# FUNDAMENTAL NUMBER TEACHING

*By*

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WITH MANY DIAGRAMS

GEORGE G. HARRAP & CO. LTD
LONDON TORONTO WELLINGTON SYDNEY

*First published in Great Britain* 1955
*by* George G. Harrap & Co. Ltd
182 High Holborn, London, W.C.1



*Composed in Baskerville type and printed by*
*William Clowes & Sons, Limited, London and Beccles*
*Made in Great Britain*

# PREFACE

This book has been written at the request of some two thousand teachers who have asked our advice from time to time in connexion with our articles in the *Teachers World*; they not only told us their difficulties but suggested the kind of help and material they needed. Some of the important results of this correspondence are contained in our book—some examples may be mentioned. (1) Number-patterns, 'figures,' diagrams to help the subnormal, dull, and backward children to visualize the composition of numbers, and number processes such as addition, subtraction, multiplication, division, etc. (2) Notation, or place-value. This cannot be taught in one lesson but needs constant teaching, so that pupils are led to understand the neatness—and, indeed, simplicity—of our decimal notation, which can express the very largest or very smallest number that can be imagined by just ten symbols (0 to 9 inclusive). In this connexion the true and important use of 0 as a place-holder is taught—this teaching is too often neglected or left too late. (3) 'Understanding' is stressed rather than 'rule of thumb.' Too many 'rules of thumb' bore intelligent children who want to think for themselves, even at the cost of a mistake. The dull child needs especially the help or confidence that understanding gives—for example, the proper understanding of place-value and notation makes almost all rules understandable to children. Explanations have been kept as simple as possible.

More ground has been covered than may seem necessary for the Junior School, but the wider the knowledge and outlook of the teacher, the more likely he is to understand the child's difficulties, and be able to explain them in words the child can understand. It is important for teachers to know and understand the two distinct techniques of teaching subtraction as well as the complementary addition method—

indeed, most alternative methods should be known. More ground has also been covered for the sake of quick, bright children. Just as the slow and dull child may tend to be neglected, so too may the very intelligent child, and the latter neglect may be more serious. Brains may be used in very unexpected and undesirable ways. It is surprising what ground an intelligent child of nine or ten can cover. Children should also be encouraged to tackle large numbers. Many, indeed, enjoy doing this. Give them when possible examples of the arithmetic used in commerce. Here is the kind of problem a railway clerk has to tackle when checking the weight and cost of goods carried by train: 1 qr 27 lb. × 132 at 27*s*. for 224 lb., and 0·813 pence per lb. above 224 lb.

In every chapter are examples of exercises (in some cases supplied by schools) that have given children real pleasure, such as successive subtraction, adding to 9, finding the quotients of long-division exercises, etc. As far as possible the work in each chapter has been carefully graded—some steps may even seem unnecessary—but these steps give the slow child confidence; his success or failure often depends on careful grading. The language of arithmetic has also been stressed; as has the keeping of notebooks by the children.

Our book is not a scholarly treatise on arithmetic but has been written mainly from the point of view of teaching children. We have read many books on the teaching of arithmetic. The two we have found by far the most helpful are—a modern book, *The Principles of Arithmetic* by Keith and Robertson (Blackie, 1951)—a scholarly, wise book—and a much older book, *The Teaching of Arithmetic* by Paul Klapper (D. Appleton-Century Co. Inc., first published 1916).

R.K.P.
M.I.R.P.

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CHAPTER I

# DEVELOPING NUMBER-IDEAS

*Number-experiences. The Ideas behind Numbers. Grouping and Counting: Number-patterns. Comparing Numbers: Number Scrapbooks. The Entity of Numbers. Games.*

## NUMBER-EXPERIENCES

Children, whether in school or out, develop number-ideas; this development is inevitable. The function of the teacher is to vary and control the child's experiences so that, as far as possible, his number-ideas should be sound, and also to give him *new* experiences, since children's number-ideas grow slowly (as, indeed, do those of adults). Think of the many interpretations of the number 12 to be learnt. Some of them are known only to scholars who have spent years in the study of mathematics. Twelve is one more than 11, one less than 13; two more than 10, two less than 14. It is one-half of 24, one-third of 36; two times 6, six times 2, four times 3. Twelve is three-fourths of 16, two-thirds of 18, three-fifths of 20. It is one dozen, and one-twelfth of a gross. Twelve is the square root of 144, and the cube root of 1728, and so on. These are enough to show that the number-ideas grow from very simple beginnings to very complicated conceptions.

Bright children enjoy being introduced to some of the interpretations of the different numbers as they progress. It is an interesting form of revision, and encourages thought.

At the beginning of the Junior School course one must find out what the number-experiences of the children are, especially those of the backward children.

## THE IDEAS BEHIND NUMBERS

Find out if children really know the idea behind 3 or 4.

What is 4? If one writes the word 'four' or the symbol '4' on the board, neither is four. Four cannot be seen, felt, or heard, but the child can look at four balls, or four sweets in a bag, or four crayons in a box, or listen to a clock striking four. What he sees are balls, sweets, crayons; what he hears are the strokes of a hammer on a bell. He does not see or hear 'four,' but four of some object or sound. If the backward child is to develop a 'fourness' idea he must have many and varied concrete experiences. This needs careful planning on the part of the teacher. To-day far too much tends to be left to chance, to incidental teaching and to free-activity periods. Reading the word four or recognizing the symbol 4 does not mean the child has any idea what four is. Very often the idea does not come until the child is six. Children in the Infant School enjoy counting and do it spontaneously, but most teachers of little children know that to possess the language of counting does not imply an understanding of its meaning.

## Grouping and Counting: Number-patterns

In the Infant School the children learn to use correctly the number symbols 1–10, to associate these symbols with the names **one**, **two**, **three**, etc., and to associate the symbols with concrete number-groups (counters) and number-patterns

| | | | | | |
|---|---|---|---|---|---|
| GROUP OR PATTERN | ● | ● ● | ● ●<br>● | ● ●<br>● ● | ● ●<br>●<br>● ● |
| NAME | ONE | TWO | THREE | FOUR | FIVE |
| SYMBOL | 1 | 2 | 3 | 4 | 5 |

Fig. 1. Basic Number-Patterns

as in Fig. 1. Figs. 2 to 6 show some number-patterns or groups used in Infant Schools. In the first year in the Junior School, and especially with backward classes, it is well worth while showing the children these groupings and discussing these with them from a different point of view from that of the Infant School. Take it for granted they know them, and let them discuss them and say which they think are most useful. They can make their own groupings, and choose one to go on the board. Point out that they are very useful for addition, subtraction, multiplication, and division. Fig. 2 is

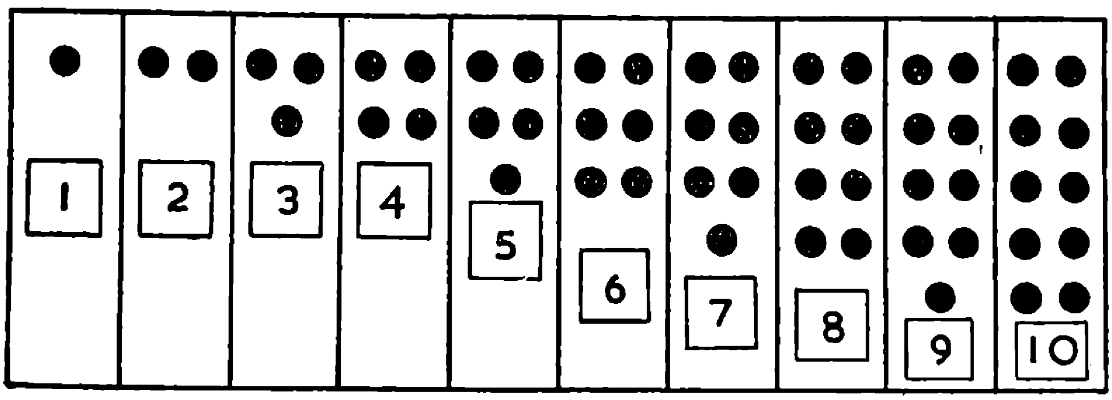



FIG. 2. THE MONTESSORI GROUPING

the **Montessori grouping**, vertical grouping in twos. This shows clearly odd and even numbers. One group is easily converted into another by addition or subtraction, and it is useful for the "two-times table" and the "table of twos."

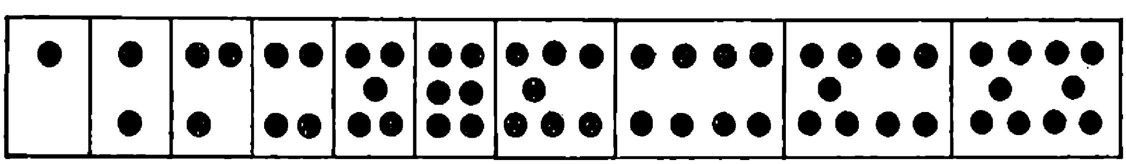


FIG. 3. MCLELLAN AND DEWEY'S GROUPING

Fig. 3, **McLellen and Dewey's grouping**, is good. Children easily visualize four, five, and six without the necessity of counting, the two fours making eight, and so on. Fig. 4, the

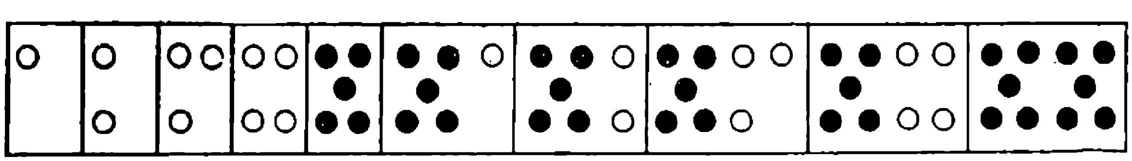


FIG. 4. THE WELLBENT GROUPING

**Welbent grouping** on the basis of five, is useful for the addition and subtraction of five and the table of fives. The children are interested in grouping the numbers in threes, as in Fig. 5, and fours, as in Fig. 6.

Explain to the children that making number-patterns with

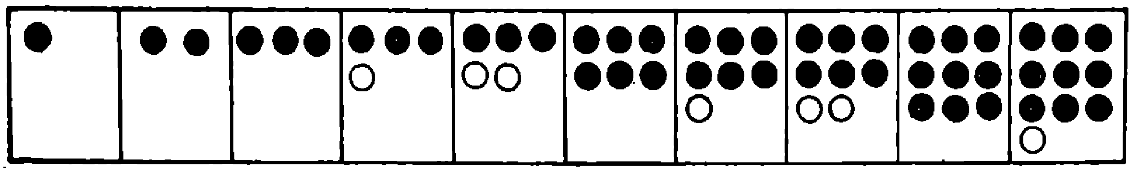


FIG. 5. GROUPING IN THREES

counters or drawing them with dots is not babyish work. The learned Greeks of long ago made number-patterns with counters. They arranged counters to represent numbers, and classified the numbers according to the shape of the group of

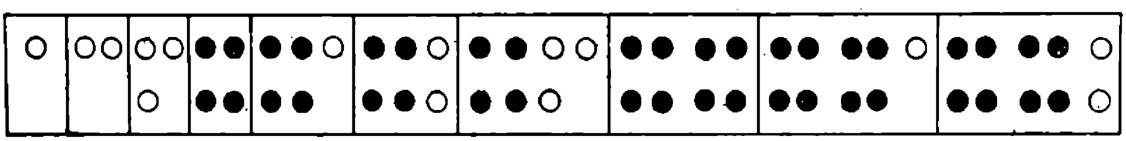


FIG. 6. GROUPING IN FOURS

counters. The numbers 1, 3, 6, 10, 15, 21 were called **triangular numbers** because they can be arranged to form triangles, as below:

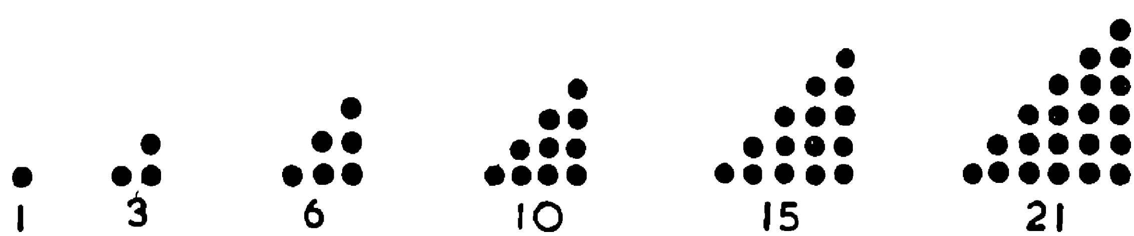



FIG. 7. TRIANGULAR NUMBERS

The numbers 1, 4, 9, 16, 25, 36 were called **square numbers** because they could be arranged to form squares or represent squares:

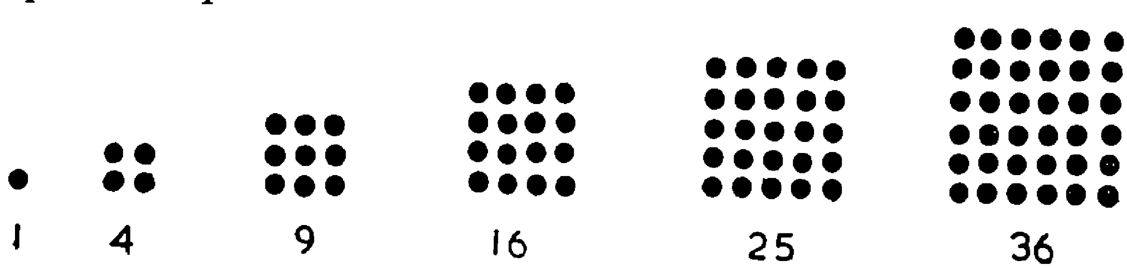



FIG. 8. SQUARE NUMBERS

The children are interested to find out that two triangular groups make a square group—thus 1 + 3 makes 4, 3 + 6 makes 9, 6 + 10 makes 16, etc. The Greeks made other number 'shapes' or 'figures,' and the interest in these shapes of **figurate** numbers went on in all parts of Europe where there were learned men. It is probably because of these **figurate** numbers that we get the word **figure** for 'number.'

Children like to see what 'figures' they can make from different numbers. Some numbers make a diamond shape, some a star, the letter T, and so on. But they like especially trying to make the triangular numbers.

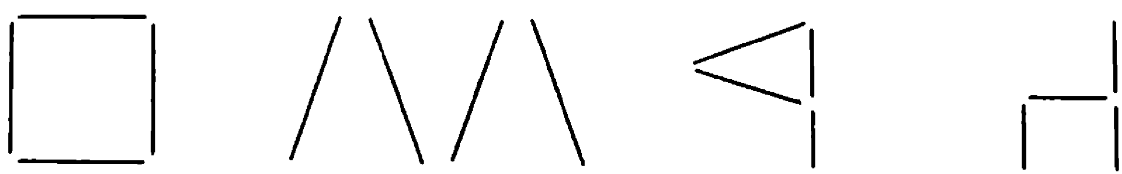


FIG. 9. PATTERNS FOR FOUR

Concrete material of all kinds may be used, especially with backward children, for making number-patterns and shapes, for counting-exercises (see Chapter II), and for addition and subtraction. Match-sticks are very useful for teaching numbers from ten onward. Unplanned objective presentation and a *limited range* of materials do not often make for effective objective teaching. The patterns made with counters or dots can be changed by giving the children new material, such as a certain number of sticks of the same length. Fig. 9 shows the shapes that can be made from four sticks. It also shows them that 4 is four 1's, two 2's, 3 and 1, and 1 more than 3.

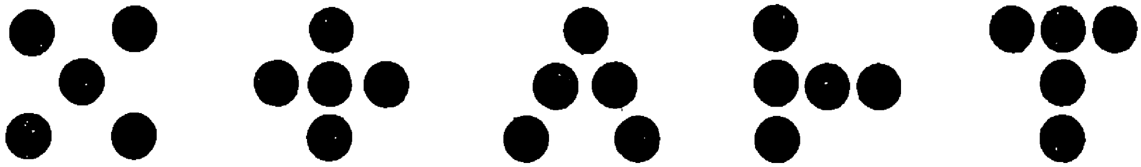


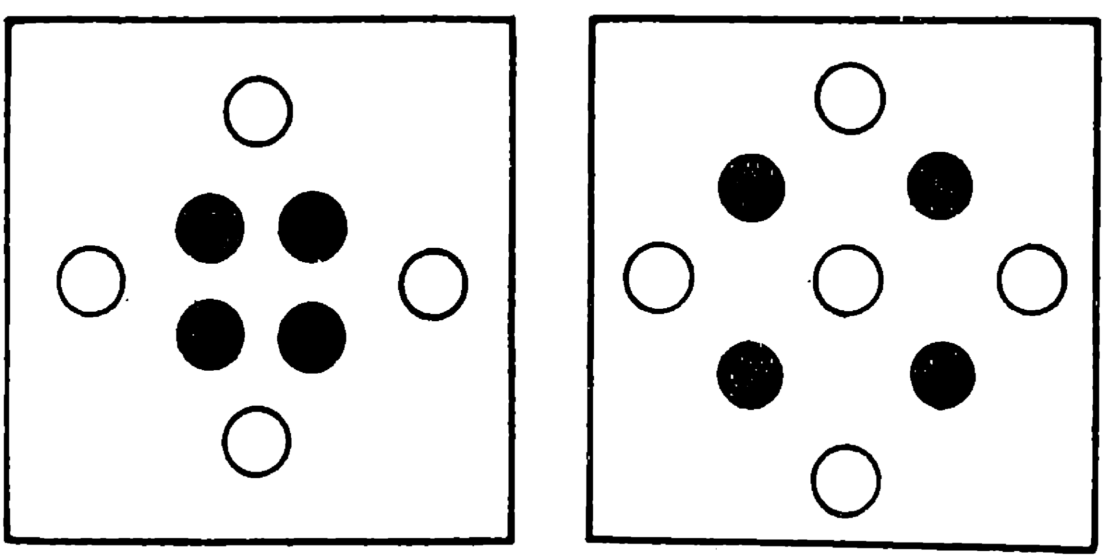
FIG. 10. PATTERNS FOR FIVE

Do not let children think of five as the domino group only. Let them find out as many different ways of arranging a group of five dots or counters as possible (Fig. 10). They can also make shapes with five sticks—*e.g.*, a saucepan, a fan, a hat with a pointed crown and broad brim, and so on.

Valuable variety can be obtained for number-experiences and counting by (1) linking the work up with games and other activities and occupations such as giving out and collecting pencils, paper, etc., arranging nature finds, playing shops, etc. All teachers know the value for this of dominoes, playing-cards, and other games; (2) varying the senses appealed to in the lesson; thus children count or add steps, sounds (the strokes of a gong), jumps, as well as things they can feel and handle.

## Comparing Numbers: Number Scrap-books

It is of great value to let both dull and bright children make scrap-books for number-pictures or -patterns. One big scrap-book representing the efforts of all the class can be made for the classroom, as well as individual books. Figs. 11 and 12



Figs. 11 and 12. Further Examples of Number-Patterns

show examples of patterns. Small circles are drawn and arranged in different ways and coloured in two colours. Quite elaborate patterns are possible with the bigger numbers. The children plan and colour their patterns on pieces of paper (squared paper is helpful), then cut them out and paste them in their scrap-books, which can be made of brown paper. Under each pattern they put the name of the number and its symbol. Some children's scrap-books are a pleasure to look at. As far as possible let them plan everything for themselves. Many children find as many patterns as possible for each number-group, leaving out 1 and 2; some put only the best pattern for each group in their books. Some make patterns also with sticks. Through these scrap-books the children learn to analyse and compare numbers. They see 9 more clearly as 5 and 4, or, if they alter the colours, as 3 and 6.

The dull child may need special help to see that 5 beads are one more than 4 beads, that 5 pencils are less than 6 pencils, that 3 balls and 2 balls make 5 balls. These are necessary sense-experiences. The child moves through three stages as he grows in ability to make comparisons. First, he compares two groups of objects which are actually present to see which includes the larger number. Second, he compares imaginary groups of objects in response to such questions as, "Which would you rather have, 8 pennies or 6 pennies?" Finally, comparisons will be made between numbers in the abstract: "Which is more, 9 or 7?"

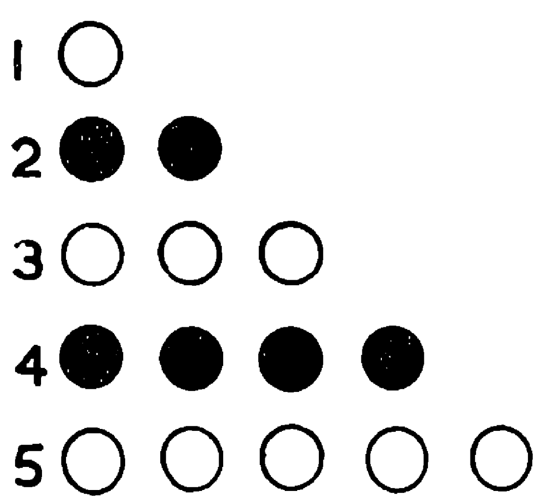


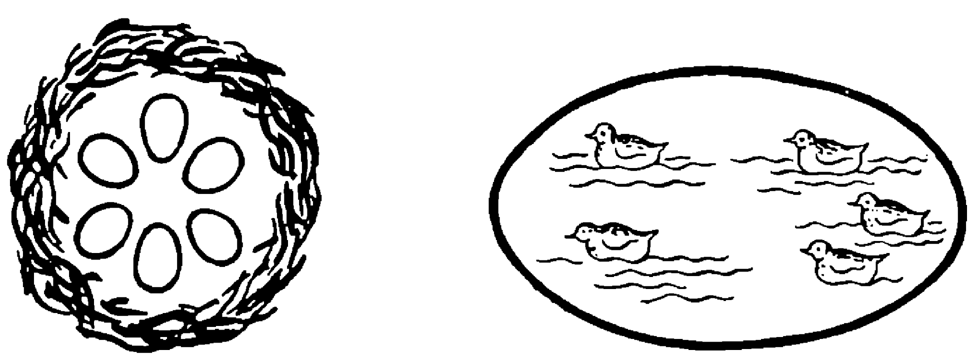

FIG. 13. NUMBER-PATTERNS FOR BACKWARD CHILDREN

The arrangement of counters of two colours in Fig. 13 (often used in Infant Schools for the comparison of numbers) is useful for backward children. Each child arranges his counters to represent the numbers—for example, from 1 to 5, as in

Fig. 13, using red and white counters for alternate numbers. The numbers are put by the side of each row. The teacher questions the children on the difference between the numbers. How many more is 5 than 2? 4 than 1? etc. Which is the greatest number? Which is the smallest? Which is the middle number? The number of 2's in each line can be found. There are no 2's in line one; one 2 in line two, and so on. Other numbers can be treated in a similar way.

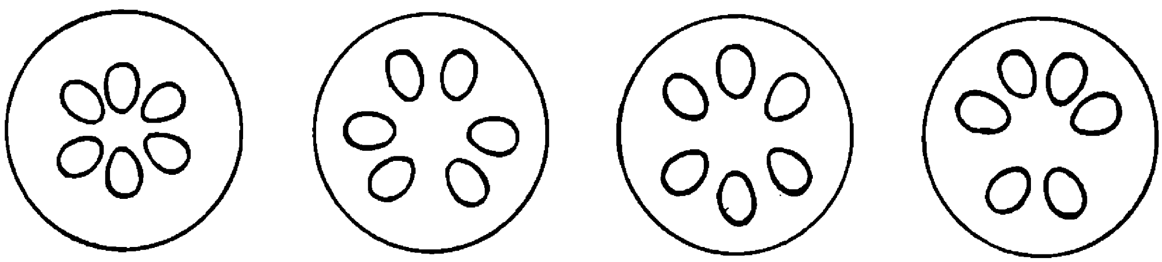
THE ENTITY OF NUMBERS

Counting is the fundamental number-experience. The child finds the names of the number-groups in Figs. 1 to 6 by counting. To get to 'six' he counts one, two, three, four, five, six. One of the main reasons for the activity of placing objects in groups to form patterns, or drawing dots of different numbers to make patterns, is to introduce the idea of the analysis (or the relationship) of numbers, that 5 is 3 and 2, 7 is 4 and 3, and so on. Through these groups children may be led away from *counting in ones*, led away from thinking, for example, that 6 is obtained by repeating a series. They must be also helped in every way to recognize 6 or any number as a whole. Six is an *entity*—a whole is itself, and not always a piling up or arranging of six *separate* things.



FIGS. 14 AND 15. DRAWINGS TO EXPLAIN NUMBER-ENTITIES

To show in a practical way the entity of six, let the children draw or model in clay a nest with six eggs in it (Fig. 14). The eggs form a group or family. These are, of course, six separate eggs, but when they are placed within the boundary of the nest the children will see them as a whole. Other good illustrations for the entity of a number are cows or sheep in a field, bulbs in a bowl, ducks on a small pond (Fig. 15), cups on a tray, little cakes on a plate, and so on. A good deal of helpful work can be done by letting children draw circles and arrange in each circle a given number of things in different ways. Placing a definite number of objects within a boundary enables children to appreciate better the entity of a number, and to analyse it, than if the objects are spread in a line along the desk. A *long line* suggests counting; one cannot see the numbers as wholes.



(*a*) (*b*) (*c*) (*d*)

FIG. 16. DRAWINGS TO EXPLAIN THE ENTITY OF SIX

Fig. 16 is a useful illustration. Fig. 16(*a*) shows 6 as a whole, an entity; Fig. 16(*b*) shows that 6 has parts, or can be broken up into parts within itself. Fig. 16(*c*) and (*d*) show that the parts can be interchanged. Fig. 16, therefore, helps pupils to realize that 6 is an entity; that it can be made of three times 2, and not only built up of 1's; that it is also made of two times 3; and finally that, as 6 is 4 and 2, it is also 2 and 4. These points are fundamental if the children are to form true concepts of number.

Children enjoy making drawings similar to those in Fig. 16 for different numbers; for example, 8 as a whole, in 2 parts, in four parts, the parts interchanged, 8 as 2 and 6. The children can make very attractive little booklets showing the entity of numbers, also booklets for number-patterns. In this way they build up a little library of gay arithmetic books and keep a record of their work. It is most important that children should themselves keep records of their work. How this can be done is suggested in the different chapters to come.

## GAMES

Games are most valuable for number-experiences. Among the most valuable are (1) games of the 'Fish-pond' type—hopscotch, number arches, skittles, spinning tops, home-made floor or table quoits, and scoring games of all kinds; (2) desk games or table games—dominoes, 'card' dominoes (playing-cards are valuable for individual and group work, since both the series and the group idea is inherent in the playing-card), group games of the Lotto and Snap type, spinning discs, games with numbered boards and dice, such as Snakes and Ladders and many similar games; (3) games children imitate from adult life. These are especially valuable, and include setting the table for varied numbers of guests, shopping, weighing and measuring, playing milkman, postman, etc. In the Junior School fewer of these games are needed. Any especially useful games for adding, subtracting, etc., are mentioned in the chapters dealing with these processes. Dominoes, Lotto, and scoring games are all used in the Junior School.

The following books contain descriptions of games used in teaching arithmetic as well as detailed accounts of work done in Infant Schools. Some of the information in these books is useful to teachers in the Junior School:

Monteith, A.: *The Teaching of Arithmetic* (Harrap, 1955).

Brideoake, E., and Groves, I. D.: *Arithmetic in Action—A Practical Approach to Infant Number Work* (University of London Press, 1948).

Hume, E. G., M.A.: *Learning and Teaching in the Infant School* (Longmans, 1948).

Williams, D.: *A Realistic Approach to Number Teaching* (Oxford University Press, 1948).

Kenwrick, E. E.: *Number in the Nursery and Infant School* (Routledge, 1949).

Serjeant, F. J.: *From Day to Day in the Infant School* (Blackie, 1952). (This contains many descriptions of useful Domino games.)

CHAPTER II

# FIRST STEPS IN ADDITION. LEARNING THE FORTY-FIVE ADDITION FACTS

*Exercises in Counting that help Addition. Teaching Addition. The Forty-five Basic Addition Facts. Charts. Addition Cards. Games. Story Exercises or Problems. Setting down Addition vertically. Speed and Testing.*

The number-groups and -patterns given in Chapter I prepare the way for both addition and subtraction. Just how much practical work has to be done in the first year in the Junior School depends on the stage at which the children are. Many children come from the Infant School knowing their number bonds (addition facts or combinations) so well that any further practical work in adding 'things' is only waste of time, and may even be a hindrance. If children are to learn to 'add' they must stop counting actual things, pictures of things, or dots, etc. Only in the case of backward or dull children must this practical work go on. We know it is true that all the child's early work should be founded on his *personal experience*, but this does not mean that all arithmetic must be concrete. It does not mean that a child must only deal with numbers of articles and never with numbers in the abstract; must always add nuts to nuts, or take apples from apples, and never add 3 to 6, or take 7 from 11.

Wise teachers have noticed that, although abstract numbers present difficulties to children, labelling quantities in an exercise does not necessarily make the exercise easier or more *real*. We have to face the fact that the fundamental operations of addition, subtraction, multiplication, and division belong to the abstract side of mathematics, and can be most simply and effectively taught in the abstract in the Junior School. Addition is the first process in which the use of objects is soon dropped.

## Exercises in Counting that help Addition

Before dealing with the teaching of addition, a few words must be said about counting. Little children enjoy counting, and do it quite spontaneously. It is valuable from the point of view of learning the names and the order of the numbers—1, 2, 3, 4, etc.—but to possess the language of counting does not imply an understanding of its meaning. This children learn by counting objects and groups (see Chapter I). Counting for children of six and seven may be developed further, and more closely related to work in addition than pure counting. Counting by 1's is really adding 1 each time, but it is of comparatively little value with regard to addition. The forms of counting closely related to work in addition include counting in 2's, 3's, and 4's. Children of six in the Infant School can do this when they have had experience in handling objects in groups. Lack of practice or proficiency in this form of counting is sometimes a cause of slowness and inaccuracy in the Junior School, since many children never seem to pass beyond the stage of counting in ones. Teachers do not always appreciate how wide the gap is between counting in ones and adding. Many pupils never bridge this gulf; they never learn to add, they never learn additive counting.

In the Infant School the children also learn to count in 'tens', and learn the convenience of counting in tens. They make up their own sets of ten from beans, beads, sticks.

The following list of *additive* counting exercises may be useful:

(1) Counting by 2's, 3's, 4's, etc., starting at 0; thus: 0, 2, 4, 6, etc.; 0, 3, 6, 9, 12, etc.
(2) Counting by 2's, 3's, 4's, etc., starting at a multiple of 2, 3, 4, etc.; thus: 4, 6, 8.
(3) Counting by 2's, 3's, 4's, etc., starting at a multiple of 2, 3, 4, etc., and *going down*; thus: 12, 10, 8, 6, 4, 2, 0.
(4) Counting by 2's, 3's, 4's, etc., starting at 1.
(5) Counting by 2's, 3's, 4's, etc., starting at any number; thus: 5, 7, 9, 11.

(6) Counting by 2's, 3's, 4's, etc., starting at any number, going down or counting backward.
(7) Counting from 10 to 100 by 10's both forward and *backward*.
(8) Counting in 10's from any number—*e.g.*, to count in 10's from 2, 12, 22, 32, 42.
(9) Counting in 5's to 100 both forward and backward.

The clock-face is a good example of counting by 5's up to 60. Let the children start from 5 and count up and down; that is, from 5 to 60 and from 60 to 5.

Steps number 3 and 6, 7, 9 pave the way for drill in subtraction.

Many teachers in Junior Schools take counting exercises in the order 10's, 2's, 5's, 3's, 4's, 6's, 8's, 9's, and 7's, an order indicated by experience rather than experiment. In the above list the easiest counting exercises come first. Counting by 8's, 9's, and 7's is most difficult for children.

In teaching children to count by 2's (not, of course, using objects) the following sense-approach may be used. The teacher writes these two lines of figures on the board for counting:

(*a*) Count 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12.
(*b*) 1, **2**, 3, **4**, 5, **6**, 7, **8**, 9, **10**, 11, **12**.

The children read the second line (*b*), emphasizing the even numbers thus: 1 (*soft*), 2 (*loud*), 3 (*soft*), etc. The teacher rings round these numbers. The children repeat them a number of times, until 2, 4, 6, 8, etc., have made a visual and oral impression. Then they count 2, 4, 6, 8, 12, the odd numbers being rubbed out. It is interesting to let children count in 2's and make marks as they emphasize the 2's thus:

• • • • • •
2 4 6 8 10 12

From these marks they will see later that 6 two's make 12. They can find how many 3's in 18 or 21 in the same way:

• • • • • • •
3 6 9 12 15 18 21

These methods of counting in 2's or 3's apply to all numbers. Counting exercises will be referred to again in connexion with the teaching of addition, subtraction, multiplication, and division.

## Teaching Addition

Addition is to save counting. When the child finds the 'how many' in 5 and 3 by saying, "*five*, six, seven, eight," he is counting. When through memory he answers the *sum* by seeing 5 and 3, or hearing "five and three," he is adding. The child first finds the 'how many' by counting and then *remembers* it to save counting.

If children are 'to add,' not count, they must learn the **basic addition facts**, also called "combinations" or "number bonds." They must know automatically that 5 and 4 are 9. The addition tables are just as important as the multiplication tables, but rarely so well taught.

## The Forty-five Basic Addition Facts

These are set out clearly in Fig. 17. Although they are not given in two forms—*e.g.*, 3 + 4 and 4 + 3—the *two forms* must be taught. Quick children may realize that 9 + 7 is the same as 7 + 9, but a child who knows 9 + 7 may fail over 7 + 9. These 45 facts when mastered are the foundation of addition; without a knowledge of these speed and accuracy cannot be attained.

Many modern writers of textbooks include among the addition facts 0 added to all the numbers from 0 to 9. It is extremely doubtful if there is any value in teaching children to add 0. We cannot *add* 'nothing,' and the child must know the meaning of **zero** or **nought**—namely, 'not anything.' Including 0 in the addition facts makes children think about it as a figure or digit, and forget it stands for 'nothing,' and, worst of all, forget its important work as a place-keeper or -holder. This question of place-keeper is fully discussed in

Chapter III. We cannot do better than quote from Keith and Robertson's *Principles of Arithmetic.*[1]

It is doubtful if the artificial type of exercise $\begin{array}{cccc} 4 & 0 & 3 & 0 \\ 0 & 7 & 0 & 8 \\ \hline \end{array}$ serves any useful purpose. It is intended to prepare the pupil for the technique of adding large numbers, but their sheer triviality should rule them out as unworthy and unnecessary. To give these so-called 'zero-combinations' equal status with the other addition combinations is to misrepresent them. However, the teacher may argue that the child coming across $\begin{array}{c} 0 \\ 7 \\ \hline \end{array}$ in an addition exercise *does* in fact pause and reflect on the result of this 'addition,' and therefore practice in the above type of admittedly artificial exercise may be helpful. In that case, the essential thing for him to realize is that these zero combinations must not in any circumstances be taught as results to be committed to memory in the same way as the other addition facts. They must be found—as all adults find them—by a moment's consideration of the meaning of the exercise.

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 9 Combinations or bonds |
| +1 | +1 | +1 | +1 | +1 | +1 | +1 | +1 | +1 | |
| 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | |
| | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 8 Combinations |
| | +2 | +2 | +2 | +2 | +2 | +2 | +2 | +2 | |
| | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | |
| | | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 7 Combinations |
| | | +3 | +3 | +3 | +3 | +3 | +3 | +3 | |
| | | 6 | 7 | 8 | 9 | 10 | 11 | 12 | |
| | | | 4 | 5 | 6 | 7 | 8 | 9 | 6 Combinations |
| | | | +4 | +4 | +4 | +4 | +4 | +4 | |
| | | | 8 | 9 | 10 | 11 | 12 | 13 | |

FIG. 17 (*a*). THE FORTY-FIVE BASIC ADDITION FACTS. (See also page 26).

