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INTUITIONISTIC THEORY OF LINEAR ORDER

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Introduction

The two principal features of Intuitionistic Mathematics are the rejection of the unlimited validity of the principle of the excluded middle, and the requirement that mathematical proofs should be constructive and not merely based on *reductio ad absurdum.* Since most of the methods of proof in classical mathematics necessitate dealing with alternatives and therefore involve, in some sense, the principle of the excluded middle, the question may be raised as regards the method that intuitionists adopt in such instances. For the intuitionists, the unending sequence of positive integers is fundamental as it is revealed in the primordial time-consciousness; and the principle of generation of these integers is essentially of the same nature as the principle of mathematical induction. Thus, sets which are effectively enumerable occupy a special place in Intuitionism, and when we speak of 'all' the elements of such sets, the word 'all' has a definite content, which it fails to carry when applied to infinite totalities of some indefinite nature. Hence the methods of proof employed in the case of enumerable sets cannot be transferred, as they are, to the case of infinite sets in general.

The proofs that intuitionists employ are generally of two sorts. (1) If a property requires to be proved, and there are only a finite number of possibilities, then the intuitionist will accept a proof as constructive if and only if a systematic proof is constructed for everyone of the possibilities, even though we may not give a systematic process to specify which of those possibilities exactly takes place. (2) In case, the number of possibilities is *enumerably* infinite, then the property is said to be proved, if it is proved for the possibility 1, and if it can be proved for the possibility $r+1$ whenever it is proved for r. The possibilities being enumerable, an exhaustion of these possibilities step-by-step is equivalent to proving the result in 'all' possibilities. Proofs of this kind are intuitionistically valid since the principle of induction used therein is a basic intuitionistic idea, even though the demand of specifiability associated with the strictest meaning of 'or' is not fully complied with.[4]

In a recent paper [3] contributed to *The Mathematics Student*, I gave an outline of intuitionistic set-theory which typically exhibits those principal features that I have just now mentioned. The present paper is devoted to an analysis of Brouwer's theory of virtual order in its formal aspects, and contains some theorems concerning the nature of the *full product* of a *fundamental series* of virtually ordered sets, which are slightly more general than those of Brouwer [1] for the full product of sets of integers. The methods of proof employed are of the two kinds above described.

§ 1. Virtual Order

1·1. *A* set P is *virtually ordered*, if for the element pairs (a, b) of P we can define an ordering relation, denoted by $a<b$, such that the following postulates are satisfied.

P_1. $r=s$, $r<s$, $r>s$ are mutually exclusive.

P_2. If $r=u$, $s=v$, and $r<s$, then $u<v$.

P_3. If $r>s$ is impossible, and $r<s$ is impossible, then $r=s$.

P_4. If $r>s$ is impossible, and $r=s$ is impossible, then $r<s$,

P_5. If $r>s$ and $s>t$, then $r>t$.

From P_1, P_3, P_4, we observe that the relations $=$, $>$, $<$ are normal, that is, identical with their double negations. For, from P_1 we see that $r=s$ implies $r \nless s$ and $r \ngtr s$; while from P_3, we see that $r \ngtr s$ and $r \nless s$ imply $r=s$; so that, $r=s$ is the same as '$r \nless s$ and $r \ngtr s$', which, being the product of two negative propositions, is normal. Similarly, $>$, $<$ are also normal.

In fact, we can replace P_3 by the statement

P_3' : $=$ is a normal relation,

and deduce P_3 as a consequence, from the other postulates and P_3'. For, if $=$ is normal, then it is identical with the absurdity of its absurdity; and by P_4, $r \ngtr s \cdot r \nless s$ imply the absurdity of the absurdity of $r=s$; that is, they imply $r=s$. So P_3 is deduced.

Similarly, P_4 can be replaced by the statement,

P_4' : $<$ is a normal relation,

and P_4 can be deduced as a consequence, from P_1, P_2, P_3 and P_4'.

It should, however, be noted that we cannot simultaneously replace P_3 and P_4 by P_3' and P_4' respectively. As an example, consider the virtually ordered set M, consisting of elements m_r. Consider the set N of pairs of elements m_r. Define an ordering relation in N thus:

$$(m_r, m_s) > (m_t, m_u)$$

if and only if $m_r > m_t$ and $m_s > m_u$.

Then, the relation $>$ in N, satisfies P_1, P_2, P_5. Also, it is normal since it is the product of two normal relations $m_r > m_t$ and $m_s > m_u$. Equality is similarly normal. Thus P_3' and P_4' are satisfied. But it does not satisfy P_4 or P_3, as in the case $m_r > m_t$, $m_s = m_u$.

1·2. Brouwer generally derives a *virtual order* in a set from some previously defined, asymmetric, transitive relation satisfying certain requirements. A formal view of his derivation is as follows:

Let the elements of a set M be the field of a binary relation $\circ\!>$ satisfying

L_1. $a \circ\!> b$, $a = b$, $a <\!\circ\, b$ are mutually exclusive.

L_2. If $a = b$ and $c = d$, then $a <\!\circ\, c$ implies $b <\!\circ\, d$.

L_3. If $a \circ\!> b$ is absurd, and $a <\!\circ\, b$ is absurd, then $a = b$.

L_4. If $a \circ\!> b$, and $b \circ\!> c$, then $a \circ\!> c$.

Denote the negation of $\circ\!>$ by $\leqslant\!\circ$, and its negation (i.e. negation of $\leqslant\!\circ$) by $>$. Then, we may shew, as follows, that the relation $>$ satisfies the conditions P_1 to P_5.

P_1: From L_1 we observe that $a \circ\!> b$, $a = b$, $a <\!\circ\, b$ are mutually exclusive, and so their double negations are also mutually exclusive. The double negation of $\circ\!>$ is $>$, and the double negation of $=$ is itself, since equality is normal by L_3. Hence $a > b$, $a = b$, $a < b$ are mutually exclusive, which is P_1.

P_2: If $a = b$, and $c = d$, then $a < c$ implies $b < d$. For, by L_2, we see that $a <\!\circ\, c$ implies $b <\!\circ\, d$, and therefore the double negation of $a <\!\circ\, c$ implies the double negation of $b <\!\circ\, d$. That is, $a < c$ implies $b < d$.

P_3: $a \geqslant b$ and $a \leqslant b$ imply $a = b$. This follows from L_3, if we note that $\geqslant$ is the same as $\circ\!\geqslant$.

P_4: $a \leqslant b$ and $a \neq b$ imply $a < b$. Since $a \leqslant b$ is the same as $a \leqslant^0 b$, suppose that $a \leqslant^0 b$ and $a \neq b$. Then, if $a \,{}^0\!\geqslant b$, by L_3 we have $a = b$, which contradicts $a \neq b$. Hence $a \,{}^0\!\geqslant b$ is impossible. So $a < b$.

P_5: If $a > b$, and $b > c$, then $a > c$. To prove this, we assume $a > b$, $b > c$, and prove that $a \neq c$ and $a \,{}^0\!\geqslant c$, which together imply $a > c$, by P_4. Firstly, $a \neq c$. For, if $a = c$, then, since $b > c$, we would have $b > a$ by P_2, which contradicts the hypothesis. Secondly, $a \,{}^0\!\geqslant c$. For, if $a <^0 c$, then $b \,{}^0\!\geqslant a$, for, otherwise $b <^0 c$ which contradicts $b > c$. On the other hand, since $a > b$, we have $a \,{}^0\!\geqslant b$. Thus the assumption $a <^0 c$ leads to $b \,{}^0\!\geqslant a$ and $a \,{}^0\!\geqslant b$; that is, $a = b$, which contradicts the hypothesis $a > b$. Thus $a <^0 c$ is impossible. So $a \,{}^0\!\geqslant c$, which is the same as $a \geqslant c$.

N.B.—Since the virtual order '$>$' is obtained only as the double negation of the given asymmetric transitive relation $\,{}^0\!>$, it follows that, even if we had taken another relation $\,{}^{00}\!>$ which implies $>$ and is implied by $\,{}^0\!>$, and which satisfies the postulates L_1 to L_4, then it follows from a known property of normal elements that the double negation of that relation also will define the same virtual order.

1·3. (1) A *virtually ordered* set is said to be *ordered*, if we have a systematic process for determining which of two *different* elements r, s is the greater.

A set is said to be *discrete* if we can determine by a systematic process whether any two given elements are equal or not.

A *discrete ordered* set is said to be *completely ordered*. This notion is introduced in order to bring out the speciality of the set of integers among virtually ordered sets in general. The principle of generation of the integers contains a general method for determining whether two integers are equal, and if not, which of them is the greater.

We can also see that a *virtually ordered* set is not necessarily *ordered*. Suppose k_1 is the rank of that digit in the decimal expansion of π at which, for the first time, the sequence 0123456789 commences. Suppose $a > b$ if k_1 exists, or it is impossible that k_1 does not exist, and $a < b$ if it is impossible for k_1 to exist. Now the set consisting of the two different elements a, b is virtually ordered. But it is not ordered since we cannot decide which of them is the greater one.

If a, b are two elements of a virtually ordered set P, then the *closed interval* $[a, b]$ means the set of elements c of P for which neither $c < a$, $c < b$ nor $c > a$, $c > b$, can hold. a and b are called the end elements of $[a, b]$.

By the *open interval* (a, b) is meant the set of elements c of P which lie between a and b; that is, firstly they are different both from a and from b, and secondly they belong to the closed interval $[a, b]$. If $a < b$, then $a < c < b$.

The virtually ordered set P is *everywhere dense* if, between any two different elements of P there exist elements of P, and *everywhere dense in a restricted sense* if, in addition, an element of the set can be constructed and, to the left and right of any element of P, there exist other elements.

