MADURAI KAMARAJ UNIVERSITY

(University with Potential for Excellence)

DISTANCE EDUCATION

B.Sc., (Mathematics)

THIRD YEAR

GRAPH THEORY

UNIT : 1 - 5 (VOLUME - 1)

Ancillary Paper - I

Recognized By DEC
www.mkudde.org

S 91

MADURAI KAMARAJ UNIVERSITY

(University with Potential for Excellence)

DISTANCE EDUCATION

B.Sc. (Mathematics)

THIRD YEAR

UNIT: 1 – 5 (Volume - 1)

Ancillary paper- I

GRAPH THEORY

Recognised by D.E.C.

www.mkudde.org

S 91

JEGN/1000 COPIES

B.Sc., Mathematics **GRAPH THEORY**

Third Year **Ancillary Paper - I**

Dear Student,

We welcome you as a Student of the Third year B.sc degree Course in mathematics.

This Ancillary paper – I deals with Graph Theory. The learning material for this paper will be supplemented by Contact seminars. ·

Learning through the Distance Education mode, as you are all aware, involves self – learning and self – assessment and in this regard you are expected to put in disciplined and dedicated effort.

On our part, we assume of our guidance and support.

With best wishes.

SYLLABUS

B.Sc., Mathematics – Third year.

Ancillary Paper – 1

Graph theory

UNIT – 1:

Introduction –The konigsberg Bridge Problem - Four Colour Problem -- Graphs - Definition and Examples with pictorial Representation – Sub graphs – Isomorphism – Degress.

UNIT – 2:

Walks and Connected graphs – Cycles in Graphs.

UNIT – 3:

Eulerian Graphs – Fleury's Graphs.

UNIT – 4:

Hamiltonian Graphs – Weighted Graphs.

UNIT – 5:

Bipartite Graphs – Marriage Problem – Trees – Connector Problem.

UNIT – 6:

Matrix representation – Vector Spaces Associated with Graphs.

UNIT – 7:

Cycle space – Cut – Set space.

UNIT – 8:

Planar Graphs – Euler formula – Platonic Solids – Dual of plane graph – characterization of Planar graphs.

UNIT – 9:

Colourings – Vertex Colouring – Edge Colouring – An Algorithm for colouring of a graph.

UNIT – 10:

Directed Graphs – Connectivity in digraphs – Strong – Orientation of Graphs – Eulerian digraphs – Tournaments.

Text Book:

1. "Invitation to Graph Theory" by Dr. S. Arumugam and S. Ramachandran, New Gamma, June 2001.

CONTENTS

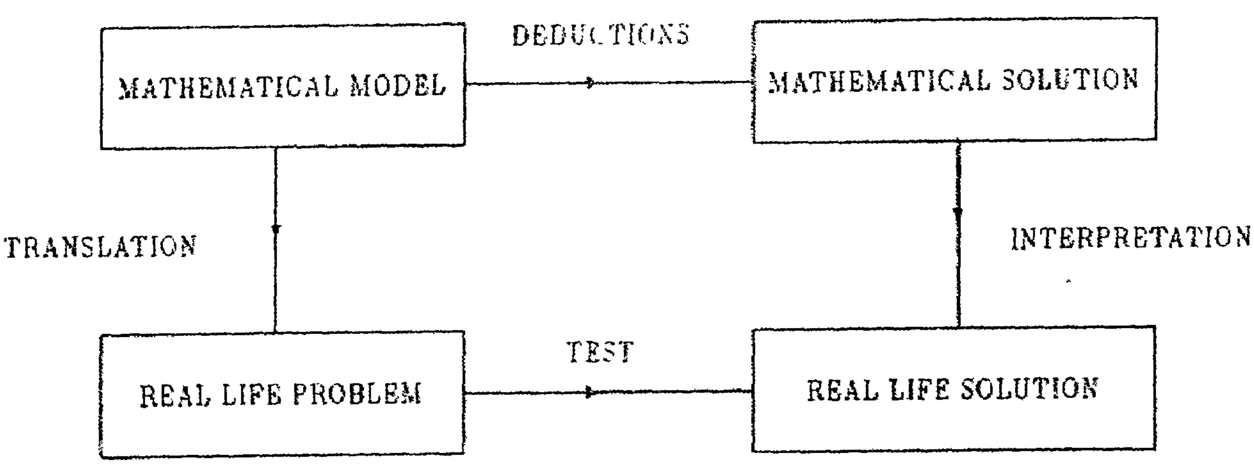
UNIT – 1

INTRODUCTION

1.0 INTRODUCTION

In the last three decades graph theory has established itself as a worthwhile mathematical discipline and three are many applications of graph theory to a wide variety of subjects which include Operations Research, Physics, Chemistry, Economics, Genetics, Sociology, Linguistics, Engineering, Computer Science etc.

The development of many branches in Mathematics has been necessitated while considering certain real life problems arising in practical life or problems arising in other sciences. Such a development may be roughly described as follows.



Graph theory also has been independently discovered many times through some puzzles that arose from the physical world, consideration of chemical isomers, electrical networks etc.

In this chapter some fundamental concepts are notations of graph theory are introduced and types of graphs and subgraphs are examined in detail with number of examples.

1.1 THE KONIGSBERG BRIDGE PROBLEM

The first paper is Graph Theory was written by Euler in 1736 when he settled the famous unsolved problem of his day, known as the **Konigsberg Bridge problem**. Konigsberg (55.2°

Space for Hints

North lattitude and 22° East longitude) is now called Kaliningrad and is in Lithuania which recently separated from U.S.S.R. The two islands and seven bridges are shown in figure 1.1

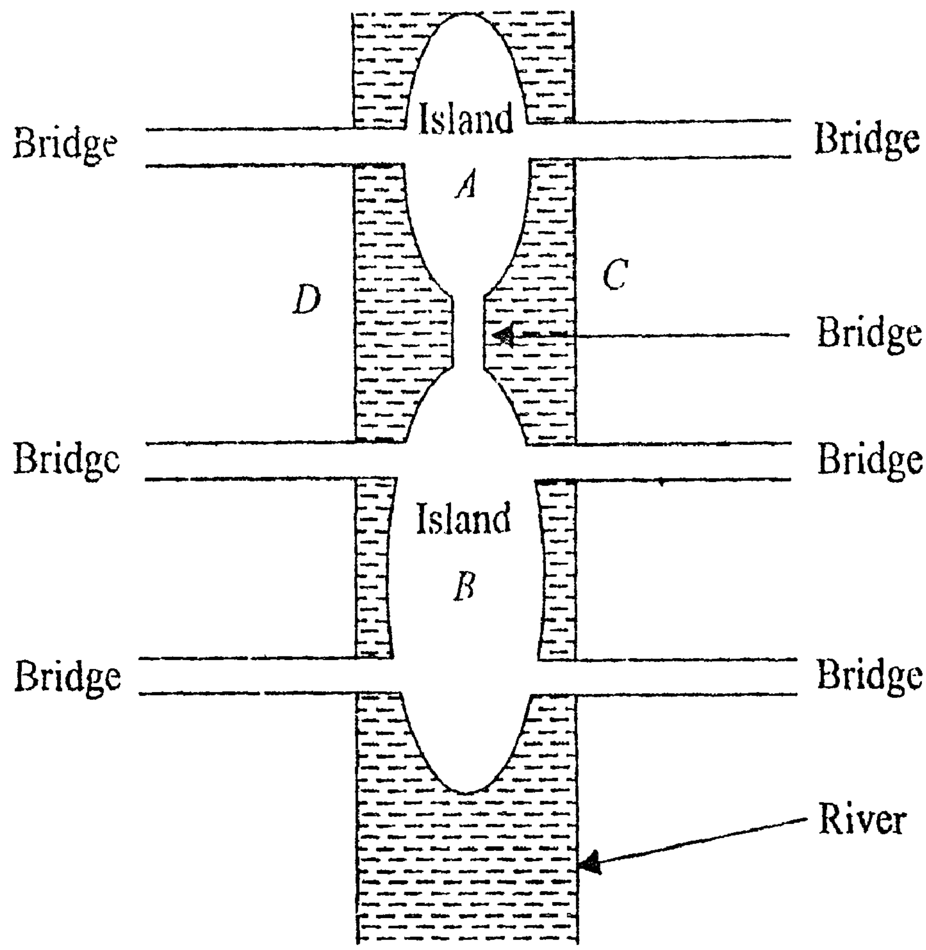


Figure 1.1

The people of Konigsberg posed the following question to famous Swiss Mathematician Leon hard Euler:

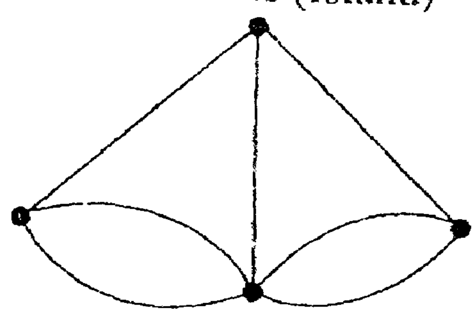
"Beginning anywhere and ending anywhere, can a person walk through the town of Konigsberg crossing all the seven bridges exactly once?"

Euler showed that such a walk is impossible. He replaced the islands A, B and the two side (banks) C and D of the river by vertices and the bridges as edges of a graph. We note then that

$$\deg(A) = 3, \qquad \deg(B) = 5,$$

$$\deg(C) = 3, \qquad \deg(D) = 3.$$

Thus the graph of the problem is as shown in figure 1.2



(Euler's graphical representation of seven bridges problem)

Figure 1.2

The Problem then reduces to

"Is there any Euler's path in the above diagram?"

To find the answer, we note that there are more than two vertices having odd degree. Hence there exists no Euler path for this graph.

The Konigsberg bridge problem is the same as the problem of drawing the above figure without lifting the pen from the paper and without retracing any line and coming back to the starting point. This problem was generalized and a necessary and sufficient condition for a graph to be so traversable has been obtained.

1.2 FOUR COLOUR PROBLEM

One of the most famous problems in Graph Theory is the **four colour problem**. The problem states that any map on a plane or on the surface of a sphere can be coloured with four colours in such a way that no two adjacent countries have the same colour. This problem can be translated as a problem in Graph theory.

We represent each country by a point and joint two points by a line if the countries are adjacent. The problem is to colour the points in such way that adjacent points have different colours. This problem was first posed in 1852 by Francis Guthrie, a post – graduate student at the University College, London. This problem was finally proved by Appel and Haken in 1976 and they have used 400 pages of arguments and about 1200 hours of computer

Space for Hints

time on some of the best computers in the world to arrive at the solution.

1.3 GRAPHS

1.3.1 DEFINTION AND EXAMPLES

Definition.1.3.1

A **graph** G consists of a pair (V(G), X(G)) where V(G) is a non-empty finite set whose elements are called **points** or **vertices** and X(G) is a set of unordered pairs of distinct elements of V(G).

The elements of X(G) are called **lines** or **edges** of the graph G. If $x = \{u, v\} \in X(G)$, the line x is said to joint u and v. We write $x = uv$ and we say that the points u and v are **adjacent.**

We also say that the point u and the line x are **incident** with each other.

If two distinct lines x and y are incident with a common point then they are called **adjacent lines.**

A graph with p points and q lines is called a (p, q) **graph.**

When there is no possibility of confusion we write $V(G) = V$ and $X(G) = X$.

Remark

It is customary to represent a graph by a diagram and refer to the diagram itself as the paragraph. Each point is represented by a small dot and each line is represented by a line segment joining the two points with which the line is incident. Thus a diagram of the graph depicts the incidents relation holding between its points and lines. In drawing a graph it is immaterial whether the lines are drawn straight or curved, long or short and what is important is the incidence relation between its points and lines.

Examples

1. Let $V = \{a, b, c, d\}$ and $X = \{\{a, b\}, \ \{a, c\}, \ \{a, d\}\}$.

$G = (V, X)$ is a $(4,3)$ graph. This graph can be represented by the diagram given in Fig.1.3

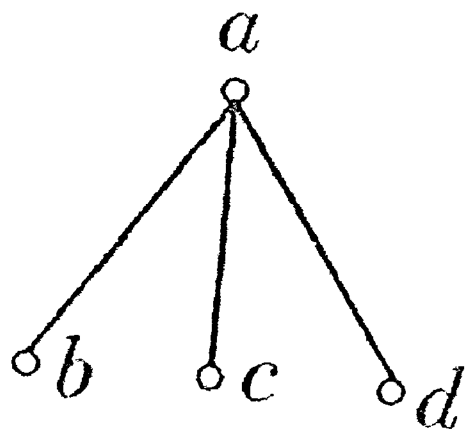


Figure 1.3

In this graph the points a and b are adjacent whereas b and c are non-adjacent.

2. Let $V = \{1,2,3,4\}$ and $X = \{\{1,2\}, \{1,3\}, \{1,4\} \ \{2,3\}, \{2,4\}, \{3,4\}$. $G = (V, X)$ is a (4, 6) graph.

This graph is represented by the diagram given in Fig1.4 Although the lines $\{1,2\}$ and $\{2,4\}$ intersect in the diagram, their intersection is not a point of the graph.

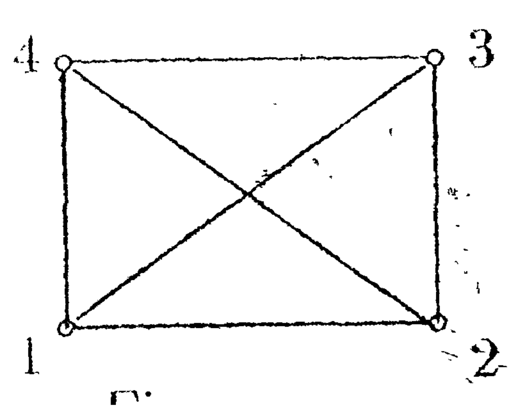


Figure 1.4

Fig.1.5 is another diagram for the graph given in Fig.1.4

Space for Hints

Space for Hints

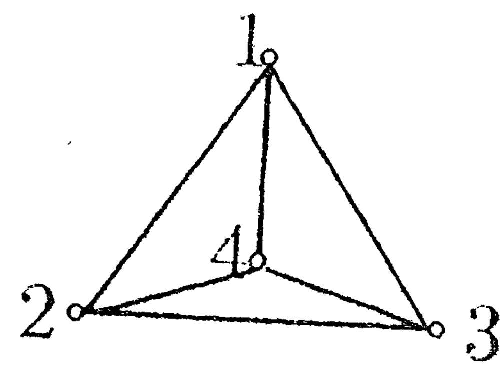


Figure 1.5

3. The (10,15) graph given in Fig.1.6 is called the **Petersen graph.**

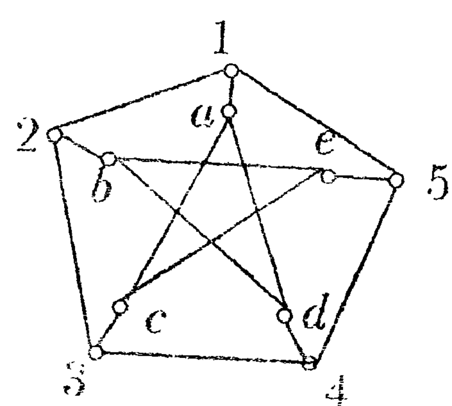


Figure 1.6

Remark

The definition of a graph does not allow more than one line joining two points. It also does not allow any line joining a point to itself. Such a line joining a point to itself is called a **loop.**

Definition.1.3.2

If more than one line joining two vertices are allowed, the resulting object is called a **multigraph**. Lines joining the same points are called **multiple lines.** If further loops are also allowed, the resulting object is called a **Pseudo graph**.

Example

Fig. 1.7 is a multigraph and Fig 1.8 is a pseudo graph. Figure

1.2 of the Konigsberg bridge problem is a multigraph.

Space for Hints

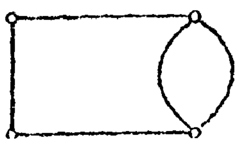


Figure 1.7

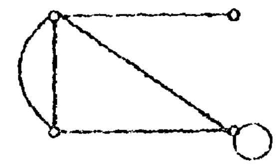


Figure 1.8

Definition: 1.3.2. (a)

The set of all vertices adjacent to a vertex v is called the neighbourhood of v and is denoted by N(v).

Example:

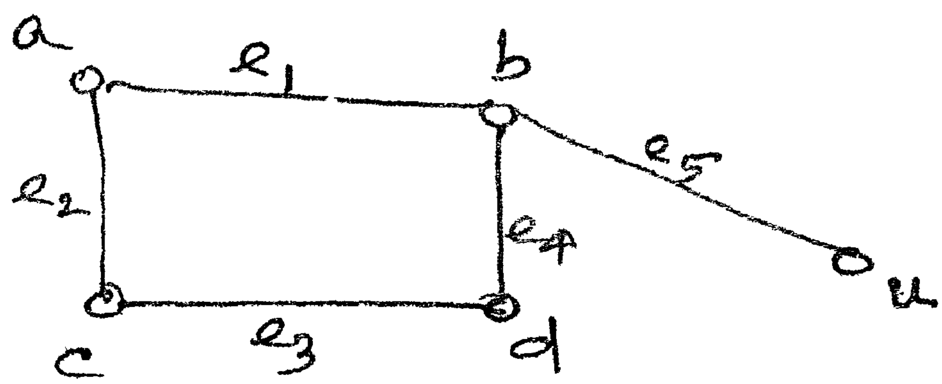


Figure 1.8(a)

Here, the neighbourhood of b is $\{a, d, u\}$ and $\{e_1, e_4, e_5\}$ are incident with b.

Remark:

Let G be a (p, q) graph. Then $q \leq \binom{p}{2}$ and $q = \binom{p}{2}$ iff any two distinct points are adjacent.

Definition: 1.3.3

A graph in which any two distinct points are adjacent is called a **complete graph**.

Space for Hints

Note:

The complete graph with p points is denoted by K_p.

Example:

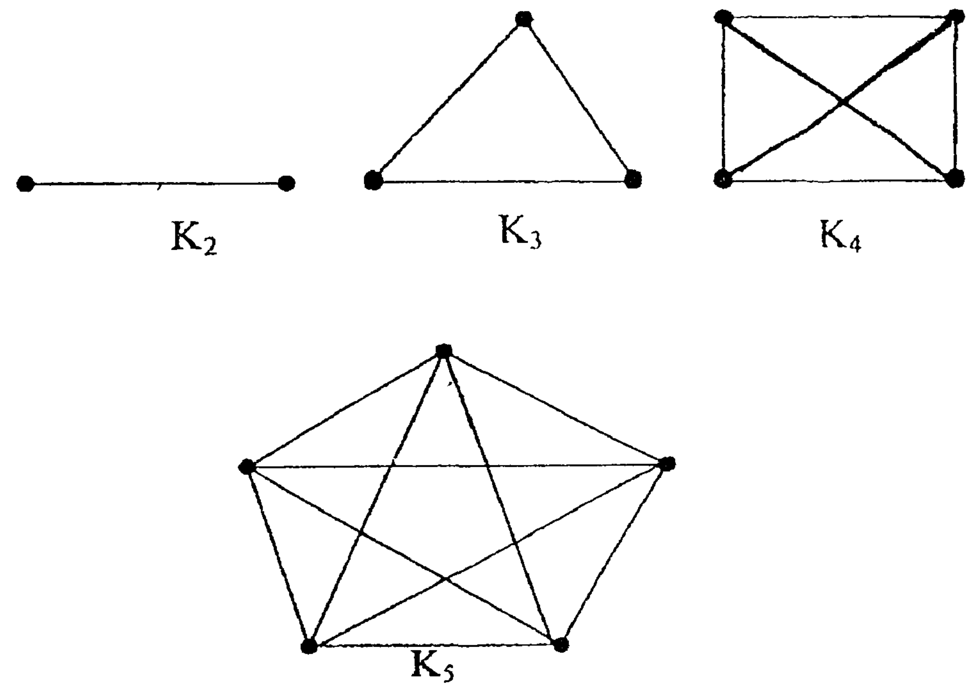


Figure 1.9

Note:

The complete graph of 5 vertices, K_5 is called **kuratowski's first graph**.

Definition: 1.3.4

A graph whose edge set is empty is called a **null graph** or a **totally disconnected graph**.

Example:

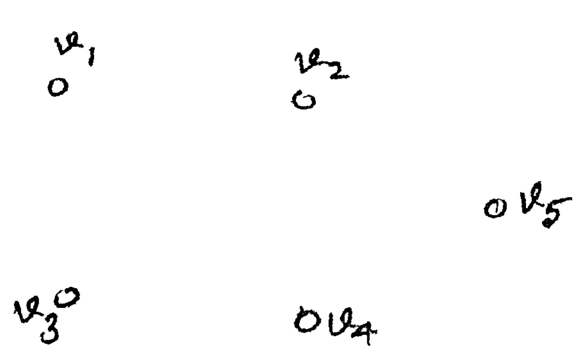


Figure 1.9 (a)

Definition: 1.3.5

Space for Hints

A graph G is called labelled if its p points are distinguished from one another by names such as $v_1, v_2, \ldots\ldots, v_p$.

The graphs given in Fig. 1.3 and 1.5 are labelled graphs and the graph in Fig 1.9 is an unlabelled graph.

Definition: 1.3.6

A graph G is called a **bigraph** or **bipartite graph** if V can be partitioned into two disjoint subsets V_1 and V_2 such that every line of G joins a point of V_1 to a point of V_2. (V_1, V_2) is called a **bipartition** of G. If further G contains ever line joining the points of V_1 to the points of V_2 then G is called a **complete bigraph.** If V_1 contains m points and V_2 contains n points then the complete bigraph G is denoted by $K_{m,n}$.

The graph given in Fig. 1.3 is $K_{1,3}$. The graph given in Fig. 1.10 is $K_{3,3}$. $K_{1,m}$ is called a **star** for $m \geq 1$.

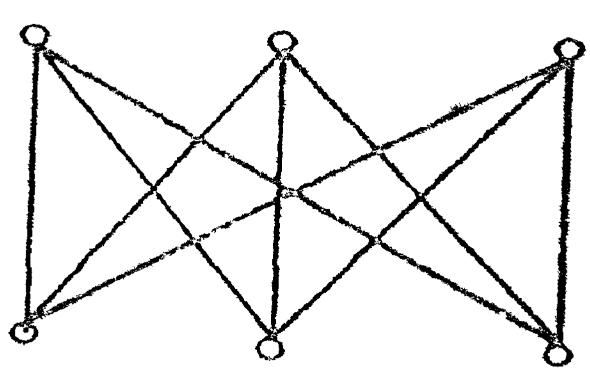


Figure 1.10

Definition: 1.3.7

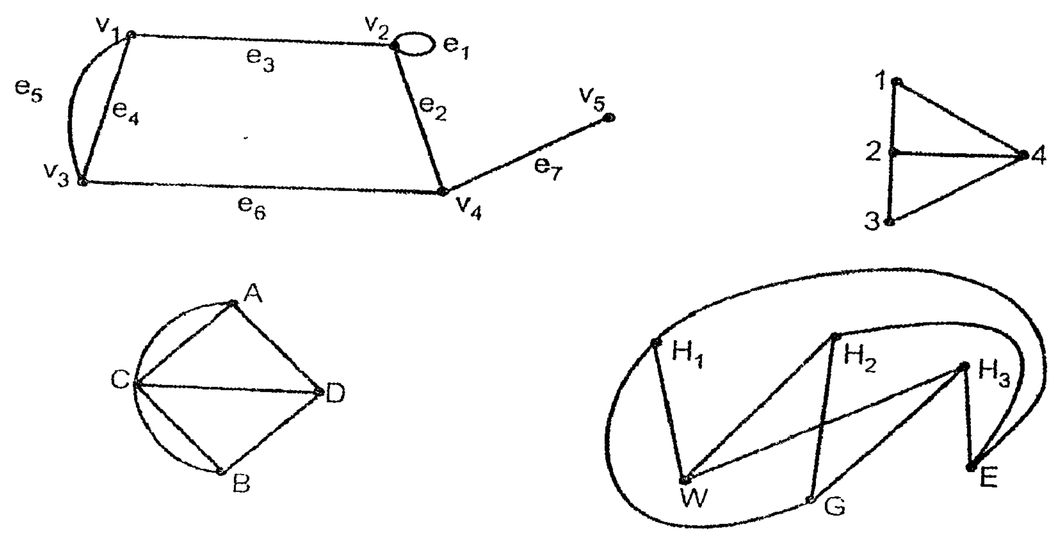
Finite and Infinite Graphs

A graph with a finite number of vertices as well as a finite

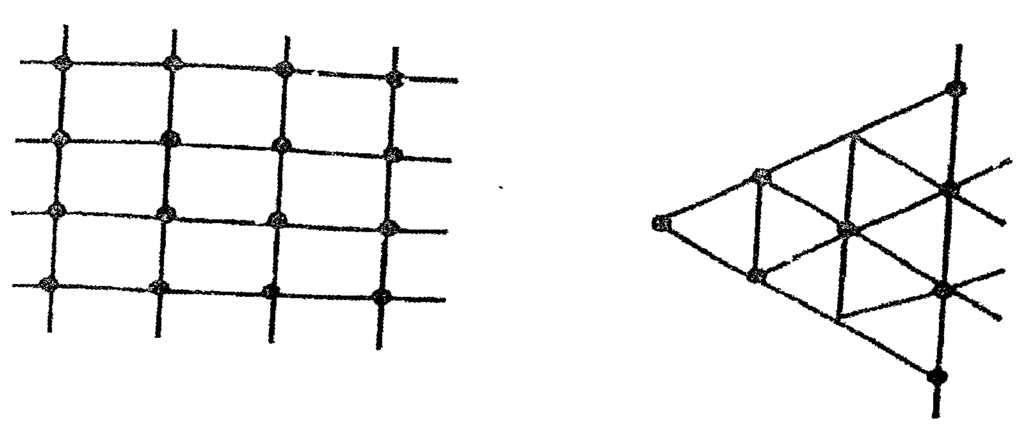
Space for Hints

number of edges is called finite graphs, otherwise it is an infinite graph.

Examples of Finite graphs



Examples of infinite graphs



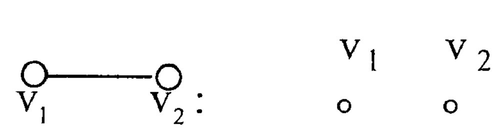
Problem: 1

Draw all graphs with 1, 2, 3 and 4 points.

