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Preface

சீரிளமைத் திறம் வியந்து செயல் மறந்து வாழ்த்துதுமே!

-மனோன்மணீயம் சுந்தரனார்

The Government of Tamil Nadu has decided to evolve a uniform system of school education in the state to ensure social justice and provide quality education to all the schools of the state. With due consideration to this view and to prepare the students to face new challenges in the field of Mathematics, this text book is well designed within the frame work of NCF2005 by the textbook committee of subject experts and practicing teachers in schools and colleges.

Mathematics is a language which uses easy words for hard ideas. With the aid of Mathematics and imagination the nano or the googolplex all things may be brought within man's domain. This laptop of handbook is an important collection of twelve topics. A brief and breezy explanation of each chapter proceeds with an introduction to the topics, significant contributions made by the great Mathematicians, concise definitions, key concepts, relevant theorems, practice problems and a brief summary at the end of the lesson written with wit and clarity to motivate the students. This book helps the student to complete the transition from usual manipulation to little rigorous Mathematics.

Real life examples quoted in the text help in the easy grasp of meaning and in understanding the necessity of mathematics. These examples will shape the abstract key concepts, definitions and theorems in simple form to understand clearly. But beyond finding these examples, one should examine the reason why the basic definitions are given. This leads to a split into streams of thought to solve the complicated problems easily in different ways.

By means of colourful visual representation, we hope the charming presents in our collection will invite the students to enjoy the beauty of Mathematics to share their views with others and to become involved in the process of creating new ideas. A mathematical theory is not to be considered complete until it has been made so clear that the student can explain it to the first man whom he or she meets on the street. It is a fact that mathematics is not a mere manipulation of numbers but an enjoyable domain of knowledge.

To grasp the meaning and necessity of Mathematics, to appreciate its beauty and its value, it is time now to learn the depth of fundamentals of Mathematics given in this text. Any one who penetrates into it will find that it proves both charming and exciting. Learning and creating Mathematics is indeed a worthwhile way to spend one's life.

Mathematics is not a magic it is a music ; play it, enjoy! bloom!! and flourish!!!

E. Chandrasekaran

and writting team

SYMBOLS

$=$	equal to	Δ	symmetric difference
$\neq$	not equal to	$\mathbb{N}$	natural numbers
$<$	less than	$\mathbb{R}$	real numbers
$\leq$	less than or equal to	$\mathbb{W}$	whole numbers
$>$	greater than	$\mathbb{Z}$	integers
$\geq$	greater than or equal to	$\triangle$	triangle
$\approx$	equavalent to	$\angle$	angle
$\cup$	union	$\perp$	perpendicular to
$\cap$	intersection	$\parallel$	parallel to
$\mathbb{U}$	universal Set	$\Longrightarrow$	implies
$\in$	belongs to	$\therefore$	therefore
$\notin$	does not belong to	$\because$	since (or) because
$\subset$	proper subset of	$\lvert\ \rvert$	absolute value
$\subseteq$	subset of or is contained in	$\simeq$	approximately equal to
$\not\subset$	not a proper subset of	$\mid$ (or) $:$	such that
$\nsubseteq$	not a subset of or is not contained in	$\equiv$ (or) $\cong$	congruent
A' (or) A^c	complement of A	$\equiv$	identically equal to
$\emptyset$ (or) $\{\ \}$	empty set or null set or void set	π	pi
$n(A)$	number of elements in the set A	$\pm$	plus or minus
$P(A)$	power set of A	■	end of the proof
$\mathrm{P}(A)$	probability of the event A		

CONTENT

1 THEORY OF SETS

No one shall expel us from the paradise that Cantor has created for us

- DAVID HILBERT

Main Targets

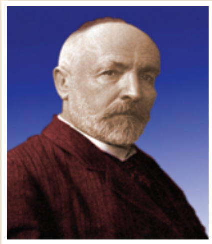
- To describe a set
- To represent sets in descriptive form, set builder form and roster form
- To identify different kinds of sets
- To understand and perform set operations
- To use Venn diagrams to represent sets and set operations
- To use the formula involving $n(A \cup B)$ simple word problems

1.1 Introduction

The concept of set is vital to mathematical thought and is being used in almost every branch of mathematics. In mathematics, sets are convenient because all mathematical structures can be regarded as sets. Understanding set theory helps us to see things in terms of systems, to organize things into sets and begin to understand logic. In chapter 2, we will learn how the natural numbers, the rational numbers and the real numbers can be defined as sets. In this chapter we will learn about the concept of set and some basic operations of set theory.

1.2 Description of Sets

We often deal with a group or a collection of objects, such as a collection of books, a group of students, a list of states in a country, a collection of coins, etc. Set may be consider of as a mathematical way of representing a collection or a group of objects.



GEORG CANTOR
(1845-1918)

*The basic ideas of set theory were developed by the German mathematician **Georg Cantor** (1845-1918).*

He worked on certain kinds of infinite series particularly on Fourier series. Most mathematicians accept set theory as a basis of modern mathematical analysis. Cantor's work was fundamental to the later investigation of Mathematical logic.

Key Concept	Set
A set is a collection of well-defined objects. The objects of a set are called elements or members of the set.	

The main property of a set in mathematics is that it is well-defined. This means that given any object, it must be clear whether that object is a member (element) of the set or not.

The objects of a set are all distinct, i.e., no two objects are the same.

Which of the following collections are well-defined?

(1) The collection of male students in your class.

(2) The collection of numbers 2, 4, 6, 10 and 12.

(3) The collection of districts in Tamil Nadu.

(4) The collection of all good movies.

(1), (2) and (3) are well-defined and therefore they are sets. (4) is not well-defined because the word good is not defined. Therefore, (4) is not a set.

Generally, sets are named with the capital letters *A*, *B*, *C*, etc. The elements of a set are denoted by the small letters *a*, *b*, *c*, etc.

Reading Notation

$\in$	'is an element of' or 'belongs to'

If x is an element of the set *A*, we write $x \in A$.

$\notin$	'is not an element of' or 'does not belong to'

If x is not an element of the set *A*, we write $x \notin A$.

For example,

Consider the set $A = \{1, 3, 5, 9\}$.

1 is an element of *A*, written as $1 \in A$

3 is an element of *A*, written as $3 \in A$

8 is not an element of *A*, written as $8 \notin A$

Example 1.1

Let $A = \{1, 2, 3, 4, 5, 6\}$. Fill in the blank spaces with the appropriate symbol $\in$ or $\notin$.

(i) 3 .... A (ii) 7 .... A

(iii) 0 ...... A (iv) 2 ...... A

Solution (i) $3 \in A$ ($\because$ 3 is an element of A)

(ii) $7 \notin A$ ($\because$ 7 is not an element of A)

(iii) $0 \notin A$ ($\because$ 0 is not an element of A)

(iv) $2 \in A$ ($\because$ 2 is an element of A)

1.3 Representation of a Set

A set can be represented in any one of the following three ways or forms.

(i) Descriptive Form

(ii) Set-Builder Form or Rule Form

(iii) Roster Form or Tabular Form

1.3.1 Descriptive Form

Key Concept **Descriptive Form**

One way to specify a set is to give a verbal description of its elements. This is known as the Descriptive form of specification.

The description must allow a concise determination of which elements belong to the set and which elements do not.

For example,

(i) The set of all natural numbers.

(ii) The set of all prime numbers less than 100.

(iii) The set of all letters in the English alphabets.

1.3.2 Set-Builder Form or Rule Form

Key Concept — **Set-Builder Form**

Set-builder notation is a notation for describing a set by indicating the properties that its members must satisfy.

Reading Notation

'|' or ':'	such that

A = $\{x : x$ is a letter in the word CHENNAI$\}$

We read it as

"A is the set of all x such that x is a letter in the word CHENNAI"

For example,

(i) $\mathbb{N} = \{x : x$ is a natural number$\}$

(ii) $P = \{x : x$ is a prime number less than $100\}$

(iii) $A = \{x : x$ is a letter in the English alphabet$\}$

1.3.3 Roster Form or Tabular Form

Key Concept — **Roster Form**

Listing the elements of a set inside a pair of braces { } is called the roster form.

For example,

(i) Let A be the set of even natural numbers less than 11.
In roster form we write $A = \{2, 4, 6, 8, 10\}$

(ii) $A = \{x : x$ is an integer and $-1 \le x < 5\}$
In roster form we write $A = \{-1, 0, 1, 2, 3, 4,\}$

Remark

(i) In roster form each element of the set must be listed exactly once. By convention, the elements in a set should NOT be repeated.

(ii) Let A be the set of letters in the word "COFFEE", i.e, A={ C, O, F, E }. So, in roster form of the set A the following are invalid.

$\{C, O, E\}$ (not all elements are listed)

$\{C, O, F, F, E\}$ (element F is listed twice)

(iii) In a roster form the elements in a set can be written in ANY order.

The following are valid roster form of the set containing the elements 2, 3 and 4.

$\{2,3,4\}$ $\{2,4,3\}$ $\{4,3,2\}$

Each of them represents the same set

(iv) If there are either infinitely many elements or a large finite number of elements, then three consecutive dots called *ellipsis* are used to indicate that the pattern of the listed elements continues, as in $\{5,6,7,\cdots\}$ or $\{3,6,9,12,15,\cdots,60\}$.

(v) Ellipsis can be used only if enough information has been given so that one can figure out the entire pattern.

Representation of sets in Different Forms

Descriptive Form	Set - Builder Form	Roster Form
The set of all vowels in English alphabet	$\{x : x$ is a vowel in the English alphabet$\}$	$\{a, e, i, o, u\}$
The set of all odd positive integers less than or equal to 15	$\{x : x$ is an odd number and $0 < x \leq 15\}$	$\{1, 3, 5, 7, 9, 11, 13, 15\}$
The set of all perfect cube numbers between 0 and 100	$\{x : x$ is a perfect cube number and $0 < x < 100\}$	$\{1, 8, 27, 64\}$

Example 1.2

List the elements of the following sets in Roster form:

(i) The set of all positive integers which are multiples of 7.

(ii) The set of all prime numbers less than 20.

Solution (i) The set of all positive integers which are multiples of 7 in roster form is $\{7, 14, 21, 28,\cdots\}$

(ii) The set of all prime numbers less than 20 in roster form is $\{2, 3, 5, 7, 11, 13, 17, 19\}$

Example 1.3

Write the set $A = \{x : x$ is a natural number $\leq 8\}$ in roster form.

Solution $A = \{x : x$ is a natural number $\leq 8\}$.

So, the set contains the elements 1, 2, 3, 4, 5, 6, 7, 8.

Hence in roster form $A = \{1, 2, 3, 4, 5, 6, 7, 8\}$

Example 1.4

Represent the following sets in set-builder form

(i) X = {Sunday, Monday, Tuesday, Wednesday, Thursday, Friday, Saturday}

(ii) $A = \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \cdots\right\}$

Solution (i) X = {Sunday, Monday, Tuesday, Wednesday, Thursday, Friday, Saturday}

The set X contains all the days of a week.

Hence in set builder form, we write

$X = \{x : x$ is a day in a week$\}$

(ii) $A = \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \cdots\right\}$. The denominators of the elements are 1, 2, 3, 4, $\cdots$

$\therefore$ The set-builder form is $A = \left\{x : x = \frac{1}{n}, n \in N\right\}$

1.3.4 Cardinal Number

Key Concept	**Cardinal Number**
The number of elements in a set is called the cardinal number of the set.	
Reading Notation	

$n(A)$	number of elements in the set A

The cardinal number of the set A is denoted by $n(A)$.

For example,

Consider that the set $A = \{-1, 0, 1, 2, 3, 4, 5\}$. The set A has 7 elements.

$\therefore$ The cardinal number of A is 7 i.e., $n(A) = 7$.

Example 1.5

Find the cardinal number of the following sets.

(i) $A = \{x : x$ is a prime factor of $12\}$

(ii) $B = \{x : x \in \mathbb{W}, x \leq 5\}$

Solution (i) Factors of 12 are 1, 2, 3, 4, 6, 12. So, the prime factors of 12 are 2, 3.

We write the set A in roster form as $A = \{2, 3\}$ and hence $n(A) = 2$.

(ii) $B = \{x : x \in \mathbb{W}, x \leq 5\}$. In Tabular form, $B = \{0, 1, 2, 3, 4, 5\}$.

The set B has six elements and hence $n(B) = 6$

1.4 Different Kinds of Sets

1.4.1 The Empty Set

Key Concept **Empty Set**

A set containing no elements is called the empty set or null set or void set.

Reading Notation

$\varnothing$ or $\{\ \}$	Empty set or Null set or Void set

The empty set is denoted by the symbol $\varnothing$ or $\{\ \}$

For example,

Consider the set $A = \{x : x < 1, x \in \mathbb{N}\}$.

There are no natural numbers which are less than 1.

$\therefore\ A = \{\ \}$

Think and Answer !
What is $n(\varnothing)$?

Note The concept of empty set plays a key role in the study of sets just like the role of the number *zero* in the study of number system.

1.4.2 Finite Set

Key Concept **Finite Set**

If the number of elements in a set is zero or finite, then the set is called a finite set.

For example,

(i) Consider the set A of natural numbers between 8 and 9.

There is no natural numbers between 8 and 9. So, $A = \{\ \}$ and $n(A) = 0$.

$\therefore\ A$ is a finite set

(ii) Consider the set $X = \{x : x$ is an integer and $-1 \le x \le 2\}$.

$X = \{-1, 0, 1, 2\}$ and $n(X) = 4$

$\therefore\ X$ is a finite set

Note The cardinal number of a finite set is finite

1.4.3 Infinite Set

Key Concept — **Infinite Set**

A set is said to be an infinite set if the number of elements in the set is not finite.

For example,

Let $\mathbb{W}$ = The set of all whole numbers. i. e., $\mathbb{W} = \{0, 1, 2, 3, \cdots\}$

The set of all whole numbers contain infinite number of elements

$\therefore$ $\mathbb{W}$ is an infinite set

Note The cardinal number of an infinite set is not a finite number.

Example 1.6

State whether the following sets are finite or infinite

(i) $A = \{x : x \text{ is a multiple of } 5,\ x \in \mathbb{N}\}$

(ii) $B = \{x : x \text{ is an even prime number}\}$

(iii) The set of all positive integers greater than 50.

Solution (i) $A = \{x : x \text{ is a multiple of } 5,\ x \in \mathbb{N}\} = \{5, 10, 15, 20, ...\}$

$\therefore$ A is an infinite set.

(ii) $B = \{x : x \text{ is even prime numbers}\}$. The only even prime number is 2

$\therefore$ $B = \{\ 2\ \}$ and hence B is a finite set.

(iii) Let X be the set of all positive integers greater than 50.

Then $X = \{51, 52, 53, ...\}$

$\therefore$ X is an infinite set.

1.4.4 Singleton Set

Key Concept — **Singleton Set**

A set containing only one element is called a singleton set

For example,

Consider the set $A = \{x : x \text{ is an integer and } 1 < x < 3\}$.

$A = \{\ 2\ \}$ i. e., A has only one element

$\therefore$ A is a singleton set

Remark

It is important to recognise that the following sets are not equal.

(i) The null set $\varnothing$

(ii) The set having the null set as its only element $\{\varnothing\}$

(iii) The set having zero as its only element $\{ 0 \}$

1.4.5 Equivalent Set

Key Concept — **Equivalent Set**

Two sets A and B are said to be equivalent if they have the same number of elements

In other words, A and B are equivalent if $n(A) = n(B)$.

Reading Notation

$\approx$	Equivalent

A and B are equivalent is written as $A \approx B$

For example,

Consider the sets $A = \{ 7, 8, 9, 10 \}$ and $B = \{ 3, 5, 6, 11 \}$.

Here $n(A) = 4$ and $n(B) = 4$ $\quad \therefore\ A \approx B$

1.4.6 Equal Sets

Key Concept — **Equal Sets**

Two sets A and B are said to be equal if they contain exactly the same elements, regardless of order. Otherwise the sets are said to be unequal.

In other words, two sets A and B, are said to be equal if

(i) every element of A is also an element of B and

(ii) every element of B is also an element of A.

Reading Notation

$=$	Equal	When two sets A and B are equal we write $A = B$.
$\neq$	Not equal	When they are unequal, we write $A \neq B$.

For example,

Consider the sets

$A = \{ a, b, c, d \}$ and $B = \{ d, b, a, c \}$

Set A and set B contain exactly the same elements $\therefore A = B$

Note

If two sets A and B are equal, then $n(A) = n(B)$.

But, if $n(A) = n(B)$, then A and B need not be equal

Thus equal sets are equivalent but equivalent sets need not be equal

Example 1.7

Let $A = \{2, 4, 6, 8, 10, 12, 14\}$ and

$B = \{x : x$ is a multiple of 2, $x \in \mathbb{N}$ and $x \leq 14\}$

State whether $A = B$ or not.

Solution $A = \{2, 4, 6, 8, 10, 12, 14\}$ and

$B = \{x : x$ is a multiple of 2, $x \in \mathbb{N}$ and $x \leq 14\}$

In roster form, $B = \{2, 4, 6, 8, 10, 12, 14\}$

Since A and B have exactly the same elements, $A = B$

1.4.7 Subset

Key Concept — Subset

A set A is a subset of set B if every element of A is also an element of B. In symbol we write $A \subseteq B$

Reading Notation

$\subseteq$	is a subset of (or) is contained in

Read $A \subseteq B$ as 'A is a subset of B' or 'A is contained in B'

$\nsubseteq$	is not a subset of (or) is not contained in

Read $A \nsubseteq B$ as 'A is not a subset of B' or 'A is not contained in B'

For example,

Consider the sets

$A = \{7, 8, 9\}$ and $B = \{ 7, 8, 9, 10 \}$

We see that every element of A is also an element of B.

$\therefore$ A is a subset of B.

i.e. $A \subseteq B$.

Note

(i) Every set is a subset of itself i.e. $A \subseteq A$ for any set A

(ii) The empty set is a subset of any set i.e., $\varnothing \subseteq A$, for any set A

(iii) If $A \subseteq B$ and $B \subseteq A$, then $A = B$.

The converse is also true i.e. if $A = B$ then $A \subseteq B$ and $B \subseteq A$

(iv) Every set (except $\varnothing$) has atleast two subsets, $\varnothing$ and the set itself.

1.4.8 Proper Subset

Key Concept **Proper Subset**

A set A is said to be a proper subset of set B if $A \subseteq B$ and $A \neq B$. In symbol we write $A \subset B$. B is called super set of A.

Reading Notation

$\subset$	is a proper subset of

Read $A \subset B$ as, A is a proper subset of B

For example,

Consider the sets $A = \{5, 7, 8\}$ and $B = \{5, 6, 7, 8\}$

Every element of A is also an element of B and $A \neq B$

$\therefore$ A is a proper subset of B

Remark

(i) Proper subsets have atleast one element less than its superset

(ii) No set is a proper subset of itself.

(iii) The empty set $\varnothing$ is a proper subset of every set except itself ($\varnothing$ has no proper subset). i.e., $\varnothing \subset A$ if A is a set other than $\varnothing$

(iv) It is important to distinguish between $\in$ and $\subseteq$. The notation $x \in A$ denotes x is an element of A. The notation $A \subseteq B$ means A is a subset of B.

Thus $\varnothing \subseteq \{a, b, c\}$ is true, but $\varnothing \in \{a, b, c\}$ is not true.

It is true that $x \in \{x\}$, but the relations $x = \{x\}$ and $x \subseteq \{x\}$ are not correct.

Example 1.8

Write $\subseteq$ or $\nsubseteq$ in each blank to make a true statement.

(a) {4, 5, 6, 7} ----- {4, 5, 6, 7, 8} (b) {a, b, c} ----- {b, e, f, g}

Solution (a) {4, 5, 6, 7} ----- {4, 5, 6, 7, 8}

Since every element of {4, 5, 6, 7} is also an element of {4, 5, 6, 7, 8}, place $\subseteq$ in the blank.

$\therefore \{4,5,6,7\} \subseteq \{4,5,6,7,8\}$

(b) The element a belongs to {a, b, c} but not to {b, e, f, g}

So, place $\nsubseteq$ in the blank

$\therefore \{a,b,c\} \nsubseteq \{b,c,f,g\}$

Example 1.9

Decide whether $\subset, \subseteq$ or both, can be placed in each blank to make a true statement.

(i) {8, 11, 13} ----- {8, 11, 13, 14}

(ii) {a, b, c} ------ {a, c, b}

Solution (i) Every element of the set {8, 11, 13} is also an element in the set {8, 11, 13, 14}

So, place $\subseteq$ in the blank

$\therefore \{8,11,13\} \subseteq \{8,11,13,14\}$

Also, the element 14 belongs to {8, 11, 13, 14} but does not belong to {8, 11, 13}

$\therefore$ {8, 11, 13} is proper subset of {8, 11, 13, 14}.

So, we can also place $\subset$ in the blank. $\therefore \{8,11,13\} \subset \{8,11,13,14\}$

(ii) Every element of {a, b, c} is also an element of {a, c, b}

and hence they are equal. So, {a, b, c} is not a proper subset of {a, c, b}

Hence we can only place $\subseteq$ in the blank.

1.4.9 Power Set

Key Concept **Power Set**

The set of all subsets of A is said to be the power set of the set A.

Reading Notation

$P(A)$	Power set of A

The power set of a set A is denoted by $P(A)$

For example,

Let $A = \{-3, 4\}$

The subsets of A are $\varnothing, \{-3\}, \{4\}, \{-3, 4\}$.

Then the power set of A is $P(A) = \{\varnothing, \{-3\}, \{4\}, \{-3, 4\}\}$

Example 1.10

Write down the power set of $A = \{3, \{4, 5\}\}$

Solution $A = \{3, \{4, 5\}\}$

The subsets of A are

$\varnothing, \{3\}, \{\{4, 5\}\}, \{3, \{4, 5\}\}$

$\therefore P(A) = \{\varnothing, \{3\}, \{\{4, 5\}\}, \{3, \{4, 5\}\}\}$

Number of Subsets of a Finite Set

For a set containing a very large number of elements, it is difficult to find the number of subsets of the set. Let us find a rule to tell how many subsets are there for a given finite set.

(i) The set $A = \varnothing$ has only itself as a subset

(ii) The set $A = \{5\}$ has subsets $\varnothing$ and $\{5\}$

(iii) The set $A = \{5, 6\}$ has subsets $\varnothing, \{5\}, \{6\}, \{5, 6\}$

(iv) The set $A = \{5, 6, 7\}$ has subsets $\varnothing, \{5\}, \{6\}, \{7\}, \{5, 6\}, \{5, 7\}, \{6, 7\}$ and $\{5, 6, 7\}$

This information is shown in the following table

Number of Elements	0	1	2	3
Number of subsets	$1 = 2^0$	$2 = 2^1$	$4 = 2^2$	$8 = 2^3$

This table suggests that as the number of elements of the set increases by one, the number of subsets doubles. i.e. the number of subsets in each case is a power of 2.

Thus we have the following generalization

The number of subsets of a set with *m* elements is 2^m

The 2^m subsets includes the given set itself.

$$n(A) = m \Rightarrow n[P(A)] = 2^m$$

$\therefore$ **The number of proper subsets of a set with *m* elements is** $2^m - 1$

Example 1.11

Find the number of subsets and proper subsets of each set

(i) $A = \{3,4,5,6,7\}$ (ii) $A = \{1,2,3,4,5,9,12,14\}$

Solution (i) $A = \{3,4,5,6,7\}$. So, $n(A) = 5$. Hence,

The number of subsets = $n[P(A)] = 2^5 = 32$.

The number of proper subsets = $2^5 - 1 = 32 - 1 = 31$

(ii) $A = \{1,2,3,4,5,9,12,14\}$. Now, $n(A) = 8$.

$\therefore$ The number of subsets $= 2^8 = 2^5 \times 2^3 = 32 \times 2 \times 2 \times 2 = 256$

The number of proper subsets = $2^8 - 1 = 256 - 1 = 255$

Exercise 1.1

1. Which of the following are sets? Justify your answer.

   (i) The collection of good books

   (ii) The collection of prime numbers less than 30

   (iii) The collection of ten most talented mathematics teachers.

   (iv) The collection of all students in your school

   (v) The collection of all even numbers

2. Let $A = \{0, 1, 2, 3, 4, 5\}$. Insert the appropriate symbol $\in$ or $\notin$ in the blank spaces

   (i) 0 ----- A (ii) 6 ----- A (iii) 3 ----- A

   (iv) 4 ----- A (v) 7 ----- A

3. Write the following sets in Set-Builder form

   (i) The set of all positive even numbers

   (ii) The set of all whole numbers less than 20

   (iii) The set of all positive integers which are multiples of 3

   (iv) The set of all odd natural numbers less than 15.

   (v) The set of all letters in the word 'TAMILNADU'

4. Write the following sets in Roster form

   (i) $A = \{x : x \in \mathbb{N},\ 2 < x \leq 10\}$

   (ii) $B = \left\{x : x \in \mathbb{Z},\ -\frac{1}{2} < x < \frac{11}{2}\right\}$

(iii) C = $\{x : x$ is a prime number and a divisor of 6$\}$

(iv) $X = \{x : x = 2^n, n \in \mathbb{N}$ and $n \leq 5\}$

(v) $M = \{x : x = 2y - 1, y \leq 5, y \in \mathbb{W}\}$

(vi) $P = \{x : x$ is an integer, $x^2 \leq 16\}$

5. Write the following sets in Descriptive form

(i) $A = \{a, e, i, o, u\}$

(ii) $B = \{1, 3, 5, 7, 9, 11\}$

(iii) $C = \{1, 4, 9, 16, 25\}$

(iv) $P = \{x : x$ is a letter in the word 'SET THEORY'$\}$

(v) $Q = \{x : x$ is a prime number between 10 and 20$\}$

6. Find the cardinal number of the following sets

(i) $A = \{x : x = 5^n, n \in \mathbb{N}$ and $n < 5\}$

(ii) $B = \{x : x$ is a consonant in English Alphabet$\}$

(iii) $X = \{x : x$ is an even prime number$\}$

(iv) $P = \{x : x < 0, x \in \mathbb{W}\}$

(v) $Q = \{x : -3 \leq x \leq 5, x \in \mathbb{Z}\}$

7. Identify the following sets as finite or infinite

(i) $A = \{4, 5, 6, ...\}$

(ii) $B = \{0, 1, 2, 3, 4, ... 75\}$

(iii) $X = \{x : x$ is a even natural number$\}$

(iv) $Y = \{x : x$ is a multiple of 6 and $x > 0\}$

(v) P = The set of letters in the word 'KARIMANGALAM'

8. Which of the following sets are equivalent?

(i) $A = \{2, 4, 6, 8, 10\}$, B = $\{1, 3, 5, 7, 9\}$

(ii) $X = \{x : x \in \mathbb{N}, 1 < x < 6\}$, $Y = \{x : x$ is vowel in the English Alphabet$\}$

(iii) $P = \{x : x$ is a prime number and $5 < x < 23\}$

$Q = \{x : x \in \mathbb{W}, 0 \leq x < 5\}$

9. Which of the following sets are equal?

(i) $A = \{1, 2, 3, 4\}$, $B = \{4, 3, 2, 1\}$

(ii) $A = \{4, 8, 12, 16\}$, $B = \{8, 4, 16, 18\}$

(iii) $X = \{2, 4, 6, 8\}$

$Y = \{x : x$ is a positive even integer $0 < x < 10\}$

(iv) $P = \{x : x$ is a multiple of 10, $x \in \mathbb{N}\}$

$Q = \{10, 15, 20, 25\ 30, ... \}$

10. From the sets given below, select equal sets.

$A = \{12, 14, 18, 22\}$, $B = \{11, 12, 13, 14\}$, $C = \{14, 18, 22, 24\}$

$D = \{13, 11, 12, 14\}$, $E = \{-11, 11\}$, $F = \{10, 19\}$, $G = \{11, -11\}$, $H = \{10, 11\}$

11. Is $\varnothing = \{\varnothing\}$? Why ?

12. Which of the sets are equal sets? State the reason.

$0, \varnothing, \{0\}, \{\varnothing\}$

13. Fill in the blanks with $\subseteq$ or $\not\subseteq$ to make each statement true.

(i) $\{3\}$ ----- $\{0, 2, 4, 6\}$ (ii) $\{a\}$ ----- $\{a, b, c\}$

(iii) $\{8, 18\}$ ----- $\{18, 8\}$ (iv) $\{d\}$ ----- $\{a, b, c\}$

14. Let $X = \{-3, -2, -1, 0, 1, 2\}$ and $Y = \{x : x$ is an integer and $-3 \leq x < 2\}$

(i) Is X a subset of Y ? (ii) Is Y a subset of X ?

15. Examine whether $A = \{x : x$ is a positive integer divisible by 3$\}$ is a subset of

$B = \{x : x$ is a multiple of 5, $x \in \mathbb{N}\}$

16. Write down the power sets of the following sets.

(i) $A = \{x, y\}$ (ii) $X = \{a, b, c\}$ (iii) $A = \{5, 6, 7, 8\}$ (iv) $A = \varnothing$

17. Find the number of subsets and the number of proper subsets of the following sets.

(i) $A = \{13, 14, 15, 16, 17, 18\}$

(ii) $B = \{a, b, c, d, e, f, g\}$

(iii) $X = \{x : x \in \mathbb{W}, x \notin \mathbb{N}\}$

18. (i) If $A = \varnothing$, find $n[P(A)]$ (ii) If $n(A) = 3$, find $n[P(A)]$

(iii) If $n[P(A)] = 512$, find $n(A)$

(iv) If $n[P(A)] = 1024$, find $n(A)$

19. If $n[P(A)] = 1$, what can you say about the set A?

20. Let $A = \{x : x \text{ is a natural number} < 11\}$

$B = \{x : x \text{ is an even number and } 1 < x < 21\}$

$C = \{x : x \text{ is an integer and } 15 \le x \le 25\}$

(i) List the elements of A, B, C

(ii) Find $n(A)$, $n(B)$, $n(C)$.

(iii) State whether the following are True (T) or False (F)

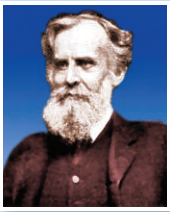
(a) $7 \in B$ ☐ (b) $16 \notin A$ ☐

(c) $\{15, 20, 25\} \subset C$ ☐ (d) $\{10, 12\} \subset B$ ☐

1.5 SET OPERATIONS

1.5.1 Venn Diagrams

We use diagrams or pictures in geometry to explain a concept or a situation and sometimes we also use them to solve problems. In mathematics, we use diagrammatic representations called Venn Diagrams to visualise the relationships between sets and set operations.



John Venn
(1834-1883)

John Venn (1834-1883) a British mathematician used diagrammatic representation as an aid to visualize various relationships between sets and set operations.

1.5.2 The Universal Set

Sometimes it is useful to consider a set which contains all elements pertinent to a given discussion.

Key Concept	**Universal Set**
The set that contains all the elements under consideration in a given discussion is called the universal set. The universal set is denoted by U.	

For example,

If the elements currently under discussion are integers, then the universal set U is the set of all integers. i.e., $U = \{n : n \in \mathbb{Z}\}$

Remark The universal set may change from problem to problem.

In Venn diagrams, the universal set is generally represented by a rectangle and its proper subsets by circles or ovals inside the rectangle. We write the names of its elements inside the figure.

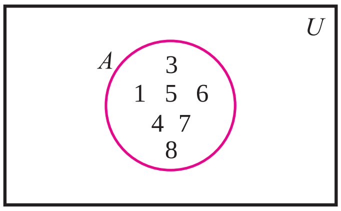


Fig. 1.1

1.5.3 Complement of a Set

Key Concept — **Complement Set**

The set of all elements of U (universal set) that are not elements of $A \subseteq U$ is called the complement of A. The complement of A is denoted by A' or A^c.

Reading Notation

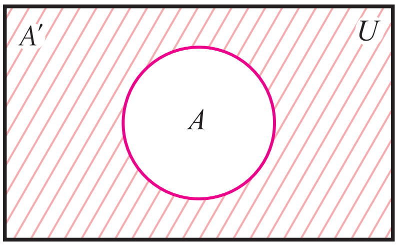
In symbol, $A' = \{x : x \in U \text{ and } x \notin A\}$

For example,

Let U $= \{a, b, c, d, e, f, g, h\}$ and $A = \{b, d, g, h\}$.

Then $A' = \{a, c, e, f\}$

In Venn diagram A', the complement of set A is represented as shown in Fig. 1.2



A' (shaded portion)
Fig. 1.2

Note (i) $(A')' = A$ (ii) $\varnothing' = U$ (iii) $U' = \varnothing$

1.5.4 Union of Two Sets

Key Concept — **Union of Sets**

The union of two sets A and B is the set of elements which are in A or in B or in both A and B. We write the union of sets A and B as $A \cup B$.

Reading Notation

$\cup$	Union

Read $A \cup B$ as 'A union B'

In symbol, $A \cup B = \{x : x \in A \text{ or } x \in B\}$

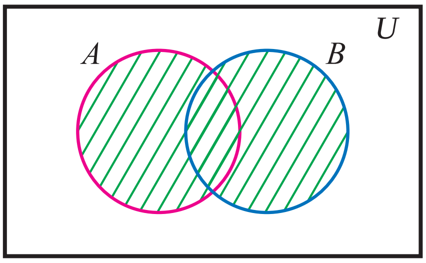
For example,

Let $A = \{11, 12, 13, 14\}$ and

$B = \{9, 10, 12, 14, 15\}$.

Then $A \cup B = \{9, 10, 11, 12, 13, 14, 15\}$

The union of two sets can be represented by a Venn diagram as shown in Fig. 1.3



$A \cup B$ (shaded portion)
Fig. 1.3

Note

(i) $A \cup A = A$ (ii) $A \cup \varnothing = A$ (iii) $A \cup A' = U$

(iv) If A is any subset of U, then $A \cup U = U$

(v) $A \subseteq B$ if and only if $A \cup B = B$ (vi) $A \cup B = B \cup A$

Example 1.12

Find the union of the following sets.

(i) $A = \{1, 2, 3, 5, 6\}$ and $B = \{4, 5, 6, 7, 8\}$

(ii) $X = \{3, 4, 5\}$ and $Y = \varnothing$

Think and Answer !
Can we say
$A \subset (A \cup B)$ and
$B \subset (A \cup B)$?

Solution (i) $A = \{1, 2, 3, 5, 6\}$ and $B = \{4, 5, 6, 7, 8\}$

1, 2, 3, 5, 6 ; 4, 5, 6, 7, 8 (repeated)

$\therefore A \cup B = \{1, 2, 3, 4, 5, 6, 7, 8\}$

(ii) $X = \{3,4,5\}$, $Y = \varnothing$. There are no elements in Y

$\therefore X \cup Y = \{3, 4, 5\}$

1.5.5 Intersection of Two Sets

Key Concept **Intersection of Sets**

The intersection of two sets A and B is the set of all elements common to both A and B. We denote it as $A \cap B$.

Reading Notation

$\cap$	Intersection

Read $A \cap B$ as 'A intersection B'

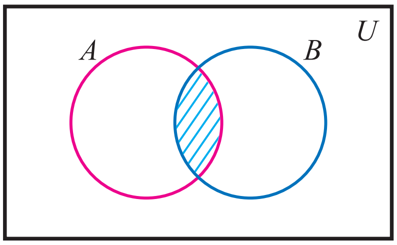
Symbolically, we write $A \cap B = \{x : x \in A \text{ and } x \in B\}$

For example,

Let $A = \{a, b, c, d, e\}$ and $B = \{a, d, e, f\}$.

$\therefore A \cap B = \{a, d, e\}$

The intersection of two sets can be represented by a Venn diagram as shown in Fig. 1.4



$A \cap B$ (shaded portion)
Fig. 1.4

Note

(i) $A \cap A = A$ (ii) $A \cap \varnothing = \varnothing$

(iii) $A \cap A' = \varnothing$ (iv) $A \cap B = B \cap A$

(v) If A is any subset of U, then $A \cap U = A$

(vi) If $A \subseteq B$ if and only if $A \cap B = A$

Think and Answer !
Can we say $(A \cap B) \subset A$ and $(A \cap B) \subset B$?

Example 1.13

Find $A \cap B$ if (i) $A = \{10, 11, 12, 13\}$, $B = \{12, 13, 14, 15\}$

(ii) $A = \{5, 9, 11\}$, $B = \varnothing$

Solution (i) $A = \{10, 11, 12, 13\}$ and $B = \{12, 13, 14, 15\}$.

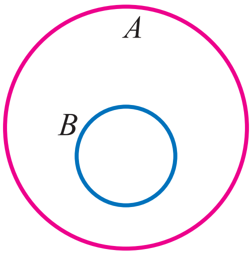
12 and 13 are common in both A and B.

$\therefore A \cap B = \{12, 13\}$

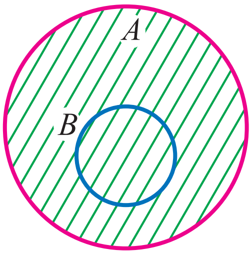
(ii) $A = \{5, 9, 11\}$ and $B = \varnothing$.

There is no element in common and hence $A \cap B = \varnothing$

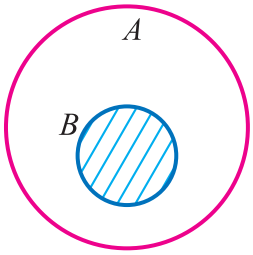
Remark When $B \subseteq A$, the union and intersection of two sets A and B are represented in Venn diagram as shown in Fig.1.6 and in Fig.1.7respectively



$B \subseteq A$
Fig.1.5



$A \cup B$ (shaded portion)
Fig.1.6



$A \cap B$ (shaded portion)
Fig.1.7

1.5.6 Disjoint Sets

Key Concept	**Disjoint Sets**

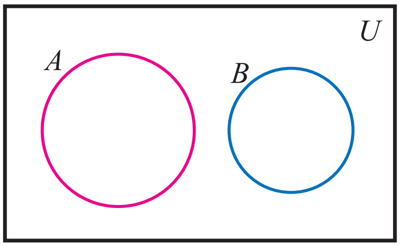
Two sets A and B are said to be disjoint if there is no element common to both A and B.

In other words, if A and B are disjoint sets, then $A \cap B = \varnothing$

For example,

Consider the sets $A = \{5,6,7,8\}$ and $B = \{11, 12, 13\}$.

We have $A \cap B = \varnothing$. So A and B are disjoint sets.



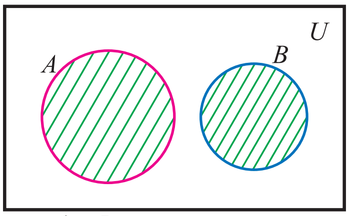
Disjoint sets
Fig.1.8

Two disjoint sets A and B are represented in Venn diagram as shown in Fig.1.8

Note

(i) The union of two disjoint sets A and B are represented in Venn diagram as shown in Fig.1.9

(ii) If $A \cap B \neq \varnothing$, then the two sets A and B are said to be overlapping sets



$A \cup B$ (shaded portion)
Fig.1.9

Example 1.14

Given the sets $A = \{4, 5, 6, 7\}$ and $B = \{1, 3, 8, 9\}$. Find $A \cap B$.

Solution $A = \{4, 5, 6, 7\}$ and $B = \{1, 3, 8, 9\}$. So $A \cap B = \varnothing$.

Hence A and B are disjoint sets.

1.5.7 Difference of Two Sets

Key Concept **Difference of two Sets**

The difference of the two sets A and B is the set of all elements belonging to A but not to B. The difference of the two sets is denoted by $A - B$.

Reading Notation

$A - B$	A difference B (or) A minus B

In symbol, we write : $A - B = \{x : x \in A \text{ and } x \notin B\}$

Similarly, we write : $B - A = \{x : x \in B \text{ and } x \notin A\}$

For example,

Consider the sets $A = \{2, 3, 5, 7, 11\}$ and $B = \{5, 7, 9, 11, 13\}$

To find $A - B$, we remove the elements of B from A.

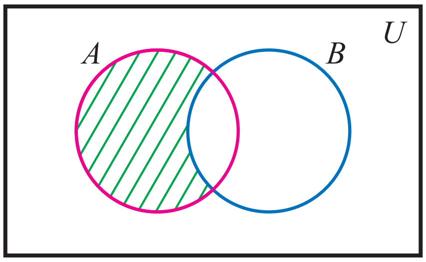
$\therefore \quad A - B = \{2, 3\}$

Note

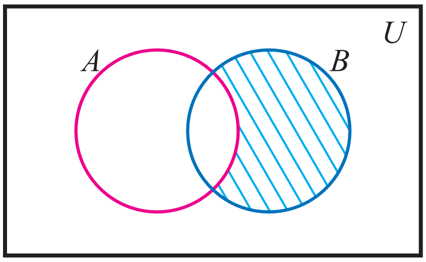
(i) Generally, $A - B \neq B - A$. (ii) $A - B = B - A \Leftrightarrow A = B$

(iii) $U - A = A'$ (iv) $U - A' = A$

The difference of two sets A and B can be represented by Venn diagram as shown in Fig.1.10 and in Fig.1.11. The shaded portion represents the difference of the two sets



A – B
Fig.1.10



B – A
Fig.1.11

Example 1.15

If $A = \{-2, -1, 0, 3, 4\}$, $B = \{-1, 3, 5\}$, find (i) $A - B$ (ii) $B - A$

Solution $A = \{-2, -1, 0, 3, 4\}$ and $B = \{-1, 3, 5\}$.

(i) $A - B = \{-2, 0, 4\}$ (ii) $B - A = \{5\}$

1.5.8 Symmetric Difference of Sets

Key Concept **Symmetric Difference of Sets**

The symmetric difference of two sets A and B is the union of their differences and is denoted by $A \Delta B$.

Reading Notation

$A \Delta B$	A symmetric B

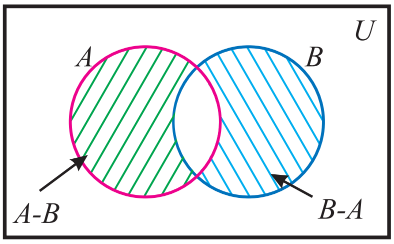
Thus, $A \Delta B = (A - B) \cup (B - A)$

For example,

Consider the sets $A = \{a, b, c, d\}$ and $B = \{b, d, e, f\}$. We have

$$A - B = \{a, c\} \text{ and } B - A = \{e, f\}$$

$$\therefore \quad A \Delta B = (A - B) \cup (B - A) = \{a, c, e, f\}$$



$A \Delta B = (A - B) \cup (B - A)$
Fig.1.12

The symmetric difference of two sets A and B can be represented by Venn diagram as shown in Fig.1.12 The shaded portion represents the symmetric difference of the two sets A and B.

Note (i) $A \triangle A = \varnothing$ (ii) $A \triangle B = B \triangle A$

(iii) From the Venn diagram 1.12, we can write

$A \triangle B = \{x : x \notin A \cap B\}$

So, we can find the elements of $A \triangle B$, by listing the elements which are not common to both A and B.

Example 1.16

If A = {2, 3, 5, 7, 11} and B = {5, 7, 9, 11, 13} , find $A \triangle B$.

Solution Given A = {2, 3, 5, 7, 11} and B = {5, 7, 9, 11, 13}. So

$A - B$ = {2, 3} and $B - A$ = {9, 13}. Hence

$A \triangle B$ = $(A - B) \cup (B - A)$ = {2, 3, 9, 13}

Exercise 1.2

1. Find $A \cup B$ and $A \cap B$ for the following sets.

   (i) $A = \{0, 1, 2, 4, 6\}$ and $B = \{-3, -1, 0, 2, 4, 5\}$

   (ii) $A = \{2, 4, 6, 8\}$ and $B = \varnothing$

   (iii) $A = \{x : x \in \mathbb{N}, x \leq 5\}$ and $B = \{x : x$ is a prime number less than 11$\}$

   (iv) $A = \{x : x \in \mathbb{N}, 2 < x \leq 7\}$ and $B = \{x : x \in \mathbb{W}, 0 \leq x \leq 6\}$

2. If $A = \{x : x$ is a multiple of 5 , $x \leq 30$ and $x \in \mathbb{N}\}$

   $B = \{1, 3, 7, 10, 12, 15, 18, 25\}$,

   Find (i) $A \cup B$ (ii) $A \cap B$

3. If $X = \{x : x = 2n, x \leq 20$ and $n \in \mathbb{N}\}$ and $Y = \{x : x = 4n, x \leq 20$ *and* $n \in \mathbb{W}\}$

   Find (i) $X \cup Y$ (ii) $X \cap Y$

4. $U = \{1, 2, 3, 6, 7, 12, 17, 21, 35, 52, 56\}$,

   P = {numbers divisible by 7}, Q = {prime numbers},

   List the elements of the set $\{x : x \in P \cap Q\}$

5. State which of the following sets are disjoint

   (i) $A = \{2, 4, 6, 8\}$; $B = \{x : x$ is an even number $< 10,\ x \in \mathbb{N}\}$

   (ii) $X = \{1, 3, 5, 7, 9\}$, $Y = \{0, 2, 4, 6, 8, 10\}$

   (iii) $P = \{x : x$ is a prime $< 15\}$

   $Q = \{x : x$ is a multiple of 2 and $x < 16\}$

   (iv) $R = \{a, b, c, d, e\}$, $S = \{d, e, a, b, c\}$

6. (i) If $U = \{x : 0 \leq x \leq 10, x \in \mathbb{W}\}$ and $A = \{x : x$ is a multiple of 3$\}$, find A'

(ii) If U is the set of natural numbers and A' is the set of all composite numbers, then what is A ?

7. If $U = \{a, b, c, d, e, f, g, h\}$, $A = \{a, b, c, d\}$ and $B = \{b, d, f, g\}$,

find (i) $A \cup B$ (ii) $(A \cup B)'$ (iii) $A \cap B$ (iv) $(A \cap B)'$

8. If $U = \{x : 1 \leq x \leq 10, x \in \mathbb{N}\}$, $A = \{1, 3, 5, 7, 9\}$ and $B = \{2, 3, 5, 9, 10\}$,

find (i) A' (ii) B' (iii) $A' \cup B'$ (iv) $A' \cap B'$

9. Given that $U = \{3, 7, 9, 11, 15, 17, 18\}$, $M = \{3, 7, 9, 11\}$ and $N = \{7, 11, 15, 17\}$,

find (i) $M - N$ (ii) $N - M$ (iii) $N' - M$ (iv) $M' - N$

(v) $M \cap (M - N)$ (vi) $N \cup (N - M)$ (vii) $n(M - N)$

10. If $A = \{3, 6, 9, 12, 15, 18\}$, $B = \{4, 8, 12, 16, 20\}$, $C = \{2, 4, 6, 8, 10, 12\}$ and $D = \{5, 10, 15, 20, 25\}$, find

(i) $A - B$ (ii) $B - C$ (iii) $C - D$ (iv) $D - A$ (v) $n(A - C)$ (vi) $n(B - A)$

11. Let $U = \{x : x$ is a positive integer less than 50$\}$, $A = \{x : x$ is divisible by 4$\}$ and

$B = \{x : x$ leaves a remainder 2 when divided by 14$\}$.

(i) List the elements of U, A and B

(ii) Find $A \cup B$, $A \cap B$, $(A \cup B)'$, $n(A \cap B)$, $A - B$ and $B - A$

12. Find the symmetric difference between the following sets.

(i) $X = \{a, d, f, g, h\}$, $Y = \{b, e, g, h, k\}$

(ii) $P = \{x : 3 < x < 9, x \in \mathbb{N}\}$, $Q = \{x : x < 5, x \in \mathbb{W}\}$

(iii) $A = \{-3, -2, 0, 2, 3, 5\}$, $B = \{-4, -3, -1, 0, 2, 3\}$

13. Use the Venn diagram 1.13 to answer the following questions

(i) List the elements of E, F, $E \cup F$ and $E \cap F$

(ii) Find $n(U)$, $n(E \cup F)$ and $n(E \cap F)$

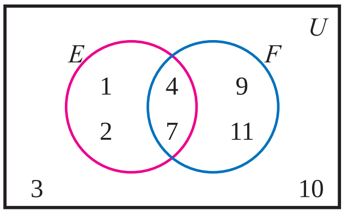


Fig. 1.13

14. Use the Venn diagram 1.14 to answer the following questions

(i) List U, G and H

(ii) Find G', H', $G' \cap H'$, $n(G \cup H)'$ and $n(G \cap H)'$

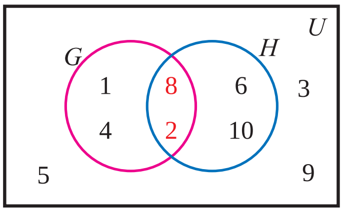


Fig. 1.14

1.6 Representation of Set Operations Using Venn Diagram

We shall now give a few more representations of set operations in Venn diagrams

(a) $A \cup B$

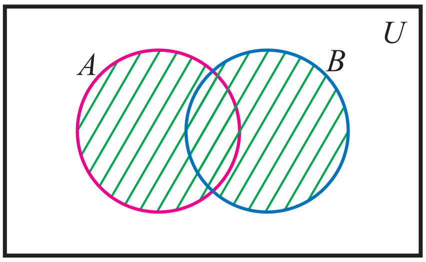


Fig. 1.15

(b) $(A \cup B)'$

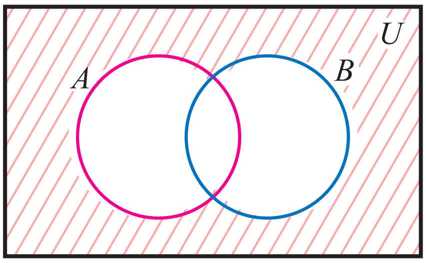
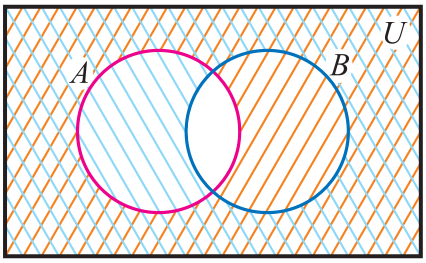


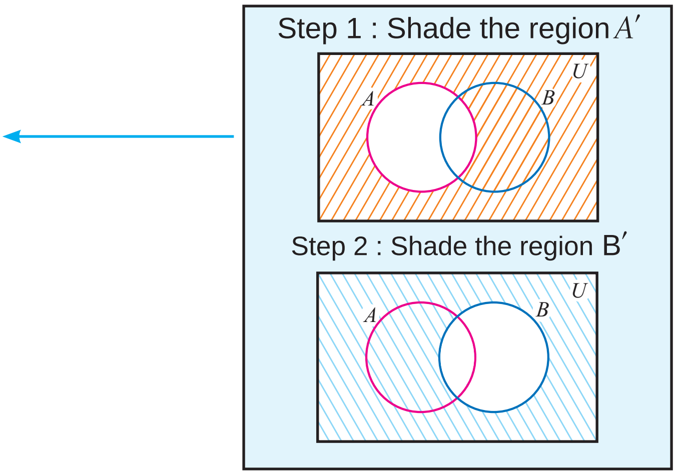
Fig. 1.16

(c) $A' \cup B'$

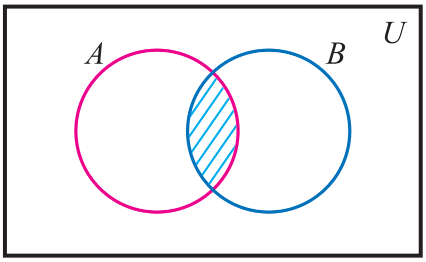


$A' \cup B'$ (shaded portion)

Fig. 1.17

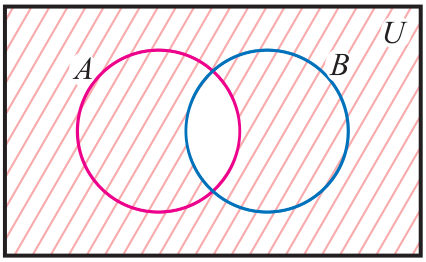


Similarly the shaded regions represent each of the following set operations.



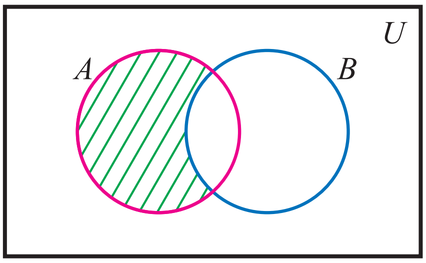
$A \cap B$ (shaded portion)

Fig. 1.18



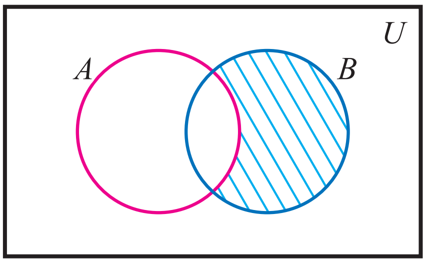
$(A \cap B)'$ (shaded portion)

Fig. 1.19



$A \cap B'$ (shaded portion)

Fig 1.20.



$A' \cap B$ (shaded portion)

Fig. 1.21

Remark

We can also make use of the following idea to represent sets and set operations in Venn diagram.

In Fig. 1.22 the sets A and B divide the universal set into four regions. These four regions are numbered for reference. This numbering is arbitrary.

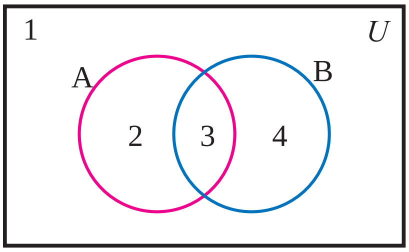


Fig. 1.22

Region 1	Contains the elements outside of both the sets A and B
Region 2	Contains the elements of the set A but not in B
Region 3	Contains the elements common to both the sets A and B.
Region 4	Contains the elements of the set B but not in A

Example 1.17

Draw a Venn diagram similar to one at the side and shade the regions representing the following sets

(i) A' (ii) B' (iii) $A' \cup B'$ (iv) $(A \cup B)'$ (v) $A' \cap B'$

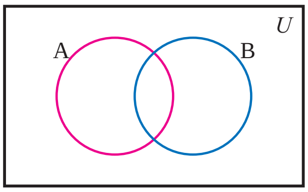
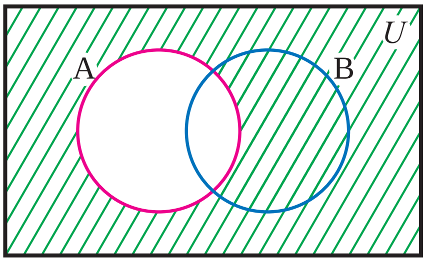


Fig. 1.23

Solution

(i) A'



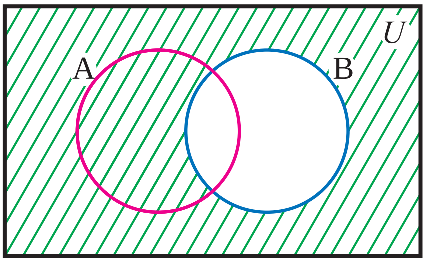
A' (shaded portion)

Fig. 1.24

Tip to shade

Set	Shaded Region
A'	1 and 4

(ii) B'



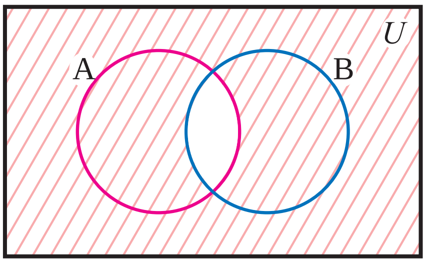
B' (shaded portion)

Fig. 1.25

Tip to shade

Set	Shaded Region
B'	1 and 2

(iii) $A' \cup B'$



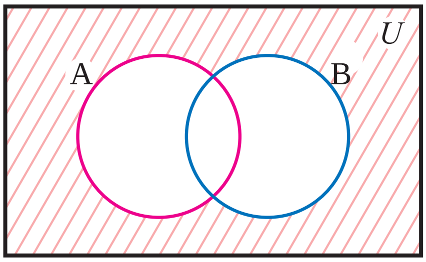
$A' \cup B'$ (shaded portion)

Fig. 1.26

Tip to shade

Set	Shaded Region
A'	1 and 4
B'	1 and 2
$A' \cup B'$	1, 2 and 4

(iv) $(A \cup B)'$

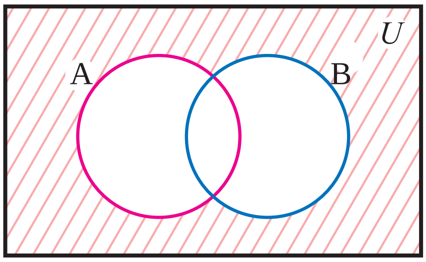


$(A \cup B)'$ (shaded portion)
Fig. 1.27

Tip to shade

Set	Shaded Region
$A \cup B$	2, 3 and 4
$(A \cup B)'$	1

(v) $A' \cap B'$



$A' \cap B'$ (shaded portion)
Fig. 1.28

Tip to shade

Set	Shaded Region
A'	1 and 4
B'	1 and 2
$A' \cap B'$	1

Important Results

For any two finite sets *A* and *B*, we have the following useful results

(i) $n(A) = n(A - B) + n(A \cap B)$

(ii) $n(B) = n(B - A) + n(A \cap B)$

(iii) $n(A \cup B) = n(A - B) + n(A \cap B) + n(B - A)$

(iv) $n(A \cup B) = n(A) + n(B) - n(A \cap B)$

(v) $n(A \cup B) = n(A) + n(B)$, when $A \cap B = \varnothing$

(vi) $n(A) + n(A') = n(U)$

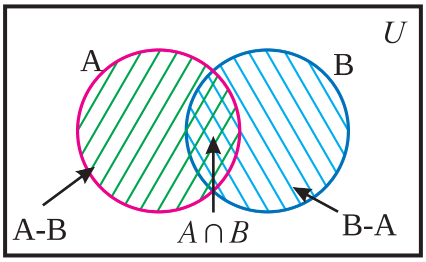


Fig. 1.29

Example 1.18

From the given Venn diagram, find the following

(i) A (ii) B (iii) $A \cup B$ (iv) $A \cap B$

Also verify that $n(A \cup B) = n(A) + n(B) - n(A \cap B)$

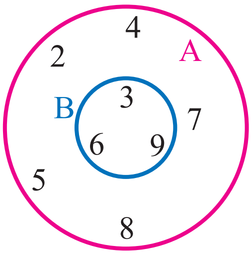


Fig. 1.30

Solution From the Venn diagram

(i) $A = \{2, 3, 4, 5, 6, 7, 8, 9\}$, (ii) $B = \{3, 6, 9,\}$,

(iii) $A \cup B = \{2, 3, 4, 5, 6, 7, 8, 9\}$ and (iv) $A \cap B = \{3, 6, 9\}$

We have $n(A) = 8,\ n(B) = 3,\ n(A \cup B) = 8,\ n(A \cap B) = 3$. Now

$$n(A) + n(B) - n(A \cap B) = 8 + 3 - 3 = 8$$

Hence, $n(A) + n(B) - n(A \cap B) = n(A \cup B)$

Example 1.19

From the given Venn diagram find

(i) A (ii) B (iii) $A \cup B$ (iv) $A \cap B$

Also verify that $n(A \cup B) = n(A) + n(B) - n(A \cap B)$

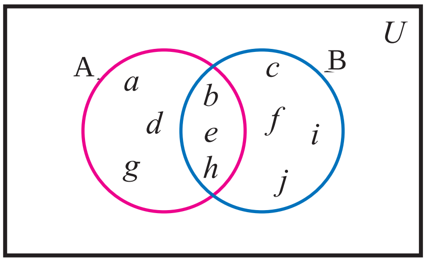


Fig. 1.31

Solution From the Venn diagram

(i) $A = \{a,b,d,e,g,h\}$ (ii) $B = \{b,c,e,f,h,i,j\}$

(iii) $A \cup B = \{a,b,c,d,e,f,g,h,i,j\}$ and (iv) $A \cap B = \{b,e,h\}$

So, $n(A) = 6,\ n(B) = 7,\ n(A \cup B) = 10,\ n(A \cap B) = 3$. Now

$n(A) + n(B) - n(A \cap B) = 6 + 7 - 3 = 10$

Hence, $n(A) + n(B) - n(A \cap B) = n(A \cup B)$

Example 1.20

If $n(A) = 12, n(B) = 17$ and $n(A \cup B) = 21$, find $n(A \cap B)$

Solution Given that $n(A) = 12, n(B) = 17$ and $n(A \cup B) = 21$

By using the formula $n(A \cup B) = n(A) + n(B) - n(A \cap B)$

$$n(A \cap B) = 12 + 17 - 21 = 8$$

Example 1.21

In a city 65% of the people view Tamil movies and 40% view English movies, 20% of the people view both Tamil and English movies. Find the percentage of people do not view any of these two movies.

Solution Let the population of the city be 100. Let T denote the set of people who view Tamil movies and E denote the set of people who view English movies. Then $n(T) = 65, n(E) = 40, n(T \cap E) = 20$. So, the number of people who view either of these movies is

$$n(T \cup E) = n(T) + n(E) - n(T \cap E)$$
$$= 65 + 40 - 20 = 85$$

Hence the number of people who do not view any of these movies is $100 - 85 = 15$

Hence the percentage of people who do not view any of these movies is 15

Aliter

From the Venn diagram the percentage of people who view at least one of these two movies is

$$45 + 20 + 20 = 85$$

Hence, the percentage of people who do not view any of these movies $= 100 - 85 = 15$

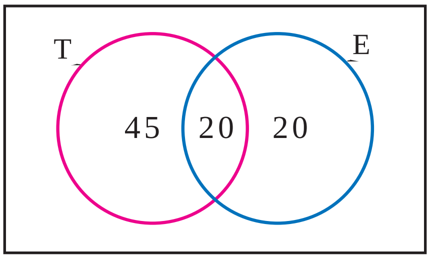


Fig. 1.32

Example 1.22

In a survey of 1000 families, it is found that 484 families use electric stoves, 552 families use gas stoves. If all the families use atleast one of these two types of stoves, find how many families use both the stoves?

Solution Let E denote the set of families using electric stove and G denote the set of families using gas stove. Then $n(E) = 484, n(G) = 552,\ n(E \cup G) = 1000$. Let x be the number of families using both the stoves . Then $n(E \cap G) = x$.

Using the result

$$n(E \cup G) = n(E) + n(G) - n(E \cap G)$$
$$1000 = 484 + 552 - x$$
$$\Rightarrow \quad x = 1036 - 1000 = 36$$

Hence, 36 families use both the stoves.

Aliter

From the Venn diagram,

$$484 - x + x + 552 - x = 1000$$
$$\Rightarrow \quad 1036 - x = 1000$$
$$\Rightarrow \quad -x = -36$$
$$x = 36$$

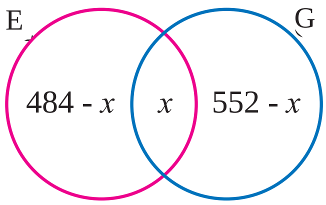


Fig. 1.33

Hence, 36 families use both the stoves.

Example 1.23

In a class of 50 students, each of the student passed either in mathematics or in science or in both. 10 students passed in both and 28 passed in science. Find how many students passed in mathematics?

Solution Let M = The set of students passed in Mathematics

S = The set of students passed in Science

Then, $n(S) = 28$, $n(M \cap S) = 10$, $n(M \cup S) = 50$

We have $n(M \cup S) = n(M) + n(S) - n(M \cap S)$

$50 = n(M) + 28 - 10$

$\Rightarrow \quad n(M) = 32$

Aliter

From the Venn diagram

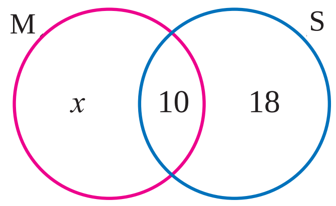


Fig. 1.34

$x + 10 + 18 = 50$

$x = 50 - 28 = 22$

Number of students passed in Mathematics $= x + 10 = 22 + 10 = 32$

Exercise 1.3

1. Place the elements of the following sets in the proper location on the given Venn diagram.

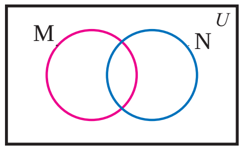


Fig. 1.35

$U = \{5, 6, 7, 8, 9, 10, 11, 12, 13\}$

$M = \{5, 8, 10, 11\}$, $N = \{5, 6, 7, 9, 10\}$

2. If A and B are two sets such that A has 50 elements, B has 65 elements and $A \cup B$ has 100 elements, how many elements does $A \cap B$ have?

3. If A and B are two sets containing 13 and 16 elements respectively, then find the minimum and maximum number of elements in $A \cup B$?

4. If $n(A \cap B) = 5$, $n(A \cup B) = 35$, $n(A) = 13$, find $n(B)$.

5. If $n(A) = 26$, $n(B) = 10$, $n(A \cup B) = 30$, $n(A') = 17$, find $n(A \cap B)$ and $n(\cup)$.

6. If $n(\cup) = 38$, $n(A) = 16$, $n(A \cap B) = 12$, $n(B') = 20$, find $n(A \cup B)$.

7. Let A and B be two finite sets such that $n(A - B) = 30$, $n(A \cup B) = 180$, $n(A \cap B) = 60$. Find $n(B)$

8. The population of a town is 10000. Out of these 5400 persons read newspaper A and 4700 read newspaper B. 1500 persons read both the newspapers. Find the number of persons who do not read either of the two papers.

9. In a school, all the students play either Foot ball or Volley ball or both. 300 students play Foot ball, 270 students play Volley ball and 120 students play both games. Find

   (i) the number of students who play Foot ball only

   (ii) the number of students who play Volley ball only

   (iii) the total number of students in the school

10. In an examination 150 students secured first class in English or Mathematics. Out of these 50 students obtained first class in both English and Mathematics. 115 students secured first class in Mathematics. How many students secured first class in English only?

11. In a group of 30 persons, 10 take tea but not coffee. 18 take tea. Find how many take coffee but not tea, if each person takes atleast one of the drinks.

12. In a village there are 60 families. Out of these 28 families speak only Tamil and 20 families speak only Urdu. How many families speak both Tamil and Urdu.

13. In a School 150 students passed X Standard Examination. 95 students applied for Group I and 82 students applied for Group II in the Higher Secondary course. If 20 students applied neither of the two, how many students applied for both groups?

14. Pradeep is a Section Chief for an electric utility company. The employees in his section cut down tall trees or climb poles. Pradeep recently reported the following information to the management of the utility.

    Out of 100 employees in my section, 55 can cut tall trees, 50 can climb poles, 11 can do both, 6 can't do any of the two. Is this information correct?

15. A and B are two sets such that $n(A-B) = 32 + x$, $n(B-A) = 5x$ and $n(A\cap B) = x$ Illustrate the information by means of a Venn diagram. Given that $n(A) = n(B)$. Calculate (i) the value of x (ii) $n(A\cup B)$.

16. The following table shows the percentage of the students of a school who participated in Elocution and Drawing competitions.

Competition	Elocution	Drawing	Both
Percentage of Students	55	45	20

Draw a Venn diagram to represent this information and use it to find the percentage of the students who

(i) participated in Elocution only

(ii) participated in Drawing only

(iii) do not participate in any one of the competitions.

17. A village has total population 2500. Out of which 1300 use brand A soap and 1050 use brand B soap and 250 use both brands. Find the percentage of population who use neither of these soaps.

Points to Remember

- A set is a well-defined collection of distinct objects
- Set is represented in three forms (i) Descriptive Form (ii) Set-builder Form (iii) Roster Form
- The number of elements in a set is said to be the cardinal number of the set.
- A set containing no element is called the empty set
- If the number of elements in a set is zero or finite, the set is called a finite set. Otherwise, the set is an infinite set.
- Two sets *A* and *B* are said to be equal if they contain exactly the same elements.
- A set *A* is a subset of a set *B* if every element of *A* is also an element of *B*.
- A set *A* is a proper subset of set *B* if $A \subseteq B$ and $A \neq B$
- The power set of the set *A* is the set of all subsets of *A*. It is denoted by *P*(*A*).
- The number of subsets of a set with *m* elements is 2^m.
- The number of proper subsets of a set with *m* elements is $2^m - 1$
- The set of all elements of the universal set that are not elements of a set A is called the complement of *A*. It is denoted by A'.
- The union of two sets *A* and *B* is the set of elements which are in *A* or in *B* or in both *A* and *B*.
- The intersection of two sets *A* and *B* is the set of all elements common to both *A* and *B*.
- If *A* and *B* are disjoint sets, then $A \cap B = \varnothing$
- The difference of two sets *A* and *B* is the set of all elements belonging to *A* but not to *B*.
- Symmetric difference of two sets *A* and *B* is defined as $A \triangle B = (A - B) \cup (B - A)$
- For any two finite sets *A* and *B*, we have

  (i) $n(A) = n(A - B) + n(A \cap B)$

  (ii) $n(B) = n(B - A) + n(A \cap B)$

  (iii) $n(A \cup B) = n(A - B) + n(A \cap B) + n(B - A)$

  (iv) $n(A \cup B) = n(A) + n(B) - n(A \cap B)$

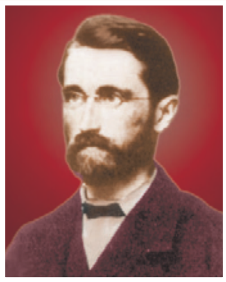
  (v) $n(A \cup B) = n(A) + n(B)$, when $A \cap B = \varnothing$

2 REAL NUMBER SYSTEM

Life is good for only two things, discovering mathematics and teaching mathematics
- SIMEON POISSON

Main Targets

- To recall Natural numbers, Whole numbers, Integers.
- To classify rational numbers as recurring / terminating decimals.
- To understand the existence of non terminating and non recurring decimals.
- To represent terminating / non terminating decimals on the number line.
- To understand the four basic operations in irrational numbers.
- To rationalise the denominator of the given irrational numbers.



RICHARD DEDEKIND
(1831-1916)

Richard Dedekind (1831-1916) belonged to an elite group of mathematicians who had been students of the legendary mathematician Carl Friedrich Gauss.

He did important work in abstract algebra, algebraic number theory and laid the foundations for the concept of the real numbers. He was one of the few mathematicians who understood the importance of set theory developed by Cantor. While teaching calculus for the first time at Polytechnic, Dedekind came up with the notion now called a Dedekind cut, a standard definition of the real numbers.

2.1 Introduction

All the numbers that we use in normal day-to-day activities to represent quantities such as distance, time, speed, area, profit, loss, temperature, etc., are called Real Numbers. The system of real numbers has evolved as a result of a process of successive extensions of the system of natural numbers. The extensions became inevitable as the science of Mathematics developed in the process of solving problems from other fields. Natural numbers came into existence when man first learnt counting. The Egyptians had used fractions around 1700 BC; around 500 BC, the Greek mathematicians led by Pythagoras realized the need for irrational numbers. Negative numbers began to be accepted around 1600 A.D. The development of calculus around 1700 A.D. used the entire set of real numbers without having defined them clearly. George Cantor can be considered the first to suggest a rigorous definition of real numbers in 1871 A.D.

In this chapter we discuss some properties of real numbers. First, let us recall various types of numbers that you have learnt in earlier classes.

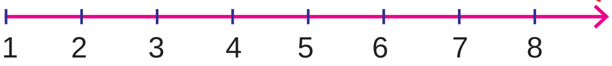
2.1.1 Natural Numbers

The counting numbers 1, 2, 3, $\cdots$ are called natural numbers.

The set of all natural numbers is denoted by $\mathbb{N}$.

i.e., $\mathbb{N} = \{1, 2, 3, \cdots\}$

The line extends endlessly only to the right side of 1.



Remark The smallest natural number is 1, but there is no largest number as it goes up continuously.

2.1.2 Whole Numbers

The set of natural numbers together with zero forms the set of whole numbers.

The set of whole numbers is denoted by $\mathbb{W}$.

$\mathbb{W} = \{ 0, 1, 2, 3, \cdots \}$

The line extends endlessly only to the right side of 0.

The smallest whole number is 0

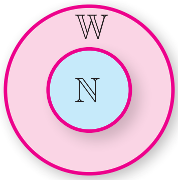
Remark

1) Every natural number is a whole number.

2) Every whole number need not be a natural number.

For, $0 \in \mathbb{W}$, but $0 \notin \mathbb{N}$

3) $\mathbb{N} \subset \mathbb{W}$



2.1.3 Integers

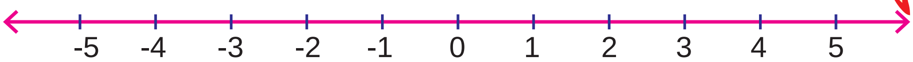
The natural numbers, their negative numbers together with zero are called integers.

$\mathbb{Z}$ is derived from the German word 'Zahlen', means 'to count'

The set of all integers is denoted by $\mathbb{Z}$

$\mathbb{Z} = \{ \cdots -3, -2, -1, 0, 1, 2, 3, \cdots\}$

The line extends endlessly on both sides of 0.



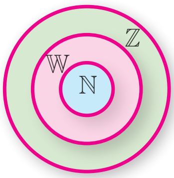
$1, 2, 3 \cdots$ are called positive integers.

$-1, -2, -3 \cdots$ are called negative integers.

Think and Answer !
Is zero a positive integer or a negative integer?

Remark

1) Every natural number is an integer.
2) Every whole number is an integer.
3) $\mathbb{N} \subset \mathbb{W} \subset \mathbb{Z}$



2.1.4 Rational Numbers

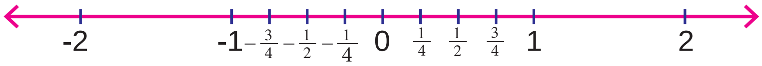
A number of the form $\frac{p}{q}$, where p and q are both integers and $q \neq 0$ is called a rational number.

For example, $3 = \frac{3}{1}, -\frac{5}{6}, \frac{7}{8}$ are rational numbers.

The set of all rational numbers is denoted by $\mathbb{Q}$.

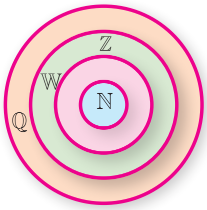
$$\mathbb{Q} = \left\{\frac{p}{q} : p \in \mathbb{Z}, q \in \mathbb{Z}, \text{and } q \neq 0\right\}$$

We find numbers in between integers



Remark

1) A rational number may be positive, negative or zero.
2) Every integer n is also a rational number, since we can write n as $\frac{n}{1}$.
3) $\mathbb{N} \subset \mathbb{W} \subset \mathbb{Z} \subset \mathbb{Q}$



Important Results

1) If a and b are two distinct rational numbers, then $\frac{a+b}{2}$ is a rational number between a and b such that $a < \frac{a+b}{2} < b$.

2) There are infinitely many rational numbers between any two given rational numbers.

Think and Answer !
Can you correlate the word ratio with rational numbers ?

Example 2.1

Find any two rational numbers between $\frac{1}{4}$ and $\frac{3}{4}$.

Solution A rational number between $\frac{1}{4}$ and $\frac{3}{4} = \frac{1}{2}\left(\frac{1}{4}+\frac{3}{4}\right) = \frac{1}{2}\,(1) = \frac{1}{2}$

Another rational number between $\frac{1}{2}$ and $\frac{3}{4} = \frac{1}{2}\left(\frac{1}{2}+\frac{3}{4}\right) = \frac{1}{2}\times\frac{5}{4} = \frac{5}{8}$

The rational numbers $\frac{1}{2}$ and $\frac{5}{8}$ lie between $\frac{1}{4}$ and $\frac{3}{4}$

Note There are infinite number of rationals between $\frac{1}{4}$ and $\frac{3}{4}$. The rationals $\frac{1}{2}$ and $\frac{5}{8}$ that we have obtained in Example 2.1 are two among them

Exercise 2.1

1. State whether the following statements are true or false.

   (i) Every natural number is a whole number.

   (ii) Every whole number is a natural number.

   (iii) Every integer is a rational number.

   (iv) Every rational number is a whole number.

   (v) Every rational number is an integer.

   (vi) Every integer is a whole number.

2. Is zero a rational number ? Give reasons for your answer.

3. Find any two rational numbers between $-\frac{5}{7}$ and $-\frac{2}{7}$.

2.2 Decimal Representation of Rational Numbers

If we have a rational number written as a fraction $\frac{p}{q}$, we get the decimal representation by long division.

When we divide p by q using long division method either the remainder becomes zero or the remainder never becomes zero and we get a repeating string of remainders.

Case (i) The remainder becomes zero

Let us express $\frac{7}{16}$ in decimal form. Then $\frac{7}{16} = 0.4375$

In this example, we observe that the remainder becomes zero after a few steps.

Also the decimal expansion of $\frac{7}{16}$ terminates.

Similarly, using long division method we can express the following rational numbers in decimal form as

$$\begin{array}{r} 0.4375 \\ 16\overline{)7.0000} \\ \underline{6\,4} \\ 60 \\ \underline{48} \\ 120 \\ \underline{112} \\ 80 \\ \underline{80} \\ 0 \end{array}$$

$$\frac{1}{2} = 0.5,\ \frac{7}{5} = 1.4,\ -\frac{8}{25} = -0.32,\ \frac{9}{64} = 0.140625,\ \frac{527}{500} = 1.054$$

In these examples, the decimal expansion terminates or ends after a finite number of steps.

Key Concept — **Terminating Decimal**

When the decimal expansion of $\frac{p}{q}$ terminates (i.e., comes to an end) the decimal expansion is called terminating.

Case (ii) The remainder never becomes zero

Does every rational number has a terminating decimal expansion? Before answering the question, let us express $\frac{5}{11}$, $\frac{7}{6}$ and $\frac{22}{7}$ in decimal form.

$$\begin{array}{r} 0.4545\cdots \\ 11\overline{)5.0000} \\ \underline{4\,4} \\ 60 \\ \underline{55} \\ 50 \\ \underline{44} \\ 60 \\ \underline{55} \\ 50 \\ \ddots \end{array} \qquad \begin{array}{r} 1.1666\cdots \\ 6\overline{)7.0000} \\ \underline{6\,0} \\ 10 \\ \underline{6} \\ 40 \\ \underline{36} \\ 40 \\ \underline{36} \\ 40 \\ \underline{36} \\ 40 \\ \ddots \end{array} \qquad \begin{array}{r} 3.142857\,142857\cdots \\ 7\overline{)22.00000000} \\ \underline{21} \\ 10 \\ \underline{7} \\ 30 \\ \underline{28} \\ 20 \\ \underline{14} \\ 60 \\ \underline{56} \\ 40 \\ \underline{35} \\ 50 \\ \underline{49} \\ 10 \\ \ddots \end{array}$$

$$\therefore \frac{5}{11} = 0.4545\cdots, \qquad \frac{7}{6} = 1.1666\cdots, \qquad \frac{22}{7} = 3.142857\,1\cdots$$

Thus, the decimal expansion of a rational number need not terminate.

In the above examples, we observe that the remainders never become zero. Also we note that the remainders repeat after some steps. So, we have a repeating (recurring) block of digits in the quotient.

Key Concept **Non-terminating and Recurring**

In the decimal expansion of $\frac{p}{q}$ when the remainder never becomes zero, we have a repeating (recurring) block of digits in the quotient. In this case, the decimal expansion is called non-terminating and recurring.

To simplify the notation, we place a bar over the first block of the repeating (recurring) part and omit the remaining blocks.

So, we can write the expansion of $\frac{5}{11}$, $\frac{7}{6}$ and $\frac{22}{7}$ as follows.

$\frac{5}{11} = 0.4545\cdots = 0.\overline{45}$, $\quad \frac{7}{6} = 1.16666\cdots = 1.1\overline{6}$

$\frac{22}{7} = 3.142857\ 1452857\ \cdots = 3.\overline{142857}$

The following table shows decimal representation of the reciprocals of the first ten natural numbers. We know that the reciprocal of a number n is $\frac{1}{n}$. Obviously, the reciprocals of natural numbers are rational numbers.

Number	Reciprocal	Type of Decimal
1	1.0	Terminating
2	0.5	Terminating
3	$0.\overline{3}$	Non-terminating and recurring
4	0.25	Terminating
5	0.2	Terminating
6	$0.1\overline{6}$	Non-terminating and recurring
7	$0.\overline{142857}$	Non-terminating and recurring
8	0.125	Terminating
9	$0.\overline{1}$	Non-terminating and recurring
10	0.1	Terminating

Thus we see that,

A rational number can be expressed by either a terminating or a non-terminating and recurring decimal expansion.

The converse of this statement is also true.

That is, if the decimal expansion of a number is terminating or non-terminating and recurring, then the number is a rational number.

We shall illustrate this with examples.

2.2.1 Representing a Terminating Decimal Expansion in the form $\frac{p}{q}$

Terminating decimal expansion can easily be expressed in the form $\frac{p}{q}$ $(p, q \in \mathbb{Z}$ and $q \neq 0)$. This method is explained in the following example

Example 2.2

Express the following decimal expansion in the form $\frac{p}{q}$, where p and q are integers and $q \neq 0$.

(i) 0.75 (ii) 0.625 (iii) 0.5625 (iv) 0.28

Solution (i) $0.75 = \frac{75}{100} = \frac{3}{4}$

(ii) $0.625 = \frac{625}{1000} = \frac{5}{8}$

(iii) $0.5625 = \frac{5625}{10000} = \frac{45}{80} = \frac{9}{16}$

(iv) $0.28 = \frac{28}{100} = \frac{7}{25}$

2.2.2 Representing a Non-terminating and Recurring Decimal Expansion in the form $\frac{p}{q}$

The expression of non-terminating and recurring decimal expansions in the form $\frac{p}{q}$ $(p, q \in \mathbb{Z}$ and $q \neq 0)$ is not quite easy and the process is explained in the next example.

Example 2.3

Express the following in the form $\frac{p}{q}$, where p and q are integers and $q \neq 0$.

(i) $0.\overline{47}$ (ii) $0.\overline{001}$ (iii) $0.5\overline{7}$ (iv) $0.2\overline{45}$ (v) $0.\overline{6}$ (vi) $1.\overline{4}$

Solution (i) Let $x = 0.\overline{47}$. Then $x = 0.474747\cdots$

Since two digits are repeating, multiplying both sides by 100, we get

$$100x = 47.474747\cdots = 47 + 0.474747\cdots = 47 + x$$

$$99x = 47$$

$$x = \frac{47}{99} \qquad \therefore\ 0.\overline{47} = \frac{47}{99}$$

(ii) Let $x = 0.\overline{001}$. Then $x = 0.001001001\cdots$

Since three digits are repeating, multiplying both sides by 1000, we get

$$1000x = 1.001001001\cdots = 1 + 0.001001001\cdots = 1 + x$$

$$1000x - x = 1$$

$$999x = 1$$

$$x = \frac{1}{999} \qquad \therefore\ 0.\overline{001} = \frac{1}{999}$$

(iii) Let $x = 0.5\overline{7}$. Then $x = 0.57777\cdots$

Multiplying both sides by 10, we get

$$10x = 5.7777\cdots = 5.2 + 0.57777\cdots = 5.2 + x$$

$$9x = 5.2$$

$$x = \frac{5.2}{9}$$

$$x = \frac{52}{90} \qquad \therefore\ 0.5\overline{7} = \frac{52}{90} = \frac{26}{45}$$

(iv) Let $x = 0.2\overline{45}$. Then $x = 0.2454545\cdots$

Multiplying both sides by 100, we get

$$100x = 24.545454\cdots = 24.3 + 0.2454545\cdots = 24.3 + x$$

$$99x = 24.3$$

$$x = \frac{24.3}{99}$$

$$0.2\overline{45} = \frac{243}{990} = \frac{27}{110}$$

(v) Let $x = 0.\overline{6}$. Then $x = 0.66666\cdots$

Multiplying both sides by 10, we get

$$10x = 6.66666\cdots = 6 + 0.6666\cdots = 6 + x$$

$$9x = 6$$

$$x = \frac{6}{9} = \frac{2}{3} \qquad \therefore\ 0.\overline{6} = \frac{2}{3}$$

(vi) Let $x = 1.\overline{5}$. Then $x = 1.55555\cdots$

Multiplying both sides by 10, we get

$$10x = 15.5555\cdots = 14 + 1.5555\cdots = 14 + x$$

$$9x = 14$$

$$x = \frac{14}{9} \qquad \therefore\ 1.\overline{5} = 1\frac{5}{9}$$

So, every number with a non-terminating and recurring decimal expansion can be expressed in the form $\frac{p}{q}$, where p and q are integers and q not equal to zero

To determine whether the decimal form of a rational number will terminate or non-terminate we can make use of the following rule.

> If a rational number $\frac{p}{q}$ can be expressed in the form $\frac{p}{2^m \times 5^n}$, where $p \in \mathbb{Z}$ and $m, n \in \mathbb{W}$, then the rational number will have a terminating decimal expansion. Otherwise, the rational number will have a non-terminating and recurring decimal expansion.

This result is based on the fact that the decimal system uses ten as its base and the prime factors of 10 are 2 and 5.

Example 2.4

Without actually dividing, classify the decimal expansion of the following numbers as terminating or non-terminating and recurring.

(i) $\frac{7}{16}$ (ii) $\frac{13}{150}$ (iii) $\frac{-11}{75}$ (iv) $\frac{17}{200}$

Solution

(i) $16 = 2^4$

$\frac{7}{16} = \frac{7}{2^4} = \frac{7}{2^4 \times 5^0}$. So, $\frac{7}{16}$ has a terminating decimal expansion.

(ii) $150 = 2 \times 3 \times 5^2$

$\frac{13}{150} = \frac{13}{2 \times 3 \times 5^2}$

Since it is not in the form $\frac{p}{2^m \times 5^n}$, $\frac{13}{150}$ has a non-terminating and recurring decimal expansion.

(iii) $\frac{-11}{75} = \frac{-11}{3 \times 5^2}$

Since it is not in the form $\frac{p}{2^m \times 5^n}$, $\frac{-11}{75}$ has a non-terminating and recurring decimal expansion.

(iv) $\frac{17}{200} = \frac{17}{8 \times 25} = \frac{17}{2^3 \times 5^2}$. So $\frac{17}{200}$ has a terminating decimal expansion.

Example 2.5

Convert $0.\overline{9}$ into a rational number.

Solution Let $x = 0.\overline{9}$. Then $x = 0.99999\cdots$

Multiplying by 10 on both sides, we get

$$10x = 9.99999\cdots = 9 + 0.9999\cdots = 9 + x$$

$$\Longrightarrow \quad 9x = 9$$

$$\Longrightarrow \quad x = 1. \text{ That is, } 0.\overline{9} = 1 \qquad (\because 1 \text{ is rational number})$$

For your Thought

We have proved $0.\overline{9} = 1$. Isn't it surprising?

Most of us think that $0.9999\cdots$ is less than 1. But this is not the case. It is clear from the above argument that $0.\overline{9} = 1$. Also this result is consistent with the fact that $3 \times 0.333\cdots = 0.999\cdots$, while $3 \times \frac{1}{3} = 1$.

Similarly, it can be shown that any terminating decimal can be represented as a non-terminating and recurring decimal expansion with an endless blocks of 9s.

For example $6 = 5.9999\cdots$, $2.5 = 2.4999\cdots$.

Exercise 2.2

1. Convert the following rational numbers into decimals and state the kind of decimal expansion.

   (i) $\frac{42}{100}$ (ii) $8\frac{2}{7}$ (iii) $\frac{13}{55}$ (iv) $\frac{459}{500}$

   (v) $\frac{1}{11}$ (vi) $-\frac{3}{13}$ (vii) $\frac{19}{3}$ (viii) $-\frac{7}{32}$

2. Without actual division, find which of the following rational numbers have terminating decimal expansion.

   (i) $\frac{5}{64}$ (ii) $\frac{11}{12}$ (iii) $\frac{27}{40}$ (iv) $\frac{8}{35}$

3. Express the following decimal expansions into rational numbers.

   (i) $0.\overline{18}$ (ii) $0.\overline{427}$ (iii) $0.\overline{0001}$

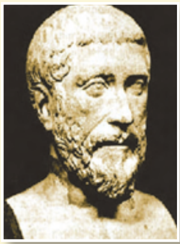
   (iv) $1.4\overline{5}$ (v) $7.\overline{3}$ (vi) $0.4\overline{16}$

4. Express $\frac{1}{13}$ in decimal form. Find the number of digits in the repeating block.

5. Find the decimal expansions of $\frac{1}{7}$ and $\frac{2}{7}$ by division method. Without using the long division method, deduce the decimal expressions of $\frac{3}{7}, \frac{4}{7}, \frac{5}{7}, \frac{6}{7}$ from the decimal expansion of $\frac{1}{7}$.

2.3 Irrational Numbers

Let us have a look at the number line again. We have represented rational numbers on the number line. We have also seen that there are infinitely many rational numbers between any two given rational numbers. In fact there are infinitely many more numbers left on the number line, which are not rationals.



Pythagoras
569BC - 479 BC

Around 400 BC, the pythagorians, followers of the famous Greek mathematician Pythagoras, were the first to discover the numbers which cannot be written in the form of a fraction. These numbers are called irrational numbers.

In other words there are numbers whose decimal expansions are non-terminating and non-recurring. Thus, there is a need to extend the system of rational numbers. Consider the following decimal expansion

$0.808008000800008\cdots$ (1) .

This is non-terminating. Is it recurring?

It is true that there is a pattern in this decimal expansion, but no block of digits repeats endlessly and so it is not recurring.

Thus, this decimal expansion is non-terminating and non-repeating (non-recurring). So it cannot represent a rational number. Numbers of this type are called irrational numbers.

Key concept	**Irrational Number**

A number having a non-terminating and non-recurring decimal expansion is called an irrational number. So, it cannot be written in the form $\frac{p}{q}$, where p and q are integers and $q \neq 0$.