This latter notion is introduced to bring out the speciality of rational numbers among everywhere-dense sets in general. This can be seen easily by considering the rational numbers as pairs of mutually prime integers. Since the integers are themselves *completely ordered*, we can decide whether two pairs are identical or different, and hence the set is *discrete*. It is also *ordered* for, given two *different* elements $\frac{p}{q}, \frac{r}{s}$, we can verify whether $ps - rq$ is positive or negative, and say that $\frac{p}{q}$ is the greater if $ps - rq$ is positive. Thus, the set of rational numbers is *completely ordered*. It is *everywhere dense* since the middle point, for instance, of any interval with rational end elements, is a rational number. It is dense in the restricted sense, since we can produce a rational number from a pair of relatively prime integers, and obtain other elements which are greater, or less than the given one.

If between two virtually orderd sets P and Q, a biunivocal correspondence is established which leaves the order invariant, then we say that P and Q possess the same *order-type*, or they are *similar*.

The set of positive integers in their natural order is said to possess the order-type ω, and ordered sets of order type ω are called *fundamental series*.

In a virtually ordered set M, we have a *divergent net of closed intervals* $i_1, i_2, \ldots$ where $i_{\nu+1} \subset i_\nu$ if to each p there is a ν_p such that p cannot belong to i_{ν_p}; and a *convergent net with kernel* q, if every two intervals of the series are different, q belongs to every interval, and any element belonging to every interval is identical with q.

An element which is the *kernel* of a convergent net is called *principal*. If all elements are principal, then M is *dense-in-itself.*

If in M it is impossible for a divergent net to exist, then M is *closed*. M is *perfect*, if it is *closed* and *dense-in-itself.*

1·4. Examples:

The set of positive integers is completely ordered and nowhere dense, while the set of rational numbers is also completely ordered but everywhere dense. The set of positive integers is closed, but not dense-in-itself.

The set of positive integers (k_n) which are the ranks of the digits at which the sequence 0123456789 commences in the decimal expansion of π, is completely ordered.

We can define a set which is everywhere dense, but not dense in a restricted sense.

Let M be the set of rational numbers greater than an element of the set (k_n). Then M is everywhere dense, but it is not dense in a restricted sense, since we cannot construct any element of the set for the simple reason that we do not know whether the set (k_n) is null or not.

§ 2. Full Product of Virtually ordered Sets

Consider a finite number or a fundamental series of virtually ordered sets $M_1, M_2, M_3, \cdots$ which are non-null. Let the order types of these be $m_1, m_2, m_3, \cdots$ Denote an element of M_ν by p_ν and consider the set $M \equiv V(\cdots\cdots, M_3, M_2, M_1)$ of complexes $f \equiv (p_1, p_2, \cdots\cdots)$. The set M is called *the full product of factors* $\cdots\cdots M_3, M_2, M_1$, when it is virtually ordered as follows:

$f' \circ\!> f''$ if there exists an r such that $p_r' > p_r''$ and $p_\nu' \geqslant p_\nu''$ for $\nu < r$. Then, the relation $\circ\!>$ satisfies L_1, L_2, L_3, L_4. Denote the double negation of $\circ\!>$ by $>$. Then the new relation '$>$' defines a virtual order, and $\geqslant$ defined as negation of $<$, is the same as $\circ\!\geqslant$.

The order type of M depends only on the order types of the M_ν's and is denoted by $V(\cdots\cdots m_3, m_2, m_1)$ and called the *full product* of factors $\cdots\cdots m_3, m_2, m_1$.

The full product is *associative*; that is, the sequence, ... M_3, M_2, M_1, can be divided into groups $N_r = (M_{\nu_{r-1}+1}, \ldots M_{\nu_r})$, where $\nu_1 = 1$, ν_2, ν_3, . . . is a finite or infinite sequence of increasing positive integers; every N_r and every product of N_r's like $V(N_r, N_{r-1}, \ldots N_1)$ can be virtually ordered according to the order-laws in M_ν. On the other hand, the original full-product M can be considered as the full product of M_ν's or as the full product of N_r's, and whichever point of view is adopted, the same virtual order results; and this is what is meant by *associativity*.

Let any element of M be expressed as $(z_1, z_2, \ldots)$ where z_ν, for any ν, is an element of the virtually ordered set N_ν. This leads to a virtual order in M, the relations of which can be denoted by $\dot{=}$, $\circ\dot{>}$, $\dot{>}$, while the relations $=$, $\circ>$, $>$ pertain to the initial virtual order.

The equivalence of $=$ and $\dot{=}$ is evident. We shall prove that $\leqslant$ and $\dot{\leqslant}$ are also equivalent.

Firstly we prove that $f' \leqslant f''$ implies $f' \dot{\leqslant} f''$. For that, suppose $f' \leqslant f''$. Then, for any r, $(z_1', z_2', \ldots z_r') \leqslant (z_1'', \ldots z_r'')$, for, $(z_1', \ldots z_r') \circ> (z_1'', \ldots z_r'')$ would imply $f' \circ> f''$ which contradicts $f' \leqslant f''$. Next assume that, if possible, $z_1' \geqslant z_1''$, $z'_{r-1} \geqslant z''_{r-1}$ $z_r' > z_r''$ hold simultaneously. Then, we have,

(i) $(z_1', \ldots z_r') <\circ (z_1'', \ldots z_r'')$ is impossible, for if it were true, then $z_\nu' <\circ z_\nu''$ for at least one $\nu \leqslant r$ which would contradict the assumption made.

(ii) $(z_1', \ldots z_r') = (z_1'', \ldots z_r'')$ is impossible, since $z_r' > z_r''$.

Thus, we have, $(z_1', \ldots z_r') > (z_1'', \ldots z_r'')$ since $\leqslant\circ$ is the same as $\leqslant$, and this contradicts the hypothesis.

Hence our assumption that

$$z_1' \geqslant z_1'', \; z_2' \geqslant z_2'', \quad z_{r-1}' \geqslant z''_{r-1}, \, z_r' > z_r''$$

hold simultaneously for any r is absurd. That is, $f' \dot{>} f''$ is absurd. Therefore $f' \dot{\leqslant} f''$.

Next, we prove that $f' \dot{\leqslant} f$ implies $f' \leqslant f''$. By the above reasoning, $f' < f''$ implies $f' \dot{\leqslant} f''$, and we know that $f' < f''$ implies

$f' \neq f''$. Thus, $f' < f''$ implies that $f' \dot{<} f''$. In other words, $f' \dot{\geqslant} f''$ implies $f' \geqslant f''$. By an interchange of f' and f'', we see that, $f' \dot{\leqslant} f''$ implies $f' \leqslant f''$.

Hence $\geqslant$ and $\dot{\geqslant}$ are equivalent. It follows, therefore, that the full product is associative,

N. B. We can define $f' \circ\circ> f''$ if $p_r' > p_r''$ and $p_{\nu'} - p_{\nu''}$ for $\nu < r$. This is obviously an asymmetric transitive relation, satisfying $L_1 - L_4$. Also $f' \circ\circ> f''$ implies $f' \circ> f''$, and the negations of both are the same. This exemplifies the remark at the end of 1·2.

§ 3. Theorems on the Full-Product

THEOREM I. *The full product of a fundamental series of virtually ordered sets is everywhere dense provided we are certain that any M_r possesses one of the two properties viz., (A) to every element of M_r we can construct a greater element, or (B) to every element of M_r we can construct a smaller element.*

Proof: Let f', f'' denote two elements of N, where $f' = (m_1', m_2', \cdots)$; $f'' = (m_1'', m_2'', \cdots\cdots)$. Assume that f' is different from f''. That is, it is impossible that $m_r' = m_r''$ for every r. So we can say that $m_1' = m_1''$, $m_2' = m_2''$, $\cdots$ $m_k' = m_k''$, and $m_{k+1}' > m_{k+1}''$ or $m_{k+1}' < m_{k+1}''$ hold simultaneously for *some* value of $k = 0$, or 1 or 2 $\cdots$, since these exhaust all the possibilities step by step, though we may not be able to *specify* the value. Let it be true for $k = \nu - 1$, where ν is not previously given, but may take any positive integral value. For every such possible value of ν, we shall show a regular method of constructing $f''' \equiv (m_1''', m_2''', \cdots\cdots)$ so as to lie between f' and f''.

The components of f''' are constructed by the following specification:

$$m_s''' = m_s' = m_s'' \text{ for } s \leqslant \nu - 1.$$

If $m_\nu' < m_\nu''$, and ν is such that $M_{\nu+1}$ possesses the property (A), then $m_\nu''' = m_\nu'$, and $m'''_{\nu+1}$ is an element of $M_{\nu+1}$ constructed such that it is $> m'_{\nu+1}$.

If $m_\nu' < m_\nu''$, and ν is such that $M_{\nu+1}$ possesses the property (B), then $m_\nu''' = m_\nu''$, and $m'''_{\nu+1}$ is an element constructed so as to be $< m_{\nu+1}''$.

If $m_\nu'' < m_\nu'$, and ν is such that $M_{\nu+1}$ possesses the property (*A*), then $m_\nu''' = m_\nu''$, $m_{\nu+1}'''$ is an element constructed so as to be $< m_{\nu+1}''$.

If $m_\nu'' < m_\nu'$, and ν is such that $M_{\nu+1}$ possesses the property (*B*), then $m_\nu''' = m_\nu'$, $m_{\nu+1}'''$ is an element constructed so as to be $> m_{\nu+1}'$.

The other places in f''' can be filled by any components.

N.B.—The proof of the theorem does not require the specifiability of ν since all the possibilities are provided for, and these possibilities can be enumerated, and there is a regular method of constructing f''' in every possible situation. Cf. *Introduction.*

THEOREM II: *The full product of a finite number or a fundamental series of dense factors is dense.*

Proof: Let $N = V(\cdots\cdot, M_2, M_1)$, where N consists of elements $n = (m_1, m_2, \cdots\cdot)$, where m_r is an element of M_r.