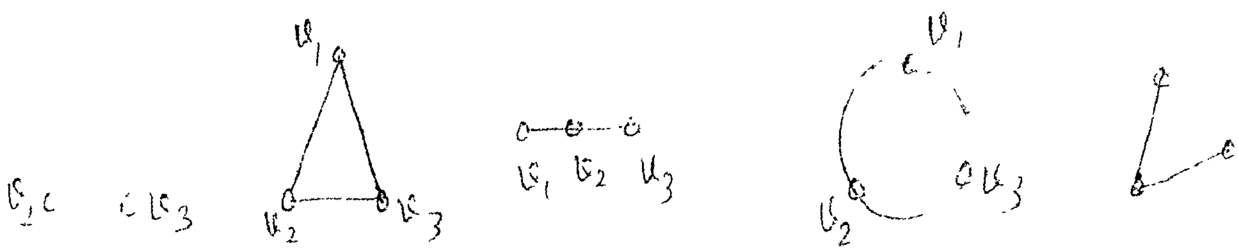
Solution:

Take point : 1 $\circ^{v}$

Points : 2



Points : 3

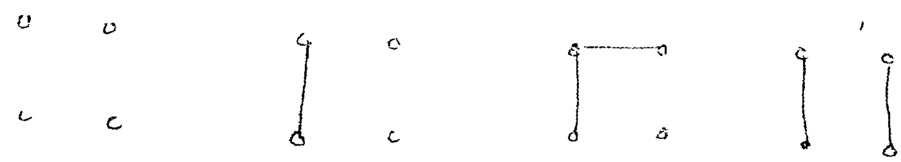


Points : 4

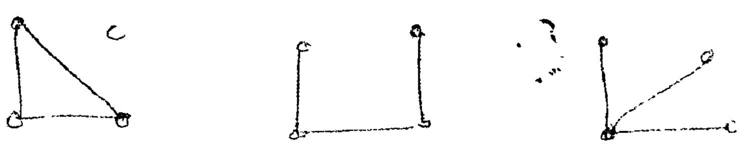
Here

P = 4;

q = 0



p = 4, q = 1   q = 2; p = 4   p = 4; q = 2

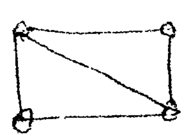


← Here p = 4;

q = 3

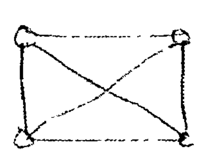
← Here p = 4;

q = 4



← Here p = 4;

q = 5



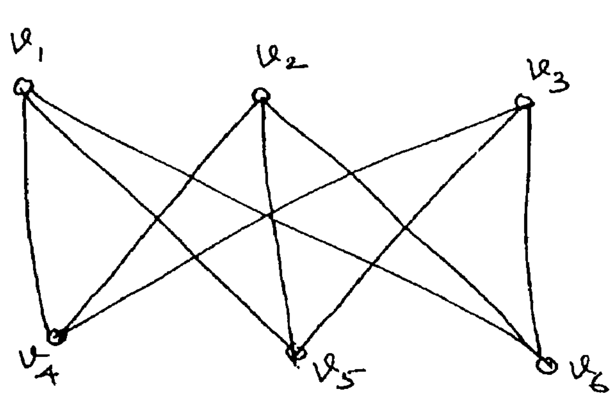
← Here p = 4;

q = 6

Problem: 2

Find the number of points and lines in $K_{m,n}$.

Solution:



$K_{3,3}$

Let $K_{n,n}$ be the complete bipartite graph.

$\therefore$ The number of points is $m+n$.

$\therefore$ The number of edges is mn.

For example: consider a graph $k_{3,3}$

$\therefore$ The number of points $= 3+3 = 6$

$\therefore$ The number of edges $=$ $3x3 = 9$

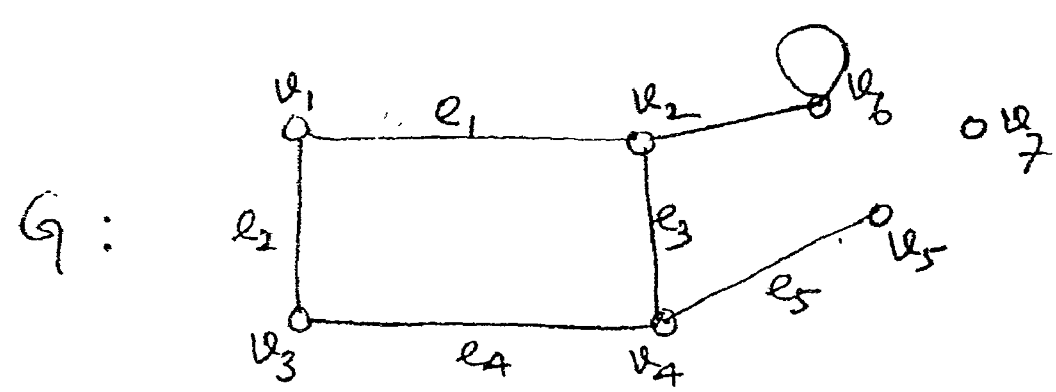
1.4 DEGREES

Definition: 1.4.1

The **degree** of a point v in a graph G is the number of lines incident with v_i. The degree of v denoted by $d_G(v_i)$ or deg v or simply $d(v_i)$.

A point v of degree 0 is called an **isolated point**. A point v of degree 1 is called an **end point** (or) **pendant vertex**.

Example:



Here $d(v_1) = d(v_3) = 2$

$d(v_2) = d(v_4) = 3$

$d(v_5) = 1$ [$\because$ Here v_5 end point]

Space for Hints

$d(v_6) = 3$ [$\because$ Here v_6 - self loop]

$d(v_7) = 0$ [$\because$ Here v_7 - isolated point]

Note: 1

An edge of a graph that joins a node to itself is called **loop** or **self loop**.

i.e., A loop is an edge (v_i, v_j) where $v_i = v_j$.

Note: 2

The degree of the vertex for self – loop is two.

Fundamental Theorem of Graph Theory:

Theorem: 1.1

The sum of the degrees of the points of a graph G is twice the number of lines. i.e., $\sum_i \deg v_i = 2q$.

Proof:

Every line of G is incident with two points. Hence every line contributes 2 to the sum of the degrees of the points.

$$\text{Hence } \sum_i \deg v_i = 2q$$

Corollary:

In any graph G the number of points of odd degree is even.

Proof:

Let $v_1, v_2, \ldots\ldots, v_k$ denote the points of odd degree and $w_1, w_2, \ldots\ldots, w_m$ denote the points of even degree in G.

Space for Hints

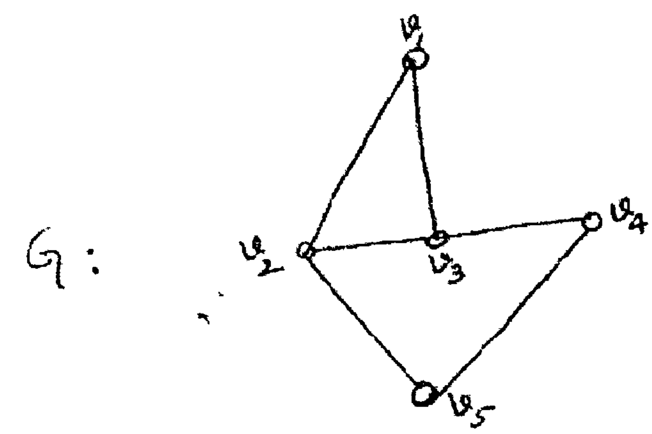
By theorem 1.1, $\sum_{i=1}^{k} \deg v_i + \sum_{i=1}^{m} \deg w_i = 2q$ which is even.

Further $\sum_{i=1}^{m} \deg w_i$ is even.

Hence $\sum_{i=1}^{k} \deg v_i$ is also even. But $\deg v_i$ is odd for each i.

Hence k must be even.

Example: 1



Let G be a graph

Number of edges in G = 6

$\therefore$ Sum of all degrees of all degree vertices = 12.

Example: 2

Consider the above example (1) figure, v_2 and v_3 have odd degree whose sum is 6.

Definition: 1.4.2

For any graph G, we define

$\delta(G) = \min\{\deg v / v \in V(G)\}$ and

$$\Delta(G) = \max\{\deg v / v \in V(G)\}.$$

If all the points of G have the same degree r then $\delta(G) = \Delta(G) = r$ and in this case G is called a **regular graph** of degree r.

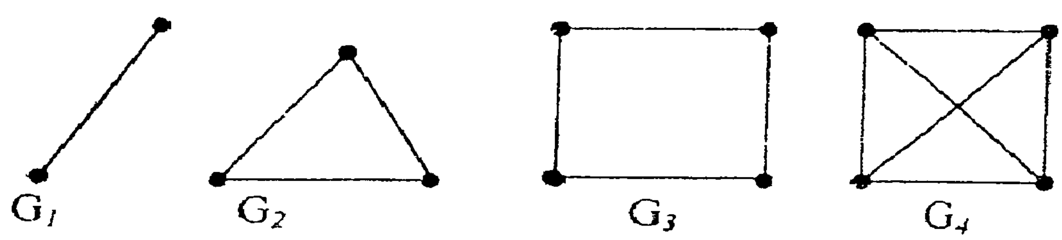
Note 1:

A regular graph of degree 3 is called a **cubic graph.**

Note 2:

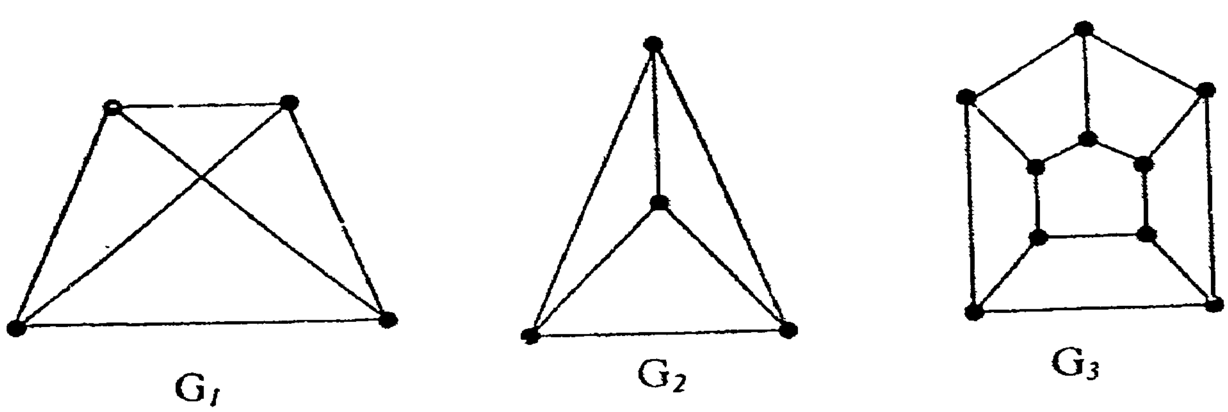
The complete graph K_p is regular of degree p – 1.

Example: 1



G_1, G_2, G_3, G_4 are regular graphs.

Example: 2



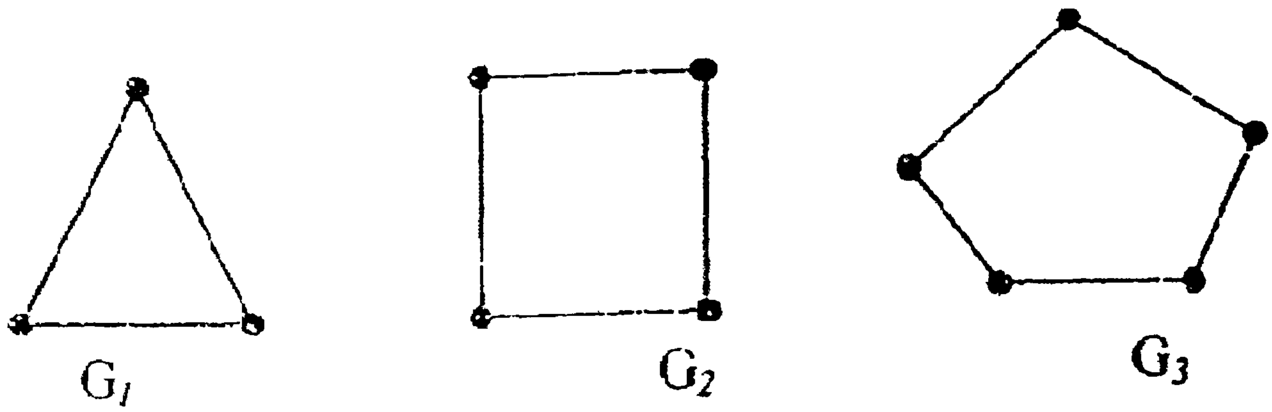
G_1, G_2, G_3 are cubic graphs.

Definition: 1.4.3

If every vertex of a graph G is of degree 2, then G is said to b a **cyclic graph** or a **cycle** or a **circuit**. Thus a cyclic graph is

Space for Hints

2 – regular.



The above cyclic graphs G_1, G_2, G_3, are usually known as a triangle, a quadrilateral and a pentagon respectively.

Theorem: 1.2

Every cubic graph has an even number of points.

Proof:

Let G be a cubic graph with p points. Then $\sum \deg v = 3p$ which is even by theorem 1.1. Hence p is even.

Problem: 1

Let G be a (p, q) graph all of whose points have degree k or k +1.

If G has $t > 0$ points of degree k, show that $t = p(k+1) - 2q$.

Solution:

Since G has t points of degree k, the remaining p – t points have degree $k+1$. Hence $\sum_{v \in V} d(v) = tk + (p-t)(k+1)$.

$$\therefore tk + (p-t)(k+1) = 2q.$$

Space for Hints

$$\therefore t = p(k+1) - 2q.$$

Problem: 2

Show that in any group of two or more people, there are always two with exactly the same number of friends inside the group.

Solution:

We construct a graph G by taking the group of people as the set of points and joining two of them if they are friends. Then $\deg v =$ number of friends of v and hence we need only to prove that at least two points of G have the same degree.

Let $V(G) = \{v_1, v_2, \ldots, v_p\}$

Clearly $0 \le \deg v_i \le p-1$ for each i.

Suppose no two points of G have the same degree. Then the degrees of $v_1, v_2, \ldots, v_p$ are the integers 0, 1, 2….., p – 1 in some order. However a point of degree p – 1 is joined to every other point of G and hence no point can have degree zero which is a contradiction. Hence there exist two points of G with equal degree.

Problem: 3

Prove that $\delta \le \frac{2q}{p} \le \Delta$.

Solution:

Let $V(G) = \{v_1, v_2, \ldots, v_p\}$.

We have $\delta \le \deg v_i \le \Delta$ for all i.

Space for Hints

$$\text{Hence } p\delta \le \sum_{i=1}^{p} \deg v_i \le p\Delta.$$

$$\therefore p\delta \le 2q \le p\Delta \qquad \text{(by theorem 1.1).}$$

$$\therefore \delta \le \frac{2q}{p} \le \Delta.$$

Problem: 4

Let G be a k – regular bigraph with bipartition (V_1, V_2) and $k > 0$. Prove that $|V_1| = V_2|$.

Solution:

Since every line of G has one end in V_1 and other end in V_2, it follows that $\sum_{v \in V_1} d(v) = \sum_{v \in V_2} d(v) = q$.

Also $d(v) = k$ for all $v \in V = V_1 \cup V_2$. Hence $\sum_{v \in V} d(v) = k \; V_1|$ and $\sum_{v \in V_2} d(v) = k|V_2|$ so that $k|V_1| = k|V_2|$.

Since $k > 0$, we have $|V_1| = |V_2|$.

Exercises:

1. Give an example of a regular graph of degree 0.
2. Give three examples for a regular graph of degree 1.
3. Give three examples for a regular graph of degree 2.
4. What is the maximum degree of any point in a graph with p points?
5. Show that a graph with p points is regular of degree p – 1 iff it is complete.
6. Let G be a graph with at least 2 points. Show that G contains two vertices of the same degree.

7. A (p, q) graph has t points of degree m and all other points are of degree n. Show that $(m-n)t+pn=2q$.

Space for Hints

1.5 SUBGRAPHS

Definition: 1.5.1

A graph $H=(V_1,X_1)$ is called a **subgraph** of $G=(V,X)$ if $V_1 \subseteq V$ and $X_1 \subseteq X$. If H is a subgraph of G we say that G is a **supergraph** of H. H is called a **spanning subgraph** of G if $V_1 = V$. H is called an **induced subgraph** of G if H is the maximal subgraph of G with point set V_1.

Thus, if H is an induced subgraph of G, two points are adjacent in H iff they are adjacent in G. If $V_2 \subseteq V$, then the induced subgraph of G with point set V_2 is called the subgraph of G induced by V_2 and is denoted by $G[V_2]$.

If $X_2 \subseteq X$, then the subgraph of G with line set X_2 and having no isolated points is called the subgraph line induced (edge induced) by X_2 and is denoted by $G[X_2]$.

Examples:

Consider the Petersen graph G given in Fig. 1.6.

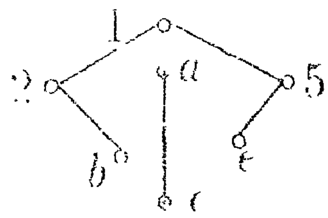


Figure 1.11

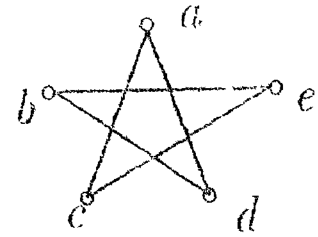


Figure 1.12

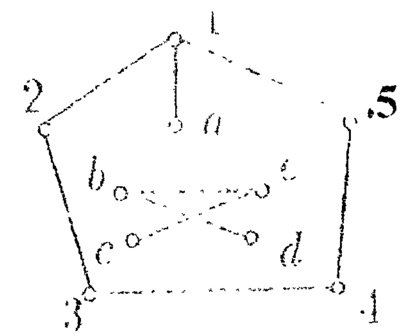


Figure 1.13

The graph given in Fig. 1.11 is a subgraph of G. The graph given in Fig. 1.12 is an induced subgraph of G. The graph given in Fig. 1.13 is a spanning subgraph of G.

Definition: 1.5.2

Let $G=(V,X)$ be a graph. Let $v_i \in V$. The subgraph of G obtained by removing the point v_i and all the lines incident with v_i is called the **subgraph obtained by the removal of the point** v_i and is denoted by $G-v_i$.

Thus if $G-v_i=(V_i,X_i)$ then $V_i=V-\{v_i\}$ and $X_i=\{x / x\in X$ and x is not incident with $v_i\}$.

Clearly $G-v_i$ is an induced subgraph of G.

Let $x_j \in X$. Then $G-x_j=(V,X-\{x_j\})$ is called the subgraph of G obtained by the removal of the line x_j. Clearly $G-x_j$ is a spanning subgraph of G which contains all the lines of G except x_j.

The removal of a set of points or lines from G is defined to be the removal of single elements in succession.

Definition: 1.5.3

Let $G=(V,X)$ be a graph. Let v_i, v_j be two points which are not adjacent in G. Then $G+v_iv_j=(V, X\cup\{v_i, v_j\})$ is called the graph obtained by **the addition of the line** v_iv_j to G.

Clearly $G+v_iv_j$ is the smallest super graph of G containing the line v_iv_j. We illustrate the concepts in figure 1.14.

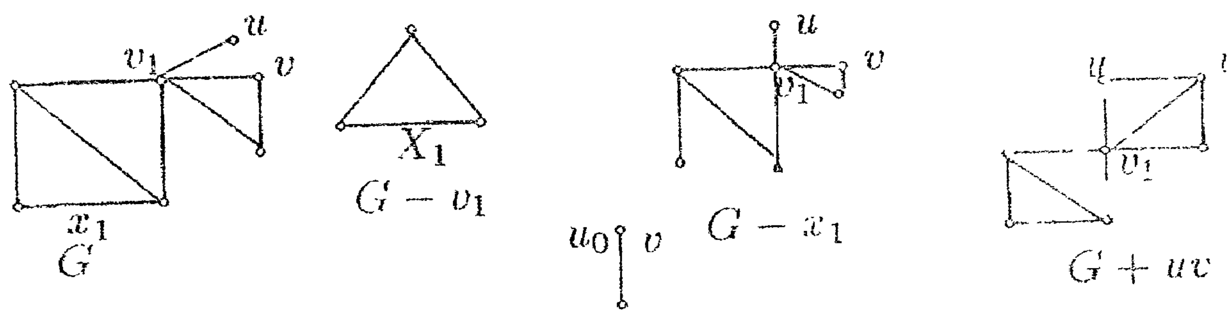


Figure 1.14

The proof given in the following theorem is typical of several proofs in graph theory.

Theorem 1.3

The maximum number of lines among all p point graphs with no triangles is $\left[\frac{p^2}{4}\right]$.

([x] denotes the greatest integer not exceeding the real number x).

Proof

The result can be easily verified for $p \le 4$.

For $p > 4$, we will prove by induction separately for odd p and for even p.

Part: 1

For odd p.

Space for Hints

Space for Hints

Suppose the result is true for all odd $p \leq 2n+1$.

Now let G be a (p,q) graph with $p = 2n + 3$ and no triangles. If $q = 0$, then $q \leq \left[\frac{p^2}{4}\right]$. Hence let $q > 0$. Let u and v be a pair of adjacent points in G. The subgraph $G' = G - \{u, v\}$ has 2n + 1 points and no triangles. Hence by introduction hypothesis,

$$q(G^1) \leq \left[\frac{(2n+1)^2}{4}\right] = \left[\frac{4n^2 + 4n + 1}{4}\right]$$

$$= \left[n^2 + n + \frac{1}{4}\right] = n^2 + n \qquad \ldots (1)$$

Since G has no triangles, no point of G' can be adjacent to both u and v in G $\ldots (2)$

Now, lines in G are of three types.

I. Lines of G' ($\leq n^2 + n$ in number by (1))

II. Lines between G' and $\{u, v\}$ ($\leq 2n + 1$ in number by (2))

III. Line uv.

Hence $q \leq (n^2 + n) + (2n + 1) + 1 = n^2 + 3n + 2$

$$= \frac{1}{4}(4n^2 + 12n + 8)$$

$$= \left(\frac{4n^2 + 12n + 9}{4} - \frac{1}{4}\right)$$

$$= \left[\frac{(2n+3)^2}{4}\right] = \left[\frac{p^2}{4}\right].$$

Also for $p = 2n + 3$, the graph $K_{n+1, n+2}$ has no triangles and has

$(n + 1)(n + 2) = n^2 + 3n + 2 = \left[\frac{p^2}{4}\right]$ lines.

Hence this maximum q is attained.

Part: 2

For even p.

Suppose the result is true for all even $p \leq 2n$.

Now let G be a (p, q) graph with $p = 2n + 2$ and no triangles. As before, let u and v be a pair of adjacent points in G and let $G' = G - \{u, v\}$.

Now G' has $2n$ points and no triangles. Hence by hypothesis,

$$q(G') \leq \left[\frac{(2n)^2}{4}\right] = n^2 \qquad \ldots (3)$$

Lines in G are of three types.

i. Lines of G' ($\leq n^2$ in number by (3))
ii. Lines between G' and $\{u, v\}$ ($\leq 2n$ in number by an argument similar to (2))
iii. Line uv.

Hence $q \leq n^2 + 2n + 1 = (n + 1)^2 = \frac{(2n + 2)^2}{4} = \left[\frac{p^2}{4}\right]$.

Hence the result holds for even p also.

We see that for $p = 2n + 2$. $K_{n+1, n+1}$ is a $\left(p, \left[\frac{p^2}{4}\right]\right)$ graph

Space for Hints

Space for Hints

without triangles.

1.6 ISOMORPHISM

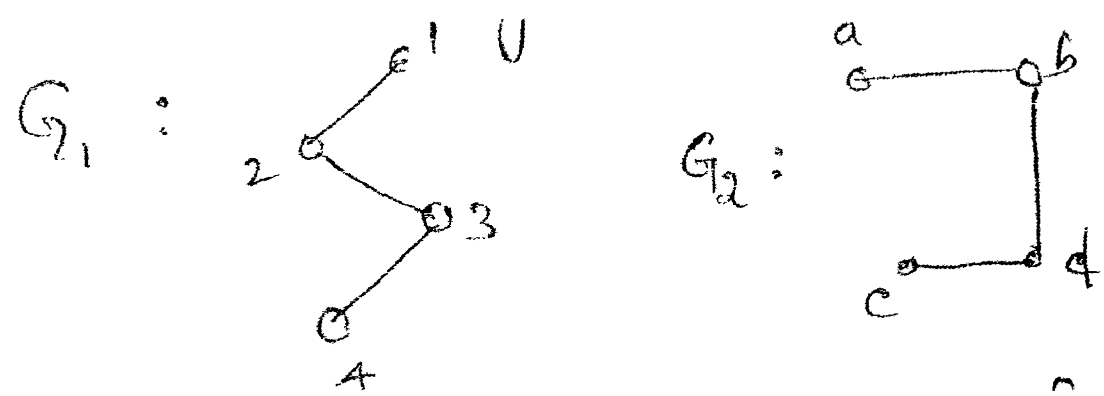
Definition: 1.6.1

Two graphs $G_1 = (V_1, X_1)$ and $G_2 = (V_2, X_2)$ are said to be **isomorphic.** If there exists a bijection $f : V_1 \rightarrow V_2$ such that u, v are adjacent in G_1 if and only if $f(u), f(v)$ are adjacent in G_2. If G_1 is isomorphic to G_2. we write $G_1 \cong G_2$. The map f is called an **isomorphism** from G_1 to G_2.

Examples

Example: 1

Consider the graphs G_1 and G_2



$V(G_1) = \{1, 2, 3, 4\}$, $V(G_2) = \{a, b, c, d\}$

$E(G_1) = \{\{1,2\}, \{2,3\}, \{3,4\}\}$ and

$E(G_2) = \{\{a,b\}, \{b,d\}, \{d,c\}\}$.

Define a function $f : V(G_1) \rightarrow V(G_2)$ as

$f(1) = a$, $f(2) = b$, $f(3) = d$, and $f(4) = c$.

f is clearly one – one and onto, hence an isomorphism.

Furthermore, $\{1,2\} \in E(G_1)$ and $\{f(1), f(2)\} = \{a, b\} \in E(G_2)$

$\{2,3\} \in E(G_1)$ and $\{f(2), f(3)\} = \{b, d\} \in E(G_2)$

$\{3,4\} \in E(G_1)$ and $\{f(3), f(4)\} = \{d, c\} \in E(G_2)$

And $\{1,2\} \notin E(G_1)$ and $\{f(1), f(3)\} = \{a, d\} \notin E(G_2)$

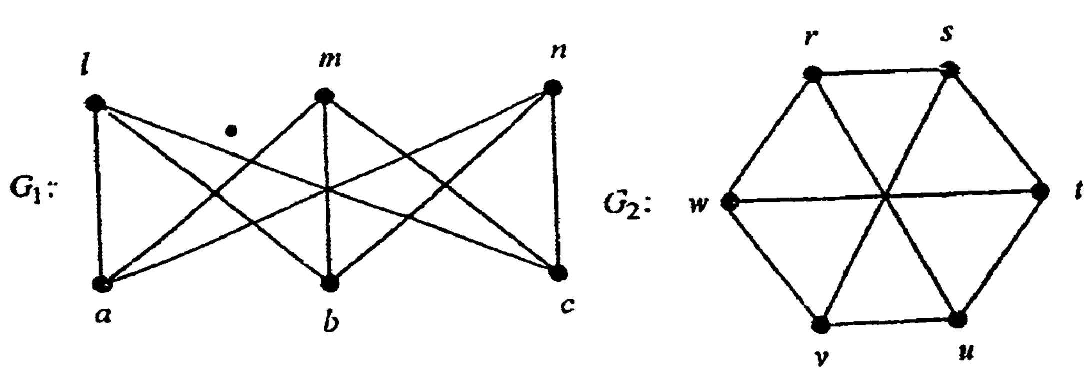
$\{1,4\} \notin E(G_1)$ and $\{f(1), f(4)\} = \{a, c\} \notin E(G_2)$

$\{2,4\} \notin E(G_1)$ and $\{f(2), f(4)\} = \{b, c\} \notin E(G_2)$

Hence f is preserves adjacency as well as non – adjacency of the vertices

$\therefore$ G_1 and G_2 are isomorphic graphs.

Example: 2



The graphs G_1 and G_2 are isomorphic because the function $f: V(G_1) \to V(G_2)$ defined by

$f(\ell) = r$, $f(m) = t$, $f(n) = v$, $f(a) = s$, $f(b) = u$, $f(c) = w$ preserves adjacency and non – adjacency among the vertices.

