplus A_r\oplus A_{r+1}.$$
$$W_r=A_1\otimes A_2\otimes\ldots\otimes A_r;\ W_{r+1}=A_1\otimes A_2\ldots\otimes A_r\otimes A_{r+1}.$$

Then, by hypothesis, V_r is the class of elements belonging to an odd number of $A_1, A_2,\ldots, A_r$; and W_r is the class of elements belonging to all but an even number of $A_1, A_2,\ldots, A_r$. We shall now show that V_{r+1} is the class of elements belonging to an odd number of $A_1, A_2,\ldots, A_{r+1}$, and W_{r+1} is the class of elements belonging to all but an even number of $A_1, A_2,\ldots, A_{r+1}$. For, from the mode of formation,

$$V_{r+1}=V_r\oplus A_{r+1};\ W_{r+1}=W_r\otimes A_{r+1}.$$

Taking the former case, it is obvious that an element belonging to an odd number of classes $A_1, A_2,\ldots, A_{r+1}$ must either belong to A_{r+1} and to an even number of classes $A_1, A_2,\ldots, A_r$, or not belong to A_{r+1} but belong to an odd number of $A_1, A_2,\ldots A_r$. In the former case, it belongs to A_{r+1} but not (by hypothesis) to V_r; and in the latter case it belongs to V_r (by hypothesis) but not to A_{r+1}. This is the same as saying that the class of elements belonging to an odd number of classes $A_1, A_2,\ldots, A_{r+1}$ is identical with the disjoint of V^r and A_{r+1}, i.e. it is identical with V_{r+1}. Thus the first part of the theorem is proved true for $r+1$ classes.

In the case of the conjoint, an element which belongs to all but an even number of classes $r+1$, must either belong to A_{r+1} and to all but an even number of classes $A_1, A_2, \ldots, A_r$, or not belong to A_{r+1} and belong to all but an odd number of $A_1, A_2,\ldots, A_r$. In other words it belongs to A_{r+1} and (by hypothesis) to W_r, or it belongs to neither A_{r+1}, nor (by hypothesis) to W_r. This is the same as saying that the class of elements belonging to all but an even number of classes $A_1, A_2,\ldots, A_{r+1}$ is identical with the conjoint of W_r and A_{r+1}, i.e. with W_{r+1}. Thus the second part of the theorem is also true for $r+1$ classes; since the theorem is true for $r=2$, the induction is complete.

Principle of Duality for $\oplus$ *and* $\otimes$.

It is well known* that the negative of any logical function $f(x_1, x_2, \ldots, x_k)$ formed by the two logical operations $+$, $\times$ (x's being arbitrary) is given by $F(x'_1, x'_2, \ldots, x'_k)$, where F is obtained by interchanging $+$ and $\times$ in f. It is easy to see that the theorem can be extended in the following form so as to include $\oplus$ and $\otimes$. The negative of a logical function $f(x_1, x_2 \ldots, x_k)$ formed by the operations $+, \times, \oplus, \otimes$ is given by $F'(x'_1, x'_2, \ldots, x'_k)$ where F' is obtained by interchanging $+$ and $\times$, $\otimes$ and $\oplus$, in f. For, it is clear that any logical function F is built up from less complex logical functions $f_1, f_2, \ldots, f_r$ by one of the four logical operations $+, \times, \oplus, \otimes$. Hence, if it can be shown that the Principle of Duality is true for f_1+f_2, $f_1 \times f_2$, $f_1 \oplus f_2$, $f_1 \otimes f_2$, whenever it holds for f_1 and f_2, then this would furnish an inductive proof of the Principle of Duality for any function F. This, however, is quite simple to establish and therefore the induction can be carried through. The values of the twelve symmetric functions given in the following table (see next section) may serve as illustrations of the Principle of Duality. The functions are easily seen to arrange themselves into six dual pairs (1, 2; 3, 9; 4, 10; 5, 7; 6, 8; 11, 12) for each of which, the Principle of Duality can be verified.

III. CERTAIN TYPES OF SYMMETRIC FUNCTIONS

The functions α_r defined in Vaidyanathaswamy (**9**) as the logical sum of logical products r at a time, of the n Boolean elements $A_1, A_2, \ldots, A_n$ can be conveniently indicated by the notation

$$\alpha_r(A_1, A_2, \ldots, A_n) = S_r(\times, + ; A_1, A_2, \ldots A_n).$$

It is clear that if instead of $\times, +$, we substitute any two of the four Boolean operations $+, \times, \oplus, \otimes$, we can form ${}_4P_2=12$ distinct symmetric functions of this type. We proceed to express all these functions in the normal form (as sums of β's). In evaluating these functions, we vary the range of r from 2 to $n-1$, since for $r=1$ and for $r=n$, one of the two operations does not come into play. The following table exhibits the functions in the normal form.

* See my paper 'On the Arithmetico-Logical Principle of Duality', *Jour. Indian Math. Soc.* (2) 1 (1935), 269-75, § 1. See also Hilbert and Ackerman (**5**), Whitehead (**10**).

No.	Elements $A_1, \ldots A_n$		Values
1	$S_r(\times, +)$	=	$\Sigma \beta_i \qquad (i=r, r+1, \ldots, n)$
2	$S_r(+, \times)$	=	$\Sigma \beta_i \qquad (i=n-r+1, \ldots, n)$
3	$S_r(\times, \oplus)$	=	$\Sigma \beta_i$ summed for values of i such that $\binom{i}{r}$ is odd.
4	$*S_r(\oplus, \times)$	=	$\begin{cases} 0 \text{ when } r \text{ is even} \\ \beta_n \text{ when } r \text{ is odd.} \end{cases}$
5	$S_r(+, \oplus)$	=	$\Sigma \beta_i$ summed for values of i such that $\binom{n}{r}-\binom{n-i}{r}$ is odd.
6	$\dagger S_r(\oplus, +)$	=	$\begin{cases} \beta_1+\beta_2+\ldots+\beta_{n-1} \text{ when } r \text{ is even.} \\ \beta_1+\beta_2+\ldots+\beta_n \text{ when } r \text{ is odd.} \end{cases}$
7	$S_r(\times, \otimes)$	=	$\Sigma \beta_i$ summed for values of i such that $\binom{n}{r}-\binom{i}{r}$ is even.
8	$\ddagger S_r(\otimes, \times)$	=	$\begin{cases} \beta_n \text{ when } r \text{ is odd.} \\ \beta_0+\beta_n \text{ when } r \text{ is even.} \end{cases}$
9	$S_r(+, \otimes)$	=	$\Sigma \beta_i$ summed for values of i such that $\binom{n-i}{r}$ is even.
10	$\Vert S_r(\otimes, +)$	=	$\begin{cases} I \text{ when } r \text{ is even.} \\ \beta_1+\beta_2+\ldots+\beta_n \text{ when } r \text{ is odd.} \end{cases}$
11	$S_r(\oplus, \otimes)$	=	$\Sigma \beta_i$ summed for values of i such that $\binom{n-i}{r}+\binom{i}{2}\binom{n-i}{r-2}+\binom{i}{4}\binom{n-i}{r-4}+\ldots$ is even.
12	$S_r(\otimes, \oplus)$	=	$\Sigma \beta_i$ summed for values of i such that $\binom{i}{r}+\binom{i}{r-2}\binom{n-i}{2}+\binom{i}{r-4}\binom{n-i}{4}+\ldots$ is odd.

NOTE.—The formulae are seen to be valid also for the end values $r=1$, $r=n$, with certain exceptions marked, for which we have as follows:

* $S_r(\oplus, \times)=$ Sum of odd β's for $r=n$.

† $S_r(\oplus, +)=$ Sum of odd β's for $r=n$.

‡ $S_r(\otimes, \times)=\Sigma\beta_i$, summed for values of i such that $n-i$ is even for $r=n$.

|| $S_r(\otimes, +)=\Sigma\beta_i$, summed for values of i such that $n-i$ is even for $r=n$.

PROOFS:

1. This follows from the definition of α_r.

2. From the alternative definition of α_r as the product of sums $n-r+1$ at a time of the A's, we have

$$S_r(+, \times)=\alpha_{n-r+1}=\beta_i \, (i=n-r+1, \ldots, n).$$

Equalities 3—12 can be proved by adopting the following general method of procedure:

The condition for β_i to occur in the normal form, is that an element E belonging to exactly i of the classes A must belong to the class specified by the symmetric function S_r. Hence, we consider an element E which belongs to exactly i of the classes A, and we determine the values of i for which it belongs to S_r. We thereby determine all the β's occurring in the normal form. Thus:

3. E belongs to the class $S_r(\times, \oplus)$, if it belongs to an odd number of sets of r classes of the type $A_1, A_2, \ldots, A_r$ (by Theorem III). But since E belongs to exactly i classes A, it is clear that it belongs to $\binom{i}{r}$ sets of r classes A. Hence, E belongs to $S_r(\times, \otimes)$ only if $\binom{i}{r}$ is odd. Thus, $S_r(\times, \oplus)=\Sigma\beta_i$ summed for values of i such that $\binom{i}{r}$ is odd.

4. The condition that E belongs to the class specified by the symmetric function $S_r(\oplus, \times)$ is that every set of r classes A has an odd number of classes (by Theorem III) in common with the i classes. When r is even, this is impossible unless for $r=n$, in which case, S_r=sum of odd β's. When r is odd, this condition is satisfied only if $i=n$; so that $S_r(\oplus, \times)=0$ if r is even and $=\beta_n$ if r is odd.

5. E belongs to the class defined by $S_r(+, \oplus)$ if it belongs to an odd number of classes of the type $A_1+A_2+\ldots+A_r$ (by Theorem III). Now, since by definition E belongs to exactly i of the classes A, there are $\binom{n-i}{r}$ sums of r classes A, not comprising any of the given i classes, i.e. E belongs to $\binom{n}{r}-\binom{n-i}{r}$ sums of r classes. Hence, for E to belong to the class $S_r(+, \otimes)$, $\binom{n}{r}-\binom{n-i}{r}$ is odd. Thus $S_r(+, \oplus)=\Sigma\beta_i$ summed for values of i such that $\binom{n}{r}-\binom{n-i}{r}$ is odd.

6. If E is to belong to $S_r(\oplus, +)$ it must belong to one at least of the disjoints of the classes A, r at a time. The condition for this is, that there should exist a set of r classes which comprise an odd number from among the i classes. This is impossible when $i=0$ for all values of r, and when $i=n$ for even values of r. In all other cases β_i occurs in the normal form. Hence, $S_r(\oplus, +)=\Sigma\beta_i$, where $i=1,2,\ldots, n-1$ when r is even and $i=1, 2,\ldots, n$ when r is odd.

7. If E is to belong to $S_r(\times, \otimes)$, it must belong (by Theorem III, to $\left\{\binom{n}{r} - \text{an even number}\right\}$ of products of r classes. But since by hypothesis E belongs to $\binom{i}{r}$ products of r classes, this can happen only if $\binom{i}{r}$ is of the form $\binom{n}{r}$ —an even number. Hence, if E is to belong to S_r, i.e. if β_i is to occur in the normal form, $\binom{n}{r}-\binom{i}{r}$ is even. Thus, $S_r(\times, \otimes)=\Sigma\beta_i$ summed for values of i such that $\binom{n}{r}-\binom{i}{r}$ is even.

8. As before, E must belong to every conjoint of r classes A if it is to belong to $S_r(\otimes, \times)$, i.e. every set of r classes A must contain an even number not included among the given i classes. If $i=n$, this condition is evidently satisfied. When $i=0$, this is satisfied only when r is even. For other values of i, it is clear that at least one set of r classes can be found, which comprise an odd number other than the i classes. Hence, $S_r(\otimes, \times)=\beta_0+\beta_n$ when r is even, and $=\beta_n$ when r is odd.

9. E belongs to the class $S_r(+, \otimes)$ if it belongs to the conjoint of sums of r classes A. But, it belongs to $\binom{n}{r}-\binom{n-i}{r}$ sums of r classes A; and this must be of the form $\binom{n}{r}$ —an even number of sums (by Theorem III). Hence, $S_r(+, \otimes)=\Sigma\beta_i$ summed for values of i such that $\binom{n-i}{r}$ is even.

10. E belongs to the class $S_r(\otimes, +)$ if it belongs to one conjoint at least of the sets of r classes A. The condition for this s that there must exist a set of r classes which has $r-2k$ classes (by Theorem III) in common with the i classes. This is always possible except for the case $i=0$, and r is odd. Hence β_i occurs in the normal form, always when r is even, and except for $i=0$ when

r is odd. Thus, $S_r(\oplus, +) = I$ when r is even and to $\beta_1+\beta_2+\dots+\beta_n$ when r is odd.

11. Since E belongs to exactly i classes, it would belong to the disjoint of a set of r classes A only if the set of r classes includes an odd number from among the i classes. The number of such sets of r classes is given by

$$x=\binom{i}{1}\binom{n-i}{r-1}+\binom{i}{3}\binom{n-i}{r-3}+\dots.$$

If E is to belong to $S_r(\oplus, \otimes)$, it is clear that it must belong to $\left\{\binom{n}{r}-\text{an even number}\right\}$ of disjoints of r classes. Hence, x is of the form $\binom{n}{r}-$ an even number. Therefore, i is such that $\binom{n}{r}-x$ is even. Thus, $S_r(\oplus, \otimes)=\Sigma\beta_i$ summed for values of i such that

$$\binom{n}{r}-\left[\binom{i}{1}\binom{n-i}{r-1}+\binom{i}{3}\binom{n-i}{r-3}+\dots\right]$$

is even, i.e.

$$\binom{n-i}{r}+\binom{n-i}{r-2}\binom{i}{2}+\binom{n-i}{r-4}\binom{i}{4}+\dots$$

is even.

12. Since E belongs to exactly i classes, it would belong to the conjoint of a set of r classes A, only if all but an even number of the r classes occur among the i classes. The number of such sets of r classes given by

$$y=\binom{i}{r}+\binom{i}{r-2}\binom{n-i}{2}+\binom{i}{r-4}\binom{n-i}{4}+\dots$$

is odd. If E is to belong to the disjoint of these conjoints of r classes, then y must be odd (Theorem III). Hence, $S_r(\otimes, \oplus)=\Sigma\beta_i$, summed for values of i such that

$$\binom{i}{r}+\binom{i}{r-2}\binom{n-i}{2}+\binom{i}{r-4}\binom{n-i}{4}+\dots$$

is odd.

The above results obtained for these twelve types of symmetric functions of n elements show that we can evaluate the same functions for any type of elements provided that the β's of these elements can be calculated. For instance, consider the symmetric functions of the $2n$ elements $A_1, A_2, \dots, A_n, A'_1, A'_2, \dots, A'_n$. To evaluate $\beta_r(A_1, A_2, \dots, A_n, A'_1, A'_2, \dots, A'_n)$, we note that any element which belongs to exactly k of the classes A, belongs by that very fact to exactly $n-k$ of the classes A'. Hence, it belongs to exactly n of the classes (A, A'). Since this is true for any element, we see that $\beta_r=0$ except for $r=n$ when $\beta_n=I$. We can now by our formulae derive the values of the twelve symmetric functions of these $2n$ arguments.

In the same manner, the evaluation of symmetric functions of α's and β's reduces to the evaluation of their β functions.

Consider any set of $k\alpha$'s—$\alpha_p, \alpha_q, \ldots, \alpha_l, \alpha_m, \ldots, \alpha_z$, where without loss of generality we can assume $p<q<\ldots<l<m<\ldots<z$. Write $B_r=\beta_r(\alpha_p, \alpha_q, \ldots, \alpha_l, \alpha_m, \ldots, \alpha_z)$. Now, since each α contains the next, B_r being the class of elements belonging to exactly n of the α's, these must necessarily be the first r α's. Thus if α_l is the rth α of the series, $B_r=\alpha_l\alpha'_m$, i.e. B_r is the sum of those β's whose suffixes fall between l and m, l included and m excluded $(l<m)$; and $B_0=\beta_0+\beta_1+\ldots+\beta_{p-1}$. We can now evaluate any symmetric function of the α's (since we know their β functions).

If in the above, we put $p=1$, $q=2, \ldots, l=r$, $m=r+1, \ldots, z=n$, B_r reduces to β_r $(r=0, 1\ldots, n)$ that is, the two sets $(\alpha_1, \alpha_2, \ldots, \alpha_n)$, $(A_1, A_2, \ldots A_n)$ have the same corresponding β functions. Hence, if F is any symmetric function of n arguments, we see that by reducing $F(\alpha_1, \alpha_2 \ldots, \alpha_n)$ and $F(A_1, A_2, \ldots, A_n)$ to the normal form, they are identical. Thus we have the following theorem.

THEOREM IV. *Any symmetric function of* $\alpha_1, \alpha_2, \ldots, \alpha_n=$ *the same symmetric function of* $A_1, A_2, \ldots, A_n$.

For instance,

$$S_r(\times, \oplus;\ \alpha_1, \alpha_2, \ldots \alpha_n)=S_r(\times, \oplus;\ A_1, A_2, \ldots, A_n)=\Sigma\beta_i$$

summed for values of i such that $\binom{i}{r}$ is odd.

We may also show that any symmetric function of $\alpha_1, \alpha_2, \ldots, \alpha_n, \alpha'_1, \alpha'_2, \ldots, \alpha'_n$ is equal to the same symmetric function of $A_1, A_2, \ldots, A_n, A'_1, A'_2, \ldots, A'_n$. For this, it is necessary and sufficient to show that the β functions of α and α' are equal to the β functions of A and A'. This is obvious since we have already shown that whatever $C_1, C_2, \ldots, C_n, C'_1, C'_2, \ldots, C'_n$ may be, $\beta_r(C_1, C_2 \ldots, C_n, C'_1, C'_2, \ldots, C'_n)=0$ or I, according as $r\neq$or$=n$. Hence it follows that the β functions of any two sets $A_1, A_2 \ldots, A_n, A'_1, A'_2, \ldots, A'_n$ and $\alpha_1, \alpha_2 \ldots, \alpha_n, \alpha'_1, \alpha'_2, \ldots, \alpha'_n$ are equal. Therefore, any the same symmetric functions of the two sets $\alpha, \alpha'; A, A'$ are equal. We can now evaluate any symmetric function of the α's and α''s.

Lastly, we may apply the same principle to evaluate symmetric functions of any $k\beta$'s of $A_1, A_2, \ldots, A_n$, e.g. of $\beta_p, \beta_q, \ldots, \beta_l, \beta_m, \ldots, \beta_z$. Write $\beta_r(\beta_p, \beta_q, \ldots, \beta_l, \beta_m, \ldots, \beta_z) = B_r$. It is evident that since the product of two distinct β's is zero, all the B's are zero except B_0 and B_1, where B_0=sum of the β's other than $\beta_p, \beta_q, \ldots$ and $B_1 = \beta_p + \beta_q + \ldots$. This enables us to evaluate any symmetric function of the β's.

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A FURTHER NOTE ON THE ZEROS OF BESSEL FUNCTIONS

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1. Even though Bourget's hypothesis* on the zeros of Bessel Functions is still unproved, it is possible to prove that J_n and J_{n+m} have no common zeros for restricted values of n and m. The object of this note is to prove the following

THEOREM 1. *J_n and J_{n+m} have no common zeros except perhaps those at the origin, provided m and n are real, $|m|<1$ and*

$$n > \max\left(\tfrac{1}{2}, \frac{m^2}{2(1-m)}\right).$$

PROOF: First we note that when $n>-1$, J_n has only real zeros. Secondly since $J_n(-z)=e^{ni\pi}J_n(z)$, it is sufficient to consider positive zeros. We start from the relation†

$$J_{n+m}(z)Y_n(z)-Y_{n+m}(z)J_n(z)$$
$$=\frac{4\sin m\pi}{\pi^2}\int_0^\infty K_m(2z\sinh t)e^{(2n+m)t}dt, \tag{1}$$

which is valid for $|m|<1$, $|\arg(z)|<\pi/2$. We suppose $z>0$. We conclude from the relation‡

$$K_m(z)=\frac{z^m\Gamma(\frac{1}{2})\cos m\pi}{2^m\Gamma(m+\frac{1}{2})}\int_0^\infty e^{-z\cosh t}\sinh^{2m}t\,dt, \tag{2}$$

which is valid for $R(m+\frac{1}{2})>0$, $|\arg(z)|<\pi/2$, that when $m\geqslant 0$, $K_m(z)$ keeps the same sign for $z>0$. Since $K_m(z)=K_{-m}(z)$, the same is true for all m positive or negative. Therefore we conclude from (1) that when $|m|<1$,

$$|J_{n+m}(z)Y_m(z)-J_n(z)Y_{n+m}(z)|>0 \tag{3}$$

for $z>0$. Now suppose that $J_n(z_0)=0$. Then by (3)

$$|J_{n+m}(z_0)Y_m(z_0)|>0$$

* G. N. Watson, *Theory of Bessel Functions*, p. 484.

† Loc. cit. p. 447.

‡ G. N. Prasad, *Spherical Harmonics*, II. 160.

provided $|Y_{n+m}(z_0)|<\infty$. Hence $J_{n+m}(z_0)\neq 0$ if $Y_m(z_0)$ and $Y_{n+m}(z_0)$ are both finite. Now it is known* that

$$J_\nu^2(z)+Y_\nu^2(z)<\frac{2}{\pi\sqrt{z^2-\nu^2}}$$

when $\nu\geqslant\frac{1}{2}$. Hence if $z_0>\max\left\{(n,\ n+m),\ \frac{1}{2}\right\}$, $Y_n(z_0)$ and $Y_{n+m}(z_0)$ are both finite. But the least positive root of $J_\nu(z)=0$ is greater† than $\sqrt{n(n+2)}$. Hence if $\sqrt{n(n+2)}>n+m$ or $n>\frac{m^2}{2(1-m)}$, the results in question hold. This proves the theorem.

1.1 Since $J_n(z)$ is finite for all n and $z>0$, the relation (3) gives immediately the following

THEOREM 2. *$Y_n(z)$ and $Y_{n+m}(z)$ have no common positive zeros except may be those at the origin, provided $|m|<1$ and n, m are real.*

* Watson, loc. cit. p. 447.
† G. N. Prasad, loc. cit. p. 154.

Errata for the paper ' On Symmetric functions of n elements in a Boolean Algebra,' by Miss S. Pankajam: Vol. II, No. 5 pp. 198-210.

Page 201 line 24 should read as $\beta_{n-1}=\Sigma A_1 A_2 \ldots A_{n-1} A'_n$

" " " 26 " $\beta_r=\Sigma A_1 A_2 \ldots A_r A'_{r+1} \ldots A'_n$

" 206 " 5 for β_i read $\Sigma\beta_i$

WARING'S PROBLEM V: ON $g(6)$

By S. S. PILLAI, Annamalainagar

[Received 20 December 1936]

1. L. E. Dickson* has proved that $g(6) \leqslant 110$. In this note I prove that
$$g(6) \leqslant 104.$$
Unless the base is indicated, the logarithm is taken to the base e.

2. In **R**(2) put $s=13$ and $n=6$.

LEMMA 1. $$\mathfrak{S}(6) > e^{-166}.$$

Let
$$\Pi_1 = \Pi\chi_p, \ [p \equiv 1, 2 \text{ or } 3 \pmod 6), \ p < 190]$$
$$\Pi_2 = \Pi\chi_p, \ [p \equiv 1 \pmod 6), \ p > 190]$$
and
$$\Pi_3 = \Pi\chi_p, \ [p \not\equiv 1, 2 \text{ or } 3 \pmod 6)].$$
From Lemma (b) in **R**(2), we have
$$\chi_p \geqslant 1 - 2/p^6 - |A(p)|.$$
But
$$|A(p)| < \frac{5^{13}}{p^{13/2}} p = \frac{5^{13}}{p^{11/2}}.$$
So
$$-\log \chi_{61} < 1/4, \ -\log \chi_{79} < 1/16, \ -\log \chi_{103} < 1/80.$$
Again from Lemma (a) in **R**(2)
$$\chi_p \geqslant 1/P^{-(r+1)}.$$
Hence
$$-\log \Pi_1 < \log\{8^8.9^9.(7.13.19.31.37.43)^7\} + \tfrac{3}{4} + \tfrac{2}{16} + \tfrac{9}{80} < 165. \quad (1)$$
$$-\log \Pi_2 < -\sum_{p>190} \log\left[1 - \frac{2}{p^6} - \frac{5^{13}}{p^{11/2}}\right]$$
$$< -\sum_{p\geq 190} \log\left(1 - \frac{1}{p^{3/2}}\right) < \cdot 3. \quad (2)$$
$$-\log \Pi_3 < -\sum_{p\geqslant 5} \log\left(1 - \frac{2}{p^6} - \frac{1}{p^{11/2}}\right)$$
$$< -\sum_{p\geqslant 5} \log\left(1 - \frac{1}{p^5}\right) < \cdot 1. \quad (3)$$

* L. E. Dickson **R**(1).

Since $165+\cdot 3+\cdot 1<166$, from (1), (2) and (3) we get the lemma.

LEMMA 2. *If* $t+33\geqslant 86$, *then every integer less than* L_t *is the sum of* $t+33$ *sixth powers, where*

$$\log_{10}\log_{10} L_t=(\cdot 079)t+\cdot 9.$$

This follows from § 3 in **R**(1).

3. We are now in a position to prove our result. We follow **R**(2). Putting $k=31$ in **R**(2), we get

$$3\sigma(1-1/6n)/2=9(1-\tfrac{1}{6})^{31}(1-\tfrac{1}{36})<1/32.5. \quad (4)$$

Assume $$P>e^{57600},$$

then $$(\log P)^{\frac{1}{2}}<P^{1/5000} \quad (5)$$

From (4) and (5) we have

$$P^{1/4n-1/4n^2-3\sigma(1-1/6n)/2}(\log P)^{-\frac{1}{2}}>P^{1/280}, \quad (6)$$

and

$$24.3^{3n/2}.n^{(3n-1)/2}.B\leqslant 24.3^9.6^{17/2}e^{166}<e^{200}. \quad (7)$$

So in **R**(2) condition *I* is satisfied.

Further when $\log P>60{,}000$ it is easy to verify that all the conditions from *A* to *I* in **R**(2) are satisfied. When $\log N_0>360{,}000+6\log 3$, *P* satisfies the above condition.

Hence from Lemma 15 in **R**(2), we get that every integer greater than β is the sum of 104 sixth powers, when

$$\log\beta>360{,}000+6\log 3. \quad (8)$$

From Lemma 2, it is easy to see, by putting $t=70$, that

$$L_{70}>\beta+1.$$

Therefore every integer less than $(\beta+1)$ is the sum of 103 sixth powers.

From (8) and (9), it follows that

$$g(6)\leqslant 104.$$

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A CORRECTION TO THE PAPER "ON $A^x - B^y = C$"

By S. S. PILLAI

[Received 17 February, 1937]

In a recent issue of the *Zentralblatt für Mathematik*, the reviewer of my paper "On $A^x - B^y = C$" (*J. I. M. S.* New Series Vol. II. No. 3) justly questions the validity of my proof of Lemma 1. The mistake can be easily rectified.

LEMMA 1. *If* $(a, b)=1$, *there are values* x *and* y *such that* (1) $x \geqslant 2$, (2) $b^y = la^x + 1$, *where* $(a, l)=1$.

PROOF: Let
$$a = p_1^{\alpha_1} \dots p_r^{\alpha_r}, \tag{3}$$
where p's are different prime factors of a. Since $(a, b)=1$, there is a y such that
$$b^y \equiv 1 \pmod{a^2}.$$
Let y_1 be the smallest value of $y \geqslant 1$ satisfying the above congruence. Then we can write b^{y_1} in the form
$$b^{y_1} = 1 + M_1 p_1^{\beta_1} \dots p_r^{\beta_r} a^{x_1}, \tag{4}$$
where $x_1 \geqslant 2$, $(M_1, a)=1$ and $\beta_s \leqslant \alpha_s - 1$ at least for one value of s.

Let $t_1 = \Pi p_s^{\alpha_s - \beta_s}$, where s runs through all values for which $\beta_s \leqslant \alpha_s - 1$. Then from (4),
$$b^{y_1 t_1} = 1 + M_1 t_1 p_1^{\beta_1} \dots p_r^{\beta_r} a^{x_1} + M . p_1^{\beta_1} \dots p_r^{\beta_r} a^{2x_1}, \tag{5}$$
$$= 1 + M_1 p_1^{\gamma_1} \dots p_r^{\gamma_r} a^{x_1+1} + M' p_1^{\gamma_1} \dots p_r^{\gamma_r} a^{x_1+2}, \text{ [for } 2x_1 \geqslant x_1+2]$$
$$= 1 + M_2 p_1^{\gamma_1} \dots p_r^{\gamma_r} a^{x_1+1}, \tag{6}$$
where γ_s is zero when $\beta_s \leqslant \alpha_s$ and $\gamma_s \leqslant \beta_s - \alpha_s$ when $\beta_s \geqslant \alpha_s + 1$ and $M_2 = M_1 + M'a$ and so $(M_2, a)=1$; for $(M_1, a)=1$. Let $t_2 = \Pi p_s^{\alpha_s - \gamma_s}$, where s runs through all values for which $\gamma_s \leqslant \alpha_s - 1$.

Then from (6),
$$b^{y_1 t_1 t_2} = 1 + t_2 M_2 p_1^{\gamma_1} \dots p_r^{\gamma_r} a^{x_1+1} + m p_1^{\gamma_1} \dots p_r^{\gamma_r} a^{2x_1+2}$$
$$= 1 + M_2 p_1^{\theta_1} \dots p_r^{\theta_r} a^{x_1+2} + m' p_1^{\theta_1} \dots p_r^{\theta_r} a^{x_1+3}$$
$$= 1 + M_3 p_1^{\theta_1} \dots p_r^{\theta_r} a^{x_1+2},$$
where, $\theta_s \leqslant \gamma_s - \alpha_s$, when $\gamma_s \geqslant \alpha_s + 1$ or else $\theta_s = 0$ and $(M_3, a)=1$; for $M_3 = M_2 + m'a$.

Repeating this process, after a finite number of steps, we arrive at the lemma.

ON THREE ORTHOGONAL CONGRUENCES OF CURVES

BY V. RANGACHARIAR, Science College, Patna

[Received 12 January, 1937]

1. In a paper on "Curvilinear Congruences" by Dr. C. E. Weatherburn, published in the *Transactions of the American Mathematical Society*, Vol. 31 (1929), it has been proved that for three orthogonal congruences of curves, the moments of any two of the congruences in the direction of the third are equal. The object of the present paper is to prove the same result by a different method and also to obtain expressions for the moments and the tendencies in terms of torsions, curvatures, normal angles and the rates of changes of these normal angles.

2. We shall speak of these three congruences as the first, the second and the third congruence. Let a, b, c denote unit tangents and let n_1, b_1; n_2, b_2; n_3, b_3 be the unit principal normals and the binormals to the three congruences respectively. Let also ω_1 be the angle in the positive direction from n_1 to b, ω_2 from n_2 to c, ω_3 from n_3 to a. Let κ_1, τ_1; κ_2, τ_2; κ_3, τ_3 be the curvatures and the torsions of the three congruences. Now the moment of the second congruence in the direction of the first is given by

$$M_{b,a}=a.(\nabla b \times b).a=\frac{db}{ds_1}\times b.a=\frac{db}{ds_1}.b\times a=-c.\frac{db}{ds_1},$$

where s_1 denotes the arc length of the first congruence.

Similarly $$M_{c,a}=a.(\nabla c\times c).a=\frac{dc}{ds_1}.b.$$

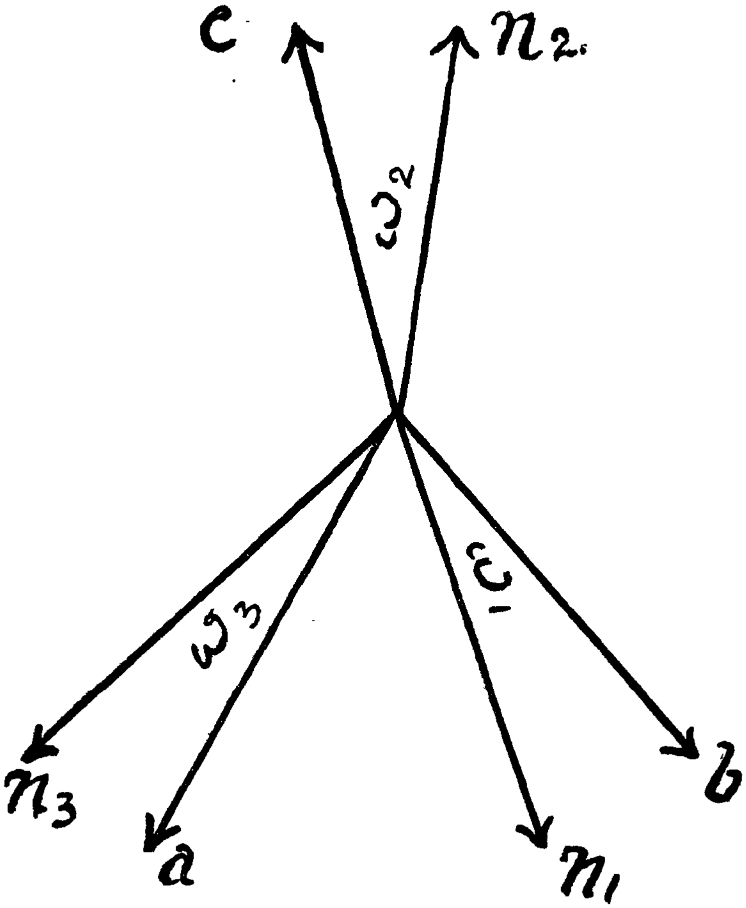
Now because $b.c=0$, $b.\frac{dc}{ds_1}=-c.\frac{db}{ds_1}$; we get

THEOREM I. *The moments of any two of the congruences in the direction of the third are equal.*

Again from the figure given below it is clear that

$$b=n_1\cos\omega_1+b_1\sin\omega_1$$

$$c=b_1\cos\omega_1-n_1\sin\omega_1.$$



Hence $$\frac{db}{ds_1} = (\tau_1 b_1 - \kappa_1 a)\cos\omega - \tau_1 n_1 \sin\omega_1 + c\frac{d\omega_1}{ds_1}$$

and $$M_{b,a} = -c.\frac{db}{ds_1} = -\left(\tau_1 + \frac{d\omega_1}{ds_1}\right).$$

Hence we get

$$M_{b,a} = M_{c,a} = -\left(\tau_1 + \frac{d\omega_1}{ds_1}\right)$$

$$M_{a,b} = M_{c,b} = -\left(\tau_2 + \frac{d\omega_2}{ds_2}\right)$$

$$M_{a,c} = M_{b,c} = -\left(\tau_3 + \frac{d\omega_3}{ds_3}\right).$$

Other results regarding three orthogonal congruences in the same paper by Dr. Weatherburn will immediately follow from these formulae.

3. The tendency of the second congruence in the direction of the first is given by

$$T_{b,\,a}=a.\nabla b.a=\frac{db}{ds_1}.a=-\kappa_1 \cos \omega_1.$$

Similarly

$$T_{c,\,a}=a.\nabla c.a=\frac{dc}{ds_1}.a$$

$$=\left\{-\tau_1 n_1 \cos \omega_1-(\tau_1 b_1-\kappa_1 a) \sin \omega_1-b\frac{d\omega_1}{ds_1}\right\}.a$$

$$=\kappa_1 \sin \omega_1.$$

Remembering that c makes an angle $\pi/2+\omega_1$ with n_1, we get

THEOREM II. *The tendency of any of the congruences in the direction of a second is the negative of the resolved part of the vector curvature of the second in its own direction.*

This theorem also follows from the result $T_{c,\,a}=a.\frac{dc}{ds_1}=-c.\frac{da}{ds_1}$. It is in complete agreement with the fact that the tendency of any curve on a surface along the orthogonal trajectory is the negative of the geodesic curvature of the orthogonal trajectory. It follows that if any of the congruences is a congruence of straight lines, it is a direction of zero tendency for each of the other two congruences.

4. We shall next find out the tendency and the moment of the second congruence in any normal direction defined by the relation $t=a \cos \theta+c \sin \theta$.

Now

$$b=n_1 \cos \omega_1+b_1 \sin \omega_1=b_3 \cos \omega_3-n_3 \sin \omega_3$$

$$c=b_1 \cos \omega_1-n_1 \sin \omega_1,$$

$$a=n_3 \cos \omega_3+b_3 \sin \omega_3$$

$$T_{b,\,t}=t.\nabla b.t$$

$$=(a \cos \theta+c \sin \theta).\nabla b.t$$

$$=\left(\cos \theta\frac{db}{ds_1}+\sin \theta\frac{db}{ds_3}\right).t$$

$$=\cos \theta\left[(\tau_1 b_1-\kappa_1 a)\cos \omega_1-\tau_1 n_1 \sin \omega_1+c\frac{d\omega_1}{ds_1}\right].t$$

$$+\sin \theta\left[-\tau_3 n_3 \cos \omega_3-(\tau_3 b_3-\kappa_3 c)\sin \omega_3-a\frac{d\omega_3}{ds_3}\right].t$$

$$=\cos \theta\left\{-\kappa_1 \cos \omega_1 \cos \theta-\sin \theta\left(\tau_3+\frac{d\omega_3}{ds_3}\right)\right\}$$

$$+\sin \theta\left\{\cos \theta\left(\tau_1+\frac{d\omega_1}{ds_1}\right)+\sin \theta\, \kappa_3 \sin \omega_3\right\}$$

$$=T_{b,\,a} \cos^2\theta+T_{b,\,c} \sin^2\theta+\cos \theta \sin \theta\,(M_{b,\,c}-M_{b,\,a}).$$

Hence a and c will be the principal directions of the normal tendency conic for the second congruence if the normal moment conic be a circle, or if they be the directions of zero moments, in which case the total moment of the second congruence is zero. In particular for a triply orthogonal normal congruence of curves any two of the congruences are principal directions for the normal tendency conic of the third congruence.

Again the moment of the second congruence in the direction of t is given by

$$\begin{aligned} M_{b,t} &= t(\nabla b \times b).t \\ &= (t.\nabla b).b \times t \\ &= (t.\nabla b).(a \sin\theta - c\cos\theta) \\ &= \sin\theta\,(t.\nabla b.a) - \cos\theta\,(t.\nabla b.c) \\ &= \sin\theta\left\{-\cos\theta\,\kappa_1\cos\omega_1 - \sin\theta\left(\tau_3 + \frac{d\omega_3}{ds_3}\right)\right\} \\ &\quad -\cos\theta\left\{\cos\theta\left(\tau_1 + \frac{d\omega_1}{ds_1}\right) + \kappa_3\sin\theta\sin\omega_3\right\} \\ &= M_{b,a}\cos^2\theta + M_{b,c}\sin^2\theta + \cos\theta\sin\theta\,(T_{b,a} - T_{c,a}). \end{aligned}$$

Hence again a and c will be the principal directions of the normal moment conic if the normal tendency conic is a circle or else they are the directions of zero tendency. In particular, for a triply orthogonal rectilinear congruence any two of the congruences are principal directions for the normal moment conic of the third congruence.

5. It is evident from the above that the normal tendency and moment conics are given by

$$T_{b,a}x^2 + T_{b,c}y^2 + xy(M_{b,c} - M_{b,a}) = 1$$
$$M_{b,a}x^2 + M_{b,c}y^2 + xy(T_{b,a} - T_{b,c}) = 1.$$

The squares of the reciprocals of the axes of the first conic—the principal tendencies—are the roots of the equation

$$\begin{vmatrix} 2(T_{b,a} - Z), & M_{b,c} - M_{b,a} \\ M_{b,c} - M_{b,a}, & 2(T_{b,c} - Z) \end{vmatrix} = 0$$

i.e., $4Z^2 - 4Z(T_{b,a} + T_{b,c}) - \left\{ (M_{b,c} - M_{b,a})^2 - 4T_{b,a}T_{b,c} \right\} = 0.$

The difference between the roots

$$= \left\{ (T_{b,a} - T_{b,c})^2 + (M_{b,a} - M_{b,c})^2 \right\}^{\frac{1}{2}}$$

From the symmetry of this result we have

THEOREM III. *The differences between the principal tendencies and moments in the normal plane of a congruence belonging to a triply orthogonal system are equal.*

A FEW SELF-RECIPROCAL FUNCTIONS

By B. M. MEHROTRA and R. V. SHASTRY

[Received 23 August 1936]

Some formulae of Ramanujan have been studied by Dr. B. M. Mehrotra* with a view to find out self-reciprocal functions. There are some more formulae in his papers from which certain other self-reciprocal functions may be derived. In this short note we bring out two such functions.