Given two elements of the set N which are different, we should give a method of constructing another element of the set between them. Let the given elements be

$$n' = (m'_1, m'_2, \cdots\cdot) \text{ and } n'' = (m_1'', m_2'', \cdots\cdot).$$

Since $n' \neq n''$, we know that $m_\nu' = m_\nu''$ *not* for all ν. Therefore, there is an r (unspecifiable, as in the argument of TH. I). Such that $m_\mu' = m_\mu''$ for $\mu \leqslant r$ and $m_{r+1}' > m_{r+1}''$ or $m_{r+1}' < m_{r+1}''$, where $r = 0$ or 1 or $2 \cdots\cdot$ Whether $m_{r+1}' > m_{r+1}''$ or $m_{r+1}' < m_{r+1}''$ we can construct an element to lie between m_{r+1}' and m_{r+1}'', since *each* M_r is dense. Call that element m_{r+1}''', so that either $m_{r+1}' < m_{r+1}''' < m_{r+1}''$, or $m_{r+1}'' < m_{r+1}''' < m_{r+1}'$. In either case construct $n''' = (m_1', m_2', \cdots m_r'\ m_{r+1}''', \cdots\cdot)$. It is clear that n''' lies between n' and n''.

THEOREM III: *The full product N is dense in itself, provided we are certain that all the factors M_ν possess either property (A) or property (B) of Th I.*

Proof: Let any element of N be denoted by $n = (m_1, m_2, \cdots)$ If M_ν has the property (*A*), construct $n'_\nu = (m_1, m_2, \cdots)$ and $n''_\nu = (m_1, m_2, \cdots m'_\nu, \cdots)$, where $m'_\nu > m_\nu$.

If M_ν has the property (*B*), construct $n'_\nu = (m_1, m_2, \cdots m''_\nu \cdots)$, where $m''_\nu < m_\nu$, and $n''_\nu = (m_1, m_2, \cdots\cdot)$.

Denote the closed interval $[n'_\nu, n''_\nu]$ by i_ν. It is clear that n is contained in every i_ν. On the other hand, if any element $k=(k_1, k_2, \cdots)$ belongs to every i_ν then we can shew that $k=n$. To prove this, it is sufficient to shew that $k_\nu = n_\nu$ for each ν. Now, if k belongs to any i_ν, then its first $\nu-1$ components are respectively equal to the first $\nu-1$ components of n. Since k is contained in every i_ν, every one of its components is equal to the corresponding component of n. That is, $k=n$.

THEOREM IV: *The full product is dense in itself if one of its factors is dense in itself.*

Proof: Suppose M_ν is dense in itself; that is, every element m_ν of M_ν is the kernel of a convergent net i_ν defined by $[m'_\nu, m''_\nu]$.

Given any element of the product, $n=(m_1, m_2, \cdots)$ we construct the convergent net I_ν, defined by $[n'_\nu, n''_\nu]$, where

$$n'_\nu \equiv (m_1, \cdots m_{\nu-1}, m'_\nu, m_{\nu+1} \cdots)$$
$$n''_\nu \equiv (m_1, \cdots m_{\nu-1}, m''_\nu, m_{\nu+1} \cdots),$$

such that n is the *kernel*. Hence the full-product is dense in itself.

THEOREM V: *If the full product of a finite number of virtually ordered sets is everywhere dense, then the left-most factor should be everywhere dense.*

Proof: Let $N \equiv V(M_r, \cdots\cdots M_1)$

Let $f' \equiv (m_1', m_2', \cdots\cdots m_r')$,

$f'' \equiv (m_1'', m_2'', \cdots\cdots m_r'')$,

be two elements of N, which are different. If N is dense, then there exists (*i.e.*, we have, for the general case a systematic process for constructing) an element $f'''=(m_1''', m_2''', \cdots)$, such that $f'<f'''<f''$ or $f''<f'''<f'$. The proof consists in showing that this process fails in the case when all the components except the last are equal in f' and f'', *unless* M_r is dense.

Suppose $m_s'=m_s''$ for $s \leqslant r-1$.

If $f'<f'''<f''$, then $f''' {}_0\!\geqslant f'$ and $f''' \leqslant_0 f''$. So, in particular, $m_1''' \geqslant m_1'$ and $m_1''' \leqslant m_1''$. But, since $m_1'=m_1''$, we have $m_1'''=m_1'$ and similarly $m_{r-1}'''=m_{r-1}'$.

Thus. $f' = (m_1', m_2', \ldots . m_{r-1}', m_r')$

$f''' = (m_1', m_2', \ldots . m_{r-1}', m_r''')$

$f'' = (m_1', m_2', \ldots . m_{r-1}', m_r'')$.

Since $f''' > f'$, $m_r''' > m_r'$, and since $f''' < f''$, $m_r''' < m_r''$,

Hence $m_r' < m_r''' < m''$.

Similarly we can prove that, if $f'' < f''' < f'$, then
$m_r'' < m_r''' < m_r'$.

Since m_r', m_r'' are any two elements whatever of M_r, and since m_r''' always lies between them, when $m'_r \neq m_r''$, we see that the assumed systematic process for constructing f''' leads to a systematic process for constructing m_r''', and therefore M_r is everywhere dense.

Ex.: If η is the order type of rational numbers, then $2 \cdot \eta$ cannot be everywhere dense.

THEOREM VI: *If the full product of a finite number or a fundamental series of factors is closed, then each of the factors is closed, provided we can construct an element of each.*

Proof: A virtually ordered set is closed, if it is impossible for a divergent net to exist in it. To prove the theorem, it is sufficient to prove that if we can construct a divergent net in any set M_ν, then we can also construct a divergent net in the full product N.

$$N \equiv V(\cdot \cdot \cdot \cdot \cdot M_\nu, \cdot \cdot \cdot \cdot \cdot, M_3, M_1).$$

Let i_{ν_1}, i_{ν_2} $\cdot \cdot \cdot \cdot \cdot$ be a divergent net of closed intervals in M_ν such that each is contained in the preceding, and such that, given any element $m_\nu(k)$ of M_ν, we can find an interval i_{ν_r}, $r = r(k)$, such that it is impossible for $m_\nu(k)$ to lie in i_ν.

Let the end-elements of i_{ν_r} be m'_{ν_r}, m''_{ν_r}. Consider the closed intervals I_{ν_r} defined by the end-elements.

$$(m_1, m_2, \cdot \cdot \cdot \cdot \cdot m_{\nu-1}, m_r', \cdot \cdot \cdot \cdot \cdot)$$

$$(m_1, m_2, \cdot \cdot \cdot \cdot \cdot m_{\nu-1}, m_{\nu_r}'', \cdot \cdot \cdot \cdot \cdot)$$

Suppose we are given any element of the product in which the first $\nu - 1$ components differ in some place; then that element cannot lie in any I_{ν_r}.

Next suppose that we are given any element of the product which has only the first $\nu-1$ components equal to $m_1 \cdot \cdot \cdot \cdot \cdot m_{\nu-1}$. Then, for any element $m_\nu(s)$ of M_ν, we can find an interval i_{ν_t}: $t=t(s)$, such that i_{ν_t} does not contain m_{ν_s}. Therefore, I_{ν_t} cannot contain the given element of the product.

Thus, whatever element of the product is given to us, we can find an interval of the sequence I_{ν_r} which does not contain that element. Thus we have constructed a divergent net in the product.

The converse of Th. VI is not true; we have instead,

THEOREM VII: *The full product is closed provided each of the factors possesses the property (A) and, in addition, the property (C) that between any two elements of any factor there exist only a finite number of elements.*

Proof: Suppose there is a divergent net of closed intervals in the product, $i_1, i_2, \cdot \cdot \cdot$

Let $_\sigma\alpha' = {}_\sigma a'_1 \; {}_\sigma a'_2 \; {}_\sigma a'_3, \cdot \cdot \cdot$

$_\sigma\alpha'' = {}_\sigma a''_1 \; {}_\sigma a''_2 \; {}_\sigma a''_3 \cdot \cdot \cdot$

be the end-elements of i_σ and, for any σ, let $_\sigma\alpha' \leqslant {}_\sigma\alpha''$. We shall now show that, given any positive integer ν, there exists a least positive integer σ_ν, such that $_{\sigma\nu}a'_\mu = {}_{\sigma\nu}a'_\mu$, for $\mu \leqslant \nu$, and therefore $_\sigma a'_\mu = {}_\sigma a''_\mu$ for $\mu \leqslant \nu$ and $\sigma \geqslant \sigma_\nu$.

We assume that $_1a'_\mu = {}_1a''_\mu$ does not hold for every $\mu \leqslant \nu$.

Therefore, $_1a'_\mu = {}_1a''_\mu$ for $\mu \leqslant r$ $(r < \nu)$ and $_1a'_{r+1} > {}_1a''_{r+1}$.

Then, $_\sigma a'_\mu = {}_\sigma a''_\mu = {}_1a'_\mu = {}_1a''_\mu$ for $\mu \leqslant r$, and

$$_\sigma a'_{r+1} \leqslant {}_1a'_{r+1}, \quad {}_\sigma a''_{r+1} \geqslant {}_1a''_{r+1}.$$

If σ is such that $_\sigma a'_{r+1} = {}_1a'_{r+1}$, $_\sigma a''_{r+1} = {}_1a''_{r+1}$, then the element $_1a''_1 \; {}_1a''_2 \; {}_1a''_3 \cdot \cdot \cdot {}_1a''_r \; (b''_{r+1}) \cdot \cdot \cdot \cdot \cdot$, where $b''_{r+1} > {}_1a''_{r+1}$, belongs to i_σ.