For example,

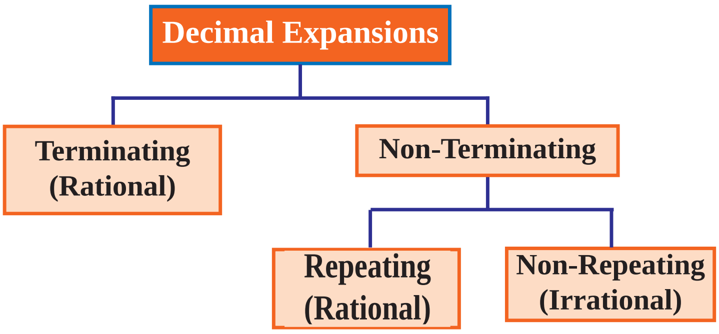
$\sqrt{2}, \sqrt{3}, \sqrt{5}, e, \pi, \sqrt{17}$, $0.2020020002\cdots$ are a few examples of irrational numbers.

Note In fact, we can generate infinitely many non-terminating and non-recurring decimal expansions by replacing the digit 8 in (1) by any natural number as we like.

Know about π: In the late 18th centurary Lambert and Legendre proved that π is irrational.

We usually take $\frac{22}{7}$ (a rational number) as an approximate value for π (an irrational number).

Classification of Decimal Expansions



2.4 Real Numbers

Key Concept — **Real Numbers**

The union of the set of all rational numbers and the set of all irrational numbers forms the set of all real numbers.

Thus, every real number is either a rational number or an irrational number.

In other words, if a real number is not a rational number, then it must be an irrational number.

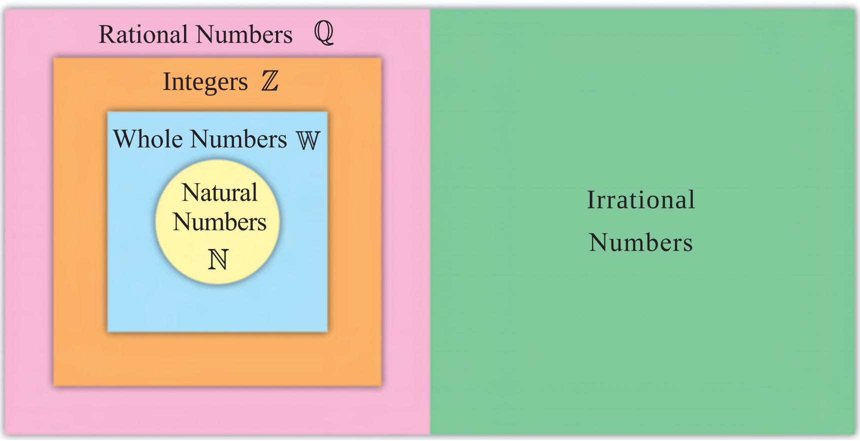
The set of all real numbers is denoted by $\mathbb{R}$.

German mathematicians, George Cantor and R. Dedekind proved independently that corresponding to every real number, there is a unique point on the real number line and corresponding to every point on the number line there exists a unique real number.

Thus, on the number line, each point corresponds to a unique real number. And every real number can be represented by a unique point on the number line.

The following diagram illustrates the relationships among the sets that make up the real numbers

REAL NUMBERS $\mathbb{R}$



Let us find the square root of 2 by long division method.

	1.4142135⋯
1	2.00 00 00 00 00
	1
24	100
	96
281	400
	281
2824	11900
	11296
28282	60400
	56564
282841	383600
	282841
2828423	10075900
	8485269
28284265	159063100
	141421325
	17641775
	⋮

$\therefore \sqrt{2} = 1.4142135\cdots$

If we continue this process, we observe that the decimal expansion has non-terminating and non-recurring digits and hence $\sqrt{2}$ is an irrational number.

Note

(i) The decimal expansions of $\sqrt{3}$, $\sqrt{5}$, $\sqrt{6}$, $\cdots$ are non-terminating and non-recurring and hence they are irrational numbers.

(ii) The square root of every positive integer is not always irrational.
For example, $\sqrt{4} = 2$, $\sqrt{9} = 3$, $\sqrt{25} = 5\cdots$. Thus $\sqrt{4}$, $\sqrt{9}$, $\sqrt{25}$, $\cdots$ are rational numbers.

(iii) The square root of every positive but a not a perfect square number is an irrational number

2.4.1 Representation of Irrational Numbers on the Number Line

Let us now locate the irrational numbers $\sqrt{2}$ and $\sqrt{3}$ on the number line.

(i) Locating $\sqrt{2}$ on the number line.

Draw a number line. Mark points O and A such that O represents the number zero and A represents the number 1. i.e., $OA = 1$ unit Draw $AB \perp OA$ such that $AB = 1$unit. Join OB.

In right triangle OAB, by Pythagorean theorem,

$$OB^2 = OA^2 + AB^2$$
$$= 1^2 + 1^2$$
$$OB^2 = 2$$
$$OB = \sqrt{2}$$

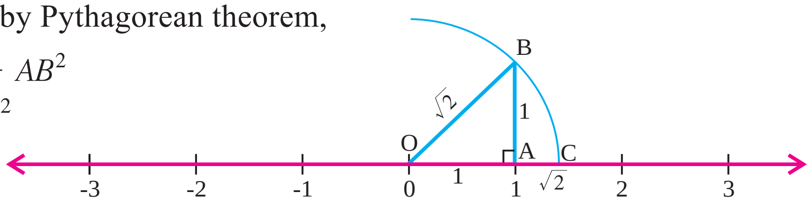


Fig. 2.6

With O as centre and radius OB, draw an arc to intersect the number line at C on the right side of O. Clearly $OC = OB = \sqrt{2}$. Thus, C corresponds to $\sqrt{2}$ on the number line.

(ii) Locating $\sqrt{3}$ on the number line.

Draw a number line. Mark points O and C on the number line such that O represents the number zero and C represents the number $\sqrt{2}$ as we have seen just above.

$\therefore$ $OC = \sqrt{2}$ unit. Draw $CD \perp OC$ such that $CD = 1$ unit. Join OD

In right triangle OCD, by Pythagorean theorem,

$$OD^2 = OC^2 + CD^2$$
$$= (\sqrt{2})^2 + 1^2 = 3$$
$$\therefore\ OD = \sqrt{3}$$

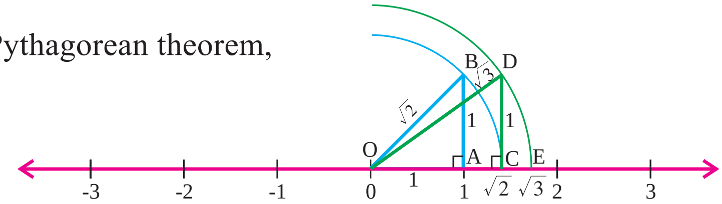


Fig. 2.7

With O as centre and radius OD, draw an arc to intersect the number line at E on the right side of O. Clearly $OE = OD = \sqrt{3}$. Thus, E represents $\sqrt{3}$ on the number line.

Example 2.6

Classify the following numbers as rational or irrational.

(i) $\sqrt{11}$ (ii) $\sqrt{81}$ (iii) 0.0625 (iv) $0.8\overline{3}$ (v) $1.505500555\cdots$

Solution

(i) $\sqrt{11}$ is an irrational number. (11is not a perfect square number)

(ii) $\sqrt{81} = 9 = \frac{9}{1}$, a rational number.

(iii) 0.0625 is a terminating decimal.

$\therefore$ 0.0625 is a rational number.

(iv) $0.8\overline{3} = 0.8333\cdots$

The decimal expansion is non-terminating and recurring.

$\therefore$ $0.8\overline{3}$ is a rational number.

(v) The decimal expression is non-terminating and non-recurring.

$\therefore$ $1.505500555\cdots$ is an irrational number.

Example 2.7

Find any three irrational numbers between $\frac{5}{7}$ and $\frac{9}{11}$.

Solution

$$\begin{array}{r} 0.714285\cdots \\ 7\,)\,5.000000 \\ 4\,9 \\ \hline 10 \\ 7 \\ \hline 30 \\ 28 \\ \hline 20 \\ 14 \\ \hline 60 \\ 56 \\ \hline 40 \\ 35 \\ \hline 50 \\ \ddots \end{array} \qquad \begin{array}{r} 0.8181\cdots \\ 11\,)\,9.0000 \\ 8\,8 \\ \hline 20 \\ 11 \\ \hline 90 \\ 88 \\ \hline 20 \\ \ddots \end{array}$$

$$\frac{5}{7} = 0.\overline{714285} \qquad \frac{9}{11} = 0.8181\cdots = 0.\overline{81}$$

To find three irrational numbers between $\frac{5}{7}$ and $\frac{9}{11}$ (i.e., between 0.714285... and 0.8181...) We find three numbers whose decimal expansions are non-terminating and non-recurring. Infact, there are infinitely many such numbers. Three such numbers are

$0.72022002220002\cdots$

$0.73033003330003\cdots$

$0.75055005550005\cdots$

Example 2.8

In the following equations determine whether x, y, z represent rational or irrational numbers.

(i) $x^3 = 8$ (ii) $x^2 = 81$ (iii) $y^2 = 3$ (iv) $z^2 = 0.09$

Solution

(i) $x^3 = 8 = 2^3$ (8 is a perfect cube)

$\Rightarrow x = 2$, a rational number.

(ii) $x^2 = 81 = 9^2$ (81 is a perfect square)

$\Rightarrow x = 9$, a rational number.

(iii) $y^2 = 3 \Rightarrow y = \sqrt{3}$, an irrational number.

(iv) $z^2 = 0.09 = \frac{9}{100} = \left(\frac{3}{10}\right)^2$

$\Rightarrow z = \frac{3}{10}$, a rational number.

Exercise 2.3

1. Locate $\sqrt{5}$ on the number line.
2. Find any three irrational numbers between $\sqrt{3}$ and $\sqrt{5}$.
3. Find any two irrational numbers between 3 and 3.5.
4. Find any two irrational numbers between 0.15 and 0.16.
5. Insert any two irrational numbers between $\frac{4}{7}$ and $\frac{5}{7}$.
6. Find any two irrational numbers between $\sqrt{3}$ and 2.
7. Find a rational number and also an irrational number between $1.1011001110001\cdots$ and $2.1011001110001\cdots$
8. Find any two rational numbers between $0.12122122212222\cdots$ and $0.2122122212222\cdots$

2.4.2 Representation of Real Numbers on the Number Line

We have seen that any real number can be represented as a decimal expansion. This will help us to represent a real number on the number line.

Let us locate 3.776 on the number line. We know that 3.776 lies between 3 and 4.

Let us look closely at the portion of the number line between 3 and 4.

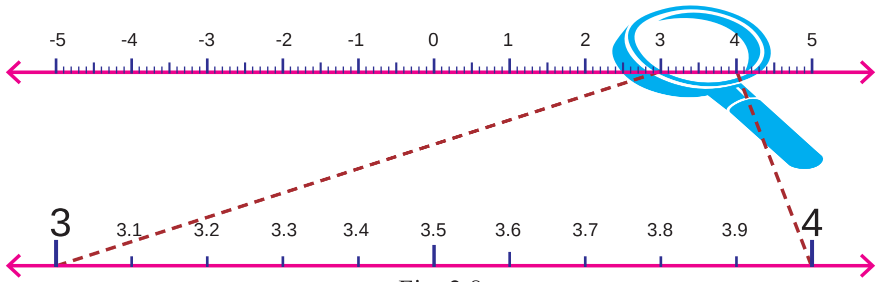


Fig. 2.8

Divide the portion between 3 and 4 into 10 equal parts and mark each point of division as in Fig 2.8. Then the first mark to the right of 3 will represent 3.1, the second 3.2, and so on. To view this clearly take a magnifying glass and look at the portion between 3 and 4. It will look like as shown in Fig. 2.8. Now 3.776 lies between 3.7 and 3.8. So, let us focus on the portion between 3.7 and 3.8 (Fig. 2.9)

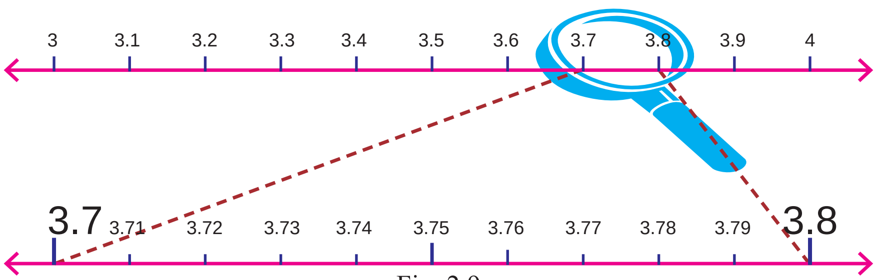


Fig. 2.9

Again divide the portion between 3.7 and 3.8 into 10 equal parts. The first mark will represent 3.71, the next 3.72, and so on. To view this portion clearly, we magnify the portion between 3.7 and 3.8 as shown in Fig 2.9

Again, 3.776 lies between 3.77 and 3.78. So, let us divide this portion into 10 equal parts. We magnify this portion, to see clearly as in Fig. 2.10.

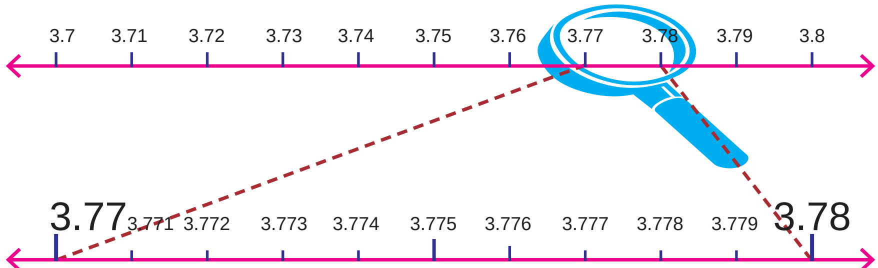


Fig. 2.10

The first mark represents 3.771, the next mark 3.772, and so on. So 3.776 is the 6^{th} mark in this sub division.

This process of visualisation of representation of numbers on the number line, through a magnifying glass is known as the process of successive magnification.

So, we can visualise the position of a real number with a terminating decimal expansion on the number line, by sufficient successive magnifications.

Now, let us consider a real number with a non-terminating recurring decimal expansion and try to visualise the position of it on the number line.

Example 2.9

Visualise $4.\overline{26}$ on the number line, upto 4 decimal places, that is upto 4.2626

Solution We locate $4.\overline{26}$ on the number line, by the process of successive magnification. This has been illustrated in Fig. 2.11

Step 1: First we note that $4.\overline{26}$ lies between 4 and 5

Step 2: Divide the portion between 4 and 5 into 10 equal parts and use a magnifying glass to visualise that $4.\overline{26}$ lies between 4.2 and 4.3

Step 3: Divide the portion between 4.2 and 4.3 into 10 equal parts and use a magnifying glass to visualise that $4.\overline{26}$ lies between 4.26 and 4.27

Step 4: Divide the portion between 4.26 and 4.27 into 10 equal parts and use a magnifying glass to visualise that $4.\overline{26}$ lies between 4.262 and 4.263

Step 5: Divide the portion between 4.262 and 4.263 into 10 equal parts and use a magnifying glass to visualise that $4.\overline{26}$ lies between 4.2625 and 4.2627

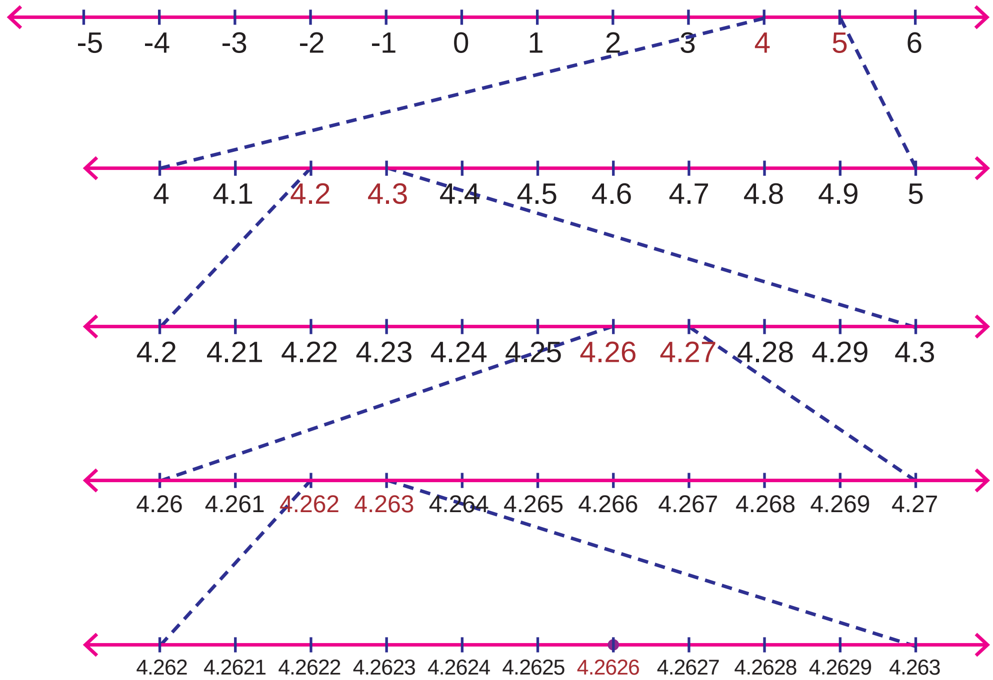


Fig. 2.11

We note that $4.\overline{26}$ is visualized closer to 4.263 than to 4.262.

The same procedure can be used to visualize a real number with a non-terminating and non-recurring decimal expansion on the number line to a required accuracy.

From the above discussions and visualizations we conclude again that every real number is represented by a unique point on the number line. Further every point on the number line represents one and only one real number.

Exercise 2.4

1. Using the process of successive magnification
   (i) Visualize 3.456 on the number line.
   (ii) Visualize $6.7\overline{3}$ on the number line, upto 4 decimal places.

2.4.3 Properties of Real Numbers

- For any two real numbers a and b, $a = b$ or $a > b$ or $a < b$
- The sum, difference, product of two real numbers is also a real number.
- The division of a real number by a non-zero real number is also a real number.
- The real numbers obey closure, associative, commutative and distributive laws under addition and under multiplication that the rational numbers obey.
- Every real number has its negative real number. The number zero is its own negative and zero is considered to be neither negative nor positive.

Further the sum, difference, product and quotient (except division by zero) of two rational numbers, will be rational number. However, the sum, difference, product and quotient of two irrational numbers may sometimes turnout to be a rational number.

Let us state the following facts about rational numbers and irrational numbers.

Key Concept
1. The sum or difference of a rational number and an irrational number is always an irrational number
2. The product or quotient of non-zero rational number and an irrational number is also an irrational number.
3. Sum, difference, product or quotient of two irrational numbers need not be irrational. The result may be rational or irrational.

Remark If a is a rational number and $\sqrt{b}$ is an irrational number then

(i) $a+\sqrt{b}$ is irrational (ii) $a-\sqrt{b}$ is irrational

(iii) $a\sqrt{b}$ is irrational (iv) $\frac{a}{\sqrt{b}}$ is irrational (v) $\frac{\sqrt{b}}{a}$ is irrational

For example, (i) $2+\sqrt{3}$ is irrational (ii) $2-\sqrt{3}$ is irrational

(iii) $2\sqrt{3}$ is irrational (iv) $\frac{2}{\sqrt{3}}$ is irrational

2.4.4 Square Root of Real Numbers

Let $a>0$ be a real number. Then $\sqrt{a}=b$ means $b^2=a$ and $b>0$.

2 is a square root of 4 because $2\times 2=4$, but -2 is also a square root of 4 because $(-2)\times(-2)=4$. To avoid confusion between these two we define the symbol $\sqrt{\ }$, to mean the principal or positive square root.

Let us now mention some useful identities relating to square roots.

	Let a and b be positive real numbers. Then
1	$\sqrt{ab}=\sqrt{a}\sqrt{b}$
2	$\sqrt{\frac{a}{b}}=\frac{\sqrt{a}}{\sqrt{b}}$
3	$(\sqrt{a}+\sqrt{b})(\sqrt{a}-\sqrt{b})=a-b$
4	$(a+\sqrt{b})(a-\sqrt{b})=a^2-b$
5	$(\sqrt{a}+\sqrt{b})(\sqrt{c}+\sqrt{d})=\sqrt{ac}+\sqrt{ad}+\sqrt{bc}+\sqrt{bd}$
6	$(\sqrt{a}+\sqrt{b})^2=a+b+2\sqrt{ab}$

Example 2.10

Give two irrational numbers so that their

(i) sum is an irrational number.

(ii) sum is not an irrational number.

(iii) difference is an irrational number.

(iv) difference is not an irrational number.

(v) product is an irrational number.

(vi) product is not an irrational number.

(vii) quotient is an irrational number.

(viii) quotient is not an irrational number.

Solution

(i) Consider the two irrational numbers $2+\sqrt{3}$ and $\sqrt{3}-2$.

Their sum $= 2+\sqrt{3}+\sqrt{3}-2 = 2\sqrt{3}$ is an irrational number.

(ii) Consider the two irrational numbers $\sqrt{2}$ and $-\sqrt{2}$.

Their sum $= \sqrt{2}+(-\sqrt{2}) = 0$ is a rational number.

(iii) Consider the two irrational numbers $\sqrt{3}$ and $\sqrt{2}$.

Their difference $= \sqrt{3}-\sqrt{2}$ is an irrational number.

(iv) Consider the two irrational numbers $5+\sqrt{3}$ and $\sqrt{3}-5$.

Their difference $= (5+\sqrt{3})-(\sqrt{3}-5) = 10$ is a rational number.

(v) Consider the irrational numbers $\sqrt{3}$ and $\sqrt{5}$.

Their product $= \sqrt{3}\times\sqrt{5} = \sqrt{15}$ is an irrational number.

(vi) Consider the two irrational numbers $\sqrt{18}$ and $\sqrt{2}$.

Their product $= \sqrt{18}\times\sqrt{2} = \sqrt{36} = 6$ is a rational number.

(vii) Consider the two irrational numbers $\sqrt{15}$ and $\sqrt{3}$.

Their quotient $= \dfrac{\sqrt{15}}{\sqrt{3}} = \sqrt{\dfrac{15}{3}} = \sqrt{5}$ is an irrational number.

(viii) Consider the two irrational numbers $\sqrt{75}$ and $\sqrt{3}$.

Their quotient $= \dfrac{\sqrt{75}}{\sqrt{3}} = \sqrt{\dfrac{75}{3}} = 5$ is a rational number.

2.5 Surds

We know that $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$ are irrational numbers. These are square roots of rational numbers, which cannot be expressed as squares of any rational number. $\sqrt[3]{2}$, $\sqrt[3]{3}$, $\sqrt[3]{7}$ etc. are the cube roots of rational numbers, which cannot be expressed as cubes of any rational number. This type of irrational numbers are called surds or radicals.

Key Concept — **Surds**

If 'a' is a positive rational number and n is a positive integer such that $\sqrt[n]{a}$ is an irrational number, then $\sqrt[n]{a}$ is called a 'surd' or a 'radical'.

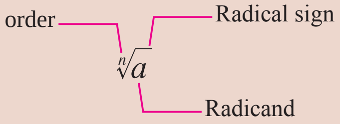
Notation

The general form of a surd is $\sqrt[n]{a}$

$\sqrt{}$ is called the *radical sign*

n is called the *order* of the radical.

a is called the *radicand*.



2.5.1 Index Form of a Surd

Think and Answer !
$a^{\frac{1}{3}}$ and a^3 differ. why?

The index form of a surd $\sqrt[n]{a}$ is $a^{\frac{1}{n}}$

For example, $\sqrt[5]{8}$ can be written in index form as $\sqrt[5]{8} = (8)^{\frac{1}{5}}$

In the following table, the index form, order and radicand of some surds are given.

Surd	Index Form	Order	Radicand
$\sqrt{5}$	$5^{\frac{1}{2}}$	2	5
$\sqrt[3]{14}$	$(14)^{\frac{1}{3}}$	3	14
$\sqrt[4]{7}$	$7^{\frac{1}{4}}$	4	7
$\sqrt{50}$	$(50)^{\frac{1}{2}}$	2	50
$\sqrt[5]{11}$	$(11)^{\frac{1}{5}}$	5	11

Remark If $\sqrt[n]{a}$ is a surd, then

(i) a is a positive rational number. (ii) $\sqrt[n]{a}$ is an irrational number.

In the table given below both the columns A and B have irrational numbers.

A	B
$\sqrt{5}$	$\sqrt{2+\sqrt{3}}$
$\sqrt[3]{7}$	$\sqrt[3]{5+\sqrt{7}}$
$\sqrt[3]{100}$	$\sqrt[3]{10-\sqrt[3]{3}}$
$\sqrt{12}$	$\sqrt[4]{15+\sqrt{5}}$

The numbers in Column A are surds and the numbers in Column B are irrationals.

Thus, every surd is an irrational number, but every irrational number need not be a surd.

2.5.2 Reduction of a Surd to its Simplest Form

We can reduce a surd to its simplest form.

For example, consider the surd $\sqrt{50}$

$$\text{Now } \sqrt{50} = \sqrt{25\times 2} = \sqrt{25}\,\sqrt{2} = \sqrt{5^2}\,\sqrt{2} = 5\sqrt{2}$$

Thus $5\sqrt{2}$ is the simplest form of $\sqrt{50}$.

2.5.3 Like and Unlike Surds

Surds in their simplest form are called like surds if their order and radicand are the same. Otherwise the surds are called unlike surds.

For example,

(i) $\sqrt{5}$, $4\sqrt{5}$, $-6\sqrt{5}$ are like surds. (ii) $\sqrt{10}$, $\sqrt[3]{3}$, $\sqrt[4]{5}$, $\sqrt[3]{81}$ are unlike surds.

2.5.4 Pure surds

A Surd is called a pure surd if its rational coefficient is unity

For example, $\sqrt{3}$, $\sqrt[3]{5}$, $\sqrt[4]{12}$, $\sqrt{80}$ are pure surds.

2.5.5 Mixed Surds

A Surd is called a mixed if its rational coefficient is other than unity

For example, $2\sqrt{3}$, $5\sqrt[3]{5}$, $3\sqrt[4]{12}$ are mixed surds.

A mixed surd can be converted into a pure surd and a pure surd may or may not be converted into a mixed surd.

For example,

(i) $\sqrt{80} = \sqrt{16\times 5} = 4\sqrt{5}$ (ii) $3\sqrt{2} = \sqrt{3^2\times 2} = \sqrt{9\times 2} = \sqrt{18}$

(iii) $\sqrt{17}$ is a pure surd, but it cannot be converted into a mixed surd.

Laws of Radicals

For positive integers m, n and positive rational numbers a, b we have

(i) $\left(\sqrt[n]{a}\right)^n = a = \sqrt[n]{a^n}$ (ii) $\sqrt[n]{a} \times \sqrt[n]{b} = \sqrt[n]{ab}$

(iii) $\sqrt[m]{\sqrt[n]{a}} = \sqrt[mn]{a} = \sqrt[n]{\sqrt[m]{a}}$ (iv) $\dfrac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\dfrac{a}{b}}$

Using (i) we have $\left(\sqrt{a}\right)^2 = a,\ \sqrt[3]{a^3} = \left(\sqrt[3]{a}\right)^3 = a$

Example 2.11

Convert the following surds into index form.

(i) $\sqrt{7}$ (ii) $\sqrt[4]{8}$ (iii) $\sqrt[3]{6}$ (iv) $\sqrt[8]{12}$

Solution In index form we write the given surds as follows

(i) $\sqrt{7} = 7^{\frac{1}{2}}$ (ii) $\sqrt[4]{8} = 8^{\frac{1}{4}}$ (iii) $\sqrt[3]{6} = 6^{\frac{1}{3}}$ (iv) $\sqrt[8]{12} = (12)^{\frac{1}{8}}$

Example 2.12

Express the following surds in its simplest form.

(i) $\sqrt[3]{32}$ (ii) $\sqrt{64}$ (iii) $\sqrt{243}$ (iv) $\sqrt[3]{256}$

Solution

(i) $\sqrt[3]{32} = \sqrt[3]{8 \times 4} = \sqrt[3]{8} \times \sqrt[3]{4} = \sqrt[3]{2^3} \times \sqrt[3]{4} = 2\sqrt[3]{4}$

(ii) $\sqrt{64} = \sqrt{8^2} = 8$

(iii) $\sqrt{243} = \sqrt{81 \times 3} = \sqrt{81} \times \sqrt{3} = \sqrt{9^2} \times \sqrt{3} = 9\sqrt{3}$

(iv) $\sqrt[3]{256} = \sqrt[3]{64 \times 4} = \sqrt[3]{64} \times \sqrt[3]{4} = \sqrt[3]{4^3} \times \sqrt[3]{4} = 4\sqrt[3]{4}$

Example 2.13

Express the following mixed surds into pure surds.

(i) $16\sqrt{2}$ (ii) $3\sqrt[3]{2}$ (iii) $2\sqrt[4]{5}$ (iv) $6\sqrt{3}$

Solution

(i) $16\sqrt{2} = \sqrt{16^2} \times \sqrt{2}$ $\quad (\because 16 = \sqrt{16^2})$

$= \sqrt{16^2 \times 2} = \sqrt{256 \times 2} = \sqrt{512}$

(ii) $3\sqrt[3]{2} = \sqrt[3]{3^3 \times 2}$ $\quad (\because 3 = \sqrt[3]{3^3})$

$= \sqrt[3]{27 \times 2} = \sqrt[3]{54}$

(iii) $2\sqrt[4]{5} = \sqrt[4]{2^4 \times 5}$ $\quad (\because 2 = \sqrt[4]{2^4})$

$= \sqrt[4]{16 \times 5} = \sqrt[4]{80}$

(iv) $6\sqrt{3} = \sqrt{6^2 \times 3}$ $\quad (\because 6 = \sqrt{6^2})$

$= \sqrt{36 \times 3} = \sqrt{108}$

Example 2.14

Identify whether $\sqrt{32}$ is rational or irrational.

Solution $\sqrt{32} = \sqrt{16 \times 2} = 4\sqrt{2}$

4 is a rational number and $\sqrt{2}$ is an irrational number.

$\therefore 4\sqrt{2}$ is an irrational number and hence $\sqrt{32}$ is an irrational number.

Example 2.15

Identify whether the following numbers are rational or irrational.

(i) $3+\sqrt{3}$ (ii) $(4+\sqrt{2})-(4-\sqrt{3})$ (iii) $\frac{\sqrt{18}}{2\sqrt{2}}$ (iv) $\sqrt{19}-(2+\sqrt{19})$
(v) $\frac{2}{\sqrt{3}}$ (vi) $\sqrt{12}\times\sqrt{3}$

Solution

(i) $3+\sqrt{3}$

3 is a rational number and $\sqrt{3}$ is irrational. Hence, $3+\sqrt{3}$ is irrational.

(ii) $(4+\sqrt{2})-(4-\sqrt{3})$

$= 4+\sqrt{2}-4+\sqrt{3} = \sqrt{2}+\sqrt{3}$, is irrational.

(iii) $\frac{\sqrt{18}}{2\sqrt{2}} = \frac{\sqrt{9\times 2}}{2\sqrt{2}} = \frac{\sqrt{9}\times\sqrt{2}}{2\sqrt{2}} = \frac{3}{2}$, is rational.

(iv) $\sqrt{19}-(2+\sqrt{19}) = \sqrt{19}-2-\sqrt{19} = -2$, is rational.

(v) $\frac{2}{\sqrt{3}}$ here 2 is rational and $\sqrt{3}$ is irrational. Hence, $\frac{2}{\sqrt{3}}$ is irrational.

(vi) $\sqrt{12}\times\sqrt{3} = \sqrt{12\times 3} = \sqrt{36} = 6$, is rational.

2.6 Four Basic Operations on Surds

2.6.1 Addition and Subtraction of Surds

Like surds can be added and subtracted.

Example 2.16

Simplify

(i) $10\sqrt{2}-2\sqrt{2}+4\sqrt{32}$

(ii) $\sqrt{48}-3\sqrt{72}-\sqrt{27}+5\sqrt{18}$

(iii) $\sqrt[3]{16}+8\sqrt[3]{54}-\sqrt[3]{128}$

Solution

(i) $10\sqrt{2} - 2\sqrt{2} + 4\sqrt{32}$

$= 10\sqrt{2} - 2\sqrt{2} + 4\sqrt{16 \times 2}$

$= 10\sqrt{2} - 2\sqrt{2} + 4 \times 4 \times \sqrt{2}$

$= (10 - 2 + 16)\sqrt{2} = 24\sqrt{2}$

(ii) $\sqrt{48} - 3\sqrt{72} - \sqrt{27} + 5\sqrt{18}$

$= \sqrt{16 \times 3} - 3\sqrt{36 \times 2} - \sqrt{9 \times 3} + 5\sqrt{9 \times 2}$

$= \sqrt{16}\sqrt{3} - 3\sqrt{36}\sqrt{2} - \sqrt{9}\sqrt{3} + 5\sqrt{9}\sqrt{2}$

$= 4\sqrt{3} - 18\sqrt{2} - 3\sqrt{3} + 15\sqrt{2}$

$= (-18 + 15)\sqrt{2} + (4 - 3)\sqrt{3} = -3\sqrt{2} + \sqrt{3}$

(iii) $\sqrt[3]{16} + 8\sqrt[3]{54} - \sqrt[3]{128}$

$= \sqrt[3]{8 \times 2} + 8\sqrt[3]{27 \times 2} - \sqrt[3]{64 \times 2}$

$= \sqrt[3]{8}\sqrt[3]{2} + 8\sqrt[3]{27}\sqrt[3]{2} - \sqrt[3]{64}\sqrt[3]{2}$

$= 2\sqrt[3]{2} + 8 \times 3 \times \sqrt[3]{2} - 4\sqrt[3]{2}$

$= 2\sqrt[3]{2} + 24\sqrt[3]{2} - 4\sqrt[3]{2}$

$= (2 + 24 - 4)\sqrt[3]{2} = 22\sqrt[3]{2}$

2.6.2 Multiplication of Surds

Product of two like surds can simplified using the following law.

$$\sqrt[n]{a} \times \sqrt[n]{b} = \sqrt[n]{ab}$$

Example 2.17

Multiply (i) $\sqrt[3]{13} \times \sqrt[3]{5}$ (ii) $\sqrt[4]{32} \times \sqrt[4]{8}$

Solution

(i) $\sqrt[3]{13} \times \sqrt[3]{5} = \sqrt[3]{13 \times 5} = \sqrt[3]{65}$

(ii) $\sqrt[4]{32} \times \sqrt[4]{8} = \sqrt[4]{32 \times 8}$

$= \sqrt[4]{2^5 \times 2^3} = \sqrt[4]{2^8} = \sqrt[4]{2^4 \times 2^4} = 2 \times 2 = 4$

2.6.3 Division of Surds

Like surds can be divided using the law

$$\frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\frac{a}{b}}$$

Example 2.18

Simplify (i) $15\sqrt{54} \div 3\sqrt{6}$ (ii) $\sqrt[3]{128} \div \sqrt[3]{64}$

Solution

(i) $15\sqrt{54} \div 3\sqrt{6}$

$$= \frac{15\sqrt{54}}{3\sqrt{6}} = 5\sqrt{\frac{54}{6}} = 5\sqrt{9} = 5 \times 3 = 15$$

(ii) $\sqrt[3]{128} \div \sqrt[3]{64}$

$$= \frac{\sqrt[3]{128}}{\sqrt[3]{64}} = \sqrt[3]{\frac{128}{64}} = \sqrt[3]{2}$$

Note When the order of the surds are different, we convert them to the same order and then multiplication or division is carried out.

Result $\sqrt[n]{a} = \sqrt[m]{a^{\frac{m}{n}}}$

For example, (i) $\sqrt[3]{5} = \sqrt[12]{5^{\frac{12}{3}}} = \sqrt[12]{5^4}$ (ii) $\sqrt[4]{11} = \sqrt[8]{11^{\frac{8}{4}}} = \sqrt[8]{11^2}$

2.6.4 Comparison of Surds

Irrational numbers of the same order can be compared. Among the irrational numbers of same order, the greatest irrational number is the one with the largest radicand.

If the order of the irrational numbers are not the same, we first convert them to the same order. Then, we just compare the radicands.

Example 2.19

Convert the irrational numbers $\sqrt{3}$, $\sqrt[3]{4}$, $\sqrt[4]{5}$ to the same order.

Solution The orders of the given irrational numbers are 2, 3 and 4.

LCM of 2, 3 and 4 is 12

$\sqrt{3} = \sqrt[12]{3^6} = \sqrt[12]{729}$

$\sqrt[3]{4} = \sqrt[12]{4^4} = \sqrt[12]{256}$

$\sqrt[4]{5} = \sqrt[12]{5^3} = \sqrt[12]{125}$

Example 2.20

Which is greater ? $\sqrt[4]{5}$ or $\sqrt[3]{4}$

Solution The orders of the given irrational numbers are 3 and 4.

We have to convert each of the irrational number to an irrational number of the same order.

LCM of 3 and 4 is 12. Now we convert each irrational number as of order 12.

$\sqrt[4]{5} = \sqrt[12]{5^3} = \sqrt[12]{125}$

$\sqrt[3]{4} = \sqrt[12]{4^4} = \sqrt[12]{256}$

$\therefore \sqrt[12]{256} > \sqrt[12]{125} \quad \Rightarrow \sqrt[3]{4} > \sqrt[4]{5}$

Example 2.21

Write the irrational numbers $\sqrt[3]{2}$, $\sqrt[4]{4}$, $\sqrt{3}$ in

(i) ascending order (ii) descending order

Solution The orders of the irrational numbers $\sqrt[3]{2}, \sqrt[4]{4}$ and $\sqrt{3}$ are 3, 4 and 2 respectively LCM of 2, 3, and 4 is 12. Now, we convert each irrational number as of order 12.

$\sqrt[3]{2} = \sqrt[12]{2^4} = \sqrt[12]{16}$

$\sqrt[4]{4} = \sqrt[12]{4^3} = \sqrt[12]{64}$

$\sqrt{3} = \sqrt[12]{3^6} = \sqrt[12]{729}$

$\therefore$ Ascending order: $\sqrt[3]{2}, \sqrt[4]{4}, \sqrt{3}$

Descending order: $\sqrt{3}, \sqrt[4]{4}, \sqrt[3]{2}$.

Exercise 2.5

1. Identify which of the following are surds and which are not with reasons.

   (i) $\sqrt{8} \times \sqrt{6}$ (ii) $\sqrt{90}$ (iii) $\sqrt{180} \times \sqrt{5}$ (iv) $4\sqrt{5} \div \sqrt{8}$ (v) $\sqrt[3]{4} \times \sqrt[3]{16}$

2. Simplify

   (i) $(10 + \sqrt{3})(2 + \sqrt{5})$ (ii) $(\sqrt{5} + \sqrt{3})^2$

   (iii) $(\sqrt{13} - \sqrt{2})(\sqrt{13} + \sqrt{2})$ (iv) $(8 + \sqrt{3})(8 - \sqrt{3})$

3. Simplify the following.

   (i) $5\sqrt{75} + 8\sqrt{108} - \frac{1}{2}\sqrt{48}$ (ii) $7\sqrt[3]{2} + 6\sqrt[3]{16} - \sqrt[3]{54}$

   (iii) $4\sqrt{72} - \sqrt{50} - 7\sqrt{128}$ (iv) $2\sqrt[3]{40} + 3\sqrt[3]{625} - 4\sqrt[3]{320}$

4. Express the following surds in its simplest form.

   (i) $\sqrt[3]{108}$ (ii) $\sqrt{98}$ (iii) $\sqrt{192}$ (iv) $\sqrt[4]{625}$

5. Express the following as pure surds.

(i) $6\sqrt{5}$ (ii) $5\sqrt[3]{4}$ (iii) $3\sqrt[4]{5}$ (iv) $\frac{3}{4}\sqrt{8}$

6. Simplify the following.

(i) $\sqrt{5}\times\sqrt{18}$ (ii) $\sqrt[3]{7}\times\sqrt[3]{8}$ (iii) $\sqrt[4]{8}\times\sqrt[4]{12}$ (iv) $\sqrt[3]{3}\times\sqrt[6]{5}$

(v) $3\sqrt{35}\div 2\sqrt{7}$ (vi) $\sqrt[4]{48}\div\sqrt[8]{72}$

7. Which is greater ?

(i) $\sqrt{2}$ or $\sqrt[3]{3}$ (ii) $\sqrt[3]{3}$ or $\sqrt[4]{4}$ (iii) $\sqrt{3}$ or $\sqrt[4]{10}$

8. Arrange in descending and ascending order.

(i) $\sqrt[3]{4}, \sqrt[4]{5}, \sqrt{3}$ (ii) $\sqrt[3]{2}, \sqrt[4]{4}, \sqrt[3]{4}$ (iii) $\sqrt[3]{2}, \sqrt[6]{3}, \sqrt[9]{4}$

2.7 Rationalization of Surds

Rationalization of Surds

When the denominator of an expression contains a term with a square root or a number under radical sign, the process of converting into an equivalent expression whose denominator is a rational number is called rationalizing the denominator.

If the product of two irrational numbers is rational, then each one is called the **rationalizing factor** of the other.

Let a and b be integers and x, y be positive integers. Then

Remark

(i) $(a+\sqrt{x})$ and $(a-\sqrt{x})$ are rationalizing factors of each other.

(ii) $(a+b\sqrt{x})$ and $(a-b\sqrt{x})$ are rationalizing factors of each other.

(iii) $\sqrt{x}+\sqrt{y}$ and $\sqrt{x}-\sqrt{y}$ are rationalizing factors of each other.

(iv) $a+\sqrt{b}$ is also called the conjugate of $a-\sqrt{b}$ and $a-\sqrt{b}$ is called the conjugate of $a+\sqrt{b}$.

(v) For rationalizing the denominator of a number, we multiply its numerator and denominator by its rationalizing factor.

Example 2.22

Rationalize the denominator of $\frac{2}{\sqrt{3}}$

Solution Multiplying the numerator and denominator of the given number by $\sqrt{3}$, we get

$$\frac{2}{\sqrt{3}} = \frac{2}{\sqrt{3}}\times\frac{\sqrt{3}}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$$

Example 2.23

Rationalize the denominator of $\frac{1}{5+\sqrt{3}}$

Solution The denominator is $5+\sqrt{3}$. Its conjugate is $5-\sqrt{3}$ or the rationalizing factor is $5-\sqrt{3}$.

$$\frac{1}{5+\sqrt{3}} = \frac{1}{5+\sqrt{3}} \times \frac{5-\sqrt{3}}{5-\sqrt{3}}$$

$$= \frac{5-\sqrt{3}}{5^2-(\sqrt{3})^2} = \frac{5-\sqrt{3}}{25-3} = \frac{5-\sqrt{3}}{22}$$

Example 2.24

Simplify $\frac{1}{8-2\sqrt{5}}$ by rationalizing the denominator.

Solution Here the denominator is $8-2\sqrt{5}$. The rationalizing factor is $8+2\sqrt{5}$

$$\frac{1}{8-2\sqrt{5}} = \frac{1}{8-2\sqrt{5}} \times \frac{8+2\sqrt{5}}{8+2\sqrt{5}}$$

$$= \frac{8+2\sqrt{5}}{8^2-(2\sqrt{5})^2} = \frac{8+2\sqrt{5}}{64-20}$$

$$= \frac{8+2\sqrt{5}}{44} = \frac{2(4+\sqrt{5})}{44} = \frac{4+\sqrt{5}}{22}$$

Example 2.25

Simplify $\frac{1}{\sqrt{3}+\sqrt{5}}$ by rationalizing the denominator.

Solution Here the denominator is $\sqrt{3}+\sqrt{5}$. So, the rationalizing factor is $\sqrt{3}-\sqrt{5}$

$$\frac{1}{\sqrt{3}+\sqrt{5}} = \frac{1}{\sqrt{3}+\sqrt{5}} \times \frac{\sqrt{3}-\sqrt{5}}{\sqrt{3}-\sqrt{5}}$$

$$= \frac{\sqrt{3}-\sqrt{5}}{(\sqrt{3})^2-(\sqrt{5})^2} = \frac{\sqrt{3}-\sqrt{5}}{3-5}$$

$$= \frac{\sqrt{3}-\sqrt{5}}{-2} = \frac{\sqrt{5}-\sqrt{3}}{2}$$

Example 2.26

If $\frac{\sqrt{7}-1}{\sqrt{7}+1}+\frac{\sqrt{7}+1}{\sqrt{7}-1} = a+b\sqrt{7}$, find the values of a and b.

Solution $\frac{\sqrt{7}-1}{\sqrt{7}+1}+\frac{\sqrt{7}+1}{\sqrt{7}-1} = \frac{\sqrt{7}-1}{\sqrt{7}+1}\times\frac{\sqrt{7}-1}{\sqrt{7}-1}+\frac{\sqrt{7}+1}{\sqrt{7}-1}\times\frac{\sqrt{7}+1}{\sqrt{7}+1}$

$$= \frac{(\sqrt{7}-1)^2}{(\sqrt{7})^2-1} + \frac{(\sqrt{7}+1)^2}{(\sqrt{7})^2-1}$$

$$= \frac{7+1-2\sqrt{7}}{7-1} + \frac{7+1+2\sqrt{7}}{7-1}$$

$$= \frac{8-2\sqrt{7}}{6} + \frac{8+2\sqrt{7}}{6}$$

$$= \frac{8-2\sqrt{7}+8+2\sqrt{7}}{6}$$

$$= \frac{16}{6} = \frac{8}{3} + 0\sqrt{7}$$

$$\therefore \frac{8}{3} + 0\sqrt{7} = a + b\sqrt{7} \implies a = \frac{8}{3},\ b = 0.$$

Exmaple 2.27

If $x = 1+\sqrt{2}$, find $\left(x - \frac{1}{x}\right)^2$

Solution $x = 1 + \sqrt{2}$

$$\implies \quad \frac{1}{x} = \frac{1}{1+\sqrt{2}}$$

$$= \frac{1}{1+\sqrt{2}} \times \frac{1-\sqrt{2}}{1-\sqrt{2}}$$

$$= \frac{1-\sqrt{2}}{1-2} = \frac{1-\sqrt{2}}{-1} = -(1-\sqrt{2})$$

$$\therefore x - \frac{1}{x} = (1+\sqrt{2}) - \{-(1-\sqrt{2})\}$$

$$= 1+\sqrt{2}+1-\sqrt{2} = 2$$

Hence, $\left(x - \frac{1}{x}\right)^2 = 2^2 = 4.$

Exercise 2.6

1. Write the rationalizing factor of the following.

(i) $3\sqrt{2}$ (ii) $\sqrt{7}$ (iii) $\sqrt{75}$ (iv) $2\sqrt[3]{5}$

(v) $5-4\sqrt{3}$ (vi) $\sqrt{2}+\sqrt{3}$ (vii) $\sqrt{5}-\sqrt{2}$ (viii) $2+\sqrt{3}$

2. Rationalize the denominator of the following

(i) $\frac{3}{\sqrt{5}}$ (ii) $\frac{2}{3\sqrt{3}}$ (iii) $\frac{1}{\sqrt{12}}$ (iv) $\frac{2\sqrt{7}}{\sqrt{11}}$ (v) $\frac{3\sqrt[3]{5}}{\sqrt[3]{9}}$

3. Simplify by rationalizing the denominator.

(i) $\frac{1}{11+\sqrt{3}}$ (ii) $\frac{1}{9+3\sqrt{5}}$ (iii) $\frac{1}{\sqrt{11}+\sqrt{13}}$ (iv) $\frac{\sqrt{5}+1}{\sqrt{5}-1}$ (v) $\frac{3-\sqrt{3}}{2+5\sqrt{3}}$

4. Find the values of the following upto 3 decimal places. Given that $\sqrt{2} \approx 1.414$, $\sqrt{3} \approx 1.732$, $\sqrt{5} \approx 2.236$, $\sqrt{10} \approx 3.162$.

(i) $\frac{1}{\sqrt{2}}$ (ii) $\frac{6}{\sqrt{3}}$ (iii) $\frac{5-\sqrt{3}}{\sqrt{3}}$ (iv) $\frac{\sqrt{10}-\sqrt{5}}{\sqrt{2}}$

(v) $\frac{3-\sqrt{5}}{3+2\sqrt{5}}$ (vi) $\frac{\sqrt{5}+\sqrt{2}}{\sqrt{5}-\sqrt{2}}$ (vii) $\frac{\sqrt{3}+1}{\sqrt{3}-1}$ (viii) $\frac{1}{\sqrt{10}+\sqrt{5}}$

5. If $\frac{5+\sqrt{6}}{5-\sqrt{6}} = a + b\sqrt{6}$ find the values of a and b.

6. If $\frac{(\sqrt{3}+1)^2}{4-2\sqrt{3}} = a + b\sqrt{3}$ find the values of a and b.

7. If $\frac{\sqrt{5}+1}{\sqrt{5}-1} + \frac{\sqrt{5}-1}{\sqrt{5}+1} = a + b\sqrt{5}$, find the values of a and b.

8. If $\frac{4+\sqrt{5}}{4-\sqrt{5}} - \frac{4-\sqrt{5}}{4+\sqrt{5}} = a + b\sqrt{5}$, find the values of a and b.

9. If $x = 2 + \sqrt{3}$, find the values of $x^2 + \frac{1}{x^2}$.

10. If $x = \sqrt{3} + 1$, find the values of $\left(x - \frac{2}{x}\right)^2$.

2.8 Division Algorithm

A series of well defined steps which gives a procedure for solving a problem is called an algorithm. In this section we state an important property of integers called the division algorithm.

As we know from our earlier classes, when we divide one integer by another non-zero integer, we get an integer quotient and a remainder (generally a rational number). Then we write

$$\text{Fraction} = \text{Quotient} + \frac{\text{remainder}}{\text{divisor}}$$

For example, $$\frac{13}{5} = 2 + \frac{3}{5} \qquad (1)$$

We can rephrase this division, totally in terms of integers, without reference to the division operation.

$13 = 2(5) + 3$

We observe that this expression is obtained by multiplying (1) by the divisor 5. We refer to this way of writing a division of integers as the division algorithm.

> If a and b are any two positive integers, then there exist two non-negative integers q and r such that $a = bq + r$, $0 \le r < b$.

In the above statement q (or) r can be zero.

Example 2.28

Using division algorithm find the quotient and remainder of the following pairs.

(i) 19, 5 (ii) 3, 13 (iii) 30, 6

Solution

(i) 19, 5

We write the given pair in the form $a = bq + r$, $0 \le r < b$ as follows.

19 = 5(3) + 4 [5 divides 19 three time and leaves the remainder 4]

$\therefore$ quotient = 3; remainder = 4

(ii) 3, 13

We write the given pair in the form $a = bq + r$, $0 \le r < b$ as follows.

3 = 13(0) + 3

$\therefore$ quotient = 0; remainder = 3

(iii) 30, 6

We write the given pair 30, 6 in the form $a = bq + r$, $0 \le r < b$ as follows.

30 = 6(5) + 0 [6 divides 30 five times and leaves the reaminder 0]

$\therefore$ quotient = 5; remainder = 0

Exercise 2.7

1. Using division algorithm, find the quotient and remainder of the following pairs.

(i) 10, 3 (ii) 5, 12 (iii) 27, 3

Points to Remember

- When the decimal expansion of $\frac{p}{q}$, $q \ne 0$ terminates i.e., comes to an end, the decimal is called a terminating decimal.
- In the decimal expansion of $\frac{p}{q}$, $q \ne 0$ when the remainder is not zero, we have a repeating (recurring) block of digits in the quotient. In this case, the decimal expansion is called non-terminating and recurring.

- If a rational number $\frac{p}{q}$, $q \neq 0$ can be expressed in the form $\frac{p}{2^m \times 5^n}$, where $p \in \mathbb{Z}$ and, $m, n \in \mathbb{W}$ then the rational number will have a terminating decimal. Otherwise, the rational number will have a non-terminating repeating (recurring) decimal.
- A rational number can be expressed by either a terminating or a non-terminating repeating decimal.
- An irrational number is a non-terminating and non-recurring decimal, i.e., it cannot be written in the form $\frac{p}{q}$, where p and q are both integers and $q \neq 0$.
- The union of all rational numbers and all irrational numbers is called the set of real numbers.
- Every real number is either a rational number or an irrational number.
- If a real number is not a rational number, then it must be an irrational number.
- The sum or difference of a rational number and an irrational number is always an irrational number
- The product or quotient of non-zero rational number and an irrational number is also an irrational number.
- Sum, difference, product or quotient of two irrational numbers need not be irrational. The result may be rational or irrational.
- If 'a' is a positive rational number and n is a positive integer such that $\sqrt[n]{a}$ is an irrational number, then $\sqrt[n]{a}$ is called a 'surd' or a 'radical'.
- For positive integers m, n and positive rational numbers a, b we have

  (i) $\left(\sqrt[n]{a}\right)^n = a = \sqrt[n]{a^n}$ (ii) $\sqrt[n]{a} \times \sqrt[n]{b} = \sqrt[n]{ab}$

  (iii) $\sqrt[m]{\sqrt[n]{a}} = \sqrt[mn]{a} = \sqrt[n]{\sqrt[m]{a}}$ (iv) $\frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\frac{a}{b}}$
- When the denominator of an expression contains a term with a square root or a number under radical sign, the process of converting to an equivalent expression whose denominator is a rational number is called rationalizing the denominator.
- If the product of two irrational numbers is rational, then each one is called the rationalizing factor of the other.
- If a and b are any two positive integers, there exist two non-negative integers q and r such that $a = bq + r$, $0 \leq r < b$. (Division Algorithm)

3 SCIENTIFIC NOTATIONS OF REAL NUMBERS AND LOGARITHMS

Seeing there is nothing that is so troublesome to mathematical practice, nor that doth more molest and hinder calculators, than the multiplications, divisions, square and cubical extractions of great numbers....
I began therefore to consider in my mind by what certain and ready art I might remove those hindrances

- JOHN NAPIER

JOHN NAPIER
(1550 - 1617)

John Napier was born in the Tower of Merciston, which is now at the center of Napier University's Merchiston campus, in 1550. Napier, who is credited with the invention of logarithms, only considered the study of mathematics as a hobby. Napier is placed within a short lineage of mathematical thinkers beginning with Archimedes and more recent geniuses, Sir Issac Newton and Albert Einstein.

Main Targets

- To represent the number in Scientific Notation.
- To convert exponential form to logarithmic form and vice-versa.
- To understand the rules of logarithms.
- To apply the rules and to use logarithmic table.

3.1 Scientific Notation

Scientists, engineers and technicians use scientific notations when working with very large or very small numbers. The speed of light is 29,900,000,000 centimeter per second; the distance of sun from earth is about 92,900,000 mile; the mass of an electron is $0 \cdot 000549$ atomic mass units. It is easier to express these numbers in a shorter way called *Scientific Notation*, thus avoiding the writing of many zeros and transposition errors.

For example,

$$29{,}900{,}000{,}000 = 299 \times 10^{8} = 2.99 \times 10^{10}$$

$$92{,}900{,}000 = 929 \times 10^{5} = 9.29 \times 10^{7}$$

$$0 \cdot 000549 = \frac{549}{1000000} = \frac{5.49}{10000}$$

$$= 5.49 \times 10^{-4}$$

That is, the very large or very small numbers are expressed as the product of a decimal number $1 \leq a < 10$ and some integral power of 10.

Key Concept **Scientific Notation**

A number N is in *scientific notation* when it is expressed as the product of a decimal number between 1 and 10 and some integral power of 10.

$$N = a \times 10^{n}, \text{ where } 1 \leq a < 10 \text{ and } n \text{ is an integer.}$$

To transform numbers from decimal notation to scientific notation, the laws of exponents form the basis for calculations using powers. Let m and n be natural numbers and a is a real number. The laws of exponents are given below:

(i) $a^m \times a^n = a^{m+n}$ (Product law)

(ii) $\dfrac{a^m}{a^n} = a^{m-n}$ (Quotient law)

(iii) $(a^m)^n = a^{mn}$ (Power law)

(iv) $a^m \times b^m = (a \times b)^m$ (Combination law)

For $a \neq 0$, we define $a^{-m} = \dfrac{1}{a^m}$, and $a^0 = 1$.

3.1.1 Writing a Number in Scientific Notation

The steps for converting a number to scientific notation are as follows:

Step 1: Move the decimal point so that there is only one non - zero digit to its left.

Step 2: Count the number of digits between the old and new decimal point. This gives n, the power of 10.

Step 3: If the decimal is shifted to the left, the exponent n is positive. If the decimal is shifted to the right, the exponent n is negative.

Example 3.1

Express 9781 in scientific notation.

Solution In integers, the decimal point at the end is usually omitted.

9 7 8 1 .
3 2 1

The decimal point is to be moved 3 places to the left of its original position. So the power of 10 is 3.

$\therefore \ 9781 = 9.781 \times 10^3$

Example 3.2

Express $0 \cdot 000432078$ in scientific notation.

Solution 0 . 0 0 0 4 3 2 0 7 8
(1 2 3 4)

The decimal point is to be moved four places to the right of its original position. So the power of 10 is – 4

$\therefore\ 0.000432078 = 4.32078 \times 10^{-4}$

Remark Observe that while converting a given number into the scientific notation, if the decimal point is moved *p* places to the left, then this movement is compensated by the factor 10^{p} ; and if the decimal point is moved *r* places to the right, then this movement is compensated by the factor 10^{-r}.

Example 3.3

Write the following numbers in scientific notation.

(i) 9345 (ii) 205852 (iii) 3449098.96

(iv) 0.0063 (v) 0.00008035 (vi) 0.000108

Solution

(i) 9345 = 9 3 4 5 . = 9.345×10^{3}, n = 3 because the decimal point is shifted (3 2 1) three places to the left.

(ii) 205852 = 2 0 5 8 5 2 . = 2.05852×10^{5}, n = 5 because the decimal (5 4 3 2 1) point is shifted five places to the left .

(iii) 3449098.96 = 3 4 4 9 0 9 8 . 9 6 = 3.44909896×10^{6}, n = 6 because (6 5 4 3 2 1) the decimal point is shifted six places to the left.

(iv) 0.0063 = 0 . 0 0 6 3 = 6.3×10^{-3}, n = –3 because the decimal point is shifted (1 2 3) three places to the right.

(v) 0.00008035 = 0 . 0 0 0 0 8 0 3 5 = 8.035×10^{-5}, n = –5 because (1 2 3 4 5) the decimal point is shifted five places to the right.

(vi) 0.000108 = 0 . 0 0 0 1 0 8 = 1.08×10^{-4}, n = –4 because the decimal (1 2 3 4) point is shifted four places to the right.

3.2 Converting Scientific Notation to Decimal Form

Often, numbers in scientific notation need to be written in decimal form. To convert scientific notation to integers we have to follow these steps.

Step 1 : Write the decimal number.

Step 2 : Move the decimal point the number of places specified by the power of ten: to the right if positive, to the left if negative. Add zeros if necessary.

Step 3 : Rewrite the number in decimal form.

Example 3.4

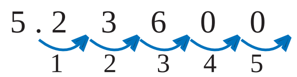
Write the following numbers in decimal form.

(i) 5.236×10^5 (ii) 1.72×10^9 (iii) 6.415×10^{-6} (iv) 9.36×10^{-9}

Solution

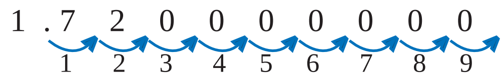
(i) 5.236

$5.236 \times 10^5 = 523600$



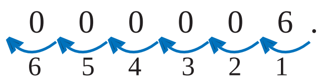
(ii) 1.72

$1.72 \times 10^9 = 1720000000$



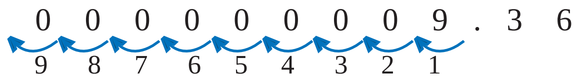
(iii) 6.415

$6.415 \times 10^{-6} = 0.000006415$



(iv) 9.36

$9.36 \times 10^{-9} = 0.00000000936$



3.2.1 Multiplication and Division in Scientific Notation

One can find the product or quotient of very large(googolplex) or very small numbers easily in scientific notion.

Example 3.5

Write the following in scientific notation.

(i) $(4000000)^3$ (ii) $(5000)^4 \times (200)^3$

(iii) $(0.00003)^5$ (iv) $(2000)^2 \div (0.0001)^4$

Solution

(i) First we write the number (within the brackets) in scientific notation.

$$4000000 = 4.0 \times 10^6$$

Now, raising to the power 3 on both sides we get,

$$\therefore (4000000)^3 = (4.0\times10^6)^3 = (4.0)^3\times(10^6)^3$$

$$= 64\times10^{18} \quad = 6.4\times10^1\times10^{18} = 6.4\times10^{19}$$

(ii) In scientific notation,

$5000 = 5.0\times10^3$ and $200 = 2.0\times10^2$.

$$\therefore (5000)^4\times(200)^3 = (5.0\times10^3)^4\times(2.0\times10^2)^3$$

$$= (5.0)^4\times(10^3)^4\times(2.0)^3\times(10^2)^3$$

$$= 625\times10^{12}\times8\times10^6 = 5000\times10^{18}$$

$$= 5.0\times10^3\times10^{18} = 5.0\times10^{21}$$

(iii) In scientific notation, $0.00003 = 3.0\times10^{-5}$

$$\therefore (0.00003)^5 = (3.0\times10^{-5})^5 = (3.0)^5\times(10^{-5})^5$$

$$= 243\times10^{-25} \quad = 2.43\times10^2\times10^{-25} = 2.43\times10^{-23}$$

(iv) In scientific notation,

$2000 = 2.0\times10^3$ and $0.0001 = 1.0\times10^{-4}$

$$\therefore (2000)^2\div(0.0001)^4 = \frac{(2.0\times10^3)^2}{(1.0\times10^{-4})^4} = \frac{(2.0)^2\times(10^3)^2}{(1.0)^4\times(10^{-4})^4}$$

$$= \frac{4\times10^6}{10^{-16}} = 4.0\times10^{6-(-16)} = 4.0\times10^{22}$$

Exercise 3.1

1. Represent the following numbers in the scientific notation.

   (i) 749300000000 (ii) 13000000 (iii) 105003

   (iv) 543600000000000 (v) 0.0096 (vi) 0.0000013307

   (vii) 0.0000000022 (viii) 0.0000000000009

2. Write the following numbers in decimal form.

   (i) 3.25×10^{-6} (ii) 4.134×10^{-4} (iii) 4.134×10^4

   (iv) 1.86×10^7 (v) 9.87×10^9 (vi) 1.432×10^{-9}

3. Represent the following numbers in scientific notation.

   (i) $(1000)^2\times(20)^6$ (ii) $(1500)^3\times(0.0001)^2$

   (iii) $(16000)^3\div(200)^4$ (iv) $(0.003)^7\times(0.0002)^5\div(0.001)^3$

   (v) $(11000)^3\times(0.003)^2\div(30000)$

3.3 Logarithms

Logarithms were originally developed to simplify complex arithmetic calculations. They were designed to transform multiplicative processes into additive ones. Before the advent of calculators, logarithms had great use in multiplying and dividing numbers with many digits since adding exponents was less work than multiplying numbers. Now they are important in nuclear work because many laws governing physical behavior are in exponential form. Examples are radioactive decay, gamma absorption, and reactor power changes on a stable period.

To introduce the notation of logarithm, we shall first introduce the exponential notation for real numbers.

3.3.1 Exponential Notation

Let a be a positive number. We have already introduced the notation a^x, where x is an integer.

We know that $a^{\frac{1}{n}}$ is a positive number whose nth power is equal to a. Now we can see how to define $a^{\frac{p}{q}}$, where p is an integer and q is a positive integer.

Notice that $\frac{p}{q} = p \times \frac{1}{q}$, so if the power rule is to hold then

$$a^{\frac{p}{q}} = \left(a^{\frac{1}{q}}\right)^p = (a^p)^{\frac{1}{q}}$$

So, we define $a^{\frac{p}{q}} = (\sqrt[q]{a})^p$. For example, $8^{\frac{3}{5}} = (\sqrt[5]{8})^3$ and $5^{\frac{-7}{3}} = (\sqrt[3]{5})^{-7}$

Thus, if $a > 0$, we have been able to give suitable meaning to a^x for all rational numbers x. Also for $a > 0$ it is possible to extend the definition of a^x to irrational exponents x so that the laws of exponents remain valid. We will not show how a^x may be defined for irrational x because the definition of a^x requires some advanced topics in mathematics. So, we accept now that, for any $a > 0$, a^x is defined for all real numbers x and satifies the laws of exponents.

3.3.2 Logarithmic Notation

If $a > 0$, $b > 0$ and $a \neq 1$, then the logarithm of b to the base a is the number to which a to be raised to obtain b.

Key Concept	**Logarithmic Notation**

Let a be a positive number other than 1 and let x be a real number (positive, negative, or zero). If $a^x = b$, we say that the exponent x is the logarithm of b to the base a and we write $x = \log_a b$.

$x = \log_a b$ is the logarithmic form of the exponential form $b = a^x$. In both the forms, the base is same.

For example,

Exponential Form	Logarithmic Form
$2^4 = 16$	$\log_2 16 = 4$
$8^{\frac{1}{3}} = 2$	$\log_8 2 = \frac{1}{3}$
$4^{-\frac{3}{2}} = \frac{1}{8}$	$\log_4\left(\frac{1}{8}\right) = -\frac{3}{2}$

Example 3.6

Change the following from logarithmic form to exponential form.

(i) $\log_4 64 = 3$ (ii) $\log_{16} 2 = \frac{1}{4}$ (iii) $\log_5\left(\frac{1}{25}\right) = -2$ iv) $\log_{10} 0.1 = -1$

Solution

(i) $\log_4 64 = 3 \implies 4^3 = 64$

(ii) $\log_{16} 2 = \frac{1}{4} \implies (16)^{\frac{1}{4}} = 2$

(iii) $\log_5\left(\frac{1}{25}\right) = -2 \implies (5)^{-2} = \frac{1}{25}$

(iv) $\log_{10} 0.1 = -1 \implies (10)^{-1} = 0.1$

Example 3.7

Change the following from exponential form to logarithmic form.

(i) $3^4 = 81$ (ii) $6^{-4} = \frac{1}{1296}$ (iii) $\left(\frac{1}{81}\right)^{\frac{3}{4}} = \frac{1}{27}$

(iv) $(216)^{\frac{1}{3}} = 6$ (v) $(13)^{-1} = \frac{1}{13}$

Solution

(i) $3^4 = 81 \implies \log_3 81 = 4$

(ii) $6^{-4} = \frac{1}{1296} \implies \log_6\left(\frac{1}{1296}\right) = -4$

(iii) $\left(\frac{1}{81}\right)^{\frac{3}{4}} = \frac{1}{27} \implies \log_{\frac{1}{81}}\left(\frac{1}{27}\right) = \frac{3}{4}$

(vi) $(216)^{\frac{1}{3}} = 6 \implies \log_{216} 6 = \frac{1}{3}$

(v) $(13)^{-1} = \frac{1}{13} \implies \log_{13}\left(\frac{1}{13}\right) = -1$

Example 3.8

Evaluate (i) $\log_8 512$ (ii) $\log_{27} 9$ (iii) $\log_{16}\left(\frac{1}{32}\right)$

Solution

(i) Let $x = \log_8 512$. Then

$8^x = 512$ (exponential form)

$8^x = 8^3 \implies x = 3$

$\therefore \log_8 512 = 3$

(ii) Let $x = \log_{27} 9$. Then

$27^x = 9$ (exponential form)

$(3^3)^x = (3)^2$ (convert both sides to base three)

$3^{3x} = 3^2 \implies 3x = 2 \implies x = \frac{2}{3}$

$\therefore \log_{27} 9 = \frac{2}{3}$

(iii) Let $x = \log_{16}\left(\frac{1}{32}\right)$. Then

$16^x = \frac{1}{32}$ (exponential form)

$(2^4)^x = \frac{1}{(2)^5}$ (convert both sides to base two)

$2^{4x} = 2^{-5} \implies 4x = -5 \implies x = -\frac{5}{4}$

$\therefore \log_{16}\left(\frac{1}{32}\right) = -\frac{5}{4}$

Example 3.9

Solve the equations (i) $\log_5 x = -3$ (ii) $x = \log_{\frac{1}{4}} 64$ (iii) $\log_x 8 = 2$

(iv) $x + 3\log_8 4 = 0$ (v) $\log_x 7^{\frac{1}{6}} = \frac{1}{3}$

Solution

(i) $\log_5 x = -3$

$5^{-3} = x$ (exponential form)

$x = \frac{1}{5^3} \implies x = \frac{1}{125}$

(ii) $x = \log_{\frac{1}{4}} 64$

$\left(\frac{1}{4}\right)^x = 64$ (exponential form)

$\frac{1}{4^x} = 4^3 \implies 4^{-x} = 4^3 \implies x = -3$

(iii) $\log_x 8 = 2$

$x^2 = 8$ (exponential form)

$x = \sqrt{8} = 2\sqrt{2}$

(iv) $x + 3\log_8 4 = 0$

$\Longrightarrow \quad -x = 3\log_8 4 = \log_8 4^3$

$\Longrightarrow \quad -x = \log_8 64 \quad \Longrightarrow (8)^{-x} = 64$ (exponential form)

$\Longrightarrow (8)^{-x} = 8^2 \quad \Longrightarrow x = -2$

(v) $\log_x 7^{\frac{1}{6}} = \frac{1}{3} \quad \Longrightarrow \quad x^{\frac{1}{3}} = 7^{\frac{1}{6}}$ (exponential form)

We write $7^{\frac{1}{6}} = \left(7^{\frac{1}{2}}\right)^{\frac{1}{3}}$. Then $x^{\frac{1}{3}} = \left(7^{\frac{1}{2}}\right)^{\frac{1}{3}}$

$\therefore\ x = 7^{\frac{1}{2}} = \sqrt{7}$

The Rules of Logarithms

1. **Product Rule:** The logarithm of the product of two positive numbers is equal to sum of their logarithms of the same base. That is,

$$\log_a(M \times N) = \log_a M + \log_a N$$

2. **Quotient Rule:** The logarithm of the quotient of two positive numbers is equal to the logarithm of the numerator minus the logarithm of the denominator to the same base. That is,

$$\log_a\left(\frac{M}{N}\right) = \log_a M - \log_a N$$

3. **Power Rule:** The logarithm of a number in exponential form is equal to the logarithm of the number multiplied by its exponent. That is,

$$\log_a (M)^n = n\log_a M$$

4. **Change of Base Rule:** If M, a and b are positive numbers and $a \neq 1$, $b \neq 1$, then

$$\log_a M = (\log_b M) \times (\log_a b)$$

Remark

(i) If a is a positive number and $a \neq 1$, $\log_a a = 1$

(ii) If a is a positive number and $a \neq 1$, $\log_a 1 = 0$

(iii) If a and b are positive numbers $a \neq 1, b \neq 1$ $(\log_a b) \times (\log_b a) = 1$ and $\log_a b = \dfrac{1}{\log_b a}$

(iv) If a and b are positive numbers and $b \neq 1$, $b^{\log_b a} = a$.

(v) If $a > 0$, $\log_a 0$ is undefined.

(vi) If b ,x and y are positive numbers other than 1 then $\log_b x = \log_b y$ if and only if $x = y$.

(vii) We are avoiding 1 in the base of all logarithms because if we consider one such logarithm, say $\log_1 9$ with 1 in the base, then $x = \log_1 9$ would give $1^x = 9$. We know that there is no real number x, such that $1^x = 9$.

Example 3.10

Simplify (i) $\log_5 25 + \log_5 625$ (ii) $\log_5 4 + \log_5\left(\frac{1}{100}\right)$

Solution

(i) $\log_5 25 + \log_5 625 = \log_5(25 \times 625)$ $[\because \log_a(M \times N) = \log_a M + \log_a N]$

$= \log_5(5^2 \times 5^4) = \log_5 5^6 = 6\log_5 5$ $[\because \log_a(M)^n = n\log_a M]$

$= 6(1) = 6$ $[\because \log_a a = 1]$

(ii) $\log_5 4 + \log_5\left(\frac{1}{100}\right) = \log_5\left(4 \times \frac{1}{100}\right)$ $[\because \log_a(M \times N) = \log_a M + \log_a N]$

$= \log_5\left(\frac{1}{25}\right) = \log_5\left(\frac{1}{5^2}\right) = \log_5 5^{-2} = -2\log_5 5$ $[\because \log_a(M)^n = n\log_a M]$

$= -2(1) = -2$ $[\because \log_a a = 1]$

Example 3.11

Simplify $\log_8 128 - \log_8 16$

Solution $\log_8 128 - \log_8 16 = \log_8 \frac{128}{16}$ $[\because \log_a\left(\frac{M}{N}\right) = \log_a M - \log_a N]$

$= \log_8 8 = 1$ $[\because \log_a a = 1]$

Example 3.12

Prove that $\log_{10} 125 = 3 - 3\log_{10} 2$

Solution $3 - 3\log_{10} 2 = 3\log_{10} 10 - 3\log_{10} 2 = \log_{10} 10^3 - \log_{10} 2^3$

$= \log_{10} 1000 - \log_{10} 8 = \log_{10} \frac{1000}{8}$

$= \log_{10} 125$

$\therefore \log_{10} 125 = 3 - 3\log_{10} 2$

Example 3.13

Prove that $\log_3 2 \times \log_4 3 \times \log_5 4 \times \log_6 5 \times \log_7 6 \times \log_8 7 = \frac{1}{3}$

Solution $\log_3 2 \times \log_4 3 \times \log_5 4 \times \log_6 5 \times \log_7 6 \times \log_8 7$

$$= (\log_3 2 \times \log_4 3) \times (\log_5 4 \times \log_6 5) \times (\log_7 6 \times \log_8 7)$$

$$= \log_4 2 \times \log_6 4 \times \log_8 6 = (\log_4 2 \times \log_6 4) \times \log_8 6 \quad [\because \log_a M = \log_b M \times \log_a b]$$

$$= \log_6 2 \times \log_8 6 = \log_8 2 = \frac{1}{\log_2 8} \quad [\because \log_a b = \frac{1}{\log_b a}]$$

$$= \frac{1}{\log_2 2^3} = \frac{1}{3\log_2 2} \quad [\because \log_a (M)^n = n\log_a M]$$

$$= \frac{1}{3} \quad [\because \log_2 2 = 1]$$

Example 3.14

Find the value of $25^{-2\log_5 3}$

Solution $25^{-2\log_5 3} = (5^2)^{-2\log_5 3} = 5^{-4\log_5 3} \quad [\because n\log_a M = \log_a M^n]$

$$= 5^{\log_5 3^{-4}} = 3^{-4} = \frac{1}{3^4} = \frac{1}{81} \quad [\because b^{\log_b a} = a]$$

Example 3.15

Solve $\log_{16} x + \log_4 x + \log_2 x = 7$

Solution $\log_{16} x + \log_4 x + \log_2 x = 7$

$$\Longrightarrow \quad \frac{1}{\log_x 16} + \frac{1}{\log_x 4} + \frac{1}{\log_x 2} = 7 \quad [\because \log_a b = \frac{1}{\log_b a}]$$

$$\frac{1}{\log_x 2^4} + \frac{1}{\log_x 2^2} + \frac{1}{\log_x 2} = 7$$

$$\frac{1}{4\log_x 2} + \frac{1}{2\log_x 2} + \frac{1}{\log_x 2} = 7 \quad [\because n\log_a M = \log_a M^n]$$

$$\left[\frac{1}{4} + \frac{1}{2} + 1\right]\frac{1}{\log_x 2} = 7 \quad \Longrightarrow \left[\frac{7}{4}\right]\frac{1}{\log_x 2} = 7$$

$$\frac{1}{\log_x 2} = 7 \times \frac{4}{7}$$

$$\log_2 x = 4 \quad [\because \log_a b = \frac{1}{\log_b a}]$$

$$2^4 = x \quad \text{(exponential form)}$$

$$\therefore \; x = 16$$

Example 3.16

Solve $\dfrac{1}{2+\log_x 10} = \dfrac{1}{3}$

Solution $\dfrac{1}{2+\log_x 10} = \dfrac{1}{3}$. Cross multiplying, we get

$$2+\log_x 10 = 3$$

$$\implies \log_x 10 = 3-2 = 1$$

$$x^1 = 10 \qquad \text{(exponential form)}$$

$$\therefore x = 10$$

Example 3.17

Solve $\log_3(\log_2 x) = 1$

Solution Let $\log_2 x = y$

Then, $\log_3 y = 1$

$$3^1 = y \qquad \text{(exponential form)}$$

$$\therefore\ y = 3$$

Put $y = 3$ in (1). Then $\log_2 x = 3$

$$2^3 = x \qquad \text{(exponential form)}$$

$$\therefore\ x = 8$$

Example 3.18

Solve $\log_2(3x-1) - \log_2(x-2) = 3$

Solution $\log_2(3x-1) - \log_2(x-2) = 3$

$$\log_2\left(\frac{3x-1}{x-2}\right) = 3 \qquad \left[\because \log_a\left(\frac{M}{N}\right) = \log_a M - \log_a N\right]$$

$$2^3 = \frac{3x-1}{x-2} \qquad \text{(exponential form)}$$

$$8 = \frac{3x-1}{x-2}$$

Cross multiplying, we get

$$8(x-2) = 3x-1 \implies 8x-16 = 3x-1$$

$$8x-3x = -1+16 \implies 5x = 15$$

$$\therefore\ x = 3$$

Example 3.19

Prove that $\log_5 1125 = 2\log_5 6 - \frac{1}{2}\log_5 16 + 6\log_{49} 7$

Solution $2\log_5 6 - \frac{1}{2}\log_5 16 + 6\log_{49} 7$

$$= \log_5 6^2 - \log_5 (16)^{\frac{1}{2}} + 3\times 2\log_{49} 7 = \log_5 36 - \log_5 4 + 3\log_{49} 7^2$$

$$= \log_5\left(\frac{36}{4}\right) + 3\log_{49} 49 = \log_5 9 + 3(1)$$

$$= \log_5 9 + 3\log_5 5 = \log_5 9 + \log_5 (5)^3$$

$$= \log_5 9 + \log_5 125 = \log_5 (9\times 125) = \log_5 1125$$

$$\therefore \log_5 1125 = 2\log_5 6 - \frac{1}{2}\log_5 16 + 6\log_{49} 7$$

Example 3.20

Solve $\log_5 \sqrt{7x-4} - \frac{1}{2} = \log_5 \sqrt{x+2}$

Solution $\log_5 \sqrt{7x-4} - \frac{1}{2} = \log_5 \sqrt{x+2}$

$$\log_5 \sqrt{7x-4} - \log_5 \sqrt{x+2} = \frac{1}{2}$$

$$\log_5\left(\frac{\sqrt{7x-4}}{\sqrt{x+2}}\right) = \frac{1}{2} \qquad \left[\because \log_a\left(\frac{M}{N}\right) = \log_a M - \log_a N\right]$$

$$\log_5\left(\frac{7x-4}{x+2}\right)^{\frac{1}{2}} = \frac{1}{2}$$

$$\frac{1}{2}\left[\log_5\left(\frac{7x-4}{x+2}\right)\right] = \frac{1}{2} \qquad [\because \log_a M^n = n\log_a M]$$

$$\log_5\left(\frac{7x-4}{x+2}\right) = 1$$

$$5^1 = \frac{7x-4}{x+2} \quad \text{(exponential form)}$$

Cross multiplying, $7x - 4 = 5(x+2)$

$$7x - 4 = 5x + 10 \Longrightarrow 7x - 5x = 10 + 4$$

$$\Longrightarrow \quad 2x = 14$$

$$\therefore \quad x = 7$$

Exercise 3.2

1. State whether each of the following statements is true or false.

(i) $\log_5 125 = 3$ (ii) $\log_{\frac{1}{2}} 8 = 3$

(iii) $\log_4 (6+3) = \log_4 6 + \log_4 3$ (iv) $\log_2\left(\frac{25}{3}\right) = \frac{\log_2 25}{\log_2 3}$

(v) $\log_{\frac{1}{3}} 3 = -1$ (vi) $\log_a (M-N) = \log_a M \div \log_a N$

2. Obtain the equivalent logarithmic form of the following.

(i) $2^4 = 16$ (ii) $3^5 = 243$ (iii) $10^{-1} = 0.1$

(iv) $8^{-\frac{2}{3}} = \frac{1}{4}$ (v) $25^{\frac{1}{2}} = 5$ (vi) $12^{-2} = \frac{1}{144}$

3. Obtain the equivalent exponential form of the following.

(i) $\log_6 216 = 3$ (ii) $\log_9 3 = \frac{1}{2}$ (iii) $\log_5 1 = 0$

(iv) $\log_{\sqrt{3}} 9 = 4$ (v) $\log_{64}\left(\frac{1}{8}\right) = -\frac{1}{2}$ (vi) $\log_{0.5} 8 = -3$

4. Find the value of the following.

(i) $\log_3\left(\frac{1}{81}\right)$ (ii) $\log_7 343$ (iii) $\log_6 6^5$

(iv) $\log_{\frac{1}{2}} 8$ (v) $\log_{10} 0.0001$ (vi) $\log_{\sqrt{3}} 9\sqrt{3}$

5. Solve the following equations.

(i) $\log_2 x = \frac{1}{2}$ (ii) $\log_{\frac{1}{5}} x = 3$ (iii) $\log_3 y = -2$

(iv) $\log_x 125\sqrt{5} = 7$ (v) $\log_x 0.001 = -3$ (vi) $x + 2\log_{27} 9 = 0$

6. Simplify the following.

(i) $\log_{10} 3 + \log_{10} 3$ (ii) $\log_{25} 35 - \log_{25} 10$

(iii) $\log_7 21 + \log_7 77 + \log_7 88 - \log_7 121 - \log_7 24$

(iv) $\log_8 16 + \log_8 52 - \frac{1}{\log_{13} 8}$

(v) $5\log_{10} 2 + 2\log_{10} 3 - 6\log_{64} 4$

(vi) $\log_{10} 8 + \log_{10} 5 - \log_{10} 4$

7. Solve the equation in each of the following.

(i) $\log_4(x+4) + \log_4 8 = 2$ (ii) $\log_6(x+4) - \log_6(x-1) = 1$

(iii) $\log_2 x + \log_4 x + \log_8 x = \frac{11}{6}$ (iv) $\log_4(8\log_2 x) = 2$

(v) $\log_{10} 5 + \log_{10}(5x+1) = \log_{10}(x+5) + 1$

(vi) $4\log_2 x - \log_2 5 = \log_2 125$ (vii) $\log_3 25 + \log_3 x = 3\log_3 5$

(viii) $\log_3(\sqrt{5x-2}) - \frac{1}{2} = \log_3(\sqrt{x+4})$

8. Given $\log_a 2 = x$, $\log_a 3 = y$ and $\log_a 5 = z$. Find the value in each of the following in terms of x, y and z.

(i) $\log_a 15$ (ii) $\log_a 8$ (iii) $\log_a 30$

(iv) $\log_a\left(\frac{27}{125}\right)$ (v) $\log_a\left(3\frac{1}{3}\right)$ (vi) $\log_a 1.5$

9. Prove the following equations.

(i) $\log_{10}1600 = 2 + 4\log_{10}2$ (ii) $\log_{10}12500 = 2 + 3\log_{10}5$

(iii) $\log_{10}2500 = 4 - 2\log_{10}2$ (iv) $\log_{10}0.16 = 2\log_{10}4 - 2$

(v) $\log_5 0.00125 = 3 - 5\log_5 10$ (vi) $\log_5 1875 = \frac{1}{2}\log_5 36 - \frac{1}{3}\log_5 8 + 20\log_{32}2$

3.4 Common Logarithms

For the purpose of calculations, the most logical number for a base is 10, the base of the decimal number system. Logarithms to the base 10 are called *common logarithms*. Therefore, in the discussion which follows, no base designation is used, i.e., $\log N$ means $\log_{10} N$. Consider the following table.

Number N	0.0001	0.001	0.01	0.1	1	10	100	1000	10000
Exponential Form of N	10^{-4}	10^{-3}	10^{-2}	10^{-1}	10^0	10^1	10^2	10^3	10^4
$\log N$	– 4	– 3	– 2	– 1	0	1	2	3	4

So, $\log N$ is an integer if N is an integral power of 10. What about logarithm of 3.16 or 31.6 or 316? For example, $3.16 = 10^{0.4997}$; $31.6 = 10^{1.4997}$; $316 = 10^{2.4997}$

Therefore, $\log 3.16 = 0.4997$; $\log 31.6 = 1.4997$; $\log 316 = 2.4997$.

Notice that logarithm of a number between 1 and 10 is a number between 0 and 1 ; logarithm of a number between 10 and 100 is a number between 1 and 2 and so on.

Every logarithm consists of an integral part called the *characteristic* and a fractional part called the *mantissa*. For example,

$\log 3.16 = 0.4997$; characteristic is 0 and mantissa is 0.4997

$\log 31.6 = 1.4997$; characteristic is 1 and mantissa is 0.4997

$\log 316 = 2.4997$; characteristic is 2 and mantissa is 0.4997

The logarithm of a number less than 1 is negative. It is convenient to keep the mantissa positive even though the logarithm is negative.

Scientific notation provides a convenient method for determining the characteristic. In scientific notation $316 = 3.16\times 10^2$. Thus, we have

$$\begin{aligned}\log 316 &= \log(3.16\times 10^2)\\ &= \log 3.16 + \log 10^2\\ &= 0.4997 + 2 = 2.4997.\end{aligned}$$

Thus, the power of 10 determines the characteristic of logarithm.

Example 3.21

Write the characteristic of the following.

(i) log 27.91 (ii) log 0.02871 (iii) log 0.000987 (iv) log 2475.

Solution

(i) In scientific notation, $27.91 = 2.791 \times 10^{1}$

$\therefore$ The characteristic is 1

(ii) In scientific notation, $0.02871 = 2.871 \times 10^{-2}$

$\therefore$ The characteristic is –2

(iii) In scientific notation, $0.000987 = 9.87 \times 10^{-4}$

$\therefore$ The characteristic is – 4

(iv) In scientific notation, $2475 = 2.475 \times 10^{3}$

$\therefore$ The characteristic is 3

The characteristic is also determined by inspection of the number itself according to the following rules.

(i) For a number greater than 1, the characteristic is positive and is one less than the number of digits before the decimal point.

(ii) For a number less than 1, the characteristic is negative and is one more than the number of zeros immediately following the decimal point. The negative sign of the characteristic is written above the characteristics as $\bar{1}, \bar{2}$, etc. For example, the characteristic of 0.0316 is $\bar{2}$.

(iii) Mantissa is always positive.

Example 3.22

Given that log 4586 = 3.6615, find (i) log 45.86 (ii) log 45860 (iii) log 0.4586 (iv) log 0.004586 (v) log 0.04586 (vi) log 4.586

Solution The mantissa of log 4586 is 0.6615. Hence,

(i) log 45.86 = 1.6615 (ii) log 45860 = 4.6615

(iii) log 0.4586 = $-1 + 0.6615 = \bar{1}.6615$ (iv) log 0.004586 = $-3 + 0.6615 = \bar{3}.6615$

(v) log 0.04586 = $-2 + 0.6615 = \bar{2}.6615$ (vi) log 4.586 = 0.6615

3.4.1 Method of Finding Logarithm

Tables of logarithm usually contain only mantissas since the characteristic can be readily determined as explained above. Note that the mantissas of logarithms of all the numbers consisting of same digits in same order but differing only in the position of decimal point are the same. The mantissas are given correct to four places of decimals.

A logarithmic table consists of three parts .

(i) First column contains numbers from 1.0, 1.2 ,1.3,... upto 9.9

(ii) Next ten columns headed by 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 contain the mantissas.

(iii) After these columns, there are again nine columns under the head mean difference. These columns are marked with serial numbers 1 to 9.

We shall explain how to find the mantissa of a given number in the following example. Suppose, the given number is 40.85. Now $40.85 = 4.085 \times 10^1$. Therefore, the characteristic is 1. The row in front of the number 4.0 in logarithmic table is given below.

											Mean Difference								
	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
4.0	.6021	.6031	.6042	.6053	.6064	.6075	.6085	.6096	.6107	.6117	1	2	3	4	5	6	8	9	10

We note the number in row beneath the digit 8 in front of $N = 4.0$. The number is 0.6107. Next the mean difference corresponding to 5 is 0.0005. Thus the required mantissa is 0.6107 + 0.0005 = 0.6112.

Hence, $\log 40.85 = 1.6112$.

3.4.2 Antilogarithms

The number whose logarithm is x, is called the antilogarithm of x and is written as antilog x. Thus, if $\log y = x$, then antilog $x = y$.

3.4.3 Method of Finding Antilogarithm

The antilogarithm of a number is found by using a table named 'ANTILOGARITHMS' given at the end of the book. This table gives the value of the antilogarithm of a number correct to four places of decimal.

For finding antilogarithm, we take into consideration only the mantissa. The characteristic is used only to determine the number of digits in the integral part or the number of zeros immediately after the decimal point.

The method of using the table of antilogarithms is the same as that of the table of logarithms discussed above.

Note Since the logarithmic table given at the end of this text book can be applied only to four digit number, in this section we approximated all logarithmic calculations to four digits

Example 3.23

Find (i) log 86.76 (ii) log 730.391 (iii) log 0.00421526

Solution

(i) $86.76 = 8.676 \times 10^1$ (scientific notation)

$\therefore$ The characteristic is 1. To find mantissa consider the number 8.676.

From the table, log 8.67 is 0.9380

Mean difference of 6 is 0.0003

$\log 8.676 = 0.9380 + 0.0003 = 0.9383$

$\therefore \log 86.76 = 1.9383$

(ii) $730.391 = 7.30391 \times 10^2$ (scientific notation)

$\therefore$ The characteristic is 2. To find mantissa consider the number 7.30391 and approximate it as 7.304 (since 9 in the fourth decimal place is greater than 5)

From the table, $\log 7.30$ is 0.8633

Mean difference of 4 is 0.0002

$\log 7.304 = 0.8633 + 0.0002 = 0.8635$

$\therefore \log 730.391 = 2.8635$

(iii) $0.00421526 = 4.21526 \times 10^{-3}$ (scientific notation)

$\therefore$ The characteristic is –3. To find mantissa consider the number 4.21526 and approximate it as 4.215 (since 2 in the fourth decimal place is less than 5).