3. The graphs given in Fig.1.4 and Fig.1.5 are isomorphic.

4. The two graphs given in Fig 1.15 are isomorphic. $f(u_i) = v_i$ is an isomorphism between these two graphs.

Space for Hints

Space for Hints

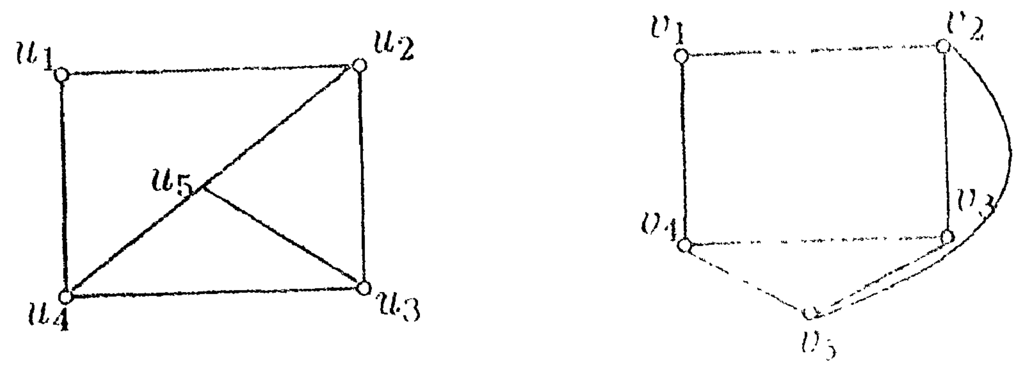


Figure 1.15

5. The three graphs given in Fig1.16 are isomorphic with each other.

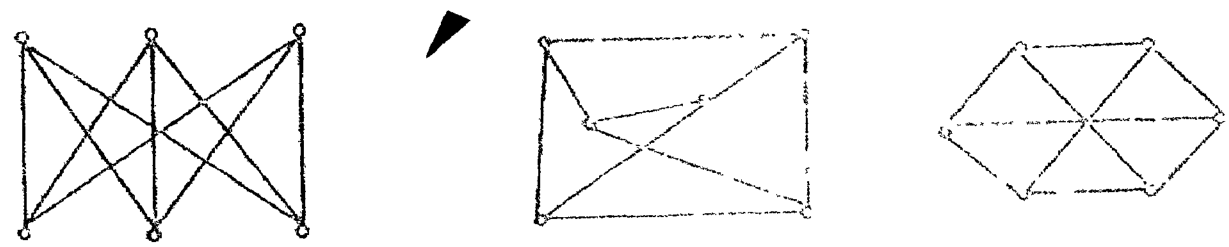


Figure 1.16

Theorem 1.4

Let f be an isomorphism of the graph $G_1 = (V_1, X_1)$ to the graph $G_2 = (V_2, X_2)$. Let $v \in V_1$. Then deg v = deg f(v).

i.e., isomorphism preserves the degree of vertices.

Proof

A point $u \in V_1$ is adjacent to v in G_1 iff f(u) is adjacent to f(v) in G_2. Also f is a bijection. Hence the number of points in V_1 which are adjacent to v is equal to the number of points in V_2 which are adjacent to f(v). Hence deg v = deg f(v).

Remark

Two isomorphic graphs have the same number of points and same number of lines. Also it follows from Theorem 1.4 that t isomorphic graphs have an equal number of points with a gi\

Space for Hints

degree. However these condition are not sufficient to ensure that two graphs are isomorphic. For example consider the two graphs given in Fig 1.17

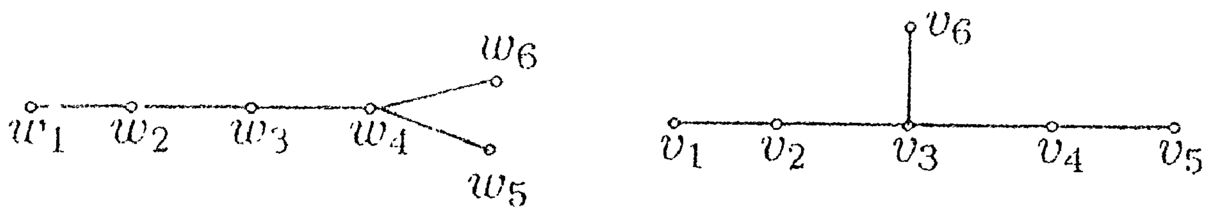


Figure 1.17

By theorem 1.4 under any isomorphism w_4 must correspond to v_3; w_1, w_5, w_6 must correspond to v_1, v_5, v_6 in some order. The remaining two points w_2, w_3 are adjacent whereas v_2, v_4 are not adjacent. Hence there does not exist an isomorphism between these two graphs. However both graphs have exactly one vertex of degree 3, three vertices of degree 1 and two vertices of degree 2.

Definition 1.6.2

An isomorphism of a graph G onto itself is called an **automorphism**

of G.

Remark

Let $\Gamma(G)$ denote the set of all automorphisms of G. Clearly the identity map $i: V \to V$ defined by $i(v) = v$ is an automorphism of G so that $i \in \Gamma(G)$. Further if α and β are automorphisms of G then $\alpha.\beta$ and α^{-} are also automorphism of G.

Hence $\Gamma(G)$ is a group and is called the **automorphism group** of G.

Definition 1.6.3

Let $G = (V, X)$ be a graph. The **complement** $\bar{G}$ of G is defined to be the graph which has V as its set of points and two points are adjacent in $\bar{G}$ iff they are not adjacent in G. G is said to be a **self complementary** graph if G is isomorphic to $\bar{G}$.

Example: (1)

<u>Complementary graph</u>

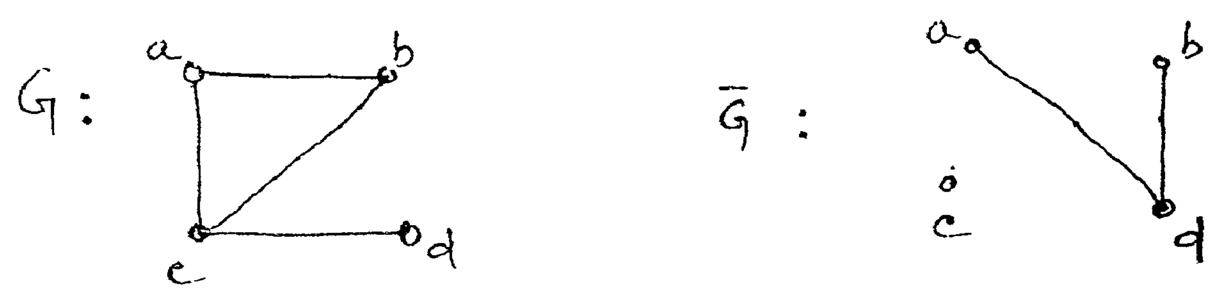


Figure 1.18 (a)

Example: (2)

<u>Self complementary graph</u>

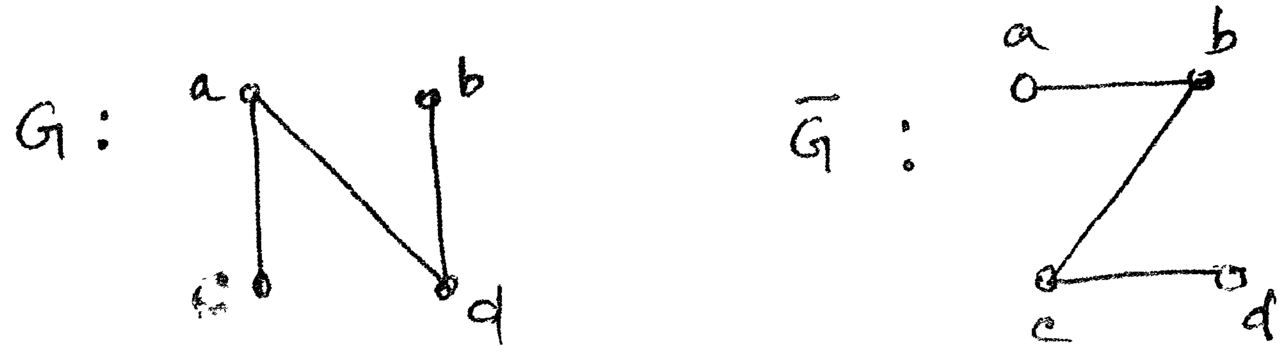


Figure 1.18 (b)

Remark:

It has been conjectured by Ulam that the collection of vertex deleted subgraphs $G - v$ determines G upto isomorphism.

Ulam's Conjecture

Space for Hints

Let G and H be two graphs with p points ($p > 2$) say $v_1, v_2, ..., v_p$ and $w_1, w_2, ..., w_p$ respectively. If for each i the subgraphs $G_i = G - v_i$ and $H_i = H - w_i$ are isomorphic, then the graphs G and H are isomorphic.

Ulam's conjecture is also known as reconstruction conjecture.

Intersection graphs and line graphs

Definition 1.6.4

Let $F = \{S_1, S_2, ..., S_p\}$ be a non – empty family of distinct non – empty subsets of a given set S. The **intersection graph** of F, denoted by $\Omega(F)$ is defined as follows:

The set of points V of $\Omega(F)$ is F itself and two points S_i, S_j are adjacent if $i \neq j$ and $S_i \cap S_j \neq \Phi$. A graph G is called an intersection graph on S if there exists a family F of subsets of S such that G is isomorphic to $\Omega(F)$.

Result: (1)

Every graph is an intersection graph.

Proof

Let $G = (V, X)$ be a graph. Let $V = \{v_1, v_2, ..., v_p\}$. Let $S = V \cup X$.

For each $v_i \in V$, let $S_i = \{v_i\} \cup \{x \in X / v_i \in x\}$.

Clearly $F = \{S_1, S_2, .., S_p\}$ is a family of distinct non –

Space for Hints

empty subsets of S.

Further if v_i, v_j are adjacent in V then $v_i v_j \in S \cap S_j$ and hence $S \cap S_j \neq \phi$.

Conversely if $S \cap S_j \neq \phi$ then element common to $S \cap S_j$ is the line joining v_i and v_j so that v, v_j are adjacent in G. Thus $f : V \to F$ defined by $f(v) = S$ is an isomorphism of G to Ω (F). Hence G is an intersection graph.

Definition 1.6.5

Let $G = (V, X)$ be a graph with $X \neq \phi$. Then X can be thought of as a family of 2 element subsets of V. The intersection graph $\Omega(X)$ is called the **line graph** of G and is denoted by L(G). Thus the points of L(G) are the lines of G and two paints in L(G) are adjacent iff the corresponding lines are adjacent in G.

A example of a graph and its line graph are given in Fig. 1.19.

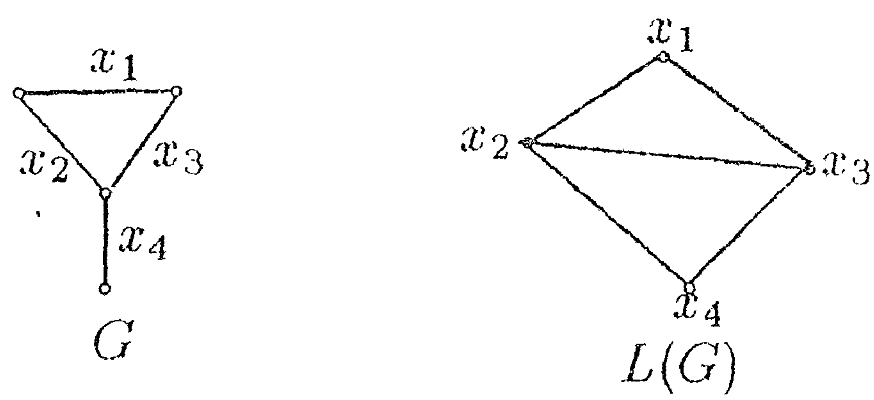


Figure 1.19

Result: 2

Let G be a (p, q) graph. Then L(G) is a (q, q_L) graph where $q_L = \frac{1}{2}\left(\sum_{i=1}^{p} d_i^2\right) - q$.

Proof:

By definition, number of points in L(G) is q.

To find the number of lines in L(G). Any two of the d_i lines incident with v_i are adjacent in L(G) and hence we get $\frac{d_i(d_i-1)}{2}$ lines in L(G).

$$\text{Hence } q_L = \sum_{i=1}^{p} \frac{d_i(d_i-1)}{2}.$$

$$= \frac{1}{2}\left(\sum_{i=1}^{p} d_i^2\right) - \frac{1}{2}\left(\sum_{i=1}^{p} d_i\right)$$

$$= \frac{1}{2}\left(\sum_{i=1}^{p} d^2\right) - \frac{1}{2}(2q) \qquad \text{(by Theorem 1.1)}$$

$$= \frac{1}{2}\left(\sum_{i=1}^{p} d_i^2\right) - q.$$

Note:

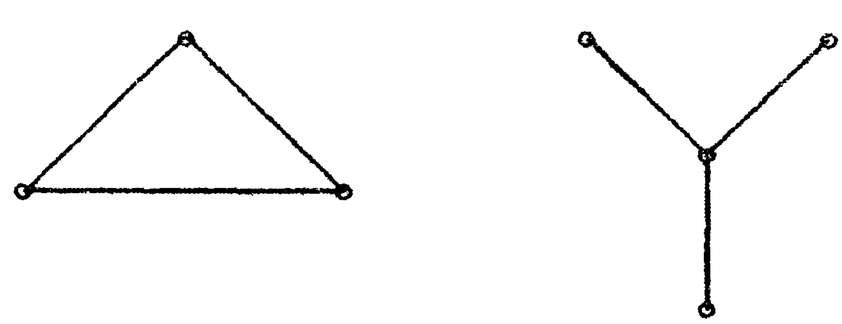


Figure 1.20

Result: 3

(Whitney). Let G and G' be connected graphs with isomorphic line graphs. Then G and G' are isomorphic unless one is K_3 and the other is $K_{1,3}$.

Space for Hints

Space for Hints

Definition: 1.6.6

A graph G is called a **line graph** if $G \cong L(H)$ for some graph H.

Example:

$K_4 - x$ is a line graph as seen in figure 1.19.

The following theorem is called Beineke's forbidden subgraph characterisation of line graphs.

Forbidden subgraph characterisation is an important and respected form of characterisation in Graph Theory.

Result: 4

(Beineke). G is a line graph iff none of the nine graphs of Fig 1.20 is an induced subgraph of G.

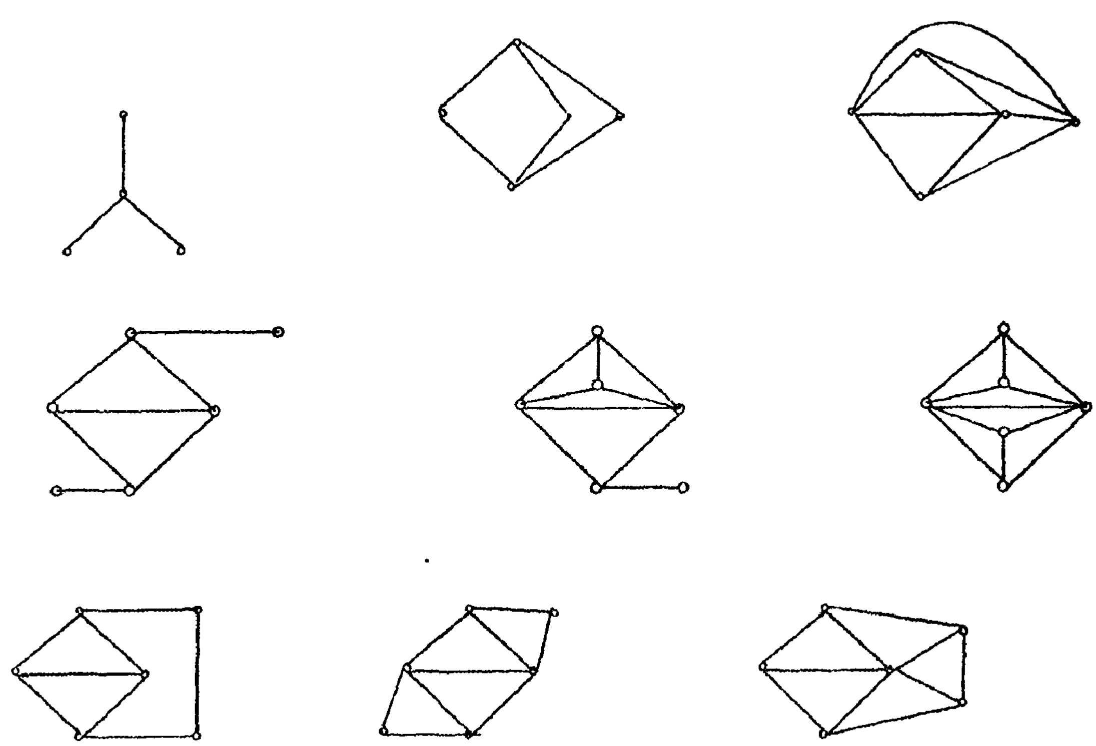


Figure 1.20

Problem: 1

Prove that any self complementary graphs has $4n$ or $4n+1$ points.

Solution:

Let $G = (V(G), X(G))$ be a self complementary graph with p points.

Since G is self complementary, G is isomorphic to $\overline{G}$.

$$\therefore \quad X(G) = X(\overline{G}) .$$

Also $X(G) + X(\overline{G}) = \binom{p}{2} = \frac{p(p-1)}{2}$.

$$\therefore \quad 2\ X\ (G) = \frac{p(p-1)}{2}$$

$$\therefore \quad |X(G)| = \frac{p(p-1)}{4} \text{ is an integer.}$$

Further one of p or p – 1 is odd

Hence p or p – 1 is a multiple of 4.

$\therefore$ p is of the form 4n or $4n+1$.

Problem: 2

Prove that $\Gamma(G) = \Gamma(\overline{G})$.

Solution:

Let $f \in \Gamma(G)$ and let $u, v \in V(G)$.

Then u, v are adjacent in $\overline{G} \Leftrightarrow u, v$ are not adjacent in G.

$\Leftrightarrow f(u), f(v)$ are not adjacent in G.

(since f is an automorphism of G)

Space for Hints

Space for Hints

$\Leftrightarrow f(u), f(v)$ are adjacent in $\overline{G}$.

Hence f is an automorphism of $\overline{G}$.

$\therefore f \in \Gamma(\overline{G})$ and hence $\Gamma(G) \subseteq \Gamma(\overline{G})$.

Similarly $\Gamma(\overline{G}) \subseteq \Gamma(G)$ so that $\Gamma(G) = \Gamma(\overline{G})$.

Problem: 3

Show that isomorphism is an equivalence relation among graphs.

Proof:

The identity map I on the vertex set V of the graph G is clearly an isomorphism of V onto itself and so $G \cong G$. That is the relation isomorphism is reflexive.

Let $G_1 \cong G_2$. So there exists an isomorphism $f : V(G_1) \to V(G_2)$. Since f is bijective $f^{-1} : V(G_2) \to V(G_1)$ exists and is bijective. Further as f preserves adjacency, f^{-1} also preserves adjacency. So f^{-1} is an isomorphism of $V(G_2)$ onto $V(G_1)$.

$\therefore G_2 \cong G_1$

(i.e) $G_1 \cong G_2 \Rightarrow G_2 \cong G_1$

So the relation is symmetric.

Let $G_1 \cong G_2$ and $G_2 \cong G_3$

$\therefore$ There exist isomorphisms $f : V(G_1) \to V(G_2)$ and $g : V(G_2) \to V(G_3)$. As f and g are bijective the composite map $gof : V(G_1) \to V(G_3)$ is also bijective. Again gof preserves adjacency

Space for Hints

as both maps f and g preserve adjacency. Thus $gof : V(G\ ) \to V(G_3)$ is a bijective map preserving adjacency. So gof is an isomorphism of $V(G_1)$ onto $V(G_3)$. That is $G_1 \cong G_3$.

Thus $G_1 \cong G_2$ and $G_2 \cong G_3 \Rightarrow G_1 \cong G_3$. So the relation is transitive.

Hence the relation of being isomorphic in the set of graphs is an equivalence relation.

Problem: 4

Show that the two graphs given in fig 1.21 are not isomorphic.

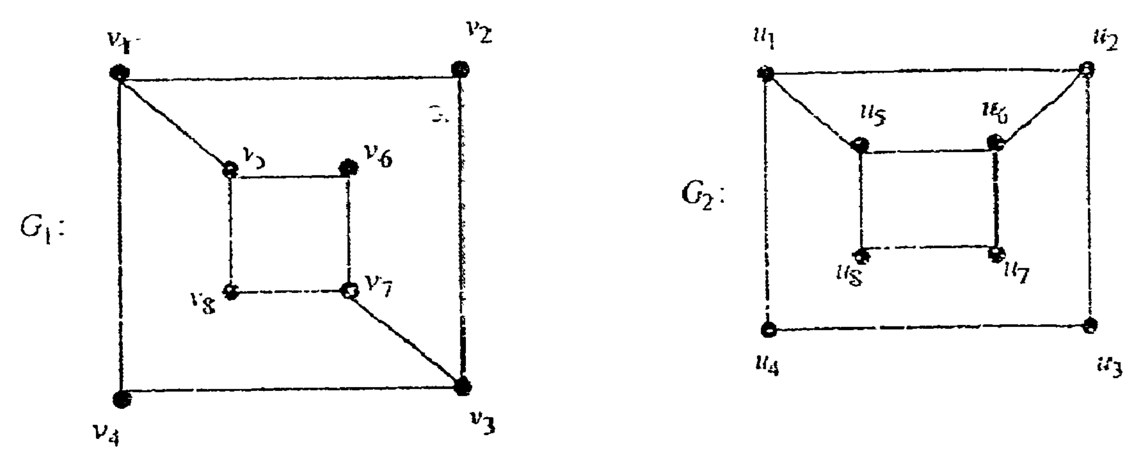


Figure 1.21

Solution:

Graphs in fig. 1.21 are not isomorphic because it is not possible to define a function between $V(G_1)$ and $V(G_2)$ preserving both adjacency and non – adjacency.

Problem: 5

Show that every simple graph on p vertices is isomorphic to a subgraph of K_p.

Space for Hints

Solution:

Let G be any simple graph on p vertices. We know K_p is a complete graph on p vertices which is the maximum simple graph on p vertices. If G is the maximum simple graph then G is isomorphic to K_p. Since K_p is a subgraph of K_p itself, we have G which is isomorphic to a subgraph of K_p. If G is any other simple graph, then number of vertices of G is less than $p-1$, and hence G is isomorphic to a proper subgraph of K_p.

Problem: 6

Prove that every induced subgraph of a complete graph is compete.

Solution:

Let $G=(V,E)$ be a complete graph and $H=(V_1,E_1)$ be an induced subgraph of G. To prove H is complete.

Since G is complete, every pair of vertices in V are adjacent. Let $V_1 \subset V$. Now, E_1 consists of those edges in E having both their ends in V_1.

Thus, every pair of vertices in V_1 must be adjacent, which implies H is complete.

Space for Hints

UNIT – 2

WALKS AND CONNECTED GRAPHS

2.0 INTRODUCTION:

In this unit we develop the basic properties of walks and connected graphs.

2.1 WALKS TRAILS AND PATHS:

Definition: 2.1.1

A walk of a graph G is an alternating sequence of points and lines $v_0, x_1, v_1, x_2, v_2, \ldots\ldots, v_{n-1}, x_n, v_n$ beginning and ending with points such that each $v_0, x_1, v_1, x_2, v_2, \ldots\ldots, v_{n-1}, x_n, v_n$ beginning and ending with points such that each line x_i is incident with v_{i-1} and v_i.

We say that the walk joins v_0 and v_n and it is called a $v_0 - v_n$ walk. v_0 is called the **initial point** and v_n is called the **terminal point** of the walk. The above walk is also denoted by $v_0, v_1, \ldots\ldots, v_n$ the lines of the walk being self evident. n, the number of lines in the walk, is called the length of this walk.

A single point is considered as a walk of length 0.

A walk is called a **trail** if all its lines are distinct and is called a **path** if all it points are distinct.

Example:

For the graph given in Fig. 2.1 $v_1, v_2, v_3, v_4, v_2, v_1, v_2, v_5$ is a walk.

Space for Hints

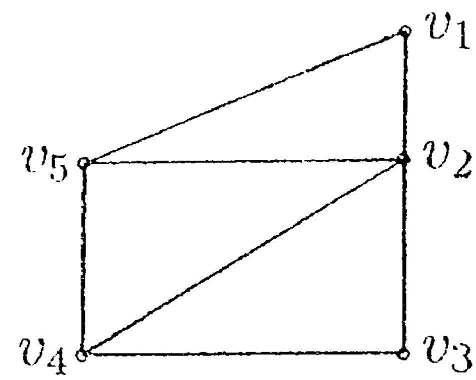


Figure 2.1

$v_1, v_2, v_4, v_3, v_2, v_5$ is a trail but not a path. v_1, v_2, v_4, v_5 is a path.

Obviously every path is a trail and a trail need not be a path.

The graph consisting of a path with n points is denoted by P_n.

Definition: 2.1.2

A $v_0 - v_n$ walk is called **closed** if $v_0 = v_n$.

A closed walk $v_0, v_1, v_2, \ldots, v_n = v_0$ in which $n \geq 3$ and $v_0, v_1, \ldots, v_{n-1}$ are distinct is called a **cycle** of length n.

The graph consisting of a cycle of length n is denoted by C_n.

C_3 is called a **triangle.**

Note:

The length of a path is the number of edges in the path. It is denoted by $\ell(p)$.

Theorem: 2.1

In a graph G, any $u - v$ walk contains a $u - v$ path.

Proof:

We prove the result by induction on the length of the walk.

Space for Hints

Any walk of length 0 or 1 is obviously a path.

Now, assume the result for all walks of length less than n.

Let $u = u_0, u_1, \ldots\ldots, u_n = v$ be a $u - v$ walk of length n.

If all the points of the walk are distinct it is already a path.

If not, there exist i and j such that $0 \leq i < j \leq n$ and $u_i = u_j$.

Now $u = u_0, \ldots\ldots, u_i, u_{j+1}, \ldots\ldots u_n = v$ is a $u - v$ walk of length less than n which by induction hypothesis conditions a $u - v$ path.

Theorem: 2.2

If $\delta \geq k$, then G has a path of length k.

Proof:

Let v_1 be an arbitrary point.

Choose v_2 adjacent to v_1.

Since $\delta \geq k$, there exists atleast $k - 1$ vertices other than v_1 which are adjacent to v_2. Choose $v_3 \neq v_1$ such that v_3 is adjacent to v_2.

In general, having chosen $v_1, v_2, \ldots\ldots, v_i$ where $1 < i \leq \delta$ there exists a point $v_{i+1} \neq v_1, v_2, \ldots\ldots, v_i$ such that v_{i+1} is adjacent to v_i. This process yields a path of length k in G.

Aliter:

Let $P = (v_0, v_1, v_2, \ldots\ldots, v_n)$ be a longest path in G. Then every vertex adjacent to v_0 lies on P.

Since $d(v_0) \geq \delta$ it follows that length of $P \geq \delta \geq k$.

Space for Hints

Hence $P_1 = (v_0, v_1, v_2, \ldots\ldots, v_k)$ is a path of length in G.

Theorem: 2.3

A closed walk of odd length contains a cycle.

Proof:

Let $v = v_0, v_1, \ldots\ldots, v_n = v$ be a closed walk of odd length.

Hence $n \geq 3$. If $n = 3$ this walk is itself the cycle C_3 and hence the result is trivial.

Now assume the result for all walks of length less than n.