Since by hypothesis, it is a divergent net, it must be possible to assign h such that i_h does not contain the above element. That is, no such i_σ can lie within i_h, so that at the h^{th} stage and after, (unspecifiable) one of the above two inequalities (viz. ${}_{\sigma}a'_{r+1} \leqslant {}_{1}a'_{r+1}$, ${}_{\sigma}a''_{r+1} \geqslant {}_{1}a''_{r+1}$) must be an equality; that is, $h\, a'_{r+1} < {}_{1}a'_{r+1}$ or ${}_{1}a''_{r+1} > {}_{1}a''_{r+1}$.

If ${}_{1}a'_{r+1} > {}_{1}a''_{r+1}$, then the reasoning can be repeated for i_h just as we did for i_1, so that the difference between the $r+1^{th}$ components of the end-elements is reduced further. Since, between any two elements, there can exist only a finite number of elements, we ultimately arrive at a stage when the $r+1^{th}$ components are equal.

If ${}_{1}a'_{r+1} = {}_{1}a''_{r+1}$, we can then find a smallest σ_{r+1} for which ${}_{\sigma_{r+1}}a'_{r+1} = {}_{\sigma_{r+1}}a''_{r+1}$, so that ${}_{\sigma_{r+1}}a'\mu = {}_{\sigma_{r+1}}a''\mu$ for $\mu \leqslant r+1$. Similarly we construct $\sigma_{r+2}, \sigma_{r+3} \cdots$, such that $\sigma_{\nu+1} \geqslant \sigma_{\nu}$ for every ν.

We have then the element ${}_{\sigma_1}{}^{a'_1}\, {}_{\sigma_2}{}^{a'_2}\, {}_{\sigma_3}{}^{a'_3} \cdots = i\omega$.

Now ${}_{\sigma}{}^{\alpha'} \leqslant {}^{i}\omega \leqslant {}_{\sigma}{}^{\alpha''}$ for every σ, which proves the impossibility of the existence of a divergent net in the product. Hence the product is closed.

§ 4. A Subset of Real Numbers

According to Brouwer, the set A of dually representable(2) real numbers is a subset of the set of real numbers. The set A is not discrete, nor is it ordered. That it is not discrete can be seen from the fact that a real number(3) can be defined which cannot be proved to be either equal to, or different from, zero. That it is also not ordered can be deduced from the fact that in the dual decimal we may not be able to decide which place is occupied by 0 and which place by 1, using the set law alone. But the set A can be virtually ordered, as we shall see presently.

Given a set of integers $a_1 a_2 \cdots a_n$, we can make it correspond to a real number which is dually representable by letting the 1's occupy the places with the ranks a_1, $a_1 + a_2$, $a_1 + a_2 + a_3$, $\cdots$ Now, the complex $(a_1, a_2, \cdots a_n)$ may be considered as an element of the full product $V(\cdots\cdots, A_2, A_1)$, (where $A_1, A_2, \cdots$ are sets of integers) which is virtually ordered as follows:

Let $f' \equiv (a_1', a_2', \cdots)$; $f'' \equiv (a_1'', a_2'', \cdots)$.

Then, $f' \circ\!> f''$, if there exists an r such that $a_r' \leqslant a_r''$, while $a_\nu' \leqslant a_\nu''$ for $\nu < r$. This relation $\circ\!>$ satisfies $L_1 - L_4$. If we define its double negation as $>$, then the full product is virtually ordered according to the relation $>$.

This product is everywhere dense by Th. I, it is dense-in-itself by Th. III, and it is closed by Th. VII[2].

I am very thankful to Dr. R. Vaidyanathaswamy for his special course of lectures on Intuitionism, of which this is a direct outcome.

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ON A THEOREM OF GROUP-THEORY CONNECTED WITH A PROBLEM ON PAPER-FOLDING AND WITH SOME OTHER PROBLEMS SOLVED AND UNSOLVED

BY

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A slip of paper $A\ A_m$ may be folded at $A_1, A_2, \ldots, A_{m-1}$ into m congruent parts; so one obtains $2m$ "fields" on the two sides of the paper. The ends A and A_m are joined without twisting the slip. Given n pairs of – say "complementary" – colours a,a'; b,b'; . . . d,d'; the 2_m fields $A_i\ A_{i+1}$ are be painted of one colour each, but fields on opposite sides of the paper have complementary colours. For slips coloured according to this rule it will be proved:

If the cyclic order of the colours on one side of the slip taken in the clockwise orientation tallies with the anti-clockwise cyclic order of the colours on the other side, then the slip can be folded so that the fields on the innerside are touching two by two, contacting fields having complementary colours.

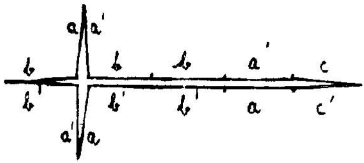
E.g. the outerside may be painted

$$a \quad a' \quad b \quad b \quad a' \quad c \quad c' \quad a \quad b' \quad b' \quad a \quad a' \quad b' \quad b$$

therefore the innerside

$$a' \quad a \quad b' \quad b' \quad a \quad c' \quad c \quad a' \quad b \quad b \quad a' \quad a \quad b \quad b'$$

If one starts from the 12th field of the inner side and proceeds to the left, one gets the same cyclic order of the colours as on the outerside from the left to right when starting from the first field. How the slip can be folded according to the proposition, is shown in the figure.



1. The above proposition on slip-colouring is a direct consequence of a theorem on groups and it is nearly equivalent with it.

Consider $a, b, \ldots, d$ as n generators of a free group F and put $a'=a^{-1}$, $b'=b^{-1}$, $\ldots$, $d'=d^{-1}$, then to every sequence of colours there corresponds an element α of F. The inverse element α^{-1} is obtained by interchanging the generators with their inverse elements and taking them from the right to the Left. A cyclic change of the order furnishes a conjugate element. If therefore α corresponds to a sequence of colours satisfying the above conditions, it is conjugate to α^{-1}. On the other hand the element a is not altered if in its representation by generators a pair like $\alpha\ \alpha^{-1}$ is inserted or omitted. If the sequence of colours corresponding to α admits a folding as proposed above, α must be conjugate and therefore be equal to the unit element 1 of F. Hence it suffices to prove that $\alpha \neq 1$ cannot be conjugate to α^{-1}. This statement is the essential part of the following theorem.

If $\alpha \neq 1$ is an element of a free group, then α^m and α^n cannot be conjugate unless $m=n$.

2. Let α^m and α^n be conjugate and β be conjugate to α, then β^m and β^n are conjugate. Hence one can suppose without loss of generality that when α is represented by the generators in the reduced form, the last term in the representation is different from the inverse of the first term. Let q be the length of the reduced representation of α, then the reduced representation of α^m and α^n are

of length $|m|\ q$ and $|n|\ q$ respectively. In each of these representations the last term is different from the inverse of the first one; hence they can be conjugate only if they differ at most by a cyclic permutation and therefore their length must be equal. Thus $|m| = |n|$.

It remains to prove that $m=n$ when $m \neq 0$. Suppose that $m = -n \neq 0$, then $\alpha^m \neq 1$ (since F is a free group) and α^m is conjugate to its inverse.

3. Suppose $\alpha \neq 1$ to be conjugate with α^{-1}. Without loss of generality we can suppose that the first and the last term of the reduced representation of α by the given free generators are non-inverse one to the other. Hence if the terms of that reduced representation are taken in their cyclic order (the 1st and the last being considered neighbours), abutting terms are not the inverse of one another. This cyclic order will be considered now. It is the same for α and for α^{-1} and a cyclic permutation C maps α on α^{-1}. If there are group elements with the required property, then there is one, say α_0, with a minimal length. Let α_0 be generated by generators $a, b, \ldots, d$ where $a^{\pm 1}$ occurs p times $b^{\pm 1}$ occurs q times, $\ldots, d^{\pm 1}$ occurs s times; without loss of generality we may suppose that $p \leqq q \leqq \cdots \leqq s$. The length of α is $l = p + q + \cdots + s$. The terms $a^{\pm 1}$ intersect the remaining cycle of terms into $p' \leqq p$ intervals; equality holds only if the terms $a^{\pm 1}$ are all isolated. If $p' = p = q = l : 2$, the element α_0 depends on a and b only and every $a^{\pm 1}$ and every $b^{\pm 1}$ stands isolated in the cyclic representation; otherwise $s - p > p'$ and there exists therefore an interval between the $a^{\pm 1}$ containing more than one term. Let δ be the word filling such an interval. The cyclic permutation C maps $a^{\pm 1}$ on $a^{\mp 1}$ and therefore maps an interval δ on an interval δ^{-1} and conversely; these intervals have no common element and are separated by other terms containing the term $a^{\pm 1}$. If one replaces every interval δ by b and every δ^{-1} by b^{-1}, one obtains a shorter expression which admits the cyclic transformation C and corresponds to a reduced word. Hence there exists a group-element $\alpha_1 \neq 1$ of a shorter length than α_0 which is conjugate to its inverse contrary to our supposition. One must therefore suppose that $p' = p = q = l : 2$. So α_0 depends on two generators a and b only and each a-term stands between two b-terms and conversely. Now it will be shown that each b-term stand between an a and an a^{-1}. Suppose there exists a sequence $\Delta = a\ b\ a \cdots a\ b\ a$ of alternating a and b, say k b's and k being the maximal number. By C the word Δ is mapped on $\Delta^{-1} = a^{-1}\ b^{-1}\ a^{-1} \cdots a^{-1}\ b^{-1}\ a^{-1}$ and conversely; thus there exists no

such sequence with more than $2k+1$ terms. Hence neither the Δ nor the Δ^{-1} can overlap in the cyclic order, nor can they abut each $a^{\pm 1}$ standing between two b-terms. If therefore Δ is replaced by b and Δ^{-1} by b^{-1}, one obtains the cyclic order of a reduced word $\alpha_2 \neq 1$ which by the cycle transformation c is mapped on α_2^{-1}. However α_2 is shorter than α_0 contrary to the supposition. Hence b cannot stand between two a's and for the same reason no term can have two equal neighbours. Obviously no reduced word consisting of one or two terms only is conjugate to its inverse. Hence every $a^{\pm 1}$ stands in the cyclic order between b and b^{-1}, every $b^{\pm 1}$ between a and a^{-1}. Therefore the cyclic order of α_0 is either that of $y^m = (ab^{-1}ab^{-1})^m$ or that of y^{-m}. But y^m is not conjugate to y^{-m}; therefore no such α_0 can exist and this finishes the proof.