[1] Blackie, 1951

$$\left.\begin{array}{ccccc} 5 & 6 & 7 & 8 & 9 \\ +5 & +5 & +5 & +5 & +5 \\ \hline 10 & 11 & 12 & 13 & 14 \end{array}\right\} \begin{array}{c} 5 \\ \text{Combinations} \end{array}$$

$$\left.\begin{array}{cccc} 6 & 7 & 8 & 9 \\ +6 & +6 & +6 & +6 \\ \hline 12 & 13 & 14 & 15 \end{array}\right\} \begin{array}{c} 4 \\ \text{Combinations} \end{array}$$

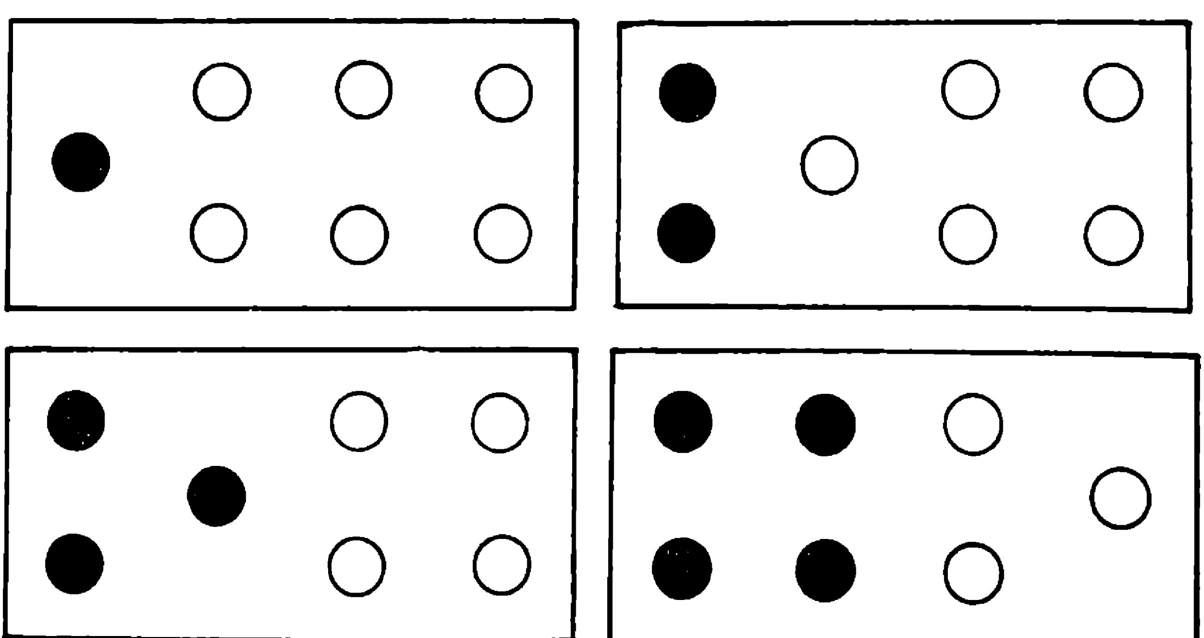
$$\left.\begin{array}{ccc} 7 & 7 & 7 \\ +7 & +8 & +9 \\ \hline 14 & 15 & 16 \end{array}\right\} \begin{array}{c} 3 \\ \text{Combinations} \end{array}$$

$$\left.\begin{array}{cc} 8 & 8 \\ +8 & +9 \\ \hline 16 & 17 \end{array}\right\} \begin{array}{c} 2 \\ \text{Combinations} \end{array}$$

$$\left.\begin{array}{c} 9 \\ +9 \\ \hline 18 \end{array}\right\} \begin{array}{c} 1 \\ \text{Combination} \end{array}$$

45 Combinations

FIG. 17 (*b*). THE FORTY-FIVE BASIC ADDITION FACTS



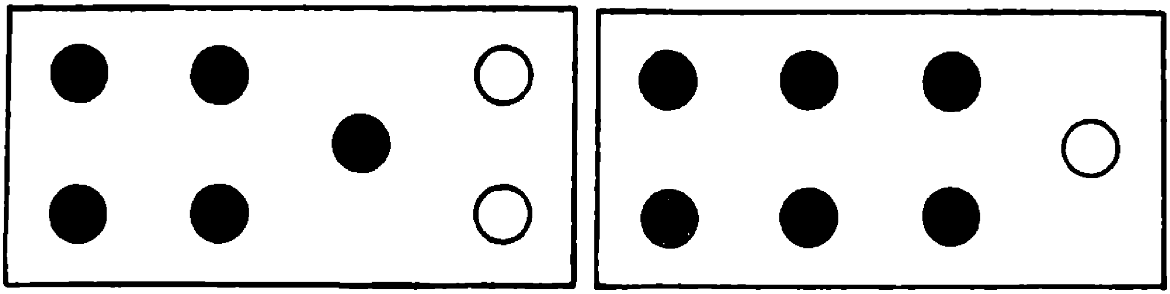


FIG. 18. THE ADDITION COMBINATIONS FOR SEVEN

The number-patterns suggested in Chapter I may be used to teach the various addition combinations to great advantage. Fig. 18 shows a good way of teaching the bonds or combinations of seven. The children draw seven small circles. Then they redraw the seven as shown, each time colouring one circle red. They notice the number of red circles at each stage, and the fact that the sum in each case is seven. The addition combinations in Fig. 18 are then summarized:

$$7 = \begin{cases} 1+6 \\ 2+5 \\ 3+4 \end{cases} \qquad 7 = \begin{cases} 4+3 \\ 5+2 \\ 6+1 \end{cases}$$

Addition tables can be constructed in this way for each number, and then learnt by heart. When a child hears or sees $3+4$ or $4+3$ he must be able to say automatically 7. It is *essential* that the child should grasp the commutative nature of addition—that $1+6$ or $6+1$ are 7; $5+2$ or $2+5$ are 7. This halves the number of facts he has to learn. But the principle is not one that can be taught in a single lesson. It is a number-idea that grows as the child studies the composition of numbers through concrete material and number-patterns. Fig. 18 can also be used for subtraction.

The commutative nature of addition is stressed if children often repeat the addition combinations for 7 and other numbers, thus:

One and six are seven.
Six and one are seven.
Two and five are seven.
Five and two are seven.
Three and four are seven.
Four and three are seven.

It is well to give drill and exercises as above on the composition of numbers from 1 to 10 first, and then from 10 to 18. Instead of drawing dots, as in Fig. 18, let the pupils use counters—for example, 11 counters to find the composition of 11. With their counters they work out the various combinations and write them in the form of tables:

$$\left.\begin{matrix} 9+2 \\ 8+3 \\ 7+4 \\ 6+5 \end{matrix}\right\} = 11 \qquad \text{or} \qquad \left.\begin{matrix} 2+9 \\ 3+8 \\ 7+4 \\ 6+5 \end{matrix}\right\} = 11$$

They do the same with the numbers 12, 13, 14, 15, 16, 17, 18. The tables they make they write in booklets or notebooks which they can illustrate. The children should discover all their addition facts experimentally, then write them out as tables and learn them.

In drilling the children on the combinations of 9, the teacher will ask varied questions such as:

5 and 4 are how many?
4 and 5 are how many?
4 and what number make 9?
5 and what number make 9?
What two numbers added together make 9?
Give another two numbers, and so on.

The subtraction combinations are just another way of stating the corresponding addition combinations. $4+?=9$, $9-4=5$. Intelligent children often never have to learn the subtraction facts. Fig. 18 may be used for subtraction.

## Charts

Fig. 19 (*a*) and (*b*) shows a useful chart for testing and revising the 45 Addition Facts or Bonds. Separate small cards, as in Fig. 19 (*b*), are prepared for the answers. A child works through the chart seeing how quickly he can cover the addition facts with the answers. He comes to $7+6$. If he knows the answer 13 at once, he quickly finds the card with 13 on it. He pins it over $7+6$ if the chart is a wall chart, or places it on 7 + 6 if the chart is flat on the table. The children may be encouraged to test themselves with it at any odd time, such as before school in the morning. They copy the facts

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 1+1 | 1+2 | 1+3 | 1+4 | 1+5 | 1+6 | 1+7 | 1+8 | 1+9 |
| 2+2 | 2+3 | 2+4 | 2+5 | 2+6 | 2+7 | 2+8 | 2+9 | 3+3 |
| 3+4 | 3+5 | 3+6 | 3+7 | 3+8 | 3+9 | 4+4 | 4+5 | 4+6 |
| 4+7 | 4+8 | 4+9 | 5+5 | 5+6 | 5+7 | 5+8 | 5+9 | 6+6 |
| 6+7 | 6+8 | 6+9 | 7+7 | 7+8 | 7+9 | 8+8 | 8+9 | 9+9 |

9

16

17

FIGS. 19 (*a*) AND (*b*). NUMBER-BOND CHART

they do not know and study them aloud at home or quietly in their spare time at school. They like to time each other to see who can give the most answers in a given time. Some aim at saying all the addition facts in 45 seconds!

A child will often bring the chart to the teacher and ask to be tested for accuracy and speed. He starts from the first sum, 1 + 1. If he fails over a sum he may be allowed to put a little mark where he stops. He moves this mark forward as he makes progress. The first line can be left out almost at once—*adding one.* This method of putting the responsibility for learning the addition facts on to the children answers very well in a good many cases. New charts should be put up from time to time with the bonds or combinations arranged in a different order. Some children unconsciously memorize the order of the answers—4, 5, 6, etc. They can say the answers perfectly if the sums are in the right order, but break down when the sums are mixed up!

It is best sometimes to let the children concentrate upon two or three combinations until they are thoroughly learnt, as $6+7$, $6+8$. Introduce many pairs of numbers in which the first number is less than the second. Children find it much easier to add $5+2$ than $2+5$; $9+2$ is easy, but $2+9$ is another matter!

## Addition Cards

These are used in a great many schools, and are essential for making the addition facts automatic. A large supply of cards about 3 in. by 2 in. are cut out. On one side is printed a

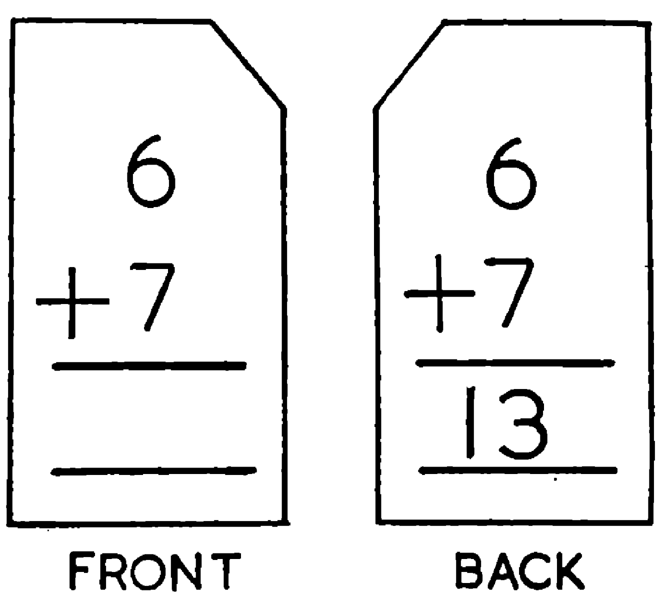



Fig. 20. An Addition Card

number-combination, and on the other side the combination with the answers as in Fig. 20. Children can help to make these cards. Dr Schonell, in *Diagnosis of Individual Difficulties in Arithmetic*[1] suggests that the top left-hand corner of each card should be cut off, so that it can easily be kept right way up. The cards can be arranged in sets according to difficulty. Numbers that add up to not more than 10 or 12 make an easy set. The teacher must arrange the sets according to the needs of the pupils.

The cards can be used by children either individually or in groups. Individual work should be encouraged in the case of many backward children. A child likes to see how quickly he can work through a set. Any combination that is not known is put aside for future study. Children are interested in this method because it is active, and it both teaches and tests. As soon as they are fairly proficient they should be encouraged to test each other.

Games of Snap help to encourage speed and accuracy. Snap can be played by pairs of children using one set of cards. Another Snap game is played with four or more children and the teacher. The children have from 12 to 20 cards each. They arrange the cards in front of them with the answer sides underneath. The teacher puts a card with a number on it—say 13—in the middle of the table, saying the number out loud in case any child cannot see it. Each child looks over his cards to see if he has a sum that makes 13—namely, 7 + 6, 8 + 5, or 9 + 4. If he has he places it by the 13, saying, "Seven plus six makes thirteen." This game is often used by teachers to test the combinations the children are learning. The child who is the first to put out his card wins a point, but it need not be a competitive game. It gives a child sufficient pleasure to see how quickly he can find the right combination.

A similar game can be played by three children—two players and one dealer. The two players have each a full set of cards, about 22. Each player spreads his cards in front of him with the answer sides underneath. The dealer has a set of cards bearing only the answers. The dealer places a card in the centre—say 11. The first player to put out 8 + 3 or 6 + 5, etc., scores a point.

[1] Oliver and Boyd, 1937

As soon as children know most of the number-bonds on Fig. 19(*a*), instead of finding the sum of any two numbers, the process can be reversed. The child takes one of the small cards—for example, the one with 16 on it. He finds out from the number-chart two numbers into which the number can be split, and writes them down thus: 16 = 9 + 7, 7 + 9, 8 + 8. They may work back through the chart in this way, beginning with 18.

18 = 9 + 9
17 = 9 + 8, 8 + 9
16 = 9 + 7, 7 + 9, 8 + 8
15 = 8 + 7, 7 + 8, 9 + 6, 6 + 9, and so on

## Games

There is much scope for ingenuity, patience, and variety of treatment when teaching the addition facts so that children do not grow weary. The following games have been found useful.

*The Game of More and Less.* This is a useful game for testing backward children. The teacher writes a number on the board (to begin with, any number from 1 to 18). Under the first number she puts a second number, either more or less than the first. She calls on a child to say if the second number is more or less than the first. If a child makes a mistake he lays out each number in counters so that he can easily see which is the greater. If a child answers correctly, "Nine is more than four," the teacher asks, "How many more?" The child knows 5 + 4 = 9, and answers at once "Five." It is interesting to test children with bigger numbers, such as 41 and 39. Some children may say 39 because 9 is more than 1. If a child says less he can find how many less by counting.

*Guess How Many.* This game is played in groups of two or more. Each child has about eight counters—match-sticks, acorns, etc.—which he keeps hidden in a box or bag. The first player takes some in his hand. If he takes two he tries to make his hand look as big as possible. He asks his companion, "How many?" If the guesser says, "Two," and it is the right number, he receives two acorns. If he says "Two" and the right number is five, because his guess is three short of the right number he must give up three acorns. He says, "Two and three are five. Here are three acorns for you." Then the next player has a turn. If he says five and the right number is seven he will say, "Five and two are seven. Here are two acorns for you." The game goes on until every one has had a certain number of guesses.

*Missing Number Game.* I am thinking of two numbers that make five. One is two. What is the other? At first the children's number-patterns help them to answer. Later, when they know their addition facts, they at once think, Three and two make five. They see 5 as 3 and 2. They like to see how quickly they can work these puzzle sums:

| | | | | | |
|---|---|---|---|---|---|
| 4 | ? | 9 | ? | 3 | ? |
| ? | 5 | ? | 5 | ? | 8 |
| 6 | 8 | 11 | 7 | 9 | 17 |

*A Card Game.* Each child has a set of cards bearing the digits from 1 to 9. Two children are chosen, each to put a card in the centre of the table. Whoever calls out the sum of the two numbers first gets them; for example, if the two numbers are 8 and 2 the child who first calls out "Ten" gets the cards. In practice it is best to let each child have *a turn* at calling out, otherwise the slow children never get a chance. It sometimes happens that two numbers are put in the pool and no one knows their sum. In this case each child takes his number back and the teacher tells them the sum which they try to memorize. When the two cards appear again, the children are keen to show they know their sum.

## Story Exercises or Problems

Give the children opportunities of applying their number-facts to everyday life, to the things they see around them, by oral work, problems or puzzles, stories, drawings, and so on. Puzzles or problems need very careful wording, or they may do more harm than good. This, for example, is an unreal problem: "Yesterday Tom had seven toy soldiers, but lost two of them. How many has he left?" He must know how many he has left in order to know he has lost two. This is a real problem: "Yesterday Tom had seven toy soldiers. To-day he can only find five of them. How many has he lost? 5 and what make 7?" Here are some more real problems: (1) It was Dick's birthday. He had four red candles on his cake, and four pink. How old was he? (2) Three boys and four girls were tidying the garden. Father said he would give them each a penny. How many pennies did he need? (3) Tom went to gather eggs in the hen-house. He found five eggs in one nest, four in another, and three in a third. How many did he bring home?

## Setting down Addition vertically

Children in their first year in the Junior School often need a good deal of practice in doing this. Let them set down the addition facts given in Fig. 17 vertically and work them; as they know most of them they can give all their attention to putting one figure exactly in under the other. This is so important when they are learning place-value.

## Speed and Testing

These remarks about speed and testing apply to every chapter in this book. For many reasons arithmetic is the easiest subject to teach, and the most enjoyed by the majority of children, *but* in far too many schools it means strain and emotional upset, because of (1) the mania for speed, (2) tests and competitive work. No subject is so easy to test as arithmetic.

For backward children and the rather slow average child, the greatest barrier to good learning progress is *speed*, because of the emotional strain. Few teachers, perhaps, realize the emotional upheaval which a child may experience when he is trying to answer questions faster than he can, feeling all the time that other children will surpass him in speed, or that he may be blamed for answering slowly. How must a child feel trying to answer when all around him hands are waving in the air? How must the slow child feel when racing with his companions in answering flash cards or groups of sums? Few can avoid errors or make correct associations if they are distraught in any way.

The learning of such basic number facts as $5 + 7 = 12$ should be a simple and satisfying task. The addition facts, the subtraction facts, the multiplication facts, could be learnt by the average and even slow child in a relatively short time if he could escape the emotional strain of quick oral tests, stop-watch exercises, competitive work or games—if no child were allowed to raise or wave his hand, or stand to indicate he can answer more quickly than another child. Neither by word, look, nor gesture should the teacher hurry the backward child. The child must concern himself *solely* with answering correctly and *give no thought to speed.* The backward child while he studies, or answers questions on what he has studied, must feel entirely at his ease, free from any inhibiting emotions. As far as possible mechanical mistakes must not occur. If a child knows that $5 + 7$ are 12 he can answer reasonably quickly, but to make the child attempt a *quick* answer if he does not know ensures neither a quick answer nor an accurate one! Time pressure forces a child to guess, and he soon falls into the guessing habit. This causes him to repeat the wrong answers as well as the right, and *incorrect associations* hamper the learning of all addition facts.

A slow child was once given special help alone in the addition facts. He worked at number-learning cards, but needed much encouragement to study the number-learning side of the card. He was always in a hurry to say the answer, right or wrong, from the testing side. He had to be taught to concern himself with answering not *quickly* but *correctly.* At last he knew all the addition facts, and his answers were automatic. Then he joined a class again which had daily speed drills with flash cards and the addition of several numbers in a column. The speed made him forget that what mattered was the right answer. He gave wrong answers, lost his nerve, and finally returned to counting. If through strain a child forgets it is much better for him to say so than to give wrong answers. If this child could have said, "I have forgotten for the moment; hurrying makes me forget," his mind would have been relieved at once. It is worth repeating again that 'mechanical mistakes must not occur.' Accuracy, at first, is far more important than speed—especially when learning the addition facts.

It is the habit of speed as well as poor reading-ability that makes children fail over "sums in words." A child sees the numbers scattered among the words. Feeling through habit that he ought to answer quickly, he takes a chance, and either adds or subtracts. He never dreams of reading through the exercise and thinking! A child trained to feel he has time, and that he must first of all be accurate, will tend to take time to do the reading and find out what he has to do. But there must be tests and worth-while games. Only by thoughtful repetition and tests can children learn by heart. Every day they must be tested on the combinations learnt the day before. Impress upon them they must be really sure they know the answers. When their turn comes, if they have any doubt they must say so. They must say, "I don't know," rather than guess. Encourage them *to beat themselves*. They can rightly boast, "I knew two bonds yesterday; I know four to-day."

Backward children should come at odd times to be tested, and should in no way be afraid or nervous over the tests—only interested to see how many they get right. Test the children in groups also, but quick children should never be tested orally with slow ones. Some children know an answer perfectly, but are slower than others at speaking.

There must be tests too on setting down exercises on paper. The addition facts must be worked as model sums in their best books, especially those involving tens and units. It is a great help if children learn to keep their figures in the right positions, and really in columns. Children like to have books in which they work *specimen* sums slowly and 'beautifully.' More will be said about children's notebooks and self-help books in the coming chapters. For more about testing see Chapter V.

CHAPTER III

# PLACE-VALUE AND THE VEXED QUESTION OF ZERO

*Zero as a Place-holder. Hints on teaching Place-value. Introducing the Hundred. Introducing the Thousand.*

## ZERO AS A PLACE-HOLDER

Before proceeding further with addition and subtraction, place-value or notation must be dealt with, partly because of the question of zero (or 0) and its right treatment. Far too often in schools this is neglected. Place-value, of course, cannot be taught in one lesson, or indeed in several lessons. It must be taught or revised in connexion with each rule learnt —addition, subtraction, multiplication, and division. Some teachers are inclined to teach notation, and then take it for granted it is known when teaching other rules. If children are to make *intelligent* progress in arithmetic they must be constantly reminded of place-value and very alive to it.

Place-value will be dealt with again in connexion with every rule, but in this chapter for the convenience of teachers we have set down notes on the teaching of place-value that cover the work done in the Junior School.

First comes the question of zero, or 0.

From the very first 0 must be treated as a place-holder and not a figure. In Arabic notation there are 9 figures, 1 to 9 inclusive, and our number system depends not only on these nine figures to show quantity, but also on the *position* in which the figures are written. Each figure can serve the double purpose of being a figure and of keeping the rest of the figures in their proper places; the two in 321 shows 2 tens and puts the 3 in the hundreds place. In writing 320 there are no units to write. Why write 0 for nothing? There is no need to indicate 'nothing' by a sign, but there is need to put the 3 and 2 in their proper places. The nought in 320 serves that important and essential purpose. A person never needs the 0 until he must write the quantity **10** or certain quantities larger than 10. He then needs the 0 to hold position. The 0 is a place-holder. If the teacher attempts to teach the zero or 0 as a symbol for *nothing*, while neglecting to call attention to the actual use of the symbol, the children may become confused. The 0 is very necessary as there are so few figures.

In a great many schools, because the teachers think that the 0 is going to cause trouble later, they set about teaching the *apparent* use of the 0 from the very beginning. They think children must add, subtract, and multiply by 0 just as they add, subtract, and multiply by any of the nine figures.

If a child has three pennies in one pocket and none in any other pockets he knows perfectly well how many pennies he has. He can count them. He does not think of counting pennies in the other pockets—there are none to count. To make him write $\begin{array}{r} 3 \\ +0 \\ \hline \end{array}$ seems no sense to him. The three pennies he has lie quite undisturbed. Writing the sum $\begin{array}{r} 3 \\ +0 \\ \hline \end{array}$ and hearing the question, "Three and nothing are how many?" may, however, distract from the actuality that the three pennies are undisturbed. To write $\begin{array}{r} 3 \\ -0 \\ \hline \end{array}$ and to say, "Nothing from three is three," is needless. If nothing has happened to the three pennies or whatever they are, there is no object in treating the situation as though something had happened.

If children are taught to write $\begin{array}{r} 0 \\ \times 3 \\ \hline 0 \end{array}$ and to say, "Three noughts are nothing" the thoughtful child may say, "Three noughts are three," and he is quite right! If you write down a 0 three times you have three 0's. When nothing is thought of as 0, it would appear that three nothings are three nothings! If the learned Greeks could not conceive of a symbol which is something for nothing there is little wonder that the primary child is confused when he is required to write and deal with 'something' as 'nothing.'

In actual fact the children will have to deal with the 0 only as a place-holder. In adding $\begin{array}{r}20\\30\\\hline\end{array}$ we actually add the 2 tens and the 3 tens just as we add 2 and 3. But we must write the answer in the tens column. Writing the 0 will put the figure 5 in its right position. In adding the figures below we

$$\begin{array}{r}25\\30\\\underline{51}\\\hline\end{array}$$

actually add the 5 and 1 in the units column. The 0 is merely there to put the 3 of thirty—3 tens—in its right place. The 0 does not stand for quantity, and so is neglected in the addition. In multiplying 40 by 2 we do not multiply 0 by 2, we multiply the 4 tens by 2, but we must set the answer down in its proper place, the tens place. Recognizing this necessity, we first write the 0 to put the 8 of the answer in its proper place.

In dividing 80 by 2, as below, we divide the 8 tens just as

$$\begin{array}{r}40\\2\overline{)80}\end{array}$$

we would divide 8 units, but we realize that the answer, 4, must go in the tens place, the same position as the 8 of the dividend, for it is 4 tens. We add the 0 to make its position clear. We do not proceed to divide 'nothing.' More will be said about this in the chapter on division.

### Hints on teaching Place-value

Children coming from the Infant School often have quite a good idea of what is meant by **tens** and **units**. Remind them that 5 is 5 ones or 5 units, 9 is 9 ones or 9 units, 1 is 1 unit, and so on. Units may be grouped together to form quite new and much bigger numbers with new names—for example, when we have **ten units** we change the name to **tens**; ten units make 1 ten. We put the 1 ten in the place to the left of the units to show it is not 1 unit but ten times bigger than a unit, as below:

| T U | | T U | |
|---|---|---|---|
| 1 | = 1 unit | 1 5 | = 1 ten 5 units = fifteen |
| 1 0 | = 1 ten | 2 9 | = 2 tens 9 units = twenty-nine |

To make sure it will not be mistaken for a unit, and will not slip into the units place, we put 0 in the units place. Nought is a place-holder, and shows clearly where the place of the tens is, left of the units. Thus we get a symbol or figure for ten—10. The children must understand clearly that **ten** units are the same as **one** ten.

Backward children may need the help of concrete material. Short sticks of some kind, such as match-sticks, straws, rods, etc., are most useful, as they can easily be tied up in bundles of ten. The compact bundle of match-sticks will help children to distinguish between 1 ten and 10 units. A child counts out a certain number of sticks—say twenty-five. He is told to tie these up as far as possible into bundles of ten. The bundles of ten are then placed in the left-hand compartment of the 'notation box,' marked T for tens; the five loose sticks over are put in the right-hand compartment, marked U for units. The children should have a great deal of practice in working exercises as above, showing 11 as 1 ten and 1 unit; 24 as 2 tens and 4 units, 48 as 4 tens and 8 units, and so on. Practice in the above way will lead the child to say, on hearing "Sixty-two," "Six tens and two units." The children can sometimes separate bundles—for example, they separate five bundles of ten, put them in the units compartment with units, and call them 54 units.

| 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 |
|---|---|---|---|---|---|---|---|---|---|
| 9 + 1 | 10 + 1 | 10 + 2 | 10 + 3 | 10 + 4 | 10 + 5 | 10 + 6 | 10 + 7 | 10 + 8 | 10 + 9 |
| 8 + 2 | 9 + 2 | 9 + 3 | 9 + 4 | 9 + 5 | 9 + 6 | 9 + 7 | 9 + 8 | 9 + 9 | |
| 7 + 3 | 8 + 3 | 8 + 4 | 8 + 5 | 8 + 6 | 8 + 7 | 8 + 8 | | | |
| 6 + 4 | 7 + 4 | 7 + 5 | 7 + 6 | 7 + 7 | | | | | |
| 5 + 5 | 6 + 5 | 6 + 6 | | | | | | | |

FIG. 21. THE 'TEENS' AND THEIR FAMILIES

Numbers such as the **-teen** and **-ty** (ten) numbers may need special attention. Fig. 21 shows a useful number chart for the 'teens' and their families. See that the children know the meaning of **teen**, ten. The teens are the numbers that contain **one** ten. The first number that ends in -teen is **thirteen,** but **ten, eleven**, and **twelve** may be included in the teens because they each contain one ten. Nineteen is the last teen. As long as a child remembers the order, **tens** then **units**, he will find little difficulty with the teens. On hearing eighteen he may say, "How many tens? One. How many over in units? Eight," and then writes eighteen. Dull children may have to be reminded that **thirteen** means three and ten, **fourteen**, four and ten, and so on. From the sound of the names fifteen, sixteen, seventeen, the child accepts and remembers the fact that $16 = 6 +$ a teen or ten, or $6 + 10$, or $10 + 6$; $17 = 7 + 10$; and so on. These bonds or combinations must be learnt from the chart, Fig. 21. Let the children make addition sums for these combinations, putting T U above each sum, thus:

| TU | TU | TU | TU |
|---|---|---|---|
| 1 0 | 1 0 | 9 | 1 0 |
| + 8 | + 1 | + 3 | + 9 |
| 1 8 | 1 1 | 1 2 | 1 9 |

After the 'teens' come the '-ty' (ten) numbers, twenty, thirty, forty, up to ninety. **Twenty** means two tens or twice ten; **thirty**, three tens, and so on. The children soon see that 'twenty-two' is a short form for twenty (two tens) and two; and so on. The sums of numbers such as $20 + 6$, $40 + 7$ are generally known to children through counting (see Chapter IV). Let the children write some -ty numbers for place-value:

| TU | TU |
|---|---|
| 3 0 | 7 0 |

### Introducing the Hundred

If necessary give the children some work with concrete things so that they have some idea how big a hundred is.

They put down 1 counter, below it they lay out 10 counters, below the 10 they lay out 100 counters—10 times the number of 10. They divide the 100 counters into ten groups of 10. The 100 counters are a hundred times more than 1. To distinguish one hundred from one ten, we put the 1 to the left of the tens place thus: $\begin{matrix}\text{H} & \text{T} & \text{U}\\ 1 & 0 & 0\end{matrix}$ and put 0's to mark the position of the tens and units. So we get this symbol or sign for one hundred—100. Impress upon the children that 100 is the same as 10 tens, 100 units. The children slowly begin to realize that moving a figure one place to the left makes it ten times bigger.

| HUNDREDS | TENS | UNITS |
|---|---|---|
| So many hundreds | So many tens | So many ones |

FIG. 22. UNITS, TENS, HUNDREDS

Let them copy Fig. 22 in their notebooks. Remind them that every figure in the units place stands for so many 'ones'; every figure in the tens place stands for so many 'tens'; every figure in the hundreds place for so many 'hundreds.' If we move a figure one place to the left we make it ten times bigger. One ten is 10 units, one hundred is 10 tens. If we put 3 in the units place it is just 3 single things. If we put 3 in the tens place it is 3 tens or 3 groups of ten—or, as we say, thirty. If we put 3 in the hundreds place it is 3 hundreds or 3 groups of a hundred.

Let the children study Fig. 23 and notice the importance of the two 0's to the right of the first 5. They tell us the 5 is in the hundreds place. Noughts to the right of a figure must never be left out. Intelligent children will see that 0 to the left of a figure may be left out. The 0 to the left of 50 is not wanted, nor the two 0's to the left of the 5 units, Fig. 23; 50

is 5 tens, as 5 is in the tens place. It is still 50 if we put 0 to the left of it in the hundreds place.

| H | T | U |
|---|---|---|
| 5 | 0 | 0 |
| 0 | 5 | 0 |
| 0 | 0 | 5 |
| 1 | 1 | 0 |
| 3 | 0 | 5 |
| 3 | 3 | 3 |

FIG. 23

Children should have plenty of practice in expressing (*a*) *words in figures* (three hundred and thirty; one hundred and ten; one hundred and one; three hundred and four; seven hundred and nine; seven hundred and nineteen; and so on) and (*b*) *figures in words* (205; 382; 110; 101; 250; 328; 300; 303).

| H | T | U |
|---|---|---|
| 2 | | |
| | 2 | |
| | | 2 |
| | 4 | |
| 3 | | |
| 1 | | 7 |
| | 9 | |

FIG. 24

Let the children say how place-value is shown in Fig. 24, where no 0's are used. Is this a better way than using 0's?

When once we have started a number—say, three hundred and something—we must put a figure in every place to the right; if there is no other figure to be put down we must put down 0's as place-holders. Remind the children that figures are place-holders as well as 0's. In 33 the 3 of the units put the 3 of the tens in its right place.

Other useful exercises to teach place-value are:

(1) Write down the hundreds, tens, and units figures in these numbers: 213, 600, 450, 302, 189, 032.
(2) Split up the following numbers (for example, 907 = 9 hundreds, 0 tens, 7 units): 970, 805, 101, 549, 300.
(3) Pick out the tens figure from each number: 427, 803, 695, 410.
(4) Write in 10's from 185 to 235, from 320 back to 270.
(5) Write in 10's from 315 to 715, from 817 back to 417.
(6) Give the children numbers like the following to decompose:

H T U
3 2 3 = 3 hundreds + 2 tens + 3 units = 300 + 20 + 3
1 0 1 = 1 hundred + 0 tens + 1 unit = 100 + 1 unit
3 8 = 3 tens + 8 units = 30 + 8

## Introducing the Thousand

← To the left ←

| Th | H | T | U |
|---|---|---|---|
| 2 | 2 | 2 | 2 |

Fig. 25. Units, Tens, Hundreds, Thousands

Let the children draw four columns as shown. Remind them that each column to the left is worth ten times as much as the same figure next to it on the right. If we put 2 in the units column it stands for 2 only, but if we put it in the tens column (the column to the left) it stands for 2 tens, or ten times 2. If we put 2 in the hundreds column it stands for 2 hundreds, or ten times 2 tens, or ten times 20, or a hundred times as much as it did in the units column; $100 \times 2 = 200$. If we go beyond the hundreds column we come to figures which are worth ten times the same number of hundreds. We call this column thousands; ten times 200 equals 2000. All this may seem much ado about nothing, but dull children need patient teaching and patient explanation. Many teachers do not realize how puzzled they are by left and right. It is worth putting an arrow on top of the figures, pointing out "To the left."

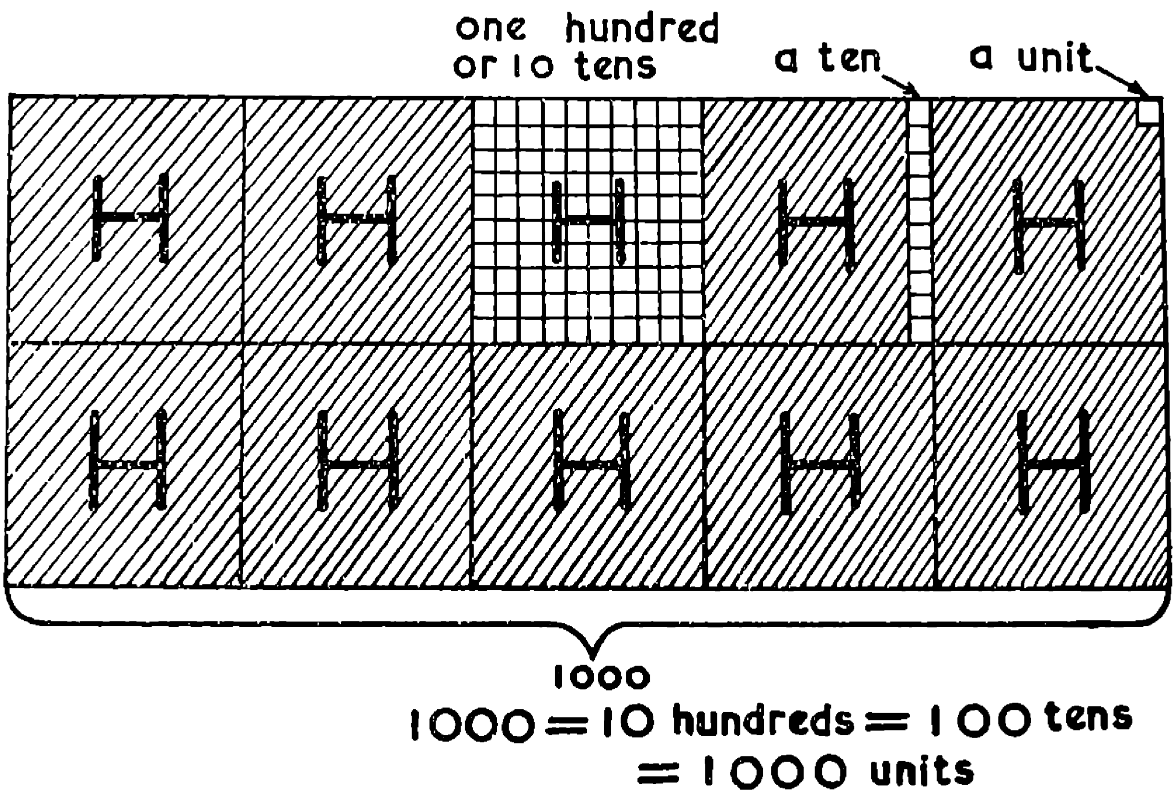



FIG. 26. DIAGRAM TO SHOW RELATIVE VALUES OF NUMBERS

Fig. 26 helps a child, especially an intelligent child, to get some idea of how much bigger 1,000 is than 1 unit. He can make this chart himself if he has a piece of squared paper ($\frac{1}{100}$ inch squares) as in Fig. 26. Each little square stands for a unit. One of these is carefully shaded. Then ten units are carefully shaded as in Fig. 26 to show one 10. Next the child marks out and shades ten of these 'tens' to show 100, and gets a big shaded square as shown, consisting of 100 little squares. He marks the square H. He sees he has ten of these big squares, and puts an H on each. He now has ten hundreds, or one thousand. He sees from this diagram how much bigger a thousand is than a unit, or a ten, or a hundred.

He can prove these facts on his diagram and write them down:

1 Th. = 10 hundreds
= 100 tens
= 1,000 units

It may help backward children if they arrange cardboard discs or sticks to show 1 unit, 1 ten, 10 tens, and 100. This shows how many more 100 is compared with a unit of one ten, and how big a thousand must be, which is ten times 100. The fact that ten groups of 100 would, as some children said, "take too long to put out," may help them to realize the magnitude of 1,000.

Give the children practice in writing numbers in words or setting them down in figures. Let them put a comma to separate the figure in the thousands place from the figure in the hundreds place—thus, 1,760, 2,040. Let them express in figures: five thousand and nine, six thousand four hundred, nine thousand, six thousand and twenty, four thousand nine hundred and ten, three thousand and seventy-three.

Let them write in words: 6,231, 8,340, 7,001, 4,030, 6,300, 9,102.

Place-value is of great importance. Children can see that the only numbers that go in the unit column are 1 to 9 inclusive; if they have 10 things the ten must be transferred to the tens column and 0 put in the units column. Only figures from 10 to 99 can go in the tens column; if they have 100 they must put 0 in the tens column and transfer the 100 to the hundreds column. If children understand place-value well, and can change one unit to another easily, as 1 ten to 10 units or 10 units to 1 ten, addition and subtraction present little difficulty.

Place-value will be dealt with again in the coming chapters from different points of view. The work in notation may be extended roughly over four years. In many Junior Schools this is the syllabus for Notation: *First Year*, age 7 +, notation to 99. *Second Year*, age 8 +, notation to 999. *Third Year*, age 9 +, notation to 9,999. *Fourth Year*, age 10 +, notation to 99,999. Intelligent children will probably get to the hundreds in the first year. No child should be kept back.

CHAPTER IV

# HARDER ADDITION

*Adding 9 and 8 to other Numbers. Simple Column Addition. Two-column Addition. Introducing Carrying Figures. Carrying the Forty-five Addition Facts to Higher Decades. Adding by Endings. Games and other Devices. The Use of Old Calendars to help Addition. The Language of Arithmetic.*

## Adding 9 and 8 to other Numbers

The hardest of the 45 Addition Facts are those adding 9 and 8 to other numbers. It is no exaggeration to say that all progress in arithmetic depends on proficiency in the 45 Addition Facts, and there is no easy way for children to master them, *but* there are several ways of helping them. Here is a suggestion to help them to add numbers to 9 and 8.