If a function $f(x)$ satisfies the singular homogeneous integral equation

$$f(x)=\sqrt{\frac{2}{\pi}}\int_0^\infty f(y)\cos xy\,dy, \tag{1}$$

$f(x)$ is said to be self-reciprocal for cosine transforms.

In the same way, a function $f(x)$ satisfying the equation

$$f(x)=\sqrt{\frac{2}{\pi}}\int_0^\infty f(y)\sin xy\,dy, \tag{2}$$

is said to be self-reciprocal for sine transforms.

Following Hardy and Titchmarsh we will denote self-reciprocal functions for sine and cosine transforms by R_s and R_c respectively.

In 1915 Ramanujan† gave the formulae:—

$$\int_0^\infty \frac{\sin \pi x^2}{\sinh \pi x}\sin 2\pi tx\,dx=\frac{\sin \pi t^2}{2\sinh \pi t} \tag{3}$$

$$\int_0^\infty \frac{\cos \pi x^2}{\cosh \pi x}\cos 2\pi tx\,dx=\frac{1+\sqrt{2}\sin \pi t^2}{2\sqrt{2}\cosh \pi t} \tag{4}$$

$$\int_0^\infty \frac{\sin \pi x^2}{\cosh \pi x}\cos 2\pi tx\,dx=\frac{-1+\sqrt{2}\cos \pi t^2}{2\sqrt{2}\cosh \pi t}. \tag{5}$$

Putting $x=ay$ in (3) we get

$$a\int_0^\infty \frac{\sin \pi a^2y^2}{\sinh \pi ay}\sin 2\pi tay\,dy=\frac{\sin \pi t^2}{2\sinh \pi t}. \tag{6}$$

* B. M. Mehrotra, 'A brief history of self-reciprocal functions'. *Jour Ind. Math. Soc.* (2) I (1935), 209-27.

† "Some definite integrals connected with Gauss's sums", *Messenger of Mathematics*, XLIV (1915), 75-85.

Again, putting $2\pi ta = x$
in (6) we get,

$$a\int_0^\infty \frac{\sin \pi a^2 y^2}{\sinh \pi a y} \sin xy\, dy = \frac{\sin (x^2/4\pi a^2)}{2 \sinh (x/2a)}. \tag{7}$$

Let $$a^2 = \frac{1}{2\pi}.$$

Then we get by simplification from (7)

$$\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\sin y^2/2}{\sinh y\sqrt{\pi/2}} \sin xy\, dy = \frac{\sin x^2/2}{\sinh x\sqrt{\pi/2}}. \tag{8}$$

Hence it follows from (2) that the function

$$\frac{\sin x^2/2}{\sinh x\sqrt{\pi/2}}$$

is R_s.

Similarly we can get from (4) and (5) the relations

$$\frac{2}{\sqrt{\pi}}\int_0^\infty \frac{\cos y^2/2}{\cosh y\sqrt{\pi/2}} \cos xy\, dy = \frac{1+\sqrt{2}\sin x^2/2}{\cosh x\sqrt{\pi/2}} \tag{9}$$

and

$$\frac{2}{\sqrt{\pi}}\int_0^\infty \frac{\sin y^2/2}{\cosh y\sqrt{\pi/2}} \cos xy\, dy = \frac{-1+\sqrt{2}\cos x^2/2}{\cosh x\sqrt{\pi/2}}. \tag{10}$$

Adding (9) and (10) and simplifying we get

$$\frac{2}{\sqrt{\pi}}\int_0^\infty \frac{\cos y^2/2+\sin y^2/2}{\cosh y\sqrt{\pi/2}} = \frac{\sin x^2/2+\cos x^2/2}{\cosh x\sqrt{\pi/2}}. \tag{11}$$

This formula proves that the function

$$\frac{\sin x^2/2+\cos x^2/2}{\cosh x\sqrt{\pi/2}}$$

is R_c.

ON SUMMATION-PROCESSES IN GENERAL

By V. GANAPATHY IYER, Madras University

[Received 5 January 1937]

I. Introduction

1. All processes which are in use to sum non-convergent sequences are particular cases of linear transformations representable by Toeplitz matrices. So far as I am aware there has been no systematic examination of the scope and limitation of processes representable by these matrices and the relation of these matrices considered as a whole to the class of all sequences on which they operate. In this paper a general study is made of these matrices without going into the special properties of any particular matrix.

1.1. It has been proved by R. P. Agnew* that a non-convergent bounded sequence could be transformed into any given bounded sequence by a Toeplitz matrix; also that any unbounded sequence could be converted into any other sequence by such a matrix. It follows from these results that any given sequence could be converted into a convergent sequence by some summation-process. In this paper it is proved that any finite or enumerably infinite set of sequences could be transformed simultaneously into convergent sequences by a Toeplitz matrix. On the other hand it is shown that no such matrix can convert even a very restricted sub-class of bounded sequences into convergent sequences simultaneously. Not even an enumerably infinite number of such matrices can effect such a transformation.

1.2. *Definitions and notation.* The field of operation is the class or the space S of all sequences $f=[x_n]$, $n=1, 2, 3, \ldots$ where x_n is a complex number. Let $f=[x_n]$, $g=[y_n]$ be two elements of S. We set

$$||f-g||=\max_{1\leqslant n<\infty} |x_n-y_n|$$

$$||f-\Theta||=||f||,$$

* R. P. Agnew, "On ranges of inconsistency of regular transformations, and allied topics", *Ann. Math.* (2) 32 (1931) 715. See also the references given there.

where $\Theta=[0]$ is the null element of S. We use the following symbols:

$S(B)$ = the class of all bounded seqences in S, that is, $||f||$ is finite;

σ_M = the class of all $f=[x_n]$ in S such that $|x_n|=M$;

Σ_M = the class of all f in S such that $||f||\leqslant M$;

$S(C)$ = the class of all convergent sequences.

The following logical relations are obviously true:

$$\sigma_M \subset \Sigma_M \subset S(B) \subset S;$$

and

$$S(C) \subset S(B).$$

Here $A \subset B$ means strict inclusion. When there is a possibility of the elements of A and B being the same we shall write $A \subseteq B$.

1.3. Our starting point is the following general theorem due to Toeplitz*.

THEOREM (A): *The necessary and sufficient conditions that*

$$g(\omega)=\sum_{k=1}^{\infty} g_k(\omega) x_k \tag{1}$$

should tend, as $\omega \to \infty$, *to the same limit as* $f=[x_n]$, *for all* $f \subset S(C)$ *are*

(*a*) $\sum_{k=1}^{\infty} |g_k(\omega)| < M$, for $\omega \geqslant \omega_0$, *M being a constant independent of* ω; *we can take* $\omega_0=0$ *without loss of generality*;

(*b*) $g_k(\omega) \to 0$, *as* $\omega \to \infty$, *for every fixed k*;

(*c*) $\sum_{k=1}^{\infty} g_k(\omega) \to 1$, *as* $\omega \to \infty$.

1.4. Setting

$$f(t)=\begin{cases} 0, & 0 \leqslant t < 1 \\ x_n, & n \leqslant t < n+1,\ n=1, 2, \ldots, \end{cases} \tag{2}$$

$$H(x, \omega)=\begin{cases} 0, & 0 \leqslant x < 1 \\ g_k(\omega), & k \leqslant x < k+1,\ k=1, 2, \ldots, \end{cases} \tag{3}$$

we can write (1) (*a*), (*b*) and (*c*) in the simpler forms

* Toeplitz, "Über allgemeine lineare Mittelbildungen", *Prace. Mat. Fiz* XXII, Varsovie (1911).

$$\left.\begin{array}{ll}\text{(i)} & g(\omega)=\int_0^\infty H(x,\omega)f(x)dx;\\ (\alpha) & \int_0^\infty |H(x,\omega)|dx<M,\ M \text{ not depending on } \omega;\\ (\beta) & \int_0^{x_0} |H(x,\omega)|dx\to 0, \text{ as } \omega\to\infty, \text{ for each fixed } x_0;\\ (\gamma) & \int_0^\infty H(x,\omega)dx\to 1, \text{ as } \omega\to\infty.\end{array}\right\} \quad (4)$$

1.5. We shall designate $f(t)$ the associated function of $f\subset S$ and $H(x, \omega)$ given by (3), a summation-function. We denote by Λ the class of all summation-functions so that for each process given by Theorem (A) there is a unique summation-function. We shall call the class D of all $f\subset S$ such that $g(\omega)$ given by (i) of (4) tends to a finite limit as $\omega\to\infty$, the domain of H. The class R of all sequences f such that $g(\omega)$ defined by (i) of (4) exists for all ω greater than some ω_0, is called the range of H. Let $X\subset S$ be a sub-class of S. We say that H sums X if $X\subseteq D$, the domain of H. Let Λ_1 be a sub-class of Λ. We say that Λ_1 sums X when X is contained in the totality of the domains of functions $H\subset\Lambda_1$. We may note in passing that the range of every $H\subset\Lambda$ contains $S(B)$.

1.6. We say that a process represented by H_1 is stronger than that represented by H_2 when $D_2\subset D_1$; in symbols $H_2\subset H_1$; we also say that H_2 is weaker than H_1. When $D_1=D_2$ and every $f\subset D_1=D_2$ is summed to the same limit by H_1 and H_2, we say that H_1 and H_2 are properly equivalent. If merely $D_1=D_2$ we say that H_1 and H_2 are improperly equivalent. We say that a process $H\subset\Lambda$ is maximal when there is no process in Λ stronger than H; a process is minimal when there is no process in Λ weaker than H. Finally we set

$$||H_1-H_2|| = \max_{(\omega)} \int_0^\infty |H_1(x,\omega)-H_2(x,\omega)|dx;$$

and denote by $I(x, \omega)$ the identity

$$I(x,\omega)=\begin{cases}1, & n\leqslant x,\ \omega<n+1,\ n=1, 2,\ldots\\ 0, & \text{otherwise.}\end{cases}$$

It is evident that $g(\omega)=f(\omega)$ when $H=I$.

1.7. Section II contains some theorems on the properties of the classes S and Λ. Section III contains the negative or the limitation theorems. Section IV contains a set of theorems leading up to the proof of the result that any enumerably infinite sub-class of S could be summed by some $H \subset \Lambda$; and also some additional theorems on the properties of functions of Λ in relation to S.

II. Properties of S and Λ

2. Some of the following theorems are, perhaps, known and might be derivable from general properties of abstract spaces. But they are stated and proved here for the sake of completeness and future reference.

2.1. Theorem 1. *Let* $f=[x_n]$, $g=[y_n]$ *be two elements of* S. *Let* $f+g=[x_n+y_n]$, $cf=[cx_n]$, c *being a constant. Let* $||f-g||$ *denote the distance between the two elements* f *and* g *of* S. *Then* S *is a linear, complete, and non-separable space with reference to this distance.**

Proof: It is evident that S is linear. To prove that it is complete we have to show that given $\{f_p\}$, $f_p=[x_n^{(p)}]$, an enumerable sequence such that $||f_p-f_q|| \to 0$ as $p, q \to \infty$, there exists an $f \subset S$ such that $||f-f_p|| \to 0$ as $p \to \infty$. By definition

$$\max_{(n)} |x_n^{(p)}-x_n^{(q)}| \to 0 \text{ as } p, q \to \infty,$$

so that, for each fixed n, $x_n^{(p)} \to x_n$, say, and

$$\max_{(n)} |x_n-x_n^{(p)}| \to 0, \text{ as } p \to \infty,$$

which is equivalent to the result desired. To prove that S is non-separable, we show that, given $\{f_p\}$, $f_p=[x_n^{(p)}]$, any enumerable set in S, there is an $f=[x_n]$ in S so that $||f-f_p|| \geqslant 1$, $p=1, 2, \ldots$. Such an element f is given by $x_n=x_n^{(n)}+1$. So the theorem is proved.

2.2. By simple modifications of the above method we can prove

Theorem 2. σ_M *and* Σ_M *are complete but non-linear and non-separable.* $S(B)$ *has the same properties as* S. $S(C)$ *is complete, linear and separable.*

3. Theorem 3. *If* $||H_1-H_2||$ *is taken as the distance in the space* Λ, *then* Λ *is non-linear, convex, complete and non-separable.*

* It is possible to define a distance in S which renders it separable. But the one adopted here is more suitable for the purposes of this paper. See S. Banach, *Operations Lineaires*, Chap. 1-4.

PROOF: If H_1 and H_2 belong to Λ, $c_1H_1+c_2H_2$ belongs to Λ if and only if $c_1+c_2=1$, in virtue of (γ) of (4). Hence Λ is non-linear but convex. To prove that it is complete, we have to show that there exists an H such that $||H-H_p||\to 0$ as $p\to\infty$, where $\{H_p\}$, $p=1, 2,\ldots$ are all in Λ satisfying the condition $||H_p-H_q||\to 0$ as $p, q\to\infty$. By definition

$$\max_{(\omega)} \int_0^\infty |H_p(x, \omega)-H_q(x, \omega)|dx \to 0 \text{ as } p, q\to\infty. \tag{5}$$

Hence, for each fixed ω, the sequence converges on the average*, so that there exists a function $H(x, \omega)$ satisfying the condition

$$\max_{(\omega)} \int_0^\infty |H(x, \omega)-H_p(x, \omega)|dx \to 0 \text{ as } p\to\infty.$$

$H(x, \omega)$, for each ω, could be determined, except for a null-set in x as the limit of a sub-sequence of $\{H_p(x, \omega)\}$ and since these are step-functions in x with saltus points at $x=1, 2,\ldots$, we find that $H(x, \omega)$ is uniquely determined. Using (5), we get

$$\int_0^\infty |H(x, \omega)|dx \leqslant \int_0^\infty |H_{p_0}(x, \omega)|dx+\varepsilon;$$

$$\int_0^{x_0} |H(x, \omega)|dx \leqslant \int_0^{x_0} |H_{p_0}(x, \omega)|dx+\varepsilon;$$

$$\overline{\lim_{\omega\to\infty}} \left| \int_0^\infty [H(x, \omega)-H_p(x, \omega)]dx \right| \leqslant \varepsilon.$$

These relations show that $H\subset\Lambda$. Hence Λ is complete. It remains to show that Λ is non-separable. Let $\{H_p\}$ be any enumerable set in Λ. We shall construct a function $H\subset\Lambda$ such that $||H-H_p||\geqslant 1$ for all $p\geqslant 1$. Let $\varepsilon(\omega)$ be a positive function of ω tending to zero as $\omega\to\infty$. With the aid of (α) of (4) we can determine a sequence of integers

$$1\leqslant\lambda_1(\omega)<\lambda_2(\omega)\ldots,<\lambda_n(\omega) \to\infty \text{ as } n\to\infty$$

such that

$$\int_{\lambda_n(\omega)}^\infty |H_n(x, \omega)|dx < \frac{\varepsilon(\omega)}{2^n} \tag{6}$$

for each n. Let $\omega_1<\omega_2\ldots<\omega_n\to\infty$ be a sequence of values of ω so chosen that $\lambda_k(\omega_n)>\lambda_k(\omega_{n-1})$ for each fixed k and for $n=2, 3,\ldots$. Let

* For the definition of convergence on the average and its consequences see Hobson, *Theory of functions of a real variable*, Vol. II.

$$K(x, \omega_n)=\begin{cases} \dfrac{1}{\lambda_{n+1}(\omega_n)-\lambda_n(\omega_n)}, & \lambda_n(\omega_n)\leqslant x<\lambda_{n+1}(\omega_n), \\ 0, \text{ otherwise.} \end{cases}$$

Then, we have

$$\int_0^\infty |K(x, \omega_n)|dx=\int_0^\infty K(x, \omega_n)dx=1$$

and

$$\int_0^{x_0} |K(x, \omega_n)|dx=0$$

for all $n\geqslant n_0$ where $\lambda_1(\omega_{n_0})>x_0$. Hence, if $B(x, \omega)=K(x, \omega_n)$, $\omega_n\leqslant\omega<\omega_{n+1}$, we find that $B\subset\Lambda$. Let

$$H(x, \omega)=H_n(x, \omega)+B(x, \omega),\ \lambda_n(\omega)\leqslant x<\lambda_{n+1}(\omega).$$

Then

$$\int_0^\infty |H(x, \omega)|dx \leqslant \int_0^\infty |B(x, \omega)|dx+\varepsilon(\omega)\left[\frac{1}{2}+\frac{1}{2^2}+\dots\right]\leqslant 1+\varepsilon(\omega);$$

$$\int_0^{x_0} |H(x, \omega)|dx \leqslant \int_0^{x_0} |B(x, \omega)|dx+\varepsilon(\omega);$$

and

$$\left|\int_0^\infty [H(x, \omega)-B(x, \omega)]dx\right|\leqslant\varepsilon(\omega).$$

These relations show that $H\subset\Lambda$. But by definition,

$$\max_{(\omega)} \int_0^\infty |H-H_n|dx\geqslant\int_{\lambda_n(\omega_n)}^{\lambda_{n+1}(\omega_n)} K(x, \omega_n)dx=1.$$

Therefore Λ is non-separable. This proves the theorem.

3.1. THEOREM 4. *The domain D of any $H\subset\Lambda$ is a linear, closed and complete manifold in S.*

PROOF: Evidently D is linear. Since $D\subset S$ and S is complete by Theorem 1, D will be complete if it is closed. To prove that D is closed, we show that $\{f_p\}$ being a sequence in D such that $||f-f_p||\to 0$ as $p\to\infty$, where $f\subset S$ then f is also contained in D. Using suffixes to denote the transforms of f_p by (i) of (4), we get

$$\max_{(\omega)} |g(\omega)-g_p(\omega)|\leqslant M||f-f_p||$$

by (α) of (4). Since $f\subset D$, $g_p(\omega)\to l_p$, a finite number, as $\omega\to\infty$. So, we get

$$\left|\overline{\lim_{\omega\to\infty}}\, g(\omega)-l_p\right|\leqslant M||f-f_p|| \tag{7}$$

that is

$$|l_p-l_q|\to 0$$

as $p, q\to\infty$. Therefore, $l_p\to l$ as $p\to\infty$. Letting $p\to\infty$ in (7) we get $g(\omega)\to l$ so that $f\subset D$. So the theorem is proved.

3.2. THEOREM 5. *Let* $H \subset \Lambda$. *Let* $\omega_1 < \omega_2 \ldots < \omega_n \ldots$ *be a sequence tending to infinity. Let*

$$K(x, \omega) = \begin{cases} 0, \ 0 \leqslant \omega < \omega_1, \\ H(x, \omega_n), \ \omega_n \leqslant \omega < \omega_{n+1}, \ n = 1, 2, \ldots \end{cases}$$

Then $K \subset \Lambda$.

PROOF: Since $H \subset \Lambda$, we have

$$\max_{(\omega)} \int_0^\infty |K(x, \omega)| dx = \max_{(n)} \int_0^\infty |H(x, \omega_n)| dx;$$

$$\overline{\lim_{\omega \to \infty}} \int_0^{x_0} |K(x, \omega)| dx = \overline{\lim_{n \to \infty}} \int_0^{x_0} |H(x, \omega_n)| dx;$$

and $$\lim_{\omega \to \infty} \int_0^\infty K(x, \omega) dx = \lim_{n \to \infty} \int_0^\infty H(x, \omega_n) dx;$$

these show that $K \subset \Lambda$.

3.3. THEOREM 6. *Let* H_1 *and* $H_2 \subset \Lambda$. *Let*

$$H_{12}(x, \omega) = \int_0^\infty H_2(x, \xi) H_1(\xi, \omega) d\xi.$$

Then $H_{12} \subset \Lambda$. *Also the domain* D_{12} *of* H_{12} *contains every* $f \subset D_2$ *which satisfies the relation*

$$\int_0^\infty H_{12}(x, \omega) f(x) dx = \int_0^\infty H_1(\xi, \omega) d\xi \left[\int_0^\infty H_2(x, \xi) f(x) dx \right]. \quad (8)$$

In particular every f *common to* D_2 *and* $S(B)$ *is in* D_{12}.

PROOF: By (3) and (α) of (4), we see that

$$|H_2(x, \xi)| \leqslant M_2$$

and therefore the integral representing H_{12} is absolutely convergent. Therefore

$$\int_0^\infty |H_{12}(x, \omega)| dx \leqslant M_1 M_2 \quad (9)$$

$$\int_0^{x_0} |H_{12}(x, \omega)| dx \leqslant M_1 \varepsilon \text{ as } \omega \to \infty$$

if ξ_0 be so chosen that

$$\int_0^{x_0} |H_2(x, \xi)| dx \leqslant \varepsilon$$

for $\xi \geqslant \xi_0$ and then ω is made to tend to infinity. Again let $f \subset D_2$ and satisfy (8). Then

$$\int_0^\infty H_{12}(x, \omega) f(x) dx = \int_0^\infty H_1(\xi, \omega)(l + \varepsilon_\xi) d\xi$$

where $\varepsilon_\xi \to 0$ as $\xi \to \infty$, since $f \subset D_2$. But $H_1 \subset \Lambda$. Hence

$$\int_0^\infty H_1(\xi, \omega) \varepsilon_\xi d\xi \to 0 \text{ as } \omega \to \infty;$$

and therefore

$$\int_0^\infty H_{12}(x,\omega)f(x)dx \to l \text{ as } \omega\to\infty.$$

Taking $f(x)\equiv 1$, we get

$$\int_0^\infty H_{12}(x,\omega)dx \to 1$$

as $\omega\to\infty$. These relations prove the theorem.

3.4. We find that the transform of every $f\subset S$ satisfying (8), by the function H_{12} is obtained by first transforming f by H_2 and the result by H_1. So we shall write H_{12} as $[H_1][H_2]$ and call it the product of H_2 by H_1.

III. Negative theorems

4. In this section we prove two theorems on the limitation of the domain of any $H\subset\Lambda$.

4.1. Theorem 7. *There exists no* $H\subset\Lambda$ *which sums* σ_M, $M>0$; *that is, given any* $H\subset\Lambda$, *there is an element* $f\subset\sigma_M$ *whch is not summable by* H.

Proof: Let $H\subset\Lambda$ be given. We can suppose $M=1$ without loss of generality. We shall construct a function $f(x)$ with the following properties:

(i) $f(x)$ is real and $|f(x)|=1$, $1\leqslant x<\infty$;

(ii) $g(\omega)$ given by (i) of (4) for this $f(x)$ does not tend to a definite limit as $\omega\to\infty$.

Since $f(x)$ is real, we can suppose that $H(x,\omega)$ is real since otherwise we can deal with the real part of $H(x,\omega)$ which posesses the same properties (α), (β), and (γ) of (4) as H itself. Let $\{\varepsilon_n\}$, $0<\varepsilon_n<1$, be a null sequence. Let ω_1 be so chosen that

$$\int_0^\infty H(x,\omega_1)dx>1-\varepsilon_1$$

which is possible by (γ) of (4). Then choose an integer Ω_1 so that

$$\int_{\Omega_1}^\infty |H(x,\omega_1)|dx<\varepsilon_1, \qquad (a_1)$$

with the aid of (α) of (4). Let

$$f(x)=1,\ 1\leqslant x<\Omega_1.$$

Then

$$\int_0^\infty H(x,\omega_1)f(x)dx>1-3\varepsilon_1$$

in virtue of (i) above.

Next choose $\omega_2 > \omega_1$, Ω_2, an integer, $> \Omega_1$, such that

$$\left.\begin{aligned}
&\int_0^\infty H(x, \omega_2)\, dx > 1 - \varepsilon_2 ;\\
&\int_0^{\Omega_1} |H(x, \omega_2)|\, dx < \varepsilon_2 ;\\
&\int_{\Omega_2}^\infty |H(x, \omega_2)|\, dx < \varepsilon_2 ;
\end{aligned}\right\} (a_2)$$

these being possible by (α), (β) and (γ) of (4).

Let

$$f(x) = -1, \ \Omega_1 \leqslant x < \Omega_2.$$

Then

$$\int_0^\infty H(x, \omega_2) f(x)\, dx < -1 + 5\varepsilon_2.$$

Having chosen $\omega_1, \omega_2, \ldots \omega_{n-1}$, $\Omega_1, \Omega_2 \ldots \Omega_{n-1}$, determine $\omega_n > \omega_{n-1}$, $\Omega_n > \Omega_{n-1}$ so that

$$\left.\begin{aligned}
&\int_0^\infty H(x, \omega_n)\, dx > 1 - \varepsilon_n ;\\
&\int_0^{\Omega n-1} |H(x, \omega_n)|\, dx < \varepsilon_n ;\\
&\int_{\Omega_n}^\infty |H(x, \omega_n)|\, dx < \varepsilon_n .
\end{aligned}\right\} (a_n)$$

Define

$$f(x) = (-1)^{n-1}, \ \Omega_{n-1} \leqslant x < \Omega_n.$$

Then

$$(-1)^{n-1} \int_0^\infty H(x, \omega_n) f(x)\, dx > 1 - 5\varepsilon_n.$$

The last inequality in (a_n), $n = 1, 2, \ldots$, shows that, for $f(x)$ defined as described

$$\overline{\lim_{\omega\to\infty}}\, g(\omega) \geqslant 1 ;$$

and

$$\underline{\lim_{\omega\to\infty}}\, g(\omega) \leqslant -1.$$

Therefore f is unsummable by H.

4.2. THEOREM 8. *No enumerable sub-class* $\{ H_p \}$, $p = 1, 2, \ldots$ *of* Λ *exists which sums* σ_M.

PROOF: Let $\{\varepsilon_n\}$, $0<\varepsilon_n<1$, be a null sequence. Choose (ω_1, Ω_1) as in (a_1), so that

$$\left.\begin{aligned} &\int_0^\infty H_1(x, \omega_1)dx > 1-\varepsilon_1; \\ &\int_{\Omega_1}^\infty |H_1(x, \omega_1)|dx < \varepsilon_1. \end{aligned}\right\} \quad (b_1)$$

Next choose $\omega_2>\omega_1$, $\Omega_2>\Omega_1$, so that for $p=1, 2$,

$$\left.\begin{aligned} &\int_0^\infty H_p(x, \omega_2)dx > 1-\varepsilon_2; \\ &\int_0^{\Omega_1} |H_p(x, \omega_2)|dx < \varepsilon_2; \\ &\int_{\Omega_2}^\infty |H_p(x, \omega_2)|dx < \varepsilon_2. \end{aligned}\right\} \quad (b_2)$$

Having chosen $\omega_1, \omega_2, \ldots \omega_{n-1}$, $\Omega_1, \Omega_2 \ldots \Omega_{n-1}$, determine $\omega_n>\omega_{n-1}$, $\Omega_n>\Omega_{n-1}$ so that for $p=1, 2, \ldots n$,

$$\left.\begin{aligned} &\int_0^\infty H_p(x, \omega_n)dx > 1-\varepsilon_n; \\ &\int_0^{\Omega_{n-1}} |H_p(x, \omega_n)|dx < \varepsilon_n; \\ &\int_{\Omega_n}^\infty |H_p(x, \omega_n)|dx < \varepsilon_n. \end{aligned}\right\} \quad (b_n)$$

This process can be continued indefinitely since all $H_p \subset \Lambda$. Let us define

$$f(x) = (-1)^{n-1}, \quad \Omega_{n-1} \leqslant x < \Omega_n,$$

where $\Omega_0=1$. The relations (b_n) show that for a given p,

$$(-1)^{n-1}\int_0^\infty H_p(x, \omega_n)f(x)dx > 1-5\varepsilon_n$$

for $n \geqslant p$. Therefore,

$$\overline{\lim_{\omega\to\infty}}\, g_p(\omega) \geqslant 1,$$

and

$$\underline{\lim_{\omega\to\infty}}\, g_p(\omega) \leqslant -1,$$

for all $p=1, 2, \ldots$. So the theorem is proved.

4.3. THEOREM 9. *No finite or enumerable sub-class of* Λ *can sum* $S(B)$.

PROOF: This follows from Theorems 7 and 8 since $\sigma_M \subset S(B)$*.

4.4. *Illustrations.* Let $1=k_0<k_1<k_2\ldots<k_n\ldots$ be a sequence of integers tending to infinity. Define $f(x)$ as follows:

$$f(x)=\begin{cases} 0, & 0\leqslant x<k_0=1 \\ (-1)^{n-1}, & k_{n-1}\leqslant x<k_n,\ n=1,2,\ldots. \end{cases} \tag{10}$$

We can verify the following statements by the method used in Theorem 7:

(i) let $\frac{k_n}{k_{n+1}}\to 0$; then $f(x)$ given by (10) is not summable by any Cesaro's mean;

(ii) let

$$H(x,\omega)=\begin{cases} 0, & 0\leqslant x<1 \\ e^{-\frac{\lambda_k}{\omega}}-e^{-\frac{\lambda_{k+1}}{\omega}}, & k\leqslant x<k+1,\ k=1,2,\ldots \end{cases}$$

be the summation-function for Abel's process, where $\lambda_1<\ldots<\lambda_n\to\infty$ as $n\to\infty$; let $\{k_n\}$ be such that $\frac{\lambda_{k_n}}{\lambda_{k_{n+1}}}\to 0$, as $n\to\infty$; then $f(x)$ given by (10) is unsummable by H;

(iii) let

$$H(x,\omega)=\begin{cases} 0, & 0\leqslant x<1, \\ e^{-\omega}\frac{\omega^{n-1}}{(n-1)!}, & n\leqslant x<n+1,\ n=1,2,\ldots \end{cases}$$

be the function for Borel's process; let $\{k_n\}$ be such that $\frac{k_{n+1}}{k_n}\geqslant 1+\theta$, $\theta>0$, then $f(x)$ given by (10) is unsummable by H.

IV. POSITIVE THEOREMS

5. In this section we shall consider the relation of elements of S to the class Λ.

5.1. THEOREM 10. *Let $f\subset S$. Let there be a function $H\subset\Lambda$ such that*

* It is known that we can associate a number $F(f)$ for every $f\subset S(B)$ with the following properties:

$$F(f)=\text{the limit of } f \text{ for all } f\subset S(C);$$
$$F(af+bg)=aF(f)+bF(g).$$

Evidently this satisfies the condition of consistency for a summation process and is a linear transformation. But from what is proved above this is not representable by a Toeplitz matrix. See Banach, *Operations Lineaires*, p. 34,

$$\left| \int_0^\infty H(x, \omega) f(x) dx \right|$$

has a finite limit point as $\omega \to \infty$. *Then there is a* $K \subset \Lambda$ *which sums* f *and satisfies the condition* $H \subseteq K$.

PROOF: By hypothesis, there is a sequence $\omega_1 < \omega_2 \ldots < \omega_n \to \infty$ such that

$$\int_0^\infty H(x, \omega_n) f(x) dx \to l,$$

a finite number, as $n \to \infty$. Let

$$K(x, \omega) = H(x, \omega_n),\ \omega_n \leqslant \omega < \omega_{n+1}.$$

Then by Theorem 5 $K \subset \Lambda$. Obviously K sums f to l and $H \subseteq K$.

5.2. THEOREM 11. *Let* $\{ f_p \}$, $p = 1, 2, \ldots$, *be an enumerable sub-class of* S. *Let there exist an* $H \subset \Lambda$ *and a sequence* $\omega_1 < \omega_2 \ldots < \omega_n \to \infty$ *such that*

$$\left| \int_0^\infty H(x, \omega_n) f_p(x) dx \right.$$

is bounded for each $p = 1, 2, \ldots$. *Then there is a* $K \subset \Lambda$ *which sums* $\{ f_p \}$ *and satisfies the condition* $H \subseteq K$.

PROOF: By using the diagonal process, we can by hypothesis, determine a sub-sequence $\omega_{k_1} < \omega_{k_2} \ldots < \omega_{k_n} \to \infty$ of $\{ \omega_n \}$ so that

$$\int_0^\infty H(x, \omega_{k_n}) f_p(x) dx \to l_p,$$

a finite number, for each $p = 1, 2, \ldots$. Let

$$K(x, \omega) = H(x, \omega_{k_n}),\ \omega_{k_n} \leqslant \omega < \omega_{k_{n+1}}.$$

We find as before $K \subset \Lambda$ and sums each f_p; also $H \subseteq K$,

5.3. THEOREM 12. *Every finite or enumerable sub-class* $S_1(B)$ *of* $S(B)$ *is summable in an infinity of ways by functions of* Λ.

PROOF. Let $\{ f_p \}$, $p = 1, 2, \ldots$ be the elements of $S_1(B)$. Let $H \subset \Lambda$. Then

$$\left| \int_0^\infty H(x, \omega) f_p(x) dx \right|$$

is bounded as $\omega \to \infty$ for each $p = 1, 2, \ldots$ Hence the theorem follows by Theorem 11.

6. The following two theorems are due to R. P. Agnew.*

THEOREM (B). *Let $f=[x_n]$ be a non-convergent bounded sequence; let $g=[y_n]$ be any other bounded sequence. Then there exists one $H \subset \Lambda$ such that $Hf=g$.*

THEOREM (C). *Let $f=[x_n]$ be any unbounded sequence; let $g=[y_n]$ be any other sequence. Then there is one $H \subset \Lambda$ such that $Hf=g$.*

6.1. We shall repeat the proof of Theorem (C) since the actual form of the function set up is required in the sequel. Let $f=[x_n]$ be the unbounded sequence. Let $\theta>0$ be given. For each $n=1, 2, \ldots$, determine a $p_n>n$ such that

$$\frac{|x_{p_n}-y_n|+|y_n-x_n|}{|x_{p_n}-x_n|} \leqslant 1+\theta.$$

This is possible because $\overline{\lim_{n\to\infty}} |x_n|=\infty$. Define $H(x, \omega)$ by

$$H(x, \omega)=\begin{cases} \dfrac{x_{p_n}-y_n}{x_{p_n}-x_n}, & n \leqslant x, \omega<n+1, n=1, 2, \ldots \\ \dfrac{y_n-x_n}{x_{p_n}-x_n}, & p_n \leqslant x<p_{n+1}, n \leqslant \omega<n+1. \\ 0, & \text{otherwise.} \end{cases}$$

Obviously $H \subset \Lambda$ and

$$Hf=\int_0^\infty H(x, \omega) f(x) dx=g(\omega)=g.$$

The following properties of $H(x, \omega)$ defined above are collected for reference; that is, the function $H(x, \omega)$ of Theorem (C) can be so chosen that

(i) $||H||<1+\theta$, where $\theta>0$ is given;

(ii) $\int_0^\infty H(x, \omega) dx=1$, for all $\omega \geqslant 1$;

(iii) $H(x, \omega)=0$, for $x<[\omega]$, the integral part of ω; (11)

(iv) $H(x, \omega)$ contains only two non-vanishing elements in each row, that is, $H(x, \omega)=0$ for each ω outside an interval of finite length in x so that the range of H (see § 1.5) is the whole of S.

6.2. We can now prove the following

THEOREM 13. *Let S_1 be any enumerable set of unbounded sequences in S. There is an $H \subset S_1$ which sums S_1.*

* See the reference given in the first footnote.

PROOF: Let $\{f_p\}$, $p=1, 2, \ldots$ be the elements of S_1. Let $\{\theta_n\}$ be a sequence of positive numbers so that $\sum_{k=1}^{\infty} \theta_n$ converges. Since f_1 is unbounded, there exists by Theorem (C), an $H_1 \subset \Lambda$ satisfying the conditions (i)-(iv) of (11) with $\theta=\theta_1$, which converts f_1 into $H_1 f_1$ so that $||H_1 f_1|| \leqslant 1$. Consider the class $\{H_1 f_p\}$, $p=1, 2, \ldots$ all of which exist by (iv) of (11). If all these belong to $S(B)$ the theorem now follows from Theorem 11. If not, there is a first p_1 so that $H_1 f_{p_1}$ is unbounded; obviously $p_1>1$. Let $H_2 \subset \Lambda$ be the function satisfying (11) with $\theta=\theta_2$, which converts $H_1 f_{p_1}$ into $H_2[H_1 f_{p_1}]$ so that $||H_2[H_1 f_{p_1}]|| \leqslant 1$. By Theorem 6, and the relations (8) and (9), we find that $[H_2][H_1]$ satisfies the following conditions:

(i) $||[H_2][H_1]|| \leqslant (1+\theta_1)(1+\theta_2)$;

(ii) $[H_2][H_1]$ satisfies (ii) and (iii) of (11);

(iii) each row of $[H_2][H_1]$ contains at most 4 non-vanishing elements; (12)

(iv) $[H_2][H_1] f_p$ belongs to $S(B)$ for $1 \leqslant p \leqslant p_1$; we shall denote by P_1 the greatest of the numbers $||[H_2][H_1] f_p||$, $1 \leqslant p \leqslant p_1$.

As before consider the class $[H_2][H_1] f_p$; if all these belong to $S(B)$ the result follows from Theorem 11. If not, we can repeat the argument. We might, after a finite number of steps, arrive at a stage when we are left with elements of $S(B)$ only, in which case Theorem 11 again applies; in the alternative case we get a sequence of integers $\{p_n\}$, $n=1, 2, \ldots$ and a sequence of functions $H_n(x, \omega)$ with the following properties:

(i) $H_n(x, \omega)$ satisfies conditions (i)-(vi) of (11) with $\theta=\theta_n$;

(ii) $H^{(n)}=[H_n][H_{n-1}]\ldots[H_1]$ converts all $\{f_p\}$, $1 \leqslant p \leqslant p_n$ into elements of $S(B)$; we shall denote by P_{n-1} the greatest of $||H^{(n)} f_p||$, $1 \leqslant p \leqslant p_n$;

(iii) $H^{(n)}$, for all n, satisfies (ii) and (iii) of (11); (13)

(iv) $H^{(n)}$ contains at most 2^n non-vanishing elements in each row, so that the range of $H^{(n)}$ is the whole of S;

(v) $||H^{(n)}|| \leqslant \prod_{k=1}^{n} (1+\theta_k)$ by (9).

We now define $H(x, \omega)$ by

$$H(x,\omega)=\begin{cases}0, & 0\leqslant\omega\leqslant 1\\ H^{(n)}(x,\omega), & n\leqslant\omega<n+1,\ n=1,\end{cases}$$

We have, by (i)-(v) of (13),

$$\int_0^\infty |H(x,\omega)|dx=\int_0^\infty |H^{(n)}(x,\omega)|dx\leqslant \prod_{k=1}^{n}(1+\theta_k), \text{ for } n\leqslant\omega<n+1;$$

$$\leqslant \prod_{k=1}^{\infty}(1+\theta_k) \text{ for all } \omega.$$

$$\int_0^{x_0} |H(x,\omega)|dx=0 \text{ for } \omega>x_0+1$$

$$\int_0^\infty H(x,\omega)dx=1 \text{ for } \omega\geqslant 1.$$

Hence $H\subset\Lambda$. Moreover, by (8) and (ii), (iv) of (13), we get, for $\omega>p_k$

$$||Hf_p||\leqslant||[H_n]\ldots[H_{k+1}]H^{(k)}f_p||\leqslant P_{k-1}\prod_{n=1}^{\infty}(1+\theta_n)$$

if
$$p_{k-1}\leqslant p<p_k.$$

Hence we can apply Theorem 11 and deduce that the result required is true.

6.3. THEOREM 14. *Let S_1 be any enumerable sub-class of S. There is an $H\subset\Lambda$ which sums S_1.*

PROOF: Let $S_1(B)$ be the component of S_1 belonging to $S(B)$. Let S_2 be the complement of $S_1(B)$ with respect to S_1. Then S_2 consists of an at most enumerably infinite set of unbounded sequences. By Theorem 13 a $K\subset\Lambda$ exists which sums S_2. The class $\{Kf_p\}$ where $\{f_p\}=S_1(B)$ is bounded. Hence by Theorem 11 there is an $H\subset\Lambda$ which sums $S_1(B)$ and satisfies the condition $K\subseteq H$. Therefore the theorem follows.