If the given walk of length n is itself a cycle there is nothing to prove. If not there exist two positive integers i and j such that $i < j, \{i, j\} \neq \{0, n\}$ and $v_i = v_j$.

Now $v_i, v_{i+1}, \ldots\ldots, v_j$ and $v = v_0, v_1, \ldots\ldots, v_i, v_{j+1}, \ldots\ldots, v_n = v$ are closed walks contained in the given walk and the sum of their lengths in n.

Since n is odd at least one of these walks is of odd length which by induction hypothesis contains a cycle.

Problem: 1

If A is the adjacency matrix of a graph with $V = \{v_1, v_2, \ldots\ldots, v_p\}$, prove that for any $n \geq 1$ the $(i, j)^{th}$ entry of A^n is the number of $v_i - v_j$ walks of length n in G.

Solution:

We prove the result by induction on n.

The number of $v_i - v_j$ walks of length 1

$$= \begin{cases} 1 & \text{if } v_i \text{ and } v_j \text{ are adjacent} \\ 0 & \text{otherwise} \end{cases}$$

$$= a_{ij}.$$

Hence the result is true for $n = 1$.

We now assume that the result is true for n – 1.

Let $A^{n-1} = (a_{ij}^{(n-1)})$ so that $a_{ij}^{(n-1)}$ is number of $v_i - v_j$ walks of length $n - 1$ in G.

Now $A^{n-1} A = (a_{ij}^{(n-1)})(a_{ij})$.

Hence $(i, j)^{th}$ entry of $A^n = \sum_{k=1}^{p} a_{ik}^{(n-1)} a_{kj} \ldots (1)$

Also every $v_i - v_j$ walk of length n in G consists of a $v_i - v_k$ walk of length n – 1 followed by a vertex v_j which is adjacent to v_k. Hence if v_j is adjacent to v_k then $a_{kj} = 1$ and $a_{ij}^{(n-1)} a_{kj}$ represents the number of $v_i - v_j$ walks of length n whose last edge is $v_k v_j$. Hence the right side of (1) gives the number of $v_i - v_j$ walks of length n in G. This completes the induction and the proof.

Exercises:

1. Give an example of a closed walk of even length which does not contain a cycle.
2. Give an example to show that the union of two distinct u – v walks need not contain a cycle.
3. Prove that the union of two distinct u – v paths contains a cycle.

Space for Hints

4. Show that if a line is in a closed trail of G then it is in a cycle of G.

2.2 CONNECTED AND COMPONENTS

Definition: 2.2.1

Two points u and v of a graph G are said to be **connected** if there exists a $u - v$ path in G.

Definition: 2.2.2

A graph G is said to be **connected** if every pair of its points are connected.

A graph which is not connected is said to be **disconnected.**

For example, for $n > 1$ the graph $\overline{K}_n$ consisting of n points and no lines is disconnected. The union of two graphs is disconnected.

It is an easy exercise to verify that connectedness of points is an equivalence relation on the set of points V. Hence V is partitioned into non – empty subsets $V_1, V_2, ..., V_n$ such that two vertices u and v are connected iff both u and v belong to the same set V_i.

Let G_i denote the induced subgraph of G with vertex set V_i. Clearly the subgraphs $G_1, G_2, ..., G_n$ are connected and are called the **components** of G.

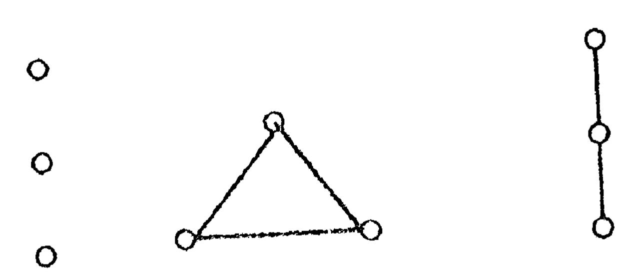


Figure 2.2

Clearly a graph G is connected iff it has exactly one component. Fig 2.2 gives a disconnected graph with 5 components.

Space for Hints

Theorem: 2.4

A graph G with p points and $\delta \geq \frac{p-1}{2}$ is connected.

Proof:

Suppose G is not connected. Then G has more than one component. Consider any component $G_1 = (V_1, X_1)$ of G.

Let $v_1 \in V_1$. Since $\delta \geq \frac{p-1}{2}$ there exist at least $\frac{p-1}{2}$ points in G_1 adjacent to v_1 and hence V_1 contains at least $\frac{p-1}{2} + 1 = \frac{p+1}{2}$ points.

Thus each component of G contains at least $\frac{p+1}{2}$ points and G has at least two components. Hence number of points in $G \geq p+1$ which is a contradiction. Hence G is connected.

Theorem: 2.5

A graph G is connected iff for any partition of V into subsets V_1 and V_2 there is a line of G joining a point of V_1 to a point of V_2.

Proof:

Suppose G is connected.

Let $V \neq V_1 \cup V_2$ be a partition of V into two subsets.

Let $u \in V_1$ and $v \in V_2$. Since G is connected, there exist a

Space for Hints

$u-v$ path in G, say, $u = v_0, v_1, v_2, ..., v_n = v$.

Let i be the least positive integer such that $v_i \in V_2$. (Such an i exists since $v_n = v \in V_2$). Then $v_{i-1} \in V_1$ and v_{i-1}, v_i are adjacent. Thus there is a line joining $v_{i-1} \in V_1$ and $v_i \in V_2$.

To prove the converse, suppose G is not connected.

Then G contains at least two components.

Let V_1 denote the set of all vertices of one component and V_2 the remaining vertices of G. Clearly $V = V_1 \cup V_2$ is a partition of V and there is no line joining any point of V_1 to any point of V_2.

Hence the theorem.

Theorem: 2.6

If G is not connected then $\overline{G}$ is connected.

Proof:

Since G is not connected, G has more than one component.

Let u, v be any two points of G. We will prove that there is a $u-v$ path in $\overline{G}$.

If u, v belong to different components in G, they are not adjacent in G and hence they are adjacent in $\overline{G}$.

If u, v lie in the same component of G, choose w in a different component.

Then u, w, v is a $u-v$ path in $\overline{G}$. Hence $\overline{G}$ is connected.

Definition: 2.2.3

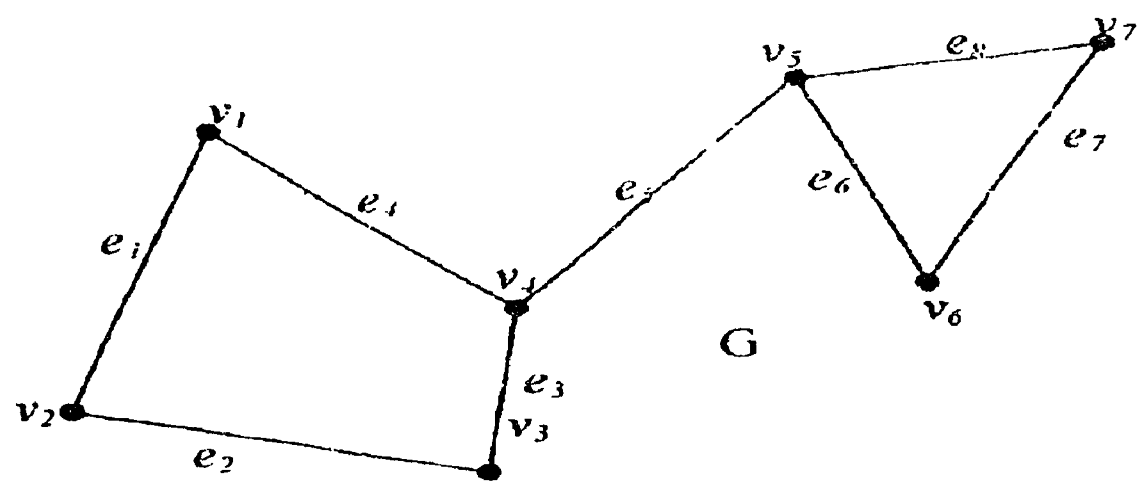
Space for Hints

For any two points u, v of a graph we define the **distance** between u and v by

$$d(u,v)=\begin{cases}\text{the length of a shortest } u-v \text{ path if such a path exists}\\ \infty \quad \text{Otherwise}\end{cases}$$

If G is a connected graph, $d(u,v)$ is always a non – negative integer. In this case d is actually a **metric** on the set of points V.

Example:



The paths connecting v_1 and v_7 are

$P_1 = v_1\ e_1\ v_2\ e_2\ v_3\ e_3\ v_4\ e_5\ v_5\ e_6\ v_6\ e_7\ v_7$

$I(P_1) = 6$

$P_2 = v_1\ e_4\ v_4\ e_5\ v_5\ e_8\ v_7$

$I(P_2) = 3$

$P_3 = v_1\ e_1\ v_2\ e_2\ v_3\ e_3\ v_4\ e_5\ v_3\ e_8\ v_7$

$I(P_3) = 5$

$P_4 = v_1\ e_4\ v_4\ e_5\ v_5\ e_6\ v_6\ e_7\ v_7$

$I(P_4) = 4$

Space for Hints

Length of shortest path = 3

$$\therefore d(v_1, v_7) = 3$$

Remarks:

(1) The distance of a vertex from itself is zero $d(u,u) = 0$

(2) If there is no path connecting two vertices u and v, then the distance between them is defined to be infinity.

(e.g)

(1) The distance of an isolated vertex from any other vertex of a graph is infinity.

(2) The distance between any two vertices of a null graph is infinity.

u – v path

In a graph G any path connecting the vertices u and v is called a u – v path. The length of the shortest u – v path is clearly the distance between u and v.

Thus $d(u,v)$ is the length of the shortest u – v path.

Result:

The distance between the vertices of a graph is a metric in the vertex set of the graph.

Proof:

The distance between two vertices u and v is the length of the shortest path between u and v and so is a non negative integer.

$$\therefore d(u,v) \geq 0 \qquad (1)$$

Distance of a vertex from itself is zero

$\therefore d(u,u) = 0$.

Also if $d(u,v) = 0$, then the vertices u and v coincide (ie) $u = v$

$\therefore d(u,v) = 0$ iff $u = v$ (2)

$d(u,v) =$ length of shortest (u-v) path

= length of shortest (v – u) path

$= d(v,u)$

$d(u,v) = d(v,u)$ (3)

Let w be any other vertex

$d(u,v) =$ length of shortest (u-v) path

$\leq$ length of shortest (u, w) path

+ length of shortest (w – v) path

$= d(u,w) + d(w,v)$

$\therefore d(u,v) \leq d(u,w) + d(w,v)$ (4)

From (1), (2), (3), (4) we find that d is a metric in the set of vertices of the graph G.

Note:

If V is the vertex set of a graph G then (V,d) is a metric space.

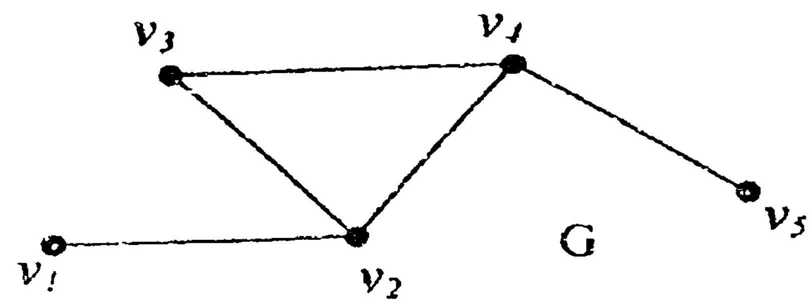
Diameter of a graph

The **diameter** of a graph G is the maximum distance

Space for Hints

Space for Hints

between any two vertices in the graph and it is denoted by $d(G)$.



$d(v\ ,v_2)=1$	$d(v\ ,v_3)=2$	$d(v_1,v_4)=2$	$d(v_1,v_5)=3$
$d(v_2,v_1)=1$	$d(v_2,v_3)=1$	$d(v_2,v_4)=1$	$d(v_2,v_5)=2$
$d(v_3,v_1)=2$	$d(v_3,v_2)=1$	$d(v_3,v_4)=1$	$d(v_3,v_5)=2$
$d(v_4,v_1)=2$	$d(v_4,v_2)=1$	$d(v_4,v_3)=1$	$d(v_4,v_5)=1$
$d(v_5,v_1)=3$	$d(v_5,v_2)=2$	$d(v_5,v_3)=2$	$d(v_5,v_4)=1$

Maximum distance between two vertices is 3. Diameter of the graph, $d(G)=3$.

Girth of a graph

The minimum of the length of the cycles in a graph G is called its **girth** and it is denoted by $g(G)$.

If the graph is free from cycles then its **girth** is zero.

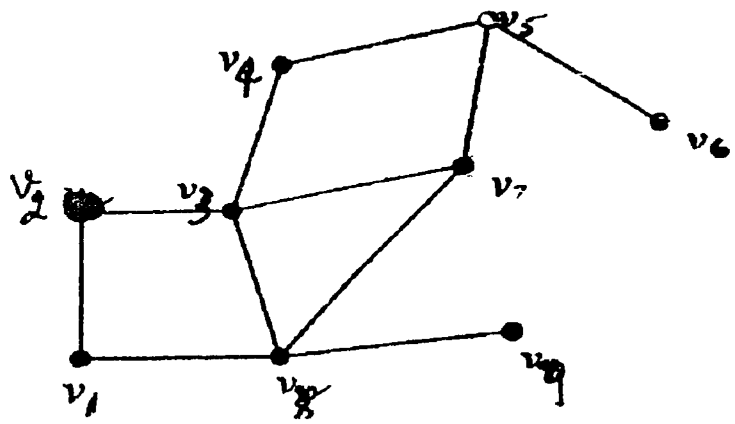
Circumference of a graph

The maximum of the lengths of cycles in a graph G is called its **circumference** and it is denoted by c(G).

If the graph is free from cycles then its circumference is zero.

Space for Hints

Example:



The cycles are

$C_1 = v_1\ v_2\ v_3\ v_8\ v_1$ $\qquad l(C_1) = 4$

$C_2 = v_3\ v_7\ v_8\ v_3$ $\qquad l(C_2) = 3$

$C_3 = v_3\ v_4\ v_5\ v_7\ v_3$ $\qquad l(C_3) = 4$

$C_4 = v_1\ v_2\ v_3\ v_7\ v_8\ v_1$ $\qquad l(C_4) = 5$

$C_5 = v_1\ v_2\ v_3\ v_4\ v_5\ v_7\ v_8\ v_1$ $\qquad l(C_5) = 7$

$C_6 = v_8\ v_3\ v_4\ v_5\ v_7\ v_8$ $\qquad l(V_6) = 5$

Minimum length of a cycle = 3

$\therefore$ grith $g(G) = 3$.

Maximum length of a cycle = 7

$\therefore$ Circumference $c(G) = 7$.

Odd and even cycles

A cycle in a graph is said to be **odd** or **even** according as its length is odd or even.

(e,g) C_2, C_4, C_5, C_6 are odd cycles.

C_1, C_3 are even cycles.

Space for Hints

Theorem: 2.7

A graph G with at least two points is bipartite iff all its cycles are of even length.

Proof:

Suppose G is a bipartite. Then V can be partitioned into two subsets V_1 and V_2 such that every line joins a point of V_1 to a point of V_2.

Now, consider any cycle $v_0, v_1, v_2, \ldots\ldots\ldots, v_n = v_0$ of length n.

Suppose $v_0 \in V$. Then $v_2, v_4, v_6, \ldots\ldots \in V_1$ and $v_1, v_3, v_5, \ldots\ldots \in V_2$. Further $v_n = v_0 \in V$ and hence n is even.

Conversely, suppose all cycles in G are of even length. We may assume without loss of generality that G is connected. (If not we consider the components of G separately).

Let $v_1 \in V$. Define $V_1 = \{v \in V / d(v, v_1) \text{ is even}\}$

$$V_2 = \{v \in V / d(v, v_1) \text{ is odd}\}.$$

Clearly, $V_1 \cap V_2 = \Phi$ and $V_1 \cup V_2 = V$.

We claim that every line of G joins a point of V_1 to a point of V_2.

Suppose two points $u, v \in V_1$ are adjacent.

Let P be a shortest $v_1 - u$ path of length m and let Q be a shortest $v_1 - v$ path of length n. Since $u, v \in V_1$ both m and n are even.

Now, let u_1 be the last point common to P and Q.

Then the $v_1 - u_1$ path along P and the $v_1 - u_1$ path along Q are both shortest paths and hence have the same length, say i.

Now the $u_1 - u$ path along P, the line uv followed by the $v - u_1$ path along Q form a cycle of length $(m - i) + 1 + (n - i) = m + n - 2i + 1$ which is odd and this is a contradiction.

Thus no two points of V_1 are adjacent. Similarly no two points of V_2 are adjacent and hence G is bipartite. Hence the theorem.

Note:

To study the measure of connectedness of a graph G we consider the minimum number of points or lines to be removed from the graph in order to disconnect it.

Definition: 2.2.4

A **cut point** of a graph G is a point whose removal increases the number of components.

A **bridge** of a graph G is a line whose removal increases the number of components.

Clearly if v is a cut point of a connected graph, G – v is disconnected.

For the graph given in Fig. 2.3, 1, 2 and 3 are cut points. The line {1, 2} and {3, 4} are bridges. 5 is non – cut point.

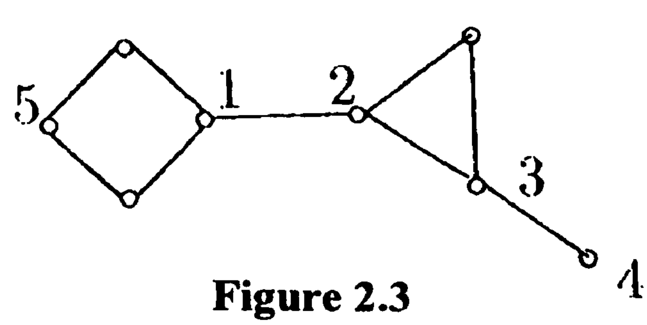


Figure 2.3

Space for Hints

Space for Hints

Theorem: 2.8

Let v be a point of a connected graph G. The following statements are equivalent.

1. v is a cut – point of G.
2. There exists a partition of V – {v} into subsets U and W such that for each $u \in U$ and $w \in W$, the point v is on every u – w path.
3. There exist two points u and w distinct from v such that v is on every u – w path.

Proof:

$(1) \Rightarrow (2)$. Since v is a cut point of $G, G-v$ is disconnected.

Hence G – v has at least two components.

Let U consist of the points of one of the components of G – v and W consist of the points of the remaining components.

Clearly $V-\{v\} = U \cup W$ is a partition of $V-\{v\}$.

Let $u \in U$ and $w \in W$. Then u and w lie in different components of $G-v$. Hence there is no $u-w$ path in $G-v$.

Therefore every $u-w$ path in G contains v.

$(2) \Rightarrow (3)$. This is trivial.

$(3) \Rightarrow (1)$. Since v is on every u – w path in G there is no u – w path in G – v. Hence G – v is not connected so that v is a cut point of G.

Theorem: 2.9

Let x be a line of a connected graph G. The following statements are equivalent.

Space for Hints

1. x is bridge of G.
2. There exists a partition of V into two subsets U and W such tat for every point $u \in U$ and $w \in W$, the line x is on every u – w path.
3. There exist two points u, w such that the line x is on every u – w path.

The proof is analogous to that of theorem 2.8 and is left as an exercise.

Theorem: 2.10

A line x of a connected graph G is a bridge iff x is not on any cycle of G.

Proof:

Let x be a bridge of G. (1)

Suppose x lies on a cycle C of G.

Let w_1 and w_2 be any two points in G.

Since G is connected, there exists a $w_1 - w_2$ path P in G.

If x is not on P, then P is a path in G – x.

If x is on P, replacing x by C – x, we obtain a $w_1 - w_2$ walk in G – x. This walk contains a $w_1 - w_2$ path in G – x. Hence G – x is connected which is a contradiction to (1).

Hence x is not on any cycle on G.

Conversely, let $x = uv$ be not on any cycle of G (2)

Suppose x is not a bridge.

Space for Hints

Hence G – x is connected.

∴ There is a u – v path in G – x.

This path together with the line $x = uv$ forms a cycle containing x and this contradicts (2). Hence x is a bridge.

Theorem: 2.11

Every non – trivial connected graphs has at least two points which are not cut points.

Proof:

Choose two points u and v such that $d(u,v)$ is maximum.

We claim that u and v are not cut points.

Suppose v is a cut point.

Hence G – v has more than one component.

Choose a point w in a component that does not contain u.

Then v lies on every u – w path and hence $d(u,w) > d(u,v)$ which is impossible.

Hence v is not a cut point.

Similarly u is not a cut point. Hence the theorem.

Blocks

Definition: 2.2.5

A connected non – trivial graph having no cut point is a **block.** A block of a graph is a subgraph that is a block and is maximal with respect to this property.

A graph and its blocks are given in Figure 2.4.

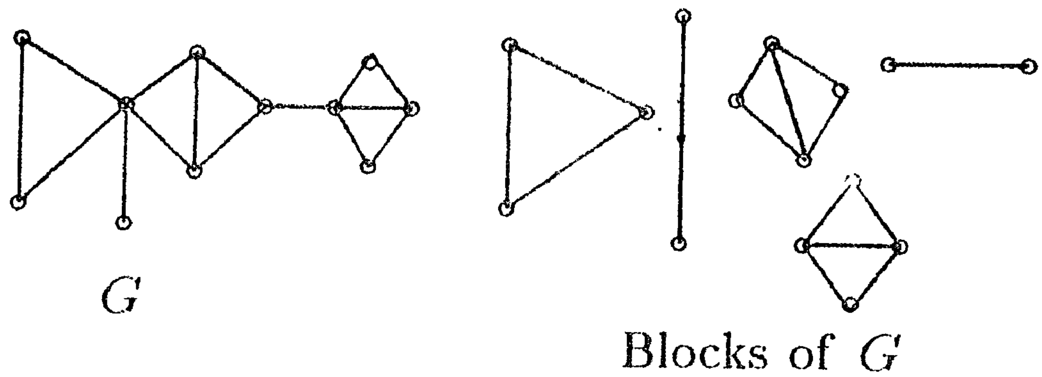


Figure 2.4

In the following theorem we give several equivalent conditions for a graph to be a block.

Theorem: 2.12

Let G be a connected graph with at least three points. The following statements are equivalent.

1. G is a block.
2. Any two points of G lie on a common cycle.
3. Any point and any line of G lie on a common cycle.
4. Any two lines of G lie on a common cycle.

Proof:

(1) $\Rightarrow$ (2) Suppose G is a block.

We shall prove by induction on the distance $d(u,v)$ between u and v, that any two vertices u and v lie on a common cycle.

Suppose $d(u,v)=1$. Hence u and v are adjacent. By hypothesis, $G \neq K_2$ and G has no cut points. Hence the line $x = uv$ is not a bridge and hence by Theorem 2.10. x is on a cycle of G.

Space for Hints

Hence the points u and v lie on a common cycle of G.

Now assume that the result is true for any two vertices at distance less than k and let $d(u,v) = k \geq 2$. Consider a $u - v$ path of length k.

Let w be the vertex that precedes v on this path.

Then $d(u,w) = k-1$

Hence by induction hypothesis there exists a cycle C that contains u and w. Now since G is a block, w is not a cut point of G and so G – w is connected.

Hence there exists $u - v$ path P not containing w.

Let v' be the last point common to P and C. (See figure 2.5). Since u is common to P and C, such a v' exists.

Now, let Q denote the u – v' path along the cycle C not containing the point w. Then, Q followed by the v' – v path along P, the line vw and the $w - u$ path along the cycle C line disjoint from Q form a cycle that contains both u and v. This completes the induction.

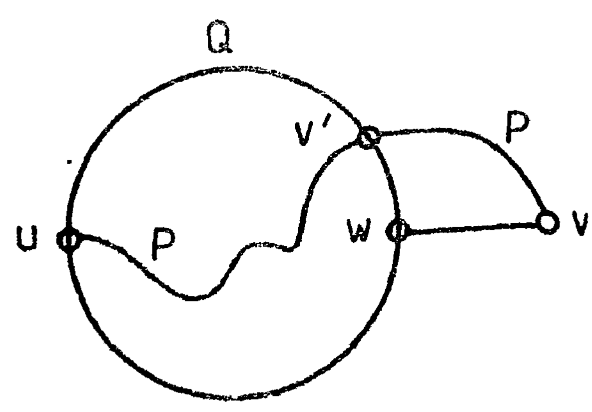


Figure 2.5

Thus any two points of G lie on a common cycle of G.

$(2) \Rightarrow (1)$. Suppose any two points of G lie on a common cycle of G. Suppose v is a cut point of G. Then there exist two points u and w distinct from v such that every u – w path contains v (Refer Theorem 2.8).

Space for Hints

Now, by hypothesis u and w lie on a common cycle and this cycle determines two u – w paths and at least one of these paths does not contain v which is a contradiction.

Hence G has no cut points so that G is a block.

$(2) \Rightarrow (3)$. Let u be a point and vw a line of G.

By hypothesis u and v lie on a common cycle C.

If w lies on C, then the line vw together with the v – w path of C containing u is the required cycle containing u and the line vw.

If w is not on C, let C' be a cycle containing u and w.

This cycle determines two w – u paths and at least one of these paths does not contain v. Denote this path by P.

Let u' be the first point common to P and C. (u' may be u itself). Then the line vw followed by the w – u' subpath of P and the u'–v path in C containing u from a cycle containing u and the line vw.

$(3) \Rightarrow (2)$ is trivial.

$(3) \Rightarrow (4)$. The proof is analogous to the proof of $(2) \Rightarrow (3)$ and is left as an exercise.

$(4) \Rightarrow (3)$ is trivial.

Connectivity

We define two parameters of a graph its connectivity and

Space for Hints

edge connectivity which measure the extent to which it is connected.

Definition: 2.2.6

The **connectivity** $\kappa = \kappa(G)$ of a graph G is the minimum number of points whose removal results in a disconnected or trivial graph. The **line connectivity** $\lambda = \lambda(G)$ of G is the minimum number of lines whose removal results in a disconnected or trivial graph.

Examples:

1. The connectivity and line connectivity of a disconnected graph is 0.
2. The connectivity of a connected graph with a cut point is 1.
3. The line connectivity of a connected graph with a bridge is 1.
4. The complete graph K_p cannot be disconnected by removing any number of points, but the removal of p – 1 points results in a travail graph. Hence $\kappa(K_p) = p - 1$.

Theorem: 2.13

For any graph $G, \kappa \leq \lambda \leq \delta$.

Proof:

We first prove $\lambda \leq \delta$. If G has no lines, $\lambda = \delta = 0$. Otherwise removal of all the lines incident with a point of minimum degree results in a disconnected graph. Hence $\lambda \leq \delta$.

Now to prove $k \leq \lambda$, we consider the following cases.

Case: (i)

G is disconnected or trivial. Then $\kappa = \lambda = 0$.

Space for Hints

Case: (ii)

G is a connected graph with a bridge x. Then $\lambda = 1$. Further in this case $G = K_2$ or one of the points incident with x is a cut point. Hence $\kappa = 1$ so that $\kappa = \lambda = 1$.

Case: (iii)

$\lambda \geq 2$. Then there exist λ lines the removal of which disconnects the graph. Hence the removal of $\lambda - 1$ of these lines results in a graph G with a bridge $x = uv$. For each of these $\lambda - 1$ lines select an incident point different from u or v. The removal of these $\lambda - 1$ points removes all the $\lambda - 1$ lines. If the resulting graph is disconnected, then $\kappa \leq \lambda - 1$. If not x is a bridge of this subgraph and hence the removal of u or v results in a disconnected or trivial graph. Hence $\kappa \leq \lambda$ and this completes the proof.