*Adding to* 9. The following method makes adding a number to 9 the simplest of the harder addition facts. The aim is to

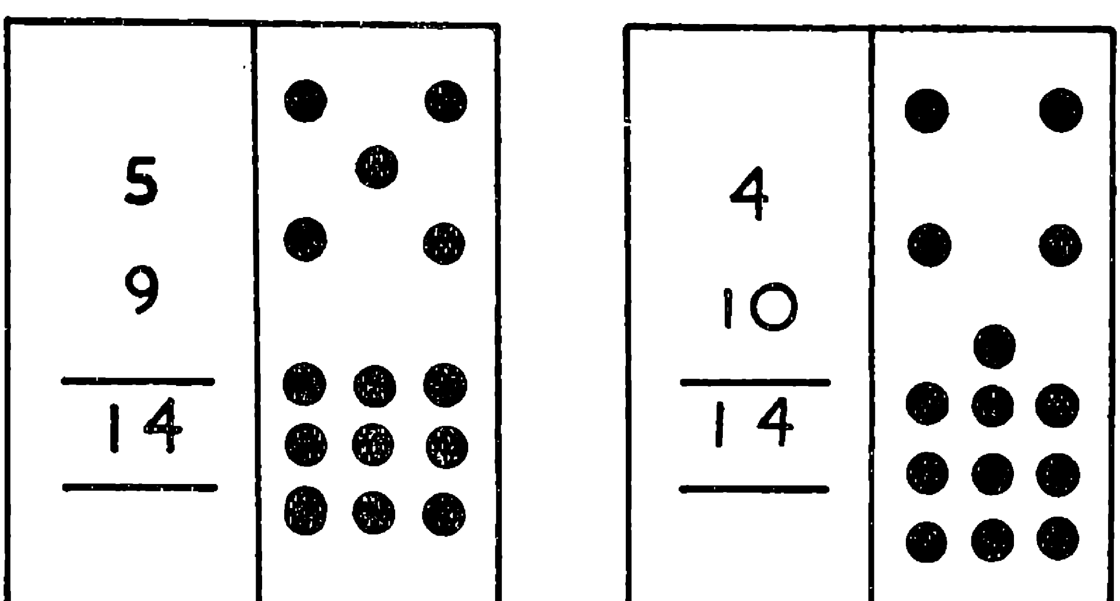



Fig. 27. Adding Nine

show the children that one less than the number added to 9 is the number of 'teen.' In Fig. 27, 4 is one less than 5, and the answer is 14. But 4 and 10 is known to them as 14, and from this they can easily see that 5 and 9 are 14.

Write on the blackboard, or upon a chart for permanent use, these examples:

| | | | | | | |
|---|---|---|---|---|---|---|
| 9 | 9 | 9 | 9 | 9 | 9 | 9 |
| +6 | +8 | +7 | +9 | +4 | +3 | +5 |
| 15 | 17 | 16 | 18 | 13 | 12 | 14 |

By pointing to 6 and then to the 5 of 15, to 8 and then the 7 of 17, to 7 and then to the 6 of 16, and so on, it is made clear that the number of the teen is one less than the number added to 9. This device greatly helps children to remember the addition facts. Give them plenty of examples from time to time, until the adding becomes automatic. Give them some sums in this form:

| | |
|---|---|
| ? | 9 |
| 9 | ? |
| 15 | 12 |

*Adding to* 8. In the same general way that adding to 9 is presented, so can adding to 8 be shown. The child knows that $10+7=17$, therefore $8+7=15$. The 5 is 2 less than the number added to 8—namely, 7.

Write on the blackboard, or upon a permanent chart:

| | | | | | | |
|---|---|---|---|---|---|---|
| 8 | 8 | 8 | 8 | 8 | 8 | 8 |
| +3 | +5 | +6 | +8 | +4 | +7 | +9 |
| 11 | 13 | 14 | 16 | 12 | 15 | 17 |

Point to the 3 and the 1 of 11, the 5 and the 3 of 13, the 6 and the 4 of 14, and so on; make clear the fact that the number of the teen is two less than the number added to 8.

## Simple Column Addition

Children soon get quick at adding columns of two figures only, but when it comes to three or more figures in a column they find it difficult. In the sum below, when the child has added 8 and 5 he has to keep the 13 in his head while he adds it to the 4. It is not easy for him at first to add 4 to a number that he cannot see. He cannot always keep the thought-numbers in his head.

4
5
8
—

The addition is begun at the bottom of each column. This single-column adding is really only oral addition with the answers not spoken but written. It helps if children have oral practice in adding a *seen* number to an *unseen* number in this way. Put some figures on the board—say, 5, 4, 6, 7, 3. Then point to 5 and say, "Add nine." The children only see 5, they think 9. Quick children will say at once 14, because they have learnt how to add 9 and 5. Other numbers are given them to add to 5. Then give them in the same way numbers to add to 4, 6, etc. Later they can be given bigger numbers to add, such as 15, 23, 14. The children will thus get accustomed to keeping big numbers in their heads. One must bear in mind that it is easy for a child to add 5 to 42, but much harder for him to add 42 to 5. The columns must be kept reasonably short, as the younger Junior's power of concentration is very limited. He tends too to lose his place in the column unless one insists that he always points with his pencil to the number to be added, and keeps it there until the number has been safely added. In some cases let him put the result of his addition by the figure. On the whole it is not wise to encourage children to search in a long column for numbers that make 10. This was once a common practice, but has fallen into disuse. Experience proves that it causes trouble.

### Two-column Addition

The children should now have a fair knowledge of place-value (see Chapter III). Through learning the table of **teens** and the **-ty** numbers they grow more familiar with tens and units. Make sure that they can add 20 + 6, 30 + 9, and so on, putting each figure in its right column. Adding 24 + 3, 35 + 4, 59 + 8, etc., is harder. Let them think 24 + 3 is 20 and 4 and 3—that is, 27. In the same way, 35 + 4 is 30 and 5 and 4—that is, 39, 3 tens and 9 units.

*Adding* 32 *and* 24. Impress upon the children that they should think of 32, not as 32 separate things or units, but as 3 groups or bundles of 10 and 2 odd ones; and to think of 24 as 2 bundles of 10 and 4 odd ones. The children add, and they see that they now have 6 units or odd ones and 5 tens, and they realize the procedure to adopt when adding such numbers—namely, add units to units, then ten to tens. They put their answer in the right columns. When there are no carrying figures these sums are quite straightforward, especially if they know their addition facts. Quick children soon pass on to 'carrying figures.'

| T | U |
|---|---|
| 3 | 2 |
| 2 | 4 |
| 5 | 6 |

INTRODUCING CARRYING FIGURES

The child finds he has 12 units. He knows this is made up of 1 ten and 2 units. He puts the unit part of the sum in the units column, and transfers the 1 ten to the tens column.

| T | U |
|---|---|
| 3 | 5 |
| 2 | 7 |
| 6 | 2 |
| 1 | |

At first, if the child finds it helpful, he should be allowed to insert the carrying figure at the foot of or at the top of the tens column, as above. Many teachers object to the use of an auxiliary, or helping figure, or crutch. But the object of addition is to make carrying a habit in order to ensure accuracy and speed. To keep a number in mind—say 7—means much more effort, more time, and more chance of error than to put it in its appropriate column. This applies especially to slow children, backward children, and those who find difficulty in working in their heads instead of on paper. *All children cannot be treated in the same way.* In some of their two-column sums children will need to use their hundreds column, as in the example given.

| H | T | U |
|---|---|---|
| | 7 | 8 |
| | 6 | 4 |
| 1 | 4 | 2 |
| | 1 | |

When the children add up their tens they may be puzzled as to what to do with fourteen. Explain to them that, as 10 units make 1 ten, so 10 tens make 100. The 14 is 10 tens and 4 tens. They put 4 tens in the tens columns; the 10 tens they call 1 hundred, and put the 1 in the hundreds column. They may read the answer as one hundred and forty-two or 1 hundred, 4 tens, and 2 units, or 14 tens and 2 units, or 142 units. From time to time lessons should be given on place-value. See Chapter III.

The children must now begin to prove all their exercises. As they add their sums by beginning at the bottom number, they can prove them by beginning at the top number.

### Carrying the Forty-five Addition Facts to Higher Decades

The children will not make much progress in addition unless they can carry each one of the forty-five combinations they know *beyond* 18. Children may recognize 6 and 7 instantly, but will hesitate when 26 and 7 have to be added. Indeed, adults often hesitate if they have to add a number like 8 to 59. Their early training in addition did not carry the association of the forty-five combinations to higher orders, so that a 9 and an 8 by force of habit would suggest a 7 in the answer.

## Adding by Endings

This is a good device for helping children to carry the forty-five combinations or addition facts to the higher orders. Write on the board the sums below:

| | | | | | | |
|---|---|---|---|---|---|---|
| (1) | 3 | 13 | 23 | 43 | 53 | 543 |
| | +4 | +4 | +4 | +4 | +4 | +4 |
| | 7 | 17 | 27 | 47 | 57 | 547 |

The children study this group of sums and see they all end the same. If they know $4+3=7$, they know $13+4=17$, and so on. Put further groups on the board for the children to see that they all end as in the primary facts; keep to the first 20 facts, and then go on to the harder combinations. Put these on the board as before, and let them study the endings and see they all end in the same way, as below:

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| (2) | 6 | 16 | 36 | 56 | (3) | 5 | 25 | 45 | 105 |
| | +9 | +9 | +9 | +9 | | +8 | +8 | +8 | +8 |
| | 15 | 25 | 45 | 65 | | 13 | 33 | 53 | 113 |

In the above examples the pupils should notice the tens digit is increased by one. Charts may be made for the harder combination facts and put on the board for reference—children like to make their own charts in their note-books and put in every decade seeing how high they can go.

If the addition facts are worked through in this way the children will have no difficulty in adding 27 and 9, 45 and 9, 35 and 6, and so on.

## Games and other Devices

(1) *The Well-known Circle* (Fig. 28). The children add the numbers in the inner circle to the numbers in the outer as quickly as the teacher points to them—*e.g.*, 9 + 7, 9 + 17, 9 + 27, etc. The numbers in the outer circle may then be rubbed out and 8, 18, 28, etc., substituted. The children draw circles of their own on paper and cut them out. They see how quickly they can work round their 'clocks.' Some children make 'clocks' for all the numbers from 4 to 7 and their decades.

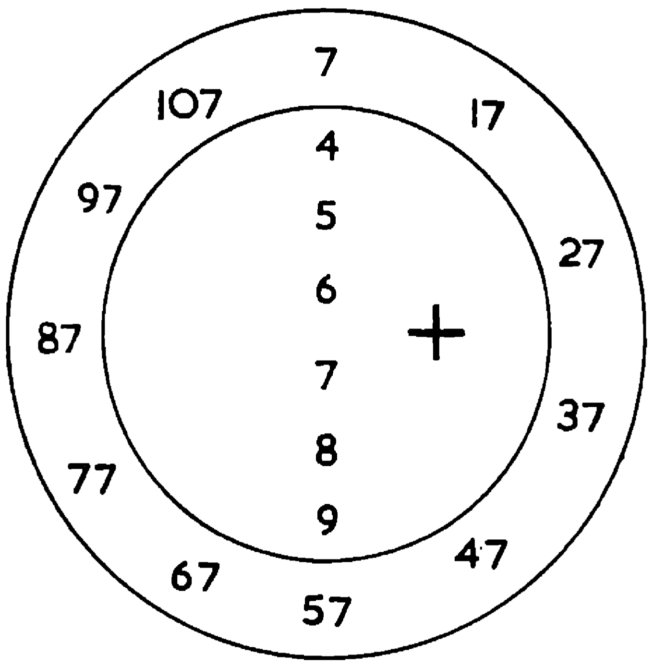



Fig. 28. The Circle

(2) *Charts for Adding*. These charts can easily be made by the children if they are given squared paper. They can be used for a great many purposes. In Fig. 29 (*a*) the chart is used for counting in 4's or adding 4's. The children colour the squares and notice the pattern made. In Fig. 29 (*b*) the chart is used for counting in 9's or adding 9. Here the adding starts with 6; 6 and 9 are 15. They shade square 15, and so on. Let them try adding 9's and beginning with 4. If they know their combination facts they see at once the next square to be shaded is 13, because 4 and 9 are 13. In this way they can revise all their addition facts.

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | **4** | 5 | 6 | 7 | **8** | 9 | 10 |
| 11 | **12** | 13 | 14 | 15 | **16** | 17 | 18 | 19 | **20** |
| 21 | 22 | 23 | **24** | 25 | 26 | 27 | **28** | 29 | 30 |
| 31 | **32** | 33 | 34 | 35 | **36** | 37 | 38 | 39 | **40** |
| 41 | 42 | 43 | **44** | 45 | 46 | 47 | **48** | 49 | 50 |
| 51 | **52** | 53 | 54 | 55 | **56** | 57 | 58 | 59 | **60** |
| 61 | 62 | 63 | **64** | 65 | 66 | 67 | **68** | 69 | 70 |
| 71 | **72** | 73 | 74 | 75 | **76** | 77 | 78 | 79 | **80** |
| 81 | 82 | 83 | **84** | 85 | 86 | 87 | **88** | 89 | 90 |
| 91 | **92** | 93 | 94 | 95 | **96** | 97 | 98 | 99 | **100** |

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | **6** | 7 | 8 | 9 | 10 |
| 11 | 12 | 13 | 14 | **15** | 16 | 17 | 18 | 19 | 20 |
| 21 | 22 | 23 | **24** | 25 | 26 | 27 | 28 | 29 | 30 |
| 31 | 32 | **33** | 34 | 35 | 36 | 37 | 38 | 39 | 40 |
| 41 | **42** | 43 | 44 | 45 | 46 | 47 | 48 | 49 | 50 |
| **51** | 52 | 53 | 54 | 55 | 56 | 57 | 58 | 59 | **60** |
| 61 | 62 | 63 | 64 | 65 | 66 | 67 | 68 | **69** | 70 |
| 71 | 72 | 73 | 74 | 75 | 76 | 77 | **78** | 79 | 80 |
| 81 | 82 | 83 | 84 | 85 | 86 | **87** | 88 | 89 | 90 |
| 91 | 92 | 93 | 94 | 95 | **96** | 97 | 98 | 99 | 100 |

FIGS. 29 (*a*) AND (*b*).
CHARTS FOR ADDING

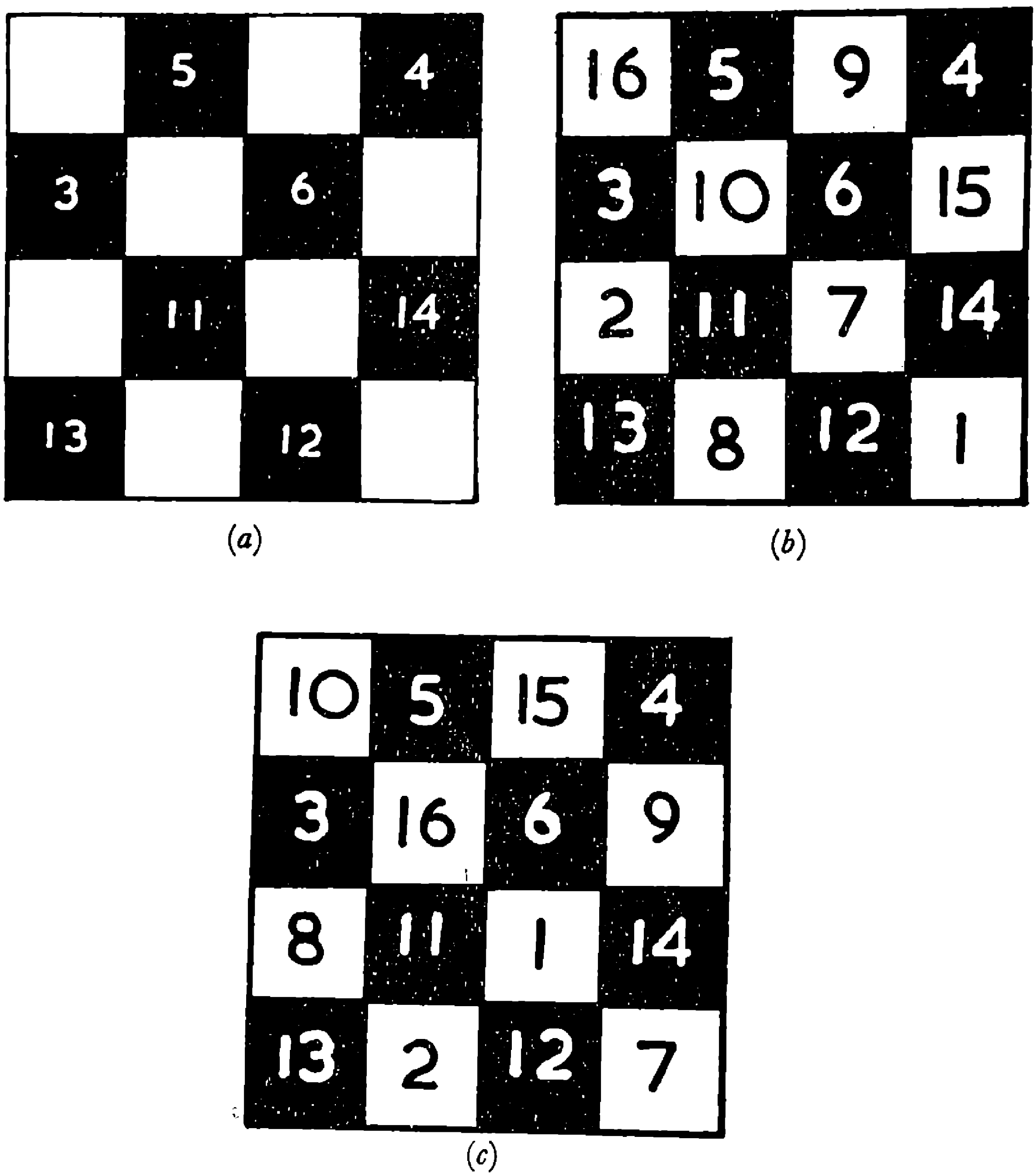
Children are much interested in adding a variety of numbers to a number in the top line and noticing what patterns are made. If they begin at 6 and add 10 successively the shaded squares form a vertical line. They soon see that successive additions of 5 give two vertical lines. It is especially useful for adding 10's. From the chart they can easily see that 10 tens make 100, or that there are 10 tens in 100. Let them find how many 5's in 100, or 20 times 5 is 100. In the same way they can find out how many 4's there are in 100. This chart thus helps multiplication and division.

As a preparation for subtraction, let them count backward or subtract. To do this they choose a number from the bottom line—say 95—and count back in 5's or 10's. Counting backward in 5's gives *two* vertical lines, counting backward in 10's gives *one* vertical line. Subtracting 7 successively from 96 gives an interesting pattern. The dullest child sees how the number 100 is built up by successive additions of one. By the addition of 10 to each number in the *top* row, the numbers 11, 12, 13, etc., are obtained. The children clearly see that 11 is made up of 10 and 1, 12 of 10 and 2, and so on. By the addition of 10 to each figure in the *second* row they get 21, 22, 23, 24, etc. They clearly see 23 is made up of 10 and 13, and so on.

All sorts of addition sums can be worked from this chart and proved—for example, each number along the top row can be added in turn to all the numbers in the vertical rows. Thus 3 can be added to all the numbers in the first vertical row, $3+1=4$, $3+11=14$, $3+21=24$, $3+31=34$. Then 4 can be added to the same vertical row, and so on. In this way practice in higher-decade addition can be obtained. Later on multiplication and division can be taught and tested in the same way.

(3) *Magic Squares or Draught-board Puzzles.* Fig. 30 (*a*), (*b*), and (*c*) shows the magic square. A square of paper is folded into 16 squares, which are ruled out in pencil. A second square is folded into 16 squares. These squares are numbered from 1 to 16 and cut out. Place squares numbered 3, 4, 5, 6, 11, 12, 13, 14, on the shaded squares as in Fig. 30 (*a*). The children now try to place the remaining squares in such a way that the numbers in each row, vertically and horizontally, add up to 34. Let the children notice that the numbers placed diagonally, 4, 6, 11, 13, in Fig. 30 (*a*) also add up to 34. Give them some hints. First they set out the remaining eight squares to be put in the rows—1, 2, 7, 8, 9, 10, 15, 16. Then they add the two numbers in the top row, 5 and 4, in

Fig. 30 (*a*), and subtract this total from 34 to get the sum of the two numbers needed in the top row, 25. The top row and the second row both need two numbers whose sum is 25.




FIGS. 30 (*a*), (*b*), AND (*c*). MAGIC SQUARES OR DRAUGHT-BOARD PUZZLES

They find 16 and 9, 10 and 25. Then they check the vertical rows to determine the position of these numbers in the horizontal rows. Fig. 30 (*b*) and (*c*) show the two solutions. Clever children see almost at once that they must subtract the sum of any two numbers from 34 to get the sum of the missing numbers. They learn a great deal from this game. It revises difficult bonds such as 16 + 9 = 25, 13 + 12 = 25.

*Another Number Square.* Draw a square, sides 3 in. Divide each side into six half-inches, and draw lines across to make 36 small squares. Shade or colour the four corner squares. Put in each square around the edges any number from 12 to 20—say 15, as in Fig. 31. Cut out twenty other squares or more (½-in. sides) and print on each a number from 1 to 9 inclusive. The children arrange these numbers on the 16 blank squares in the middle so that their sum will be the number in the border, whether adding across or up and down. In Fig. 31 seven small squares have been placed in position. Children much enjoy making this puzzle and working it out. They make a puzzle square for all the numbers from 12 to 20 inclusive. They soon begin to see 15 as 8 + 2 + 1 + 4 or 9 + 1 + 2 + 3 or 7 + 3 + 2 + 3, etc.

| | | | | | |
|---|---|---|---|---|---|
| | 15 | 15 | 15 | 15 | |
| 15 | 2 | 7 | 4 | 2 | 15 |
| 15 | 8 | | | | 15 |
| 15 | 1 | | | | 15 |
| 15 | 4 | | | | 15 |
| | 15 | 15 | 15 | 15 | |

FIG. 31. ANOTHER NUMBER SQUARE

When children have a good grasp of addition they should be able to pass on to exercises including hundreds (see Chapter III) with little difficulty. If they can carry from the unit column to the ten column successfully they can carry from the ten column to the hundred, but they must realize that **1** hundred is not only one hundred, it is **10** tens, and **100** units. These two ideas should be kept in the child's mind: (1) the grouping of tens—the standard grouping; (2) the value of position or place.

## The Use of Old Calendars to help Addition

Fig. 32 shows a good use for old monthly calendars. Collect the calendar leaves, and encourage children to bring them to school as they tear them off at home. If necessary paste them on paper or thin card to strengthen them. The children add the columns, first beginning with Sundays, and write the answers as shown. Quick children are keen to add them crosswise. In this case let them put down the sum of the units when they have added them, as in Fig. 32. Then add

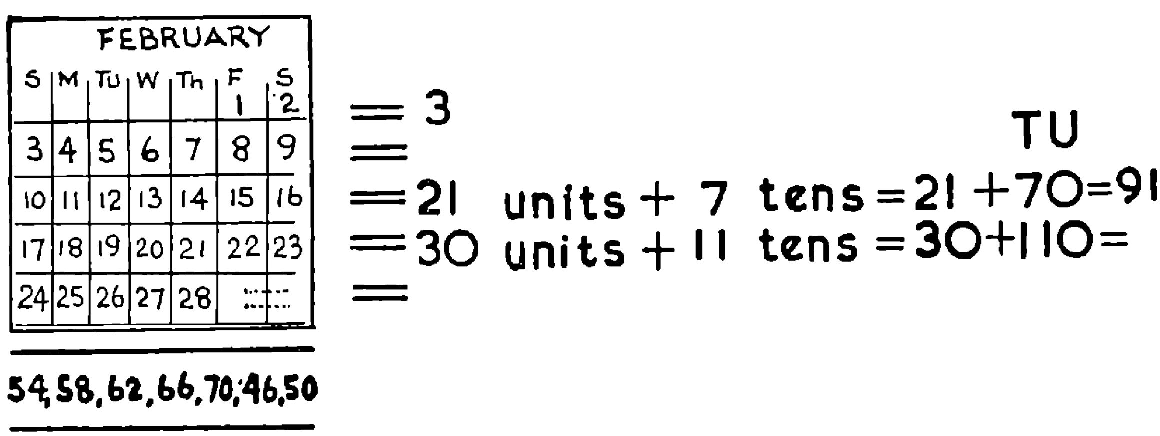



Fig. 32. Use of Old Calendars

the tens, and put down the answer, finally adding the two results. This helps to show whether the children really understand place-value. Children enjoy using them because they are a novelty. Incidentally, they become interested in calendars, and find out many things about them. If they have a calendar for a year showing all the months, they like to add up all the Sundays or Mondays in a year, and find perhaps there are 52!

Give many addition exercises in words, *e.g.*, find the sum of twenty-seven, thirteen, and one hundred and forty-eight.

To increase the children's understanding of the meaning and use of addition, simple problems including addition should be given. Most standard textbooks contain graduated examples of word sums or puzzles or problems, as they are variously called.

### The Language of Arithmetic

Finally, children must be very sure of the meaning of all words used. From the very start they must learn to grasp the language of arithmetic. **Sum** is the total amount after adding. It does not mean an **exercise** to be worked but an **answer**. Every conceivable type of exercise should not be called a "sum." **Plus**, **total**, **equal**, are all words children should be very sure about. "Find the **total** of 2, 3, 6," means the same as "Find the sum of 2, 3, 6." As soon as possible children should begin to build up both a class dictionary and their own individual dictionaries.

# CHAPTER V

# SUBTRACTION

*Addition and Subtraction. Varied Forms of Drill in Subtraction. The Three Main Ideas in Subtraction. Devices and Games for teaching the Younger Juniors. Twenty Steps in teaching Subtraction.*

## Addition and Subtraction

Addition and subtraction must go on side by side. Subtraction should be taught almost simultaneously with addition, as we have shown. Children who know that $4+2=6$ and $2+4=6$ should also know that $6-4=2$; that 4 is two less than 6, for $4+2=6$. If a backward child still cannot manage subtraction up to 10 in the Junior School he must have more work with counters. Give him, for example, some red and white counters and let him make patterns in this way for 7. For 7 he will use 4 red, 3 white; 2 red, 5 white; 3 red, 4 white; 1 red, 6 white. As each pattern is made he copies it on paper and writes beside it the sum of the red and white counters used. Then he takes some of the counters away and sets down the results. This work is on the lines of the work done in Chapter I.

It is clear that every addition result gives two results in subtraction:

$$7+5=12$$

Gives

$$12-5=7$$
$$12-7=5$$

The questions "7 and what makes 12?" and "7 from 12 leaves what?" are identical questions. A child knows 7 from 12 leaves 5, not because any subtraction has been carried out, but because he knows that 7 and 5 are 12.

## Varied Forms of Drill in Subtraction

The 45 addition facts afford an effective basis for teaching subtraction. The thorough knowledge of $9+6$ means drills, not only in $6+9$, but also in $9+?=15$, $?+9=15$, $?+6=15$, and later in $15-9=?$, $15-6=?$ The child therefore tends to acquire the basis of subtraction incidentally while he learns numbers and the basic facts in addition. If he knows these facts well he finds subtraction tables much easier to learn. Indeed, intelligent children rarely have to learn them.

Care should be taken to use each combination in as many forms as possible—for example, drill in $16-7$ must be given in these varied forms:

(1) $7+9=?$ (2) $9+7=?$ (3) $7+?=16$ (4) $?+7=16$
(5) $9+?=16$ (6) $?+9=16$ (7) $16-7=?$ (8) $16-9=?$
(9) $16-?=9$ (10) 16 is how much greater than 9? than 7?
(11) 9 is how much less than 16? 7 is how much less than 16?

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| (12) | 16 | 16 | ? | ? | 7 | 9 | 16 | 16 |
| | −7 | −9 | +7 | +9 | −? | +? | −? | −? |
| | ? | ? | 16 | 16 | 16 | 16 | 7 | 9 |

The teacher must be on his guard against stereotyped forms in practice work. They do not encourage thought or understanding.

## The Three Main Ideas in Subtraction

There are three basic ideas in subtraction, or three aspects of subtraction. These are:

(1) The "Take away idea" or **direct subtraction**—9 less 3 is what? 3 from 9, or 9 take away 3, leaves what?

(2) The "adding idea"—what number added to 4 gives 10? This is called **complementary addition** or **inverse addition** and is really not subtraction. There is no idea of taking away, or rather, perhaps, it is kept in the background, and the main notion is that of adding. It is the method of subtraction used by all those who have to do a good deal of reckoning—bankers,

accountants, and, of course, shopkeepers when they give change to customers. Most adults use both inverse addition and direct subtraction mentally. When the two numbers are fairly equal, as in giving change, they use inverse addition, but when big numbers are being subtracted, or one number is much larger than the other, they use direct subtraction, as it is said to be quicker.

(3) The "difference idea" or comparison—8 is how many more than 3? 5 is how many less than 9? The children often discover this idea for themselves. In playing games they often compare the counters or marbles they have won. "You have only 6 marbles, I have 10. I have four more than you." Very often children in the Infant School become aware of the main ideas underlying subtraction by their practical work. Junior children become familiar with complementary or inverse additions by their puzzle exercises, such as:

| | | | |
|---|---|---|---|
| 7 | 8 | ? | ? |
| ? | ? | 5 | 6 |
| 11 | 13 | 9 | 13 |

Looking at the above puzzles in turn, a child thinks, Seven and what are eleven? Then he answers his own question, Seven and four are eleven, stressing the four and writing it down in the blank space. He does the same with the other examples. In each case he gets the answer from his knowledge of addition. Later on when he meets these numbers in subtraction exercises, as 11 – 7, 13 – 8, he knows the answer at once. Children will also learn unconsciously inverse addition in their shopping games. It is probably best, however, to begin with the 'taking away' idea, or direct subtraction. It is clearer to the children. But the three aspects of subtraction help each other, and give form and purpose to the different drills and exercises suggested for the subtraction tables. The child is not necessarily troubled with the names of the different aspects, but the teacher must have them in mind in order to vary his examples. Most arithmetic textbooks have varied examples of subtraction called "Story Exercises and Problems," or "Puzzles," that bring in the three aspects of subtraction, thus: (1) There were 18 eggs in a basket. The grocer sold a dozen. How many were left? (12 from 18 leaves what?) (2) Jane is saving up to buy a doll which costs tenpence. She has fourpence. How much more must she save? (What must be added to four to make ten?) (3) Jack was eight. Jill was four. How much younger was Jill than Jack? (Eight is how much more than four?)

## Devices and Games for teaching the Younger Juniors

(1) Let the children *collect all the subtraction facts* for 6 or any number above 6, thus: $6-6=0$ (this is a legitimate use of 0, apart from its use as a place-holder), $6-5=1$, $6-4=2$, $6-3=3$, $6-2=4$, $6-1=5$.

(2) Writing addition and subtraction facts side by side. This interests children very much. Again it helps them to see the connexion between addition and subtraction, thus:

| | |
|---|---|
| $7=6+1$ | $7-1=6$ |
| $7=5+2$ | $7-2=5$ |
| $7=4+3$ | $7-3=4$ |
| $7=3+4$ | $7-4=3$ |
| $7=2+5$ | $7-5=2$ |
| $7=1+6$ | $7-6=1$ |

(3) Let the children find subtraction exercises that give the same answers—for example, the answer 4, $6-2=4$, $11-7=4$. Tell them sometimes to find a certain number of subtraction facts for one number—for example, seven subtraction facts for 8. Alert children often discover an easy way of finding a great number of subtraction exercises with the same answer, say 8 (or any number), thus: $9-1=8$, $10-2=8$, $11-3=8$, $12-4=8$, $19-11=8$, $20-12=8$, and so on. Anything a child finds out for himself is generally well remembered.

Younger children enjoy having subtraction examples given them with answers either 4 or 5 or any two numbers. They sort them into two groups according to the answers. Sometimes there can be three different answers. Children make examples for their companions to sort.

| Up to 10 | |
|---|---|
| 3 | 7 |
| 4 | 6 |
| 8 | |
| 7 | |
| 5 | |
| 6 | |

(*a*)

| Up to 30 |
|---|
| 10 |
| 24 |
| 18 |
| 17 |
| 21 |
| 16 |

(*b*)

FIG. 33 (*a*) AND (*b*)

(4) Fig. 33 (*a*) and (*b*) show an interesting way of setting down addition and subtraction exercises. Children copy the columns of figures and by each they put a figure that when added to it makes it equal to the number at the top. Just what numbers are placed at the top depends on what stage the children have reached. In the first year they may be able to carry on the making up of numbers to 10. This exercise is also of value because it links addition and subtraction, 3 and ? = 10, 10 − 3 = 7.

(5) **Successive subtraction** interests and amuses children. A number, say 3, is subtracted from 18, and then from each answer in turn as long as possible, thus:

$$\begin{array}{r} 18 \\ -\ 3 \\ \hline 15 \end{array} \quad \begin{array}{r} 15 \\ -\ 3 \\ \hline 12 \end{array} \quad \begin{array}{r} 12 \\ -\ 3 \\ \hline 9 \end{array} \quad \begin{array}{r} 9 \\ -\ 3 \\ \hline 6 \end{array} \quad \begin{array}{r} 6 \\ -\ 3 \\ \hline 3 \end{array} \quad \begin{array}{r} 3 \\ -\ 3 \\ \hline 0 \end{array}$$

These exercises are quickly set—indeed, the children can set their own. They like to begin with big numbers, as 92 − 11, and work down to 0.

## TWENTY STEPS IN TEACHING SUBTRACTION

In order to make the drills and tests more *varied* and to see that no difficulties are passed over but all tested in turn, it is helpful for teachers to consider these Twenty Steps, each step presenting a difficulty.

*Step I.* In this step only numbers *smaller than* 10 are dealt with. This stage is covered in the Infant School. In the case of the exercise below, point out the use of 0. 6 from 6 leaves nothing. An *empty* space best shows 'nothing,' but the symbol for nothing, 0, shows the exercise has been worked.

```
 TU
  6
- 6
───
  0
```

*Step II.* This shows the extended minuend. One ten in the tens column, and no figure in the subtrahend greater than the one above in the minuend. This gives practice in putting units exactly in the units place—5, for example, exactly under 9.

```
 TU      TU      TU      TU
 19      15      18      13
- 5     - 2     - 6     - 3
───     ───     ───     ───
```

The above step helps them to realize that you cannot subtract nothing. If there is no figure to subtract from the 1 ten, then the 1 ten remains, and must be put down in the answer.

*Step III.* This step differs very slightly from Step II in that the extended minuend may be any number. Again the children see that, if there is nothing to subtract (an empty space implies a 0), the figure remains the same and is just copied down. This is an easy step.

```
 TU      TU      TU      TU
 78      95      87      79
- 6     - 5     - 3     - 5
───     ───     ───     ───
         90
```

*Step IV.* Two-place subtrahend as well as two-place minuend, no figure in the minuend smaller than the figure below it in the subtrahend.

| | | | |
|---|---|---|---|
| T U | T U | T U | T U |
| 8 9 | 9 7 | 9 8 | 8 8 |
| −8 5 | −5 7 | −6 3 | −5 8 |
| 4 | 4 0 | 3 5 | 3 0 |

In the first exercise the empty space stands for nothing. Why is there no need to put a 0? It is not needed as a place-holder. We never write 4 units as 04. In the second exercise there are no units, but a 0 must be put to keep the 4 tens of the answer in its right place. We say 5 tens from 9 tens leave 4 tens.

*Step V.* Three-place subtrahend and three-place minuend. No figure in the minuend smaller than the figure below it in the subtrahend. This is a very easy step, except that it introduces **hundreds**.

| | | | |
|---|---|---|---|
| H T U | H T U | H T U | H T U |
| 3 8 5 | 5 8 9 | 6 7 2 | 9 7 8 |
| −3 5 2 | −3 8 3 | −5 7 2 | −9 7 5 |

*Step VI.* The secondary combinations, subtracting figures 1 to 9 from the 'teens.' This is a very important step. These are **basic facts**, and should be learnt by heart.

| | | | |
|---|---|---|---|
| T U | T U | T U | T U |
| 1 3 | 1 5 | 1 8 | 1 6 |
| − 8 | − 6 | − 9 | − 7 |

The above leads up to the hardest steps—subtracting greater lower figures from lesser upper ones. In subtracting units from the 'teens' the child gets into the 'habit of thought' of subtracting units from a ten and units. When a child later has to subtract 9 from 25 he will tend to think naturally, 9 from 15, taking a ten from the tens column and leaving one ten.

Methods of subtraction are discussed in the next chapter—the decomposition method and the equal additions method.

*Step VII.* Two-place minuend and one place subtrahend, the top unit figure smaller than the bottom.

| | | | |
|---|---|---|---|
| TU | TU | TU | TU |
| 43 | 54 | 57 | 93 |
| − 5 | − 6 | − 8 | − 7 |

Here the children take a ten from the tens column and add 10 units to the units. They make use of the basic facts in Step VI.

*Step VIII.* Two-place minuend and two-place subtrahend, top unit figures smaller than the bottom.

| | | | |
|---|---|---|---|
| TU | TU | TU | TU |
| 88 | 76 | 53 | 32 |
| − 39 | − 38 | − 25 | − 24 |

The above is an easy step, and the children work the exercises quickly if they know their subtraction facts.