6.4. Combining Theorems 4 and 14 we get

THEOREM 15. *Any linear, closed and separable sub-class of S is summable by some $H\subset\Lambda$.*

7. The following theorem gives in a general form the principle employed* by N. Wiener to prove Tauberian theorems.

* See N. Wiener, "Tauberian Theorems", *Annals of Math.* (2) 33 (1932), 25, Theorem 8.

THEOREM 16. *Let Λ_1 be any sub-class of Λ. Let $\overline{\Lambda}_1$ denote the closed linear extension of Λ_1; that is, all $H \subset \Lambda$ such that*

(i) $H = \sum_{k=1}^{p} c_k H_k$, *where* $\sum_{k=1}^{p} c_k = 1$ *and* $H_1, \ldots H_p$ *are all in* Λ_1;

(ii) all H such that $||H-H_p|| \to 0$ as $p \to \infty$, where H_p is of form (i) for each p.

Let $f \subset S(B)$ be summable by each function in Λ_1. Then every H in $\overline{\Lambda}_1$ also sums f.

PROOF: It is evident that any function of form (*i*) sums f. Let H be any other function of $\overline{\Lambda}_1$. Then there exists a sequence $\{H_p\}$ each of which is of form (*i*) such that $||H-H_p|| \to 0$ as $p \to \infty$. Let $g_p(\omega) = H_p f$ and $g(\omega) = Hf$. Then

$$\max_{(\omega)} |g(\omega) - g_p(\omega)| \leqslant ||f|| \times ||H-H_p||$$

and therefore

$$\left| \overline{\lim_{\omega\to\infty}} g(\omega) - l_p \right| \leqslant ||f|| \times ||H-H_p||$$

where $g_p(\omega) \to l_p$ as $\omega \to \infty$. Hence $|l_p - l_q| \to 0$ as $p, q \to \infty$, and so $l_p \to l$ as $p \to \infty$. Thus we get

$$\left| \overline{\lim_{\omega\to\infty}} g(\omega) - l \right| \leqslant 0$$

or, $g(\omega) \to l$ which proves the theorem.

7.1. THEOREM 17. *There is no maximal process in Λ.*

PROOF: Let $H \subset \Lambda$ be given. By Theorem 7, there is at least one $f \subset \sigma_M$ which is not summable by H. But

$$\left| \int_0^\infty H(x, \omega) f(x)\, dx \right|$$

is bounded. Hence by Theorem 10 there exists a $K \subset \Lambda$ which sums f and satisfies the condition $H \subset K$. So there is no maximal process in Λ.

7.2. THEOREM 18. *All minimal processes are properly equivalent to $I(x, \omega)$.*

PROOF: Let H be a minimal process in Λ. We have always $S(C) \subseteq D$. But $I(x, \omega)$ has domain $S(C)$ and so by the definition of a minimal process $D \subseteq S(C)$. Hence $D = S(C)$ and therefore

$H(x, \omega)$ is properly equivalent to $I(x, \omega)$ by Theorem (A). The following is an instance of a minimal process not identical with I.*

$$H(x, \omega)=e^{-\frac{\lambda_k}{\omega}}-e^{-\frac{\lambda_{k+1}}{\omega}},\ k\leqslant x<k+1,\ k=1, 2, \ldots,$$

where
$$\frac{\lambda_{k+1}}{\lambda_k}\geqslant 1+\theta,\ \theta>0.$$

Additional note: The distance $||f-g||$ adopted in this paper suffers from a technical objection that it is not always finite for *every* couple (f, g) of elements in S; and it might be thought that the properties, especially those of section II, depend on this factor. If we take the distance

$$D(f, g)=\max_{1\leqslant n\leqslant\infty}\frac{|x_n-y_n|}{1+|x_n-y_n|},$$

the objection mentioned above does not arise and it is easy to see that all the properties where the distance enters in this paper remain true if $D(f, g)$, instead of $||f-g||$, is adopted as the distance in S. But $D(f, g)=D(f-g, 0)$ is not linear in the sense that $D(cf, cg)=|c|D(f, g)$ while $||f-g||$ is linear which makes the latter useful for manipulation especially when the sequences in $S(B)$ for which $||f-g||$ is always finite, are considered. Therefore $||f-g||$ has been adopted as the distance in the body of this paper.

* See Hardy and Littlewood, "A further note on the converse of Abel's theorem", (1926) *Proc. L.M.S.* (2), 25, 219-236.

THE ALGEBRA OF QUADRATIC RESIDUES

By R. VAIDYANATHASWAMY

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I. In a recent paper* I shewed that the classes $C_1, C_2, \ldots$ into which the elements of a group G fall with respect to any group of automorphisms of G, combine among themselves by the group-operation in the sense that each element of any class C_k occurs the same number γ_{ij}^k of times among the elements which result on combining each element of C_i with each element of C_j by the group-operation. A simple case of this general principle was studied in the paper referred to, namely the classes of a cyclic group with respect to its total group of automorphisms. The present paper is devoted to another simple case of the same principle.

Consider a cyclic group of order p (an odd prime), which we may regard without loss of generality as the additive group of the p residue-classes mod p. The automorphisms of this cyclic group are obtained by multiplying the residue-classes by a number prime to p. From the existence of a primitive root mod p, it follows that the group of automorphisms is a multiplicative cyclic group G_{p-1} of order $p-1$. From the parity of $p-1$, it follows that the squares of the elements of G_{p-1} form a cyclic sub-group $G_{(p-1)/2}$ of order $(p-1)/2$. The automorphisms of $G_{(p-1)/2}$ are evidently obtained by multiplying the residue-classes mod p by a *quadratic residue* of p. Now the p numbers mod p fall into *two* classes K_1, K_0 with respect to G_{p-1}, where K_1 consists of zero only and K_0 consists of the remaining $p-1$ numbers. It is clear that the class K_0 subdivides into two equal sub-classes R_0, L_0 with respect to the subgroup $G_{(p-1)/2}$, where R_0 consists of the $(p-1)/2$ quadratic residues of p, while L_0 consists of the $(p-1)/2$ non-residues. By our general principle the three classes K_1, R_0, L_0 must combine among themselves by the group-operation which is here addition. Hence we must have equations of the form:

* 'A remarkable property of Integers mod N and its bearing on Group-Theory', *Proc. Ind. Acad. of Sciences,* Vol. 5. No. 1. Jan. 1937.

$$K_1^2=K_1;\ K_1R_0=R_0;\ K_1L_0=L_0;$$
$$R_0^2=a_1K_1+a_0R_0+a_0'L_0$$
$$R_0L_0=b_1K_1+b_0R_0+b_0'L_0$$
$$L_0^2=c_1K_1+c_0R_0+c_0'L_0,$$

where a's, b's, c's are positive integers to be determined. Here a product of the form R_0L_0 denotes the set of numbers obtained by adding each number of R_0 to each number of L_0.

II. We first establish the relations:

$$a_0'=c_0;\ c_0'=a_0;\ b_0'=b_0;\ a_1=c_1.$$

These follow very simply from the property of quadratic residues. For, the expression for R_0^2 signifies that any particular residue α can be expressed in exactly a_0 ways in the form $\beta+\gamma$ where β and γ are residues. Since any non-residue α' can be obtained by multiplying α by a suitably chosen non-residue, it follows that α' can be expressed in exactly a_0 ways in the form $\beta'+\gamma'$ where β' and γ' are non-residues; in other words $c_0'=a_0$. A similar argument shews that $a_0'=c_0$, $b_0'=b_0$ and $a_1=c_1$. Thus the algebra of the three classes takes the form:

$$K_1^2=K_1;\ K_1R_0=R_0;\ K_1L_0=L_0;$$
$$R_0^2=a_1K_1+a_0R_0+c_0L_0$$
$$R_0L_0=b_1K_1+b_0R_0+b_0L_0$$
$$L_1^2=a_1K_1+c_0R_0+a_0L_0.$$

By comparing the total number of elements on the two sides of these equations, we have:

$$\left(\frac{p-1}{2}\right)^2=a_1+(a_0+c_0)\frac{p-1}{2}=b_1+b_0(p-1)$$

or

$$a_1=\frac{p-1}{2}\left\{\frac{p-1}{2}-(a_0+c_0)\right\};\ b_1=\frac{p-1}{2}\left\{\frac{p-1}{2}-2b_0\right\}. \qquad (1)$$

Further this algebra of classes is associative since the group-operation is so. Hence, comparing the coefficient of K_1 in $R_0^2.L_0$ and in $R_0.R_0L_0$, we have

$$a_0b_1+c_0a_1=b_0a_1+b_0b_1. \qquad (2)$$

III. Suppose now that $p=1 \bmod 4$. In this case $(p-1)/2$ is even and so

$$(-1)^{\frac{p-1}{2}}=1.$$

Hence -1 is a residue. Therefore the $(p-1)/2$ numbers of R_0 or of L_0 consist of pairs of numbers differing only in sign. Therefore zero occurs precisely $(p-1)/2$ times in R_0^2 and in L_0^2 and it does not occur at all in R_0L_0. Hence

$$a_1=(p-1)/2;\ b_1=0.$$

Substituting in equations (1) and (2), we have

$$a_0+c_0=\frac{p-3}{2};\ b_0=\frac{p-1}{4}=c_0.$$

Thus when $p=1$ mod 4, the algebra of the three classes is:

$$\left.\begin{aligned} R_0^2&=\frac{p-1}{2}K_1+\frac{p-5}{4}R_0+\frac{p-1}{4}L_0\\ R_0L_0&=\frac{p-1}{4}(R_0+L_0)\\ L_0^2&=\frac{p-1}{2}K_1+\frac{p-1}{4}R_0+\frac{p-5}{4}L_0. \end{aligned}\right\} \qquad (3)$$

IV. When $p=-1$ mod 4, -1 is a non-residue. Hence L_0 is obtained by changing the signs of the numbers of R_0. Hence zero does not occur in R_0^2, L_0^2, and occurs $(p-1)/2$ times in R_0L_0. Hence

$$a_1=0;\ b_1=(p-1)/2.$$

Substituting in equations (1) and (2), we have:

$$a_0=b_0=(p-3)/4;\ c_0=(p+1)/4.$$

Thus when $p=-1$ mod 4,

$$\left.\begin{aligned} R_0^2&=\frac{p-3}{4}R_0+\frac{p+1}{4}L_0\\ R_0L_0&=\frac{p-1}{2}K_1+\frac{p-3}{4}\{R_0+L_0\}\\ L_0^2&=\frac{p+1}{4}R_0+\frac{p-3}{4}L_0. \end{aligned}\right\} \qquad (4)$$

From (3) and (4), we have immediately the theorem of Hermite:

If p is an odd prime and a a fixed number $\neq 0$ mod p, then for $x=1, 2, 3, \ldots, p-1$ mod p, x^2+a will represent non-residues as often as, or once oftener than residues, according as a is a non-residue or a residue.

V. To obtain the similar algebra mod p^n, we note first of all that since there is a primitive root mod p^n, the group of residue classes mod p^n prime to p is a cyclic (multiplicative) group of order $\phi(p^n)$. Hence the subgroup of quadratic residues is of order $\frac{1}{2}\phi(p^n)$. With respect to the total group of automorphisms the additive cyclic group of all residue classes mod p^n falls into $n+1$ classes $K_0, K_1, \ldots, K_n$, where K_i consists of all numbers mod p^n whose g.c.d. with p^n is p^i $(i=0, 1, \ldots, n)$. In particular K_n consists of the zero residue class only, while K_0 consists of the $\phi(p^n)$ residue classes prime to p^n. From the formula of my paper, the classes K combine by the scheme:

$$(A)\quad \left| \begin{aligned} &K_iK_j=\phi(p^{n-j})K_i \;(j>i) \\ &K_i^2=\phi(p^{n-i})\{1-1/(p-1)\}K_i \\ &\qquad +\phi(p^{n-i})\{K_{i+1}+K_{i+2}+\ldots+K_n\}. \end{aligned} \right.$$

If now, instead of the total group of automorphisms, we consider the classes with respect to the subgroup of automorphisms corresponding to multiplication by the $\frac{1}{2}\phi(p^n)$ quadratic residues, it is clear that we would have $2n+1$ classes in all, each class K_i other than K_n now dividing into two equinumerous subclasses R_i, L_i such that p^iq (q prime to p) belongs to R_i or L_i according as q is a quadratic residue or non-residue mod p^{n-i}. According to our general principle the $2n+1$ classes (K_n, R_i, L_i) combine among themselves by addition. It will be shewn now that they combine according to the scheme:

$$(B)\quad \left\{ \begin{aligned} &K_n^2=K_n;\; K_nR_i=R_i;\; K_nL_i=L_i; \\ &R_iR_j=R_iL_j=\tfrac{1}{2}\phi(p^{n-j})R_i;\; L_iR_j=L_iL_i=\tfrac{1}{2}\phi(p^{n-j})L_i.\,(j>i) \\ &R_i^2=\{\tfrac{1}{4}\phi(p^{n-i})-p^{n-i-1}\}R_i+\tfrac{1}{4}\phi(p^{n-i})L_i \\ &\qquad +\tfrac{1}{2}\phi(p^{n-i})\{K_{i+1}+K_{i+2}+\ldots+K_n\} \\ &R_iL_i=\tfrac{1}{4}\phi(p^{n-i})K_i \\ &L_i^2=\tfrac{1}{4}\phi(p^{n-i})R_i+\{\tfrac{1}{4}\phi(p^{n-i})-p^{n-i-1}\}L_i+\tfrac{1}{2}\phi(p^{n-i}) \\ &\times\{K_{i+1}+K_{i+2}+\ldots+K_n\}\,(p=1 \bmod 4;\, i=0, 1, \ldots, n-1). \\ &R_i^2=\{\tfrac{1}{4}\phi(p^{n-i})-\tfrac{1}{2}p^{n-i-1}\}R_i+\{\tfrac{1}{4}\phi(p^{n-i})+\tfrac{1}{2}p^{n-i-1}\}L_i \\ &R_iL_i=\{\tfrac{1}{4}\phi(p^{n-i})-\tfrac{1}{2}p^{n-i-1}\}K_i \\ &\qquad +\tfrac{1}{2}\phi(p^{n-i})\{K_{i+1}+K_{i+2}+\ldots+K_n\} \\ &L_i^2=\{\tfrac{1}{4}\phi(p^{n-i})+\tfrac{1}{2}p^{n-i-1}\}R_i+\{\tfrac{1}{4}\phi(p^{n-i})-\tfrac{1}{2}p^{n-i-1}\}L_i \\ &\qquad (p=-1 \bmod 4;\, i=0, 1, \ldots, n-1). \end{aligned} \right.$$

VI. The first three equations involving K_n are obvious. The next four equations giving the products of classes with *different* indices may be derived from the following known theorem.

THEOREM I. *A number is a quadratic residue mod p^n if and only if it is a quadratic residue mod p.*

For N to be a quadratic residue mod p^n it is of course *necessary* that it be a quadratic residue mod p; the theorem asserts that it is also *sufficient*. To see this, we observe that there are $\frac{1}{2}\phi(p)$ quadratic residues mod p and that there are therefore $\frac{1}{2}\phi(p).p^{n-1}=\frac{1}{2}\phi(p^n)$ residue classes mod p^n which are equal mod p to quadratic residues of p. The quadratic residues mod p^n must accordingly be sought among these $\frac{1}{2}\phi(p^n)$ residue classes. Since however there are $\frac{1}{2}\phi(p^n)$ distinct quadratic residues of p^n, it follows at once that every number which is a quadratic residue mod p is also a quadratic residue mod p^n.

COR. *If a number is a quadratic residue of p^n, it is a residue of every power of p.*

This theorem leads immediately to the expressions (B) for R_iL_j etc. For, by equations (A), when $j>i$ the result of adding a number of K_i to a number of K_j is a number of K_i. Hence R_iL_j is of the form aR_i+bL_i. But if q_1 is a residue, and q_2 a non-residue of p^{n-i}, we cannot have an equation of the form

$$p^iq_2=p^iq_1+p^jl;$$

for, this would imply $p^iq_2=p^iq_1$ mod p^{i+1} and therefore $q_2=q_1$mod p; this is impossible since q_1, q_2 being respectively a residue and a non-residue mod p^{n-i} are by Theorem I a residue and non-residue respectively mod p. Hence $b=0$, so that

$$R_iL_j=R_iR_j=aR_i.$$

Since R_i, L_j contain respectively $\frac{1}{2}\phi(p^{n-i})$, $\frac{1}{2}\phi(p^{n-j})$ numbers it follows that $a=\frac{1}{2}\phi(p^{n-j})$. In the same manner, we have:

$$L_iR_j=L_iL_j=\frac{1}{2}\phi(p^{n-j})L_i \ (j>i).$$

VII. To establish the remaining equations (B), we must discuss separately the cases $p=1$ mod 4 and $p=-1$ mod 4. If $p=1$ mod 4, -1 is a residue of p and therefore of p^k. Hence the numbers of R_i or L_i $(i=0, 1,\ldots, n-1)$ fall into pairs which differ only in sign. Hence the zero residue class mod p^n (i.e. K_n)

occurs exactly $\frac{1}{2}\phi(p^{n-i})$ times in the set R_i^2 or the set L_i^2, and it does not occur at all in the set R_iL_i.

Further, if a is any number of R_i, it follows from the expressions for R_iL_j, R_iR_j, that $a+\lambda p^{i+1}$ is in R_i for any integer λ. Hence it follows that *there are precisely* p^{n-t-i} *numbers* x *in* R_i *such that* $a+x=0 \bmod p^{i+t}$ $(t>0)$.

These numbers x are those that satisfy:

$$x=-a+a_1p^{i+t}+a_2p^{i+t+1}+\ldots+a_{n-i-t}p^{n-1} (\text{mod } p^n).$$

As $a_1, a_2, \ldots, a_{n-i-t}$ range independently over the distinct numbers mod p, we get the p^{n-i-t} distinct numbers x mod p^n. Hence the number of numbers x in R_i for which $a+x=0 \bmod p^{i+t}$ and $\neq 0 \bmod p^{i+t+1}$, is

$$p^{n-t-i}-p^{n-t-i-1}=\phi(p^{n-i-t}).$$

Also, since $a_1, \ldots, a_{n-i-t}$ are arbitrary numbers mod p, it is clear that if b is *any* given number divisible by p^{i+t}, there is just one number x in R_i for which

$$a+x=b \ (\text{mod } p^n).$$

Hence $a+x$ must represent each number of the class K_{i+t} just once for the $\phi(p^{n-i-t})$ values of x in R_i (which have been shewn to exist) for which $a+x$ is divisible by p^{i+t} but not by p^{i+t+1}. By making a vary over the $\frac{1}{2}\phi(p^{n-i})$ numbers of R_i, it follows that each number of K_{i+t} can be represented in exactly $\frac{1}{2}\phi(p^{n-i})$ ways as the sum of two numbers of R_i; in other words K_{i+t} occurs with the coefficient $\frac{1}{2}\phi(p^{n-i})$ in R_i^2. By multiplication by a non-residue mod p^n, it follows that K_{i+t} occurs with the same coefficient in L_i^2 also. Thus K_{i+t} occurs with the coefficient $\phi(p^{n-i})$ in $R_i^2+L_i^2$; since however by equations (A) it occurs with the coefficient $\phi(p^{n-i})$ in K_i^2, follows that no number of K_{i+t} $(t=1, 2, \ldots)$ can occur in R_iL_i. It follows that $R_iL_i=aR_i+bL_i$. If however x, x' are numbers of R_i and y, y' of L_i, any representation of the form:

$$x'=x+y$$

leads, on multiplication by a fixed quadratic non-residue of p^n, to just one representation of the form

$$y'=x+y;$$

and vice versa. Hence $a=b$; on comparing the number of elements on both sides we see that

$$R_iL_i=\tfrac{1}{4}\phi(p^{n-i})\left\{R_i+L_i\right\}=\tfrac{1}{4}\phi(p^{n-i})K_i.$$

The same reasoning shews that the coefficients of R_i, L_i in R_i^2 must be respectively equal to the coefficients of L_i, R_i in L_i^2. We have thus arrived at the equations:

$$R_i^2 = aR_i + bL_i + \tfrac{1}{2}\phi(p^{n-i})\left\{ K_{i+1} + K_{i+2} + \ldots + K_n \right\}$$

$$R_iL_i = \tfrac{1}{4}\phi(p^{n-i})K_i$$

$$L_i^2 = bR_i + aL_i + \tfrac{1}{2}\phi(p^{n-i})\left\{ K_{i+1} + \ldots + K_n \right\},\ p = 1 \bmod 4.$$

By comparing the coefficients of K_n in $R_i^2.L_i$ and in $R_iL_i.L_i$ we have

$$b.\tfrac{1}{2}\phi(p^{n-i}) = \tfrac{1}{4}\phi(p^{n-i}).\tfrac{1}{2}\phi(p^{n-i}).$$

Thus $$b = \tfrac{1}{4}\phi(p^{n-i}).$$

Lastly the coefficient of K_i in K_i^2 in equations (A) is

$$\phi(p^{n-i})\left\{ 1 - 1/(p-1) \right\}.$$

The coefficient of K_i in $R_i^2 + 2R_iL_i + L_i$ is $a + b + \frac{1}{2}\phi(p^{n-i})$. Hence

$$a = \tfrac{1}{4}\phi(p^{n-i}) - p^{n-i-1}.$$

This establishes equations (B) for $p = 1 \bmod 4$.

VIII. If $p = -1 \bmod 4$, -1 is a non-residue of p and therefore of every power of p. Hence the numbers of L_i are obtained by changing the signs of those of R_i. Hence the zero residue class (i.e K_n) occurs precisely $\frac{1}{2}\phi(p^{n-i})$ times in R_iL_i, and it does not occur at all in R_i^2, L_i^2. By the method of VII we may prove that a number of K_{i+t} can be expressed in exactly $\frac{1}{2}\phi(p^{n-i})$ ways as the sum of a number of R_i and a number of L_i, while it cannot be expressed at all as a sum of two numbers of R_i or of two numbers of L_i. Also by the method of multiplying an additive representation by a non-residue of p^n, we may shew that the coefficients of R_i, L_i in R_i^2 are respectively equal to the coefficients of L_i, R_i in L_i^2, and that the coefficients of R_i, L_i in R_iL_i are equal. Thus when $p = -1 \bmod 4$, the algebra takes the form:

$$R_i^2 = aR_i + cL_i$$

$$R_iL_i = bK_i + \tfrac{1}{2}\phi(p^{n-i})\left\{ K_{i+1} + K_{i+2} + \ldots + K_n \right\}$$

$$L_i^2 = cR_i + aL_i.$$

By equating the number of elements on both sides in the second equation, we have

$$\tfrac{1}{4}\left\{ \phi(p^{n-i}) \right\}^2 = b\phi(p^{n-i}) + \tfrac{1}{2}\phi(p^{n-i}).p^{n-i-1}.$$

Hence $$b=\tfrac{1}{4}\phi(p^{n-i})-\tfrac{1}{2}p^{n-i-1}.$$

By equating the coefficients of K_n in $R_i^2L_i$ and $R_i.R_iL_i$, we see that $a=b$. By equating the number of elements on both sides of the first equation, we see that $c=\frac{1}{4}\phi(p^{n-i})+\frac{1}{2}p^{n-i-1}$. Thus the equations (B) are established for $p=-1 \bmod 4$.

IX. *The Algebra of Quadratic Residues mod* 2^n, $(n>2)$. As in the case of p^n, the 2^n residue classes mod 2^n fall into $n+1$ classes $K_0, K_1, \ldots, K_n$ with respect to the total group of automorphisms. Since $\phi(2)=1$, $\phi(4)=2$, the classes K_n, K_{n-1} consist each of a single residue class, while K_{n-2} contains two residue classes. The algebra of these classes, obtained by putting $p=2$ in equations (A) is:*

$$(C)\begin{cases} K_iK_j=\phi(2^{n-j})K_i\,, \quad (j>i) \\ K_i^2=\phi(2^{n-i})\left\{K_{i+1}+K_{i+2}+\ldots+K_n\right\}. \end{cases}$$

In the case of 2^n, the group of automorphisms (i.e. the multiplicative group of the odd residue classes) is no longer cyclic if $n>2$, but is the direct product of the group of order two composed of $(-1, 1)$ with the cyclic group $(1, 5, 5^2, \ldots)$ of order 2^{n-2}†. The subgroup formed by the squares of all odd residue classes, that is, the subgroup of quadratic residues of 2^n, is therefore the cyclic group $(1, 5^2, 5^4, \ldots)$ of order 2^{n-3}. Hence, with respect to the subgroup of quadratic residues, the group of the 2^{n-1} odd residue classes falls into 4 subclasses, say, R, L, M, N, each consisting of 2^{n-3} numbers. Here the class R is the subgroup of quadratic residues; the class L is the set $(-1, -5^2, -5^4, \ldots)$ obtained by multiplying the quadratic residues by -1; the class M is the set $(5, 5^3, 5^5 \ldots)$; the class N is the set $(-5, -5^3, -5^5, \ldots)$ obtained by multiplying M by -1. When $n=3$, the four classes each consist of a single residue class; namely R is 1 mod 8, L is -1 mod 8, M is -3 mod 8 and N is 3 mod 8. When $n=2$, that is, for the modulus 4, there are only two odd residue classes 1 and 3; of these 1 should be considered to represent both R and M, while 3 represents both L and N.

* The equations (A) are valid for $p=2$ also, since they are obtained from the general formula for γ_{ij}^k in the paper 'On a remarkable property etc.' *loc. cit.*

† Compare E. Hecke, *Theorie der Algebraischen Zahlen*, (1923). pp. 49, 50.

Consider now the classes into which the 2^n numbers mod 2^n fall with respect to the subgroup of automorphisms defined by the quadratic residues. It is clear that K_i for $i<n-2$, falls now into four equinumerous subclasses R_i, L_i, M_i, N_i, where for example R_i consists of numbers of the form ($2^i\times$an odd number of type R mod 2^{n-i}). The class K_{n-2} consisting of the two numbers 2^{n-2} and 3.2^{n-2}, now subdivides into two subclasses which may be denoted by R_{n-2} and L_{n-2}, the former being composed of 2^{n-2} and the latter of 3.2^{n-2}. The two classes K_{n-1}, K_n each containing a single number, do not subdivide. Thus for the subgroup of quadratic residues, we have the $4(n-1)$ classes:

$$K_n, K_{n-1}, R_{n-2}, L_{n-2}, R_i, L_i, M_i, N_i \ (i=n-3, n-4\ldots, 1, 0).$$

If P is any of these $4(n-1)$ classes it is clear that $K_nP=P$, It is also clear that the four classes (or numbers) $K_n, K_{n-1}, R_{n-2}, L_{n-2}$ combine in the same way as the four residue classes mod 4. Thus:

$$R_{n-2}^2=L_{n-2}^2=K_{n-1}\,;\ K_{n-1}R_{n-2}=L_{n-2}\,;\ K_{n-1}L_{n-2}=R_{n-2}\,;$$
$$R_{n-2}L_{n-2}=K_n\,;\ K_{n-1}^2=K_n.$$

THEOREM II. *If q is odd, $q+8K$ belongs to the same class mod 2^n as q; $q+2$ belongs to N_0, R_0, L_0, M_0 when q belongs respectively to R_0, L_0, M_0, N_0.*

This theorem corresponds to Theorem I for the modulus p^n. The proof is on the same lines. A number of the form, $\pm 5^{2K}$, $\pm 5^{2K+1}$ mod 2^n is of the same form mod 8. *The converse is also true*; for, there are 2^{n-3} numbers mod 2^n which are of any one of these forms, and there are also 2^{n-3} numbers mod 2^n which are equal to a given number mod 8. It follows that q belongs to R_0, L_0, M_0 or N_0 mod 2^n, according as $q=1, -1, -3$, or 3 mod 8.

The second part of the theorem follows by reduction to mod 8. Since $1+2=3$, $2+3=-3$, $2-3=-1$, $2-1=1$ mod 8, it follows that addition of 2 permutes cyclically the classes $R_0N_0M_0L_0$ mod 8. Hence by the first part of the theorem it permutes cyclically the classes $R_0N_0M_0L_0$ mod 2^n. Hence also the addition of 4 interchanges the classes R_0 and M_0, as well as the classes N_0 and L_0.

It follows that if $j>i+2$, and T represents indifferently R, L, M or N, the sum of a number 2^ia of R_i and a number of T_j, is of the form $2^i(a+8K)$ and hence belongs to R_i. We have there fore:

$$(D)\begin{cases} R_iT_j=\tfrac{1}{4}\phi(2^{m-j})R_i;\ L_iT_j=\tfrac{1}{4}\phi(2^{m-j})L_i; \\ \qquad\qquad M_iT_j=\tfrac{1}{4}\phi(2^{n-j})M_i; \\ N_iT_j=\tfrac{1}{4}\phi(2^{m-j})N_i;\ n-2>j>i+2; \\ R_iR_{n-2}=R_i \text{ and similar equations } (i<n-4); \\ R_iK_{n-1}=R_i \text{ and similar equations } (i<n-3). \end{cases}$$

To discuss the case $j=i+2$, we use the fact that the addition of 4 interchanges the classes (R_0, M_0) and (L_0, N_0) mod 2^{n-i}. Hence, if T represents indifferently R, L, M, or N,

$$(D_1)\begin{cases} R_iT_{i+2}=2^{n-i-5}M_i;\ M_iT_{i+2}=2^{n-i-5}R_i; \\ L_iT_{i+2}=2^{n-i-5}N_i;\ N_iT_{i+2}=2^{n-i-5}L_i;\ i<n-4; \\ R_{n-4}R_{n-2}=R_{n-4}L_{n-2}=M_{n-4} \text{ and similar equations}; \\ R_{n-3}K_{n-1}=M_{n-3} \text{ and similar equations.} \end{cases}$$

For the case $j=i+1$, we use the fact that the addition of 2 permutes cyclically the classes $R_0N_0M_0L_0$ mod 2^{n-i}. Hence:

$$(D_2)\begin{cases} R_iT_{i+1}=2^{n-i-4}N_i:\ N_iT_{i+1}=2^{n-i-4}M_i; \\ \qquad\qquad M_iT_{i+1}=2^{n-i-4}L_i; \\ L_iT_{i+1}=2^{n-i-4}R_i;\ i<n-3; \\ R_{n-3}R_{n-2}=R_{n-3}L_{n-2}=N_i; \text{ and similar equations.} \end{cases}$$

From (D), (D_1), (D_2) the equation $K_iK_j=\phi(2^{n-j})K_i$ of (C) may be verified.

X. We next consider the combination of classes with the *same* index. The results follow simply by reduction to mod 8.

Since the numbers of R_0, L_0, M_0, N_0 mod 2^n are those that are equal to 1, -1, -3, 3 mod 8, the sum of two numbers of R_0 or of M_0, as well as the sum of a number of L_0 and a number of N_0 are those of the form $2(4m+1)$ mod 2^n. It follows easily that

$$R_0^2=M_0^2=L_0N_0=2^{n-3}(R_1+M_1);$$
$$L_0^2=N_0^2=R_0M_0=2^{n-3}(L_1+N_1);$$

in the same way, the sum of a number of R_0 and a number of N_0 or of a number of L_0 and a number of M_0, is of the form $4\times$an odd number. It follows easily that

$$L_0M_0=R_0N_0=2^{n-3}K_2.$$

Lastly, if a is any number R_0, then the numbers of L_0 are of the form $-a+8K$ mod 2^n; $(K=1, 2, \ldots 2^{n-3})$. Hence by adding a to all numbers of L_0 we get once each multiple of 8 mod 2^n.

Hence

$$R_0L_0=M_0N_0=2^{n-3}(K_3+K_4+\ldots+K_n).$$

These equations give also the result of combining R_i, L_i, M_i, N_i. We have:

$$(E)\quad\begin{cases} R_i^2=M_i^2=L_iN_i=2^{n-i-3}(R_{i+1}+M_{i+1}) \\ L_i^2=N_i^2=R_iM_i=2^{n-i-3}(L_{i+1}+N_{i+1}) \\ L_iM_i=R_iN_i=2^{n-i-3}K_{i+2} \\ R_iL_i=M_iN_i=2^{n-i-3}(K_{i+3}+K_{i+4}+\ldots+K_n). \end{cases}$$

The equation $K_i^2=\phi(2^{n-i})\left\{K_{i+1}+K_{i+2}+\ldots+K_n\right\}$ of (C) may be verified from (E).

ON THE GROUP-OPERATIONS OF A BOOLEAN ALGEBRA

By R. VAIDYANATHASWAMY

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1. We owe to Bernstein* the result that every Boolean Algebra can be exhibited as a group, and that all the group-operations R are commutative, and of the form:

$$xRy = A(x \oplus y) + A'(x \otimes y),$$

where x, y are arbitrary elements of the Boolean Algebra, A' is the negative of the element A which defines R, and the operations $\oplus$ of *disjunction*† and $\otimes$ of *conjunction*, are defined by:

$$x \oplus y = xy' + x'y$$
$$x \otimes y = xy + x'y'.$$

Bernstein's treatment however, does not seem to me to bring out the essential simplicity of the situation, or to reveal the inevitability of the result. I attempt here a more illuminating treatment of the problem of determining all the group-operations of a Boolean Algebra. The main result which will be proved, may be stated thus:

A Boolean Algebra admits essentially only one group-operation; we mean by this that all its group-operations are 'paraphrases' (in a sense to be explained) of any one of them.

2. *Disjunction as a group-operation of the Boolean Algebra.* We first prove

THEOREM I. *There exists one and only one group-operation of a Boolean Algebra having zero as its identity-element; this group-operation is disjunction, and is therefore a commutative operation.*

* B. A. Bernstein 'Operations with respect to which the elements of a Boolean Algebra form a group', *Trans. Am. Math. Soc.* 26 (1924) 171-174; also *ibid* 27 (1925) p. 600.

† The nomenclature is the same as in Miss S. Pankajam's paper 'On symmetric functions of n elements in a Boolean Algebra', *Jour. Ind. Math. Soc.* Vol. II. 198-210 (1937). This paper may also be referred to, for properties of Conjunction and Disjunction.

For, the most general Boolean function of two variables x, y has the form:

$$R(x, y) \equiv axy + bxy' + cx'y + dx'y' + ex + fx' + gy + hy' + k.$$

Hence the most general Boolean operation xRy, which leads from any two given elements x, y to a definite third element z of the Boolean Algebra, may be identified with this function. Since by hypothesis, zero is the identity-element of the operation R, we have:

$$xR0 = (b+e)x + (d+f)x' + (h+k) \equiv x \qquad (1)$$

$$0Ry = (c+g)y + (d+h)y' + (f+k) \equiv y. \qquad (2)$$

Multiplying (1) by x' we have

$$(d+f+h+k)x' = 0 \text{ for all } x.$$

In particular, putting $x=0$, we have $d=f=h=k=0$. The equations (1) and (2) therefore become:

$$xR0 = (b+e)x \equiv x \text{ for all } x$$

$$0Ry = (c+g)y \equiv y \text{ for all } y.$$

Hence $$b+e=c+g=1. \qquad (3)$$

Again since R is a group-operation, with zero as the identity-element, every element, in particular, the universe, must have a unique inverse in R. Hence there is a unique y such that

$$1Ry = (a+g)y + by' + e = 0. \qquad (4)$$

Hence $e=0$; substituting in (3), we have $b=1$. Hence the equation (4) implies that the equations

$$by' = y' = 0; \ (a+g)y = 0$$

have a unique solution, namely $y=1$. This is possible only if $(a+g)=0$, that is, only if $a=g=0$. Substituting in (3), we have $c=1$. Hence $R(x, y)$ has the form:

$$R(x, y) = xy' + x'y = x \oplus y.$$

Thus xRy reduces to the *disjoint* of x and y. It is well known on the other hand, that disjunction is in fact a group-operation of the Boolean Algebra, with respect to which each element has the order two, and constitutes therefore its own inverse. Our theorem is thus proved.

3. *The Paraphrases of a group-operation.* Now, let G be any given set of elements, finite or infinite in number. Let it be supposed further that we are given an operation R with respect to which G is a group (not necessarily Abelian). If e is the identity-element of R, we may indicate this fact by writing R in the form R_e.

If a be an arbitrary element of G, we shall now shew that we can derive from R_e, a new group-operation R_a which has a for its identity-element. For given a, R_a is unique and will be called a *paraphrase of* R_e. There are thus as many paraphrases of R_e as there are elements of G; these paraphrases are all different from one another, as their identity-elements are different.

For any two elements x, y of G, we define R_a by

$$xR_ay = xR_ea'R_ey, \tag{3}$$

where a' is the inverse element of a with respect to the group-operation R_e. *The R_a thus defined is a group-operation of G.* For,

(1) R_a is associative. For, R_e being given to be a group-operation, is associative. It is clear from the expression (3) of R_a in terms of R_e, that R_a is also associative.

(2) R_a has the identity-element a; for,

$$x.R_a.a = xR_ea'R_ea = xR_ee \text{ (since } a' \text{ is the inverse of } a \text{ in } R_e)$$
$$= x \text{ (since } e \text{ is the identity-element of } R_e),$$

and $\quad a.R_a.y = aR_ea'R_ey = eR_ey = y.$

(3) Any element x of G has the unique inverse $aR_ex'R_ea$, with respect to the operation R_a, where x' is the unique inverse of x (known to exist) with respect to R_e. For,

$$xR_aaR_ex'R_ea = xR_ea'R_eaR_ex'R_ea \text{ (by (3))}$$
$$= xR_eeR_ex'R_ea = xR_ex'R_ea = eR_ea$$
$$= a, \text{ the identity-element of } R_a;$$

and $\quad aR_ex'R_eaR_ax = aR_ex'R_eaR_ea'R_ex$ (by (3))

$$= aR_ex'R_eeR_ex = aR_ex'R_ex = aR_ee = a.$$

Thus R_a is a new group-operation of G with a as the identity-element.

THEOREM II. *The relation 'is a paraphrase of' is reflexive, symmetric and transitive. In other words, if* a *is put equal to* e, *the paraphrase R_a becomes identical with R_e. Further if R_a is a paraphrase of R_e, then R_e is a paraphrase of R_a, namely that paraphrase which has* e *as its identity-element. Lastly if R_a is a paraphrase of R_e, then any two paraphrases of R_e, R_a respectively are identical, if they have the same identity-element.*

The proof of the first statement is immediate; for, on putting $a=e$ on the right side of (3), it becomes $xR_eeR_ey = xR_ey$. To prove the second statement, let T_e be the paraphrase of R_a which has e for its identity-element. Then, by (3)

$xT_ey = xR_ae'R_ay$, where e' is the inverse of e in R_a.

$=xR_aaR_eeR_eaR_ay$ (since $aR_ex'R_ea$ is the inverse of x in R_a)
$=xR_aaR_eaR_ay=xR_ea'R_eaR_eaR_ay$ (by (3))
$=xR_eaR_ea'R_ey$ (by (3)) $=xR_ey$.

This proves that T_e is identical with R_e. Lastly let $R_b(b\neq a)$ be a paraphrase of R_e and let T_b be that paraphrase of R_a which has b as its identity-element. We have:

$xT_by=xR_ab_1R_ay$ (by (3), b_1 being the inverse of b in R_a)
$=xR_aaR_eb'R_eaR_ay$ (where b' is the inverse of b in R_e)
$=xR_ea'R_eaR_eb'R_eaR_ea'R_ey$ (by (3))
$=xR_eb'R_ey=xR_by$ (by (3)).

Thus T_b is identical with R_b, proving the transitivity of the paraphrase relation.

From Theorem II, it follows that, by the usual method of abstraction in Mathematics, we can conceive of a group-operation and all its paraphrases as constituting a single abstract entity. Thus the paraphrases of R_e should not be viewed as distinct in essence from it.