From the table, $\log 4.21$ is 0.6243

Mean difference of 5 is 0.0005

$\log 4.215 = 0.6243 + 0.0005 = 0.6248$

$\therefore \log 0.00421526 = -3 + 0.6248 = \bar{3}.6248$

Example 3.24

Find (i) antilog 1.8652 (ii) antilog 0.3269 (iii) antilog $\bar{2}.6709$

Solution

(i) Characteristic is 1. So, the number contains two digits in its integral part. Mantissa is 0.8652.

From the table, antilog 0.865 is 7.328

Mean difference of 2 is 0.003

antilog $0.8652 = 7.328 + 0.003 = 7.331$

$\therefore$ antilog 1.8652 = 73.31

(ii) Characteristic is 0. So, the number contains one digit in its integral part. Mantissa is 0.3269

From the table, antilog of 0.326 is 2.118

Mean difference of 9 is 0.004

$\therefore$ antilog 0.3269 = 2.118 + 0.004 = 2.122

(iii) Characteristic is -2. So, the number contains one zero immediately following the decimal point. Mantissa is 0.6709

From the table, antilog 0.670 is 4.677

Mean difference of 9 is 0.010

antilog $0.6709 = 4.677 + 0.010 = 4.687$

$\therefore$ antilog $\bar{2}.6709 = 0.04687$

Example: 3.25

Find (i) 42.6×2.163 (ii) 23.17×0.009321

Solution

(i) Let $x = 42.6 \times 2.163$. Taking logarithm on both sides, we get

$$\log x = \log(42.6 \times 2.163)$$

$$= \log 42.6 + \log 2.163$$

$$= 1.6294 + 0.3351 = 1.9645$$

$$\therefore\ x = \text{antilog } 1.9645 = 92.15$$

(ii) Let $x = 23.17 \times 0.009321$. Taking logarithm on both sides, we get

$$\log x = \log(23.17 \times 0.009321) = \log 23.17 + \log 0.009321$$

$$= 1.3649 + \bar{3}.9694 = 1 + 0.3649 - 3 + 0.9694$$

$$= -2 + 1.3343 = -2 + 1 + 0.3343$$

$$= -1 + 0.3343 = \bar{1}.3343$$

$$\therefore \quad x = \text{antilog } \bar{1}.3343 = 0.2159$$

Example: 3.26

Find the value of (i) $(36.27)^6$ (ii) $(0.3749)^4$ (iii) $\sqrt[5]{0.2713}$

Solution

(i) Let $x = (36.27)^6$. Taking logarithm on both sides, we get

$$\log x = \log(36.27)^6 = 6\log 36.27 = 6(1.5595) = 9.3570$$

$$\therefore x = \text{antilog } 9.3570 = 2275000000$$

(ii) Let $x = (0.3749)^4$. Taking logarithm on both sides, we get

$$\log x = \log(0.3749)^4 = 4\log 0.3749 = 4(\bar{1}.5739) = 4(-1 + 0.5739)$$

$$= -4 + 2.2956 = -4 + 2 + 0.2956 = -2 + 0.2956 = -2.2956$$

$$= \bar{2}.2956$$

$$\therefore \; x = \text{antilog } \bar{2}.2956 = 0.01975$$

(iii) Let $x = \sqrt[5]{0.2713} = (0.2713)^{\frac{1}{5}}$. Taking logarithm on both sides, we get

$$\log x = \log(0.2713)^{\frac{1}{5}} = \frac{1}{5}\log 0.2713$$

$$= \frac{1}{5}(\bar{1}.4335) = \frac{-1 + 0.4335}{5}$$

$$= \frac{(-1-4) + 4 + 0.4335}{5}$$

$$= \frac{-5 + 4.4335}{5} = \frac{-5}{5} + \frac{4.4335}{5}$$

$$= -1 + 0.8867 = \bar{1}.8867$$

$$\therefore \; x = \text{antilog } \bar{1}.8867 = 0.7703$$

Example : 3.27

Simplify (i) $\dfrac{(46.7)\times\sqrt{65.2}}{(2.81)^3\times(4.23)}$ (ii) $\dfrac{(84.5)^4\times\sqrt[3]{0.0064}}{(72.5)^2\times\sqrt{62.3}}$

Solution

(i) Let $x = \dfrac{(46.7)\times\sqrt{65.2}}{(2.81)^3\times(4.23)} = \dfrac{46.7\times(65.2)^{\frac{1}{2}}}{(2.81)^3\times 4.23}$

Taking logarithm on both sides, we get

$$\log x = \log\left[\frac{46.7\times(65.2)^{\frac{1}{2}}}{(2.81)^3\times 4.23}\right]$$

$$= \log 46.7 + \log\,(65.2)^{\frac{1}{2}} - \log(2.81)^3 - \log 4.23$$

$$= \log 46.7 + \frac{1}{2}\log 65.2 - 3\log 2.81 - \log 4.23$$

$$= 1.6693 + \frac{1}{2}(1.8142) - 3(0.4487) - 0.6263$$

$$= 1.6693 + 0.9071 - 1.3461 - 0.6263$$

$$= 2.5764 - 1.9724 = 0.6040$$

$$\therefore x = \text{antilog}\,0.6040 = 4.018$$

(ii) Let $x = \dfrac{(84.5)^4\times\sqrt[3]{0.0064}}{(72.5)^2\times\sqrt{62.3}} = \dfrac{(84.5)^4\times(0.0064)^{\frac{1}{3}}}{(72.5)^2\times(62.3)^{\frac{1}{2}}}$

Taking logarithm on both sides, we get

$$\log x = \log\left[\frac{(84.5)^4\times(0.0064)^{\frac{1}{3}}}{(72.5)^2\times(62.3)^{\frac{1}{2}}}\right]$$

$$= \log\,(84.5)^4 + \log(0.0064)^{\frac{1}{3}} - \log\,(72.5)^2 - \log(62.3)^{\frac{1}{2}}$$

$$= 4\log 84.5 + \frac{1}{3}\log 0.0064 - 2\log 72.5 - \frac{1}{2}\log 62.3$$

$$= 4(1.9269) + \frac{1}{3}(\bar{3}.8062) - 2(1.8603) - \frac{1}{2}(1.7945)$$

$$= 7.7076 + \frac{1}{3}(-3 + 0.8062) - 3.7206 - 0.8973$$

$$= 3.0897 + (-1+0.2687) = 3+0.0897-1+0.2687$$

$$= 2+0.3584 = 2.3584$$

$$\therefore x = \text{antilog}2.3584 = 228.2$$

Example 3.28

Find the value of $\log_4 13.26$

Solution $\log_4 13.26 = \log_{10} 13.26 \times \log_4 10$ [$\because \log_a M = \log_b M \times \log_a b$]

$= \log_{10} 13.26 \times \dfrac{1}{\log_{10} 4}$ [$\because \log_a b = \dfrac{1}{\log_b a}$]

$= \dfrac{1.1225}{0.6021} = x$ (say)

Then $x = \dfrac{1.1225}{0.6021}$. Taking logarithm on both sides, we get

$$\log x = \log\left(\frac{1.1225}{0.6021}\right)$$

$$= \log 1.1225 - \log 0.6021 = 0.0503 - \bar{1}.7797$$

$$= 0.0503 - (-1 + 0.7797) = 0.0503 + 1 - 0.7797$$

$$= 1.0503 - 0.7797 = 0.2706$$

$$\therefore x = \text{antilog } 0.2706 = 1.865$$

Exercise: 3.3

1. Write each of the following in scientific notation:

(i) 92.43   (ii) 0.9243   (iii) 9243

(iv) 924300   (v) 0.009243   (vi) 0.09243

2. Write the characteristic of each of the following

(i) log 4576   (ii) log 24.56   (iii) log 0.00257

(iv) log 0.0756   (v) log 0.2798   (vi) log 6.453

3. The mantissa of log 23750 is 0.3576. Find the value of the following.

(i) log 23750   (ii) log 23.75   (iii) log 2.375

(iv) log 0.2375   (v) log 23750000   (vi) log 0.00002375

4. Using logarithmic table find the value of the following.

(i) log 23.17   (ii) log 9.321   (iii) log 329.5

(iv) log 0.001364   (v) log 0.9876   (vi) log 6576

5. Using antilogarithmic table find the value of the following.

(i) antilog 3.072   (ii) antilog 1.759   (iii) antilog $\bar{1}.3826$

(iv) antilog $\bar{3}.6037$ (v) antilog 0.2732   (vi) antilog $\bar{2}.1798$

6. Evaluate:

(i) 816.3×37.42 (ii) $816.3 \div 37.42$ (iii) 0.000645×82.3

(iv) $0.3421 \div 0.09782$ (v) $(50.49)^5$ (vi) $\sqrt[3]{561.4}$

(vii) $\dfrac{175.23 \times 22.159}{1828.56}$ (viii) $\dfrac{\sqrt[3]{28} \times \sqrt[5]{729}}{\sqrt{46.35}}$ (ix) $\dfrac{(76.25)^3 \times \sqrt[3]{1.928}}{(42.75)^5 \times 0.04623}$

(x) $\sqrt[3]{\dfrac{0.7214 \times 20.37}{69.8}}$ (xi) $\log_9 63.28$ (xii) $\log_3 7$

Points to Remember

- A number N is in scientific notation when it is expressed as the product of a decimal number $1 \leq a < 10$ and some integral power of 10.

  $N = a \times 10^n$, where $1 \leq a < 10$ and n is an integer.
- If $a^x = b\,(a > 0, a \neq 1)$, then x is said to be the logarithm of b to the base a, which is written $x = \log_a b$.
- Product rule : $\log_a(M \times N) = \log_a M + \log_a N$
- Quotient rule : $\log_a\left(\dfrac{M}{N}\right) = \log_a M - \log_a N$
- Power rule : $\log_a (M)^n = n \log_a M$
- Change of base rule : $\log_a M = \log_b M \times \log_a b,\ a \neq 1, b \neq 1$

4 ALGEBRA

Mathematics is as much an aspect of culture as it is a collection of algorithms

- CARL BOYER

Main Targets

- To classify polynomials.
- To use Remainder Theorem.
- To use Factor Theorem.
- To use algebraic identities.
- To factorize a polynomial.
- To solve linear equations in two variables.
- To solve linear inequation in one variable.

DIOPHANTUS

(200 to 284 A.D. or 214 to 298 A.D.)

Diophantus was a Hellenistic mathematician who lived circa 250 AD, but the uncertainty of this date is so great that it may be off by more than a century. He is known for having written Arithmetica, a treatise that was originally thirteen books but of which only the first six have survived. Arithmetica has very little in common with traditional Greek mathematics since it is divorced from geometric methods, and it is different from Babylonian mathematics in that Diophantus is concerned primarily with exact solutions, both determinate and indeterminate, instead of simple approximations

4.1 Introduction

The language of algebra is a wonderful instrument for expressing shortly, perspicuously, suggestively and the exceedingly complicated relations in which abstract things stand to one another. The history of algebra began in ancient Egypt and Babylon, where people learned to solve linear ($ax = b$) and quadratic ($ax^2 + bx = c$) equations, as well as indeterminate equations such as $x^2 + y^2 = z^2$, whereby several unknowns are involved. Algebra has been developed over a period of 4000 years. But, only by the middle of the 17th Century the representation of elementary algebraic problems and relations looked much as it is today. By the early decades of the twentieth century, algebra had evolved into the study of axiomatic systems. This axiomatic approach soon came to be called modern or abstract algebra. Important new results have been discovered, and the subject has found applications in all branches of mathematics and in many of the sciences as well.

4.2 Algebraic Expressions

An algebraic expression is an expression formed from any combination of numbers and variables by using the operations of addition, subtraction, multiplication, division, exponentiation (raising powers), or extraction of roots.

For instance, $7, x, 2x-3y+1, \frac{5x^3-1}{4xy+1}, \pi r^2$ and $\pi r\sqrt{r^2+h^2}$ are algebraic expressions. By an algebraic expression in certain variables, we mean an expression that contains only those variables. A constant, we mean an algebraic expression that contains no variables at all. If numbers are substituted for the variables in an algebraic expression, the resulting number is called the value of the expression for these values of variables.

If an algebraic expression consists of part connected by plus or minus signs, it is called an algebraic sum. Each part, together with the sign preceding it is called a term. For instance, in the algebraic sum $3x^2y - \frac{4xz^2}{y} + \pi x^{-1}y$, the terms are $3x^2y$, $-\frac{4xz^2}{y}$ and $\pi x^{-1}y$.

Any part of a term that is multiplied by the remaining part of the term is called the coefficient of the remaining part. For instance, in the term $-\frac{4xz^2}{y}$, the coefficient of $\frac{z^2}{y}$ is $-4x$, whereas the coefficient of $\frac{xz^2}{y}$ is –4. A coefficient such as –4, which involves no variables, is called a numerical coefficient. Terms such as $5x^2y$ and $-12x^2y$, which differ only in their numerical coefficients, are called like terms or similar terms.

An algebraic expression such as $4\pi r^2$ can be considered as an algebraic expression consisting of just one term. Such a one-termed expression is called a monomial. An algebraic expression with two terms is called a binomial, and an algebraic expression with three terms is called a trinomial. For instance, the expression $3x^2+2xy$ is a binomial, whereas $-2xy^{-1}+3\sqrt{x}-4$ is a trinomial. An algebraic expression with two or more terms is called a multinomial.

4.3 Polynomials

A polynomial is an algebraic expression, in which no variables appear in denominators or under radical signs, and all variables that do appear are powers of positive integers. For instance, the trinomial $-2xy^{-1}+3\sqrt{x}-4$ is not a polynomial; however, the trinomial $3x^2y^4+\sqrt{2}\,xy-\frac{1}{2}$ is a polynomial in the variables x and y. A term such as $-\frac{1}{2}$ which contains no variables, is called a constant term of the polynomial. The numerical coefficients of the terms in a polynomial are called the coefficients of the polynomial. The coefficients of the polynomial above are 3, $\sqrt{2}$ and $-\frac{1}{2}$.

The degree of a term in a polynomial is the sum of the exponents of all the variables in that term. In adding exponents, one should regard a variable with no exponent as being power one. For instance, in the polynomial $9xy^7-12x^3yz^2+3x-2$, the term $9xy^7$ has degree 1 + 7 = 8, the term $-12x^3yz^2$ has degree 3 + 1 + 2 = 6, and the term $3x$ has degree one. The constant term is always regarded as having degree zero.

The degree of the highest degree term that appears with nonzero coefficients in a polynomial is called the degree of the polynomial.

For instance, the polynomial considered above has degree 8. Although the constant monomial 0 is regarded as a polynomial, this particular polynomial is not assigned a degree.

4.3.1 Polynomials in One Variable

In this section we consider only polynomials in one variable.

Key Concept — **Polynomial in One Variable**

A polynomial in one variable x is an algebraic expression of the form

$$p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0, \quad a_n \neq 0$$

where $a_0, a_1, a_2, \cdots, a_{n-1}, a_n$ are constants and n is a non negative integer.

Here n is the degree of the polynomial and $a_1, a_2, \cdots, a_{n-1}, a_n$ are the coefficients of $x, x^2, \cdots x^{n-1}, x^n$ respectively. a_0 is the constant term. $a_n x^n, a_{n-1}x^{n-1}, \cdots, a_2 x^2, a_1 x, a_0$ are the terms of the polynomial $p(x)$.

For example, in the polynomial $5x^2 + 3x - 1$, the coefficient of x^2 is 5, the coefficient of x is 3 and –1 is the constant term. The three terms of the polynomial are $5x^2, 3x$ and -1.

4.3.2 Types of Polynomials

Key Concept — **Types of Polynomials Based on Number of Terms**

Monomial

Polynomials which have only one term are known as monomials.

Binomial

Polynomials which have only two terms are called binomials.

Trinomial

Polynomials which have only three terms are named as trinomials.

Note

1. A binomial is the sum of two monomials of different degrees.
2. A trinomial is the sum of three monomials of different degrees.
3. A polynomial is a monomial or the sum of two or more monomials.

Key Concept | **Types of Polynomials Based on the Degree**

Constant Polynomial

A polynomial of degree zero is called a constant polynomial.

General form : $p(x) = c$, where c is a real number.

Linear Polynomial

A polynomial of degree one is called a linear polynomial.

General form : $p(x) = ax+b$, where a and b are real numbers and $a \neq 0$.

Quadratic Polynomial

A polynomial of degree two is called a quadratic polynomial.

General form: $p(x)=ax^2 + bx + c$ where a, b and c are real numbers and $a \neq 0$.

Cubic Polynomial

A polynomial of degree three is called a cubic polynomial.

General form : $p(x) = ax^3 + bx^2 + cx + d$, where a, b, c and d are real numbers and $a \neq 0$.

Example 4.1

Classify the following polynomials based on number of terms.

(i) $x^3 - x^2$ (ii) $5x$ (iii) $4x^4 + 2x^3 + 1$ (iv) $4x^3$

(v) $x + 2$ (vi) $3x^2$ (vii) $y^4 + 1$ (viii) $y^{20} + y^{18} + y^2$

(ix) 6 (x) $2u^3 + u^2 + 3$ (xi) $u^{23} - u^4$ (xii) y

Solution

$5x$, $3x^2$, $4x^3$, y and 6 are monomials because they have only one term.

$x^3 - x^2$, $x + 2$, $y^4 + 1$ and $u^{23} - u^4$ are binomials as they contain only two terms.

$4x^4 + 2x^3 + 1$, $y^{20} + y^{18} + y^2$ and $2u^3 + u^2 + 3$ are trinomials as they contain only three terms.

Example 4.2

Classify the following polynomials based on their degree.

(i) $p(x) = 3$ (ii) $p(y) = \frac{5}{2}y^2 + 1$ (iii) $p(x) = 2x^3 - x^2 + 4x + 1$

(iv) $p(x) = 3x^2$ (v) $p(x) = x + 3$ (vi) $p(x) = -7$

(vii) $p(x) = x^3 + 1$ (viii) $p(x) = 5x^2 - 3x + 2$ (ix) $p(x) = 4x$

(x) $p(x) = \frac{3}{2}$ (xi) $p(x) = \sqrt{3}x + 1$ (xii) $p(y) = y^3 + 3y$

Solution

$p(x) = 3,\ p(x) = -7,\ p(x) = \frac{3}{2}$ are constant polynomials.

$p(x) = x + 3,\ p(x) = 4x,\ p(x) = \sqrt{3}x + 1$ are linear polynomials, since the highest degree of the variable x is one.

$p(x) = 5x^2 - 3x + 2,\ p(y) = \frac{5}{2}y^2 + 1,\ p(x) = 3x^2$ are quadratic polynomials, since the highest degree of the variable is two.

$p(x) = 2x^3 - x^2 + 4x + 1,\ p(x) = x^3 + 1,\ p(y) = y^3 + 3y$ are cubic polynomials, since the highest degree of the variable is three.

Exercise: 4.1

1. State whether the following expressions are polynomials in one variable or not. Give reasons for your answer.

   (i) $2x^5 - x^3 + x - 6$ (ii) $3x^2 - 2x + 1$ (iii) $y^3 + 2\sqrt{3}$

   (iv) $x - \frac{1}{x}$ (v) $\sqrt[3]{t} + 2t$ (vi) $x^3 + y^3 + z^6$

2. Write the coefficient of x^2 and x in each of the following.

   (i) $2 + 3x - 4x^2 + x^3$ (ii) $\sqrt{3}x + 1$ (iii) $x^3 + \sqrt{2}x^2 + 4x - 1$

   (iv) $\frac{1}{3}x^2 + x + 6$

3. Write the degree of each of the following polynomials.

   (i) $4 - 3x^2$ (ii) $5y + \sqrt{2}$ (iii) $12 - x + 4x^3$ (iv) 5

4. Classify the following polynomials based on their degree.

   (i) $3x^2 + 2x + 1$ (ii) $4x^3 - 1$ (iii) $y + 3$

   (iv) $y^2 - 4$ (v) $4x^3$ (vi) $2x$

5. Give one example of a binomial of degree 27 and monomial of degree 49 and trinomial of degree 36.

4.3.3 Zeros of a Polynomial

Consider the polynomial $p(x) = x^2 - x - 2$. Let us find the values of $p(x)$ at $x = -1$, $x = 1$ and $x = 2$.

$p(-1) = (-1)^2 - (-1) - 2 = 1 + 1 - 2 = 0$

$p(1) = (1)^2 - 1 - 2 = 1 - 1 - 2 = -2$

$p(2) = (2)^2 - 2 - 2 = 4 - 2 - 2 = 0$

That is, 0, –2 and 0 are the values of the polynomial $p(x)$ at $x = -1$, 1 and 2 respectively.

If the value of a polynomial is zero for some value of the variable then that value is known as zero of the polynomial.

Since $p(-1) = 0$, $x = -1$ is a zero of the polynomial $p(x) = x^2 - x - 2$.

Similarly, $p(2) = 0$ at $x = 2$, $\Longrightarrow$ 2 is also a zero of $p(x)$.

Key Concept	**Zeros of Polynomial**
Let $p(x)$ be a polynomial in x. If $p(a) = 0$, then we say that a is a zero of the polynomial $p(x)$.	

Note Number of zeros of a polynomial $\leq$ the degree of the polynomial. Carl Friedrich Gauss (1777-1855) had proven in his doctoral thesis of 1798 that the polynomial equations of any degree n must have exactly n solutions in a certain very specific sense. This result was so important that it became known as the fundamental theorem of algebra. The exact sense in which that theorem is true is the subject of the other part of the story of algebraic numbers.

Example 4.3

If $p(x) = 5x^3 - 3x^2 + 7x - 9$, find (i) $p(-1)$ (ii) $p(2)$

Solution Given that $p(x) = 5x^3 - 3x^2 + 7x - 9$

(i) $p(-1) = 5(-1)^3 - 3(-1)^2 + 7(-1) - 9 = -5 - 3 - 7 - 9$

$\therefore\ p(-1) = -24$

(ii) $p(2) = 5(2)^3 - 3(2)^2 + 7(2) - 9 = 40 - 12 + 14 - 9$

$\therefore\ p(2) = 33$

Example 4.4

Find the zeros of the following polynomials.

(i) $p(x) = 2x - 3$ (ii) $p(x) = x - 2$

Solution

(i) Given that $p(x) = 2x - 3 = 2\left(x - \frac{3}{2}\right)$.We have

$p\left(\frac{3}{2}\right) = 2\left(\frac{3}{2} - \frac{3}{2}\right) = 2(0) = 0$

Hence $x = \frac{3}{2}$ is the zero of $p(x)$.

(ii) Given that $p(x) = x - 2$. Now,

$p(2) = 2 - 2 = 0$

Hence $x = 2$ is the zero of $p(x)$.

4.3.4 Roots of a Polynomial Equations

Let $p(x)$ be a polynomial expression in x. Then $p(x) = 0$ is called a polynomial equation in x.

Consider the polynomial $p(x) = x - 1$. Clearly 1 is the zero of the polynomial $p(x) = x - 1$.Now, consider the polynomial equation $p(x) = 0$, that is, $x - 1 = 0$. Now, $x - 1 = 0$ implies $x = 1$. The value $x = 1$ is called the root of the polynomial equation $p(x) = 0$.

Hence zeros of a polynomial are the roots of the corresponding polynomial equation.

Key Concept	**Root of a Polynomial Equation**
If $x = a$ satisfies the polynomial equation $p(x) = 0$, then $x = a$ is called a root of the polynomial equation $p(x) = 0$.	

Example 4.5

Find the roots of the following polynomial equations

(i) $x - 6 = 0$ (ii) $2x + 1 = 0$

Solution

(i) Given that $x - 6 = 0 \Longrightarrow x = 6$

$\therefore$ $x = 6$ is a root of $x - 6 = 0$

(ii) Given that $2x + 1 = 0 \Longrightarrow 2x = -1 \Longrightarrow x = -\frac{1}{2}$

$\therefore$ $x = -\frac{1}{2}$ is a root of $2x + 1 = 0$

Example 4.6

Verify whether the following are roots of the polynomial equations indicated against them.

(i) $2x^2 - 3x - 2 = 0$; $x = 2, 3$

(ii) $x^3 + 8x^2 + 5x - 14 = 0$; $x = 1, 2$

Solution

(i) Let $p(x) = 2x^2 - 3x - 2$.

$p(2) = 2(2)^2 - 3(2) - 2 = 8 - 6 - 2 = 0$

$\therefore$ $x = 2$ is a root of $2x^2 - 3x - 2 = 0$

But $p(3) = 2(3)^2 - 3(3) - 2 = 18 - 9 - 2 = 7 \neq 0$

$\therefore$ $x = 3$ is not a root of $2x^2 - 3x - 2 = 0$

(ii) Let $p(x) = x^3 + 8x^2 + 5x - 14$

$p(1) = (1)^3 + 8(1)^2 + 5(1) - 14 = 1 + 8 + 5 - 14 = 0$

$\therefore\quad x = 1$ is a root of $x^3 + 8x^2 + 5x - 14 = 0$

But $p(2) = (2)^3 + 8(2)^2 + 5(2) - 14 = 8 + 32 + 10 - 14 = 36 \neq 0$

$\therefore\quad x = 2$ is not a root of $x^3 + 8x^2 + 5x - 14 = 0$

Exercise 4.2

1. Find the zeros of the following polynomials

   (i) $p(x) = 4x - 1$ (ii) $p(x) = 3x + 5$ (iii) $p(x) = 2x$ (iv) $p(x) = x + 9$

2. Find the roots of the following polynomial equations.

   (i) $x - 3 = 0$ (ii) $5x - 6 = 0$ (iii) $11x + 1 = 0$ (iv) $-9x = 0$

3. Verify Whether the following are roots of the polynomial equations indicated against them.

   (i) $x^2 - 5x + 6 = 0$; $x = 2, 3$ (ii) $x^2 + 4x + 3 = 0$; $x = -1, 2$

   (iii) $x^3 - 2x^2 - 5x + 6 = 0$; $x = 1, -2, 3$ (iv) $x^3 - 2x^2 - x + 2 = 0$; $x = -1, 2, 3$

4.4 Remainder Theorem

Remainder Theorem

Let $p(x)$ be any polynomial and a be any real number. If $p(x)$ is divided by the linear polynomial $x - a$, then the remainder is $p(a)$.

Note

1. If $p(x)$ is divided by $(x + a)$, then the remainder is $p(-a)$.
2. If $p(x)$ is divided by $(ax - b)$, then the remainder is $p\left(\frac{b}{a}\right)$.
3. If $p(x)$ is divided by $(ax + b)$, then the remainder is $p\left(-\frac{b}{a}\right)$.
4. Here $-a$, $\frac{b}{a}$ and $-\frac{b}{a}$ are the zeros of the divisors $x + a$, $ax - b$ and $ax + b$ respectively.

Example 4.7

Find the remainder when $4x^3 - 5x^2 + 6x - 2$ is divided by $x - 1$.

Solution Let $p(x) = 4x^3 - 5x^2 + 6x - 2$. The zero of $x - 1$ is 1.

When $p(x)$ is divided by $(x - 1)$ the remainder is $p(1)$. Now,

$p(1) = 4(1)^3 - 5(1)^2 + 6(1) - 2$

$= 4 - 5 + 6 - 2 = 3$

$\therefore$ The remainder is 3.

Example 4.8

Find the remainder when $x^3 - 7x^2 - x + 6$ is divided by $(x + 2)$.

Solution Let $p(x) = x^3 - 7x^2 - x + 6$. The zero of $x + 2$ is -2.

When $p(x)$ is divided by $x + 2$, the remainder is $p(-2)$. Now,

$$\begin{aligned} p(-2) &= (-2)^3 - 7(-2)^2 - (-2) + 6 \\ &= -8 - 7(4) + 2 + 6 \\ &= -8 - 28 + 2 + 6 = -28 \end{aligned}$$

$\therefore$ The remainder is -28

Example 4.9

Find the value of a if $2x^3 - 6x^2 + 5ax - 9$ leaves the remainder 13 when it is divided by $x - 2$.

Solution Let $p(x) = 2x^3 - 6x^2 + 5ax - 9$.

When $p(x)$ is divided by $(x - 2)$ the remainder is $p(2)$.

$$\begin{aligned} \text{Given that } p(2) &= 13 \\ \Longrightarrow \quad 2(2)^3 - 6(2)^2 + 5a(2) - 9 &= 13 \\ 2(8) - 6(4) + 10a - 9 &= 13 \\ 16 - 24 + 10a - 9 &= 13 \\ 10a - 17 &= 13 \\ 10a &= 30 \\ \therefore \quad a &= 3 \end{aligned}$$

Example 4.10

Find the remainder when $x^3 + ax^2 - 3x + a$ is divided by $x + a$.

Solution

Let $p(x) = x^3 + ax^2 - 3x + a$.

When $p(x)$ is divided by $(x + a)$ the remainder is $p(-a)$.

$$p(-a) = (-a)^3 + a(-a)^2 - 3(-a) + a = -a^3 + a^3 + 4a = 4a$$

$\therefore$ The remainder is $4a$.

Example 4.11

Find the remainder when $f(x) = 12x^3 - 13x^2 - 5x + 7$ is divided by $(3x + 2)$.

Solution $f(x) = 12x^3 - 13x^2 - 5x + 7$.

When $f(x)$ is divided by $(3x + 2)$ the remainder is $f\left(-\frac{2}{3}\right)$. Now,

$$f\left(-\frac{2}{3}\right) = 12\left(-\frac{2}{3}\right)^3 - 13\left(-\frac{2}{3}\right)^2 - 5\left(-\frac{2}{3}\right) + 7$$

$$= 12\left(-\frac{8}{27}\right) - 13\left(\frac{4}{9}\right) + \frac{10}{3} + 7$$

$$= -\frac{32}{9} - \frac{52}{9} + \frac{10}{3} + 7 = \frac{9}{9} = 1$$

$\therefore$ The remainder is 1.

Example 4.12

If the polynomials $2x^3 + ax^2 + 4x - 12$ and $x^3 + x^2 - 2x + a$ leave the same remainder when divided by $(x - 3)$, find the value of a. Also find the remainder.

Solution Let $p(x) = 2x^3 + ax^2 + 4x - 12$,

$q(x) = x^3 + x^2 - 2x + a$

When $p(x)$ is divided by $(x - 3)$ the remainder is $p(3)$. Now,

$$p(3) = 2(3)^3 + a(3)^2 + 4(3) - 12$$

$$= 2(27) + a(9) + 12 - 12$$

$$= 54 + 9a \qquad (1)$$

When $q(x)$ is divided by $(x - 3)$ the remainder is $q(3)$. Now,

$$q(3) = (3)^3 + (3)^2 - 2(3) + a$$

$$= 27 + 9 - 6 + a$$

$$= 30 + a \qquad (2)$$

Given that $p(3) = q(3)$. That is,

$$54 + 9a = 30 + a \quad \text{(By (1) and (2))}$$

$$9a - a = 30 - 54$$

$$8a = -24$$

$$\therefore a = -\frac{24}{8} = -3$$

Substituting $a = -3$ in $p(3)$, we get

$$p(3) = 54 + 9(-3) = 54 - 27 = 27$$

$\therefore$ The remainder is 27.

Exercise 4.3

1. Find the remainder using remainder theorem, when
   (i) $3x^3 + 4x^2 - 5x + 8$ is divided by $x - 1$
   (ii) $5x^3 + 2x^2 - 6x + 12$ is divided by $x + 2$
   (iii) $2x^3 - 4x^2 + 7x + 6$ is divided by $x - 2$
   (iv) $4x^3 - 3x^2 + 2x - 4$ is divided by $x + 3$
   (v) $4x^3 - 12x^2 + 11x - 5$ is divided by $2x - 1$
   (vi) $8x^4 + 12x^3 - 2x^2 - 18x + 14$ is divided by $x + 1$
   (vii) $x^3 - ax^2 - 5x + 2a$ is divided by $x - a$
2. When the polynomial $2x^3 - ax^2 + 9x - 8$ is divided by $x - 3$ the remainder is 28. Find the value of a.
3. Find the value of m if $x^3 - 6x^2 + mx + 60$ leaves the remainder 2 when divided by $(x + 2)$.
4. If $(x - 1)$ divides $mx^3 - 2x^2 + 25x - 26$ without remainder find the value of m.
5. If the polynomials $x^3 + 3x^2 - m$ and $2x^3 - mx + 9$ leave the same remainder when they are divided by $(x - 2)$, find the value of m. Also find the remainder.

4.5 Factor Theorem

Factor Theorem

Let $p(x)$ be a polynomial and a be any real number. If $p(a) = 0$, then $(x - a)$ is a factor of $p(x)$.

Note If $(x - a)$ is a factor of $p(x)$, then $p(a) = 0$

Example 4.13

Determine whether $(x - 5)$ is a factor of the polynomial $p(x) = 2x^3 - 5x^2 - 28x + 15$.

Solution By factor theorem, if $p(5) = 0$, then $(x - 5)$ is a factor of $p(x)$. Now,

$$p(5) = 2(5)^3 - 5(5)^2 - 28(5) + 15$$

$$= 2(125) - 5(25) - 140 + 15$$

$$= 250 - 125 - 140 + 15 = 0$$

$\therefore$ $(x - 5)$ is a factor of $p(x) = 2x^3 - 5x^2 - 28x + 15$.

Example 4.14

Determine whether $(x-2)$ is a factor of the polynomial $2x^3-6x^2+5x+4$.

Solution Let $p(x)=2x^3-6x^2+5x+4$. By factor theorem,$(x-2)$ is a factor of $p(x)$ if $p(2)=0$. Now,

$$p(2) = 2(2)^3-6(2)^2+5(2)+4 = 2(8)-6(4)+10+4$$
$$= 16-24+10+4 = 6 \neq 0$$

$\therefore$ $(x-2)$ is not a factor of $2x^3-6x^2+5x+5$.

Example 4.15

Determine whether $(2x-3)$ is a factor of $2x^3-9x^2+x+12$.

Solution Let $p(x)=2x^3-9x^2+x+12$. By factor theorem, $(2x-3)$ is a factor of $p(x)$ if $p\left(\frac{3}{2}\right)=0$. Now,

$$p\left(\frac{3}{2}\right) = 2\left(\frac{3}{2}\right)^3-9\left(\frac{3}{2}\right)^2+\frac{3}{2}+12 = 2\left(\frac{27}{8}\right)-9\left(\frac{9}{4}\right)+\frac{3}{2}+12$$
$$= \frac{27}{4}-\frac{81}{4}+\frac{3}{2}+12 \qquad = \frac{27-81+6+48}{4} = 0$$

$\therefore$ $(2x-3)$ is a factor of $2x^3-9x^2+x+12$.

Example 4.16

Determine the value of m if $(x-1)$ is a factor of x^3+5x^2+mx+4.

Solution Let $p(x)=x^3+5x^2+mx+4$. Since $(x-1)$ is a factor of $p(x)$, the remainder $p(1)=0$. Now,

$$p(1)=0 \Longrightarrow (1)^3+5(1)^2+m(1)+4=0$$
$$\Longrightarrow 1+5+m+4 = 0$$
$$m+10 = 0$$
$$\therefore\ m = -10$$

Exercise 4.4

1. Determine whether $(x+1)$ is a factor of the following polynomials

   (i) $6x^4+7x^3-5x-4$ (ii) $2x^4+9x^3+2x^2+10x+15$

   (iii) $3x^3+8x^2-6x-5$ (iv) $x^3-14x^2+3x+12$

2. Determine whether $(x+4)$ is a factor of $x^3+3x^2-5x+36$.

3. Using factor theorem show that $(x-1)$ is a factor of $4x^3-6x^2+9x-7$.

4. Determine whether $(2x+1)$ is a factor of $4x^3+4x^2-x-1$.

5. Determine the value of p if $(x+3)$ is a factor of $x^3-3x^2-px+24$.

4.6 Algebraic Identities

Key Concept — **Algebraic Identities**

An identity is an equality that remains true regardless of the values of any variables that appear within it.

We have learnt the following identities in class VIII. Using these identities let us solve some problems and extend the identities to trinomials and third degree expansions.

$(a+b)^2 \equiv a^2 + 2ab + b^2$	$(a+b)(a-b) \equiv a^2 - b^2$
$(a-b)^2 \equiv a^2 - 2ab + b^2$	$(x+a)(x+b) \equiv x^2 + (a+b)x + ab$

Example 4.17

Expand the following using identities

(i) $(2a+3b)^2$ (ii) $(3x-4y)^2$ (iii) $(4x+5y)(4x-5y)$ (iv) $(y+7)(y+5)$

Solution

(i) $(2a+3b)^2 = (2a)^2 + 2(2a)(3b) + (3b)^2$

$= 4a^2 + 12ab + 9b^2$

(ii) $(3x-4y)^2 = (3x)^2 - 2(3x)(4y) + (4y)^2$

$= 9x^2 - 24xy + 16y^2$

(iii) $(4x+5y)(4x-5y) = (4x)^2 - (5y)^2$

$= 16x^2 - 25y^2$

(iv) $(y+7)(y+5) = y^2 + (7+5)y + (7)(5)$

$= y^2 + 12y + 35$

4.6.1 Expansion of the Trinomial $(x \pm y \pm z)^2$

$$(x+y+z)^2 = (x+y+z)(x+y+z)$$

$$= x(x+y+z) + y(x+y+z) + z(x+y+z)$$

$$= x^2 + xy + xz + yx + y^2 + yz + zx + zy + z^2$$

$$= x^2 + y^2 + z^2 + 2xy + 2yz + 2zx$$

$$(x+y+z)^2 \equiv x^2 + y^2 + z^2 + 2xy + 2yz + 2zx$$

(ii) $(x-y+z)^2 = [x+(-y)+z]^2$

$= x^2+(-y)^2+z^2+2(x)(-y)+2(-y)(z)+2(z)(x)$

$= x^2+y^2+z^2-2xy-2yz+2zx$

$(x-y+z)^2 \equiv x^2+y^2+z^2-2xy-2yz+2zx$

In the same manner we get the expansion for the following

(iii) $(x+y-z)^2 \equiv x^2+y^2+z^2+2xy-2yz-2zx$

(iv) $(x-y-z)^2 \equiv x^2+y^2+z^2-2xy+2yz-2zx$

Example 4.18

Expand (i) $(2x+3y+5z)^2$ (ii) $(3a-7b+4c)^2$ (iii) $(3p+5q-2r)^2$

(iv) $(7l-9m-6n)^2$

Solution

(i) $(2x+3y+5z)^2 = (2x)^2+(3y)^2+(5z)^2+2(2x)(3y)+2(3y)(5z)+2(5z)(2x)$

$= 4x^2+9y^2+25z^2+12xy+30yz+20zx$

(ii) $(3a-7b+4c)^2$

$= (3a)^2+(-7b)^2+(4c)^2+2(3a)(-7b)+2(-7b)(4c)+2(4c)(3a)$

$= 9a^2+49b^2+16c^2-42ab-56bc+24ca$

(iii) $(3p+5q-2r)^2$

$= (3p)^2+(5q)^2+(-2r)^2+2(3p)(5q)+2(5q)(-2r)+2(-2r)(3p)$

$= 9p^2+25q^2+4r^2+30pq-20qr-12rp$

(iv) $(7l-9m-6n)^2$

$= (7l)^2+(-9m)^2+(-6n)^2+2(7l)(-9m)+2(-9m)(-6n)+2(-6n)(7l)$

$= 49l^2+81m^2+36n^2-126lm+108mn-84nl$

4.6.2 Identities Involving Product of Binomials $(x+a)(x+b)(x+c)$

$(x+a)(x+b)(x+c) = [(x+a)(x+b)](x+c)$

$= [x^2+(a+b)x+ab](x+c)$

$= x^3+(a+b)x^2+abx+cx^2+c(a+b)x+abc$

$= x^3+(a+b+c)x^2+(ab+bc+ca)x+abc$

$$(x+a)(x+b)(x+c) \equiv x^3+(a+b+c)x^2+(ab+bc+ca)x+abc$$

4.6.3 Expansion of $(x \pm y)^3$

In the above identity by substituting $a = b = c = y$, we get

$$(x+y)(x+y)(x+y) = x^3 + (y+y+y)x^2 + [(y)(y)+(y)(y)+(y)(y)]x + (y)(y)(y)$$

$$(x+y)^3 = x^3 + (3y)x^2 + (3y^2)x + y^3$$

$$= x^3 + 3x^2y + 3xy^2 + y^3$$

$$(x+y)^3 \equiv x^3 + 3x^2y + 3xy^2 + y^3$$

(or) $$(x+y)^3 \equiv x^3 + y^3 + 3xy(x+y)$$

Replacing y *by* $-y$ in the above identity, we get

$$(x-y)^3 \equiv x^3 - 3x^2y + 3xy^2 - y^3$$

(or) $$(x-y)^3 \equiv x^3 - y^3 - 3xy(x-y)$$

Using these identities of 4.4.2 and 4.4.3, let us solve the following problems.

Example 4.19

Find the product of (i) $(x+2)(x+5)(x+7)$

(ii) $(a-3)(a-5)(a-7)$

(iii) $(2a-5)(2a+5)(2a-3)$

Solution

(i) $(x+2)(x+5)(x+7)$

$$= x^3 + (2+5+7)x^2 + [(2)(5)+(5)(7)+(7)(2)]x + (2)(5)(7)$$

$$= x^3 + 14x^2 + (10+35+14)x + 70$$

$$= x^3 + 14x^2 + 59x + 70$$

(ii) $(a-3)(a-5)(a-7) = [a+(-3)][a+(-5)][a+(-7)]$

$$= a^3 + (-3-5-7)a^2 + [(-3)(-5)+(-5)(-7)+(-7)(-3)]a + (-3)(-5)(-7)$$

$$= a^3 - 15a^2 + (15+35+21)a - 105$$

$$= a^3 - 15a^2 + 71a - 105$$

(iii) $(2a-5)(2a+5)(2a-3) = [2a+(-5)][2a+5][2a+(-3)]$

$$= (2a)^3 + (-5+5-3)(2a)^2 + [(-5)(5)+(5)(-3)+(-3)(-5)](2a) + (-5)(5)(-3)$$

$$= 8a^3 + (-3)4a^2 + (-25-15+15)2a + 75$$

$$= 8a^3 - 12a^2 - 50a + 75$$

Example 4.20

If $a + b + c = 15$, $ab + bc + ca = 25$ find $a^2 + b^2 + c^2$.

Solution We have $(a+b+c)^2 = a^2 + b^2 + c^2 + 2(ab + bc + ca)$. So,

$$15^2 = a^2 + b^2 + c^2 + 2(25)$$

$$225 = a^2 + b^2 + c^2 + 50$$

$$\therefore a^2 + b^2 + c^2 = 225 - 50 = 175$$

Example 4.21

Expand (i) $(3a+4b)^3$ (ii) $(2x-3y)^3$

Solution

(i) $(3a+4b)^3 = (3a)^3 + 3(3a)^2(4b) + 3(3a)(4b)^2 + (4b)^3$

$= 27a^3 + 108a^2b + 144ab^2 + 64b^3$

(ii) $(2x-3y)^3 = (2x)^3 - 3(2x)^2(3y) + 3(2x)(3y)^2 - (3y)^3$

$= 8x^3 - 36x^2y + 54xy^2 - 27y^3$

Example 4.22

Evaluate each of the following using suitable identities.

(i) $(105)^3$ (ii) $(999)^3$

Solution

(i) $(105)^3 = (100+5)^3$

$= (100)^3 + (5)^3 + 3(100)(5)(100+5)$ $(\because (x+y)^3 = x^3 + y^3 + 3xy(x+y))$

$= 1000000 + 125 + 1500(105)$

$= 1000000 + 125 + 157500 = 1157625$

(ii) $(999)^3 = (1000-1)^3$

$= (1000)^3 - (1)^3 - 3(1000)(1)(1000-1)$

$(\because (x-y)^3 = x^3 - y^3 - 3xy(x-y))$

$= 1000000000 - 1 - 3000(999)$

$= 1000000000 - 1 - 2997000 = 997002999$

Some Useful Identities involving sum ,difference and product of x and y

$$x^3 + y^3 \equiv (x+y)^3 - 3xy(x+y)$$

$$x^3 - y^3 \equiv (x-y)^3 + 3xy(x-y)$$

Let us solve some problems involving above identities.

Example 4.23

Find $x^3 + y^3$ if $x + y = 4$ and $xy = 5$

Solution We know that $x^3 + y^3 = (x + y)^3 - 3xy(x + y)$

$$\therefore \quad x^3 + y^3 = (4)^3 - 3(5)(4) = 64 - 60 = 4$$

Example 4.24

Find $x^3 - y^3$ if $x - y = 5$ and $xy = 16$

Solution We know that $x^3 - y^3 = (x - y)^3 + 3xy(x - y)$

$$\therefore \quad x^3 - y^3 = (5)^3 + 3(16)(5) = 125 + 240 = 365$$

Example 4.25

If $x + \frac{1}{x} = 5$, find the value of $x^3 + \frac{1}{x^3}$

Solution We know that $x^3 + y^3 = (x + y)^3 - 3xy(x + y)$

$$\therefore \quad x^3 + \frac{1}{x^3} = \left(x + \frac{1}{x}\right)^3 - 3\left(x + \frac{1}{x}\right)$$

$$= (5)^3 - 3(5) = 125 - 15 = 110$$

Example 4.26

If $y - \frac{1}{y} = 9$, find the value of $y^3 - \frac{1}{y^3}$

Solution We know that, $x^3 - y^3 = (x - y)^3 + 3xy(x - y)$

$$\therefore \quad y^3 - \frac{1}{y^3} = \left(y - \frac{1}{y}\right)^3 + 3\left(y - \frac{1}{y}\right)$$

$$= (9)^3 + 3(9) = 729 + 27 = 756$$

The following identity is frequently used in higher studies

$$x^3 + y^3 + z^3 - 3xyz \equiv (x + y + z)(x^2 + y^2 + z^2 - xy - yz - zx)$$

Note If $x + y + z = 0$ then $x^3 + y^3 + z^3 = 3xyz$

Example 4.27

Simplify $(x + 2y + 3z)(x^2 + 4y^2 + 9z^2 - 2xy - 6yz - 3zx)$

Solution We know that, $(x + y + z)(x^2 + y^2 + z^2 - xy - yz - zx) = x^3 + y^3 + z^3 - 3xyz$

$$\therefore \quad (x + 2y + 3z)(x^2 + 4y^2 + 9z^2 - 2xy - 6yz - 3zx)$$

$$= (x + 2y + 3z)[x^2 + (2y)^2 + (3z)^2 - (x)(2y) - (2y)(3z) - (3z)(x)]$$

$$= (x)^3 + (2y)^3 + (3z)^3 - 3(x)(2y)(3z)$$

$$= x^3 + 8y^3 + 27z^3 - 18xyz$$

Example 4.28

Evaluate $12^3 + 13^3 - 25^3$

Solution Let $x = 12,\ y = 13,\ z = -25$. Then

$$x + y + z = 12 + 13 - 25 = 0$$

If $x + y + z = 0$, then $x^3 + y^3 + z^3 = 3xyz$.

$$\therefore\ 12^3 + 13^3 - 25^3 = 12^3 + 13^3 + (-25)^3 = 3(12)(13)(-25) = -11700$$

Exercise 4.5

1. Expand the following

   (i) $(5x + 2y + 3z)^2$ (ii) $(2a + 3b - c)^2$ (iii) $(x - 2y - 4z)^2$ (iv) $(p - 2q + r)^2$

2. Find the expansion of

   (i) $(x + 1)(x + 4)(x + 7)$ (ii) $(p + 2)(p - 4)(p + 6)$

   (iii) $(x + 5)(x - 3)(x - 1)$ (iv) $(x - a)(x - 2a)(x - 4a)$

   (v) $(3x + 1)(3x + 2)(3x + 5)$ (vi) $(2x + 3)(2x - 5)(2x - 7)$

3. Using algebraic identities find the coefficients of x^2 term, x term and constant term.

   (i) $(x + 7)(x + 3)(x + 9)$ (ii) $(x - 5)(x - 4)(x + 2)$

   (iii) $(2x + 3)(2x + 5)(2x + 7)$ (iv) $(5x + 2)(1 - 5x)(5x + 3)$

4. If $(x + a)(x + b)(x + c) \equiv x^3 - 10x^2 + 45x - 15$ find $a + b + c$, $\frac{1}{a} + \frac{1}{b} + \frac{1}{c}$ and $a^2 + b^2 + c^2$.

5. Expand : (i) $(3a + 5b)^3$ (ii) $(4x - 3y)^3$ (iii) $\left(2y - \frac{3}{y}\right)^3$

6. Evaluate : (i) 99^3 (ii) 101^3 (iii) 98^3 (iv) 102^3 (v) 1002^3

7. Find $8x^3 + 27y^3$ if $2x + 3y = 13$ and $xy = 6$.

8. If $x - y = -6$ and $xy = 4$, find the value of $x^3 - y^3$.

9. If $x + \frac{1}{x} = 4$, find the value of $x^3 + \frac{1}{x^3}$.

10. If $x - \frac{1}{x} = 3$, find the value of $x^3 - \frac{1}{x^3}$.

11. Simplify : (i) $(2x + y + 4z)(4x^2 + y^2 + 16z^2 - 2xy - 4yz - 8zx)$

    (ii) $(x - 3y - 5z)(x^2 + 9y^2 + 25z^2 + 3xy - 15yz + 5zx)$

12. Evaluate using identities : (i) $6^3 - 9^3 + 3^3$ (ii) $16^3 - 6^3 - 10^3$

4.7 Factorization of Polynomials

We have seen how the distributive property may be used to expand a product of algebraic expressions into sum or difference of expressions.

For example,

(i) $x(x+y) = x^2 + xy$

(ii) $x(y-z) = xy - xz$

(iii) $a(a^2 - 2a + 1) = a^3 - 2a^2 + a$

Now, we will learn how to convert a sum or difference of expressions into a product of expressions.

Now, consider $ab + ac$. Using the distributive law, $a(b+c) = ab + ac$, by writing in the reverse direction $ab + ac$ is $a(b+c)$.This process of expressing $ab + ac$ into $a(b+c)$ is known as factorization. In both the terms, ab and ac 'a' is the common factor. Similarly,

$5m + 15 = 5(m) + 5(3) = 5(m+3)$.

In $b(b-5) + g(b-5)$ clearly $(b-5)$ is a common factor.

$b(b-5) + g(b-5) = (b-5)(b+g)$

Example 4.29

Factorize the following

(i) $pq + pr - 3ps$ (ii) $4a - 8b + 5ax - 10bx$ (iii) $2a^3 + 4a^2$ (iv) $6a^5 - 18a^3 + 42a^2$

Solution

(i) $pq + pr - 3ps = p(q + r - 3s)$

(ii) $4a - 8b + 5ax - 10bx = (4a - 8b) + (5ax - 10bx)$

$= 4(a-2b) + 5x(a-2b) = (a-2b)(4+5x)$

(iii) $2a^3 + 4a^2$

Highest common factor is $2a^2$

$\therefore 2a^3 + 4a^2 = 2a^2(a+2)$.

(iv) $6a^5 - 18a^3 + 42a^2$

Highest common factor is $6a^2$

$\therefore 6a^5 - 18a^3 + 42a^2 = 6a^2(a^3 - 3a + 7)$

4.7.1 Factorization Using Identities

(i) $a^2 + 2ab + b^2 \equiv (a+b)^2$

(ii) $a^2 - 2ab + b^2 \equiv (a-b)^2$ (or) $a^2 - 2ab + b^2 \equiv (-a+b)^2$

(iii) $a^2 - b^2 \equiv (a+b)(a-b)$

(iv) $a^2 + b^2 + c^2 + 2ab + 2bc + 2ca \equiv (a+b+c)^2$

Example 4.30

Factorize (i) $4x^2 + 12xy + 9y^2$ (ii) $16a^2 - 8a + 1$ (iii) $9a^2 - 16b^2$

(iv) $(a+b)^2 - (a-b)^2$ (v) $25(a+2b-3c)^2 - 9(2a-b-c)^2$ (vi) $x^5 - x$

Solution

(i) $4x^2 + 12xy + 9y^2 = (2x)^2 + 2(2x)(3y) + (3y)^2 = (2x+3y)^2$

(ii) $16a^2 - 8a + 1 = (4a)^2 - 2(4a)(1) + (1)^2 = (4a-1)^2$ or $(1-4a)^2$

(iii) $9a^2 - 16b^2 = (3a)^2 - (4b)^2 = (3a+4b)(3a-4b)$

(iv) $(a+b)^2 - (a-b)^2 = [(a+b)+(a-b)][(a+b)-(a-b)]$

$= (a+b+a-b)(a+b-a+b) = (2a)(2b) = (4)(a)(b)$

(v) $25(a+2b-3c)^2 - 9(2a-b-c)^2 = [5(a+2b-3c)]^2 - [3(2a-b-c)]^2$

$= [5(a+2b-3c) + 3(2a-b-c)][5(a+2b-3c) - 3(2a-b-c)]$

$= (5a+10b-15c+6a-3b-3c)(5a+10b-15c-6a+3b+3c)$

$= (11a+7b-18c)(-a+13b-12c)$

(vi) $x^5 - x = x(x^4-1) = x[(x^2)^2 - (1)^2]$

$= x(x^2+1)(x^2-1) = x(x^2+1)[(x)^2 - (1)^2]$

$= x(x^2+1)(x+1)(x-1)$

4.7.2 Factorization Using the Identity

$a^2 + b^2 + c^2 + 2ab + 2bc + 2ca \equiv (a+b+c)^2$

Example 4.31

Factorize $a^2 + 4b^2 + 36 - 4ab - 24b + 12a$

Solution $a^2 + 4b^2 + 36 - 4ab - 24b + 12a$ can be written as

$(a)^2 + (-2b)^2 + (6)^2 + 2(a)(-2b) + 2(-2b)(6) + 2(6)(a) = (a-2b+6)^2$ or

$(-a)^2 + (2b)^2 + (-6)^2 + 2(-a)(2b) + 2(2b)(-6) + 2(-6)(-a) = (-a+2b-6)^2$

That is $(a-2b+6)^2 = [(-1)(-a+2b-6)^2] = (-1)^2(-a+2b-6)^2 = (-a+2b-6)^2$

Example 4.32

Factorize $4x^2 + y^2 + 9z^2 - 4xy + 6yz - 12zx$

Solution $4x^2 + y^2 + 9z^2 - 4xy + 6yz - 12zx$

$= (2x)^2 + (-y)^2 + (-3z)^2 + 2(2x)(-y) + 2(-y)(-3z) + 2(-3z)(2x)$

$= (2x-y-3z)^2$ or $(-2x+y+3z)^2$

4.7.3 Factorization of $x^3 + y^3$ and $x^3 - y^3$

We have $x^3 + 3x^2y + 3xy^2 + y^3 = (x+y)^3$. So,

$$x^3 + y^3 + 3xy(x+y) = (x+y)^3$$

$$\Longrightarrow x^3 + y^3 = (x+y)^3 - 3xy(x+y)$$

$$= (x+y)[(x+y)^2 - 3xy]$$

$$= (x+y)(x^2 + 2xy + y^2 - 3xy)$$

$$= (x+y)(x^2 - xy + y^2)$$

$$x^3 + y^3 \equiv (x+y)(x^2 - xy + y^2)$$

We have $x^3 - 3x^2y + 3xy^2 - y^3 = (x-y)^3$. So,

$$x^3 - y^3 - 3xy(x-y) = (x-y)^3$$

$$\Longrightarrow x^3 - y^3 = (x-y)^3 + 3xy(x-y)$$

$$= (x-y)[(x-y)^2 + 3xy]$$

$$= (x-y)(x^2 - 2xy + y^2 + 3xy)$$

$$= (x-y)(x^2 + xy + y^2)$$

$$x^3 - y^3 \equiv (x-y)(x^2 + xy + y^2)$$

Using the above identities let us factorize the following expressions.

Example 4.33

Factorize (i) $8x^3 + 125y^3$ (ii) $27x^3 - 64y^3$

Solution

(i) $8x^3 + 125y^3 = (2x)^3 + (5y)^3$

$$= (2x+5y)[(2x)^2 - (2x)(5y) + (5y)^2]$$

$$= (2x+5y)(4x^2 - 10xy + 25y^2)$$

(ii) $27x^3 - 64y^3 = (3x)^3 - (4y)^3$

$$= (3x-4y)[(3x)^2 + (3x)(4y) + (4y)^2]$$

$$= (3x-4y)(9x^2 + 12xy + 16y^2)$$

Exercise 4.6

1. Factorize the following expressions:

   (i) $2a^3 - 3a^2b + 2a^2c$ (ii) $16x + 64x^2y$ (iii) $10x^3 - 25x^4y$

   (iv) $xy - xz + ay - az$ (v) $p^2 + pq + pr + qr$

2. Factorize the following expressions:

   (i) $x^2 + 2x + 1$ (ii) $9x^2 - 24xy + 16y^2$

   (iii) $b^2 - 4$ (iv) $1 - 36x^2$

3. Factorize the following expressions:

   (i) $p^2 + q^2 + r^2 + 2pq + 2qr + 2rp$ (ii) $a^2 + 4b^2 + 36 - 4ab + 24b - 12a$

   (iii) $9x^2 + y^2 + 1 - 6xy + 6x - 2y$ (iv) $4a^2 + b^2 + 9c^2 - 4ab - 6bc + 12ca$

   (v) $25x^2 + 4y^2 + 9z^2 - 20xy + 12yz - 30zx$

4. Factorize the following expressions:

   (i) $27x^3 + 64y^3$ (ii) $m^3 + 8$ (iii) $a^3 + 125$

   (iv) $8x^3 - 27y^3$ (v) $x^3 - 8y^3$

4.7.4 Factorization of the Quadratic Polynomials of the type $ax^2 + bx + c;\ a \neq 0$

So far we have used the identities to factorize certain types of polynomials. In this section we will learn, without identities how to resolve quadratic polynomials into two linear polynomials when (i) $a = 1$ and (ii) $a \neq 1$

(i) Factorizing the quadratic polynomials of the type $x^2 + bx + c$.

suppose $(x + p)$ and $(x + q)$ are the two factors of $x^2 + bx + c$.Then we have

$$x^2 + bx + c = (x + p)(x + q)$$
$$= x(x + p) + q(x + p)$$
$$= x^2 + px + qx + pq$$
$$= x^2 + (p + q)x + pq$$

This implies that the two numbers p and q are chosen in such way that $c = pq$ and $b = p + q$.

to get $x^2 + bx + c = (x + p)(x + q)$

We use this basic idea to factorize the following problems

For example,

(1) $x^2 + 8x + 15 = (x+3)(x+5)$

here $c = 15 = 3 \times 5$ and $3 + 5 = 8 = b$

(2) $x^2 - 5x + 6 = (x-2)(x-3)$

here $c = 6 = (-2)\times(-3)$ and $(-2)+(-3) = -5 = b$

(3) $x^2 + x - 2 = (x+2)(x-1)$

here $c = -2 = (+2)\times(-1)$ and $(+2)+(-1) = 1 = b$

(4) $x^2 - 4x - 12 = (x-6)(x+2)$

here $c = -12 = (-6)\times(+2)$ and $(-6)+(+2) = -4 = b$

In the above examples the constant term is split into two factors such that their sum is equal to the coefficients of x.

Example 4.34

Factorize the following.

(i) $x^2 + 9x + 14$ (ii) $x^2 - 9x + 14$ (iii) $x^2 + 2x - 15$ (iv) $x^2 - 2x - 15$

Solution

(i) $x^2 + 9x + 14$

To factorize we have to find p and q, such that $pq = 14$ and $p + q = 9$.

$$x^2 + 9x + 14 = x^2 + 2x + 7x + 14$$
$$= x(x+2) + 7(x+2)$$
$$= (x+2)(x+7)$$
$$\therefore x^2 + 9x + 14 = (x+7)(x+2)$$

Factors of 14	Sum of factors
1, 14	15
2, 7	9
The required factors are 2, 7	

(ii) $x^2 - 9x + 14$

To factorize we have to find p and q such that $pq = 14$ and $p + q = -9$

$$x^2 - 9x + 14 = x^2 - 2x - 7x + 14$$
$$= x(x-2) - 7(x-2)$$
$$= (x-2)(x-7)$$
$$\therefore x^2 - 9x + 14 = (x-2)(x-7)$$

Factors of 14	Sum of factors
–1, –14	–15
–2, –7	–9
The required factors are –2, –7	

(iii) $x^2 + 2x - 15$

To factorize we have to find p and q, such that $pq = -15$ and $p + q = 2$

$$x^2 + 2x - 15 = x^2 - 3x + 5x - 15$$
$$= x(x-3) + 5(x-3)$$
$$= (x-3)(x+5)$$
$$\therefore\ x^2 + 2x - 15 = (x-3)(x+5)$$

Factors of –15	Sum of factors
–1, 15	14
–3, 5	2
The required factors are –3, 5	

(iv) $x^2 - 2x - 15$

To factorize we have to find p and q, such that $pq = -15$ and $p + q = -2$

$$x^2 - 2x - 15 = x^2 + 3x - 5x - 15$$
$$= x(x+3) - 5(x+3)$$
$$= (x+3)(x-5)$$
$$\therefore\ x^2 - 2x - 15 = (x+3)(x-5)$$

Factors of –15	Sum of factors
1, –15	–14
3, –5	–2
The required factors are 3, –5	

(ii) Factorizing the quadratic polynomials of the type $ax^2 + bx + c$.

Since a is different from 1, the linear factors of $ax^2 + bx + c$ will be of the form $(rx + p)$ and $(sx + q)$.

Then, $ax^2 + bx + c = (rx+p)(sx+q)$
$$= rsx^2 + (ps + qr)x + pq$$

Comparing the coefficients of x^2, we get $a = rs$. Similarly, comparing the coefficients of x, we get $b = ps + qr$. And, on comparing the constant terms, we get $c = pq$.

This shows us that b is the sum of two numbers ps and qr, whose product is $(ps)\times(qr) = (pr)\times(sq) = ac$

Therefore, to factorize $ax^2 + bx + c$, we have to write b as the sum of two numbers whose product is ac (= b)

The following steps to be followed to factorize $ax^2 + bx + c$

Step1 : Multiply the coefficient of x^2 and constant term (= ac) .

Step2 : Split this product into two factors such that their sum is equal to the coefficient of x.

Step3 : The terms are grouped into two pairs and factorize.

Example 4.35

Factorize the following

(i) $2x^2 + 15x + 27$ (ii) $2x^2 - 15x + 27$

(iii) $2x^2 + 15x - 27$ (iv) $2x^2 - 15x - 27$

Solution

(i) $2x^2 + 15x + 27$

Coefficient of $x^2 = 2$; constant term = 27

Their product $= 2 \times 27 = 54$

Coefficient of $x = 15$

$\therefore$ product = 54; sum = 15

Factors of 54	Sum of factors
1, 54	55
2, 27	29
3, 18	21
6, 9	15
The required factors are 6, 9	

$2x^2 + 15x + 27 = 2x^2 + 6x + 9x + 27$

$= 2x(x + 3) + 9(x + 3)$

$= (x + 3)(2x + 9)$

$\therefore\ 2x^2 + 15x + 27 = (x + 3)(2x + 9)$

(ii) $2x^2 - 15x + 27$

Coefficient of $x^2 = 2$; constant term = 27

Their product $= 2 \times 27 = 54$

Coefficient of $x = -15$

$\therefore$ product = 54 ; sum = -15

Factors of 54	Sum of factors
–1, –54	–55
–2, –27	–29
–3, –18	–21
–6, –9	–15
The required factors are –6, –9	

$2x^2 - 15x + 27 = 2x^2 - 6x - 9x + 27$

$= 2x(x - 3) - 9(x - 3)$

$= (x - 3)(2x - 9)$

$\therefore\ 2x^2 - 15x + 27 = (x - 3)(2x - 9)$

(iii) $2x^2 + 15x - 27$

Coefficient of $x^2 = 2$; constant term $= -27$

Their product $= 2 \times -27 = -54$

Coefficient of $x = 15$

$\therefore$ product = –54; sum = 15

Factors of –54	Sum of factors
–1, 54	53
–2, 27	25
–3, 18	15
The required factors are –3, 18	

$$2x^2 + 15x - 27 = 2x^2 - 3x + 18x - 27$$
$$= x(2x - 3) + 9(2x - 3)$$
$$= (2x - 3)(x + 9)$$
$$\therefore\ 2x^2 + 15x - 27 = (2x - 3)(x + 9)$$

(iv) $2x^2 - 15x - 27$

Coefficient of $x^2 = 2$; constant term $= -27$

Their product $= 2 \times -27 = -54$

Coefficient of $x = -15$

$\therefore$ product = –54; sum = –15

Factors of –54	Sum of factors
1, –54	–53
2, –27	–25
3, –18	–15
The required factors are 3, –18	

$$2x^2 - 15x - 27 = 2x^2 + 3x - 18x - 27$$
$$= x(2x + 3) - 9(2x + 3)$$
$$= (2x + 3)(x - 9)$$
$$\therefore\ 2x^2 - 15x - 27 = (2x + 3)(x - 9)$$

Example 4.36

Factorize $(x + y)^2 + 9(x + y) + 8$

Solution Let $x + y = p$

Then the equation is $p^2 + 9p + 8$

Coefficient of $p^2 = 1$; constant term = 8

Their product= $1 \times 8 = 8$

Coefficient of p= 9

Factors of 8	Sum of factors
1, 8	9
The required factors are 1, 8	

$\therefore$ product = 8; sum = 9

$$p^2 + 9p + 8 = p^2 + p + 8p + 8$$
$$= p(p + 1) + 8(p + 1)$$
$$= (p + 1)(p + 8)$$

substituting, $p = x + y$

$$\therefore\ (x + y)^2 + 9(x + y) + 8 = (x + y + 1)(x + y + 8)$$

Example 4.37

Factorize : (i) $x^3 - 2x^2 - x + 2$ (ii) $x^3 + 3x^2 - x - 3$

Solution

(i) Let $p(x) = x^3 - 2x^2 - x + 2$

$p(x)$ is a cubic polynomial, so it may have three linear factors.

The constant term is 2. The factors of 2 are –1, 1, –2 and 2.

$p(-1) = (-1)^3 - 2(-1)^2 - (-1) + 2 = -1 - 2 + 1 + 2 = 0$

$\therefore$ $(x + 1)$ is a factor of $p(x)$.

$p(1) = (1)^3 - 2(1)^2 - 1 + 2 = 1 - 2 - 1 + 2 = 0$

$\therefore$ $(x - 1)$ is a factor of $p(x)$.

$p(-2) = (-2)^3 - 2(-2)^2 - (-2) + 2 = -8 - 8 + 2 + 2 = -12 \neq 0$

$\therefore$ $(x + 2)$ is not a factor of $p(x)$.

$p(2) = (2)^3 - 2(2)^2 - 2 + 2 = 8 - 8 - 2 + 2 = 0$

$\therefore$ $(x - 2)$ is a factor of $p(x)$.

The three factors of $p(x)$ are $(x + 1), (x - 1)$ and $(x - 2)$

$\therefore$ $x^3 - 2x^2 - x + 2 = (x + 1)(x - 1)(x - 2)$.

Another method

$$x^3 - 2x^2 - x + 2 = x^2(x - 2) - 1(x - 2)$$

$$= (x - 2)(x^2 - 1)$$

$$= (x - 2)(x + 1)(x - 1) \qquad [(\because a^2 - b^2 = (a + b)(a - b)]$$

(ii) Let $p(x) = x^3 + 3x^2 - x - 3$

$p(x)$ is a cubic polynomial, so it may have three linear factors.

The constant term is –3. The factors of –3 are –1, 1,–3 and 3.

$p(-1) = (-1)^3 + 3(-1)^2 - (-1) - 3 = -1 + 3 + 1 - 3 = 0$

$\therefore (x + 1)$ is a factor of $p(x)$.

$p(1) = (1)^3 + 3(1)^2 - 1 - 3 = 1 + 3 - 1 - 3 = 0$

$\therefore (x - 1)$ is a factor of $p(x)$.

$p(-3) = (-3)^3 + 3(-3)^2 - (-3) - 3 = -27 + 27 + 3 - 3 = 0$

$\therefore (x + 3)$ is a factor of $p(x)$.

The three factors of $p(x)$ are $(x + 1), (x - 1)$ and $(x + 3)$

$\therefore x^3 + 3x^2 - x - 3 = (x + 1)(x - 1)(x + 3)$.

Exercise 4.7

1. Factorize each of the following.

(i) $x^2 + 15x + 14$ (ii) $x^2 + 13x + 30$ (iii) $y^2 + 7y + 12$

(iv) $x^2 - 14x + 24$ (v) $y^2 - 16y + 60$ (vi) $t^2 - 17t + 72$

(vii) $x^2 + 14x - 15$ (viii) $x^2 + 9x - 22$ (ix) $y^2 + 5y - 36$

(x) $x^2 - 2x - 99$ (xi) $m^2 - 10m - 144$ (xii) $y^2 - y - 20$

2. Factorize each of the following.

(i) $3x^2 + 19x + 6$ (ii) $5x^2 + 22x + 8$ (iii) $2x^2 + 9x + 10$

(iv) $14x^2 + 31x + 6$ (v) $5y^2 - 29y + 20$ (vi) $9y^2 - 16y + 7$

(vii) $6x^2 - 5x + 1$ (viii) $3x^2 - 10x + 8$ (ix) $3x^2 + 5x - 2$

(x) $2a^2 + 17a - 30$ (xi) $11 + 5x - 6x^2$ (xii) $8x^2 + 29x - 12$

(xiii) $2x^2 - 3x - 14$ (xiv) $18x^2 - x - 4$ (xv) $10 - 7x - 3x^2$

3. Factorize the following

(i) $(a+b)^2 + 9(a+b) + 14$ (ii) $(p-q)^2 - 7(p-q) - 18$

4. Factorize the following

(i) $x^3 + 2x^2 - x - 2$ (ii) $x^3 - 3x^2 - x + 3$

(iii) $x^3 + x^2 - 4x - 4$ (iv) $x^3 + 5x^2 - x - 5$

4.8 Linear Equations

Recall the linear equations in one variable is of the form $ax + b = 0$, where a, b are constants and $a \neq 0$.

For example, solving $3x + 2 = 8$

$\Longrightarrow 3x = 8 - 2 \Longrightarrow 3x = 6 \Longrightarrow x = \frac{6}{3} \Longrightarrow x = 2$

In fact a linear equation in one variable has a unique solution.

4.8.1 Pair of Linear Equations in Two Variables

In general linear equation in two variables x and y is of the form $ax + by = c$ where a, b and c are constants and $a \neq 0$, $b \neq 0$.

Let us consider a pair of linear equations in two variables x and y.

$$a_1x + b_1y = c_1 \qquad (1)$$

$$a_2x + b_2y = c_2 \qquad (2)$$

Where a_1, a_2, b_1, b_2, c_1 and c_2 are constants and $a_1 \neq 0$, $b_1 \neq 0$, $a_2 \neq 0$ and $b_2 \neq 0$.

If an ordered pair (x_0, y_0) satisfies both the equations, then (x_0, y_0) is called a solution of these equations. Hence, solving these equations involves the method of finding the ordered pair (x_0, y_0) that satisfies both the equations.

The substitution method, the elimination method and the cross-multiplication method are some of the methods commonly used to solve the system of equations.

In this chapter we consider only the substitution method to solve the linear equations in two variables.

Substitution method

In this method, one of the two variables is expressed interms of the other, using either of the equations. It is then substituted in the other equation and solved.

Example 4.38

Solve the following pair of equations by substitution method.

$2x + 5y = 2$ and $x + 2y = 3$

Solution We have $2x + 5y = 2$ (1)

$x + 2y = 3$ (2)

Equation (2) becomes, $x = 3 - 2y$ (3)

Substituting x in (1) we get, $2(3 - 2y) + 5y = 2$

$$\Longrightarrow 6 - 4y + 5y = 2$$

$$-4y + 5y = 2 - 6$$

$$\therefore\ y = -4$$

Substituting $y = -4$ in (3), we get, $x = 3 - 2(-4) = 3 + 8 = 11$

$\therefore$ The solution is $x = 11$ and $y = -4$

Example 4.39

Solve $x + 3y = 16,\ 2x - y = 4$ by using substitution method.

Solution

We have $x + 3y = 16$ (1)

$2x - y = 4$ (2)

Equation (1) becomes, $x = 16 - 3y$ (3)

Substituting x in (2) we get, $2(16 - 3y) - y = 4$

$$\Longrightarrow 32 - 6y - y = 4$$

$$-6y - y = 4 - 32$$

$$-7y = -28$$

$$y = \frac{-28}{-7} = 4$$

Substituting $y = 4$ in (3) we get, $x = 16 - 3(4)$

$$= 16 - 12 = 4$$

$\therefore$ The solution is $x = 4$ and $y = 4$.

Example 4.40

Solve by substitution method $\frac{1}{x} + \frac{1}{y} = 4$ and $\frac{2}{x} + \frac{3}{y} = 7, \quad x \neq 0, y \neq 0$

Solution

Let $\frac{1}{x} = a$ and $\frac{1}{y} = b$

The given equations become

$$a + b = 4 \qquad (1)$$

$$2a + 3b = 7 \qquad (2)$$

Equation (1) becomes $b = 4 - a$ (3)

Substituting b in (2) we get, $2a + 3(4 - a) = 7$

$$\Longrightarrow 2a + 12 - 3a = 7$$

$$2a - 3a = 7 - 12$$

$$-a = -5 \Longrightarrow a = 5$$

Substituting $a = 5$ in (3) we get, $b = 4 - 5 = -1$

But $\frac{1}{x} = a \Rightarrow x = \frac{1}{a} = \frac{1}{5}$

$\frac{1}{y} = b \Rightarrow y = \frac{1}{b} = \frac{1}{-1} = -1$

$\therefore$ The solution is $x = \frac{1}{5}, y = -1$

Example 4.41

The cost of a pen and a note book is ₹ 60. The cost of a pen is ₹ 10 less than that of a notebook. Find the cost of each.

Solution

Let the cost of a pen = ₹ x

Let the cost of a note book = ₹ y

From given data we have

$$x + y = 60 \qquad (1)$$

$$x = y - 10 \qquad (2)$$

Substituting x in (1) we get, $y - 10 + y = 60$

$\Longrightarrow y + y = 60 + 10 \Longrightarrow 2y = 70$

$\therefore y = \frac{70}{2} = 35$

Substituting $y = 35$ in (2) we get, $x = 35 - 10 = 25$

$\therefore$ The cost of a pen is ₹ 25.

The cost of a note book is ₹ 35.

Example 4.42

The cost of three mathematics books and four science books is ₹ 216. The cost of three mathematics books is the same as that of four science books. Find the cost of each book.

Solution

Let the cost of a mathematics book be ₹ x and cost of a science book be ₹ y.

By given data,

$3x + 4y = 216$ (1)

$3x = 4y$ (2)

The equation (2) becomes, $x = \frac{4y}{3}$ (3)

Substituting x in (1) we get, $3\left(\frac{4y}{3}\right) + 4y = 216$

$\Longrightarrow 4y + 4y = 216 \Longrightarrow 8y = 216$

$\therefore y = \frac{216}{8} = 27$

substituting $y = 27$ in (3) we get, $x = \frac{4(27)}{3} = 36$

$\therefore$ The cost of one mathematics book = ₹ 36.

The cost of one science book = ₹ 27.

Example 4.43

From Dharmapuri bus stand if we buy 2 tickets to Palacode and 3 tickets to Karimangalam the total cost is Rs 32, but if we buy 3 tickets to Palacode and one ticket to Karimangalam the total cost is Rs 27. Find the fares from Dharmapuri to Palacode and to Karimangalam.

Solution

Let the fare from Dharmapuri to Palacode be ₹ x and to Karimangalam be ₹ y.

From the given data, we have

$2x + 3y = 32$ (1)

$3x + y = 27$ (2)

Equation (2) becomes, $y = 27 - 3x$ (3)

Substituting y in (1) we get, $2x + 3(27 - 3x) = 32$

$$\Longrightarrow 2x + 81 - 9x = 32$$

$$2x - 9x = 32 - 81$$

$$-7x = -49$$

$$\therefore x = \frac{-49}{-7} = 7$$

Substituting $x = 7$ in (3) we get, $y = 27 - 3(7) = 27 - 21 = 6$

$\therefore$ The fare from Dharmapuri to Palacode is ₹ 7 and to Karimangalam is ₹ 6.

Example 4.44

The sum of two numbers is 55 and their difference is 7. Find the numbers .

Solution

Let the two numbers be x and y, where $x > y$

By the given data, $x + y = 55$ (1)

$x - y = 7$ (2)

Equation (2) becomes, $x = 7 + y$ (3)

Substituting x in (1) we get, $7 + y + y = 55$

$$\Longrightarrow 2y = 55 - 7 = 48$$

$$\therefore y = \frac{48}{2} = 24$$

Substituting $y = 24$ in (3) we get, $x = 7 + 24 = 31$.

$\therefore$ The required two numbers are 31 and 24.

Example 4.45

A number consist of two digits whose sum is 11. The number formed by reversing the digits is 9 less than the original number. Find the number.

Solution

Let the tens digit be x and the units digit be y. Then the number is $10x + y$.

Sum of the digits is $x + y = 11$ (1)

The number formed by reversing the digits is $10y + x$.

Given data, $(10x + y) - 9 = 10y + x$

$\Longrightarrow 10x + y - 10y - x = 9$

$9x - 9y = 9$

Dividing by 9 on both sides, $x - y = 1$ (2)

Equation (2) becomes $x = 1 + y$ (3)

Substituting x in (1) we get, $1 + y + y = 11$

$\Longrightarrow 2y + 1 = 11$

$2y = 11 - 1 = 10$

$\therefore y = \frac{10}{2} = 5$

Substituting $y = 5$ in (3) we get, $x = 1 + 5 = 6$

$\therefore$ The number is $10x + y = 10(6) + 5 = 65$

4.9 Linear Inequations in One Variable

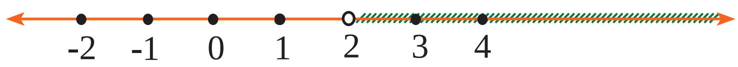
We know that $x + 4 = 6$ is a linear equation in one variable. Solving we get $x = 2$. There is only one such value for x in a linear equation in one variable.

Let us consider,

$x + 4 > 6$

ie $x > 6 - 4$

$x > 2$



So any real number greater than 2 will satisfy this inequation. We represent those real numbers in the number line.

Unshaded circle indicates that point is not included in the solution set.

Example 4.46

Solve $4(x - 1) \leq 8$

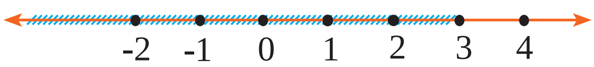
Solution

$4(x - 1) \leq 8$

Dividing by 4 on both sides,

$x - 1 \leq 2$

$\Longrightarrow x \leq 2 + 1 \Longrightarrow x \leq 3$



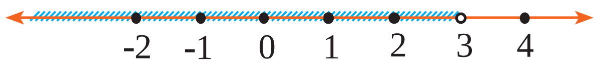
The real numbers less than or equal to 3 are solutions of given inequation.

Shaded circle indicates that point is included in the solution set.

Example 4.47

Solve $3(5-x)>6$

Solution We have, $3(5-x)>6$



Dividing by 3 on both sides, $5-x>2$

$\Longrightarrow -x>2-5 \Longrightarrow -x>-3$

$\therefore\ x<3$ (See remark given below)

The real numbers less than 3 are solutions of given inequation.

Remark (i) $-a>-b \Longrightarrow a<b$ (ii) $a<b \Longrightarrow \frac{1}{a}>\frac{1}{b}$ where $a\neq 0,\ b\neq 0$

(iii) $a<b \Longrightarrow ka<kb$ for $k>0$ (iv) $a<b \Longrightarrow ka>kb$ for $k<0$

Example 4.48

Solve $3-5x\leq 9$

Solution We have, $3-5x\leq 9$

$\Longrightarrow -5x\leq 9-3 \Longrightarrow -5x\leq 6$

$\Longrightarrow 5x\geq -6 \Longrightarrow x\geq -\frac{6}{5} \Longrightarrow x\geq -1.2$

The real numbers greater than or equal to -1.2 are solutions of given inequation.

Exercise 4.8

1. Solve the following equations by substitution method.

(i) $x+3y=10;\ 2x+y=5$ (ii) $2x+y=1;\ 3x-4y=18$

(iii) $5x+3y=21;\ 2x-y=4$ (iv) $\frac{1}{x}+\frac{2}{y}=9;\ \frac{2}{x}+\frac{1}{y}=12\quad (x\neq 0, y\neq 0)$

(v) $\frac{3}{x}+\frac{1}{y}=7;\ \frac{5}{x}-\frac{4}{y}=6\ (x\neq 0, y\neq 0)$

2. Find two numbers whose sum is 24 and difference is 8.

3. A number consists of two digits whose sum is 9. The number formed by reversing the digits exceeds twice the original number by 18. Find the original number.

4 Kavi and Kural each had a number of apples . Kavi said to Kural “If you give me 4 of your apples, my number will be thrice yours”. Kural replied “If you give me 26, my number will be twice yours”. How many did each have with them?.

5. Solve the following inequations.

(i) $2x+7>15$ (ii) $2(x-2)<3$ (iii) $2(x+7)\leq 9$ (iv) $3x+14\geq 8$

Points to Remember

- A polynomial in one variable x is an algebraic expression of the form $p(x)= a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0$, $a_n \neq 0$ where $a_0, a_1, a_2, \cdots, a_{n-1}, a_n$ are constants and n is a non negative integer .
- Let $p(x)$ be a polynomial. If $p(a) = 0$ then we say that a is a zero of the polynomial $p(x)$
- If $x = a$ satisfies the polynomial equation $p(x) = 0$ then $x = a$ is called a root of the polynomial equation $p(x) = 0$.
- Remainder Theorem : Let $p(x)$ be any polynomial and a be any real number. If $p(x)$ is divided by the linear polynomial $x - a$, then the remainder is $p(a)$.
- Factor Theorem : Let $p(x)$ be a polynomial and a be any real number. If $p(a) = 0$, then $(x-a)$ is a factor of $p(x)$.

- $(x + y + z)^2 \equiv x^2 + y^2 + z^2 + 2xy + 2yz + 2zx$
- $(x + y)^3 \equiv x^3 + y^3 + 3xy(x + y)$ $\quad$ $x^3 + y^3 \equiv (x + y)(x^2 - xy + y^2)$
- $(x - y)^3 \equiv x^3 - y^3 - 3xy(x - y)$ $\quad$ $x^3 - y^3 \equiv (x - y)(x^2 + xy + y^2)$
- $x^3 + y^3 + z^3 - 3xyz \equiv (x + y + z)(x^2 + y^2 + z^2 - xy - yz - zx)$**t**
- $(x + a)(x + b)(x + c) \equiv x^3 + (a + b + c)x^2 + (ab + bc + ca)x + abc$

5 COORDINATE GEOMETRY

I hope that posterity will judge me kindly, not only as to the things which I have explained, but also as to those which I have intentionally omitted so as to leave to others the pleasure of discovery

\- RENE DESCARTES

DESCARTES
(1596-1650)

Descartes (1596-1650) has been called the father of modern philosophy, perhaps because he attempted to build a new system of thought from the ground up, emphasized the use of logic and scientific method, and was "profoundly affected in his outlook by the new physics and astronomy." Descartes went far past Fermat in the use of symbols, in 'Arithmetizing' analytic geometry, in extending it to equations of higher degree. The fixing of a point position in the plane by assigning two numbers - coordinates - giving its distance from two lines perpendicular to each other, was entirely Descartes' invention.

Main Targets

- To understand Cartesian coordinate system
- To identify abscissa, ordinate and coordinates of a point
- To plot the points on the plane
- To find the distance between two points

5.1 Introduction

Coordinate Geometry or Analytical Geometry is a system of geometry where the position of points on the plane is described using an ordered pair of numbers called coordinates. This method of describing the location of points was introduced by the French mathematician René Descartes (Pronounced "day CART"). He proposed further that curves and lines could be described by equations using this technique, thus being the first to link algebra and geometry. In honour of his work, the coordinates of a point are often referred to as its Cartesian coordinates, and the coordinate plane as the Cartesian Coordinate Plane. The invention of analytical geometry was the beginning of modern mathematics.

In this chapter we learn how to represent points using cartesian coordinate system and derive formula to find distance between two points in terms of their coordinates.

5.2 Cartesian Coordinate System

In the chapter on *Real Number System*, you have learnt` how to represent real numbers on the *number line*. Every real number, whether rational, or irrational, has a unique location on the number line. Conversely, a point *P* on a number line can be specified by a real number *x* called its *coordinate*. Similarly, by using a Cartesian coordinate system we can specify a point *P* in the plane with two real numbers, called its coordinates.

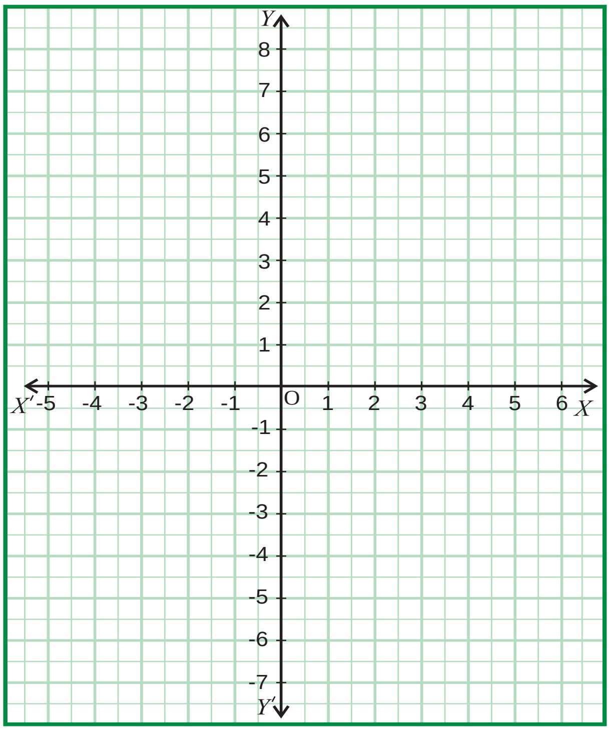


Fig. 5.1

A Cartesian coordinate system or rectangle coordinate system consists of two perpendicular number lines, called *coordinate axes*. The two number lines intersect at the zero point of each as shown in the Fig. 5.1 and this point is called *origin* '*O*'. Generally the horizontal number line is called the *x*-axis and the vertical number line is called the *y*-axis. The *x* coordinate of a point to the right of the *y*-axis is positive and to the left of *y*-axis is negative. Similarly, the *y* coordinate of a point above the *x*-axis is positive and below the *x*-axis is negative. We use the same scale (that is, the same unit distance) on both the axes.

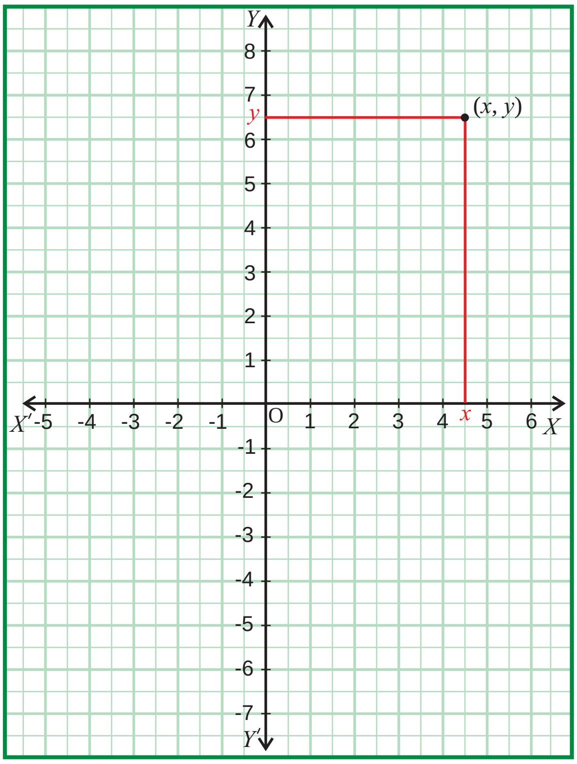


Fig. 5.2

5.2.1 Coordinates of a Point

In Cartesian system, any point *P* in the plane is associated with an ordered pair of real numbers. To obtain these number, we draw two lines through the point *P* parallel (and hence perpendicular) to the axes. We are interested in the coordinates of the points of intersection of the two lines with the axes. There are two coordinates: *x*-coordinate on the *x*-axis and *y*-coordinate on the *y*-axis. The *x*-coordinate is called the *abscissa* and the *y*-coordinate is called the *ordinate* of the point at hand. These two numbers associated with the point *P* are called *coordinates* of *P*. They are usually written as (*x*, *y*), the abscissa coming first, the ordinate second.

Remarks

1. In an ordered pair (a, b), the two elements a and b are listed in a specific order. So the ordered pairs (a, b) and (b, a) are not equal, i.e., $(a,b) \neq (b,a)$.
2. Also $(a_1,b_1) = (a_2,b_2)$ is equivalent to $a_1 = a_2$ and $b_1 = b_2$
3. The terms *point* and *coordinates of a point* are used interchangeably.

5.2.2 Identifying the *x*-coordinate

The *x*-coordinate or abscissa, of a point is the value which indicates the distance and direction of the point to the right or left of the *y*-axis. To find the *x*-coordinate of a point *P*:

(i) Drop a perpendicular from the point *P* to the *x*-axis.

(ii) The number where the line meets the *x*-axis is the value of the *x*-coordinate.

In Fig. 5.3., the *x*-coordinate of *P* is 1 and the *x*-coordinate of *Q* is 5.