*Step IX.* Three-place minuend and two-place subtrahend, top unit figure smaller than the bottom. An easy step that gives more practice in taking one ten from the tens column.

| | | | |
|---|---|---|---|
| HTU | HTU | HTU | HTU |
| 282 | 374 | 467 | 831 |
| − 39 | − 48 | − 39 | − 13 |

*Step X.* Three-place minuend and two-place subtrahend. Top figures in the tens and units columns smaller than the bottom figures. Subtraction from the extended minuend.

| | | | |
|---|---|---|---|
| 326 | 435 | 732 | 612 |
| − 37 | − 56 | − 44 | − 25 |

*Step XI.* Here there is no figure in the answers in the hundreds place, or none in both the tens and hundreds places. The absent figure is an implied 0.

| | | | |
|---|---|---|---|
| 336 | 547 | 163 | 641 |
| − 327 | − 528 | − 139 | − 638 |

*Step XII.* The implied 0 in the answer comes under the extended minuend.

| | | | |
|---|---|---|---|
| 1 6 3 | 1 8 7 | 1 2 3 | 1 5 6 |
| 7 4 | – 9 9 | 2 7 | – 6 5 |

*Step XIII.* Three digits in both minuend and subtrahend. Top figures in the tens and units places smaller than the bottom figures.

| | | | |
|---|---|---|---|
| 5 3 8 | 7 6 4 | 8 4 3 | 6 4 6 |
| – 4 3 9 | – 5 9 8 | – 6 9 4 | – 2 6 8 |

*Step XIV.* This is the first of the so-called seven 0 difficulties. The unit digit in the subtrahend is a nought. An easy step.

| | | | |
|---|---|---|---|
| 5 8 | 6 3 | 9 2 | 7 7 |
| – 4 0 | – 3 0 | – 2 0 | – 7 0 |

*Step XV.* There are no unit figures in the minuend and subtrahend, or no units and no hundreds in the minuend and subtrahend.

| | | | |
|---|---|---|---|
| 8 0 | 3 0 | 7 0 0 | 9 0 0 |
| – 2 0 | – 1 0 | – 3 0 0 | – 5 0 0 |

*Step XVI.* A 0 in the units place of the minuend.

| | | | |
|---|---|---|---|
| 4 6 0 | 7 5 0 | 5 3 0 | 4 2 0 |
| – 1 9 2 | – 2 9 5 | – 3 9 6 | – 2 9 1 |

*Step XVII.* Noughts in the units and tens places of the minuend.

| | | | |
|---|---|---|---|
| 3 0 0 | 7 0 0 | 4 0 0 | 8 0 0 |
| – 1 3 0 | – 4 2 7 | – 2 8 1 | – 7 5 5 |

*Step XVIII.* A 0 in the tens place in the minuend.

| | | | |
|---|---|---|---|
| 6 0 5 | 7 0 8 | 7 0 2 | 3 0 5 |
| – 4 9 6 | – 3 9 9 | – 5 9 5 | – 1 9 6 |

*Step XIX.* A 0 in the tens place in the subtrahend.

| | | | |
|---|---|---|---|
| 5 3 2 | 6 3 7 | 5 5 8 | 3 6 3 |
| - 2 0 4 | - 4 0 9 | - 3 0 9 | - 1 0 5 |

*Step XX.* A 0 in the tens place in the minuend and subtrahend.

| | | | |
|---|---|---|---|
| 3 0 7 | 5 0 8 | 3 0 4 | 6 0 2 |
| - 1 0 9 | - 3 0 9 | - 2 0 5 | - 2 0 6 |

These twenty steps, especially the last fourteen, are useful when setting tests for children of from eight to nine years of age. They also enable the teacher to be sure that children revise and practise every type of subtraction exercise. It is difficult to give useful *varied* exercises unless one has a list of all the possible steps. Each step, of course, has its own slight variations. Tests can be quickly prepared by taking two or more exercises from each step. Such tests help to find out the weakness of individual children; they also help children to keep in mind what they have learnt. The headings Th. H. T. U. should be used until they are thoroughly familiar to the child.

Children should prove all their exercises by adding the subtrahend and the minuend. Methods of subtraction are discussed in the next chapter.

## CHAPTER VI

# METHODS OF SUBTRACTION

*The Decomposition or Reduction Method. Subtraction by Equal Additions. Advantages and Disadvantages of the Two Methods. Carrying to Higher Decades. Complementary Addition.*

Before we consider in detail the two techniques or methods of subtraction most used in schools—Equal Additions and Decomposition—it must be emphasized that the words 'borrowing' or 'paying back' should not be used in either method, because *no borrowing takes place.*

### The Decomposition or Reduction Method

The decomposition of numbers has already been dealt with in Chapter III in connexion with place-value. The children know that 1 ten can be decomposed into 10 units. The first essential in using this method is that the process of decomposition shall be thoroughly understood and learnt. It is a useful process to know, and helps children to get a clearer idea of place-value.

In the example below the child rewrites 83 as shown in (*b*):

| (*a*) | (*b*) | (*c*) |
|---|---|---|
| T U | T U | T U |
| 8 3 | 7 13 | 7 [13] |
| − 2 5 | − 2 5 | − 2 5 |

The pupils' usual form of writing the exercise will then be as in (*c*). At first the child must make these adjustments and say, "I cannot take 5 from 3. Take one of the 8 tens, leaving 7 tens. Decompose it into ten units, making thirteen units." Then he subtracts. The auxiliary figures, or 'crutches,' used in subtraction are temporary aids, but very important at first.

Intelligent children soon discard them. P. B. Ballard looks with disfavour on these temporary aids on the grounds that children are learning an arrangement which must eventually be discarded. Teachers of backward children, however, know that the 'crutches' are essential. P. Klapper, in his most useful and thought-provoking book *The Teaching of Arithmetic*,[1] makes a good case out for their use, saying wisely that the less intelligent pupils "should not be made the victims of fixed pedagogical procedures." Plenty of exercises like the following should be given: Rearrange the number 62 so that it will be possible to take away 39.

The step to exercises involving three digit numbers can be made without much difficulty. But exercises with 0's in them may cause trouble (see Chapter IV for examples). The child beginning with the units columns hesitates when he sees *no* tens in the tens column. Remind him of his lessons on place-value and notation (Chapter III), and that the number 703 can be read as 70 tens and 3 units, so he has plenty of tens from which to take one, and the exercise

$$\begin{array}{r} 7\ 0\ 3 \\ -\ 2\ 5\ 7 \\ \hline \end{array} \quad \text{becomes:} \quad \begin{array}{r} 6\ 9\ {}^{1}3 \\ -\ 2\ 5\ 7 \\ \hline \end{array}$$

It will help the children to have some 'number-decomposition' lessons; thus the number 324 may be read as:

(1) 3 hundreds, 2 tens, 4 units
(2) 32 tens, 4 units
(3) 31 tens, 14 units

according to the readjustment needed.

The number 7,003 may be read as:

(1) 7 thousand and 3 units
(2) 70 hundreds and 3 units
(3) 700 tens and 3 units
(4) 699 tens and 13 units

Children soon learn to change 1,000 into $\overset{H}{9}\ \overset{T}{9}\ \overset{U}{{}^{1}0}$ (see Chapter III).

[1] Appleton–Century, 1934

Teaching subtraction in the above way gives the teacher an excellent opportunity for revising place-value and notation. Children soon learn that every figure in the units column stands for so many 1's, every figure in the tens column for so many 10's, and every figure in the hundreds column for so many 100's, *but* they are not so ready to see that one hundred is the same as 10 tens, and that if necessary 10 can be put in the tens column, or that one thousand is the same as 10 hundreds, or that one ten is the same as 10 units.

**Decomposition** is very similar to **reduction**. Children will meet reduction when they are learning compound quantities—see Chapter X. Just as 1 hundred can be decomposed into 10 tens or 100 units, so £1 can be decomposed into 20 shillings or 240 pence.

## Subtraction by Equal Additions

The idea of subtraction by equal addition is based on the fundamental notion of the **difference** between two numbers. Intelligent children will be interested in the explanation, although they will fail to understand it entirely.

The *difference* between 6 and 2 is 4, or $6-2=4$. "The difference between two numbers is unaltered if the *same* number is added to each." (This statement is simply put, therefore less exactly put.) While it is axiomatic to the adult mind that "if equal numbers be added to unequal numbers, the difference between the unequals remains the same," it is very difficult for children to understand generalizations. However, the children can prove the statements above by adding in turn 2, 3, 5, 8, etc., to the minuend and subtrahend of $6-2=4$, getting $8-4=4$; $9-5=4$; $11-7=4$; $14-10=4$. They see the difference is always the same as $6-2=4$.

Concrete illustrations—that is, using counters, etc.—are best avoided. They are rarely satisfactory, but many illustrations of a somewhat practical nature can be given:

(1) If a girl has threepence more than her sister she will still have threepence more if each is given twopence.

(2) The difference in age between two children remains the same as years advance.
(3) The difference in height between two children remains the same if they both stand on the *same* bench or the *same* chair.

Children like to discuss these illustrations and think of others themselves. Now let us apply this principle of equal additions to a subtraction exercise 63 − 28, as below.

$$\begin{array}{r} 6\,{}^{13} \\ {}^{3}\not{2}\;8 \\ \hline \end{array}$$

Eight from three shows more units are needed. We know we can add *any* number of units we please to the upper number of units, as long as we treat the bottom number in the same way according to the principles of Equal Additions. The number that best meets all cases is ten. Therefore ten is added to the upper number in the form of 10 units, and to the lower number in the form of 1 ten. Hence we arrive at the form above. Three digit exercises are then taken that involve further equal addition to both numbers—10 tens to the upper number, 1 hundred to the lower, and so on. Noughts present little difficulty. In the exercise given below the pupil, using no unnecessary words, says:

| | |
|---|---|
| 5003 | Eight from thirteen, five |
| − 2678 | Eight from ten, two |
| —— | Seven from ten, three |
| 2325 | Three from five, two |

The children soon fall into the habit of thinking, Add ten to the top line and one to the bottom, and the process becomes absolutely mechanical, so that mistakes are rare.

### ADVANTAGES AND DISADVANTAGES OF THE TWO METHODS

The Equal Additions Method has the advantages of being *easily taught* (apart from any explanation); it is a *quick method*; the *O's* cause *less difficulty*; and the work is *more accurate*. Tests have proved that children using the Equal Additions

Method get more exercises right than those using the Decomposition Method. But the tests only apply to natural numbers. When it comes to compound quantities, £ *s.* *d.*, yd, ft, in., etc., more mistakes are likely. The dull child, when he cannot take 9*d.* from 4*d.*, persists in adding 10. Many teachers complain of the difficulty of passing from natural numbers to compound quantities. Children taught the Decomposition Method find little difficulty:

£100 0*s.* 0*d.* – £76 13*s.* 8½*d.* becomes naturally
£99 19*s.* 12*d.* – £76 13*s.* 8½*d.*

There is no strain on their understanding, as there is when they are told to add 12*d.* to the top line and 1*s.* to the bottom. From where do the 12 pennies come, and the 1*s.*? Money to a child is something concrete. He has had dealings with it. The thoughtful child was content to add an abstract ten to the top line and one to the bottom, but he hesitates about money. Perhaps it savours of sharp practice!

The Equal Additions Method is difficult when it comes to the subtraction of vulgar fractions (see Chapter XIII).

Another disadvantage of the Equal Additions Method is that it cannot really be explained in a satisfactory manner to Junior children; there can be no concrete illustrations for dull children, the additions are "purely hypothetical and the process abstract and artificial." It is like nothing they have experienced before in the Infant School. On the other hand, the Decomposition Method can be explained; it links up with the work in the Infant School, and with the children's shopping experiences. It is not an entirely new rule. Concrete examples can be given, and it is easy to pass from this method to Complementary Addition if necessary. The disadvantages of the Decomposition Method are that it is slower, more mistakes are made (perhaps because the children are thinking!), and the 0's may cause more trouble.

### Carrying to Higher Decades

Every effort must be made to carry the **45 Subtraction Facts** to **higher decades**, as in addition. Many of the devices used for addition can be used for subtraction. The wheels shown in Fig. 34 and 35 are useful.

Fig. 34 shows subtraction in the same decade: 8 − 2, 18 − 12, 28 − 22, etc. Fig. 35 shows subtraction of a digit from any decade: 9 from 16, 26, 36, etc.; 16, 66, 96, etc. The children

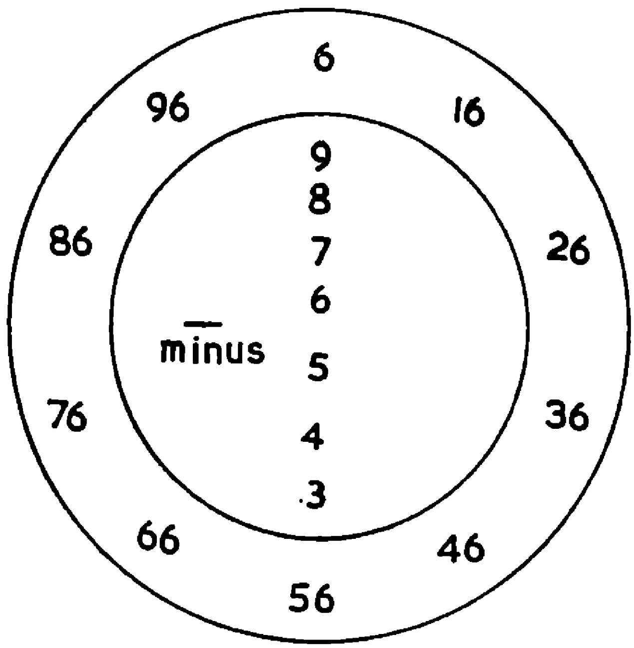



FIG. 34. CIRCLE FOR SUBTRACTION IN THE SAME DECADE

like to see how quickly they can work round the circle subtracting each figure in the inner circle from one in the outer circle. Sometimes, in the case of backward children, let them copy each exercise down thus: $\begin{array}{r} 18 \\ -\ 12 \\ \hline \end{array}$. Many children need a great deal of practice in setting down exercises for themselves *if they have no exact copy.* Even such a simple task as copying an exercise from the Circle tests the understanding of the backward child. One must beware of too many devices that save children from *copying down examples,* and leave only the answer to be written down. Some of these devices are good, but they should *not* be used in the lowest classes, and never *always* used. Children need practice in copying figures, both from sight and from dictation. Above all, they need practice in keeping the figures in the units, tens, and hundreds columns exactly under each other. Badly set down exercises are a fruitful cause of many mistakes. As the exercises grow harder it is increasingly important that children should prove their answers right by adding them to the subtrahend to get the minuend. This ensures accurate work and understanding.

It is a good plan too to let each child make a sample note-book for subtraction in which he puts down very carefully

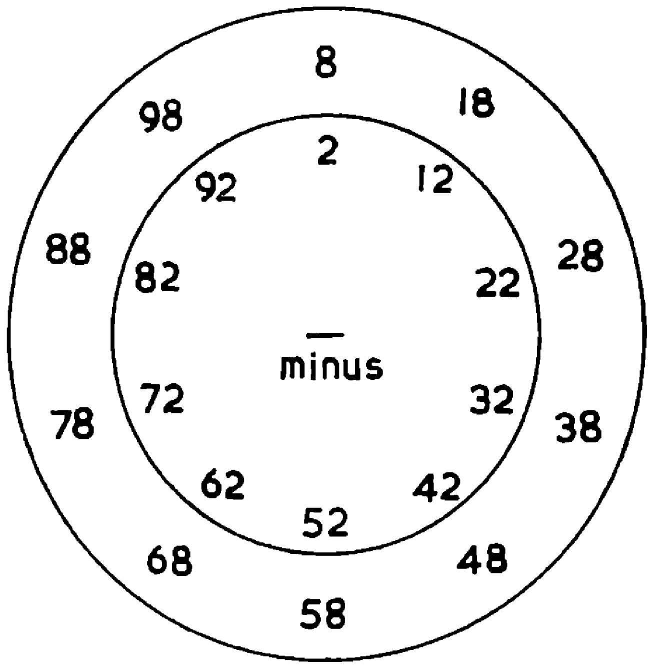



FIG. 35. CIRCLE FOR SUBTRACTION IN ANY DECADE

examples of different subtraction exercises. Children like these booklets, and they are a record of their work, and models of perfection!

COMPLEMENTARY ADDITION

A word must be said in conclusion on the method of **Complementary Addition.** This is a completely different method of subtraction which is popular in America and on the Continent, but it is not usually taught in Primary Schools in this country. In this method the idea of 'taking away' or 'difference' is put in the background and the essential notion is that of **addition**. Children unconsciously use the method in their puzzle exercises **in addition** and subtraction.

| | | | |
|---|---|---|---|
| 7 | ? | 7 | 9 |
| + ? | + 4 | − ? | − ? |
| 16 | 9 | 3 | 5 |

Both adults and children use it when giving change. When a child is being taught short division he is introduced to it informally. Thus in the first stage of the exercise

7)478

the child says, "How many sevens in forty-seven? Six. Six sevens are forty-two, *and five* are forty-seven."

Subtraction by Complementary Addition is worked in this way:

```
823
165
---
658
```

Five and 'the number to be found' gives an answer whose unit figure is 3; the figure must be 13. Five and what make 13? Obviously 8. In the addition (5 + 8 = 13) there is 1 to carry. Add it to the 6. Seven and what make 12? Clearly 5. Carry 1. Two and what make 8? Six. Children who have learnt the decomposition method naturally say, "5 and 8 make 13; 6 and 5 make 11; 1 and 7 make 8."

Exercises of the following type are of practical value, and illustrate the advantage of this method. Find the missing numbers in this addition exercise:

| | |
|---|---|
| 134 | Adding downward |
| 469 | 4; 13; 15; 16; and **3** are 19. Carry 1 |
| 352 | 4; 10; 15; 20; and **2** are 22. Carry 2 |
| 251 | 3; 7; 10; 12; and **4** are 16 |
| ... | The figures in heavy type are put in their proper places |
| 1629 | as they are said. |

The above exercise with sums of money is used in the world of commerce to-day. The bank clerk uses this process many times daily in checking up his clients' pass-book, thus: Mr W. has £1,635 in the bank. On March 1 he drew out £493; June 1, £176; August 4, £230. How much has he left?

| | |
|---|---|
| £493 | 3; 9; and **6** are 15. Carry 1 |
| 176 | 10; 17; 20; and **3** are 23. Carry 2 |
| 230 | 6; 7; 9; and **7** are 16 |
| ... | |
| 1635 | |

This method makes it possible to take a sum of numbers from one number in a single step. It is therefore a quick and efficient method for people who have a good deal of computation to do. It regards subtraction as no new process but as an aspect of addition. Bankers, shopkeepers, accountants, statisticians, astronomers, business-men of all kinds, make use of it. It is also used in work involving logarithmic tables.

Most educationists appreciate this method; that it is seldom taught in schools is perhaps because it does not lend itself so easily to the subtraction of British weights and measures. Intelligent children enjoy finding the missing numbers. It is an interesting puzzle to them. In the Junior School one must cater for these children and give them a wide outlook. They become excellent 'at arithmetic' from the point of view of accuracy and speed, but they may get no idea of the delights of mathematics.

The study of arithmetic introduces them to new words and ideas, and the pupils' understanding of these words increases their vocabulary in a purposeful way. Let them add to their dictionaries whenever possible, and keep notebooks about 'the language of arithmetic.' In it, for example, they can put all

11 take away 7<br>
11 minus 7<br>
11 − 7<br>
11 less 7<br>
Take 7 from 11<br>
Subtract 7 from 11<br>
Difference between 7 and 11

the different ways of expressing subtraction, as shown above. At the end of a year, or at the end of their stay in the Junior School, the notebooks of the intelligent children are often like little textbooks.

CHAPTER VII

# THE MULTIPLICATION TABLES

*Making the Tables in Two Forms. The Multiplication Sign and what it means. Learning the Tables by Heart. The Use of Number Charts.*

Adding is not only closely linked with subtraction but with multiplication—indeed, it is the basis of multiplication. "Multiplication is that particular case of addition in which the numbers to be added are all the same."

In the Infants' School the children have picked up many ideas about multiplication. In adding doubles they have learnt the 2-times table. In making number patterns and analyzing numbers they have found out that 8 can be arranged as 2 fours or as 4 twos, and so on (see Chapter I). Another informal preparation for multiplication is found in **counting** in twos, threes, fours, etc. The child who counts 3, 6, 9, 12, 15, 18, 21, and so on is learning the endings of the multiplication table of Three Times.

The **multiplication tables** should be developed by the children themselves, so that they understand the reason and the system of the combinations. Attempts to learn the tables from a wall-chart or the printed versions found in table books, or on the cover of exercise-books, are of little value. From this traditional method the children sometimes get the impression that the tables consist of long lists of new and *unconnected* number facts. This impression is both harmful and far from the truth.

## Making the Tables in Two Forms

It can be taken for granted that the '2-times table' is already known to them, for these are just the doubles of the 45 Addition Facts they have already learnt.

| | | | | | | |
|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 | 9 |
| 1 | 2 | 3 | 4 | 5 | 6 | 9 |
| 2 | 4 | 6 | 8 | 10 | 12 | 18 |

Let the children begin by making the 3-times table. They write down and find by addition the following sums:

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 7 | 8 | 9 |
| 1 | 2 | 3 | 4 | 5 | 7 | 8 | 9 |
| 1 | 2 | 3 | 4 | 5 | 7 | 8 | 9 |
| 3 | 6 | 9 | 12 | 15 | 21 | 24 | 27 |

Instead of saying 1 and 1 and 1 are 3, 2 and 2 and 2 are 6, lead the children to see that they can just as well say, "Three 1's are 3, three 2's are 6." Each number has been written *three times,* so we can also say, "Three times 1 are 3." By remembering the answers given below, the children see that much time is saved when the sum of three equal numbers is needed. If a pencil costs 4*d.*, it is easy to find the cost of 3 pencils.

When the **3-times table** has been made the children write it down in the well-known form:

3
times
1 are 3
2 are 6
3 are 9

There is no need for them to keep their copy. Let them compile it again when they need it. It is a *great help* to the mastering of a table if it is built up again on *many* occasions. While they learn their tables remind them frequently that multiplication is a special way of adding when all the numbers are the same. This point can be brought out when asking questions on the tables—for example, "What are three times four?" When the child answers correctly, "Twelve, ask him, "Why twelve?" to get the reply, "Three fours added together make twelve."

*Making the 'Table of Threes,' or the Times-*3 *Table.* Besides the 3-times table given above, there is another form of the table to be learnt—namely, the **Table of threes**, or the **Times-3 table.** This table is needed for division. It can be learnt after the first table, the 3-times table, or with it. Fig. 36 shows how the children build it up. It can be written down in table form:

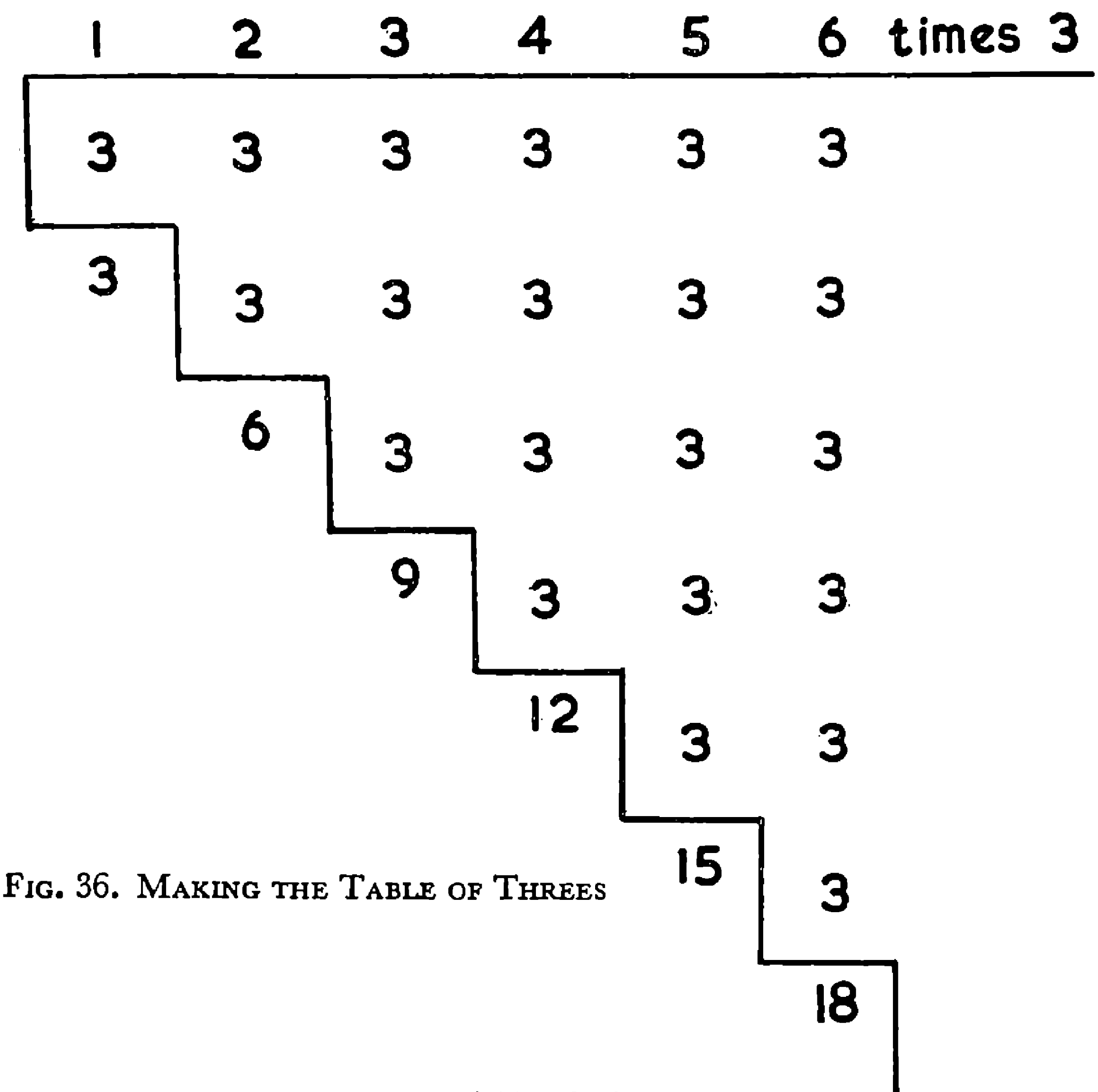



FIG. 36. MAKING THE TABLE OF THREES

*Table of Threes*

1 three is 3
2 threes are 6
3 threes are 9
4 threes are 12

And so on.

It is a help to understanding if the two tables are written down side by side, as below. The children notice the difference between the two tables, but the *answers* are the same.

| | |
|---|---|
| 3 times 1 are 3 | 1 three is 3 |
| 3 ,, 2 ,, 6 | 2 threes are 6 |
| 3 ,, 3 ,, 9 | 3 ,, ,, 9 |
| 3 ,, 4 ,, 12 | 4 ,, ,, 12 |
| 3 ,, 12 ,, 36 | 12 ,, ,, 36 |
| 3-times table | Table of 3's or Times 3 |

In the case of backward or dull children, the setting down of the tables, as in Fig. 37 and 38, is helpful. If necessary the children can build these tables up concretely with beads or

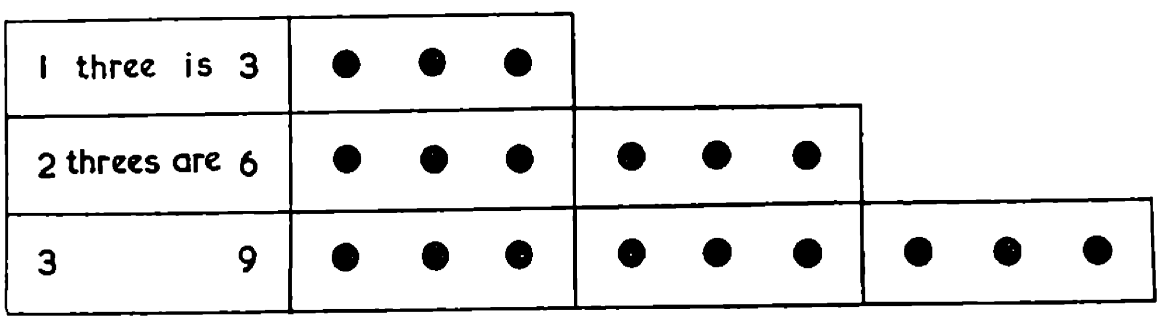



FIG. 37

counters. Even intelligent children see better the difference between 3 times 1 and 1 times 3, although the answers are the same, when they plan out the tables as in Fig. 37 and 38. In the first, Fig. 37, there are three single ones. In the second table, Fig. 38, there is one group of three. The children like drawing these diagrams, and make them for other tables. There is no need for them to complete the whole

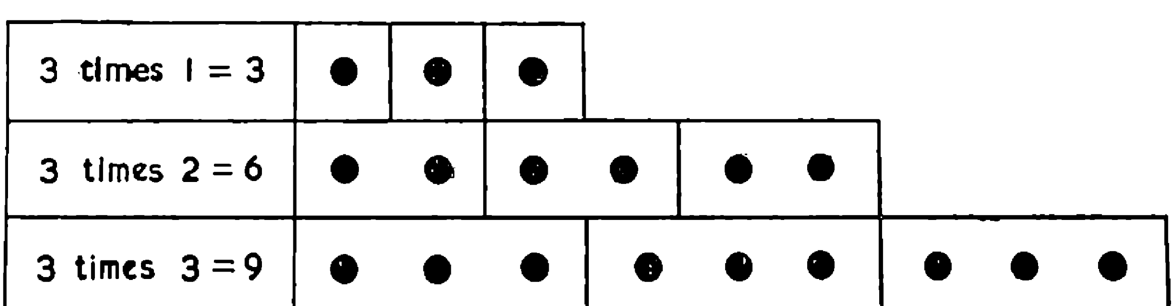



FIG. 38

table, especially with the bigger numbers. A few lines are enough to make sure the child has the right idea. The learning of *both* forms of the table is essential, because the association of these tables is necessary to the full understanding of the law of the commutation of factors that 2 times 3 gives the same answer as 3 times 2. The two aspects of the 9-times table and the table of 9's gives visual help to a great extent:

| | |
|---|---|
| 9 times 5 = 45 | 5 nines = 45 |
| 9 times 6 = 54 | 6 nines = 54 |
| 9 times 7 = 63 | 7 nines = 63 |

The close connexion between these two tables can also be shown by setting down the results thus:

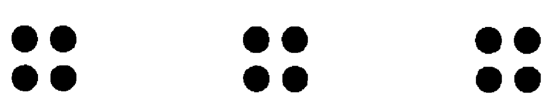
Three ones = one 3 = 3
Three twos = two 3's = 6
Three threes = three 3's = 9
Three fours = four 3's = 12

It does not matter which form of the multiplication table is learnt first. The second form, as in the table of threes, Fig. 38, enables every child to construct any required table without help, by the process of equal additions. Multiplication and division facts are thus taught together: $3 \times 4 = 12$; $12 \div 3 = 4$.

## The Multiplication Sign and What it means

There is often some controversy as to the exact meaning of $4 \times 3$. It is of some importance that this contraction should be read correctly as four **multiplied by** three. The symbol $\times$ is *never* equivalent to '**times.**' In this example 4 is called the **multiplicand**, and 3 the **multiplier**.

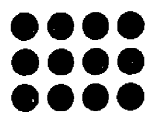
$4 \times 3$ is not the same as $3 \times 4$, as can be shown by counters. Thus $4 \times 3$ appears as 3 heaps each containing 4 counters, thus:



But $3 \times 4$ appears as 4 heaps each containing 3 counters, thus:

Although 4 × 3 is not the same as 3 × 4, they both give the same total—12.

Here is another diagram that interests children, to show that 4 × 3 gives the same total as 3 × 4:



How many counters are there in the above group? First count the rows. There are 3 rows each containing 4 counters, 4 + 4 + 4 or 4 × 3 = 12. Next count by columns. There are 4 columns, each containing 3 counters, 3 + 3 + 3 + 3 or 3 × 4 = 12.

A knowledge of this property of multiplication that 4 × 3 = 3 × 4, or, in words, that the multiplier and multiplicand may be interchanged, enables one to choose the simpler number as multiplier. This property, however, is not quickly realized by children. They may, for example, answer readily to "four eights?" but will hesitate over "eight fours?" Every opportunity should be taken to remind them of this property, because it halves the multiplication facts to be learns. When they learn 2 eights are 16, they should also learn 8 twos are 16. The above diagram for 4 × 3 = 3 × 4 should often be made for the multiplication facts that are being learnt. To teach every combination with its commutation is generally a great help, 6 × 7 with 7 × 6.

## Learning the Tables by Heart

There is no royal road to the mastery of the multiplication tables, *but they must be learned.* It is most unfair to the child not to insist on this learning. It is not enough for the children to know how to make their tables, they must know them. When the question "Nine eights?" is asked they must answer automatically, "Seventy-two." Here are some suggestions for the teaching of the tables, some good and some not so good. An open mind must be kept, because children vary so; some of the following suggestions may help a certain type of child but not others. Variety, new illustrations, new exercises, arouse interest. So any one of these devices is worth trying at least once.

(1) *The Whole Class chant the Table in Chorus.* There are many arguments against this method. This kind of repetition is almost useless unless the children repeating the chant have the *desire* to learn. It only shows that some children really know their tables. It does not always help the backward ones. On the other hand, and especially with younger children, the sixes and sevens, to march round the room chanting the tables and stamping the foot at a difficult combination can be very effective. Just as children like the lilt and rhythm of 2, 4, 6, 8, 10, so the sound of "three ones are three, three twos are six, three threes are nine," etc., appeals to them. Beginning with the same figure helps the rhythm, starting with a different figure each time makes the rhythm more jerky. Therefore it is best to begin with the first form of the tables (see beginning of chapter). The younger the children are, the more they enjoy learning by heart, and one must take advantage of this. The older the child gets, the harder it is for him to learn his tables. He begins to feel it is impossible, and no longer desires to learn. Younger children enjoy singing the tables, and this is worth trying. The great advantage of reciting tables aloud is that it leads to a *tongue* and *ear* memory that powerfully aids the eye memory when a pupil needs to recall a number fact. A child once said, "My tables sing me to sleep." He was accustomed to say his tables in bed, and to see how many he could say before he went to sleep!

(2) *Individual Repetition.* This is of great value. A child repeating a table to himself will tend to concentrate on trying to remember the combinations that trouble him. But it is dull work. For part of a lesson the children should be allowed to scatter and repeat aloud to themselves the table they are learning. Sound helps memory. Sometimes group repetition is effective if the group is kept small enough, so that it is almost individual repetition. The group takes pride in every member knowing the table well. But these groups should not be forced on children. Encourage them to repeat their

tables at all sorts of odd times. Suggest they lull themselves to sleep by saying their tables!

(3) *Children should be frequently tested on the tables* they are learning in a variety of ways. (*a*) Individual pupils should be asked to repeat a table. (*b*) The whole class should be asked to write it down. (*c*) Sometimes let them write down the results or 'stations' of a table—for example, the 'stations' of the 6-times table are, 6, 12, 18, 24, etc. (*d*) Let them write a table down beginning at a certain combination—say, four sixes in the 4-times table—or at a particular 'station'—say, 12 in the 4-times table. (*e*) Ask varied questions round the class, such as:

Four nines? Nine fours?
What two numbers multiplied together give 36?
Find two other numbers that multiplied together make 36.
What is 4 multiplied by to get 36?
What is 9 multiplied by to get 36?
How many fours in 36?
How many nines in 36?
36 is made up of how many nines?
36 is equal to ? × ?

At regular intervals the children should be required to write down the answers to questions such as the above on slips of paper. These are collected and looked at by the teacher, so that he can see what multiplication facts cause trouble to individual pupils. Notice that division facts are taught at the same time.

(4) *Missing Number Drills.* Just as the 'missing number' drills in addition give valuable drill in addition and prepare the way for subtraction, so similar drills in multiplication help to fix the multiplication facts and prepare for division. $3 \times ? = 15$, $? \times 8 = 56$, $? \times ? = 24$, and so on.

(5) *Noticing the Patterns of the Tables and Interesting Facts about them.* Learning the tables is dull work, even drudgery, so the teacher must be ready to point out anything interesting about them. In the 2-times table the pattern of the unit figures is 2, 4, 6, 8, 0, then again 2, 4, 6, 8, 0, and so on. In

the five-times table, the products end in 0 when the multiplicand is even, 5 when it is odd. In the eleven-times table, the product merely repeats the multiplicand, "eleven 5's are 55."

The **nine-times** table can be made to interest children. Let them write out the table from 9 times 1 to 9 times 10. In this table the digits of the products add up to nine. Thus in the column below 9 = 9, 1 and 8 = 9, 2 and 7 = 9, 3 and 6 = 9, 4 and 5 = 9, 5 and 4 = 9, 6 and 3 = 9, 7 and 2 = 9, 8 and 1 = 9, 9 and 0 = 9. Then let them notice that the unit digits decrease by 1 from 9 to 8, 7, 6, 5, 4, etc., to 0.

9
18
27
36
45
54
63
72
81
90

The digits in the second column, the tens column, increase by 1 from 0 to 1, 2, 3, 4, 5, 6, etc., to 9. Interested children may like to consider 99 and 108. If the figures in 108 are set down as 1 + 0 + 8 they equal 9. Some may want to continue the table and see what happens—for example, in the case of 9 times 13 = 117. Here again 1 + 1 + 7 = 9. By continuing the table further something else interesting will be discovered. Children like to write out the 9-times table and find out all these interesting facts for themselves. By adding the digits they can check any product of the 9-times table.

Let them practise the tables by not writing or saying the whole table, but just the products or 'stations' or 'stopping-places,' as they like to call them—9, 18, 27, 36, 45, 54, etc.; this is really counting in 9's. It is a good plan to let the children do this with all the tables. The 'stations' of the 4-times table are easy, 4, 8, 12, because counting in 4's is easy. The 'stations' of the 6-times tables are harder, because it is not so easy to count in 6's—6, 12, 18, 24, 30 (for counting see Chapter II).

(6) *The well-known Circle or Clock-face* (Fig. 28, Chapter IV) can be used with advantage for drill in the multiplication tables. The multiplier is put in the centre, the numbers to be multiplied are placed round the rim, in any order. It is especially useful for the younger children. They like to see how quickly they can work round the clock. Other devices used for the addition combinations are also used, the ladder, the staircase, the relay race, the railway sleeper, etc. A simple form of drill is to write a row of figures on the blackboard—say, 7, 4, 6, 3, 8, 9, 5, 12, 11. As the teacher points to a number in the list individual pupils multiply it by 6, if the 6-times table is being studied.

(7) *Ways of teaching one Particular Combination*—a difficult one, such as $7 \times 6$. Let the children count by 6's and then by 7's, emphasizing 42 in each series:

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 6, | 12, | 18, | 24, | 30, | 36, | **42**, | 48 |
| 7, | 14, | 21, | 28, | 35, | **42**, | 49, | 56 |

Prove the commutation $6 \times 7 = 7 \times 6 = 42$. Arrange 6 rows of 7 counters (see page 84).

Whenever a pupil comes across a product which he cannot remember he should *make* the table containing it—that is, go back to first principles. Every product has in the beginning been discovered by them. They are not learning by heart anything they do not understand.

## The Use of Number Charts

The following charts are of great importance and of great interest to children: **Table Square**, which is shown in Fig. 39; and **Charts for all the Tables**, of which an example is given in Fig. 40. Although in the early stages no permanent copy of the tables should be kept, a permanent record is useful when the knowledge of the tables is being revised. Moreover, the making of a compact summary of all the tables in the form shown in Fig. 39 is itself a very instructive exercise.