It is clear that, if the group-operation R_e is commutative, all its paraphrases are also commutative. More generally, we have

THEOREM III. *If a set of elements constitute a group G under the operation R_e and a group G' under the paraphrase R_a of R_e then the groups G, G' are simply isomorphic.*

For, make any element x of G correspond to the element aR_ex of G'.* This correspondence is obviously reversible and (1, 1). Also it institutes an isomorphism between G, G'; for the element aR_exR_ey of G' which corresponds to xR_ey of G, is identical with $aR_exR_aaR_ey$, since

$$aR_exR_aaR_ey=aR_exR_ea'R_eaR_ey \text{ (by (3))}=aR_exR_ey.$$

This isomorphism supports the view that the paraphrases of a group-operation are not essentially different from it.

The symmetry of the paraphrase relation shews also

THEOREM IV. *If R, R' are two different group-operations with the same identity-element, no paraphrase of R can be identical with a paraphrase of R'.*

* The Correspondence: any element x of $G\rightarrow$ the element xR_ea of G' may also be shewn to be an isomorphism of G, G'.

4. *The group-operations of the Boolean Algebra.* We have seen that a Boolean Algebra possesses only one (Boolean) group-operation, namely disjunction, with zero as identity-element. We may accordingly denote disjunction by D_0. Hence all the paraphrases of D_0 are group-operations of the Boolean Algebra. Since D_0 is commutative, all its paraphrases are also commutative. We have

THEOREM V. *Conjunction is a paraphrase of disjunction; namely it is that paraphrase D_1 of D_0 which has the universe as its identity-element. Hence Conjunction is a group-operation.*

For, the paraphrase D_1 which has the universe as its identity-element, is given by:

$$xD_1y = xD_0 1' D_0 y \text{ (by (3), } 1' \text{ being the inverse of 1 in } D_0\text{)}$$
$$= xD_0 1 D_0 y = x'D_0 y = xy + x'y' = x \otimes y.$$

Thus D_1 is identical with conjunction.

THEOREM VI. *The general group-operation of Bernstein is identical with the general paraphrase D_a of D_0.*

For,

$$xD_ay = xD_0 a D_0 y \text{ (by (3) since } a \text{ is its own inverse in } D_0\text{)}$$
$$= x \oplus a \oplus y = a'(x \oplus y) + a(x \otimes y),$$

which is the general form of Bernstein's group-operation. Lastly we have

THEOREM VII. *A Boolean Algebra has no group-operations other than the paraphrases D_a.*

For, suppose the Boolean Algebra has a group-operation Δ_a which has the identity-element a, and is different from D_a. Let Δ_0 be the paraphrase of Δ_a which has the identity-element zero. We know already that D_0 is the paraphrase of D_a having zero as its identity-element. Since Δ_a is different from D_a by hypothesis, it follows from Theorem IV that the paraphrases Δ_0, D_0 should be different from one another. We would thus have a (Boolean) group-operation Δ_0 which has the identity-element zero, and is different from disjunction. This is impossible by Theorem I. Thus the group-operations of the Boolean Algebra consist only of disjunction and its paraphrases. The point to note is that the existence of many group-operations is not a peculiar property of the Boolean Algebra; by the theory of paraphrasing, it is already implied in its possessing a single group-operation (*e.g.* disjunction or conjunction).

ON A NET OF TETRAHEDRA ASSOCIATED WITH A SPACE CUBIC CURVE

By K. RANGASWAMI, M.A., M.Sc.

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1. It is well known that the locus of a point the feet of the perpendiculars from which to the faces of a tetrahedron lie in a plane is a quadrinodal cubic surface, having nodes at the vertices of the tetrahedron and containing its six edges. This locus* known as the Steiner's pedal locus of the tetrahedron is the isogonal transformation of the plane at infinity with respect to the tetrahedron and contains ∞^2 twisted cubics through the four nodes which are the transforms of the lines at infinity. Among these, *the cubics that are the transforms of the tangents to the absolute conic Ω have the peculiar property of meeting the plane at infinity in the vertices of a triangle circumscribed to the absolute conic.*

A special feature of any twisted cubic of this type, namely, that the feet of the perpendiculars from any point of the cubic to the faces of any inscribed tetrahedron lie in a plane was noticed by H. W. Richmond† who also extended the result to a norm-curve in an Euclidean space of n-dimensions. However, the converse problem of specifying those inscribed tetrahedra of a twisted cubic whose corresponding pedal locus contains the cubic, appears not to have been attempted so far. If for simplicity we call an inscribed tetrahedron Δ of a twisted cubic Γ a *pedal tetrahedron* if the feet of the perpendiculars from *any* point of the cubic to the faces of the tetrahedron lie in a plane, the main purpose of this paper is to show that *the totality of pedal tetrahedra of a twisted cubic constitutes a net.* It will be found that Richmond's result appears as a special case of our main theorem.

2. Let a, b, c, d be the lines of intersection of the plane at infinity with the faces of Δ and t a tangent to the absolute conic Ω.

* Baker, *Principles of Geometry*, Vol. IV, 12-27.

† *Proc. Camb. Phil. Soc.* 22, 34-38.

Consider the net (N) of quadrics apolar* to Δ and outpolar to Ω. The quadrics of this net will meet the plane at infinity in a net (n) of conics which will be outpolar to Ω and also to any inconic of the quadrilateral δ formed by the lines a, b, c, d. Hence the three pairs of opposite vertices of any quadrilateral circumscribed to Ω and to any inconic of δ will be degenerate class conics inpolar to the conics of the net (n) and therefore will be isogonal conjugates† with respect to Δ. It follows therefore that if S is the unique conic touching a, b, c, d, t and if t_1, t_2, t_3 be the three common tangents of S and Ω other than t, the isogonal transformation of the line t will be a twisted cubic through the vertices of Δ and the points $(t_2, t_3), (t_3, t_1), (t_1, t_2)$. We have thus the result:

The isogonal transformation af any tangent to the absolute conic is a twisted cubic through the vertices of Δ which meets the plane at infinity in the vertices of a triangle circumscribed to the absolute conic. (2.1)

Now, the seven linearly independent class quadrics inpolar to the net (N) of order quadrics may be taken to be the six pairs of points consisting of any two vertices of Δ and the absolute conic Ω. Hence any class quadric inpolar to the net (N) must be linearly dependent on these. Thus, a pair of points p, p' conjugate to all the quadrics of the net, i.e. a pair of isogonal conjugates for Δ, must belong to this linear system and hence must be the foci of a quadric of revolution‡ inscribed in Δ. In particular if p' is at infinity one of the quadrics of the net (N) has its centre at p, and in this case p is the finite focus of an inscribed paraboloid of revolution. Consequently the feet of the perpendiculars from p on the faces of Δ lie in a plane. Hence we see that:

If the feet of the perpendiculars from p on the faces of Δ lie on a plane then p is the centre of a quadric of the net (N).

3. Let P, Q, R be the points in which the twisted cubic Γ meets the plane at infinity and Δ any inscribed tetrahedron. Now in order that a point O on Γ should have a pedal plane for Δ it is

* The term 'apolar' is here used to mean 'having Δ for a self-polar tetrahedron',

† Two points are said to be isogonal conjugates with respect to a tetrahedron Δ when they are conjugate with respect to the net of quadrics having Δ for a self-polar tetrahedron and outpolar to the 'Absolute'.

‡ If the tangential equations of the vertices of Δ be $A=0$, $B=0$, $C=0$, $D=0$, those of a pair of isogonal conjugates be $P=0, P'=0$ and the tangential equation of the absolute conic be $\Omega=0$, there is a relation of the form

$$\lambda PP'+\Omega=\lambda_{AD}AD+\ldots+\lambda_{BC}BC+\ldots .$$

necessary and sufficient that the unique order-quadric apolar to the two tetrahedra Δ and $OPQR$ must be also outpolar to Ω. Taking the familiar mode of representing the inscribed tetrahedra* of a twisted cubic in [3] by the points of [4], the quadrics inpolar to Γ and outpolar to Ω will be represented by the lines of a linear complex in [4]. The unique tetrahedron which is in general inscribed in Γ and circumscribed to Ω corresponds to the singular point of the linear complex. Since $OPQR$ is a pedal tetrad on Γ, by varying O on Γ, we have a pencil of pedal tetrads which is represented in [4] by a line l of the linear complex.

Now, it is easily seen that if Δ is a pedal tetrad, any two points and therefore every point on Γ should have pedal planes for Δ. Thus by our representation the problem of determining the pedal tetrads on Γ is equivalent to that of finding the points Δ in [4] the line joining which to any point on l is a complex line. It follows, therefore, that all such points lie in the polar plane of l with respect to the linear complex which is also the plane determined by l and the singular point ω of the linear complex. We have thus proved the main result of this paper, viz.:

The totality of pedal tetrads on a cubic curve Γ *constitutes a net.* (3.1)

We note further that if Δ be any tetrahedron of this net, the pencil of tetrahedra determined by Δ and $OPQR$ will also be pedal tetrads.

Proceeding to the case when the triangle PQR is circumscribed to the absolute conic Ω, every tetrahedron $OPQR$ is circumscribed to Ω, so that there is a pencil of inscribed tetrahedra of Γ circumscribed to Ω. Hence by a known theorem† there must be a net of inscribed tetrahedra of Γ also circumscribed to Ω. Consequently the linear complex in [4] has a plane ω of singular points so that the line joining *any* point Δ of [4] to any point of the singular plane is necessarily a complex line. Hence in this case every tetrad of Γ is a pedal tetrad—which is Richmond's result stated in the beginning.

4. We will now proceed to specify geometrically the net of pedal tetrads on the cubic curve Γ. Now if Δ is a pedal tetrahedron on Γ, it is easily seen from (2.2) that among the quadrics of the net (N) there is a pencil whose centres lie on Γ.

* R. Vaidyanathaswamy, 'On the rational Norm curve II', *Jour. Lond. Math. Soc.* 7, 54-5.

† On the rational Norm curve II, loc. cit. p. 54.

The quadrics of this pencil meet the plane at infinity in a pencil γ of conics contained in the net (n) and having PQR for a self-polar triangle. Since the quadrics of the net (N) are apolar to Δ the three pairs of opposite edges of Δ meet the plane at infinity in three pairs of points conjugate for the conics of the net (n) and therefore for the conics of the pencil γ. We thus infer that the pairs of opposite edges of Δ meet the plane at infinity in pairs of isogonal conjugate points* with respect to the triangle PQR.

Conversely, let p, p' be a pair of isogonal conjugate points with respect to the triangle PQR and Δ the inscribed tetrahedron formed by drawing the chords of Γ through p, p'. Now it is easily seen from the characteristic property of the twisted cubic, that QR, RP, PQ determine related ranges with the faces of Δ. Hence if the faces of Δ meet the plane at infinity in the lines a, b, c, d there is a definite conic σ touching a, b, c, d and also the sides of the triangle PQR. Thus the pencil γ being outpolar to the linearly independent conics σ, Ω and (p, p') is contained in the net (n). This implies that there is a pencil of quadrics of the net (N). whose centres lie on Γ; in other words, Δ is a pedal tetrahedron of Γ. We have therefore the result:

The vertices of a pedal tetrahedron are the extremities of chords of Γ through a pair of isogonal conjugate points with respect to the triangle PQR. (4.1)

* 'Isogonal conjugates' in the non-euclidean plane with Ω as the absolute conic.

THE MULTIPLICATIVE ARITHMETIC FUNCTIONS CONNECTED WITH A FINITE ABELIAN GROUP

By T. VENKATARAYUDU, M.A., University of Madras

Introduction.

An arithmetic function $f(N)$ is multiplicative* if

$$f(MN)=f(M)f(N)$$

whenever the integers M, N are relatively prime. The function $F(N)$ defined by the equation

$$F(N)=\sum f_1(\delta)f_2\left(\frac{N}{\delta}\right)$$

summed for all divisors δ of N is called the *composite* of the two arithmetic functions f_1, f_2. If f_1 and f_2 are multiplicative, it is easy to see that their composite F (represented by $f_1.f_2$) is also multiplicative.

Analogous definitions may be given for functions of several arguments. Thus the arithmetic function $f(M_1, M_2,\ldots M_r)$ is said to be multiplicative if

$$f(M_1N_1, M_2N_2,\ldots M_rN_r)=f(M_1, M_2,\ldots M_r)f(N_1, N_2,\ldots N_r)$$

whenever the products $M_1M_2\ldots M_r$, $N_1N_2\ldots N_r$ are relatively prime. The notion of composition may be obviously extended to functions of several arguments.

The function of r arguments $M_1, M_2,\ldots M_r$ which takes the value 1 for all values of its arguments is denoted by $E(M_1, M_2\ldots M_r)$, The composite of $f(M_1, M_2,\ldots M_r)$ and $E(M_1, M_2,\ldots M_r)$

is given by $\quad (f\cdot E)(M_1, M_2,\ldots M_r)=\Sigma f(\delta_1, \delta_2,\ldots \delta_r)$

summed for all divisors δ_i of M_i $(i=1, 2,\ldots r)$. We call $f\cdot E$ as the *integral* of f.

It is known that given an arithmetic function $f(N)$ there exists a unique arithmetic function $\psi(N)$ such that the composite $f.\psi$ vanishes for all values of its argument other than 1 and takes

* *See* Dr. R. Vaidyanathaswamy "The theory of multiplicative arithmetic functions" *Trans. Amer. Math. Soc.*, Vol. 33. pp. 579-662. (1931).

the value 1 when the argument is equal to 1. We call ψ the *inverse function* of f and denote it by f^{-1}. It is easy to see that f^{-1} is also a multiplicative function.

The multiplicative arithmetic function $f(N)$ is said to be a *rational integral function* of degree r if $f^{-1}(N)$ vanishes for all values of N divisible by an $(r+1)$th power.

In the present paper we consider the four multiplicative arithmetic functions $m(d_1, d_2)$, $m(d)$, $M(d)$, $\Gamma(d_1, d_2, \ldots d_r)$ (defined below) connected with a finite Abelian group.

I. THE MULTIPLICATIVE FUNCTIONS.

Let G be an Abelian group of finite order N. We require the known*

LEMMA: An element S of order d_1d_2 in G where $(d_1, d_2)=1$ can be uniquely expressed as the product of two elements of orders d_1 and d_2 respectively.

PROOF: If $$S^{d_2}=M, S^{d_1}=N$$
and $$ad_1+bd_2=1$$
M^b, N^a are of orders d_1 and d_2. Then $S=M^b N^a$ is the required expression.

If possible, let $S=M_1N_1$ where M_1, N_1 are of orders d_1 and d_2.

$$S^{d_2}=M_1^{d_2}=M.$$

Therefore $$M^b=M_1^{bd_2}=M_1^{1-ad_1}=M_1.$$

Similarly $N^a=N_1$. Hence the representation is unique.

We denote by $G(d_1, d_2)$ the totality of the elements in G whose orders divide d_1 and are multiples of d_2. Let $m(d_1, d_2)$ denote the number of elements in $G(d_1, d_2)$.

THEOREM 1: $m(d_1, d_2)$ is a multiplicative arithmetic function of the two arguments d_1, d_2.

PROOF: Let the product $G(d_1, d_2).G(d'_1, d'_2)$ represent the totality of the elements obtained by multiplying each element of $G(d_1, d_2)$ with each element of $G(d'_1, d'_2)$. We first show that

$$G(d_1, d_2).G(d'_1, d'_2)=G(d_1d'_1, d_2d'_2) \qquad (1)$$

whenever $(d_1d_2, d'_1d'_2)=1$.

Evidently $G(d_1, d_2).G(d'_1, d'_2)$ is contained in $G(d_1d'_1, d_2d'_2)$. Again if S is an element in $G(d_1d'_1, d_2d'_2)$ of order d we

* Burnside. Theory of groups (1911), pp. 15-16.

can write $d=\delta\delta'$ uniquely where δ is a divisor of d_1 and a multiple of d_2 and δ' is a divisor of d'_1 and a multiple of d'_2. By the lemma we can express S as S_1S_2 where S_1 is of order δ and S_2 is of order δ'. Thus $G(d_1d'_1, d_2d'_2)$ is contained in the product $G(d_1, d_2).G(d'_1, d'_2)$. Hence (1) is established and therefore, $m(d_1, d_2)m(d'_1, d'_2)=m(d_1d'_1, d_2d'_2)$* whenever $(d_1d_2, d'_1d'_2)=1$.

THEOREM 2. If $G(d)$ denotes the totality of the elements in G whose orders divide d and $m(d)$ is the number of elements in $G(d)$, $m(d)$ is a multiplicative arithmetic function.

PROOF: This follows immediately from Theorem 1 for

$$m(d)=m(d, 1)\dagger$$

THEOREM 3. If $C(d)$ is the class of elements of order d in G and if $M(d)$ denotes the number of elements in $C(d)$, $M(d)$ is a multiplicative function.

PROOF: This can be proved directly by showing

$$C(dd')=C(d)C(d') \quad \text{when } (d, d')=1.$$

or thus:

$$m(d)=\Sigma M(\delta) \text{ summed for all divisors } \delta \text{ of } d.$$

$m(d)$ is therefore the integral of $M(d)$. Since $m(d)$ is multiplicative so also is $M(d)$.

It is well known‡ that every Abelian group is the direct product of cyclic groups of prime power orders. Let a be the greatest of the exponents of the prime powers. $m(d)$, $M(d)$ vanish for values of d divisible by an $(a+1)$th power and therefore $m(d)$, $M(d)$ are inverses of rational integral functions of degree a. Also $m(d)=0=M(d)$ if d is prime to N.

I showed elsewhere|| that the classes $C(d)$ combine among themselves by multiplication. That is, if $C(d_i)C(d_j)$ represents the totality of the elements obtained by multiplying each element of $C(d_i)$ with each element of $C(d_j)$ then each element of $C(d_k)$ occurs the same number of times $\gamma(d_i, d_j, d_k)$ in the product $C(d_i)$ $C(d_j)$. Let $\Gamma(d_1, d_2, \ldots d_r)$ denote the number of times the identity class (=element) occurs in the product $C(d_1)C(d_2)\ldots C(d_r)$

* Since $m(d_1, d_2)$ vanishes whenever d_2 is not a divisor of d_1 it is an *ordinal function. See* Dr. R. Vaidyanathaswamy loc. cit. p. 656.

† $m(d)$ is the derivative $D_{d_2}[m(d_1, d_2)]$ *Cf.* Dr. R. Vaidyanathaswamy, loc. cit., p. 623.

‡ Speiser "Theorie der Gruppen von Endlicher ordnung", p. 50.

|| "On the linear algebra of classes of elements in a finite Abelian group", under publication in the *Proc. Ind. Acad. Sc.*

and $\gamma(d_1, d_2 \ldots d_r)$ denote the number of times the class $C(d_r)$ occurs in the product $C(d_1).C(d_2)\ldots C(d_{r-1})$. Since the inverses of the elements in a class $C(d)$ form the class $C(d)$ itself it is easy to see that

$$M(d_r).\gamma(d_1, d_2, \ldots d_r) = \Gamma(d_1, d_2, \ldots d_r).$$

The function $\Gamma(d_1, d_2, \ldots d_r)$ is clearly symmetric and we have

THEOREM 4. $\Gamma(d_1, d_2, \ldots d_r)$ is a symmetric multiplicative function of the r arguments $d_1, d_2, \ldots d_r$.

PROOF: By definition $\Gamma(d_1d'_1, d_2d'_2, \ldots d_rd'_r)$ = the number of times the identity element occurs in the product $C(d_1d'_1)C(d_2d'_2)\ldots C(d_rd'_r)$. If the products $d_1d_2\ldots d_r$, $d'_1d'_2\ldots d'_r$ are relatively prime $\Gamma(d_1d'_1, d_2d'_2, \ldots d_rd'_r)$ = the number of times the identity occurs in

$$C(d_1)C(d_2)\ldots C(d_r).C(d'_1)C(d'_2)\ldots C(d'_r).$$

i.e. in

$$\left\{ \Sigma\gamma(d_1, d_2, \ldots d_r, d_k)C(d_k). \right\}\left\{ \Sigma\gamma(d'_1, d'_2, \ldots d'_r, d'_k)C(d'_k) \right\}$$
$$= \gamma(d_1, d_2, \ldots d_r, 1).\gamma(d'_1, d'_2, \ldots d'_r, 1).$$
$$= \Gamma(d_1, d_2, \ldots d_r).\Gamma(d'_1, d'_2, \ldots d'_r).$$

Hence $\Gamma(d_1, d_2, \ldots d_r)$ is a multiplicative function.

Cor. $\gamma(d_1, d_2, \ldots d_r)$ is multiplicative.

II. EXPRESSIONS FOR THE MULTIPLICATIVE FUNCTIONS.

Since $m(d)$, $M(d)$, $m(d_1, d_2)$, $\Gamma(d_1, d_2, \ldots d_r)$ are multiplicative it is enough if we find their values when the arguments are powers of a single prime p. Hence the functions depend only on the sub-group of G, consisting of all elements in G whose orders are powers of p (the sylow component of G belonging to the prime p). Let the type of this group be $(\alpha_1 \geqslant \alpha_2 \geqslant \ldots \geqslant \alpha_r)$.

$m(p^\beta) = p^{k\beta+\alpha_{k+1}+\ldots+\alpha_r}$ where k is the number of α's $\geqslant \beta$* (1)

Therefore $\quad M(p^\beta) = p^{k(\beta-1)+\alpha_{k+1}+\ldots+\alpha_r}(p^k-1)$ (2)

From (1) and (2) the value of $m(p^{a_1}, p^{a_2})$, $a_1 > a_2$ can be obtained

$$m(p^{a_1}, p^{a_2}) = m(p^{a_1}) - m(p^{a_2}) + M(p^{a_2}) \qquad (3)$$

Expression for $\gamma(p^{a_i}, p^{a_j}, p^{a_k})$:—I have shown that

$$\gamma(d_i, d_j, d_k) = \sum m[g(t_i, t_j)]\mu\left(\frac{d_i}{t_i}\right)\mu\left(\frac{d_j}{t_j}\right)$$

* Miller, Blichfeldt, and Dickson "Finite groups", p. 93.

summed for divisors t_i, t_j of d_i, d_j such that the l.c.m. of $t_i . t_j$ is a multiple of d_k. If we take $d_i = p^{a_i}$, $d_j = p^{a_j}$, $d_k = p^{a_k}$ we have the following cases:

(1) When $a_i > a_j$; $a_i > a_k$

$$\gamma(p^{a_i}, p^{a_j}, p^{a_k}) = 0.$$

(2) When $a_i > a_j$; $a_i = a_k$

$$\gamma(p^{a_i}, p^{a_j}, p^{a_k}) = m(p^{a_j}) - m(p^{a_j - 1}) = M(p^{a_j}).$$

(3) When $a_i = a_j$; $a_i > a_k$

$$\gamma(p^{a_i}, p^{a_j}, p^{a_k}) = m(p^{a_i}) - m(p^{a_i - 1}) = M(p^{a_i}).$$

(4) When $a_i = a_j = a_k$

$$\gamma(p^{a_i}, p^{a_j}, p^{a_k}) = m(p^{a_i}) - 2m(p^{a_i - 1})$$

$$= M(p^{a_i}).\left[1 - \frac{m(p^{a_i - 1})}{M(p^{a_i})}\right].$$

We can now state the value of $\gamma(d_i, d_j, d_k)$ in the general case viz.

$$\gamma(d_i, d_j, d_k) = M[g(d_i, d_j)]\Pi\left[1 - \frac{m(p^{a-1})}{M(p^a)}\right], \text{ or } 0$$

according as the l.c.m. of every two of d_i, d_j, d_k is the same or not and the product extending over the common block factors* of d_i, d_j, d_k. $\gamma(p^{a_i}, p^{a_j}, p^{a_k})$ can be directly obtained by a method which we use below but the above form suggests an important class of multiplicative arithmetic functions.

Expression for $\Gamma(p^{a_1}, p^{a_2}, \ldots p^{a_r})$.

Since $\Gamma(p^{a_1}, p^{a_2}, \ldots p^{a_r})$ is symmetric we can take

$$a_1 \geqslant a_2 \geqslant \ldots \geqslant a_r.$$

$$C(p^{a_1})C(p^{a_2})\ldots C(p^{a_r}) =$$

$$\left\{ G(p^{a_1}) - G(p^{a_1 - 1}) \right\}\left\{ G(p^{a_2}) - G(p^{a_2 - 1}) \right\}\ldots$$
$$\times\left\{ G(p^{a_r}) - G(p^{a_r - 1}) \right\}$$

Case i. $a_1 \neq a_2$. $C(p^{a_2}), C(p^{a_3}) \ldots C(p^{a_r})$ are complexes (See my paper loc. cit.) in $G(p^{a_1})$ and $G(p^{a_1 - 1})$. Hence

$$C(p^{a_1}).C(p^{a_2})\ldots C(p^{a_r}) = M(p^{a_2}).M(p^{a_3})\ldots M(p^{a_r}).C(p^{a_1}).$$

* My paper, loc. cit.

Case ii, $a_1=a_2=\ldots=a_k=a\neq a_{k+1}$.

In this case $C(p^{a_1}).C(p^{a_2})\ldots C(p^{a_r})$

$$=M(p^{a_{k+1}})M(p^{a_{k+2}})\ldots M(p^{a_r}).\{G(p^a)-G(p^{a-1})\}^k$$

$$=M(p^{a_{k+1}})\ldots M(p^{a_r}).\Big[G(p^a)-\binom{k}{1}G(p^a)^{k-1}G(p^{a-1})+\ldots$$

$$+(-1)^{k-1}\binom{k}{k-1}G(p^{a-1})^{k-1}G(p^a)+(-1)^kG(p^{a-1})^k\Big]$$

$$=M(p^{a_{k+1}})\ldots M(p^{a_r}).\Big[m(p^a)^{k-1}G(p^a)$$

$$-\binom{k}{1}m(p^a)^{k-2}.\,m(p^{a-1}).G(p^a)+\ldots$$

$$+(-1)^{k-1}km(p^{a-1})^{k-1}G(p^a)+(-1)^k m(p^{a-1})^{k-1}.G(p^{a-1})\Big].$$

$$=M(p^{a_{k+1}})\ldots M(p^{a_r}).\Big[\frac{\{m(p^a)-m(p^{a-1})\}^k+(-1)^{k-1}m(p^{a-1})^k}{m(p^a)}$$

$$\times G(p^a)+(-1)^km(p^{a-1})^{k-1}G(p^{a-1})\Big]$$

$$=M(p^{a_{k+1}})\ldots M(p^{a_r}).\Big[\frac{M(p^a)^k+(-1)^{k-1}m(p^{a-1})^k}{m(p^a)}G(p^a)$$

$$+(-1)^km(p^{a-1})^{k-1}G(p^{a-1})\Big]$$

Hence $\Gamma(p^{a_1}, p^{a_2},\ldots p^{a_r})=0$ if $a_1\neq a_2$,

$$=M(p^{a_{k+1}})\ldots M(p^{a_r}).\Big[\frac{M(p^a)^k+(-1)^{k-1}.m(p^{a-1})^k}{m(p^a)}$$

$$+(-1)^k.m(p^{a-1})^{k-1}\Big]$$

if $a=a_1=\ldots=a_k\neq a_{k+1}$.

My thanks are due to Dr. R. Vaidyanathaswamy for his kind help with the manuscript.

ON SOME THEOREMS CONCERNING DETERMINANTAL SYMMETRIC FUNCTIONS

By M. ZIA-UD-DIN, M.A.Ph.D. (Wales), Bahawalpur

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1. In a previous paper* a fundamental theorem was given by the author by means of which the general isobaric determinants are expressed as a sum of bi-alternant symmetric functions. In this paper I shall deal with the generalisation of the theorem and certain deductions that follow from it. The *theorem* is:

If the bi-alternant $h\begin{pmatrix} 0 & p & q\dots \\ 0 & 1 & 2\dots \end{pmatrix}$ be expanded in terms of monomial symmetric functions $\Sigma\, a^i b^j c^k \dots$, as

$$h\begin{pmatrix} 0 & p & q\dots \\ 0 & 1 & 2\dots \end{pmatrix} = \dots + \rho_{ijk\dots} P(ijk\dots) + \dots;\ \text{then}$$

$$h\begin{pmatrix} \alpha & \beta & \gamma\dots \\ 0 & p & q\dots \end{pmatrix} = \dots + \rho_{ijk\dots} \sum_{ijk\dots} h\begin{pmatrix} \alpha-i & \beta-j & \gamma-k\dots \\ 0 & 1 & 2\ \dots \end{pmatrix}$$

where Σ indicates permutation of $i, j, k, \dots$ and summation, $\alpha, \beta, \gamma, \dots; p, q, r, \dots$ are in ascending order of magnitude and the expansion being indicated by a single typical term.

2. Before proceeding to the generalisation of the above theorem, a theorem due to D. E. Littlewood will be deduced here.

Following Littlewood and Richardson,† we have

$$\{\lambda/\mu\} = \left| h_{\lambda_s - \mu_t - s + t} \right| = h\begin{pmatrix} \lambda_1+n-1 & \lambda_2+n-2\dots\lambda_{n-1}+1 & \lambda_n \\ \mu_1+n-1 & \mu_2+n-2\dots\mu_{n-1}+1 & \mu_n \end{pmatrix}$$

where (λ) denotes the partition $(\lambda_1, \lambda_2, \dots \lambda_n)$ and (μ) the partition $(\mu_1, \mu_2, \dots \mu_n)$, $\lambda_1 \geqslant \lambda_2 \geqslant \dots \geqslant \lambda_n$, $\mu_1 \geqslant \mu_2 \geqslant \dots \geqslant \mu_n$; s denotes the row and t the column of the matrix from which the element is taken, the determinant having n rows and columns; and

$$\{\lambda\} = \left| h_{\lambda_s - s + t} \right| = \frac{A(\alpha\ \beta\ \gamma\dots)}{A(0\ 1\ 2\dots)}$$

By taking $\mu_n = 0$, we can write

* Zia-ud-Din, *Proc. Edin. Math. Soc.* (2) 4 (1934) 47-52.
† *See* D. E. Littlewood, *Proc. London, M. S.* (2) 40 (1936) 60.

$$\left\{\lambda/\mu\right\}=h\begin{pmatrix}\lambda_n & \lambda_{n-1}+1 & \lambda_{n-2}+2\ldots\lambda_1+n-1\\ 0 & \mu_{n-1}+1 & \mu_{n-2}+2\ldots\mu_1+n-1\end{pmatrix}$$

$$=\ldots+\rho_{ijk\ldots}\sum_{ijk\ldots} h\begin{pmatrix}\lambda_n-i & \lambda_{n-1}+1-j & \lambda_{n-2}+2-k\ldots\\ 0 & 1 & 2\ \ \ldots\end{pmatrix}$$

Put $$h\begin{pmatrix}\lambda_n-i & \lambda_{n-1}+1-j\ldots\\ 0 & 1\ \ \ldots\end{pmatrix}=\left\{\nu\right\} \text{ and}$$

$$\Sigma\, \rho_{i'j'k'\ldots}P(i'j'k'\ldots)=\left\{\mu\right\},$$

(Σ denoting summation for different values of $i', j', k',\ldots$, including $i, j, k,\ldots$).

Since*

$$A(\alpha_1\beta_1\gamma_1\ldots)P(\alpha_2\beta_2\gamma_2\ldots)=A(\alpha_1+\alpha_2^{+},\ \beta_1+\beta_2^{+},\ \gamma_1+\gamma_2^{+},\ldots)$$

therefore the coefficient of

$$A(\lambda_n\lambda_{n-1}+1\ldots\lambda_1+n-1) \text{ in}$$

$A(\lambda_n-i\ \lambda_{n-1}+1-j\ \lambda_{n-2}+2-k\ldots)\Sigma\, \rho_{i'j'\ldots}P(i'j'k'\ldots)$, that is in $\left\{\nu\right\}A(012\ldots)\left\{\mu\right\}$ is clearly $\rho_{ijk\ldots}$. Hence $\rho_{ijk\ldots}$ is the coefficient of $\left\{\lambda\right\}$ in the product $\left\{\nu\right\}\left\{\mu\right\}$.

Similarly by assigning different values to $i, j, k,\ldots$ in $\left\{\nu\right\}$, the coefficient of $\left\{\lambda\right\}$ in $\left\{\nu\right\}\left\{\mu\right\}$ will evidently be $\rho_{ijk\ldots}$ according to those values.

Hence we obtain Littlewood's† Theorem

$$\left\{\lambda/\mu\right\}=\left|{}^{h}\lambda_s-\mu_t-s+t\right|=\sum g_{\nu\mu\lambda}\left\{\nu\right\}$$

where $g_{\nu\mu\lambda}$ is the coefficient of $\left\{\lambda\right\}$ in the product $\left\{\nu\right\}\left\{\mu\right\}$.

3. The theorem of expressibility of general isobarics as a sum of simple-isobarics or S-functions can be generalised as

THEOREM. *To a rational integral identity of t variables.*

$$A(\lambda\mu\nu\ldots)=A(lmn\ldots)P(pqr\ldots)$$
$$\pm A(l'm'n'\ldots)P(p'q'r'\ldots)+\ldots, \qquad I$$

where $(l+m+n+\ldots)+(p+q+r+\ldots)=\lambda+\mu+\nu+\ldots,$

* Zia-ud-Din *l.c.* p. 49.

N.B. $A(\alpha\beta\gamma\ldots)$ denotes the Alternant $|a^{\alpha}b^{\beta}c^{\gamma}\ldots|$ and the associated permanent which forms the monomial symmetric functions $\Sigma a^{\alpha}b^{\beta}c^{\gamma}\ldots$, is denoted by $P(\alpha\beta\gamma\ldots)$ and it will be termed as *Monomial Permanent.*

† D. E. Littlewood, *l.c.* p. 61, Theorem VIII.

$\lambda, \mu, \nu \ldots, l, m, n, \ldots$ etc being in ascending order of magnitude, there corresponds the identity in isobaric determinants of t-th order,

$$h\begin{pmatrix} \alpha & \beta & \gamma & \ldots \\ \lambda & \mu & \nu & \ldots \end{pmatrix} = \sum h\begin{pmatrix} \alpha-p & \beta-q & \gamma-r & \ldots \\ l & m & n & \ldots \end{pmatrix}$$

$$\pm \sum h\begin{pmatrix} \alpha-p' & \beta-q' & \gamma-r' & \ldots \\ l' & m' & n' & \ldots \end{pmatrix} + \ldots \qquad II$$

where Σ denotes permutation of $p, q, r, \ldots$ etc. and summation.

Proof. In the alternants and monomial symmetric functions, use, instead of $a, b, c, \ldots$ the operational letters $A, B, C, \ldots$ which are such that $A^k h_\alpha = h_{\alpha-k}$, $B^k h_\beta = h_{\beta-k}$, $C^k h_\gamma = h_{\gamma-k}$, and so on. Thus identity I can be operationally written as,

$$H(\lambda\mu\nu\ldots) = H(lmn\ldots)\Sigma A^p B^q C^r \ldots$$
$$\pm H(l'm'n'\ldots)\Sigma A^{p'} B^{q'} C^{r'} \ldots + \ldots$$

Performing the operations on $h_\alpha h_\beta h_\gamma \ldots$, we obtain the identity II,

$$h\begin{pmatrix} \alpha & \beta & \gamma & \ldots \\ \lambda & \mu & \nu & \ldots \end{pmatrix} = \sum_{pqr\ldots} h\begin{pmatrix} \alpha-p & \beta-q & \gamma-r & \ldots \\ l & m & n & \ldots \end{pmatrix}$$

$$\pm \sum_{p'q'r'\ldots} h\begin{pmatrix} \alpha-p' & \beta-q' & \gamma-r' & \ldots \\ l' & m' & n' & \ldots \end{pmatrix}$$

$$+ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

which proves the Theorem.

4. The identity II does not depend merely on the fact that h's are symmetric functions, but we could quite well use determinantal elements. Thus the identity in pure determinants corresponding to

$$A(026) = A(015)P(011) - A(014)P(111), \qquad (1)$$

can be written as,

$$\begin{vmatrix} a_0 & b_0 & c_0 \\ a_2 & b_2 & c_2 \\ a_6 & b_6 & c_6 \end{vmatrix} = \begin{vmatrix} a_0 & b_1 & c_1 \\ a_1 & b_2 & c_2 \\ a_5 & b_6 & c_6 \end{vmatrix} + \begin{vmatrix} a_1 & b_0 & c_1 \\ a_2 & b_1 & c_2 \\ a_6 & b_5 & c_6 \end{vmatrix}$$
$$+ \begin{vmatrix} a_1 & b_1 & c_0 \\ a_2 & b_2 & c_1 \\ a_6 & b_6 & c_5 \end{vmatrix} - \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_5 & b_5 & c_5 \end{vmatrix}$$

which can be verified on expansion.

Let the operation of increasing by k the suffixes in the s-th row of the determinant $|a_0 b_1 c_2 \ldots|$ of n-th order be denoted by E_s^k

and that of increasing by σ the suffixes in column t by F_t^τ. These operators will be called *Displacement Operators* as they correspond to moving rows and columns about.

Now taking the operand of 3rd order

$$\begin{vmatrix} a_0 & b_0 & c_0 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}$$

the operational identity corresponding to (1) is

$$\begin{aligned} E_1^0E_2^1E_3^4 &= E_1^0E_2^0E_3^3(F_1^0F_2^1F_3^1+F_1^1F_2^0F_3^1+F_1^1F_2^1F_3^0) \\ &\quad -E_1^0E_2^0E_3^2(F_1^1F_2^1F_3^1), \\ &= E_1^0E_2^0E_3^3(\Sigma F_1^0F_2^1F_3^1)-E_1^0E_2^0E_3^2(F_1^1F_2^1F_3^1). \end{aligned} \qquad (2)$$

Generally corresponding to identity I of Art 3, between the products of alternants and monomial Permanents there is an identity in pure determinants,

$$E_1^\lambda E_2^{\mu-1}E_3^{\nu-2}\ldots = E_1^l E_2^{m-1}E_3^{n-2}\ldots\Big(\sum_{pqr\ldots} F_1^pF_2^qF_3^r\ldots\Big)+\ldots; \qquad III$$

when the operand is of the type.

$$| \; a_0 \quad b_1 \quad c_2 \ldots |.$$

4.1. As an example we can apply III to the Wronskians, say,

$$\begin{vmatrix} f_1 & f_2 & f_3\ldots \\ f'_1 & f'_2 & f'_3\ldots \\ f''_1 & f''_2 & f''_3\ldots \\ \ldots & \ldots & \ldots \end{vmatrix}$$

dashes denoting differentiations.

Thus (2) may be written as

$$\begin{vmatrix} f_1 & f_2 & f_3 \\ f''_1 & f''_2 & f''_3 \\ f_1^{vi} & f_2^{vi} & f_3^{vi} \end{vmatrix} = \begin{vmatrix} f_1 & f'_2 & f'_3 \\ f'_1 & f''_2 & f''_3 \\ f_1^{v} & f_2^{vi} & f_3^{vi} \end{vmatrix}$$

$$+\begin{vmatrix} f'_1 & f_2 & f'_3 \\ f''_1 & f'_2 & f''_3 \\ f_1^{vi} & f_2^{v} & f_3^{vi} \end{vmatrix} + \begin{vmatrix} f'_1 & f'_2 & f_3 \\ f''_1 & f''_2 & f'_3 \\ f_1^{vi} & f_2^{vi} & f_3^{v} \end{vmatrix}$$

$$-\begin{vmatrix} f'_1 & f'_2 & f'_3 \\ f''_1 & f''_2 & f''_3 \\ f_1^{v} & f_2^{v} & f_3^{v} \end{vmatrix}$$

which again may be verified.