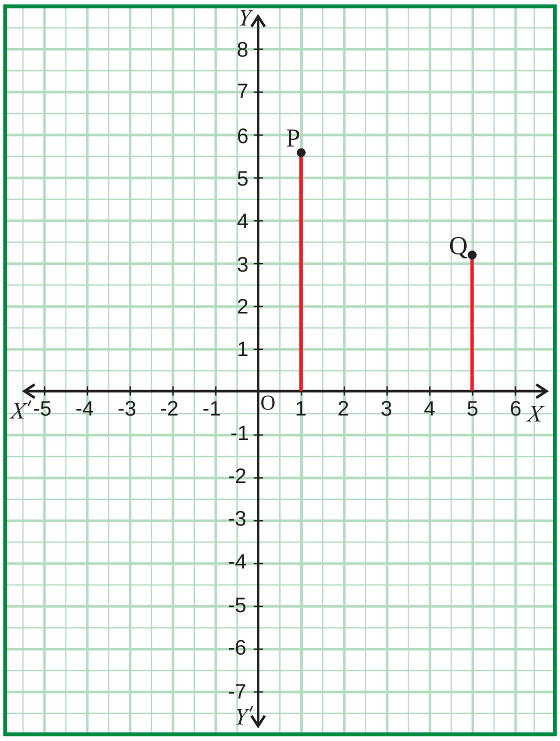


Fig. 5.3

5.2.3 Identifying the *y*-coordinate

The *y*-coordinate, or ordinate, of a point is the value which indicates the distance and direction of the point above or below the *x*-axis. To find the *y*-coordinate of a point *P*:

(i) Drop a perpendicular from the point *P* to the *y*-axis.

(ii) The number where the line meets the *y*-axis is the value of the *y*-coordinate.

In Fig. 5.4., the *y*-coordinate of *P* is 6 and the *y*-coordinate of *Q* is 2.

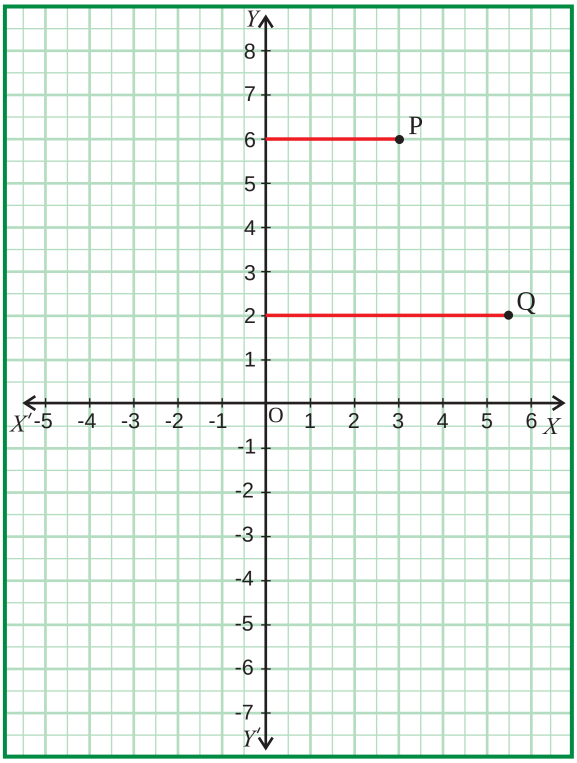


Fig. 5.4

Note (i) For any point on the x-axis, the value of y-coordinate (ordinate) is zero.

(ii) For any point on the y-axis, the value of x-coordinate (abscissa) is zero.

5.2.4 Quadrants

A plane with the rectangular coordinate system is called the cartesian plane. The coordinate axes divide the plane into four parts called quadrants, numbered counter-clockwise for reference as shown in Fig. 5.5. The x coordinate is positive in the I and IV quadrants and negative in II and III quadrants. The y coordinate is positive in I and II quadrants and negative in III and IV quadrants. The signs of the coordinates are shown in parentheses in Fig. 5.5.

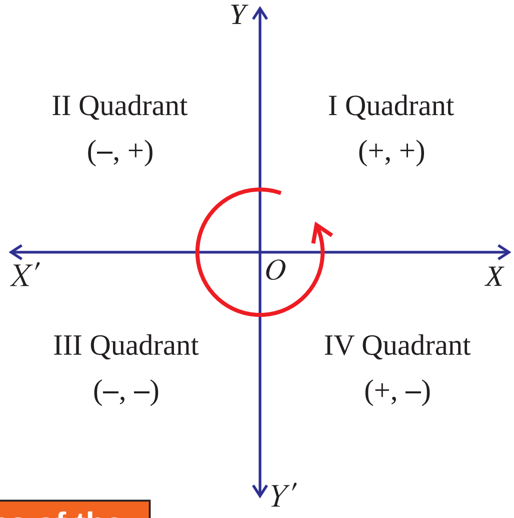


Fig. 5.5

Region	Quadrant	Nature of x, y	Signs of the coordinates
XOY	I	$x > 0, y > 0$	+, +
X′OY	II	$x < 0, y > 0$	–, +
X′OY′	III	$x < 0, y < 0$	–, –
XOY′	IV	$x > 0, y < 0$	+, –

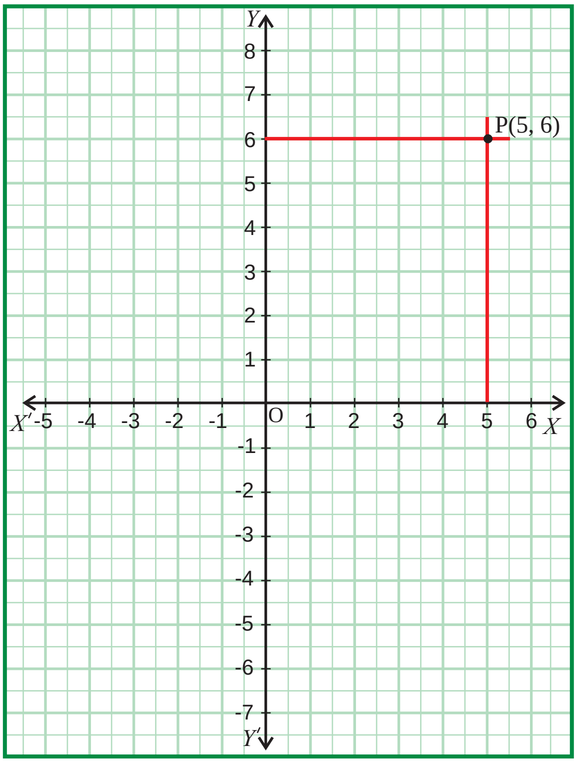


Fig. 5.6

5.2.5 Plotting Points in Cartesian Coordinate System

Let us now illustrate through an example how to plot a point in Cartesian coordinate system. To plot the point (5, 6) in cartesian coordinate system we follow the x-axis until we reach 5 and draw a vertical line at $x = 5$. Similarly, we follow the y-axis until we reach 6 and draw a horizontal line at $y = 6$. The intersection of these two lines is the position of (5, 6) in the cartesian plane.

That is we count from the origin 5 units along the positive direction of x-axis and move along the positive direction of y-axis through 6 units and mark the corresponding point. This point is at a distance of 5 units from the y-axis and 6 units from the x-axis. Thus the position of (5, 6) is located in the cartesian plane.

Example 5.1

Plot the following points in the rectangular coordinate system.

(i) $A\ (5, 4)$ (ii) $B\ (-4, 3)$ (iii) $C\ (-2, -3)$ (iv) $D\ (3, -2)$

Solution (i) To plot (5,4), draw a vertical line at $x = 5$ and draw a horizontal line at $y = 4$.

The intersection of these two lines is the position of (5, 4) in the Cartesian plane. Thus, the point A (5, 4) is located in the Cartesian plane.

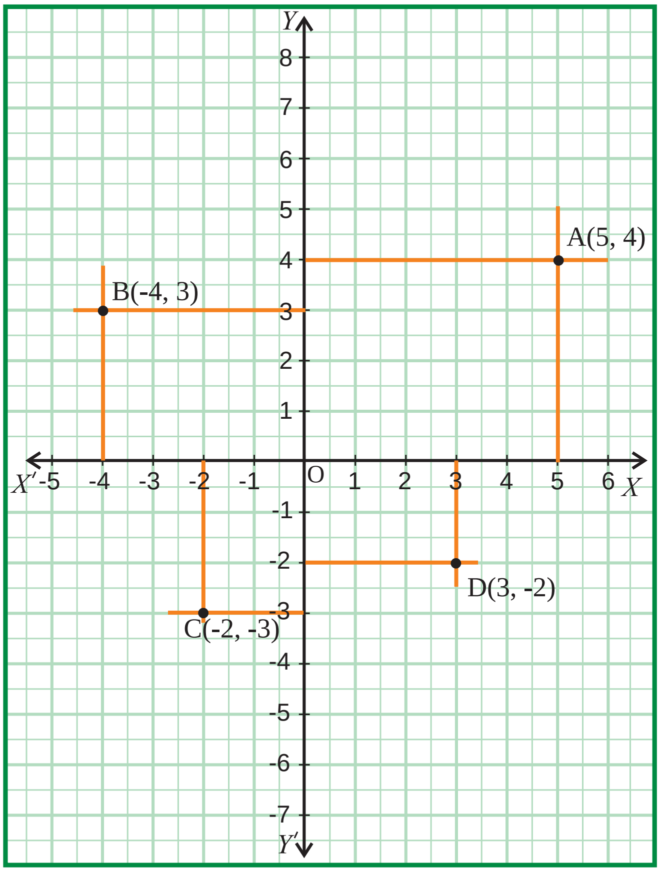


Fig. 5.7

(ii) To plot $(-4, 3)$, draw a vertical line at $x = -4$ and draw a horizontal line at $y = 3$. The intersection of these two lines is the position of $(-4, 3)$ in the Cartesian plane. Thus, the point $B(-4, 3)$ is located in the Cartesian plane.

(iii) To plot $(-2, -3)$, draw a vertical line at $x = -2$ and draw a horizontal line at $y = -3$. The intersection of these two lines is the position of $(-2, -3)$ in the Cartesian plane. Thus, the point $C(-2, -3)$ is located in the Cartesian plane.

(iv) To plot $(3, -2)$, draw a vertical line at $x = 3$ and draw a horizontal line at $y = -2$. The intersection of these two lines is the position of $(3, -2)$ in the Cartesian plane Thus, the point $D\ (3, -2)$ is located in the Cartesian plane.

Example 5.2

Locate the points (i) (3, 5) and (5, 3) (ii) (– 2, – 5) and (– 5, – 2) in the rectangular coordinate system.

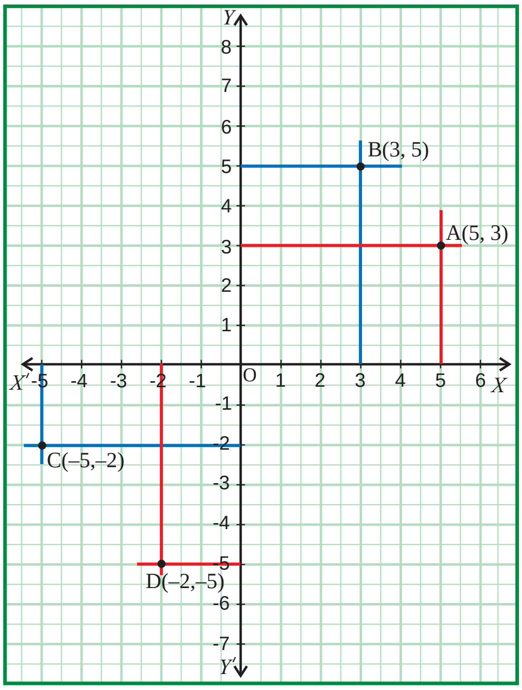


Fig. 5.8

Remark Observe that if we interchange the abscissa and ordinate of a point, then it may represent a different point in the Cartesian plane.

Example 5.3

Plot the points (– 1, 0), (2, 0), (– 5, 0) and (4, 0) in the cartesian plane.

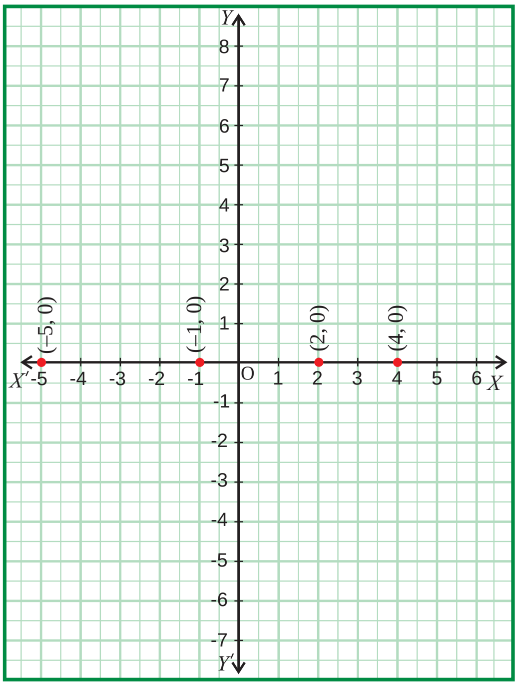


Fig. 5.9

Example 5.4

Plot the points (0, 4), (0, −2), (0, 5) and (0, −4) in the cartesian plane.

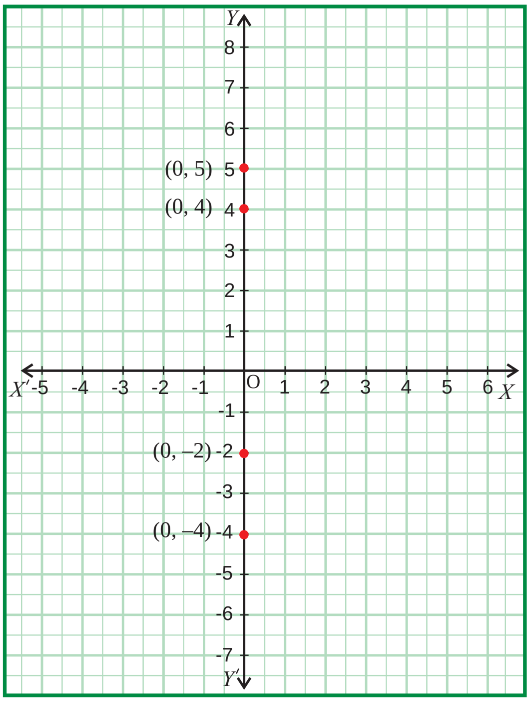


Fig. 5.10

Example 5.5

Plot the points (i) (−1, 2), (ii) (−4, 2), (iii) (4, 2) and (iv) (0, 2). What can you say about the position of these points?

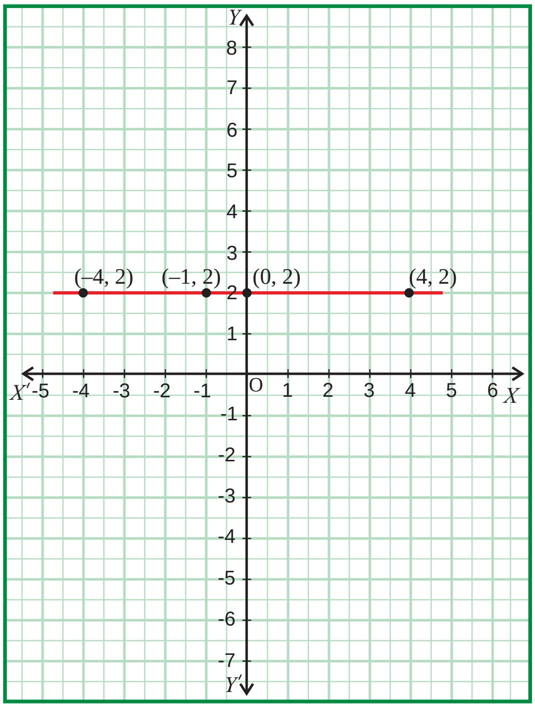


Fig. 5.11

When you join these points, you see that they lie on a line which is parallel to *x*-axis.

Remark For points on a line parallel to *x*-axis, the *y* coordinates are equal.

Example 5.6

Identify the quadrants of the points *A* (2, 3), *B* (– 2, 3), *C* (– 2, – 3) and *D* (2, –3). Discuss the type of the diagram by joining all the points.

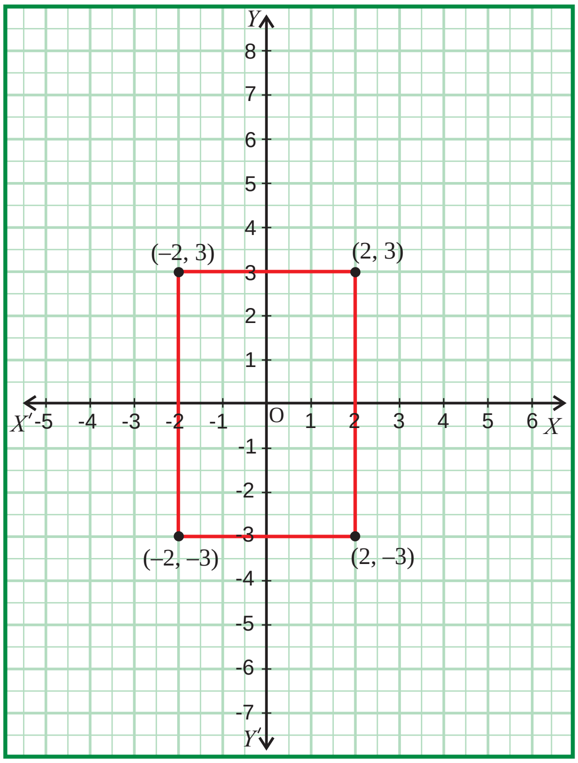


Fig. 5.12

Solution

Point	A	B	C	D
Quadrant	I	II	III	IV

ABCD is a rectangle

Can you find the length, breath and area of the rectangle?

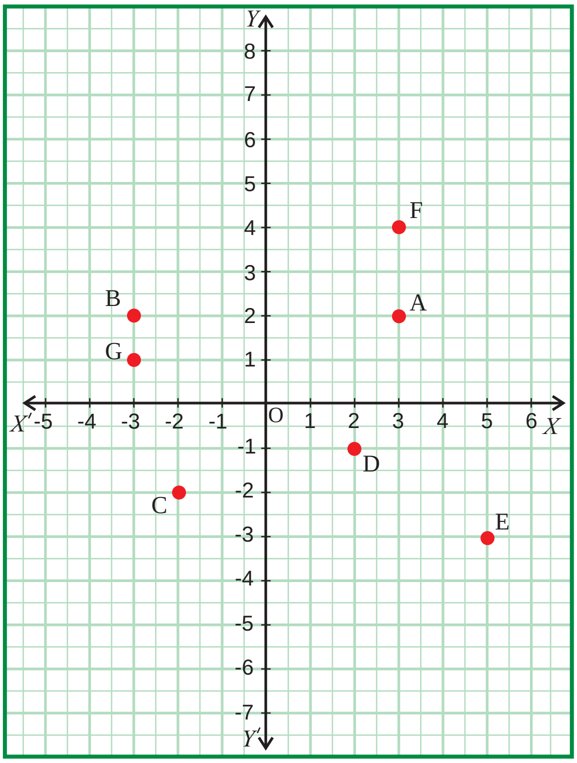


Fig. 5.13

Example 5.7

Find the coordinates of the points shown in the Fig. 5.13., where each square is a unit square.

Solution Consider the point *A*. *A* is at a distance of 3 units from the origin along the positive direction of *x*-axis and 2 units distance from the origin along the positive direction of *y*-axis. Hence the coordinates of *A* are (3, 2).

Similarly, *B* is (– 3, 2), *C* is (– 2, – 2), *D* is (2, – 1), *E* is (5, – 3), *F* is (3, 4) and *G* is (– 3, 1)

Exercise 5.1

1. State whether the following statements are true / false .

   (i) (5, 7) is a point in the IV quadrant.

   (ii) $(-2, -7)$ is a point in the III quadrant.

   (iii) $(8, -7)$ lies below the x-axis.

   (iv) (5, 2) and $(-7, 2)$ are points on the line parallel to y-axis.

   (v) $(-5, 2)$ lies to the left of y-axis.

   (vi) (0, 3) is a point on x-axis.

   (vii) $(-2, 3)$ lies in the II quadrant.

   (viii) $(-10, 0)$ is a point on x-axis.

   (ix) $(-2, -4)$ lies above x-axis.

   (x) For any point on the x-axis its y coordinate is zero.

2. Plot the following points in the coordinate system and specify their quadrant.

   (i) (5, 2) (ii) $(-1, -1)$ (iii) (7, 0) (iv) $(-8, -1)$ (v) $(0, -5)$

   (vi) (0, 3) (vii) $(4, -5)$ (viii) (0, 0) (ix) (1, 4) (x) $(-5, 7)$

3. Write down the abscissa for the following points.

   (i) $(-7, 2)$ (ii) (3, 5) (iii) $(8, -7)$ (iv) $(-5, -3)$

4. Write down the ordinate of the following points.

   (i) (7, 5) (ii) (2, 9) (iii) $(-5, 8)$ (iv) $(7, -4)$

5. Plot the following points in the coordinate plane.

   (i) (4, 2) (ii) $(4, -5)$ (iii) (4, 0) (iv) $(4, -2)$

   How is the line joining them situated?

6. The ordinates of two points are each -6. How is the line joining them related with reference to x-axis?

7. The abscissa of two points is 0. How is the line joining situated?

8. Mark the points A $(-3, 4)$, B (2, 4), C $(-3, -1)$ and D $(2, -1)$ in the cartesian plane. State the figure obtained by joining *A and B*, *B and C*, *C and D* ,and *D and A.*

9. With rectangular axes plot the points O (0, 0), A (5, 0), B (5, 4). Find the coordinate of point C such that $OABC$ forms a rectangle.

10. In a rectangle $ABCD$, the coordinates of A, B and D are (0, 0) (4, 0) (0, 3). What are the coordinates of C?

5.3 Distance between any Two Points

One of the simplest things that can be done with analytical geometry is to calculate the distance between two points. The distance between two points A and B is usually denoted by AB.

5.3.1 Distance between two points on coordinate axes

If two points lie on the x-axis, then it is easy to find the distance between them because the distance is equal to the difference between x coordinates. Consider the two points $A\,(x_1, 0)$ and $B\,(x_2, 0)$ on the x-axis.

The distance of B from A $= AB = OB - OA$

$$= x_2 - x_1 \quad \text{if } x_2 > x_1$$

$$= x_1 - x_2 \quad \text{if } x_1 > x_2$$

$$\therefore \; AB = |x_2 - x_1|$$

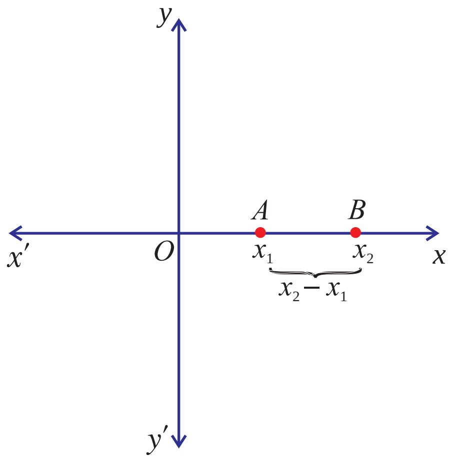


Fig. 5.14

Similarly, if two points lie on y axis, then the distance between them is equal to the difference between the y coordinates. Consider two points $A(0, y_1)$ and $B(0, y_2)$. These two points lie on the y axis.

The distance of B from A $= AB = OB - OA$

$$= y_2 - y_1 \quad \text{if } y_2 > y_1$$

$$= y_1 - y_2 \quad \text{if } y_1 > y_2$$

$$\therefore \; AB = |y_2 - y_1|$$

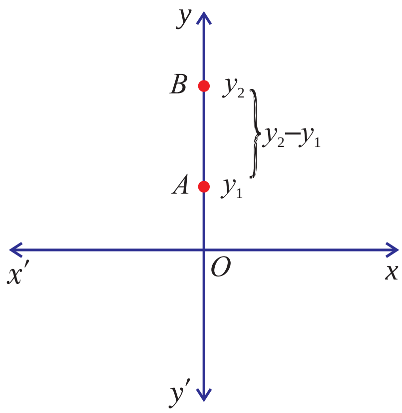


Fig. 5.15

5.3.2 Distance between two points on a line parallel to coordinate axes

Consider the points $A\,(x_1, y_1)$ and $B(x_2, y_1)$. Since the y ordinates are equal, the two points lie on a line parallel to x-axis. Draw AP and BQ perpendicular to x-axis. Distance between A and B is equal to distance between P and Q. Hence

Distance AB = Distance PQ $= |x_1 - x_2|$

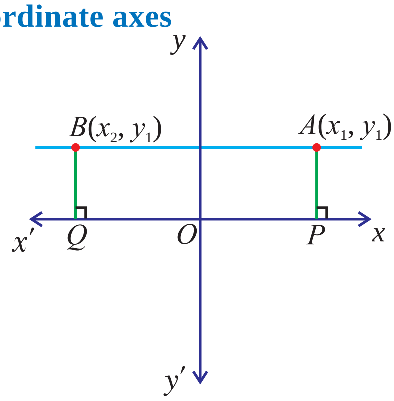


Fig. 5.16

Now consider the points $A(x_1, y_1)$ and $B(x_1, y_2)$ that lie on a line parallel to y-axis. Draw AP and BQ perpendicular to y-axis. The distance between A and B is equal to the distance between P and Q. Hence

Distance AB = Distance PQ = $|y_1 - y_2|$

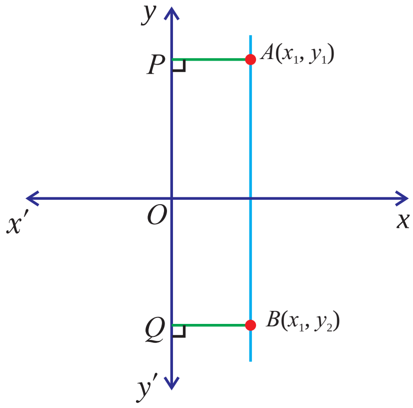


Fig. 5.17

Remark The distance between two points on a line parallel to the coordinate axes is the absolute value of the difference between respective coordinates.

5.3.3 Distance between two points:

Let $A(x_{1,}\ y_1)$ and $B(x_2\ y_2)$ be any two points in the plane. We shall now find the distance between these two points.

Let P and Q be the foot of the perpendiculars from A and B to the x-axis respectively. AR is drawn perpendicular to BQ. From the diagram,

$AR = PQ = OQ - OP = x_2 - x_1$ and

$BR = BQ - RQ = y_2 - y_1$

From right triangle ARB

$$AB^2 = AR^2 + RB^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$$

(By Pythagoras theorem)

i. e., $AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

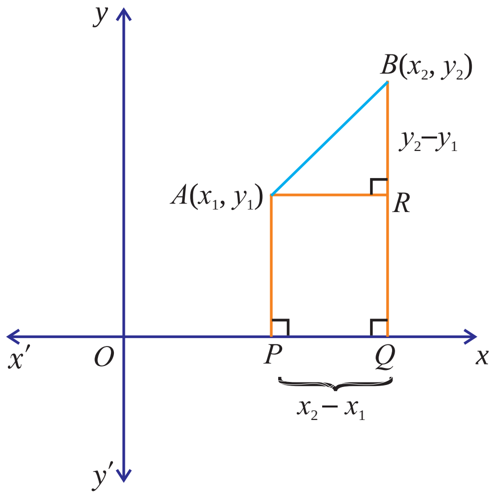


Fig. 5.18

Hence the distance between the points A and B is

$$AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Key Concept	**Distance Between Two Points**

Given the two points (x_1, y_1) and (x_2, y_2), the distance between these points is given by the formula:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Remark (i) This formula holds good for all the above cases.

(ii) The distance of the point $P(x_1, y_1)$ from the origin O is $OP = \sqrt{x_1^2 + y_1^2}$

Example 5.8

Find the distance between the points $(-4, 0)$ and $(3, 0)$

Solution The points $(-4, 0)$ and $(3, 0)$ lie on the x-axis. Hence

$$d = |x_1 - x_2| = |3 - (-4)| = |3 + 4| = 7$$

Aliter :

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{(3 + 4)^2 + 0^2} = \sqrt{49} = 7$$

Example 5.9

Find the distance between the points $(-7, 2)$ and $(5, 2)$

Solution The line joining $(5, 2)$ and $(-7, 2)$ is parallel to x axis. Hence, the distance

$$d = |x_1 - x_2| = |-7 - 5| = |-12| = 12$$

Aliter :

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{(5 + 7)^2 + (2 - 2)^2} = \sqrt{12^2} = \sqrt{144} = 12$$

Example 5.10

Find the distance between the points $(-5, -6)$ and $(-4, 2)$

Solution Using the distance formula $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$, we find

$$d = \sqrt{(-4 + 5)^2 + (2 + 6)^2} = \sqrt{1^2 + 8^2} = \sqrt{1 + 64} = \sqrt{65}$$

Example 5.11

Find the distance between the points $(0, 8)$ and $(6, 0)$

Solution The distance between the points $(0, 8)$ and $(6, 0)$ is

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$= \sqrt{(6 - 0)^2 + (0 - 8)^2} = \sqrt{36 + 64} = \sqrt{100} = 10$$

Aliter :

Let A and B denote the points $(6, 0)$ and $(0, 8)$ and let O be the origin. The point $(6, 0)$ lies on the x-axis and the point $(0, 8)$ lies on the y-axis. Since the angle between coordinate axes is right angle, the points A, O and B form a right triangle. Now $OA = 6$ and $OB = 8$. Hence, using Pythagorean Theorem

$$AB^2 = OA^2 + OB^2 = 36 + 64 = 100.$$

$$\therefore\ AB = \sqrt{100} = 10$$

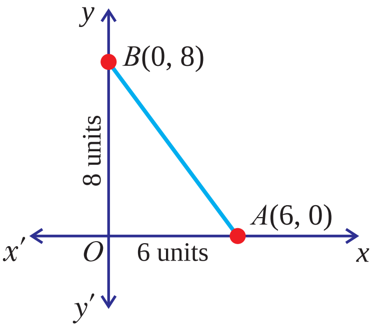


Fig. 5.19

Example 5.12

Find the distance between the points (–3, – 4), (5, – 7)

Solution The distance between the points (–3, – 4), (5, – 7) is

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$
$$= \sqrt{(5+3)^2 + (-7+4)^2} = \sqrt{8^2 + 3^2} = \sqrt{64+9} = \sqrt{73}$$

Example 5.13

Show that the three points (4, 2), (7, 5) and (9, 7) lie on a straight line.

Solution Let the points be A (4, 2), B (7, 5) and C (9, 7). By the distance formula

$$AB^2 = (4-7)^2 + (2-5)^2 = (-3)^2 + (-3)^2 = 9 + 9 = 18$$

$$BC^2 = (9-7)^2 + (7-5)^2 = 2^2 + 2^2 = 4 + 4 = 8$$

$$CA^2 = (9-4)^2 + (7-2)^2 = 5^2 + 5^2 = 25 + 25 = 50$$

So, $AB = \sqrt{18} = \sqrt{9 \times 2} = 3\sqrt{2}$; $BC = \sqrt{8} = \sqrt{4 \times 2} = 2\sqrt{2}$;

$CA = \sqrt{50} = \sqrt{25 \times 2} = 5\sqrt{2}$.

This gives $AB + BC = 3\sqrt{2} + 2\sqrt{2} = 5\sqrt{2} = AC$. Hence the points A, B, and C are collinear.

Example 5.14

Determine whether the points are vertices of a right triangle A (– 3, – 4) B (2, 6) and C (– 6, 10)

Solution Using the distance formula $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$, we get

$$AB^2 = (2+3)^2 + (6+4)^2 = 5^2 + 10^2 = 25 + 100 = 125$$

$$BC^2 = (-6-2)^2 + (10-6)^2 = (-8)^2 + 4^2 = 64 + 16 = 80$$

$$CA^2 = (-6+3)^2 + (10+4)^2 = (-3)^2 + (14)^2 = 9 + 196 = 205$$

i. e., $AB^2 + BC^2 = 125 + 80 = 205 = CA^2$

Hence ABC is a right angled triangle since the square of one side is equal to sum of the squares of the other two sides.

Example 5.15

Show that the points $(a, a), (-a, -a)$ and $(-a\sqrt{3}, a\sqrt{3})$ form an equilateral triangle.

Solution Let the points be represented by $A\,(a, a)$, $B\,(-a, -a)$ and $C(-a\sqrt{3}, a\sqrt{3})$. Using the distance formula $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$, we have

$$AB = \sqrt{(a+a)^2 + (a+a)^2}$$
$$= \sqrt{(2a)^2 + (2a)^2} = \sqrt{4a^2 + 4a^2} = \sqrt{8a^2} = 2\sqrt{2}\,a$$
$$BC = \sqrt{(-a\sqrt{3}+a)^2 + (a\sqrt{3}+a)^2} = \sqrt{3a^2 + a^2 - 2a^2\sqrt{3} + 3a^2 + a^2 + 2a^2\sqrt{3}}$$
$$= \sqrt{8a^2} = \sqrt{4 \times 2a^2} = 2\sqrt{2}\,a$$
$$CA = \sqrt{(a+a\sqrt{3})^2 + (a-a\sqrt{3})^2} = \sqrt{a^2 + 2a^2\sqrt{3} + 3a^2 + a^2 - 2a^2\sqrt{3} + 3a^2}$$
$$= \sqrt{8a^2} = 2\sqrt{2}\,a$$
$$\therefore\ AB = BC = CA = 2\sqrt{2}\,a.$$

Since all the sides are equal the points form an equilateral triangle.

Example 5.16

Prove that the points (− 7, −3), (5, 10), (15, 8) and (3, − 5) taken in order are the corners of a parallelogram.

Solution Let A, B, C and D represent the points (− 7, − 3), (5, 10), (15, 8) and (3, − 5) respectively. Using the distance formula $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$, we find

$$AB^2 = (5+7)^2 + (10+3)^2 = 12^2 + 13^2 = 144 + 169 = 313$$
$$BC^2 = (15-5)^2 + (8-10)^2 = 10^2 + (-2)^2 = 100 + 4 = 104$$
$$CD^2 = (3-15)^2 + (-5-8)^2 = (-12)^2 + (-13)^2 = 144 + 169 = 313$$
$$DA^2 = (3+7)^2 + (-5+3)^2 = 10^2 + (-2)^2 = 100 + 4 = 104$$

So, $AB = CD = \sqrt{313}$ and $BC = DA = \sqrt{104}$

i.e,. The opposite sides are equal. Hence $ABCD$ is a parallelogram.

Example 5.17

Show that the following points (3, − 2), (3, 2), (− 1, 2) and (− 1, − 2) taken in order are vertices of a square.

Solution Let the vertices be taken as A (3, − 2), B (3, 2), C (− 1, 2) and D (− 1, − 2)

$$AB^2 = (3-3)^2 + (2+2)^2 = 4^2 = 16$$
$$BC^2 = (3+1)^2 + (2-2)^2 = 4^2 = 16$$
$$CD^2 = (-1+1)^2 + (2+2)^2 = 4^2 = 16$$
$$DA^2 = (-1-3)^2 + (-2+2)^2 = (-4)^2 = 16$$

$$AC^2 = (3+1)^2 + (-2-2)^2 = 4^2 + (-4)^2 = 16 + 16 = 32$$

$$BD^2 = (3+1)^2 + (2+2)^2 = 4^2 + 4^2 = 16 + 16 = 32$$

$AB = BC = CD = DA = \sqrt{16} = 4$. (That is, all the sides are equal.)

$AC = BD = \sqrt{32} = 4\sqrt{2}$. (That is, the diagonals are equal.)

Hence the points A, B, C and D form a square.

Example 5.18

Let P be a point on the perpendicular bisector of the segment joining (2, 3) and (6, 5). If the abscissa and the ordinate of P are equal, find the coordinates of P.

Solution Let the point be $P(x, y)$. Since the abscissa of P is equal to its ordinate, we have $y = x$. Therefore, the coordinates of P are (x, x). Let A and B denote the points (2, 3) and (6, 5). Since P is equidistant from A and B, we get $PA = PB$. Squaring on both sides, we get $PA^2 = PB^2$.

i.e.,
$$(x-2)^2 + (x-3)^2 = (x-6)^2 + (x-5)^2$$
$$x^2 - 4x + 4 + x^2 - 6x + 9 = x^2 - 12x + 36 + x^2 - 10x + 25$$
$$2x^2 - 10x + 13 = 2x^2 - 22x + 61$$
$$22x - 10x = 61 - 13$$
$$12x = 48$$
$$x = \frac{48}{12} = 4$$

Therefore, the coordinates of P are (4, 4)

Example 5.19

Show that (4, 3) is the centre of the circle which passes through the points (9, 3), (7, −1) and (1, −1). Find also its radius.

Solution Suppose C represents the point (4, 3). Let P, Q and R denote the points (9, 3), (7, −1) and (1, −1) respectively. Using the distance formula $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$, we get

$$CP^2 = (9-4)^2 + (3-3)^2 = 5^2 = 25$$

$$CQ^2 = (7-4)^2 + (-1-3)^2 = 3^2 + (-4)^2 = 9 + 16 = 25$$

$$CR^2 = (4-1)^2 + (3+1)^2 = 3^2 + 4^2 = 9 + 16 = 25$$

So, $CP^2 = CQ^2 = CR^2 = 25$ or $CP = CQ = CR = 5$. Hence the points P, Q, R are on the circle with centre at (4, 3) and its radius is 5 units.

Example 5.20

If the point (α, β) is equidistant from (3, $-$4) and (8, $-$5), show that $5\alpha - \beta - 32 = 0$.

Solution Let P deonte the point (α, β). Let A and B represent the points (3, $-$4) and (8, $-$5) respectively. Since P is equidistant from A and B, we have $PA = PB$ and hence $PA^2 = PB^2$.
Using the distance formula $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$, we have

$$(\alpha - 3)^2 + (\beta + 4)^2 = (\alpha - 8)^2 + (\beta + 5)^2$$

$$\alpha^2 - 6\alpha + 9 + \beta^2 + 8\beta + 16 = \alpha^2 - 16\alpha + 64 + \beta^2 + 10\beta + 25$$

$$-6\alpha + 8\beta + 25 + 16\alpha - 10\beta - 89 = 0$$

$$10\alpha - 2\beta - 64 = 0$$

Dividing throughout by 2, we get $5\alpha - \beta - 32 = 0$

Example 5.21

Show that S (4, 3) is the circum-centre of the triangle joining the points A (9, 3), B (7, $-$1) and C (1, $-$1)

Solution $SA = \sqrt{(9-4)^2 + (3-3)^2} = \sqrt{25} = 5$

$$SB = \sqrt{(7-4)^2 + (-1-3)^2} = \sqrt{25} = 5$$

$$SC = \sqrt{(1-4)^2 + (-1-3)^2} = \sqrt{25} = 5$$

$$\therefore SA = SB = SC.$$

It is known that the circum-center is equidistant from all the vertices of a triangle. Since S is equidistant from all the three vertices, it is the circum-centre of the triangle ABC.

Exercise 5.2

1. Find the distance between the following pairs of points.

(i) (7, 8) and ($-$2, $-$3) (ii) (6, 0) and ($-$2, 4)

(iii) ($-$3, 2) and (2, 0) (iv) ($-$2, $-$8) and ($-$4, $-$6)

(v) ($-$2, $-$3) and (3, 2) (vi) (2, 2) and (3, 2)

(vii) $(-2, 2)$ and $(3, 2)$ (viii) $(7, 0)$ and $(-8, 0)$

(ix) $(0, 17)$ and $(0, -1)$ (x) $(5, 7)$ and the origin.

2. Show that the following points are collinear.

(i) $(3, 7)$, $(6, 5)$ and $(15, -1)$ (ii) $(3, -2)$, $(-2, 8)$ and $(0, 4)$

(iii) $(1, 4)$, $(3, -2)$ and $(-1, 10)$ (iv) $(6, 2)$, $(2, -3)$ and $(-2, -8)$

(v) $(4, 1)$, $(5, -2)$ and $(6, -5)$

3. Show that the following points form an isosceles triangle.

(i) $(-2, 0)$, $(4, 0)$ and $(1, 3)$ (ii) $(1, -2)$, $(-5, 1)$ and $(1, 4)$

(iii) $(-1, -3)$, $(2, -1)$ and $(-1, 1)$ (iv) $(1, 3)$, $(-3, -5)$ and $(-3, 0)$

(v) $(2, 3)$, $(5, 7)$ and $(1, 4)$

4. Show that the following points form a right angled triangle.

(i) $(2, -3)$, $(-6, -7)$ and $(-8, -3)$ (ii) $(-11, 13)$, $(-3, -1)$ and $(4, 3)$

(iii) $(0, 0)$, $(a, 0)$ and $(0, b)$ (iv) $(10, 0)$, $(18, 0)$ and $(10, 15)$

(v) $(5, 9)$, $(5, 16)$ and $(29, 9)$

5. Show that the following points form an equilateral triangle.

(i) $(0, 0)$, $(10, 0)$ and $(5, 5\sqrt{3})$ (ii) $(a, 0), (-a, 0)$ and $(0, a\sqrt{3})$

(iii) $(2, 2), (-2, -2)$ and $(-2\sqrt{3}, 2\sqrt{3})$ (iv) $(\sqrt{3}, 2), (0, 1)$ and $(0, 3)$

(v) $(-\sqrt{3}, 1), (2\sqrt{3}, -2)$ and $(2\sqrt{3}, 4)$

6. Show that the following points taken in order form the vertices of a parallelogram.

(i) $(-7, -5)$, $(-4, 3)$, $(5, 6)$ and $(2, -2)$ (ii) $(9, 5)$, $(6, 0)$, $(-2, -3)$ and $(1, 2)$

(iii) $(0, 0)$, $(7, 3)$, $(10, 6)$ and $(3, 3)$ (iv) $(-2, 5)$, $(7, 1)$, $(-2, -4)$ and $(7, 0)$

(v) $(3, -5)$, $(-5, -4)$, $(7, 10)$ and $(15, 9)$

7. Show that the following points taken in order form the vertices of a rhombus.

(i) $(0, 0)$, $(3, 4)$, $(0, 8)$ and $(-3, 4)$ (ii) $(-4, -7)$, $(-1, 2)$, $(8, 5)$ and $(5, -4)$

(iii) $(1, 0)$, $(5, 3)$, $(2, 7)$ and $(-2, 4)$ (iv) $(2, -3)$, $(6, 5)$, $(-2, 1)$ and $(-6, -7)$

(v) $(15, 20)$, $(-3, 12)$, $(-11, -6)$ and $(7, 2)$

8. Examine whether the following points taken in order form a square.

   (i) $(0, -1)$, $(2, 1)$, $(0, 3)$ and $(-2, 1)$ (ii) $(5, 2)$, $(1, 5)$, $(-2, 1)$ and $(2, -2)$

   (iii) $(3, 2)$, $(0, 5)$, $(-3, 2)$ and $(0, -1)$ (iv) $(12, 9)$, $(20, -6)$, $(5, -14)$ and $(-3, 1)$

   (v) $(-1, 2)$, $(1, 0)$, $(3, 2)$ and $(1, 4)$

9. Examine whether the following points taken in order form a rectangle.

   (i) $(8, 3)$, $(0, -1)$, $(-2, 3)$ and $(6, 7)$ (ii) $(-1, 1)$, $(0, 0)$, $(3, 3)$ and $(2, 4)$

   (iii) $(-3, 0)$, $(1, -2)$, $(5, 6)$ and $(1, 8)$

10. If the distance between two points $(x, 7)$ and $(1, 15)$ is 10, find x.

11. Show that $(4, 1)$ is equidistant from the points $(-10, 6)$ and $(9, -13)$.

12. If two points $(2, 3)$ and $(-6, -5)$ are equidistant from the point (x, y), show that $x + y + 3 = 0$.

13. If the length of the line segment with end points $(2, -6)$ and $(2, y)$ is 4, find y.

14. Find the perimeter of the triangle with vertices (i) $(0, 8)$, $(6, 0)$ and origin ; (ii) $(9, 3)$, $(1, -3)$ and origin

15. Find the point on the y-axis equidistant from $(-5, 2)$ and $(9, -2)$ (Hint: A point on the y-axis will have its x coordinate as zero).

16. Find the radius of the circle whose centre is $(3, 2)$ and passes through $(-5, 6)$.

17. Prove that the points $(0, -5)$ $(4, 3)$ and $(-4, -3)$ lie on the circle centred at the origin with radius 5.

18. In the Fig. 5.20, PB is perpendicular segment from the point A (4,3). If $PA = PB$ then find the coordinates of B.

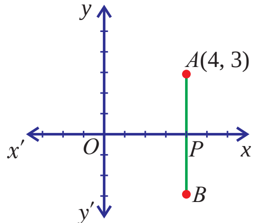


Fig. 5.20

19. Find the area of the rhombus $ABCD$ with vertices A $(2, 0)$, B $(5, -5)$, C $(8, 0)$ and D $(5, 5)$ [Hint: Area of the rhombus $ABCD = \frac{1}{2}d_1 d_2$]

20. Can you draw a triangle with vertices $(1, 5)$ $(5, 8)$ and $(13, 14)$? Give reason.

21. If origin is the centre of a circle with radius 17 units, find the coordinates of any four points on the circle which are not on the axes. (Use the Pythagorean triplets)

22. Show that (2, 1) is the circum-centre of the triangle formed by the vertices (3, 1), (2, 2) and (1, 1)

23. Show that the origin is the circum-centre of the triangle formed by the vertices (1, 0), (0, −1) and $\left(\frac{-1}{2}, \frac{\sqrt{3}}{2}\right)$

24. If the points $A(6, 1)$, $B(8, 2)$, $C(9, 4)$ and $D(p, 3)$ taken in order are the vertices of a parallelogram, find the value of p using distance formula.

25. The radius of the circle with centre at the origin is 10 units. Write the coordinates of the point where the circle intersects the axes. Find the distance between any two of such points.

Points to Remember

- Two perpendicular lines are needed to locate the position of a point in a plane.
- In rectangular coordinate systems one of them is horizontal and the other is vertical.
- These two horizontal and vertical lines are called the coordinate axes (x-axis and y-axis)
- The point of intersection of x-axis and y-axis is called the origin with coordinates (0, 0)
- The distance of a point from y-axis is x coordinate or abscissa and the distance of the point from x-axis is called y coordinate or ordinate.
- y coordinate of the points on x-axis is zero.
- x coordinate of the points on y-axis is zero.
- y coordinate of the points on the horizontal lines are equal.
- x coordinate of the points on the vertical lines are equal.
- If x_1 and x_2 are the x coordinates of two points on the x-axis, then the distance between them is $|x_1 - x_2|$
- If y_1 and y_2 are the y coordinates of two points on the y-axis, then the distance between the point is $|y_1 - y_2|$
- Distance between (x_1, y_1) and the origin is $\sqrt{x_1^2 + y_1^2}$
- Distance between the two points (x_1, y_1) and (x_2, y_2) is $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

6 TRIGONOMETRY

There is perhaps nothing which so occupies the middle position of mathematics as trigonometry.

– J.F. HERBART

Main Targets

- To understand Trigonometric Ratios
- To understand Trigonometric Ratios of Complementary Angles
- Method of Using Trigonometric Table

ARYABHATTA
(A.D. 476 – 550)

The first use of the idea of 'sine' in the way we use it today was in the work Aryabhatiyam by Aryabhatta, in A.D. 500. Aryabhatta was the first of great Indian Mathematicians. He lived at Kusumapura or Pataliputra in ancient Magadha or modern Patna in Bihar State. The time of birth of Aryabhatta may be fixed at Mesa-Sankranti on March 21, A.D 476. At the age of 23 years Aryabhatta wrote at least two books on astronomy (1) Aryabhatta (2) Aryabhatta-Siddhanta. The Aryabhatta deals with both mathematics and astronomy.

6.1 Introduction

The word *trigonometry* is a derivation from the Greek language and means *measurement of triangles*. This is because trigonometry was initially used to study relationships between different sides of a given triangle. **Hipparchus**, a Greek astronomer and mathematician developed the subject trigonometry and the first trigonometric table was compiled by him. He is now known as "**the Father of Trigonometry**". Trigonometry is an ancient mathematical tool with many applications, even in our modern world. Ancient civilizations used right triangle trigonometry for the purpose of measuring angles and distances in surveying land and astronomy. Trigonometry can be applied in the fields of navigation, planetary motion, and vibrations (sound waves, guitar strings), to name a few.

6.2 Trigonometric Ratios

6.2.1 Angle

We begin this section with the definition of an angle, which will involve several terms and their definitions.

Key Concept **Angle**

An angle is a portion of the 2-dimensional plane which resides between two different directed line segments. The starting position of the angle is known as the *initial side* and the ending position of the angle is known as the *terminal side*. The point from which both of the directed line segments originate is known as the *vertex* of the angle.

See the Fig. 6.1 below for a visual example of an angle.

Here the ray OA is rotated about the point O to the position OB to generate the angle AOB denoted by $\angle AOB$. OA is the initial side, OB is the terminal side and O is the vertex of the angle. We will often use Greek letters to denote angles, such as θ, α, β, etc.,

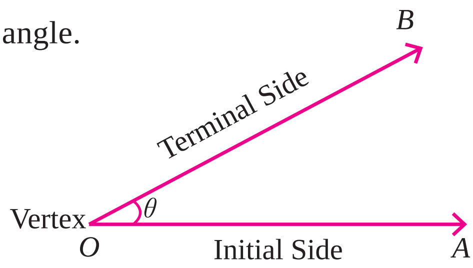


Fig. 6.1

A more common unit of measurement for an angle is the *degree*. This unit was used by the Babylonians as early as 1000 B.C. One degree (written 1°) is the measure of an angle generated by $\frac{1}{360}$ of one revolution.

6.2.2 Pythagoras Theorem

The Pythagoras theorem is a tool to solve for unknown values on right triangle.

Pythagoras Theorem: The square of the length of the hypotenuse of a right triangle is equal to the sum of the squares of the other two sides.

This relationship is useful in solving many problems and in developing trigonometric concepts.

6.2.3 Trigonometric Ratios

Consider the right triangle in the Fig. 6.2. In the right triangle, we refer to the lengths of the three sides according to how they are placed in relation to the angle θ.

- The side that is opposite to the right angle is called the *Hypotenuse*. This is the longest side in a right triangle.
- The side that is opposite to the angle θ is called the *Opposite side*.
- The side that runs alongside the angle θ and which is not the Hypotenuse is called the *Adjacent side*.

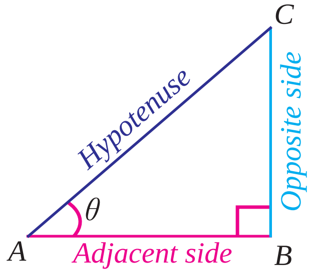
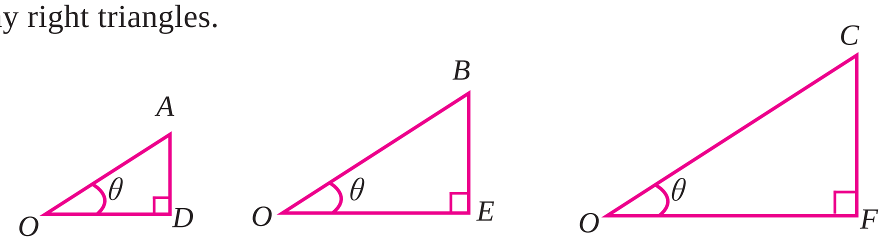


Fig. 6.2

When trigonometry was first developed it was based on *similar right triangles.* All right triangles that have a common acute angle are similar. So, for a given acute angle θ, we have many right triangles.



For each triangle above, the ratios of the corresponding sides are equal.

For example,

$$\frac{AD}{OA} = \frac{BE}{OB} = \frac{CF}{OC}; \qquad \frac{OD}{OA} = \frac{OE}{OB} = \frac{OF}{OC}$$

That is, the ratios depend only on the size of θ and not on the particular right triangle used to compute the ratios. We can form six ratios with the sides of a right triangle. Long ago these ratios were given names.

The ratio $\frac{\textit{Opposite side}}{\textit{Hypotenuse}}$ is called *sine* of angle θ and is denoted by $\sin\theta$

The ratio $\frac{\textit{Adjacent side}}{\textit{Hypotenuse}}$ is called *cosine* of angle θ and is denoted by $\cos\theta$

The ratio $\frac{\textit{Opposite side}}{\textit{Adjacent side}}$ is called *tangent* of angle θ and is denoted by $\tan\theta$

The ratio $\frac{\textit{Hypotenuse}}{\textit{Opposite side}}$ is called *cosecant* of angle θ and is denoted by $\operatorname{cosec}\theta$

The ratio $\frac{\textit{Hypotenuse}}{\textit{Adjacent side}}$ is called *secant* of angle θ and is denoted by $\sec\theta$

The ratio $\frac{\textit{Adjacent side}}{\textit{Opposite side}}$ is called *cotangent* of angle θ and is denoted by $\cot\theta$

Key Concept | **Trigonometric Ratios**

Let θ be an acute angle of a right triangle. Then the six trigonometric ratios of θ are as follows

$$\sin\theta = \frac{\textit{Opposite side}}{\textit{Hypotenuse}} \qquad \operatorname{cosec}\theta = \frac{\textit{Hypotenuse}}{\textit{Opposite side}}$$

$$\cos\theta = \frac{\textit{Adjacent side}}{\textit{Hypotenuse}} \qquad \sec\theta = \frac{\textit{Hypotenuse}}{\textit{Adjacent side}}$$

$$\tan\theta = \frac{\textit{Opposite side}}{\textit{Adjacent side}} \qquad \cot\theta = \frac{\textit{Adjacent side}}{\textit{Opposite side}}$$

Reciprocal Relations

The trigonometric ratios cosec θ, sec θ and cot θ are receprocals of sin θ, cos θ and tan θ respectively.

$$\sin\theta = \frac{1}{\text{cosec}\,\theta} \qquad \cos\theta = \frac{1}{\sec\theta} \qquad \tan\theta = \frac{1}{\cot\theta}$$

$$\text{cosec}\,\theta = \frac{1}{\sin\theta} \qquad \sec\theta = \frac{1}{\cos\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$

Remark

1. The basic trigonometric ratios sin θ, cos θ and tan θ are connected by the relation $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$
2. When calculating the trigonometric ratios of an acute angle θ, you may use any right triangle which has θ as one of the angles.
3. Since we defined the trigonometric ratios in terms of ratios of sides, you can think of the units of measurement for those sides as cancelling out in those ratios. This means that the values of the trigonometric functions are unitless numbers.

Example 6.1

Find the six trigonometric ratios of the angle θ in the right triangle *ABC*, as shown at right.

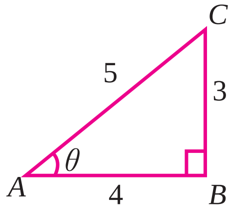


Fig. 6.3

Solution From the Fig. 6.3, the opposite side = 3, the adjacent side = 4 and the hypotenuse = 5.

$$\sin\theta = \frac{BC}{AC} = \frac{3}{5} \qquad \text{cosec}\,\theta = \frac{AC}{BC} = \frac{5}{3}$$

$$\cos\theta = \frac{AB}{AC} = \frac{4}{5} \qquad \sec\theta = \frac{AC}{AB} = \frac{5}{4}$$

$$\tan\theta = \frac{BC}{AB} = \frac{3}{4} \qquad \cot\theta = \frac{AB}{BC} = \frac{4}{3}$$

Example 6.2

In the right triangle *PQR* as shown at right, find the six trigonometric ratios of the angle θ.

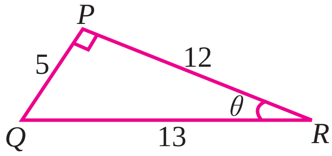


Fig. 6.4

Solution From the Fig. 6.4 the opposite side = 5, the adjacent side = 12 and the hypotenuse = 13.

$$\sin\theta = \frac{PQ}{RQ} = \frac{5}{13} \qquad \text{cosec}\,\theta = \frac{RQ}{PQ} = \frac{13}{5}$$

$$\cos\theta = \frac{PR}{RQ} = \frac{12}{13} \qquad \sec\theta = \frac{RQ}{PR} = \frac{13}{12}$$

$$\tan\theta = \frac{PQ}{PR} = \frac{5}{12} \qquad \cot\theta = \frac{PR}{PQ} = \frac{12}{5}$$

Example 6.3

From the Fig. 6.5, find the six trigonometric ratios of the angle θ

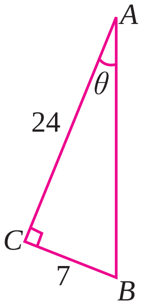


Fig. 6.5

Solution From the Fig. 6.5, $AC = 24$ and $BC = 7$. By Pythagoras theorem

$$AB^2 = BC^2 + CA^2 = 7^2 + 24^2 = 49 + 576 = 625$$

$$\therefore AB = \sqrt{625} = 25$$

We now use the three sides find the six trigonometric ratios of angle θ

$$\sin\theta = \frac{BC}{AB} = \frac{7}{25} \qquad \text{cosec}\,\theta = \frac{AB}{BC} = \frac{25}{7}$$

$$\cos\theta = \frac{AC}{AB} = \frac{24}{25} \qquad \sec\theta = \frac{AB}{AC} = \frac{25}{24}$$

$$\tan\theta = \frac{BC}{AC} = \frac{7}{24} \qquad \cot\theta = \frac{AC}{BC} = \frac{24}{7}$$

Example 6.4

In ΔABC, right angled at B, $15 \sin A = 12$. Find the other five trigonometric ratios of the angle A. Also find the six ratios of the angle C

Solution Given that $15 \sin A = 12$, so $\sin A = \frac{12}{15}$. Let us consider ΔABC (see Fig. 6.6), right angled at B, with $BC = 12$ and $AC = 15$. By Pythagoras theorem

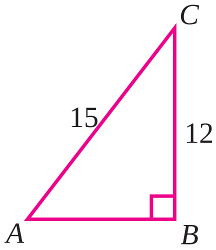


Fig. 6.6

$$AC^2 = AB^2 + BC^2$$

$$15^2 = AB^2 + 12^2$$

$$AB^2 = 15^2 - 12^2 = 225 - 144 = 81$$

$$\therefore AB = \sqrt{81} = 9$$

We now use the three sides to find the six trigonometric ratios of angle A and angle C.

$$\cos A = \frac{AB}{AC} = \frac{9}{15} \qquad \sin C = \frac{AB}{AC} = \frac{9}{15}$$

$$\tan A = \frac{BC}{AB} = \frac{12}{9} \qquad \cos C = \frac{BC}{AC} = \frac{12}{15}$$

$$\text{cosec}\,A = \frac{AC}{BC} = \frac{15}{12} \qquad \tan C = \frac{AB}{BC} = \frac{9}{12}$$

$$\sec A = \frac{AC}{AB} = \frac{15}{9} \qquad \text{cosec}\,C = \frac{AC}{AB} = \frac{15}{9}$$

$$\cot A = \frac{AB}{BC} = \frac{9}{12} \qquad \sec C = \frac{AC}{BC} = \frac{15}{12}$$

$$\cot C = \frac{BC}{AB} = \frac{12}{9}$$

Example 6.5

In $\triangle PQR$, right angled at Q, PQ=8 and PR=17. Find the six trigonometric ratios of the angle P

Solution Given that PQR is a right triangle, right angled at Q, (see Fig. 6.7), PQ=8 and PR=17. By Pythagoras theorem,

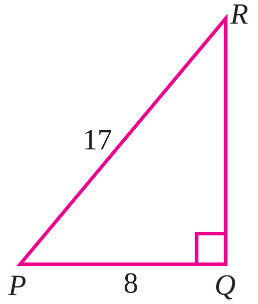


Fig. 6.7

$$PR^2 = PQ^2 + QR^2$$
$$17^2 = 8^2 + QR^2$$
$$QR^2 = 17^2 - 8^2$$
$$= 289 - 64 = 225$$
$$\therefore \quad QR = \sqrt{225} = 15$$

We now use the lengths of the three sides to find the six trigonometric ratios of angle P

$$\sin P = \frac{RQ}{PR} = \frac{15}{17} \qquad \operatorname{cosec} P = \frac{PR}{RQ} = \frac{17}{15}$$
$$\cos P = \frac{PQ}{PR} = \frac{8}{17} \qquad \sec P = \frac{PR}{PQ} = \frac{17}{8}$$
$$\tan P = \frac{RQ}{PQ} = \frac{15}{8} \qquad \cot P = \frac{PQ}{RQ} = \frac{8}{15}$$

Example 6.6

If $\cos A = \frac{35}{37}$, find $\frac{\sec A + \tan A}{\sec A - \tan A}$.

Solution Given that $\cos A = \frac{35}{37}$. Let us consider $\triangle ABC$ (see Fig. 6.8), $\angle B = 90^\circ$, with $AB = 35$ and $AC = 37$. By Pythagoras theorem, we have

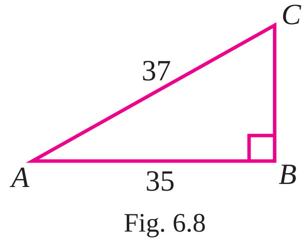


Fig. 6.8

$$AC^2 = AB^2 + BC^2$$
$$37^2 = 35^2 + BC^2$$
$$BC^2 = 37^2 - 35^2$$
$$= 1369 - 1225 = 144$$
$$\therefore BC = \sqrt{144} = 12$$

$$\sec A = \frac{AC}{AB} = \frac{37}{35}, \qquad \tan A = \frac{BC}{AB} = \frac{12}{35}$$

Now, $\sec A + \tan A = \frac{37}{35} + \frac{12}{35} = \frac{49}{35}$, $\qquad \sec A - \tan A = \frac{37}{35} - \frac{12}{35} = \frac{25}{35}$

$$\therefore \frac{\sec A + \tan A}{\sec A - \tan A} = \frac{\frac{49}{35}}{\frac{25}{35}} = \frac{49}{35} \times \frac{35}{25} = \frac{49}{25}$$

Example 6.7

If $\tan\theta = \frac{20}{21}$, show that $\frac{1-\sin\theta+\cos\theta}{1+\sin\theta+\cos\theta} = \frac{3}{7}$.

Solution Given that $\tan\theta = \frac{20}{21}$. Let us consider the right triangle *ABC* (see Fig. 6.9), with $AB = 21$ and $BC = 20$. By Pythagoras theorem, we have

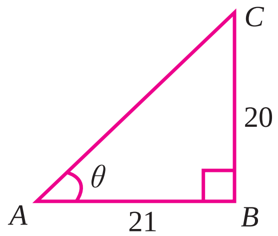


Fig. 6.9

$$AC^2 = AB^2 + BC^2 = 20^2 + 21^2 = 400 + 441 = 841.$$

$$\therefore AC = \sqrt{841} = 29.$$

$$\sin\theta = \frac{BC}{AC} = \frac{20}{29}, \qquad \cos\theta = \frac{AB}{AC} = \frac{21}{29}$$

$$1-\sin\theta+\cos\theta = 1 - \frac{20}{29} + \frac{21}{29} = \frac{29-20+21}{29} = \frac{30}{29}$$

$$1+\sin\theta+\cos\theta = 1 + \frac{20}{29} + \frac{21}{29} = \frac{29+20+21}{29} = \frac{70}{29}$$

$$\frac{1-\sin\theta+\cos\theta}{1+\sin\theta+\cos\theta} = \frac{\frac{30}{29}}{\frac{70}{29}} = \frac{30}{29}\times\frac{29}{70} = \frac{30}{70} = \frac{3}{7}$$

6.3 Trigonometric Ratios of Some Special Angles

For certain special angles such as $30^\circ, 45^\circ$ and 60°, which are frequently seen in applications, we can use geometry to determine the trigonometric ratios.

6.3.1 Trigonometric Ratios of 30° and 60°

Let $\triangle ABC$ be an equilateral triangle whose sides have length *a* (see Fig. 6.10). Draw $AD \perp BC$, then *D* bisects the side *BC*. So, $BD = DC = \frac{a}{2}$ and $\angle BAD = \angle DAC = 30^\circ$. Now, in right triangle *ADB*, $\angle BAD = 30^\circ$ and $BD = \frac{a}{2}$. So,

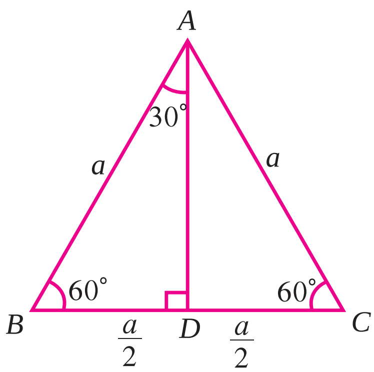


Fig. 6.10

$$AB^2 = AD^2 + BD^2$$

$$a^2 = AD^2 + \left[\frac{a}{2}\right]^2$$

$$AD^2 = a^2 - \frac{a^2}{4} = \frac{3a^2}{4}$$

$$\therefore AD = \frac{\sqrt{3}}{2}a$$

Hence, we can find the trigonometric ratios of angle 30° from the right triangle BAD

$$\sin 30^\circ = \frac{BD}{AB} = \frac{\frac{a}{2}}{a} = \frac{1}{2} \qquad \operatorname{cosec} 30^\circ = \frac{1}{\sin 30^\circ} = 2$$

$$\cos 30^\circ = \frac{AD}{AB} = \frac{\frac{\sqrt{3}}{2}a}{a} = \frac{\sqrt{3}}{2} \qquad \sec 30^\circ = \frac{1}{\cos 30^\circ} = \frac{2}{\sqrt{3}}$$

$$\tan 30^\circ = \frac{BD}{AD} = \frac{\frac{a}{2}}{\frac{\sqrt{3}}{2}a} = \frac{1}{\sqrt{3}} \qquad \cot 30^\circ = \frac{1}{\tan 30^\circ} = \sqrt{3}$$

In $\triangle ABD$, $\angle ABD = 60^\circ$. So, we can determine the trigonometric ratios of angle 60°

$$\sin 60^\circ = \frac{AD}{AB} = \frac{\frac{\sqrt{3}}{2}a}{a} = \frac{\sqrt{3}}{2} \qquad \operatorname{cosec} 60^\circ = \frac{1}{\sin 60^\circ} = \frac{2}{\sqrt{3}}$$

$$\cos 60^\circ = \frac{BD}{AB} = \frac{\frac{a}{2}}{a} = \frac{1}{2} \qquad \sec 60^\circ = \frac{1}{\cos 60^\circ} = 2$$

$$\tan 60^\circ = \frac{AD}{BD} = \frac{\frac{\sqrt{3}}{2}a}{\frac{a}{2}} = \sqrt{3} \qquad \cot 60^\circ = \frac{1}{\tan 60^\circ} = \frac{1}{\sqrt{3}}$$

6.3.2 Trigonometric Ratio of 45°

If an acute angle of a right triangle is 45°, then the other acute angle is also 45°. Thus the triangle is isosceles. Let us consider the triangle ABC with $\angle B = 90^\circ$, $\angle A = \angle C = 45^\circ$. Then $AB = BC$. Let $AB = BC = a$. By Pythagoras theorem,

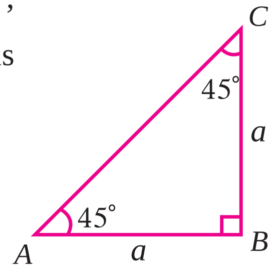


Fig. 6.11

$$AC^2 = AB^2 + BC^2$$
$$= a^2 + a^2 = 2a^2$$
$$\therefore AC = a\sqrt{2}$$

From Fig. 6.11, we can easily determine the trigonometric ratios of 45°

$\sin 45^\circ = \dfrac{BC}{AC} = \dfrac{a}{a\sqrt{2}} = \dfrac{1}{\sqrt{2}}$	$\operatorname{cosec} 45^\circ = \dfrac{1}{\sin 45^\circ} = \sqrt{2}$
$\cos 45^\circ = \dfrac{AB}{AC} = \dfrac{a}{a\sqrt{2}} = \dfrac{1}{\sqrt{2}}$	$\sec 45^\circ = \dfrac{1}{\cos 45^\circ} = \sqrt{2}$
$\tan 45^\circ = \dfrac{BC}{AB} = \dfrac{a}{a} = 1$	$\cot 45^\circ = \dfrac{1}{\tan 45^\circ} = 1$

6.3.3 Trigonometric Ratios of 0°and 90°

Consider Fig. 6.12 which shows a circle of radius 1 unit centered at the orgin. Let P be a point on the circle in the first quadrant with coordinates (x, y).

We drop a perpendicular PQ from P to the x-axis in order to form the right triangle OPQ. Let $\angle POQ = \theta$, then

$$\sin\theta = \frac{PQ}{OP} = \frac{y}{1} = y \text{ (} y \text{ coordinate of } P\text{)}$$

$$\cos\theta = \frac{OQ}{OP} = \frac{x}{1} = x \text{ (} x \text{ coordinate of } P\text{)}$$

$$\tan\theta = \frac{PQ}{OQ} = \frac{y}{x}$$

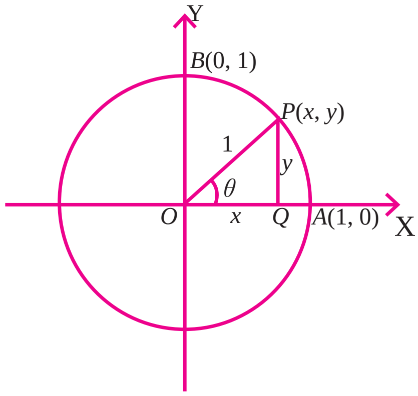


Fig. 6.12

If OP coincides with OA, then angle $\theta = 0^\circ$. Since the coordinates of A are (1, 0), we have

$\sin 0^\circ = 0$ (y coordinate of A)	$\operatorname{cosec} 0^\circ$ is not defined
$\cos 0^\circ = 1$ (x coordinate of A)	$\sec 0^\circ = 1$
$\tan 0^\circ = \dfrac{\sin 0^\circ}{\cos 0^\circ} = \dfrac{0}{1} = 0.$	$\cot 0^\circ$ is not defined

If OP coincides with OB, then angle $\theta = 90^\circ$. Since the coordinates of B are (0, 1), we have

$\sin 90^\circ = 1$ (y coordinate of B)	$\operatorname{cosec} 90^\circ = 1$
$\cos 90^\circ = 0$ (x coordinate of B)	$\sec 90^\circ$ is not defined
$\tan 90^\circ = \dfrac{\sin 90^\circ}{\cos 90^\circ} = \dfrac{1}{0}$ is not defined.	$\cot 90^\circ = 0$

The six trigonometric ratios of angles $0°, 30°, 45°, 60°$ and $90°$ are provided in the following table.

ratio \ angle θ	$0°$	$30°$	$45°$	$60°$	$90°$
$\sin\theta$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
$\tan\theta$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	not defined
$\operatorname{cosec}\theta$	not defined	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	1
$\sec\theta$	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	not defined
$\cot\theta$	not defined	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0

Example 6.8

Evaluate $\sin^2 45° + \tan^2 45° + \cos^2 45°$.

Solution We know, $\sin 45° = \frac{1}{\sqrt{2}}$, $\tan 45° = 1$ and $\cos 45° = \frac{1}{\sqrt{2}}$

$$\therefore \sin^2 45° + \tan^2 45° + \cos^2 45° = \left(\frac{1}{\sqrt{2}}\right)^2 + (1)^2 + \left(\frac{1}{\sqrt{2}}\right)^2$$

$$= \frac{1}{2} + 1 + \frac{1}{2} = 2$$

Note We write $(\sin\theta)^2$ as $\sin^2\theta$

Example 6.9

Evaluate $\frac{12\cos^2 30° - 2\tan^2 60°}{4\sec^2 45°}$.

Solution We know, $\cos 30° = \frac{\sqrt{3}}{2}$, $\tan 60° = \sqrt{3}$ and $\sec 45° = \sqrt{2}$

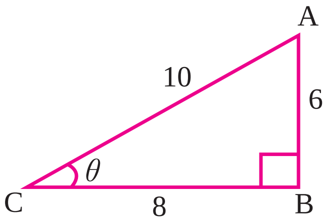
$$\therefore \frac{12\cos^2 30° - 2\tan^2 60°}{4\sec^2 45°} = \frac{\left(12\times\left(\frac{\sqrt{3}}{2}\right)^2\right) - (2\times(\sqrt{3})^2)}{4\times(\sqrt{2})^2}$$

$$= \frac{\left(12\times\frac{3}{4}\right) - (2\times 3)}{4\times 2}$$

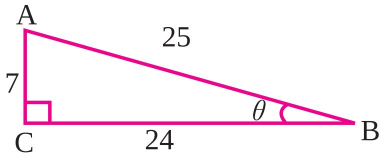
$$= \frac{9-6}{8} = \frac{3}{8}$$

Exercise 6.1

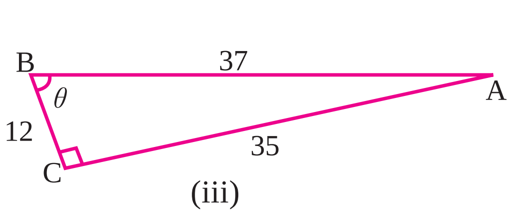
1. From the following diagrams, find the trigonometric ratios of the angle θ



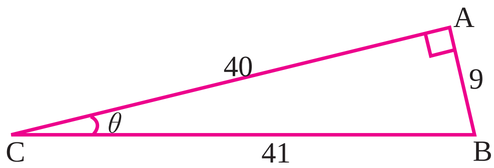
(i)



(ii)



(iii)



(iv)

2. Find the other trigonometric ratios of the following

(i) $\sin A = \frac{9}{15}$ (ii) $\cos A = \frac{15}{17}$ (iii) $\tan P = \frac{5}{12}$

(iv) $\sec\theta = \frac{17}{8}$ (v) $\operatorname{cosec}\theta = \frac{61}{60}$ (vi) $\sin\theta = \frac{x}{y}$.

3. Find the value of θ, if

(i) $\sin\theta = \frac{1}{\sqrt{2}}$ (ii) $\sin\theta = 0$ (iii) $\tan\theta = \sqrt{3}$ (iv) $\cos\theta = \frac{\sqrt{3}}{2}$.

4. In $\triangle ABC$, right angled at B, $AB = 10$ and $AC = 26$. Find the six trigonometric ratios of the angles A and C.

5. If $5\cos\theta - 12\sin\theta = 0$, find $\frac{\sin\theta + \cos\theta}{2\cos\theta - \sin\theta}$.

6. If $29\cos\theta = 20$, find $\sec^2\theta - \tan^2\theta$.

7. If $\sec\theta = \frac{26}{10}$, find $\frac{3\cos\theta + 4\sin\theta}{4\cos\theta - 2\sin\theta}$.

8. If $\tan\theta = \frac{a}{b}$, find $\sin^2\theta + \cos^2\theta$.

9. If $\cot\theta = \frac{15}{8}$, evaluate $\frac{(1+\sin\theta)(1-\sin\theta)}{(1+\cos\theta)(1-\cos\theta)}$.

10. In triangle PQR, right angled at Q, if $\tan P = \frac{1}{\sqrt{3}}$ find the value of

(i) $\sin P \cos R + \cos P \sin R$ (ii) $\cos P \cos R - \sin P \sin R$.

11. If $\sec\theta = \frac{13}{5}$, show that $\frac{2\sin\theta - 3\cos\theta}{4\sin\theta - 9\cos\theta} = 3$.

12. If $\sec A = \frac{17}{8}$, prove that $1 - 2\sin^2 A = 2\cos^2 A - 1$.

13. Evaluate.

(i) $\sin 45^\circ + \cos 45^\circ$

(ii) $\sin 60^\circ \tan 30^\circ$

(iii) $\dfrac{\tan 45^\circ}{\tan 30^\circ + \tan 60^\circ}$

(iv) $\cos^2 60^\circ \sin^2 30^\circ + \tan^2 30^\circ \cot^2 60^\circ$

(v) $6\cos^2 90^\circ + 3\sin^2 90^\circ + 4\tan^2 45^\circ$

(vi) $\dfrac{4\cot^2 60^\circ + \sec^2 30^\circ - 2\sin^2 45^\circ}{\sin^2 60^\circ + \cos^2 45^\circ}$

(vii) $\dfrac{\tan^2 60^\circ + 4\cos^2 45^\circ + 3\sec^2 30^\circ + 5\cos^2 90^\circ}{\operatorname{cosec} 30^\circ + \sec 60^\circ - \cot^2 30^\circ}$

(viii) $4(\sin^4 30^\circ + \cos^4 60^\circ) - 3(\cos^2 45^\circ - \sin^2 90^\circ)$.

14. Verify the following equalities.

(i) $\sin^2 30^\circ + \cos^2 30^\circ = 1$

(ii) $1 + \tan^2 45^\circ = \sec^2 45^\circ$

(iii) $\cos 60^\circ = 1 - 2\sin^2 30^\circ = 2\cos^2 30^\circ - 1$

(iv) $\cos 90^\circ = 1 - 2\sin^2 45^\circ = 2\cos^2 45^\circ - 1$

(v) $\dfrac{\cos 60^\circ}{1 + \sin 60^\circ} = \dfrac{1}{\sec 60^\circ + \tan 60^\circ}$

(vi) $\dfrac{1 - \tan^2 60^\circ}{1 + \tan^2 60^\circ} = 2\cos^2 60^\circ - 1$

(vii) $\dfrac{\sec 30^\circ + \tan 30^\circ}{\sec 30^\circ - \tan 30^\circ} = \dfrac{1 + \sin 30^\circ}{1 - \sin 30^\circ}$

(viii) $\tan^2 60^\circ - 2\tan^2 45^\circ - \cot^2 30^\circ + 2\sin^2 30^\circ + \frac{3}{4}\operatorname{cosec}^2 45^\circ = 0$

(ix) $4\cot^2 45^\circ - \sec^2 60^\circ + \sin^2 60^\circ + \cos^2 60^\circ = 1$

(x) $\sin 30^\circ \cos 60^\circ + \cos 30^\circ \sin 60^\circ = \sin 90^\circ$.

6.4 Trigonometric Ratios for Complementary Angles

Two acute angles are complementary to each other if their sums are equal to 90°. In a right triangle the sum of the two acute angles is equal to 90°. So, the two acute angles of a right triangle are always complementary to each other.

Let ABC be a right triangle, right angled at B (see Fig. 6.13). If $\angle ACB = \theta$, then $\angle BAC = 90^\circ - \theta$ and hence the angles $\angle BAC$ and $\angle ACB$ are complementary.

We have

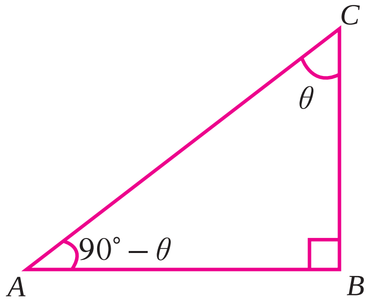


Fig. 6.13

$$\left.\begin{aligned} \sin\theta &= \frac{AB}{AC} & \operatorname{cosec}\theta &= \frac{AC}{AB} \\ \cos\theta &= \frac{BC}{AC} & \sec\theta &= \frac{AC}{BC} \\ \tan\theta &= \frac{AB}{BC} & \cot\theta &= \frac{BC}{AB} \end{aligned}\right\} \quad (1)$$

Similarly, for the angle $(90^\circ - \theta)$, we have

$$\left.\begin{aligned} \sin(90^\circ - \theta) &= \frac{BC}{AC} & \operatorname{cosec}(90^\circ - \theta) &= \frac{AC}{BC} \\ \cos(90^\circ - \theta) &= \frac{AB}{AC} & \sec(90^\circ - \theta) &= \frac{AC}{AB} \\ \tan(90^\circ - \theta) &= \frac{BC}{AB} & \cot(90^\circ - \theta) &= \frac{AB}{BC} \end{aligned}\right\} \quad (2)$$

Comparing the equations in (1) and (2) we get,

$$\begin{aligned} \sin\theta &= \frac{AB}{AC} = \cos(90^\circ - \theta) & \operatorname{cosec}\theta &= \frac{AC}{AB} = \sec(90^\circ - \theta) \\ \cos\theta &= \frac{BC}{AC} = \sin(90^\circ - \theta) & \sec\theta &= \frac{AC}{BC} = \operatorname{cosec}(90^\circ - \theta) \\ \tan\theta &= \frac{AB}{BC} = \cot(90^\circ - \theta) & \cot\theta &= \frac{BC}{AB} = \tan(90^\circ - \theta) \end{aligned}$$

Key Concept   Trigonometric Ratios of Complementary Angles

Let θ be an acute angle of a right triangle. Then we have the following identities for trigonometric ratios of complementary angles.

$$\begin{aligned} \sin\theta &= \cos(90^\circ - \theta) & \operatorname{cosec}\theta &= \sec(90^\circ - \theta) \\ \cos\theta &= \sin(90^\circ - \theta) & \sec\theta &= \operatorname{cosec}(90^\circ - \theta) \\ \tan\theta &= \cot(90^\circ - \theta) & \cot\theta &= \tan(90^\circ - \theta) \end{aligned}$$

Example 6.10

Evaluate $\dfrac{\cos 56^\circ}{\sin 34^\circ}$.

Solution The angles 56° and 34° are complementary. So, using trigonometric ratios of complementary angles $\cos 56^\circ = \cos(90^\circ - 34^\circ) = \sin 34^\circ$. Hence $\dfrac{\cos 56^\circ}{\sin 34^\circ} = \dfrac{\sin 34^\circ}{\sin 34^\circ} = 1$

Example 6.11

Evaluate $\dfrac{\tan 25^\circ}{\cot 65^\circ}$

Solution We write $\tan 25^\circ = \tan(90^\circ - 65^\circ) = \cot 65^\circ$. Hence,

$$\frac{\tan 25^\circ}{\cot 65^\circ} = \frac{\cot 65^\circ}{\cot 65^\circ} = 1$$

Example 6.12

Evaluate $\frac{\cos 65^\circ \sin 18^\circ \cos 58^\circ}{\cos 72^\circ \sin 25^\circ \sin 32^\circ}$.

Solution Using trigonometric ratios of complementary angles, we get

$$\cos 65^\circ = \cos(90^\circ - 25^\circ) = \sin 25^\circ,$$

$$\sin 18^\circ = \sin(90^\circ - 72^\circ) = \cos 72^\circ$$

$$\cos 58^\circ = \cos(90^\circ - 32^\circ) = \sin 32^\circ.$$

$$\therefore \quad \frac{\cos 65^\circ \sin 18^\circ \cos 58^\circ}{\cos 72^\circ \sin^\circ 25^\circ \sin 32^\circ} = \frac{\sin 25^\circ \cos 72^\circ \sin 32^\circ}{\cos 72^\circ \sin 25^\circ \sin 32^\circ} = 1$$

Example 6.13

Show that $\tan 35^\circ \tan 60^\circ \tan 55^\circ \tan 30^\circ = 1$.

Solution We have $\tan 35^\circ = \tan(90^\circ - 55^\circ) = \cot 55^\circ$

$$\tan 60^\circ = \tan(90^\circ - 30^\circ) = \cot 30^\circ$$

$$\therefore \; \tan 35^\circ \tan 60^\circ \tan 55^\circ \tan 30^\circ = \cot 55^\circ \cot 30^\circ \tan 55^\circ \tan 30^\circ$$

$$= \frac{1}{\tan 55^\circ} \times \frac{1}{\tan 30^\circ} \times \tan 55^\circ \times \tan 30^\circ = 1$$

Example 6.14

If $\operatorname{cosec} A = \sec 25^\circ$, find A.

Solution We have $\operatorname{cosec} A = \sec(90^\circ - A)$. So,

$$\sec(90^\circ - A) = \sec 25^\circ \implies 90^\circ - A = 25^\circ$$

$$\therefore \; A = 90^\circ - 25^\circ = 65^\circ$$

Note In Example 6.14, the value of A is obtained not by cancelling sec on both sides but using uniqueness of trigonometric ratios for acute angles. That is, if α and β are acute angles,

$$\sin\alpha = \sin\beta \implies \alpha = \beta$$

$$\cos\alpha = \cos\beta \implies \alpha = \beta, \text{ etc.}$$

Example 6.15

If $\sin A = \cos 33^\circ$ find A

Solution We have $\sin A = \cos(90 - A)$. So,

$$\cos(90^\circ - A) = \cos 33^\circ \implies 90^\circ - A = 33^\circ$$

$$\therefore \; A = 90^\circ - 33^\circ = 57^\circ$$

Exercise 6.2

1. Evaluate

(i) $\frac{\sin 36^\circ}{\cos 54^\circ}$ (ii) $\frac{\operatorname{cosec} 10^\circ}{\sec 80^\circ}$ (iii) $\sin\theta \sec(90^\circ - \theta)$

(iv) $\frac{\sec 20^\circ}{\operatorname{cosec} 70^\circ}$ (v) $\frac{\sin 17^\circ}{\cos 73^\circ}$ (vi) $\frac{\tan 46^\circ}{\cot 44^\circ}$.

2. Simplify

(i) $\cos 38^\circ \cos 52^\circ - \sin 38^\circ \sin 52^\circ$ (ii) $\frac{\cos 80^\circ}{\sin 10^\circ} + \cos 59^\circ \operatorname{cosec} 31^\circ$

(iii) $\frac{\sin 36^\circ}{\cos 54^\circ} - \frac{\tan 54^\circ}{\cot 36^\circ}$ (iv) $3\frac{\tan 67^\circ}{\cot 23^\circ} + \frac{1}{2}\frac{\sin 42^\circ}{\cos 48^\circ} + \frac{5}{2}\frac{\operatorname{cosec} 61^\circ}{\sec 29^\circ}$

(v) $\frac{\cos 37^\circ}{\sin 53^\circ} \times \frac{\sin 18^\circ}{\cos 72^\circ}$ (vi) $2\frac{\sec(90^\circ - \theta)}{\operatorname{cosec}\theta} + 7\frac{\cos(90^\circ - \theta)}{\sin\theta}$

(vii) $\frac{\sec(90^\circ - \theta)}{\sin(90^\circ - \theta)} \times \frac{\cos\theta}{\tan(90^\circ - \theta)} - \sec\theta$ (viii) $\frac{\sin 35^\circ}{\cos 55^\circ} + \frac{\cos 55^\circ}{\sin 35^\circ} - 2\cos^2 60^\circ$

(ix) $\cot 12^\circ \cot 38^\circ \cot 52^\circ \cot 60^\circ \cot 78^\circ$.

3. Find A if

(i) $\sin A = \cos 30^\circ$ (ii) $\tan 49^\circ = \cot A$ (iii) $\tan A \tan 35^\circ = 1$

(iv) $\sec 35^\circ = \operatorname{cosec} A$ (v) $\operatorname{cosec} A \cos 43^\circ = 1$ (vi) $\sin 20^\circ \tan A \sec 70^\circ = \sqrt{3}$.

4. Show that

(i) $\cos 48^\circ - \sin 42^\circ = 0$ (ii) $\cos 20^\circ \cos 70^\circ - \sin 70^\circ \sin 20^\circ = 0$

(iii) $\sin(90^\circ - \theta)\tan\theta = \sin\theta$ (iv) $\frac{\cos(90^\circ - \theta)\tan(90^\circ - \theta)}{\cos\theta} = 1$.

6.5 Method of Using Trigonometric Table

We have computed the trigonometric ratios for angles 0°, 30°, 45°, 60° and 90°. In our daily life, we come across situations, wherein we need to solve right triangles which have angles different from 0°, 30°, 45°, 60° and 90°. To apply the results of trigonometric ratios to these situations, we need to know the values of trigonometric ratios of all the acute angles. Trigonometrical tables indicating approximate values of sines, cosines and tangents of all the acute angles have been provided at the end of the book.

To express fractions of degrees, One degree is divided into 60 minutes and One minute is divided into 60 seconds. One minute is denoted by $1'$and One second is denoted by $1''$.

Therefore, $1^\circ = 60'$ and $1' = 60''$

The trigonometrical tables give the values, correct to four places of decimals of all the three tirgonometric ratios for angles from 0° to 90° spaced at intervals of $6'$. A trigonometric table consists of three parts.

(i) A column on the extreme left which contains degrees from 0° to 90°

(ii) Ten columns headed by $0', 6', 12', 18', 24', 30', 36', 42', 48'$ and $54'$ respectively

(iii) Five columns under the head *Mean difference* and these five columns are headed by $1'$, $2'$, $3'$, $4'$ and $5'$

The ten columns mentioned in (ii) provide the values for sine, cosine and tangent of angles in multiple of $6'$. For angles containing other numbers of minutes, the appropriate adjustment is obtained from the mean difference columns.

The mean difference is to be added in the case of sine and tangent, while it is to be subtracted in the case of cosine.

Example 6.16

Find the value of $\sin 46^\circ 51'$

Solution The relevant part of the sine table is given below.

	$0'$	$6'$	$12'$	$18'$	$24'$	$30'$	$36'$	$42'$	$48'$	$54'$	Mean Diff.				
	0.0°	0.1°	0.2°	0.3°	0.4°	0.5°	0.6°	0.7°	0.8°	0.9°	1	2	3	4	5
46°									0.7290				6		

Write $46^\circ 51' = 46^\circ 48' + 3'$. From the table we have

$$\sin 46^\circ 48' = 0.7290$$

Mean Difference for $3' = 0.0006$

$\therefore\ \sin 46^\circ 51' = 0.7290 + 0.0006 = 0.7296$

Example 6.17

Find the value of $\cos 37^\circ 16'$

Solution

	$0'$	$6'$	$12'$	$18'$	$24'$	$30'$	$36'$	$42'$	$48'$	$54'$	Mean Diff.				
	0.0°	0.1°	0.2°	0.3°	0.4°	0.5°	0.6°	0.7°	0.8°	0.9°	1	2	3	4	5
37°			0.7695											7	

Write $37^\circ 16' = 37^\circ 12' + 4'$. From the table

$$\cos 37^\circ 12' = 0.7695$$

Mean Difference for $4' = 0.0007$

Since $\cos\theta$ decreases from 1 to 0 as θ increases from 0° to 90°, we must subtract the Mean Difference.

$\therefore\ \cos 37^\circ 16' = 0.7695 - 0.0007 = 0.7688$

Example 6.18

Find the value of $\tan 25^\circ 15'$

Solution

	$0'$	$6'$	$12'$	$18'$	$24'$	$30'$	$36'$	$42'$	$48'$	$54'$	Mean Diff.
	0.0°	0.1°	0.2°	0.3°	0.4°	0.5°	0.6°	0.7°	0.8°	0.9°	1 2 3 4 5
25°			0.4706								11

Write $25^\circ 15' = 25^\circ 12' + 3'$. From the table

$$\tan 25^\circ 12' = 0.4706$$

Mean Difference for $3' = 0.0011$

$\therefore\ \tan 25^\circ 15' = 0.4706 + 0.011 = 0.4717$

Example 6.19

If $\sin\theta = 0.0958$, find the angle θ.