Each child makes a Table Square. Squared paper may be used, or the child rules off a piece of paper in squares, 12 squares by 12 squares, or 144 squares. The children write the numbers 1–12 across the top and down the left-hand side; as shown in Fig. 39. Each table is then built up in turn, the child beginning with 2, and multiplying each figure in the top row by 2, setting down the results in the second row, as in Fig. 39. Then 3 is used in the same way. When the Table

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
| 2 | 4 | 6 | 8 | 10 | 12 | 14 | 16 | 18 | 20 | 22 | 24 |
| 3 | 6 | 9 | 12 | 15 | 18 | 21 | | | | | |
| 4 | | | | | | | | | | | |
| 5 | | | | | | | | | | | |
| 6 | | | | | | | | | | | |
| 7 | | | | | | | | | | | |
| 8 | | | | | | | | | | | |
| 9 | | | | | | | 72 | | | | |
| 10 | | | | | | | | | | | |
| 11 | | | | | | | | | | | |
| 12 | | | | | | | | | | | 144 |

FIG. 39. CHART FOR ALL TABLES

Square is complete with all the tables the children prove its value by seeing how quickly they can find a product. Supposing they want the answer to $8 \times 9$. Point out to the children that the multiplier is found in the left-hand column—namely, 9—the multiplicand in the row along the top—namely, 8. By running one's fingers along the 9 line and down the 8 line, it is found they meet in square 72, which gives the desired product $8 \times 9 = 72$. This can also be tested by the number of 9's in the rows.

1 row of 9 squares $=9$ squares
2 rows of 9 squares $=18$ squares
3 rows of 9 squares $=27$ squares,
etc.

The table can be used in several other different ways, and for more advanced work—for example, for finding common multiples of pairs of numbers (such as 4 and 3), for finding pairs of factors of such numbers as 36, 24, etc., for finding the squares of numbers up to 12. The making of the Table Square can be fully justified because it gives the children a useful introduction to the art of using a table of reference.

*A Chart for learning Tables and finding Table Patterns (Fig. 40).* This is an easier chart than Fig. 39. All the numbers from 1 to 100 or to 120 are written down on squared paper, as in Fig. 39. The children may make eleven charts, one for each table from 2 to 12. They find the pattern by saying the table, and colouring the squares that contain the 'stations' or 'stopping-places' of the table. Fig. 40 shows the pattern made by the 6-times table. It goes up to 13 × 6.

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | **6** | 7 | 8 | 9 | 10 |
| 11 | **12** | 13 | 14 | 15 | 16 | 17 | **18** | 19 | 20 |
| 21 | 22 | 23 | **24** | 25 | 26 | 27 | 28 | 29 | **30** |
| 31 | 32 | 33 | 34 | 35 | **36** | 37 | 38 | 39 | 40 |
| 41 | **42** | 43 | 44 | 45 | 46 | 47 | **48** | 49 | 50 |
| 51 | 52 | 53 | **54** | 55 | 56 | 57 | 58 | 59 | **60** |
| 61 | 62 | 63 | 64 | 65 | **66** | 67 | 68 | 69 | 70 |
| 71 | **72** | 73 | 74 | 75 | 76 | 77 | **78** | 79 | 80 |
| 81 | 82 | 83 | 84 | 85 | 86 | 87 | 88 | 89 | 90 |
| 91 | 92 | 93 | 94 | 95 | 96 | 97 | 98 | 99 | 100 |

FIG. 40. CHART FOR LEARNING TABLES

The 'stations' of the 10-times and 5-times tables make vertical lines. The child sees clearly from the table that two tens are twenty, three tens are thirty, and so on. The 5-times table makes two vertical lines. It is interesting to put related tables on one chart, as the 2-times, 4-times, 8-times. These tables make four vertical lines.

The related tables 3-times, 6-times, and 9-times give diagonal lines. The 12-times table gives diagonal lines going in the opposite direction to the diagonal line of the 9-times table. The pattern of the 11-times table interests children.

Exercises in short multiplication such as $71 \times 3$, $243 \times 7$, may be introduced (see next chapter) before the children know perfectly the tables concerned (see next chapter).

Give the children easy problems to bring out the *meaning* and *use* of multiplication.

(1) If one loaf of bread costs 4*d.*, how much will 7 loaves cost?

(2) There are 6 rows of children in a class. In each row there are 7 children. How many children in the class?

(3) Tom had to buy 7 stamps. Each cost 3*d.* How many pennies did he need?

(4) These problems introduce simple proportion or ratio.

Be sure the children begin to know the language of multiplication; **multiplier, product, factors** or makers. $3 \times 4 = 12$, or four times three make 12. 3 and 4 are factors of 12. See Chapter VIII.

CHAPTER VIII

# STEPS IN MULTIPLICATION

*Short Multiplication: Six Steps. Multiplying by* 10, 100, *etc. Multiplying by* 20, 30, 40 . . . 90. *Focusing Attention on the Carrying Figure. The Ratio Idea. Long Multiplication: Graded Steps. The Language of Multiplication. Tables again.*

As the children learn their tables they should set them down in vertical form, as they will most often meet them. It is, of course, another way of practising the tables.

## SHORT MULTIPLICATION: SIX STEPS

(1) $\begin{array}{r} 221 \\ \times 4 \\ \hline 884 \end{array}$ No carrying. Each multiplication is complete.

(2) $\begin{array}{r} 922 \\ \times 4 \\ \hline 3688 \end{array}$ The hundreds digit is so large that when multiplied it gives two places.

(3) $\begin{array}{r} 324 \\ \times 3 \\ \hline 972 \end{array}$ Carrying is introduced in one digit, but the carrying figure is 1.

(4) $\begin{array}{r} 212 \\ \times 6 \\ \hline 1272 \end{array}$ Steps (2) and (3) are combined.

| | | |
|---|---|---|
| (5) | 432<br>×6<br>2592 | Carrying in more than one place, but the carrying figure still only 1. |
| (6) | 645<br>×6<br>3870 | Carrying one or more times, but the carrying figure may be more than 1. |

The above exercises give good practice in the tables. Backward children should continue to label their columns H.T.U., etc., so that the figures are kept in the right places. Let them often read their answers in words—*e.g.*, three thousand eight hundred and seventy.

The case of 702 × 3 needs special thought because of the 0 in the multiplicand. In multiplication as in addition, the 0 difficulty is best got over by letting the children pause for a moment to think of the meaning of 0. There is nothing in the tens place. So no multiplication is possible. They write down 0 and multiply the next figure by 3.

In the case of 906 × 7 the child says or thinks, "Seven sixes? Forty-two. Write down 2. Carry four." He sees the 0 and knows that no multiplication is to be done. He puts down the 4 tens carried from the units. "Seven nines? Sixty-three. Write down 63."

Let the children practise these exercises:

| HTU | HTU | HTU | HTU | HTU | HTU |
|---|---|---|---|---|---|
| 3 0 | 4 0 | 5 0 | 6 0 | 7 0 | 8 0 |
| × 7 | × 9 | × 4 | × 8 | × 6 | × 5 |
| 2 1 0 | 3 6 0 | 2 0 0 | 4 8 0 | 4 2 0 | 4 0 0 |

Here again the child sees that as there is nothing in the units place no multiplication is possible. He writes down 0 so that he will know where to put the tens when he multiplies them. The answer can be read as "two hundred and ten" or "twenty-one tens." If the children know their tables they can write down the answer at once.

MULTIPLYING BY 10, 100, ETC.

For obvious reasons, this is worth spending some time over. Some teachers—chiefly, perhaps, inexperienced teachers—teach the easy but 'meaningless' rule 'to multiply by 10 you add 0.' The teacher must think of the kind of work his children will be doing later. He must teach, as far as possible, nothing that later on will require to be untaught. When the children come to decimals, 23·4 × 10, the 'meaningless' rule of 'adding 0' breaks down.

Let the children set down on squared paper the two examples given below, 47 × 10 and 234 × 10.

| Th | H | T | U |
|---|---|---|---|
| | | 4 | 7 |
| | X | 1 | 0 |
| | 4 | 7 | 0 |

| | Th | H | T | U |
|---|---|---|---|---|
| | | 2 | 3 | 4 |
| | | X | 1 | 0 |
| | 2 | 3 | 4 | 0 |

FIG. 41. MULTIPLYING BY TEN

Take the first example, (*a*). Remind the children that the 4 and the 7 of the multiplicand have certain 'ranks' because of their positions; the 7 that of units, the 4 that of tens; 10 units make 1 ten. The tens are therefore bigger and of higher rank than the units. The hundreds are bigger still, for 1 hundred is 10 tens, or 100 units. If we alter the position of a figure and move it one place **to the left** we 'promote' it; in (*a*), for example, the 7 units become 7 tens, and the 4 tens become 4 hundreds. Each figure becomes ten times bigger when it is moved one place to the left. **To multiply by 10 is therefore to move each figure one place to the left.** A 0 is inserted in the vacant units place to keep 7 in the tens place.

Hence we have this rule: **To multiply a number by 10,**

**move the digits of the number one place to the left, and put a 0 in the vacant units place.**

The idea of promotion is of great importance. If the children grasp it they will have little difficulty in multiplication. Similar rules may be developed for multiplication by 100, 1,000, etc., thus: **To multiply a number by 100, move the digits of the number two places to the left, and put 0's in the vacant units and tens places.**

MULTIPLYING BY 20, 30, 40 . . 90

To multiply by 20, move each digit one place to the left, at the same time multiplying by 2 because there are 2 tens, as in (*a*) below. Then insert a 0 in the vacant units place.

| (*a*) Th | H | T | U | | (*b*) Th | H | T | U |
|---|---|---|---|---|---|---|---|---|
| | 3 | 6 | 4 | | | 3 | 6 | 4 |
| | | 2 | 0 | | | | 2 | 0 |
| 7 | 2 | 8 | | | 7 | 2 | 8 | 0 |

It is wise to let the children multiply and promote first, and then fill in the stop-gap 0 afterwards, because the promotion of the digits is of first importance. There is no question of multiplying by the 0 of 20. The problem is to multiply by 20 or 2 tens. In the case of backward children, they may be allowed to put in the stop-gap 0, and then to multiply by 2, each figure falling into its promoted place.

The 0 in the units place should present no difficulty in the exercise given below. It must be moved one place to the left with the other digits, as in the example below.

| | Th | H | T | U | | | Th | H | T | U |
|---|---|---|---|---|---|---|---|---|---|---|
| | | 7 | 5 | 0 | | | | 7 | 5 | 0 |
| | | | 3 | 0 | | | | | 3 | 0 |
| 2 | 2 | 5 | 0 | | | 2 | 2 | 5 | 0 | 0 |

The gap left in the units place is filled by another 0.

So we get this rule: "To multiply by 20 (30, 40, etc.) multiply by 2 (or 3, or 4, etc.), at the same time moving the digits of the result one place to the left, put a 0 in the vacant units place." The same rule applies when multiplying by 200, 300, etc., except, of course, that the figures are moved two places to the left.

## Focusing Attention on the Carrying Figures

Before going on to Long Multiplication, it is necessary to point out the slight difference of method between addition and multiplication. In addition the carrying figure is added at the beginning of the operation, and the child has formed the habit of adding the number carried to the first figure of the column to be added. In multiplication the carrying figure is added *at the end* of the multiplication operation. This presents two difficulties: (1) the child has to form a new habit; (2) he has to hold in mind the number to be carried while finding the product. This requires more concentration than is needed in addition. If possible the child should be shown *why* he must add after multiplying, and then *form* the *habit* of doing it.

Perhaps the best way to show *why* is to find an answer by both addition and multiplication. Thus find $364 \times 3$ in two ways:

$$\begin{array}{r} 364 \\ 364 \\ 364 \\ \hline 1092 \\ \hline \end{array} \qquad \begin{array}{r} 364 \\ \times 3 \\ \hline 1092 \\ \hline \end{array}$$

Although in adding the child carries 1 to 6 and adds it before adding the other two 6's, he sees that it is really added to the **three sixes** (3 sixes are 18, and 1 are 19; $7+6+6$ is 19); but in multiplying he must first *find* the **three sixes** before adding. If he adds first he gets 7, and three sevens are 21! In the **addition sum** the three sixes are given; in the **multiplication sum** they have to be found. There are two types of oral drill to help children to form the habit of multiplying first. Here are the two types: (*a*) $3 \times 2 + 1 = 7$. (*b*) Add 2 to $3 \times 4$

=14. Let the children say their tables adding a certain number to the answer before they give it, thus:

Three 1's are 3 + 2 = 5
Three 2's are 6 + 3 = 9

## Ratio Idea

Give the children problems as in Chapter VII to show the meaning and use of multiplication. The ratio or proportion idea can be introduced: 2 apples cost 4*d*., 4 apples cost . . . ? 2 loaves cost 8*d*., 6 loaves cost . . . ? Six loaves are three times as many as two loaves, therefore they will cost three times as much. The problem of ratio is thus easily introduced. "How many times as large (long, heavy, expensive, etc.) is one thing as another." Tom is 5 years old, Bill is 10 years old. Bill is twice as old as Tom or Tom is half as old as Bill. Nan is 4 years old, Jill is 12 years old. Jill is—times as old as Nan.

## Long Multiplication: Graded Steps

(1) *Multiplying by Numbers of Two Digits.* So far the children have had no example of such multiplication as 45 × 24. First take a practical example—a greengrocer has 24 boxes each containing 45 pears. How many pears has he altogether? Through their problem exercises in connexion with short multiplication, the children know that the answer is found by multiplying 45 by 24. Let them arrange the boxes in two groups of 20 and 4.

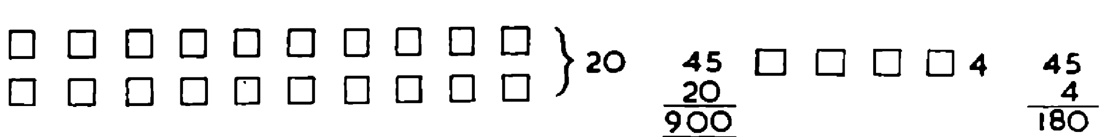



Fig. 42

They know how to multiply by 20 and by 4, so they quickly find the number of pears in each group. They see these must be added together.

$$\begin{array}{r} 900 \\ 180 \\ \hline 1080 \text{ pears} \\ \hline \end{array}$$

Make sure that the children understand how the multiplier 24 has been analysed into partial multipliers that they know how to use—24 has become $20+4$. The next step is to put the working shown above into a more compact form, thus:

$$\begin{array}{r} 45 \\ 24 \\ \hline 900 = 45 \times 20 \\ 180 = 45 \times \phantom{2}4 \\ \hline 1080 = 45 \times 24 \\ \hline \end{array}$$

For practice the pupils multiply any number by 14, 16, 17, 18, 19, and then by 34, 46, 57, 79, etc.

*Multiplying by Numbers of Three Digits.* In all the early lessons, every line of the working is argued out carefully. Thus the first line of working in (*a*) below is got by multiplying by 600, promoting the digits of the result two places, and then inserting two 0's, and so on. This is a hard example to show the method of working:

$$\begin{array}{r} 2462 \\ 673 \\ \hline 1477200 = 2462 \times 600 \\ 172340 = 2462 \times \phantom{6}70 \\ 7386 = 2462 \times \phantom{67}3 \\ \hline (a) \quad 1656926 = 2462 \times 673 \\ \hline \end{array}$$

In the example (*b*) worked below, point out to the children (1) that the 0's at the end of the different lines of working may be left out because they are not wanted in the answer; (2) the first figure of each line of working **is always put exactly under the particular multiplier in use**; this is a foolproof rule. Pupils like best to put the 0's in, because this helps them to keep the figures exactly under each other.

```
       2462
        673
    -------
    14772
     17234
       7386
    -------
(b) 1656926
    -------
```

A great many schools teach the children to begin multiplying with the unit figure. They say it is easier. This is difficult to prove, for, if the two forms are carefully analysed, it is clear that each entails exactly the same steps. Moreover, beginning with the highest figure is more 'sensible,' for one at once gets some idea of the total. Later, if methods of approximation are used such as contracted multiplication of decimals, it becomes essential to deal with successive partial products in **descending** order of magnitude.

A great deal of trouble is often made over exercises like (*c*) because of the 0's in the multiplier. There is, of course, no

```
          12415
          40206
    ---------
    49660
      24830
        74490
    ---------
(c) 499157490
    ---------
```

question of multiplying by 0. The multiplicand has to be multiplied by **forty thousand**, then by **two hundred,** and lastly by **six.** The method is practically foolproof if the children remember that the first product of every line of working comes in under the particular multiplier in use as in (*c*). This example is, of course, a difficult one, but the procedure is the same in easier graduated examples such as the following, in each of which a 0 difficulty occurs:

| | | | | | | |
|---|---|---|---|---|---|---|
| 334 | 204 | 386 | 3800 | 2004 | 2356 | 207 |
| 206 | 236 | 500 | 540 | 1236 | 2004 | 604 |

Make it clear to the children which are the multipliers. In the first example they are multiplying by 200 and then by six, and so on.

Continue to give them problems to bring out the meaning and use of multiplication.

(1) There are 24 biscuits in a pound. A grocer buys 112 pounds. How many biscuits has he?
(2) A bus holds 32 passengers. How many passengers would 48 buses hold?
(3) There are 52 weeks in a year. How many weeks has Dick lived if he is seven? How many weeks have you lived? How many months?
(4) A man walks 1 mile in 15 minutes. How long will it take him to walk 43 miles?

## The Language of Multiplication

The children learn a great many new words through multiplication, and one must see they understand and make use of them. They appreciate the meaning of **to multiply**—to **produce** large numbers. The answer to a multiplication exercise is called the *product*. **Multiple** and **factor** are useful words to know. Tell the children that 8 is a multiple of 2 because 8 is made by **multiplying** 2 by 4. Let them find multiples of 4: 8, 12, 16, 20 Similarly multiples of 6 are 12, 18, 24, 30, 36, 42, 48, 54 the list is endless. Children like to see how far they can go. It is a good way to revise the tables to let them find a given number of multiples of certain numbers—*e.g.*, five multiples of 9. Quick children may find 117 (9 × 13) for a multiple. It is important they should know that the two numbers that are multiplied together to make—say, 16, 2 × 8, are called *factors* of 16, that is, *makers* of 16; 2 × 10 are the factors or makers of 20. Again, the tables can be revised in a new way by letting them find factors of 18, 36, 48, 108, etc. (see Chapter XII). The children have many words to add to their arithmetic dictionaries, **multiplier**, **multiplicand**, **product**, and so on. Let the children themselves make a list of all the ways of setting down multiplication exercises.

TABLES AGAIN

(1) 6 multiplied by 7 = ? (2) Multiply 6 by 7. (3) What is the **product** of 6 and 7? (4) The product of 6 and 7 is . . . (5) Three times 7 is . . . (6) Three sevens are . . . (7) Find the product of 7 and 9. (8) 3 × 7 = ? (9) 8 × ? = 32 (10) ? × ? = 64. (11) Find the factors of 16, 22, 48, 72, 49.

# CHAPTER IX

# DIVISION

*Preparation for Division. Two Aspects of Division. Drill Charts for Quotients and Remainders. Graded Steps. Division by* 10, 100, 1000. *Division by* 20, 30 . . 90. *Divisors of Two or more Digits. Dividing by the 'Teens,'* 13 *to* 19. *A Drill Table for dividing by the Teens,* 13 *to* 19 *Inclusive. Trial Quotients. Some more Drills and Devices. The Language of Division. Division by Factors.*

## PREPARATION FOR DIVISION

Division, although considered the most difficult of the fundamental operations, is generally liked by the children. It causes less strain to the child than do other exercises of equal length—for example, multiplication—because of the *variety* of the work involved. It combines multiplication and subtraction with division. Tests tend to show that children make more mistakes in division than in the 'other operations,' but this is because of *weakness* in these 'other operations,' weakness in the multiplication or subtraction tables. This weakness shows very readily in pupils' work in division. It is hopeless to teach division if the children do not know thoroughly their multiplication facts: **Division is the test of the mastery of fundamentals.**

From the very beginning division may be taught with multiplication. From his earlier lessons in the Infant School the child has learnt to realize that the answer to a division exercise may be found by calculating the number of smaller quantities contained in the larger. With this important and essential clue to help him, as well as a sound knowledge of multiplication facts, the child from the Infant School does not find division within the limits of 20 too difficult. An example, such as $18 \div 3$, he interprets as "How many threes in

18?" If he knows $3 \times 6 = 18$ as a multiplication fact, he should be able to give the answer $18 \div 3 = 6$. If his memory fails him, let him count by threes to 18 – 3, 6, 9, 12, 15, 18, thus proving there are six 3's in 18.

In the case of very backward children, let them find with the help of counters "How many 2's in 10?" and state the result thus: "With ten counters I can make 5 groups of 2." Do the same with a great variety of numbers. Give them questions like these: What two numbers multiplied together make 12? What two make 18? (Take all the **pairs** in each case.) How many 3's in 15? How many 5's in 25? How many 4's in 28? etc.

Careful and varied drill in the multiplication tables makes possible the ready mastery of combinations in divisions. The child who has been taught:

$5 \times 9 = ?$ $? \times 9 = 45$ $5 \times ? = 45$ $45 = 5 \times ?$ $45 = ? \times 9$

can readily make the transition to

$45 \div 9 = ?$ $45 \div 5 = ?$ $45 \div ? = 9$ $45 \div ? = 5$

The diagrams, Fig. 36, 37, 38, in Chapter VII, 'Multiplication,' will help backward children.

Both forms of the multiplication tables must be revised, not only the "3-times, 4-times," etc., but especially the "table of threes" or "table of fours," etc. This latter table is perhaps the most needed. It should be revised thus:

1 four is 4, or 4 is 1 four
2 fours are 8, or 8 is 2 fours
3 fours are 12, or 12 is 3 fours
4 fours are 16, or 16 is 4 fours

### Two Aspects of Division

Before going any further these two aspects of division must be considered:

(1) **Quotition** or **Measuring.** This interprets division not only as inverse multiplication but mainly as *repeated subtraction.*

(2) **Partition** or **Sharing** regards division only as the inverse process to multiplication. While these two forms of division must be understood by the teacher so as to enable her to give more varied examples, the children have not got to learn two different processes. **Division** is the one mental activity behind both.

(1) *Quotition or Measuring.* Here are some simple examples:
"How many times can I take 4 sticks from 16 sticks?"
"How many groups of 4 sticks must I take to have 24 sticks?"
"How many 4-pint basins can I fill with 24 pints of milk?"
"How many children will get 2*d.* each out of 16*d.*?"

Let us consider this example: If 48 apples are arranged into groups of 6 apples, how many such groups can be formed? To find the answer the child must know how many 6's are in 48. He must measure 48 by 6. This example may be worked with counters, thus: from a heap of 48 counters 6 are counted off and set aside, then another 6 from the main group are put beside it, then another, until the stock is exhausted. This is clearly repeated subtraction. The result shows that, if 48 counters are separated into equal groups of 6, the number of groups is 8; in abstract terms, the number of 6's in 48 is 8. The process of finding the number of 6's in 48 is called **division**. (It is inverse multiplication, $6 \times ? = 48$, and written $48 \div 6 = 8$.)

It is clear that when we divide one quantity into smaller quantities we are **measuring**; we are asking the question **how many times** (Latin *quot*) one quantity is contained in another. So this aspect of division is rightly called **measurement** or **quotition** or division, properly speaking.

(2) *Partition or Sharing.* Here are some simple examples. I put 20 pints of milk into 5 basins; how much do I put into each? If I have 16*d.*, how many pennies can I give to 8 children? How many pennies can I give to 8 children if I have 16*d.*? How many girls can have 6 biscuits from a bag containing 54?

In partition the exercise $48 \div 6$ is given the concrete meaning. If 48 apples are separated into 6 equal groups, how many *apples* will be in each group? The children can do the actual **sharing** with counters. This is most simply done by 'dealing' out the counters one by one into 6 equal groups as below.

SIX GROUPS

| 1st | 2nd | 3rd | 4th | 5th | 6th | |
|---|---|---|---|---|---|---|
| ● | ● | ● | ● | ● | ● | |
| ● | ● | ● | ● | ● | ● | |
| ● | ● | ● | ● | ● | ● | and so on |

The first six counters are placed separately on the table, a second row is added, then a third, and so on. When the counters give out it will be found that there are 8 counters in each of the 6 groups. Each deal requires that 6 counters be drawn from the stock; hence each group contains as many counters as there are 6's in 48. In other words, the question "How many counters are in each group?" becomes again "How many 6's in 48?" The answer is 8.

In Partition, or Sharing, when we divide a concrete quantity by an abstract number, we are sharing that quantity among a given number of people or things and finding the share of each, or *parting* it among a certain number. This is sharing, or partition. The answer is a 'concrete' quantity—so many pennies, so many apples, etc. In the case of measuring the result is an 'abstract' number of times.

Many teachers think **sharing** is the most helpful interpretation of division, and make all their first exercises sharing ones, since children are used to 'sharing' cards, counters, etc., among themselves. Others prefer to begin with the 'measuring' type, and give as their reasons that, when the division tables are introduced with those of multiplication, 'measuring' is more evident than 'sharing'; also sharing, carrying with it the notion of fractions, is more advanced than measuring. Most teachers make no distinction but let the children work both types of problems. However, children perform the mental operation of division at the call of either type of question. Insistence on their result being an answer to *the given question* prevents blunder. In the Infant School division is learnt in a simple way as sharing and grouping.

### Graded Steps

Give the children plenty of oral work, such as: How many 6's in 12, 18, 24, 30, 36? Soon such questions present little difficulty. Questions like the following require special attention: How many 4's in 17, 19, 22, 25, 39? In the case of "How many fours in seventeen?" the child must think of the station in the 4-times table that comes nearest to 17. He sets out the stations, 4, 8, 12, 16. Clearly 16 is nearest to 17; 17 is 4 fours and 1 over.

A certain amount of written work must be done as soon as possible, both for the purpose of giving tests and to make the pupils familiar with the ways of setting down division exercises, but oral work at first is of great importance.

It is a mistake to regard long division and short division as two distinct rules. They are not. There is, of course, only *one* rule, and that is long division. Short division is long division with more of the work done in the head and less on paper; logically long division comes first.

The main teaching-difficulty in division is the question of gradation. Here are some important steps to be observed—some are very easy, and can be fairly quickly passed over—but they are definite steps.

(1) *Divisors of One Digit.* The first step involves division of a number by another number consisting of *one digit* only. In all the first examples the number of tens and units involved should be such as to admit of exact sharing with no remainder.

Units only in the dividend:

$$(a)\quad \begin{array}{r} 2\\ 4\overline{)\,8} \\ 8 \\ - \end{array}\qquad \begin{array}{r} 2\\ 3\overline{)\,9} \\ 9 \end{array}\quad \text{or}\quad (b)\quad \begin{array}{r} 4\underline{)\,8} \\ 2 \end{array}\qquad \begin{array}{r} 3\underline{)\,9} \\ 3\end{array}$$

Examples (*a*) show clearly the relationship between multiplication and division. Multiplying the divisor by the quotient proves the answer is correct. What the children say or think as they work their division exercises is very important. They should say, "How many fours in eight?" or, very briefly, "Fours in eight?" They should not say "Fours *into* eight?" or "Four divided into eight?" which might mean 4 divided into 8 parts!

(2) *Two-digit Dividends; No Remainders.* The number of possible examples is limited, but enough should be given to fix in the children's minds the procedure: divide the tens first and then the units, thus—How many 3's in 6 tens? 20, because 20 threes are 60. There are 3 units over. 3's in 3, 1. At an early stage similar *exact* division examples involving hundreds may be introduced.

```
    21
 3) TU
    63
    60
    —
     3
     3
     -
```

(3) *Introducing Remainders.* Steps (*a*), (*b*), (*c*), (*d*), (*e*).

(*a*) In this step (*a*) there is an exact division of the tens, but the division of the units results in a remainder. This step causes little difficulty. In every step let the children prove their answers. Multiply the divisor by the quotient and add the remainder, if any. This gives the dividend: $21 \times 3 + 2 = 65$.

```
(a) 21
 3) TU
    65
    60
    —
     5
     3
    —
     2 R
```

(*b*) In step (*b*) the division of the tens results in the remainder of 1 ten, which must be decomposed into 10 units and divided along with the existing units, thus making 14 units, see examples (*b*). Children who have learnt the decomposition method of subtraction find this no difficulty. The decomposition method is really only needed in the case of short division, see example (*b*) below. In long division, the 1 ten over is considered as 10 units, the 4 units are brought down beside it, thus making 14 units.

(*b*)

```
   24
3) 74
   6
   —
   14
   12
   —
    2 R
```

(*b*)

$$3)\ \underline{7\,{}^{1}4}$$
$$\;\;2\ 4\quad 2\ \text{R}$$

(*c*) In step (*c*) the dividends are increased to include three-digit and larger numbers. There may be remainders from the hundreds, or tens, or units. Examples are 742 ÷ 6, 736 ÷ 5, 735 ÷ 4, 854 ÷ 7.

(*d*) In step (*d*) the hundreds digit in the dividend is too small to contain the divisor. So the one hundred must be decomposed and be called ten 10's. There are no twos in 1; a 0 goes in the hundreds place in the quotient. How many twos in 12 tens? 6. 6 goes in the tens place in the quotient. Only 6 units remain to be divided. Twos in 6? Three. Although there is no need to put a 0 in the hundreds place, it helps the children at first to put the 6 exactly in the tens place.

```
   0 6 3
2) H T U
   1 2 6
   1 2
   ———
       6
       6
```

(*e*) In this step a 0 must be put in the quotient. The working goes like this:

| | | |
|---|---|---|
| 2 0 6<br>4) HTU<br>8 2 7<br>8<br>——<br>2 7<br>2 4<br>——<br>3 R | or 2 0 6 3R<br>4) HTU<br>8 2²7 | 4's in 8? 2<br>4's in 2? 0<br>4's in 27? 6 and 3 over |

The work soon becomes automatic, and children begin to use the 4-times table instead of the **table of fours**, thus:

| | |
|---|---|
| 2 4 3 1R<br>4) 9¹7¹3 | Four **2's**, 8 and 1 over<br>Four **4's**, 16 and 1 over<br>Four **3's**, 12 and 1 over |

The figures in black type are put down **in the quotient** as they are said.

## Drill Charts for Quotients and Remainder

These charts help to make the work automatic. As seen in the simple exercises above, the pupils must not only know the

A B

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 5 | 39 | 27 | 43 | 19 | 16 | 31 | 23 | 47 |
| 10 | 6 | 37 | 30 | 22 | 40 | 34 | 32 | 17 |
| 11 | 42 | 21 | 29 | 45 | 7 | 48 | 25 | 18 |
| 36 | 49 | 24 | 8 | 14 | 44 | 35 | 12 | 41 |
| 46 | 33 | 26 | 13 | 38 | 20 | 28 | 15 | 9 |

C D

Fig. 43. Drill Chart for multiplying by Five

tables but they must know "how many and how many remaining" for each divisor. Thus in dividing by 5 he will meet 'How many 5's?' in any number from 5 to 49 inclusive. So drill in the mere division (or multiplication tables) is not enough for efficient work in written division. A chart may be made for each divisor from 2 to 9. Fig. 43 shows a chart for 5. In making this chart 45 numbers are needed. Since the factors of 45 are 5 and 9, draw a rectangle divided into squares, 9 squares by 5 squares to make 45. Then, beginning with the divisor 5 in the upper left-hand corner, scatter the rest of the numbers in any of the squares, as in Fig. 43. Ring round 5, or colour or shade the square to show 5 is the divisor. In using the chart the pupil gives the **quotient** and the **remainder**. Thus when the teacher points to 18 the child says, "3 and 3 remaining."

This chart gives more thorough drill on *all* the needed facts than is possible if written exercises are made up at random. To make a chart for division by 7 a rectangle 9 squares by 7 is needed. These 63 squares are filled by the numbers from 7 to 69 put in in any order. A chart for 9 will have a rectangle 9 squares by 9, and the 81 squares will be filled up by the numbers 9 to 89, and so on for any number. These charts are for short division. Children like to test themselves with them, and to test each other. Sometimes they like to cross out all the numbers that come in the table of fives and see how many are left.

With the help of the charts children should know well how to divide by the table numbers from 1–9 inclusive, using the long division or short division form. In the case of short division it is best to place the quotient *above* the dividend. This makes the transition to long division easier. Many teachers think that the short form is better when beginning with young children, because there are fewer figures to set down and the working is simpler if they understand decomposition. This is, of course, true. The children must work a great many exercises in the long division form before division of two or more numbers begins. Both forms, however, must be known by the children.

DIVISION BY 10, 100, 1,000

Remind the children of their exercises in place-value (see Chapter III). 326 is read as 3 hundred 2 tens 6 units or, if one likes, 32 tens 6 units. So 326 ÷ 10 (how many 10's in 326?) gives the answer 32 and 6 over.

The short or long form of division may also be used thus:

```
     032 and 6 over          10's in 32?  3 and 2 over
10) 326
     30                      10's in 26?  2 and 6 over
     —
      26
      20
      —
       6
```

Whichever method is used, on inspecting the answers a quick way of dividing by 10 can be seen—separate off the unit digit as remainder, and the other digits in the right order give the quotient:

762 ÷ 10 = 76 and 2 over
1351 ÷ 10 = 135 and 1 over
79 ÷ 10 = 7 and 9 over

Intelligent children may see that each figure is moved one place to the *right*, and is ten times *smaller*. The rule for multiplying by 10 is to move each figure one place to the *left*. Why not apply the inverse rule to division and move each figure of the dividend one place to the *right*? But there is a snag about this. When the figures are moved to the left there is a definite place with a name to which each figure can go, but when they are moved to the right to what place is the units figure to go? The children have no description of the place, and they will not learn about it until the decimal notation for fractions is taught them. Then they will know the figure in the units place moves to the tenths place:

```
  U       U
762 ÷ 10 = 76·2
```

So for the time being an adequate rule is to separate off the units digit as remainder; then the rest of the digits give the quotient. Division by a hundred can be dealt with in the same way; separate off the last *two* digits of the dividend as the remainder, and the remaining digits in order are the quotients.

```
 ThH T U
  1 7 6 2 ÷ 100
=     1 7  and 62 over
```

The children see the division has not really been completed, that the 62 over has not been divided by 100. They learn how to complete their division exercises when they learn fractions and decimal fractions (see Chapter XIV).

### Division by 20, 30 . . 90

Set aside the unit digit as before, and then divide by 2, 3, 4, etc., thus:

```
2̸0) 153̸
    ____
     7      13
```

Many teachers let the children cross off the 0 and last figure to show division by ten has taken place. The 1 ten over goes in front of the 3 units to make 13 over. Give the children special practice with the divisor 20, as it is required for reducing shillings to pounds.

Encourage them to divide right away by 20, 30, 90, etc., using their multiplication facts:

```
    007  R12        005  R8         002  R6
20) 152         30) 158         90) 186
```

### Divisors of Two or more Digits

When the children can divide by the digits 1–9, by 10 and the multiples of 10, they are ready for 2-figure divisors. The work must be carefully graduated. Instead of going on from 10 and using the sequence 11, 12, 13, 14, 15, 16, 17, 18, 19, 21, 22, 23 . . . 28, etc., as the order for divisor, miss out some numbers and use the following numbers first: **11, 12,**

**21, 22, 31, 32, 41, 42, 19, 29, 39,** then go on to 24, 36, 47, etc., later. This rearrangement is because the divisors in heavy type present less difficulty to the children when they are finding an approximate quotient.

The dividend at first should not be more than **three** digits, and many **one-step** examples of the following type should be worked by the children to make them gradually familiar with the appearance of the long-division setting. It helps backward children if the same divisor is kept to throughout in each of the early lessons.

| (*a*) | (*b*) | (*c*) |
|---|---|---|
| T U | H T U | H T U |
| 0 4 | 0 0 8 | 0 0 6 |
| 21) 8 7 | 21) 1 7 9 | 31) 2 0 5 |
| 8 4 | 1 6 8 | 1 8 6 |
| 3 | 1 1 | 1 9 |

It is essential in long division to put the figures in the quotient in their right column. Therefore at first 0's may be used. The child says, "21's in 8? None. Put 0 in the tens column. 21's in 87? 4 and 3 over."

In (*b*) he says, "21's in 1? None. Put 0 in the hundreds column. 21's in 17? None. Put 0 in the tens column. 21's in 179? 8 and 11 over."

The above exercises are fairly easy because the children know how to divide by 20 and 30. The number of twenties in 87 is 4, and 7 over. Next two-step examples are introduced:

| | |
|---|---|
| 026 | 0065 |
| 31) 823 | 41) 2699 |
| 62 | 246 |
| 203 | 239 |
| 186 | 205 |
| 17 | 34 |

The practice of crossing out the digits as they are used is necessary. Its use is not so obvious in the examples given, but in the case of larger dividends the crossing out is important, especially for pupils who cannot keep their working in rigid columns.

It is essential that children should achieve many successes with the rule in its simplest form to give them confidence. The children often feel helpless at finding a trial quotient.

### Dividing by the 'Teens,' 13 to 19

These exercises children often find difficult. It is harder to choose the correct quotient figure when the left-hand figure is one and the right-hand figure 2 or more. The children have to try out several possible quotients. 18 is nearly 20. There are 3 twenties in 60; perhaps there are 3 eighteens in 63. There are, and 9 over. How many twenties in 90? 4. Perhaps there are 4 eighteens in 90? There

```
     35
18) 630
    54
    --
     90
    --
```

are more than 4. Try 5. Another difficult example is 736 ÷ 16. 16 is near to 20. How many twenties are there in 73?

Teach children not to use the **first figure** of the divisor to obtain a quotient; thus, in 29)4156, 4 divided by 2 gives 2, but this is sure to be wrong. Much better would be 30's in 41. The greatest difficulty for children in long division is to approximate the correct quotient digit *in a reasonable time.*

### A Drill Table for dividing by the Teens, 13 to 19 Inclusive

To divide by the teens easily a child must have practice in multiplying at sight all numbers from 13 to 19 by all one-figured numbers. The drill tables may be used for a few minutes daily drill, and pinned up in the classroom for children to consult. Let them also make their own tables:

DRILL TABLES

Give product at sight

| | | | | | | |
|---|---|---|---|---|---|---|
| 2 × 13 | 2 × 14 | 2 × 15 | 2 × 16 | 2 × 17 | 2 × 18 | 2 × 19 |
| 3 × 13 | 3 × 14 | 3 × 15 | 3 × 16 | 3 × 17 | 3 × 18 | 3 × 19 |
| 4 × 13 | 4 × 14 | | | | | |
| to | to | to | to | to | to | to |
| 9 × 13 | 9 × 14 | 9 × 15 | 9 × 16 | 9 × 17 | 9 × 18 | 9 × 19 |

Let the children also write these tables in the form below and see how far they can go:

26 is 2 thirteens or 2 thirteens is 26
39 is 3 thirteens or 3 thirteens is 39
52 is 4 thirteens or 4 thirteens is 52, etc.