Remark:

The inequalities in Theorem 2.13 are often strict. For the graph given in Fig. 2.6, $\kappa = 2, \lambda = 3$ and $\delta = 4$.

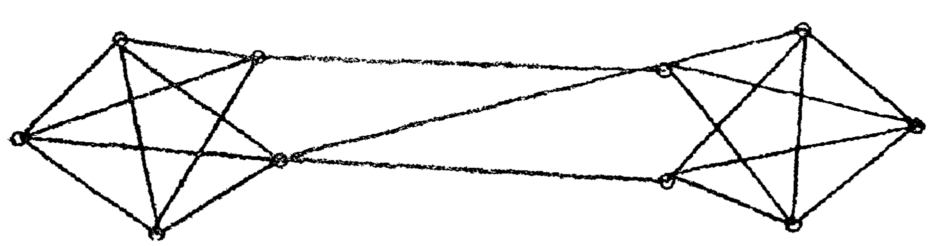


Figure 2.6

Definition: 2.2.7

A graph G is said to be n – **connected** if $\kappa(G) \geq n$ and n – **line connected** if $\lambda(G) \geq n$.

Thus a nontrivial graph is 1 – connected iff it is connected.

A nontrivial graph is 2 – connected iff it is a block having

Space for Hints

more than one line. Hence K_2 is the only block which is not 2 – connected.

Problems: 1

Prove that there is no 3 – connected graph with 7 edges.

Solution:

Suppose G is a 3 – connected graph with 7 edges.

G has 7 edges $\Rightarrow p \geq 5$.

Now $q \geq \frac{3p}{2}$ (by problem 1)

$\therefore q \geq \frac{15}{2}$.

$\therefore q \geq 8$ which is a contradiction.

Hence there is no 3 – connected graph with 7 edges.

Problem: 2

Give examples to show that there are walks that are not trails and trails that are not paths.

Solution:

In the graph G (Figure 2.7)

a – g walk: $ae\ be_9ge_7ce_5fe_4de_3ce_2be_9g\ a$

a – b walk: $ae_{10}ge_8fe_6be_2ce_7ge\ _0ae_1b$

These are walks but not trails since in a – g walk the edge e_9 is repeated and in a – b walk the edge e_{10} is repeated.

a – g trail: $ae_1be_7fe_5ce_3de_4fe_8g$.

This is a trail but not path since the vertex f is repeated.

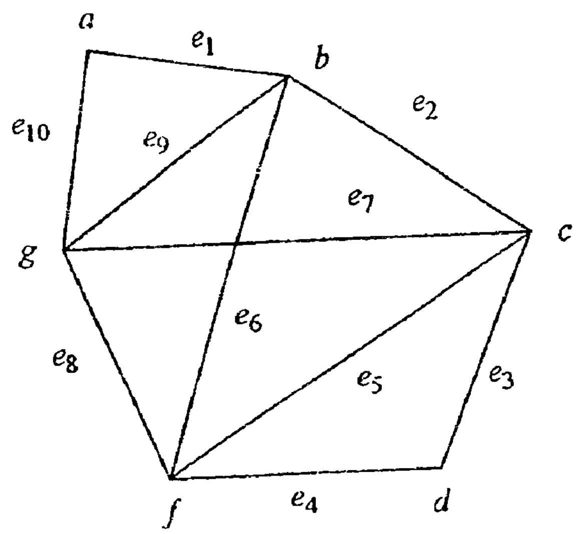


Figure 2.7

Problem: 3

Prove that the relation "is connected to" is an equivalence relation on the vertex set of a given graph.

Solution:

(i) Every vertex u of a graph G is connected to itself. So the relation is Reflexive.

(ii) When u is connected to v, there is a path from u to v. Then we consider the same path from v to u also. So v is connected to u. Hence the relation is symmetric.

(iii) When u is connected to v and v is connected to w then there are u – v and v – w paths in G. The union of u – v and v – w path is a u – w path in G and hence u is connected to w so that the relation is transitive.

Thus the relation "is Connected to" is an equivalence relation.

Space for Hints

Space for Hints

Problem: 4

Show that if G is a connected graph of order p. Then size of G is atleast p – 1 (i.e.,) $q \geq p-1$.

Solution:

Let G be a connected graph. Consider a path in G which connects any two vertices of G. The length of this path is atleast p – 1 (because it includes all the vertices) so that the number of edges is atleast $p-1$.

Since all the edges are involved in this path, the size of the graph is atleast $p-1$. (i.e.) $q \geq p-1$.

Problem: 5

If G contains no odd degree vertices then G contain no bridges.

Solution:

Let G be a graph such that the degree of each vertex is even. We claim that there is no bridge in the graph. If possible, let $e = xy$ be a bridge of the graph G. Then the graph G – e is disconnected. However we claim that there is a x – y path in G – e.

Since the degree of each and every vertex other than x and y is of even degree in G – e, whenever we enter into a vertex through an edge for a path starting form vertex x, we come out from the vertex through some other edge and there by we reach the vertex y.

This is a contradiction to the fact that the graph G – e is disconnected. Hence there cannot be any bridge in the graph G.

Problem: 6

Prove that if G is a k – connected graph then $q \geq \frac{Pk}{2}$

Solution:

Since G is k – connected, $k(G) \geq k$ But

$$\delta(G) \geq \lambda(G) \geq k(G)$$

$$\therefore \delta(G) \geq k$$

$$q = \frac{1}{2}\sum d(v)$$

$$\geq \frac{1}{2}p\delta(G) \text{ [since } d(v) > \delta(G)\text{]}$$

$$\geq \frac{pk}{2}$$

$\therefore q \geq 8$ which is a contradiction.

Hence there is no 3 – connected graph with 7 edges.

Space for Hints

Unit – 3

EULERIAN GRAPHS

3.0 INTRODUCTION

The concepts Eulerian trials and Hamiltonian cycles mainly deal with the nature of connectivity in graphs. These concepts have applications to the area of puzzle and games. In this chapter we discuss the relation between a local property, namely, degree of a vertex and global properties like the existence of Eulerian or Hamiltonian cycles. Euler (1736) formulated the concept of Eulerian trail when he solved the problem of the Konigsberg bridge. We see that there are elegant characterizations for Eulerian graphs whereas there are no such characterizations for Hamiltonian graphs.

3.1 EULERIAN GRAPHS

Definition: 3.1.1

A closed trail containing all points and lines is called an **Eulerian trail.** A graph having an Eulerian trail is called an **Eulerian graph.**

Remark:

Obviously in an Eulerian graph, for every pair of points u and v there exists at least two edge – disjoint u – v trails and consequently there are at least two edge – disjoint u – v paths.

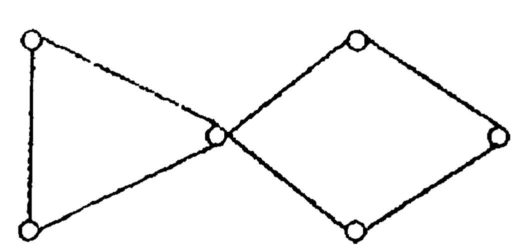


Figure 3.1

The graph given in Fig. 3.1 is an Eulerian graph.

First we prove a simple lemma that is needed in the proof of the main theorem.

Lemma: 3.1

If G is a graph in which the degree of every vertex is at least two then G contains a cycle.

Proof:

Construct a sequence $v, v_1, v_2, ...,$ of vertices as follows. Choose any vertex v. Let v_1 be any vertex adjacent to v. Let v_2 be any vertex adjacent to v_1 other than v. At any stage, if vertex $v_i, i \geq 2$ is already chosen, then choose v_{i+1} to be any vertex adjacent to v_i other than v_{i-1} is always guaranteed.

Since G has only a finite number of vertices, at some stage we have to choose a vertex which has been chosen before.

Let v_k be the first such vertex and let $v_k = v_i$ where $i < k$. Then $v_i, v_{i+1}, ..., v_k$ is a cycle.

The following theorem answers the problem. In what type of graph G is it possible to find a closed trail running through every edge of G?

Theorem: 3.2

The following statements are equivalent for a connected graph G.

(1) G is Eulerian.

(2) Every point of G has even degree.

Space for Hints

Space for Hints

(3) The set of edges of G can be partitioned into cycles.

Proof:

$(1) \Rightarrow (2)$: Let T be an Eulerian trial in G, with origin (and terminus) u. Each time a vertex v occurs in T in a place other than the origin and terminus, two of the edges incident with v are accounted for.

Since an Eulerian trail contains every edges of G, $d(v)$ is even for every $v \neq u$. For u, one of the edges incident with u is accounted for by the origin of T, another by the terminus of T and others are accounted for in pairs.

Hence $d(u)$ is also even.

$(2) \Rightarrow (3)$: Since G is connected and nontrivial every vertex of G has degree at least 2. Hence G contains a cycle Z. The removal of the lines of Z results in a spanning subgraph G_1 in which again every vertex has even degree. In G_1 has no edges, then all the lines of G form one cycle and hence (3) holds.

Otherwise, G_1 has a cycle Z_1. Removal of the lines of Z_1 from G_1 results in spanning subgraph G_2 in which every vertex has even degree. Continuing the above process, when a graph G_n with no edge is obtained, we obtain a partition of the edges of G into n cycles.

$(3) \Rightarrow (1)$: If the partition has only one cycle, then G is obviously Eulerian, since it is connected. Otherwise let $Z_1, Z_2, \ldots\ldots, Z_n$ be the cycles forming a partition of the lines of G. Since G is connected there exists a cycle $Z_i \neq Z_1$ having a common point v_1 with Z_1. Without loss of generality, let it be Z_2. The walk beginning at v_1 and consisting of the cycles Z_1 and Z_2 in succession is a closed trail containing the edges of these two cycles. Continuing this process, we can construct a closed trail containing all the edges of G. Hence G is Eulerian.

Space for Hints

Note:

The above theorem and its proof hold for pseudo graphs (graphs having loops and multiple edges) also. Even otherwise, a pseudo graph G^* becomes a graph G when we introduce two points of degree 2 on each loop and a point of degree 2 on every other edge. Every vertex of G is of even degree iff every vertex of G* is of even degree. Also G has a closed trail running through every edge iff every vertex of G* is of even degree.

The proof of the above theorem gives a method for finding an Eulerian trial when such a trail exists.

Konigsberg Bridge Problem:

The "graph" of the Konigsberg bridges (Fig. 1.2) has vertices of odd degree. Hence it cannot have a closed trail running through every edge. Hence one cannot walk through each of the Konigsberg bridges exactly once and come back of the starting place.

Corollary: 1

Let G be a connected graph with exactly $2n(n \geq 1)$, odd vertices. Then the edge set of G can be partitioned into n open trails.

Proof:

Let the odd vertices of G be labelled $v_1, v_2, \ldots\ldots, v_n$; $w_1, w_2, \ldots\ldots, w_n$ in any arbitrary order. Add n edges to G between

Space for Hints

the vertex pairs $(v_1, w_1), (v_2, w_2), \ldots\ldots, (v_n, w_n)$ to form a new graph G' (G' may be a multigraph). No two of these n edges are incident with same vertex. Further every vertex of G' is of even degree and hence G' has an Eulerian trail T. If the n edges that we added to G are now removed from T, it will split into n open trails (since no two of these edges are adjacent). These are open trails in G and form a partition of the edges of G.

Corollary: 2

Let G be a connected graph with exactly two odd vertices. Then G has an open trail containing all the vertices and edges of G.

Proof:

This is only a particular case of Corollary 1.

Obviously the open trail mentioned in corollary 2 begins at one of the odd vertices and ends at the other.

Arbitrarily Traceable Graphs

Consider the graph G given below

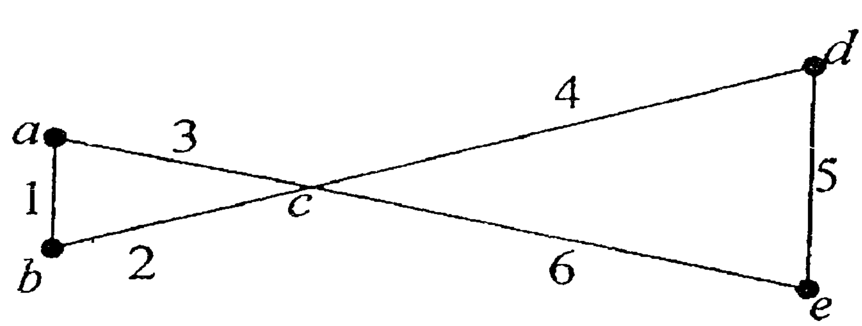


Figure 3.2

Clearly it is an Eulerian graph. So it is possible to trace all the edges exactly once and reach the starting vertex. Is it possible to do this exercise starting from any vertex. Suppose we start from the vertex a and come to c after tracing the edges 1 and 2. How to proceed further? There are three choices the edges 3, 4 and 6.

Suppose we take 3, tracing this edge we reach the vertex a and we cannot proceed further without retracing the edges. Thus starting from a we cannot arbitrarily choose the edges to trace all the edges exactly one. We say the graph is not arbitrarily traceable from a.

Suppose we start from c we can choose arbitrarily any edge incident at c and trace the graph. So we say the graph is arbitrarily traceable from the vertex c.

Thus an Eulerian graph is said to be **arbitrarily traceable** from a vertex v, if an Euler tour can be obtained starting from v and choosing edges arbitrarily while tracing the graph.

We give the following results without proof

(1) An Eulerian graph is arbitrarily traceable from the vertex v if and only if every cycle contains v.

(2) If G is arbitrarily traceable from v then v has maximum degree.

(3) If G is arbitrarily traceable from v then either v is the only cut vertex or G has no cut vertices.

Note:

. If a graph is arbitrarily traversable from a vertex then it is obviously Eulerian.

The graph in Fig. 3.1 is arbitrarily traversable from v. From no other point it is arbitrarily traversable.

Space for Hints

Space for Hints

3.2 FLEURY'S GRAPHS

Theorem: 3.3

An Eulerian graph G is arbitrary traversable from a vertex v in G iff every cycle in G contains v.

There is a good algorithm, due to Fleury, to construct an Eulerian trail in an Eulerian graph.

Fleury's Algorithm

To construct an Euler tour in an Eulerian graph G.

The algorithm is given in the following steps.

Step: 1

Choose an arbitrary vertex v_0. Take $W_0 = v_0$

Step: 2

Suppose that the trail

$W_i = v_0 e_1 v_1 e_2 v_2, \ldots\ldots, e_i v_i$ has been chosen. Then choose e_{i+1} from $E(G) - \{e_1, e_2, \ldots\ldots, e_i\}$ in such a way that

(i) e_{i+1} is incident on v_i.

(ii) e_{i+1} is not a cut edge of $G_i = G - \{e_1, e_2, \ldots\ldots, e_i\}$ unless no other edge incident on v_i is available.

Step: 3

Stop when step (2) cannot be further employed.

The trail obtained by these steps in the Eulerian graph G is an Euler trail.

Theorem: 3.4

Any trail constructed by Fleury's algorithm in an Eulerian graph is Eulerian.

Proof:

Let G be an Eulerian graph and $W_n = v_0e_1v_1e_2v_2,........,e_nv_n$ be a trail constructed by Fleury's algorithm. The algorithm terminates at v_n and so the degree of v_n in $G_n = G - \{e_1, e_2,........, e_n\}$ is zero (otherwise the algorithm will not terminate at v_n'. We claim that v_n coincides with some $v_i (0 \le i \le n-1)$.

If v_n does not coincide with any $v_i (0 \le i \le n-1)$ then W_n contains only one edge e_n incident on v_n in G. Since the degree of v_n in G is even there must be at least one edge incident on v_n in G_n. So the degree of v_n in G_n is not zero. This is a contradiction to the fact that the degree of v_n in G_n is zero.

So v_n must coincide with some $v_i (0 \le i \le n-1)$. If $v_n = v_i$ $i \ne 0$ then the degree of v_i in the trail W_n is odd because when the trail visits v_i first there is one edge incident on it and every future visit adds two edges to v_i so that the number of edges incident on v_i is odd. As the degree of v_i in G is even, there is (at least) an edge in G not covered by W_n.

This is a contradiction as the algorithm terminates. So $v_n = v_i$ with $i = 0$ and so $v_n = v_0$. Therefore the trail W_n is a closed in G.

In order to prove that W_n is an Eulerian trail we have

Space for Hints

further to prove that it contains all the edges of G. For this it is enough we prove that the edge set of G_n is empty.

Suppose the edge set of G_n is not empty (i.e.) $E(G_n) \neq \Phi$.

Let S be the set of all vertices of degree greater than zero in G_n Clearly S is not empty b our assumption that G_n has edges.

Since v_n is of degree zero in G_n we find $v_n \notin S$. Therefore $v_n \in S'$ which is the vertex set of $G - S$.

Let m be the largest integer such that $v_m \in S$ so that $v_{m+1} \notin S$. $\therefore v_{m+1} \in S'$.

Further the vertices v $(m+1 \le i < n)$ belong to S'. Thus $[S, S']$ is a partition of the vertex set of G.

Now the edge e_{m+1} is an edge of G_m in the bipartition $[S, S']$. If $e_k = v_{k-1}v_k$ is any other edge of $[S, S']$ in G_m then $k > m+1$ and $v_{k-1} \in S$.

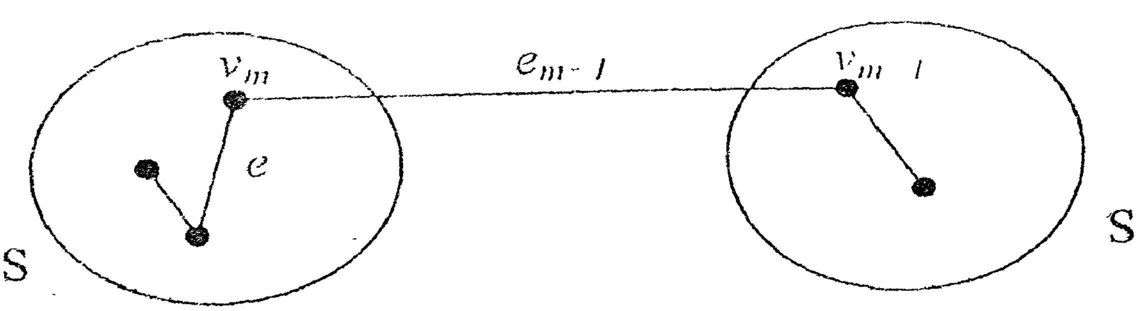


Figure 3.3

This implies that m is not the largest integer such that $v_m \in S$. This is a contradiction. So $e_{m+1} = v_m\ v_{m+1}$ is the only edge of G_m is $[S, S']$ and so e_{m+1} is a cut edge of G_m.

Let e be any other edge incident on v_m (such an edge exists

Space for Hints

because $d(v_m)$ is even), then e must also be a cut edge of G_n, for otherwise e would have been chosen instead of e_{m+1} (in step 2). Further every cut edge of G_m is also a cut edge of $G_m[S]$. Clearly the edge set of $G_n[S]$ is a subset of the edge set of $G_n[S]$.

$$\text{(i.e) } E(G_n[S]) \subseteq E(G_m[S])$$

If some edge of $G_m[S]$ is not the edge of $G_l[S]$ it must be some $e_k, m+1 \leq k$. But no such edge has both end vertices in S. So the edge set of $G_n[S]$ must be equal to edge set of $G_m[S]$.

$$\text{(i.e) } E(G_n[S] = E(G_m[S])$$

This implies $G_n[S] = G_m[S]$.

As W_n being a closed trail covers an even number of edges incident at each vertex v_i, $0 \leq i \leq n-1$, each vertex v is of even degrees in $G_n[S] = G_m[S]$. Hence $G_m[S]$ is the union of edge – disjoint cycles ($\therefore$ by problem (1)). Therefore no edge of $G_m[S]$ is a cut edge. This is a contradiction to the conclusion that e_{m+1} is a cut edge of G_m.

Therefore there is no edge e_{m+1} in G_n.

$$\therefore E(G_n) = \Phi.$$

Hence the **trail** W_n in G is a closed Eulerian trail and so is an Euler tour.

Problem: 1

Show that a simple connected Eulerian graph is the union of edge disjoint cycles.

Space for Hints

Let G be a simple connected Eulerian graph. We know each vertex is of even degree. Therefore the minimum degree of the graph is $\delta \geq 2$. So G contains cycles. Let C_1 be one such cycle. Remove the edges of C_1 from G. The degree of each vertex of the remaining graph G_1 is also even. If there are no edges in G_1, there is nothing to prove. If not, $\delta \geq 2$ in G, and so G_1 contains a cycle C_2. Remove the edges of C_2 from G_1 and proceed as before. As G is a finite graph the process will result in a finite number of cycles $C_1, C_2, ..., C_n$ which are all edge disjoint.

Hence $E(G) = E(C_1) \cup E(C_2) \cup .... \cup E(C_n)$.

Problem: 2

If a connected graph G has 2k vertices of odd degrees then there are k edge disjoint trails $Q_1, Q_2, ..., Q_k$ in G such that

$$E(G) = E(Q_1) \cup E(Q_1) \cup .... \cup E(Q_k)$$

Let the odd vertices of the given graph G be $v_1, v_1, v_1, ..., v_{2k}$. Let v be a new vertex. Join vv , $vv_2, vv_3, ..., vv_{2k}$ by edges so that we get a newgraph G' with vertex set $V(G') = V(G) \cup \{v\}$ and edge set $E(G') = E(G) \cup \{vv_1, vv_2, ..., vv_{2k}\}$

Clearly all the vertices of G_1 are of even degrees and so G'is an Eulerian graph. Therefore it is the union of edge disjoint cycles having a common vertex at v.

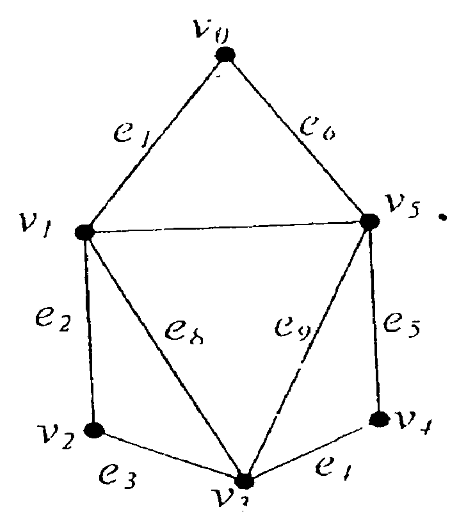
Each cycle has two new edges incident at v. Therefore the removal of all the 2k edges decomposes G into k trails which are all edge disjoint. Let them be $Q_1, Q, ..., Q_k$ in G so that

$$E(G) = E(Q\ ) \cup E(Q_2) \cup ....\cup E(Q_k)$$

Space for Hints

Problem: 3

Using Fleury's algorithm find an Euler tour of the following graph.



Take $W_0 = v_0$

$W_1 = v_0 e_1 v_1$

At v_1 none of the other edges e_2, e_8, e_7 is a cut edge of $G_1 = G - e_1$. Take any edge. Let us take e_2.

$W_2 = v_0 e_1 v_1 e_2 v_2$

At v_2, the only other edge is e_3 and it is not a cut edge of $G_2 = G - \{e_1, e_2\}$. Take e_3

$W_3 = v_0 e_1 v_1 e_2 v_2 e_3 v_3$.

At v_3, the other edges are e_8, e_9, e_4. None of them is a cut edge of $G_3 = G - \{e_1, e_2, e_3\}$. Take any edge. Let us take e_4.

$W_4 = v_0 e_1 v_1 e_2 v_2 e_3 v_3 e_4 v_4$.

At v_4 the only other edge is e_5 and it is not a cut edge of

Space for Hints

G_4. $G_4-\{e_1,e_2,e_3,e_4\}$. Take e_5

$$W_5=v_0e_1v_1e_2v_2e_3v_3e_4v_4e_5v_5.$$

At v_5 the other edges are e_6,e_7,e_9. Of these e_6 is a cut edge of $G_5=G-\{e_1,e_2,e_3,e_4,e_5\}$.

Choose e_7 which is not a cut edge.

$$W_6=v_0e_1v_1e_2v_2e_3v_3e_4v_4e_5v_5e_7v_2.$$

At v_2 the only other edge is e_8 and it is not a cut edge of $G_6=G-\{e_1,e_2,e_3,e_4,e_5,e_7\}$.

Take e_8

$$W_7=v_0e_1v_1e_2v_2e_3v_3e_4v_4e_5v_5e_7v_2e_8v_3.$$

At v_3 the only other vertex is e_9 which is not a cut edge of $G_7=G-\{e_1,e_2,e_3,e_4,e_5,e_7,e_8\}$.

Take e_9.

$$W_8=v_0e_1v_1e_2v_2e_3v_3e_4v_4e_5v_5e_7v_2e_8v_3e_9v_5$$

The only edge remaining is e_6 at v_5.

Take e_6

$$W_9=v_0e_1v_1e_7v_2e_3v_3e_4v_4e_5v_5e_7v_2e_8v_3e_9v_5e_6v_0$$

Hence W_9 is the required Euler tour.

Problem: 4

For what values of n, is k_n Eulerian?

Solution:

In the complete graph K_n, the degree of every vertex is n – 1. Therefore K_n is Eulerian if and only if n – 1 is even; that is if and only if n is odd.

Problem: 5

For what values of m and n is $K_{m,n}$ Eulerian?

Solution:

$\therefore$ $K_{m,n}$ is Eulerian $\Leftrightarrow$ m and n are even.

For example: $K_{2,4}$

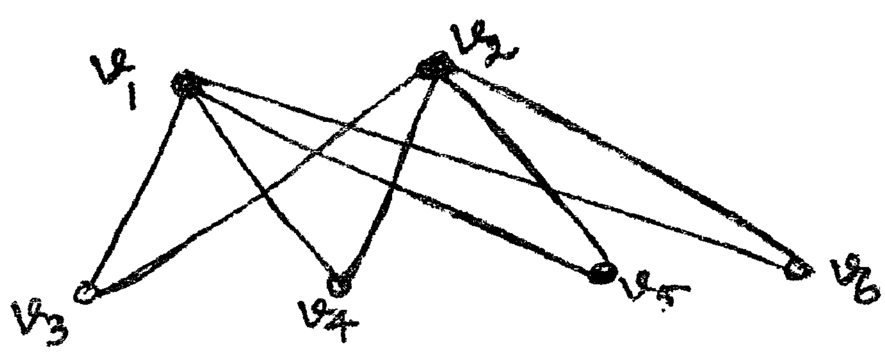


Figure 3.4

Space for Hints

Unit – 4

HAMILTONIAN GRAPHS

4.0 INTRODUCTION

In 1859, Sir William Hamilton devised a mathematical game on the graph of the dodecahedron (Fig 4.1). In this the first player sticks five pins 1, 2, 3, 4 and 5 in any five consecutive vertices and the second player is required to complete the path so formed to a spanning cycle. In the case of the dodecahedron, the completion is always possible. This game led to the concept of Hamiltonian graphs.