EXTENSIONS OF SOME SELF-RECIPROCAL FUNCTIONS

By R. S. VARMA

1. In a recent paper* I have shown that the functions

$$x^{\nu+\frac{1}{2}}e^{\frac{1}{4}x^2}D_{-2\nu-3}(x) \qquad R(\nu)>-1$$

and

$$x^{\nu-\frac{1}{2}}e^{\frac{1}{4}x^2}D_{-2\nu}(x) \qquad R(\nu)>-\tfrac{1}{2}$$

are self-reciprocal in the Hankel-transform of order ν, that is,

$$x^{\nu+\frac{1}{2}}e^{\frac{1}{4}x^2}D_{-2\nu-3}(x)=\int_0^\infty \sqrt{(xy)}J_\nu(xy)\,y^{\nu+\frac{1}{2}}e^{\frac{1}{4}y^2}D_{-2\nu-3}(y)\,dy. \quad (1)$$

and

$$x^{\nu-\frac{1}{2}}e^{\frac{1}{4}x^2}D_{-2\nu}(x)=\int_0^\infty \sqrt{(xy)}J_\nu(xy)\,y^{\nu-\frac{1}{2}}e^{\frac{1}{4}y^2}D_{-2\nu}(y)\,dy. \quad (2)$$

The object of the present paper is to amplify and generalise these results. Thus in § 2 of this paper I have investigated the integral

$$\int_0^\infty y^{n+1}J_n(xy)\,e^{\frac{1}{4}y^2}D_{-m}(y)\,dy. \quad (3)$$

$$=\frac{x^n\Gamma(2n+2)}{2^{n+1}\Gamma(m)\Gamma(n+1)}\Big[2^{\frac{1}{2}m-n-1}\Gamma(\tfrac{1}{2}m-n-1)$$

$$\times{}_1F_1(n+\tfrac{3}{2};2+n-\tfrac{1}{2}m;\tfrac{1}{2}x^2)$$

$$+x^{m-2n-2}\frac{\Gamma(1+n-\frac{1}{2}m)\Gamma(\frac{1}{2}m+\frac{1}{2})}{\Gamma(n+\frac{3}{2})}{}_1F_1(\tfrac{1}{2}m+\tfrac{1}{2};-n+\tfrac{1}{2}m;\tfrac{1}{2}x^2)\Big]$$

$$R(m)>0,\ R(n+1)>0 \text{ and } R(n-m+\tfrac{1}{2})<0$$

which gives the generalisation of the integral equation (1). The extension of the integral equation (2) is furnished by the following integral, deduced by me elsewhere† by a different method,

* R. S. Varma, "Some functions which are self-reciprocal in the Hankel-transform", *Proc. Lond. Math. Soc.* (2) 42 (1936), 9-17.

† R. S. Varma, "An infinite integral involving Bessel functions and parabolic cylinder functions", *Proc. Camb. Phil. Soc.* 33 (1937), 210-11.

$$\int_0^\infty y^{n-\frac{1}{2}}e^{\frac{1}{4}y^2}J_{n-\frac{1}{2}}(xy)D_{-m}(y)dy \tag{4}$$

$$=\frac{(2x)^{n-\frac{1}{2}}\Gamma(n)}{2\sqrt{\pi}\Gamma(m)}\Big[2^{\frac{1}{2}m-n}\Gamma(\tfrac{1}{2}m-n)\,{}_1F_1(n;1+n-\tfrac{1}{2}m;\tfrac{1}{2}x^2)$$

$$+x^{m-2n}\frac{\Gamma(n-\frac{1}{2}m)\Gamma(\frac{1}{2}m)}{\Gamma(n)}\,{}_1F_1(\tfrac{1}{2}m;1-n+\tfrac{1}{2}m;\tfrac{1}{2}x^2)\Big]$$

$$R(m)>0,\ R(n)>0 \text{ and } R(n-m-1)<0$$

since, by the help of the known result*

$$D_n(x)=\frac{\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{2}-\frac{1}{2}n)}2^{\frac{1}{2}n}e^{-\frac{1}{4}x^2}\,{}_1F_1(-\tfrac{1}{2}n;\tfrac{1}{2};\tfrac{1}{2}x^2)$$

$$+\frac{\Gamma(-\frac{1}{2})}{\Gamma(-\frac{1}{2}n)}2^{\frac{1}{2}n-\frac{1}{2}}xe^{-\frac{1}{4}x^2}\,{}_1F_1(\tfrac{1}{2}-\tfrac{1}{2}n;\tfrac{3}{2};\tfrac{1}{2}x^2)$$

it reduces, for $m=2\nu$ and $n=\nu+\frac{1}{2}$, to (2). The integral (3) is important from another point of view as well. It gives with the help of the integral (4) the following

THEOREM: *If*

$$f(x)=x^{\nu-\frac{1}{2}}e^{\frac{1}{4}x^2}D_{-2\nu-2}(x)$$

and

$$g(x)=\frac{x^{\nu+\frac{1}{2}}e^{\frac{1}{4}x^2}D_{-2\nu-1}(x)}{2\nu+1}$$

then $f(x)$ *and* $g(x)$ *are* J_ν*-transforms of each other, provided* $R(\nu)>-\frac{1}{2}$.

This Theorem is established in § 3. Finally in §§ 5-6, the values of the integrals

$$\int_0^\infty y^{2\nu+1}e^{\frac{1}{4}y^2\left(1-\frac{1}{2p^2}\right)}I_\nu\left(\frac{y^2}{8p^2}\right)D_{-m}(y)dy$$

$$\int_0^\infty y^{2\nu}e^{\frac{1}{4}y^2\left(1-\frac{1}{2p^2}\right)}I_\nu\left(\frac{y^2}{8p^2}\right)D_{-m}(y)dy$$

are given, by the help of (3) and (4) respectively, in terms of hypergeometric series.

2. Using Whittaker's integral† for $D_n(x)$, viz.,

$$D_n(x)=\frac{1}{\Gamma(-n)}e^{-\frac{1}{4}x^2}\int_0^\infty e^{-tx-\frac{1}{2}t^2}t^{-n-1}dt \qquad R(n)<0$$

we have

* Whittaker and Watson, *Modern Analysis* (4th Edition), p. 347.

† E. T. Whittaker, "On the functions associated with the parabolic cylinder in harmonic analysis", *Proc. Lond. Math. Soc.* (1), 35 (1903), 417-427.

$$\int_0^\infty y^{n+1} J_n(xy) e^{\frac{1}{4}y^2} D_{-m}(y)\,dy$$

$$=\frac{1}{\Gamma(m)}\int_0^\infty y^{n+1} J_n(xy)\,dy \int_0^\infty e^{-ty-\frac{1}{2}t^2} t^{m-1}\,dt \qquad R(m)>0$$

$$=\frac{1}{\Gamma(m)}\int_0^\infty e^{-\frac{1}{2}t^2} t^{m-1}\,dt \int_0^\infty e^{-ty} y^{n+1} J_n(xy)\,dy$$

$$=\frac{x^n\Gamma(2n+2)}{2^n\Gamma(m)\Gamma(n+1)}\int_0^\infty \frac{t^m e^{-\frac{1}{2}t^2}}{(t^2+x^2)^{n+\frac{3}{2}}}\,dt \qquad R(n+1)>0$$

since

$$\int_0^\infty e^{-ty} y^{n+1} J_n(xy)\,dy$$

$$=\frac{x^n}{2^n t^{2n+2}}\cdot\frac{\Gamma(2n+2)}{\Gamma(n+1)}\,{}_2F_1\left(n+1,\, n+\tfrac{3}{2}\,;\, n+1\,;-\frac{x^2}{t^2}\right)$$

$$=\frac{t x^n \Gamma(2n+2)}{2^n\Gamma(n+1)(t^2+x^2)^{n+\frac{3}{2}}}.$$

Using the known result

$$2\int_0^\infty \frac{x^{m-1}e^{-\frac{1}{2}x^2}}{(x^2+a^2)^n}\,dx = 2^{\frac{1}{2}m-n}\Gamma(\tfrac{1}{2}m-n)\,{}_1F_1(n\,;\,1+n-\tfrac{1}{2}m\,;\,\tfrac{1}{2}a^2)$$

$$+a^{m-2n}\frac{\Gamma(n-\frac{1}{2}m)\Gamma(\frac{1}{2}m)}{\Gamma(n)}\,{}_1F_1(\tfrac{1}{2}m\,;\,1-n+\tfrac{1}{2}m\,;\,\tfrac{1}{2}a^2)$$

$$R(m)>0$$

we at once arrive at the integral (3).

If we put $n=\nu$ and $m=2\nu+3$ and make use of the relation (5), we obtain the integral equation (2) as a particular case of the integral (3).

To justify the change in the order of integration, suppose that

$$\theta(t)=t^{m-1}e^{-\frac{1}{2}t^2}\int_0^\infty e^{-ty}y^{n+1}J_n(xy)\,dy$$

$$=\frac{\Gamma(2n+2)x^n t^m e^{-\frac{1}{2}t^2}}{2^n\Gamma(n+1)(t^2+x^2)^{n+\frac{3}{2}}}$$

and $$\phi(y)=y^{n+1}J_n(xy)\int_0^\infty e^{-ty-\frac{1}{2}t^2}t^{m-1}\,dt$$

For real values of x, $\theta(t)$ is uniformly convergent for the unlimited range $t\geqslant 0$ provided that $R(m-1)>0$ and $R(n+1)>0$. Since for all values of n,

$$J_n(x)=\sum_{r=0}^{\infty}\frac{(-1)^r x^{n+2r}}{2^{n+2r}r!\Gamma(n+r+1)}$$

$\phi(y)$ converges uniformly in $y\geqslant 0$, provided that $R(n+\frac{1}{2})>0$ and $R(m)>0$. Again consider the integral

$$I=\int_{\tau}^{\infty}|y^{n+1}J_n(xy)|dy\int_0^{\infty}e^{-ty-\frac{1}{2}t^2}t^{m-1}dt$$

where τ is large.

Now for large values of y,

$$J_n(y)=O(y^{-\frac{1}{2}})$$

and

$$e^{-ty}<(ty)^{-d-1},$$

where $R(d)>0$.

If we choose $d=n+1$, where $R(n+1)>0$, I is less than a constant multiple of

$$x^{-\frac{1}{2}}\int_{\tau}^{\infty}y^{-\frac{3}{2}}dy\int_0^{\infty}e^{-\frac{1}{2}t^2}t^{m-n-3}dt$$

which tends to zero when $R(m-n-2)>0$.

Hence the inversion is justified if $R(m-1)>0$, $R(n+\frac{1}{2})>0$ and $R(m-n-2)>0$. But by the theory of analytic continuation, the integral (3) is true for the more extensive ranges of m and n stated in § 1.

3. For $n=\nu$ and $m=2\nu+1$, (3) gives

$$\int_0^{\infty}\sqrt{xy}\,J_\nu(xy)\frac{y^{\nu+\frac{1}{2}}e^{\frac{1}{4}y^2}D_{-2\nu-1}(y)}{2\nu+1}dy$$

$$=x^{\nu-\frac{1}{2}}e^{\frac{1}{4}x^2}D_{-2\nu-2}(x) \qquad (6)$$

$$R(\nu)>-\tfrac{1}{2}.$$

and by putting $n=\nu+\frac{1}{2}$ and $m=2\nu+2$ in (4), we obtain

$$\int_0^{\infty}\sqrt{xy}\,J_\nu(xy)y^{\nu-\frac{1}{2}}e^{\frac{1}{4}y^2}D_{-2\nu-2}(y)dy$$

$$=\frac{x^{\nu+\frac{1}{2}}e^{\frac{1}{4}x^2}D_{-2\nu-1}(x)}{2\nu+1} \qquad R(\nu)>-\tfrac{1}{2}. \qquad (7)$$

The integrals (6) and (7) at once lead us to the important Theorem enunciated in § 1.

4. We shall now give a few more interesting special cases of our general results (3) and (4).

Thus for $n=-\frac{1}{2}$ and $n=\frac{1}{2}$, the integral (3) reduces, by virtue of the known relations,

$$J_{-\frac{1}{2}}(z)=\left(\frac{2}{\pi z}\right)^{\frac{1}{2}}\cos z, \text{ and } J_{\frac{1}{2}}(z)=\left(\frac{2}{\pi z}\right)^{\frac{1}{2}}\sin z$$

to

$$\int_0^\infty \cos xy\, e^{\frac{1}{4}y^2}D_{-m}(y)\,dy=\frac{1}{2\Gamma(m)}\Big[\Gamma(\tfrac{1}{2}-\tfrac{1}{2}m)\Gamma(\tfrac{1}{2}+\tfrac{1}{2}m)\,x^{m-1}e^{\frac{1}{2}x^2}$$
$$+2^{\frac{1}{2}m-\frac{3}{2}}\Gamma(\tfrac{1}{2}m-\tfrac{1}{2})\,{}_1F_1(1\,;\tfrac{3}{2}-\tfrac{1}{2}m\,;\tfrac{1}{2}x^2)\Big],$$

valid when $R(m)>0$, odd integral values of $R(m)$ being excluded, and

$$\int_0^\infty y\sin xy\, e^{\frac{1}{4}y^2}D_{-m}(y)\,dy=\frac{x}{\Gamma(m)}\Big[2^{\frac{1}{2}m-\frac{3}{2}}\Gamma(\tfrac{1}{2}m-\tfrac{3}{2})$$
$$\times{}_1F_1(2\,;\tfrac{5}{2}-\tfrac{1}{2}m\,;\tfrac{1}{2}x^2)$$
$$+x^{m-3}\Gamma(\tfrac{3}{2}-\tfrac{1}{2}m)\Gamma(\tfrac{1}{2}+\tfrac{1}{2}m)\,{}_1F_1(\tfrac{1}{2}+\tfrac{1}{2}m\,;\tfrac{1}{2}m-\tfrac{1}{2}\,;\tfrac{1}{2}x^2)\Big],$$

true when $R(m)>1$, odd integral values of $R(m)$ being excluded.

The integral (4) for $n=1$ gives

$$\int_0^\infty y^{-1}e^{\frac{1}{4}y^2}\sin xy D_{-m}(y)\,dy=\frac{x}{2\Gamma(m)}\Big[\Gamma(\tfrac{1}{2}m)\Gamma(1-\tfrac{1}{2}m)\,x^{m-2}e^{\frac{1}{2}x^2}$$
$$+2^{\frac{1}{2}m-1}\Gamma(\tfrac{1}{2}m-1)\,{}_1F_1(1\,;2-\tfrac{1}{2}m\,;\tfrac{1}{2}x^2)\Big],$$

valid when $R(m)>0$, even integral values of $R(m)$ being excluded.

5. Writing 2ν for n in (3), we get

$$\int_0^\infty y^{2\nu+1}J_{2\nu}(xy)\,e^{\frac{1}{4}y^2}D_{-m}(y)\,dy \tag{8}$$
$$=\frac{x^{2\nu}\Gamma(4\nu+2)}{2^{2\nu+1}\Gamma(m)\Gamma(2\nu+1)}\Big[2^{\frac{1}{2}m-2\nu-1}\Gamma(\tfrac{1}{2}m-2\nu-1)$$
$${}_1F_1(2\nu+\tfrac{3}{2}\,;2+2\nu-\tfrac{1}{2}m\,;\tfrac{1}{2}x^2)$$
$$+x^{m-4\nu-2}\frac{\Gamma(1+2\nu-\frac{1}{2}m)\Gamma(\frac{1}{2}m+\frac{1}{2})}{\Gamma(2\nu+\frac{3}{2})}\,{}_1F_1(\tfrac{1}{2}m+\tfrac{1}{2}\,;-2\nu+\tfrac{1}{2}m\,;\tfrac{1}{2}x^2)\Big]$$

Multiply both sides of (8) by $e^{-p^2x^2}$ and integrate with respect to x between the limits 0 and ∞. The left hand side of (8) then becomes

$$\int_0^\infty e^{-p^2x^2}dx\int_0^\infty y^{2\nu+1}J_{2\nu}(xy)\,e^{\frac{1}{4}y^2}D_{-m}(y)\,dy$$

which, on changing the order of integration—a process obviously permissible—and then using the known integral*

$$\int_0^\infty J_{2\nu}(at)\exp(-p^2t^2)\,dt=\frac{\sqrt{\pi}}{2p}\exp\left(-\frac{a^2}{8p^2}\right)I_\nu\left(\frac{a^2}{8p^2}\right)$$

reduces to

$$\frac{\sqrt{\pi}}{2p}\int_0^\infty y^{2\nu+1}e^{\frac{1}{4}y^2\left(1-\frac{1}{2p^2}\right)}I_\nu\left(\frac{y^2}{8p^2}\right)D_{-m}(y)\,dy.$$

Now it is easy to prove that

$$\int_0^\infty x^l e^{-p^2x^2}\,{}_1F_1(a;b;\tfrac{1}{2}x^2)\,dx$$

$$=\frac{\Gamma(\frac{1}{2}l+\frac{1}{2})}{2p^{l+1}}\,{}_2F_1\left(a,\tfrac{1}{2}l+\tfrac{1}{2};b;\frac{1}{2p^2}\right)$$

$$2p^2>1.$$

By the help of this the right side of (8) yields

$$\frac{\Gamma(4\nu+2)}{2^{2\nu+1}\Gamma(m)\Gamma(2\nu+1)}\left[2^{\frac{1}{2}m-2\nu-1}\frac{\Gamma(\frac{1}{2}m-2\nu-1)\Gamma(\nu+\frac{1}{2})}{2p^{2\nu+1}}\right.$$

$$\times{}_2F_1\left(2\nu+\tfrac{3}{2},\nu+\tfrac{1}{2};2+2\nu-\tfrac{1}{2}m;\frac{1}{2p^2}\right)$$

$$+\frac{\Gamma(1+2\nu-\frac{1}{2}m)\Gamma(\frac{1}{2}m+\frac{1}{2})\Gamma(\frac{1}{2}m-\nu-\frac{1}{2})}{2p^{m-2\nu-1}\Gamma(2\nu+\frac{3}{2})}$$

$$\left.\times{}_2F_1\left(\tfrac{1}{2}m+\tfrac{1}{2},\tfrac{1}{2}m-\nu-\tfrac{1}{2};-2\nu+\tfrac{1}{2}m;\frac{1}{2p^2}\right)\right].$$

It follows therefore that, when $R(m)>0$, $R(2\nu+1)>0$ and $R(2\nu-m+1)<0$

$$\int_0^\infty y^{2\nu+1}e^{\frac{1}{4}y^2\left(1-\frac{1}{2p^2}\right)}I_\nu\left(\frac{y^2}{8p^2}\right)D_{-m}(y)\,dy$$

$$=\frac{\Gamma(4\nu+2)}{\sqrt{\pi}2^{2\nu+1}\Gamma(m)\Gamma(2\nu+1)}\left[2^{\frac{1}{2}m-2\nu-1}\frac{\Gamma(\frac{1}{2}m-2\nu-1)\Gamma(\nu+\frac{1}{2})}{p^{2\nu}}\right.$$

$${}_2F_1\left(2\nu+\tfrac{3}{2},\nu+\tfrac{1}{2};2+2\nu-\tfrac{1}{2}m;\frac{1}{2p^2}\right)$$

$$+\frac{\Gamma(1+2\nu-\frac{1}{2}m)\Gamma(\frac{1}{2}m+\frac{1}{2})\Gamma(\frac{1}{2}m-\nu-\frac{1}{2})}{p^{m-2\nu-2}\Gamma(2\nu+\frac{3}{2})}$$

$$\left.{}_2F_1\left(\tfrac{1}{2}m+\tfrac{1}{2},\tfrac{1}{2}m-\nu-\tfrac{1}{2};-2\nu+\tfrac{1}{2}m;\frac{1}{2p^2}\right)\right]$$

$$2p^2>1.$$

* Watson, *Bessel Functions*, p. 394.

6. Writing the integral (4) in the form

$$\int_0^\infty y^{2\nu} e^{\frac{1}{4}y^2} J_{2\nu}(xy) D_{-m}(y)\,dy$$

$$=\frac{(2x)^{2\nu}\Gamma(2\nu+\frac{1}{2})}{2\sqrt{\pi}\Gamma(m)}\Big[2^{\frac{1}{2}m-2\nu-\frac{1}{2}}\Gamma(\tfrac{1}{2}m-2\nu-\tfrac{1}{2})$$

$$\times {}_1F_1(2\nu+\tfrac{1}{2};\tfrac{3}{2}+2\nu-\tfrac{1}{2}m;\tfrac{1}{2}x^2)$$

$$+x^{m-4\nu-1}\frac{\Gamma(2\nu+\frac{1}{2}-\frac{1}{2}m)\Gamma(\frac{1}{2}m)}{\Gamma(2\nu+\frac{1}{2})}{}_1F_1(\tfrac{1}{2}m;\tfrac{1}{2}-2\nu+\tfrac{1}{2}m;\tfrac{1}{2}x^2)\Big]$$

and proceeding with it in the manner of § 5, we obtain

$$\int_0^\infty y^{2\nu} e^{\frac{1}{4}y^2\left(1-\frac{1}{2p^2}\right)} I_\nu\left(\frac{y^2}{8p^2}\right) D_{-m}(y)\,dy \qquad 2p^2>1.$$

$$=\frac{2^{2\nu-1}\Gamma(2\nu+\frac{1}{2})}{\pi\Gamma(m)}\Big[2^{\frac{1}{2}m-2\nu-\frac{1}{2}}\frac{\Gamma(\frac{1}{2}m-2\nu-\frac{1}{2})\Gamma(\nu+\frac{1}{2})}{p^{2\nu}}$$

$$\times {}_2F_1\left(2\nu+\tfrac{1}{2},\nu+\tfrac{1}{2};\tfrac{3}{2}+2\nu-\tfrac{1}{2}m;\frac{1}{2p^2}\right)$$

$$+\frac{\Gamma(2\nu+\frac{1}{2}-\frac{1}{2}m)\Gamma(\frac{1}{2}m)\Gamma(\frac{1}{2}m-\nu)}{p^{m-2\nu-1}\Gamma(2\nu+\frac{1}{2})}$$

$$\times {}_2F_1\left(\tfrac{1}{2}m,\tfrac{1}{2}m-\nu;\tfrac{1}{2}+\tfrac{1}{2}m-2\nu;\frac{1}{2p^2}\right)\Big],$$

valid when $R(m)>0$, $R(2\nu+\frac{1}{2})>0$ and $R(2\nu-m)<0$.

"ON SOME k_n-FUNCTION FORMULAE"

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1. Introduction:—The object of this paper is to collect a number of results involving k_n-functions. These formulae do not seem to have been noted explicitly as yet. Some of them are obtained by the methods of operational calculus and others are derived from formulae involving $W_{k,m}$ or $M_{k,m}$ functions and Laguerre or Sonine polynomials given by Erdelyi,* Howell,† Bailey‡ and Bateman.§ The k_n-functions are related to other functions by the relations

$$W_{\nu,\frac{1}{2}}(t) = \Gamma(\nu+1)k_{2\nu}(t/2) \tag{1}$$

$$M_{n,\frac{1}{2}}(t) = (-1)^{n-1}k_{2n}(t/2) \tag{2}$$

$$k_{2n}(x) = (-1)^{n-1}(2x).\frac{e^{-x}}{n}.L_{n-1}^{(1)}(2x) \tag{3}$$

$$T_1^{n-1}(2s) = \frac{e^s k_{2n}(s)}{(2s)(n-1)!}. \tag{4}$$

The Theorems in the operational calculus, which are made use of in this paper, are given by Nissen K. F.‖ Only the results are given in this paper and lengthy algebra has been suppressed.

* (i) "Funktionalrelationen mit konfluenthypergeometrischen Funktionen", *Math. Zeit.*, 42, 125-43.

(ii) "Über eine Methode zur Gewinnung von Funktionalbezeihungen zwischen konfluenten hypergeometrischen Funktionen", *Monatshefte für Mathematik und Physik*, 45, 31-52.

(iii) Über die erzeugende Funktion der Jacobischen Polynome, *Math. Zeit.*. 40, 693-702.

† "A note on Laguerre Polynomials", *Phil. Mag.*, (7) 23 (1937), Supp. number.

‡ "On the product of two Legendre Polynomials with different arguments", *Proc. Lond. Math. Soc.* (2), 41, 215-220.

§ The Partial Differential Equations in Mathematical Physics, pp. 451-459. Bateman has collected in this book results involving Sonine Polynomials due to Wilson, Koshliakov and other authors.

‖ "A contribution to the Symbolic Calculus", *Phil. Mag.*. (7) 20 (1935), 977-997.

2. THE FORMULAE :—

(1) $$(-1)^{m+n-2} m!\, n!\, k_{2n}(x) k_{2m}(x) = 4x^2 \sum_{s=0}^{\infty} \frac{\Gamma(n+s+1)\Gamma(m+s+1)\Gamma(m+n+1)}{s!\,(s+2)\Gamma(m+n+2+2s)} (2x)^{2s+1} \times L_{n+m}^{2s+1}(2x)$$

(2) $$k_{2m}(x) k_{2n}(x) = (-1)^{m+n-2} 2x \frac{1}{m!\,n!} \sum_{s=0}^{\infty} (-1)^{2s+1} \frac{\Gamma(n+s+1)\Gamma(m+s+1)}{s!\,\Gamma(s+2)} \times L_{m+n+2s+1}^{-1-2s}(2x)$$

(3) $$\sum_{n=1}^{\infty} n t^{n-1} k_{2n}(x) k_{2n}(y) = 2\sqrt{xy} \frac{t^{-\frac{1}{2}}}{1-t} \exp\left\{-(x+y) \cdot \frac{1+t}{1-t}\right\} I_1\left\{\frac{4\sqrt{xyt}}{1-t}\right\};\ |t|<|$$

(4) $$\int_0^{\infty} \frac{e^{-t} J_1(2\sqrt{2xt})\, k_{2n}(t)\, dt}{\sqrt{2t}} = \frac{(-1)^{n-1} x^{n-\frac{1}{2}} e^{-x}}{2(n!)}$$

(5) $$\int_0^{\infty} e^{-t} (2t)^{\frac{n}{2}-1} J_{2-n}(4\sqrt{xt})\, k_{2n}(t)\, dt = \frac{e^{-x}}{2} (2x)^{\frac{n}{2}-1} k_{2n}(x)$$

(6) $$n\rho \int_0^{\infty} \frac{e^{\lambda^2 \tau} . e^{-\frac{\rho^2}{8\tau}} . \tau^{-n} k_{2n}\left(\frac{\rho^2}{8\tau}\right) d\tau}{\rho^2} = -\int_0^{\infty} \frac{v^{2n} J_1(\rho v)\, dv}{v^2+\lambda^2}$$

(7) $$k_{2n}(x+y) = \frac{2(-1)^{n-1}(x+y) e^{y-x}}{n} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} (2y)^m L_{n-1}^{1+m}(2x)$$

(8) $$n \int_0^{x} e^{\frac{(\xi+x)}{2}} . k_{2n}\left(\frac{x-\xi}{2}\right) d\xi = (-1)^{n-1} x e^{x} \int_0^{\infty} e^{-t} . t^{n-1} . I_2(2\sqrt{tx})\, dt$$

(9) $$k_{2n}(x) = (n-1)!\, e^{-x/2} . \sum_{m=0}^{n-1} \frac{2^{n-m}}{m!\,(n-1-m)!} k_{2n-2m}(x/2)$$

(10) $$\int_0^{1} e^{xy} k_{2n}(xy) (1-y)^{\rho-1} dy = \frac{(n-1)!\,\Gamma(\rho)(-1)^{n-1}}{\Gamma(\rho+n+1)} L_{n-1}^{(\rho+1)}(2x);\ \rho>0$$

(11) $$\int_0^s e^{s/2} k_{2n+2}\left(\frac{s-t}{2}\right) k_{2\nu+2}(t/2)\,dt$$

$$=\Gamma(n+\nu+1)(-1)^{n+\nu}\frac{s^3 L^3_{n+\nu}(s)}{\Gamma(4+n+s)}$$

(12) $$\int_0^\infty e^{\frac{-s(2x-a-b)}{2}} \frac{k_{2n+2}\left(\frac{as}{2}\right) k_{2\nu+2}\left(\frac{bs}{2}\right) ds}{s}$$

$$=(-1)^{n+\nu}\frac{\Gamma(n+\nu+2)ab(x-a)^n(x-b)^\nu}{\Gamma(n+2)\Gamma(\nu+2)x^{n+\nu+2}}$$

$$\times F\left(-n,-\nu;-1-n-\nu;\frac{x(x-a-b)}{(x-a)(x-b)}\right);\ a>0,\ b>0$$

(13) $$\int_0^\pi U_{2n}(\sqrt{s}\cos\psi)\sin^2\psi\,d\psi=\frac{\pi}{2}\cdot\frac{(2n)!}{n!}\frac{e^{s/2}}{s}k_{2n+2}(s/2)$$

where $$U_n(x)=(-1)^n e^{x^2}\frac{d^n}{dx^n}(e^{-x^2})$$

(14) $$\frac{2(p-1)^{n-1}}{p^{m-2}(1+p)^{n+1}} \doteqdot \int_0^\infty \left(\frac{x}{s}\right)^{\frac{m}{2}} J_m(2\sqrt{xs})k_{2n}(s)\,ds.$$

When $m=1$, this gives

$$(-1)^{n-1}k_{2n}(x)=\int_0^\infty\left(\frac{x}{s}\right)^{\frac{1}{2}} J_1(2\sqrt{xs})k_{2n}(s)\,ds$$

(15) $$\frac{2p\{1-\sqrt{1+p^2}\}^{n-1}}{\{1+\sqrt{1+p^2}\}^{n+1}} \doteqdot k_{2n}(x)-\int_0^x k_{2n}(\xi)\frac{\xi J_1(\sqrt{x^2-\xi^2})\,d\xi}{\sqrt{x^2-\xi^2}}$$

$$\doteqdot k_{2n}(x)-\int_0^x k_{2n}\{\sqrt{x^2-s^2}\}J_1(s)\,ds$$

(16) $$\frac{2p^2(p-p^2-1)^{n-1}}{(p+1+p^2)^{n+1}} \doteqdot \int_0^x J_0\{2\sqrt{s(x-s)}\}k_{2n}(s)\,ds$$

(17) $$\int_0^x (\xi-x)^m k_{2n}(\xi)\,d\xi$$

$$=\Pi(m)(-1)^{n-1}\int_0^\infty\left(\frac{x}{s}\right)^{\frac{m}{2}+1} J_{m+2}(2\sqrt{xs})k_{2n}(s)\,ds;\ m>0$$

(18) $$e^{-s/2}\frac{s^2}{n}\cdot\frac{d}{ds}\left[\frac{e^{s/2}.k_{2n+2}(s/2)}{s}\right] \doteqdot k_{2n}(s/2)+k_{2n+2}(s/2)$$

(19) $$\frac{d^p}{ds^p}\left[\frac{e^{s/2}.k_{2n+2}(s/2)}{s.n!}\right]=T^{n-p}_{1+p}(s)$$

(20) $$\frac{d^p}{ds^p}\left[\frac{e^{s/2}k_{2n+2}(s/2)}{n!}\right]=s^{1-p}(-1)^n\frac{L_n^{1-p}(s)}{\Gamma(2-p+n)}$$

(21) $$s^2T_2^n(s)=\frac{e^{s/2}}{n!}k_{2n+4}(s/2)+\frac{e^{s/2}}{n!}k_{2n+2}(s/2)$$

(22) $$\frac{e^{s/2}k_{2n}(s/2)}{s(n-1)!}=(n+1)T_2^{n-1}(s)+T_2^{n-2}(s)$$

(23) $$\frac{k_{2n}(s/2)e^{s/2}}{(n-2)!}=(s-n)T_2^{n-2}(s)-T_2^{n-3}(s)$$
$$=(s-2)T_2^{n-2}(s)-sT_3^{n-3}(s)$$

(24) $$(n+2)\frac{d}{ds}\left[\frac{e^{s/2}k_{2n+4}(s/2)}{(n+1)!\,s}\right]$$
$$=\frac{e^{s/2}k_{2n+2}(s)}{n!\,s}-\frac{d}{ds}\left[\frac{e^{s/2}.k_{2n+2}(s/2)}{s.n!}\right]$$

(25) $$\frac{d}{ds}\left[\frac{s^{-1}e^sk_{2n+4}(s/2)}{(n+1)k_{2n+2}(s/2)}\right]=e^ss^{-2}\left[\frac{e^{s/2}k_{2n+2}(s/2)}{n!\,s}\right]^{-2}$$
$$\times\left[(n+1)\left\{\frac{e^{s/2}k_{2n+4}(s/2)}{(n+1)!\,s}\right\}^2\right.$$
$$\left.+\left\{\frac{e^{s/2}k_{2n+2}(s/2)}{n!\,s}+(n+1)\frac{e^{s/2}k_{2n+4}(s/2)}{n!\,s}\right\}^2\right].$$

TRAJECTORIES AND LINES OF FORCE IN A RIEMANNIAN SPACE*

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Kasner† has proved the following Theorems relating to the curvatures of the Trajectories and Lines of force in a plane :—

THEOREM I. The curvature of the Trajectory obtained by starting a particle from rest in any field of force is one-third the curvature of the line of force through the given point.

THEOREM II. If the line of force has contact of nth order with the tangent line, the trajectory produced by starting a particle from rest will also have contact of the nth order and the limiting ratio of the departure of the trajectory to the departure of the line of force from the common tangent will be $1:2n+1$.

THEOREM III. If a particle is projected in the direction of the force with a speed different from zero, the initial curvature will be zero and the infinitesimal departure from the common tangent will vary inversely as the square of the speed. That is $d\gamma/ds$ varies as $1/v^2$.

THEOREM IV. The single infinity of paths obtained by starting at a given point in the force direction with varying speeds under the conditions of Theorem II will have contact of order $(n+1)$ with the common tangent, and will give departures from the common tangent varying inversely as the square of the speed; except for the single path due to zero speed for which case the contact will be of the nth order and the departure ratio will be of the form $1:2n+1$.

THEOREM V. If R_0, the resistance due to zero speed at a point does not vanish, as in the case of sliding friction, the ratio

* I express my grateful thanks to Professor A. Narasinga Rao who suggested this problem and under whose guidance this investigation was carried out.

† Edward Kasner. 'General Theorems on Trajectories and Lines of Force', *Proceedings of the National Acadamy of Sciences U.S.A.*, 20 (1934), 130-135.

of the initial curvatures of the trajectory and the line of force for a particle starting from rest is $1 : 3+2R_0/F$, where F is the acting force at the position where the particle starts from rest.

The object of this paper is to extend the above results to a Riemannian space of n-dimensions.

1. Let $(x^1, x^2, \ldots x^n)$ be the co-ordinates of a point in a Riemannian n-space. Then $(\dot{x}^1, \dot{x}^2, \ldots \dot{x}^n)$ where dots denote differentiation with respect to the time t are the generalised velocities and $\delta\dot{x}^i/\delta t$ $(i=1, 2, \ldots n)$, where $\delta/\delta t$ is the covariant time flux operator, are the accelerations. Then if f^i denote the component of the accelerations, we have

$$f^i = \frac{\delta \dot{x}^i}{\delta t} = \ddot{x}^i + \Gamma^i_{jl}\dot{x}^j\dot{x}^l$$

$$= \ddot{s}\lambda^i_0 + \kappa_1 \dot{s}^2 \lambda^i_1, ^* \qquad (1\cdot1)$$

where λ^i_0, λ^i_1 are the components of the unit tangent and the first normal to the trajectory and κ_1 is its first curvature.

2. We now prove the following:

LEMMA. *If C and $\bar{C}$ be two curves having contact of order p at a point O, then*

$$\left(\frac{d^n}{ds^n}\right)_0 = \left(\frac{d^n}{d\sigma^n}\right)_0 \qquad (n=1, 2, \ldots p),$$

where s and σ are the arc lengths measured along C and $\bar{C}$ respectively and the suffix 0, denotes the value at the point O.

Let P and Q be two points at equal infinitesimal arc lengths s from O measured along the curves. Then C and $\bar{C}$ are said to have contact of order p if PQ is an infinitesimal of order s^{p+1}.†

Then, if ϕ is any function of position, we have

$$\frac{d\phi}{ds} = \frac{d\phi}{d\sigma}\frac{d\sigma}{ds} + \frac{d\phi}{d\eta}\frac{d\eta}{ds}, \quad \text{where } \eta = PQ.$$

When C and $\bar{C}$ touch at O, $\frac{d\eta}{ds} \rightarrow$ zero and since $s=\sigma$ we have $\frac{d\sigma}{ds} = 1$ always. Hence

*J. L. Synge 'On the Geometry of Dynamics', *Phil. Trans. Royal Society of London*, A 226 (1926).

†A. J. McConnell, 'The contact of curves in a Riemannian space', *Proc. Lond. Math. Soc.* (1927), 512.

$$\left(\frac{d}{ds}\right)_0 = \left(\frac{d}{d\sigma}\right)_0 \tag{2·1}$$

Putting $\frac{d}{ds}$ in the form $\frac{d}{ds}=\frac{d\sigma}{ds}A+\frac{d\eta}{ds}B$, where $A=\frac{d}{d\sigma}$ and $B=\frac{d}{d\eta}$ we have, as a result of successive differentiations,

$$\frac{d^n}{ds^n}=\sum_{r=0}^{n-1}\binom{n-1}{r}\frac{d^{r+1}\sigma}{ds^{r+1}}A^{(n-r-1)}+\sum_{r=0}^{n-1}\binom{n-1}{r}\frac{d^{r+1}\eta}{ds^{r+1}}B^{(n-r-1)}, \tag{2·2}$$

where $A^{(r)}$, $B^{(r)}$ stand for the rth derivative of the expressions with respect to s.

Since $\sigma=s=$ the length measured along both the curves we have

$$\frac{d^r\sigma}{ds^r}=0 \quad (r=2, 3,\ldots n) \tag{2·3}$$

Also, when C and $\bar{C}$ have contact of order p, η is an infinitesimal of order $s^{(p+1)}$ and hence $d^r\eta/ds^r$ is an infinitesimal of order s^{p+1-r} and vanishes if $r<p+1$

i.e. we have $\quad d^r\eta/ds^r \to 0 \quad (r=1, 2,\ldots p)$ (2·4)

Then

$$\left(\frac{d^n}{ds^n}\right)_0=A_0^{(n-1)}. \quad (n=1, 2,\ldots p) \tag{2·5}$$

the other terms vanishing as a result of (2·3) and (2·4).

Also

$$A^{(n-1)}=\frac{d\sigma}{ds}\frac{d^n}{d\sigma^n}+\sum_{r=2}^{n-1}\frac{d^r\sigma}{ds^r}C_r+\sum_{r=1}^{n-1}\frac{d^r\eta}{ds^r}D_r$$

so that

$$A_0^{(n-1)}=\left(\frac{d^n}{d\sigma^n}\right)_0 \quad (n=1, 2,\ldots,p) \tag{2·6}$$

From (2·5) and (2·6) we have the Lemma.

3. McConnell* has shown that the necessary and sufficient conditions for two curves to have contact of order p, are

$$\lambda_0^{i(n)} \quad (n=0, 1,\ldots p-1)$$

where $\lambda_0^{i(n)}=\frac{\delta^n\lambda_0^i}{\delta s^n}$, should be the same for both the curves. (3·1)

* Loc. cit. p. 513.

Now

$$\lambda_0^{i(n)}=\sum_{r=0}^{n-1}\binom{n-1}{r}\kappa_1^{(r)}\lambda_1^{i(n-r-1)} \tag{3·2}$$

So, if $\kappa_1, \kappa'_1 \ldots \kappa_1^{(m-2)}$ are all zero for both the curves, we have $\lambda_0^{i(n)}=0$ ($n=0, 1, 2, \ldots, \ldots m-1$) for both the curves by (3·2) and hence by (3·1) they have contact of order m. (3·3)

We shall make use of this result (3·3) in all the stages of our investigation. It must be noted that these are merely sufficient conditions.

4. Let C be the trajectory and $\bar{C}$ the line of force through the point O. Let f^r denote the contravariant components of the acceleration vector and Q^r those of the force vector. Also let λ^r_μ, $\bar{\lambda}^r_\mu$ ($\mu=0, 1, \ldots n-1$) denote the unit tangent and the $(n-1)$ normals to C and $\bar{C}$ respectively. Then

$$f^r=\ddot{s}\lambda_0^r+\kappa_1\dot{s}^2\lambda_1^r=Q^r=f\bar{\lambda}_0^r \tag{4.1}$$

where f is the magnitude of the force vector. If the tangential and normal components be denoted by T and N respectively, we have

$$f^r=T\lambda_0^r+N\lambda_1^r=f\bar{\lambda}_0^r.$$

CASE I. SUPPOSE A PARTICLE STARTS FROM REST AT THE POINT O.