Solution From the sine table, we find 0.0958 is corresponding to $\sin 5^\circ 30'$.

$$\Longrightarrow \sin 5^\circ 30' = 0.0958$$

$$\therefore\ \theta = 5^\circ 30'$$

Example 6.20

If $\sin\theta = 0.0987$, find the angle θ.

Solution From the sine table, we find the value 0.0993 is corresponding to $\sin 5^\circ 42'$ and 0.0006 is corresponding to $2'$. So,

$$\sin 5^\circ 40' = \sin 5^\circ 42' - \text{Mean Difference for } 2'$$

$$= 0.0993 - 0.0006 = 0.0987$$

$$\therefore \theta = 5^\circ 40'$$

Example 6.21

Find the angle θ if $\tan\theta = 0.4040$

Solution From the tangent table, we find the value 0.4040 is corresponding to $\tan 22^\circ 0'$.

$$\Longrightarrow \tan 22^\circ = 0.4040$$

$$\therefore \theta = 22^\circ$$

Example 6.22

Simplify $\sin 30^\circ 30' + \cos 5^\circ 33'$.

Solution From the sine table $\sin 30^\circ 30' = 0.5075$. And from the cosine table $\cos 5^\circ 30' = 0.9954$ and Mean Difference for $3' = 0.0001$. So,

$$\cos 5^\circ 33' = 0.9954 - 0.0001 = 0.9953$$

$$\therefore \sin 30^\circ 30' + \cos 5^\circ 33' = 0.5075 + 0.9953 = 1.5028$$

Example 6.23

Simplify $\cos 70^\circ 12' + \tan 48^\circ 54'$.

Solution From the cosine and tangent tables, we find

$$\cos 70^\circ 12' = 0.3387, \quad \tan 48^\circ 54' = 1.1463$$

$$\therefore \cos 70^\circ 12' + \tan 48^\circ 54' = 0.3387 + 1.1463 = 1.4850$$

Example 6.24

Find the area of the right triangle given in Fig. 6.14.

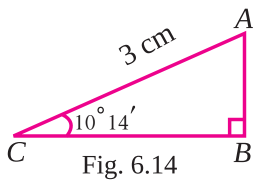


Fig. 6.14

Solution From the Fig. 6.14., $\sin\theta = \frac{AB}{AC} \Rightarrow \sin 10^\circ 14' = \frac{AB}{3}$

From the sine table, $\sin 10^\circ 12' = 0.1771$ and Mean Difference for $2' = 0.0006$

$$\therefore \sin 10^\circ 14' = 0.1777 \Longrightarrow 0.1777 = \frac{AB}{3}$$

$$\therefore AB = 0.1777 \times 3 = 0.5331 \text{ cm}$$

$$\cos\theta = \frac{BC}{AC} \Longrightarrow \cos 10^\circ 14' = \frac{BC}{3}$$

From the cosine table, $\cos 10^\circ 12' = 0.9842$ and Mean Difference for $2' = 0.0001$

$$\therefore \cos 10^\circ 14' = 0.9842 - 0.0001 = 0.9841$$

$$0.9841 = \frac{BC}{3}$$

$$\therefore BC = 0.9841 \times 3 = 2.9523 \text{ cm}$$

Area of the right rriangle $= \frac{1}{2}bh = \frac{1}{2} \times 2.9523 \times 0.5331$

$$= 0.786935565$$

$\therefore$ Area of the triangle is 0.7869 cm^2 (approximately)

Example 6.25

Find the length of the chord of a circle of radius 6 cm subtending an angle of 165° at the centre.

Solution Let AB be the chord of a circle of radius 6 cm with O as centre. Draw $OC \perp AB$. Therefore C is the mid point of AB and $\angle AOB = 165^\circ$. Then

$$\angle AOC = \frac{165^\circ}{2} = 82^\circ 30'$$

In the right triangle OCA,

$$\sin 82^\circ 30' = \frac{AC}{OA} \Longrightarrow AC = \sin 82^\circ 30' \times OA$$

$$AC = 0.9914 \times 6 = 5.9484 \text{ cm}$$

$\therefore$ Length of the chord is $5.9484 \times 2 = 11.8968$ cm

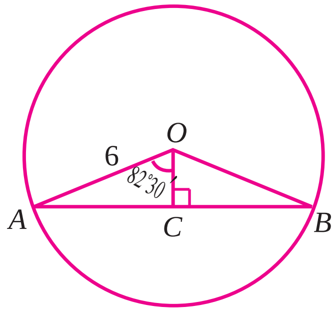


Fig. 6.15

Example 6.26

Find the length of the side of a regular polygon of 9 sides inscribed in a circle of radius 8 units.

Solution Let AB be a side of the regular polygon with 9 sides in the circle of radius 8 units.

If O is a centre of the circle, then $\angle AOB = \dfrac{360^\circ}{9} = 40^\circ$. Draw $OC \perp AB$ then

$$\angle AOC = \frac{40^\circ}{2} = 20^\circ$$

$$\sin 20^\circ = \frac{AC}{OA} = \frac{AC}{8}$$

$$\text{i.e., } 0.3420 = \frac{AC}{8}$$

$$AC = 0.3420 \times 8 = 2.736$$

$\therefore$ Length of the side $AB = 2 \times AC = 2 \times 2.736 = 5.472$ units

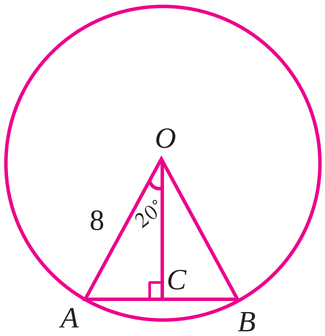


Fig. 6.16

Example 6.27

Find the radius of the incircle of a regular hexagon of side 6 cm.

Solution Let AB be the side of the regular hexagon and let O be the centre of the incircle. Draw $OC \perp AB$. If r is the radius of the circle, then $OC = r$. So,

$\angle AOB = \dfrac{360^\circ}{6} = 60^\circ$

$\therefore \angle AOC = \dfrac{60^\circ}{2} = 30^\circ$

$\therefore \tan 30^\circ = \dfrac{AC}{r}$

i.e., $\dfrac{1}{\sqrt{3}} = \dfrac{3}{r}$

$\therefore r = 3 \times 1.732 = 5.196$ cm

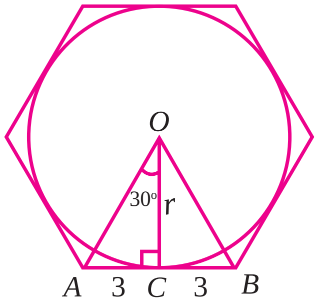


Fig. 6.17

Hence, radius of incircle is 5.196 cm

Exercise 6.3

1. Find the value of the following.

(i) $\sin 26^\circ$ (ii) $\cos 72^\circ$ (iii) $\tan 35^\circ$ (iv) $\sin 75^\circ 15'$

(v) $\sin 12^\circ 12'$ (vi) $\cos 12^\circ 35'$ (vii) $\cos 40^\circ 20'$ (viii) $\tan 10^\circ 26'$

(ix) $\cot 20^\circ$ (x) $\cot 40^\circ 20'$

2. Find the value of θ, if

(i) $\sin\theta = 0.7009$ (ii) $\cos\theta = 0.9664$ (iii) $\tan\theta = 0.3679$

(iv) $\cot\theta = 0.2334$ (v) $\tan\theta = 63.6567$

3. Simplify, using trigonometric tables

(i) $\sin 30^\circ 30' + \cos 40^\circ 20'$ (ii) $\tan 45^\circ 27' + \sin 20^\circ$

(iii) $\tan 63^\circ 12' - \cos 12^\circ 42'$ (iv) $\sin 50^\circ 26' + \cos 18^\circ + \tan 70^\circ 12'$

(v) $\tan 72^\circ + \cot 30^\circ$

4. Find the area of the right triangle with hypotenuse 20 cm and one of the acute angle is 48°

5. Find the area of the right triangle with hypotenuse 8 cm and one of the acute angle is 57°

6. Find the area of the isosceles triangle with base 16 cm and vertical angle $60^\circ 40'$

7. Find the area of the isosceles triangle with base 15 cm and vertical angle 80°

8. A ladder makes an angle 30° with the floor and its lower end is 12 m away from the wall. Find the length of the ladder.

9. Find the angle made by a ladder of length 4 m with the ground if its one end is 2 m away from the wall and the other end is on the wall.

10. Find the length of the chord of a circle of radius 5 cm subtending an angle of 108° at the centre.

11. Find the length of the side of regular polygon of 12 sides inscribed in a circle of radius 6 cm

12. Find the radius of the incircle of a regular hexagon of side 24 cm.

Points to Remember

- **Pythagoras Theorem:**

  The square of the length of the hypotenuse of a right triangle is equal to the sum of the squares of the other two sides.

- **Trigonometric Ratios:**

  Let θ be an acute angle of a right triangle. Then the six trigonometric ratios of θ are as follows.

  $\sin\theta = \dfrac{\textit{Opposite side}}{\textit{Hypotenuse}}$ $\qquad$ $\text{cosec}\,\theta = \dfrac{\textit{Hypotenuse}}{\textit{Opposite side}}$

  $\cos\theta = \dfrac{\textit{Adjacent side}}{\textit{Hypotenuse}}$ $\qquad$ $\sec\theta = \dfrac{\textit{Hypotenuse}}{\textit{Adjacent side}}$

  $\tan\theta = \dfrac{\textit{Opposite side}}{\textit{Adjacent side}}$ $\qquad$ $\cot\theta = \dfrac{\textit{Adjacent side}}{\textit{Opposite side}}$

- **Reciprocal Relations:**

  $\sin\theta = \dfrac{1}{\text{cosec}\,\theta}$ $\qquad$ $\cos\theta = \dfrac{1}{\sec\theta}$ $\qquad$ $\tan\theta = \dfrac{1}{\cot\theta}$

  $\text{cosec}\,\theta = \dfrac{1}{\sin\theta}$ $\qquad$ $\sec\theta = \dfrac{1}{\cos\theta}$ $\qquad$ $\cot\theta = \dfrac{1}{\tan\theta}$

- **Trigonometric Ratios of Complementary Angles:**

  Let θ be an acute angle of a right triangle. Then we have the following identities for trigonometric ratios of complementary angles.

  $\sin\theta = \cos(90^\circ - \theta)$ $\qquad$ $\text{cosec}\,\theta = \sec(90^\circ - \theta)$

  $\cos\theta = \sin(90^\circ - \theta)$ $\qquad$ $\sec\theta = \text{cosec}(90^\circ - \theta)$

  $\tan\theta = \cot(90^\circ - \theta)$ $\qquad$ $\cot\theta = \tan(90^\circ - \theta)$

7 GEOMETRY

Truth can never be told so as to be understood,
and not to be believed

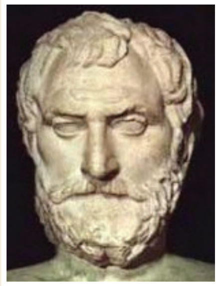
- William Blake

Main Targets

- To recall the basic concepts of geometry.
- To understand theorems on parallelograms.
- To understand theorems on circles.

7.1 Introduction

The very name ***Geometry*** is derived from two greek words meaning *measurement of earth*. Over time geometry has evolved into a beautifully arranged and logically organized body of knowledge. It is concerned with the properties of and relationships between points, lines, planes and figures. The earliest records of geometry can be traced to ancient Egypt and the Indus Valley from around 3000 B.C. Geometry begins with undefined terms, definitions, and assumptions; these lead to theorems and constructions. It is an abstract subject, but easy to visualize, and it has many concrete practical applications. Geometry has long been important for its role in the surveying of land and more recently, our knowledge of geometry has been applied to help build structurally sound bridges, experimental space stations, and large athletic and entertainment arenas, just to mention a few examples. The geometrical theorem of which a particular case involved in the method just described in the first book of Euclid's *Elements***.**

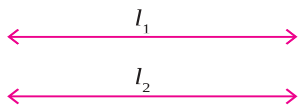
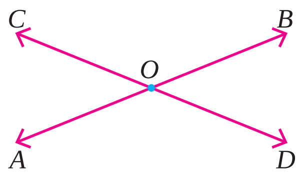
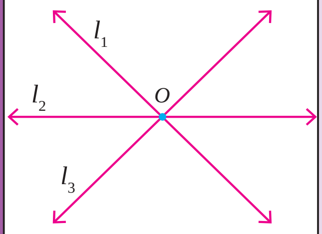
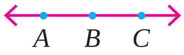


THALES
(640 - 546 BC)

Thales (pronounced THAY-lees) was born in the Greek city of Miletus. He was known for his theoretical and practical understanding of geometry, especially triangles. He established what has become known as Thales' Theorem, whereby if a triangle is drawn within a circle with the long side as a diameter of the circle then the opposite angle will always be a right angle. Thales used geometry to solve problems such as calculating the height of pyramids and the distance of ships from the shore. He is credited with the first use of deductive reasoning applied to geometry, by deriving four corollaries to Thales' Theorem. As a result, he has been hailed as the first true mathematician and is the first known individual to whom a mathematical discovery has been attributed. He was one of the so-called Seven Sages or Seven Wise Men of Greece, and many regard him as the first philosopher in the Western tradition.

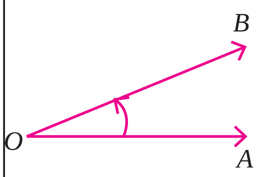
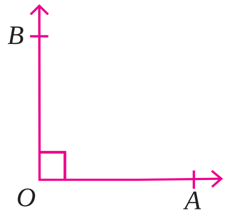
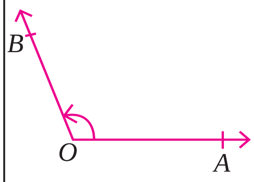
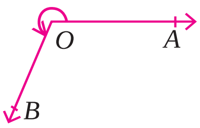
7.2 Geometry Basics

The purpose of this section is to recall some of the ideas that you have learnt in the earlier classes.

Term	Diagram	Description
Parallel lines		Lines in the same plane that do not intersect are called parallel lines. The distance between two parallel lines always remains the same.
Intersecting lines		Two lines having a common point are called intersecting lines. The point common to the two given lines is called their point of intersection. In the figure, the lines AB and CD intersect at a point O.
Concurrent lines		Three or more lines passing through the same point are said to be concurrent. In the figure, lines l_1, l_2, l_3 pass through the same point O and therefore they are concurrent.
Collinear points		If three or more points lie on the same straight line, then the points are called collinear points. Otherwise they are called non-collinear points.

7.2.1 Kinds of Angle

Angles are classified and named with reference to their degree of measure.

Name	Acute Angle	Right Angle	Obtuse Angle	Reflex Angle
Diagram				
Measure	$\angle AOB < 90^\circ$	$\angle AOB = 90^\circ$	$90^\circ < \angle AOB < 180^\circ$	$180^\circ < \angle AOB < 360^\circ$

Complementary Angles

Two angles are said to be complementary to each other if sum of their measures is 90°

For example, if $\angle A = 52^\circ$ and $\angle B = 38^\circ$, then angles $\angle A$ and $\angle B$ are complementary to each other.

Supplementary Angles

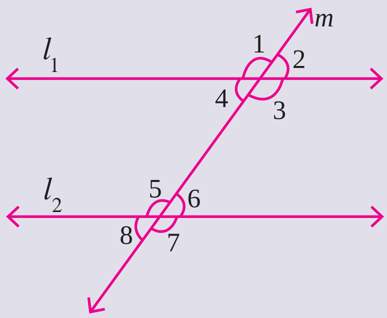
Two angles are said to be supplementary to each other if sum of their measures is 180°.

For example, the angles whose measures are 112° and 68° are supplementary to each other.

7.2.2 Transversal

A line that intersects two or more lines at distinct points is called a transversal.

Suppose a transversal intersects two parallel lines. Then:

Name	Angle	Diagram
Vertically opposite angles are equal	$\angle 1 = \angle 3$, $\angle 2 = \angle 4$, $\angle 5 = \angle 7$, $\angle 6 = \angle 8$	
Corresponding angles are equal.	$\angle 1 = \angle 5$, $\angle 2 = \angle 6$, $\angle 3 = \angle 7$, $\angle 4 = \angle 8$	
Alternate interior angles are equal.	$\angle 3 = \angle 5$, $\angle 4 = \angle 6$	
Alternate exterior angles are equal.	$\angle 1 = \angle 7$, $\angle 2 = \angle 8$	
Consecutive interior angles are supplementary.	$\angle 3 + \angle 6 = 180°$; $\angle 4 + \angle 5 = 180°$	

7.2.3 Triangles

The sum of the angles of a triangle is 180°.

In the Fig. 7.1., $\angle A + \angle B + \angle C = 180°$

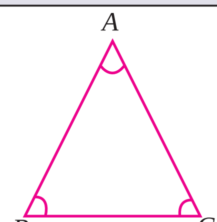


Fig. 7.1

Remarks

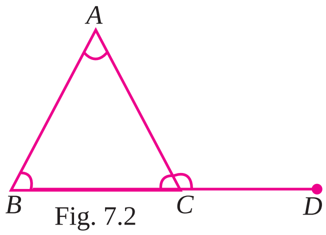


Fig. 7.2

(i) If a side of a triangle is produced , then the exterior angle so formed is equal to the sum of its interior opposite angles.

$\angle ACD = \angle BAC + \angle ABC$

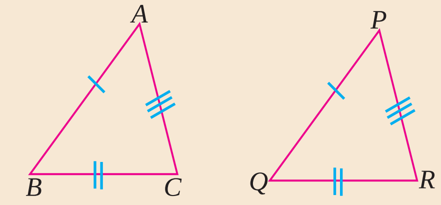
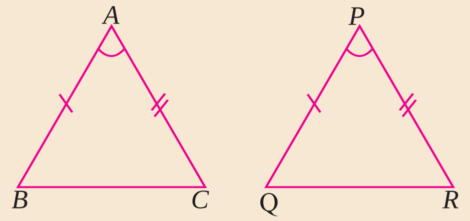
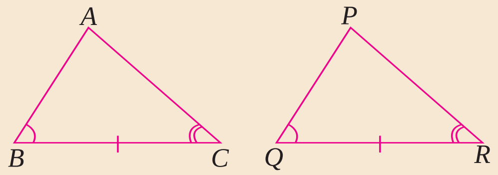
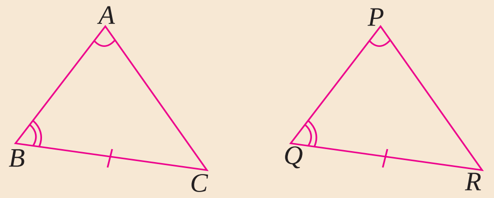
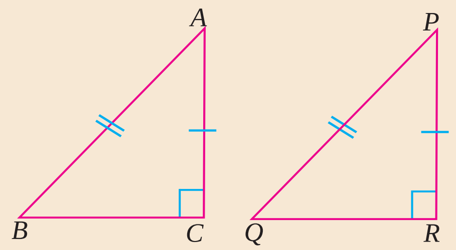
(ii) An exterior angle of a triangle is greater than either of the interior opposite angles.

(iii) In any triangle, the angle opposite to the largest side has the greatest angle.

Congruent Triangles

Two triangles are congruent if and only if one of them can be made to superpose on the other, so as to cover it exactly.

For congruence, we use the symbol ‘≡’

	Description	Diagram
SSS	If the three sides of a triangle are equal to three sides of another triangle, then the two triangles are congruent.	 $\Delta ABC \equiv \Delta PQR$
SAS	If two sides and the included angle of a triangle are equal to two sides and the included angle of another triangle, then the two triangles are congruent.	 $\Delta ABC \equiv \Delta PQR$
ASA	If two angles and the included side of a triangle are equal to two angles and the included side of another triangle, then the two triangles are congruent.	 $\Delta ABC \equiv \Delta PQR$
AAS	If two angles and any side of a triangle are equal to two angles and a side of another triangle, then the two triangles are congruent.	 $\Delta ABC \equiv \Delta PQR$
RHS	If one side and the hypotenuse of a right triangle are equal to a side and the hypotenuse of another right triangle, then the two triangles are congruent	 $\Delta ABC \equiv \Delta PQR$

Exercise 7.1

1. Find the complement of each of the following angles.

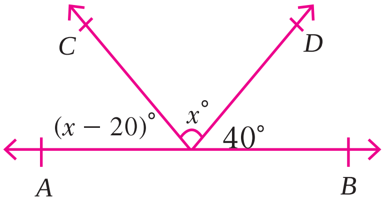
   (i) 63° (ii) 24° (iii) 48° (iv) 35° (v) 20°

2. Find the supplement of each of the following angles.

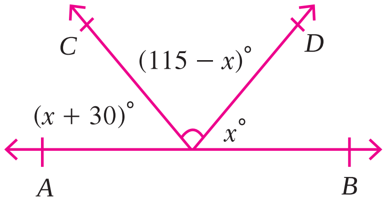
   (i) 58° (ii) 148° (iii) 120° (iv) 40° (v) 100°

3. Find the value of x in the following figures.

(i)



(ii)



4. Find the angles in each of the following.

i) The angle which is two times its complement.

ii) The angle which is four times its supplement.

iii) The angles whose supplement is four times its complement.

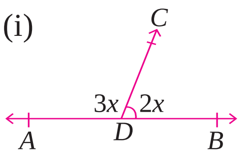
iv) The angle whose complement is one sixth of its supplement.

v) Supplementary angles are in the ratio 4:5

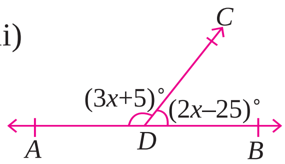
vi) Two complementary angles are in the ratio3:2

5. Find the values of x, y in the following figures.

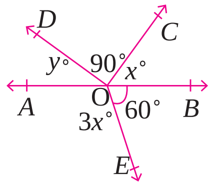
(i)



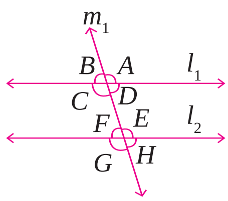
(ii)



(iii)

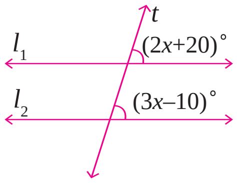


6. Let $l_1 \parallel l_2$ and m_1 is a transversal . If $\angle F = 65^\circ$, find the measure of each of the remaining angles.

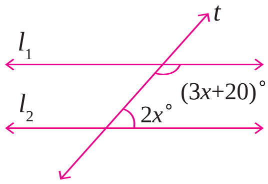


7. For what value of x will l_1 and l_2 be parallel lines.

(i)



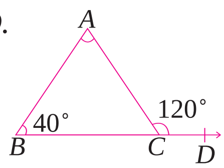
(ii)



8. The angles of a triangle are in the ratio of 1:2:3. Find the measure of each angle of the triangle.

9. In $\triangle ABC$, $\angle A + \angle B = 70^\circ$ and $\angle B + \angle C = 135^\circ$. Find the measure of each angle of the triangle.

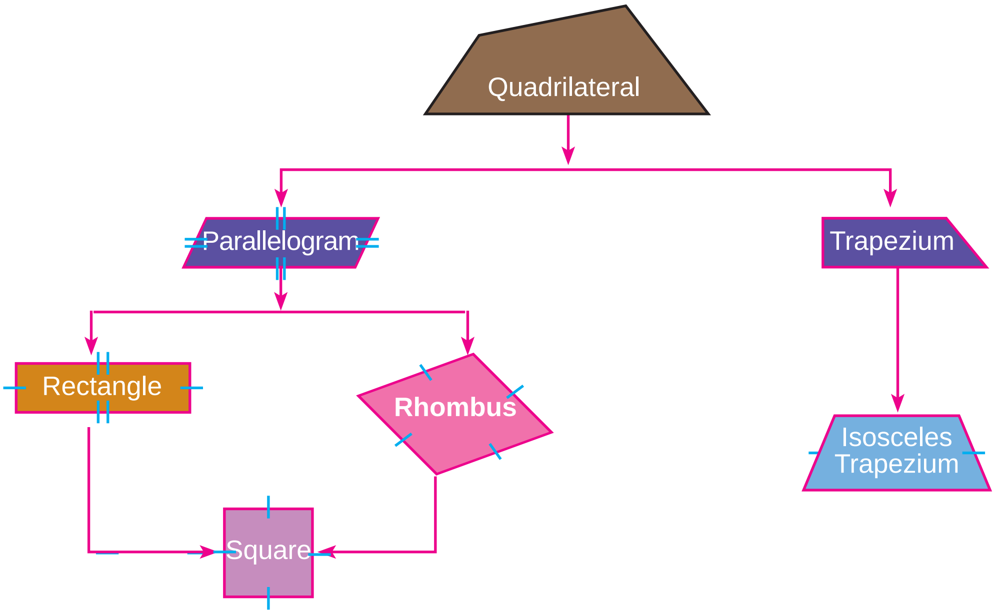
10. In the given figure at right, side BC of $\triangle ABC$ is produced to D. Find $\angle A$ and $\angle C$.



7.3 Quadrilateral

A closed geometric figure with four sides and four vertices is called a quadrilateral.

The sum of all the four angles of a quadrilateral is 360°.



7.3.1 Properties of Parallelogram, Rhombus and Trapezium

Parallelogram	Sides	Opposite sides are parallel and equal.
	Angles	Opposite angles are equal and sum of any two adjacent angles is 180°.
	Diagonals	Diagonals bisect each other.
Rhombus	Sides	All sides are equal and opposite sides are parallel.
	Angles	Opposite angles are equal and sum of any two adjacent angles is 180°.
	Diagonals	Diagonals bisect each other at right angles.
Trapezium	Sides	One pair of opposite sides is parallel
	Angles	The angles at the ends of each non-parallel side are supplementary
	Diagonals	Diagonals need not be equal.
Isosceles Trapezium	Sides	One pair of opposite sides is parallel, the other pair of sides is equal in length.
	Angles	The angles at the ends of each parallel side are equal.
	Diagonals	Diagonals are equal in length.

Note
(i) A rectangle is an equiangular parallelogram
(ii) A rhombus is an equilateral parallelogram
(iii) A square is an equilateral and equiangular parallelogram.
(iv) Thus a square is a rectangle, a rhombus and a parallelogram.

7.4 Parallelogram

A quadrilateral in which the opposites sides are parallel is called a parallelogram.

7.4.1 Properties of Parallelogram

Property 1 : In a parallelogram, the opposite sides are equal.

Given : $ABCD$ is a parallelogram. So, $AB \parallel DC$ and $AD \parallel BC$

To prove : $AB = CD$ and $AD = BC$

Construction : Join BD

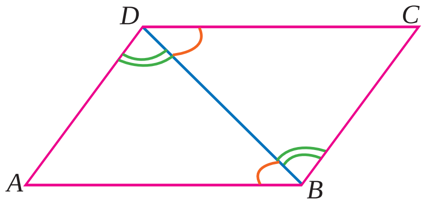


Fig. 7.3

Proof :

Consider the ΔABD and the ΔBCD.

(i) $\angle ABD = \angle BDC$ ($AB \parallel DC$ and BD is a transversal. So, alternate interior angles are equal.)

(ii) $\angle BDA = \angle DBC$ ($AD \parallel BC$ and BD is a transversal. So, alternate interior angles are equal.)

(iii) BD is common side

$\therefore \Delta ABD \equiv \Delta BCD$ (By ASA property)

Thus, $AB = DC$ and $AD = BC$ (Corresponding sides are equal) ■

Converse of Property 1: If the opposite sides of a quadrilateral are equal, then the quadrilateral is a parallelogram.

Property 2 : In a parallelogram, the opposite angles are equal.

Given : $ABCD$ is a parallelogram, where $AB \parallel DC$, $AD \parallel BC$

To prove : $\angle ABC = \angle ADC$ and $\angle DAB = \angle BCD$

Construction : Join BD

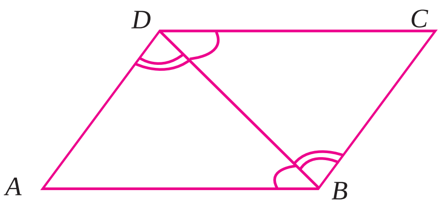


Fig. 7.4

Proof :

(i) $\angle ABD = \angle BDC$ ($AB \parallel DC$ and BD is a transversal. So, alternate interior angles are equal.)

(ii) $\angle DBC = \angle BDA$ ($AD \parallel BC$ and BD is a transversal. So, alternate interior angles are equal.)

(iii) $\angle ABD + \angle DBC = \angle BDC + \angle BDA$

$\therefore \angle ABC = \angle ADC$

Similarly, $\angle BAD = \angle BCD$ ■

Converse of Property 2: If the opposite angles in a quadrilateral are equal, then the quadrilateral is a parallelogram.

Property 3 : The diagonals of a parallelogram bisect each other.

Given : $ABCD$ is a parallelogram, in which $AB \parallel DC$ and $AD \parallel BC$

To prove : M is the midpoint of diagonals AC and BD .

Proof :

Consider the ΔAMB and ΔCMD

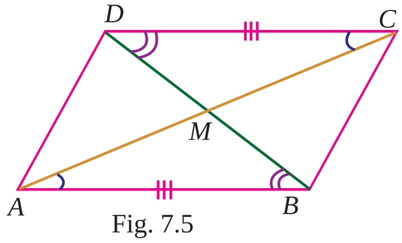


Fig. 7.5

(i) $AB = DC$ Opposite sides of the parallelogram are equal

(ii) $\angle MAB = \angle MCD$ Alternate interior angles ($\because AB \parallel DC$)

$\angle ABM = \angle CDM$ Alternate interior angles ($\because AB \parallel DC$)

(iii) $\Delta AMB \equiv \Delta CMD$ (By ASA property)

$\therefore AM = CM$ and $BM = DM$

i.e., M is the mid point of AC and BD

$\therefore$ The diagonals bisect each other ■

Converse of Property 3: If the diagonals of a quadrilateral bisect each other, then the quadrilateral is a parallelogram.

Note

(i) A diagonal of a parallelogram divides it into two triangles of equal area.

(ii) A parallelogram is a rhombus if its diagonals are perpendicular.

(iii) Parallelograms on the same base and between the same parallels are equal in area.

Example 7.1

If the measures of three angles of a quadrilateral are 100°, 84°, 76°,find the meausre of fourth angle.

Solution Let the measure of the fourth angle be x°.

The sum of the angles of a quadrilateral is 360°. So,

$$100^\circ + 84^\circ + 76^\circ + x^\circ = 360^\circ$$
$$260^\circ + x^\circ = 360^\circ$$
i.e., $$x^\circ = 100^\circ$$

Hence, the measure of the fourth angle is 100°.

Example 7.2

In the parallelogram *ABCD* if $\angle A = 65^\circ$, find $\angle B$, $\angle C$ and $\angle D$.

Solution Let *ABCD* be a parallelogram in which $\angle A = 65^\circ$.

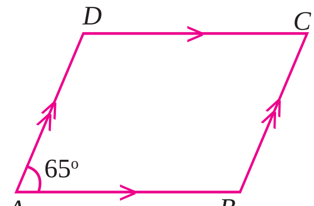


Fig. 7.6

Since $AD \parallel BC$ we can treat AB as a transversal. So,

$$\angle A + \angle B = 180^\circ$$
$$65^\circ + B = 180^\circ$$
$$\angle B = 180^\circ - 65^\circ$$
$$\angle B = 115^\circ$$

Since the opposite angles of a parallelogram are equal, we have

$$\angle C = \angle A = 65^\circ \text{ and } \angle D = \angle B = 115^\circ$$

Hence, $\angle B = 115^\circ$, $\angle C = 65^\circ$ and $\angle D = 115^\circ$

Example 7.3

Suppose *ABCD* is a rectangle whose diagonals *AC* and *BD* intersect at *O*. If $\angle OAB = 62^\circ$, find $\angle OBC$.

Solution The diagonals of a rectangle are equal and bisect each other. So,

$OA = OB$ and $\angle OBA = \angle OAB = 62^\circ$

Since the measure of each angle of rectangle is 90°

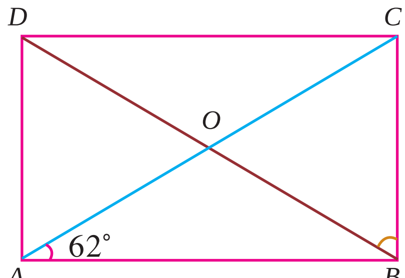


Fig. 7.7

$$\angle ABC = 90^\circ$$
$$\angle ABO + \angle OBC = 90^\circ$$
$$62^\circ + \angle OBC = 90^\circ$$
$$\angle OBC = 90^\circ - 62^\circ$$
$$= 28^\circ$$

Example 7.4

If $ABCD$ is a rhombus and if $\angle A = 76^\circ$, find $\angle CDB$.

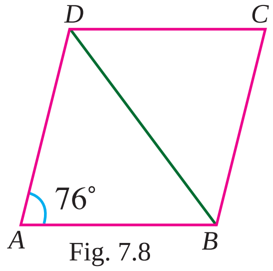


Fig. 7.8

Solution $\angle A = \angle C = 76^\circ$ (Opposite angles of a rhombus)

Let $\angle CDB = x^\circ$. In $\triangle$ CDB, $CD = CB$

$$\angle CDB = \angle CBD = x^\circ$$

$$\angle CDB + \angle CBD + \angle DCB = 180^\circ \text{ (Angles of a triangle)}$$

$$2x^\circ + 76^\circ = 180^\circ \implies 2x = 104^\circ$$

$$x^\circ = 52^\circ$$

$$\therefore \angle CDB = 52^\circ$$

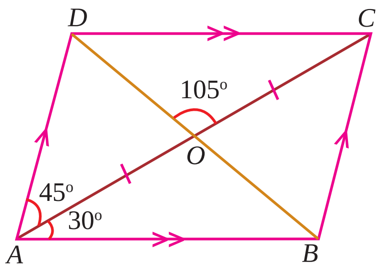
Exercise 7.2

1. In a quadrilateral $ABCD$, the angles $\angle A$, $\angle B$, $\angle C$ and $\angle D$ are in the ratio 2:3:4:6. Find the measure of each angle of the quadrilateral.

2. Suppose $ABCD$ is a parallelogram in which $\angle A = 108^\circ$. Calculate $\angle B$, $\angle C$ and $\angle D$.

3. In the figure at right, $ABCD$ is a parallelogram $\angle BAO = 30^\circ$, $\angle DAO = 45^\circ$ and $\angle COD = 105^\circ$. Calculate

   (i) $\angle ABO$ (ii) $\angle ODC$ (iii) $\angle ACB$ (iv) $\angle CBD$



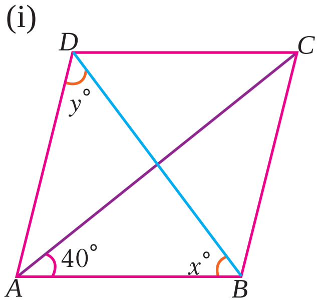
4. Find the measure of each angle of a parallelogram, if larger angle is 30° less than twice the smaller angle.

5. Suppose $ABCD$ is a parallelogram in which $AB = 9$ cm and its perimeter is 30 cm. Find the length of each side of the parallelogram.

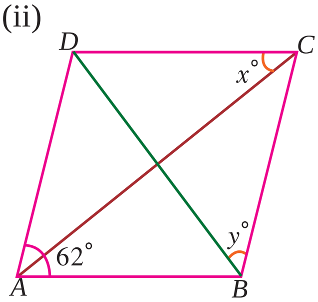
6. The length of the diagonals of a rhombus are 24 cm and 18 cm. Find the length of each side of the rhombus.

7. In the following figures, $ABCD$ is a rhombus. Find the values of x and y.

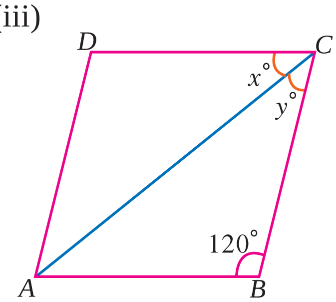
(i)



(ii)

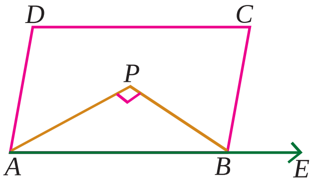


(iii)



8. The side of a rhombus is 10 cm and the length of one of the diagonals is 12 cm. Find the length of the other diagonal.

9. In the figure at the right, *ABCD* is a parallelogram in which the bisectors of $\angle A$ and $\angle B$ intersect at the point *P*. Prove that $\angle APB = 90^\circ$.



7.5 Circles

Locus

Locus is a path traced out by a moving point which satisfies certain geometrical conditions.

For example, the locus of a point equidistant from two fixed points is the perpendicular bisector of the line segment joining the two points.

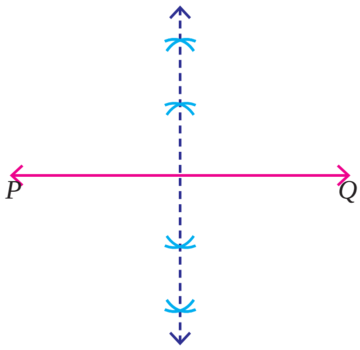


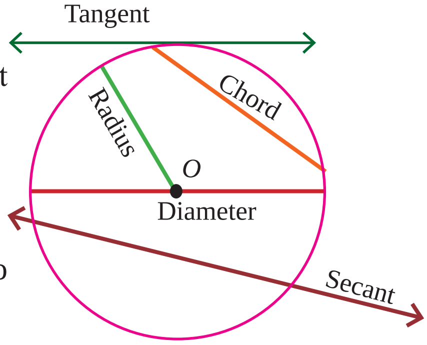
Fig. 7.9

Circles

The locus of a point which moves in such that the distance from a fixed point is always a constant is a circle.

The fixed point is called its centre and the constant distance is called its radius.

The boundary of a circle is called its circumference.



Chord

A chord of a circle is a line segment joining any two points on its circumference.

Diameter

A diameter is a chord of the circle passing through the centre of the circle.

Diameter is the longest chord of the circle.

Secant

A line which intersects a circle in two distinct points is called a secant of the circle.

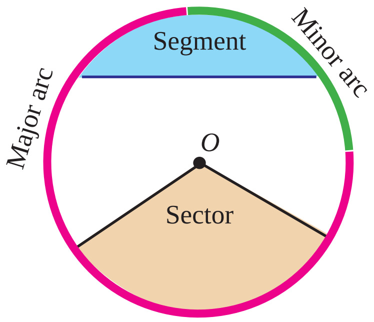


Fig. 7.10

Tangent

A line that touches the circle at only one point is called a tangent to the circle.

The point at which the tangent meets the circle is its point of contact.

Arc of a Circle

A continuous piece of a circle is called an arc of the circle.

The whole circle has been divided into two pieces, namely, major arc, minor arc.

Concentric Circles

Circles which have the same centre but different radii are called concentric circles.

In the given figure, the two circles are concentric circles having the same centre O but different radii r and R respectively.

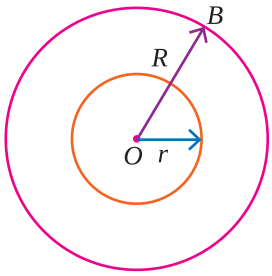


Fig. 7.11

Congruent Arcs

Two arcs $\overset{\frown}{AB}$ and $\overset{\frown}{CD}$ of a circle are said to be congruent if they substend same angle at the centre and we write $\overset{\frown}{AB} \equiv \overset{\frown}{CD}$. So,

$$\overset{\frown}{AB} \equiv \overset{\frown}{CD} \iff m\overset{\frown}{AB} = m\overset{\frown}{CD} \iff \angle AOB = \angle COD$$

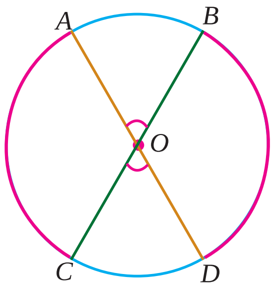


Fig. 7.12

7.5.1 Properties of Chords of a Circle

Result

Equal chords of a circle subtend equal angles at the centre.

In the Fig. 7.13., chord AB = chord $CD \implies \angle AOB = \angle COD$

Converse of the result

If the angles subtended by two chords at the centre of a circle are equal, then the chords are equal.

$\angle AOB = \angle COD \implies$ chord AB = chord CD

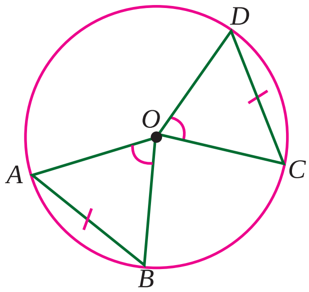


Fig. 7.13

Theorem 1

Perpendicular from the centre of a circle to a chord bisects the chord.

Given : A circle with centre O and AB is a chord of the circle other than the diameter and $OC \perp AB$

To prove: $AC = BC$

Construction: Join OA and OB

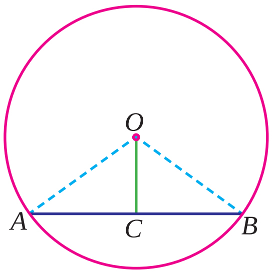


Fig. 7.14

Proof:

In Δs OAC and OBC

(i) $OA = OB$ (Radii of the same circle.)

(ii) OC is common

(iii) $\angle OCA = \angle OCB$ (Each 90° ,since $OC \perp AB$.)

(iv) $\Delta OAC \equiv \Delta OBC$ (RHS congruency.)

$\therefore AC = BC$ ■

Converse of Theorem 1 : The line joining the centre of the circle and the midpoint of a chord is perpendicular to the chord.

Theorem 2

Equal chords of a circle are equidistant from the centre.

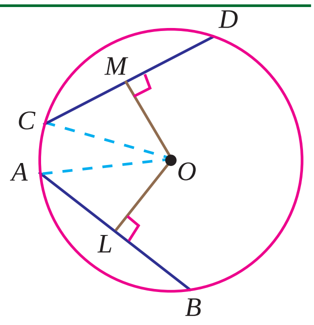


Fig. 7.15

Given: A cirle with centre O and radius r such that chord AB = chord CD.

To prove: $OL = OM$

Construction: Draw $OL \perp AB$ and $OM \perp CD$. Join OA and OC

Proof:

(i) $AL = \frac{1}{2}AB$ and $CM = \frac{1}{2}CD$ (Perpendicular from the centre of a circle to the chord bisects the chord.)

$AB = CD \Longrightarrow \frac{1}{2}AB = \frac{1}{2}CD \Longrightarrow AL = CM$

(ii) $OA = OC$ (radii)

(iii) $\angle OMC = \angle OLA$ (Each 90°)

(iii) $\Delta OLA \equiv \Delta OMC$ (RHS congruence.)

$\therefore\ OL = OM$

Hence AB and CD are equidistant from O .

Converse of Theorem 2 : The chords of a circle which are equidistant from the centre are equal.

Example 7.5

A chord of length 16 cm is drawn in a circle of radius 10 cm. Find the distance of the chord from the centre of the circle.

Solution AB is a chord of length 16 cm

C is the midpoint of AB.

OA is the radius of length 10 cm

$AB = 16$ cm

$AC = \frac{1}{2} \times 16 = 8$ cm

$OC = 10$ cm

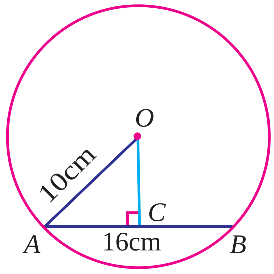


Fig. 7.16

In a right triangle OAC.

$OC^2 = OA^2 - AC^2$

$= 10^2 - 8^2 = 100 - 64 = 36$ cm

$\therefore OC = 6$ cm

Hence, the distance of the chord from the centre is 6 cm.

Example 7.6

In two concentric circles, chord AB of the outer circle cuts the inner circle at C and D. Prove that $AC = BD$.

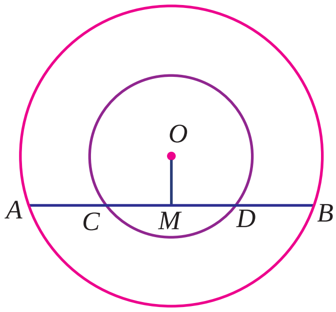


Fig. 7.17

Solution Given: Chord AB of the outer circle cuts the inner circle at C and D.

To prove: $AC = BD$

Construction: Draw $OM \perp AB$

Proof :

Since $OM \perp AB$ (by construction)

OM also $\perp CD$ ($ACDB$ is a line)

In the outer circle

$AM = BM$ (1) ($\because$ OM bisects the chord AB)

In the inner circle

$CM = DM$ (2) ($\because$ OM bisects the chord CD)

From (1) and (2), we get

$AM - CM = BM - DM$

$AC = BD$

7.5.2 Angles Subtended by Arcs

Theorem 3

The angle subtended by an arc of a circle at the centre is double the angle subtended by it at any point on the remaining part of the circle.

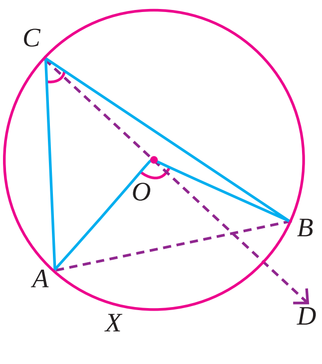


Fig. 7.18

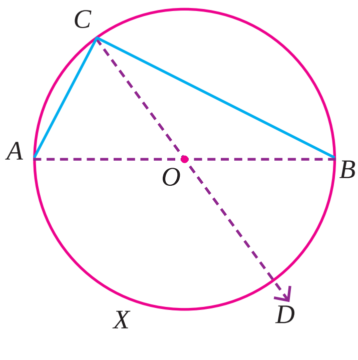


Fig. 7.19

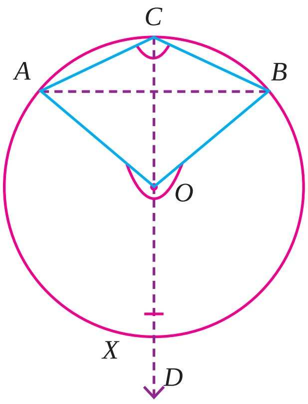


Fig. 7.20

Given : O is the centre of the circle. AXB is the arc. $\angle AOB$ is the angle subtended by the arc $\overset{\frown}{AXB}$ at the centre. $\angle ACB$ is the angle subtended by the arc $\overset{\frown}{AXB}$ at a point on the remaining part of the circle.

To prove : $\angle AOB = 2\angle ACB$

Construction : Join CO and produce it to D

Proof :

(i) $OA = OC$ (radii)

(ii) $\angle OCA = \angle OAC$ (angles opposite to equal sides are equal.)

(iii) In ΔAOC
$\angle AOD = \angle OCA + \angle OAC$ (exterior angles of a triangle = sum of interior opposite angles.)

(iv) $\angle AOD = \angle OCA + \angle OCA$ (substituting $\angle OAC$ by $\angle OCA$)

(v) $\angle AOD = 2\angle OCA$ (by addition)

(vi) similarly in ΔBOC
$\angle BOD = 2\angle OCB$

(vii) $\angle AOD + \angle BOD = 2\angle OCA + 2\angle OCB$
$= 2(\angle OCA + \angle OCB)$ ($\because \angle AOD + \angle BOD = \angle AOB$
$\angle OCA + \angle OCB = \angle ACB$)

(viii) $\angle AOB = 2\angle ACB$ ■

Note
(i) An angle inscribed in a semicircle is a right angle.
(ii) Angles in the same segment of a circle are equal.

7.5.3 Cyclic Quadrilaterals

Theorem 4

Opposite angles of a cyclic quadrilateral are supplementary (or)
The sum of opposite angles of a cyclic quadrilateral is 180°

Given : O is the centre of circle. $ABCD$ is the cyclic quadrilateral.

To prove : $\angle BAD + \angle BCD = 180^\circ$, $\angle ABC + \angle ADC = 180^\circ$

Construction : Join OB and OD

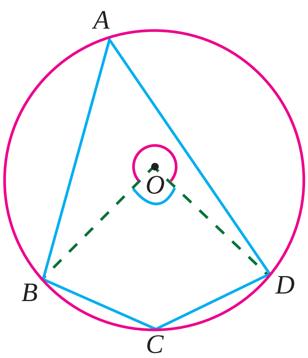


Fig. 7.21

Proof:

(i) $\angle BAD = \frac{1}{2}\angle BOD$ (The angle substended by an arc at the centre is double the angle on the circle.)

(ii) $\angle BCD = \frac{1}{2}$ reflex $\angle BOD$

(iii) $\therefore \angle BAD + \angle BCD = \frac{1}{2}\angle BOD + \frac{1}{2}$ reflex $\angle BOD$
(add (i) and (ii))

i.e., $\angle BAD + \angle BCD = \frac{1}{2}(\angle BOD + \text{reflex } \angle BOD)$

i.e., $\angle BAD + \angle BCD = \frac{1}{2}(360^\circ)$ (Complete angle at the centre is 360°)

i.e., $\angle BAD + \angle BCD = 180^\circ$

(iv) Similarly $\angle ABC + \angle ADC = 180^\circ$ ■

Converse of Theorem 4 : If a pair of opposite angles of a quadrilateral is supplementary, then the quadrilateral is cyclic.

Theorem 5 (Exterior - angle property of a cyclic quadrilateral)

If one side of a cyclic quadrilateral is produced then the exterior angle is equal to the interior opposite angle.

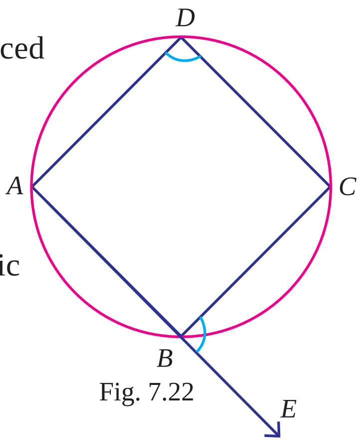


Fig. 7.22

Given : A cyclic quadrilateral $ABCD$, whose side AB is produced to E.

To prove : $\angle CBE = \angle ADC$

Proof :

(i) $\angle ABC + \angle CBE = 180^\circ$ (linear pair)

(ii) $\angle ABC + \angle ADC = 180^\circ$ (Opposite angles of a cyclic quadrilateral)

from (i)and (ii)

(iii) $\angle ABC + \angle CBE = \angle ABC + \angle ADC$

(iv) $\therefore \angle CBE = \angle ADC$ ■

Example: 7.7

Find the value of x in the following figure.

(i)

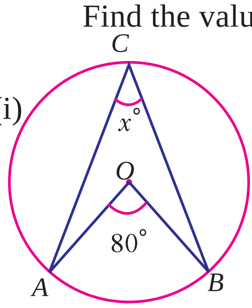


Fig. 7.23

(ii)

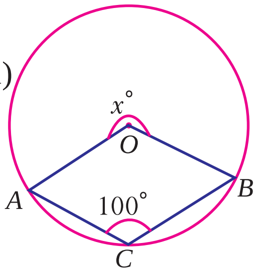


Fig. 7.24

(iii)

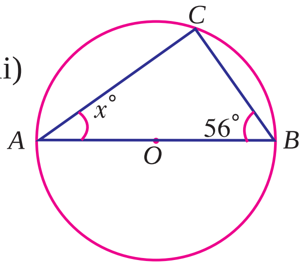


Fig. 7.25

(iv)

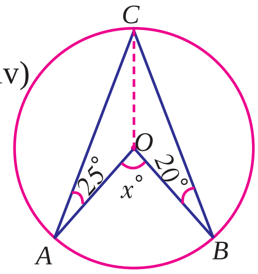
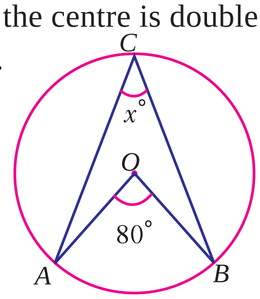


Fig. 7.26

Solution Using the theorem the angle subtended by an arc of a circle at the centre is double the angle subtended by it at any point on the remaining part of the circle.



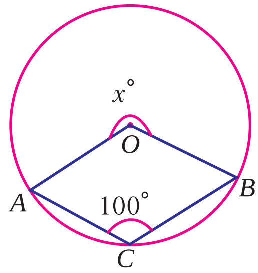
(i) $\angle AOB = \frac{1}{2}\angle ACB$

$\angle ACB = \frac{1}{2}\angle AOB$

$= \frac{1}{2} \times 80^\circ = 40^\circ$

(ii) $reflex \angle AOB = 2 \angle ACB$

$x = 2 \times 100^\circ = 200^\circ$



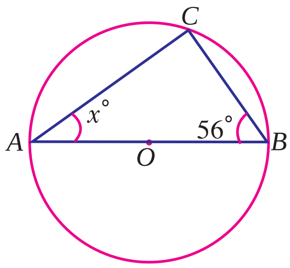
(iii) $\angle ABC + \angle BCA + \angle CAB = 180^\circ$

$56^\circ + 90^\circ + \angle CAB = 180^\circ$

($\because \angle BCA$ = angle on a semicircle = 90°)

$\angle CAB = 180^\circ - 146^\circ$

$x = 34^\circ$



(iv) $OA = OB = OC$ (radius)

$\angle OCA = \angle OAC = 25^\circ$

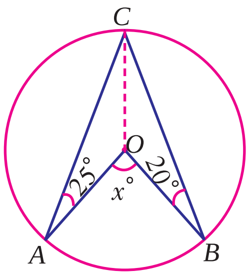
$\angle OBC = \angle OCB = 20^\circ$

$\angle ACB = \angle OCA + \angle OCB$

$= 25^\circ + 20^\circ = 45^\circ$

$AOB = 2 \angle ACB$

$x = 2 \times 45^\circ = 90^\circ$



Example 7.8

In the Fig.7.27, O is the centre of a circle and $\angle ADC = 120^\circ$. Find the value of x.

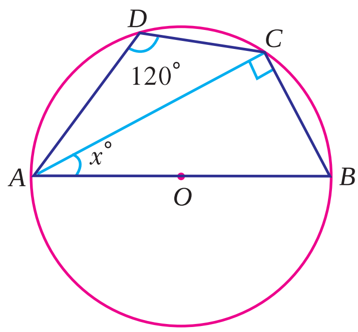


Fig. 7.27

Solution $ABCD$ is a cyclic quadrilateral.

we have

$\angle ABC + \angle ADC = 180^\circ$

$\angle ABC = 180^\circ - 120^\circ = 60^\circ$

Also $\angle ACB = 90^\circ$ (angle on a semi circle)

In $\triangle ABC$ we have

$\angle BAC + \angle ACB + \angle ABC = 180^\circ$

$\angle BAC + 90^\circ + 60^\circ = 180^\circ$

$\angle BAC = 180^\circ - 150^\circ = 30^\circ$

Exercise 7.3

1. The radius of a circle is 15 cm and the length of one of its chord is 18 cm. Find the distance of the chord from the centre.

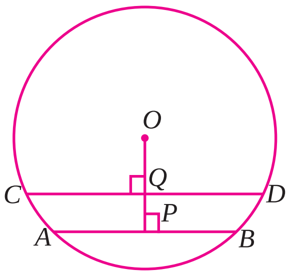
2. The radius of a circles 17 cm and the length of one of its chord is 16 cm. Find the distance of the chord from the centre.

3. A chord of length 20 cm is drawn at a distance of 24 cm from the centre of a circle. Find the radius of the circle.

4. A chord is 8 cm away from the centre of a circle of radius 17 cm. Find the length of the chord.

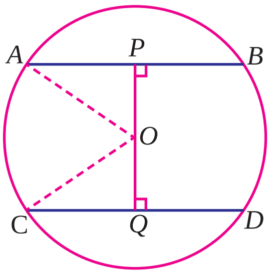
5. Find the length of a chord which is at a distance of 15 cm from the centre of a circle of radius 25 cm.

6. In the figure at right, *AB* and *CD* are two parallel chords of a circle with centre *O* and radius 5 cm such that *AB* = 6 cm and *CD* = 8 cm. If $OP \perp AB$ and $CD \perp OQ$ determine the length of *PQ*.



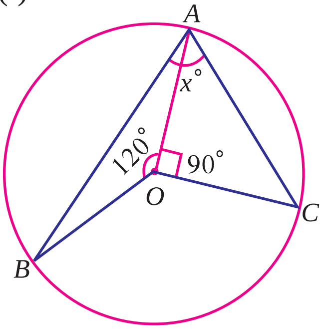
7. *AB* and *CD* are two parallel chords of a circle which are on either sides of the centre. Such that *AB* = 10 cm and *CD* = 24 cm. Find the radius if the distance between *AB* and *CD* is 17 cm.

8. In the figure at right, *AB* and *CD* are two parallel chords of a circle with centre *O* and radius 5 cm. Such that *AB* = 8 cm and *CD* = 6 cm. If $OP \perp AB$ and $OQ \perp CD$ determine the length *PQ*.

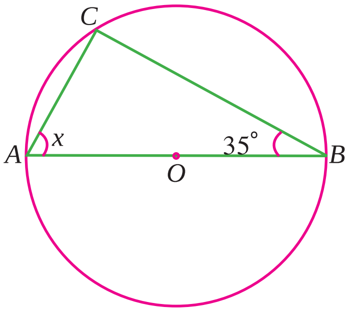


9. Find the value of x in the following figures.

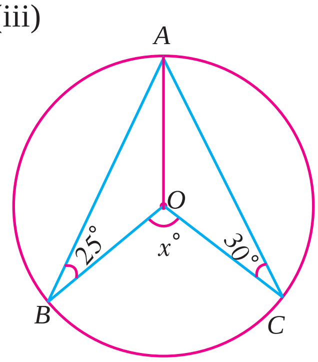
(i)



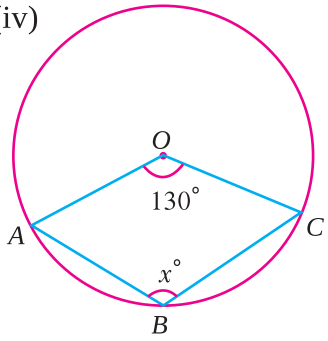
(ii)



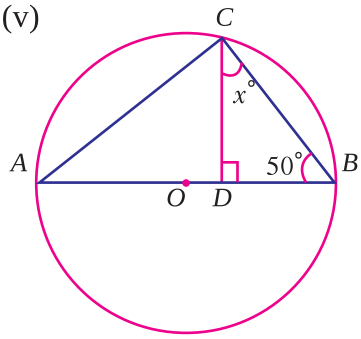
(iii)



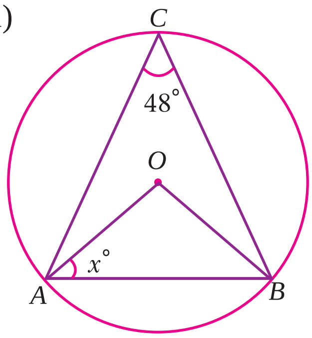
(iv)



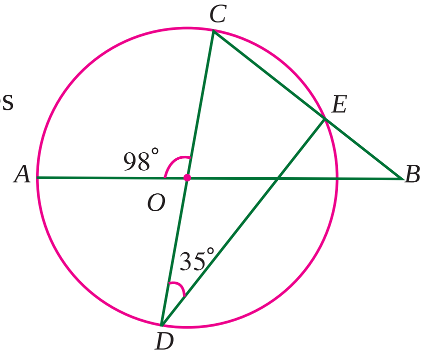
(v)

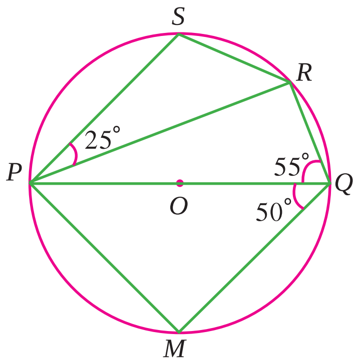


(vi)



10. In the figure at right, AB and CD are straight lines through the centre O of a circle. If $\angle AOC = 98^\circ$ and $\angle CDE = 35^\circ$ find (i) $\angle DCE$ (ii) $\angle ABC$

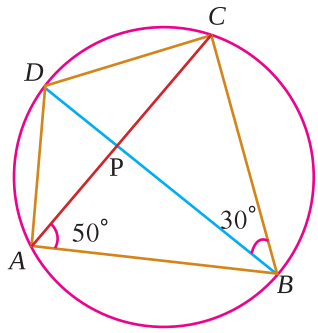


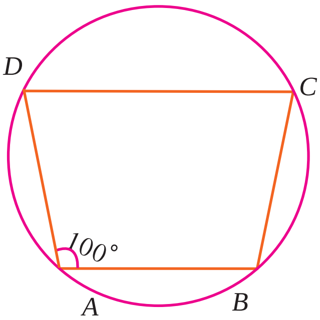
11. 

In the figure at left, PQ is a diameter of a circle with centre O. If $\angle PQR = 55^\circ$, $\angle SPR = 25^\circ$ and $\angle PQM = 50^\circ$. Find (i) $\angle QPR$, (ii) $\angle QPM$ and (iii) $\angle PRS$.

12. In the figure at right, $ABCD$ is a cyclic quadrilateral whose diagonals intersect at P such that $\angle DBC = 30^\circ$ and $\angle BAC = 50^\circ$.

Find (i) $\angle BCD$ (ii) $\angle CAD$

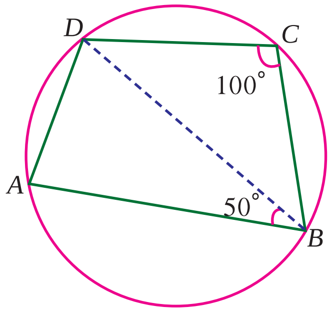


13. 

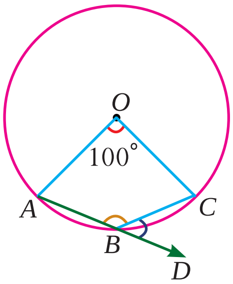
In the figure at left , $ABCD$ is a cyclic quadrilateral in which $AB \parallel DC$. If $\angle BAD = 100^\circ$

find (i) $\angle BCD$ (ii) $\angle ADC$ (iii) $\angle ABC$.

14. In the figure at right, $ABCD$ is a cyclic quadrilateral in which $\angle BCD = 100^\circ$ and $\angle ABD = 50^\circ$ find $\angle ADB$



15. In the figure at left, O is the centre of the circle, $\angle AOC = 100^\circ$ and side AB is produced to D.



Find (i) $\angle CBD$ (ii) $\angle ABC$

Points to remember

- In a parallelogram the opposite sides are equal.
- In a parallelogram, the opposite angles are equal.
- The diagonals of a parallelogram bisect each other.
- A rectangle is an equiangular parallelogram
- A rhombus is an equilateral parallelogram
- A square is an equilateral and equiangular parallelogram. Thus a square is a rectangle, a rhombus and a parallelogram.
- Each diagonal divides the parallelogram into two congruent triangles.
- A parallelogram is rhombus if its diagonals are perpendicular.
- Parallelograms on the same base and between the same parallels are equal in area.
- A diagonal of a parallelogram divides it into two triangles of equal area.
- Equal chords of a circle subtend equal angles at the centre.
- If two arcs of a circle are congruent then the corresponding chords are equal.
- Perpendicular from the centre of a circle to a chord bisects the chord.
- Equal chords of a circle are equidistant from the centre.
- The angle substended by an are of a circle at the centre is double the angel subtended by it at any point on the remaining part of the circle.
- The angle in a semi circle is a right angle.
- Angle in the same segment of a circle are equal.
- The sum of either pair of opposite angle of opposite angles of a cyclic quadrilateral is 180°
- If one side of a cyclic quadrilateral is produced then the exterior angle is equal to the interior opposite angle.

8 MENSURATION

The most beautiful plane figure is – the circle and the most beautiful solid figure – the sphere

- PYTHAGORAS

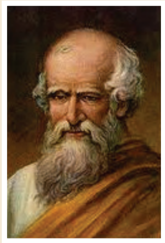
Main Targets

- To find the length of arc, area and perimeter of sectors of circles.
- To find the surface area and volume of cubes.
- To find the surface area and volume of cuboids.

8.1 Introduction

Every day, we see various shapes like triangles, rectangles, squares, circles, spheres and so on all around us, and we are already familiar with some of their properties: like area and perimeter. The part of Mathematics that deals with measurements of geometrical shapes is known as *Mensuration*. It is considered very important because there are various fields of life where geometry is considered as an important field of study.

Perimeter, Area and Volume plays a vital role in architecture and carpentry. Perimeter, Area and volume can be used to analyze real-world situations. It is necessary for everyone to learn formulas used to find the perimeter ,areas of two-dimensional figures and the surface areas and volumes of three dimensional figures for day- to-day life. In this chapter we deal with arc length and area of sectors of circles and area and volume of cubes and cuboids.



Archimedes
287 - 212 B.C.

One of the very great mathematicians of all time was Archimedes, a native of the Greek city of Syracus on the island of Sicily. He was born about 287 B.C. It was Archimedes who inaugurated the classical method of computing π by the use of regular polygons inscribed in and circumscribed about a circle. He is responsible for the correct formulas for the area and volume of a sphere. He calculated a number of interesting curvilinear areas, such as that of a parabolic segment and of a sector of the now so called Archimedean spiral. In a number of his works he laid foundations of mathematical physics.

8.2 Sectors

Two points P and Q on a circle with centre O determine an arc PQ denoted by $\overset{\frown}{PQ}$, an angle $\angle POQ$ and a sector POQ. The arc starts at P and goes counterclockwise to Q along the circle. The sector POQ is the region bounded by the arc $\overset{\frown}{PQ}$ and the radii OP and OQ. As Fig.8.1 shows, the arcs $\overset{\frown}{PQ}$ and $\overset{\frown}{QP}$ are different.

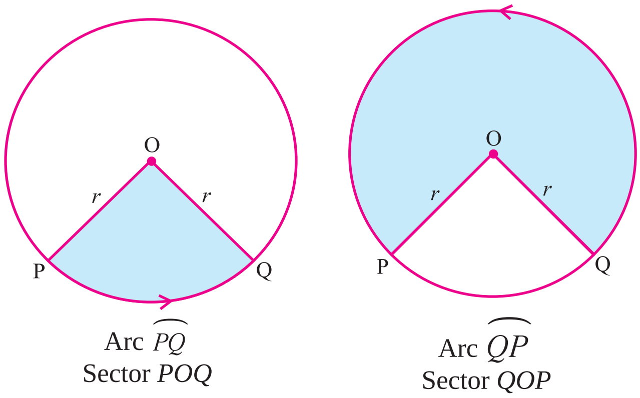


Fig. 8.1

Key Concept	**Sector**
A sector is the part of a circle enclosed by any two radii of the circle and their intercepted arc.	

8.2.1 Central Angle or Sector Angle of a Sector

Key Concept	**Central Angle**
Central Angle is the angle subtended by the arc of the sector at the centre of the circle in which the sector forms a part.	

In fig.8.2, the angle subtended by the arc $\overset{\frown}{PQ}$ at the centre is θ. So the central angle of the sector POQ is θ.

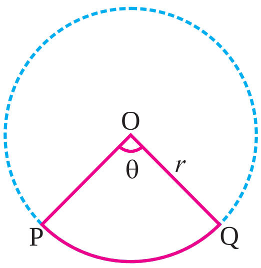


Fig. 8.2

For example,

1. A semi- circle is a sector whose central angle is 180°.
2. A quadrant of a circle is a sector whose central angle is 90°.

8.2.2 Length of Arc (Arc Length) of a Sector

In fig.8.3, arc length of a sector POQ is the length of the portion on the circumference of the circle intercepted between the bounding radii (OP and OQ) and is denoted by l.

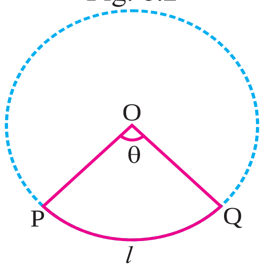


Fig. 8.3

For example,

1. Length of arc of a circle is its circumference. i.e., $l=2\pi r$ units, where r is the radius.

2. Length of arc of a semicircle is $l = 2\pi r \times \frac{180}{360} = \pi r$ units, where r is the radius and the central angle is 180°.

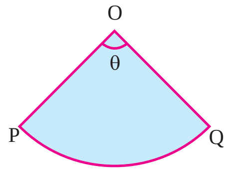
3. Length of arc of a quadrant of a circle is $l = 2\pi r \times \frac{90}{360} = \frac{\pi r}{2}$ units, where r is the radius and the central angle is 90°.

Key Concept	**Length of Arc**
If θ is the central angle and r is the radius of a sector, then its arc length is given by $l = \frac{\theta}{360} \times 2\pi r$ units	

8.2.3 Area of a Sector

Area of a sector is the region bounded by the bounding radii and the arc of the sector.

For Example,



1. Area of a circle is πr^2 square units.

2. Area of a semicircle is $\frac{\pi r^2}{2}$ square units.

3. Area of a quadrant of a circle is $\frac{\pi r^2}{4}$ square units.

Key Concept	**Area of a Sector**
If θ is the central angle and r is the radius of a sector, then the area of the sector is $\frac{\theta}{360} \times \pi r^2$ square units.	

Let us find the relationship between area of a sector, its arc length l and radius r.

$$\text{Area} = \frac{\theta}{360} \times \pi r^2$$

$$= \frac{\theta}{360} \times \frac{2\pi r}{2} \times r$$

$$= \frac{1}{2} \times \left(\frac{\theta}{360} \times 2\pi r\right) \times r$$

$$= \frac{1}{2} \times lr$$

Area of sector = $\frac{lr}{2}$ square units.

8.2.4 Perimeter of a Sector

The perimeter of a sector is the sum of the lengths of all its boundaries. Thus, perimeter of a sector is $l + 2r$ units.

Key Concept	**Perimeter of a Sector**
If l is the arc length and r is the radius of a sector, then its perimeter P is given by the formula $P = l + 2r$ units.	

For example,

1. Perimeter of a semicircle is $(\pi + 2)r$ units.

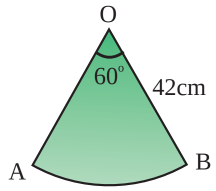
2. Perimeter of a quadrant of a circle is $\left(\frac{\pi}{2} + 2\right)r$ units.

Note

1. Length of an arc and area of a sector are proportional to the central angle.
2. As π is an irrational number, we use its approximate value $\frac{22}{7}$ or 3.14 in our calculations.

Example 8.1

The radius of a sector is 42 cm and its sector angle is 60°. Find its arc length, area and perimeter.



Solution Given that radius $r = 42$ cm and $\theta = 60^\circ$. Therefore,

$$\text{length of arc } l = \frac{\theta}{360} \times 2\pi r$$

$$= \frac{60}{360} \times 2 \times \frac{22}{7} \times 42 = 44 \text{ cm},$$

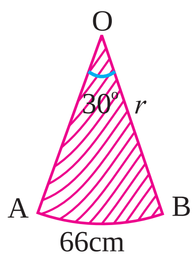
$$\text{Area of the sector} = \frac{lr}{2} = \frac{44 \times 42}{2} = 924 \text{ cm}^2,$$

$$\text{Perimeter} = l + 2r$$

$$= 44 + 2(42) = 128 \text{ cm}.$$

Example 8.2

The arc length of a sector is 66 cm and the central angle is 30°. Find its radius.



Solution Given that $\theta = 30^\circ$ and $l = 66$ cm. So,

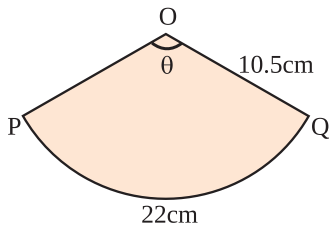
$$\frac{\theta}{360} \times 2\pi r = l$$

$$\text{i. e., } \frac{30}{360} \times 2 \times \frac{22}{7} \times r = 66$$

$$\therefore r = 66 \times \frac{360}{30} \times \frac{1}{2} \times \frac{7}{22} = 126 \text{ cm}$$

Example 8.3

The length of arc of a sector is 22 cm and its radius is 10.5 cm. Find its central angle.



Solution Given that $r = 10.5$ cm and $l = 22$ cm.

$$\frac{\theta}{360} \times 2\pi r = l$$

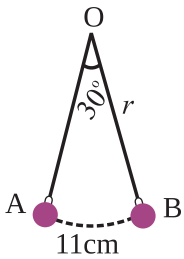
$$\text{i. e., } \frac{\theta}{360} \times 2 \times \frac{22}{7} \times 10.5 = 22$$

$$\therefore \theta = 22 \times 360 \times \frac{1}{2} \times \frac{7}{22} \times \frac{1}{10.5} = 120^\circ$$

Example 8.4

A pendulum swings through an angle of 30° and describes an arc length of 11 cm. Find the length of the pendulum.

Solution If the pendulum swings once, then it forms a sector and the radius of the sector is the length of the pendulum. So,



$$\theta = 30^\circ, l = 11 \text{ cm}$$

Using the formula $\frac{\theta}{360} \times 2\pi r = l$, we have

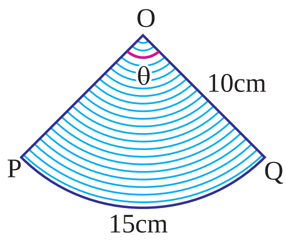
$$\frac{30}{360} \times 2 \times \frac{22}{7} \times r = 11$$

$$\therefore r = 11 \times \frac{360}{30} \times \frac{1}{2} \times \frac{7}{22} = 21 \text{ cm}$$

Example 8.5

The radius and length of arc of a sector are 10 cm and 15 cm respectively.Find its perimeter.

Solution Given that $r = 10$ cm, $l = 15$ cm



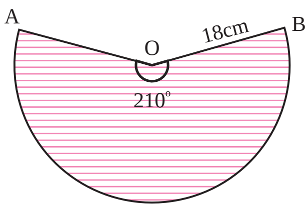
Perimeter $= l + 2r = 15 + 2(10)$

$= 15 + 20 = 35$ cm

Example 8.6

Find the perimeter of a sector whose radius and central angle are 18 cm and 210° respectively.

Solution Given that $r = 18$ cm, $\theta = 210^\circ$. Hence,



$$l = \frac{\theta}{360} \times 2\pi r$$

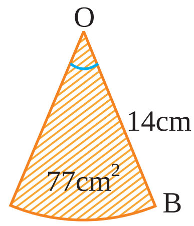
$$= \frac{210}{360} \times 2 \times \frac{22}{7} \times 18 = 66 \text{ cm}$$

$\therefore$ Perimeter $= l + 2r = 66 + 2(18) = 66 + 36 = 102$ cm

Example 8.7

The area of a sector of a circle of radius 14 cm is 77 cm^2. Find its central angle.

Solution Given that $r = 14$ cm, area $= 77$ cm^2



$$\frac{\theta}{360} \times \pi r^2 = \text{Area of the sector}$$

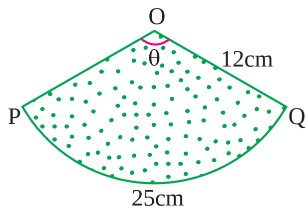
$$\frac{\theta}{360} \times \frac{22}{7} \times 14 \times 14 = 77$$

$$\therefore \theta = \frac{77 \times 360 \times 7}{22 \times 14 \times 14} = 45^\circ$$

Example 8.8

Calculate the area of a sector whose radius and arc length are 6 cm and 20 cm respectively.

Solution Given that $r = 6$ cm, $l = 20$ cm



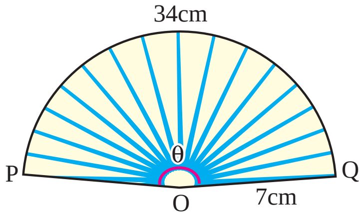
$$\text{Area} = \frac{lr}{2} \text{ square units}$$

$$= \frac{20 \times 6}{2} = 60 \text{ cm}^2$$

Example 8.9

If the perimeter and radius of a sector are 38 cm and 9 cm respectively, find its area.

Solution Given, $r = 9$ cm, perimeter $= 38$ cm



$$\text{Perimeter} = l + 2r = 38$$

i.e., $l + 18 = 38$

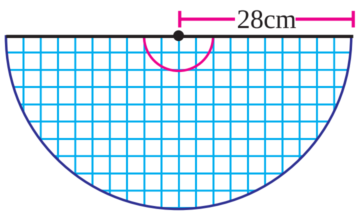
$$l = 38 - 18 = 20 \text{ cm}$$

$$\therefore \text{Area} = \frac{lr}{2} = \frac{20 \times 9}{2} = 90 \text{ cm}^2$$

Example 8.10

Find the perimeter and area of a semicircle of radius 28 cm.

Solution



$$\text{Perimeter} = (\pi + 2)\, r = (\frac{22}{7} + 2)\, 28 = 144 \text{ cm}$$

$$\text{Area} = \frac{\pi r^2}{2} = \frac{22}{7} \times \frac{28 \times 28}{2} = 1232 \text{ cm}^2$$

Example 8.11

Find the radius, central angle and perimeter of a sector whose arc length and area are 27.5 cm and 618.75 cm^2 respectively.

Solution Given that l = 27.5 cm and Area = 618.75 cm^2. So,

$$\text{Area} = \frac{lr}{2} = 618.75 \text{ cm}^2$$

$$\text{i.e. } \frac{27.5 \times r}{2} = 618.75$$

$$\therefore r = 45 \text{ cm}$$

Hence, perimeter is $l + 2r = 27.5 + 2(45) = 117.5$cm

Now, arc length is given by $\frac{\theta}{360} \times 2\pi r = l$

i.e. $\frac{\theta}{360} \times 2 \times \frac{22}{7} \times 45 = 27.5$

$\therefore \theta = 35°$

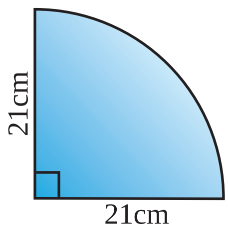
Example 8.12

Calculate the area and perimeter of a quadrant of a circle of radius 21 cm.

Solution Given that r = 21 cm, θ = 90°

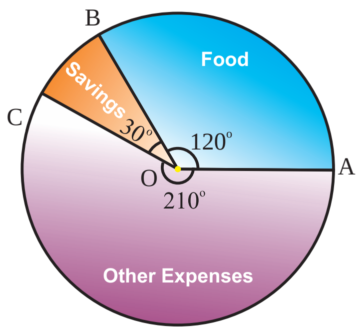
$$\text{Perimeter} = \left(\frac{\pi}{2} + 2\right)r = \left(\frac{22}{7 \times 2} + 2\right) \times 21 = 75 \text{ cm}$$

$$\text{Area} = \frac{\pi r^2}{4} = \frac{22}{7 \times 4} \times 21 \times 21 = 346.5 \text{ cm}^2$$



Example 8.13

Monthly expenditure of a person whose monthly salary is ₹ 9,000 is shown in the adjoining figure. Find the amount he has (i) spent for food (ii) in his savings



Solution Let ₹ 9,000 be represented by the area of the circle, i. e., $\pi r^2 = 9000$

(i) Area of sector $AOB = \frac{\theta}{360} \times \pi r^2$

$$= \frac{210}{360} \times 9000 = 5{,}250$$

Amount spent for food is ₹ 5,250.

(ii) Area of sector $BOC = \frac{\theta}{360} \times \pi r^2$

$$= \frac{30}{360} \times 9{,}000 = 750$$

Amount saved in savings is ₹ 750.

Example 8.14

Three equal circles of radius 3 cm touch one another. Find the area enclosed by them.

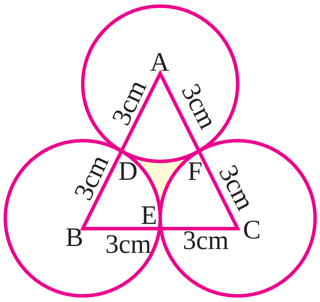
Solution Since the radius of the circles are equal and the circles touch one another, in the figure, *ABC* is an equilateral triangle and the area of the sectors *DAF*, *DBE* and *ECF* are equal. Hence,

Area enclosed = area of the equilateral triangle *ABC* − 3 times area of the sector

$$= \frac{\sqrt{3}}{4}a^2 - 3\times\frac{\theta}{360}\times\pi r^2$$

$$= \frac{\sqrt{3}}{4}\times 6\times 6 - 3\times\frac{60}{360}\times\frac{22}{7}\times 3\times 3$$

$$= 9\sqrt{3} - \frac{99}{7} = 15.59 - 14.14 = 1.45 \text{ cm}^2$$



$\therefore$ Area = 1.45 cm^2

Example 8.15

Find the area of the shaded portion in the following figure $[\pi = 3.14]$

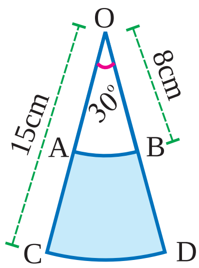
Solution Let *R* and *r* denote the radius of sector *COD* and sector A*OB* respectively.

Area of the shaded portion = Area of sector C*OD* − Area of sector A*OB*

$$= \frac{\theta}{360}\times\pi R^2 - \frac{\theta}{360}\times\pi r^2$$

$$= \frac{30}{360}\times 3.14\times 15\times 15 - \frac{30}{360}\times 3.14\times 8\times 8$$

$$= 58.875 - 16.747 = 42.128 \text{ cm}^2$$



Exercise: 8.1

1. Find the arc length, area and perimeter of the sector with

   (i) radius 21 cm and central angle 60° (ii) radius 4.9 cm and central angle 30°

   (iii) radius 14 cm and sector angle 45° (iv) radius 15 cm and sector angle 63°

   (v) radius 21 dm and sector angle 240°

2. (i) Find the angle subtended by an arc 88 cm long at the centre of a circle of radius 42 cm.

   (ii) The arc length of a sector of a circle of radius 14 cm is 22 cm. Find its central angle.

   (iii) Find the radius of a sector of a circle having a central angle 70° and an arc length of 44 cm.

3. Find the area and perimeter of the sector with

(i) radius 10 cm and arc length 33 cm.

(ii) radius 55 cm and arc length 80 cm.

(iii) radius 12 cm and arc length 15.25 cm.

(iv) radius 20 cm and arc length 25 cm.

4. (i) Find the arc length of the sector of radius 14 cm and area 70 cm^2

(ii) Find the radius of the sector of area 225 cm^2 and having an arc length of 15 cm

(iii) Find the radius of the sector whose central angle is 140° and area 44 cm^2.

5. (i) The perimeter of a sector of a circle is 58 cm. Find the area if its diameter is 9 cm.

(ii) Find the area of a sector whose radius and perimeter are 20 cm and 110 cm respectively.

6. Find the central angle of a sector of a circle having

(i) area 352 cm^2 and radius 12 cm

(ii) area 462 cm^2 and radius 21 cm

7. (i) Calculate the perimeter and area of the semicircle whose radius is 14 cm.

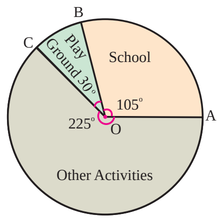
(ii) Calculate the perimeter and area of a quadrant circle of radius 7 cm.

8. (i) Calculate the arc length of a sector whose perimeter and radius are 35 cm and 8 cm respectively.

(ii) Find the radius of a sector whose perimeter and arc length are 24 cm and 7 cm respectively.

9. Time spent by a student in a day is shown in the figure. Find how much time is spent in

(i) school (ii) play ground (iii) other activities



10. Three coins each 2 cm in diameter are placed touching one another. Find the area enclosed by them.

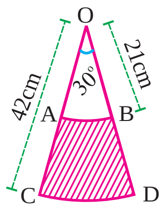
11. Four horses are tethered with ropes measuring 7 m each to the four corners of a rectangular grass land 21 m $\times$ 24 m in dimension. Find

(i) the maximum area that can be grazed by the horses and

(ii) the area that remains ungrazed.

12. Find the area of card board wasted if a sector of maximum possible size is cut out from a square card board of size 24 cm.

13. Find the area of the shaded portion in the adjoining figure



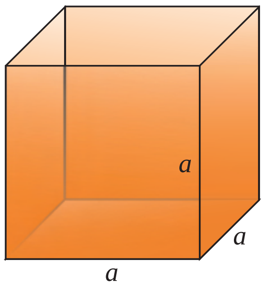
14. Find the radius, central angle and perimeter of a sector whose length of arc and area are 4.4 m and 9.24 m^2 respectively.

8.3 Cubes

You have learnt that a cube is a solid having six square faces. In this section you will learn about surface area and volume of a cube. Example: Die.

8.3.1 Surface Area of a Cube

The sum of the areas of all the six equal faces is called the *Total Surface Area* (T.S.A) of the cube.



In the adjoining figure, let the side of the cube measure a units. Then the area of each face of the cube is a^2 square units. Hence, the total surface area is $6a^2$ square units.

In a cube, if we don't consider the top and bottom faces, the remaining area is called the *Lateral Surface Area* (L.S.A). Hence, the lateral surface area of the cube is $4a^2$ square units.

Key Concept	**Surface Area of Cube**
Let the side of a cube be a units. Then: (i) The Total Surface Area (T.S.A) = $6a^2$ square units. (ii) The Lateral Surface Area (L.S.A) = $4a^2$ square units.	

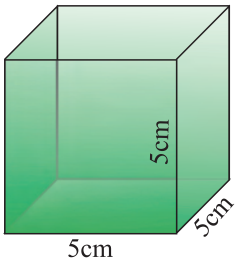
8.3.2 Volume of a Cube

Key Concept	**Volume of Cube**
If the side of a cube is a units, then its volume V is given by the formula $V = a^3$ cubic units	

Note Volume can also be defined as the number of unit cubes required to fill the entire cube.

Example 8.16

Find the L.S.A, T.S.A and volume of a cube of side 5 cm.



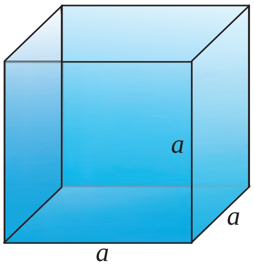
Solution L.S.A $= 4a^2 = 4(5^2) = 100$ sq. cm

T.S.A $= 6a^2 = 6\,(5^2) = 150$ sq. cm

Volume $= a^3 = 5^3 = 125$ cm^3

Example: 8.17

Find the length of the side of a cube whose total surface area is 216 square cm.



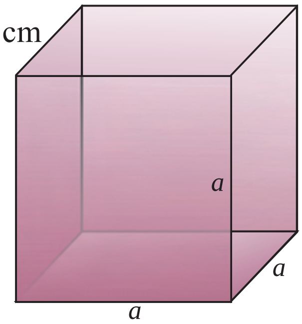
Solution Let a be the side of the cube. Given that T.S.A = 216 sq. cm

i. e., $6a^2 = 216 \implies a^2 = \frac{216}{6} = 36$

$\therefore \quad a = \sqrt{36} = 6$ cm

Example 8.18

A cube has a total surface area of 384 sq. cm. Find its volume.



Solution Let a be the side of the cube. Given that T.S.A = 384 sq. cm

$6a^2 = 384 \implies a^2 = \frac{384}{6} = 64$

$\therefore \; a = \sqrt{64} = 8$ cm

Hence, Volume $= a^3 = 8^3 = 512$ cm^3

Example 8.19

A cubical tank can hold 27,000 litres of water. Find the dimension of its side.

Solution Let a be the side of the cubical tank. Volume of the tank is 27,000 litres. So,

$V = a^3 = \frac{27,000}{1,000} m^3 = 27\, m^3 \qquad \therefore\; a = \sqrt[3]{27} = 3\, m$

Exercise 8.2

1. Find the Lateral Surface Area (LSA), Total Surface Area (TSA) and volume of the cubes having their sides as

(i) 5.6 cm (ii) 6 dm (iii) 2.5 m (iv) 24 cm (v) 31 cm

2. (i) If the Lateral Surface Area of a cube is 900 cm^2, find the length of its side.

(ii) If the Total Surface Area of a cube is 1014 cm^2, find the length of its side.

(iii) The volume of the cube is 125 dm^3. Find its side.

3. A container is in the shape of a cube of side 20 cm. How much sugar can it hold?
4. A cubical tank can hold 64,000 litres of water. Find the length of the side of the tank.
5. Three metallic cubes of side 3 cm, 4 cm and 5 cm respectively are melted and are recast into a single cube. Find the total surface area of the new cube.
6. How many cubes of side 3 cm are required to build a cube of side 15 cm?
7. Find the area of card board required to make an open cubical box of side 40 cm. Also find the volume of the box.
8. What is the total cost of oil in a cubical container of side 2 m if it is measured and sold using a cubical vessel of height 10 cm and the cost is ₹ 50 per measure.
9. A container of side 3.5m is to be painted both inside and outside. Find the area to be painted and the total cost of painting it at the rate of ₹ 75 per square meter.

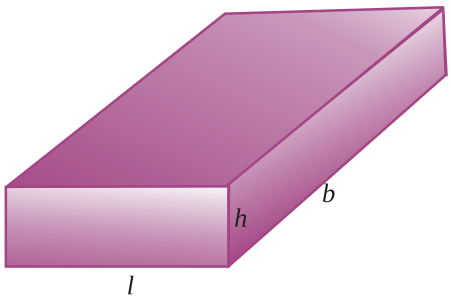
8.4 Cuboids

A cuboid is a three dimensional solid having six rectangular faces.

Example: Bricks, Books etc.,

8.4.1 Surface Area of a Cuboid

Let l, b and h be the length, breadth and height of a cuboid respectively. To find the total surface area, we split the faces into three pairs.



(i) The total area of the front and back faces is $lh + lh = 2lh$ square units.

(ii) The total area of the side faces is $bh + bh = 2bh$ square units.

(iii) The total area of the top and bottom faces is $lb + lb = 2lb$ square units.

The Lateral Surface Area (L.S.A) = $2(l + b)h$ square units.

The Total Surface Area (T.S.A) = $2(lb + bh + lh)$ square units.

Key Concept — **Surface Area of a Cuboid**

Let l, b and h be the length, breadth and height of a cuboid respectively. Then

(i) The Lateral Surface Area (L.S.A) = $2(l + b)h$ square units

(ii) The Total Surface Area (T.S.A) = $2(lb + bh + lh)$ sq. units

Note L.S.A. is also equal to the product of the perimeter of the base and the height.