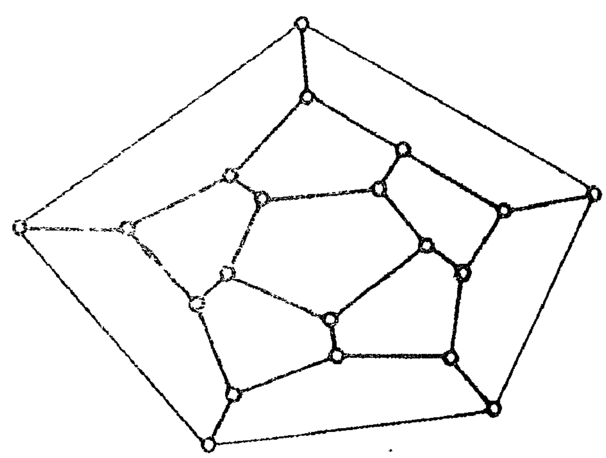


Figure 4.1

4.1 HAMILTONIAN GRAPHS

Definition: 4.1.1

A spanning cycle in a graph is called a **Hamiltonian cycle.**

A graph having a Hamiltonian cycle is called a **Hamiltonian graph.**

Several necessary or sufficient conditions for Hamiltonian graphs exist, but no elegant characterization of Hamiltonian graphs is known.

Definition: 4.1.2

A block with two non adjacent vertices of degree 3 and all other vertices of degree 2 is called a **theta graph.**

Thus a theta graph consists of two vertices of degree 3 and three disjoint paths joining them, each of length at least 2. The graph given in Fig. 4.2 is a theta graph.

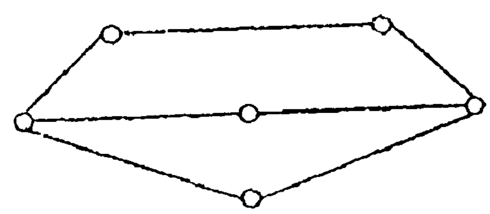


Figure 4.2

A theta graph is obviously non Hamiltonian and every non Hamiltonian 2 – connected graph has a theta subgraph.

Theorem: 4.1

Every Hamiltonian graph is 2 – connected.

Proof:

Let G be a Hamiltonian graph and let Z be a Hamiltonian cycle in G. For any vertex v of G, Z – v is connected and hence G – v is also connected. Hence G has no cut points and thus G is 2 – connected.

The following theorem gives a simple and useful necessary condition for Hamiltonian graphs.

Theorem: 4.2

If G is Hamiltonian, then for every nonempty proper subset S of $V(G), \omega(G-S) \leq |S|$. where $\omega(H)$ denotes the number of

Space for Hints

components in any graph H.

Proof:

Let Z be a hamiltonian cycle of G. Let S be any nonempty proper subset of V(G).

Now, $\omega(Z-S) \leq |S|$. Also $Z-S$ is a spanning subgraph of $G-S$ and hence $\omega(G-S) \leq \omega(Z-S)$. Hence $\omega(G-S) \leq S$.

Note: 1

The above theorem is useful is showing that some graphs are non Hamiltonian. For example, consider the complete bipartite graph $K_{m,n}$ with $m < n$.

Let (V_1, V_2) be a bipartition of the graph with $V_1 |= m$ and $| V_2 |= n$. The graph $K_{m,n} - V_1$ is the totally disconnected graph with n points.

Hence $\omega(K_{m,n} - V_1) = n > m = V_1 |$.

$\therefore K_{m,n}$ is non Hamiltonian.

Note: 2

The converse of the above theorem is not true. For example the Petersen graph (Figure 1.6) satisfies the condition of the theorem but is non Hamiltonian.

We now discuss some sufficient conditions for a graph G to be Hamiltonian.

Theorem: 4.3 **Dirac Theorem**

Statement:

If G is a graph with $p \geq 3$ vertices and $\delta \geq p/2$, then G is Hamiltonian.

Proof:

Suppose the theorem is false. Let G be a maximal (with respect to number of edges) non Hamiltonian graph with p vertices and $\delta \geq p/2$.

Since $p \geq 3$, G cannot be complete.

Let u and v be non adjacent vertices in G. By the choice of $G, G+uv$ is Hamiltonian. Moreover, since G is non Hamiltonian, each Hamiltonian cycle of G + uv must contain the lien uv.

Thus G has a spanning path $v_1, v_2, \ldots\ldots, v_p$ with origin $u = v_1$ and terminus $v = v_p$.

Let $S = \{v_i \;\; uv_{i+1} \in E\}$ and $T = \{v_i \;\; i < p \text{ and } \; v_i v \in E\}$ where E is the edge set of G.

Clearly $v_p \notin S \cup T$ and hence $S \cup T < p$ (1)

Again if $v_i \in S \cap T$, then $v_1, v_2, \ldots\ldots, v_i v_p v_{p-1} \ldots\ldots v_{i+1} v_1$ is a Hamiltonian cycle in G, contrary to the assumption (Refer Fig. 4.3).

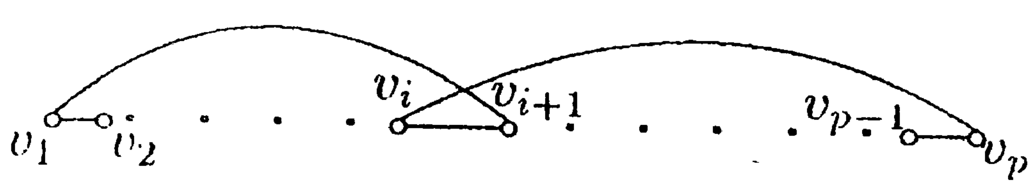


Figure 4.3

Hence $S \cap T = \Phi$ so that $|S \cap T| = 0$ (2)

Also by the definition of S and T, $d(u) = |S|$ and $d(v) = |T$.

Space for Hints

Hence by (1) and (2), $d(u)+d(v) = |S| + |T|$

$= |S \cup T| < p$.

Thus $d(u)+d(v) < p$.

But since $\delta \geq p/2$, we have, $d(u)+d(v) \geq p$ which gives a contradiction. Hence the theorem.

Lemma: 4.4

Let G be a graph with p points and let u and v be nonadjacent points in G such that $d(u)+d(v) \geq p$. Then G is Hamiltonian iff $G+uv$ is Hamiltonian.

Proof:

If G is Hamiltonian, then obviously $G+uv$ is also Hamiltonian.

Conversely, suppose that $G+uv$ is Hamiltonian, but G is not.

Then, as in the proof of Theorem 4.3, we obtain $d(u)+d(v) < p$.

This contradicts the hypothesis that $d(u)+d(v) \geq p$.

Thus $G+uv$ is Hamiltonian implies G is Hamiltonian.

Note:

The above lemma motivates the following definition of closure.

4.1.3 Definition

The **closure** of a graph G with p points is the graph obtained from G by repeatedly joining pairs of nonadjacent vertices whose degree sum is at least p until no such pair remains. The closure of G is denoted by $c(G)$.

Theorem: 4.5

Space for Hints

$c(G)$ is well defined.

Proof:

Let G have p vertices. Let G_1 and G_2 be two graphs obtained from G by repeatedly joining pairs of nonadjacent vertices whose degree sum is at least p until no such pair remains. Let $x_1, x_2, \ldots\ldots, x_m$ and $y_1, y_2, \ldots\ldots, y_n$ be the sequences of edges added to G in obtaining G_1 and G_2 respectively.

We claim that $\{x_1, x_2, \ldots\ldots, x_m\} = \{y_1, y_2, \ldots, y_n\}$

If possible let $x_{i+1} = uv$ be the first edge in the sequence $\{x_1, x_2, \ldots\ldots, x_m\}$ that is not an edge of G_2.

Let $H = G + \{x_1, x_2, \ldots\ldots, x\}$. Since uv is the next edge to be added to H in the process of constructing G_1, we have

$$d_H(u) + d_H(v) \geq p. \qquad (1)$$

Also by the choice of x_{i+1}, H is a subgraph of G_2.

Hence $d'(u) \geq d_H(u)$ and $d'(v) \geq d_H(v)$, where $d'(u)$ and $d'(v)$ denote degrees of u and v in G_2. Hence (1) implies $d'(u) + d'(v) \geq p$.

Hence by the definition of G_2, u and v must be adjacent in G_2. This is contradiction, since u and v are not adjacent in G_2.

Hence each x_i is an edge of G_2.

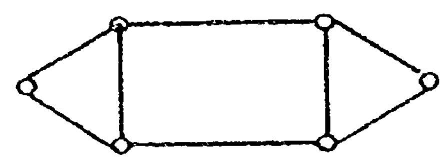
Similarly we can be prove that each y_i is an edge of G_1.

Space for Hints

Hence $G = G_2$. Thus $c(G)$ is unique and hence is well defined.

Example:

Figure 4.4 illustrate the construction of the closure of a graph G on six vertices. In this case $c(G)$ is complete.



G

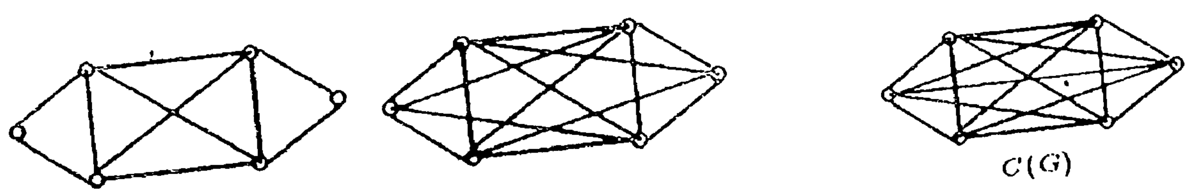


Figure 4.4

The following theorem is a consequence of Lemma 4.4

Theorem: 4.6

A graph is Hamiltonian iff its closure is Hamiltonian.

Proof:

Let $x_1, x_2, \ldots\ldots\ldots, x_n$ be the sequence of edges added to G in obtaining $c(G)$.

Let $G_1, G_2, \ldots\ldots, G_n = c(G)$ be the successive graphs obtained. Applying Lemma 4.4 repeatedly,

G is Hamiltonian $\Leftrightarrow$ G_1 is Hamiltonian

$\Leftrightarrow$ G_2 is Hamiltonian

....

....

$\Leftrightarrow$ $G_n = c(G)$ is Hamiltonian.

Space for Hints

The following corollary is only a particular case of the above theorem.

Corollary:

Let G be a graph with at least 3 points. If $c(G)$ is complete, then G is Hamiltonian.

The above theorem and corollary are often useful in showing that a given graph is Hamiltonian.

For example the graph G given in Figure 4.4 is Hamiltonian.

Note:

If $\delta \geq \frac{p}{2}$, then $c(G)$ is complete. Hence if $p \geq 3$, G is Hamiltonian.

Thus $\delta \geq \frac{p}{2}$ and $p \geq 3 \Rightarrow$ G is Hamiltonian.

This is Dirac's Theorem proved before (Theorem 4.3).

Theorem: 4.7

Chavatal Theorem

Statement: Let G be a graph with degree sequence $(d_1, d_2, \ldots\ldots, d_p)$, where $d_1 \leq d_2 \leq \ldots\ldots \leq d_p$ and $p \geq 3$. Suppose that for ever value of m less than $\frac{p}{2}$, either $d_m > m$ or $d_{p-m} \geq p-m$.

(i.e., there is no value of m less than $\frac{p}{2}$ for which $d_m \leq m$ and $d_{p-m} < p-m$). Then G is Hamiltonian.

Space for Hints

Proof:

Let G satisfy the hypothesis of the theorem.

We claim that c(G) is complete. Let u denote the degree of a vertex v in $c(G)$ by $d'(v)$.

If possible, let $c(G)$ be not complete.

Now let u and v be two nonadjacent vertices in $c(G)$ with

$$d'(u) \leq d'(v). \qquad (1)$$

and $d'(u)+d'(v)$ as large as possible. Let $d'(u)=m$. Since no two nonadjacent points in $c(G)$ can have degree sum p or more, we have $d'(u)+d'(v)<p$

$$\therefore d'(v) < p - d'(u)$$

$$\therefore d'(v) < p - m \qquad (2)$$

Now, let S denote the set of vertices in $V-\{v\}$ which are not adjacent to v in $c(G)$. Let T denote the set of vertices in $V-\{u\}$ which are not adjacent to u in $c(G)$.

Clearly $S = p-1-d'(v)$ and $T = p-1-d'(u)$ (3)

Also by the choice of u and v, each vertex in S has degree atmost $d'(u)$ and each vertex in $T \cup \{u\}$ has degree at most $d'(v)$.

Putting (2) in the first equation of (3) we get $S > p-1-(p-m) = m-1$. Hence $S \geq m$. Hence c(G) has at least m points with degree $\leq m$. (4)

From (3), $T = p-1-m$. Since each vertex in $T \cup \{u\}$ has degree $\leq d'(v)$, this implies that $c(G)$ has at least $p-m$ vertices of degree $\leq d'(v)$. Therefore by (2), $c(G)$ has at least $p-m$ vertices of degree $< p-m$. (5)

Because G is a spanning subgraph of $c(G)$, degree of each point in G cannot exceed that in $c(G)$. Hence statements (4) and (5) hold in the case of G also.

Hence $d_m < m$ and $d_{p-m} < p-m$. Also by (1) and (2), $m < p/2$. This contradicts the hypothesis on G.

$\therefore$ $c(G)$ is complete. Hence G is Hamiltonian.

Remark:

There does not appear to be any relationship between Eulerian and Hamiltonian graphs. This is illustrated by the following examples.

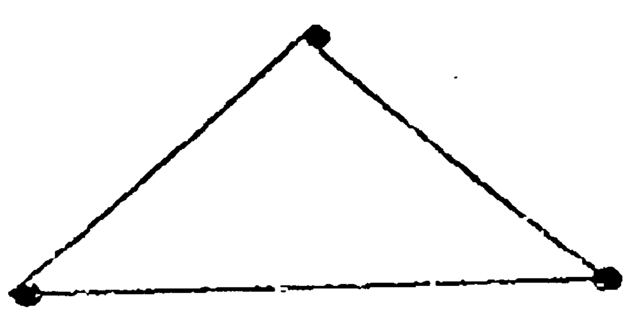
Example: 1

Give an example of a graph which is both Eulerian and Hamiltonian.

Solution:

The complete graph K_3 is both Eulerian and Hamiltonian

K_3



Example: 2

Give an example of a graph which is Eulerian but not Hamiltonian.

Space for Hints

Solution:

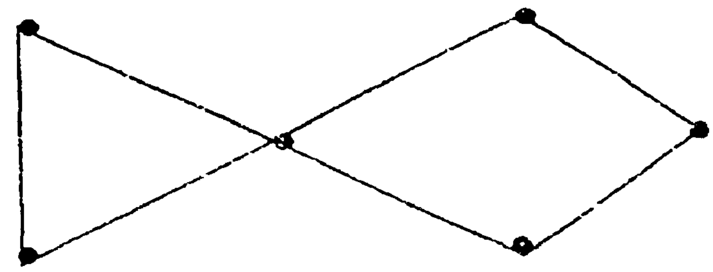


Figure 4.5

The graph in figure (4.5) is Eulerian but not Hamiltonian.

Example: 3

Give an example of a graph which is not Eulerian but Hamiltonian.

Solution:

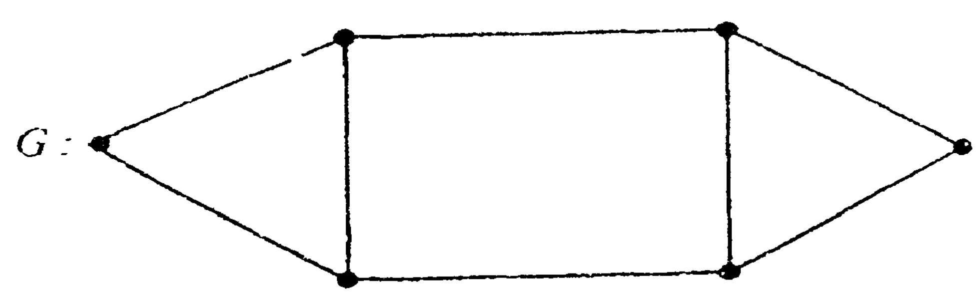


Figure 4.6

The graph G in figure (4.6) is Hamiltonian because the closure of G is complete.

But G is not Eulerian, Since there are vertices of odd degree.

Example: 4

Give an example of a graph that is neither Eulerian nor Hamiltonian.

Space for Hints

Solution:

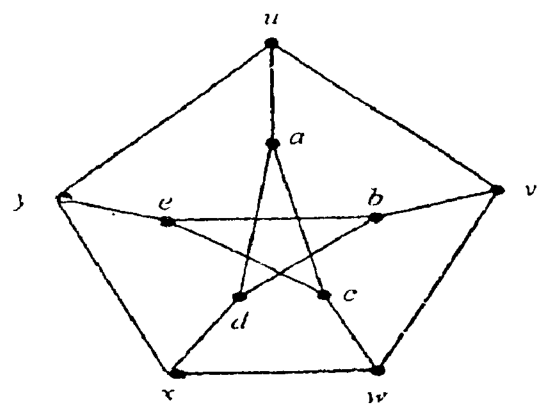


Figure 4.7

The Petersen graph given in figure (4.7) is neither Eulerian nor Hamiltonian.

Problem: 1

Show that $K_{m,1}$ is non Hamiltonian.

Proof:

Consider the complete bipartite graph $K_{m,n}$ with $m < n$. Let V_1 and V_2 be the two partitions of the graph with $|V_1| = m$ and $|V_2| = n$. The graph $K_{m,n} - V_1$ is disconnected with n vertices.

$\therefore$ The number of components in $K_{m,n} - V_1 = n$

Thus the necessary condition of a graph to be Hamiltonian is not satisfied. Thus $K_{m,n}$ is non – Hamiltonian.

Problem: 2

Find the closure of the graph given in following figure 4.8

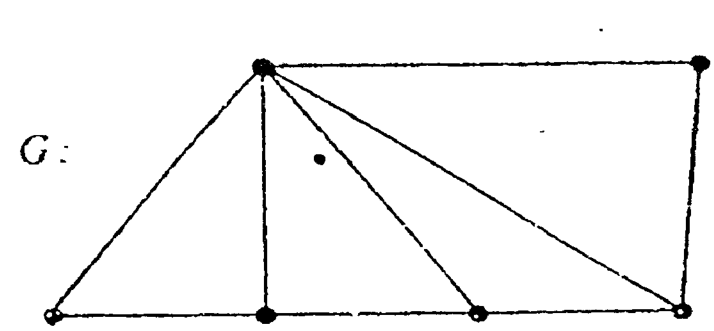


Figure 4.8

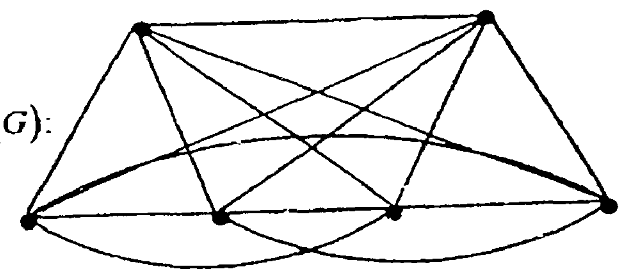


Figure 4.8 (a)

Problem: 3

If G is bipartite graph with odd number of vertices then prove that G is non – Hamiltonian.

Solution:

Suppose G is Hamiltonian, then G has a Hamiltonian cycle C. Since G is bipartite, C is of even length. So, the number of vertices on C is even. Since C is a spanning cycle, the number of vertices of G is even. This is a contradiction to our assumption. Hence the solution.

Problem: 4

Show that the Petersen graph is non Hamiltonian.

Solution:

Let us label the vertices as in Fig 4.9. If the Petersen graph G has a Hamiltonian cycle C, then $G - E(C)$ must be a regular spanning subgraph of degree 1. (A regular spanning subgraph of degree 1 is called a 1 – factor).

Let us search for all 1 – factor in G and show that none of them arise out of a Hamiltonian cycle of G.

Case: 1

Consider the subset $A = \{1a, 2b, 3c, 4d, 5e\}$ of the edge set of G.

Clearly A is a 1 – factor of G, but G – A is the union of two disjoint cycles and hence is not a Hamiltonian cycle of G.

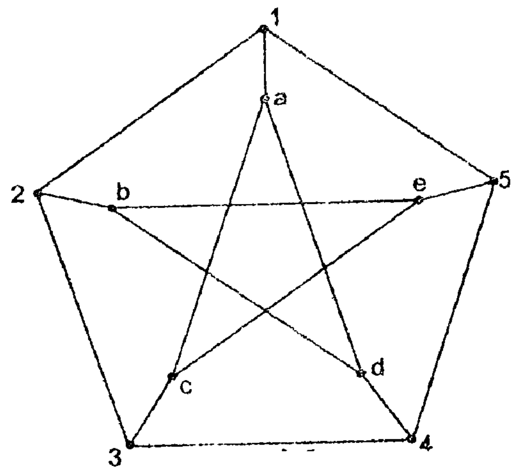


Figure 4.9

Case: 2

If the 1 – factor contains 4 edges from A, then the only line passing through the remaining two points must also be included in the 1 – factor, so that we again get A.

Case: 3

If a 1 – factor contains just 3 edges from A, then two such choices can be made.

Subcase: 3A

Let the 1 – factor contain 1a, 2b and 3c. Now the subgraph induced by the remaining four points is a P_4 (a path with 4 points) whose unique 1 – factor is $\{4d, 5e\}$. Thus the 1 – factor of G considered becomes A.

Subcase: 3B

Let the 1 – factor contain 1a, 2b and 4d. Here again the remaining four points induce P_4, whose unique 1 – factor is

Space for Hints

Space for Hints

$\{3c,5e\}$. Thus the 1 – factor of G considered becomes A.

Case: 4

If a 1 – factor contains just 2 edges from A, then again two such choices are possible.

Subcase: 4A

Let the 1 – factor contain 1a and 2b. In the subgraph induced by the remaining 6 points, point d has degree one and hence any 1 – factor of that subgraph must contain edge 4d. Thus case 3 is repeated.

Subcase: 4B

Let the 1 – factor contain 1a and 3b. In the subgraph induced by the remaining 6 points, point 2 has degree one and hence any 1 – factor of that subgraph must contain edge 2b. Thus case 3 is repeated.

Case: 5

Let a 1 – factor contain just one edge of A, say 1a. If it a contains one more edge from A, then one of the earlier cases will be repeated. Hence we have to choose the other four edges of this 1 – factor from two paths, each of length 3. (The paths are cebd and 2345). Hence the 1 – factor is $B = \{1a, ce, bd, 23, 45\}$. Now $G - B$ is again union of two disjoint cycles, and not a Hamiltonian cycle.

Case: 6

Suppose there exists a 1 – factor that does not contain any edge from A. It can contain at most two edges from the cycle 123451 and at most two edges from the cycle acebda. Hence it can contain at most four edges.

Hence there does not exist such a 1 – factor.

Space for Hints

Since the above 6 cases cover all possible types of 1 – factors, we see that G has no 1 – factor arising out of a Hamiltonian cycle.

Hence G has no Hamiltonian cycle.

Thus G is not Hamiltonian.

Problem: 5

Give an example of a non Hamiltonian graph with n vertices and $\frac{(n-1)(n-2)}{2}+1$ edges.

Solution:

Consider the complete graph of (n – 1) vertices K_{n-1}. The number of edges is $(n-1)c_2 = \frac{(n-1)(n-2)}{2}$.

Take another vertex v and join it with any one vertex of k_{n-1}.

Now the resulting graph G has n vertices and $\frac{(n-1)(n-2)}{2}+1$ edges.

For the graph G, $S = 1 < \frac{n}{2}$.

So by Dirac's theorem G is not Hamiltonian.

4.2 WEIGHTED GRAPH

There are two practical problems associated with Eulerian trails and Hamiltonian cycles. To describe these we need the following concepts.

Space for Hints

Definition: 4.2.1

If a real number w is attached to an edge e of a graph G then w is called the **weight** of the edge e we may write w(e). If real numbers (weights) are attached to all the edges of the graph G then it is said to be a weighted graph. The sum of the weights of all the edges is called the weight of the graph and it is denoted by $W(G)$.

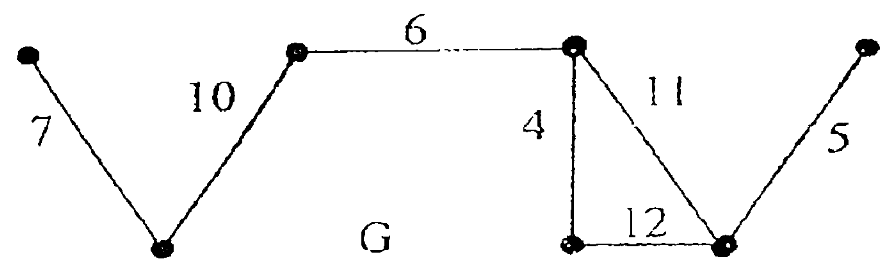


Figure 70

$$W(G) = 7 + 10 + 6 + 4 + 12 + 11 + 5 = 55$$

Note: 1

Any spanning tree of a weighted graph is called a weighted spanning tree.

Note: 2

The weights of different spanning trees of the same weighted graph need not be the same.

Note: 3

Note (1) and Note (2) are discussed in unit.

Problem: 1

Chinese postman problem:

A postman starting from the post office walks along the streets of his area to deliver the postal articles to the addresses and returns to the post office. The problem of choosing a route, which covers all the locations, assigned to him such that the walking distance is minimum is known as the Chinese postman problem.

Let G be the graph whose vertices are the different locations (addresses) and the edges are the paths connecting the locations. Let the weights of the edges be the distances between the locations. Now the Chinese postman problem is equivalent to the problem of finding an optimal tour of the weighted graph G. If G is an Eulerian graph then any tour of G is an optimal tour, because no edge is retraced. Therefore the Chinese postman problem reduces to the problem of finding an Eulerian tour in an Eulerian graph.

Fleury's algorithm gives an effective and elegant method to solve this problem.

Problem: 2

Travelling salesman problem:

The travelling salesman problem (TSP) is one of the many unsolved problem in Graph Theory.

A travelling salesman wishes to visit a number of towns to promote his business and to return to his starting place. If the time of travel (or the travelling expenses) between the towns are known, how should he plan his travel so that he visits each town exactly once and returns to his starting place in the shortest time (or with minimum expenses) possible. This is known as the Travelling Salesman Problem (TSP).

Space for Hints

UNIT – 5

BIPARTITE GRAPHS

5.0 INTRODUCTION

In this unit we deal with a Bipartite graphs. Matching and Marriage problem. Also introduce the concept of trees and connector problem.

5.1 BIPARTITE GRAPHS

Definition: 5.1.1

Let G be a simple graph whose vertex set is V. If V can be partitioned into two non empty sets X and Y such that no two vertices of X are adjacent and no two vertices of Y are adjacent so that each edge of G has one end vertex in X and the other in Y, then G is called a **bipartite** graph or a **bigraph**.

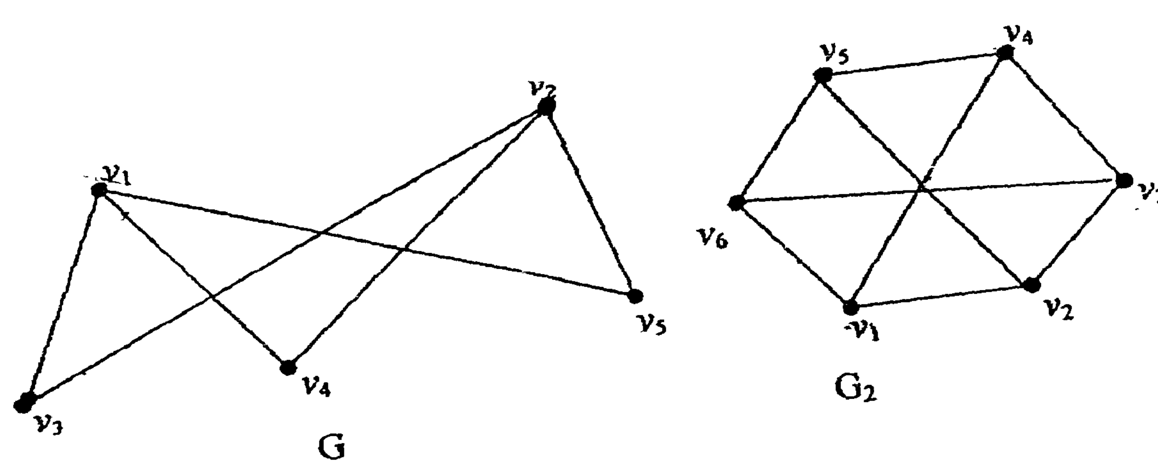


Figure 5.1

$X = \{v_1, v_2\}$ $\qquad$ $X = \{v_1, v_3, v_5\}$.