4. It is of a particular interest to state whether in a group G, where every element $\neq 1$ is of infinite order, none of these elements is conjugate to its inverse element. In this case, G satisfies two necessary conditions for being able to be "ordered". Groups are often considered as topological spaces *i.e.* one introduces in them a notion of "neighbourhood" satisfying the axioms of topology.

These neighbourhoods are supposed to be transformed into neighbourhoods by multiplication with group-elements from the left as well as from the right side. In general a group can be made a topological space in various essentially different ways. One may ask * about the groups which can be made linear topological spaces, *i.e.* in which a transitive relation $\lessgtr$ can be introduced such that when $a \neq b$ *either* $a<b$ and $b>a$, or $b<a$ and $a>b$ hold and $a<b$, $b<c$, $d' \leqq d$ imply $a<c$, $d'a<db$, $ad'<bd'$ and $b^{-1}<a^{-1}$. In particular $a<1$, $b<1$ imply $ab<1$, $ba<1$, $a^{-1}>1$, $1>a>a^2>\ldots$ and $k^{-1}ak<1$ for every k. The converse holds for $a>1$, $b>1$. This shows already that an element $\neq 1$ of an ordered group can neither have a finite order nor can it be conjugate to its inverse element. A free group with one generator can be ordered but although these two necessary conditions are satisfied, it is not yet known whether the same proposition holds for free groups with more than one generator. If a group is ordered, all its subgroups are ordered. Now a free group with two generators has subgroups with any finite or enumerable infinite number of generators and obviously the converse holds. Hence it would suffice to decide the question for free groups with two generators.

* see: Proceedings Indian Ac. of Science Vol. XVI (1942) p. 256-263 and Vol. XVII (1943) p. 199-201.

INSTABILITY OF VARIABLE STARS AND THE CEPHEID THEORY OF THE ORIGIN OF THE SOLAR SYSTEM.

BY

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To the common star-gazer, there never occurs any doubt regarding the constancy of the light that reaches him from the age-long stars. There are, however, many stars which have failed to conceal their variability from the practised eye of the astronomer observing them night after night. Such stars, the radiations from which do not remain constant are known as variable stars.

The variable stars fall into two broad groups:—one consisting of those whose intrinsic light actually does not vary; the other includes those which are intrinsically variable. The stars of the former class appear variable to us because they consist of two components, one eclipsing the other at regular intervals. These are the eclipsing binaries. The other class consists of stars whose light variation the astronomer has not yet succeeded in explaining fully. This class includes stars varying irregularly and also those whose brightness goes through a cycle in definite periods. These former stars are known as "cataclysmic variables". Among these we find stars of the type R. Coronae Boraealis which become suddenly faint, regaining their former brightness slowly, and stars known as Nova, Supernova, and Subnova which suddenly flare out, come into prominence from invisibility and then gradually fade out until they become again as faint as they were. These are giant stars, but there are also dwarf novae, e.g., the S. S. Cygni stars on the other hand, which have the further interesting characteristic that while the usual novae and supernovae flare up only once in their career and are generally lost afterwards into oblivion, the S.S. Cygni stars repeat their display at regular intervals. Besides these, there are other stars which are bright and faint in an erratic manner including the nebular variables, the irregular variables, with nebular spectra and stars with variable atmospheres.

Next we come to the most important group of variable stars viz., those whose brightness goes through a cycle in definite periods. This group of stars includes:—(a) Long-period variables, with periods ranging from 32 days to 560 days, the best known among which is Mira Ceti, in the constellation of (Ceti) the whale. The variation of its brightness was first noted by Fabricus in 1596 A. D. When brightest, it is a second magnitude star visible to the naked eye, then it fades till in its faintest condition, it is a telescopic star of the ninth or tenth magnitude. When next brightest, it is about of the third magnitude. On an average the period is 330 days; but there is no constancy regarding the value of the period or the actual magnitude in its maximum.

(b) Cepheid variables, of which the typical star is δ- Cephei. The periods of these range from 3 days to 32 days. The variation of these stars are very regular—indeed so regular that their periods have become standard scales in stellar measurements. This group of stars comes first in order of importance among the variable stars, the next being the novae. The third group of the periodic variables is

(c) The Cluster Variables, with periods ranging from 5 hours to 23 hours. The typical among these is R. R Lyrae. These are found in large numbers in globular clusters.

A very important characteristic of the long-period stars is that they are all supergiant stars. Cepheids, too, are giant stars, and a characteristic difference of these from the ordinary stars is in the fact that they are more homogeneous than ordinary stars. In fact, Gamow remarks that the Cepheid variables are in the transition stage of merging into the main sequence of stars and that the variation is due to "instability during transition from the giant branch into the main sequence." [2]

Among the characteristics of Cepheid variables are:

(1) They are giant stars, but are much more luminous than the normal giants of their type.

(2) As the period increases, the star becomes more and more red.

(3) The logarithms of the period and the magnitude are linearly related.

(4) The rate of brightening is more rapid than the rate of darkening.

(5) There is a marked spherical symmetry. The variation would appear much the same from whatever direction it may be observed.

(6) The spectral lines indicate that there is an approach and recession of the light source, following the same cycle as the variation in brightness. Getting showed that the luminosity attains its maximum value a little before the velocity of approach becomes maximum.

(7) The spectral lines are very sharp and much less diffuse than that which would be expected from a rotational theory of Cepheid variation.

(8) Cepheids are more homogeneous than ordinary stars. The condensation of mass in Cepheids in the centre is less than that in ordinary stars.

Based on these observed facts, theories were formulated to explain Cepheid variation. The original binary star hypothesis in which the variation of light was explained to be due to the motion of a bright primary in a resisting medium round the common centre of gravity of the bright primary and the relatively dark secondary has to be ruled out since it would need the secondary being too near the primary to give a dynamically possible configuration. The fission theory of Jeans, in which we have an envelope of gas surrounding a rotating liquid core near the point of fissional break-up fails to explain spherical symmetry. The theory that holds the ground today is the Pulsation theory conceived by Ritter, Plummer and Shapley and given a mathematical start by Eddington, being later developed by many investigators including

Edgar and Sterne. In this theory we imagine a spherical gaseous equilibrium configuration oscillating symmetrically and radially, the adiabatic condition holding during the oscillation. Naturally the brightness will vary as the gas-sphere gets compressed or expanded. But one difficulty arises—we would expect maximum brightness when the star is compressed and we know that as in simple harmonic oscillation the velocity is maximum after a quarter period when the oscillating mass passes through the equilibrium position. But, as we have observed, greatest brightness is in the same phase with the greatest velocity of approach. Further the theory has not yet explained the Period-Luminosity law. Also according to Edgar's calculations the oscillations would die out in an unacceptably short period of the order of 10^4 years. Thus to maintain the oscillations we must have other sources of energy which according to Gamow are "the hydrogen reactions with Li, Be and Bo." These are the observational difficulties. Let us now turn to the dynamical ones.

Eddington, Edgar, and Sterne considered the stars oscillating with small amplitude. For Sterne's three stellar models Bhatnagar was the first to take into consideration the square of the amplitude, and he showed that the consideration of such terms makes the oscillations unstable. This forms the basis of a recent and entirely novel theory of the solar system, by A. C. Banerji [1], who considers the radial oscillations of two stellar models: (1) a homogeneous star (2) a star in which the density varies as the p'th power of the distance, (where p is a positive integer excluding 1 and 3) except in a small finite core of constant density, taking into consideration the square of the amplitude in both the cases. This in both the cases leads to instability.

Imagine now a stable Cepheid pulsating with small amplitude. If now a star passes by, not necessarily too near, then due to the tidal effect the oscillations would become larger and larger. Instability would set in, and hence the Cepheid would break up, resulting in ejection of matter of the Sun's mass, from it. The planets are supposed to form from the ribbon attached to the Sun's mass, and the visiting star is supposed to impart sufficient angular momentum to the planets. Banerji's calculations have shown that a parent Cepheid of about nine times the Sun's mass, the Sun carrying away two-fifths of the energy of the parent Cepheid after formation, would suffice for the formation of the planetary system.

A striking feature of this theory is that it makes the minimum possible number of assumptions and easily explains the usual difficulties. A distant encounter between two stars is definitely much more probable than the accumulation of the special conditions needed for the theory of Jeans and Jeffreys, Russell and Lyttleton, and at the same time it is known that nearly five per cent of all stars are variable. [3] Further for every breaking up of one Cepheid we shall have two planetary systems, one for Sun, the other for the parent Cepheid. These interesting features not only remove the cause of the comment [5] "the solar system had a very narrow escape from never coming into existence" but makes the existence of other planetary worlds definitely more probable.

From the instability of oscillation of slowly rotating models H. K. Sen has shown that a more or less homogeneous star, the instability would lead to fissional break up with the formation of a binary system, while for a centrally condensed core with an extended tenuous envelope, there would be equatorial break up with the formation of the spiral arms. Following Banerji's method, Sen by taking different stellar models has found oscillations stable only for the homogeneous star, which tallies with observation.