TRIAL QUOTIENTS. SOME MORE DRILLS AND DEVICES

(1) Put certain numbers on the board, such as 19, 59, 50, 8, 29, 40, 45, 75, 39, 37. The children see how near they can get to these numbers without passing them in the 6-times table. They write down how many are left over in the following manner. **19**: 6 threes are 18 and 1 over. **59**: 6 nines are 54 and 5 over. Let the pupils go through the tables from 6 to 12 in this way.

(2) *Chart for Trial Quotients* (Divisor 7; Fig. 44). A chart is drawn on the board or on a piece of stiff paper, as in Fig. 44; 7 is put at the top of the first column on the left, and underneath are the 'stations,' or endings, of the 7-times table—14, 21, 28, 35, 42, 49, 56, 63, etc. Fill up the spaces to the right as shown. The chart is used as follows. The teacher points to 34. The child looks at the 'station' 28 and says, "Four and six over,"—that is, in 34 there are 4 sevens and 6 over. The teacher points to 52, the child looks at station 49 and says, "Seven and three over."

| | DIVISOR SEVEN | | | | | |
|---|---|---|---|---|---|---|
| 7 | 8 | 9 | 10 | 11 | 12 | 13 |
| 14 | 15 | 16 | 17 | 18 | 19 | 20 |
| 21 | 22 | 23 | 24 | 25 | 26 | 27 |
| 28 | 29 | 30 | 31 | 32 | 33 | 34 |
| 35 | 36 | 37 | 38 | 39 | 40 | 41 |
| 42 | 43 | 44 | 45 | 46 | 47 | 48 |
| 49 | 50 | 51 | 52 | 53 | 54 | 55 |
| 56 | 57 | 58 | 59 | 60 | 61 | 62 |
| 63 | 64 | 65 | 66 | 67 | 68 | 69 |
| 70 | 71 | 72 | 73 | 74 | 75 | 76 |
| 77 | 78 | 79 | 80 | 81 | 82 | 83 |
| 84 | 85 | 86 | 87 | 88 | 89 | 90 |

FIG. 44. DRILL CHART FOR DIVIDING BY SEVEN

(3) *Another Device for finding the Digits of the Quotient.* If a child has to work exercises with the divisor 34, it helps him if he makes a table for this particular divisor, as below.

| | 34's |
|---|---|
| 1 | 34 |
| 2 | 68 |
| 3 | 102 |
| 4 | 136 |
| 5 | 170 |
| 6 | 204 |
| etc. | |

If an exercise such as 73 ÷ 34 has to be worked, a glance at the table shows that the number of 34's lies between 2 and 3. In fact, 73 is twice 34 and 5 over. If the exercise is 189 ÷ 34, a glance at the chart shows that the number of 34's lies between 5 and 6. It is 5. There are five 34's in 189 and 19 over. Let the children work many examples with the divisor 34; first of all one-step examples, and later two-step. Impress upon the children that the quotient figures, 1, 2, 3, 4, etc., and the multiples of 34—namely, 34, 68, 102, etc.—are all found in the table of 34's, and that all they have to do is to subtract.

It is wise to keep to the same divisor for the whole lesson. The next day they compile a table for a new divisor. They need this help while they are getting used to the *form* of long division.

```
      200
34) 6822
     68
     ——
       22
```

*Noughts* should not cause much trouble in division. In the example given, the child says, "34's in 6? None." (They may put a 0 in the quotient if it helps them to keep the figures in rigid columns, but the 0 is of no use.)

Impress upon them that they are finding how many 34's there are in 68 **hundreds**, and the 2 must go above the 8. They go on, "34's in 2? None. 34's in 22? None." These noughts must appear as they are place-holders; without them the answer would be 2 units, not hundreds. Impress upon them that the **last figure or nought of the quotient must always appear above the last digit or nought of the dividend.** There must be *no* empty spaces above the **last figures** of the dividend—that is, to the right.

All division exercises should be checked by the children. There are two checks: (1) Multiply divisor and quotient to obtain dividend. (2) Divide dividend by quotient to obtain divisor.

### The Language of Division

```
          Quotient
Divisor ) Dividend
```

As in the case of all rules, attention must be paid to the 'language of division.' The more a word means to a child, the more confidence he feels in his work. (1) The number to be measured or divided is the most important number and is called the **dividend.** (2) The number that does the *work* of division or dividing is called the **divisor.** (3) The share, portion, ration, measure, or quota that results from division is called the **quotient.** (4) The part that cannot be divided or shared equally is called the **remainder.**

The children will find a great deal to put in their 'Division Notebooks!'

### Division by Factors

Some teachers favour division by factors because divisors of two figures may be needed before the child has learnt long division, such divisors as 14, 16, 28, are necessary for the Table of Weights (see Chap. XI). But many teachers think it is a non-essential stage in division. However, in certain circumstances it is simpler and neater than long division and not difficult to understand if children really understand what division means. It appeals to the thoughtful child. In any case, we introduce children to division by factors when we teach them the short method of dividing by 20, 30, etc. To divide by 20 is to divide by 10 and 2; $10 \times 2 = 20$.

Let the children prove by using matchsticks or counters that $39 \div 28$ is the same as $39 \div 7$ and then by 4. First they find with their matchsticks how many 28's are in 39 and how many over. They find one 28 and 11 over. Next they divide by the factors 7 and 4. First they divide the 39 sticks into groups of 7, and find 5 groups of 7 with **4 units over.** Next they divide the 5 groups of 7 into groups of 4; they find 1 group of 4 and 1 group of **7 over.** 7 over and 4 over make 11 over. Then they work the exercise as below, looking at the layout of their counters as they do so:

$$28\left\{\begin{array}{l|l} 7 & 39 \\ \hline 4 & 5\ (7\text{'s}),\quad 4\text{ over} \\ \hline & 1\ (28\text{'s})\quad 1\ (7\text{'s})\text{ over} \end{array}\right.\left.\begin{array}{l} \\ \\ \end{array}\right\} 11\text{ over}$$

This was once taught as a mechanical rule: multiply the last partial remainder by the first divisor and add the first partial remainder.

CHAPTER X

# UNITS OF MEASUREMENT. COMPOUND QUANTITIES

*Measuring. Tables of Weights and Measures. Reduction (or Decomposition or Conversion). Addition, Subtraction, Multiplication, and Division: Compound Quantities, beginning with Money.*

## MEASURING

Children naturally know a good deal more about counting than about measurement. In counting we are finding the number of separate things in a group. How many beads in a box? The results of counting can be expressed exactly. It is a simple operation.

In the physical world there are things that have to be **measured**: we need to know **how long** some things are, such as a piece of tape, a room, or a road, or the exact weight of something, such as sugar or coal. To measure we must have something with which to measure, a stick of a certain length to measure length, a weight to measure weight, but the length or weight we measure with must be a **fixed** or **standard** quantity. Finding by experiment or calculation how often the standard quantity is contained in the given quantity is the process known as measuring. The standard quantity is called a **unit.**

There are three very different kinds of quantity to be measured, **length**, **weight** (or mass), and **time**. These three quantities have no connexion with each other. The unit of length is the **foot**; the unit of weight, the **pound** (lb.); the unit of time is the **second**. These three units are fundamental units. There are other units derived from these —the **unit of area**, which is the **square** whose side is the unit of length, the **unit of money**, or **unit of value**; the **pound sterling** is defined in terms of a weight of gold of a certain fineness, and so on.

There are at least three important steps in the teaching of measuring that help children to get a clear idea of the units of measure. These are generally covered in the Infant School.

(1) *General Comparison.* The lengths and weights of objects should be compared—"This book is heavy." "My ball is lighter than yours." Lines of varying length are drawn on the board, and their relative lengths are judged. The distances between objects in the room are compared. One distance may be twice as long as another. Cups and boxes may be used in the judging exercises. "My box holds more than yours." "This jug holds **two** cupfuls of water." These judging exercises train the senses, and give a feeling for **more**, **less**, **larger**, **smaller**, **heavier**, **lighter**, **shorter**, **taller**. This work is often done as language work in the Infant School.

(2) In the second step defined but varying units are used. A *step* to measure the length of the room, a *stick* or a *piece of string*, a *cup*, *pebbles*, or *lumps of clay* for weights may be the measuring agents. Unconventional measures used by younger children are described in *Arithmetic in Action*, by E. Brideoake and I. D. Groves.[1] This is a useful book that covers the practical work done in the Infant School. It is well for the teacher in the Junior School to know what goes on in the Infant School.

(3) The use of **fixed** and **legally defined units**—that is, standard measurements. This brings us to the work of the Junior School and the Tables of Weights and Measures.

## Tables of Weights and Measures

Only tables of "weights and measures" should be taught that are used by people in the course of their daily work, and all units must be taught *through use*. It is true psychologically

[1] University of London Press, 1948.

that number is not a sense-fact, but the number idea had its origin in *sense-experiences*. Children should learn **inch**, **foot**, **yard**, **pound**, **pint**, etc., by using these units and measuring with them in a variety of situations. Most children think of a foot as twelve inches, but the association is often purely *verbal*. Few of these units call up to children the actual measures that they represent. It sometimes happens that children who have used them in the arithmetic lessons for perhaps three years cannot give approximately the height of a door, the width of a window, etc. The children in a top class in a Grammar School when asked on one occasion the height of an ordinary room gave answers varying from 25 ft to 60 ft! Unless units are learned through use, the tables based on them are associations that never function intelligently.

A knowledge of a variety of important units should always precede the formation of a table. The complete tables should come later, and they should be formulated by the children. "Weights and Measures" cannot be taught as independent branches of arithmetic, they must be incorporated in work in fractions, in decimals, and in fundamental operations of whole numbers.

Below are the tables needed in the Primary School.

(1) *Table of Money or Table of Value.*

| | | | |
|---|---|---|---|
| 2 | halfpennies ($\frac{1}{2}d.$) | = | 1 penny (*d.*) |
| 12 | pence | = | 1 shilling (*s.*) |
| 20 | shillings | = | 1 pound (£) |

The symbols £ *s. d.* are contractions for the Latin equivalent, *librae*, *solidi*, *denarii*.

(2) *Table of Length.*

| | | | |
|---|---|---|---|
| 12 | inches (in.) | = | 1 foot (ft) |
| 3 | feet | = | 1 yard (yd) |
| 22 | yards | = | 1 chain (ch.) |
| 10 | chains | = | 1 furlong (fur.) |
| 8 | furlongs | = | 1 mile (ml.) |
| 1,760 | yards | = | 1 mile |

(The plurals of these contractions have an 's' only if the singular contraction ends with the last letter of the uncontracted word and if the latter has an 's' in the plural, thus yd, yds; qr, qrs; but the plural of in. is in. and of lb. is lb., etc. This applies to all the tables.

(3) *Table of Weight.*

| | | |
|---|---|---|
| 16 ounces (oz.) | = | 1 pound (lb.) |
| 14 pounds | = | 1 stone (st.) |
| 28 lb. or 2 stones | = | 1 quarter (qr) |
| 112 lb. or 4 quarters | = | 1 hundredweight (cwt) |
| 20 hundredweights | = | 1 ton |

(4) *Table of Liquid Capacity.*

| | | |
|---|---|---|
| 4 gills | = | 1 pint (pt) |
| 2 pints | = | 1 quart (qt) |
| 4 quarts | = | 1 gallon (gal.) |

Local use should determine whether children are taught about gills.

4 gills make 1 pint. (The cookery-book measure.)
2 gills make 1 pint. (The milk measure in some localities.)

(5) *Table of Time.*

| | | |
|---|---|---|
| 60 seconds (sec.) | = | 1 minute (min.) |
| 60 minutes | = | 1 hour (hr) |
| 24 hours | = | 1 day (hours of daylight and darkness) |
| 365 days | = | 1 year (yr) |
| 366 days | = | 1 leap year |
| 7 days | = | 1 week |

All these tables should be built up experimentally, as far as possible. In tables (1) and (5), Value and Time, the thing to be measured is *abstract*, but children are so familiar with some of the units used that, although 'money' is a measure of *value*, they accept it as part of everyday life, and as the means of buying and selling. Time, again, is intangible, and yet children easily take for granted the time labels they hear used.

In tables (2), (3), and (4) the thing to be measured is *visible*. There are two essential ideas. (1) The physical property to be measured, the **length of a table**. (2) The measurer or unit that is used, a **foot.**

A thoughtful consideration of the first should precede the teaching of the second.

## Reduction (or Decomposition or Conversion)

Formal written work involving compound quantities begins generally with exercises in reduction. A child will not make much progress in the manipulation of compound quantities until he is thoroughly familiar with the **process of reduction**. Reduction exercises are not an *end in themselves*, and must not, therefore, be over-stressed. Their practical use rests on the fact that they simplify calculations. To look far ahead, $5\frac{1}{4}$ ft is a much simpler quantity to deal with than 1 yd, 2ft, 3 in.

The term **reduction** covers two different types of exercise:

(*a*) Reduce £3 10*s*. 6*d*. to pence.
(*b*) Reduce 1,692 halfpence to £ *s*. *d*.

The term **reduction** often bothers children, especially intelligent children. In (*a*) the **denomination**, or **name**, of the coins is '**reduced**' from pounds to pence! In (*b*) the **number** of coins is reduced from 1,692 coins to 19 'coins.' Many teachers now use the term **conversion**. In setting exercises the confusing word 'reduce' is generally changed to 'bring,' 'change,' or 'express.' If reduction is taken in detail with money, there should be no need to spend so long over the other compound quantities or measures.

**Introduce reduction by simple oral exercises**—thus, "Change 2*s*. 5*d*. to pence." "Express 35*d*. in shillings." "If you have 35*s*., how many pounds have you?" and so on. One must constantly bear in mind that the change-over from natural numbers—units, tens, hundreds, thousands—to new units—pence, shillings, pounds; yards, feet, inches, etc.—is sure to cause some confusion at first, especially in adding and subtraction. Whenever possible impress upon children that 'a unit is anything we use to measure with.' It may be a shilling; when a boy says, "I have saved four shillings," he is measuring his money by the unit **one shilling**. Very often we want to change the unit of measurement. The boy who saves 6*d.* a week may want to reckon or measure his money in sixpences and say, "I have saved eight sixpences. Eight sixpences are four shillings." Here sixpence is his unit of measurement. The greengrocer says, "Potatoes are twopence a lb." Tom thinks, How many pounds can I buy for my shilling? He measures his shilling by 2*d.* A year may be a unit of measurement as, "I am nine years old to-day," and so on. It makes no difference whether we call a shilling a shilling, 2 sixpences, 4 threepences, 12 pence, 24 halfpence. The shilling is exactly the same, but measured in different units. In the same way the pound can be measured in different units, in sixpences, shillings, half-crowns, crowns, ten shillings, etc.

The children need much practice in using the 12-times table to change pence into shillings and shillings into pence. It is worth while letting them revise the analyse of 12—*i.e.*, $1+11=12$, $2+10=12$, $3+9=12$, etc.—so that the conversion of pence to shillings can be made by building up to 12. The children have to realize, as soon as possible, that the addition and subtraction of pence is a 'twelve system' and must not be confused with the tens system of numbers.

It is also helpful if they make an Addition Table for 12 and learn it:

| | | | |
|---|---|---|---|
| $13=12+1$ | $16=12+4$ | $19=12+7$ | $22=12+10$ |
| $14=12+2$ | $17=12+5$ | $20=12+8$ | $23=12+11$ |
| $15=12+3$ | $18=12+6$ | $21=12+9$ | $24=12+12$ |

From the above table they make a pence table to 24, thus 13*d.* = 1*s.* 1*d.*, 14*d.* = 1*s.* 2*d.*, and so on.

Changing shillings to pounds or pounds to shillings is comparatively easy. The children work orally such examples as 80*s.* = ?, 24*s.* = ?, 36*s.* = ? How many shillings in £180?

The 12-times table needs the most revision. It must be learnt both ways; 7 twelves being 84 means that 7 shillings contains 84 pence and that 84 pence makes 7 shillings.

Begin with easy written examples changing pounds to shillings, shillings to pence, etc. Then give exercises including three successive units. This is how the work is set down:

(*a*) How many pennies are there in £5 14*s*. 8*d*.?

| £ | | *s*. | *d*. |
|---|---|---|---|
| 5 | –20 → | 14 | 8 |
| 20 | | 100 | 1368 |
| 100*s*. | | 114 | 1376 |
| | | 12 | |
| | | 1368*d*. | |

The column headings, £ *s*. *d*., should be widely spaced in order to avoid the overlapping of the working in adjacent columns. The conversion factors may be inserted between the column headings. They are not really needed in money exercises, but they become necessary when the newer tables of weights and measures are being used.

(*b*) Change 625*d*. into £ *s*. *d*.

| | | | |
|---|---|---|---|
| 12 | 625*d*. | | |
| 2~~0~~ | 5[1]~~2~~*s*. | leaving | 1*d*. |
| | £2 | ,, | 12*s*. |

The 2*s*. over is written down when crossed off, and the 1 ten-shilling note after the division by 2.
For dividing by 20, see Chapter XIX.

Give the children plenty of word exercises or problems, so that they see the use of reduction. As far as possible, practical examples should be given. It is as essential that a child should know *when* to apply a particular process as *how*. Here are some examples:

(1) A baker made 200 penny buns. How much were they all worth, in shillings and pence?

(2) A firm bought 500 penny stamps. How much did they cost?

(3) A newsagent made 28*s.* 9*d.* from the sale of a three-penny newspaper. How many did he sell?

(4) A cinema has 820 seats at a shilling and 360 seats at two shillings. Supposing it was full, how much money would be taken?

Encourage the children to have "Reduction Notebooks," in which they keep charts and tables that help them with reduction or **changing the unit.** Charts, for example, to show (1) the composition of a shilling—2 sixpences, 4 three-pences, 12 pence. (2) The composition of five shillings. (3)

<table>
<tr><td colspan="8">£1</td></tr>
<tr><td colspan="4">10/-</td><td colspan="4">10/-</td></tr>
<tr><td colspan="2">5/-</td><td colspan="2">5/-</td><td colspan="2">5/-</td><td colspan="2">5/-</td></tr>
<tr><td>2/6</td><td>2/6</td><td>2/6</td><td>2/6</td><td>2/6</td><td>2/6</td><td>2/6</td><td>2/6</td></tr>
</table>

FIG. 45. THE COMPOSITION OF £1

The composition of ten shillings. (4) The composition of £1 (Fig. 45). These charts are especially useful for backward children. They may be able to think out charts for themselves. The following useful shillings and pence tables can go in their notebooks.

| | |
|---|---|
| 12*d.* = 1*s.* 0*d.* | 36*d.* = 3*s.* 0*d.* |
| 18*d.* = 1*s.* 6*d.* | 40*d.* = 3*s.* 4*d.* |
| 20*d.* = 1*s.* 8*d.* | 48*d.* = 4*s.* 0*d.* |
| 24*d.* = 2*s.* 0*d.* | 50*d.* = 4*s.* 2*d.* |
| 30*d.* = 2*s.* 6*d.* | 60*d.* = 5*s.* 0*d.* |

Then there is the half-crown table and other tables for their notebooks. The 8-times table is needed for half-crowns.

ADDITION, SUBTRACTION, MULTIPLICATION, DIVISION: COMPOUND QUANTITIES, BEGINNING WITH MONEY

(1) *Compound Addition.* Begin in easy steps with backward children. (*a*) Simple examples to show that the problem of adding 3*d.* to 6*d.* or 7*d.* to 4*d.* is the same as that of adding 3 units to 6 units, or 7 units to 4 units. (*b*) Similar exercises involving the conversion of the sum, say 15*d.*, into the more normal form, 1*s.* 3*d.*

| |
|---|
| *d.* |
| 6 |
| 9 |
| 1*s.* 3*d.* |

This can be followed by more difficult sums of the same type. (*c*) Examples involving two denominations but no conversion (that is, carrying) necessary. This is like adding natural numbers, but instead of only adding units to units, and tens to tens, pence are added to pence, and then shillings to shillings. (*d*) Similar examples involving conversion, and therefore carrying. Here there may be trouble. The pence are added first. Then the sum found is changed into shillings as far as possible. Just as in the natural numbers the sum of the units is changed into tens, so the sum of the pence must be changed into groups of 12, or shillings. (*e*) Similar examples involving halfpennies. There should be no separate column for halfpennies. They clearly belong to the pence column. (*f*) Examples involving pounds and all the cases already mentioned. This is how the sums are set down and worked:

| £ | *s.* | *d.* |
|---|---|---|
| 6 | 11 | 4 |
| 7 | 16 | 5 |
| 11 | 8 | 6 |
| | 13 | 5 |
| 6 | 18 ← | 10 |
| 33 | 8 | 6 |
| | 20 \| 68 | 12 \| 30 |
| | £3 8*s.* | 2*s.* 6*d.* |

The sum of each column is found and set down under the answer space, as above; it is then converted to the next higher unit.

*Carrying from the Shillings Column, working in Ten-shilling Notes.* This is an easy, quick way. Children add the units column as before, getting 28 shillings. They change these into ten-shilling notes, getting 2 and 8*s*. over. They put down the 8*s*. as above in the units place, and carry the 2 tens to the tens column; 2 tens and 4 tens make 6 tens, 6 ten-shilling notes are £3.

(2) *Compound Subtraction.* Children brought up on the Equal Additions Method tend to go on adding ten to the top line and ten to the bottom.

This is a difficult example:

| | | |
|---|---|---|
| | 20 | 12 |
| £ | *s*. | *d*. |
| 8 | 13 | 3 |
| ~~2~~3 | 1~~6~~7 | 8 |
| 5 | 16 | 7 |

*A child using the Equal Additions Method* says, "8 from 3, I cannot. Add 12*d*. 8 from 15, 7."

"17 from 13, I cannot. Add 20*s*. 17 from 33? This is difficult for a child to do in his head. Better let him say:

"17 from 13, I cannot. Add 20*s*. 17 from 20, 3 and 13, 16."

It may also be easier in the case of the pence to say, "8 from 12 is 4, plus 3 is 7."

Thoughtful children are often puzzled as to the source of the extra pence and shillings, so conveniently added to the top and bottom line. They accept the addition of 10 when dealing with natural numbers because it is outside their experience, but they themselves have 'much experience in money matters,' shopping for their parents, adding up money when saving to buy something, subtracting to see how much they have left, and so on. The Equal Additions Method plays no part in their daily life. They 'decompose' or 'convert' their money if they want to **subtract**—that is, to give change. A child using the Decomposition Method, as shown below, says: "8 from 3, I cannot. Take 12*d.* from 13*s.* 8 from 12 leaves 4, and 3 is 7."

"16 from 12, I cannot. Take 20*s.* from £8. 16 from 20 leaves 4, and 12 is 16."

"2 from 7, 5."

| | | |
|---|---|---|
| | 20 | 12 |
| £ | *s.* | *d.* |
| ~~8~~7 | 1~~3~~2 | 3 |
| 2 | 16 | 8 |
| 5 | 16 | 7 |

If a child is puzzled by subtracting first from 12 and then adding 3, explain that, if one has a shilling and three pennies in one purse and buys an article costing eightpence, one gives the shilling only and adds the change—namely, fivepence—to the threepence one already has. So in the above exercise it is easier to subtract the 8 from the 12 (already conveniently displayed above the pence column), and then add the 3 to the result.

If a child knows well the relations of the different units in the compound exercises, he will be able to translate the units into terms of a single unit or any unit he likes. This may often mean patient, careful teaching. It is no use the children knowing reduction or decomposition, unless one makes use of it. Encourage the children to think of £100 as £99 19*s.* 12*d.* if it makes subtraction easier. In an example such as 4*s.* 1*d.* – 3*s.* 10½*d.*, the children think of 4*s.* 1*d.* as 3*s.* 13*d.*

It is easy to reconcile and combine the complementary method of subtraction with the decomposition method. In the shopping activities in the Infant School, 'customers' and 'shopkeepers' are trained by the wise teacher to use both methods; for example, a 'customer' offers a shilling for a doll priced 5*d.* Fivepence must be taken from the shilling, and therefore the shilling changed to twelve pence, but the 'shopkeeper,' in giving change, builds up the shilling as he counts on from the 5*d.* he has deducted, 6, 7, 8, 9, 10, 11, 12, laying down 7 pennies. Five from twelve leaves seven.

All that has been said about the subtraction of money applies also to the other weights and measures, so if the children know the tables there are no new difficulties. (See coming sections.)

(3) *Compound Multiplication.* With compound multiplication there are several entirely different methods that can be used. None of them are really easy, because they need so much care and concentration. The 'Direct Method' is best for the Junior School. It is a straightforward method, and an extension of the method one would naturally employ in multiplying a simple sum such as 2*s.* 5*d.* by 6. The method can, of course, be used with all the measures—weight, length, time, etc. The next best method is 'Practice.' Practice may be used at a later stage after fractions have been taught, and when difficult cost prices are being considered. Practice, however, is probably best left out in the Junior School.

Begin with simple examples, many of them oral. Here are some written examples:

| | | | | | | |
|---|---|---|---|---|---|---|
| *d.* | *s.* | *d.* | *s.* | *d.* | *s.* | *d.* |
| 9 | 2 | 5 | 5 | $2\frac{1}{2}$ | 2 | 8 |
| ×8 | | 8 | | 9 | | 11 |

It may be necessary to emphasize that any shillings from the pence column must be kept in mind until the shillings are multiplied. Dull children may have to be reminded what is the greatest number of pence and shillings that is allowed in the answer. This is to counter the often strongly formed habit of carrying ten. Examples must now be given to include pounds with one-figured shillings only. Finally, two-figured shillings are included, but the multiplier is still only *one* figure. The exercise below shows the direct method with a one-figure multiplier:

| £ | s. | d. |
|---|---|---|
| 7 | 15 | 9 × 5 |
| 5 | 5 | 5 |
| 38 | 18 | 9 |

| | | |
|---|---|---|
| 3 | 3s. | 12 \| 45 |
| 35 | 75 | 3s. 9d. |
| 38 | 20̸ \| 71 8̸ | |
| | £3 18s. | |

It may help backward children at first if the multiplier is repeated in each column. The child multiplies the number of pence by 5 and puts the product at the foot of the pence column. This product is then changed or reduced to shillings. The 3*s.* is carried to the shillings' column, 15*s.* is multiplied by 5 and set down under it, and so on. As a good deal of detail has to appear in the working of long multiplication exercises, the exercises must be well spaced across the page to avoid confusion. This is specially the case with two-figured multipliers as given below. In the case of two-figured multipliers it is wise to repeat the multiplier.

| £ | s. | d. |
|---|---|---|
| 7 | 16 | 8 × 35 |
| 35 | 35 | 35 |
| 274 | 3 | 4 |

| | | |
|---|---|---|
| 2 9 | 2 3 | 12 \| 2 8 40 |
| 2 4 5 | 4 8 0 | 2 3s. 4d. |
| | 8 0 | |
| 2 7 4 | | |
| | 20̸ \| 5 18 3̸ | |
| | £29 3s. | |

Point out that in the pence column, it is quicker to multiply by 8 than 35, and in the pounds column by 7 than 35. Remind the children that $8 \times 35 = 35 \times 8$ and $7 \times 35 = 35 \times 7$.

Many teachers think it simpler if the children do all the individual multiplication first. There is nothing against this method, and it looks simpler:

| £ | s. | d. |
|---|---|---|
| 7 | 15 | 9 |
| | | × 5 |
| | | |
| 35 | 75 | 45 |

When the multiplication is done, all that remains is a little reduction. Change the 45*d.* to shillings, change the *total* shillings to pounds, and add them to the pounds. Below is a harder example worked in this way:

| £ | s. | d. |
|---|---|---|
| 7 | 16 | 8 × 35 |
| 35 | 35 | 35 |
| 274 | 3 | 4 |

| | | |
|---|---|---|
| 2 4 5 | 4 8 0 | 12 \| 2 8 4 0 |
| 2 9 | 8 0 | 2 3*s.* 4*d.* |
| 2 7 4 | 5 6 0 | |
| | 2 3 | |
| | 20 \| 5 18 3 | |
| | £29 3*s.* | |

Remind the children to choose **the top figure as multiplier** if it is smaller. Some teachers favour the Top-line Method for Long Multiplication. In this method the top figures are **always** used as multipliers. Give the children examples of problems or "word exercises" so that they will see the need of multiplication of money. Encourage them to choose their own multipliers in Long Multiplication top or bottom. This encourages thought. Here are two problems:

If 25 workmen each receive £5 15*s.* 9*d.* per week, find the total cost of their wages. How much will 8 articles cost at £4 10*s.* 6*d.* each?

(4) *Compound Division.* One must keep in mind the two interpretations of division, **partition** or **sharing**, **quotition** or **measuring**. When we come to compound division, these two points of view give rise to two quite different kinds of problem. The children have worked multiplication exercises like this: "If one article costs 10*s.* 7*d.*, how much will 15 cost?"

$$10s.\ 7d. \times 15 = £7\ 18s.\ 9d.$$

If we try to state the opposite to the above, or an inverse problem to the above, we get two different problems:

(*a*) If 15 articles cost £17 18*s.* 9*d.*, what does one cost?
(*b*) How many articles each costing £1 3*s.* 11*d.* can be bought with £17 18*s.* 9*d.*?

The first problem is **sharing**. £7 18*s.* 9*d.* is **shared** among 15 articles.

The second problem is **measuring**. How many times is £1 3*s.* 11*d.* contained in £17 18*s.* 9*d.*?

It need not be explained to Junior children that these problems are inverse forms of the same problems in multiplication. However, all children should be introduced to problems of both types.

In the case of problem (*a*), let the children begin with simple examples of sharing. Share 2*s.* 8*d.* among 8 boys. The only practical method of sharing this sum of money is to change the shillings to pence and divide by 8. In the case of sharing 19*s.* 7*d.* among 8 boys, they see they must first share the shillings. Divide the shillings by 8. Each boy gets 2*s.*, and there are 3*s.* over. Change the shillings to pence, add 7*d.*, divide by 8. The pupil soon sees the only practical way to share money is first to share the pounds, then the shillings, and finally the pence.

Here is Exercise (*a*) above, a sharing problem.

| | | | |
|---|---|---|---|
| | £ | *s.* | d. |
| | 1 | 3 | 11 cost of each |
| 15 | 17 | 18 | 9 |
| | 15 | 40 | 156 |
| | — | — | — |
| | 2 | 58 | 165 |
| | 20 | 45 | 15 |
| | — | — | — |
| | 40*s.* | 13 | 15 |
| | | 12 | 15 |
| | | — | — |
| | | 156*d.* | |

The children first divide the pounds. They change the £2 over into shillings, then share the total number of shillings, 58, and so on. They see the method is an extension of ordinary long division as applied to natural numbers (see Chapter IX). There are really **three** separate divisions, first the pounds, then the shillings, then the pence. Division corresponds to direct multiplication, where there are **three** separate multiplications. When setting examples give many that do not work out evenly, thus: "If £54 15*s.* 4*d.* is divided equally among 18 people, how much will each get, and what is left over?"

**Problem** (*b*) **is one of measuring.** How many articles each costing £1 3*s.* 11*d.* can be bought with £17 18*s.* 9*d.*?

Begin as usual with simpler problems of the same kind: How many apples each costing 2*d.* can be bought with 10*d.*? How many articles each costing 4*d.* can be bought with 8*d.*; 1*s.*; 2*s.* 8*d.*; 3*s.* 5*d.*; etc.? From these examples the children will see that the questions really are, "How many fourpences are there in 8*d.*; 1*s.*; etc.?" Then go on to harder examples: "How many articles at 2*s.* 3*d.* can be bought for 15*s.*?" Then change both quantities to pence. From talks over many examples the children begin to see how to tackle these problems. They must bring both quantities to a convenient **unit** —that is, they must bring both quantities to the **same name**, or **denomination**, and find out how many times the smaller quantity is contained in the larger.

At first the children use ordinary standard units such as pence. Later they will see how to use more uncommon units —for example, in the problem, "How many articles each worth 3*s.* 6*d.* can be bought for £1 2*s.* 6*d.*?" both sums of money can be reduced to sixpences. The lower the unit is— for example, pence—the more work there is to be done. Children who can readily see one unit in terms of another, shillings in terms of sixpences, shillings and pence in terms of fourpences, as 2*s.* 4*d.* = 7 fourpences, often find short cuts. "How many half-crown books can be bought for £1?" The quick children think at once of £1 as eight half-crowns.

Below is how the children will work problem (*b*) above.

| £ | *s.* | *d.* | | £ | *s.* | *d.* |
|---|---|---|---|---|---|---|
| 1 | 3 | 11 | into | 17 | 18 | 9 |
| 20 | 20 | 276 | | 20 | 340 | 4296 |
| | 23 | 287 | | 340 | 358 | 4305 |
| | 12 | | | | 12 | |
| | 276 | | | | 4296 | |

| *d.* | | *d.* |
|---|---|---|
| 287 | into | 4305 |

```
                                             £  s.  d.
          1 5 articles can be bought for    17  18  9
287 ) 4 3 0 5
      2 8 7
      -------
      1 4 3 5
      1 4 3 5
```

CHAPTER XI

# COMPOUND QUANTITIES (continued)

*Measuring Length or Distance. Longer Distances. Longer Measures of Length used in the Past. Measuring Area or Surface. Area without counting. Volume. Measuring Weight. Building up a Table of Pounds and Ounces. Using Heavier Weights. Capacity. Time.*

## MEASURING LENGTH OR DISTANCE (for table, see Chapter X)

Here the first units to be taught, or in some cases revised, are the **inch, foot, yard.** Draw a line an inch long in red on a piece of stiff paper or card and pin it on the notice board, so that the children get familiar with the length of an inch. In the case of backward children, let them have long strips of stiff paper half an inch broad; from these they cut off a little inch ruler, and with it cut off from the rest of the strip the following lengths— 1 in., 2 in., 3 in., 4 in., 5 in., 6 in.


FIG. 46

Inch-squared paper may also be provided for the marking off of inches. With these strips they can measure many small things in inches, and build up their tables. Have a rod one foot long in the class-room; also give each child a plain strip of cardboard one foot in length. On these strips they mark 12 inches with the help of their inch-strips. They can see 12 inches make 1 foot. They put their inch-strips together in as many ways as possible to make twelve—for example, 5 in. + 4 in. + 3 in.

As far as possible, let the children make their own rulers.

It is a valuable practical exercise for learning the size of the units, inch and foot.

The children should often estimate length by *observation.* They write their estimates or 'guesses' down and then check them carefully with their rulers. A similar oral exercise such as this should be taken regularly.

Let them measure a number of things of interest to them. They should keep a record of these in booklets or notebooks called *My Book of Measurements* or *Things I have measured,* thus:

(1) The length of a pencil. 7 in.
(2) The size of my rubber. 1 in. wide. $1\frac{1}{4}$ in. long.

Some of the measurements they make they will find useful to them. It is worth knowing that the width, or diameter, of a halfpenny is 1 in. Even if one has no ruler one can always measure an inch with a halfpenny. Children like to enter the measurements of all the coins; the penny is $1\frac{1}{4}$ in., the sixpence is $\frac{3}{4}$ in. They soon see the need of marking their rulers to show half-inches and quarter-inches, etc. Ruler work prepares the way for fractions—that is, for the *practical* use of fractions.

The children are able to do a great deal with the foot rule. They measure doors, windows, tables, blackboards, their height, and so on. The yard measure comes next, marked off first only in feet. The children, with the help of their foot rulers placed end to end, see that 3 feet make 1 yard. For measuring yards the children use yard-sticks, tape measures, a string knotted the right length, etc. Continue to let them estimate length *by sight,* and then by measurement. Interest in the yard can be maintained in connexion with their games, the rounders' base and other game-pitches.

All that the children have learnt about the addition, subtraction, multiplication, and division of money applies equally well to the Table of Length—yards, feet, and inches—and they have little difficulty in working through the same graded steps. In fact the two measurements are often taught together, 12 in. make 1 ft; 12*d.* make 1*s.*

It interests children to know the units of measurement used long ago when there were no rulers. For short measurements, the units were naturally parts of the body.

(1) The **digit** was the breadth of the forefinger (about $\frac{1}{2}$ to $\frac{3}{4}$ inch).

(2) The **palm** or **hand-breadth** was the width across the open hand at the base of the fingers, the widest part of the palm (about 4 inches). It is still used for measuring the heights of horses.

(3) The **span**—the greatest stretch of the hand from the top of the thumb to the tip of the little finger (about 9 inches).

(4) The **foot** was the length of the foot, and became the chief measure of length. It varied in length from 10 to 14 inches. The average length of a man's foot is about 10 inches. The Roman 'foot' was not quite so long as ours. It was divided into twelve parts called *unciæ* (*uncia* is the Latin word for twelfth part, hence our word **inch**). An **inch** (twelfth) on the body was the thumb's breadth, or the distance between the tip and the first joint of the thumb.

(5) The **cubit** was the length of the forearm from the point of the elbow to the end of the middle finger (about 20 inches).

(6) The **yard** was the distance from the point of the nose to the end of the outstretched arm (about 35 inches). Women to-day estimate a yard of material by 'nosing it.' Children enjoy finding all these measurements on their own bodies, and enter them in their notebooks.

The unit of length to-day is the Imperial standard yard, the distance between the centres of two gold pins in a platinum bar, measured when the bar is at 62° F. The standard yard is kept with other standards under lock and key in the Standards Department of the Board of Trade. Inspectors of weights and measures go about the country from time to time to see that every one is using the standard yard and no other.

### Longer Distances

As far as possible children should have a mental picture of each unit of length. This is comparatively easy with the shorter distances. The length of a cricket pitch, 22 yards, will give them a 'picture' of a chain. The chain measure (Gunter's chain), the surveyor's metal measuring instrument, may be used in simple outdoor work if the children can be organized in small groups.

The mile should be discussed in a practical way, attention should be drawn to public notices or signposts, milestones, speed-limit signs, etc. The distance between two well-known local landmarks one mile apart is useful to illustrate the length of a mile. They should know that a mile is about the distance that one can walk in 15 or 20 minutes. It helps children to grasp the distance of a mile or furlong if they try to measure out their distance. Even if the playground is not very large, the children can find out how many times they must walk around it to walk a chain, a furlong, or a mile. They enjoy walking a 'mile' in the playground in the dinner hour! For the older children the teacher should have a map of the roads in the neighbourhood, and with the children's help find some interesting walks of certain lengths. What interesting places are two miles, three miles, one mile, from the school? Each child finds out how long it takes him to walk a mile, and thus long distances can be expressed *in time*. A child who says, "It takes me about one hour to walk to the Park," is also saying it is about two miles to the Park.