$Y = \{v_3, v_4, v_5\}$ $\qquad$ $Y = \{v_2, v_4, v_6\}$

G_1 and G_2 are bipartite graphs.

A bipartite graph G in which the vertex set is partitioned into two sets X and Y is denoted by (X, Y) (i.e) $G = (X, Y)$. We also say that (X, Y) is a **bipartition** of the graph G.

Space for Hints

Definition: 5.1.2

Complete bipartite graph

Let (X, Y) be a bipartite graph. If each vertex of X is adjacent to each vertex of Y then the graph is said to be a **complete bipartite graph**. If the vertex subsets X and Y have m and n vertices, then the complete bipartite graph is denoted by $K_{m,n}$ or $K_{n,m}$.

Remarks:

In a complete bipartite graph $K_{m,n}$.

(i) The number of vertices is $m+n$. $\upsilon(K_{m,n}) = m+n$.

(ii) The number of edges is mn. $\varepsilon(K_{m,n}) = mn$

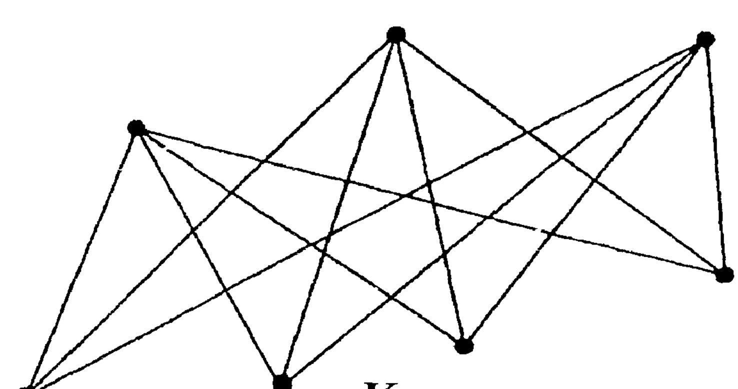


Figure 5.2

(iii) There is no isolated vertex.

(iv) $K_{m,n}$ is not a complete graph in the usual sense.

(v) The degree of a vertex m or n.

(vi) The complex bipartite graph $K_{3,3}$ is called **Kuratowski's second graph.**

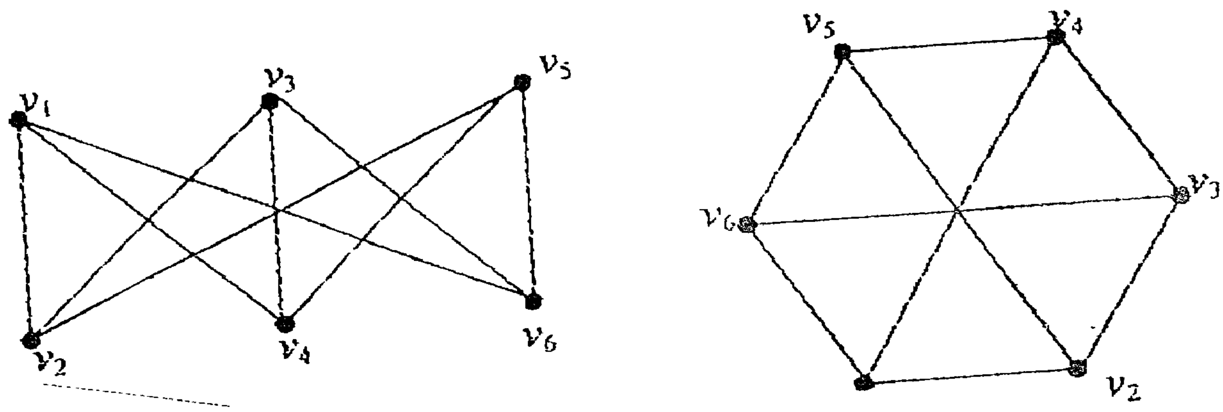


Figure 5.3

Note:

(1) The number of vertices is denoted by υ,

(2) The number of edges is denoted by ε.

Theorem: 5.1

A graph is bipartite if and only if it contains no odd cycle.

Proof:

Let G be a bipartite graph. We have to prove that it contains no odd cycle.

Since G is bipartite the vertex set is partitioned into two non empty set X and Y such that every edge of G has one vertex in X and the other vertex is Y.

Consider the cycle.

$C = v_1 v_2 v_3 ...... v_n v_1$

Let $v_1 \in X$ so that $v_2 \in Y$, $v_3 \in X$, $v_4 \in Y$ ......

In general $v_k \in X$ if k is odd and $v_k \in Y$ if k is even.

For the cycle C the last vertex $v_1 \in X$ and so the preceding vertex $v_1 \in Y$. Therefore n is even and so the cycle C has an even number of vertices and consequently it has an even number of edges. Therefore the cycle C is an even cycle. Since the cycle C is arbitrary it follows that every cycle is even. Hence the graph G has no odd cycle.

Conversely,

Let the graph G be without odd cycles. We have to prove that G is bipartite.

Let v be any vertex of G.

Let $X = \{x \in G \ \ d(x, v) = \text{even}\}$.

$Y = \{x \in G\} \mid d(x, v) = \text{odd}\}$

Clearly X and Y are non empty disjoint subset of the vertex set of G.

We now prove that no two vertices of X are adjacent.

Suppose the vertices u, $v \in X$ be adjacent so that uv is an edge.

Now the x – u path, x – v path and the edge uv together form a cycle.

Length of the cycle $= d(x, u) + d(x, v) + 1$

$= \text{even} + \text{even} + 1$

$= \text{odd}.$

Space for Hints

Space for Hints

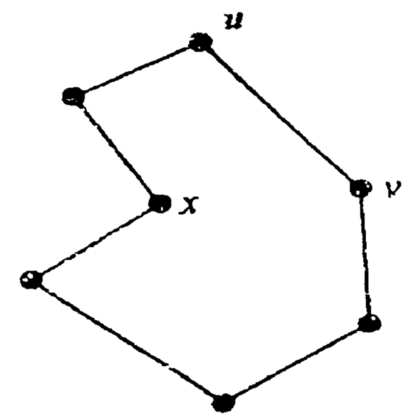


Figure 5.4

$\therefore$ This cycle is an odd cycle.

This is a contradiction to the assumption that the graph is without odd cycles. $\therefore$ No two vertices of X are adjacent. Again we prove that no two vertices of Y are adjacent. Suppose two vertices $u, v \in Y$ be adjacent so that uv is an edge.

Consider any vertex $y \in Y$.

Now the (y-u) path the (y-v) path and the edge uv together form a cycle.

Length of this cycle $= d(y, u) + d(y, v) + 1$

$= \text{odd} + \text{odd} + 1$

$= \text{odd}$

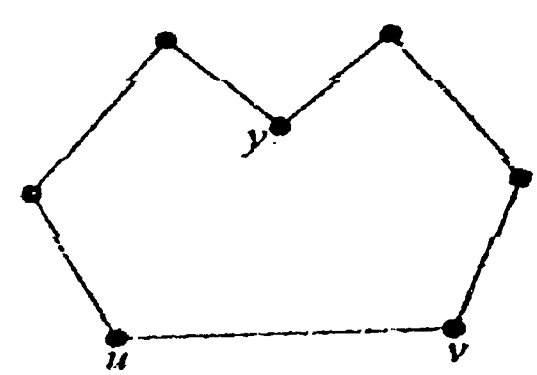


Figure 5.5

(i.e.) This cycle is an odd cycle. It is a contradiction to the assumption that the graph is without odd cycles.

Therefore no two vertices of Y are adjacent.

Thus the vertex set of the graph G is divided into the non empty disjoint subsets X, Y such that no two vertices of X are adjacent and no two vertices of Y are adjacent.

$\therefore (X, Y)$ is bipartition of G and so that graph G is a bipartite graph.

6.1 MATCHING

Definition: 5.1.3

A subset M of E is called a **matching** in G if no two of the edges in M are adjacent. The two ends of an edge in M are said to be matched under M.

Example:

In the graph G of figure 5.6 the sets $M_1 = \{e_6, e_8\}$.

G :

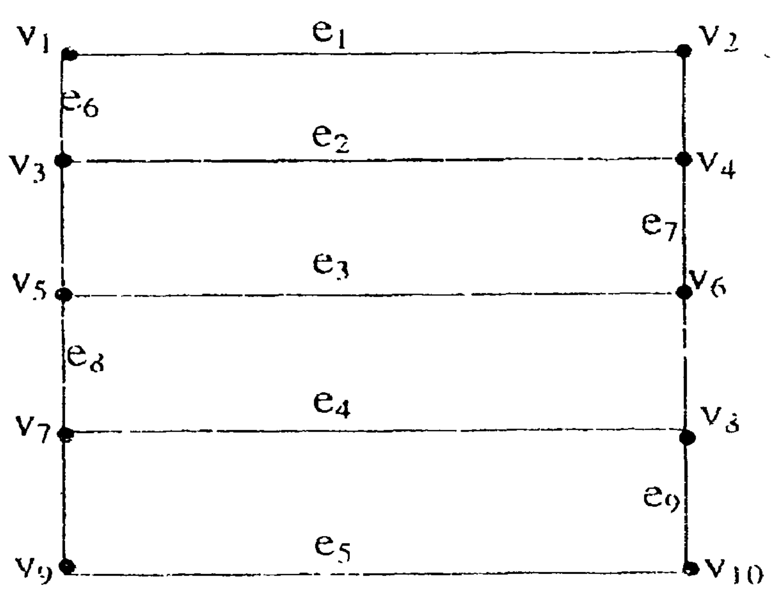


Figure 5.6

$M_2 = \{e_6, e_7, e_8, e_9\}$ and $M_3 = \{e_1, e_2, e_3, e_4, e_5\}$ are all matching's.

Space for Hints

Definition: 5.1.4

A matching M **saturates** a vertex v if one edge of M is incident with v. Also, we say v is **M – saturated**. Otherwise, v is **M – unsaturated.**

Example:

In the graph G of figure 5.6, v_1 is both M_1 - saturated and M_2 - saturated: v_4 is M_2 - saturated but M_1 - unsaturated: but M_3 saturates every vertex of G.

Definition: 5.1.5

If M is a matching in G such that every vertex of G is M – saturated then M is called a **perfect** matching.

Example:

The matching M_3 of G of figure 5.6 is a perfect matching where as M_1 and M_2 are not perfect.

Note:

If G has a perfect matching then p is even.

Definition: 5.1.6

A matching M is called a **maximal matching** of G if there is no matching M' of G such that $M' \supset M$.

Remark:

Note that two maximal matching need not have same cardinality.

Example:

Space for Hints

In the graph G of figure 5.7, $M = \{e_1, e_6, e_3\}$ and $M_2 = \{e_5, e_3\}$ are maximal matchings.

Definition: 5.1.7

A matching M of G is called a **Maximum matching** if G has no matching M' with $M' > M$. The number of edges in a maximum matching of G is called as the **matching number** of G.

G:

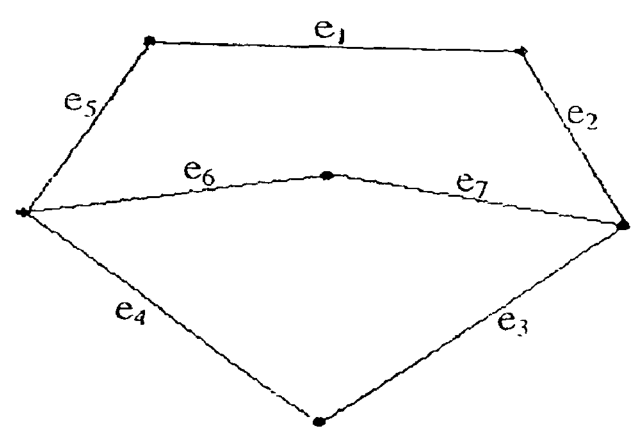


Figure 5.7

We note that $M_1 = \{e_1, e_6, e_3\}$ is a maximum matching of G of figure 5.7, but $M_2 = \{e_5, e_3\}$ is not a maximum matching, though it is a maximal matching of G of figure 5.7. Clearly every perfect matching is maximum; but maximum matching's need not be perfect.

Example:

Consider the star $K_{1,6}$ and in general K_{1p}. Here any maximum matching contains only one edge and hence it is not perfect.

Space for Hints

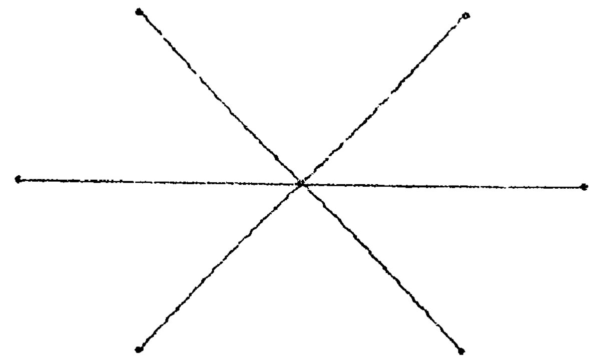


Figure 5.8 $K_{1,6}$

Definition: 5.1.8

Let M be the matching in G. An **M – alternating path** in G is a path whose edges are alternately in E\M and M.

Example:

In the graph G of figure 5.9, if we consider the matching $M = \{e_1, e_2\}$ then the path $v_1v_2v_6v_5v_3$ is an M – alternating path.

Definition: 5.1.9

Let M be a matching in G. An M – **augmenting path** is an M – alternating path whose origin and terminus and M – unsaturated.

Example:

In the graph G of figure 5.9, if we consider the matching $M = \{e_1, e_2\}$ then the path $v_1v_2v_6v_5v_3v_4$ is an M – augmenting path.

Note:

1. In M – augmenting path initial and final edges are in E\M.
2. An M – alternating path whose initial and final edges are in E\M, need not be an M – augmenting path.

G:

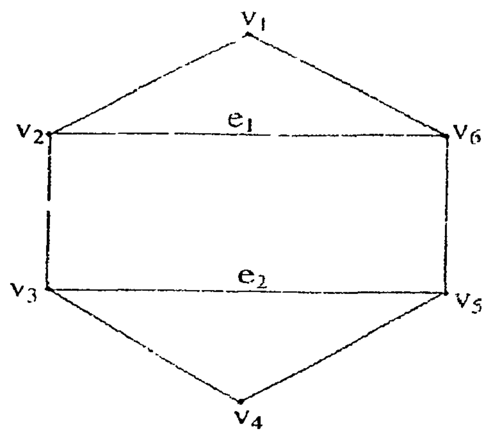


Figure 5.9

Theorem: 5.2

Let M_1 and M_2 be two matching's in a graph G. Let $M_1 \Delta M_2 = (M - M_2) \cup (M_2 - M_1)$ be the symmetric difference of M and M_2. Let $H = G[M - M_2]$ be the graph of G induced by $M \Delta M_2$. Then each component of H is either an even cycle with edges alternately in M and M_2 or a path P with edges alternately in M and M_2 such that the origin and the terminus of P are unsaturated in M_1 or M_2.

Proof:

Let v be any point in H. Since M_1 and M_2 are matchings in G, at most one line of M and at most one line of M_2 are incident with v. Hence the degree of v in H is either 1 or 2. Hence it follows that the components of H must be as described in theorem.

Example: 1

Consider the graph G_1 given in Figure 5.10. $M_1 = \{v_1v_2, v_6v_3, v_5v_4\}$ is a perfect matching in G_1. Also $M_2 = \{v\ v_3, v_6v_5\}$ is a matching in G_1. However M_2 is not a perfect matching. The points v_2 and v_4 are not M_2 - saturated.

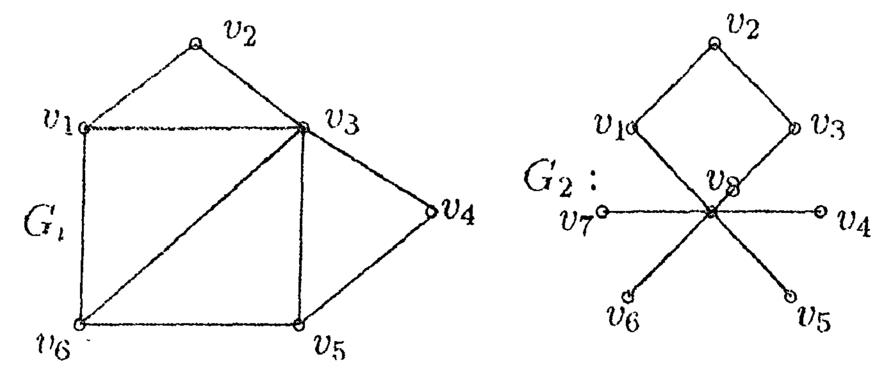


Figure 5.10

For the graph $G_2, M = \{v_8v_4, v_1v_2\}$ is a maximum matching but is not a perfect matching.

Example: 2

For the graph G_1 given in Fig 5.10

$$M_1 \Delta M_2 = \{v_1v_2, v_6v_3, v_5v_4, v_1v_3, v_6v_5\}$$

the graph $H_1 = G_1[M_1\Delta M_2]$ is given in Fig. 5.11.

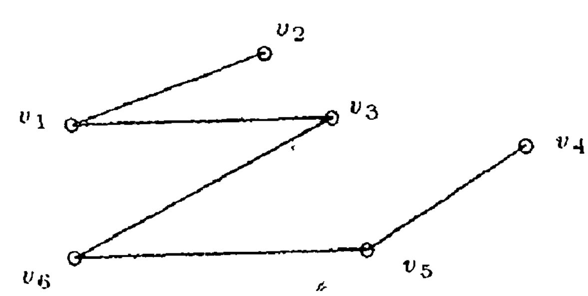


Figure 5.11

Clearly H_1 is a path whose edges are alternately in M_1 or M_2. The origin v_2 and terminus v_4 are both M_2 - unsaturated.

The following theorem due to Berge gives a characterization of maximum matching.

Theorem: 5.3

Berge Theorem

Statement: A matching M in a graph G is a maximum matching if and only if G contains no M – augmenting path.

Proof:

Let M be a maximum matching in G. Suppose G contains an M – augmenting path $P = (v_0, v_1, v_2, \ldots\ldots, v_{2k+1})$.

By definition of M – augmenting path the lines $v_0v_1, v_2v_3, \ldots\ldots, v_{2k}v_{2k+1}$ are not in M and the lines $v_1v_2, v_3v_4, \ldots\ldots, v_{2k-1}v_{2k}$ are in M. Hence

$$M' = M - \{v_1v_2, v_3v_4, \ldots\ldots, v_{2k-1}v_{2k}\} \cup \{v_0v_1, v_2v_3, \ldots\ldots, v_{2k}v_{2k+1}\}$$

is a matching in G and $M' = M + 1$, which is a contradiction, since M is a maximum matching. Hence G has no M – augmenting path.

Conversely, suppose G has no M – augmenting path. If M is not a maximum matching in G then there exists a matching M' of G such that $M' > M$.

Let $H = G[M \Delta M']$. By theorem 5.2, each component of H is either an even cycle with edges alternately in M and M' or a path P with edges alternately in M and M' such that the origin and the terminus of P are unsaturated in M or M'. Clearly any component of H which is a cycle contains equal number of edges from M and M'.

Since $M' > M$ there exists at least one component of H which is a path whose first and last edges are from M'. Thus the origin and terminus of P and M' – saturated in H and hence they are M – unsaturated in G. Thus P is an M – augmenting path in G, which is a contradiction. Hence M is a maximum matching in G.

Space for Hints

Problems:

Problem: 1

For what values of n does the complete graph K_n have perfect matching.

Solution:

Clearly any graph with p odd has no perfect matching. Also the complete graph K_n has a perfect matching if n is even. For example if $V(K_n) = \{1,2,\ldots\ldots., n\}$ then $\{12,34,\ldots..,(n-1)n\}$ is a perfect matching of K_n. Thus K_n has a perfect matching if and only if n is even.

Problem: 2

Show that a tree has at most one perfect matching.

Solution:

Let T be a tree. Suppose T has two perfect matching's say M_1 and M_2. Then degree of every vertex in $H = T[M_1 \Delta M_2]$ is 2. Hence every component of H is an even cycle with edges alternately in M and M_2. This is a contradiction, since T has no cycles. Therefore T has at most one perfect matching.

Problem: 3

Find the number of perfect matching's in the complete bipartite graph $K_{n,n}$.

Solution:

Let $A = \{x_1, x_2, \ldots\ldots.., x_n\}$ and $B = \{y_1, y_2, \ldots\ldots.., y_n\}$ be a bipartition of $K_{n,n}$.

We observe that any matching of $K_{n,n}$ that saturates every vertex of A is a perfect matching. Now the vertex x_1 can be saturated in n ways by choosing any one of the edges $x_1y_1, x_1y_2, \ldots, x_1y_n$. Having saturated x_1, the vertex x_2 can be saturated in n – 1 ways. In general having saturated $x_1, x_2, \ldots, x_i$ the next vertex x_{i+1} can be saturated in $n-i$ ways. Hence the number of perfect matching's in $K_{n,n}$ is $n(n-1)\ldots 2.1 = n!$

Problem: 4

Find the number of perfect matching's in the complete graph K_{2n}.

Solution:

Let $V(K_{2n}) = \{v_1, v_2, \ldots, v_{2n}\}$. The vertex v_1 can be saturated in $2n-1$ ways by choosing any line e_1 incident at v_1. In general having chosen the edges $e_1, e_2, \ldots, e_k$ a vertex v which is not saturated by any of the edges $e_1, e_2, e_3, \ldots, e_k$ can be saturated in $2n-(2k+1)$ ways. We obtain a perfect matching after the choice of n lines in the above process.

Hence the number of perfect matching's in K_{2n}.

$$= 1.3.5\ldots\ldots(2n-1)$$

$$= \frac{1.2.3.4.5\ldots\ldots(2n-1)(2n)}{2.4.6\ldots.2n}$$

$$= \frac{(2n)!}{2^n n!}.$$

Space for Hints

Space for Hints

5.2 THE MARRIAGE PROBLEM

Let $A = \{x_1, x_2, \ldots\ldots, x_n\}$ be a set of n boys and $B = \{y_1, y_2, \ldots\ldots, y_m\}$ be a set of m girls in a village. Each boy has one or more girl friends. Under what conditions can we arrange marriage in such a way that each boy marries one of his girl friends? This problem is known as the marriage problem.

We now obtain a graph theoretical formulation of the above problem. Let G be the bipartite graph with partition (A, B) such that x_i is joined to y_j if and only if y_j is a girl friend of x_i. The marriage problem is equivalent to finding the conditions under which G has a matching that saturates every vertex of A.

For example, suppose there are five boys b_1, b_2, b_3, b_4 and b_5 and six girls g_1, g_2, g_3, g_4, g_5 and g_6 with their relationship as follows.

$b_1 \rightarrow \{g_1, g_2, g_3\} = S_1$

$b_2 \rightarrow \{g_1, g_3\} = S_2$

$b_3 \rightarrow \{g_4, g_5\} = S_3$

$b_4 \rightarrow \{g_3\} = S_4$

$b_5 \rightarrow \{g_4, g_5, g_6\} = S_5$

The bipartite graph representing this situation is shown in figure 5.12

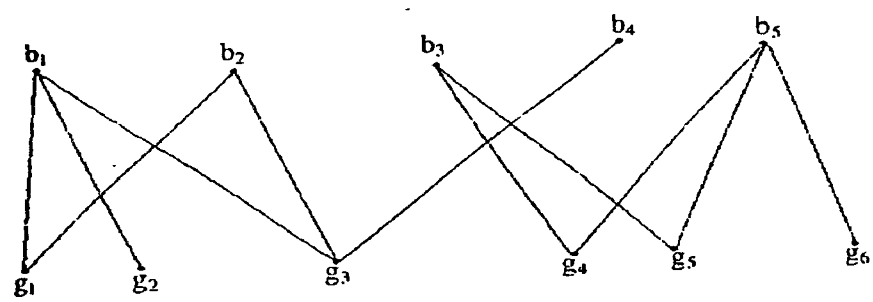


Figure 5.12

One of the solutions to this example is, b_1 to marry g_2, b_2 to marry g ,b_3 to marry g_4, b_4 to marry g_3 and b_5 to marry g_5.

Now, we present a necessary and sufficient condition for the existence of a solution to the above marriage problem. First given by P. Hall.

Definition: 5.2.1

Neighbour set

Let S be a subset of the vertex set of a graph G. Those vertices of G which are adjacent to the vertices in S are called the **neighbours** of the vertices in S. The set of these vertices is called the **neighbour set** of S in G and is denoted by $N_G(S)$ or $N(S)$.

Example:

Let $S = \{v_1, v_5, v_7\}$. We get $N(S) = \{v_2, v_3, v_4, v_6, v_8\}$.

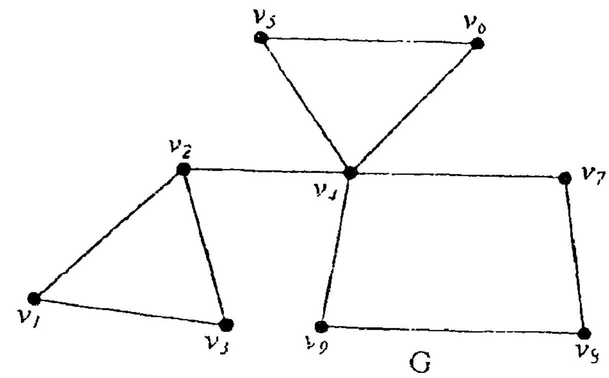


Figure 5.13

Theorem: 5.4

Hall's marriage theorem

Statement: Let G be a bipartite graph with bipartition (A, B). Then G has a matching that saturates all the vertices of A if and only if $N(S) \geq S$, for every subset S of A.

Proof:

Suppose G has a matching M that saturates all the vertices in A. Let $S \subseteq A$. Then every vertex is S is matched under M to a vertex in $N(S)$ and two distinct vertices of S are matched to two distinct vertices of $N(S)$. Hence it follows that $N(S) \geq S$.

Conversely, suppose $N(S) \geq S$ for all $S \subseteq A$. We wish to show that G contains a matching which saturates every vertex in A. Suppose G has no such matching.

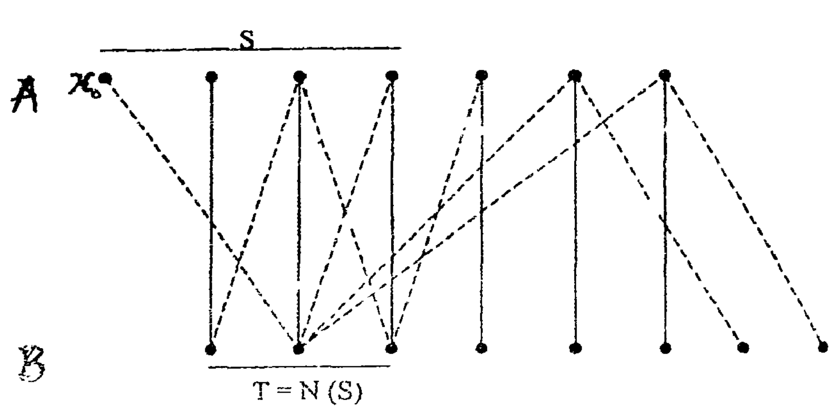


Figure 5.14

Let M* be a maximum matching in G. By assumption there exists a vertex $x_0 \in A$ which is M* - unsaturated. Let $Z = \{v \in V(G) /$ there exists a M* - alternating path connecting x_0 and $v\}$

Since M* is a maximum matching, by Berge's theorem, G has no M* - augmenting path and hence x_0 is the only M* - unsaturated vertex in Z.