So far we have seen the consequences of a steady cumulative effect of tidal forces or rotational momentum on the production of instability in a pulsating star— there is no cataclysmic variation anywhere. Let us now take the case of an intrinsically unstable configuration whose equilibrium is exploded by the lightning visit of an extraneous agency. Thus for example the vast untapped energy of the interior of stars may be released in a cataclysmic penetration of a planet into the stellar interior leading to the blazing of a nova. There are of course better theories to explain the formation of a nova—thus suppose that a dwarf is passing through an envelope of nebulous matter; it goes on gathering mass round it like a blanket with the resultant storage of energy in the interior. This process obviously cannot go on indefinitely—a time comes— say in a century—when the whole blazes into a nova. Nebulous envelopes have been observed round novae that have burst. The abundance of dwarfs, and the nebulae and the observed frequency of novae supply good confirmation of the theory.

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NUMERICAL NIGHTMARES

BY

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If you wish to tempt your friend into a foolish answer, you may propose to him one of those problems, ancient or modern, which will test his appreciation of the frightful rapidity with which certain numbers grow. Ten to one, he will fail miserably and will leave you in a hurry on the pleasant errand of making others look as silly as he himself did.

Double, double, never mind trouble

Perhaps you have heard the story of the inventor of the game of Chess who when asked by the King of his land to name his own reward said :—" O Great Monarch whose generosity is the envy and the despair of the other monarchs in the seven worlds above and the seven worlds below, grant me but this boon. For the first square in yonder chessboard give me one grain of paddy, for the second square two grains, for the third square four grains, and so on doubling each time till all the 64 squares are exhausted." The King was disappointed at this apparently modest request and offered to begin with a bag instead of a single grain, but the inventor said he would be satisfied with the single grain. And so the King passed the necessary orders to his Minister in Charge of Charities who, in his turn, commanded the State Computer to make the necessary calculations. After figuring it all out, the Computer declared that it amounted to $2^{64}-1=18446744073709551615$ grains and that there was not enough paddy in the land to carry out the order; for if it were all spread uniformly over the 10,000 square miles of the kingdom, it would have to be piled so high that the tallest temple gopurams would be submerged. The story does not state how the King met the situation. Perhaps he bestowed on the inventor half his kingdom and the hand of his daughter *Dhanyasindhu* (Ocean of Grain) marriage.

My next example of doubling numbers should endear me to the missionaries of birth control and make their patron Saint Malthus sit up in his grave—if he should be still there. If a man has two

* This is adapted from the text of a Radio Talk broadcast from the Madras Station of All India Radio under School Broadcasts, and is published with the permission of the A. I. R. Madras " "

children and each of them two children and so on in succession, what will be the total progeny after say 70 generations? An easy calculation will show that in 32 generations they will amount to the present human population of the globe. 20 generations later they will have "multiplied and replenished the earth" to the extent of having just enough standing room only, with a modest square foot per person. Thereafter we shall have to arrange them in columns piled one over the other. In 20 generations more each of these human columns will have reached a height of a thousand miles. It would be cruel to carry on these calculations farther.

Powerful Numbers

If your friend and victim has not meanwhile run away or murdered you peacefully, you may next ask him whether he has any idea of the biggest number which may be formed with 4 figures. He will answer that it is 9999, and will be quite right if the figures are to be written down as usual. But, if they could be used as powers unimaginably big numbers which stagger the imagination may be written down. Thus with two nines we have 9^9 which means $9\times9\times9\times9$...nine times. This number which I shall call A has the value 387,420,489. With three nines we may form 9^{9^9} which means 9^A. I shall call this number B. It begins with 428,124,773,175,747,048,036, 987,115 and ends with the digits 89. I hope my considerate reader will not insist on all the figures being given for there are about 300 million of them and if printed in full it would not only occupy the whole of this issue of *The Mathematics Student* to the exclusion of all other matter, but also require all such issues for the next 300 years, and I am afraid you may consider the article somewhat long! Written on a strip of paper so as to be readable, it would require a strip a thousand miles long, while printed in book form it would make a small library of 42 volumes similar to those of the *Encyclopaedia Britannica.* All this using only three nines. With four nines we have the number $9^{9^{9^9}}=C$ which is the same as 9^B. If we could compress all the 42 volumes referred to above into a grain of sand and close-pack such grains so that they formed a sphere extending up the farthest nebulae faintly visible in our biggest telescopes, we shall still have a library which is hopelessly inadequate for expressing this number C. In fact even if this huge sphere were again compressed to the size of a grain of sand and again such grains close-packed to the size of the big sphere, we shall not have made a serious attempt to have a library adequate for this number. But why attempt the impossible. As the Upanishadic Seers have it, "words return back baffled along with the mind without having reached it." It is a definite number all the same, and to be condemned to its calculation is the equivalent of the Eternal Hell in the mathematician's *inferno*, reserved only for those who have committed the five deadly sins against mathematics (such as division by zero etc.).

Kasner's Googol

Prof. Edward Kasner of the Columbia University has coined the word "googol" for 1 followed by a hundred zeros. It is a pigmy compared with the number $9^{9^{9}}$, but is itself so huge that we shall never require it in any counting or measurement. The largest figure in finance was probably the total number of paper marks in circulation in Germany at the peak of the inflation when it was said you could go to the market with the money carried in a basket and bring the vegetables in your purse. Even so it was a figure of only 20 digits and so, very much less than a googol. The total number of sand particles on the Madras Beach—including, of course both the Tamil and Andhra sands in the reckoning—would be a still smaller figure. The total number of drops of water in the ocean is less than a googol, being a miserable figure with about 28 digits. If all the matter not only in the Earth but throughout the Stars, the Milky Way and all the Nebulae in the Universe were converted into atoms and then into electrons, how many would they amount to? Eddington has calculated this by making use of Einstein's Theory of Relativity and his estimate is a figure with 80 digits. So you see we have still not reached the googol. How much vaster is $9^{9^{9}}$, which contains nearly three hundred million digits while the googol has only a hundred and one!

The next time you write to thank anyone, send him $9^{9^{9}}$ thanks, or if you are of a miserly temperament, send him at least a googol of them. To convey "a thousand thanks" when these larger consignments cost no more is a wanton waste of the milk of human kindness.

ON THE DIFFERENTIAL EQUATION $f^r(x)=f(1/x)$

BY

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DR. LUDWICK SILBERSTEIN *, has solved the hystero-differential equation $f'(x)=f(1/x)$. In this paper the solution of the hystero-differential equation $f^r(x)=f(1/x)$ is discussed and that of

$$f''(x)=f(1/x)$$

is given as an illustration.

1. Our equation is $$f^r(x)=f(1/x) \tag{1}$$

Let us assume that, $$f(x)=x^m+\lambda x^n \tag{2}$$

where λ is independent of x. Substituting, we have

$$m(m-1)\ .\ .\ .\ (m-r+1)x^{m-r}+\lambda n(n-1)\ .\ .\ .\ (n-r+1)x^{n-r}=x^{-m}+\lambda x^{-n} \tag{3}$$

from which we obtain,

$$\left.\begin{aligned} &m+n=r\ ;\ \lambda=m(m-1)\ .\ .\ .\ (m-r+1) \\ &\lambda n(n-1)\ .\ .\ .\ (n-r+1)=1 \end{aligned}\right\} \tag{4}$$

By eliminating λ and n we get

$$m(m-1)^2\ .\ .\ .\ (m-r+1)^2(m-r)=(-1)^r \tag{5}$$

The equation (5) may be re-written,

$$\left.\begin{aligned} &[y+(r-1)]^2[y+2(r-2)]^2\ .\ .\ .\ [y+\tfrac{1}{2}(r-1)\cdot\tfrac{1}{2}(r+1)]^2y=(-1)^r\ (r\ \text{odd}) \\ &[y+(r-1)]^2[y+2(r-2)]^2\ .\ .\ .\ [y+(\tfrac{1}{2}r)^2]y=(-1)^r \qquad (r\ \text{even}) \end{aligned}\right\} \tag{6}$$

where $y=m^2-mr$.

Therefore we see that the values of m depend upon the solution of an equation of rth degree in y. Let $y=y_1, y_2, \ .\ .\ .\ y_r$ be the solutions of the equation (6). (6)

Now if $y=y_s$ be a solution of (6) we get $$m^2-mr=y_s \tag{7}$$

Let m_{1s}, m_{2s} be the solutions and $\lambda_{1s}, \lambda_{2s}$ be the corresponding values of λ from (4): Thus we get two solutions corresponding to $y=y_s$ where $m_{1s}+m_{2s}=r$. It can be easily shown that these two solutions are identical since $\lambda_{1s}\lambda_{2s}=1$.

* Silberstein *Phil. Mag.* Vol. 30 (1940), 185–87

$\therefore$ The complete solution of the equation $f^r(x) = f(1/x)$ may be written

$$f(x)=\sum a_s U_s (s=1, 2, 3 \ldots r); \quad U_s = x^{m_{1s}} + \lambda_{1s} x^{m_{2s}}, \ldots \tag{8}$$

where the m, s are the solutions of the equation $m^2 - mr = y_s$, and a's are the r arbitrary constants.