A useful rough measurement for the children to use, when measuring long distances like the length of a road or a large playground when making maps, is the length of their *stride* or *pace* (a single step in walking or running is about 30 in.). A quick way to find the length of one's stride is to take twenty ordinary strides or paces; measure the distance walked and divide by twenty. The result gives roughly the length of one's stride. Children like counting their steps when walking, and measuring distances in this way.

Throughout their time in the Junior School the children should continue to estimate lengths and distances by sight and check by measuring. Some children keep records of their estimates and see if they improve. They use these headings:

*Thing Measured. What I thought. What My Measure said.*

Encourage the children to enter in their notebooks certain heights that are useful to know for the sake of comparison, such as the height of an average man, a tree, a room, an ordinary house, a modern skyscraper, a tall cathedral like St Paul's, and so on. The following average **heights** are useful to know: an average room, 8 ft 6 in. or 8½ ft; a two-story house, 35 ft; a three-storey house, 45 ft; an ordinary church, about 150 ft. If children know the height of an average room, then 20 ft will call up a distance over twice the height of a room. If they know the height of their church—about 150 ft—they realize better how tall St Paul's is; 365 ft is 2½ times higher than an average church. Many skyscrapers in New York are over 700 ft high—that is, twice the height of St Paul's. Salisbury Cathedral to tip of spire is 404 ft high. Children may draw the heights to scale.

## Longer Measures of Length used in the Past

(1) The **bowshot**—the distance an arrow can be shot from its bow (Old English longbow 400 yd; modern bow 424 yd).

(2) The **furlong** (furrow-long), the length of a furrow in the common field, about 220 yd. Horse or ox must have a pause for breath after the plough has been pulled a certain distance; this decided the length of the furrow.

(3) The **mile,** the Roman measure of 1,000 paces (Latin *mille*, thousand). The Roman mile was about 1,618 yd.

(4) **League,** said to be the distance a man can see on an open plain; about 3 miles. It is a varying measure of road distance, usually about three miles. But it has never been fixed by law, and to-day it is only used poetically.

(5) **Fathom**, the distance between the finger-tips of the two hands when both arms are outstretched, which in a well-proportioned man is practically the same as his height. It is now a measure of 6 ft and only used for depths, chiefly when sounding.

### Measuring Area or Surface

Every child has an idea of what is meant by **surface.** The possibility of 'covering' a surface—for example, by painting, dusting, scrubbing, ironing, etc.—distinguishes **surface** from **boundary line.** The table of area is not a new table; it is based on the table of length. **The unit of area is the square whose side is the unit of length.** This unit is less easily manipulated than the inch or foot, because it is not readily adaptable to irregular surfaces. A **square inch** is the amount of surface covered by a square of 1-inch side. Let the children make a number of inch-squares of stiff-paper or cardboard. With these let them measure the surface of a book, a piece of blotting-paper, a sheet of paper, a mat, etc. One is not concerned at first with accurate measurements, or with the rule for finding area, but rather with giving children right conceptions. Intelligent children sometimes discover for themselves the rule. They have got to get used to measuring with a **square** instead of a **line.** The unit is square in shape because that has been found to be convenient, but areas can be covered by a variety of unit-shapes, by triangles, by oblongs, by hexagons, but not by circles. As the units of measurement are squares, the areas of rectangular surfaces are the easiest to find. Give them, or let them draw, rectangles of different sizes, 4 in. by 5 in.; 4 in. by 9 in.; 5 in. by 5 in.; etc. They find how many squares 'big' each is by dividing each side into inches and drawing lines across to make inch-squares. They count the squares to find the area. In this way they get more accustomed to measuring surfaces in 'squares.' At first finding the area is counting in squares. **Paper or cardboard foot-squares** are cut in the same way. Children find the connexion between the square foot and a square inch by covering the square foot by the inch-square unit. They use their foot-square units for finding roughly the area of the blackboard, mats, hearthrugs, table-tops, etc. These exercises are to help children to visualize an inch-square and a foot-square.

A **yard-square** of cardboard should be shown the children and by superimposing foot-squares on it they can find the connexion between the two units. Various surfaces can be measured in square feet and square yards—for example, flower beds, classrooms, parts of corridors, and so on.

### Area without Counting

So far the children have found area by counting or **adding** squares, but they soon find that measuring big areas by counting is slow. Can we find the number without counting or adding? The children know that multiplication is a quick way of adding. They study Fig. 47, a rectangle 4 in. by 3in. How can they find the area of this without counting or adding squares? There are four squares along the top—the length; and three down one side—the width. $4 \times 3 = 12$; they see this is the same number of squares as they get by counting. They find the area of many rectangles in this way, and prove they are right by drawing and counting the squares.

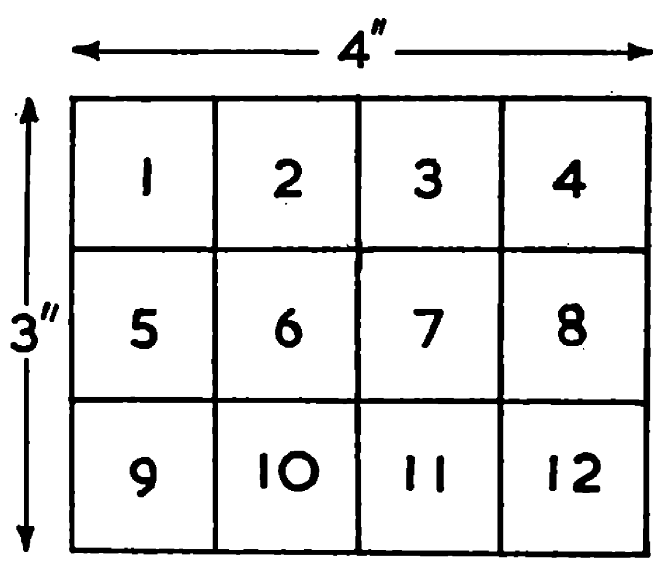



Fig. 47

Therefore, to find the area of a rectangle by arithmetic, the number of inches, feet, or yards in the length are multiplied by the number of inches, feet, or yards in the breadth. Remind the children that **numbers** are multiplied, not inches, feet, or yards. Since inches, feet, and yards are called units, we can put the rule shortly in this way: "To find the area of any rectangle, multiply the number of units in the length by the number of the **same kind of units** in the breadth." Or, to put it shorter still: Area = units in length × units in width or breadth.

The children must have plenty of examples to work, both straightforward examples and problems such as the following:

(1) Find the area of a rectangle 2 ft by 10 in.
(2) A lawn is 12 yd long and 6 yd wide. How much will it cost to turf it, at one shilling a square yard?
(3) Give the children rectangles with no measurements on them, and ask them to find the area. This means they must measure the length and breadth.
(4) Give them rectangles with sides marked off in inches, not numbered. The children have to count the inches to find the area.
(5) What is the area of a garden 13 yd 2 ft broad and 23 yd 2 ft long?
(6) A square table is 60 in. each side. What is its area in square feet?

The children *make* the square-measure table:

144 sq. in. = 1 sq. ft
9 sq. ft = 1 sq. yd

The find the areas of gardens, etc., of different shapes, as in Fig. 48.

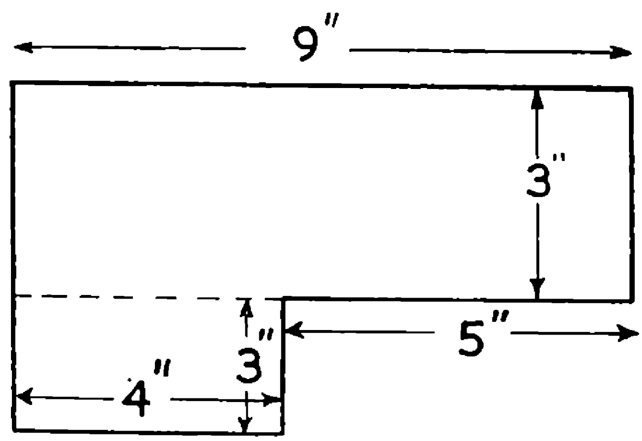



FIG. 48

Finally, make sure that the children realize that:

1 square inch is the amount of surface covered by a square of 1-inch side, an inch-square.
1 square foot is the amount of surface covered by a square of 1-foot side, a foot-square.
1 square yard is the amount of surface covered by a square of 1-yard side, a yard-square.

In the case of an irregular surface, one of the measures, say the foot-square, may be torn or cut into two or more pieces and the pieces placed so as to form an irregular shape, but still to cover 1 square foot of area.

Carpets, walls, floors, etc., are generally measured in square feet; areas of playgrounds, gardens, etc., in square yards.

The **acre** and the **square mile** may be introduced and connected with some **local areas**. The Romans took as their unit of size the amount of land that a pair of oxen could plough in a day, and this is the origin of our word **acre**, derived from the Latin word *ager*, a field, which has, of course, given us also the word *agriculture*.

At first the **acre** was a definite shape, due to the Anglo-Saxon custom of dividing land into strips for cultivation. The original **acre** was a furlong in length, so that the plough went from one end to the other without turning; it was one-tenth as wide. The furlong being 220 yards long, the size of the standard acre was obviously 220 yd by 22 yd, and the 'acre's breadth' is therefore the old equivalent of our modern chain.

In our table to-day 1 acre = 4,840 sq. yd—that is, 220 × 22.

It is because it began as a strip that the acre is not the square of any of our units of distance, and seems an oddity in our area table among the square inches, square feet, etc.

In connexion with length and area, there should be some drawing to scale of the playground or garden, as in Fig. 48, roads to school, classroom, etc. Examples should be worked to show a clear distinction between **length** and **area.**

### VOLUME

Volume again does not introduce any new tables; it is based on the square inch. The unit of measurement in the case of area was the **inch-square**; the unit of measurement in the case of volume is the **inch-cube**. The measurement of volume seldom presents much difficulty if the measurement of area has been placed on a sound basis. The children must be quite clear on the distinction between **lines**, **surfaces**, and **solids**, and between **length**, **area**, and **volume**.

The children examine some inch-cubes. (They can make paper or Plasticene cubes, or they may be bought in some toy-shops, or sometimes borrowed from an Infant School.) They notice a cube has six equal sides, each side an inch square, and the length of each edge is one inch. Collect, with the help of the children, a number of rectangular boxes, as far as possible measuring an exact number of inches each way. If 8 cubes fit in a box, we say it is **8 inch-cubes** big, or, as is more generally said, **8 cubic inches**.

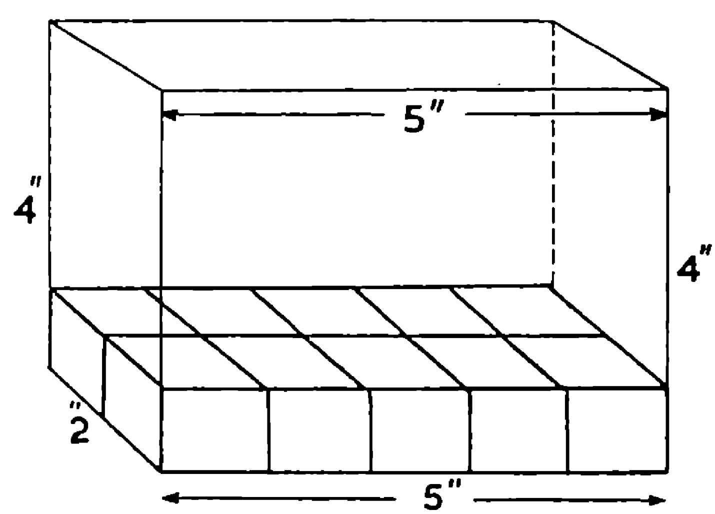



FIG. 49

Give them boxes, as in Fig. 49, 5 in. long, 2 in. wide, 4 in. deep (or high), and tell them to find the volume. With the help of their cubes they find the boxes are 40 inch-cubes big, or 40 cubic in. When they have found the 'capacity' or 'volume' in this way by counting cubes, tell them to try to find a quick way of working this without using cubes. If necessary, give them some hints. How many cubes just cover the bottom of the box? Does this remind you of anything you have already learnt? Quick children will see that the **number of cubes** which just covers the base or bottom of the box is exactly the same as the number of **square units** in the area of the base. There are ten inch-units in the area of the base, therefore there are ten inch-cubes. To fill the box, four layers of cubes are needed, each containing ten cubes; that is, 40 cubes are needed. Therefore, if we multiply the number of cubes which cover the base by the number of layers that fill the box, we get the number of cubes in the box, —*i.e.*, the cubic content or **volume** of the box.

Therefore, volume = *area* of base × *height.*

Encourage them to think:

(1) **Area of base** is the number of square units in the base—that is, 10.

(2) **Height** is the number of units in the height—that is, 4. Thus the volume, or **capacity**, of the box in Fig. 50 is 10 × 4 = 40 cubic in. The children might be told that any solid shaped like a rectangular box is called a **cuboid**, sometimes a useful term. **Capacity, volume**, or **cubic content** all mean much the same.

Give the children plenty of varied examples:

(1) Find the volume of this rectangular box or cuboid—length 4 in., breadth 2 in., height 3 in.

(2) How many 1-inch cubes are in a 2-inch cube, or in a 12-inch cube?

(3) How many cubic inches are in a box 4 in. long, 5 in. wide, 3 in. high?

(4) A tin of sweets takes up 24 cubic in. and measures 24 inch-cubes. How many tins can be packed into a box 9 in. by 6 in. by 8 in.?

(5) The area of the base of a box is 20 sq. in., the height is 2 in. Find the volume.

(6) The contents of six full boxes (cuboids) 3 in. by 5 in. by 6 in. just fill a large box. What is the capacity of the large box?

Cubic inches and cubic feet as units for the measurement of volume are comparatively modern. The first mention of a cubic inch is in a statute of 1824. Builders sometimes speak of a yard of gravel when they mean a cubic yard.

## Measuring Weight

Children experience weight at quite an early age. They know some things are light and some heavy. They should be familiar with weight as a property of things, and as distinct from **size**. But on the whole measuring weight is difficult for children. They rarely see 'weights' or things being weighed in these days of mechanical weighing. Another difficulty is the use of fractional terms. Ounces are generally spoken of as parts of a pound. Before they work any examples on paper, they must have some idea of the meaning of **weight** and of the different units of weight. This they get through experimental weighing using a pair of scales—*not* a modern spring balance or the steelyard from the medical room. As far as possible, estimating weight should precede actual weighing. Let a child hold a pound weight in his hand, and then replace it by a pile of books judged to weigh a pound. Weight is more difficult to estimate than either length or capacity. **Comparison** with something is generally necessary. A child can be expected to feel which is the heavier of **two** articles, but to estimate the weight of an object without a pound weight is very difficult, and not too much time should be spent over it. It helps children if they are encouraged at home to hold and notice the weight of 1 lb. of sugar, a 2-lb. packet of sugar, a $\frac{1}{2}$-lb. packet of sugar, 1 lb. of jam, 1 lb. of biscuits, etc.

## Building up a Table of Pounds and Ounces

Let the children have a 1-oz. weight and 1-lb. weight. As they need a good many 1-oz. weights they can make them from clay or Plasticene or fill little paper bags with sand. With these 1 oz. weights they find out how many are needed to balance 1 lb. Thus they learn that 16 oz. = 1 lb. They make this table:

16 oz. = 1 lb.
8 oz. = $\frac{1}{2}$ lb.
4 oz. = $\frac{1}{4}$ lb.
12 oz. = $\frac{3}{4}$ lb.

To enable any number of ounces up to 8 oz. ($\frac{1}{2}$ lb.) to be weighed only four weights are needed—1 oz., 2 oz., 3 oz., 4 oz. The children can prove this.

It is a good exercise to let children divide a 1-lb. packet of bran or sawdust into $\frac{1}{2}$-lb. packets without a weight. A thoughtful child takes two bags or boxes of the same size and divides the bran as carefully as he can between them. He puts them on the scales. If they balance each other, he knows each weighs $\frac{1}{2}$ lb.

Let the children notice the difference in size between a pound of sawdust and a pound of sand, a pound of feathers and a pound of sand. **Weight** (or mass) has little to do with size or volume.

Let the children weigh a variety of small things. They are interested in the weight of coins, especially when they find out that three pennies weigh 1 oz. With three pennies they always have an ounce weight. Six pennies give them a 2-oz. weight, and half a crown a weight of $\frac{1}{2}$ oz. A brick weighs about 7 lb. Encourage the children to make attractive note-books in which they write down all the interesting things they discover. On one or more pages they keep a record of the things they weigh—penknife, rubber, reel of cotton, fountain-pen, etc. They are as much interested in doing this as they were in Infant School days in playing shops. Indeed, shops and the money element distracts children's minds from the idea of weight.

They have separate pages for lists of things sold by (1) **the pound**: tea, sugar, butter, etc.; (2) **the half-pound**: tea, sugar, sweets, cheese, butter, etc.; (3) **the quarter-pound**: sweets, wool, tobacco, etc. This helps them to realize that ounces are generally used only as **parts** of a pound. Let them if possible have a pair of letter scales and weigh letters and small parcels. The six weights used for letters are: 8 oz., 4 oz., 2 oz., 1 oz., and two weights each ½ oz.

### Using Heavier Weights

Not much practical work is possible with children, but the younger ones can be made familiar with commodities sold in large quantities, and given some idea of the magnitude of a **hundredweight** and a **ton.**

The children like to weigh themselves frequently; and they sometimes try to keep a record of their variations in weight from month to month. This brings in the use of a heavier weight, **the stone**; 14 pounds are 1 stone (st.). The weekly increase in the weight of a puppy or kitten is of great interest to children. Potatoes are sold by the stone.

A coal-cart carrying a ton of coal may be familiar to some children. A hundredweight of clay is sometimes bought for school use, and is useful for giving an idea of a **hundredweight** (cwt). The coalman again, with his load of sacks, helps to introduce the idea of the hundredweight. His load of a ton is divided into 20 sacks, each holding a hundredweight.

Gradually the children build up their table of weights to 20 cwt = 1 ton. (See Chapter X.)

The great weight of a ton can be brought home to children if they add their own weights, and find how many children are needed to balance a quarter of a ton, a half, or one whole ton.

Some of the words in the Table of Weights are confusing to children. The **quarter** often confuses children, because they are so familiar with the colloquial use of 'quarter' for a quarter of a pound and quarter of a stone, etc. A **quarter** in weight is a quarter of a hundredweight. The word **pound** is confusing because it stands for both a pound weight and a sum of money; a **pound** note is worth 20 shillings. The Roman pound was a pound of silver, a definite amount of money. Thus the word **pound** came to mean both a weight and a money value. The word itself is from a Latin word meaning weight. Our abbreviations, lb. and £, come from the Latin word *libra*, meaning a balance (a two-scaled balance). The Roman pound weight was divided into 12 *unciæ*, hence our ounce (cf. **inch,** the twelfth part of a foot).

The stone was a good 'natural' weight, but it varied a great deal until the standard imperial stone of 14 pounds was introduced and all the other 'stones' abolished. The abbreviations are worth explaining to children, especially that for **hundredweight**, which weight was once equal to 100 pounds. In our abbreviation **cwt** the 'c' stands for the Latin word *centum*, one hundred, and 'wt' for weight. As early as Queen Elizabeth's reign a hundredweight became fixed as 112 lb.

The method of dealing with reduction, and addition, subtraction, multiplication, and division is the same as that for money sums (Chapter X). The new units bother the children unless they know their tables well. A variety of exercises will be found in good modern textbooks. The textbook, however, should be a useful servant and not a tyrannical master. Every teacher needs to add his own exercises.

Children of nine and ten may be taught:

112 lb. = 1 cwt
Then 28 lb. = 1 qr (involving short division by 4)
and 2,240 lb. = 1 ton (involving multiplication by 20)

The **Table of Weight** may cause trouble on account of the bigger divisors. Children who have not learned long division are not able to divide mentally, or by short division, by 16 and 28. To divide by these numbers let them use factors (see Chapter IX). The factors of 28 are 7 × 4. The working is shown below.

| | | |
|---|---|---|
| 28 { 7 | 39 lb. to qr | |
| | 5 groups of 7 lb. and 4 lb. over | } 11 lb. |
| 4 | 1 qr and 1 group of 7 lb. over | |

or more shortly.

| | | | | |
|---|---|---|---|---|
| 28 | 7 | 39 lb. to qr | | |
| | 4 | 5 (7's) | 4 lb. | 11 lb. |
| | | 1 qr | 1(7's) lb. | |

CAPACITY

Capacity means the power of holding—that is, how much a cup, jug, bottle, etc., can hold of any liquid, whether milk, petrol, water. Not a great deal of practical work is necessary in the Junior School. In the case of backward children, let them have pint and quart milk-bottles and a gallon tin, all clearly labelled on the outside, with which they can experiment. The measuring of milk, oil, vinegar, petrol, and so on may be discussed. The children find by experiment that:

3 school milk-bottles make 1 pint
2 half-pints make 1 pint
2 pints make 1 quart
4 quarts make 1 gallon

The fact that 'quart' stands for 'quarter,' and that a quart is a quarter of a gallon, may be pointed out. The children learn that the unit of measurement is the pint. They find out by experiment that a large cup or tumbler holds or measures half a pint. A tea-cup holds a quarter of a pint. Jugs hold half a pint, one pint, or one quart; a pail or kettle holds a number of pints or quarts. In this practical way children will acquire exact knowledge of the first two units of the liquid measure of capacity. The part of the capacity table containing pecks, bushels, and quarters has been omitted because these units apply to the dry measure of grain, seeds, soft fruit, etc., and not to liquids. This part of the table of capacity need not be taught, but there should be some discussion of the use that is made of these measures for dry materials, especially of the way in which they are used *locally*. Similarly, local use should determine whether children are taught about gills. 4 gills make 1 pint (the cookery book measure), but 2 gills make 1 pint is the milk measure in some localities. Cream is often sold by the gill.

TIME

A clock-face with movable hands is indispensable for teaching 'time' and the divisions on a clock-face. Place the clock-face in a prominent place where the children can examine it and talk about it in their free periods. They can also make their own clock-faces from stiff paper. A circle cut out and folded in half, and then in half again, gives the hours 12, 6; 3, 9. The children notice 12 is the highest point on the clock's face, and 6 is the lowest. The missing numbers are found by cutting three segments that fit into each quarter of the circle. Let the children notice that one hand is **short** and one **long**. The short hand is the hour hand. The children move the hour hand to the different hours. It is essential that the hands are moved clockwise from **left to right**. They find there are twelve hours on the face. It helps dull children if the short hand or hour hand is used alone at first. The children tell the hour by looking at the short hand. As they move it round they say, "It is just past three o'clock." Then they move it farther again and say, "It is nearly four o'clock." Next explain the minutes. There are 60 minutes round the face of the clock. The long hand, the quick hand, is the minute hand and points out the minutes. It moves from minute to minute and tells how long a minute is. Let the children count the minutes as the long hand moves through five minutes, fifteen minutes, etc. It goes through 60 min. as it moves from 12 round to 12. While the long hand goes from 12 to 12 the short hand goes from one hour to the next, say from 3 to 4, very slowly. It has to take 60 minutes to do this. When it is four o'clock the little hand is exactly on 4, and the long hand on 12. In this way the children can prove that 60 minutes make 1 hour. Teach them landmarks—a quarter of an hour, half an hour, and three-quarters of an hour. The long hand tells when it is quarter past the hour (15 min. past), half-past the hour (30 min. past), and quarter to (15 min. to) the hour. The difference between 'past the hour' and 'to the hour' must be explained. They set their clocks at quarter-past four, quarter to four, etc.

Remind the children that, although only 12 hours are shown on the clock's face, there are 24 hours in one day. The children move the hour hand through 24 hours or one day—that is, they make two complete turns with it.

The initials A.M. and P.M. must be explained. The letters A.M. are for *ante meridiem*, meaning "before noon." *Ante* is a Latin word meaning *before*, *meridiem* is two Latin words—*medius*, middle, plus *dies*, day; thus *ante meridiem* means before noon or midday. It is from 12 o'clock midnight to 12 o'clock midday. The letters P.M. are for *post meridiem*. *Post* is a Latin word meaning after. *Post meridiem* means afternoon or midday—that is, from 12 o'clock midday to 12 o'clock midnight.

Much practice should be given on the two ways of expressing time—namely, twenty minutes to seven and 6.40; 6.35 and twenty-five minutes to seven.

How long is a minute? Sixty seconds make 1 minute. The children count slowly 60 seconds as the minute hand moves from one minute division to the next. If they count correctly, when they get to 60 the minute hand should be on the next minute division. From their experiments with clock-faces, etc., they write out the Table of Time (see Chapter X).

Children get some idea of estimating time by sitting perfectly still for one minute, then two minutes, by timing themselves over different pieces of work, and by using a sandglass. It is important to encourage children to time their work. Their ideas of time are very vague. They frequently say, "I was ages doing my homework," when they were really less than half an hour. Let them find out in minutes how long they are in school from nine o'clock to break-time, how long it takes them to walk to school, and so on. Familiarity with the units of time can also be obtained by the use of railway or bus time-tables, B.B.C. programmes, and so on. Interesting exercises can be set on the reading of time-tables of local rail and bus services. Local routes that accord with the children's experiences should be used at first, as the times and distances concerned are of more reasonable dimensions. In setting exercises such as, "Tom took from Oct. 23 to Nov. 5 to make his aeroplane. How many days did he take? How many weeks?" care must be taken to state exactly whether the days named are included or excluded in the period to be measured.

A useful calendar for long periods of time is one where all the months are displayed on one sheet. This provides excellent exercises. The children find the months with 30 days and those with 31 days. They count how many days or weeks it is to their next birthday; how many weeks in a year; how many days in a year; and so on. Then they learn the rhyme:

> Thirty days has September,
> April, June, and November, etc.

All the exercises given on the measure of value (money exercises) can be adapted to the measure of time. Every opportunity should be taken to show the use of the word *unit*—for example, "Change the unit so that all these times are expressed in hours: (*a*) 2 days, 3 hours. (*b*) 3 days, 14 hours." "Change the unit so that these hours are expressed in days: 48 hours; 173 hours; 192 hours."

Exercises should be given to test the children's understanding of the two ways of expressing time, 10.36 or twenty-four minutes to eleven, thus:

A train starts from X at 10.36 and reaches Y at nine minutes to eleven. How long did the journey take?

The children think ten thirty-six is twenty-four minutes to eleven. What is the difference between twenty-four minutes to eleven and nine minutes to eleven?

CHAPTER XII

# PROPERTIES OF NUMBERS AND FRACTIONS

*Multiples. Factors, Divisors, or Measures. Tests of Divisibility. Prime Numbers. Fractions. More Exercises on naming and writing Fractions. The Fundamental Property of Fractions; Equivalent Fractions.*

When children begin the systematic study of fractions in the Junior School, they have already picked up in the Infant School and at home some useful ideas about them. Some of them are quite familiar with the idea of 'half an apple,' a 'half-pint of milk,' 'a quarter of a pound of tea.' They probably look upon the last two as units in themselves, units that are clearly smaller than the pint and the pound, but they know little about the relationship between **the part and the whole.** As a rule very few children have a sound basis of knowledge on which to build. Indeed, it is best to suppose they have not, and to do some revising.

These properties of number are worth teaching or revising before beginning the study of fractions. Children must be able to recognize and understand **factors, multiples, common multiples, odd and even** numbers, **prime numbers,** and so on. See that they can find quickly:

(*a*) **The double of any number** (6 is the double of 3). Give them a list of numbers to double.

(*b*) **The half of numbers** such as 2, 4, 6, 8. In this connexion remind them of **even** and **odd** numbers. **Even numbers** can be divided exactly by 2, as 2, 4, 6, 8. **Odd numbers** cannot be divided exactly by 2; there is always a remainder: $3 \div 2 = 1$ and 1 over. They write the numbers from 1 to 20, and ring round the odd numbers. Lead them to see that any number, however big, that ends in an **even** number is divisible by 2—for example, 3,594. Such numbers, in other words, can be easily **halved.**

Although L.C.M. and H.C.F. are not usually taught in the Junior School, since denominators are such that they can be found by the Inspection Method, for the sake of the more complete understanding of numbers, and above all for the enjoyment of those intelligent Juniors (too often neglected) who are interested in number in the **abstract** and do not like money exercises or any kind of weights and measures, we give the following suggestions, with which we hope teachers will experiment. Some suggestions are on a par with number squares or magic squares (Chapter IV). They have given our children so much pleasure that we hope they will give pleasure to others. For finding common denominators by the Inspection Method see Chapter XIII.

### MULTIPLES

A **multiple** of a number, say 4, is the number we get when 4 is multiplied by another number, as $4 \times 8 = 32$. 32 is a multiple of 4. The children find multiples of 4. They multiply 4 by each number in turn.

(*a*) 4, 8, 12, 16, 20, 24, 28, 32, 36 . . .

They can easily find multiples of 4 by saying the Table of Fours. They see that the multiples of 4 or any number are endless. It is obvious that every multiple of 4 is exactly divisible by 4.

Let them find the multiples of 6.

(*b*) 6, 12, 18, 24, 30, 36, 42, 48 . .

Next they look at their two long lines of multiples, (*a*) and (*b*), and see if any are common to 4 and 6. These they ring round —namely 12, 24, 36. These numbers are said to be **common multiples** of 4 and 6. They can now find the smallest number which is a Common Multiple of 4 and 6—namely, 12. Therefore 12 is the **Least Common Multiple**, or the smallest number that is exactly divisible by each of these numbers, 4 and 6. As the list of multiples of 4 and 6 is endless, the list of common multiples of 4 and 6 is endless, so we cannot have a Greatest Common Multiple. Point out the common multiples of 4 and 6 are exactly divisible by 4 and 6.

Children are much interested in finding the multiples of numbers say from 2 to 20, then finding common multiples, and the Least Common Multiple of three or more numbers. This is a valuable exercise. It also revises the tables.

### Factors, Divisors, or Measures

If a number is exactly divisible by a second number, the second number is said to be a factor or measure or divisor of the first number. 24 is exactly divisible by 6, therefore 6 is a factor of 24; 7 is a factor or divisor of 28; and so on. Let the children write down all the factors or divisors of two numbers. Choose 72 and 84 first, as they have so many factors.

The factors of 72 are 1, 2, 3, 4, 6, 8, 9, 12, 18, 24, 36, 72.
The factors of 84 are 1, 2, 3, 4, 6, 7, 12, 14, 21, 28, 42, 84.

They find the numbers 1, 2, 3, 4, 6, 12, appear in both lists, they are therefore **Common Factors** or **Common Divisors** of 72 and 84. The largest of these common factors is 12, and is called the **Highest Common Factor** or **Greatest Common Divisor** or **Greatest Common Measure** of 72 and 84.

It is convenient to be able to recognize quickly whether or not a number is exactly divisible by any of the smaller numbers 2, 3, 4, 5–12. There are simple tests of divisibility that Junior children can well understand and even enjoy.

### Some Tests of Divisibility

(1) *Divisibility by* 2. All numbers ending in **even** numbers are exactly divisible by 2—*e.g.*, 5,796.

(2) *Divisibility by* 3 *and* 9. (*a*) All numbers are exactly divisible by 3 if the sum of their digits is divisible by 3—*e.g.*, 1,095: S = 15, a multiple of 3. (*b*) All numbers are exactly divisible by 9 if the sum of their digits is divisible by 9—*e.g.*,

2,889: S = 27, a multiple of 9, therefore 2,889 is exactly divisible by 9.

(3) *Divisibility by* 4 *and* 8. (*a*) All numbers are exactly divisible by 4 if the last two numbers are divisible by 4, *e.g.*, 5,300. Now, 100 is a multiple of 4, therefore 53 × 100, or 5,300, is a multiple of 4, and so exactly divisible by 4. In the case of 8,528, 28 is exactly divisible by 4. (*b*) Numbers are exactly divisible by 8 if the last *three* figures are divisible by 8—for example, 158,376: 376 is exactly divisible by 8, hence so is 158,376.

(4) *Divisibility by* 5 *and* 10. (*a*) Numbers are exactly divisible by 5 if they end in 0 or 5. (*b*) They are exactly divisible by 10 if they end in 0.

(5) *Divisibility by* 11. A number is exactly divisible by 11 if the difference between the sum of the alternate digits, beginning with the units digit, and the sum of the alternate digits, beginning with the tens digit, is equal to 11, or a multiple of 11, or 0. Take as an illustration of this the number 356,686. The sum of alternate digits beginning with the units digit 6 is 17. The sum of alternative digits beginning with the tens digit is 17. The difference is 0, so that the number is exactly divisible by 11. Let the children try these numbers: 190,718 and 609,180.

## Prime Numbers

A number such as 36 can be expressed as a product of two numbers in *various* ways:

$$36 = 1 \times 36 = 3 \times 12 = 4 \times 9 = 6 \times 6$$

Such a number as 36 is called a **composite** number. A number such as 13, which can be expressed as a product in one way only—namely, 1 × 13—is called a **prime number.** Intelligent pupils will understand the definition **a prime number is a number which has no factors except**

**itself and unity.** Let children find prime numbers 1, 2, 3, 5, 7, 11, etc.

To find prime numbers in a given range—say, 41 to 100—one should proceed as below.

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 41 | ~~42~~ | 43 | ~~44~~ | ~~45~~ | ~~46~~ | 47 | ~~48~~ | ~~49~~ | ~~50~~ |
| ~~51~~ | ~~52~~ | 53 | ~~54~~ | ~~55~~ | ~~56~~ | ~~57~~ | ~~58~~ | 59 | ~~60~~ |
| 61 | ~~62~~ | ~~63~~ | ~~64~~ | ~~65~~ | ~~66~~ | 67 | ~~68~~ | ~~69~~ | ~~70~~ |
| 71 | ~~72~~ | 73 | ~~74~~ | ~~75~~ | ~~76~~ | ~~77~~ | ~~78~~ | 79 | ~~80~~ |
| ~~81~~ | ~~82~~ | 83 | ~~84~~ | ~~85~~ | ~~86~~ | ~~87~~ | ~~88~~ | 89 | ~~90~~ |
| ~~91~~ | ~~92~~ | 93 | ~~94~~ | ~~95~~ | ~~96~~ | 97 | ~~98~~ | ~~99~~ | ~~100~~ |

Begin by crossing out all the even numbers—that is, multiples of 2 or numbers exactly divisible by 2. Next find the first number on the list which is divisible by 3—namely 42; cross this off again, and also every third number thereafter—namely, 45, 48, 51, 57, and so on. Next cross out the first multiple of 5, and every fifth number thereafter. Lastly cross out the first multiple of 7, and every seventh number thereafter. There is no need to go further. The prime numbers are 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97.

Children like to cross out the figures with crayons of different colours, but the result is often messy and untidy. The above exercise is not only useful for revising the tables but a knowledge of the prime numbers helps the pupils' work in vulgar fractions. Another interesting and instructive exercise that pupils enjoy and that will help their work in fractions is to make a list of all the factors or divisors of a given number, say 96.

$$\begin{aligned} 96 &= 1 \times 96 \\ &= 2 \times 48 \\ &= 3 \times 32 \\ &= 4 \times 24 \\ &= 6 \times 16 \\ &= 8 \times 12 \end{aligned}$$

It is easy to see how this table is constructed. It shows clearly all the factors of 96: 1, 2, 3, 4, 6, 8, 12, 16, 24, 32, 48, 96.

## FRACTIONS

If children have some understanding of the **properties** of numbers, it helps them in their work with fractions. Practical work with fractions begins with **measurements**.

Fractions are primarily associated with measurable quantities, and children are best taught fractions at first in connexion with **measurements**. Let them have a set of suitable shapes—squares, circles, rectangles—marked in such a way

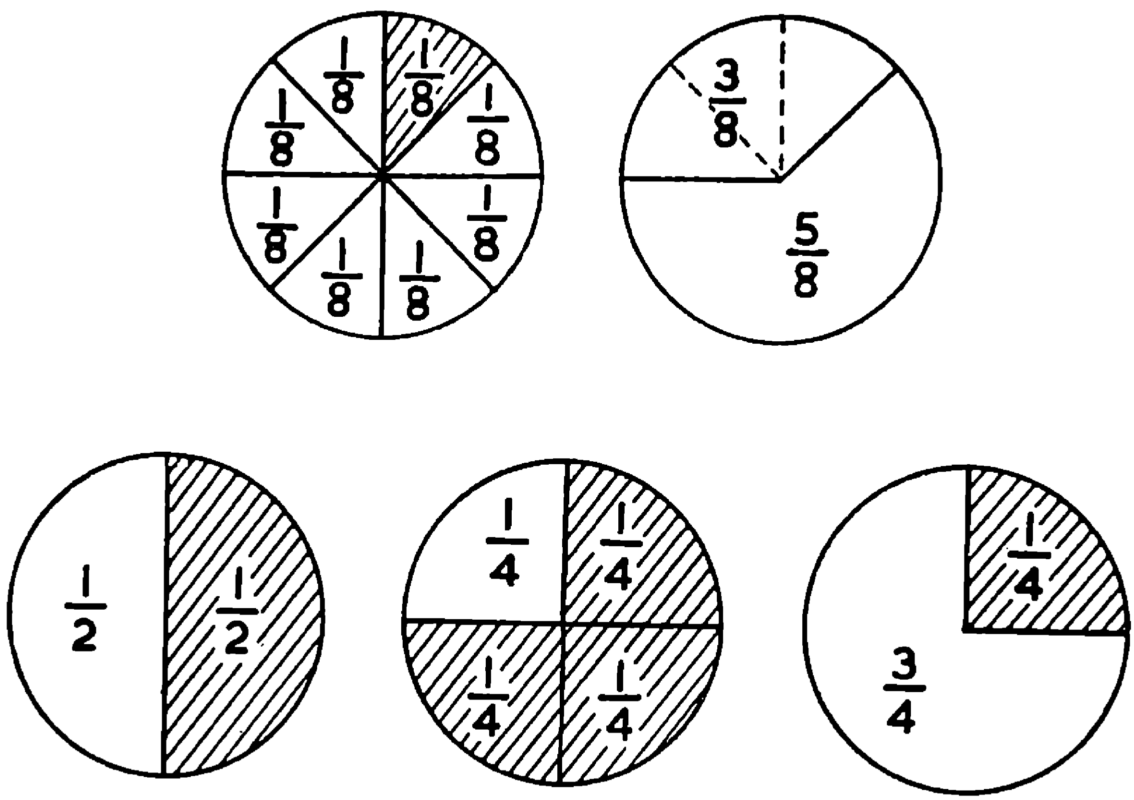



FIG. 50

that their division into convenient numbers of equal parts can easily be effected, as in Fig. 50. The children study these, and draw and cut out circles for themselves which are folded into halves, $\frac{2}{2}=1$ whole, $\frac{2}{4}=\frac{1}{2}$, $\frac{8}{8}=1$ whole. The teacher should stress the fact that there are, for example, six sixths in the whole object or quantity, and that they are all **equal** in size.