Let $S = Z \cap A$ and $T = Z \cap B$. Clearly $x_0 \in S$ and every vertex of $S - \{x_0\}$ is matched under M* with a vertex of T.

Thus $$T = S - 1 \qquad (2)$$

We now claim that $N(S) = T$. Clearly from the definition of T, we have

$$T \subseteq N(S) \qquad (3)$$

Now, let $v \in N(S)$. Hence there exists $u \in S$ such that v is adjacent to u. Since $S = Z \cap A$ it follows that $u \in Z$.

Hence there exists an M* - alternating path P

$$(x_0, y_1, x_1, y_2, \ldots\ldots, x_{k-1}, y_k, u)$$

If v lies on P, then clearly $v \in Z \cap B = T$. Suppose v does not lie on P. Now the edge $y_k u \in M^*$. Hence the edge uv is not in M*. Hence the path P_1 consisting of P followed by the edge uv is an M* - alternating path. Hence $v \in Z \cap B = T$.

Thus $$N(S) \subseteq T \qquad (4)$$

From (3) and (4) we have $$N(S) = T \qquad (5)$$

From (2) and (5) we have $N(S) = T = S - 1 < S$ which is a contradiction. Hence the theorem.

Remark:

Hall's theorem answers the marriage problem. The marriage problem with n boys has a solution if and only if for every k with $1 \leq k \leq n$, every set of k boys has collectively at least k girl friends.

The following is an important consequence of Hall's marriage theorem.

Theorem: 5.5

Let G be a k – regular bipartite graph with $k > 0$. Then G has a perfect matching.

Proof:

Let (V, V_2) be a bipartition of G. Since each edge of G has one end in V_1 and there are k edges incident with each vertex of V, we have $q = k \ V_1$.

By similar argument $q = k \ V_2$, so that $k \ V = k \ V_2$. Since $k > 0$ we get $V_1 = V_2$.

Now let $S \subseteq V_1$. Let E_1 denote the set of all edges incident with vertices $N(S)$. Since G is k – regular, $E_1 = k \ S$ and $E_2 = k \ N(S)$.

Also by definition of N(S), we have $E_1 \subseteq E_2$, and hence it follow that $k \ S \leq k \ N(S)$. Thus $N(S) \geq S$.

Hence by Hall's theorem, G has a matching M that saturates every vertex in V_1. Since $V_1 = V_2$, M also saturates all the vertices of V_2. Thus M is a perfect matching.

Space for Hints

5.3 TREES

Definition: 5.3.1

A graph that contains no cycles is called an **acyclic graph.** A connected acyclic graph is called a **tree.**

Any graph without cycles is also called a **forest** so that the components of a forest are trees.

Example:

Draw all the trees with 6 vertices.

Solution:

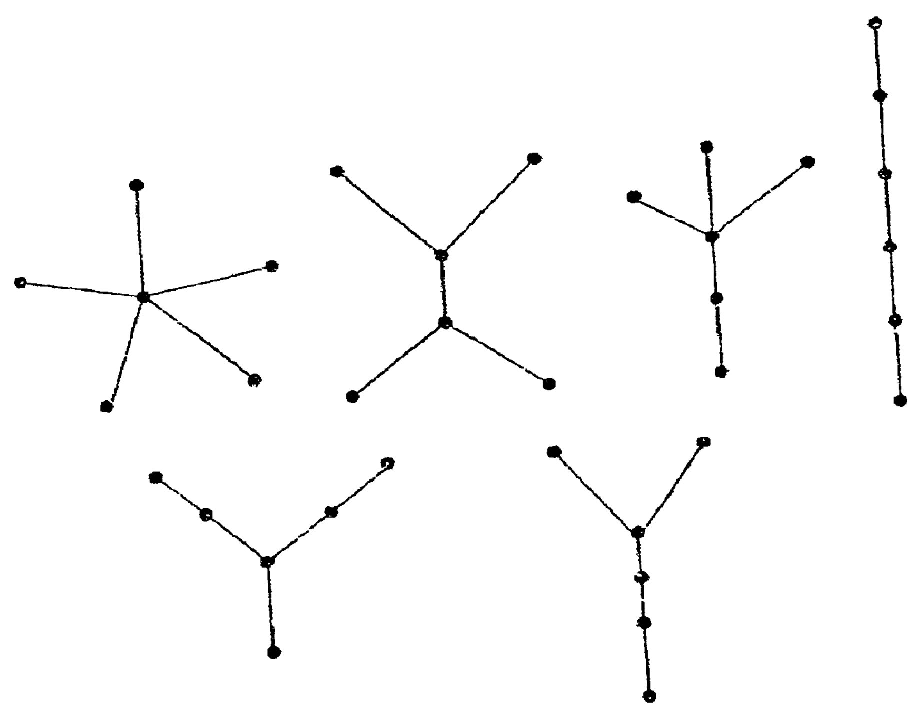


Figure 5.15

Theorem: 5.6

Let G be a (p, q) graph. The following statements are equivalent.

1. G is a tree.
2. Every two points of G are joined by a unique path.
3. G is connected and $p = q + 1$.

Space for Hints

4. G is acyclic and $p = q + 1$.

Proof:

$1 \Rightarrow 2$: Let u, v be any two points of G.

Since G is connected there exists a $u - v$ path in G.

Now suppose there exist two distinct $u - v$ paths.

$P_1 : u = v_0, v_1, v_2, \ldots, v_n = v$ and

$P_2 : u = w_0, w_1, w_2, \ldots, w_m = v$.

Let i be the least positive integer such that $1 \le i < m$ and $w_i \notin P_1$ (such an i exists since P_1 and P_2 are distinct).

Hence $w_{i-} \in P \cap P_2$.

Let j be the least positive integer such that $i < j \le m$ and $w_j \in P_1$ Then the $w_{i-1} - w_j$ path along P_2 followed by the w_j w_{i-1} path along P_1 form a cycle which is a contradiction.

Hence there exists a unique $u - v$ path in G.

$2 \Rightarrow 3$: Clearly G is connected.

We prove $p = q + 1$ by induction on p.

This is trivial for a connected graph with 1 or 2 points.

Assume the result for graphs with fewer than p points.

Let G be a graph with p points. Let $x = uv$ be any line G.

Since there exists a unique $u - v$ path in G, G – x is a disconnected graph with exactly two components G_1 and G_2.

Let G be a (p_1, q_1) graph and G_2 a (p_2, q_2) graph.

Then $p_1 + p_2 = p$ and $q_1 + q_2 = q - 1$.

Further by induction hypothesis $p_1 = q_1 + 1$ and $p_2 = q_2 + 1$.

$$\begin{aligned} \text{Hence } p &= p_1 + p_2 \\ &= q_1 + q_2 + 2 \\ &= q - 1 + 2 \\ &= q + 1. \end{aligned}$$

$3 \Rightarrow 4$: We must prove that G is acyclic.

Suppose G contains a cycle of length n.

There are n points and n lines on this cycle. Fix a point u on the cycle. Consider any one of the remaining $p - n$ points not on the cycle, say v.

Since G is connected we can find a shortest $u - v$ path in G. Consider the line on this shortest path incident with v. The p – n lines thus obtained are all distinct.

Hence $q \geq (p - n) + n = p$ which is a contradiction since $q + 1 = p$. Thus G is a cyclic.

$4 \Rightarrow 1$: Since G is acyclic to prove that G is a tree we need only to prove that G is connected.

Suppose G is not connected. Let $G_1, G_2, \ldots, G_k \, (k \geq 2)$ be the components of G.

Since G is acyclic each of these components is a tree.

Space for Hints

Hence $q_i + 1 = p_i$ where G_i is a (p_i, q_i) graph.

$$\therefore \sum_{i=1}^{k} (q_i + 1) = \sum_{i=1}^{k} p_i.$$

i.e., $q + k = p$ and $k \geq 2$, which is a contradiction.

Hence G is connected.

This completes the proof.

Corollary:

Every non – trivial tree G has atleast two vertices of degree 1.

Proof:

Since G is non – trivial, $d(v) \geq 1$ for all points v. Also $\sum d(v) = 2q = 2(p-1) = 2p-2$.

Hence $d(v) = 1$ for at least two vertices.

Definition: 5.3.2

Spanning Tree:

A spanning subgraph of a graph that is a tree is called a spanning tree.

Example:

Draw all the spanning trees of the graph given in fig 5.16

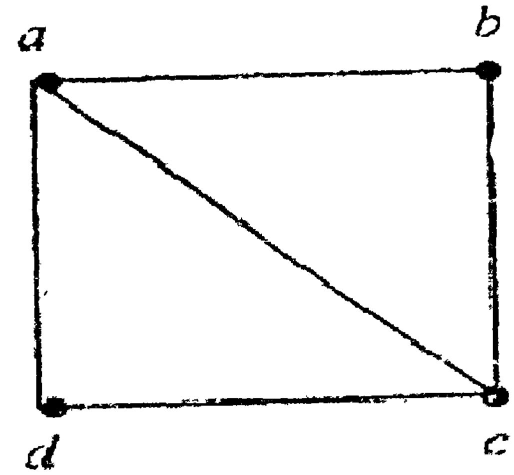


Figure 5.16

Solution:

The spanning trees of the given graph are

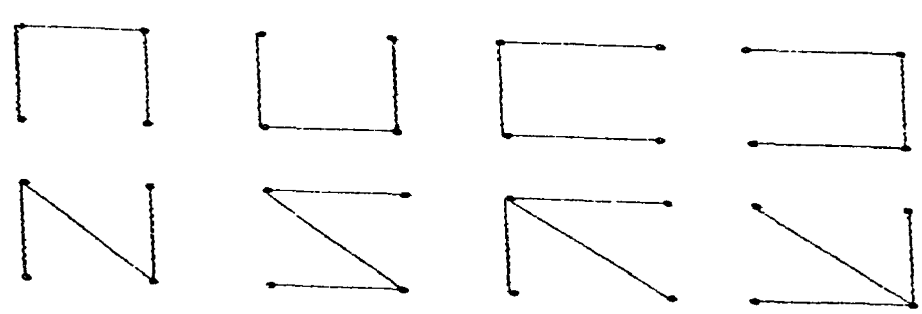


Figure 5.17

Theorem: 5.7

Every connected graph has a spanning tree.

Proof:

Let G be a connected graph. Let T be a minimal connected spanning subgraph of G. Then for any line x of T, $T-x$ is disconnected and hence x is a bridge of T.

Hence T is acyclic.

Further T is connected and hence is a tree.

Space for Hints

Corollary:

Let G be a (p, q) connected graph. Then $q \geq p - 1$.

Proof:

Let T be a spanning tree of G. Then the number of lines in T is $p - 1$. Hence $q \geq p - 1$.

Theorem: 5.8

Let T be a spanning tree of a connected graph G.

Let $x = uv$ be an edge of G not in T. Then $T + x$ contains a unique cycle.

Proof:

Since T is acyclic every cycle in $T + x$ must contain x. Hence there exists a one to one correspondence between cycles in $T + x$ and $u - v$ paths in T. As there is a unique $u - v$ path in tree T, there is a unique cycle in $T + x$.

Definition: 5.3.3

Eccentricity, Radius, Centre

Let G be any graph. Consider any vertex $u \in G$ and its distances from all other vertices. The maximum of these distances is called the **eccentricity** of the vertex u and it is denoted by $e(u)$.

Consider the eccentricities of all the vertices of a graph G. The minimum of these eccentricities is called the **radius** of the graph and it is denoted b $r(G)$.

If a vertex u of a graph G is such that its eccentricity is equal to the radius of the graph (i.e) $e(u) = r(G)$ then the vertex u is called a

central point. The set of all central points of the graph is called the **centre** of the graph and it is denoted by $c(G)$.

Space for Hints

Example:

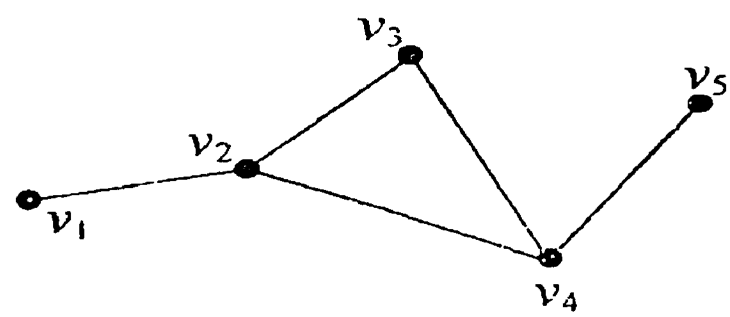


Figure 5.17

Distance of v_1	Distance of v_2
$d(v_1, v_2) = 1$	$d(v_2, v_1) = 1$
$d(v_1, v_3) = 2$	$d(v_2, v_3) = 1$
$d(v_1, v_4) = 2$	$d(v_2, v_4) = 1$
$d(v_1, v_5) = 3$	$d(v_2, v_5) = 2$.
Eccentricity $e(v_1) = 3$	$e(v_2) = 2$

Distance of v_3	Distance of v_4	Distance of v_5
$d(v_3, v_1) = 2$	$d(v_4, v_1) = 2$	$d(v_5, v_1) = 3$
$d(v_3, v_2) = 1$	$d(v_4, v_2) = 1$	$d(v_5, v_2) = 2$
$d(v_3, v_4) = 1$	$d(v_4, v_3) = 1$	$d(v_5, v_3) = 2$
$d(v_3, v_5) = 2$	$d(v_4, v_5) = 1$	$d(v_5, v_4) = 1$
$e(v_3) = 2$	$e(v_4) = 2$	$e(v_5) = 3$

The minimum of all eccentricities is 2.

$\therefore$ Radius of the graph is $r(G) = 2$.

We find $e(v_2) = e(v_3) = e(v_4) = r(G)$

$\therefore$ v_2, v_3, v_4 are central points of G.

So the set $\{v_2, v_3, v_4\}$ is the centre of the graph s

(ie) $c(G) = \{v_2, v_3, v_4\}$

Note:

The above definition (5.3.3) is also stated as

Let v be a point in a connected graph G. The **eccentricity** $e(v)$ of v is defined by $e(v) = \max\{d(u, v) \;\; u \in V(G)\}$.

The **radius** $r(G)$ is defined by $r(G) = \min\{e(v) \;\; v \in V(G)\}$.

v is called a **central point** if $e(v) = r(G)$ and the set of all central points is called the **centre** of G.

Theorem: 5.8

Every tree has a centre consisting of either one point or two adjacent points.

Proof:

The result is obvious for the tree K and K_2.

Now, let T be any tree with $p \geq 2$ points.

T has at least two end points and maximum distance from a given point u to any other point v occurs only when v is an end point. Now delete all the end points from T. The resulting graph T' is also a

Space for Hints

tree and the eccentricity of each point in T' is exactly one less than the eccentricity of the same point in T. Hence T and T' have the same centre.

In the process of removing end points is repeated, we obtain successive trees having the same centres as T and we eventually obtain a tree which is either K_1 or K_2.

Hence the centre of T consists of either one point or two adjacent points.

5.4 CONNECTOR PROBLEM

Consider the construction of railway lines connecting different cities. It is desirable that the cost of construction is minimum. The problem of designing a railway net – work to minimize the total cost of construction is one of the many problems known as **connector problems.**

Each city can be considered as the vertex of a weighted graph G and the railway lines connecting the cities as edges. The cost of construction of the railway line between the cities v_i and v_j is c_{ij} taken as the weight of the edge $v_i v_j$. Then the connector problem is equivalent to finding the connected spanning subgraph with minimum weight. Further the minimum weight spanning subgraph is clearly a spanning tree. This minimum weight spanning tree of a weighted graph is called an optimal tree. Hence connector problem is the same as finding an **optimal tree** of a weighted graph.

Note:

(1) The optimal tree of a weighted graph is not unique.

(2) If the weights of all the edges of a weighted graph are equal

Space for Hints

then any spanning tree is an optimal tree.

They are a number of methods to find an optimal tree of a weighted graph. We discuss only **Krushkal's algorithm** and **Prim's algorithm.**

5.4.1 KRUSHKAL'S ALGORITHM

Krushkal's algorithm to find an optimal tree of a weighted graph.

Let G be a weighted graph whose edge set is E. Let w_k be the weight of the edge e_k for $k = 1,2,3,.......$The algorithm of finding an optimal tree that is a spanning tree of minimum weight is given in the following steps.

Step: 1

Choose a link e_1 of minimum weight w_1.

Step: 2

Having chosen $\{e_1, e_2, ..., e_i\}$ choose the edge e_{i+1} such that

(i) $e_{i+1} \in E - \{e_1, e_2, ..., e_i\}$

(ii) $\{e_1, e_2, ..., e_i, e_{i+1}\}$ is acyclic.

(iii) The weight of e_{i+1} is least among the remaining edges.

Step: 3

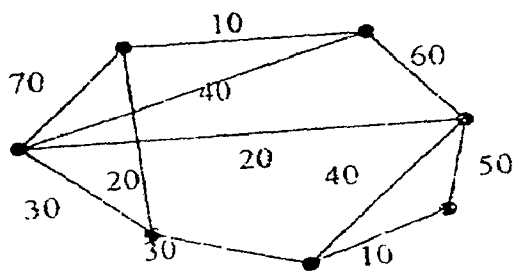
Stop when step (2) cannot be further employed.

When the algorithm terminates we get a spanning tree with minimum weight and hence an optimal tree of the graph.

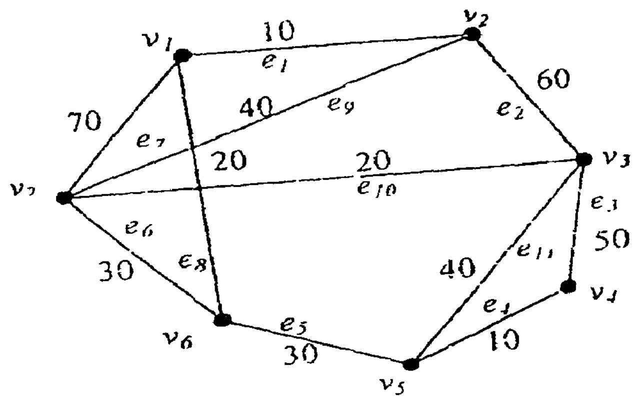
Problem:

Using Krushkal's algorithm find an optimal tree of the weighted graph G given below.

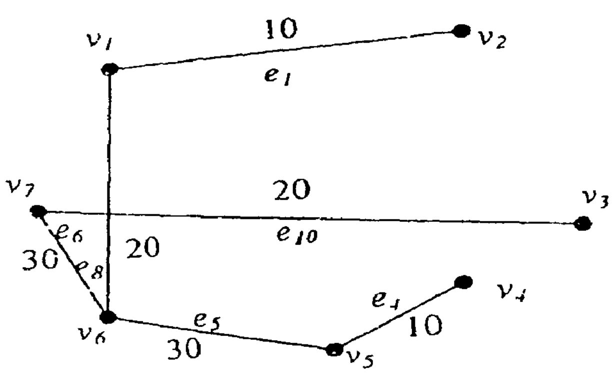
Let us label the vertices and edges of the graph as follows.



G.



Choose the edge e with minimum weight $w_1 = 10$. From $E-\{e_1\}$ choose e_4 with minimum weight $w_4 = 10$. Clearly $\{e_1, e_4\}$ is acyclic. From $E-\{e\ , e_4\}$ choose e_8 with minimum weight $w_8 = 20$ clearly $\{e_1, e_4, e_8\}$ is acyclic.



From $E-\{e_1, e_4, e_8\}$ choose e_{10} with minimum weight $w_{10} = 20$. Clearly $\{e_1, e_4, e_8, e_{10}\}$ is acyclic. From $E-\{e_1, e_4, e_8, e_{10}\}$ choose e_5 with minimum weight $w_5 = 30$. Clearly $(e_1, e_4, e_8, e_{10}, e_5)$ is acyclic. From $E-\{e_1, e_4, e_8, e_{10}, e_5\}$ choose e_6 with minimum weight $w_6 = 30$. Clearly $\{e_1, e_4, e_8, e_{10}, e_5, e_6\}$ is acyclic.

Now the algorithm terminates as 6 edges have been chosen in a graph of 7 vertices.

$\therefore$ The optimal spanning tree which has been constructed

Space for Hints

has the edge set $\{e_1, e_4, e_8, e_{10}, e_5, e_6\}$.

Weight of the optimal tree $T = w_1 + w_4 + w_8 + w_{10} + w_5 + w_6$.

$= 10 + 10 + 20 + 20 + 30 + 30$

$= 120$.

Theorem: 5.9

Any spanning tree constructed by Krushkal's algorithm in a weighted graph is an optimal tree.

Proof:

Let G be a weighted graph of n vertices. Let T be a spanning tree constructed by Krushkal's algorithm. We have to prove that T is an optimal tree, that is T is a spanning tree of minimum weight.

If possible let T be not an optimal tree. Therefore there are spanning trees of smaller weights than T. Any two spanning trees of a graph have some edges in common (why?). Let k be the largest number of edges common with T and any other optimal tree. Let T' be an optimal tree with k edges common with T and $T' \neq T$. Let the common edges be $e_1, e_2, \ldots\ldots\ldots, e_k$ taken in ascending order of weights.

So we can take

$T = \{e_1, e_2, \ldots\ldots\ldots, e_k, e_{k+1}, e_{k+2}, \ldots\ldots\ldots, e_{n-1}\}$

$T' = \{e_1, e_2, \ldots\ldots\ldots, e_k, e'_{k+1}, e'_{k+2}, \ldots\ldots\ldots, e'_{n-1}\}$

Since T' is a spanning tree and $e_{k+1} \notin T'$ we find $T' + e_{k+1}$ contains a unique cycle C(say). Clearly C is not contained in T. So the cycle C contains an edge $e \in T'$ such that $e'_i \notin T$. Deleting e'_j from the cycle C we get a spanning tree.

Space for Hints

$T_\circ = T' + e_{k+1} - e_j$.

$\therefore W(T_\circ) = W(T') + w(e_{k+1}) - w(e_j)$ (1)

As $T_\circ$ is a spanning tree and T' is an optimal tree

$W(T') \leq W(T_\circ)$ (2)

From (1) and (2)

$w(e_{k+1}) \geq w(e_j)$

By our choice of e_{k+1} and e step 2 of the algorithm e' e_{k+} is of smaller weight than e_j

$\therefore w(e_{k+1}) \leq w(e_j')$.

So we get $w(e_{k+1}) = w(e_j)$.

Using this in (1) $W(T_\circ) = W(T')$

Since T' is an optimal tree it follows that $T_\circ$ is also an optimal tree.

Now $T_\circ = T' + e_{k+1} - e_j$

$\therefore e_1, e_2, \ldots\ldots, e_k, e_{k+1} \in T_\circ$.

Already $e_1, e_2, \ldots\ldots\ldots, e_k, e_{k+1} \in T$

Thus we get an optimal tree $T_\circ$ with $k+1$ edges common with T. This is a contradiction to our hypothesis. So our assumption that T is not optimal is wrong. Hence T is an optimal tree.

Space for Hints

5.4.2 Prim's Algorithm

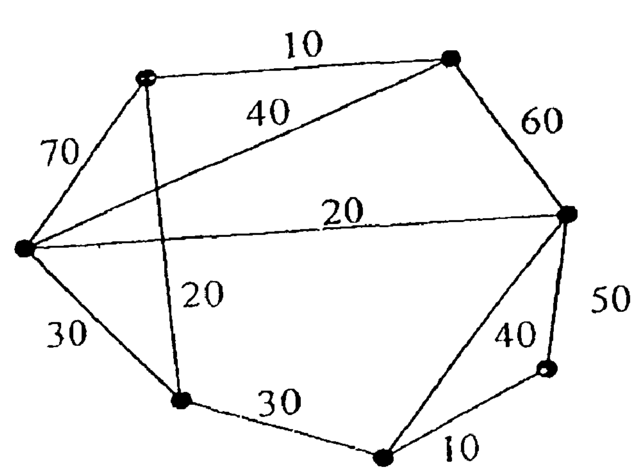
Let G be a graph of n vertices. Select arbitrarily any vertex. Call it v_1. Choose the edge of minimum weight among the edges incident at v_1`.Call the other end of this edge as v_2. Consider the tree $\{v_1, v_2\}$ as a subgrpah of G. Choose the edge of minimum weight among the edges incident on v_1 and v_2 and such that the order end vertex v_3 of this edge is in G - $\{v_1, v_2\}$. Consider the tree $\{v_1, v_2, v_3\}$ as a subgraph of G. Choose the edge of minimum weight among the edges incident on v_1, v_2 and v_3 and such that the other end vertex v_4 of this edge is in $G - \{v_1, v_2, v_3\}$. Proceeding like this we get a tree $\{v_1, v_2, v_3, \ldots\ldots, v_n\}$. This tree is clearly a spanning tree of minimum weight of the graph G and so is an optimal tree of G.

Remarks:

Prim's algorithm is more efficient that Krushkal's algorithm, for we need not check connectivity and cyclicity of the construction at any stage.

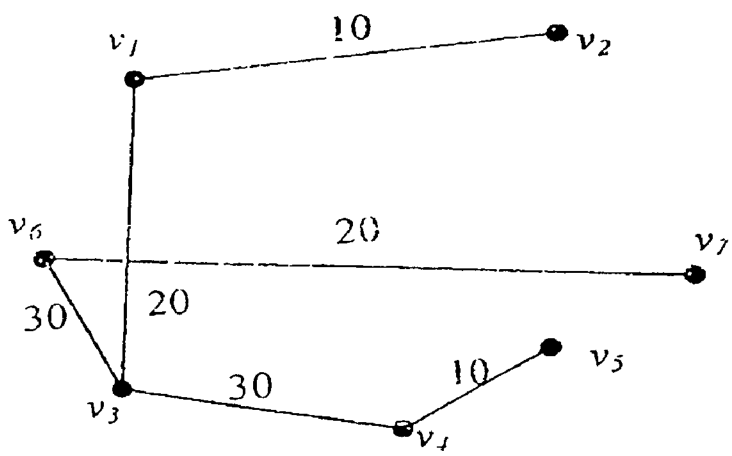
Problems:

1. Using Prims algorithm find an optimal tree of the graph G given below:



Solution:

Following the algorithm step by step we get the following optimal tree.



Weight of optimal tree

$$= w(v_1, v_2) + w(v_1 v_2) + w(v_3 v_4) + w(v_4 v_5) +$$

$$+ w(v_3 v_6) + w(v_6 v_7).$$

$$= 10 + 20 + 30 + 10 + 30 + 20$$

$$= 120$$

2. If $\delta \geq 2$ in any connected graph G, prove that G contains a cycle.

Solution:

Given G is a connected graph with $\delta \geq 2$. If G does not contain cycles then G is a connected acyclic graph and so it is a tree.

Therefore there are atleast two pendant vertices in G.

It is a contradiction to the data that the minimum degree of any vertex is $\delta \geq 2$.

Hence G contains a cycle.

Space for Hints