8.4.2 Volume of a Cuboid

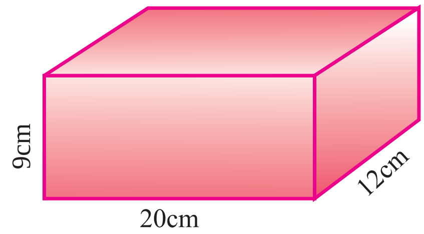
Key Concept	Volume of a Cuboid
If the length, breadth and height of a cuboid are l, b and h respectively, then the volume V of the cuboid is given by the formula $V = l \times b \times h$ cu. units	

Example: 8.20

Find the total surface area of a cuboid whose length, breadth and height are 20 cm, 12 cm and 9 cm respectively.

Solution

Given that l = 20 cm, b = 12 cm, h = 9 cm



$\therefore$ T.S.A $= 2\,(lb + bh + lh)$

$= 2[(20\times 12) + (12\times 9) + (20\times 9)]$

$= 2(240 + 108 + 180)$

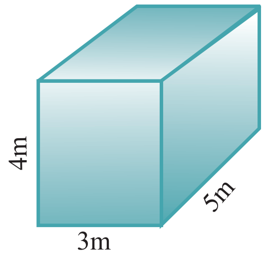
$= 2\times 528$

$= 1056 \text{ cm}^2$

Example: 8.21

Find the L.S.A of a cuboid whose dimensions are given by $3\text{m}\times 5\text{m}\times 4\text{m}$.

Solution Given that l = 3 m, b = 5 m, h = 4 m



L.S.A $= 2(l + \text{b})\text{h}$

$= 2 \times (3+5) \times 4$

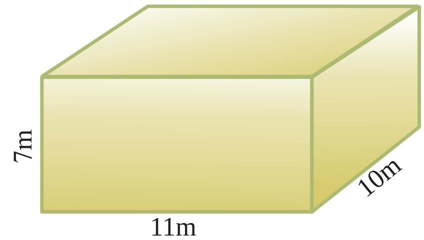
$= 2\times 8\times 4$

$= 64$ sq. m

Example: 8.22

Find the volume of a cuboid whose dimensions are given by 11 cm, 10 cm and 7 cm.

Solution Given that l = 11 m, b = 10 m, h = 7 m



volume $= lbh$

$= 11\times 10\times 7$

$= 770$ cu.cm

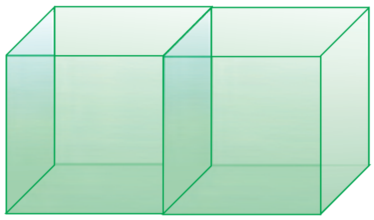
Example: 8.23

Two cubes each of volume 216 cm^3 are joined to form a cuboid as shown in the figure. Find the T.S.A of the resulting cuboid.

Solution Let the side of each cube be a. Then $a^3 = 216$

$\therefore$ a = $\sqrt[3]{216} = 6$ cm

Now the two cubes of side 6 cm are joined to form a cuboid. So,



$\therefore\ l = 6 + 6 = 12$ cm, $b = 6$ cm, $h = 6$ cm

$\therefore$ T.S.A $= 2\,(lb + bh + lh)$

$= 2\,[(12 \times 6) + (6 \times 6) + (12 \times 6)]$

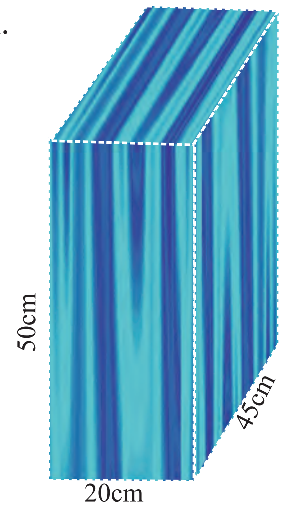
$= 2\,[72 + 36 + 72]$

$= 2 \times 180 = 360\ cm^2$

Example 8.24

Johny wants to stitch a cover for his C.P.U whose length, breadth and height are 20 cm, 45 cm and 50 cm respectively. Find the amount he has to pay if it costs ₹ 50/sq. m

Solution The cover is in the shape of a one face open cuboidal box.



$l = 20$ cm $= 0.2$ m, b = 45 cm =0.45 m, h = 50 cm = 0.5 m

$\therefore$ Area of cloth required = L.S.A + area of the top

$= 2\,(l + b)\,h + lb$

$= 2\,(0.2 + 0.45)\,0.5 + (0.2 \times 0.45)$

$= 2 \times 0.65 \times 0.5 + 0.09$

$= 0.65 + 0.09$

$= 0.74$ sq.m

Given that cost of 1 sq. m of cloth is ₹ 50

$\therefore$ cost of 0.74 sq.m of cloth is $50 \times 0.74 =$ ₹ 37.

Example: 8.25

Find the cost for filling a pit of dimensions 5m $\times$ 2m $\times$ 1m with soil if the rate of filling is ₹ 270 per cu. m

Solution The pit is in the shape of a cuboid having l = 5m, b = 2m and h = 1m.

$\therefore$volume of the pit = volume of the cuboid

= lbh

= $5 \times 2 \times 1$

= 10 cu.m

Given that cost for filling 1 cu. m is ₹ 270

$\therefore$ cost for filling 10 cu. m is

270×10 = ₹ 2700

Exercise 8.3

1. Find the L.S.A, T.S.A and volume of the cuboids having the length, breadth and height respectively as

   (i) 5 cm, 2 cm , 11cm (ii) 15 dm, 10 dm, 8 dm

   (iii) 2 m, 3 m, 7 m (iv) 20 m, 12 m, 8 m

2. Find the height of the cuboid whose length, breadth and volume are 35 cm, 15 cm and 14175 cm^3 respectively.

3. Two cubes each of volume 64 cm^3 are joined to form a cuboid. Find the L.S.A and T.S.A of the resulting solid.

4. Raju planned to stitch a cover for his two speaker boxes whose length, breadth and height are 35 cm, 30 cm and 55 cm respectively. Find the cost of the cloth he has to buy if it costs ₹ 75 per sq.m.

5. Mohan wanted to paint the walls and ceiling of a hall. The dimensions of the hall is $20\text{m} \times 15\text{m} \times 6\text{m}$. Find the area of surface to be painted and the cost of painting it at ₹ 78 per sq. m.

6. How many hollow blocks of size $30\text{cm} \times 15\text{cm} \times 20\text{cm}$ are needed to construct a wall 60m in length, 0.3m in breadth and 2m in height.

7. Find the cost of renovating the walls and the floor of a hall that measures $10\text{m}\times45\text{m}\times6\text{m}$ if the cost is ₹ 48 per square meter.

Points to Remember

- A sector is the part of a circle enclosed by any two radii of the circle and their intercepted arc.
- Central Angle is the angle subtended by the arc of the sector at the centre of the circle in which the sector forms a part.
- If θ is the central angle and r is the radius of a sector, then its arc length is given by $l=\frac{\theta}{360}\times 2\pi r$ units
- If θ is the central angle and r is the radius of a sector, then the area of the sector is $\frac{\theta}{360}\times \pi r^2$ square units.
- If l is the arc length and r is the radius of a sector, then its perimeter P is given by the formula $P = l + 2r$ units.
- Let the side of a cube be a units. Then:
  (i) The Total Surface Area (T.S.A) = $6a^2$ square units.
  (ii) The Lateral Surface Area (L.S.A) = $4a^2$ square units.
- If the side of a cube is a units, then its volume V is given by the formula, $V = a^3$ cubic units
- Let l, b and h be the length, breadth and height of a cuboid respectively. Then:
  (i) The Lateral Surface Area (L.S.A) = $2(l + b)h$ squre units
  (ii) The Total Surface Area (T.S.A) = $2(lb + bh + lh)$ sq. units
- If the length, breadth and height of a cuboid are l, b and h respectively, then the volume V of the cuboid is given by the formula $V = l\times b\times h$ cu. units

9 PRACTICAL GEOMETRY

Main Targets

- To construct the Circumcentre
- To construct the Orthocentre
- To construct the Incentre
- To construct the Centroid

9.1 Introduction

The fundamental principles of geometry deal with the properties of points, lines, and other figures. Practical Geometry is the method of applying the rules of geometry to construct geometric figures. “Construction” in Geometry means to draw shapes, angles or lines accurately. The geometric constructions have been discussed in detail in Euclid’s book ‘Elements’. Hence these constructions are also known as Euclidean constructions. These constructions use only a compass and a straightedge (i.e. ruler). The compass establishes equidistance and the straightedge establishes collinearity. All geometric constructions are based on those two concepts.

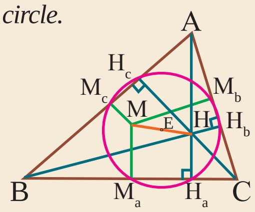
It is possible to construct rational and irrational numbers using a straightedge and a compass as seen in chapter II. In 1913 the Indian Mathematical Genius, Ramanujan gave a geometrical construction for $355/113 = \pi$. Today with all our accumulated skill in exact measurements, it is a noteworthy feature when lines driven through a mountain meet and make a tunnel. How much more wonderful is it that lines, starting at the corner of a perfect square, should be raised at a certain angle and successfully brought to a point, hundreds of feet aloft! For this, and more, is what is meant by the building of a pyramid:

LEONHARD EULER
1707 - 1783

The Swiss mathematician Leonhard Euler lived during the 18th century. Euler wrote more scientific papers than any mathematician before or after him. For Euler, mathematics was a tool to decipher God’s design of our world. With every new discovery, he felt a step nearer to understanding nature and by this understanding God.

Euler even found a new theorem in Euclidean geometry, a field which had been looked at as completed. Here’s a short explanation of this theorem:

The three altitudes of a triangle meet in point H, and the three perpendicular bisectors in point M. Point E in the middle of the line between H and M is the center of a circle on which are all the inter-sections of the altitudes and the perpendicular bisectors with the triangle. This circle known as 9 points circle.



In class VIII we have learnt the construction of triangles with the given measurements. In this chapter we learn to construct centroid, ortho-centre, in-centre and circum-centre of a triangle.

9.2 Special line segments within Triangles

First let us learn to identify and to construct

(i) Perpendicular bisector to a given line segment

(ii) Perpendicular from an external point to a given line

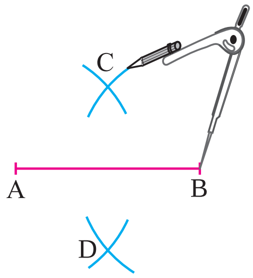
(iii) Bisector of a given angle and

(iv) Line joining a given external point and the midpoint of a given line segment.

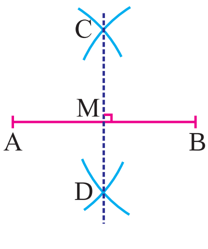
9.2.1 Construction of the Perpendicular Bisector of a given line segment

Step1 : Draw the given line segment AB.

Step 2 : With the two end points A and B of the line segment as centres and more than half the length of the line segment as radius draw arcs to intersect on both sides of the line segment at C and D



Step 3 : Join C and D to get the perpendicular bisector of the given line segment AB.

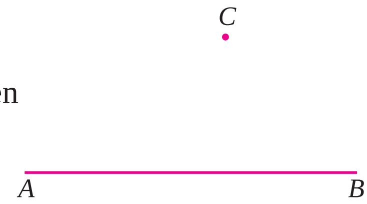


Key Concept — **Perpendicular Bisector**

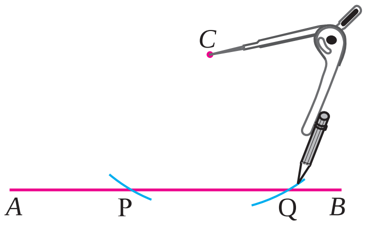
The line drawn perpendicular through the midpoint of a given line segment is called the perpendicular bisector of the line segment.

9.2.2 Construction of Perpendicular from an External Point to a given line

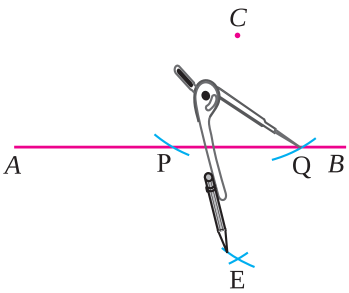
Step 1 : Draw the given line *AB* and mark the given external point *C*.



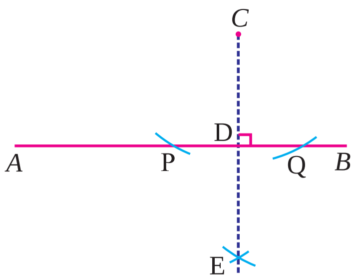
Step 2 : With *C* as centre and any convenient radius draw arcs to cut the given line at two points *P* and *Q*.



Step 3 : With *P* and *Q* as centres and more than half the distance between these points as radius draw two arcs to intersect each other at *E*.

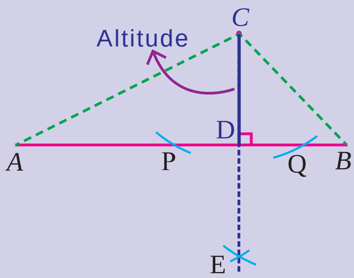


Step 4 : Join *C* and *E* to get the required perpendicular line.

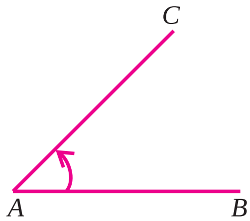


Key Concept **Altitude**

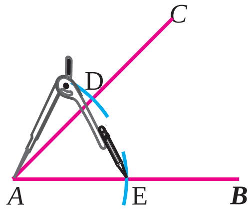
In a triangle, an altitude is the line segment drawn from a vertex of the triangle perpendicular to its opposite side.



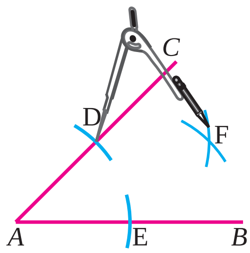
9.2.3 Construction of Angle Bisector



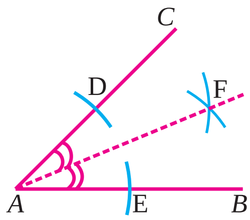
Step 1 : Draw the given angle $\angle CAB$ with the given measurement.



Step 2 : With A as centre and a convenient radius draw arcs to cut the two arms of the angle at D and E.



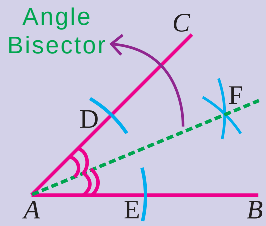
Step 3 : With D and E as centres and a suitable radius draw arcs to intersect each other at F.



Step 4 : Join A and F to get the angle bisector AF of $\angle CAB$.

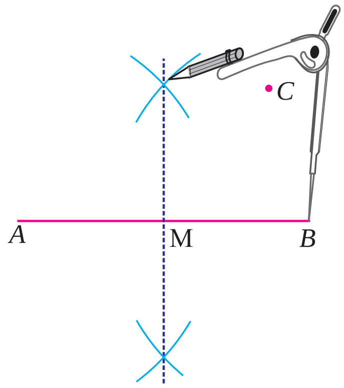
Key Concept — **Angle Bisector**

The line which divides a given angle into two equal angles is called the angle bisector of the given angle.

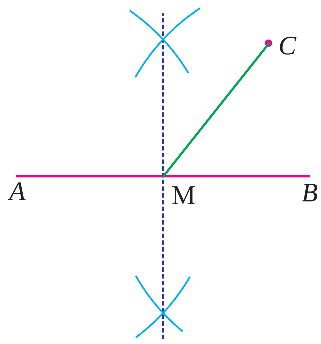


9.2.4 Construction of Line Joining a External Point and the Midpoint of a Line Segment

Step 1 : Draw a line segment *AB* with the given measurement and mark the given point *C* (external point).



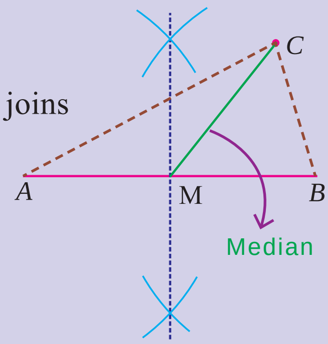
Step 2 : Draw the perpendicular bisector of AB and mark the point of intersection *M* which is the mid point of line segment.



Step 3 : Join *C* and *M* to get the required line.

Key Concept **Median**

In a triangle, a median is the line segment that joins a vertex of the triangle and the midpoint of its opposite side.



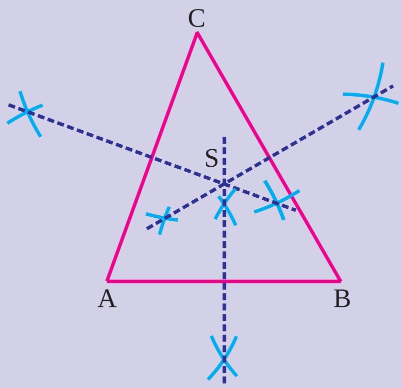
9.3 The Points of Concurrency of a Triangle

As we have already learnt how to draw the Perpendicular Bisector, Altitude, Angle Bisector and Median, now let us learn to locate the Circumcentre, Othocentre, Incentre and Centroid of a given triangle.

9.3.1 Construction of the Circumcentre of a Triangle

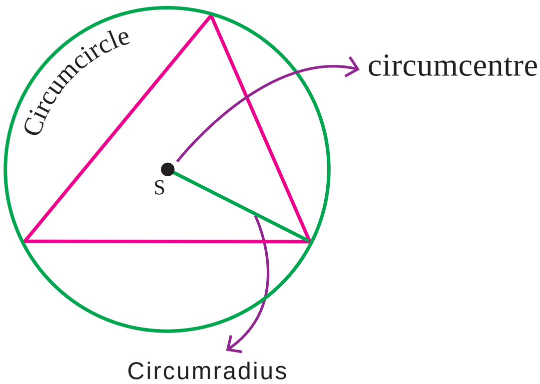
Key Concept — **Circumcentre**

The point of concurrency of the perpendicular bisectors of the sides of a triangle is called the circumcentre and is usually denoted by *S*.



Circumcircle

The circle drawn with *S* (circumcentre) as centre and passing through all the three vertices of the triangle is called the circumcircle.

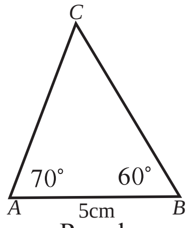


Circumradius

The radius of a circumcircle is called circumradius of the triangle. In other words, the distance between the circumcentre *S* and any vertex of the triangle is the circumradius.

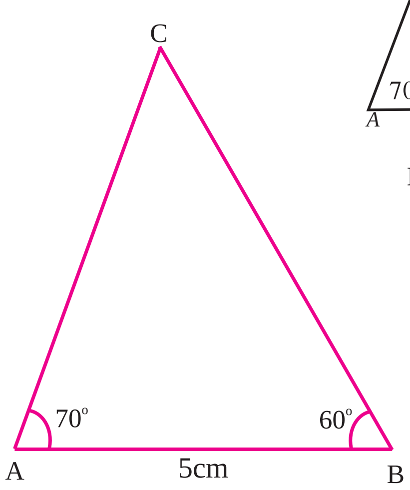
Example 9.1

Construct the circumcentre of the ΔABC with AB = 5cm, $\angle A = 70^\circ$ and $\angle B = 60^\circ$. Also draw the circumcircle and find the circumradius of the ΔABC.



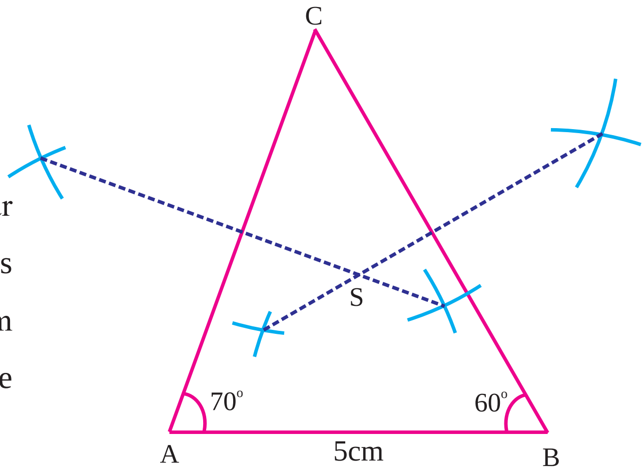
Rough Diagram

Solution

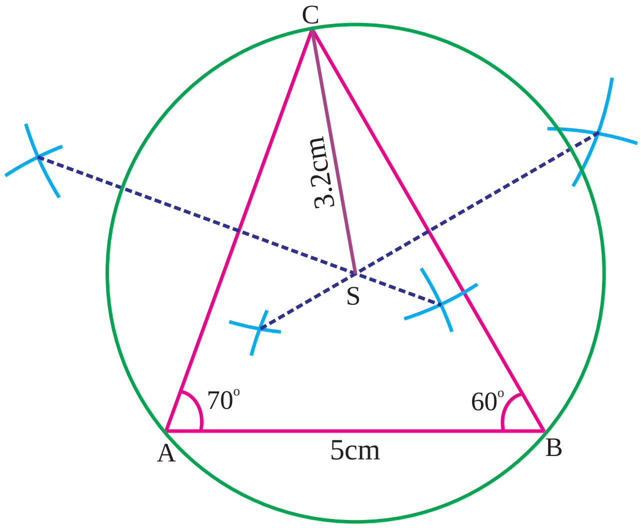


Step 1 : Draw the ΔABC with the given measurements.

\



Step2 : Construct the perpendicular bisectors of any two sides (AC and BC) and let them meet at S which is the circumcentre.



Step 3 : With S as centre and $SA = SB = SC$ as radius draw the circumcircle to pass through A, B and C.

Circumradius = 3. 2cm

Remark
1. The circumcentre of an acute angled triangle lies inside the triangle.
2. The circumcentre of a right triangle is at the midpoint of its hypotenuse.
3. The circumcentre of an obtuse angled triangle lies outside the triangle.

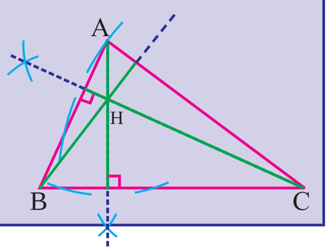
Exercise 9.1

1. Construct ΔPQR with $PQ = 5$cm, $\angle P = 100^\circ$ and $PR = 5$cm and draw its circumcircle.
2. Draw the circumcircle for
   (i) an equilateral triangle of side 6cm
   (ii) an isosceles right triangle having 5cm as the length of the equal sides.
3. Draw ΔABC, where $AB = 7$cm, $BC = 8$cm and $\angle B = 60^\circ$ and locate its circumcentre.
4. Construct the right triangle whose sides are 4.5cm, 6cm and 7.5cm. Also locate its circumcentre.

9.3.2 Construction of the Orthocentre of a Triangle

Key Concept — **Orthocentre**

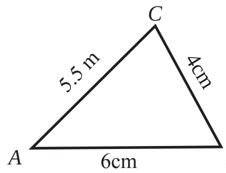
The point of concurrency of the altitudes of a triangle is called the orthocentre of the triangle and is usually denoted by *H*.



Example 9.2

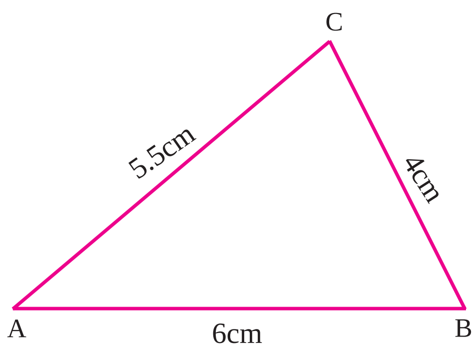
Construct ΔABC whose sides are $AB = 6$cm, $BC = 4$cm and $AC = 5.5$cm and locate its orthocentre.

Solution

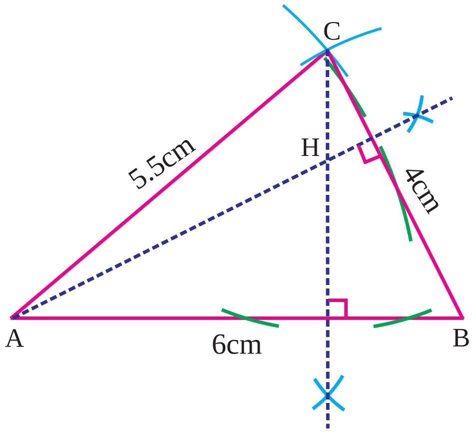


Rough Diagram

Step 1 : Draw the ΔABC with the given measurements.



Step 2 : Construct altitudes from any two vertices (A and C) to their opposite sides (BC and AB respectively)



The point of intersection of the altitudes H is the orthocentre of the given ΔABC

Remark

1. Three altitudes can be drawn in a triangle.
2. The orthocentre of an acute angled triangle lies inside the triangle.
3. The orthocentre of a right triangle is the vertex of the right angle.
4. The orthocentre of an obtuse angled triangle lies outside the triangle.

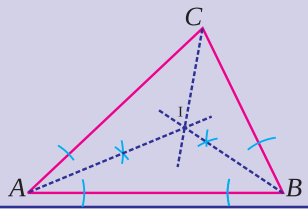
Exercise 9.2

1. Draw ΔABC with sides $AB = 8$cm, $BC = 7$cm and $AC = 5$cm and construct its orthocentre.
2. Construct the orthocentre of ΔLMN, where $LM = 7$cm, $\angle M = 130^\circ$ and $MN = 6$cm
3. Construct an equilateral triangle of sides 6cm and locate its orthocentre.
4. Draw and locate the orthocentre of a right triangle PQR right angle at Q, with PQ = 4.5cm, RS = 6cm
5. Construct an isosceles triangle ABC with AB = BC and $\angle B = 80^\circ$ of sides 6cm and locate its orthocentre.

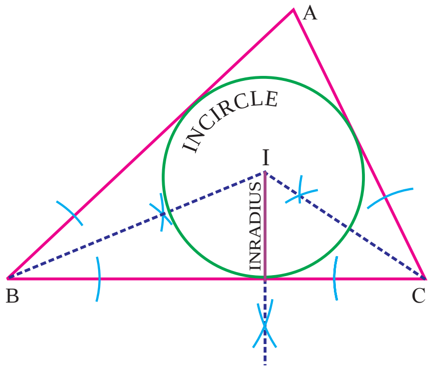
9.3.3 Construction of the Incentre of a Triangle

Key Concept **Incentre**

The point of concurrency of the internal angle bisectors of a triangle is called the incentre of the triangle and is denoted by I.



Incircle The circle drawn with the incentre (I) as centre and touching all the three sides of a triangle is the incircle of the given triangle.



Inradius The radius of the incircle is called the inradius of the triangle.

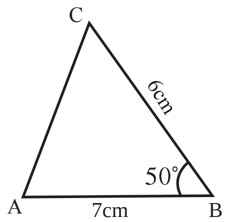
(or)

It is the shortest distance of any side of the triangle from the incentre I.

Example 9.3

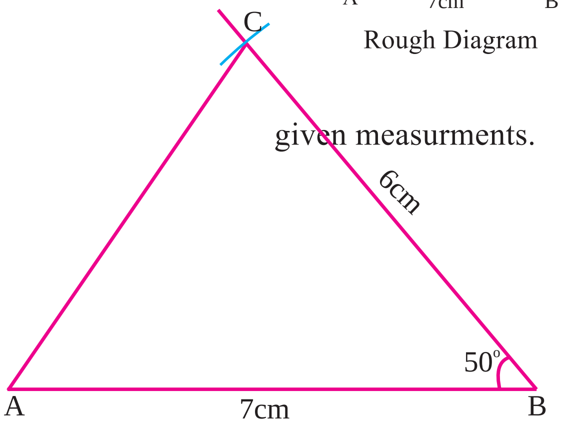
Construct the incentre of ΔABC with $AB = 7\text{cm}$, $\angle B = 50^o$ and $BC = 6\text{cm}$. Also draw the incircle and measure its inradius.

Solution

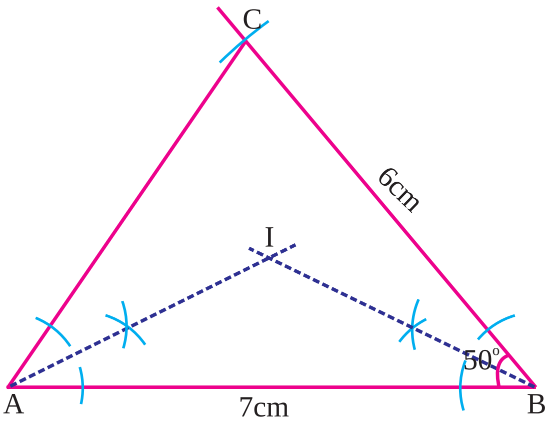


Rough Diagram

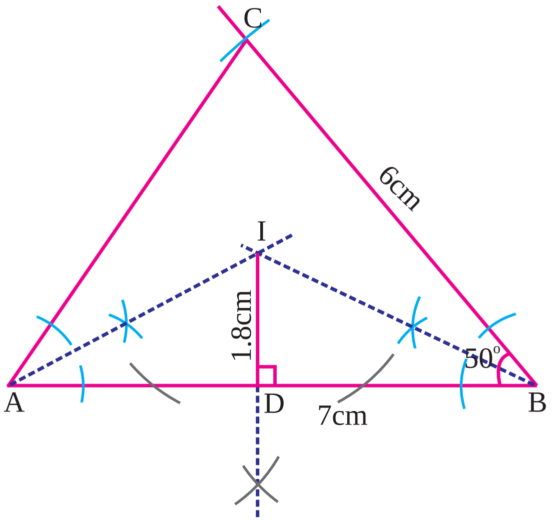
Step 1 : Draw the ΔABC with the given measurments.



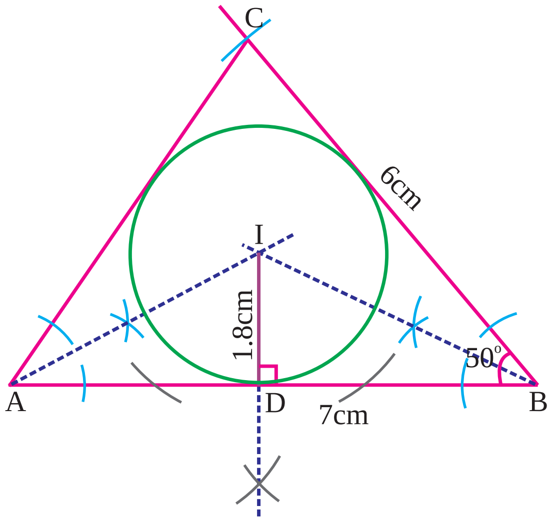
Step 2 : Construct the angle bisectors of any two angles (A and B) and let them meet at I. Then I is the incentre of ΔABC



Step 3 : With *I* as an external point drop a perpendicular to any one of the sides to meet at *D*.



Step 4 : With *I* as centre and *ID* as radius draw the circle. This circle touches all the sides of the triangle.



Inradius = 1.8 cm

Remark The incentre of any triangle always lies inside the triangle.

Exercise 9.3

1. Draw the incircle of ΔABC, where $AB = 9$ cm, $BC = 7$cm, and $AC = 6$cm.

2. Draw the incircle of ΔABC in which $AB = 6$ cm, $AC = 7$ cm and $\angle A = 40^\circ$. Also find its inradius.

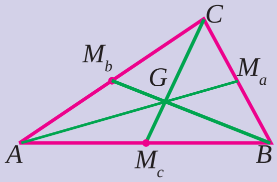
3. Construct an equilateral triangle of side 6cm and draw its incircle.

4. Construct ΔABC in which $AB = 6$ cm, $AC = 5$ cm and $\angle A = 110^\circ$. Locate its incentre and draw the incircle.

9.3.4 Construction of the Centroid of a Triangle.

Key Concept — **Centroid**

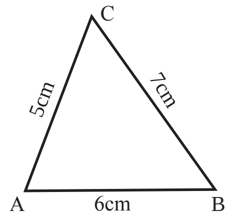
The point of concurrency of the medians of a triangle is called the Centroid of the triangle and is usually denoted by G.



Example 9.4

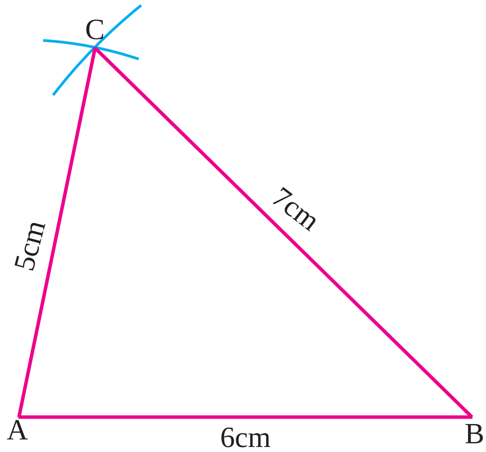
Construct the centroid of ΔABC whose sides are AB = 6cm, BC = 7cm, and AC = 5cm.

Solution

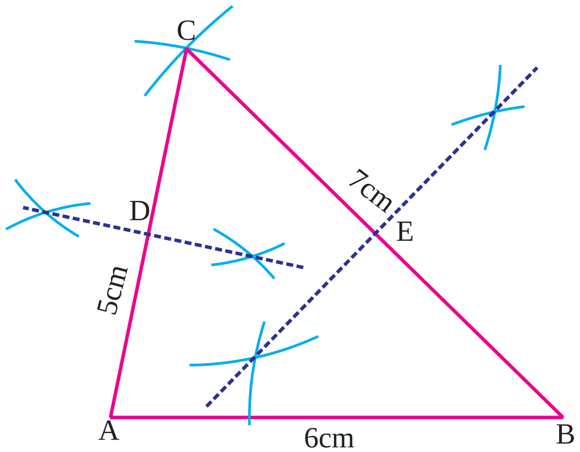


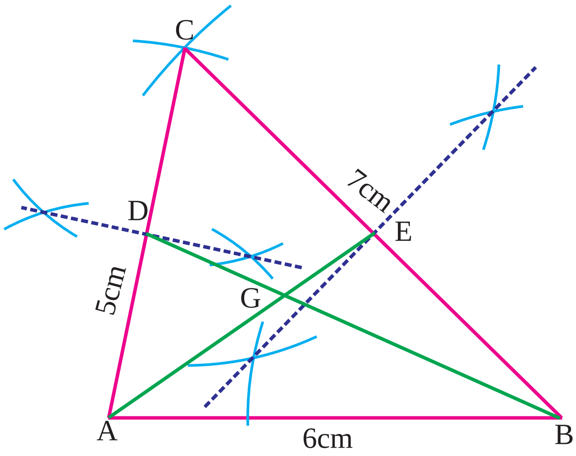
Rough Diagram

Step 1 : Draw ΔABC using the given measurements.



Step 2 : Construct the perpendicular bisectors of any two sides (AC and BC) to find the mid points D and E of AC and BC respectively .





Step 3 : Draw the medians AE and BD and let them meet at G.

The point G is the centroid of the given ΔABC

Remark (i) Three medians can be drawn in a triangle.

(ii) The centroid divides a median in the ratio 2:1 from the vertex.

(iii) The centroid of any triangle always lie inside the triangle.

Exercise 9.4

1. Construct the ΔABC such that AB = 6cm, BC = 5cm and AC = 4cm and locate its centroid.

2. Draw and locate the centroid of of triangle LMN with LM = 5.5cm, $\angle M = 100^{\circ}$ MN = 6.5cm.

3. Draw a equilateral triangle of side 7.5cm and locate the centriod.

4. Draw the right triangle whose sides are 3cm, 4cm and 5cm and construct its centroid.

5. Draw the ΔPQR, where $PQ = 6cm$, $\angle P = 110^{o}$ and QR = 8cm and construct its centroid.

10 GRAPHS

A mathematical theory can be regarded as perfect only if you are prepared to present its contents to the first man in the street – D.HILBERT

Main Targets

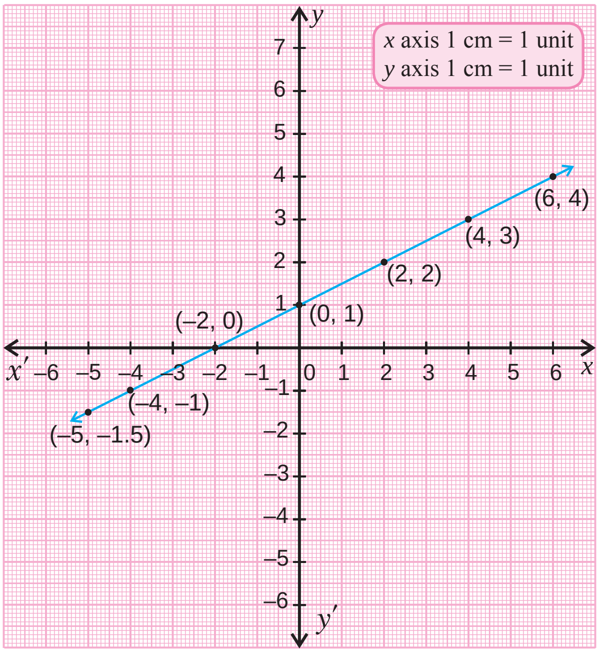
- To understand the concept of graph.
- To graph linear equations.
- To solve linear equations in two variables.

10.1 Introduction

This chapter covers the basic ideas of graphs. Almost everyday you see diagrams and graphs in newspapers, magazines, books etc. The purpose of the graph is to show numerical facts in visual form so that they can be understood quickly, easily and clearly. In this chapter you will learn how to use graphs to give a visual representation of the relationship between two variables and find solutions of equations in two variables.

10.2 Linear Graph

An equation such as $x - 2y = -2$ is an example of a linear equation in the two variables x and y. A solution of this equation is an ordered pair of numbers (x_0, y_0) so that x_0 and y_0 satisfy the equation $x - 2y = -2$, in the sense that $x_0 - 2y_0 = -2$. We observe that in this situation, it is easy to find all the solutions with a prescribed first number x_0 or a prescribed second number y_0. Relative to a pair of coordinate axes in the plane, the collection of all the points (x_0, y_0) in the coordinate plane so that each pair (x_0, y_0) is a solution of the equation $x - 2y = -2$ is called the *graph* of $x - 2y = -2$ in the plane. Using the above method of getting all the solutions of the equation $x - 2y = -2$, we can plot as many points of the graph as we please to get a good idea of the graph. For example, the adjacent picture contains the following points (given by the dots) on the graph, going from left to right: $(-5,-1.5)$, $(-4,-1)$, $(-2, 0)$, $(0, 1)$, $(2, 2)$, $(4, 3)$, $(6, 4)$.



These points strongly suggest that the graph of $x - 2y = -2$ is a straight line.

Thus, a first degree equation in two variables always represents a straight line. Hence we can take general equation of a straight line as $ax + by + c = 0$, with at least one of a or b not equal to zero. For the sake of simplicity to draw lines in graphs we consider $y = mx + c$ as another simple form of the equation of straight line. For each value of x, the equation $y = mx + c$ gives a value of y and we can obtain an ordered pair (x, y).

Note The general equation of a straight line is $ax + by + c = 0$

(i) If $c = 0$, then the equation becomes $ax + by = 0$ and the line passes through the origin

(ii) If $a = 0$, then the equation becomes $by + c = 0$ and the line is parallel to x-axis

(iii) If $b = 0$, then the equation becomes $ax + c = 0$ and the line is parallel to y-axis

10.2.1 Procedure to Draw a Linear Graph

When graphing an equation, we usually begin by creating a table of x and y values. We do this by choosing three x values and computing the corresponding y values. Although two points are sufficient to sketch the graph of a line, we usually choose three points so that we can check our work.

Step 1: Using the given equation construct a table of with x and y values.
Step 2: Draw x-axis and y-axis on the graph paper.
Step 3: Select a suitable scale on the coordinate axes.
Step 4: Plot the points
Step 5: Join the points and extend it to get the line.

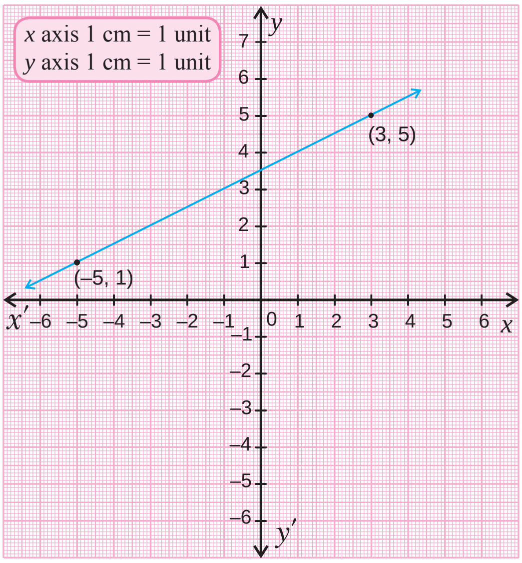
10.2.2 Draw Straight Lines

Example 10.1

Draw the graph of the line joining the points (3, 5) and (−5, 1)

Solution

1. Draw the x-axis and y-axis on a graph sheet with 1 cm = 1 unit on both axes.
2. We plot the two given points (3, 5), (−5, 1) on the graph sheet.
3. We join the points by a line segment and extend it on either direction.
4. We get the required linear graph.

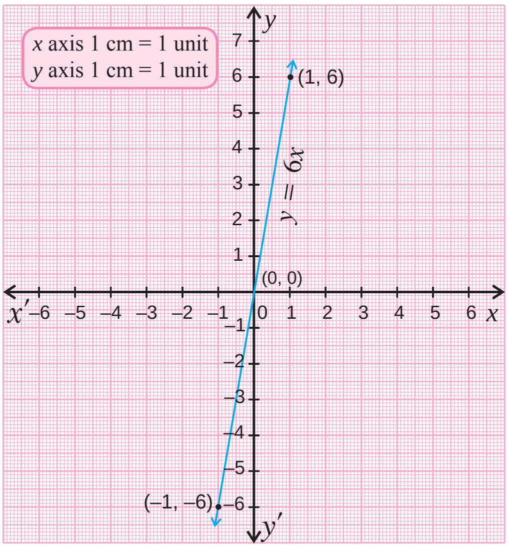


Example 10.2

Draw the graph of $y = 6x$

Solution Substituting the values $x = -1, 0, 1$ in the equation of the line, we find the values of y as follows

$y = 6x$			
x	-1	0	1
y	-6	0	6



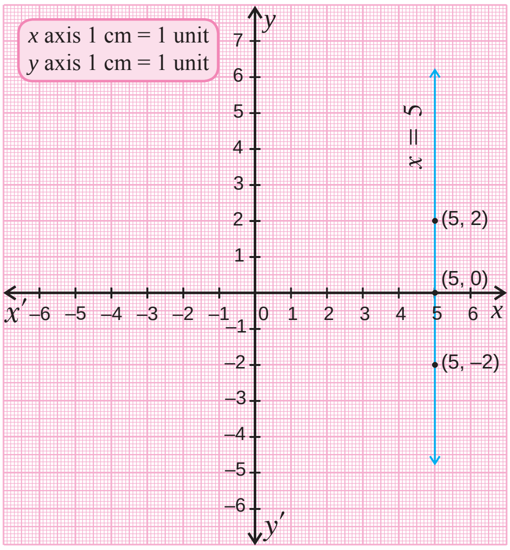
In a graph, plot the points $(-1, -6)$, $(0, 0)$ and $(1, 6)$ and draw a line passing through the plotted points. This is the required linear graph.

Example 10.3

Draw the graph of $x = 5$

Solution The line $x = 5$ is parallel to y-axis. On this line $x = 5$, a constant. So, any point on this line is of the form $(5, y)$. Taking the values $y = -2, 0, 2$ we get the points $(5, -2)$, $(5, 0)$ and $(5, 2)$.

$x = 5$			
x	5	5	5
y	-2	0	2



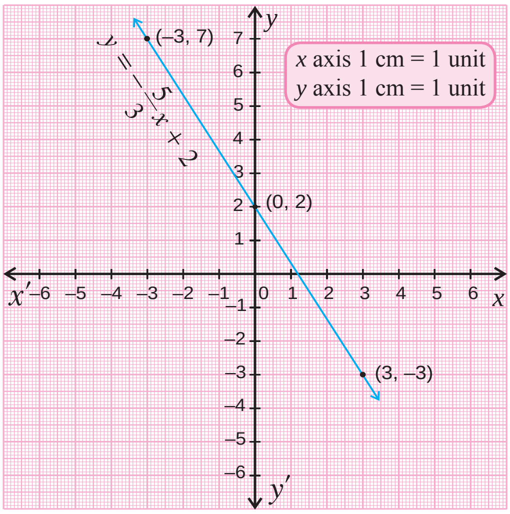
In a graph sheet, plot these points and draw a line passing through the points. Thus we get the required linear graph.

Example 10.4

Draw the graph of the line $y = -\frac{5}{3}x + 2$.

Solution Substituting $x = -3, 0, 3$ in the equation of the line, we find the values of y as follows

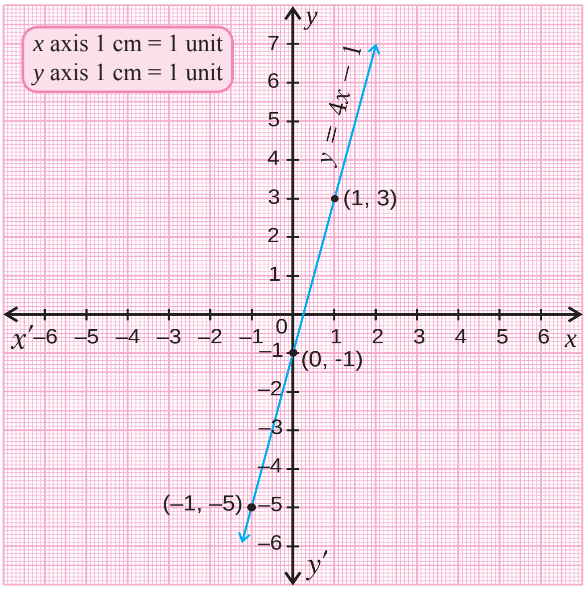
$y = -\frac{5}{3}x + 2$			
x	-3	0	3
$-\frac{5}{3}x$	5	0	-5
$y = -\frac{5}{3}x + 2$	7	2	-3



Plot the points $(-3, 7)$, $(0, 2)$ and $(3, -3)$ and draw a line passing through the plotted points. This is the required graph of the equation $y = \frac{-5}{3}x + 2$.

Example 10.5

Draw the graph of $y = 4x - 1$.



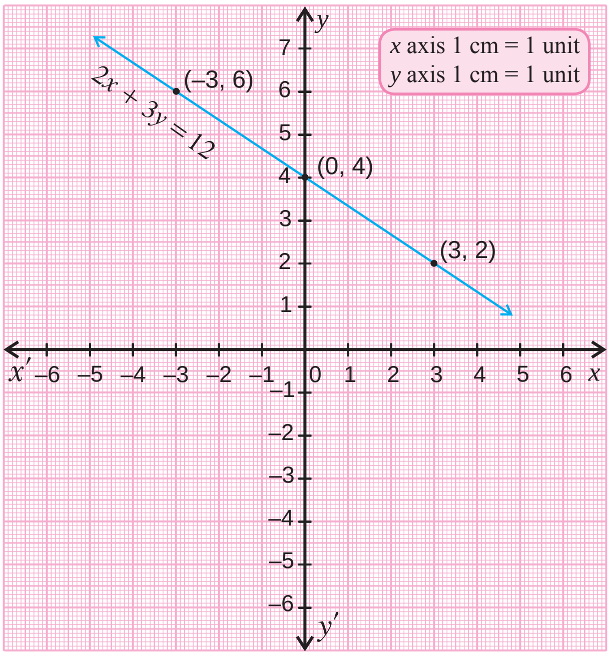
Solution Substituting the values $x = -1, 0, 1$ in the given equation of line, we find the values of y as follows

$y = 4x - 1$			
x	-1	0	1
$4x$	-4	0	4
$y = 4x - 1$	-5	-1	3

Plot the points $(-1, -5)$, $(0, -1)$ and $(1, 3)$ in a graph sheet and draw a line passing through the plotted points. We now get the required linear graph.

Example 10.6

Draw the graph of $2x + 3y = 12$



Solution First, we rewrite the equation $2x + 3y = 12$ in the form of $y = mx + c$.

$2x + 3y = 12$ implies $y = -\frac{2}{3}x + 4$

Substituting $x = -3, 0, 3$ in the above equation, we find the values of y as follows

$y = -\frac{2}{3}x + 4$			
x	-3	0	3
$-\frac{2}{3}x$	2	0	-2
$y = -\frac{2}{3}x + 4$	6	4	2

Plot the points $(-3, 6)$, $(0, 4)$ and $(3, 2)$ and draw a line passing through these points. Now we get the required graph.

Exercise 10.1

1. Draw the linear graph joining the points

(i) $(2, 3)$ and $(-6, -5)$ (ii) $(-2, -4)$ and $(-1, 6)$

(iii) $(5, -7)$ and $(-1, 5)$ (iv) $(-3, 9)$ and $(5, -6)$ (v) $(4, -5)$ and $(6, 10)$

2. Draw the graph of the following

(i) $y = 5$ (ii) $y = -6$ (iii) $x = 3$

(iv) $x = -5$ (v) $2x + 7 = 0$ (vi) $6 + 3y = 0$

3. Draw the graph of the following

(i) $y = 4x$ (ii) $3x + y = 0$ (iii) $x = -2y$

(iv) $y - 3x = 0$ (v) $9y - 3x = 0$

4. Draw the linear graph of the following equations

(i) $y = 3x + 1$ (ii) $4y = 8x + 2$ (iii) $y - 4x + 3 = 0$

(iv) $x = 3y + 3$ (v) $x + 2y - 6 = 0$ (vi) $x - 2y + 1 = 0$

(vii) $3x + 2y = 12$

5. Draw the graph of the equation $y = mx + c$, where

(i) $m = 2$ and $c = 3$ (ii) $m = -2$ and $c = -2$ (iii) $m = -4$ and $c = 1$

(iv) $m = 3$ and $c = -4$ (v) $m = \frac{1}{2}$ and $c = 3$ (vi) $m = \frac{-2}{3}$ and $c = 2$

10.3 Application of Graphs

By a system of linear equations in two variables we mean a collection of more than one linear equations in two variables. The solutions of system of linear equations is the set of ordered pairs that satisfy all the equations in that system. In this section you will learn to solve graphically a pair of two linear equations in two variables.

Here three cases arise:

(i) The two graphs coincide, that is, the graphs are one and the same. In this case there are infinitely many solutions.

(ii) The two graphs do not coincide but they are parallel. That is, do not meet at all. So, there is no common point and hence there is no solution.

(iii) The two graphs intersect exactly at one point. In this case the equations have a unique solution.

Example 10.7

Solve graphically the pair of equations $x + 2y = 4;\ 2x + 4y = 8$.

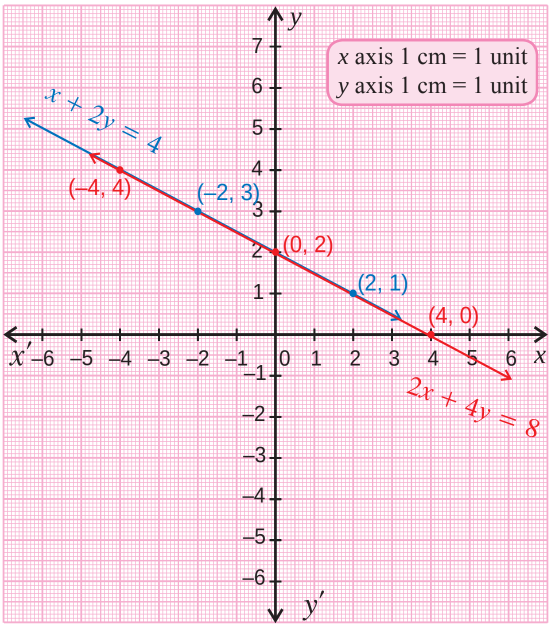
Solution We find three points for each equation, by choosing three valuesof x and computing the corresponding y values. We show our results in tables.

Line 1: $x + 2y = 4$

$$2y = -x + 4 \implies y = -\frac{x}{2} + 2$$

Substituting $x = -2, 0, 2$ in the above equation, we get the corresponding y values as

$y = -\frac{x}{2} + 2$			
x	-2	0	2
$-\frac{x}{2}$	1	0	-1
$y = -\frac{x}{2} + 2$	3	2	1



Line 2: $2x + 4y = 8$

$$4y = -2x + 8 \implies y = -\frac{x}{2} + 2$$

Substituting $x = -2, 0, 2$ in the above equation, we get y values as

$y = -\frac{x}{2} + 2$			
x	-4	0	4
$-\frac{x}{2}$	2	0	-2
$y = -\frac{x}{2} + 2$	4	2	0

We plot these points in a graph paper and draw the lines. Then we find that both the lines coincide. Any point on one line is also a point on the other. That is all points on the line are common points. Therefore each point on the line is a solution.Hence there are infinitely many solution

Example 10.8

Solve graphically $x - 3y = 6;\ x - 3y + 9 = 0$

Solution let us find three points for each equation, by choosing three x values and computing the corresponding y values. We present our results in the tables.

Line 1: $x - 3y = 6$

$3y = x - 6 \implies y = \frac{x}{3} - 2$

Substituting $x = -3, 0, 3$ in the above equation, we get the values of y as follows

$y = \frac{x}{3} - 2$			
x	-3	0	3
$\frac{x}{3}$	-1	0	1
$y = \frac{x}{3} - 2$	-3	-2	-1

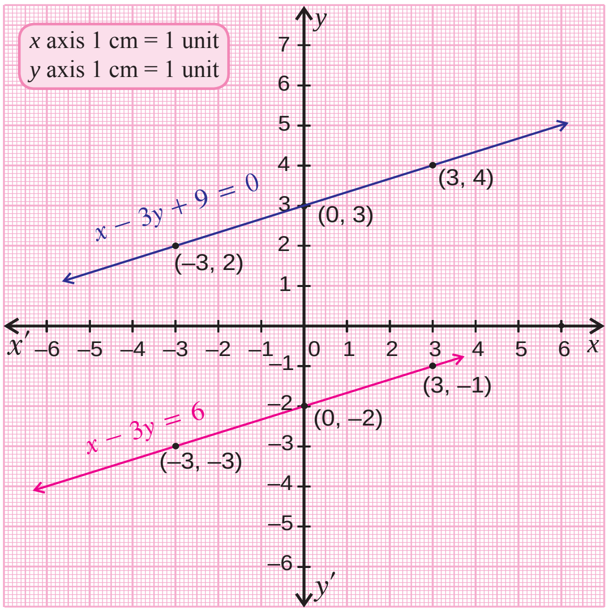
Line 2: $x - 3y + 9 = 0$

$$3y = x + 9$$

$$y = \frac{x}{3} + 3$$

Substituting $x = -3, 0, 3$ in above equation, we get

$y = \frac{x}{3} + 3$			
x	-3	0	3
$\frac{x}{3}$	-1	0	1
$y = \frac{x}{3} + 3$	2	3	4



We plot the points $(-3, -3)$ $(0, -2)$ and $(3, -1)$ in the graph sheet and draw the line through them. Next, we plot the points $(-3, 2)$ $(0, 3)$ and $(3, 4)$ in the same graph sheet and draw the line through them. We find that the two graphs are parallel. That is no point is common to both lines. So, the system of equations has no solution.

Example 10.9

Solve graphically the equations $2x - y = 1; x + 2y = 8$

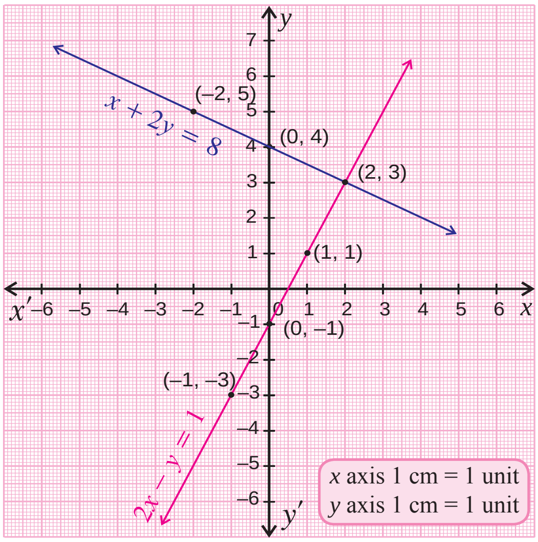
Solution We find three points for each equation, by choosing three values of x and computing the corresponding y values. We'll put our results in tables.

Line 1: $2x - y = 1$

$$y = 2x - 1$$

Substituting $x = -1, 0, 1$ in the above equation, we find

$y = 2x - 1$			
x	-1	0	1
$2x$	-2	0	2
$y = 2x - 1$	-3	-1	1



Line 2: $x + 2y = 8$

$$2y = -x + 8 \Longrightarrow y = -\frac{x}{2} + 4$$

Substituting $x = -2, 0, 2$ in above equation, we get

$y = -\frac{x}{2} + 4$			
x	-2	0	2
$-\frac{x}{2}$	1	0	-1
$y = -\frac{x}{2} + 4$	5	4	3

We plot the points $(-1, -3)$, $(0, -1)$ and $(1, 1)$ in a graph sheet and draw the line through them. Next, we plot the points $(-2, 5)$, $(0, 4)$ and $(2, 3)$ in the same graph sheet and draw the line through them. We find that the two graphs are intersecting at the point (2, 3). Hence, the system of equations has only one solution (unique solution) and the solution is $x=2$, $y=3$

Therefore the solution is (2, 3)

Exercise 10.2

Solve Graphically the following pairs of equations.

1. $3x - y = 0;\ x - 2 = 0$
2. $2x + y = 4;\ 4x + 2y = 8$
3. $2x = y + 1;\ x + 2y - 8 = 0$
4. $x + y = 5;\ x - y = 1$
5. $x - 2y = 6;\ x - 2y = -6$
6. $4x - y - 5 = 0;\ x + y - 5 = 0$
7. $3x + 2y = 4;\ 9x + 6y - 12 = 0$
8. $y = 2x + 1;\ y + 3x - 6 = 0$
9. $y - 2x + 2 = 0;\ y = 4x - 4$
10. $x - y = 0;\ y + 3 = 0$
11. $2x - 4 = 0;\ 4x + y + 4 = 0$
12. $\frac{x}{2} + \frac{y}{4} = 1;\ \frac{x}{2} + \frac{y}{4} = 2$

11 STATISTICS

"Statistical thinking today is as necessary for efficient citizenship as the ability to read and write"

Herbert. G. Wells

Main Targets

- To draw Histogram, Frequency Polygon
- To find Measures of Central Tendency : Mean, Median and Mode

11.1 Introduction

The subject statistics comprises the collection, organization, presentation, analysis and interpretation of data, assists in decision making. In the earlier classes you have studied about the collection of statistical data through primary sources and secondary sources. The data collected through these sources may contain a large number of numerical facts. These numerical facts must be arranged and presented in a tabular form in an orderly way before analysis and interpretation.

In case of some investigations, the classification and tabulation will give a clear picture of the significance of the data arranged so that no further analysis is required. However, these forms of presentation do not always prove to be interesting to the common man. One of the most convincing and appealing ways, in which statistical results may be presented is through diagrams and graphs.

11.2 Graphical Representation of Frequency Distribution

It is often said that "one picture is worth a thousand words." Indeed, statisticians have employed graphical techniques to more vividly describe the data. In particular, histograms and polygons are used to describe quantitative data that have been grouped into frequency, or percentage distributions.

A frequency distribution is organizing of raw data in tabular form, using classes and frequencies. A frequency distribution can be represented graphically by

(i) Histogram (ii) Frequency polygon

(iii) Smoothened frequency curve and (iv) Ogive or Cumulative frequency curve.

In this chapter we see the first two types of graphs, other two will be discussed in higher classes.

11.2.1 Histogram

Out of several methods of graphical representation of a frequency distribution, histogram is the most popular and widely used method. A histogram is a two dimensional graphical representation of continuous frequency distribution. In a histogram, rectangles are drawn such that the areas of the rectangles are proportional to the corresponding frequencies.

To draw a histogram with equal class intervals

1. Mark the intervals on the horizontal axis and the frequencies on the vertical axis.
2. The scales for both the axes need not be the same.
3. Class intervals must be exclusive. If the intervals are in inclusive form, convert them to the exclusive form.
4. Draw rectangles with class intervals as bases and the corresponding frequencies as lengths. The class limits are marked on the horizontal axis and the frequency is marked on the vertical axis. Thus a rectangle is constructed on each class interval.

Remark A histogram is similar to a bar graph. However, a histogram utilizes classes (intervals) and frequencies while a bar graph utilizes categories and frequencies . Histograms are used only for continuous data that is grouped.

11.2.2 Frequency Polygon

A frequency polygon uses the mid-point of a class interval to represent all the data in that interval. It is constructed by taking mid-points of class intervals on the horizontal axis and the frequencies on the vertical axis and joining these points. The two extremes are joined with the base in such a way that they touch the horizontal axis at half the distance of class interval outside the extreme points.

If we have to construct histogram and frequency polygon both, first draw the histogram and then join the mid-points of the tops of all the rectangles and finally the extreme points with the points outside the extreme rectangles.

Remark A histogram is often drawn as a guide, so that a frequency polygon can be drawn over the top.

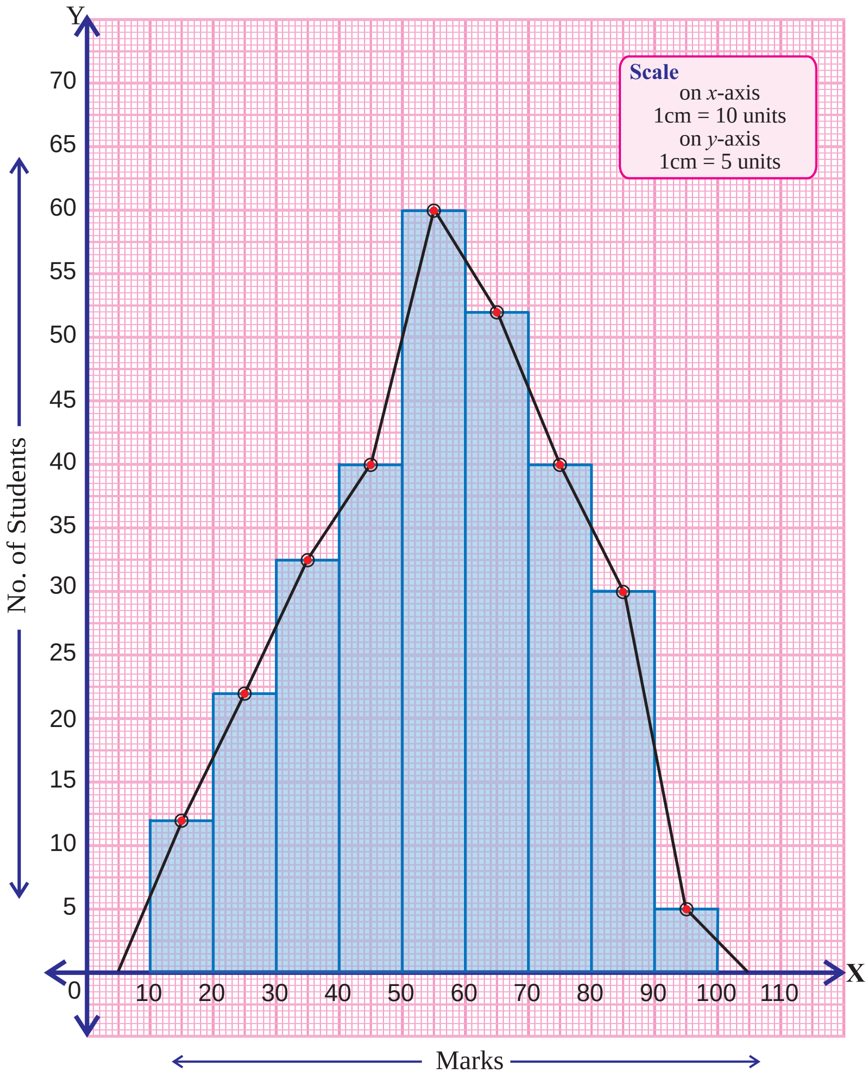
Example 11.1

Draw a histogram and frequency polygon to represent the following data.

Marks	10-20	20-30	30-40	40-50	50-60	60-70	70-80	80-90	90-100
No. of Students	12	22	35	40	60	52	40	30	5

Solution First we draw the histogram and then by joining the midpoints of the tops of the rectangles we draw the frequency polygon.

Histogram and Frequency Polygon



In the above example, the intervals are exclusive. Now, let us consider an example with inclusive intervals.

Example 11.2

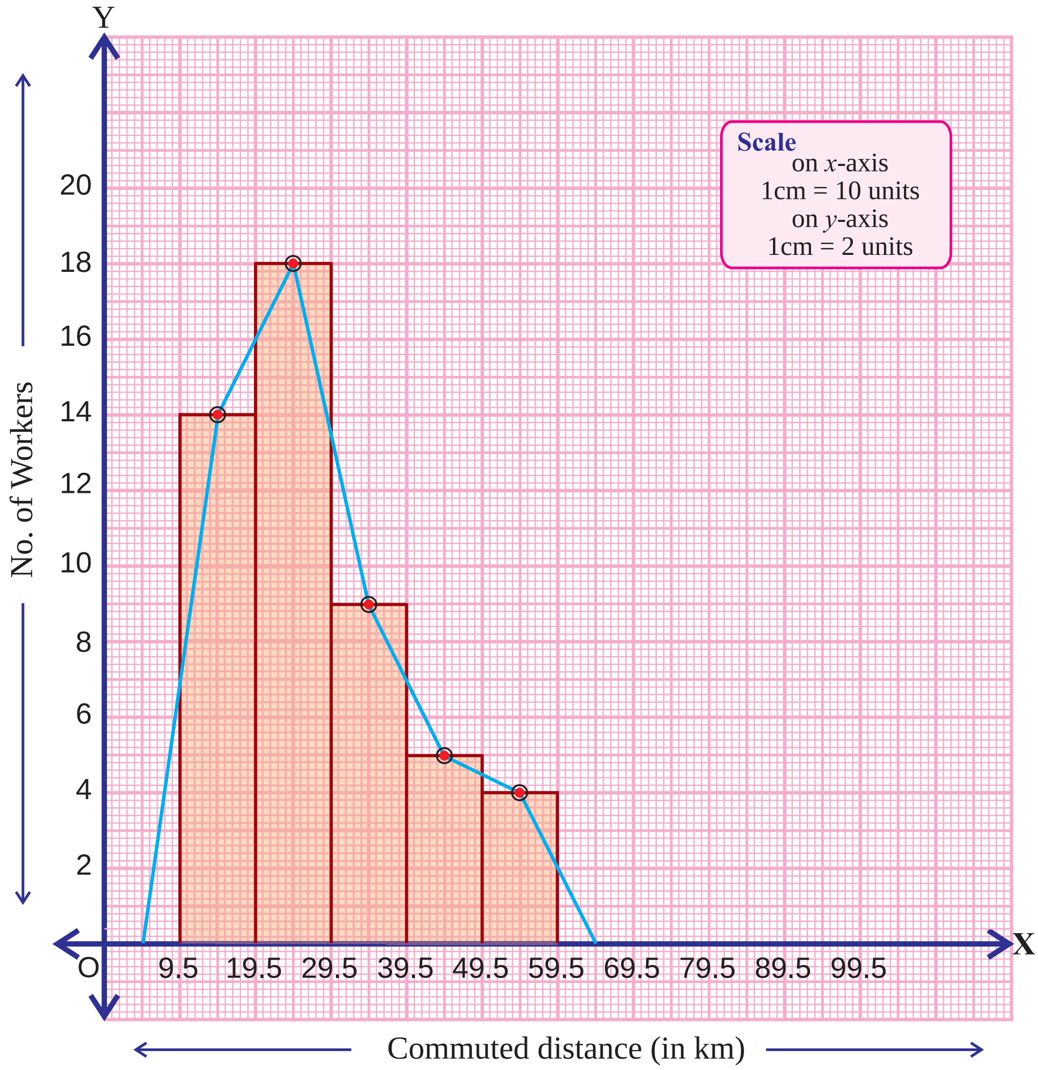
A survey was conducted in a small industrial plant having 50 workers to find the number of km each person commuted to work and the details are given below. Represent it by a histogram and frequency polygon.

Commuted Distance (km)	50-59	40-49	30-39	20-29	10-19
No. of Workers	4	5	9	18	14

Solution In the given table the class intervals are inclusive. So we convert them to the exclusive form and arrange the class intervals in ascending order.

Commuted Distance (km)	9.5-19.5	19.5-29.5	29.5-39.5	39.5-49.5	49.5-59.5
No. of Workers	14	18	9	5	4

Histogram and Frequency Polygon



Note (i) The class intervals are made continuous and then the histogram is constructed.

(ii) If the scale along the horizontal axis does not start at the origin, a zig - zag curve is shown near the origin.

11.2.3 Histogram with Varying Base Width

Consider the following frequency distribution:

Time (seconds)	40-60	60-70	70-80	80-85	85-90	90-120
Frequency	100	60	90	70	60	90

The class interval 40-60 appears to be most popular, as it has the highest frequency. Note that this frequency 100 is spread across a time of 20 seconds. Although the class interval 80-85 has only a frequency of 70, this frequency is spread across a time of only 5 seconds.

So, we need to take into account the width of each class interval, before we draw the histogram, otherwise the histogram would not represent the data set correctly.

We do this by calculating the frequency density and modifying length of the rectangle.

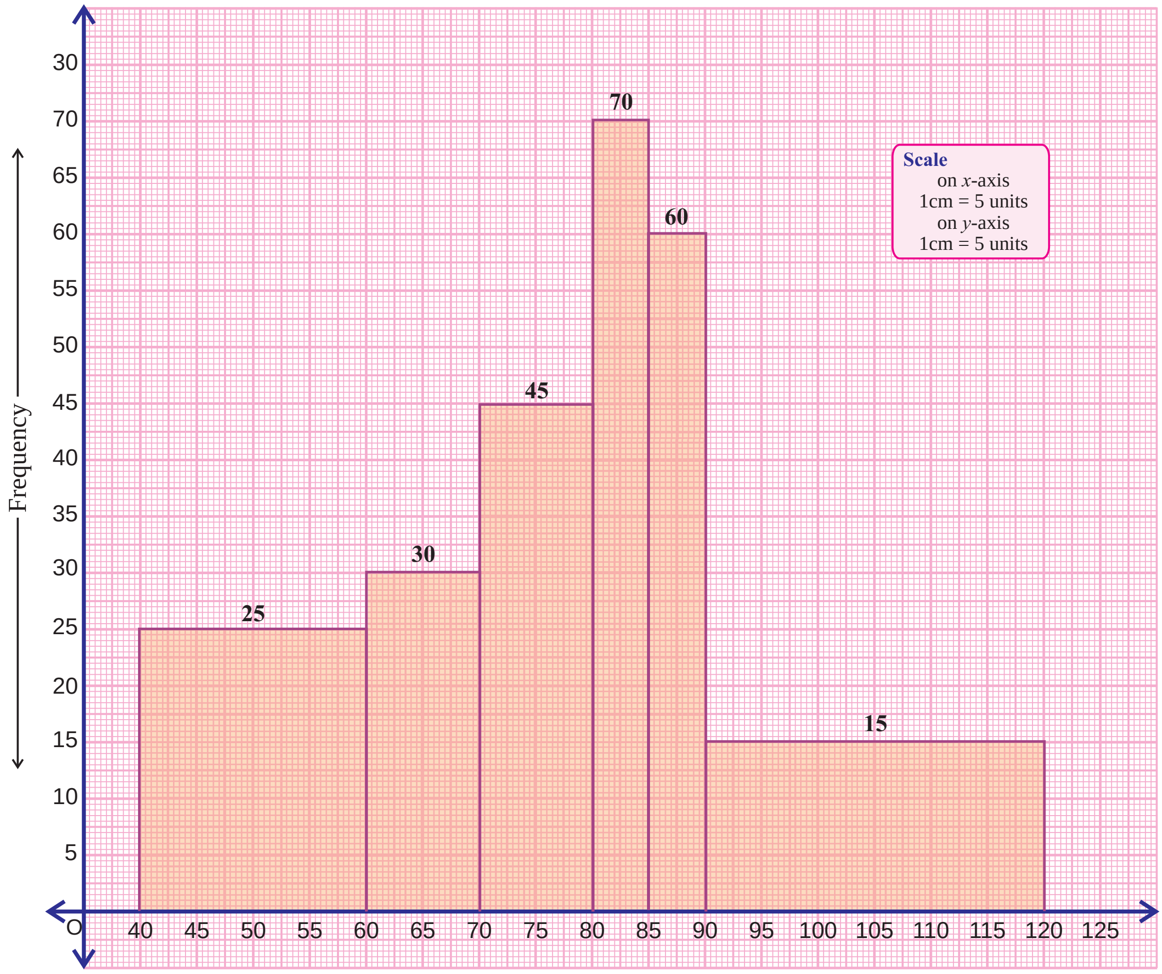
Key Concept **Frequency Density**

Frequency density = Frequency ÷ class width

If C denotes the minimum class width of the data set, then the length of the rectangle is given by

$$\text{Length of the rectangle} = \frac{\text{frequency}}{\text{Class width}} \times C$$

Time (Seconds)	40-60	60-70	70-80	80-85	85-90	90-120
Frequency	100	60	90	70	60	90
Class Width	20	10	10	5	5	30
Length of the rectangle	$\frac{100}{20}\times 5 = 25$	$\frac{60}{10}\times 5 = 30$	$\frac{90}{10}\times 5 = 45$	$\frac{70}{5}\times 5 = 70$	$\frac{60}{5}\times 5 = 60$	$\frac{90}{30}\times 5 = 15$



Time (in Seconds)

Example 11.3

Draw a histogram to represent the following data set.

Marks	0-10	10-20	20-40	40-50	50-60	60-70	70-90	90-100
No. of students	4	6	14	16	14	8	16	5

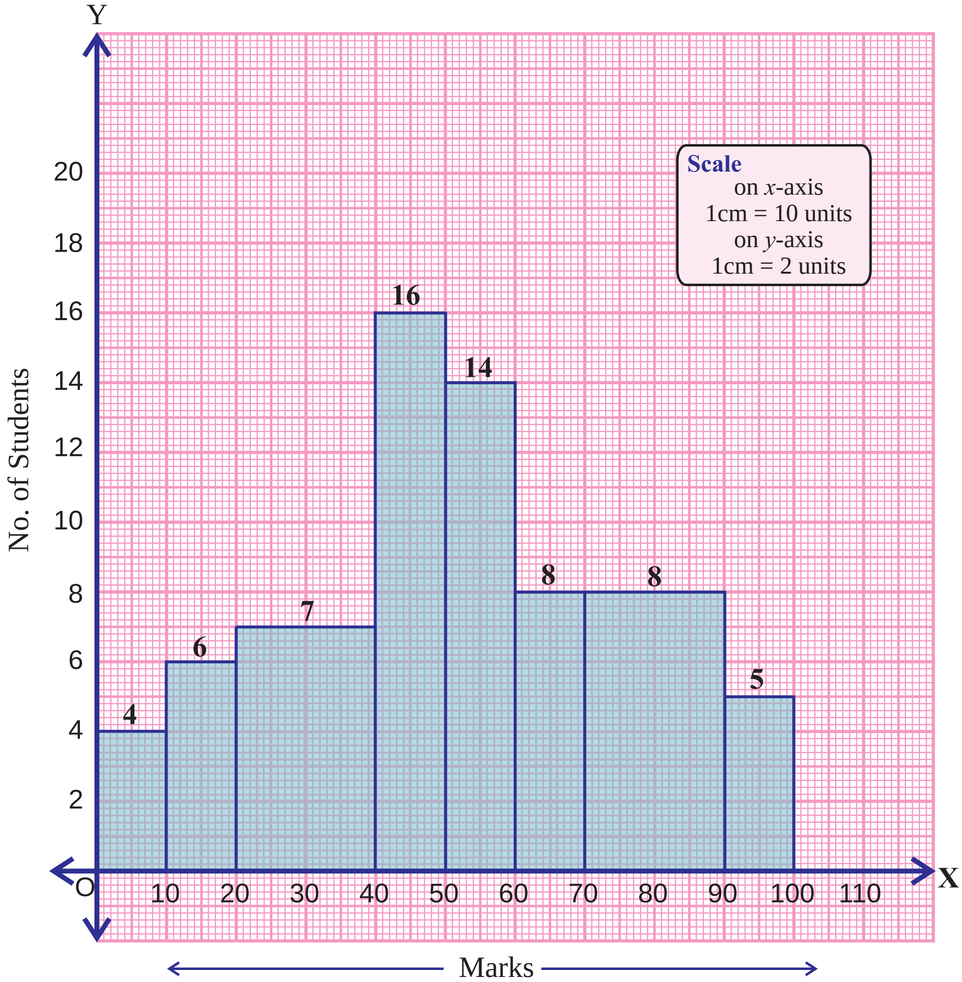
Solution The minimum of the class widths of the data set is 10. So, we draw rectangles with class intervals as bases and the lengths of the rectangles given by

$$\text{length of rectanlge} = \frac{\text{frequency}}{\text{class width}} \times 10.$$

Thus, the histogram can be drawn as follows.

Marks	0-10	10-20	20-40	40-50	50-60	60-70	70-90	90-100
No. of students	4	6	14	16	14	8	16	5
Class width	10	10	20	10	10	10	20	10
Length of the rectangle	$\frac{4}{10}\times 10 = 4$	$\frac{6}{10}\times 10 = 6$	$\frac{14}{20}\times 10 = 7$	$\frac{16}{10}\times 10 = 16$	$\frac{14}{10}\times 10 = 14$	$\frac{8}{10}\times 10 = 8$	$\frac{16}{20}\times 10 = 8$	$\frac{5}{10}\times 10 = 5$

Histogram with Varying Base Length



Exercise 11.1

1. Draw a histogram for the following distribution.

Class Interval	0-10	10-30	30-45	45-50	50-60
Frequency	8	28	18	6	10

2. Draw a histogram for the monthly wages of the workers in a factory as per data given below.

Monthly wages (₹)	2000-2200	2200-2400	2400-2800	2800-3000	3000-3200	3200-3600
No. of workers	25	30	50	60	15	10

3. The following distribution gives the mass of 48 objects measured to the nearest gram. Draw a histogram to illustrate the data.

Mass in (gms)	10-19	20-24	25-34	35-49	50-54
No. of objects	6	4	12	18	8

4. Draw a histogram to represent the following data.

Class interval	10-14	14-20	20-32	32-52	52-80
Frequency	5	6	9	25	21

5. The age (in years) of 360 patients treated in the hospital on a particular day are given below.

Age in years	10-20	20-30	30-50	50-60	60-70
No. of patients	80	50	80	120	30

Draw a histogram for the above data.

Measures of Central Tendency

One of the main objectives of statistical analysis is to get a single value that describes the characteristic of the entire data. Such a value is called the central value and the most commonly used measures of central tendencies are Arithmetic Mean, Median and Mode.

11.3 Mean

11.3.1 Arithmetic Mean - Raw Data

The arithmetic mean is the sum of a set of observations, positive, negative or zero, divided by the number of observations. If we have n real numbers $x_1, x_2, x_3 ...., x_n$, then their arithmetic mean, denoted by $\overline{x}$, is given by

$$\overline{x} = \frac{x_1 + x_2 + \cdots + x_n}{n} \text{ or } \overline{x} = \frac{1}{n}\sum_{i=1}^{n} x_i \text{ or } \overline{x} = \frac{\sum x}{n}$$

Remark $\overline{x} = \frac{\sum x}{n} \implies n\overline{x} = \sum x$. That is,

Total number of observations $\times$ Mean = Sum of all observations

Think and Answer !

Can a person of height 5 feet, who does not know swimming wade through a river which has an average depth of 4 feet to the other bank?

Example 11.4

Find the arithmetic mean of the marks 72, 73, 75, 82, 74 obtained by a student in 5 subjects in an annual examination.

Solution Here $n = 5$

$$\overline{x} = \frac{\sum x}{n} = \frac{72+73+75+82+74}{5} = \frac{376}{5} = 75.2$$

$\therefore$ Mean $= 75.2$

Example 11.5

The mean of the 5 numbers is 32. If one of the numbers is excluded, then the mean is reduced by 4. Find the excluded number.

Solution

Mean of 5 numbers $= 32.$

Sum of these numbers $= 32 \times 5 = 160$ $\quad (\because n\overline{x} = \sum x)$

Mean of 4 numbers $= 32 - 4 = 28$

Sum of these 4 numbers $= 28 \times 4 = 112$

Excluded number = (Sum of the 5 given numbers) − (Sum of the 4 numbers)

$= 160 - 112 = 48$

11.3.2 Arithmetic Mean - Ungrouped Frequency Distribution

The mean (or average) of the observations $x_1, x_2, x_3 ... x_n$ with frequencies $f_1, f_2, f_3 .. f_n$ respectively is given by

$$\overline{x} = \frac{f_1x_1 + f_2x_2 + \cdots + f_nx_n}{f_1 + f_2 + \cdots + f_n} = \frac{\sum_{i=1}^{n} f_ix_i}{\sum_{i=1}^{n} f_i}$$

The above formula, more briefly, is written as $\overline{x} = \frac{\sum fx}{\sum f}$

Example 11.6

Obtain the mean of the following data.

x	5	10	15	20	25
f	3	10	25	7	5

Solution

x	f	fx
5	3	15
10	10	100
15	25	375
20	7	140
25	5	125
Total	$\sum f = 50$	$\sum fx = 755$

$$\text{Mean} = \frac{\sum fx}{\sum f} = \frac{755}{50} = 15.1$$

Mean = 15.1

11.3.3 Arithmetic Mean - Grouped Frequency Distribution

Consider the following frequency table.

Class interval (Marks)	0-10	10-20	20-30	30-40	40-50
Frequency (No. of students)	3	4	3	7	8

The first entry of the table says that 3 children got less than 10 marks but does not say anything about the marks got by the individuals. Now, for each class interval we require a point which could serve as the representative of the class interval. In the interval 0-10, we assume it as 5. That is, it is assumed that the frequency of each class interval is centered around its mid-point. Thus the mid-point or the class mark of each class can be chosen to represent the observations falling in that class.

Class mark = $\frac{UCL + LCL}{2}$ (UCL= Upper Class Limit, LCL = Lower Class Limit)

Using the above formula the class marks for each of the class intervals are found out and are represented as x

$\overline{x} = \frac{\sum fx}{\sum f}$ can be used to find the mean of the grouped data.

In grouped frequency distribution, arithmetic mean may be computed by applying any one of the following methods:

(i) Direct Method

(ii) Assumed Mean Method

(iii) Step Deviation Method

Direct Method

When direct method is used, the formula for finding the arithmetic mean is

$$\overline{x} = \frac{\sum fx}{\sum f},$$

where x is the midpoint of the class interval and f is the frequency.

Steps :

(i) Obtain the midpoint of each class and denote it by x.

(ii) Multiply these midpoints by the respective frequency of each class and obtain the total of fx.

(ii) Divide $\sum fx$ by the sum $\sum f$ of the frequencies to obtain Mean.

Example 11.7

From the following table compute arithmetic mean by direct method.

Marks	0-10	10-20	20-30	30-40	40-50	50-60
No. of students	5	10	25	30	20	10

Solution

Marks	Midpoint (x)	No. of students (f)	fx
0-10	5	5	25
10-20	15	10	150
20-30	25	25	625
30-40	35	30	1050
40-50	45	20	900
50-60	55	10	550
		$\sum f = 100$	$\sum fx = 3300$

$$\bar{x} = \frac{\sum fx}{\sum f} = \frac{3300}{100} = 33$$

$\therefore$ Mean $= 33$

Assumed Mean Method

When assumed mean method is used, arithmetic mean is computed by applying the following formula.

$$\bar{x} = A + \frac{\sum fd}{\sum f},$$

where A is the assumed mean and $d = x - A$ is the deviation of midpoint x from assumed mean A.

Steps:

(i) Choose A as the assumed mean.

(ii) Find the deviation, $d = x - A$ for each class

(iii) Multiply the respective frequencies of each class by their deviations and obtain $\sum fd$.

(iv) Apply the formula $\bar{x} = A + \frac{\sum fd}{\sum f}$

Example 11.8

Calculate the arithmetic mean by assumed mean method for the data given in the above example.

Solution Let the assumed mean be A = 35

Marks	Mid-value (x)	No of students (f)	$d = x - 35$	fd
0-10	5	5	-30	-150
10-20	15	10	-20	-200
20-30	25	25	-10	-250
30-40	35	30	0	0
40-50	45	20	10	200
50-60	55	10	20	200
		$\sum f = 100$		$\sum fd = -200$

$$\bar{x} = A + \frac{\sum fd}{\sum f}$$

$$= 35 + \left(\frac{-200}{100}\right) = 35 - 2 = 33$$

Step Deviation Method

In order to simplify the calculation, we divide the deviation by the width of the class intervals. i.e calculate $\frac{x-A}{c}$ and then multiply by c in the formula for getting the mean of the data.

$$\bar{x} = A + \frac{\sum fd}{\sum f} \times c$$

Solution: Width of the class interval is c = 10

Marks	Mid-value	No of students f	$d = \frac{x-35}{10}$	fd
0-10	5	5	-3	-15
10-20	15	10	-2	-20
20-30	25	25	-1	-25
30-40	35	30	0	0
40-50	45	20	1	20
50-60	55	10	2	20
		$\sum f = 100$		$\sum fd = -20$

$$\bar{x} = A + \frac{\sum fd}{\sum f} \times c \quad = 35 - \left(\frac{20}{100} \times 10\right) = 35 - 2 = 33$$

$$\therefore \text{Mean} \quad = 33$$

11.3.4 Properties of Mean

Property 1

Sum of the deviations taken from the arithmetic mean is zero.

If $x_1, x_2, x_3 \ldots x_n$ are n observations with mean $\overline{x}$ then $(x_1 - \overline{x}) + (x_2 - \overline{x}) + \ldots + (x_n - \overline{x}) = 0$

For example, the mean of 6, 8, 9, 14, 13 is 10. Consider the deviation of each observation from arithmetic mean.

Sum of the deviations from arithmetic mean is

$(6 - 10) + (8 - 10) + (9 - 10) + (14 - 10) + (13 - 10)$

$$= -4 + (-2) + (-1) + 4 + 3 = -7 + 7 = 0$$

Hence, from the above example, we observe that sum of the deviations from the arithmetic mean is zero.

Property 2

If each observation is increased by k then the mean of the new observations is the original mean increased by k.

i.e., suppose the mean of n observations $x_1, x_2, x_3 \ldots x_n$ is $\overline{x}$. Each observation is increased by k, then the mean of the new observation is $(\overline{x} + k)$.

For example, consider five numbers x_1, x_2, x_3, x_4 and x_5 whose mean is 20.

$$\text{i.e., } \overline{x} = \frac{x_1 + x_2 + x_3 + x_4 + x_5}{5} = 20$$

If each of the number is increased by 5, then the new numbers are

$x_1 + 5, x_2 + 5, x_3 + 5, x_4 + 5$ and $x_5 + 5$

Mean of these new numbers is

$$\frac{x_1 + 5 + x_2 + 5 + x_3 + 5 + x_4 + 5 + x_5 + 5}{5}$$

$$= \frac{x_1 + x_2 + x_3 + x_4 + x_5 + 25}{5}$$

$$= \frac{x_1 + x_2 + x_3 + x_4 + x_5}{5} + \frac{25}{5} = 20 + 5$$

= original mean + the increased value.

Hence, the original mean is increased by 5.

Property 3

If each observation is decreased by k, then the mean of the new observations is original mean decreased by k.

i.e., suppose the mean of n observations is $\overline{x}$. If each observation is decreased by k, then the mean of the new observation is $\overline{x} - k$.

For example, consider five numbers x_1, x_2, x_3, x_4 and x_5 whose mean is 20.

i.e., $\dfrac{x_1 + x_2 + x_3 + x_4 + x_5}{5} = 20$

If each of the number is decreased by 5, then the new numbers are

$$x_1 - 5, x_2 - 5, x_3 - 5, x_4 - 5, x_5 - 5.$$

$$\text{New mean} = \frac{x_1 - 5 + x_2 - 5 + x_3 - 5 + x_4 - 5 + x_5 - 5}{5}$$

$$= \frac{x_1 + x_2 + x_3 + x_4 + x_5}{5} - \frac{25}{5}$$

$$= 20 - 5$$

= original mean - the decreased value.

Hence, the original mean is decreased by 5.

Property 4

If each observation is multiplied by k, $k \neq 0$, then the mean of the new observation is the original mean multiplied by k

i.e., suppose the mean of n observations $x_1, x_2, x_3 \ldots x_n$ is $\overline{x}$. If each observation is multiplied by k, $k \neq 0$, then the mean of the new observations is $k\overline{x}$.

For example, consider five numbers x_1, x_2, x_3, x_4 and x_5 whose mean is 20.

Mean of these numbers $= \dfrac{x_1 + x_2 + x_3 + x_4 + x_5}{5} = 20$

If each data is multiplied by 5, then the new observations are $5x_1, 5x_2, 5x_3, 5x_4, 5x_5$.

$$\text{New mean} = \frac{5x_1 + 5x_2 + 5x_3 + 5x_4 + 5x_5}{5}$$

$$= \frac{5(x_1 + x_2 + x_3 + x_4 + x_5)}{5} = 5\ (20)$$

= Five times the original mean.

Hence, the new mean is 5 times its original mean.

Property 5

If each observation is divided by k, $k \neq 0$, then the mean of new observations is the original mean divided by k.

i.e., suppose the mean of the n observations $x_1, x_2, x_3, x_4, x_5 \ldots x_n$ is $\overline{x}$. If each observation is divided by k, where $k \neq 0$, then the mean of the new observation is $\frac{\overline{x}}{k}$.