2. *Solution of $f''(x)=f(1/x)$*

In this case equations (4), (6) and (7) reduce to

$$\left.\begin{aligned} &\lambda = m^2 - m; \\ &m^2 - 2n = y; \\ &y^2 + y - 1 = 0; \end{aligned}\right\} \tag{9}$$

From the last equation we obtain,

$$y = (-1 \pm \sqrt{5})/2. \tag{10}$$

$$\therefore \text{ either, } m^2 - 2m = \frac{-1+\sqrt{5}}{2} = y_1 \text{ or } m^2 - 2m = \frac{-1-\sqrt{5}}{2} = y_2. \tag{11}$$

from which we obtain,

$$\left.\begin{aligned} m_{11} &= \tfrac{1}{2}(2 + \sqrt{2+2\sqrt{5}}) \\ m_{21} &= \tfrac{1}{2}(2 - \sqrt{2+2\sqrt{5}}) \\ m_{12} &= \tfrac{1}{2}(2 + \sqrt{2-2\sqrt{5}}) \\ m_{22} &= \tfrac{1}{2}(2 - \sqrt{2-2\sqrt{5}}) \\ \lambda_{11} &= \tfrac{1}{2}(1 + \sqrt{5} + \sqrt{2+2\sqrt{5}}) \\ \lambda_{12} &= \tfrac{1}{2}(1 - \sqrt{5} + \sqrt{2-2\sqrt{5}}) \end{aligned}\right\} \tag{12}$$

Using these values of m and λ we get the complete solution of the differential equation.

I am indebted to Mr. G. R. Seth, lecturer Hindu College Delhi, for various suggestions and improvements in the preparation of this paper.

NOTES AND DISCUSSIONS

Cylindrical projection and rolling

1. The axes being rectangular, the two generators of the right circular cylinder: $y^2+z^2-z=0$, lying in the XOZ plane are given by the equations:

$$y=0,\ z=0;\ \text{and}\ y=0,\ z=1.$$

The former is the X-axis. We shall call the latter the 'second' generator.

If $P(x, y, 0)$ be any point in the XOY plane and the perpendicular from P to the second generator meet the cylinder in the points Q and R of which the point R is on the second generator, then Q shall be called *the projection of P on the cylinder.*

It is easy to show that the co-ordinates of Q are $\left(x, \frac{y}{1+y^2}, \frac{y^2}{1+y^2}\right)$; so that if the point P traces the curve: $y=f(x)$, $z=0$; then the point Q traces the curve:

$$z=yf(x),\ y^2+z^2-z=0$$

Moreover, the projection of the point $P_1(x, 1/y, 0)$ on the cylinder is the point $Q_1\left(x, \frac{y}{1+y^2}, \frac{1}{1+y^2}\right)$, which is the reflection of the point Q in the plane $z=\frac{1}{2}$. It would thus be seen that the projections of the two curves: $y=f(x)$, $z=0$; and $y=1/f(x)$, $z=0$ on the cylinder are reflections of each other in the plane $z=\frac{1}{2}$.

2. The cylindrical projection of the curve: $y=f(x)$, $z=0$, may be obtained by rolling or wrapping round the cylinder, the curve:

$$y=\tan^{-1} f(x), \qquad z=0,$$

in such a manner that the axis of x remains fixed.

"Rolling" transforms the point $(x, y, 0)$ in the XOY plane, into the point $(x, \sin y \cos y, \sin^2 y)$ on the cylinder.

The curve: $$y=\phi(x),\ z=0.$$

is thus transformed into the curve:

$$z=y \tan \phi(x), \quad y^2+z^2-z=0$$

by rolling.

3. In particular, the projection of $y=\tan x$, $z=0$, is obtained by rolling round the cylinder, the straight line $y=x$, $z=0$. It is noteworthy that while the tangent graph is discontinuous, its projection on the cylinder is a continuous curve.

The projections of the sine and cosecant; and the cosine and secant graphs are equally interesting. They are not only reflections of each other in the plane $z=\frac{1}{2}$, but lie completely on either side of it on the cylinder, touching each other on the two generators of the cylinder lying in the plane $z=\frac{1}{2}$.

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Notes on Ellipse

This note establishes certain simple relation and constructions (associated with ellipse) that the writer has found out. All quantities whose constructions have been given are associated with a point whose radius vector (r_1) is known.

Let P be a point on an ellipse (axes a, b), having S_1, S_2 as foci, c as centre, and TPT' as tangent at P.

Let $r_1,\ r_2,\ r = S_1P,\ S_2P,\ CP$;

$p_1,\ p_2,\ p =$ perpendiculars on TPT' from S_1, S_2, C;

$\alpha = \angle TPS_1 = \angle T'PS_2$; $f^2 = r_1 r_2$

We define ω by $f = a \sin \omega$. ... (A)

It is easily seen that $2p = p_1 + p_2$.

$\therefore\quad 2p = p_1 + p_2 = r_1 \sin \alpha + r_2 \sin \alpha = 2a \sin \alpha$

$\therefore\quad p = a \sin \alpha$ ... (1)

From the pedal equation of the ellipse we have

$$\frac{b^2}{p_1^2} = \frac{2a}{r_1} - 1 = \frac{r_2}{r_1} \qquad \text{... } (B)$$

Hence $2p = b\,\{\sqrt{(r_1/r_2)} + \sqrt{(r_2/r_1)}\} = b\,\dfrac{r_1 + r_2}{\sqrt{r_1 r_2}} = \dfrac{2ab}{f}$

$\therefore\quad pf = ab$ ... (2)

Substituting from (*A*) and (1) in (2) we get

$$p \sin \omega = b \qquad \dots \quad (3)$$

$$f \sin \alpha = b \qquad \dots \quad (4)$$

Again, by differentiating logarithmically the result (*B*) we have

$$-2\frac{dp_1}{p_1} = \frac{dr_2}{r_2} - \frac{dr_1}{r_1} = -dr_1\left(\frac{1}{r_1} + \frac{1}{r_2}\right) \text{ since } dr_1 + dr_2 = 0.$$

$$\therefore \quad 2\frac{dp_1}{p_1} = \frac{2a\,dr_1}{r_1 r_2} \quad \therefore \quad \rho = r_1\frac{dr_1}{dp_1} = \frac{r_1 r_2}{a}\cdot\frac{r_1}{p_1}$$

$$i.e. \quad \rho = \frac{r_1^2 r_2}{a}\cdot\frac{1}{b}\sqrt{\frac{r_2}{r_1}} = \frac{(r_1 r_2)^{3/2}}{ab} = \frac{f^3}{ab} \qquad \dots \quad (5)$$

Constructions:—

With $AB = 2a$ as diameter draw a semicircle. Let O be its centre. Cut off AN on AB making $AN = r_1$, then $BN = r_2$.

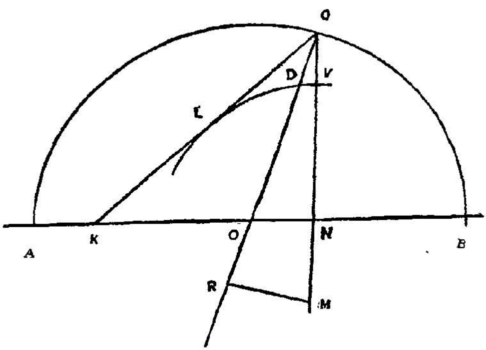
At N erect perpendicular to AB cutting the circle at Q. Then by elementary geometry $NQ^2 = AN \cdot BN = r_1 \cdot r_2$ $\therefore$ $NQ = f$ ... (i)

$$\therefore \quad \text{Sin } \angle NOQ = \frac{NQ}{OQ} = \frac{f}{a} = \text{Sin } \omega \quad \therefore \quad \angle NOQ = \omega \qquad \dots \quad \text{(ii)}$$

With centre N and radius b draw a circle cutting NQ at V. Draw QL tangent to this circle from Q.

$$\text{Then } \sin \angle NQL = \frac{LN}{NQ} = \frac{b}{f} = \sin \alpha, \text{ by (4)}$$

$$\therefore \quad \angle NQL = \alpha \qquad \dots \quad \text{(iii)}$$



Draw VD perpendicular to VN cutting OQ at D.

$\therefore$ $OD \cdot \sin \omega = b$.

Hence by (3) we have $OD = p$ . . . . . (*iv*)

Produce QL to meet AB at K.

Cut off $QM = KQ$ on QN produced, and from M drop the perpendicular MR on QO (produced if necessary) then, using (A), (1), (2) we have

$$QR = QM \cdot \cos \angle OQN = KQ \cdot \sin \omega$$

$$= \frac{f}{\sin \alpha} \cdot \sin \omega = \frac{f}{a \sin \alpha} \cdot a \sin \omega = \frac{f^2}{p} = \frac{f^3}{pf} = \frac{f^3}{ab} = \rho \qquad \ldots \quad (v)$$

Also

$$\left.\begin{aligned} r_1^2 &= r^2 + a^2 e^2 - 2aer \cos \theta \\ r_2^2 &= r^2 + a^2 e^2 + 2aer \cos \theta \end{aligned}\right\} \text{ where } \theta = \angle S_1CP.$$

Adding $r_1^2 + r_2^2 = 2r^2 + 2a^2e^2 = 2r^2 + 2a^2 - 2b^2$.

Adding $2f^2$ or $2r_1 r_2$ to both sides, we have

$$(r_1 + r_2)^2 = 2r^2 + 2a^2 - 2b^2 + 2f^2$$

or $\quad 4a^2 = 2r^2 + 2a^2 - 2b^2 + 2f^2$

$\therefore \quad r^2 = a^2 + b^2 - f^2 = a^2 - a^2 \sin^2 \omega + b^2 = a^2 \cos^2 \omega + b^2$

But $a \cos \omega = OQ \cdot \cos \angle QON = ON$

$\therefore \quad r^2 = ON^2 + VN^2 = OV^2 \quad i.e. \quad OV = r \qquad \ldots \quad (vi)$

GAGANBIHARI BANDYOPADHYAYA.