Then $$N_0=(\kappa_1\dot{s}^2)_0=0$$

Hence $$(\lambda_0^r)_0=(\bar{\lambda}_0^r)_0$$

and $$T_0=f_0 \tag{4·2}$$

It follows that the trajectory and the line of force touch at O. Hence we have,

$$\left(\frac{\delta f^r}{\delta s}\right)_0=\left(\frac{\delta f^r}{\delta \sigma}\right)_0 \text{ by the Lemma}$$

$$=\left(\frac{\delta Q^r}{\delta \sigma}\right)_0 \text{ since } f^r=Q^r \text{ always.} \tag{4·3}$$

Now

$$\left.\begin{aligned}\frac{\delta f^r}{\delta s}&=(T'-\kappa_1 N)\lambda_0^r+(\kappa_1 T+N')\lambda_1^r+\kappa_2 N\lambda_2^r\\ \frac{\delta Q^r}{\delta \sigma}&=f'\bar{\lambda}_0^r+f\bar{\kappa}_1\bar{\lambda}_1^r\end{aligned}\right\} \tag{4·4}$$

$N_0=0$ and $N'=(\kappa_1 v^2)'=\kappa'_1 v^2+2\kappa_1 T$, and so $N'_0=2\kappa_1 f_0$. Hence we have

$$\left.\begin{aligned}\left(\frac{\delta f^r}{\delta s}\right)_0 &= T'_0\lambda_0^r+3\kappa_1 f_0\lambda_1^r\\ \left(\frac{\delta Q^r}{\delta \sigma}\right)_0 &= f'_0\bar{\lambda}_0^r+f_0\bar{\kappa}_1\bar{\lambda}_1^r\end{aligned}\right\} \qquad (4{\cdot}5)$$

Now $\left(\frac{\delta f^r}{\delta s}\right)_0$ and $\left(\frac{\delta Q^r}{\delta \sigma}\right)_0$ are identical vectors by (4·3) and by the former of the equations (4·5) they lie in the osculating plane of the trajectory. Also since $(\lambda_0^r)_0=(\bar{\lambda}_0^r)_0$ by (4·2) and $\lambda_1^r, \bar{\lambda}_1^r$ are vectors perpendicular to them and lying in the same plane, we have

$$(\lambda_1^r)_0=(\bar{\lambda}_1^r)_0 \qquad (4{\cdot}6)$$

Hence equating the co-efficients we have

$$3\kappa_1 f_0=\bar{\kappa}_1 f_0 \text{ i.e. } \kappa_1=\bar{\kappa}_1/3 \text{ if } f_0\neq\text{zero}.$$

So we get

THEOREM 1. *The first curvature of a trajectory obtained when a particle starts from rest in any non-vanishing field of force is one-third the first curvature of the line of force through that point.*

5. When $\bar{\kappa}_1=0$, we have $\kappa_1=0$. Then by (3·3), the trajectory and the line of force have contact of order 2. So $f_0^{r(2)}=Q_0^{r(2)}$

$$\left.\begin{aligned}f_0^{r(2)} &= T''_0\lambda_0^r+5\kappa'_1 f_0\lambda_1^r\\ Q_0^{r(2)} &= f''_0\bar{\lambda}_0^r+\bar{\kappa}'_1 f_0\bar{\lambda}_1^r\end{aligned}\right\} \qquad (5{\cdot}1)$$

Equating the co-efficients of λ_1^r and $\bar{\lambda}_1^r$ we have

$$\kappa'_1=\bar{\kappa}'_1/5 \text{ when } f_0\neq\text{zero}. \qquad (5{\cdot}2)$$

Generalising from this, we have the following

THEOREM 2. *If a particle starts from rest at a point O and if $\bar{\kappa}_1$ and its first $(p-2)$ derivatives be all zero at O, then $\kappa_1, \kappa'_1 \ldots \kappa_1^{(p-2)}$ are also zero, so that the trajectory and line of force have contact of order p. Then $\kappa_1^{(p-1)}=\bar{\kappa}_1^{(p-1)}/(2p+1)$*

We shall prove this by the method of induction.

Assuming the theorem to be true for values upto $(p-1)$ we shall prove it holds good for p. We have $\bar{\kappa}_1, \bar{\kappa}'_1 \ldots \bar{\kappa}^{(p-3)}$ are all zero and $\kappa_1, \kappa'_1 \ldots \kappa_1^{(p-3)}$ are also zero. Hence they have contact of order $(p-1)$ and $\kappa_1^{(p-2)}=\bar{\kappa}_1^{(p-2)}/(2p-1)$. Now if $\bar{\kappa}_1^{(p-2)}=0$

we have $\kappa_1^{(p-2)}$ is also zero so that $\bar{\kappa}_1, \bar{\kappa}'_1 \ldots \bar{\kappa}_1^{(p-2)}$ and $\kappa_1, \kappa'_1 \ldots \kappa_1^{(p-2)}$ are all zero. Then by (3·3), the trajectory and the line of force have contact of order p. Hence $f_0^{r(p)} = Q_0^{r(p)}$.

Also

$$f^{r(p)} = \lambda_0^r [T^{(p)} + A_m \kappa_1^{(m)}] + \lambda_1^r [B_m \kappa_1^{(m)} + (2p+1)\kappa_1^{(p-1)} T + \kappa_1^{(p)} v^2] + \lambda_2^r [C_m \kappa_1^{(m)} + p\kappa_2 \kappa_1^{(p-1)} v^2] + \lambda_s^r [D_m^s \kappa_1^{(m)}] \tag{5·3}$$

$(m=0, 1, \ldots p-2)$, $(s=3, 4, \ldots p+1)$, $D_m^n = 0$, $m > 1$ and $D_m^{n+1} = 0$, $m > 0$

Also

$$Q^{r(p)} = \bar{\lambda}_0^r [f^{(p)} + \bar{A}_m \bar{\kappa}_1^{(m)} + \bar{\lambda}_1^r [\bar{B}_m \bar{\kappa}_1^{(m)} + f \bar{\kappa}_1^{(p-1)}] + \bar{\lambda}_s^r [\bar{C}_m^s \bar{\kappa}_1^{(m)}] \tag{5·4}$$

$(m=0, 1 \ldots p-2)$, $(s=2, 3, \ldots p)$, $\bar{C}_m^{n-1} = 0$, $m > 1$ and $\bar{C}_m^n = 0$, $m > 0$

Since $\kappa_1^{(m)} = 0 = \bar{\kappa}_1^{(m)}$ $(m=0, 1, \ldots, p-2)$

we have

$$\left.\begin{aligned} f_0^{r(p)} &= \lambda_0^r [T^{(p)}] + \lambda_1^r [(2p+1)\kappa_1^{(p-1)} T] \\ Q_0^{r(p)} &= \bar{\lambda}_0^r [f_0^{(p)}] + \bar{\lambda}_1^r [f_0 \bar{\kappa}_1^{(p-1)}] \end{aligned}\right\} \tag{5·5}$$

Equating the co-efficients of λ_1^r and $\bar{\lambda}_1^r$ we get

$$\kappa_1^{(p-1)} = \bar{\kappa}_1^{(p-1)}/(2p+1), \quad f_0 \neq \text{zero}.$$

Hence the Theorem.

6. We have till now been discussing the motion of a particle that starts from rest at the point O. Now let us consider

Case II. The motion of a particle that is projected in the force-direction with a non-zero velocity v.

Since the infinitesimal displacement is in the direction of the force $N_0 = (\kappa_1 v^2)_0 = 0$ and hence $(\kappa_1)_0 = 0$ (6·1)

Also the line of force and the trajectory touch at O and hence $f_0^{r(1)} = Q_0^{r(1)}$,

$$\left.\begin{aligned} f_0^{r(1)} &= T'_0 \lambda_0^r + \kappa'_1 v^2 \lambda_1^r \\ Q_0^{r(1)} &= f'_0 \bar{\lambda}_0^r + \bar{\kappa}_1 f_0 \bar{\lambda}_1^r \end{aligned}\right\}. \tag{6·2}$$

Equating the coefficients of λ_1^r and $\bar{\lambda}_1^r$ we have

$$\kappa'_1 = f_0 \bar{\kappa}_1 / v^2 = A/v^2 \tag{6·3}$$

where A is a constant, since f_0 and $\bar{\kappa}_1$ have fixed non-zero values at O. Hence we have

THEOREM 3. *When a particle is projected in the direction of the line of force with a non-zero velocity v, its first curvature vanishes and the first derivative of the first curvature varies inversely as the square of the velocity.*

7. If $\bar{\kappa}_1=0$ we have $\kappa'_1=0$. Then since $\kappa_1=0$ and $\bar{\kappa}_1=0$ the trajectory and the line of force have contact of order 2, (by 3·3) and hence

$$f_0^{r(2)}=Q_0^{r(2)}$$

$$\left.\begin{aligned} f_0^{r(2)}&=T''_0\lambda_0^r+\kappa_1''v^2\lambda_1^r \\ Q_0^{r(2)}&=f''_0\bar{\lambda}_0^r+\bar{\kappa}'_1f_0\bar{\lambda}_1^r \end{aligned}\right\} \tag{7·1}$$

We get

$$\kappa''_1=f_0\bar{\kappa}'_1/v^2=A/v^2 \tag{7·2}$$

$$\text{if } \bar{\kappa}'_1\neq\text{zero}.$$

Hence if $\bar{\kappa}'_1\neq$zero while $\bar{\kappa}=$zero, we get $\kappa_1=0$, $\kappa'_1=0$. The trajectory and the line of force have contact of order 2 and κ''_1 varies inversely as the square of the velocity. Generalising this we have

THEOREM 4. *If $\bar{\kappa}_1^{(p-1)}$ is the first derivative of $\bar{\kappa}_1$ that does not vanish, then $\kappa_1, \kappa'_1..,\kappa_1^{(p-1)}$ all vanish. The trajectory and the line of force have contact of order p and $\kappa_1^{(p)}=A/v^2$ where $A=f_0\bar{\kappa}_1^{(p-1)}$.*

Let us as before prove this by induction. Suppose the Theorem to be true for values up to $(p-1)$. Then $\bar{\kappa}_1,\bar{\kappa}'_1...\bar{\kappa}_1^{(p-3)}$ are all zero and κ_1, $\kappa'_1...\kappa_1^{(p-2)}$ all vanish. The trajectory and the line of force have contact of order $(p-1)$ and $\kappa_1^{(p-1)}=A/v^2$ where $A=f_0\bar{\kappa}_1^{(p-2)}$.

Now suppose $\bar{\kappa}_1^{(p-2)}=0$. Then $\kappa_1^{(p-1)}=0$. Since $\bar{\kappa}_1$, $\bar{\kappa}'_1...\bar{\kappa}_1^{(p-2)}$ and κ_1, $\kappa'_1...\kappa_1^{(p-2)}$ all vanish, the trajectory and the line of force have contact of order p (by 3·3). So $f_0^{r(p)}=Q_0^{r(p)}$. From (5·3) and (5·4) we have, making use of the above conditions

$$\left.\begin{aligned} f_0^{r(p)}&=T^{(p)}\lambda_0^r+[\kappa_1^{(p)}v^2]\lambda_1^r \\ Q_0^{r(p)}&=f^{(p)}\bar{\lambda}_0^r+[f_0\bar{\kappa}_1^{(p-1)}]\bar{\lambda}_1^r \end{aligned}\right\} \tag{7·3}$$

Equating the coefficients of λ_1^r and $\bar{\lambda}_1^r$ we get

$$\kappa_1^{(p)}=f_0\bar{\kappa}_1^{(p-1)}/v^2=A/v^2 \tag{7·4}$$

Hence the Theorem.

8. *Motion in a Resisting Medium.* The equations of motion in a resisting medium are

$$f^r = \ddot{s}\lambda_0^r + \kappa_1 \dot{s}^2 \lambda_1^r = Q^r + R\lambda_0^r \tag{8.1}$$

i.e. $$(\ddot{s} - R)\lambda_0^r + \kappa_1 \dot{s}^2 \lambda_1^r = f\bar{\lambda}_0^r.$$

Denoting the left hand member as $\bar{f}^r$ we have

$$\bar{f}^r = Q^r \tag{8.2}$$

Denoting $\ddot{s} - R = T$ and $\kappa_1 \dot{s}^2 = N$

$$\left.\begin{aligned} \bar{f}^r &= T\lambda_0^r + N\lambda_1^r \\ Q^r &= f\bar{\lambda}_0^r \end{aligned}\right\} \tag{8.3}$$

CASE I. THE PARTICLE STARTS FROM REST AT THE POINT O.

$N_0 = \kappa_1 \dot{s}^2 = 0$ and hence $T_0 = f_0$ and $\lambda_0^r = \bar{\lambda}_0^r$. The trajectory and the line of force touch each other at O. Hence $\bar{f}_0^{r(1)} = Q_0^{r(1)}$.

$$\bar{f}^{r(1)} = (T' - \kappa_1 N)\lambda_0^r + (\kappa_1 T + N')\lambda_1^r + \kappa_2 N \lambda_2^r$$

$$N' = \kappa'_1 v^2 + \kappa_1 (v^2)' = \kappa'_1 v^2 + 2\kappa_1 \ddot{s} = \kappa'_1 v^2 + 2\kappa_1 (T + R).$$

Hence

$$\left.\begin{aligned} \bar{f}_0^{r(1)} &= T'\lambda_0^r + \kappa_1 (3T + 2R)\lambda_1^r \\ Q_0^{r(1)} &= f'\bar{\lambda}_0^r + f_0 \bar{\kappa}_1 \bar{\lambda}_1^r \end{aligned}\right\} \tag{8.4}$$

Equating the coefficients of λ_1^r and $\bar{\lambda}_1^r$ we get

$$\kappa_1 (3T_0 + 2R_0) = \bar{\kappa}_1 f_0$$

i.e. $$\kappa_1 : \bar{\kappa}_1 = 1 : 3 + \frac{2R_0}{f_0} \tag{8.5}$$

Hence we nave the analogue of Theorem I for a resisting medium.

THEOREM 5. *If a particle starts from rest in any non-vanishing field of force, the first curvature of the trajectory and that of the line of force are in the ratio* $1 : 3 + 2R_0/f_0$.

9. If $\bar{\kappa}_1 = 0$ then κ_1 is also equal to zero. The trajectory and the line of force have contact of order 2 by (3·3) and hence $\bar{f}_0^{r(2)} = Q_0^{r(2)}$.

$$\left.\begin{aligned} \bar{f}_0^{r(2)} &= T''_0 \lambda_0^r + \kappa'_1 (5f_0 + 4R_0)\lambda_1^r \\ Q_0^{r(2)} &= f''_0 \bar{\lambda}_0^r + \bar{\kappa}'_1 f_0 \bar{\lambda}_1^r \end{aligned}\right\} \tag{9.1}$$

Equating the coefficients of λ_1^r and $\bar{\lambda}_1^r$ we get

$$\kappa'_1 (5f_0 + 4R_0) = \bar{\kappa}'_1 f_0$$

and hence at O

$$\kappa'_1 : \bar{\kappa}'_1 = 1 : 5 + \frac{4R_0}{f_0} \tag{9.2}$$

Now we shall prove the following Theorem by induction.

THEOREM 6. *If a particle starts from rest at a point O and if* $\bar{\kappa}_1, \bar{\kappa}'_1 \ldots \bar{\kappa}_1^{(p-2)}$ *are all zero, then* $\kappa_1, \kappa'_1 \ldots \kappa_1^{(p-2)}$ *are also zero, so that the trajectory and the line of force have contact of order* p. *Then*

$$\kappa_1^{(p-1)} : \bar{\kappa}_1^{(p-1)} = 1 : (2p+1) + \frac{2pR_0}{f_0}.$$

Following the usual mode of proof, by assuming that this is true for values up to $(p-1)$ we are led to the condition that the trajectory and the line of force have contact of order p. Then $\bar{f}_0^{r(p)} = Q_0^{r(p)}$.

$$\begin{aligned}\bar{f}^{r(p)} &= \lambda_0^r[T^{(p)} + A_m\kappa_1^{(m)}] \\ &+ \lambda_1^r\Big[B_m\kappa_1^{(m)} + \kappa_1^{(p)}v^2 + \kappa_1^{(p-1)}\{(2p+1)T + 2pR\}\Big] \\ &+ \lambda_2^r[C_m\kappa_1^{(m)} + p\kappa_2\kappa_1^{(p-1)}v^2] \\ &+ \lambda_s^r[D_m^s\kappa_1^{(m)}] \qquad (9{\cdot}3)\end{aligned}$$

$$(m=0, 1, \ldots p-2),\ (s=3, 4, \ldots p+1),\ D_m^n=0,\ m>1 \text{ and } D_m^{n+1}=0,\ m>0.$$

Also

$$Q^{r(p)} = \bar{\lambda}_0^r[f^{(p)} + \bar{A}_m\bar{\kappa}_1^{(m)}] + \bar{\lambda}_1^r[\bar{B}_m\bar{\kappa}_1^{(m)} + f\bar{\kappa}_1^{(p-1)}] + \bar{\lambda}_s^r[\bar{C}_m^s\bar{\kappa}_1^{(m)}] \qquad (9{\cdot}4)$$

$$(m=0, 1, \ldots p-2),\ (s=2, 3, \ldots p),\ \bar{C}_m^{n-1}=0,\ m>1 \text{ and } \bar{C}_m^n=0,\ m>0.$$

Since $\kappa_1^{(m)} = 0 = \bar{\kappa}_1^{(m)}$ $(m=0, 1, \ldots p-2)$

we have

$$\left.\begin{aligned}\bar{f}_0^{r(p)} &= T^{(p)}\lambda_0^r + [(2p+1)T_0 + 2pR_0]\kappa_1^{(p-1)}\lambda_1^r \\ Q_0^{r(p)} &= f_0^{(p)}\bar{\lambda}_0^r + f_0\bar{\kappa}_1^{(p-1)}\bar{\lambda}_1^r\end{aligned}\right\} \qquad (9{\cdot}5)$$

As usual, equating the coefficients of λ_1^r and $\bar{\lambda}_1^r$ we get

$$\kappa_1^{(p-1)} : \bar{\kappa}_1^{(p-1)} = 1 : (2p+1) + \frac{2pR_0}{f_0}. \qquad (9{\cdot}6)$$

10. For a particle projected with a velocity v in the direction of the line of force we have from equations (9·3) and (9·4), and the fact that R occurs as a coefficient of $\kappa_1^{(p-1)}$ which vanishes, that $\bar{f}_0^{r(p)}$ and $Q_0^{r(p)}$ reduce to those in (7·3). Hence Theorems 3 and 4 hold good even in a resisting medium. Hence we note that the initial curvature of a free particle is influenced by the resistance of the medium when $R_0 \neq 0$ whereas, if the particle is projected with a velocity v, its presence is not felt as far as the initial curvature and its derivatives are concerned.

A PROPERTY OF THE ZEROS OF THE SUCCESSIVE DERIVATIVES OF INTEGRAL FUNCTIONS

By V. GANAPATHY IYER, Madras University

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1. The following result has been proved by Takenaka:*

THEOREM 1. *Let $f(z)$ be an integral function of order one and type not exceeding σ. Let $[\alpha_n]$ be a sequence such that*

$$\overline{\lim_{n\to\infty}} |\alpha_n| = L < \tfrac{1}{\sigma}\log 2. \tag{1}$$

Let $f^{(n)}(\alpha_n)=0$, $n=0, 1, 2\ldots[f^{(o)}(z)\equiv f(z)]$. Then $f(z)\equiv 0$.

1.1. The function $\sin z - \cos z$ shows that $\log 2$ in (1) cannot be replaced by any number greater than $\pi/4$, and it has been conjectured† that this might be the best possible result and this conjecture has been proved true‡ when the numbers $[\alpha_n]$ are real. In this paper I show that when $f(z)$ is odd or even, $\log 2$ in (1) can be replaced by $\log(2+\sqrt{3}) > 1 > \pi/4$; the function $\sin z$ shows that probably $\pi/2$ is the "best possible constant" in the case of odd or even functions. The method used closely resembles that employed by Takenaka to prove Theorem 1. The result can also be proved by a modification of the proof given by J. M. Whittaker§ for Theorem 1.

2. We prove

THEOREM 2. *Let $f(z)$ be an even or odd integral function of order one and type not exceeding σ. Let $[\alpha_n]$ be a sequence such that*

$$\overline{\lim_{\omega\to\infty}} |\alpha_n| = L < \frac{\log(2+\sqrt{3})}{\sigma}. \tag{2}$$

If $f^{(n)}(\alpha_n)=0$, $n=0, 1, 2,\ldots$, then $f(z)\equiv 0$.

* *Proc. Physico. Math. Soc.* Japan, 14 (1932), 529-42.

† *Cf.* J. M. Whittaker, *Interpolatory Function Theory*, Camb. Tract, No. 33, p. 45.

‡ Schoenberg, *Trans. Amer. Math. Soc.*, 40 (1936), 12-23.

§ loc. cit.

2.1. Theorem 2 is derived from

THEOREM 3. *Let $f(z)$ be an even integral function of order one and type $\sigma<1$. Let $[\alpha_{2n}]$, $n=0, 1, \ldots$ be such that*

$$|\alpha_{2n}|\leqslant L<\log (2+\sqrt{3}). \tag{3}$$

Then, for all finite z,

$$f(z)=\sum_{0}^{\infty} f^{(2n)}(\alpha_{2n})p_{2n}(z). \tag{4}$$

where

$p_0(z)\equiv 1$, $p_{2n}(z)$

$$=\int_{\alpha_0}^{z} dt_1\int_{0}^{t_1} dt_2\int_{\alpha_2}^{t_2} dt_3 \int_{0}^{t_3} dt_4\ldots\int_{\alpha_{2n-2}}^{t_{2n-2}} dt_{2n-1}\int_{0}^{t_{2n-1}} dt_{2n} \int_{\alpha_{2n}}^{t_{2n}} dt_{2n+1} \tag{5}$$

for $n\geqslant 1$.

2.2. To prove Theorem 3 we need the following

LEMMA. *Let $[\alpha_{2n}]$, $n=0, 1, \ldots$, satisfy the condition (3). Let $\phi(z)$ be an even function regular in $|z|\leqslant R(>1)$. Then*

$$\phi(z)=\sum_{0}^{\infty} c_{2n}z^{2n} \cosh \alpha_{2n}z, \tag{6}$$

the series on the right side of (6) converging absolutely and uniformly for $|z|\leqslant r<1$.

PROOF. Let

$$\phi(z)=\sum_{0}^{\infty} a_{2n}z^{2n}. \tag{7}$$

Comparing (6) and (7) formally, we get

$$\left.\begin{aligned} &c_0=a_0,\\ &c_{2n}=a_{2n}-c_{2n-2}\frac{(\alpha_{2n-2})^2}{2!}-\ldots-\frac{c_0\alpha_0^{2n}}{2n!},\ n\geqslant 1. \end{aligned}\right\} \tag{8}$$

Now, given $[\alpha_{2n}]$, $[a_{2n}]$, $n=0, 1, 2, \ldots$, suppose that the $[c_{2n}]$ are calculated from (8). If for this $[c_{2n}]$, the series (6) converges absolutely and uniformly in some circle $|z|<\rho>0$, we get by (7) and (8),

$$\phi^{(2n)}(0)=\psi^{(2n)}(0),$$

where $\psi(z)$ denotes the sum of the series (6) in $|z|<\rho$. Hence $\phi(z)\equiv\psi(z)$. Therefore, if it is shown that the series in (6) converges absolutely and uniformly for $|z|\leqslant r<1$, the Lemma would be proved. First we note that, in virtue of (3),

$$\lim_{n\to\infty} |z^{2n}\cosh \alpha_{2n}z|^{\frac{1}{2n}}=|z|. \tag{9}$$

Next, we show that for a properly chosen k, we must have

$$|c_{2s}|\leqslant k, \tag{10}$$

for $s=0, 1, 2, \ldots$ Suppose that for some k, (10) holds for $s=0, 1, 2, \ldots, n-1$. Then (8) gives

$$\begin{aligned}|c_{2n}|&\leqslant|a_{2n}|+k\left[\frac{L^2}{2!}+\frac{L^4}{4!}+\ldots+\frac{L^{2n}}{2n!}\right]\\&\leqslant|a_{2n}|+k\ (\cosh L-1).\end{aligned} \tag{11}$$

Now, if $M(R)=\max\limits_{|z|\leqslant R}|\phi(z)|$, we have

$$|a_{2n}|\leqslant\frac{M(R)}{R^{2n}}\to 0, \tag{12}$$

as $n\to\infty$, since $R>1$. Since $L<\log\ (2+\sqrt{3})$, we get

$$\cosh L-1<\cosh \log\ (2-\sqrt{3})-1=1. \tag{13}$$

Combining (11), (12) and (13), we conclude that there exists a constant k depending only on L, R and $M(R)$, such that the relations

$$|c_{2s}|\leqslant k,\ s=0, 1, \ldots n-1$$

involve

$$|c_{2n}|\leqslant k.$$

Hence by induction, (10) holds for a properly chosen k and the Lemma follows from (9) and (10).

2.3. *Proof of Theorem* 3. By (5), $p_{2n}^{(2s-1)}(0)=0$, $s=1, \ldots n$. Therefore we can write

$$p_{2n}(z)=\sum_{\nu=0}^{n}\lambda_{2\nu}^{(n)}\frac{z^{2\nu}}{2\nu!}. \tag{14}$$

Let

$$\pi_{2n}(z)=\sum_{\nu=0}^{n}\lambda_{2\nu}^{(n)}z^{2\nu}. \tag{15}$$

Then if Γ denotes any circle round $z=0$,

$$p_{2n}(z)=\frac{1}{2\pi i}\int_{\Gamma}\frac{\pi_{2n}(s)}{s}\cosh\frac{z}{s}ds. \tag{16}$$

From (5) and (16) we get

$$\frac{1}{2\pi i}\int_{\Gamma}\frac{\pi_{2n}(s)}{s^{2\nu+1}}\cosh\frac{\alpha_{2\nu}}{s}ds=\begin{matrix}0, \ \nu\neq n,\\ 1, \ \nu=n.\end{matrix}\Bigg\} \tag{17}$$

For a given z, $\cosh xz$ is regular for all x; hence putting $x=1/s$ and applying the lemma we get

$$\cosh\frac{z}{s}=\sum_{0}^{\infty}\frac{c_{2n}(z)}{s^{2n}}\cosh\frac{\alpha_{2n}}{s}, \tag{18}$$

which converges absolutely and uniformly for $|s|\geqslant d>1$. Taking $|s|=d$ for Γ in (16) and using (17), we get from (18), that

$$c_{2n}(z)=p_{2n}(z),\ n=0,\ 1,\ 2,\ldots$$

so that, for $|s|>1$,

$$\cosh\frac{z}{s}=\sum_{0}^{\infty}\frac{p_{2n}(z)}{s^{2n}}\cosh\frac{\alpha_{2n}}{s}. \tag{19}$$

Now let

$$f(z)=\sum_{0}^{\infty}\frac{b_{2n}}{2n!}z^{2n}. \tag{20}$$

and

$$\phi(z)=\sum_{0}^{\infty}b_{2n}z^{2n}. \tag{21}$$

Since $\overline{\lim_{n\to\infty}}|b_{2n}|^{\frac{1}{2n}}=\sigma<1$, $\phi(z)$ is regular in some circle $C:|z|\leqslant d>1$.

Taking C for Γ, we get from (19), (20) and (21),

$$f(z)=\frac{1}{2\pi i}\int_{C}\frac{\phi(s)}{s}\cosh\frac{z}{s}ds$$

$$=\sum_{n=0}^{\infty}p_{2n}(z)\left\{\frac{1}{2\pi i}\int_{C}\frac{\phi(s)}{s^{2n+1}}\cosh\frac{\alpha_{2n}}{s}ds\right\}$$

$$=\sum_{0}^{\infty}p_{2n}(z)f^{(2n)}(\alpha_{2n}),$$

which proves the Theorem.

2.4. *Proof of Theorem* 2. Suppose first that

$$\left.\begin{array}{l} f(z) \text{ is even, } \sigma<1, \\ |\alpha_{2n}\leqslant L<\log(2+\sqrt{3}),\ n=0,1,2,\ldots \end{array}\right\} \qquad (22)$$

In this case the Theorem follows at once from the relation (4) since the hypothesis of Theorem 3 holds and

$$f^{(2n)}(\alpha_{2n})=0,\ n=0,1,2,\ldots.$$

Next suppose

$$\left.\begin{array}{l} f(z) \text{ is even} \\ |\alpha_{2n}|\leqslant L<\dfrac{\log(2+\sqrt{3})}{\sigma},\ n=0,1,2\ldots \end{array}\right\} \qquad (23)$$

while $f(z)$ is of type not exceeding σ. Let

$$g(z)=f\left(\frac{z}{\sigma+\varepsilon}\right),\ \beta_{2n}=(\sigma+\varepsilon)\alpha_{2n}$$

where $\varepsilon>0$. Then the type of $g(z)$ does not exceed $\dfrac{\sigma}{\sigma+\varepsilon}<1$ while $g^{(2n)}(\beta_{2n})=0$; moreover

$$|\beta_{2n}|\leqslant(\sigma+\varepsilon)L<\frac{\sigma+\varepsilon}{\sigma}\log(2+\sqrt{3})<\log(2+\sqrt{3}),$$

if $\varepsilon>0$ is small enough. Hence $g(z)$ satisfies a hypothesis of the type (22) so that $g(z)\equiv 0$ which involves $f(z)\equiv 0$. Next suppose that

$$\overline{\lim}\ |\alpha_n|=L<\frac{\log(2+\sqrt{3})}{\sigma}$$

while $f(z)$ is even and of type not exceeding σ. Let $\eta>0$ be such that

$$|\alpha_n|\leqslant L+\eta<\frac{\log(2+\sqrt{3})}{\sigma}\ \text{ for } n\geqslant 2n_0.$$

Then $f^{(2n_0)}(z)=g(z)$ satisfies (23) with $\beta_n=\alpha_{n+2n_0}$. Hence $f^{(2n_0)}(z)\equiv 0$ and since $f^{(2s)}(\alpha_{2s})=0$, $s=0,1,\ldots n_0-1$ while $f^{(2s-1)}(0)=0$, $s=1,\ldots n_0$, we get $f(z)\equiv 0$. Hence, Theorem 2 holds when $f(z)$ is even. If $f(z)$ is odd, then $g(z)=f'(z)$ is even and satisfies the conditions of Theorem 2 with $\beta_n=\alpha_{n+1}$. Hence $f'(z)\equiv 0$; therefore $f(z)\equiv 0$ since it is odd. This proves Theorem 2.

3. For functions regular in a finite circle the following result has been proved by Kakeya.*

* *See* the reference in footnote 1, p. 125.

THEOREM 4. *Let $f(z)$ be regular in $|z|<R$. Let $[\alpha_n]$ be a sequence such that*

$$\overline{\lim_{n\to\infty}}\, n|\alpha_n| = L < R \log 2.$$

Then $f^{(n)}(\alpha_n)=0$, $n=0, 1, 2, \ldots$, involves $f(z)\equiv 0$.

3.1. By modifying Kakeya's method, or using the functions

$$\tfrac{1}{2}z^{2n}\left[\frac{1}{(1-\alpha_{2n}z)^{2n+1}}+\frac{1}{(1+\alpha_{2n}z)^{2n+1}}\right]$$

instead of $z^{2n}\cosh \alpha_{2n}z$ and proceeding as in the proof of Theorem 2 we can establish

THEOREM 5. *Let $f(z)$ be an even or odd function regular in $|z|<R$. Let $[\alpha_n]$ be such that*

$$\overline{\lim_{n\to\infty}}\, n|\alpha_n| = L < R \log (2+\sqrt{3}).$$

Then $f^{(n)}(\alpha_n)=0$ involves $f(z)\equiv 0$.

3.2. It may be noted that in Theorems 2 and 5 it is sufficient to suppose that

$$f^{(2n)}(\alpha_{2n})=0,\ n=0, 1, 2\ldots$$

when $f(z)$ is even and

$$f^{2n+1}(\alpha_{2n+1})=0,\ n=0, 1, 2, \ldots$$

when $f(z)$ is odd, since the remaining derivatives in the respective cases vanish at $z=0$.

ON THE AUTOMORPHISMS OF THE VECTOR RING MOD $(M_1, M_2, \ldots, M_n)$

By T. VENKATARAYUDU, M.A., University of Madras.

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Introduction.

Let $M_1, M_2, \ldots, M_n$ be a set of n integers. The residue classes with respect to any integer $M_r (r=1, 2, \ldots, n)$ form a ring in which the sum and the product of two residue classes m_1 mod (M_r), m_2 mod (M_r) are (m_1+m_2) mod (M_r) and m_1m_2 mod (M_r) respectively. Each M_r therefore defines a residue class ring mod (M_r). The vector compound* of the n residue class rings mod (M_1), mod $(M_2), \ldots,$ mod (M_n) has been called the vector ring mod $(M_1, M_2, \ldots, M_n)$. It consists of the totality of the vectors

$$(m_1 \text{ mod } M_1, m_2 \text{ mod } M_2, \ldots, m_n \text{ mod } M_n)$$

which we simply write as

$$(m_1, m_2, \ldots, m_n) \text{ mod } (M_1, M_2, \ldots, M_n).$$

The sum and the product of two vectors

$$(m_1, m_2, \ldots, m_n) \text{ mod } (M_1, M_2, \ldots, M_n)$$

and $$(m'_1, m'_2, \ldots, m'_n) \text{ mod } (M_1, M_2, \ldots, M_n)$$

are defined as

$$(m_1+m'_1, m_2+m'_2, \ldots, m_n+m'_n) \text{ mod } (M_1, M_2, \ldots, M_n)$$

and $$(m_1m'_1, m_2m'_2, \ldots, m_nm'_n) \text{ mod } (M_1, M_2, \ldots, M_n),$$

respectively.

Let p^a be an elementary block factor† in exactly r numbers of the set $M_1, M_2, \ldots, M_n$ and let S_r be the symmetric permutation group of degree r. The principal result of the present paper is

Theorem 1. *The automorphisms of the vector ring mod $(M_1, M_2, \ldots, M_n)$ form a group which is abstractly equivalent to the direct product of symmetric permutation groups of the form S_r.*

I. The direct compound of two vector rings.

Let A denote the vector ring mod $(M_1M'_1, M_2M'_2, \ldots, M_nM'_n)$ and A_1 and A_2 the vector rings mod $(M_1, M_2, \ldots, M_n)$ and

* See my paper "On the significance and the extension of the Chinese Remainder Theorem", *Jour. Ind. Math. Soc.* (2) 2 (1936), 99.

† An elementary block factor of N is a prime-power factor p^r of N, which is relatively prime to its complementary factor N/p^r.

$(M'_1, M'_2, \ldots, M'_n)$ respectively. If the two products $M_1M_2\ldots M_n$, $M'_1M'_2\ldots M'_n$ are relatively prime, then by the Chinese Remainder Theorem any vector of A corresponds uniquely to two vectors; one of A_1 and the other of A_2 and conversely. If $(m_1, m_2, \ldots, m_n)$ mod $(M_1, M_2, \ldots, M_n)$ and $(m'_1, m'_2, \ldots, m'_n)$ mod $(M'_1, M'_2, \ldots, M'_n)$ are any two vectors of A_1 and A_2 respectively, the corresponding vector of A is given by $(a_1, a_2, \ldots, a_n)$ mod $(M_1M'_1, M_2M'_2, \ldots, M_nM'_n)$ where

$$a_r \equiv m_r \text{ mod } (M_r)$$
$$\equiv m'_r \text{ mod } (M'_r),$$

and the Chinese Remainder Theorem uniquely determines the vector $(a_1, a_2, \ldots, a_n)$ mod $(M_1M'_1, M_2M'_2, \ldots, M_nM'_n)$ from the two vectors $(m_1, m_2, \ldots, m_n)$ mod $(M_1, M_2, \ldots, M_n)$ and $(m'_1, m'_2, \ldots, m'_n)$ mod $(M'_1, M'_2, \ldots, M'_n)$ and conversely. Further the correspondence is such that the sum and the product of two vectors of A correspond respectively to the sum and the product of the corresponding vectors of A_1 and A_2. We then say that the vector ring A is the *direct compound* of the two vector rings A_1 and A_2. We say that the vector ring mod $(q_1, q_2, \ldots, q_n)$ is a *primary vector ring* belonging to the prime p if $q_1, q_2, \ldots, q_n$ are powers of the prime p. We have therefore now proved

THEOREM 2. *The vector ring mod $(M_1, M_2, \ldots, M_n)$ is the direct compound of primary vector rings belonging to the prime divisors of the product $M_1M_2\ldots M_n$.*

II. THE GROUP OF AUTOMORPHISMS OF THE VECTOR RING.

A one to one correspondence (denoted in symbols by $\rightarrow$) between the elements of the ring A is said to define an automorphism of A if for every element a of A there corresponds a single element a' of A such that if

$$a \rightarrow a', \; b \rightarrow b'$$

then $$a+b \rightarrow a'+b' \text{ and } ab \rightarrow a'b'.$$

We have now the fundamental

THEOREM 3. *The automorphisms of the vector ring mod $(M_1, M_2, \ldots, M_n)$ form a group.*

PROOF: (1) Let η and θ be any two automorphisms of the vector ring. Then if we define

$$\eta\Big(\theta(a)\Big) = \eta\theta(a) \tag{1}$$

we will now show that $\eta\theta$ is also an automorphism. For

$$a \to \theta(a) \to \eta\big(\theta(a)\big) = \eta\theta(a)$$
$$b \to \theta(b) \to \eta\big(\theta(b)\big) = \eta\theta(b)$$

then $a+b \to \theta(a)+\theta(b)=\theta(a+b)$ since θ is an automorphism (2);

$$a+b \to \eta\big(\theta(a)\big)+\eta\big(\theta(b)\big)=\eta\big(\theta(a)+\theta(b)\big) \text{ since } \eta \text{ is an automorphism.}$$
$$=\eta\big(\theta(a+b)\big) \text{ by (2)}$$
$$=\eta\theta(a+b) \text{ by (1).}$$

Similarly $\quad ab \to \theta(a).\theta(b)=\theta(ab)$

$$\to \eta\big(\theta(a)\big).\eta\big(\theta(b)\big)=\eta\big(\theta(a).\theta(b)\big)$$
$$=\eta\big(\theta(ab)\big)=\eta\theta(ab).$$

Hence $\eta\theta$ is an automorphism.

(2) For every automorphism θ of A there is an inverse automorphism denoted by θ^{-1} such that if

$$a \to a' \text{ by } \theta, \ a' \to a \text{ by } \theta^{-1}.$$

(3) The identical automorphism is the automorphism by which every element of A corresponds to itself.

(4) Finally the automorphisms are associative.

The automorphisms therefore form a group.

If two groups G_1 and G_2 have no common elements except the identical element and if each element of G_1 is permutable with each element of G_2, then the group obtained by combining in every possible way the elements of the two groups G_1 and G_2 is called the *direct product* of G_1 and G_2. An analogous definition may be given for the direct product of several groups.