For example, consider five numbers x_1, x_2, x_3, x_4 and x_5 whose mean is 20. So,

$$\overline{x} = \frac{x_1 + x_2 + x_3 + x_4 + x_5}{5} = 20$$

Now we divide each number by 5. Let $y_1 = \frac{x_1}{5}$, $y_2 = \frac{x_2}{5}$, $y_3 = \frac{x_3}{5}$, $y_4 = \frac{x_4}{5}$ and $y_5 = \frac{x_5}{5}$. Then

$$\overline{y} = \frac{\frac{x_1}{5} + \frac{x_2}{5} + \frac{x_3}{5} + \frac{x_4}{5} + \frac{x_5}{5}}{5}$$

$$= \frac{1}{5}\left(\frac{x_1 + x_2 + x_3 + x_4 + x_5}{5}\right) = \frac{1}{5}(20)$$

$$= \frac{1}{5}(\overline{x})$$

New mean is the original mean divided by 5.

Example 11.9

The mean mark of 100 students was found to be 40. Later on, it was found that a score of 53 was misread as 83. Find the correct mean corresponding to the correct score.

Solution Given that the total number of students n = 100, $\overline{x} = 40$. So,

$$\sum x = \overline{x} \times n = 40 \times 100 = 4000$$

Correct $\sum x$ = Incorrect $\sum x$ − wrong item + correct item.

$$= 4000 - 83 + 53 = 3970$$

$$\text{Correct } \overline{x} = \frac{\text{correct} \sum x}{n}$$

$$= \frac{3970}{100} = 39.7$$

Hence the correct mean is 39.7.

Exercise 11.2

1. Obtain the mean number of bags sold by a shopkeeper on 6 consecutive days from the following table

Days	Monday	Tuesday	Wednesday	Thursday	Friday	Saturday
No. of bags sold	55	32	30	25	10	20

2. The number of children in 10 families in a locality are 2, 4, 3, 4, 1, 6, 4, 5, x, 5. Find x if the mean number of children in a family is 4

3. The mean of 20 numbers is 59. If 3 is added to each number what will be the new mean?

4. The mean of 15 numbers is 44. If 7 is subtracted from each number what will be the new mean?

5. The mean of 12 numbers is 48. If each numbers is multiplied by 4 what will be the new mean?

6. The mean of 16 numbers is 54. If each number is divided by 9 what will be the new mean?

7. The mean weight of 6 boys in a group is 48 kg. The individual weights of 5 of them are 50kg, 45kg, 50kg, 42kg and 40kg. Find the weight of the sixth boy.

8. Using assumed mean method find the mean weight of 40 students using the data given below.

weights in kg.	50	52	53	55	57
No. of students	10	15	5	6	4

9. The arithmetic mean of a group of 75 observations was calculated as 27. It was later found that one observation was wrongly read as 43 instead of the correct value 53. Obtain the correct arithmetic mean of the data.

10. Mean of 100 observations is found to be 40. At the time of computation two items were wrongly taken as 30 and 27 instead of 3 and 72. Find the correct mean.

11. The data on number of patients attending a hospital in a month are given below. Find the average number of patients attending the hospital in a day.

No. of patients	0-10	10-20	20-30	30-40	40-50	50-60
No. of days attending hospital	2	6	9	7	4	2

12. Calculate the arithmetic mean for the following data using step deviation method.

Marks	0-10	10-20	20-30	30-40	40-50	50-60
No. of students	8	15	22	20	10	5

13. In a study on patients, the following data were obtained. Find the arithmetic mean.

Age (in yrs)	10-19	20-29	30-39	40-49	50-59
No. of patients	1	0	1	10	13

14. The total marks obtained by 40 students in the Annual examination are given below

Marks	150 - 200	200 - 250	250 - 300	300 - 350	350 - 400	400 - 450	450 - 500
Students	2	3	12	10	4	6	3

Using step deviation method to find the mean of the above data.

15. Compute the arithmetic mean of the following distribution.

Class Interval	0 - 19	20 - 39	40 - 59	60 - 79	80 - 99
Frequency	3	4	15	14	4

11.4 Median

Median is defined as the middle item of the given observations arranged in order.

11.4.1 Median - Raw Data

Steps:

(i) Arrange the n given numbers in ascending or descending order of magnitude.

(ii) When n is odd, $\left(\frac{n+1}{2}\right)^{th}$ observation is the median.

(iii) When n is even the median is the arithmetic mean of the two middle values.

i.e., when n is even,

Median = Mean of $\left(\frac{n}{2}\right)^{th}$ and $\left(\frac{n}{2}+1\right)^{th}$ observations.

Example 11.10

Find the median of the following numbers

(i) 24, 22, 23, 14, 15, 7, 21 (ii) 17, 15, 9, 13, 21, 32, 42, 7, 12, 10.

Solution

(i) Let us arrange the numbers in ascending order as below.

7, 14, 15, 21, 22, 23, 24

Number of items $n = 7$

$$\text{Median} = \left(\frac{n+1}{2}\right)^{th} \text{ observation} \quad (\because n \text{ is odd})$$

$$= \left(\frac{7+1}{2}\right)^{th} \text{ observation}$$

$$= 4^{th} \text{ observation} = 21$$

(ii) Let us arrange the numbers in ascending order

7, 9, 10, 12, 13, 15, 17, 21, 32, 42.

Number of items $n = 10$

Median is the mean of $\left(\frac{n}{2}\right)^{th}$ and $\left(\frac{n}{2}+1\right)^{th}$ observations. $(\because n$ is even)

$$\left(\frac{n}{2}\right)^{th} \text{ observation} = \left(\frac{10}{2}\right)^{th} \text{ observation} = 5^{th} \text{ observation} = 13$$

$\left(\frac{n}{2}+1\right)^{th}$ observation= 6^{th} observation = 15.

$$\therefore \text{Median} = \frac{13+15}{2} = 14$$

11.4.2 Median - Ungrouped Frequency Distribution

(i) Arrange the data in ascending or descending order of magnitude.

(ii) Construct the cumulative frequency distribution.

(iii) If n is odd, then Median = $\left(\frac{n}{2}+1\right)^{th}$ term.

(iv) If n is even, then Median = $\dfrac{\left\{\left(\frac{n}{2}\right)^{th} term + \left(\frac{n}{2}+1\right)^{th} term\right\}}{2}$

Example 11.11

Calculate the median for the following data.

Marks	20	9	25	50	40	80
No. of students	6	4	16	7	8	2

Solution Let us arrange marks in ascending order.

Marks	f	cf
9	4	4
20	6	10
25	16	26
40	8	34
50	7	41
80	2	43
	n = 43	

Here, n = 43, which is odd

Position of median $= \left(\frac{n+1}{2}\right)^{th}$ observation .

$= \left(\frac{43+1}{2}\right)^{th}$ observation.

$= 22^{nd}$ observation.

The above table shows that all items from 11 to 26 have their value 25. So, the value of 22^{nd} item is 25.

$\therefore$ Median = 25.

Example 11.12

Find the median for the following distribution.

Value	1	2	3	4	5	6
f	1	3	2	4	8	2

Solution

Value	f	cf
1	1	1
2	3	4
3	2	6
4	4	10
5	8	18
6	2	20
	n = 20	

$n = 20$ (even)

Position of the median $= \left(\frac{20+1}{2}\right)^{\text{th}}$ th observation

$= \left(\frac{21}{2}\right)^{\text{th}}$ th observation $= (10.5)^{\text{th}}$ observation

The median then, is the average of the tenth and the eleventh items. The tenth item is 4, the eleventh item is 5.

Hence median $= \frac{4+5}{2} = \frac{9}{2} = 4.5.$

11.4.3 Median - Grouped Frequency Distribution

In a grouped frequency distribution, computation of median involves the following steps.

(i) Construct the cumulative frequency distribution.

(ii) Find $\frac{N}{2}^{th}$ term.

(iii) The class that contains the cumulative frequency $\frac{N}{2}$ is called the median class.

(iv) Find the median by using the formula:

$$\text{Median} = l + \frac{\frac{N}{2} - m}{f} \times c,$$

where l = Lower limit of the median class, f = Frequency of the median class

c = Width of the median class, N = The total frequency

m = cumulative frequency of the class preceeding the median class

Example 11.13

Find the median for the following distribution.

Wages (Rupees in hundreds)	0-10	10-20	20-30	30-40	40-50
No of workers	22	38	46	35	20

Solution

Wages	f	cf
0-10	22	22
10-20	38	60
20-30	46	106
30-40	35	141
40-50	20	161
	N = 161	

Here, $\frac{N}{2} = \frac{161}{2} = 80.5$. Median class is 20-30.

Lower limit of the median class $l = 20$

Frequency of the median class $f = 46$

Cumulative frequency of the class preceeding the median class $m = 60$

Width of the class $c = 10$

$$\text{Median} = l + \frac{\frac{N}{2} - m}{f} \times c$$

$$= 20 + \frac{80.5 - 60}{46} \times 10 = 20 + \frac{10}{46} \times 20.5$$

$$= 20 + \frac{205}{46} = 20 + 4.46 = 24.46$$

$$\therefore \text{Median} = 24.46$$

Example 11.14

Find the median for the following data.

Marks	11-15	16-20	21-25	26-30	31-35	36-40
Frequency	7	10	13	26	9	5

Solution

Since the table is given in terms of inclusive type we convert it into exclusive type.

Marks	f	cf
10.5- 15.5	7	7
15.5-20.5	10	17
20.5-25.5	13	30
25.5-30.5	26	56
30.5-35.5	9	65
35.5-40.5	5	70
	N = 70	

$$N = 70, \ \frac{N}{2} = \frac{70}{2} = 35$$

Median class is 25.5-30.5

Lower limit of the median class l = 25.5

Frequency of the median class f = 26

Cumulative frequency of the preceding median class m = 30

Width of the median class c = 30.5 − 25.5 = 5

$$\text{Median} = l + \frac{\frac{N}{2} - m}{f} \times c$$

$$= 25.5 + \frac{35 - 30}{26} \times 5 = 25.5 + \frac{25}{26} = 26.46$$

Exercise 11.3

1. Find the median of the following data.

   (i) 18,12,51,32,106,92,58

   (ii) 28,7,15,3,14,18,46,59,1,2,9,21

2. Find the median for the following frequency table.

Value	12	13	15	19	22	23
Frequency	4	2	4	4	1	5

3. Find the median for the following data.

Height (ft)	5-10	10-15	15-20	20-25	25-30
No of trees	4	3	10	8	5

4. Find the median for the following data.

Age group	0-9	10-19	20-29	30-39	40-49	50-59	60-69
No. of persons	4	6	10	11	12	6	1

5. Calculate the median for the following data

Class interval	1 - 5	6 - 10	11 - 15	16 - 20	21 - 25	26 - 30	31 - 35
Frequency	1	18	25	26	7	2	1

6. The following table gives the distribution of the average weekly wages of 800 workers in a factory. Calculate the median for the data given below.

Wages (₹ in hundres)	20 - 25	25 - 30	30 - 35	35 - 40	40 - 45	45 - 50	50 - 55	55 - 60
No. of persons	50	70	100	180	150	120	70	60

11.5 Mode

The Mode of a distribution is the value at the point around which the items tend to be most heavily concentrated.

11.5.1 Mode - Raw Data

In a raw data, mode can be easily obtained by arranging the observations in an array and then counting the number of times each observation occurs.

For example, consider a set of observations consisting of values 20,25,21,15,14,15.

Here, 15 occurs twice where as all other values occur only once. Hence mode of this data = 15.

Remark Mode can be used to measure quantitative as well as qualitative data. If a printing press turns out 5 impressions which we rate very sharp, sharp, sharp, sharp and Blurred, then the model value is sharp.

Example 11.15

The marks of ten students in a mathematics talent examination are 75,72,59,62, 72,75,71,70,70,70. Obtain the mode.

Solution Here the mode is 70, since this score was obtained by more students than any other.

Note A distribution having only one mode is called unimodal.

Example 11.16

Find the mode for the set of values 482,485,483,485,487,487,489.

Solution In this example both 485 and 487 occur twice. This list is said to have two modes or to be bimodal.

Note

(i) A distribution having two modes is called bimodal.

(ii) A distribution having three modes is called trimodal.

(iii) A distribution having more than three modes is called multimodal.

11.5.2 Mode - Ungrouped Frequency Distribution

In a ungrouped frequency distribution data the mode is the value of the variable having maximum frequency.

Example 11.17

A shoe shop in Chennai sold hundred pairs of shoes of a particular brand in a certain day with the following distribution.

Size of shoe	4	5	6	7	8	9	10
No of pairs sold	2	5	3	23	39	27	1

Find the mode of the following distribution.

Solution Since 8 has the maximum frequency with 39 pairs being sold the mode of the distribution is 8.

11.5.3 Mode - Grouped Frequency Distribution

In case of a grouped frequency distribution, the exact values of the variables are not known and as such it is very difficult to locate mode accurately. In such cases, if the class intervals are of equal width an appropriate value of the mode may be determined by using the formula

$$\text{Mode} = l + \left(\frac{f - f_1}{2f - f_1 - f_2}\right) \times c,$$

where l = lower limit of the modal class

f = frequency of modal class

c = class width of the modal class

f_1 = frequency of the class just preceeding the modal class.

f_2 = frequency of the class succeeding the modal class.

Example 11.18

Calculate the mode of the following data.

Size of item	10-15	15-20	20-25	25-30	30-35	35-40	40-45	45-50
No of items	4	8	18	30	20	10	5	2

Solution

Size of the item	f
10-15	4
15-20	8
20-25	18
25-30	30
30-35	20
35-40	10
40-45	5
45-50	2

Modal class is 25-30 since it has the maximum frequency.

Lower limit of the modal class $l = 25$

Frequency of the modal class $f = 30$

Frequency of the preceding the modal class $f_1 = 18$

Frequency of the class reducing the modal class $f_2 = 20$

Class width $c = 5$

$$\text{Mode} = l + \left(\frac{f - f_1}{2f - f_1 - f_2}\right) \times c$$

$$= 25 + \left(\frac{30 - 18}{60 - 18 - 20}\right) \times 5 = 25 + \frac{12 \times 5}{22}$$

$$= 25 + \frac{60}{22} = 25 + 2.73 = 27.73$$

$$\text{Mode} = 27.73$$

Exercise 11.4

1. The marks obtained by 15 students of a class are given below. Find the modal marks.
42,45,47,49,52,65,65,71,71,72,75,82,72,47,72

2. Calculate the mode of the following data.

Size of shoe	4	5	6	7	8	9	10
No. of Pairs sold	15	17	13	21	18	16	11

3. The age (in years) of 150 patients getting medical treatment in a hospital in a month are given below. Obtain its mode.

Age (yrs)	10-20	20-30	30-40	40-50	50-60	60-70
No of patients	12	14	36	50	20	18

4. For the following data obtain the mode.

Weight (in kg)	21-25	26-30	31-35	36-40	41-45	46-50	51-55	56-60
No of students	5	4	3	18	20	14	8	3

5. The ages of children in a scout camp are 13, 13, 14, 15, 13, 15, 14, 15, 13, 15 years. Find the mean, median and mode of the data.

6. The following table gives the numbers of branches and number plants in a garden of a school.

No. of branches	2	3	4	5	6
No. of plants	14	21	28	20	17

Calculate the mean, median and mode of the above data.

7. The following table shows the age distribution of cases of a certain disease reported during a year in a particular city.

Age in year	5 - 14	15 - 24	25 - 34	35 - 44	45 - 54	55 - 64
No. of cases	6	11	12	10	7	4

Obtain the mean, median and mode of the above data.

8. Find the mean, mode and median of marks obtained by 20 students in an examination. The marks are given below.

Marks	0 - 10	10 - 20	20 - 30	30 - 40	40 - 50
No. of students	1	4	5	8	2

ACTIVITY

1. Find the mean of 10,20,30,40 and 50.

   * Add 10 to each value and find the mean.
   * Subtract 10 from each value and find the mean.
   * Multiply each value by 10 and find the mean.
   * Divide each value by 10 and find the mean.

   Make a general statement about each situation and compare it with the properties of mean.

2. Give specific examples of your own in which,

   (i) The median is preferred to arithmetic mean.

   (ii) Mode is preferred to median.

   (iii) Median is preferred to mode.

Points to Remember

The mean for grouped data

- The direct method

$$\bar{x} = \frac{\sum fx}{\sum f}$$

- The assumed mean method

$$\bar{x} = A + \frac{\sum fd}{\sum f}$$

- The step deviation method

$$\bar{x} = A + \frac{\sum fd}{\sum f} \times C$$

- The cumulative frequency of a class is the frequency obtained by adding the frequencies of all up to the classes preceeding the given class.

- The median for grouped date can be found by using the formula

$$median = l + \frac{\frac{N}{2} - m}{f} \times c$$

- The mode for the grouped data can be found by using the formula

$$\text{mode} = l + \left(\frac{f - f_1}{2f - f_1 - f_2} \right) \times c$$

12 PROBABILITY

All business proceeds on beliefs, or judgments of probabilities, and not on certainty - CHARLES ELIOT

Main Targets

- To understand repeated experiments and observed frequency approach of Probability
- To understand Empirical Probability

Richard Von Mises
(1883-1953)

The statistical, or empirical, attitude toward probability has been developed mainly by R.F. Fisher and R. Von Mises. The notion of sample space comes from R.Von Mises. This notion made it possible to build up a strictly mathematical theory of probability based on measure theory. Such an approach emerged gradually in the last century under the influence of many authors. An axiomatic treatment representing the modern development was given by A. Kolmogorov.

12.1 Introduction

From dawn to dusk any individual makes decisions regarding the possible events that are governed at least in part by chance. Few examples are: "Should I carry an umbrella to work today?", "Will my cellphone battery last until tonight?", and "Should I buy a new brand of laptop?". Probability provides a way to make decisions when the person is uncertain about the things, quantities or actions involved in the decision. Though probability started with gambling, it has been used extensively, in the fields of Physical Sciences, Commerce, Biological Sciences, Medical Sciences, Insurance, Investments, Weather Forecasting and in various other emerging areas.

Consider the statements:

- ***Probably*** Kuzhalisai will stand first in the forth coming annual examination.
- ***Possibly*** Thamizhisai will catch the train today.
- The prices of essential commodities are ***likely*** to be stable.
- There is a ***chance*** that Leela will win today's Tennis match.

The words "***Probably***", "***Possibly***" , "***Likely***", "***Chance***" , etc., will mean "the lack of certainty" about the events mentioned above. To measure "the lack of certainty or uncertainty", there is no perfect yardstick, i.e., uncertainty is not perfectly quantifiable one. But based on some assumptions uncertainty can be measured mathematically. This numerical measure is referred to as probability. It is a purposeful technique used in decision making depending on, and changing with, experience. Probability would be effective and useful even if it is not a single numerical value.

12.2 Basic Concepts and Definitions

Before we start the theory on Probability, let us define some of the basic terms required for it.

- Experiment
- Random Experiment
- Trial
- Sample Space
- Sample Point
- Events

Key Concept	**Experiment**
An ***experiment*** is defined as a process whose result is well defined	

Experiments are classified broadly into two ways:

EXPERIMENT

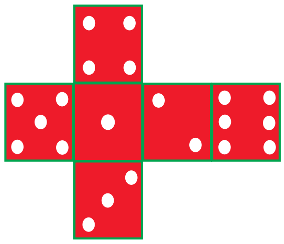
DETERMINISTIC

RANDOM

1. ***Deterministic Experiment*** : It is an experiment whose outcomes can be predicted with certainty, under identical conditions.

For example, in the cases-when we heat water it evaporates, when we keep a tray of water into the refrigerator it freezes into ice and while flipping an unusual coin with heads on both sides getting head - the outcomes of the experiments can be predicted well in advance. Hence these experiments are deterministic.

2. ***Random Experiment*** : It is an experiment whose all possible outcomes are known, but it is not possible to predict the exact outcome in advance.



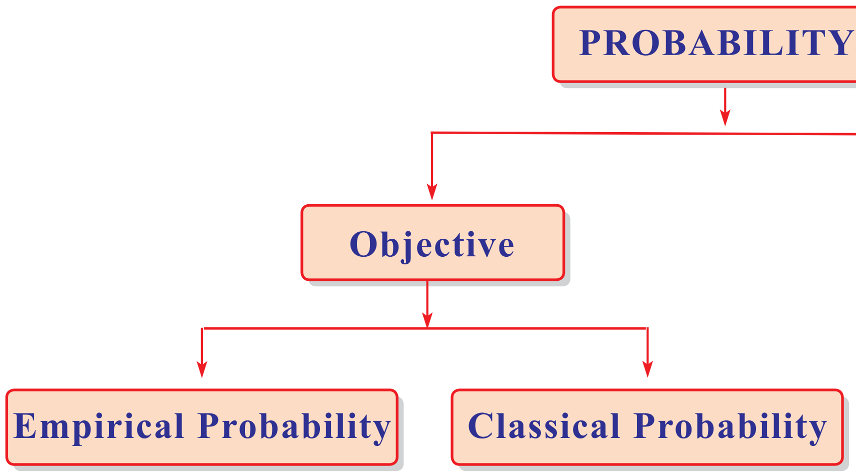
For example, consider the following experiments:

(i) A coin is flipped (tossed)

(ii) A die is rolled.

These are random experiments, since we cannot predict the outcome of these experiments.

Key Concept		
Trial	A Trial is an action which results in one or several outcomes.	For example, " Flipping" a coin and "Rolling" a die are trials
Sample Space	A sample space S is the set of all possible outcomes of a random experiment.	For example, While flipping a coin the sample space, S = { Head, Tail} While rolling a die, sample space S = { 1, 2, 3, 4, 5, 6}
Sample Point	Each outcome of an experiment is called a sample point.	While flipping a coin each outcome {Head}, {Tail} are the sample points. While rolling a die each outcome, {1} {2} {3} {4} {5} and {6} are are corresponding sample points
Event	Any subset of a sample space is called an event.	For example, When a die is rolled some of the possible events are {1, 2, 3}, {1, 3}, {2, 3, 5, 6}



12.3 Classification of Probability

According to various concepts of probability, it can be classified mainly in to three types as given below:

(1) Subjective Probability

(2) Classical Probability

(3) Empirical Probability

12.3.1 Subjective Probability

Subjective probabilities express the strength of one's belief with regard to the uncertainties. It can be applied especially when there is a little or no direct evidence about the event desired, there is no choice but to consider indirect evidence, educated guesses and perhaps intuition and other subjective factors to calculate probability .

12.3.2 Classical Probability

Classical probability concept is originated in connection with games of chance. It applies when all possible outcomes are equally likely. If there are n equally likely possibilities of which one must occur and s of them are regarded as favorable or as a *success* then the probability of a *success* is given by $({}^{s}\!/\!_{n})$.

12.3.3 Empirical Probability

It relies on actual experience to determine the likelihood of outcomes.

12.4 Probability - An Empirical Approach

In this chapter, we shall discuss only about empirical probability. The remaining two approaches would be discussed in higher classes. *Empirical* or *experimental* or *Relative frequency Probability* relies on actual experience to determine the likelihood of outcomes.

Empirical approach can be used whenever the experiment can be repeated many times and the results observed. Empirical probability is the most accurate scientific 'guess' based on the results of experiments about an event.

For example, the decision about people buying a certain brand of a soap, cannot be calculated using classical probability since the outcomes are not equally likely. To find the probability for such an event, we can perform an experiment such as you already have or conduct a survey. This is called collecting experimental data. The more data we collect the better the estimate is.

Key Concept **Empirical Probability**

Let m be the number of trials in which the event E happened (number of observations favourable to the event E) and n be the total number of trials (total number of observations) of an experiment. The empirical probability of happening of an event E, denoted by P(E), is given by

$$P(E) = \frac{\textit{Number of trials in which the event happened}}{\textit{Total number of trials}}$$

(or)

$$P(E) = \frac{\textit{Number of favourable observations}}{\textit{Total number of observations}}$$

(or)

$$P(E) = \frac{m}{n}$$

Clearly $0 \leq m \leq n \implies 0 \leq \frac{m}{n} \leq 1$, hence $0 \leq P(E) \leq 1$.

$$\mathbf{0 \leq P(E) \leq 1}$$

i.e.the probability of happening of an event always lies from 0 to 1.

Probability is its most general use is a measure of our degree of confidence that a thing will happen. If the probability is 1.0, we know the thing happen certainly, and if probability is high say 0.9, we feel that the event is likely to happen. A probability of 0.5 denotes that the event is a equally likely to happen or not and one of 0 means that it certainly will not. This interpretation applied to statistical probabilities calculated from frequencies is the only way of expecting what we know of the individual from our knowledge of the populations.

Remark

If $P(E) = 1$ then E is called **Certain event** or **Sure event.**

If $P(E) = 0$ then E is known is an **Impossible event**.

If $P(E)$ is the probability of an event, then the probability of not happening of E is denoted by $P(E')$ or $P(\overline{E})$

We know, $P(E) + P(E') = 1; \Rightarrow P(E') = 1 - P(E)$

$$P(E') = 1 - P(E)$$

We shall calculate a few typical probabilities, but it should be kept in mind that numerical probabilities are not the principal object of the theory. Our aim is to learn axioms, laws, concepts and to understand the theory of probability easily in higher classes.

Illustration

A coin is flipped several times. The number of times head and tail appeared and their ratios to the number of flips are noted below.

Number of Tosses (n)	Number of Heads (m_1)	P(H) = $\frac{m_1}{n}$	Number of Tails (m_2)	P(T) = $\frac{m_2}{n}$
50	29	$\frac{29}{50}$	21	$\frac{21}{50}$
60	34	$\frac{34}{60}$	26	$\frac{26}{60}$
70	41	$\frac{41}{70}$	29	$\frac{29}{70}$
80	44	$\frac{44}{80}$	36	$\frac{36}{80}$
90	48	$\frac{48}{90}$	42	$\frac{42}{90}$
100	52	$\frac{52}{100}$	48	$\frac{48}{100}$

From the above table we observe that as we increase the number of flips more and more, the probability of getting of heads and the probability of getting of tails come closer and closer to each other.

Activity (1) Flipping a coin:

Each student is asked to flip a coin for 10 times and tabulate the number of heads and tails obtained in the following table.

Outcome	Tally Marks	Number of heads or tails for 10 flips.
Head		
Tail		

Repeat the experiment for 20, 30, 40, 50 times and tabulate the results in the same manner as shown in the above example. Write down the values of the following fractions.

$$\frac{\text{Number of times head turn up}}{\text{Total number of times the coin is flipped}} = \frac{\square}{\square}$$

$$\frac{\text{Number of times tail turn up}}{\text{Total number of times the coin is flipped}} = \frac{\square}{\square}$$

Activity (2) Rolling a die:

Roll a die 20 times and calculate the probability of obtaining each of six outcomes.

Outcome	Tally Marks	Number of outcome for 20 rolls.	$\frac{\text{No. of times corresponding outcomes come up}}{\text{Total no. of times the die is rolled}}$
1			
2			
3			
4			
5			
6			

Repeat the experiment for 50, 100 times and tabulate the results in the same manner.

Activity (3) Flipping two coins:

Flip two coins simultaneously 10 times and record your observations in the table.

Outcome	Tally	Number of outcomes for 10 times	$\frac{\text{No. of times corresponding outcomes comes up}}{\text{Total no. of times the two coins are flipped}}$
Two Heads			
One head and one tail			
No head			

In Activity (1) each flip of a coin is called a trial. Similarly in Activity (2) each roll of a die is called a trial and each simultaneous flip of two coins in Activity (3) is also a trial.

In Activity (1) the getting a head in a particular flip is an event with outcome “head”. Similarly, getting a tail is an event with outcome tail.

In Activity (2) the getting of a particular number say “ 5” is an event with outcome 5.

The value $\dfrac{\text{Number of heads comes up}}{\text{Total number of times the coins fliped}}$ is called an experimental or empirical probability.

Example 12.1

A manufacturer tested 1000 cell phones at random and found that 25 of them were defective. If a cell phone is selected at random, what is the probability that the selected cellphone is a defective one.

Solution Total number of cell phones tested = 1000 i.e., n = 1000

Let E be the event of selecting a defective cell phone.

n(E) = 25 i.e., m = 25

$$P(E) = \frac{\text{Number of defective cellphones}}{\text{Total number of cellphones tested}}$$

$$= \frac{m}{n} = \frac{25}{1000} = \frac{1}{40}$$

Example 12.2

In T-20 cricket match, Raju hit a "Six" 10 times out of 50 balls he played. If a ball was selected at random find the probability that he would not have hit a "Six".

Solution Total Number of balls Raju faced = 50i.e., n = 50

Let E be the event of hit a "Six" by Raju

$n(E)$ = 10 i.e., m=10

$$P(E) = \frac{\text{Number of times Raju hits a "Six"}}{\text{Total number of balls faced}}$$

$$= \frac{m}{n} = \frac{10}{50} = \frac{1}{5}$$

P (Raju does not hit a Six) = $P(E') = 1 - P(E)$

$$= 1 - \frac{1}{5} = \frac{4}{5}$$

Example 12.3

The selection committee of a cricket team has to select a team of players. If the selection is made by using the past records scoring more than 40 runs in a match, then find the probability of selecting these two players whose performance are given below?

The performance of their last 30 matches are

Name of the player	More than 40 runs
Kumar	20 times
Kiruba	12 times

Solution Total number of matches observed = 30 i.e., $n = 30$

Let E_1 be the event of Kumar scoring more than 40 runs.

$$n(E_1) = 20 \qquad \text{i.e., } m_1 = 20$$

Let E_2 be the event of Kiruba scoring more than 40 runs.

$$n(E_2) = 12 \qquad \text{i.e., } m_2 = 12$$

$$P(E_1) = \frac{m_1}{n} = \frac{20}{30}$$

$$P(E_2) = \frac{m_2}{n} = \frac{12}{30}$$

The probability of Kumar being selected is = $\frac{20}{30} = \frac{2}{3}$

The probability of Kiruba being selected is = $\frac{12}{30} = \frac{2}{5}$

Example 12.4

On a particular day a policeman observed vehicles for speed check. The frequency table shows the speed of 160 vehicles that pass a radar speed check on dual carriage way.

Speed (Km/h)	20-29	30-39	40-49	50-59	60-69	70 & above
No. of Vehicles	14	23	28	35	52	8

Find the probability that the speed of a vehicle selected at random is

(i) faster than 70 *km/h*. (ii) between 20 - 39 *km/h*.

(iii) less than 60 *km/h*. (iv) between 40 - 69 *km/h*.

Solution

(i) Let E_1 be the event of a vehicle travelling faster than 70 *km/h*.

$$n(E_1) = 8 \qquad \text{i.e. } m_1 = 8$$

Total number of vehicles = 160. i.e. $n = 160$

$$P(E_1) = \frac{m_1}{n} = \frac{8}{160} = \frac{1}{20}$$

(ii) Let E_2 be the event of a vehicle travelling the speed between 20 - 39 *km/h*.

$$n(E_2) = 14+23 = 37 \qquad \text{i.e. } m_2 = 37$$

$$P(E_2) = \frac{m_2}{n} = \frac{37}{160}$$

(iii) Let E_3 be the event of a vehicle travelling the speed less than 60 *km/h*.

$$n(E_3) = 14+23+28+35 = 100 \qquad \text{i.e. } m_3 = 100$$

$$P(E_3) = \frac{m_3}{n} = \frac{100}{160} = \frac{5}{8}$$

(iv) Let E_4 be the event of a vehicle travelling the speed between 40-69 *km/h*.

$$n(E_4) = 28+35+52 = 115 \quad \text{i.e. } m_4 = 115$$

$$P(E_4) = \frac{m_4}{n} = \frac{115}{160} = \frac{23}{32}$$

Example 12.5

A researcher would like to determine whether there is a relationship between a student's interest in statistics and his or her ability in mathematics. A random sample of 200 students is selected and they are asked whether their ability in mathematics and interest in statistics is low, average or high. The results were as follows:

		Ability in mathematics		
		Low	Average	High
Interest in statistics	Low	60	15	15
	Average	15	45	10
	High	5	10	25

If a student is selected at random, what is the probability that he / she
(i) has a high ability in mathematics (ii) has an average interest in statistics
(iii) has a high interest in statistics (iv) has high ability in mathematics and high interest in statistics and (v) has average ability in mathematics and low interest in statistics.

Solution

Total number of students = 80+70+50=200. i.e. $n = 200$

(i) Let E_1 be the event that he/she has a high ability in mathematics .

$$n(E_1) = 15+10+25 = 50 \quad \text{i.e. } m_1 = 50$$

$$P(E_1) = \frac{m_1}{n} = \frac{50}{200} = \frac{1}{4}$$

(ii) Let E_2 be the event that he/she has an average interest in statistics.

$$n(E_2) = 15+45+10 = 70 \quad \text{i.e. } m_2 = 70$$

$$P(E_2) = \frac{m_2}{n} = \frac{70}{200} = \frac{7}{20}$$

(iii) Let E_3 be the event that he/she has a high interest in statistics.

$$n(E_3) = 5+10+25 = 40 \quad \text{i.e. } m_3 = 40$$

$$P(E_3) = \frac{m_3}{n} = \frac{40}{200} = \frac{1}{5}$$

(iv) Let E_4 be the event has high ability in mathematics and high interest in statistics.

$$n(E_4) = 25 \qquad \text{i.e. } m_4 = 25$$

$$P(E_4) = \frac{m_4}{n} = \frac{25}{200} = \frac{1}{8}$$

(v) Let E_5 be the event has has average ability in mathematics and low interest in statistics.

$$n(E_5) = 15 \qquad \text{i.e. } m_5 = 15$$

$$P(E_5) = \frac{m_5}{n} = \frac{15}{200} = \frac{3}{40}$$

Example 12.6

A Hospital records indicated that maternity patients stayed in the hospital for the number of days as shown in the following.

No. of days stayed	3	4	5	6	more than 6
No. of patients	15	32	56	19	5

If a patient was selected at random find the probability that the patient stayed

(i) exactly 5 days (ii) less than 6 days

(iii) at most 4 days (iv) at least 5 days

Solution

Total number of patients of observed = 127 i.e., $n = 127$

(i) Let E_1 be the event of patients stayed exactly 5 days.

$$n(E_1) = 56 \qquad \text{i.e., } m_1 = 56$$

$$P(E_1) = \frac{m_1}{n} = \frac{56}{127}$$

(ii) Let E_2 be the event of patients stayed less than 6 days.

$$n(E_2) = 15 + 32 + 56 = 103 \quad \text{i.e., } m_2 = 103$$

$$P(E_2) = \frac{m_2}{n} = \frac{103}{127}$$

(iii) Let E_3 be the event of patients stayed atmost 4 days (3 and 4 days only).

$$n(E_3) = 15 + 32 = 47 \qquad \text{i.e., } m_3 = 47$$

$$P(E_2) = \frac{m_3}{n} = \frac{47}{127}$$

(iv) Let E_4 be the event of patients stayed atleast 5 days (5, 6 and 7 days only).

$$n(E_4) = 56 + 19 + 5 = 80 \qquad \text{i.e., } m_4 = 80$$

$$P(E_4) = \frac{m_4}{n} = \frac{80}{127}$$

Exercise 12.1

1. A probability experiment was conducted. Which of these cannot be considered as a probability of an outcome?

   i) 1/3 ii) – 1/5 iii) 0.80 iv) – 0.78 v) 0

   vi) 1.45 vii) 1 viii) 33% ix) 112%

2. Define: i) experiment ii) deterministic experiment iii) random experiment iv) sample space v) event vi) trial

3. Define empirical probability.

4. During the last 20 basket ball games, Sangeeth has made 65 and missed 35 free throws. What is the empirical probability if a ball was selected at random that Sangeeth make a foul shot?

5. The record of a weather station shows that out of the past 300 consecutive days, its weather was forecasted correctly 195 times. What is the probability that on a given day selected at random, (i) it was correct (ii) it was not correct.

6. Gowri asked 25 people if they liked the taste of a new health drink. The responses are,

Responses	Like	Dislike	Undecided
No. of people	15	8	2

   Find the probability that a person selected at random

   (i) likes the taste (ii) dislikes the taste (iii) undecided about the taste

7. In the sample of 50 people, 21 has type "O" blood, 22 has type "A" blood, 5 has type "B" blood and 2 has type "AB" blood. If a person is selected at random find the probability that

   (i) the person has type "O" blood (ii) the person does not have type "B" blood
   (iii) the person has type "A" blood (iv) the person does not have type "AB" blood.

8. A die is rolled 500 times. The following table shows that the outcomes of the die.

Outcomes	1	2	3	4	5	6
Frequencies	80	75	90	75	85	95

   Find the probability of getting an outcome (i) less than 4 (ii) less than 2
   (iii) greater than 2 (iv) getting 6 (v) not getting 6.

9. 2000 families with 2 children were selected randomly, and the following data were recorded.

Number of girls in a family	2	1	0
Number of families	624	900	476

Find the probability of a family, chosen at random, having (i) 2 girls (ii) 1 girl (iii) no girl

10. The follwing table gives the lifetime of 500 CFL lamps.

Life time (months)	9	10	11	12	13	14	more than 14
Number of Lamps	26	71	82	102	89	77	53

A bulb is selected at random. Find the probability that the life time of the selected bulb is

(i) less than 12 months (ii) more than 14 months

(iii) at most 12 months (iv) at least 13 months

11. On a busy road in a city the number of persons sitting in the cars passing by were observed during a particular interval of time. Data of 60 such cars is given in the following table.

No. of persons in the car	1	2	3	4	5
No. of Cars	22	16	12	6	4

Suppose another car passes by after this time interval. Find the probability that it has

(i) only 2 persons sitting in it (ii) less than 3 persons in it

(iii) more than 2 persons in it (iv) at least 4 persons in it

12. Marks obtained by Insuvai in Mathematics in ten unit tests are listed below.

Unit Test	I	II	III	IV	V	VI	VII	VIII	IX	X
Marks obtained (%)	89	93	98	99	98	97	96	90	98	99

Based on this data find the probability that in a unit test Insuvai get

(i) more than 95% (ii) less than 95% (iii) more than 98%

13. The table below shows the status of twenty residents in an apartment

Status / Gender	College Students	Employees
Male	5	3
Female	4	8

If one of the residents is chosen at random, find the probability that the chosen resident will be (i) a female (ii) a college student (iii) a female student (iv) a male employee

14. The following table shows the results of a survey of thousand customers who bought a new or used cars of a certain model

Type \ Satisfaction level	Satisfied	Not Satisfied
New	300	100
Used	450	150

If a customer is selected at random, what is the probability that the customer

(i) bought a new car (ii) was satisfied (iii) bought an used car but not satisfied

15. A randomly selected sample of 1,000 individuals were asked whether they were planning to buy a new cellphone in the next 12 months. A year later the same persons were interviewed again to find out whether they actually bought a new cellphone. The response of both interviews is given below

	Buyers	Non-buyers
Plan to buy	200	50
No plan to buy	100	650

If a person was selected at random, what is the probability that he/she (i) had a plan to buy

(ii) had a plan to buy but a non-buyer (iii) had no plan to buy but a buyer.

16. The survey has been undertaken to determine whether there is a relationship between the place of residence and ownership of an automobile. A random sample of car owners, 200 from large cities, 150 from suburbs and 150 from rural areas were selected and tabulated as follow

Car ownership \ Type of Area	Large city	Suburb	Rural
Own a foreign car	90	60	25
Do not own a foreign car	110	90	125

If a car owner was selected at random, what is the probability that he/she

(i) owns a foreign car.

(ii) owns a foreign car and lives in a suburb.

(iii) lives in a large city and does not own a foreign car.

(iv) lives in large city and owns a foreign car.

(v) neither lives in a rural area nor owns a foreign car.

17. The educational qualifications of 100 teachers of a Government higher secondary school are tabulated below

Education / Age	M.Phil	Master Degree Only	Bachelor Degree Only
below 30	5	10	10
30 - 40	15	20	15
above 40	5	5	15

If a teacher is selected at random what is the probability that the chosen teacher has (i) master degree only (ii) M.Phil and age below 30 (iii) only a bachelor degree and age above 40 (iv) only a master degree and in age 30-40 (v) M.Phil and age above 40

18. A random sample of 1,000 men was selected and each individual was asked to indicate his age and his favorite sport. The results were as follows.

Sports / Age	Volleyball	Basket ball	Hockey	Football
Below 20	26	47	41	36
20 - 29	38	84	80	48
30 - 39	72	68	38	22
40 - 49	96	48	30	26
50 and above	134	44	18	4

If a respondent is selected at random, what is the probability that

(i) he prefers Volleyball (ii) he is between 20 - 29 years old

(iii) he is between 20 and 29 years old and prefers Basketball

(iv) he doesn't prefer Hockey (v) he is at most 49 of age and prefers Football.

19. On one Sunday Muhil observed the vehicles at a Tollgate in the NH-45 for his science project about air pollution from 7 am. to 7 pm. The number of vehicles crossed are tabulated below.

Time interval / Vehicles	7 a.m. to 11 a.m.	11 a.m. to 3 p.m.	3 p.m. to 7 p.m.
Bus	300	120	400
Car	200	130	250
Two Wheeler	500	250	350

A vehicle is selected at random. Find the probability that the vehicle chosen is a

(i) a bus at the time interval 7 a.m. to 11 a.m. (ii) a car at the time interval 11 a.m. to 7 p.m.

(iii) a bus at the time interval 7 a.m. to 3 p.m. (iv) a car at the time interval 7 a.m. to 7 p. m.

(v) not a two wheeler at the time interval 7 a.m. to 7 p.m.

Points to remember

- Uncertainty or probability can be measured numerically.
- Experiment is defined as a process whose result is well defined.
- Deterministic Experiment : It is an experiment whose outcomes can be predicted with certainty, under identical conditions.
- Random Experiment is an experiment whose all possible outcomes are known, but it is not possible to predict the exact outcome in advance.
- A trial is an action which results in one or several outcomes.
- A sample space S is a set of all possible outcomes of a random experiment.
- Each outcome of an experiment is called a sample point.
- Any subset of a sample space is called an event.
- Classification of probability
  (1) Subjective probability (2) Classical probability (3) Empirical probability
- The empirical probability of happening of an event E, denoted by P(E), is given by

$$\mathrm{P}(E) = \frac{\textit{Number of trials in which the event happened}}{\textit{Total number of trials}}$$

$$\text{(or)}\ \ \mathrm{P(E)} = \frac{\textit{Number of favourable observations}}{\textit{Total number of observations}} \qquad \text{(or)}\ \ \mathrm{P}(E) = \frac{m}{n}$$

- $0 \le P(E) \le 1$
- $P(E') = 1 - P(E)$, where E' is the complementary event of E.

MULTIPLE CHOICE QUESTIONS

1. Theory of Sets

1. If $A = \{5, \{5, 6\}, 7\}$, which of the following is correct?

   (A) $\{5,6\} \in A$ (B) $\{5\} \in A$ (C) $\{7\} \in A$ (D) $\{6\} \in A$

2. If $X = \{a, \{b, c\}, d\}$, which of the following is a subset of X?

   (A) $\{a, b\}$ (B) $\{b, c\}$ (C) $\{c, d\}$ (D) $\{a, d\}$

3. Which of the following statements are true?

   (i) For any set *A*, *A* is a proper subset of *A*

   (ii) For any set *A*, $\varnothing$ is a subset of *A*

   (iii) For any set *A*, *A* is a subset of *A*

   (A) (i) and (ii) (B) (ii) and (iii) (C) (i) and (iii) (D) (i) (ii) and (iii)

4. If a finite set A has *m* elements, then the number of non-empty proper subsets of *A* is

   (A) 2^m (B) $2^m - 1$ (C) 2^{m-1} (D) $2(2^{m-1} - 1)$

5. The number of subsets of the set $\{10, 11, 12\}$ is

   (A) 3 (B) 8 (C) 6 (D) 7

6. Which one of the following is correct?

   (A) $\{x : x^2 = -1, x \in \mathbb{Z}\} = \varnothing$ (B) $\varnothing = 0$

   (C) $\varnothing = \{0\}$ (D) $\varnothing = \{\varnothing\}$

7. Which one of the following is incorrect?

   (A) Every subset of a finite set is finite

   (B) $P = \{x : x - 8 = -8\}$ is a singleton set

   (C) Every set has a proper subset

   (D) Every non - empty set has at least two subsets, $\varnothing$ and the set itself

8. Which of the following is a correct statement?

   (A) $\varnothing \subseteq \{a, b\}$ (B) $\varnothing \in \{a, b\}$ (C) $\{a\} \in \{a, b\}$ (D) $a \subseteq \{a, b\}$

9. Which one of the following is a finite set?

   (A) $\{x : x \in \mathbb{Z}, x < 5\}$ (B) $\{x : x \in \mathbb{W}, x \geq 5\}$

   (C) $\{x : x \in \mathbb{N}, x > 10\}$ (D) $\{x : x \text{ is an even prime number}\}$

10. Given $A = \{5,6,7,8\}$. Which one of the following is incorrect?

(A) $\varnothing \subseteq A$ (B) $A \subseteq A$ (C) $\{7,8,9\} \subseteq A$ (D) $\{5\} \subset A$

11. If $A = \{3,4,5,6\}$andB $= \{1,2,5,6\}$, then $A \cup B =$

(A) $\{1,2,3,4,5,6\}$ (B) $\{1,2,3,4,6\}$ (C) $\{1,2,5,6\}$ (D) $\{3,4,5,6\}$

12. The number of elements of the set $\{x : x \in \mathbb{Z}, x^2 = 1\}$ is

(A) 3 (B) 2 (C) 1 (D) 0

13. If $n(X) = m,\ n(Y) = n$ and $n(X \cap Y) = p$ then $n(X \cup Y) =$

(A) $m + n + p$ (B) $m + n - p$ (C) $m - p$ (D) $m - n + p$

14. If $U = \{1,2,3,4,5,6,7,8,9,10\}$ and $A = \{2,5,6,9,10\}$ then A' is

(A) $\{2,5,6,9,10\}$ (B) $\varnothing$ (C) $\{1,3,5,10\}$ (D) $\{1,3,4,7,8\}$

15. If $A \subseteq B$, then $A - B$ is

(A) B (B) A (C) $\varnothing$ (D) $B - A$

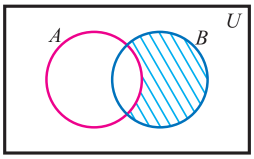
16. If A is a proper subset of B, then $A \cap B =$

(A) A (B) B (C) $\varnothing$ (D) $A \cup B$

17. If A is a proper subset of B, then $A \cup B$

(A) A (B) $\varnothing$ (C) B (D) $A \cap B$

18. The shaded region in the adjoint diagram represents

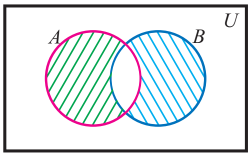


(A) $A - B$ (B) A' (C) B' (D) $B - A$

19. If $A = \{a,b,c\},\ B = \{e,f,g\}$, then $A \cap B =$

(A) $\varnothing$ (B) A (C) B (D) $A \cup B$

20. The shaded region in the adjoining diagram represents



(A) $A - B$ (B) $B - A$ (C) $A \triangle B$ (D) A'

2. Real Numbers

21. A number having non terminating and recurring decimal expansion is

(A) an integer (B) a rational number

(C) an irrational number (D) a whole number

22. If a number has a non-terminating and non-recurring decimal expansion, then it is

(A) a rational number (B) a natural number

(C) an irrational number (D) an integer.

23. Decimal form of $-\frac{3}{4}$ is

(A) – 0.75 (B) – 0.50 (C) – 0.25 (D) – 0.125

24. The $\frac{p}{q}$ form of $0.\bar{3}$ is

(A) $\frac{1}{7}$ (B) $\frac{2}{7}$ (C) $\frac{1}{3}$ (D) $\frac{2}{3}$

25. Which one of the following is not true?

(A) Every natural number is a rational number

(B) Every real number is a rational number

(C) Every whole number is a rational number

(D) Every integer is a rational number.

26. Which one of the following has a terminating decimal expansion?

(A) $\frac{5}{32}$ (B) $\frac{7}{9}$ (C) $\frac{8}{15}$ (D) $\frac{1}{12}$

27. Which one of the following is an irrational number?

(A) π (B) $\sqrt{9}$ (C) $\frac{1}{4}$ (D) $\frac{1}{5}$

28. Which of the following are irrational numbers?

(i) $\sqrt{2+\sqrt{3}}$ (ii) $\sqrt{4+\sqrt{25}}$ (iii) $\sqrt[3]{5+\sqrt{7}}$ (iv) $\sqrt{8-\sqrt[3]{8}}$

(A) (ii),(iii) and (iv) (B) (i),(ii) and (iv)

(C) (i),(ii) and (iii) (D) (i),(iii) and (iv)

29. Which one of the following is not a surd?

(A) $\sqrt[3]{8}$ (B) $\sqrt[3]{30}$ (C) $\sqrt[5]{4}$ (D) $\sqrt[8]{3}$

30. The simplest form of $\sqrt{50}$ is

(A) $5\sqrt{10}$ (B) $5\sqrt{2}$ (C) $10\sqrt{5}$ (D) $25\sqrt{2}$

31. $\sqrt[4]{11}$ is equal to

(A) $\sqrt[8]{11^2}$ (B) $\sqrt[8]{11^4}$ (C) $\sqrt[8]{11^8}$ (D) $\sqrt[8]{11^6}$

32. $\frac{2}{\sqrt{2}}$ is equal to

(A) $2\sqrt{2}$ (B) $\sqrt{2}$ (C) $\frac{\sqrt{2}}{2}$ (D) 2

33. The ratioanlising factor of $\frac{5}{\sqrt[3]{3}}$ is

(A) $\sqrt[3]{6}$ (B) $\sqrt[3]{3}$ (C) $\sqrt[3]{9}$ (D) $\sqrt[3]{27}$

34. Which one of the following is not true?

(A) $\sqrt{2}$ is an irrational number

(B) $\sqrt{17}$ is a irrational number

(C) $0.10110011100011110\cdots$ is an irrational number

(D) $\sqrt[4]{16}$ is an irrational number

35. The order and radicand of the surd $\sqrt[8]{12}$ are respectively

(A) 8,12 (B) 12,8 (C) 16,12 (D) 12,16

36. The surd having radicand 9 and order 3 is

(A) $\sqrt[9]{3}$ (B) $\sqrt[3]{27}$ (C) $\sqrt[3]{9}$ (D) $\sqrt[3]{81}$

37. $5\sqrt[3]{3}$ represents the pure surd

(A) $\sqrt[3]{15}$ (B) $\sqrt[3]{375}$ (C) $\sqrt[3]{75}$ (D) $\sqrt[3]{45}$

38. Which one of the following is not true?

(A) $\sqrt{2}$ is an irrational number

(B) If a is a rational number and $\sqrt{b}$ is an irrational number then $a\sqrt{b}$ is irrational number

(C) Every surd is an irrational number.

(D) The square root of every positive integer is always irrational

39. Which one of the following is not true?

(A) When x is not a perfect square, $\sqrt{x}$ is an irrational number

(B) The index form of $\sqrt[m]{x^n}$ is $x^{\frac{n}{m}}$

(C) The radical form of $\left(x^{\frac{1}{n}}\right)^{\frac{1}{m}}$ is $\sqrt[mn]{x}$

(D) Every real number is an irrational number

40. $(\sqrt{5}-2)(\sqrt{5}+2)$ is equal to

(A) 1 (B) 3 (C) 23 (D) 21

3. Scientific Notations of Real Numbers and Logarithms

41. The scientific notation of 923.4 is

(A) 9.234×10^{-2} (B) 9.234×10^{2} (C) 9.234×10^{3} (D) 9.234×10^{-3}

42. The scientific notation of 0.00036 is

(A) 3.6×10^{-3} (B) 3.6×10^{3} (C) 3.6×10^{-4} (D) 3.6×10^{4}

43. The decimal form of 2.57×10^{3} is

(A) 257 (B) 2570 (C) 25700 (D) 257000

44. The decimal form of 3.506×10^{-2} is

(A) 0.03506 (B) 0.003506 (C) 35.06 (D) 350.6

45. The logarithmic form of $5^2 = 25$ is

(A) $\log_5 2 = 25$ (B) $\log_2 5 = 25$ (C) $\log_5 25 = 2$ (D) $\log_{25} 5 = 2$

46. The exponential form of $\log_2 16 = 4$ is

(A) $2^4 = 16$ (B) $4^2 = 16$ (C) $2^{16} = 4$ (D) $4^{16} = 2$

47. The value of $\log_{\frac{3}{4}}\left(\frac{4}{3}\right)$ is

(A) – 2 (B) 1 (C) 2 (D) –1

48. The value of $\log_{49} 7$ is

(A) 2 (B) $\frac{1}{2}$ (C) $\frac{1}{7}$ (D) 1

49. The value of $\log_{\frac{1}{2}} 4$ is

(A) – 2 (B) 0 (C) $\frac{1}{2}$ (D) 2

50. $\log_{10} 8 + \log_{10} 5 - \log_{10} 4 =$

(A) $\log_{10} 9$ (B) $\log_{10} 36$ (C) 1 (D) –1

4. Algebra

51. The coefficients of x^2 and x in $2x^3 - 3x^2 - 2x + 3$ are respectively

(A) 2,3 (B) –3,–2 (C) –2,–3 (D) 2,–3

52. The degree of the polynomial $4x^2 - 7x^3 + 6x + 1$ is

(A) 2 (B) 1 (C) 3 (D) 0

53. The polynomial $3x - 2$ is a

(A) linear polynomial (B) quadratic polynomial

(C) cubic polynomial (D) constant polynomial

54. The polynomial $4x^2 + 2x - 2$ is a

(A) linear polynomial (B) quadratic polynomial

(C) cubic polynomial (D) constant polynomial

55. The zero of the polynomial $2x - 5$ is

(A) $\frac{5}{2}$ (B) $-\frac{5}{2}$ (C) $\frac{2}{5}$ (D) $-\frac{2}{5}$

56. The root of the polynomial equation $3x - 1 = 0$ is

(A) $x = -\frac{1}{3}$ (B) $x = \frac{1}{3}$ (C) $x = 1$ (D) $x = 3$

57. The roots of the polynomial equation $x^2 + 2x = 0$ are

(A) $x = 0, 2$ (B) $x = 1, 2$ (C) $x = 1, -2$ (D) $x = 0, -2$

58. If a polynomial $p(x)$ is divided by $(ax + b)$, then the remainder is

(A) $p\left(\frac{b}{a}\right)$ (B) $p\left(-\frac{b}{a}\right)$ (C) $p\left(\frac{a}{b}\right)$ (D) $p\left(-\frac{a}{b}\right)$

59. If the polynomial $x^3 - ax^2 + 2x - a$ is divided $(x - a)$, then remainder is

(A) a^3 (B) a^2 (C) a (D) $-a$

60. If $(ax - b)$ is a factor of $p(x)$, then

(A) $p(b) = 0$ (B) $p\left(-\frac{b}{a}\right) = 0$ (C) $p(a) = 0$ (D) $p\left(\frac{b}{a}\right) = 0$

61. One of the factors of $x^2 - 3x - 10$ is

(A) $x - 2$ (B) $x + 5$ (C) $x - 5$ (D) $x - 3$

62. One of the factors of $x^3 - 2x^2 + 2x - 1$ is

(A) $x - 1$ (B) $x + 1$ (C) $x - 2$ (D) $x + 2$

63. The expansion of $(x + 2)(x - 1)$ is

(A) $x^2 - x - 2$ (B) $x^2 + x + 2$ (C) $x^2 + x - 2$ (D) $x^2 - x + 2$

64. The expansion of $(x + 1)(x - 2)(x + 3)$ is

(A) $x^3 + 2x^2 - 5x - 6$ (B) $x^3 - 2x^2 + 5x - 6$

(C) $x^3 + 2x^2 + 5x - 6$ (D) $x^3 + 2x^2 + 5x + 6$

65. $(x-y)(x^2+xy+y^2)$ is equal to

(A) x^3+y^3 (B) x^2+y^2 (C) x^2-y^2 (D) x^3-y^3

66. Factorization of x^2+2x-8 is

(A) $(x+4)(x-2)$ (B) $(x-4)(x+2)$ (C) $(x+4)(x+2)$ (D) $(x-4)(x-2)$

67. If one of the factors of $x^2-6x-16$ is $(x+2)$ then other factor is

(A) $x+5$ (B) $x-5$ (C) $x+8$ (D) $x-8$

68. If $(2x+1)$ and $(x-3)$ are the factors of ax^2-5x+c, then the values of a and c are respectively

(A) 2,3 (B) –2,3 (C) 2,–3 (D) 1,–3

69. If $x+y=10$ and $x-y=2$, then value of x is

(A) 4 (B) –6 (C) –4 (D) 6

70. The solution of $2-x<5$ is

(A) $x>-3$ (B) $x<-3$ (C) $x>3$ (D) $x<3$

5. Coordinate Geometry

71. The point (–2,7) lies is the quadrant

(A) I (B) II (C) III (D) IV

72. The point (x,0) where $x<0$ lies on

(A) OX (B) OY (C) OX' (D) OY'

73. For a point A (a,b) lying in quadrant III

A) $a>0, b<0$ (B) $a<0, b<0$ C) $a>0, b>0$ D) $a<0, b>0$

74. The diagonal of a square formed by the points (1,0) (0,1) (–1,0) and (0,–1) is

A) 2 B) 4 C) $\sqrt{2}$ D) 8

75. The triangle obtained by joining the points A (–5,0) B (5,0) and C (0,6) is

A) an isosceles triangle B) right triangle

C) scalene triangle D) an equilateral triangle

76. The distance between the points (0,8) and (0,–2) is

A) 6 B) 100 C) 36 D) 10

77. (4,1), (–2,1), (7,1) and (10,1) are points

(A) on x axis (B) on a line parallel to x axis

(C) on aline paralled to y axis (D) on y axis

78. The distance between the points (a, b) and (–a, –b) is

(A) 2a (B) 2b (C) 2a + 2b (D) $2\sqrt{a^2 + b^2}$

79. The relation between p and q such that the point (p,q) is equidistant from (–4, 0) and (4,0) is

(A) p = 0 (B) q = 0 (C) p + q = 0 (D) p + q = 8

80. The point which is on y axis with ordinate –5 is

(A) (0, –5) (B) (–5, 0) (C) (5, 0) (D) (0, 5)

6. Trigonometry

81. The value of $\sin^2 60^\circ + \cos^2 60^\circ$ is equal to

(A) $\sin^2 45^\circ + \cos^2 45^\circ$ (B) $\tan^2 45^\circ + \cot^2 45^\circ$

(C) $\sec^2 90^\circ$ (D) 0

82. If $x = \dfrac{2\tan 30^\circ}{1 - \tan^2 30^\circ}$, then the value of x is

(A) $\tan 45^\circ$ (B) $\tan 30^\circ$ (C) $\tan 60^\circ$ (D) $\tan 90^\circ$

83. The value of $\sec^2 45^\circ - \tan^2 45^\circ$ is equal to

(A) $\sin^2 60^\circ - \cos^2 60^\circ$ (B) $\sin^2 45^\circ + \cos^2 60^\circ$

(C)$\sec^2 60^\circ - \tan^2 60^\circ$ (D) 0

84. The value of $2\sin 30^\circ \cos 30^\circ$ is equal to

(A) $\tan 30^\circ$ (B) $\cos 60^\circ$ (C) $\sin 60^\circ$ (D) $\cot 60^\circ$

85. The value of $\operatorname{cosec}^2 60^\circ - 1$ is equal to

(A) $\cos^2 60^\circ$ (B) $\cot^2 60^\circ$ (C) $\sec^2 60^\circ$ (D) $\tan^2 60^\circ$

86. $\cos 60^\circ \cos 30^\circ - \sin 60^\circ \sin 30^\circ$ is equal to

(A) $\cos 90^\circ$ (B) $\operatorname{cosec} 90^\circ$ (C) $\sin 30^\circ + \cos 30^\circ$ (D) $\tan 90^\circ$

87. The value of $\frac{\sin 27^\circ}{\cos 63^\circ}$ is

(A) 0 (B) 1 (C) $\tan 27^\circ$ (D) $\cot 63^\circ$

88. If $\cos x = \sin 43^\circ$, then the value of x is

(A) 57° (B) 43° (C) 47° (D) 90°

89. The value of $\sec 29^\circ - \operatorname{cosec} 61^\circ$ is

(A) 1 (B) 0 (C) $\sec 60^\circ$ (D) $\operatorname{cosec} 29^\circ$

90. If $3x \operatorname{cosec} 36^\circ = \sec 54^\circ$, then the value of x is

(A) 0 (B) 1 (C) $\frac{1}{3}$ (D) $\frac{3}{4}$

91. The value of $\sin 60^\circ \cos 30^\circ + \cos 60^\circ \sin 30^\circ$ is equal to

(A) $\sec 90^\circ$ (B) $\tan 90^\circ$ (C) $\cos 60^\circ$ (D) $\sin 90^\circ$

92. If $\cos A \cos 30^\circ = \frac{\sqrt{3}}{4}$, then the measure of A is

(A) 90° (B) 60° (C) 45° (D) 30°

93. The value of $\tan 26^\circ \cot 64^\circ$ is

(A) $\frac{1}{2}$ (B) $\frac{\sqrt{3}}{2}$ (C) 0 (D) 1

94. The value of $\sin 60^\circ - \cos 30^\circ$ is

(A) 0 (B) $\frac{1}{\sqrt{2}}$ (C) $\frac{\sqrt{3}}{2}$ (D) 1

95. The value of $\cos^2 30^\circ - \sin^2 30^\circ$ is

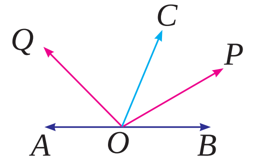
(A) $\cos 60^\circ$ (B) $\sin 60^\circ$ (C) 0 (D) 1

7. Geometry

96. An angle is equal to one third of its supplement its measure is equal to

(A) 40° (B) 50° (C) 45° (D) 55°

97. In the given figure, OP bisect $\angle BOC$ and OQ bisect $\angle AOC$. Then $\angle POQ$ is equal to



(A) 90° (B) 120°

(C) 60° (D) 100°

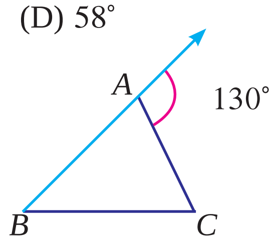
98. The complement of an angle exceeds the angle by 60°. Then the angle is equal to

(A) 25° (B) 30° (C) 15° (D) 35°

99. Find the measure of an angle, if six times its complement is 12° less than twice its supplement.

(A) 48° (B) 96° (C) 24° (D) 58°

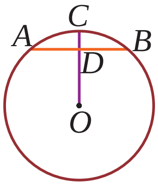
100. In the given figure, $\angle B : \angle C = 2:3$, Find $\angle B$



(A) 120° (B) 52°

(C) 78° (D) 130°

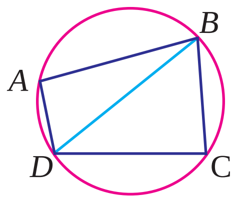
101. O is the centre of the circle. AB is the chord and D is mid-point of AB. If the length of CD is 2cm and the length of chord is12 cm, what is the radius of the circle



(A) 10cm (B) 12cm

(C) 15cm (D) 18cm

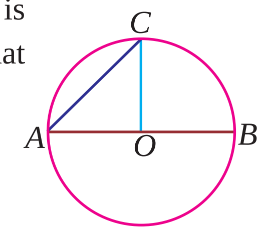
102. $ABCD$ is a cyclic quadrilateral. Given that $\angle ADB + \angle DAB = 120°$ and $\angle ABC + \angle BDA = 145°$. Find the value of $\angle CDB$



(A) 75° (B) 115°

(C) 35° (D) 45°

103. In the given figure, AB is one of the diameters of the circle and OC is perpendicular to it through the center O. If AC is $7\sqrt{2}\,cm$, then what is the area of the circle in cm^2



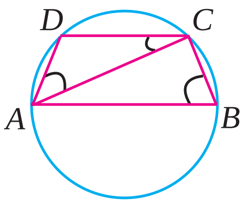
(A) 24.5 (B) 49

(C) 98 (D) 154

104. $ABCD$ is a parllellogram, E is the mid-point of AB and CE bisects $\angle BCD$. Then $\angle DEC$ is

(A) 60° (B) 90° (C) 100° (D) 120°

105. In the given figure, AB is a diameter of the circle and points C and D are on the circumference such that $\angle CAD = 30°$ and $\angle CBA = 70°$ what is the measure of ACD?



(A) 40° (B) 50°

(C) 30° (D) 90°

106. Angle in a semi circle is

(A) obtuse angle (B) right angle (C) an acute angle (D) supplementary

107. Angle in a minor segment is

(A) an acute angle (B) an obtuse angle (C) a right angle (D)a reflexive angle

108. In a cyclic quadrilateral *ABCD*, $\angle A = 5x, \angle C = 4x$ the value of x is

(A) 12° (B) 20° (C) 48° (D) 36°

109. Angle in a major segment is

(A) an acute angle (B) an obtuse angle

(C) a right triangle (D) a reflexive angle

110. If one angle of a cyclic quadrilateral is 70°, then the angle opposite to it is

(A) 20° (B) 110° (C) 140° (D) 160°

8. Mensuration

111. The length of the arc of a sector having central angle 90° and radius 7 cm is

(A) 22 cm (B) 44 cm (C) 11 cm (D) 33 cm

112. If the radius and arc length of a sector are 17 cm and 27 cm respectively, then the perimeter is

(A) 16 cm (B) 61 cm (C) 32 cm (D) 80 cm

113. If the angle subtended by the arc of a sector at the center is 90°, then the area of the sector in square units is

(A) $2\pi r^2$ (B) $4\pi r^2$ (C) $\frac{\pi r^2}{4}$ (D) $\frac{\pi r^2}{2}$

114. Area of a sector having radius 12 cm and arc length 21 cm is

(A) 126 cm^2 (B) 252 cm^2 (C) 33 cm^2 (D) 45 cm^2

115. The area of a sector with radius 4 cm and central angle 60° is

(A) $\frac{2\pi}{3}cm^2$ (B) $\frac{4\pi}{3}cm^2$ (C) $\frac{8\pi}{3}cm^2$ (D) $\frac{16\pi}{3}cm^2$

116. If the area and arc length of the sector of a circle are 60 cm^2 and 20 cm respectively, then the diameter of the circle is

(A) 6 cm (B) 12 cm (C) 24 cm (D) 36 cm

117. The perimeter of a sector of a circle is 37cm. If its radius is 7cm, then its arc length is

(A) 23 cm (B) 5.29 cm (C) 32 cm (D) 259 cm

118. A solid having six equal square faces is called a

(A) cube (B) cuboid (C) square (D) rectangle

119. The quantity of space occupied by a body is its

(A) area (B) length (C) volume (D) T.S.A

120. The LSA of a cube of side 1dm is

(A) 16 dm^2 (B) 4 dm^2 (C) 2 dm^2 (D) 1 dm^2

11. Statistics

121. The mean of the first 10 natural numbers is

(A) 25 (B) 55 (C) 5.5 (D) 2.5

122. The Arithmetic mean of integers from –5 to 5 is

(A) 3 (B) 0 (C) 25 (D) 10

123. If the mean of x, $x + 2$, $x + 4$, $x + 6$, $x + 8$ is 20 then x is

(A) 32 (B) 16 (C) 8 (D) 4

124. The mode of the data 5, 5, 5, 5, 5, 1, 2, 2, 3, 3, 3, 4, 4, 4, 4 is

(A) 2 (B) 3 (C) 4 (D) 5

125. The median of 14, 12, 10, 9, 11 is

(A) 11 (B) 10 (C) 9.5 (D) 10.5

126. The median of 2, 7, 4, 8, 9, 1 is

(A) 4 (B) 6 (C) 5.5 (D) 7

127. The mean of first 5 whole number is

(A) 2 (B) 2.5 (C) 3 (D) 0

128. The Arithmetic mean of 10 number is –7. If 5 is added to every number, then the new Arithmetic mean is

(A) –2 (B) 12 (C) –7 (D) 17

129. the Arithmetic mean of all the factors of 24 is

(A) 8.5 (B) 5.67 (C) 7 (D) 7.5

130. The mean of 5 numbers is 20. If one number is excluded their mean is 15. Then the excluded number is

(A) 5 (B) 40 (C) 20 (D) 10.

12. Probability

131. Probability of sure event is

(A) 1 (B) 0 (C) $\frac{1}{2}$ (D) 2

132. Which one can represent a probability of an event

(A) $\frac{7}{4}$ (B) – 1 (C) $-\frac{2}{3}$ (D) $\frac{2}{3}$

133. Probability of impossible event is

(A) 1 (B) 0 (C) $\frac{1}{2}$ (D) – 1

134. Probability of any event x lies

(A) $0 < x < 1$ (B) $0 \leq x < 1$ (C) $0 \leq x \leq 1$ (D) $1 < x < 2$

135. $P(E')$ is

(A) $1 - P(E)$ (B) $P(E) - 1$ (C) 1 (D) 0

ANSWERS

Exercise 1.1

1. (i) Not a set (ii) Set (iii) Not a set (iv) Set (v) Set

2. (i) $0 \in A$ (ii) $6 \notin A$ (iii) $3 \in A$ (iv) $4 \in A$ (v) $7 \notin A$

3. (i) $\{x : x \text{ is a positive even number}\}$ (ii) $\{x : x \text{ is a whole number and } x < 20\}$

(iii) $\{x : x \text{ is a multiple of } 3\}$ (iv) $\{x : x \text{ is an odd natural number and } x < 15\}$

(v) $\{x : x \text{ is a letter in the word 'TAMILNADU'}\}$

4. (i) $A = \{3, 4, 5, 6, 7, 8, 9, 10\}$ (ii) $B = \{0, 1, 2, 3, 4, 5\}$ (iii) $C = \{2, 3\}$

(iv) $X = \{2, 4, 8, 16, 32\}$ (v) $M = \{-1, 1, 3, 5, 7, 9\}$

(vi) $P = \{-4, -3, -2, -1, 0, 1, 2, 3, 4\}$

5. (i) A = The set of all vowels in the English alphabet

(ii) B = The set of all odd natural numbers less than or equal to 11

(iii) C = The set of all square numbers less than 26.

(iv) P = The set of all letters in the word 'SET THEORY'

(v) Q = The set of all prime numbers between 10 and 20

6. (i) 4 (ii) 21 (iii) 1 (iv) 0 (v) 9 **7.** (i) infinite (ii) finite (iii) infinite (iv) infinite (v) finite

8. (i) equivalent (ii) not equivalent (iii) equivalent

9. (i) equal (ii) not equal (iii) equal (iv) not equal **10.** $B = D$ and $E = G$

11. No, $\varnothing$ contains no element but $\{\varnothing\}$ contains one element,

12. Each one is different from others.

0 is an integer. It is not a set — $\varnothing$ contains no element

$\{0\}$ contains one element i.e., 0. — $\{\varnothing\}$ contains one element, i.e., the null set

13. (i) $\nsubseteq$ (ii) $\subseteq$ (iii) $\subseteq$ (iv) $\nsubseteq$ **14.** (i) X is not a subset of Y (ii) Y is a subset of X

15. A is not a subset of B

16. (i) $P(A) = \{\varnothing, \{x\}, \{y\}, \{x,y\}\}$ (ii) $P(A) = \{\varnothing, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, \{a,b,c\}\}$

(iii) $P(A) = \{\varnothing, \{5\}, \{6\}, \{7\}, \{8\}, \{5,6\}, \{5,7\}, \{5,8\}, \{6,7\}, \{6,8\}, \{7,8\}, \{5,6,7\}, \{5,6,8\}, \{5,7,8\}, \{6,7,8\}, \{5,6,7,8\}\}$

17. (i) 64, 63 (ii) 128, 127 (iii) 2, 1 **18.** (i) (*a*) 1 (*b*) 8 (ii) 9 (iii) 10 **19.** A is the empty set

20. (i) a) $A=\{1,2,3,4,5,6,7,8,9,10\}$ b) $B=\{2,4,6,8,10,12,14,16,18,20\}$

c) $C=\{15,16,17,18,19,20,21,22,23,24,25\}$

(ii) a) $n(A)=10$ b) $n(B)=10$ c) $n(C)=11$ (iii) a) F b) T c) T d) T

Exercise 1.2

1. (i) $A\cup B=\{-3,-1,0,1,2,4,5,6\}$, $A\cap B=\{0,2,4\}$ (ii) $A\cup B=\{2,4,6,8\}$

(iii) $A\cup B=\{1,2,3,4,5,7\}$, $A\cap B=\{2,3,5\}$

(iv) $A\cup B=\{0,1,2,3,4,5,6,7\}$, $A\cap B=\{3,4,5,6\}$

2. (i) $A\cup B=\{1,3,5,7,10,12,15,18,20,25,30\}$ (ii) $A\cap B=\{10,15,25\}$

3. (i) $X\cup Y=\{0,2,4,6,8,1012,14,16,18,20\}$, $X\cup Y=\{4,8,12,16,20\}$

4. (i) $\{7\}$ **5.** (ii) X and Y are disjoint sets

6. (i) $A'=\{0,1,2,4,5,7,8,10\}$ (ii) A is the set of all prime numbers

7. (i) $A\cup B=\{a,b,c,d,f,g\}$ (ii) $(A\cup B)'=\{e,h\}$ (iii) $A\cap B=\{b,d\}$

(iv) $(A\cap B)'=\{a,c,e,f,g,h\}$ **8.** (i) $A'=\{2,4,6,8,10\}$ (ii) $B'=\{1,4,6,7,8\}$

(iii) $A'\cup B'=\{1,2,4,6,8,10\}$ (iv) $A'\cap B'=\{4,6,8\}$

9. (i) $M-N=\{3,9\}$ (ii) $N-M=\{15,17\}$ (iii) $N'-M=\{18\}$ (iv) $M'-N=\{18\}$

(v) $M\cap(M-N)=\{3,9\}$ (vi) $N\cup(N-M)=\{7,11,15,17\}$ (vii) $n(M-N)=2$

10. (i) $A-B=\{3,6,9,15,18\}$ (ii) $B-C=\{16,20\}$ (iii) $C-D=\{2,4,6,8,12\}$

(iv) $D-A=\{5,10,20,25\}$ (v) $n(A-C)=4$

11. (i) (a) $\text{U}=\{1,2,3\cdots 49\}$, (b) $A=\{4,8,12,16,20,24,28,32,36,40,44,48\}$

(c) $B=\{16,30,44\}$ (ii) (a) $A\cup B=\{4,8,12,16,20,24,28,30,32,36,40,44,48\}$

(b) $A\cap B=\{16,44\}$, (c) $n(A\cup B)=13$, (d) $n(A\cap B)=2$

12. (i) $X\triangle Y=\{a,b,d,e,f,k\}$ (ii) $P\triangle Q=\{0,1,2,3,5,6,7,8\}$

(iii) $A\triangle B=\{-4,-2,-1,5\}$

13. (i) (a) $\text{U}=\{1,2,3,4,7,9,10,11\}$, (b) $E=\{1,2,4,7\}$, (c) $F=\{4,7,9,11\}$

(d) $E\cup F=\{1,2,4,7,9,11\}$ (e) $E\cap F=\{4,7\}$

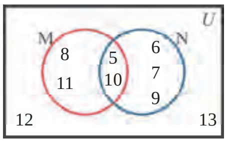
(ii) (a) $n(\cup)=8$, (b) $n(E\cup F)=6$, (c) $n(E\cap F)=2$

14. (i) (a) $\text{U}=\{1,2,3,4,5,6,8,9,10\}$, (b) $G=\{1,2,4,8\}$, (c) $H=\{2,6,8,10\}$

(ii) (a) $G'=\{3,5,6,9,10\}$, (b) $H'=\{1,3,4,5,9\}$, (c) $n(G\cup H)'=3$

(d) $n(G\cap H)'=7$ (e) $G'\cap H'=\{3,5,9\}$

Exercise 1.3

1. 

2. $n(A \cap B) = 15$ **3.** 16, 29 **4.** $n(B) = 27$

5. $n(A \cap B) = 6,\ n(\cup) = 43$ **6.** $n(A \cup B) = 22$ **7.** 150 **8.** 1400

9. (i) 180 (ii) 150 (iii) 450 **10.** 35 **11.** 12 **12.** 12 **13.** 47 **14.** Yes, correct

15. (i) $x = 8$ (ii) $n(A \cup B) = 88$ **16.** (i) 35 (ii) 25 (iii) 20 **17.** 16%

Exercise 2.1

1. (i) True (ii) False (iii) True (iv) False (v) False (vi) False

2. Yes, For $0 = \frac{0}{1} = \frac{0}{2} = \frac{0}{3} = \frac{0}{-1} = \cdots$ **3.** $-\frac{4}{7}, -\frac{3}{7}$

Exercise 2.2

1. (i) 0.42, terminating (ii) $8.\overline{285714}$, nonterminating and recurring

(iii) $0.2\overline{36}$,non-terminating and recurring (iv) 0.918, terminatting

(v) $0.\overline{09}$,non-terminating and recurring

(vi) $-0.\overline{230769}$,non-terminating and recurring

(vii) $6.\overline{3}$, non-terminating and recurring (viii) -0.21875, terminating

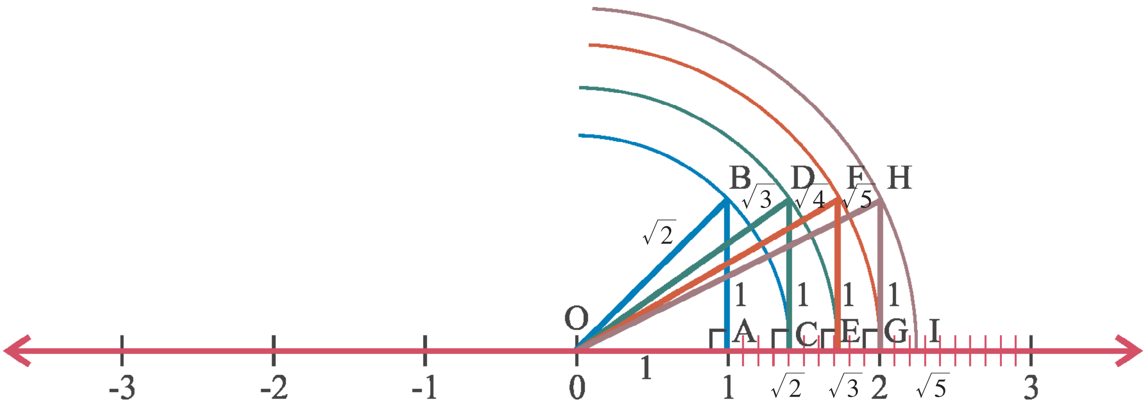
2. (i) terminating (ii) non-terminating (iii) terminating (iv) non-terminating

3. (i) $\frac{2}{11}$, (ii) $\frac{427}{999}$, (iii) $\frac{1}{9999}$, (iv) $\frac{16}{11}$, (v) $\frac{22}{3}$, (vi) $\frac{206}{495}$ **4.** $0.\overline{076923}$, 6

5. $\frac{1}{7} = 0.\overline{142857},\ \frac{2}{7} = 0.\overline{285714},\ \frac{3}{7} = 0.\overline{428571},\ \frac{4}{7} = 0.\overline{571428},$

$\frac{5}{7} = 0.\overline{714285},\ \frac{6}{7} = 0.\overline{857142}$

Exercise 2.3

1. 

2. 1.83205$\cdots$, 1.93205$\cdots$, 2.03205$\cdots$

3. 3.10110011100011110$\cdots$, 3.2022002220002222$\cdots$

4. 0.1510100110001110$\cdots$, 0.1530300330003330$\cdots$

5. 0.58088008880$\cdots$, 0.59099009990$\cdots$

6. 1.83205$\cdots$, 1.93205$\cdots$

7. One rational number : 1.102, An irrational number : 1.9199119991119$\cdots$

8. 0.13, 0.20 [Note: Questions from 2 to 8 will have infinitely many solutions]

Exercise 2.5

1. (i) Surd (ii) Surd (iii) not a surd (iv) Surd (v) not a surd

2. (i) $20+10\sqrt{5}+2\sqrt{3}+\sqrt{15}$ (ii) $8+2\sqrt{15}$ (iii) 11 (iv) 61

3. (i) $71\sqrt{3}$ (ii) $16\sqrt[3]{2}$ (iii) $-37\sqrt{2}$ (iv) $3\sqrt[3]{5}$ **4.** (i) $3\sqrt[3]{4}$ (ii) $7\sqrt{2}$ (iii) $8\sqrt{3}$ (iv) 5

5. (i) $\sqrt{180}$ (ii) $\sqrt[3]{500}$ (iii) $\sqrt[4]{405}$ (iv) $\sqrt{\frac{9}{2}}$ **6.** (i) $3\sqrt{10}$ (ii) $2\sqrt[3]{7}$ (iii) $2\sqrt[4]{6}$

(iv) $\sqrt[6]{45}$ (v) $\frac{3}{2}\sqrt{5}$ (vi) $\sqrt[8]{32}$ **7.** (i) $\sqrt[3]{3}>\sqrt{2}$ (ii) $\sqrt[3]{3}>\sqrt[4]{4}$ (iii) $\sqrt[4]{10}>\sqrt{3}$

8. (i) descending order : $\sqrt{3},\ \sqrt[3]{4},\ \sqrt[4]{5}$, ascending order : $\sqrt[4]{5},\ \sqrt[3]{4},\ \sqrt{3}$

(ii) descending order : $\sqrt[3]{4},\ \sqrt[4]{4},\ \sqrt[3]{2}$, ascending order : $\sqrt[3]{2},\ \sqrt[4]{4},\ \sqrt[3]{4}$

(iii) descending order : $\sqrt[3]{2},\ \sqrt[6]{3},\ \sqrt[9]{4}$, ascending order : $\sqrt[9]{4},\ \sqrt[6]{3},\ \sqrt[3]{2}$

Exercise 2.6

1. (i) $\sqrt{2}$ (ii) $\sqrt{7}$ (iii) $\sqrt{3}$ (iv) $\sqrt[3]{25}$ (v) $5+4\sqrt{3}$ (vi) $\sqrt{2}-\sqrt{3}$ (vii) $\sqrt{5}+\sqrt{2}$

(viii) $2-\sqrt{3}$ **2.** (i) $\frac{3\sqrt{5}}{5}$ (ii) $\frac{2\sqrt{3}}{9}$ (iii) $\frac{\sqrt{3}}{6}$ (iv) $\frac{2\sqrt{77}}{11}$ (v) $\sqrt[3]{15}$

3. (i) $\frac{11-\sqrt{3}}{118}$ (ii) $\frac{3-\sqrt{5}}{12}$ (iii) $\frac{\sqrt{13}-\sqrt{11}}{2}$ (iv) $\frac{3+\sqrt{5}}{2}$ (v) $\frac{17\sqrt{3}-21}{71}$

4. (i) 0.707 (ii) 3.464 (iii) 1.887 (iv) 0.655 (v) 0.102 (vi) 4.441

(vii) 3.732 (viii) 0.185 **5.** $a=\frac{31}{19},\ b=\frac{10}{19}$ **6.** $a=7,\ b=4$

7. $a=3,\ b=0$ **8.** $a=0,\ b=\frac{16}{11}$ **9.** 14 **10.** 4

Exercise 2.7

1. 3, 1 **2.** 0, 5 **3.** 9, 0

Exercise 3.1

1. (i) 7.493×10^{11} (ii) 1.3×10^{7} (iii) 1.05003×10^{5} (iv) 5.436×10^{14}
(v) 9.6×10^{-3} (vi) 1.3307×10^{-6} (vii) 2.2×10^{-9} (viii) 9.0×10^{-13}

2. (i) 0.00000325 (ii) 0.0004134 (iii) 41340 (iv) 18600000
(v) 9870000000 (vi) 0.000000001432

3. (i) 6.4×10^{13} (ii) 3.375×10^{1} (iii) 2.56×10^{3} (iv) 6.9984×10^{-28} (v) 3.993×10^{2}

Exercise 3.2

1. (i) True (ii) False (iii) False (iv) False (v) True (vi) False

2. (i) $\log_2 16 = 4$ (ii) $\log_3 243 = 5$ (iii) $\log_{10} 0.1 = -1$ (iv) $\log_8\left(\frac{1}{4}\right) = -\frac{2}{3}$
(v) $\log_{25} 5 = \frac{1}{2}$ (vi) $\log_{12}\left(\frac{1}{144}\right) = -2$

3. (i) $6^3 = 216$ (ii) $9^{\frac{1}{2}} = 3$ (iii) $5^0 = 1$ (iv) $(\sqrt{3})^4 = 9$ (v) $(64)^{-\frac{1}{2}} = \frac{1}{8}$ (vi) $(.5)^{-3} = 8$

4. (i) -4 (ii) 3 (iii) 5 (iv) -3 (v) -4 (vi) 5

5. (i) $x = \sqrt{2}$ (ii) $x = \frac{1}{125}$ (iii) $y = \frac{1}{9}$ (iv) $x = \sqrt{5}$ (v) $x = 10$ (vi) $x = -\frac{4}{3}$

6. (i) $\log_{10} 9$ (ii) $\log_{25}\left(\frac{7}{2}\right)$ (iii) 2 (iv) 2 (v) $\log_{10}\left(\frac{72}{25}\right)$ (vi) 1

7. (i) $x = -2$ (ii) $x = 2$ (iii) $x = 2$ (iv) $x = 4$ (v) $x = 3$ (vi) $x = 5$ (vii) $x = 5$
(viii) $x = 7$ **8.** (i) $y + z$ (ii) $3x$ (iii) $x + y + z$ (iv) $3(y - z)$ (v) $x - y + z$ (vi) $y - x$

Exercise 3.3

1. (i) 9.243×10^{1} (ii) 9.243×10^{-1} (iii) 9.243×10^{3} (iv) 9.243×10^{5} (v) 9.243×10^{-3}
(vi) 9.243×10^{-2} **2.** (i) 3 (ii) 1 (iii) –3 (iv) –2 (v) –1 (vi) 0

3. (i) 4.3576 (ii) 1.3576 (iii) 0.3576 (iv) $\bar{1}.3576$ (v) 7.3576 (vi) $\bar{5}.3576$

4. (i) 1.3649 (ii) 0.9694 (iii) 2.5179 (iv) $\bar{3}.1348$ (v) $\bar{1}.9946$ (vi) 3.8180

5. (i) 1180 (ii) 57.41 (iii) 0.2413 (iv) 0.004015(v) 1.876 (vi) 0.01513

6. (i) 30550 (ii) 21.82 (iii) 0.05309 (iv) 3.497 (v) 328100000 (vi) 8.249
(vii) 2.122 (viii) 1.666 (ix) 0.08366 (x) 0.5948 (xi) 1.888 (xii) 1.772

Exercise 4.1

1. (i) Polynomial in one variable (ii) Polynomial in one variable.
(iii) Polynomial in one variable
(iv) Since the exponent of x is not a whole number is not a polynomial.
(v) Since the exponent of t is not a whole number is not a polynomial.
(vi) Polynomial in three variable

2. (i) –4, 3 (ii) $0, \sqrt{3}$ (iii) $\sqrt{2}, 4$ (iv) $\frac{1}{3}, 1$ **3.** (i) 2 (ii) 1 (iii) 3 (iv) 0

4. (i) quadratic polynomial (ii) cubic polynomial (iii) linear polynomial

(iv) quadratic polynomial (v) cubic polynomial (vi) linear polynomial

5. $ax^{27} + b,\ \ cx^{49},\ \ lx^{36} + mx^{35} + nx^2$

Exercise 4.2

1. (i) $x = \frac{1}{4}$ (ii) $x = -\frac{5}{3}$ (iii) $x = 0$ (iv) $x = -9$

2. (i) $x = 3$ (ii) $x = \frac{6}{5}$ (iii) $x = -\frac{1}{11}$ (iv) $x = 0$

3. (i) $x = 2$ is a root, $x = 3$ is a root (ii) $x = -1$ is a root, $x = 2$ is not a root

(iii) $x = 1$ is a root, $x = -2$ is a root, $x = 3$ is a root

(iv) $x = -1$ is a root, $x = 2$ is a root, $x = 3$ is not a root

Exercise 4.3

1. (i) 10 (ii) –8 (iii) 20 (iv) –145 (v) –2 (vi) 26 (vii) $-3a$

2. $a = 5$ **3.** $m = 13$ **4.** $m = 3$ **5.** $m = 5$, remainder is 15.