A Neglected equation in Analytical Conics

A writer in the *Mathematical Gazette* * speaks of the equation to the chord of

$$s \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \dagger$$

with a given mid-point as a much neglected equation. Curiously enough he approaches the equation through another even more neglected. The latter equation, that of the chord joining two given points on a conic, furnishes a ready interpretation of the "general equations" connected with $s = 0$ and will be found helpful in a rapid preliminary survey of the conic regarded merely as a curve of the second degree. The indifference to this equation shown by our text-books on Analytical Conics is surely not merited.

Let us write

$$s_{pq} = ax_p x_q + h(x_p y_q + x_q y_p) + \cdots + g(x_p + x_q) + \cdots + c = s_{qp}$$

* February 1942 *Gazette*, p. 51.

† Needless to say, the arguments presented in this note are applicable to any m of homogeneous co-ordinates provided C in paragraph three is replaced by Cl e $l = 0$ represents the line at infinity.

and denote by s_{pp}, s_p respectively the expressions obtained from s_{pq} by changing the suffix q into p and dropping the suffix q.*

The equation to a straight line can be related to the conic $s=0$ and to two given points 1 (x_1, y_1) and 2 (x_2, y_2) by writing it in the form $As_1+Bs_2+C=0$. If the points are on both the straight line and the conic, $As_{11}+Bs_{21}+C=0=As_{12}+Bs_{22}+C$ and $s_{11}=0=s_{22}$ so that $C/A = C/B = -s_{12}$. Hence the equation to the chord joining the points 1 and 2 on $s=0$ is

$$s_1+s_2=s_{12}.$$

By letting the point 2 tend to the position of 1 in the above, we obtain the equation to the tangent at 1 to the conic in the form $2s_1=s_{11}$ or $s_1=0$.

If the points $2(x_2, y_2)$ and $3(x_3, y_3)$ on the conic are connected by a chord whose mid-point is $1(x_1, y_1)$, the equation to the chord is $s_2+s_3=s_{23}$ which, after replacing s_2+s_3 by $2s_1$, becomes $2s_1=k$ (constant) $=2s_{11}$. Thus the equation to a chord of $s=0$ can be expressed in terms of the co-ordinates of its mid-point 1:

$$s_1 = s_{11}.$$

Let now 1 be any fixed point dividing a variable chord through it and through 2, 3 on the conic in the ratio $\lambda:1$. Then, remembering the relation between the co-ordinates of 1 and those of 2, 3, we can write the equation $s_1=0$ in the form $(s_2+\lambda s_3)/(1+\lambda)=0$ and show that $s_1=0$ always passes through the intersection of the tangents $s_2=0$ and $s_3=0$. Hence the polar of 1 w.r.t. $s=0$, defined in the usual manner, is given by

$$s_1 = 0.$$

The Harmonic Property of Pole and Polar can be readily established by pursuing further the argument in the last paragraph. Let $s_1=0$ divide the chord joining 2 and 3 in the ratio μ. Then $\mu = -s_{12}/s_{13}$ which, in virtue of the relation $s_1\equiv(s_2+\lambda s_3)/(1+\lambda)$, leads to

$$\mu = -\frac{s_{22}+\lambda\, s_{32}}{s_{23}+\lambda\, s_{33}} = -\lambda.$$

The reader will no doubt recall that the shortest method of obtaining the pair of tangents from (x_1, y_1) to $s=0$ takes as starting point the equation $s+ks_1{}^2=0$. This method and our discussion both owe their simplicity to a systematic exploitation of the formal relations between s_{pq}, s_p and s.

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C. T. RAJAGOPAL

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* This notation is due to A. Robson. (February 1942 *Gazette*, p. 49.)

ANNOUNCEMENTS AND NEWS

The following gentlemen have been admitted as members of the Indian Mathematical Society.

S. A. Hamid Esq., M.A. (Cantab), P. E. S., Government College, Lahore.

D. R. Jain Esq., M.A., Government College, Hoshiarpur.

Gian Chand Esq., M.A., " " "

Dr. S. Chowla, Ph.D. (Cantab), Government College, Lahore.

A. M. Krishnamurti Esq., M.A., National College, Trichinopoly.

Arrangements are in progress for the Thirteenth Biennial Conference of the Indian Mathematical Society which is to meet at Annamalainagar on the 28th, 29th and 30th of December 1943. The dates have been so fixed that the delegates proceeding to the Indian Science Congress at Trivandrum, will be able to halt at Annamalainagar *en route*, participate in the activities of the Conference, and still be in time for the Science Congress. All those desirous of attending the Mathematical Conference are requested to communicate with the Local Secretary, Dr. A. Narasinga Rao, Dean, Faculty of Science, Annamalai University, Annamalainagar, stating the date of arrival and the kind of food they will require.

The "Mathematical Exposition" will be opened on the afternoon of the 28th December. Besides models, pictures, charts and other material intended to illustrate interesting results in Mathematics and the richness and variety of its applications to life situations, there will be a "Book Section" for exhibiting text-books on Collegiate and Higher Mathematics and its applications. All books, intended for the exhibition may be sent to the Local Secretary to reach him at least a week before the Conference. Those willing to send charts, diagrams or models should communicate with the Local Secretary.

An appeal for pictorial and other material for the Mathematical Exposition has met with a very generous response from the U. S. A. The authorities of the popular Mathematical Journal, *Scripta Mathematica* have been pleased to present a large number of portraits, of Mathematicians, Physicists and Philosophers as also a complete set of their "Scripta Mathematica Library" publications. Messrs. Simon and Schuster have presented their fascinating books: Bell: *Men of Mathematics*, and Kasner and Newman: *Mathematics* [and] *the Imagination*, while Messrs. H. G. Lieber have sent four [deligh]tful booklets on Relativity, Non-Eucledean Geometry, Algebra and Galoisian groups which mark a new era in simple and attractive presentation of mathematical topics.

The Editors of the Mathematical Reviews, Brown University Providence, R. I., U. S. A., are finding it difficult to procure for purposes of review, research papers published in India. They request the cooperation of Indian Mathematicians by sending them promptly reprints of their papers.

The National Academy of Sciences, India, at its 12th Annual Session held at Allahabad awarded the Gold Medal offered by Dr. Panna Lal, Adviser to the Governor of the U. P. to Dr. Ram Behari, Professor of Mathematics, St. Stephen's College, Delhi, for his group of papers on "Differential Geometry" which have been assessed as the best papers in Mathematics published in the Proceedings of the Academy during the last five years.

Non-Solar Planetary Systems: Till recently there was no evidence of planetary systems belonging to stars other than our Sun, but the *Observatory* of May 1943 gives details regarding two cases of visual binaries where parallax observations have led to the inference that a third invisible companion exists sufficiently small to be classified as a planet. In the system 61 *Cygni* (period 720 years) the deviations from the Keplerian motion have a period of 4·9 years, and the invisible companion must be of mass 0·016 ⊙ with a semi-major axis of 2·4 astronomical units and a highly eccentric orbit ($e = \cdot 7$). In the other case 70 *Ophiuchi* (period 88 years), the deviations have a period of 17 years and the disturbing planet should have a mass 0·012 ⊙ or ·008. according as it is considered as belonging to one or the other of the two components of the binary system. In both cases the perturbing mass is less than 1/9th of the mass of the lightest known star (*Kruger* 60B = 0·14 ⊙).

Prof. Earl Raymond Hedrick of the Brown University well known author of text books, died in February 1943.

MATHEMATICAL GREETINGS TO THE NEW YEAR

O Mother Earth, greetings to you with all hearty good wishes, ere you begin your next course on a long elliptic journey of 292 million miles. This is your one thousand nine hundred and forty third course in the heavens, reckoned from the date when the Son of God went back to the Father in Heaven.

Mortal man keeps count of the successive steps of your career by an elaborate machinery which he calls the 'Calendar,' printing it year in and year out with dim awareness of your mathematical regularities. Wise men know that the next year of grace 1943 is identical in dates and week-days with those of several years that have gone before and several others yet to come.

In illustration thereof, it may be pointed out that the calendars of the years 1909, 1915, 1926, 1937, **1943**, 1954, 1965, 1971, 1982, 1993, 1999 are identical! Note that the coming year is the fifth spoke in the wheel of the 6–11–11 cycle.

Ye aspirants to centenarianism realise the value of the New Year calendar till the penultimate year of this century.

Ye men of the year of grace 1937 that have preserved untarnished the calendar of that year celebrate its re-incarnation in the New Year.

Ye number-intoxicated children of Mother Earth, rejoice in the birth of the New Year, and propitiate her in her manifold mathematical manifestations as the glorious product of two primes (29×67) the illustrious union of three $(643 + 647 + 653)$, the powerful combination of a plane and a solid number $(40^2 + 7^3)$, the harmonious whole of four beautiful squares $(1^2 + 6^2 + 15^2 + 41^2)$, or a square and two cubes $(10^2 + 8^3 + 11^3)$ the lawful predecessor of the sum of two cubes $(12^3 + 6^3)$ and the product of a cube and a fifth power $(2^3 \times 3^5)$.

Ye philosophers, contemplate on this fresh section of the Eternal Element of Time. and surrender your all in humble homage to the DIVINE IMMINENCE.

A. A. K.

GLEANINGS

"Mathematics, to my way of thinking, is the most general of all subjects. Everything else is more special. There is no school subject that has a richer profusion of applications. There is nothing that travels over the whole domain of human knowledge as does mathematics. There is no surer way to unlock all sorts of doors than mathematics. It won't get you all the way, but it will get you into places where you can't enter by any other method. It will supplement all other types of investigations and help get at profounder truths than will be possible without its aid."

Prof. HOTELLING.

A knowledge of mathematics is one of the foundations on which the operations of modern armies are based. It cannot take the place of study of tactics and strategy, but now that armies are so thoroughly mechanized. It is impossible to visualize their successful operation unless the officers and men of which they are composed have a through groundwork in mathematical training.

Major BARRETT.