THEOREM 4. *The group of automorphisms of the vector ring mod $(M_1, M_2, \ldots, M_n)$ is the direct product of the groups of automorphisms of the component primary vector rings.*

PROOF: Let A be the given vector ring and $A_1, A_2, \ldots, A_k$ be the component primary vector rings. A vector of A_r may be taken as

$$(a_{r1}, a_{r2}, \ldots, a_{rn}) \bmod (q_{r1}, q_{r2}, \ldots, q_{rn}),$$

where q_{rs} is the elementary block factor of M_s belonging to the prime p_r (for $s=1, 2, \ldots, n$). Let η_r be an automorphism of the vector ring A_r by which

$$(a_{r1}, a_{r2}, \ldots, a_{rn}) \bmod (q_{r1}, q_{r2}, \ldots, q_{rn}) \to$$
$$(a'_{r1}, a'_{r2}, \ldots, a'_{rn}) \bmod (q_{r1}, q_{r2}, \ldots, q_{rn}).$$

Obviously A admits an automorphism by which the elements of A_r undergo the automorphism η_r while the elements of $A_1, A_2, \ldots, A_{r-1}, A_{r+1}, \ldots, A_k$ undergo the identical automorphism. If $(a_1, a_2, \ldots, a_n) \bmod (M_1, M_2, \ldots, M_n)$ is a vector of A and if by the Chinese Remainder Theorem it corresponds to the vector $(a_{t1}, a_{t2}, \ldots, a_{tn}) \bmod (q_{t1}, q_{t2}, \ldots, q_{tn})$ of $A_t (t=1, 2, \ldots, k)$, the vector corresponding to $(a_1, a_2, \ldots, a_n) \bmod (M_1, M_2, \ldots, M_n)$ by means of the above automorphism of A is that vector of A which by the Chinese Remainder Theorem corresponds to the vectors

$$(a_{t1}, a_{t2}, \ldots, a_{tn}) \bmod (q_{t1}, q_{t2}, \ldots, q_{tn}) \text{ of } A_t \text{ for } t \neq r$$

and

$$(a'_{r1}, a'_{r2}, \ldots, a'_{rn}) \bmod (q_{r1}, q_{r2}, \ldots, q_{rn}) \text{ of } A_r \text{ i.e. for } t=r.$$

Secondly, if η_s is any automorphism of the vector ring $A_s (s \neq r)$, η_r is clearly permutable with η_s. Hence the groups of automorphisms of $A_1, A_2, \ldots, A_k$ are such that in any two of the groups the elements of the one are permutable with the elements of the other and the identity is the only common element between them. Hence the group generated by the elements of the groups of automorphisms of $A_1, A_2, \ldots, A_k$ is their direct product. Now any automorphism of A is a product of automorphisms of $A_1, A_2, \ldots, A_k$ and conversely. The group of automorphisms of A is therefore the direct product of the groups of automorphisms of the component vector rings $A_1, A_2, \ldots, A_k$.

THEOREM 5. *The identical automorphism is the only automorphism of the residue class (scalar) ring mod (M).*

PROOF: In any automorphism the identity element corresponds to the identity element itself. Since the residue class ring mod (M) is additively cyclic* and is generated by the identity class 1 mod (M), every element of A must therefore correspond to itself in any possible automorphism. Hence the identical automorphism is the only automorphism of the residue class ring mod (M).

THEOREM 6. *The group of automorphisms of the vector ring mod $(p^a, p^a, \ldots, p^a)$ (n in number) is abstractly identical with the symmetric permutation group on n symbols.*

PROOF: Let e_t denote the vector $(0, 0, \ldots, 1, 0, \ldots, 0) \bmod (p^a, p^a, \ldots, p^a)$ where the tth element only is 1 and the remain-

* My thanks are due to Dr. R. Vaidyanathaswamy for pointing out to me that this property is not true in a general residue class ring with respect to an ideal modulus, when I was attempting to generalize the results of this paper to the general commutative ring under certain conditions.

ing elements are zero. Now $e_1, e_2, \ldots, e_n$ form a set of basis vectors of the vector ring. An automorphism is therefore completely determined if the elements corresponding to $e_1, e_2, \ldots, e_n$ by the automorphism are determined.

Let $e_t \to (\alpha_1, \alpha_2, \ldots, \alpha_n) \bmod (p^a, p^a, \ldots, p^a)$ be any automorphism. Since any power of e_t is e_t itself

$$\alpha_s^2 \equiv \alpha_s \bmod (p^a) \quad (s=1, 2, \ldots, n).$$

Therefore α_s is either $\equiv 0 \bmod (p^a)$ or $\equiv 1 \bmod (p^a)$ for $s=1,2,\ldots,n$

i.e. $\qquad e_t \to$ a sum of different e's.

Further, no two different e's correspond to sums of e's containing the same e, for if

$$e_t \to e_k + \ldots$$

$$e_{t'} \to e_k + \ldots$$

$e_t e_{t'} =$ the zero vector $\to$ a vector different from the zero vector. But in any automorphism the zero vector must correspond to the zero vector. An e cannot therefore correspond to the zero vector. Hence in any automorphism any e must correspond to just one e only. Thus corresponding to any possible automorphism we can associate with it a permutation on the n symbols $e_1, e_2, \ldots, e_n$. Obviously for every permutation there corresponds an automorphism of the vector ring. Further the product of two automorphisms corresponds to the product of the two corresponding permutations of $e_1, e_2, \ldots, e_n$. Hence the group of automorphisms of the given vector ring is equivalent to the symmetric group of degree n.

THEOREM 7. *The group of automorphisms of the primary vector ring mod* $(q_1, q_2, \ldots, q_n)$ *(i.e. the q's are powers of a single prime p) where $r_1, r_2, \ldots, r_k$ of the q's are equal, is equivalent to the direct product of the symmetric groups of degrees $r_1, r_2, \ldots, r_k$.*

PROOF: With the same notation as in Theorem 6 we have $e_1, e_2, \ldots, e_n$ as a set of basis vectors of the vector ring mod $(q_1, q_2, \ldots, q_n)$. If the r_1 q's equal to q_a are $q_a, q_b, q_c, \ldots,$ we put $e_a, e_b, e_c, \ldots,$ into a class and thus divide the n e's into k classes containing $r_1, r_2, \ldots, r_k$ e's respectively. Here in any automorphism of the vector ring any e must correspond to another e in the same class and two e's belonging to different classes cannot correspond to each other. If S_t is the symmetric permutation group on the r_t e's of the sth class and S the direct product of $S_1, S_2, \ldots, S_k$, then any element of S defines as in Theorem 6 an automorphism of the vector ring and any possible automorphism must correspond to an element of S. Further the correspondence defines an isomorphism, i.e. the product of two permutations corresponds to the product of automorphisms. Hence the group of automorphisms of the given vector ring is equivalent to S. Our principal Theorem 1 enunciated in the introduction follows from Theorems 4 and 7.

III. CONNECTION WITH THE EXTENSION OF THE CHINESE REMAINDER THEOREM.

The present discussion of the automorphisms of the vector ring mod $(M_1, M_2, \ldots, M_n)$ is in fact a continuation of my previous paper "On the Significance and the Extension of the Chinese Remainder Theorem" and is closely related with it. It has been shown there that if $G_1, G_2, \ldots, G_n$ are the successive g.c.d's of $M_1, M_2, \ldots, M_n$ and if p^a is an elementary block factor in exactly r numbers of the set $M_1, M_2, \ldots, M_n$, the number of representations of the integers M_r, G_r $(r=1, 2, \ldots, n)$ as products of mutually relatively prime factors in the form

$$\left.\begin{array}{l} M_r = M_{r1} M_{r2} \ldots M_{rn} \\ G_r = M_{1r} M_{2r} \ldots M_{nr} \end{array}\right\} \qquad (1)$$

is the product of a number of factors of the form $r!$.

If p^a is an elementary block factor in the r M's, say $M_p, M_q, M_r, \ldots$ and in the r consecutive G's, say $G_k, G_{k+1}, \ldots, G_{k+r-1}$ we take p^a as an elementary block factor in r numbers of the set $M_{rs}(r, s=1, 2, \ldots, n)$ which have the first suffix as one of the r numbers $p, q, r, \ldots$ and the second suffix as one of the r numbers $k, k+1, \ldots, k+r-1$. If S_r is the symmetric group of permutations on r letters and if one way of choosing such r numbers of the set M_{rs} is given, then corresponding to any permutation of S_r there is another way of choosing the r numbers M_{rs} by making the corresponding permutation on the r first suffixes of the given r numbers M_{rs}. By taking all the elementary block factors in $M_1, M_2, \ldots, M_n$ we obtain as above all the elementary block factors of $M_{rs}(r, s= 1, 2, \ldots, n)$. If one representation of the integers M_r, G_r in the form (1) is given and if S is the direct product of the symmetric groups of the form S_r, corresponding to every element of S there is another representation of the integers $M_r, G_r (r=1, 2, \ldots, n)$ in the form (1). By Theorem 1 the group S is simply isomorphic with the group of automorphisms of the vector ring mod $(M_1, M_2, \ldots, M_n)$. Hence a one to one correspondence can thus be established between the representations of the form (1) and the automorphisms of the vector ring mod $(M_1, M_2, \ldots, M_n)$.

Given the different representations of the integers M_r, G_r in the form (1), we can obtain the automorphisms of the vector ring by means of the Extension of the Chinese Remainder Theorem. We know that the matrix of transformation

$$B=\begin{vmatrix} \beta_{11} & \beta_{12}\ldots\beta_{1n} \\ \beta_{21} & \beta_{22}\ldots\beta_{2n} \\ \ldots\ldots & \ldots\ldots \\ \ldots\ldots & \ldots\ldots \\ \beta_{n1} & \beta_{n2}\ldots\beta_{nn} \end{vmatrix}$$

by which we obtain the vector $(m_1, m_2,\ldots,m_n) \bmod (M_1, M_2,\ldots,M_n)$ from the corresponding vector $(g_1, g_2,\ldots,g_n) \bmod (G_1, G_2,\ldots,G_n)$ is completely determined from the set of integers M_{rs} of (1) by means of the congruences

$$\beta_{rs}\equiv 0 \bmod \left(\frac{M_r}{M_{rs}}\right)$$

and

$$\equiv 1 \bmod (M_{rs}).$$

Let $M_{rs}(r, s=1, 2,\ldots, n)$ be a fixed representation of the form (1), and let M'_{rs} be any other representation of the same form. Let B' be the matrix of linear transformation determined by the numbers $M'_{rs}(r, s=1, 2,\ldots, n)$. Let $(m_1, m_2,\ldots, m_n) \bmod (M_1, M_2,\ldots,M_n)$ and $(m'_1, m'_2,\ldots,m'_n) \bmod (M_1, M_2,\ldots,M_n)$ be the vectors obtained from the same vector $(g_1, g_2,\ldots, g_n) \bmod (G_1, G_2,\ldots, G_n)$ by B and B' respectively. Now the corres- dence

$$(m_1, m_2,\ldots, m_n) \bmod (M_1, M_2,\ldots, M_n)\rightarrow$$
$$(m'_1, m'_2,\ldots, m'_n) \bmod (M_1, M_2,\ldots, M_n)$$

obviously defines an automorphism of the vector ring $\bmod (M_1, M_2,\ldots, M_n)$. By varying M'_{rs} we obtain as above all the automorphisms of the vector ring $\bmod (M_1, M_2,\ldots, M_n)$.

In conclusion I may state that the automorphisms which we have been considering in this paper are *ring automorphisms,* i.e. a one to one correspondence between the elements of the vector ring by which the sum and the product of two elements correspond respectively to the sum and the product of the two corresponding elements. On the other hand a one to one correspondence between the elements of the vector ring by which the sum of two elements corresponds to the sum of the two corresponding elements is a *group automorphism* of the vector ring. The discussion of the group automorphisms of the vector ring $\bmod (M_1, M_2,\ldots, M_n)$ is identical with the discussion of the automorphisms of a multiplicative Abelian group which is the direct product of cyclic groups of orders $M_1, M_2,\ldots, M_n$.

A NOTE ON HARMONIC CURVES

BY C. N. SRINIVASIENGAR, Bangalore.

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1. The equations of a rational curve which lies on a quadric surface and whose tangents belong to a linear complex can be expressed in the form*

$$x:y:z:w=t^{m+n}:t^m:t^n:1 \tag{1}$$

where m and n are integers prime to each other. Such curves are called *harmonic curves* and form a particular class of W-curves given by

$$x:y:z:w=t^p:t^m:t^n:1$$

where p, m, n are integers. Conversely, if a W-curve belongs to a linear complex, it is a harmonic curve. Putting in equations (1) $m=2$, $n=1$ we have the twisted cubic, while $m=3$, $n=1$ give the twisted quartic with two inflexions.

We shall assume $m>n$. The curve (1) lies on the scroll

$$y^{m+n}w^{m-n}=z^{m+n}x^{m-n} \tag{2}$$

which has two directrices forming multiple lines of orders $m+n$ and $m-n$. Expressing the co-ordinates of any point on (2) in terms of two parameters, and calculating the second order magnitudes L, M, N, it is easy to prove that the asymptotic curves of (2)† are the intersections of (2) with the system of quadrics $yz=A^2xw$, viz. the curves

$$x:y:z:w=t^{m+n}:At^m:At^n:1. \tag{3}$$

Any generator of the scroll intersects any of the asymptotic curves in two points which harmonically separate the points where the generator meets the two directrix lines. For, the generator

$$y=p^{m-n}z;\ x=p^{m+n}w$$

meets the curve where

$$p^{m-n}z^2-A^2p^{m+n}w^2=0$$

and the directrices where $zw=0$.

* *Encycl. der Math. Wiss.*, III C 9, Art. 58 (pp. 1372-73); also R. Vaidyanathaswamy, Question 1269 *J. I. M. S.* (*Vide* Solution by A. Narasinga Rao *J. I. M. S.* Vol. 17, Part II, p. 9).

† Compare A. Wiman, *Ark. Mat. Astron. Fys.* 25 A (1935); *Zentralblatt Für. Math.* Band 13, p. 223.

The reciprocal of the scroll (2) is a scroll of the same nature. The Hessian consists of the planes $y=0$, $z=0$ each counted $2(m+n-1)$ times, and $x=0$, $w=0$ each counted $2(m-n-1)$ times.

2. Let t_1, t_2, t_3, t_4 be the parameters of four fixed points on the harmonic curve (1). The equation of the plane through the point t and the generator $x=\lambda z$;$y=\lambda w$ of the quadric through the curve is

$$x-\lambda z=t^n(y-\lambda w),$$

while the plane through t and the generator $x=\mu y$; $z=\mu w$ is

$$x-\mu y=t^m(z-\mu w).$$

Hence we get the result:

Four fixed points on the curve joined to any generator of the quadric containing the curve form a pencil of planes whose cross-ratio is constant for all generators of the same system. The value of the cross-ratio is that of the four values t_1^m, t_2^m, t_3^m, t_4^m for one system, and t_1^n, t_2^n, t_3^n, t_4^n for the other system.

3. If a surface of degree N possesses a multiple line of order $N-1$ (and is therefore ruled) with two "pinch-points" at either of which all the tangent planes coincide, the asymptotic curves are harmonic curves. For the equation of the surface can be expressed in the form $y^{N-1}w=z^{N-1}x$. If N is even, we can take $m=N/2$, $n=(N/2)-1$ and write $y^{m+n}w=z^{m+n}x$. Equations (3) give the asymptotic curves. If N is odd, we can still take equations (3) with these values of m and n, in other words it is directly verified that the asymtotic curves are

$$x:y:z:w=t^{2N-2}:At^N:At^{N-2}:1.$$

The order of the asymptotic curves on $y^{N-1}w=z^{N-1}x$ is thus $N-1$ or $2N-2$ according as N is even or odd. The cases $N=2$ and $N=3$ give the quadric and the cubic scroll of the first type respectively. $N=4$ gives the scroll $xz^3=yw^3$ which is a degenerate case of the quartic scroll with two directrix lines one of which is a triple line [Type IV-A (Edge) ; Type II (Salmon)]. There are two types of quintic harmonic curves, viz.

$$x:y:z:w=t^5:t^4:t:1,$$

and

$$x:y:z:w=t^5:t^3:t^2:1.$$

The second of these corresponds to the value $N=6$.

4. Putting $n=1$ in (1), we get the curve

$$x:y:z:w=t^{m+1}:t^m:t:1$$

which has two points given by $t=0$ and $t=\infty$ at either of which the tangent has contact of order m with the curve. Each of these points counts as $m-2$ inflexions. Conversely, if a curve of order $m+1$ has two such points, the curve is a harmonic curve (Marletta). The scroll (2) now becomes $y^{m+1}w^{m-1}=z^{m+1}x^{m-1}$.

Likewise, for the scroll $y^{m+1}w^{m}=z^{m+1}x^{m}$, the asymptotic curves are given by

$$x:y:z:w=t^{2m+2}:At^{2m+1}:At:1.$$

There is only one type of sextic harmonic curve, and this characterises the asymptotic curves of $y^3w^2=z^3x^2$.

5. The surface $x^{m-1}z+x^{m-2}yw=y^m$ where m is a positive integer has $x=0$, $y=0$ as a multiple line of order $m-1$ with a pinch-point $(0, 0, 1, 0)$ at which all the nodal planes coincide. Any plane through the multiple line contains one generator, and the multiple line itself counts as the generator corresponding to the plane $x=0$. The line counts as two coincident directrices. I shall call this the *Cayley scroll of order* m, since $m=3$ gives the familiar Cayley's cubic scroll. *The reciprocal of a Cayley scroll of order* m *is another Cayley scroll of order* m, the reciprocal of the above surface with respect to $x^2+y^2+z^2+w^2=0$ being $z^{m-2}(xz+yw)+w^m=0$. The Hessian consists of the pinch-plane $x=0$ counted $4(m-2)$ times.

The asymptotic curves of Cayley's scroll of order m *are harmonic curves of order* m, *each of which is a curve of type* $(1, m-1)$ *lying on a quadric surface. Any generator of the scroll meets any of the curves in a single point.*

Taking $$x:y:z:w=r:pr:p^m r-p:1$$
as the equations of the surface in terms of two parameters and calculating the second order magnitudes L, M, N, we get the asymptotic lines as

$$x:y:z:w=2:2t:(2-m)t^m+ct:mt^{m-1}-c. \tag{4}$$

Any one of these curves can be reduced by a simple transformation to

$$X:Y:Z:W=2:2t:mt^{m-1}:(2-m)t^m$$

and is hence a harmonic curve, the tangents at $t=0$ and $t=\infty$ having $(m-1)$-pointic contact with the curve.

Putting $m=4$, we get the surface $y^4=x^2(xz+yw)$. This scroll is not included in Salmon's classification of quartic scrolls with a triple line.* It corresponds to a sub-case of the equation $u_4=zu_3+wv_3$ where u_3 and v_3 have a common square factor, but Salmon's form $x^2y^2=(ax+by)^2(xz+yw)$ does not cover this case; for in Salmon's form there is no plane $y=mx$ which does not contain another generator, in other words the double line does not count also as a generator.

The linear complex to which the curve (4) belongs is given by $q-q'=cr'$ where p, q, r, p', q', r' denote line-coordinates. The complexes $q-q'=cr'$ and $q-q'=-cr'$ may be called *mutually apolar*.†

The cross-ratio of the points in which any generator meets four given asymptotic curves is equal to the cross-ratio of the corresponding linear complexes, viz. $(c_1c_2c_3c_4)$.

The points where a generator of the Cayley scroll of order m intersects a pair of asymptotic curves whose linear complexes are mutually apolar form an involution one of whose double points is on the directrix line. The other double point lies on the asymptotic curve corresponding to $c=0$.

6. The surface of **5** can be generalised.

The asymptotic curves on $x^{m-n}(zx+wy)^n=y^{m+n}$ *where* m *and* n *are integers are harmonic curves given by*

$$x:y:z:w=2n:2nt^n:(n-m)t^{m+n}-ct^n:(m+n)t^m+c.$$

If $n=1$, we get the surface of **5**. Any generator meets an asymptotic curve in a single point either if $n=1$ or if $m=1$. The cross-ratio properties of **5** hold in either of the cases.

The reciprocal surface is again of the same nature as the original. With reference to the quadric $x^2+y^2+z^2+w^2=0$, the reciprocal is $(xz+yw)^n z^{m-n}=(-1)^n w^{m+n}$

Special cases of interest other than those coming under **5** are the following:

$(a)\, m=1, n=2$. $(zx+wy)^2=xy^3$. This is a quartic scroll with two intersecting double lines. This surface can be included in Type II(C) (Edge) which is a scroll with a double conic and a double line which in the general case is not in the plane of the conic. The equation of the scroll can be written‡

$$(cyz-bzx+axy+zw-xw)^2=xz(ax-by+cz)^2.$$

* Salmon: *Analytical Geometry of three Dimensions.* §§ 546-48.

† A Narasinga Rao. *loc. cit.*, p. 11.

‡ W. L. Edge: *Theory of Ruled Surfaces.* p. 64.

When $a=b=0$, this reduces to the present form, with a change of variables. The double conic now breaks up into two lines one of which coincides with the double line *i.e.* the double curve consists of two straight lines in a plane one of which has to be counted twice. It may also be observed that the elliptic quartic scroll Type VI B (Edge) degenerates into the present surface when several of the coefficients vanish.

(*b*) $m=3$, $n=2$. The quintic $x(zx+wy)^2=y^5$ has a triple line and a double line meeting it, and is necessarily a special case of a surface with a triple line and a double conic.

(*c*) $m=1$, $n=3$. $(zx+wy)^3=x^2y^4$.

7. The following are some special properties of the skew quartic with two inflexions.

If A, B, C, D be the points on the curve the osculating planes at which pass through a given point O, and if A', B', C', D' be the points at which these osculating planes meet the curve again, then the five points O, A', B', C', D' are coplanar. The relation between the two tetrads of points $ABCD$ and $A'B'C'D'$ is reciprocal. AA', BB', CC', DD' are generators of the cubic scroll associated with the curve.

These properties are easily verified. In the general case, for a curve of order n belonging to a linear complex, if we draw the n osculating planes that pass through a given point O of general position, these planes will meet the curve again in $n(n-3)$ points which when joined to O form a cone of order $n-3$. To prove this, we observe that if we project the curve from O on any plane, we get a plane n-*ic* having n collinear inflexions. The inflexional tangents at these points meet the curve again in points lying on a $(n-3)$-*ic*.

The sextic developable forming the tangent surface of the curve $x:y:z:w=t^4:t^3:t:1$ is the envelope of the plane $x-2ty+2t^3z-t^4w=0$ and is given by $I^3-27J^2=0$ where

$$I=yz-xw;\; J=\tfrac{1}{4}(xz^2-wy^2).$$

The surface $J=0$ is the unique cubic scroll of the first species that passes through the curve (and has it as an asymptotic curve), while $I=0$ is the quadric through the curve. If we form the equation whose roots in t are the cubes of the roots of $x-2ty+2t^3z-t^4w=0$, it will be found that the J of the first equation is a factor of the J of the transformed equation also. Hence, when the cross-ratio $(t_1t_2t_3t_4)=-1$, where t_1, t_2, t_3, t_4 are the roots of $x_1-2ty_1+2t^3z_1-t^4w_1=0$, the cross-ratio $(t_1^3\, t_2^3\, t_3^3\, t_4^3)$ is also equal to -1. Using the property of 2, we have therefore the result:

The feet of the osculating planes drawn to a twisted quartic with two inflexions, from any point of the associated cubic scroll form when joined to a generator of either system of the quadric passing through the curve, a harmonic pencil of planes.

As a special case we may take the four points in which the curve is met by any plane through the simple directrix of the cubic scroll.

Similarly, we get an equianharmonic pencil of planes by joining to any generator of either system of the quadric the feet of the osculating planes from any point on the quadric.

ON CERTAIN SYMMETRIC FUNCTIONS OF NUMBERS PRIME TO m

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1. Notation. All letters denote integers unless stated otherwise.

(i) $\Sigma_r(a, b, c, \ldots)$ denotes the sum of all the products of $a, b, c, \ldots$ taken r at a time.

(ii) $P(n) = P(n/d+1)$ means the product of all primes of the form $n/d+1$.

E.g. $$P(8) = (\tfrac{8}{2}+1)(\tfrac{8}{4}+1) = 15.$$

(iii) $h = \phi(m)$ and p is always a prime.

(iv) $a_1, a_2, a_3, \ldots, a_h$ is the least positive reduced residue system modulo m.

(v) S_r stands for $\sum_{i=1}^{\phi(m)} a_i^r$.

(vi) π is the least prime in m.

1.1. *Lagrange stated and proved that for a prime p

$$\Sigma_k(1, 2, 3, \ldots, p-1) \equiv 0 \pmod p \text{ for } 1 \leqslant k < p-1.$$

In Theorem I a congruence of similar character for any modulus m will be proved. It may be noted that the method of proof provides a more elegant proof of Lagrange's result than has ever been given.

2. Theorem I.

$$\Sigma_k(a_1, a_2, \ldots, a_h) \equiv 0 (\textit{mod } m) \textit{ provided } \Big(m, P(k)\Big) = 1.$$

Proof: There exists an integer ξ such that $(m, \xi^k - 1) = 1$, which is the †common solution of the congruences

$$x \equiv c_t \pmod{p_t}$$

* L. E. Dickson, *History of Theory of Numbers*, Vol. I, p. 99.

† That such a solution exists follows from Landau: *Vorlesungen über Zahlentheorie*, Band 1, Satz 70.

where $p_t | m$ and c_t is the primitive root modulo p_t. If m and $\xi^k - 1$ have a factor p_x in common, $p_x - 1 | k$, $p_x | m$ which is in contradiction with the condition $\left(P(k), m\right) = 1$. Now then

$$\xi^k \Sigma_k(a_1, a_2, \ldots, a_h) \equiv \Sigma_k(a_1\xi, a_2\xi, \ldots, a_h\xi)$$
$$\equiv \Sigma_k(a_1, a_2, \ldots, a_h) \pmod{m}.$$

$$(\xi^k - 1)\, \Sigma_k(a_1, a_2, \ldots, a_h) \equiv 0 \pmod{m},$$

whence it follows $\Sigma_k(a_1, a_2, \ldots, a_h) \equiv 0 \pmod{m}$.

2.1. THEOREM II.

$$\Sigma_k(a_1^r, a_2^r, \ldots, a_h^r) \equiv 0 \ (mod\ m) \ if \ \left(P(rk), m\right) = 1.$$

PROOF: Just as in the case of Theorem I it can be shown that it is possible to choose a number ξ such that $\xi^{rk} \not\equiv 1 \pmod{p}$, for $p | m$.

$$\xi^{rk} \Sigma_k(a_1^r, a_2^r, \ldots, a_h^r) \equiv \Sigma_k(\overline{a_1\xi^r}, \overline{a_2\xi^r}, \ldots, \overline{a_h\xi^r})$$
$$\equiv \Sigma_k(a_1^r, a_2^r, \ldots, a_h^r) \pmod{m}.$$

Hence the result.

3. Wolstenholme's theorem was generalised by Leudesdorf (1889) as follows:

$$If\ (m, 6) = 1,\ \sum_{i=1}^{h} \frac{1}{a_i} \equiv 0 \ (mod\ m^2).$$

What is meant, in fact, is that $\Sigma_{h-1}(a_1, a_2, \ldots, a_h) \equiv 0 \pmod{m^2}$ provided $(m, 6) = 1$. With this as basis I proceed to examine whether $\Sigma_k(a_1, a_2, \ldots, a_h) \equiv 0 (m^2)$ holds for any values of k other than $h - 1$.

THEOREM III. $\Sigma_k(a_1, a_2, \ldots, a_h) \equiv 0 \pmod{m^2}$ *if k is odd and* $1 < k \begin{matrix} < \pi - 1 \\ or \\ > h - \pi + 2 \end{matrix}$

3.1. *LEMMA: *Whether r is positive or negative*

(i) $S_r \equiv 0$ *(mod m) if r is even, $p - 1 \nmid r$, for $p | m$ and $2 \nmid m$*

(ii) $S_r \equiv 0$ *(mod m^2) if r is odd and $p - 1 \nmid r - 1$, for $p | m$, $2 \nmid m$.*

PROOF OF THE LEMMA: Case (i). Let the canonical representation of m be $\prod_{i=1}^{s} p_i^{\alpha_i}$. For $1 \leqslant i \leqslant s$,

* Proved also by N. Rama Rao in a different way in his paper on Leudesdorf's Theorem. *Journ. L.M.S.*, July 1937.

$$*a_1^r+a_2^r+\ldots+a_h^r\equiv\left\{\frac{h}{\phi(p_i^{\alpha_i})}\right\}\left\{c_{i1}^r+c_{i2}^r+\ldots+c_{i\phi(p_i^{\alpha_i})}^r\right\}(\mathrm{mod}\ p_i^{\alpha_i})$$

where $c_{i1}, c_{i2}, \ldots, c_{i\phi(p_i^{\alpha_i})}$ is a reduced residue system modulo $p_i^{\alpha_i}$. And if g_i is the primitive root modulo $p_i^{\alpha_i}$

$$S_r\equiv\phi\left(\frac{m}{p_i^{\alpha_i}}\right)\sum_{t=1}^{\phi(p_i^{\alpha_i})} g_i^{rt}\ (\mathrm{mod}\ p_i^{\alpha_i})$$

$$\equiv\phi\left(\frac{m}{p_i^{\alpha_i}}\right)\left\{\frac{g_i^r(g_i^{\phi(p_i^{\alpha_i}).r}-1)}{g_i^r-1}\right\}\equiv 0\ (\mathrm{mod}\ p_i^{\alpha_i})$$

since $g_i^r\not\equiv 1\ (\mathrm{mod}\ p_i)$, for otherwise $p_i-1|r$ which contradicts the hypothesis.

Case (ii).

$$S_r\equiv\sum_{\substack{i=1\\a_i<m/2}}^{h/2}\left\{a_i^r+(m-a_i)^r\right\}$$

$$\equiv 2rm\sum_{i=1}^{h/2} a_i^{r-1}\ (\mathrm{mod}\ m^2).$$

It need only be shown that

$$\sum_{\substack{i=1\\a_i<m/2}}^{h/2} a_i^{r-1}\equiv 0\ (\mathrm{mod}\ m)$$

which is equivalent to (since m is odd)

$$\sum_{i=1}^{h} a_i^{r-1}\equiv 0\ (\mathrm{mod}\ m),$$

for

$$\sum_{i=1}^{h} a_i^{r-1}\equiv\sum_{\substack{i=1\\a_i<m/2}}^{h/2}\left\{a_i^{r-1}+(m-a_i)^{r-1}\right\}$$

$$\equiv 2\sum_{\substack{i=1\\a_i<m/2}}^{h/2} a_i^{r-1}\ (\mathrm{mod}\ m).$$

* *See* Landau, *loc. cit.*, Satz 74.

Let the canonical representation of m be $\prod_{i=1}^{s} p_i^{\alpha_i}$ and let g_i be the primitive root modulo $p_i^{\alpha_i}$. For $1 \leqslant i \leqslant s$,

$$\sum_{i=1}^{h} a_i^{r-1} \equiv \phi\left(\frac{m}{p_i^{\alpha_i}}\right)\left\{c_{i_1}^{r-1}+c_{i_2}^{r-1}+\ldots+c_{i\phi(p_i^{\alpha_i})}^{r-1}\right\} \pmod{p^{\alpha_i}},$$

where $c_{i_1}, c_{i_2}, \ldots, c_{i\phi(p_i^{\alpha_i})}$ is the reduced residue system modulo $p_i^{\alpha_i}$.

$$\sum_{i=1}^{h} a_i^{r-1} \equiv \phi\left(\frac{m}{p_i^{\alpha_i}}\right) \sum_{t=1}^{\phi(p_i^{\alpha_i})} g_i^{(r-1)t}$$

$$\equiv \phi\left(\frac{m}{p_i^{\alpha_i}}\right)\left\{\frac{g_i^{r-1}\left(g_i^{\phi(p^{\alpha_i})(r-1)}-1\right)}{g_i^{r-1}-1}\right\}$$

$$\equiv 0 \pmod{p_i^{\alpha_i}} \text{ since } g_i^{r-1} \not\equiv 1 \pmod{p_i}$$

for otherwise $p_i - 1 \mid r-1$.

3.2. PROOF OF THE THEOREM: *Newton showed that

$$1-\Phi(1)y+\Phi(2)y^2-\ldots+\Phi(h)y^h = \exp.\left[-\chi_1 y-\tfrac{1}{2}\chi_2 y^2-\ldots-\tfrac{1}{r}\chi_r y^r-\ldots\right], \quad (1)$$

where $\Phi(r)=\Sigma_r(\alpha_1, \alpha_2, \ldots, \alpha_h)$ and $\chi_r = \sum_{i=1}^{h} \alpha_i^r$. The right side on expansion becomes

$$1-\chi_1 y-\left[\tfrac{1}{2}\chi_2-\tfrac{1}{1.2}\chi_1^2\right]y^2-\left[\tfrac{1}{3}\chi_3-\tfrac{1}{1.2}\chi_1\chi_2+\tfrac{1}{1.2.3}\chi_1^3\right]y^3$$

$$-\left[\tfrac{1}{4}\chi_4-\tfrac{1}{3}\chi_1\chi_3+\tfrac{1}{4}\chi_1^2\chi_2+\tfrac{1}{2.2^2}\chi_2^2-\tfrac{1}{1.2.3.4}\chi_1^4\right]y^4-\ldots. \quad (1)$$

We see that on putting $\alpha_i = a_i$, $\Phi(r)$ becomes $\Sigma_r(a_1, a_2, \ldots, a_h)$ and χ_r becomes S_r. On putting $\alpha_i = 1/a_i$, $\Phi(r)$ becomes $\Sigma_r(1/a_1, 1/a_2, \ldots, 1/a_h)$ and χ_r becomes S_{-r}.

If $t = m_1 a + m_2 b + m_3 c + \ldots$, the coefficient of $y^t \chi_a^{m_1} \chi_b^{m_2} \chi_c^{m_3} \ldots$ will be

$$\frac{(-1)^{m_1+m_2+m_3+\ldots}}{a^{m_1} b^{m_2} c^{m_3} \ldots m_1!\, m_2!\, m_3! \ldots}.$$ Since $\frac{t!}{(m_1 a)!\,(m_2 b)!\ldots}$ is an

* Burnside and Panton, *Theory of Equations*, Vol. I, 170-1.

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integer, so is $\frac{t!}{a^{m_1}b^{m_2}\ldots m_1!\, m_2!\ldots}$. Comparing coefficients in the equation (1)

$$\Phi(k)=1/k!\, F(\chi_1, \chi_2, \ldots, \chi_k) \qquad (2)$$

where $F(\chi_1, \chi_2, \ldots, \chi_k)$ is a polynomial in $\chi_1, \chi_2, \ldots, \chi_k$ with *integral* coefficients. Also for all k, $F(\chi_1, \chi_2, \ldots, \chi_k)$ contains χ_k only in the first degree while all the other terms are of a degree higher than the first.

Taking $\alpha_i=a_i$, $(1\leqslant i\leqslant h)$ in (2),

$$\Sigma_k{}^*=1/k!\, F(S_1, S_2, \ldots, S_k).$$

Hence if $k(>1)$ is odd and $<\pi-1$, by the lemma

$$\Sigma_k\equiv 0 \pmod{m^2},$$

since $(m, k!)=1$ and $p-1\nmid k$ for $p|m$, $p-1\nmid k-1$ for $p|m$.

3.3. Taking $\alpha_i=1/a_i$, $(1\leqslant i\leqslant h)$ in (2),

$$\Sigma_k(1/a_1, 1/a_2, \ldots, 1/a_h)=1/k!\, F(S_{-1}, S_{-2}, \ldots, S_{-k}).$$

Hence if k is odd, $1<k<\pi-2$

$$\Sigma_k(1/a_1, 1/a_2, \ldots, 1/a_h)\equiv 0 \pmod{m^2}$$

since $(k!, m)=1$, $p-1\nmid -k-1$, $p-1\nmid -k$ for $p|m$, which result is equivalent to

$$\Sigma_k\equiv 0 \pmod{m^2} \text{ for } k>h-\pi+2.$$

3.4. COR. If $m=p$

$$\Sigma_k(1, 2, 3, \ldots, p-1)\equiv 0 \pmod{p^2} \text{ for } 1<k<p,\ 2\nmid k.$$

3.5. EXAMPLES: (1) If $(m, 6)=1$, π must be >3.

$$\Sigma_k\equiv 0 \pmod{m^2} \text{ if } k>h-\pi+2$$
$$i.e. \text{ if } k>h-5+2$$
$$\text{or if } k>h-3.$$

Hence if $k=h-1$, $\Sigma_k\equiv 0 \pmod{m^2}$ which is Wolstenholme's Theorem as generalised by Leudesdorf.

(2) To determine the nature of m in order that

$$\sum\frac{1}{a_1a_2a_3}\equiv 0\ (m^2).$$

This is the same as $\Sigma_{h-3}\equiv 0\ (m^2)$. This holds if $h-3>h-\pi+2$, *i.e.* if $\pi>5$. Hence a sufficient condition is that $(m, 30)=1$.

* $\Sigma_k(a_1, a_2, \ldots, a_h)$ is denoted by Σ_k for brevity.

4. REMARKS: It is interesting to note that the following congruences and many others can be discussed in a similar way. For simplicity, let us assume that $p, q, r, \ldots$ are either all positive or all negative.

(1) $\Sigma a_1^p a_2^q \equiv 0 \pmod{m^2}$, if $2 \nmid m$, $p+q$ is odd and (i) $p+q$ is less than $\pi - 1$ in case p and q are positive (ii) $|p+q| < \pi - 2$ in case p and q are negative.

(2) $\Sigma a_1^p a_2^q a_3^r \equiv 0 \pmod{m^2}$ if $2 \nmid m$, $p+q+r$ is odd and (i) $p+q+r < \pi - 1$ in case p, q, and r are positive, (ii) $|p+q+r| < \pi - 2$ in case p, q, r are negative.

The proofs depend on the identities:

(1) $\Sigma a_1^p a_2^q = s_p s_q - s_{p+q}$

(2) $\Sigma a_1^p a_2^q a_3^r = s_p s_q s_r - s_{p+q} s_r - s_{q+r} s_p - s_{r+p} s_q + 2 s_{p+q+r}$.

ON THE EXTENSION OF A THEOREM OF CARATHEODORY IN THE THEORY OF FOURIER SERIES

By S. MINAKSHISUNDARAM, Madras University.

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I

1. The aim of this paper is to prove certain results which are converses of the following theorem in the theory of Fourier Series.

THEOREM A. *Let* $\{f_k(x)\}$, $0\leqslant x\leqslant 2\pi$, $k=1, 2\ldots$

be a uniformly bounded sequence of measurable functions convergent almost everywhere to a measurable function $F(x)$, *and let*

$$f_k(x)\sim\tfrac{1}{2}a_0^{(k)}+\sum_{n=1}^{\infty}(a_n^{(k)}\cos nx+b_n^{(k)}\sin nx)$$

$$F(x)\sim\tfrac{1}{2}A_0+\sum_{n=1}^{\infty}(A_n\cos nx+B_n\sin nx),$$

then $a_n^{(k)}\to A_n$, $b_n^{(k)}\to B_n$ *as* $k\to\infty$, *for every fixed n.*

1.1. It might be noted that this theorem is analogous to a theorem of Weierstrass on a sequence of analytic functions converging uniformly on a contour within the domain of their existence. A theorem converse to Theorem A has been stated and proved by Caratheodory,* which runs thus:

THEOREM B. *Let* $\{f_k(x)\}$, $0\leqslant x\leqslant 2\pi$,

be a uniformly bounded and non-decreasing sequence of functions. Then if $a_n^{(k)}\to A_n$ *and* $b_n^{(k)}\to B_n$ *as* $k\to\infty$, (A_n, B_n) *are the Fourier coefficients of a non-decreasing function* $F(x)$, *and* $f_k(x)\to F(x)$ *at every point* x, $0<x<2\pi$ *at which* $F(x)$ *is continuous.*

* See A. Zygmund, "*Trigonometrical Series*", p. 82.

Theorem 1 of this paper is a generalisation of Theorem A and the remaining ones are of the same type as Theorem B.