Exercise 4.4

1. (i) Factor (ii) Factor (iii) Not a factor (iv) Not a factor

2. Not a factor **4.** Factor **5.** $p = 10$

Exercise 4.5

1. (i) $25x^2 + 4y^2 + 9y^2 + 20xy + 12yz + 30zx$ (ii) $4a^2 + 9b^2 + c^2 + 12ab - 6bc - 4ca$

(iii) $x^2 + 4y^2 + 16z^2 - 4xy + 16yz - 8zx$ (iv) $p^2 + 4q^2 + r^2 - 4pq - 4qr + 2rp$

2. (i) $x^3 + 12x^2 + 39x + 28$ (ii) $p^3 + 4p^2 - 20p - 48$ (iii) $x^3 + x^2 - 17x + 15$

(iv) $x^3 - 7ax^2 + 14a^2x - 8a^3$ (v) $27x^3 + 72x^2 + 51x + 10$ (vi) $8x^3 - 36x^2 - 2x + 105$

3. (i) 19, 111, 189 (ii) –7, 2, 40 (iii) 60, 142, 105 (iv) –100, –5, 6 **4.** –10, –3, 10

5. (i) $27a^3 + 135a^2b + 225ab^2 + 125b^3$ (ii) $64x^3 - 144x^2y + 108xy^2 - 27y^3$

(iii) $8y^3 - 36y + \frac{54}{y} - \frac{27}{y^3}$ **6.** (i) 970299 (ii) 1030301 (iii) 941192 (iv) 1061208

(v) 1006012008 **7.** 793 **8.** –288 **9.** 52 **10.** 36

11. (i) $8x^3 + y^3 + 64z^3 - 24xyz$ (ii) $x^3 - 27y^3 - 125z^3 - 45xyz$ **12.** (i) –486 (ii) 2880

Exercise 4.6

1. (i) $a^2(2a - 3b + 2c)$ (ii) $16x(1 + 4xy)$ (iii) $5x^3(2 - 5xy)$

(iv) $(y - z)(x + a)$ (v) $(p + q)(p + r)$

2. (i) $(x+1)^2$ (ii) $(3x-4y)^2$ (iii) $(b+2)(b-2)$ (iv) $(1+6x)(1-6x)$

3. (i) $(p+q+r)^2$ (ii) $(a-2b-6)^2$ (iii) $(3x-y+1)^2$

(iv) $(2a-b+3c)^2$ (v) $(5x-2y-3z)^2$

4. (i) $(3x+4y)(9x^2-12xy+16y^2)$ (ii) $(m+2)(m^2-2m+4)$

(iii) $(a+5)(a^2-5a+25)$ (iv) $(2x-3y)(4x^2+6xy+9y^2)$

(v) $(x-2y)(x^2+2xy+4y^2)$

Exercise 4.7

1. (i) $(x+1)(x+14)$ (ii) $(x+3)(x+10)$ (iii) $(y+3)(y+4)$

(iv) $(x-2)(x-12)$ (v) $(y-6)(y-10)$ (vi) $(t-8)(t-9)$

(vii) $(x-1)(x+15)$ (viii) $(x-2)(x+11)$ (ix) $(y-4)(y+9)$

(x) $(x+9)(x-11)$ (xi) $(m+8)(m-18)$ (xii) $(y+4)(y-5)$

2. (i) $(3x+1)(x+6)$ (ii) $(5x+2)(x+4)$ (iii) $(x+2)(2x+5)$

(iv) $(14x+3)(x+2)$ (v) $(5y-4)(y-5)$ (vi) $(9y-7)(y-1)$

(vii) $(3x-1)(2x-1)$ (viii) $(3x-4)(x-2)$ (ix) $(3x-1)(x+2)$

(x) $(2a-3)(a+10)$ (xi) $(x+1)(11-6x)$ (xii) $(8x-3)(x+4)$

(xiii) $(x+2)(2x-7)$ (xiv) $(9x+4)(2x-1)$ (xv) $(1-x)(3x+10)$

3. (i) $(a+b+2)(a+b+7)$ (ii) $(p-q+2)(p-q-9)$

4. (i) $(x+1)(x-1)(x+2)$ (ii) $(x+1)(x-1)(x-3)$

(iii) $(x+1)(x+2)(x-2)$ (vi) $(x+1)(x-1)(x+5)$

Exercise 4.8

1. (i) $x=1, y=3$ (ii) $x=2, y=-3$ (iii) $x=3, y=2$ (iv) $x=\frac{1}{5}, y=\frac{1}{2}$

(v) $x=\frac{1}{2}, y=1$ **2.** 16, 8 **3.** 27 **4.** 50, 22

5. (i) $x>4$ (ii) $x<3.5$ (iii) $x\le -2.5$ (iv) $x\ge -2$

Exercise 5.1

1. (i) False (ii) True (iii) True (iv) False (v) True (vi) False (vii) True (viii) True

(ix) False (x) True **2.** (i) I (ii) III (iii) on *x* axis (iv) III (v) on *y* axis

(vi) on y axis (vii) IV (viii) origin (ix) I (x) II **3.** (i) –7 (ii) 3 (iii) 8 (iv) –5

4. (i) 5 (ii) 9 (iii) 8 (iv) –4 **5.** parallel to *y* axis **6.** parallel to *x* axis **7.** *y* axis

8. *ABCD* is a rectangle **9.** (0,4) **11.** (4,3)

Exercise 5.2

1. (i) $\sqrt{202}$ (ii) $4\sqrt{5}$ (iii) $\sqrt{29}$ (iv) $2\sqrt{2}$ (v) $5\sqrt{2}$ (vi) 1 (vii) 5

(viii) 15 (ix) 18 (x) $\sqrt{74}$ **10.** 7, –5 **13.** –10, –2 **14.** (i) 24 (ii) $10 + 4\sqrt{10}$

15. (0,–7) **16.** $4\sqrt{5}$ **18.** (4,–3) **19.** 30 **20.** No, collinear points

21. (8,–15) (–8,–15) (–8,15)(8, 15) **24.**11, 7 **25.** 20

Exercise 6.1

1. (i) $\sin\theta = \frac{6}{10}$, $\cos\theta = \frac{8}{10}$, $\tan\theta = \frac{6}{8}$, $\operatorname{cosec}\theta = \frac{10}{6}$, $\sec\theta = \frac{10}{8}$, $\cot\theta = \frac{8}{6}$

(ii) $\sin\theta = \frac{7}{25}$, $\cos\theta = \frac{24}{25}$, $\tan\theta = \frac{7}{24}$, $\operatorname{cosec}\theta = \frac{25}{7}$, $\sec\theta = \frac{25}{24}$, $\cot\theta = \frac{24}{7}$

(iii) $\sin\theta = \frac{35}{37}$, $\cos\theta = \frac{12}{37}$, $\tan\theta = \frac{35}{12}$, $\operatorname{cosec}\theta = \frac{37}{35}$, $\sec\theta = \frac{37}{12}$, $\cot\theta = \frac{12}{35}$

(iv) $\sin\theta = \frac{9}{41}$, $\cos\theta = \frac{40}{41}$, $\tan\theta = \frac{9}{40}$, $\operatorname{cosec}\theta = \frac{41}{9}$, $\sec\theta = \frac{41}{40}$, $\cot\theta = \frac{40}{9}$

2. (i) $\cos A = \frac{12}{15}$, $\tan A = \frac{9}{12}$, $\operatorname{cosec} A = \frac{15}{9}$, $\sec A = \frac{15}{12}$, $\cot A = \frac{12}{9}$

(ii) $\sin A = \frac{8}{17}$, $\tan A = \frac{8}{15}$, $\operatorname{cosec} A = \frac{17}{8}$, $\sec A = \frac{17}{15}$, $\cot A = \frac{15}{8}$

(iii) $\sin P = \frac{5}{13}$, $\cos P = \frac{12}{13}$, $\operatorname{cosec} P = \frac{13}{5}$, $\sec P = \frac{13}{12}$, $\cot P = \frac{12}{5}$

(iv) $\sin\theta = \frac{15}{17}$, $\cos\theta = \frac{8}{17}$, $\tan\theta = \frac{15}{8}$, $\operatorname{cosec}\theta = \frac{17}{15}$, $\cot\theta = \frac{8}{15}$

(v) $\sin\theta = \frac{60}{61}$, $\cos\theta = \frac{11}{61}$, $\tan\theta = \frac{60}{11}$, $\sec\theta = \frac{61}{11}$, $\cot\theta = \frac{11}{60}$

(vi) $\cos\theta = \frac{\sqrt{y^2 - x^2}}{y}$, $\tan\theta = \frac{x}{\sqrt{y^2 - x^2}}$, $\operatorname{cosec}\theta = \frac{y}{x}$,

$\sec\theta = \frac{y}{\sqrt{y^2 - x^2}}$, $\cot\theta = \frac{\sqrt{y^2 - x^2}}{x}$

3. (i) 45° (ii) 0° (iii) 60° (iv) 30°

4. $\sin A = \frac{24}{26}$, $\cos A = \frac{10}{26}$, $\tan A = \frac{24}{10}$, $\operatorname{cosec} A = \frac{26}{24}$, $\sec A = \frac{26}{10}$, $\cot A = \frac{10}{24}$

$\sin C = \frac{10}{26}$, $\cos C = \frac{24}{26}$, $\tan C = \frac{10}{24}$, $\operatorname{cosec} C = \frac{26}{10}$, $\sec C = \frac{26}{24}$, $\cot C = \frac{24}{10}$

5. $\frac{17}{19}$ **6.** 1 **7.** $\frac{-63}{4}$ **8.** 1 **9.** $\frac{225}{64}$ **10.** (i) 1 (ii) 0

13. (i) $\sqrt{2}$ (ii) $\frac{1}{2}$ (iii) $\frac{\sqrt{3}}{4}$ (iv) $\frac{25}{144}$ (v) 7 (vi) $\frac{4}{3}$ (vii) 9 (viii) 2

Exercise 6.2

1. (i) 1 (ii) 1 (iii) 1 (iv) 1 (v) 1 (vi) 1

2. (i) 0 (ii) 2 (iii) 0 (iv) 6 (v) 1 (vi) 9 (vii) 0 (viii) $\frac{3}{2}$ (ix) $\frac{1}{\sqrt{3}}$

3. (i) 60° (ii) 41° (iii) 55° (iv) 55° (v) 47° (vi) 60°

Exercise 6.3

1. (i) 0.4384 (ii) 0.3090 (iii) 0.7002 (iv) 0.9670 (v) 0.2113 (vi) 0.9760

(vii) 0.7623 (viii) 0.1841 (ix) 2.7475 (x) 1.1778 **2.** (i) $44^\circ 30'$ (ii) $14^\circ 54'$

(iii) $20^\circ 12'$ (iv) $76^\circ 30'$ (v) $89^\circ 6'$ **3.** (i) 1.2698 (ii) 1.3579 (iii) 1.0042

(iv) 4.4996 (v) 4.8098 **4.** 99.4134 cm^2 **5.** 14.6278 cm^2 **6.** 109.376 cm^2

7. 67.0389 cm^2 **8.** 13.8568 m **9.** 60° **10.** 8.09 cm **11.** 3.1056 cm **12.** 20.784 cm

Exercise 7.1

1. (i) 27° (ii) 66° (iii) 42° (iv) 55° (v) 70° **2.** (i) 122° (ii) 32° (iii) 60° (iv) 140

(v) 80° **3.** (i) 80° (ii) 35° **4.** (i) 30° (ii) 36° (iii) 60° (iv) 72° (v) $80^\circ, 100^\circ$

(vi) $54^\circ, 36^\circ$ **5.** (i) 36° (ii) 40° (iii) $40^\circ, 50^\circ$

6. (i) $\angle A = \angle C = \angle E = \angle G = 115^\circ$, $\angle B = \angle D = \angle H = 65^\circ$ **7.** (i) 30° (ii) 32°

8. $30^\circ, 60^\circ, 90^\circ$ **9.** $45^\circ, 25^\circ, 110^\circ$ **10.** $80^\circ, 60^\circ$

Exercise 7.2

1. $48^\circ, 72^\circ, 96^\circ, 144^\circ$ **2.** $72^\circ, 108^\circ, 72^\circ$ **3.** (i) 45° (ii) 45° (iii) 45° (iv) 60°

4. 70°, 110°, 70°, 110° **5.** $l = 9, b = 6$ **6.** 15

7. (i) 50°, 50° (ii) 31°, 59° (iii) 30°, 30° **8.** 12

Exercise 7.3

1. 12cm **2.** 15cm **3.** 26cm **4.** 30cm **5.** 40cm **6.** 1cm **7.** 13cm **8.** 7cm

9. (i) 75° (ii) 55° (iii) 110° (iv) 115° (v) 40° (vi) 42° **10.** (i) 55°, (ii) 43°

11. (i) 35° (ii) 40° (iii) 30° **12.** (i) 100° (ii) 30° **13.** (i) 80° (ii) 80° (iii) 100

14. 50° **15.** (i) 50° (ii) 130°

Exercise 8.1

1. (i) 22 cm, 231 cm^2, 64 cm (ii) 2.57 cm, 6.3 cm, 12.37 cm (iii) 11 cm, 77 cm^2, 39 cm

(iv) 16.5 cm, 123.75 cm^2, 46.5 cm (v) 88 dm, 924 dm^2, 130 dm

2. (i) 120° (ii) 90° (iii) 36 cm **3.** (i) 165 cm, 53 cm (ii) 2200 cm^2, 190 cm

(iii) 91.5 cm^2, 39.25cm (iv) 250 cm^2,65 cm **4.** (i) 10 cm (ii) 30 cm (iii) 6 cm

5. (i) 110.25 cm^2 (ii) 700 cm^2 **6.** (i) 280° (ii) 120° **7.** (i) 72 cm, 308cm^2 (ii) 25m, 38.5 m^2

8. (i) 19 cm, (ii) 8.5 cm **9.** (i) 7 hrs, (ii) 2 hrs, (iii) 15 hrs **10.** 0.16 cm^2

11. (i) 154 m^2, (ii) 350 m^2, **12.** 123.84 cm^2 **13.** 346.5 cm^2 **14.** 4.2 m, 60°,12.8 m

Exercise 8.2

1. (i) 125.44 cm^2, 188.16 cm^2, 175.62 cm^3, (ii) 144 dm^2, 216 dm^2, 216 dm^3

(iii) 25 m^2, 37.5 m^2, 15.625 m^3, (iv) 2304 cm^2,3456 cm^2, 13824 cm^3

(v) 3844 cm^2,5766 cm^2, 29791 cm^3 **2.** (i) 15 cm (ii) 13 cm (iii) 5 dm

3. 8000 $cm^{3,}$ **4.** 4 m **5.** 216 cm^2 **6.** 125 cubes **7.** 8000 cm^2, 64000 cm^3

8. Rs. 4,00,000 **9.** 147 m^2, Rs. 11,025

Exercise 8.3

1. (i) 154 cm^2, 174 cm^2, 110 cm^3 (ii) 400 dm^2, 700 dm^2, 1200 dm^3

(iii) 70 m^2, 82 m^2, 42 m^3 (iv) 512 m^2,992 m^2, 1920 m^3

2. 27 cm **3.** 96 cm^2, 160 cm^2 **4.** Rs. 123 **5.** 720 m^2 , Rs.56,160,

6. 4000 hollow blocks, **7.** Rs. 53,280

Exercise 10.2

1. (2, 6) **2.** many solutions **3.** (2, 3) **4.** (3, 2) **5.** no solution

6. (2, 3) **7.** many solutions **8.** (1, 3) **9.** (1, 0) **11.** (–3, –3)

11. (2, –12) **12.** no solution

Exercise 11.1

1. 4, 7, 6, 6, 5 **2.** 25, 30, 25, 60, 15, 5 **3.** 3, 4, 6, 6, 8 **4.** 5, 4, 3, 5, 3 **5.** 80, 50, 40, 120, 30

Exercise 11.2

1. 28.67 **2.** 6 **3.** 62 **4.** 37 **5.** 192 **6.** 6 **7.** 61kg **8.** 52.58 **9.** 27.13

10. 40.18 **11.** 28.67 **12.** 28 **13.** 48.1 **14.** 326.25 **15.** 55.5

Exercise 11.3

1. (i) 51 (ii) 14.5 **2.** 17 **3.** 19 **4.** 34.05 **5.** 14.7 **6.** 40

Exercise 11.4

1. 72 **2.** 7 **3**. 43.18 **4**. 41.75

Q. No.	Mean	Median	Mode
5.	14	14	13,15
6.	4.05	4	4
7.	32.1	31.2	27.8
8.	28	30	33.3

Exercise 12.1

1. (ii) $\frac{-1}{5}$ (iv) -0.78 (vi) 1.45 (ix) 112% **4.** $\frac{13}{20}$ **5.** (i) $\frac{13}{20}$ (ii) $\frac{7}{20}$

6. (i) $\frac{3}{5}$ (ii) $\frac{8}{25}$ (iii) $\frac{2}{25}$ **7.** (i) $\frac{21}{50}$ (ii) $\frac{9}{10}$ (iii) $\frac{11}{25}$ (iv) $\frac{24}{25}$

8. (i) $\frac{49}{100}$ (ii) $\frac{4}{25}$ (iii) $\frac{69}{100}$ (iv) $\frac{19}{100}$ (v) $\frac{81}{100}$ **9.** (i) $\frac{39}{125}$ (ii) $\frac{9}{20}$ (iii) $\frac{119}{500}$

10. (i) $\frac{179}{500}$ (ii) $\frac{53}{500}$ (iii) $\frac{201}{500}$ (iv) $\frac{219}{500}$ **11.** (i) $\frac{4}{15}$ (ii) $\frac{19}{30}$ (iii) $\frac{11}{30}$ (iv) $\frac{1}{6}$

12. (i) $\frac{7}{10}$ (ii) $\frac{3}{10}$ (iii) $\frac{1}{5}$ **13.** (i) $\frac{3}{5}$ (ii) $\frac{9}{20}$ (iii) $\frac{1}{5}$ (iv) $\frac{3}{20}$

14. (i) $\frac{2}{5}$ (ii) $\frac{3}{4}$ (iii) $\frac{9}{20}$ **15.** (i) $\frac{1}{4}$ (ii) $\frac{1}{20}$ (iii) $\frac{1}{10}$

16. (i) $\frac{7}{20}$ (ii) $\frac{3}{25}$ (iii) $\frac{11}{50}$ (iv) $\frac{9}{50}$ (v) $\frac{3}{4}$

17. (i) $\frac{7}{20}$ (ii) $\frac{1}{20}$ (iii) $\frac{3}{20}$ (iv) $\frac{1}{5}$ (v) $\frac{1}{20}$

18. (i) $\frac{183}{500}$ (ii) $\frac{1}{4}$ (iii) $\frac{21}{250}$ (iv) $\frac{793}{100}$ (v) $\frac{33}{250}$

19. (i) $\frac{3}{25}$ (ii) $\frac{19}{125}$ (iii) $\frac{21}{125}$ (iv) $\frac{29}{125}$ (v) $\frac{14}{25}$

Multiple Choice

1	A	28	D	55	A	82	C	109	A
2	D	29	A	56	B	83	C	110	B
3	B	30	B	57	D	84	C	111	C
4	D	31	A	58	B	85	B	112	B
5	B	32	B	59	C	86	A	113	C
6	A	33	C	60	D	87	B	114	A
7	C	34	D	61	C	88	C	115	C
8	A	35	A	62	A	89	B	116	B
9	D	36	C	63	C	90	C	117	A
10	C	37	B	64	A	91	D	118	C
11	A	38	D	65	D	92	B	119	C
12	B	39	D	66	A	93	D	120	B
13	B	40	B	67	D	94	A	121	C
14	D	41	B	68	C	95	A	122	B
15	C	42	C	69	D	96	D	123	B
16	A	43	B	70	A	97	B	124	D
17	C	44	A	71	B	98	A	125	A
18	D	45	C	72	C	99	D	126	C
19	A	46	A	73	B	100	B	127	A
20	C	47	D	74	A	101	A	128	A
21	B	48	B	75	A	102	D	129	D
22	C	49	A	76	D	103	D	130	B
23	A	50	C	77	B	104	D	131	A
24	C	51	B	78	D	105	C	132	D
25	B	52	C	79	A	106	B	133	A
26	A	53	A	80	A	107	B	134	C
27	A	54	B	81	A	108	B	135	A

LOGARITHM TABLE

											Mean Difference								
	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
1.0	0.0000	0.0043	0.0086	0.0128	0.0170	0.0212	0.0253	0.0294	0.0334	0.0374	4	8	12	17	21	25	29	33	37
1.1	0.0414	0.0453	0.0492	0.0531	0.0569	0.0607	0.0645	0.0682	0.0719	0.0755	4	8	11	15	19	23	26	30	34
1.2	0.0792	0.0828	0.0864	0.0899	0.0934	0.0969	0.1004	0.1038	0.1072	0.1106	3	7	10	14	17	21	24	28	31
1.3	0.1139	0.1173	0.1206	0.1239	0.1271	0.1303	0.1335	0.1367	0.1399	0.1430	3	6	10	13	16	19	23	26	29
1.4	0.1461	0.1492	0.1523	0.1553	0.1584	0.1614	0.1644	0.1673	0.1703	0.1732	3	6	9	12	15	18	21	24	27
1.5	0.1761	0.1790	0.1818	0.1847	0.1875	0.1903	0.1931	0.1959	0.1987	0.2014	3	6	8	11	14	17	20	22	25
1.6	0.2041	0.2068	0.2095	0.2122	0.2148	0.2175	0.2201	0.2227	0.2253	0.2279	3	5	8	11	13	16	18	21	24
1.7	0.2304	0.2330	0.2355	0.2380	0.2405	0.2430	0.2455	0.2480	0.2504	0.2529	2	5	7	10	12	15	17	20	22
1.8	0.2553	0.2577	0.2601	0.2625	0.2648	0.2672	0.2695	0.2718	0.2742	0.2765	2	5	7	9	12	14	16	19	21
1.9	0.2788	0.2810	0.2833	0.2856	0.2878	0.2900	0.2923	0.2945	0.2967	0.2989	2	4	7	9	11	13	16	18	20
2.0	0.3010	0.3032	0.3054	0.3075	0.3096	0.3118	0.3139	0.3160	0.3181	0.3201	2	4	6	8	11	13	15	17	19
2.1	0.3222	0.3243	0.3263	0.3284	0.3304	0.3324	0.3345	0.3365	0.3385	0.3404	2	4	6	8	10	12	14	16	18
2.2	0.3424	0.3444	0.3464	0.3483	0.3502	0.3522	0.3541	0.3560	0.3579	0.3598	2	4	6	8	10	12	14	15	17
2.3	0.3617	0.3636	0.3655	0.3674	0.3692	0.3711	0.3729	0.3747	0.3766	0.3784	2	4	6	7	9	11	13	15	17
2.4	0.3802	0.3820	0.3838	0.3856	0.3874	0.3892	0.3909	0.3927	0.3945	0.3962	2	4	5	7	9	11	12	14	16
2.5	0.3979	0.3997	0.4014	0.4031	0.4048	0.4065	0.4082	0.4099	0.4116	0.4133	2	3	5	7	9	10	12	14	15
2.6	0.4150	0.4166	0.4183	0.4200	0.4216	0.4232	0.4249	0.4265	0.4281	0.4298	2	3	5	7	8	10	11	13	15
2.7	0.4314	0.4330	0.4346	0.4362	0.4378	0.4393	0.4409	0.4425	0.4440	0.4456	2	3	5	6	8	9	11	13	14
2.8	0.4472	0.4487	0.4502	0.4518	0.4533	0.4548	0.4564	0.4579	0.4594	0.4609	2	3	5	6	8	9	11	12	14
2.9	0.4624	0.4639	0.4654	0.4669	0.4683	0.4698	0.4713	0.4728	0.4742	0.4757	1	3	4	6	7	9	10	12	13
3.0	0.4771	0.4786	0.4800	0.4814	0.4829	0.4843	0.4857	0.4871	0.4886	0.4900	1	3	4	6	7	9	10	11	13
3.1	0.4914	0.4928	0.4942	0.4955	0.4969	0.4983	0.4997	0.5011	0.5024	0.5038	1	3	4	6	7	8	10	11	12
3.2	0.5051	0.5065	0.5079	0.5092	0.5105	0.5119	0.5132	0.5145	0.5159	0.5172	1	3	4	5	7	8	9	11	12
3.3	0.5185	0.5198	0.5211	0.5224	0.5237	0.5250	0.5263	0.5276	0.5289	0.5302	1	3	4	5	6	8	9	10	12
3.4	0.5315	0.5328	0.5340	0.5353	0.5366	0.5378	0.5391	0.5403	0.5416	0.5428	1	3	4	5	6	8	9	10	11
3.5	0.5441	0.5453	0.5465	0.5478	0.5490	0.5502	0.5514	0.5527	0.5539	0.5551	1	2	4	5	6	7	9	10	11
3.6	0.5563	0.5575	0.5587	0.5599	0.5611	0.5623	0.5635	0.5647	0.5658	0.5670	1	2	4	5	6	7	8	10	11
3.7	0.5682	0.5694	0.5705	0.5717	0.5729	0.5740	0.5752	0.5763	0.5775	0.5786	1	2	3	5	6	7	8	9	10
3.8	0.5798	0.5809	0.5821	0.5832	0.5843	0.5855	0.5866	0.5877	0.5888	0.5899	1	2	3	5	6	7	8	9	10
3.9	0.5911	0.5922	0.5933	0.5944	0.5955	0.5966	0.5977	0.5988	0.5999	0.6010	1	2	3	4	5	7	8	9	10
4.0	0.6021	0.6031	0.6042	0.6053	0.6064	0.6075	0.6085	0.6096	0.6107	0.6117	1	2	3	4	5	6	8	9	10
4.1	0.6128	0.6138	0.6149	0.6160	0.6170	0.6180	0.6191	0.6201	0.6212	0.6222	1	2	3	4	5	6	7	8	9
4.2	0.6232	0.6243	0.6253	0.6263	0.6274	0.6284	0.6294	0.6304	0.6314	0.6325	1	2	3	4	5	6	7	8	9
4.3	0.6335	0.6345	0.6355	0.6365	0.6375	0.6385	0.6395	0.6405	0.6415	0.6425	1	2	3	4	5	6	7	8	9
4.4	0.6435	0.6444	0.6454	0.6464	0.6474	0.6484	0.6493	0.6503	0.6513	0.6522	1	2	3	4	5	6	7	8	9
4.5	0.6532	0.6542	0.6551	0.6561	0.6571	0.6580	0.6590	0.6599	0.6609	0.6618	1	2	3	4	5	6	7	8	9
4.6	0.6628	0.6637	0.6646	0.6656	0.6665	0.6675	0.6684	0.6693	0.6702	0.6712	1	2	3	4	5	6	7	7	8
4.7	0.6721	0.6730	0.6739	0.6749	0.6758	0.6767	0.6776	0.6785	0.6794	0.6803	1	2	3	4	5	5	6	7	8
4.8	0.6812	0.6821	0.6830	0.6839	0.6848	0.6857	0.6866	0.6875	0.6884	0.6893	1	2	3	4	4	5	6	7	8
4.9	0.6902	0.6911	0.6920	0.6928	0.6937	0.6946	0.6955	0.6964	0.6972	0.6981	1	2	3	4	4	5	6	7	8
5.0	0.6990	0.6998	0.7007	0.7016	0.7024	0.7033	0.7042	0.7050	0.7059	0.7067	1	2	3	3	4	5	6	7	8
5.1	0.7076	0.7084	0.7093	0.7101	0.7110	0.7118	0.7126	0.7135	0.7143	0.7152	1	2	3	3	4	5	6	7	8
5.2	0.7160	0.7168	0.7177	0.7185	0.7193	0.7202	0.7210	0.7218	0.7226	0.7235	1	2	2	3	4	5	6	7	7
5.3	0.7243	0.7251	0.7259	0.7267	0.7275	0.7284	0.7292	0.7300	0.7308	0.7316	1	2	2	3	4	5	6	6	7
5.4	0.7324	0.7332	0.7340	0.7348	0.7356	0.7364	0.7372	0.7380	0.7388	0.7396	1	2	2	3	4	5	6	6	7

LOGARITHM TABLE

											Mean Difference								
	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
5.5	0.7404	0.7412	0.7419	0.7427	0.7435	0.7443	0.7451	0.7459	0.7466	0.7474	1	2	2	3	4	5	5	6	7
5.6	0.7482	0.7490	0.7497	0.7505	0.7513	0.7520	0.7528	0.7536	0.7543	0.7551	1	2	2	3	4	5	5	6	7
5.7	0.7559	0.7566	0.7574	0.7582	0.7589	0.7597	0.7604	0.7612	0.7619	0.7627	1	2	2	3	4	5	5	6	7
5.8	0.7634	0.7642	0.7649	0.7657	0.7664	0.7672	0.7679	0.7686	0.7694	0.7701	1	1	2	3	4	4	5	6	7
5.9	0.7709	0.7716	0.7723	0.7731	0.7738	0.7745	0.7752	0.7760	0.7767	0.7774	1	1	2	3	4	4	5	6	7
6.0	0.7782	0.7789	0.7796	0.7803	0.7810	0.7818	0.7825	0.7832	0.7839	0.7846	1	1	2	3	4	4	5	6	6
6.1	0.7853	0.7860	0.7868	0.7875	0.7882	0.7889	0.7896	0.7903	0.7910	0.7917	1	1	2	3	4	4	5	6	6
6.2	0.7924	0.7931	0.7938	0.7945	0.7952	0.7959	0.7966	0.7973	0.7980	0.7987	1	1	2	3	3	4	5	6	6
6.3	0.7993	0.8000	0.8007	0.8014	0.8021	0.8028	0.8035	0.8041	0.8048	0.8055	1	1	2	3	3	4	5	5	6
6.4	0.8062	0.8069	0.8075	0.8082	0.8089	0.8096	0.8102	0.8109	0.8116	0.8122	1	1	2	3	3	4	5	5	6
6.5	0.8129	0.8136	0.8142	0.8149	0.8156	0.8162	0.8169	0.8176	0.8182	0.8189	1	1	2	3	3	4	5	5	6
6.6	0.8195	0.8202	0.8209	0.8215	0.8222	0.8228	0.8235	0.8241	0.8248	0.8254	1	1	2	3	3	4	5	5	6
6.7	0.8261	0.8267	0.8274	0.8280	0.8287	0.8293	0.8299	0.8306	0.8312	0.8319	1	1	2	3	3	4	5	5	6
6.8	0.8325	0.8331	0.8338	0.8344	0.8351	0.8357	0.8363	0.8370	0.8376	0.8382	1	1	2	3	3	4	4	5	6
6.9	0.8388	0.8395	0.8401	0.8407	0.8414	0.8420	0.8426	0.8432	0.8439	0.8445	1	1	2	2	3	4	4	5	6
7.0	0.8451	0.8457	0.8463	0.8470	0.8476	0.8482	0.8488	0.8494	0.8500	0.8506	1	1	2	2	3	4	4	5	6
7.1	0.8513	0.8519	0.8525	0.8531	0.8537	0.8543	0.8549	0.8555	0.8561	0.8567	1	1	2	2	3	4	4	5	5
7.2	0.8573	0.8579	0.8585	0.8591	0.8597	0.8603	0.8609	0.8615	0.8621	0.8627	1	1	2	2	3	4	4	5	5
7.3	0.8633	0.8639	0.8645	0.8651	0.8657	0.8663	0.8669	0.8675	0.8681	0.8686	1	1	2	2	3	4	4	5	5
7.4	0.8692	0.8698	0.8704	0.8710	0.8716	0.8722	0.8727	0.8733	0.8739	0.8745	1	1	2	2	3	4	4	5	5
7.5	0.8751	0.8756	0.8762	0.8768	0.8774	0.8779	0.8785	0.8791	0.8797	0.8802	1	1	2	2	3	3	4	5	5
7.6	0.8808	0.8814	0.8820	0.8825	0.8831	0.8837	0.8842	0.8848	0.8854	0.8859	1	1	2	2	3	3	4	5	5
7.7	0.8865	0.8871	0.8876	0.8882	0.8887	0.8893	0.8899	0.8904	0.8910	0.8915	1	1	2	2	3	3	4	4	5
7.8	0.8921	0.8927	0.8932	0.8938	0.8943	0.8949	0.8954	0.8960	0.8965	0.8971	1	1	2	2	3	3	4	4	5
7.9	0.8976	0.8982	0.8987	0.8993	0.8998	0.9004	0.9009	0.9015	0.9020	0.9025	1	1	2	2	3	3	4	4	5
8.0	0.9031	0.9036	0.9042	0.9047	0.9053	0.9058	0.9063	0.9069	0.9074	0.9079	1	1	2	2	3	3	4	4	5
8.1	0.9085	0.9090	0.9096	0.9101	0.9106	0.9112	0.9117	0.9122	0.9128	0.9133	1	1	2	2	3	3	4	4	5
8.2	0.9138	0.9143	0.9149	0.9154	0.9159	0.9165	0.9170	0.9175	0.9180	0.9186	1	1	2	2	3	3	4	4	5
8.3	0.9191	0.9196	0.9201	0.9206	0.9212	0.9217	0.9222	0.9227	0.9232	0.9238	1	1	2	2	3	3	4	4	5
8.4	0.9243	0.9248	0.9253	0.9258	0.9263	0.9269	0.9274	0.9279	0.9284	0.9289	1	1	2	2	3	3	4	4	5
8.5	0.9294	0.9299	0.9304	0.9309	0.9315	0.9320	0.9325	0.9330	0.9335	0.9340	1	1	2	2	3	3	4	4	5
8.6	0.9345	0.9350	0.9355	0.9360	0.9365	0.9370	0.9375	0.9380	0.9385	0.9390	1	1	2	2	3	3	4	4	5
8.7	0.9395	0.9400	0.9405	0.9410	0.9415	0.9420	0.9425	0.9430	0.9435	0.9440	0	1	1	2	2	3	3	4	4
8.8	0.9445	0.9450	0.9455	0.9460	0.9465	0.9469	0.9474	0.9479	0.9484	0.9489	0	1	1	2	2	3	3	4	4
8.9	0.9494	0.9499	0.9504	0.9509	0.9513	0.9518	0.9523	0.9528	0.9533	0.9538	0	1	1	2	2	3	3	4	4
9.0	0.9542	0.9547	0.9552	0.9557	0.9562	0.9566	0.9571	0.9576	0.9581	0.9586	0	1	1	2	2	3	3	4	4
9.1	0.9590	0.9595	0.9600	0.9605	0.9609	0.9614	0.9619	0.9624	0.9628	0.9633	0	1	1	2	2	3	3	4	4
9.2	0.9638	0.9643	0.9647	0.9652	0.9657	0.9661	0.9666	0.9671	0.9675	0.9680	0	1	1	2	2	3	3	4	4
9.3	0.9685	0.9689	0.9694	0.9699	0.9703	0.9708	0.9713	0.9717	0.9722	0.9727	0	1	1	2	2	3	3	4	4
9.4	0.9731	0.9736	0.9741	0.9745	0.9750	0.9754	0.9759	0.9763	0.9768	0.9773	0	1	1	2	2	3	3	4	4
9.5	0.9777	0.9782	0.9786	0.9791	0.9795	0.9800	0.9805	0.9809	0.9814	0.9818	0	1	1	2	2	3	3	4	4
9.6	0.9823	0.9827	0.9832	0.9836	0.9841	0.9845	0.9850	0.9854	0.9859	0.9863	0	1	1	2	2	3	3	4	4
9.7	0.9868	0.9872	0.9877	0.9881	0.9886	0.9890	0.9894	0.9899	0.9903	0.9908	0	1	1	2	2	3	3	4	4
9.8	0.9912	0.9917	0.9921	0.9926	0.9930	0.9934	0.9939	0.9943	0.9948	0.9952	0	1	1	2	2	3	3	4	4
9.9	0.9956	0.9961	0.9965	0.9969	0.9974	0.9978	0.9983	0.9987	0.9991	0.9996	0	1	1	2	2	3	3	3	4

ANTI LOGARITHM TABLE

											Mean Difference								
	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
0.00	1.000	1.002	1.005	1.007	1.009	1.012	1.014	1.016	1.019	1.021	0	0	1	1	1	1	2	2	2
0.01	1.023	1.026	1.028	1.030	1.033	1.035	1.038	1.040	1.042	1.045	0	0	1	1	1	1	2	2	2
0.02	1.047	1.050	1.052	1.054	1.057	1.059	1.062	1.064	1.067	1.069	0	0	1	1	1	1	2	2	2
0.03	1.072	1.074	1.076	1.079	1.081	1.084	1.086	1.089	1.091	1.094	0	0	1	1	1	1	2	2	2
0.04	1.096	1.099	1.102	1.104	1.107	1.109	1.112	1.114	1.117	1.119	0	1	1	1	1	2	2	2	2
0.05	1.122	1.125	1.127	1.130	1.132	1.135	1.138	1.140	1.143	1.146	0	1	1	1	1	2	2	2	2
0.06	1.148	1.151	1.153	1.156	1.159	1.161	1.164	1.167	1.169	1.172	0	1	1	1	1	2	2	2	2
0.07	1.175	1.178	1.180	1.183	1.186	1.189	1.191	1.194	1.197	1.199	0	1	1	1	1	2	2	2	2
0.08	1.202	1.205	1.208	1.211	1.213	1.216	1.219	1.222	1.225	1.227	0	1	1	1	1	2	2	2	3
0.09	1.230	1.233	1.236	1.239	1.242	1.245	1.247	1.250	1.253	1.256	0	1	1	1	1	2	2	2	3
0.10	1.259	1.262	1.265	1.268	1.271	1.274	1.276	1.279	1.282	1.285	0	1	1	1	1	2	2	2	3
0.11	1.288	1.291	1.294	1.297	1.300	1.303	1.306	1.309	1.312	1.315	0	1	1	1	2	2	2	2	3
0.12	1.318	1.321	1.324	1.327	1.330	1.334	1.337	1.340	1.343	1.346	0	1	1	1	2	2	2	2	3
0.13	1.349	1.352	1.355	1.358	1.361	1.365	1.368	1.371	1.374	1.377	0	1	1	1	2	2	2	3	3
0.14	1.380	1.384	1.387	1.390	1.393	1.396	1.400	1.403	1.406	1.409	0	1	1	1	2	2	2	3	3
0.15	1.413	1.416	1.419	1.422	1.426	1.429	1.432	1.435	1.439	1.442	0	1	1	1	2	2	2	3	3
0.16	1.445	1.449	1.452	1.455	1.459	1.462	1.466	1.469	1.472	1.476	0	1	1	1	2	2	2	3	3
0.17	1.479	1.483	1.486	1.489	1.493	1.496	1.500	1.503	1.507	1.510	0	1	1	1	2	2	2	3	3
0.18	1.514	1.517	1.521	1.524	1.528	1.531	1.535	1.538	1.542	1.545	0	1	1	1	2	2	2	3	3
0.19	1.549	1.552	1.556	1.560	1.563	1.567	1.570	1.574	1.578	1.581	0	1	1	1	2	2	3	3	3
0.20	1.585	1.589	1.592	1.596	1.600	1.603	1.607	1.611	1.614	1.618	0	1	1	1	2	2	3	3	3
0.21	1.622	1.626	1.629	1.633	1.637	1.641	1.644	1.648	1.652	1.656	0	1	1	2	2	2	3	3	3
0.22	1.660	1.663	1.667	1.671	1.675	1.679	1.683	1.687	1.690	1.694	0	1	1	2	2	2	3	3	3
0.23	1.698	1.702	1.706	1.710	1.714	1.718	1.722	1.726	1.730	1.734	0	1	1	2	2	2	3	3	4
0.24	1.738	1.742	1.746	1.750	1.754	1.758	1.762	1.766	1.770	1.774	0	1	1	2	2	2	3	3	4
0.25	1.778	1.782	1.786	1.791	1.795	1.799	1.803	1.807	1.811	1.816	0	1	1	2	2	2	3	3	4
0.26	1.820	1.824	1.828	1.832	1.837	1.841	1.845	1.849	1.854	1.858	0	1	1	2	2	3	3	3	4
0.27	1.862	1.866	1.871	1.875	1.879	1.884	1.888	1.892	1.897	1.901	0	1	1	2	2	3	3	3	4
0.28	1.905	1.910	1.914	1.919	1.923	1.928	1.932	1.936	1.941	1.945	0	1	1	2	2	3	3	4	4
0.29	1.950	1.954	1.959	1.963	1.968	1.972	1.977	1.982	1.986	1.991	0	1	1	2	2	3	3	4	4
0.30	1.995	2.000	2.004	2.009	2.014	2.018	2.023	2.028	2.032	2.037	0	1	1	2	2	3	3	4	4
0.31	2.042	2.046	2.051	2.056	2.061	2.065	2.070	2.075	2.080	2.084	0	1	1	2	2	3	3	4	4
0.32	2.089	2.094	2.099	2.104	2.109	2.113	2.118	2.123	2.128	2.133	0	1	1	2	2	3	3	4	4
0.33	2.138	2.143	2.148	2.153	2.158	2.163	2.168	2.173	2.178	2.183	0	1	1	2	2	3	3	4	4
0.34	2.188	2.193	2.198	2.203	2.208	2.213	2.218	2.223	2.228	2.234	1	1	2	2	3	3	4	4	5
0.35	2.239	2.244	2.249	2.254	2.259	2.265	2.270	2.275	2.280	2.286	1	1	2	2	3	3	4	4	5
0.36	2.291	2.296	2.301	2.307	2.312	2.317	2.323	2.328	2.333	2.339	1	1	2	2	3	3	4	4	5
0.37	2.344	2.350	2.355	2.360	2.366	2.371	2.377	2.382	2.388	2.393	1	1	2	2	3	3	4	4	5
0.38	2.399	2.404	2.410	2.415	2.421	2.427	2.432	2.438	2.443	2.449	1	1	2	2	3	3	4	4	5
0.39	2.455	2.460	2.466	2.472	2.477	2.483	2.489	2.495	2.500	2.506	1	1	2	2	3	3	4	5	5
0.40	2.512	2.518	2.523	2.529	2.535	2.541	2.547	2.553	2.559	2.564	1	1	2	2	3	4	4	5	5
0.41	2.570	2.576	2.582	2.588	2.594	2.600	2.606	2.612	2.618	2.624	1	1	2	2	3	4	4	5	5
0.42	2.630	2.636	2.642	2.649	2.655	2.661	2.667	2.673	2.679	2.685	1	1	2	2	3	4	4	5	6
0.43	2.692	2.698	2.704	2.710	2.716	2.723	2.729	2.735	2.742	2.748	1	1	2	3	3	4	4	5	6
0.44	2.754	2.761	2.767	2.773	2.780	2.786	2.793	2.799	2.805	2.812	1	1	2	3	3	4	4	5	6
0.45	2.818	2.825	2.831	2.838	2.844	2.851	2.858	2.864	2.871	2.877	1	1	2	3	3	4	5	5	6
0.46	2.884	2.891	2.897	2.904	2.911	2.917	2.924	2.931	2.938	2.944	1	1	2	3	3	4	5	5	6
0.47	2.951	2.958	2.965	2.972	2.979	2.985	2.992	2.999	3.006	3.013	1	1	2	3	3	4	5	5	6
0.48	3.020	3.027	3.034	3.041	3.048	3.055	3.062	3.069	3.076	3.083	1	1	2	3	4	4	5	6	6
0.49	3.090	3.097	3.105	3.112	3.119	3.126	3.133	3.141	3.148	3.155	1	1	2	3	4	4	5	6	6

ANTI LOGARITHM TABLE

											Mean Difference								
	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
0.50	3.162	3.170	3.177	3.184	3.192	3.199	3.206	3.214	3.221	3.228	1	1	2	3	4	4	5	6	7
0.51	3.236	3.243	3.251	3.258	3.266	3.273	3.281	3.289	3.296	3.304	1	2	2	3	4	5	5	6	7
0.52	3.311	3.319	3.327	3.334	3.342	3.350	3.357	3.365	3.373	3.381	1	2	2	3	4	5	5	6	7
0.53	3.388	3.396	3.404	3.412	3.420	3.428	3.436	3.443	3.451	3.459	1	2	2	3	4	5	6	6	7
0.54	3.467	3.475	3.483	3.491	3.499	3.508	3.516	3.524	3.532	3.540	1	2	2	3	4	5	6	6	7
0.55	3.548	3.556	3.565	3.573	3.581	3.589	3.597	3.606	3.614	3.622	1	2	2	3	4	5	6	7	7
0.56	3.631	3.639	3.648	3.656	3.664	3.673	3.681	3.690	3.698	3.707	1	2	3	3	4	5	6	7	8
0.57	3.715	3.724	3.733	3.741	3.750	3.758	3.767	3.776	3.784	3.793	1	2	3	3	4	5	6	7	8
0.58	3.802	3.811	3.819	3.828	3.837	3.846	3.855	3.864	3.873	3.882	1	2	3	4	4	5	6	7	8
0.59	3.890	3.899	3.908	3.917	3.926	3.936	3.945	3.954	3.963	3.972	1	2	3	4	5	5	6	7	8
0.60	3.981	3.990	3.999	4.009	4.018	4.027	4.036	4.046	4.055	4.064	1	2	3	4	5	6	6	7	8
0.61	4.074	4.083	4.093	4.102	4.111	4.121	4.130	4.140	4.150	4.159	1	2	3	4	5	6	7	8	9
0.62	4.169	4.178	4.188	4.198	4.207	4.217	4.227	4.236	4.246	4.256	1	2	3	4	5	6	7	8	9
0.63	4.266	4.276	4.285	4.295	4.305	4.315	4.325	4.335	4.345	4.355	1	2	3	4	5	6	7	8	9
0.64	4.365	4.375	4.385	4.395	4.406	4.416	4.426	4.436	4.446	4.457	1	2	3	4	5	6	7	8	9
0.65	4.467	4.477	4.487	4.498	4.508	4.519	4.529	4.539	4.550	4.560	1	2	3	4	5	6	7	8	9
0.66	4.571	4.581	4.592	4.603	4.613	4.624	4.634	4.645	4.656	4.667	1	2	3	4	5	6	7	9	10
0.67	4.677	4.688	4.699	4.710	4.721	4.732	4.742	4.753	4.764	4.775	1	2	3	4	5	7	8	9	10
0.68	4.786	4.797	4.808	4.819	4.831	4.842	4.853	4.864	4.875	4.887	1	2	3	4	6	7	8	9	10
0.69	4.898	4.909	4.920	4.932	4.943	4.955	4.966	4.977	4.989	5.000	1	2	3	5	6	7	8	9	10
0.70	5.012	5.023	5.035	5.047	5.058	5.070	5.082	5.093	5.105	5.117	1	2	4	5	6	7	8	9	11
0.71	5.129	5.140	5.152	5.164	5.176	5.188	5.200	5.212	5.224	5.236	1	2	4	5	6	7	8	10	11
0.72	5.248	5.260	5.272	5.284	5.297	5.309	5.321	5.333	5.346	5.358	1	2	4	5	6	7	9	10	11
0.73	5.370	5.383	5.395	5.408	5.420	5.433	5.445	5.458	5.470	5.483	1	3	4	5	6	8	9	10	11
0.74	5.495	5.508	5.521	5.534	5.546	5.559	5.572	5.585	5.598	5.610	1	3	4	5	6	8	9	10	12
0.75	5.623	5.636	5.649	5.662	5.675	5.689	5.702	5.715	5.728	5.741	1	3	4	5	7	8	9	10	12
0.76	5.754	5.768	5.781	5.794	5.808	5.821	5.834	5.848	5.861	5.875	1	3	4	5	7	8	9	11	12
0.77	5.888	5.902	5.916	5.929	5.943	5.957	5.970	5.984	5.998	6.012	1	3	4	5	7	8	10	11	12
0.78	6.026	6.039	6.053	6.067	6.081	6.095	6.109	6.124	6.138	6.152	1	3	4	6	7	8	10	11	13
0.79	6.166	6.180	6.194	6.209	6.223	6.237	6.252	6.266	6.281	6.295	1	3	4	6	7	9	10	11	13
0.80	6.310	6.324	6.339	6.353	6.368	6.383	6.397	6.412	6.427	6.442	1	3	4	6	7	9	10	12	13
0.81	6.457	6.471	6.486	6.501	6.516	6.531	6.546	6.561	6.577	6.592	2	3	5	6	8	9	11	12	14
0.82	6.607	6.622	6.637	6.653	6.668	6.683	6.699	6.714	6.730	6.745	2	3	5	6	8	9	11	12	14
0.83	6.761	6.776	6.792	6.808	6.823	6.839	6.855	6.871	6.887	6.902	2	3	5	6	8	9	11	13	14
0.84	6.918	6.934	6.950	6.966	6.982	6.998	7.015	7.031	7.047	7.063	2	3	5	6	8	10	11	13	15
0.85	7.079	7.096	7.112	7.129	7.145	7.161	7.178	7.194	7.211	7.228	2	3	5	7	8	10	12	13	15
0.86	7.244	7.261	7.278	7.295	7.311	7.328	7.345	7.362	7.379	7.396	2	3	5	7	8	10	12	13	15
0.87	7.413	7.430	7.447	7.464	7.482	7.499	7.516	7.534	7.551	7.568	2	3	5	7	9	10	12	14	16
0.88	7.586	7.603	7.621	7.638	7.656	7.674	7.691	7.709	7.727	7.745	2	4	5	7	9	11	12	14	16
0.89	7.762	7.780	7.798	7.816	7.834	7.852	7.870	7.889	7.907	7.925	2	4	5	7	9	11	13	14	16
0.90	7.943	7.962	7.980	7.998	8.017	8.035	8.054	8.072	8.091	8.110	2	4	6	7	9	11	13	15	17
0.91	8.128	8.147	8.166	8.185	8.204	8.222	8.241	8.260	8.279	8.299	2	4	6	8	9	11	13	15	17
0.92	8.318	8.337	8.356	8.375	8.395	8.414	8.433	8.453	8.472	8.492	2	4	6	8	10	12	14	15	17
0.93	8.511	8.531	8.551	8.570	8.590	8.610	8.630	8.650	8.670	8.690	2	4	6	8	10	12	14	16	18
0.94	8.710	8.730	8.750	8.770	8.790	8.810	8.831	8.851	8.872	8.892	2	4	6	8	10	12	14	16	18
0.95	8.913	8.933	8.954	8.974	8.995	9.016	9.036	9.057	9.078	9.099	2	4	6	8	10	12	15	17	19
0.96	9.120	9.141	9.162	9.183	9.204	9.226	9.247	9.268	9.290	9.311	2	4	6	8	11	13	15	17	19
0.97	9.333	9.354	9.376	9.397	9.419	9.441	9.462	9.484	9.506	9.528	2	4	7	9	11	13	15	17	20
0.98	9.550	9.572	9.594	9.616	9.638	9.661	9.683	9.705	9.727	9.750	2	4	7	9	11	13	16	18	20
0.99	9.772	9.795	9.817	9.840	9.863	9.886	9.908	9.931	9.954	9.977	2	5	7	9	11	14	16	18	20

NATURAL SINES

Degree	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	Mean Difference				
	0.0°	0.1°	0.2°	0.3°	0.4°	0.5°	0.6°	0.7°	0.8°	0.9°	1	2	3	4	5
0	0.0000	0.0017	0.0035	0.0052	0.0070	0.0087	0.0105	0.0122	0.0140	0.0157	3	6	9	12	15
1	0.0175	0.0192	0.0209	0.0227	0.0244	0.0262	0.0279	0.0297	0.0314	0.0332	3	6	9	12	15
2	0.0349	0.0366	0.0384	0.0401	0.0419	0.0436	0.0454	0.0471	0.0488	0.0506	3	6	9	12	15
3	0.0523	0.0541	0.0558	0.0576	0.0593	0.0610	0.0628	0.0645	0.0663	0.0680	3	6	9	12	15
4	0.0698	0.0715	0.0732	0.0750	0.0767	0.0785	0.0802	0.0819	0.0837	0.0854	3	6	9	12	15
5	0.0872	0.0889	0.0906	0.0924	0.0941	0.0958	0.0976	0.0993	0.1011	0.1028	3	6	9	12	14
6	0.1045	0.1063	0.1080	0.1097	0.1115	0.1132	0.1149	0.1167	0.1184	0.1201	3	6	9	12	14
7	0.1219	0.1236	0.1253	0.1271	0.1288	0.1305	0.1323	0.1340	0.1357	0.1374	3	6	9	12	14
8	0.1392	0.1409	0.1426	0.1444	0.1461	0.1478	0.1495	0.1513	0.1530	0.1547	3	6	9	12	14
9	0.1564	0.1582	0.1599	0.1616	0.1633	0.1650	0.1668	0.1685	0.1702	0.1719	3	6	9	12	14
10	0.1736	0.1754	0.1771	0.1788	0.1805	0.1822	0.1840	0.1857	0.1874	0.1891	3	6	9	12	14
11	0.1908	0.1925	0.1942	0.1959	0.1977	0.1994	0.2011	0.2028	0.2045	0.2062	3	6	9	11	14
12	0.2079	0.2096	0.2113	0.2130	0.2147	0.2164	0.2181	0.2198	0.2215	0.2233	3	6	9	11	14
13	0.2250	0.2267	0.2284	0.2300	0.2317	0.2334	0.2351	0.2368	0.2385	0.2402	3	6	8	11	14
14	0.2419	0.2436	0.2453	0.2470	0.2487	0.2504	0.2521	0.2538	0.2554	0.2571	3	6	8	11	14
15	0.2588	0.2605	0.2622	0.2639	0.2656	0.2672	0.2689	0.2706	0.2723	0.2740	3	6	8	11	14
16	0.2756	0.2773	0.2790	0.2807	0.2823	0.2840	0.2857	0.2874	0.2890	0.2907	3	6	8	11	14
17	0.2924	0.2940	0.2957	0.2974	0.2990	0.3007	0.3024	0.3040	0.3057	0.3074	3	6	8	11	14
18	0.3090	0.3107	0.3123	0.3140	0.3156	0.3173	0.3190	0.3206	0.3223	0.3239	3	6	8	11	14
19	0.3256	0.3272	0.3289	0.3305	0.3322	0.3338	0.3355	0.3371	0.3387	0.3404	3	5	8	11	14
20	0.3420	0.3437	0.3453	0.3469	0.3486	0.3502	0.3518	0.3535	0.3551	0.3567	3	5	8	11	14
21	0.3584	0.3600	0.3616	0.3633	0.3649	0.3665	0.3681	0.3697	0.3714	0.3730	3	5	8	11	14
22	0.3746	0.3762	0.3778	0.3795	0.3811	0.3827	0.3843	0.3859	0.3875	0.3891	3	5	8	11	14
23	0.3907	0.3923	0.3939	0.3955	0.3971	0.3987	0.4003	0.4019	0.4035	0.4051	3	5	8	11	14
24	0.4067	0.4083	0.4099	0.4115	0.4131	0.4147	0.4163	0.4179	0.4195	0.4210	3	5	8	11	13
25	0.4226	0.4242	0.4258	0.4274	0.4289	0.4305	0.4321	0.4337	0.4352	0.4368	3	5	8	11	13
26	0.4384	0.4399	0.4415	0.4431	0.4446	0.4462	0.4478	0.4493	0.4509	0.4524	3	5	8	10	13
27	0.4540	0.4555	0.4571	0.4586	0.4602	0.4617	0.4633	0.4648	0.4664	0.4679	3	5	8	10	13
28	0.4695	0.4710	0.4726	0.4741	0.4756	0.4772	0.4787	0.4802	0.4818	0.4833	3	5	8	10	13
29	0.4848	0.4863	0.4879	0.4894	0.4909	0.4924	0.4939	0.4955	0.4970	0.4985	3	5	8	10	13
30	0.5000	0.5015	0.5030	0.5045	0.5060	0.5075	0.5090	0.5105	0.5120	0.5135	3	5	8	10	13
31	0.5150	0.5165	0.5180	0.5195	0.5210	0.5225	0.5240	0.5255	0.5270	0.5284	2	5	7	10	12
32	0.5299	0.5314	0.5329	0.5344	0.5358	0.5373	0.5388	0.5402	0.5417	0.5432	2	5	7	10	12
33	0.5446	0.5461	0.5476	0.5490	0.5505	0.5519	0.5534	0.5548	0.5563	0.5577	2	5	7	10	12
34	0.5592	0.5606	0.5621	0.5635	0.5650	0.5664	0.5678	0.5693	0.5707	0.5721	2	5	7	10	12
35	0.5736	0.5750	0.5764	0.5779	0.5793	0.5807	0.5821	0.5835	0.5850	0.5864	2	5	7	10	12
36	0.5878	0.5892	0.5906	0.5920	0.5934	0.5948	0.5962	0.5976	0.5990	0.6004	2	5	7	9	12
37	0.6018	0.6032	0.6046	0.6060	0.6074	0.6088	0.6101	0.6115	0.6129	0.6143	2	5	7	9	12
38	0.6157	0.6170	0.6184	0.6198	0.6211	0.6225	0.6239	0.6252	0.6266	0.6280	2	5	7	9	11
39	0.6293	0.6307	0.6320	0.6334	0.6347	0.6361	0.6374	0.6388	0.6401	0.6414	2	4	7	9	11
40	0.6428	0.6441	0.6455	0.6468	0.6481	0.6494	0.6508	0.6521	0.6534	0.6547	2	4	7	9	11
41	0.6561	0.6574	0.6587	0.6600	0.6613	0.6626	0.6639	0.6652	0.6665	0.6678	2	4	7	9	11
42	0.6691	0.6704	0.6717	0.6730	0.6743	0.6756	0.6769	0.6782	0.6794	0.6807	2	4	6	9	11
43	0.6820	0.6833	0.6845	0.6858	0.6871	0.6884	0.6896	0.6909	0.6921	0.6934	2	4	6	8	11
44	0.6947	0.6959	0.6972	0.6984	0.6997	0.7009	0.7022	0.7034	0.7046	0.7059	2	4	6	8	10

NATURAL SINES

Degree	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	Mean Difference				
	0.0°	0.1°	0.2°	0.3°	0.4°	0.5°	0.6°	0.7°	0.8°	0.9°	1	2	3	4	5
45	0.7071	0.7083	0.7096	0.7108	0.7120	0.7133	0.7145	0.7157	0.7169	0.7181	2	4	6	8	10
46	0.7193	0.7206	0.7218	0.7230	0.7242	0.7254	0.7266	0.7278	0.7290	0.7302	2	4	6	8	10
47	0.7314	0.7325	0.7337	0.7349	0.7361	0.7373	0.7385	0.7396	0.7408	0.7420	2	4	6	8	10
48	0.7431	0.7443	0.7455	0.7466	0.7478	0.7490	0.7501	0.7513	0.7524	0.7536	2	4	6	8	10
49	0.7547	0.7559	0.7570	0.7581	0.7593	0.7604	0.7615	0.7627	0.7638	0.7649	2	4	6	8	9
50	0.7660	0.7672	0.7683	0.7694	0.7705	0.7716	0.7727	0.7738	0.7749	0.7760	2	4	6	7	9
51	0.7771	0.7782	0.7793	0.7804	0.7815	0.7826	0.7837	0.7848	0.7859	0.7869	2	4	5	7	9
52	0.7880	0.7891	0.7902	0.7912	0.7923	0.7934	0.7944	0.7955	0.7965	0.7976	2	4	5	7	9
53	0.7986	0.7997	0.8007	0.8018	0.8028	0.8039	0.8049	0.8059	0.8070	0.8080	2	3	5	7	9
54	0.8090	0.8100	0.8111	0.8121	0.8131	0.8141	0.8151	0.8161	0.8171	0.8181	2	3	5	7	8
55	0.8192	0.8202	0.8211	0.8221	0.8231	0.8241	0.8251	0.8261	0.8271	0.8281	2	3	5	7	8
56	0.8290	0.8300	0.8310	0.8320	0.8329	0.8339	0.8348	0.8358	0.8368	0.8377	2	3	5	6	8
57	0.8387	0.8396	0.8406	0.8415	0.8425	0.8434	0.8443	0.8453	0.8462	0.8471	2	3	5	6	8
58	0.8480	0.8490	0.8499	0.8508	0.8517	0.8526	0.8536	0.8545	0.8554	0.8563	2	3	5	6	8
59	0.8572	0.8581	0.8590	0.8599	0.8607	0.8616	0.8625	0.8634	0.8643	0.8652	1	3	4	6	7
60	0.8660	0.8669	0.8678	0.8686	0.8695	0.8704	0.8712	0.8721	0.8729	0.8738	1	3	4	6	7
61	0.8746	0.8755	0.8763	0.8771	0.8780	0.8788	0.8796	0.8805	0.8813	0.8821	1	3	4	6	7
62	0.8829	0.8838	0.8846	0.8854	0.8862	0.8870	0.8878	0.8886	0.8894	0.8902	1	3	4	5	7
63	0.8910	0.8918	0.8926	0.8934	0.8942	0.8949	0.8957	0.8965	0.8973	0.8980	1	3	4	5	6
64	0.8988	0.8996	0.9003	0.9011	0.9018	0.9026	0.9033	0.9041	0.9048	0.9056	1	3	4	5	6
65	0.9063	0.9070	0.9078	0.9085	0.9092	0.9100	0.9107	0.9114	0.9121	0.9128	1	2	4	5	6
66	0.9135	0.9143	0.9150	0.9157	0.9164	0.9171	0.9178	0.9184	0.9191	0.9198	1	2	3	5	6
67	0.9205	0.9212	0.9219	0.9225	0.9232	0.9239	0.9245	0.9252	0.9259	0.9265	1	2	3	4	6
68	0.9272	0.9278	0.9285	0.9291	0.9298	0.9304	0.9311	0.9317	0.9323	0.9330	1	2	3	4	5
69	0.9336	0.9342	0.9348	0.9354	0.9361	0.9367	0.9373	0.9379	0.9385	0.9391	1	2	3	4	5
70	0.9397	0.9403	0.9409	0.9415	0.9421	0.9426	0.9432	0.9438	0.9444	0.9449	1	2	3	4	5
71	0.9455	0.9461	0.9466	0.9472	0.9478	0.9483	0.9489	0.9494	0.9500	0.9505	1	2	3	4	5
72	0.9511	0.9516	0.9521	0.9527	0.9532	0.9537	0.9542	0.9548	0.9553	0.9558	1	2	3	3	4
73	0.9563	0.9568	0.9573	0.9578	0.9583	0.9588	0.9593	0.9598	0.9603	0.9608	1	2	2	3	4
74	0.9613	0.9617	0.9622	0.9627	0.9632	0.9636	0.9641	0.9646	0.9650	0.9655	1	2	2	3	4
75	0.9659	0.9664	0.9668	0.9673	0.9677	0.9681	0.9686	0.9690	0.9694	0.9699	1	1	2	3	4
76	0.9703	0.9707	0.9711	0.9715	0.9720	0.9724	0.9728	0.9732	0.9736	0.9740	1	1	2	3	3
77	0.9744	0.9748	0.9751	0.9755	0.9759	0.9763	0.9767	0.9770	0.9774	0.9778	1	1	2	3	3
78	0.9781	0.9785	0.9789	0.9792	0.9796	0.9799	0.9803	0.9806	0.9810	0.9813	1	1	2	2	3
79	0.9816	0.9820	0.9823	0.9826	0.9829	0.9833	0.9836	0.9839	0.9842	0.9845	1	1	2	2	3
80	0.9848	0.9851	0.9854	0.9857	0.9860	0.9863	0.9866	0.9869	0.9871	0.9874	0	1	1	2	2
81	0.9877	0.9880	0.9882	0.9885	0.9888	0.9890	0.9893	0.9895	0.9898	0.9900	0	1	1	2	2
82	0.9903	0.9905	0.9907	0.9910	0.9912	0.9914	0.9917	0.9919	0.9921	0.9923	0	1	1	2	2
83	0.9925	0.9928	0.9930	0.9932	0.9934	0.9936	0.9938	0.9940	0.9942	0.9943	0	1	1	1	2
84	0.9945	0.9947	0.9949	0.9951	0.9952	0.9954	0.9956	0.9957	0.9959	0.9960	0	1	1	1	2
85	0.9962	0.9963	0.9965	0.9966	0.9968	0.9969	0.9971	0.9972	0.9973	0.9974	0	0	1	1	1
86	0.9976	0.9977	0.9978	0.9979	0.9980	0.9981	0.9982	0.9983	0.9984	0.9985	0	0	1	1	1
87	0.9986	0.9987	0.9988	0.9989	0.9990	0.9990	0.9991	0.9992	0.9993	0.9993	0	0	0	1	1
88	0.9994	0.9995	0.9995	0.9996	0.9996	0.9997	0.9997	0.9997	0.9998	0.9998	0	0	0	0	0
89	0.9998	0.9999	0.9999	0.9999	0.9999	1.0000	1.0000	1.0000	1.0000	1.0000	0	0	0	0	0

NATURAL COSINES

(Numbers in mean difference columns to be subtracted, not added)

Degree	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	Mean Difference				
	0.0°	0.1°	0.2°	0.3°	0.4°	0.5°	0.6°	0.7°	0.8°	0.9°	1	2	3	4	5
0	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9999	0.9999	0.9999	0	0	0	0	0
1	0.9998	0.9998	0.9998	0.9997	0.9997	0.9997	0.9996	0.9996	0.9995	0.9995	0	0	0	0	0
2	0.9994	0.9993	0.9993	0.9992	0.9991	0.9990	0.9990	0.9989	0.9988	0.9987	0	0	0	1	1
3	0.9986	0.9985	0.9984	0.9983	0.9982	0.9981	0.9980	0.9979	0.9978	0.9977	0	0	1	1	1
4	0.9976	0.9974	0.9973	0.9972	0.9971	0.9969	0.9968	0.9966	0.9965	0.9963	0	0	1	1	1
5	0.9962	0.9960	0.9959	0.9957	0.9956	0.9954	0.9952	0.9951	0.9949	0.9947	0	1	1	1	2
6	0.9945	0.9943	0.9942	0.9940	0.9938	0.9936	0.9934	0.9932	0.9930	0.9928	0	1	1	1	2
7	0.9925	0.9923	0.9921	0.9919	0.9917	0.9914	0.9912	0.9910	0.9907	0.9905	0	1	1	2	2
8	0.9903	0.9900	0.9898	0.9895	0.9893	0.9890	0.9888	0.9885	0.9882	0.9880	0	1	1	2	2
9	0.9877	0.9874	0.9871	0.9869	0.9866	0.9863	0.9860	0.9857	0.9854	0.9851	0	1	1	2	2
10	0.9848	0.9845	0.9842	0.9839	0.9836	0.9833	0.9829	0.9826	0.9823	0.9820	1	1	2	2	3
11	0.9816	0.9813	0.9810	0.9806	0.9803	0.9799	0.9796	0.9792	0.9789	0.9785	1	1	2	2	3
12	0.9781	0.9778	0.9774	0.9770	0.9767	0.9763	0.9759	0.9755	0.9751	0.9748	1	1	2	3	3
13	0.9744	0.9740	0.9736	0.9732	0.9728	0.9724	0.9720	0.9715	0.9711	0.9707	1	1	2	3	3
14	0.9703	0.9699	0.9694	0.9690	0.9686	0.9681	0.9677	0.9673	0.9668	0.9664	1	1	2	3	4
15	0.9659	0.9655	0.9650	0.9646	0.9641	0.9636	0.9632	0.9627	0.9622	0.9617	1	2	2	3	4
16	0.9613	0.9608	0.9603	0.9598	0.9593	0.9588	0.9583	0.9578	0.9573	0.9568	1	2	2	3	4
17	0.9563	0.9558	0.9553	0.9548	0.9542	0.9537	0.9532	0.9527	0.9521	0.9516	1	2	3	3	4
18	0.9511	0.9505	0.9500	0.9494	0.9489	0.9483	0.9478	0.9472	0.9466	0.9461	1	2	3	4	5
19	0.9455	0.9449	0.9444	0.9438	0.9432	0.9426	0.9421	0.9415	0.9409	0.9403	1	2	3	4	5
20	0.9397	0.9391	0.9385	0.9379	0.9373	0.9367	0.9361	0.9354	0.9348	0.9342	1	2	3	4	5
21	0.9336	0.9330	0.9323	0.9317	0.9311	0.9304	0.9298	0.9291	0.9285	0.9278	1	2	3	4	5
22	0.9272	0.9265	0.9259	0.9252	0.9245	0.9239	0.9232	0.9225	0.9219	0.9212	1	2	3	4	6
23	0.9205	0.9198	0.9191	0.9184	0.9178	0.9171	0.9164	0.9157	0.9150	0.9143	1	2	3	5	6
24	0.9135	0.9128	0.9121	0.9114	0.9107	0.9100	0.9092	0.9085	0.9078	0.9070	1	2	4	5	6
25	0.9063	0.9056	0.9048	0.9041	0.9033	0.9026	0.9018	0.9011	0.9003	0.8996	1	3	4	5	6
26	0.8988	0.8980	0.8973	0.8965	0.8957	0.8949	0.8942	0.8934	0.8926	0.8918	1	3	4	5	6
27	0.8910	0.8902	0.8894	0.8886	0.8878	0.8870	0.8862	0.8854	0.8846	0.8838	1	3	4	5	7
28	0.8829	0.8821	0.8813	0.8805	0.8796	0.8788	0.8780	0.8771	0.8763	0.8755	1	3	4	6	7
29	0.8746	0.8738	0.8729	0.8721	0.8712	0.8704	0.8695	0.8686	0.8678	0.8669	1	3	4	6	7
30	0.8660	0.8652	0.8643	0.8634	0.8625	0.8616	0.8607	0.8599	0.8590	0.8581	1	3	4	6	7
31	0.8572	0.8563	0.8554	0.8545	0.8536	0.8526	0.8517	0.8508	0.8499	0.8490	2	3	5	6	8
32	0.8480	0.8471	0.8462	0.8453	0.8443	0.8434	0.8425	0.8415	0.8406	0.8396	2	3	5	6	8
33	0.8387	0.8377	0.8368	0.8358	0.8348	0.8339	0.8329	0.8320	0.8310	0.8300	2	3	5	6	8
34	0.8290	0.8281	0.8271	0.8261	0.8251	0.8241	0.8231	0.8221	0.8211	0.8202	2	3	5	7	8
35	0.8192	0.8181	0.8171	0.8161	0.8151	0.8141	0.8131	0.8121	0.8111	0.8100	2	3	5	7	8
36	0.8090	0.8080	0.8070	0.8059	0.8049	0.8039	0.8028	0.8018	0.8007	0.7997	2	3	5	7	9
37	0.7986	0.7976	0.7965	0.7955	0.7944	0.7934	0.7923	0.7912	0.7902	0.7891	2	4	5	7	9
38	0.7880	0.7869	0.7859	0.7848	0.7837	0.7826	0.7815	0.7804	0.7793	0.7782	2	4	5	7	9
39	0.7771	0.7760	0.7749	0.7738	0.7727	0.7716	0.7705	0.7694	0.7683	0.7672	2	4	6	7	9
40	0.7660	0.7649	0.7638	0.7627	0.7615	0.7604	0.7593	0.7581	0.7570	0.7559	2	4	6	8	9
41	0.7547	0.7536	0.7524	0.7513	0.7501	0.7490	0.7478	0.7466	0.7455	0.7443	2	4	6	8	10
42	0.7431	0.7420	0.7408	0.7396	0.7385	0.7373	0.7361	0.7349	0.7337	0.7325	2	4	6	8	10
43	0.7314	0.7302	0.7290	0.7278	0.7266	0.7254	0.7242	0.7230	0.7218	0.7206	2	4	6	8	10
44	0.7193	0.7181	0.7169	0.7157	0.7145	0.7133	0.7120	0.7108	0.7096	0.7083	2	4	6	8	10

NATURAL COSINES

(Numbers in mean difference columns to be subtracted, not added)

Degree	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	Mean Difference				
	0.0°	0.1°	0.2°	0.3°	0.4°	0.5°	0.6°	0.7°	0.8°	0.9°	1	2	3	4	5
45	0.7071	0.7059	0.7046	0.7034	0.7022	0.7009	0.6997	0.6984	0.6972	0.6959	2	4	6	8	10
46	0.6947	0.6934	0.6921	0.6909	0.6896	0.6884	0.6871	0.6858	0.6845	0.6833	2	4	6	8	11
47	0.6820	0.6807	0.6794	0.6782	0.6769	0.6756	0.6743	0.6730	0.6717	0.6704	2	4	6	9	11
48	0.6691	0.6678	0.6665	0.6652	0.6639	0.6626	0.6613	0.6600	0.6587	0.6574	2	4	7	9	11
49	0.6561	0.6547	0.6534	0.6521	0.6508	0.6494	0.6481	0.6468	0.6455	0.6441	2	4	7	9	11
50	0.6428	0.6414	0.6401	0.6388	0.6374	0.6361	0.6347	0.6334	0.6320	0.6307	2	4	7	9	11
51	0.6293	0.6280	0.6266	0.6252	0.6239	0.6225	0.6211	0.6198	0.6184	0.6170	2	5	7	9	11
52	0.6157	0.6143	0.6129	0.6115	0.6101	0.6088	0.6074	0.6060	0.6046	0.6032	2	5	7	9	12
53	0.6018	0.6004	0.5990	0.5976	0.5962	0.5948	0.5934	0.5920	0.5906	0.5892	2	5	7	9	12
54	0.5878	0.5864	0.5850	0.5835	0.5821	0.5807	0.5793	0.5779	0.5764	0.5750	2	5	7	9	12
55	0.5736	0.5721	0.5707	0.5693	0.5678	0.5664	0.5650	0.5635	0.5621	0.5606	2	5	7	10	12
56	0.5592	0.5577	0.5563	0.5548	0.5534	0.5519	0.5505	0.5490	0.5476	0.5461	2	5	7	10	12
57	0.5446	0.5432	0.5417	0.5402	0.5388	0.5373	0.5358	0.5344	0.5329	0.5314	2	5	7	10	12
58	0.5299	0.5284	0.5270	0.5255	0.5240	0.5225	0.5210	0.5195	0.5180	0.5165	2	5	7	10	12
59	0.5150	0.5135	0.5120	0.5105	0.5090	0.5075	0.5060	0.5045	0.5030	0.5015	3	5	8	10	13
60	0.5000	0.4985	0.4970	0.4955	0.4939	0.4924	0.4909	0.4894	0.4879	0.4863	3	5	8	10	13
61	0.4848	0.4833	0.4818	0.4802	0.4787	0.4772	0.4756	0.4741	0.4726	0.4710	3	5	8	10	13
62	0.4695	0.4679	0.4664	0.4648	0.4633	0.4617	0.4602	0.4586	0.4571	0.4555	3	5	8	10	13
63	0.4540	0.4524	0.4509	0.4493	0.4478	0.4462	0.4446	0.4431	0.4415	0.4399	3	5	8	10	13
64	0.4384	0.4368	0.4352	0.4337	0.4321	0.4305	0.4289	0.4274	0.4258	0.4242	3	5	8	11	13
65	0.4226	0.4210	0.4195	0.4179	0.4163	0.4147	0.4131	0.4115	0.4099	0.4083	3	5	8	11	13
66	0.4067	0.4051	0.4035	0.4019	0.4003	0.3987	0.3971	0.3955	0.3939	0.3923	3	5	8	11	14
67	0.3907	0.3891	0.3875	0.3859	0.3843	0.3827	0.3811	0.3795	0.3778	0.3762	3	5	8	11	14
68	0.3746	0.3730	0.3714	0.3697	0.3681	0.3665	0.3649	0.3633	0.3616	0.3600	3	5	8	11	14
69	0.3584	0.3567	0.3551	0.3535	0.3518	0.3502	0.3486	0.3469	0.3453	0.3437	3	5	8	11	14
70	0.3420	0.3404	0.3387	0.3371	0.3355	0.3338	0.3322	0.3305	0.3289	0.3272	3	5	8	11	14
71	0.3256	0.3239	0.3223	0.3206	0.3190	0.3173	0.3156	0.3140	0.3123	0.3107	3	6	8	11	14
72	0.3090	0.3074	0.3057	0.3040	0.3024	0.3007	0.2990	0.2974	0.2957	0.2940	3	6	8	11	14
73	0.2924	0.2907	0.2890	0.2874	0.2857	0.2840	0.2823	0.2807	0.2790	0.2773	3	6	8	11	14
74	0.2756	0.2740	0.2723	0.2706	0.2689	0.2672	0.2656	0.2639	0.2622	0.2605	3	6	8	11	14
75	0.2588	0.2571	0.2554	0.2538	0.2521	0.2504	0.2487	0.2470	0.2453	0.2436	3	6	8	11	14
76	0.2419	0.2402	0.2385	0.2368	0.2351	0.2334	0.2317	0.2300	0.2284	0.2267	3	6	8	11	14
77	0.2250	0.2233	0.2215	0.2198	0.2181	0.2164	0.2147	0.2130	0.2113	0.2096	3	6	9	11	14
78	0.2079	0.2062	0.2045	0.2028	0.2011	0.1994	0.1977	0.1959	0.1942	0.1925	3	6	9	11	14
79	0.1908	0.1891	0.1874	0.1857	0.1840	0.1822	0.1805	0.1788	0.1771	0.1754	3	6	9	11	14
80	0.1736	0.1719	0.1702	0.1685	0.1668	0.1650	0.1633	0.1616	0.1599	0.1582	3	6	9	12	14
81	0.1564	0.1547	0.1530	0.1513	0.1495	0.1478	0.1461	0.1444	0.1426	0.1409	3	6	9	12	14
82	0.1392	0.1374	0.1357	0.1340	0.1323	0.1305	0.1288	0.1271	0.1253	0.1236	3	6	9	12	14
83	0.1219	0.1201	0.1184	0.1167	0.1149	0.1132	0.1115	0.1097	0.1080	0.1063	3	6	9	12	14
84	0.1045	0.1028	0.1011	0.0993	0.0976	0.0958	0.0941	0.0924	0.0906	0.0889	3	6	9	12	14
85	0.0872	0.0854	0.0837	0.0819	0.0802	0.0785	0.0767	0.0750	0.0732	0.0715	3	6	9	12	15
86	0.0698	0.0680	0.0663	0.0645	0.0628	0.0610	0.0593	0.0576	0.0558	0.0541	3	6	9	12	15
87	0.0523	0.0506	0.0488	0.0471	0.0454	0.0436	0.0419	0.0401	0.0384	0.0366	3	6	9	12	15
88	0.0349	0.0332	0.0314	0.0297	0.0279	0.0262	0.0244	0.0227	0.0209	0.0192	3	6	9	12	15
89	0.0175	0.0157	0.0140	0.0122	0.0105	0.0087	0.0070	0.0052	0.0035	0.0017	3	6	9	12	15

NATURAL TANGENTS

Degree	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	Mean Difference				
	0.0°	0.1°	0.2°	0.3°	0.4°	0.5°	0.6°	0.7°	0.8°	0.9°	1	2	3	4	5
0	0.0000	0.0017	0.0035	0.0052	0.0070	0.0087	0.0105	0.0122	0.0140	0.0157	3	6	9	12	15
1	0.0175	0.0192	0.0209	0.0227	0.0244	0.0262	0.0279	0.0297	0.0314	0.0332	3	6	9	12	15
2	0.0349	0.0367	0.0384	0.0402	0.0419	0.0437	0.0454	0.0472	0.0489	0.0507	3	6	9	12	15
3	0.0524	0.0542	0.0559	0.0577	0.0594	0.0612	0.0629	0.0647	0.0664	0.0682	3	6	9	12	15
4	0.0699	0.0717	0.0734	0.0752	0.0769	0.0787	0.0805	0.0822	0.0840	0.0857	3	6	9	12	15
5	0.0875	0.0892	0.0910	0.0928	0.0945	0.0963	0.0981	0.0998	0.1016	0.1033	3	6	9	12	15
6	0.1051	0.1069	0.1086	0.1104	0.1122	0.1139	0.1157	0.1175	0.1192	0.1210	3	6	9	12	15
7	0.1228	0.1246	0.1263	0.1281	0.1299	0.1317	0.1334	0.1352	0.1370	0.1388	3	6	9	12	15
8	0.1405	0.1423	0.1441	0.1459	0.1477	0.1495	0.1512	0.1530	0.1548	0.1566	3	6	9	12	15
9	0.1584	0.1602	0.1620	0.1638	0.1655	0.1673	0.1691	0.1709	0.1727	0.1745	3	6	9	12	15
10	0.1763	0.1781	0.1799	0.1817	0.1835	0.1853	0.1871	0.1890	0.1908	0.1926	3	6	9	12	15
11	0.1944	0.1962	0.1980	0.1998	0.2016	0.2035	0.2053	0.2071	0.2089	0.2107	3	6	9	12	15
12	0.2126	0.2144	0.2162	0.2180	0.2199	0.2217	0.2235	0.2254	0.2272	0.2290	3	6	9	12	15
13	0.2309	0.2327	0.2345	0.2364	0.2382	0.2401	0.2419	0.2438	0.2456	0.2475	3	6	9	12	15
14	0.2493	0.2512	0.2530	0.2549	0.2568	0.2586	0.2605	0.2623	0.2642	0.2661	3	6	9	12	16
15	0.2679	0.2698	0.2717	0.2736	0.2754	0.2773	0.2792	0.2811	0.2830	0.2849	3	6	9	13	16
16	0.2867	0.2886	0.2905	0.2924	0.2943	0.2962	0.2981	0.3000	0.3019	0.3038	3	6	9	13	16
17	0.3057	0.3076	0.3096	0.3115	0.3134	0.3153	0.3172	0.3191	0.3211	0.3230	3	6	10	13	16
18	0.3249	0.3269	0.3288	0.3307	0.3327	0.3346	0.3365	0.3385	0.3404	0.3424	3	6	10	13	16
19	0.3443	0.3463	0.3482	0.3502	0.3522	0.3541	0.3561	0.3581	0.3600	0.3620	3	7	10	13	16
20	0.3640	0.3659	0.3679	0.3699	0.3719	0.3739	0.3759	0.3779	0.3799	0.3819	3	7	10	13	17
21	0.3839	0.3859	0.3879	0.3899	0.3919	0.3939	0.3959	0.3979	0.4000	0.4020	3	7	10	13	17
22	0.4040	0.4061	0.4081	0.4101	0.4122	0.4142	0.4163	0.4183	0.4204	0.4224	3	7	10	14	17
23	0.4245	0.4265	0.4286	0.4307	0.4327	0.4348	0.4369	0.4390	0.4411	0.4431	3	7	10	14	17
24	0.4452	0.4473	0.4494	0.4515	0.4536	0.4557	0.4578	0.4599	0.4621	0.4642	4	7	11	14	18
25	0.4663	0.4684	0.4706	0.4727	0.4748	0.4770	0.4791	0.4813	0.4834	0.4856	4	7	11	14	18
26	0.4877	0.4899	0.4921	0.4942	0.4964	0.4986	0.5008	0.5029	0.5051	0.5073	4	7	11	15	18
27	0.5095	0.5117	0.5139	0.5161	0.5184	0.5206	0.5228	0.5250	0.5272	0.5295	4	7	11	15	18
28	0.5317	0.5340	0.5362	0.5384	0.5407	0.5430	0.5452	0.5475	0.5498	0.5520	4	8	11	15	19
29	0.5543	0.5566	0.5589	0.5612	0.5635	0.5658	0.5681	0.5704	0.5727	0.5750	4	8	12	15	19
30	0.5774	0.5797	0.5820	0.5844	0.5867	0.5890	0.5914	0.5938	0.5961	0.5985	4	8	12	16	20
31	0.6009	0.6032	0.6056	0.6080	0.6104	0.6128	0.6152	0.6176	0.6200	0.6224	4	8	12	16	20
32	0.6249	0.6273	0.6297	0.6322	0.6346	0.6371	0.6395	0.6420	0.6445	0.6469	4	8	12	16	20
33	0.6494	0.6519	0.6544	0.6569	0.6594	0.6619	0.6644	0.6669	0.6694	0.6720	4	8	13	17	21
34	0.6745	0.6771	0.6796	0.6822	0.6847	0.6873	0.6899	0.6924	0.6950	0.6976	4	9	13	17	21
35	0.7002	0.7028	0.7054	0.7080	0.7107	0.7133	0.7159	0.7186	0.7212	0.7239	4	9	13	18	22
36	0.7265	0.7292	0.7319	0.7346	0.7373	0.7400	0.7427	0.7454	0.7481	0.7508	5	9	14	18	23
37	0.7536	0.7563	0.7590	0.7618	0.7646	0.7673	0.7701	0.7729	0.7757	0.7785	5	9	14	18	23
38	0.7813	0.7841	0.7869	0.7898	0.7926	0.7954	0.7983	0.8012	0.8040	0.8069	5	9	14	19	24
39	0.8098	0.8127	0.8156	0.8185	0.8214	0.8243	0.8273	0.8302	0.8332	0.8361	5	10	15	20	24
40	0.8391	0.8421	0.8451	0.8481	0.8511	0.8541	0.8571	0.8601	0.8632	0.8662	5	10	15	20	25
41	0.8693	0.8724	0.8754	0.8785	0.8816	0.8847	0.8878	0.8910	0.8941	0.8972	5	10	16	21	26
42	0.9004	0.9036	0.9067	0.9099	0.9131	0.9163	0.9195	0.9228	0.9260	0.9293	5	11	16	21	27
43	0.9325	0.9358	0.9391	0.9424	0.9457	0.9490	0.9523	0.9556	0.9590	0.9623	6	11	17	22	28
44	0.9657	0.9691	0.9725	0.9759	0.9793	0.9827	0.9861	0.9896	0.9930	0.9965	6	11	17	23	29

NATURAL TANGENTS

Degree	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	Mean Difference				
	0.0°	0.1°	0.2°	0.3°	0.4°	0.5°	0.6°	0.7°	0.8°	0.9°	1	2	3	4	5
45	1.0000	1.0035	1.0070	1.0105	1.0141	1.0176	1.0212	1.0247	1.0283	1.0319	6	12	18	24	30
46	1.0355	1.0392	1.0428	1.0464	1.0501	1.0538	1.0575	1.0612	1.0649	1.0686	6	12	18	25	31
47	1.0724	1.0761	1.0799	1.0837	1.0875	1.0913	1.0951	1.0990	1.1028	1.1067	6	13	19	25	32
48	1.1106	1.1145	1.1184	1.1224	1.1263	1.1303	1.1343	1.1383	1.1423	1.1463	7	13	20	27	33
49	1.1504	1.1544	1.1585	1.1626	1.1667	1.1708	1.1750	1.1792	1.1833	1.1875	7	14	21	28	34
50	1.1918	1.1960	1.2002	1.2045	1.2088	1.2131	1.2174	1.2218	1.2261	1.2305	7	14	22	29	36
51	1.2349	1.2393	1.2437	1.2482	1.2527	1.2572	1.2617	1.2662	1.2708	1.2753	8	15	23	30	38
52	1.2799	1.2846	1.2892	1.2938	1.2985	1.3032	1.3079	1.3127	1.3175	1.3222	8	16	24	31	39
53	1.3270	1.3319	1.3367	1.3416	1.3465	1.3514	1.3564	1.3613	1.3663	1.3713	8	16	25	33	41
54	1.3764	1.3814	1.3865	1.3916	1.3968	1.4019	1.4071	1.4124	1.4176	1.4229	9	17	26	34	43
55	1.4281	1.4335	1.4388	1.4442	1.4496	1.4550	1.4605	1.4659	1.4715	1.4770	9	18	27	36	45
56	1.4826	1.4882	1.4938	1.4994	1.5051	1.5108	1.5166	1.5224	1.5282	1.5340	10	19	29	38	48
57	1.5399	1.5458	1.5517	1.5577	1.5637	1.5697	1.5757	1.5818	1.5880	1.5941	10	20	30	40	50
58	1.6003	1.6066	1.6128	1.6191	1.6255	1.6319	1.6383	1.6447	1.6512	1.6577	11	21	32	43	53
59	1.6643	1.6709	1.6775	1.6842	1.6909	1.6977	1.7045	1.7113	1.7182	1.7251	11	23	34	45	56
60	1.7321	1.7391	1.7461	1.7532	1.7603	1.7675	1.7747	1.7820	1.7893	1.7966	12	24	36	48	60
61	1.8040	1.8115	1.8190	1.8265	1.8341	1.8418	1.8495	1.8572	1.8650	1.8728	13	26	38	51	64
62	1.8807	1.8887	1.8967	1.9047	1.9128	1.9210	1.9292	1.9375	1.9458	1.9542	14	27	41	55	68
63	1.9626	1.9711	1.9797	1.9883	1.9970	2.0057	2.0145	2.0233	2.0323	2.0413	15	29	44	58	73
64	2.0503	2.0594	2.0686	2.0778	2.0872	2.0965	2.1060	2.1155	2.1251	2.1348	16	31	47	63	78
65	2.1445	2.1543	2.1642	2.1742	2.1842	2.1943	2.2045	2.2148	2.2251	2.2355	17	34	51	68	85
66	2.2460	2.2566	2.2673	2.2781	2.2889	2.2998	2.3109	2.3220	2.3332	2.3445	18	37	55	73	92
67	2.3559	2.3673	2.3789	2.3906	2.4023	2.4142	2.4262	2.4383	2.4504	2.4627	20	40	60	79	99
68	2.4751	2.4876	2.5002	2.5129	2.5257	2.5386	2.5517	2.5649	2.5782	2.5916	22	43	65	87	108
69	2.6051	2.6187	2.6325	2.6464	2.6605	2.6746	2.6889	2.7034	2.7179	2.7326	24	47	71	95	119
70	2.7475	2.7625	2.7776	2.7929	2.8083	2.8239	2.8397	2.8556	2.8716	2.8878	26	52	78	104	131
71	2.9042	2.9208	2.9375	2.9544	2.9714	2.9887	3.0061	3.0237	3.0415	3.0595	29	58	87	116	145
72	3.0777	3.0961	3.1146	3.1334	3.1524	3.1716	3.1910	3.2106	3.2305	3.2506	32	64	96	129	161
73	3.2709	3.2914	3.3122	3.3332	3.3544	3.3759	3.3977	3.4197	3.4420	3.4646	36	72	108	144	180
74	3.4874	3.5105	3.5339	3.5576	3.5816	3.6059	3.6305	3.6554	3.6806	3.7062	41	81	122	163	204
75	3.7321	3.7583	3.7848	3.8118	3.8391	3.8667	3.8947	3.9232	3.9520	3.9812	46	93	139	186	232
76	4.0108	4.0408	4.0713	4.1022	4.1335	4.1653	4.1976	4.2303	4.2635	4.2972	53	107	160	213	267
77	4.3315	4.3662	4.4015	4.4373	4.4737	4.5107	4.5483	4.5864	4.6252	4.6646					
78	4.7046	4.7453	4.7867	4.8288	4.8716	4.9152	4.9594	5.0045	5.0504	5.0970					
79	5.1446	5.1929	5.2422	5.2924	5.3435	5.3955	5.4486	5.5026	5.5578	5.6140					
80	5.6713	5.7297	5.7894	5.8502	5.9124	5.9758	6.0405	6.1066	6.1742	6.2432					
81	6.3138	6.3859	6.4596	6.5350	6.6122	6.6912	6.7720	6.8548	6.9395	7.0264					
82	7.1154	7.2066	7.3002	7.3962	7.4947	7.5958	7.6996	7.8062	7.9158	8.0285					
83	8.1443	8.2636	8.3863	8.5126	8.6427	8.7769	8.9152	9.0579	9.2052	9.3572					
84	9.5144	9.6768	9.8448	10.0187	10.1988	10.3854	10.5789	10.7797	10.9882	11.2048					
85	11.4301	11.6645	11.9087	12.1632	12.4288	12.7062	12.9962	13.2996	13.6174	13.9507					
86	14.3007	14.6685	15.0557	15.4638	15.8945	16.3499	16.8319	17.3432	17.8863	18.4645					
87	19.0811	19.7403	20.4465	21.2049	22.0217	22.9038	23.8593	24.8978	26.0307	27.2715					
88	28.6363	30.1446	31.8205	33.6935	35.8006	38.1885	40.9174	44.0661	47.7395	52.0807					
89	57.2900	63.6567	71.6151	81.8470	95.4895	114.5887	143.2371	190.9842	286.4777	572.9572					