II

2. In what follows, when we write $f(x)\varepsilon L^{(r)}$, we mean that $f(x)$ belongs to the class of functions the rth powers of the modulus of which are integrable in the sense of Lebesgue, and invariably we put

$$M_r[f]=M_r[f;ab]=\left\{\int_b^a |f(x)|^r dx\right\}^{1/r}$$

In addition to the integral analogues of Hölder's and Minkowski's inequalities in series, we shall be using the following lemmas.*

LEMMA 1. *If* $f(x)\varepsilon L^r$, $r\geqslant 1$ *and*

$$f(x)\sim\tfrac{1}{2}a_0+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)$$

then $$M_r[f-\sigma_n]\to 0 \text{ as } n\to\infty$$

where

$$\sigma_n=\sigma_n(x)=\frac{s_0+s_1+\ldots+s_n}{n+1}$$

$$s_n=s_n(x)=\tfrac{1}{2}a_0+\sum_{\nu=1}^{n}(a_\nu\cos\nu x+b_\nu\sin\nu x).$$

LEMMA 2. *A necessary and sufficient condition that the Trigonometrical Series*

$$\tfrac{1}{2}a_0+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)$$

should be the Fourier Series of a function $f(x)\varepsilon L^r$, $r>1$, *is that*

$$M_r[\sigma_n]=O(1) \text{ as } n\to\infty.$$

LEMMA 3. *A necessary and sufficient condition that the Trigonometrical Series of Lemma 2 may be the Fourier series of a function* $f(x)\varepsilon L$ *is that*

$$M[\sigma_m-\sigma_n]\to 0, \text{ as } m,\ n\to\infty,\ M=M_1.$$

* For a full account of these lemmas see A. Zygmund, *Trigonometrical Series*, Ch. iv §§ 4.2, 4.3 and 4.4.

3. We now proceed to state and prove the main results of our paper.

THEOREM 1. *Let $f_k(x)\varepsilon L^{(r)}$, $r\geqslant 1$, $k=1, 2,\ldots$ and let the sequence converge in the mean to a function $F(x)\varepsilon L^{(r)}$, that is to say*

$$\lim_{k\to\infty} M_r[f_k-F]=0;$$

then, if

$$f_k(x)\sim\tfrac{1}{2}a_0^{(k)}+\sum_{n=1}^{\infty}(a_n^{(k)}\cos nx+b_n^{(k)}\sin nx)$$

$$F(x)\sim\tfrac{1}{2}A_0+\sum_{n=1}^{\infty}(A_n\cos nx+B_n\sin nx)$$

$$a_n^{(k)}\to A_n \text{ and } b_n^{(k)}\to B_n \text{ as } k\to\infty$$

for every fixed n.

PROOF: From Hölder's inequality, it follows that

$$\left|a_n^{(k)}-A_n\right|\leqslant\frac{1}{2\pi}M_r[f_k-F].M_{r'}[\cos nx]$$

$$=O\left\{M_r[f_k-F]\right\}$$

$$\to 0 \text{ as } k\to\infty,$$

where $1/r+1/r'=1$.

Therefore $a_n^{(k)}\to A_n$ as $k\to\infty$. Similarly $b_n^{(k)}\to B_n$ as $k\to\infty$.

3.1. But the converse of this theorem is not always true. In fact, if the Fourier coefficients $\left\{a_n^{(k)}, b_n^{(k)}\right\}$ of $f_k(x)$ tend to (A_n, B_n) respectively as $k\to\infty$, the series

$$\tfrac{1}{2}A_0+\sum_{n=1}^{\infty}(A_n\cos nx+B_n\sin nx)$$

may cease to be a Fourier series as can be seen from the following example. The series

$$\sum_{n=1}^{\infty}\frac{\sin nx}{\left\{\log(n+1)\right\}^{1+1/k}}$$

for every $k>0$ can be proved* to be the Fourier series of an integrable function, while the series obtained by letting $k \to \infty$ in every term is

$$\sum_{n=1}^{\infty} \frac{\sin nx}{\log (n+1)}$$

which, it is known, is not a Fourier series, since the series got by integrating term by term diverges definitely to $+\infty$, for $x=0$.

And even if it is known that $\{A_n, B_n\}$ are the Fourier coefficients of a function $F(x)$, we cannot say that $\{f_k(x)\}$ converges almost everywhere or even in the mean to $F(x)$. But with some additional conditions we can prove a theorem similar to Theorem B. We do this in the next section.

III

4. We first prove

THEOREM 2. *Let* $f_k(x) \varepsilon L^{(r)}$, $r \geqslant 1$, $k=1, 2, \ldots$

and $$\chi_n^{(k)} = M_r[f_k - \sigma_n^{(k)}].$$

$$\sin x + \sin 2x + \ldots + \sin nx = \frac{\cos \frac{1}{2}x - \cos (n+\frac{1}{2})x}{2 \sin \frac{1}{2}x}$$

$$= \frac{1 - \cos (n+\frac{1}{2})x}{2 \sin \frac{1}{2}x} - \frac{1}{2} \tan \frac{1}{4}x$$

$$= \delta_n(x) - \frac{1}{2} \tan \frac{1}{4}x \text{ (say)}.$$

Now the coefficients of the series tend monotonically to zero, so that the series converges for every x. Also by Abel's method of partial summation the series is equal to

$$\sum_{\iota=1}^{\infty} \Delta \left(\frac{1}{\{\log (n+1)\}^{1+1/k}} \right) \delta_n(x) - \tfrac{1}{2} a_1 \tan \tfrac{1}{4}x.$$

The terms of the series are non-negative and therefore the series is absolutely convergent. Also since

$$\int_0^{\pi} \delta_n(x)\,dx = O(\log n), \text{ and since}$$

$$\sum_{n=1}^{\infty} \Delta \left(\frac{1}{\{\log (n+1)\}^{1+1/k}} \right) \log n$$

converges, we infer that the above Trigonometric series is a Fourier series for every $k>0$.

Also let the Fourier coefficients $\{a_n^{(k)}, b_n^{(k)}\}$ *of* $f_k(x)$ *tend to* (A_n, B_n) *respectively as* $k\to\infty$, *for every fixed* n. *Then a necessary and sufficient condition that* $\{A_n, B_n\}$ *may be the Fourier coefficients of a function* $F(x)\varepsilon L^{(r)}$ *which is the limit in the mean of* $\{f_k(x)\}$ *is that*

$$\lim_{n\to\infty} \overline{\lim_{k\to\infty}}\, \chi_n^{(k)}=0.$$

PROOF: It might be noted that, by Lemma 1, $\chi_n^{(k)}\to 0$ as $n\to\infty$ for every fixed k. Therefore $\lim\limits_{k\to\infty}\lim\limits_{n\to\infty}\chi_n^{(k)}=0$; so that the condition of the theorem is equivalent to

$$\lim_{k\to\infty}\lim_{n\to\infty}\chi_n^{(k)}=0=\lim_{n\to\infty}\overline{\lim_{k\to\infty}}\,\chi_n^{(k)}.$$

We shall first prove the necessity of the condition. If

$$\tfrac{1}{2}A_0+\sum_{n=1}^{\infty}(A_n\cos nx+B_n\sin nx)$$

is the Fourier series of a function $F(x)\varepsilon L^{(r)}$ to which $\{f_k(x)\}$ converges in the mean, we observe by writing

$$\begin{aligned} f_k(x)-\sigma_n^{(k)}=f_k(x)&-F(x)\\ &+F(x)-\sigma_n(x)\\ &+\sigma_n(x)-\sigma_n^{(k)}(x) \end{aligned}$$

that

$$\chi_n^{(k)}\leqslant M_r[f_k-F]+M_r[F-\sigma_n]+M_r[\sigma_n-\sigma_n^{(k)}].$$

We first let $k\to\infty$. The first and the third expressions on the right side will tend to zero by hypotheses. That the first expression tends to zero is obvious, while in the third, since the polynomials $\sigma_n^{(k)}(x)$ involve the Fourier coefficients $(a_n^{(k)}, b_n^{(k)})$ linearly, $\sigma_n^{(k)}\to\sigma_n$ uniformly in $0\leqslant x\leqslant 2\pi$, for every fixed n, as $k\to\infty$; and *a fortiori* $M_r[\sigma_n-\sigma_n^{(k)}]\to 0$ as $k\to\infty$. Therefore,

$$\overline{\lim_{k\to\infty}}\,\chi_n^{(k)}\leqslant M_r[F-\sigma_n].$$

And now letting $n\to\infty$, we note that

$$\lim_{n\to\infty}\overline{\lim_{k\to\infty}}\,\chi_n^{(k)}=0.$$

To prove sufficiency, we observe that if

$$\lim_{n\to\infty} \overline{\lim_{k\to\infty}} \chi_n^{(k)}=0, \text{ then}$$

$$\lim_{m,\, n\to\infty} \overline{\lim_{k\to\infty}} M_r[\sigma_m^{(k)}-\sigma_n^{(k)}]=0$$

for

$$\sigma_m^{(k)}-\sigma_n^{(k)}=\sigma_m^{(k)}(x)-f_k(x)+f_k(x)-\sigma_n^{(k)}(x)$$

and

$$M_r[\sigma_m^{(k)}-\sigma_n^{(k)}]\leqslant\chi_m^{(k)}+\chi_n^{(k)}.$$

From this we easily deduce, since $\sigma_m^{(k)}$ and $\sigma_n^{(k)}$ tend to σ_m and σ_n respectively as $k\to\infty$ uniformly in $0\leqslant x\leqslant 2\pi$, that

$$M_r[\sigma_m-\sigma_n]\to 0 \text{ as } m,\, n\to\infty.$$

A fortiori $\qquad M_r[\sigma_n]=O(1)$ as $n\to\infty$.

Applying Lemma 2 if $r>1$, and Lemma 3 if $r=1$ we conclude that

$$\tfrac{1}{2}A_0+\sum_{n=1}^{\infty}(A_n\cos nx+B_n\sin nx)$$

is the Fourier series of a function $F(x)\varepsilon L^{(r)}$ to which $\sigma_n(x)$ converges in the mean. Now

$$\begin{aligned} f_k(x)-F(x)=f_k(x)&-\sigma_n^{(k)}(x)\\ &+\sigma_n^{(k)}(x)-\sigma_n(x)\\ &+\sigma_n(x)-F(x)\end{aligned}$$

so that

$$M_r[f_k-F]\leqslant M_r[f_k-\sigma_n^{(k)}]+M_r[\sigma_n^{k}-\sigma_n]+M_r[\sigma_n-F].$$

Letting $k\to\infty$ first and then $n\to\infty$, we note that the left side tends to zero as $k\to\infty$; and our theorem is completely proved.

4.1. The following theorem can be easily deduced from Theorem 2 as a corollary.

THEOREM 3, *Let* $f_k(x)\varepsilon L^{(r)}$, $r\geqslant 1$; $k=1, 2,\ldots$ *so that if*

$$f_k(x)\sim\tfrac{1}{2}a_0^{(k)}+\sum_{n=1}^{\infty}(a_n^{(k)}\cos nx+b_n^{(k)}\sin nx)$$

$$M_r[f_k-\sigma_n^{(k)}]\to 0, \text{ as } n\to\infty$$

by *Lemma* 1. *If this convergence is uniform with respect to k and if* $\{a_n^{(k)}, b_n^{(k)}\}$ *tend to* $\{A_n, B_n\}$ *respectively for every fixed n, as* $k\to\infty$, *then* $\{A_n, B_n\}$ *are the Fourier coefficients of a function* $F(x)\varepsilon L^{(r)}$ *and*

$$\lim_{k\to\infty} \text{ in the mean } f_k(x)=F(x).$$

PROOF: By hypothesis

$$\lim_{n\to\infty} \chi_n^{(k)}=0 \text{ uniformly in } k.$$

Therefore, given a positive number δ however small, we can find integers p and q, so that for $k>p$ and $n>q$

$$\chi_n^{(k)}<\delta.$$

This in fact implies that

$$\lim_{n,\, k\to\infty} \chi_n^{(k)}=0$$

and in particular that

$$\lim_{n\to\infty}\lim_{k\to\infty} \chi_n^{(k)}=0.$$

Now Theorem 3 follows from Theorem 2.

REMARK: Apart from the change of the order of repeated limits, the proofs of Theorems 2 and 3, depend on the use of Lemmas 1, 2 and 3. But these lemmas, it is known, remain true if we replace Cesaro's means of the first order by Cesaro's means of any order $\alpha>0$. So the above theorems remain true even when we have Cesaro's means of order α, $\alpha>0$.

THE FOURIER SERIES OF A SEQUENCE OF FUNCTIONS

BY S. MINAKSHISUNDARAM, Madras University.

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1. Let $\{f_k(x)\}$ be a sequence of functions of class L^r, $r \geqslant 1$, defined in $0 \leqslant x \leqslant 2\pi$. Let

$$f_k(x) \sim \tfrac{1}{2} a_0^{(k)} + \sum_{n=1}^{\infty} (a_n^{(k)} \cos nx + b_n^{(k)} \sin nx) \tag{1}$$

be the Fourier series of $f_k(x)$, $k=1, 2, \ldots$. Let $S_n^{(k)}(x)$ denote the sum to $(n+1)$ terms and $\sigma_n^{(k)}(x)$ the Cesaro's mean of the first order of the series (1). We write

$$\chi_n^{(k)} = M_r\{f_k - \sigma_n^{(k)}\} = \left\{ \frac{1}{2\pi} \int_0^{2\pi} |f_k(x) - \sigma_n^{(k)}(x)|^r dx \right\}^{1/r}.$$

In my earlier paper,* I have proved the following

THEOREM A. *Let $f_k(x) \varepsilon L^r$, $r \geqslant 1$, in $0 \leqslant x \leqslant 2\pi$, $k=1, 2 \ldots$. Let $a_n^{(k)} \to A_n$, $b_n^{(k)} \to B_n$, as $k \to \infty$ for each n. A necessary and sufficient condition that (A_n, B_n) should be the Fourier series of a function $F(x) \varepsilon L^r$ which is the limit in the mean of $\{f_k(x)\}$ is that*

$$\lim_{n\to\infty} \lim_{k\to\infty} \chi_n^{(k)} = 0.$$

1.1. In this paper, I propose to consider the changes necessary in Theorem A when $\sigma_n^{(k)}$ is replaced by $S_n^{(k)}$, the partial sums. Firstly it may be noted that Theorem A remains true when $r > 1$, since the corresponding lemmas on which its proof is based, remain true when partial sums replace Cesaro's means. When $r=2$, the analogue of Theorem A takes an elegant form in virtue of the Parseval and Riesz-Fischer theorems. If

$$\rho_n^{(k)} = \{a_n^{(k)}\}^2 + \{b_n^{(k)}\}^2, \quad \Delta_n^{(k)} = \sum_{\nu=0}^{n} \rho_\nu^{(k)}, \quad \rho_n = A_n^2 + B_n^2,$$

the result takes the following form.

* See pp. 314-20 of this issue.

THEOREM 1. *Let $f_k(x)$ belong to L^2, $k=1, 2, \ldots,$ and $a_n^{(k)} \to A_n$, $b_n^{(k)} \to B_n$ as $k \to \infty$. A necessary and sufficient condition that (A_n, B_n) should be Fourier coefficients of a function $F(x) \varepsilon L^2$ to which $\{ f_k(x) \}$ converges in the mean is**

$$\lim_{n\to\infty} \left\{ \lim_{k\to\infty} \Delta_n^{(k)} \right\} = \lim_{k\to\infty} \left\{ \lim_{n\to\infty} \Delta_n^{(k)} \right\} \tag{2}$$

1.2. When $\rho_n^{(k)} \to \rho_n$ monotonically as $k \to \infty$ and either $\sum_1^\infty \rho_n$ or $\lim_{k\to\infty} \sum_1^\infty \rho_n^{(k)}$ is finite, it is easy to see that (2) holds in virtue of a classical theorem on double series with positive terms. Hence under these conditions Theorem 1 remains true.

2. When $r=1$, Theorem A ceases to be true as it stands when $S_n^{(k)}$ replaces $\sigma_n^{(k)}$. However, an examination of the proof of Theorem A will show that the condition in question is sufficient though not necessary. The corresponding result with the necessary modifications runs as follows:

THEOREM 2. *Let $f_k(x) \varepsilon L$, $k=1, 2, \ldots$ and $a_n^{(k)} \to A_n$, $b_n^{(k)} \to B_n$ as $k \to \infty$. Let*

$$\delta_n^{(k)}(p) = \frac{1}{2\pi} \int_0^{2\pi} |f_k(x) - S_n^{(k)}(x)|^p dx, \; 0 < p \leqslant 1.$$

Then,

(α) *if* $\lim_{n\to\infty} \lim_{k\to\infty} \delta_n^{(k)}(1) = 0$, (A_n, B_n) *are the Fourier coefficients of a function $F(x) \varepsilon L$ such that*

$$\lim_{k\to\infty} \int_0^{2\pi} |F(x) - f_k(x)| dx = 0;$$

(β) *if, in addition, $|f_k| \log^+ |f_k| \varepsilon L$, then the condition in (α) is also necessary;*

* In the usual notation, $\lim_{m\to\infty} \lim_{n\to\infty} S_{m,n}$ does not imply that $\lim_{n\to\infty} S_{m,n}$ exists for each m. What is meant is that all the quantities $\overline{\underline{\lim}}_{m\to\infty} \overline{\underline{\lim}}_{n\to\infty} S_{m,n}$ are equal. In the above, $\{ \quad \}$ is used to denote that each limit indicated exists separately.

(γ) *if, instead of* (α), *we have*

$$\lim_{n\to\infty}\lim_{k\to\infty}\delta_n^{(k)}(p)=0$$

for some p *in* $0<p<1$, *then the partial sums* $S_n(x)$ *of the series*

$$\tfrac{1}{2}A_0+\sum_{n=1}^{\infty}(A_n\cos nx+B_n\sin nx) \tag{3}$$

converge in the mean to a function $F(x)\varepsilon L^p$ *which is the limit in the mean of* $f_k(x)$, *that is*

$$\lim_{k\to\infty}\int_0^{2\pi}|F(x)-f_k(x)|^p dx=0;$$

(δ) *if, under* (γ), *the series* (3) *is known to be the Fourier series of a function* $\phi(x)\varepsilon L$, *then* $F(x)\equiv\phi(x)$.

2.1. The proof of Theorem 2 except (δ) is, more or less, on the same lines as that of Theorem A if we use the following lemmas.*

LEMMA 1. *If* $f(x)$ *and* $|f|\log^+|f|$ *belong to* L, *then*

$$\int_0^{2\pi}|f(x)-S_n(x)|dx\to 0 \text{ as } n\to\infty.$$

LEMMA 2. *If* $f(x)$ *belongs to* L *and* $0<p<1$, *then*

$$\int_0^{2\pi}|f(x)-S_n(x)|^p dx\to 0 \text{ as } n\to\infty.$$

In these lemmas, $S_n(x)$ denotes the sum to $(n+1)$ terms of the Fourier series of $f(x)$.

2.2. The only point that requires examination in Theorem 2 is (δ). Here, we have only to note that

$$\int_0^{2\pi}|F(x)-\phi(x)|^p dx\leqslant\int_0^{2\pi}|F(x)-S_n(x)|^p dx+\int_0^{2\pi}|\phi(x)-S_n(x)|^p dx$$

since† $0<p<1$. By (γ) and Lemma (2), we conclude that

$$\int_0^{2\pi}|F(x)-\phi(x)|^p dx=0,$$

so that $F(x)\equiv\phi(x)$.

* A. Zygmund, *Trigonometrical Series*, p. 153.
† Minkowski's inequality for $0<p<1$.

SOME UNIQUENESS THEOREMS FOR FUNCTIONS OF CLASS L_p

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1. *Introduction.* Let $\{\phi_n(x)\}$, $n=1, 2, \ldots$, be a sequence of measurable functions of a real variable x defined almost everywhere in an interval $a \leqslant x \leqslant b$. Let $f(x)$ be a function belonging to the Lebesgue class $L_p (p>0)$ and let $\{\phi_n\}$ be such that all the integrals

$$c_n(f) = \int_a^b f(x)\phi_n(x)dx \tag{1}$$

exist as L-integrals. In many problems, it is important to find out whether there exists a function $f(x)$ satisfying the enumerable infinity of relations (1) when the sequences $\{\phi_n\}$ and $\{c_n(f)\}$ are given. For instance, when

$$c_n(f)=0, n=1, 2 \ldots \tag{2}$$

the problem is to determine whether a function $f(x)$ exists orthogonal to all the functions of the sequence $\{\overline{\phi_n}(x)\}$. If the only function $f(x)$ satisfying (2) is the nul function,* the sequence $\{\overline{\phi}_n\}$ is said to be closed with respect to the class L_p. Again, when

$$\phi_n(x) = x^{n-1}, n=1, 2, \ldots,$$

the question raised is the well-known problem of moments. The object of this paper is to discuss certain cases of the general problem mentioned above when $\{\phi_n\}$ is a sequence derived from an integral function and $c_n(f)$ is bounded or decreases or increases according to some specified law. Throughout this paper, it is supposed that the interval (a, b) in question is finite and the class

* $f(x)$ is said to be a nul function when it is zero almost everywhere in (a, b).

L_p is such that $p \geqslant 1$.* Since a function belonging to $L_p(p \geqslant 1)$ in a finite interval also belongs to $L_1 \equiv L$, we confine ourselves to the class L in the statement of the theorems although all the results hold for any class $L_p(p \geqslant 1)$.

1.1. *Definition and notation.* Let $g(z)$ be an integral function and $M(r, g) = \max_{|z| \leqslant r} |g(z)|$. The order ρ and the type $k(g)$ of $g(z)$ are defined by

$$\rho = \overline{\lim_{r\to\infty}} \frac{\log\log M(r, g)}{\log r}; \; k(g) = \overline{\lim_{r\to\infty}} \frac{\log M(r, g)}{r^{\rho}}.$$

When $k(g)=0$, $g(z)$ is said to be of order ρ and minimal type.

1.2. Let $\{ \lambda_n \}$ be a sequence such that

$$0 < \lambda_1 < \lambda_2 \ldots < \lambda_n \to \infty \text{ as } n \to \infty.$$

Let $n(r)$ be the number of λ_n's not exceeding r. If

$$\lim_{r\to\infty} \frac{n(r)}{r^{\rho}} = D$$

we shall say that the sequence $\{ \lambda_n \}$ is measurable† of order ρ and density D. When $\rho=1$, the index† of condensation of $\{ \lambda_n \}$ is given by

$$\delta = \overline{\lim_{n\to\infty}} \frac{1}{\lambda_n} \log \left| \frac{1}{\sigma'(\lambda_n)} \right|.$$

When $\rho \neq 1$, we denote by $\delta(\rho)$ the index of condensation of $\{ \lambda_n^{\rho} \}$ which is of order one and density D.

1.3. Let

$$g(z) = c_0 + c_1 z + c_2 z^2 + \ldots \tag{3}$$

be an integral function. Let $\{ n_j \}$, $j=1, 2, \ldots$, be those indices in (3) whose coefficients c_{n_j} do not vanish. We shall say that

* This is to ensure that all the integrals (1) exist when $\{ \phi_n \}$ is derived from an integral function.

† For the definitions of measurability and the index of condensation, see V. Bernstein, *Series de Dirichlet*, Borel Tracts (1933), 22-27. The index is non-negative and is always zero when $\underline{\lim}_{n\to\infty} (\lambda_{n+1} - \lambda_n) > 0$.

$g(z)$ is *a complete integral function* when $\sum_{j=1}^{\infty} \frac{1}{n_j}$ diverges.*
When no coefficient vanishes or when only a finite number of coefficients vanish, the integral function is always complete since $\sum_{1}^{\infty} \frac{1}{n}$ diverges. The significance of this definition will become clear from the following

LEMMA.† *Let $g(z)$ be any complete integral function. Let $f(x)$ belong to $L_p (p \geqslant 1)$ in (a, b) and*

$$G(z) \equiv \int_a^b f(x) g(zx) dx \equiv 0. \tag{4}$$

Then $f(x)$ is a nul function in (a, b).

PROOF: Substituting (3) in (4), we get

$$c_n \int_a^b f(x) x^n dx = 0, \; n = 0, 1, 2, \ldots . \tag{5}$$

Let $\{n_j\}$ be the indices of the non-vanishing coefficients in (3). Then, by (5)

$$\int_a^b f(x) x^{n_j} dx = 0, \; j = 1, 2, \ldots \tag{6}$$

since $c_{n_j} \neq 0$. But, by the definition of a complete integral function, $\sum^{\infty} \frac{1}{n_j}$ diverges so that, by a theorem due to Muntz,‡ the relations (6) involve the consequence that $f(x)$ is nul in (a, b). This proves the lemma. It may be noted, in virtue of Muntz's theorem mentioned above, that $f(x)$ need not be nul when $g(z)$ is not a complete integral function. The lemma just proved is the basis of all the later results of this paper.

2. We now proceed to the theorems of this paper. First, we consider integral functions of order one.

* We suppose $n_1 < n_2 < \ldots$ and $n_1 \neq 0$ since if $n_1 = 0$, $\sum_{1}^{\infty} 1/n_j = \infty$.

† Dr. Szasz has kindly pointed out to me that the above lemma is not entirely new but has been noticed earlier by Hille and Tamarkin, *Acta Mathematica*, 57 (1931), p. 70, Lemma 11.1.

‡ See Kaczmarz and Steinhaus, *Theorie der Orthogonal Riehen*, p. 92.

THEOREM 1. *Let $g(z)$ be a complete integral function of order one and type not exceeding one. Let it be bounded on the imaginary axis. Let $\{\lambda_n\}$ be a measurable sequence of order one, density $D>0$, and with index of condensation zero. Let $f(x)$ belong to L in $(-a, a)$ and*

$$c_n(f)=\int_{-a}^{a} f(x)g(\lambda_n x)\,dx,\ n=\pm 1, \pm 2, \ldots$$

where $\lambda_{-n}=-\lambda_n$. If

$$c_n(f)=O(|\lambda_n|^k), \tag{7}$$

for some fixed k and $a<\pi D$, then $f(x)$ is nul in $(-a, a)$.

PROOF: There is no loss of generality if the theorem is proved under the hypothesis that k is a positive integer. Let

$$G(z)=\int_{-a}^{a} f(x)g(zx)\,dx. \tag{8}$$

In virtue of the hypothesis on $g(z)$, we conclude that $G(z)$ is an integral function of order one, type not exceeding a and bounded along the imaginary axis. Moreover, by (7)

$$G(\lambda_n)=O(|\lambda_n|^k),\ n\to\pm\infty$$

so that the integral function

$$G_1(z)=\frac{G(z)-G(0)-G'(0)z-\ldots-\dfrac{G^{(k-1)}(0)}{(k-1)!}z^{k-1}}{z^k}$$

satisfies the conditions:

(i) $G_1(z)$ is of order one and type not exceeding a;

(ii) $G_1(z)$ is bounded along the imaginary axis;

and (iii) $G_1(\lambda_n)=O(1)$ as $n\to\pm\infty$.

Moreover, $a<\pi D$ by hypothesis. Therefore, in virtue of a theorem I have proved elsewhere,* the function $G_1(z)$ must reduce to a constant so that $G(z)$ reduces to a polynomial of degree not exceeding k. Hence, differentiating (8) $(k+1)$ times, we get

$$G^{(k+1)}(z)\equiv\int_{-a}^{a} x^{k+1}f(x)g^{k+1}(zx)\,dx\equiv 0.$$

Since $g(z)$ is complete, the same is true of $g^{k+1}(z)$. Hence by the lemma of § 1.3, $x^{k+1}f(x)$ is nul in $(-a, a)$ and therefore $f(x)$ is also nul in $(-a, a)$.

* *Annals of Mathematics,* Vol. 38 (1937), 313, Theorem 2.

2.1. Theorem 1 can be stated in another form which is, perhaps, more significant.

THEOREM 2. *Let* $g(z)$, $\{\lambda_n\}$ *be as in Theorem* 1. *Let* $f(x)$ *be a non-nul function of* L *in* $(-a, a)$. *Then, however large* k *be,*

$$\overline{\lim_{n\to\infty}} \frac{1}{\lambda_n^k} \left| \int_{-a}^{a} f(x) g(\pm\lambda_n x) dx \right| = \infty,$$

provided $a < \pi D$.

2.2. As a special case, we can take $g(z) = e^z$. Then we shall have

THEOREM 3. *Let* $0 < a < \pi$ *and* $f(x)$ *be a non-nul function of* L *in* $(-a, a)$. *Then*

$$\overline{\lim_{n\to\infty}} \frac{1}{n^k} \left| \int_{-a}^{a} f(x) e^{\pm nx} dx \right| = \infty,$$

however large k *be.*

3. We now consider sequences derived from integral functions of order other than one. Here, the statement of the results analogous to Theorem 1 becomes complicated except when $0 < \rho \leqslant \frac{1}{2}$. Therefore we confine ourselves to this case.

THEOREM 4. *Let* $g(z)$ *be a complete integral function of order* $\rho (0 < \rho \leqslant \frac{1}{2})$ *and type not exceeding one: Let* $\{\lambda_n\}$ *be a measurable sequence of order* ρ *and density* $D > 0$ *such that* $\delta(\rho) \leqslant \pi D \cot \pi\rho$. *Let* $f(x)$ *belong to* L *in* $(-a, a)$ *and*

$$c_n(f) = \int_{-a}^{a} f(x) g(\lambda_n x) dx, \; n = \pm 1, \pm 2, \ldots$$

where, as before, $\lambda_{-n} = -\lambda_n$. *If*

$$c_n(f) = O(|\lambda_n|^k)$$

for some fixed k *and* $a^\rho < \pi D \operatorname{cosec} \pi\rho$, *then* $f(x)$ *is a nul function.*

PROOF: As before, we set

$$G(z) = \int_{-a}^{a} f(x) g(zx) dx.$$

Then

(i) $G(z)$ is an integral function of order ρ and type not exceeding a^ρ;

(ii) $G(\lambda_n) = O(|\lambda_n|^k)$;

(iii) and, by hypothesis, $a^\rho < \pi D$ cosec $\pi\rho$ and $\delta(\rho) \leqslant \pi D \cot \pi\rho$.

sing these properties of $G(z)$ and proceeding as in Theorem 1, we conclude, by using a theorem* analogous to that used in the proof of Theorem 1, that $G(z)$ is a polynomial of degree k at most. Hence, as in Theorem 1, we find that $f(x)$ is a nul function.

4. We now consider cases where $c_n(f)$ tends to zero according to some law. We prove the following general

THEOREM 5. *Let $g(z)$ be any complete integral function of finite order $\rho>0$ and type not exceeding one. Let $\{\lambda_n\}$ be a measurable sequence of order ρ and density D. Let $f(x)$ belong to L in $(-a, a)$ and*

$$c_n(f)=\int_{-a}^{a} f(x)g(\lambda_n x)\,dx,\ n=1, 2,\ldots$$

Let

$$\overline{\lim_{n\to\infty}}\ \frac{1}{\lambda_n^\rho} \log |c_n(f)|=-d<0$$

and

$$\eta(\rho)=\begin{cases}\min.\ \{d-\delta(\rho),\ \pi D\},\ \rho\geqslant\frac{1}{2};\\ \min.\ \{d-\delta(\rho)+\pi D\cot \pi\rho,\ \pi D \operatorname{cosec} \pi\rho\},\ 0<\rho<\frac{1}{2}.\end{cases}$$

Then, if $a^\rho<\eta(\rho)$, $f(x)$ is a nul function.

PROOF: Let

$$G(z)=\int_{-a}^{a} f(x)g(zx)\,dx.$$

Then, $G(z)$ is an integral function of order ρ and type not exceeding a^ρ. By hypothesis,

$$\overline{\lim_{n\to\infty}}\ \frac{1}{\lambda_n^\rho} \log |G(\lambda_n)|=-d$$

and $a^\rho<\eta$. Hence it follows† from a known result that $G(z)$ is identically zero. Hence, as in § 2, $f(x)$ is a nul function.

4.1. When $g(z)$ is at most of order ρ and minimal type the interval $(-a, a)$ in Theorem 5 can be replaced by any finite interval. Hence we get

* *Annals of Mathematics*, *l.c.*, p. 317, Theorem 5. It must be noted that the quantity $\delta(\rho)$ used in that place is equal to the '$\delta(\rho)$' in this paper diminished by $\pi D \cot \pi\rho$ (See *Proc. Lond. Math. Soc.* (2), 43 (1937), 64 Lemma 1).

† V. Ganapathy Iyer, *Quarterly Journal of Mathematics*, (Oxford Series) 8 (1937), p. 136, Theorem 4, p. 138, Theorem 8 and remarks in § 5.1.

THEOREM 6. *Let $g(z)$ be a complete integral function of order ρ and minimal type. Let $\{\lambda_n\}$ be as in Theorem 5 and*

$$c_n(f)=\int_a^b f(x)g(\lambda_n x)dx,\ n=1,2,\ldots .$$

where (a, b) is any finite interval. Then, if η (as defined in Theorem 5) be positive, $f(x)$ is a nul function.

4.2. Adopting the alternative enunciation of Theorem 2, we get as a special case of Theorem 5 with $\delta(\rho)=0$, $d=\pi D$,

THEOREM 7. *Let $g(z)$, $\{\lambda_n\}$ be as in Theorem 5 and $\delta(\rho)=0$. Let $f(x)$ belong to L in $(-a, a)$ and*

$$c_n(f)=\int_{-a}^{a} f(x)g(\lambda_n x)dx,\ n=1,2,\ldots .$$

Then, if $f(x)$ is a non-nul function in $(-a, a)$,

$$\overline{\lim_{n\to\infty}}\frac{1}{\lambda_n^\rho}\log|c_n(f)|\geqslant\begin{cases}-\pi D\operatorname{cosec}\pi\rho & (0<\rho<\frac{1}{2}),\\ -\pi D & (\rho\geqslant\frac{1}{2}).\end{cases}\tag{9}$$

provided

$$a^\rho<\begin{cases}\pi D\operatorname{cosec}\pi\rho & (0<\rho<\frac{1}{2}),\\ \pi D & (\rho\geqslant\frac{1}{2}).\end{cases}\tag{10}$$

PROOF: If

$$\overline{\lim_{n\to\infty}}\frac{1}{\lambda_n^\rho}\log|c_n(f)|=-d$$

and does not satisfy (9), then it is easy to see by using (10) that $a^\rho<\eta$ where η is defined as in Theorem 5 so that $f(x)$ is a nul function which is contrary to the hypothesis.

4.3. The following special case of Theorem 7 with $g(z)=e^{iz}$, $\lambda_n=n$, is worthy of notice.

THEOREM 8. *Let $0<a<\pi$. Then*

$$\overline{\lim_{n\to\infty}}\frac{1}{n}\log\left|\int_{-a}^{a} f(x)e^{inx}dx\right|\geqslant-\pi,\ n=1,2,\ldots$$

where $f(x)$ is any non-nul function of x in $(-a, a)$.

4.4. REMARK: Theorem 8 is interesting when compared with the properties of the Fourier coefficients of a function of L in $(-\pi, \pi)$. Let

$$c_n(f, a)=\int_{-a}^{a} f(x)e^{inx}dx$$

where $f(x)$ belongs to L in $(-\pi, \pi)$ and $0<a\leqslant\pi$. By the Riemann-Lebesgue theorem,*

$$c_n(f, a)\to 0 \text{ as } n\to\infty.$$

It is known* that when $a=\pi$, that is, when $c_n(f)$ are the actual Fourier coefficients of f in $(-\pi, \pi)$, nothing further could be stated regarding the order of decrease of $c_n(f, \pi)$. On the other hand, Theorem 8 gives a lower bound for the order of decrease of $c_n(f, a)$ when $0<a<\pi$ and $f(x)$ is *any* non-nul function in $(-a, a)$. That this ceases to be true when $a=\pi$ can also be seen by considering the series

$$f(x)=\sum_{n=1}^{\infty} \exp(-n \log n)e^{-inx}. \tag{11}$$

The right side of (11) converges absolutely and uniformly for all x and the sum $f(x)$ is a continuous periodic function of x with period 2π. Since $f(0)\neq 0$, $f(x)$ is not a nul function while

$$c_n(f, \pi)=2\pi \exp(-n \log n)$$

so that

$$\lim_{n\to\infty} \frac{1}{n} \log |c_n(f, \pi)|=-\infty.$$

Hence Theorem 8 ceases to be true when $a=\pi$. In this sense, Theorem 8 and hence Theorem 5 also, is a best possible result. It may also be noted that Theorem 8 remains true if $(-a, a)$ is replaced by any interval of length *less* than 2π since $|e^{ix}|=1$ when x is real.

* *Cf* Titchmarsh, *Theory of Functions*, p. 403, § 13.21 and p. 425, § 13.7.

A THEOREM ON CONGRUENCE

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1. The object of this note is to prove the following

THEOREM 1. *Let p be an odd prime. Then*

$$1^{p-1}+2^{p-1}+\ldots+(p-1)^{p-1}-p-(p-1)!\equiv 0 \pmod{p^2}. \quad (1)$$

1.1. The point in the above theorem is seen if we write the left side of (1) in the form

$$(1^{p-1}-1)+(2^{p-1}-1)+\ldots+[(p-1)^{p-1}-1]-[(p-1)!+1]$$

which is $\equiv 0 \pmod{p}$ by Fermat's and Wilson's theorems. The above theorem states that the left side of (1) is divisible by p^2.

1.2. To prove the above we first establish the following.

LEMMA: *Let p, s, r be integers. Then**

$$\binom{p-s}{r}-(-1)^r\binom{r+s-1}{r}\equiv 0 \pmod{d},$$

where d is any prime factor of p greater than r.

PROOF: We have

$$\binom{p-s}{r}-(-1)^r\binom{r+s-1}{r}$$

$$=\frac{1}{r!}\Big\{(p-s)(p-\overline{s+1})\ldots(p-\overline{s+r-1})$$

$$-(-1)^r s(s+1)\ldots(s+r-1)\Big\}$$

$$=\frac{1}{r!}M(p) \quad (2)$$

where $M(p)$ means a multiple of p. The lemma follows immediately from (2).

1.3. PROOF OF THEOREM 1. We have the identity

$$(n-1)^{n-1}-\binom{n-1}{1}(n-2)^{n-1}+\ldots$$

$$+(-1)^{n-2}\binom{n-1}{n-2}1^{n-1}=(n-1)!, \quad (3)$$

* $\binom{n}{r}$ means the binomial coefficient $\dfrac{n(n-1)\ldots(n-r+1)}{r!}$.

for, the left side of (3) is the coefficient of $\frac{t^{n-1}}{(n-1)!}$ in

$$e^{(n-1)t}-\binom{n-1}{1}e^{(n-2)t}+\ldots$$
$$+(-1)^{n-2}\binom{n-1}{n-2}e^{t}=(e^{t}-1)^{n-1}-(-1)^{n-1};$$

that is $(n-1)!$. Now (3) can be written as

$$(n-1)^{n-1}+(n-2)^{n-1}+\ldots+1^{n-1}$$
$$-\Big\{[(n-2)^{n-1}-1]\Big[\binom{n-1}{1}+1\Big]-[(n-3)^{n-1}-1]\Big[\binom{n-1}{2}-1\Big]$$
$$+\ldots+(-1)^{n-3}[1^{n-1}-1]\Big[\binom{n-1}{n-2}-(1)^{n-2}\Big]\Big\}$$
$$-\Big\{\binom{n-1}{1}-\binom{n-2}{2}\ldots+(-1)^{n-3}\binom{n-1}{n-2}+n-2\Big\}$$
$$=(n-1)^{n-1}+(n-2)^{n-1}+\ldots+1^{n-1}-M(n^2)-n+(1-1)^{n-1}$$
$$=(n-1)!, \tag{4}$$

since $\binom{n-s}{s}-(-1)^{s}\equiv 0 \pmod n$, n being an odd prime, by the lemma and $x^{n-1}-1\equiv 0 \pmod n$ when x is prime to n by Fermat's theorem. The result follows from (4).

2. THEOREM 2. *Let m and n be prime. Then*

$$\binom{m+n-p}{r}-\binom{m-p}{r}-\binom{n-p}{r}$$
$$+(-1)^{r}\binom{r+p-1}{r}\equiv 0 \pmod{mn},$$

provided m and n are greater than r.

PROOF: We have

$$\binom{m+s}{r}-\binom{s}{r}\equiv 0 \pmod m \tag{5}$$

as in the lemma of § 1.2 and by that lemma

$$\binom{m-p}{r}-(-1)^{r}\binom{r+p-1}{r}\equiv 0 \pmod m. \tag{6}$$

and hence taking $s=n-p$ in (5), we get, by (5) and (6)

$$\binom{m+n-p}{r}-\binom{m-p}{r}-\binom{n-p}{r}$$
$$+(-1)^{r}\binom{r+p-1}{r}\equiv 0 \pmod m.$$

Similarly the same expression $\equiv 0 \pmod n$. So the theorem is proved.

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