A rectangle such as the figure below is excellent for explaining fractions:

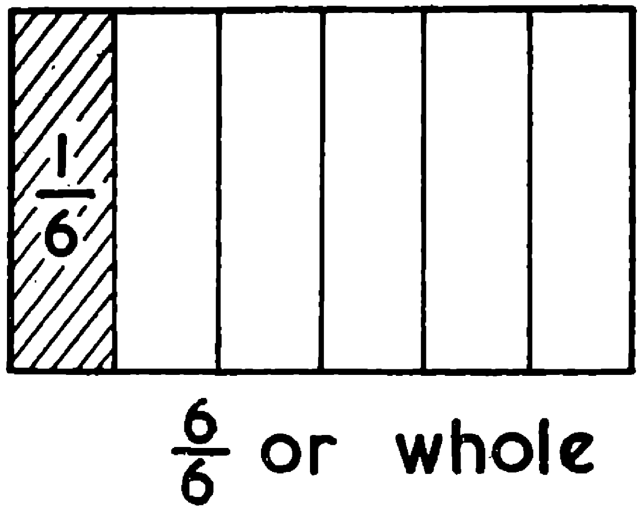



FIG. 51

The children have their own strips of paper which they fold into six equal parts, cutting off any part that is over. Each part is called a sixth or one-sixth of AB.

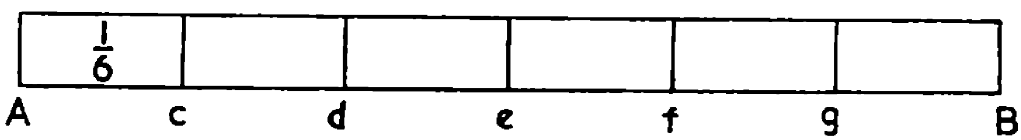



FIG. 52

They find out that

Ac or cd, etc. $=\frac{1}{6}$ of AB
Ad or df, etc. $=\frac{2}{6}$ of AB
Ae or eB, etc. $=\frac{3}{6}$ of AB
Af or dB, etc. $=\frac{4}{6}$ of AB
Ag or cB, etc. $=\frac{5}{6}$ of AB
AB $=\frac{6}{6}$

It is from illustrations such as the above that children learn most quickly how to name and write fractions.

## More Exercises on naming and writing Fractions

Taking the 'sharing' view of division, the exercise $24 \div 6$ may be looked upon as meaning "How many apples will each of 6 boys receive if 24 apples are shared equally among them?" The sharing divides the dividend (24) into 6 equal shares of 4 apples. The share that falls to one boy is called **a sixth** or

**one-sixth** of the apples. One-sixth of 24 apples is 4 apples, or in abstract terms

$$\tfrac{1}{6} \text{ of } 24 = 4$$

If two boys put their shares together, their joint share is ${}_6$, and

$$\tfrac{2}{6} \text{ of } 24 = 8$$

In the same way, the pupils can interpret $\frac{3}{6}$, $\frac{4}{6}$, and $\frac{5}{6}$ of 24.

One of the commonest types of exercise which the pupil will have to work is finding a stated fraction of some given quantity, as

(*a*) $\frac{3}{8}$ of £56 (*b*) $\frac{5}{12}$ of 3 tons, 5 cwt

In teaching the above examples, the **spoken name** of the fraction should be emphasized, **three-**eighths of £56. By emphasizing particularly the 3, the pupils may realize the need to know **one-**eighth first.

The working is as follows:

$$\begin{aligned} \tfrac{1}{8} \text{ of } £56 &= £7 \\ \tfrac{3}{8} \text{ of } £56 &= £7 \times 3 \\ &= £21 \end{aligned}$$

In the case of $\frac{5}{12}$ of 2 tons 8 cwt, it is easier to reduce 2 tons, 8 cwt to cwt.

Thus,

$$\begin{aligned} 2 \text{ tons, } 8 \text{ cwt} &= 48 \text{ cwt} \\ \tfrac{1}{12} \text{ of } 48 \text{ cwt} &= 4 \text{ cwt} \\ \tfrac{5}{12} \text{ of } 48 \text{ cwt} &= 20 \text{ cwt} \\ &= 1 \text{ ton} \end{aligned}$$

In these exercises the necessary reductions should be as simple as possible, so that the children are able to give their whole minds to the new fractional ideas. As soon as possible the words **numerator** and **denominator** should be taught. From the practical work they have done, they know that the figure under the line tells us, or **denotes**, the name of the fraction—that is, into how many equal parts a whole is divided. It is called the **denominator**, which just means the **naming** part.

Let the pupils say the names of fractions with these denominations:

$\overline{4}$ $\overline{7}$ $\overline{12}$ $\overline{3}$ $\overline{2}$ $\overline{10}$ $\overline{5}$

The number above the line serves to count or enumerate the number of equal parts to be used. This number is called the **numerator** (num(b)erator). Thus the fraction $\frac{3}{8}$ means "From the 8 equal parts into which the thing was divided 3 must be taken."

$$\frac{\text{numerator}}{\text{denominator}} \text{ or } \frac{\text{numbering figure}}{\text{naming figure}} \text{ or } \frac{\text{how many}}{\text{what name}} \text{ or } \frac{2}{\text{thirds}}$$

Revise fractions of pounds, hours, feet, etc., such as $\frac{1}{2}$ lb., $\frac{1}{4}$ hour, $1\frac{1}{2}$ in. Pupils find from experiment that, the larger the denominator is, the smaller are the parts; thus $\frac{1}{8}$ of a circle is much smaller than $\frac{1}{2}$ of a circle. Let them arrange these fractions in order of size, beginning with the smallest:

$\frac{1}{8}$ $\frac{1}{4}$ $\frac{1}{7}$ $\frac{1}{2}$ $\frac{1}{3}$ $\frac{1}{5}$ $\frac{1}{9}$

## The Fundamental Property of Fractions. Equivalent Fractions

The pupils must be taught to realize that $\frac{1}{2}=\frac{2}{4}=\frac{4}{8}$, etc., or in other words, that if both denominators and numerators are multiplied by the **same** number, the fraction remains unchanged in value. If this is understood, addition and subtraction are easy. One way to help understanding is to let the children study a ruler divided into halves, quarters, eighths, and state various lengths on it in as many ways as possible, thus

1 half-inch = 2 quarter-inches = 4 eighths of an inch.
Or, $\frac{1}{2}=\frac{2}{4}=\frac{4}{8}$; $\frac{1}{4}=\frac{2}{8}$; $\frac{1}{2}=\frac{4}{8}$; $\frac{3}{4}=\frac{6}{8}$

Another set of equivalent fractions can be found by examining a ruler graduated in halves, quarters, and twelfths, or thirds, sixths, twelfths. They find that

$$\tfrac{1}{2}=\tfrac{6}{12};\quad \tfrac{1}{4}=\tfrac{3}{12};\quad \tfrac{3}{4}=\tfrac{9}{12};\quad \tfrac{3}{6}=\tfrac{1}{2}$$

They work exercises on their discoveries:

How many eighths in $\frac{1}{4}$, $\frac{2}{4}$, $\frac{3}{4}$?

Copy and fill in the spaces $\frac{1}{4}=\frac{?}{8}$, $\frac{3}{4}=\frac{?}{8}$, $\frac{1}{4}=\frac{?}{12}$, $36=\frac{?}{18}$

Another help to understanding equivalent fractions is the following easy exercises (oral or written) on finding the value of fractions of varied quantities.

How many pennies in $\frac{1}{3}$ of a shilling? 4
How many pennies in $\frac{2}{6}$ of a shilling? 4

From these answers it is seen at once that $\frac{1}{3}=\frac{2}{6}$. They find fractions of yards, feet, and other measures in the same way. The chart shown in Fig. 53 is often a help to backward children.

From illustrations such as those below, and others supplied by the teacher, the children will begin to see that, in an equation such as $\frac{3}{4}=\frac{9}{12}$, the 9 and the 12 may be obtained by multiplying the 3 and 4 of $\frac{3}{4}$ by 3. Other equivalent fractions may be found for $\frac{3}{4}$, such as $\frac{21}{28}$, the 21 and 28 being obtained by multiplying the 3 and 4 by 7. They have now learnt the **fundamental property of fractions** given below, although they may not realize it:

**A fraction is unaltered in value if the numerator and denominator are multiplied by the same number.**

The converse of the above is, of course, true. **A fraction is unaltered in value if the numerator and the denominator are each divided by the same number.** The numbers at the top and bottom of the fraction are called "terms." $\frac{1}{2}$ is a fraction in its lowest terms; $\frac{24}{48}$ is the same fraction in higher terms.

Fractions can thus be raised to higher terms or reduced to lower terms. As far as possible the children must be helped to realize that in the process of raising a fraction from lower to higher terms, **the fraction itself has not been multiplied**

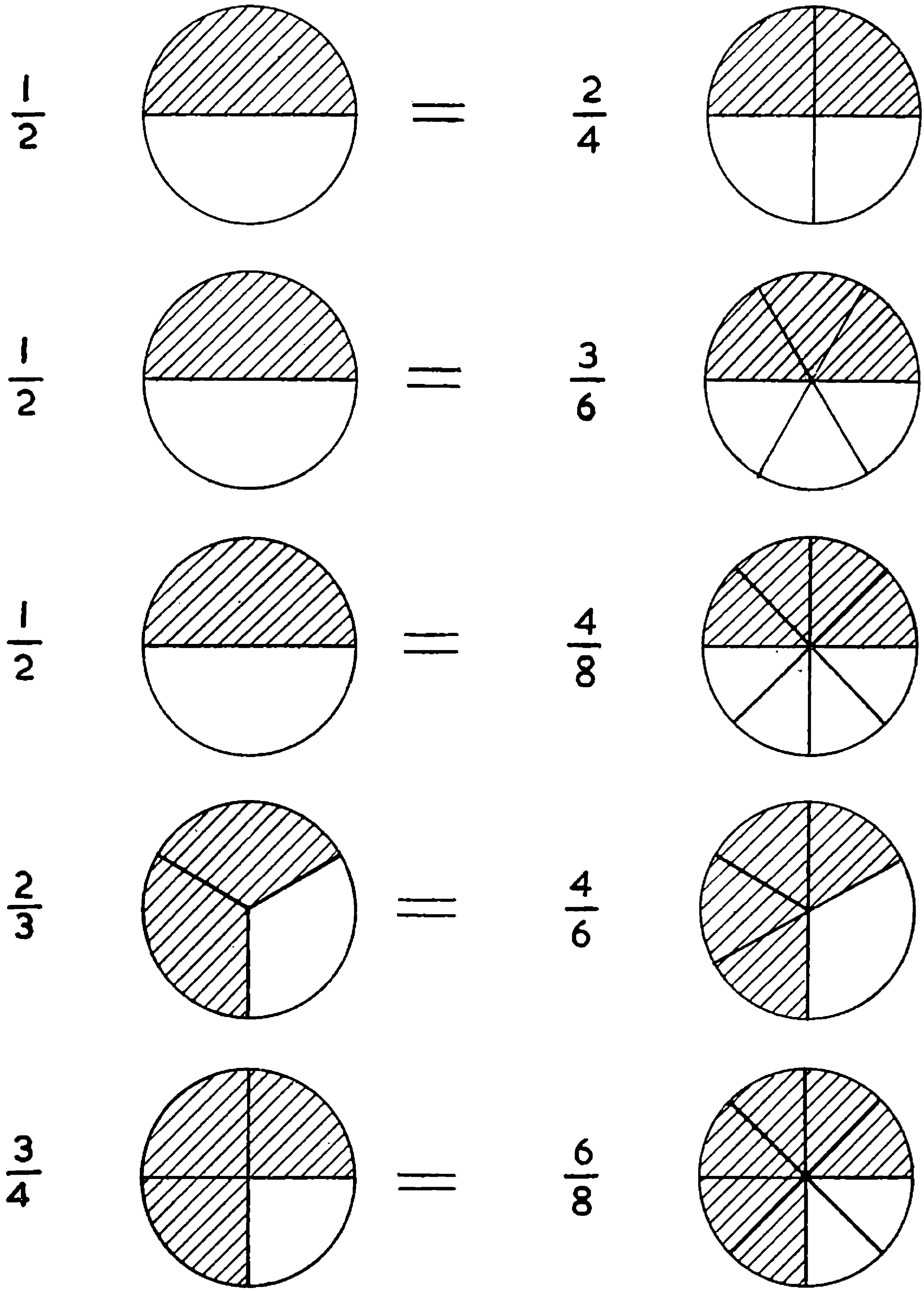



FIG. 53. EQUIVALENT FRACTIONS

**by anything.** It has only been transformed by having its numerator and denominator multiplied by the same number. $\frac{24}{48}$ is still $\frac{1}{2}$. Again, in the process of reducing a fraction to lower terms **the fraction itself has not been divided by anything.**

The following exercises are useful:

(1) Write a series of fractions each equal to $\frac{2}{3}$.
The children multiply both the numerator and denominator by 2, 3, 4, etc., in turn getting $\frac{2}{3}=\frac{4}{6}=\frac{6}{9}=\frac{8}{12}$

(2) Write down a fraction which is equal in value to $\frac{2}{3}$ and has the denominator 15.

The problem is set down like this:

$$\frac{2}{3}=\frac{?}{15}$$

The child thinks, 15 is obtained by multiplying 3 by 5, therefore we must multiply the numerator 2 by 5. Hence

$$\frac{2}{3}=\frac{2\times 5}{15}=\frac{10}{15}$$

(3) Express the fraction $\frac{36}{48}$ in its simplest or lowest form.

The child sees at once that 2 is a common factor of 36 and 48 (see Factors, p. 157), hence $\frac{36}{48}$ becomes $\frac{18}{24}$. 2 is again a common factor of both numbers, and we get $\frac{9}{12}$. 3 is a common factor of 9 and 12, and the fraction is reduced to $\frac{3}{4}$.

The quickest way to reduce $\frac{36}{48}$ (or any fraction) to its lowest terms or simplest form is to find the Greatest Common Factor or Highest Common Factor (H.C.F.) of 36 and 48; the greatest number that will go evenly into 36 and 48 is 12.

CHAPTER XIII

# VULGAR FRACTIONS (continued)

*Proper and Improper Fractions. The Addition of Fractions. The Subtraction of Fractions. Multiplication of Fractions. Division involving Fractions. Easy Percentages.*

## PROPER AND IMPROPER FRACTIONS

Let the children measure strips of paper in lengths over two inches but less than three inches, as AB below. They may say, "Two inches long and something over." They need another unit instead of an inch to measure the something over.

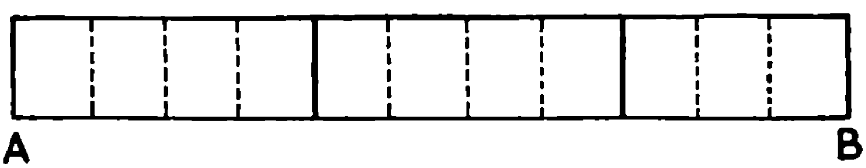



FIG. 54

They try a quarter and find there are $\frac{3}{4}$ inches over. The line is therefore 2 inches + $\frac{3}{4}$ inches, or $2\frac{3}{4}$ inches long. If the whole line is measured in quarter-inches, the length is 11 quarter-inches. Just as we write $\frac{3}{4}$ for 3 quarter-inches, so we can write $\frac{11}{4}$ for 11 quarter-inches. We have now two measures for the same line AB, and we can say:

$$2\frac{3}{4} = \frac{11}{4}$$

A number like $2\frac{3}{4}$, made up of a fractional part added to a whole number, is called a **mixed number**. Fractions like $\frac{11}{4}$, $\frac{47}{8}$, $\frac{2}{2}$, in which the numerator is the same as or greater than the denominator, are called **improper fractions**, because they are not fractions in the real sense of the word. All fractions in which the numerator is less than the denominator are called **proper fractions**. They cannot be changed into whole numbers. A true fraction is a part only. $\frac{11}{4}$ is more than a part. It is two wholes and three parts.

Further diagrams similar to the above may be used for the children to find such equivalences as:

1 inch $= \frac{4}{4}$ inches, 2 inches $= \frac{8}{4}$, 3 inches $= \frac{12}{4}$, $1\frac{3}{4}$ inches $= \frac{7}{4}$ inches.

The quantity measured need not be always a length. It can be bars of chocolate, pounds, gallons.

Conversion from mixed numbers to improper fractions soon becomes a simple matter. To express $7\frac{3}{4}$ as an improper fraction, the child thinks of the problem of measuring a length in inches and quarters. Each whole inch is equivalent to 4 quarters, so 7 inches are equivalent to 28 quarters. Hence $7\frac{3}{4} = 7 + 3$ quarters

$$= 28 \text{ quarters} + 3 \text{ quarters}$$
$$= 31 \text{ quarters}$$
$$= \frac{31}{4}$$

In expressing $\frac{53}{8}$ as a mixed number, a child may think, $\frac{53}{8}$ means some whole numbers and a fraction. The whole number will be $\frac{8}{8}$, therefore how many 8's are there in 53? Six 8's and $\frac{5}{8}$ over—that is $6\frac{5}{8}$. It will help the children if they look at a ruler marked in inches and eighths of an inch. Each inch has eight parts, and therefore equals $\frac{8}{8}$. To find how many inches there are in 53 eighths, they divide by 8. They soon see the quickest way to change an improper fraction to a mixed number is to divide the numerator by the denominator.

## The Addition of Fractions

Adding fractions with the same denominators or names presents little difficulty.

(*a*) $\frac{5}{12} + \frac{4}{12} = \frac{9}{12} = \frac{3}{4}$ (*b*) $\frac{7}{12} + \frac{5}{12} = \frac{12}{12} = 1$

Stress what the pupils are adding:

**five** twelfths and **four** twelfths

Just as 5 sweets and 4 sweets make 9 sweets, or 7*d.* and 5*d.* mean 12*d.*, and so on, so 5 twelfths and 4 twelfths mean 9 twelfths. Give them simple concrete examples that they can verify.

(a) $\frac{3}{8}$ ft + $\frac{1}{8}$ ft + $\frac{5}{8}$ ft = $\frac{9}{8}$ ft = $1\frac{1}{8}$ ft
(b) $1\frac{1}{2}$ ft + $1\frac{1}{2}$ ft = 3 ft = 1 yd
(c) $\frac{3}{16}$ lb. + $\frac{5}{16}$ lb. = $\frac{8}{16}$ lb. = $\frac{1}{2}$ lb.
(d) $\frac{1}{4}$ hr × $\frac{3}{4}$ hr = $\frac{4}{4}$ hr = 1 hr

Impress upon the children that fractions of the same quantity and the same names are being added. With the help of their rulers let them add $\frac{5}{16}$ inches and $\frac{3}{16}$ inches and so on. Now take the case of $\frac{2}{3}+\frac{1}{4}$, both fractions of the same quantity but with different names or denominators. First they must be expressed as fractions of the same name. Any fraction to be equal to $\frac{2}{3}$ must have for its denominator a number that is a multiple of 3, and any fraction to be equal to $\frac{3}{4}$ must have as denominator a number that is a multiple of 4. To bring these fractions to the same name we must find a multiple common to 3 and 4. The multiples of 3 are—3, 6, 9, 12 . The multiples of 4 are—4, 8, 12 There is no need to go farther. 12 is the common multiple of 3 and 4. The problem now is to find the new numerators.

$$\frac{2}{3}=\frac{?}{12} \qquad \frac{3}{4}=\frac{?}{12}$$

The children have already worked examples like the above, and the sum becomes $\frac{8}{12}+\frac{9}{12}=\frac{17}{12}=1\frac{5}{12}$.

The children work a number of examples. Very often they can guess the common multiple. The denominators most used will be those found on ordinary rulers—for example, **halves, quarters, eighths, sixteenths**; **thirds, sixths, twelfths**; **fifths, tenths, twentieths** (for money); **hundredths** in the case of percentages. But children enjoy varied numbers. Give them hints on finding the L.C.M. by **inspection**. Suppose the denominators are 5, 2, 4, 10. They select the **largest** number, 10; 2 and 5 are contained in it, 4 is not. Find the next multiple of 10, 20; 20 is the L.C.M. Some children like to cross out 5 and 2.

The form $\frac{8+9}{12}=\frac{17}{12}$ should not be used for beginners, as it tends to obscure the real procedure. Each fraction should be shown separately in terms of the new denominator. It is best too when examples are being worked on the board, or when the children work them, for each equivalent to be written immediately below its counterpart.

In the case of mixed numbers, the whole numbers should be added first and then the fractions—thus:

$$\begin{aligned} & 2\tfrac{3}{8}+3\tfrac{4}{5}+1\tfrac{3}{10} \\ =\ & 6\tfrac{15}{40}+\tfrac{32}{40}+\tfrac{12}{40} \\ =\ & 6\tfrac{59}{40}=7\tfrac{19}{40} \end{aligned}$$

## The Subtraction of Fractions

The pupil who understands the process of adding fractions should have little difficulty in subtraction. Just as in the subtraction of whole numbers the procedure was "units from units, tens from tens," so now the procedure is "fraction from fraction, whole number from whole number." In subtractions involving the type $5\frac{1}{3}-\frac{2}{3}$, where a bigger fraction has to be taken from a smaller, it is better to adopt the Decomposition Method and write the exercise as $4\frac{4}{3}-\frac{2}{3}$, rather than the Equal Addition Method $5\frac{4}{3}-1\frac{2}{3}$. (See *The Primary School in Scotland*, Scottish Education Department, 1950.)

## Multiplication of Fractions

Remind the children that $5 \times 4$ really means $5+5+5+5$—that is, it is the **sum** of 4 numbers, each of which is 5. **Multiplying a fraction by a whole number** thus presents little difficulty:

$$\tfrac{3}{8}\times 5=\tfrac{3}{8}+\tfrac{3}{8}+\tfrac{3}{8}+\tfrac{3}{8}+\tfrac{3}{8}=\tfrac{15}{8}$$

or, more quickly, $\frac{3}{8}+5=\frac{3\times 5}{8}=\frac{15}{8}=1\frac{7}{8}$

Similarly, $\frac{3}{8}\times 7=\frac{3\times 7}{8}=\frac{21}{8}=2\frac{5}{8}$. Give them many examples.

Let them do some practical work with their rulers. Thus

3 times $\frac{3}{10}$ of an inch $=\frac{9}{10}$ of an inch, or $\frac{3}{10}$ inch $\times 3=\frac{9}{10}$

The line AB in Fig. 55 represents 2 inches, and is divided into 10 parts. Also AC $=\frac{3}{10}$, and AD $=3$ times AC $=\frac{3}{10}=3=\frac{9}{10}$.

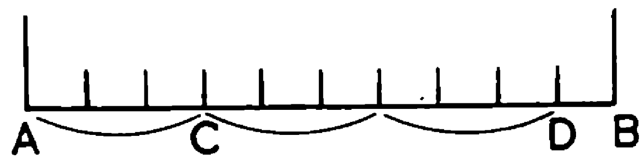



FIG. 55.

Another way to state the above is to say that:

$$AC = \tfrac{1}{3} \text{ of } AD, \text{ that is } \tfrac{3}{10} = \tfrac{1}{3} \text{ of } \tfrac{9}{10}$$

Interesting results are got from multiplying a fraction by a whole number; thus from the result that $\frac{3}{4} \times 4 = 3$ we get

$$\tfrac{1}{4} \text{ of } 3 = \tfrac{3}{4}$$

This shows the idea of $\frac{3}{4}$ in a new light which is of great importance. It can be explained by letting pupils share 3 bars of chocolate among 4 boys. Strips of paper represent the bars, conveniently marked off in quarters. The diagram Fig. 56 will make the new idea still clearer. So far the children have thought of $\frac{3}{4}$ of any quantity as 3 times the fourth part of that quantity. The result obtained above shows that we may also take $\frac{3}{4}$ of any quantity to mean the fourth part of three times that quantity.

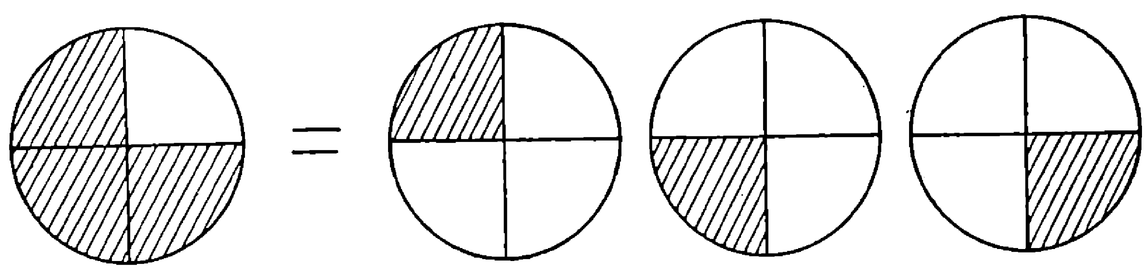


FIG. 56

*Multiplying a whole Number by a Fraction (for example,* $6 \times \frac{1}{3}$*).* Notice that $6 \times \frac{1}{3}$ is not a case of 'multiplication,' in any sense of the word as we have so far used it.

$6 \times 5$ *is* an addition sum—namely, $6 + 6 + 6 + 6 + 6$.
$6 \times 3$ *is* another sum—namely, $6 + 6 + 6$.
$6 \times \frac{1}{3}$ *is not* a sum.

The very idea of multiplication (see Chapter VIII) implies that the multiplier is a whole number, but the idea of a fractional multiplier is quite new. Is one likely to meet the combination $6 \times \frac{1}{2}$ in one's daily life? Suppose the cost of ribbon is 6*d.* per yard. The cost of five yards would be 6*d.* $\times$ 5; the cost of 3 yd is 6*d.* $\times 3$; and so on. The cost of any number of yards is 6*d.* multiplied by the number of yards. The same mode of expression is used if the quantities involved are fractions, so the cost of $\frac{1}{2}$ yd is 6*d.* $\times \frac{1}{2}$; the cost of $\frac{1}{3}$ yd is 6*d.* $\times \frac{1}{3}$. The cost of $\frac{1}{2}$ yd of ribbon is plainly 3*d.*, found by taking $\frac{1}{2}$ of 6*d.*; the cost of $\frac{1}{3}$ yd is 2*d.*, found by taking $\frac{1}{3}$ of 6d. It is a great advantage if we take $6 \times \frac{1}{2}$ and $6 \times \frac{1}{3}$ to mean $\frac{1}{2}$ of 6 and $\frac{1}{3}$ of 6. Similarly it is desirable to take $12 \times \frac{3}{4}$ to mean $\frac{3}{4}$ of 12, $15 \times \frac{3}{5}$ to mean $\frac{3}{5}$ of 15.

Children will not understand fully the above reasoning. The child's conception of all that the term 'multiplication' implies undergoes a change when he reaches multiplication by a fraction, and finds out that multiplication does not always result in a *larger* answer. It is probably sufficient to state to the children that to multiply any quantity by $\frac{2}{3}$ means to find $\frac{2}{3}$ of that quantity.

Give them plenty of easy mixed examples, such as $\frac{1}{3}$ of 9, $9 \times \frac{1}{3}$, $\frac{2}{3}$ of 9, $\frac{1}{2}$ of 7, $\frac{4}{5}$ of 15.

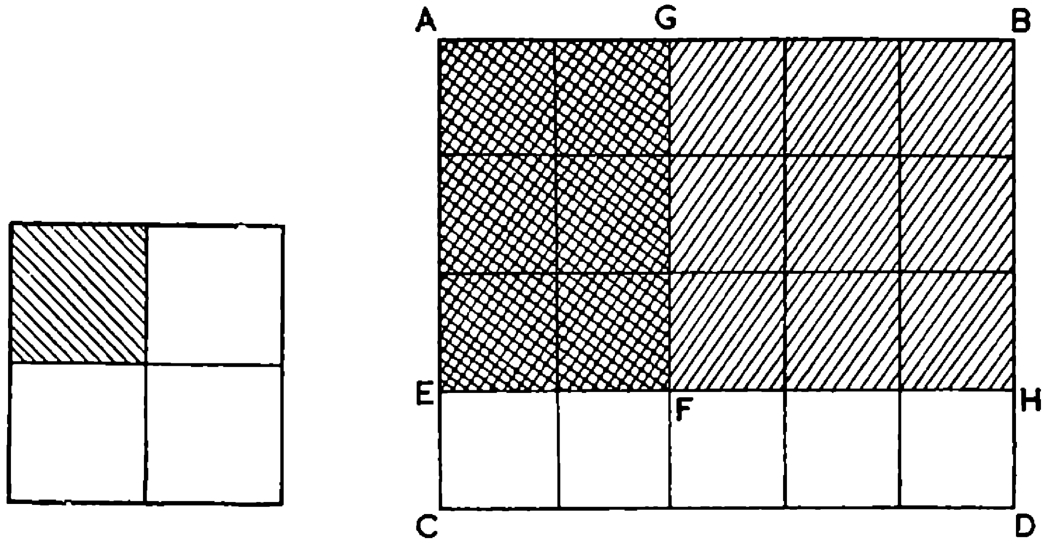
*Multiplying Fractions by Fractions.* To multiply $\frac{5}{6}$ by $\frac{2}{3}$ means to find $\frac{2}{3}$ of $\frac{5}{6}$.

$$\begin{aligned} \text{Thus, } \tfrac{5}{6} \times \tfrac{2}{3} &= \tfrac{2}{3} \text{ of } \tfrac{5}{6} \\ &= 2 \text{ times } \tfrac{1}{3} \text{ of } \tfrac{5}{6} \\ &= 2 \text{ times } \tfrac{5}{3 \times 6} \\ &= \tfrac{2 \times 5}{3 \times 6} = \tfrac{10}{18} = \tfrac{5}{9}. \end{aligned}$$

$$\text{More simply, } \tfrac{5}{6} \times \tfrac{2}{3} = \tfrac{5 \times 2}{6 \times 3} = \tfrac{10}{18} = \tfrac{5}{9}.$$

Thus the children see that, to multiply fractions by fractions, one multiplies together the numerators, and then the denominators, to make a *new* fraction. As with all fractions, the answer must be given in its lowest form or terms.

Fig. 57 (*a*) and (*b*) illustrates the above process. (*a*) $\frac{1}{2}$ of $\frac{1}{2} = \frac{1}{4}$ is obvious. In (*b*) the diagram shows exactly how $\frac{2}{5}$ of $\frac{3}{4}$ is found. A rectangular figure ABCD is drawn. It is divided into 5 equal parts by vertical lines, and into 4 equal parts by horizontal lines.




FIGS. 57 (*a*) AND (*b*)

Then clearly ABEH $= \frac{3}{4}$ of ABCD
also AGEF $= \frac{2}{5}$ of ABEH
$= \frac{2}{5}$ of $\frac{3}{4}$ of ABCD

The part AGEF may be expressed as $\frac{2\times 3}{5\times 4}$ of the whole ABCD, for the whole rectangle is divided into $5 \times 4$ or 20 equal parts, and AGEF contains $2 \times 3$ or 6 of them. Hence $\frac{2}{5}$ of $\frac{3}{4} = \frac{2\times 3}{5\times 4} = \frac{6}{20} = \frac{3}{10}$.

*Multiplying Mixed Numbers* (*for example*, $4\frac{2}{3} \times 3\frac{3}{4}$). The simplest way is to express $4\frac{2}{3}$ and $3\frac{3}{4}$ as improper fractions, and then to apply the general multiplication rule—multiply numerators together, and denominators likewise, to form a new fraction.

$$3\frac{2}{3} \times 1\frac{3}{4} = \frac{11}{3} \times \frac{7}{4} = \frac{77}{12} = 6\frac{5}{12}.$$

In the case of $\frac{3}{5} \times \frac{2}{3} = \frac{3\times 2}{5\times 3} = \frac{6}{15} = \frac{2}{5}$ clearly it would have been quicker to have removed the common factor 3 before multiplication.

In $1\frac{7}{11} \times 2\frac{4}{9} = \frac{\overset{2}{\cancel{18}}}{\underset{1}{\cancel{11}}} \times \frac{\overset{2}{\cancel{22}}}{\underset{1}{\cancel{9}}} = 4$, the cancellation marks explain themselves. At first it is better not to use the word **cancel.** Impress upon the child that he must, where possible, divide **numerator** and **denominator by the same numbers**, and in every case he must put the quotient he gets above or below the number, even when the quotient is 1.

This will help him to avoid blunders such as $\frac{3}{4} \times \frac{4}{3} = 0$.

## Division involving Fractions

*The Division of a Fraction by a Whole Number.* If a piece of ribbon $\frac{5}{8}$ yd long is divided into 3 equal parts, what is the length of each part? From a practical point of view the length of each part is $\frac{1}{3}$ of $\frac{5}{8}$ of a yard. But we already know that $\frac{1}{3}$ of $\frac{5}{8}$ is the meaning of $\frac{5}{8} \times \frac{1}{3}$. Hence

$$\tfrac{5}{8} \div 3 = \tfrac{5}{8} \times \tfrac{1}{3} = \tfrac{5}{24}.$$

The exercise $17\frac{1}{2} \div 4$ may be dealt with in the same way:

$$17\tfrac{1}{2} \div 4 = \tfrac{35}{2} \div 4 = \tfrac{1}{4} \text{ of } \tfrac{35}{2} = \tfrac{35}{2} \times \tfrac{1}{4} = \tfrac{35}{8} = 4\tfrac{3}{8}.$$

*The Division of a Fraction by a Fraction*—for example, $\frac{2}{3} \div \frac{3}{8}$. This may be interpreted to mean, "How many pieces of ribbon each $\frac{3}{8}$ yd long can be cut from a length $\frac{2}{3}$ yd long?" We must compare these two lengths so that we can see how many times the second length is contained in the first. To do this we bring the fractions to the same denominators, thus:

$$\tfrac{2}{3} \div \tfrac{3}{8} = \tfrac{16}{24} \div \tfrac{9}{24}.$$

The problem now becomes, "How many lengths, each $\frac{9}{24}$ of a yard, can be cut from a length $\frac{16}{24}$ of a yard?" The answer is clearly found by dividing 16 by 9, thus:

$$16 \div 9 = \tfrac{16}{9} = 1\tfrac{7}{9}.$$

Put down briefly in fractional form, the exercise becomes:

$$\tfrac{2}{3} \div \tfrac{3}{8} = \tfrac{16}{24} \div \tfrac{9}{24} = \tfrac{16}{9} = 1\tfrac{7}{9}.$$

The children see that $\dfrac{16}{24} \div \dfrac{9}{24}$ is really $\dfrac{16}{\cancel{24}_{1}} \times \dfrac{\overset{1}{\cancel{24}}}{9} = 1\tfrac{7}{9}$.

From the above examples, they find the general rule: **To divide by a fraction, multiply by the fraction inverted.**

Remind them that this rule covers the case when the divisor is a whole number:

$$\frac{5}{8} \div 4 = \frac{5}{8} \times \frac{1}{4}.$$

It helps children if the whole number is written as an improper fraction, $4 = \frac{4}{1}$. Let them work many examples.

Here are three further examples to show how the general rule above may be interpreted in terms of concrete things.

(1) *Dividing a Whole Number by a Fraction.* This has not yet been considered. Take the example $9 \div \frac{3}{4}$. This may be taken to mean, "How many pieces each $\frac{3}{4}$ yard long can be cut from a 9-yard length?"

Let the pupils imagine that the 9-yard length is marked off into quarter-yards. Then clearly in 9 yards there are $9 \times 4$ quarters—that is, 36 quarter-yards. The problem now is to find how many pieces, each $\frac{3}{4}$ yd long, can be cut from 36 quarter-yards. It is easy to see that the required number is found by dividing 36 by 3.

Hence $9 \div \frac{3}{4} = \frac{9}{1} \times \frac{4}{3} = \frac{36}{3} = 12$.

(2) *Dividing a Mixed Number by a Fraction—e.g.*, $3\frac{3}{4} \div \frac{2}{5}$. This example is worked in the same way as the one above. How many lengths, each $\frac{2}{5}$ yard long, can be cut from $3\frac{3}{4}$ yards? Changing the mixed number $3\frac{3}{4}$ to an improper fraction $\frac{15}{4}$, the problem becomes

$$\frac{15}{4} \div \frac{2}{5} = \frac{15}{4} \times \frac{5}{2} = \frac{75}{8} = 9\frac{3}{8}.$$

(3) *Dividing Smaller Fractions by Greater—e.g.*, $\frac{2}{7} \div \frac{3}{4}$. This is an interesting example, because the divisor is greater than the dividend. $\frac{3}{4}$ is clearly bigger than $\frac{2}{7}$. We cannot use the interpretation given previously. We could not find how many $\frac{3}{4}$ yd there were in $\frac{2}{7}$ of a yard. In the case $3 \div 4$, $3 \div 4$ is the fraction that 3 is of 4—namely, $\frac{3}{4}$. $7 \div 14$ is the fraction that 7 is of 14—namely, $\frac{7}{14} = \frac{1}{2}$. Therefore the question we ask in the case of $\frac{2}{7} \div \frac{3}{4}$ is, "How does $\frac{2}{7}$ compare with $\frac{3}{4}$?" meaning, "What fraction of $\frac{3}{4}$ is $\frac{2}{7}$?" As before, for comparison bring the fractions to the same denominator, $\frac{8}{28} \div \frac{21}{28}$. The

question now is, "What fraction is $\frac{8}{28}$ of $\frac{21}{28}$?" and the answer is plainly the same fraction that 8 is of 21—namely, $\frac{8}{21}$. So:

$$\frac{2}{7} \div \frac{3}{4} = \frac{2}{7} \times \frac{4}{3} = \frac{8}{21}$$

## Easy Percentages

A percentage is a special kind of fraction whose denominator is always 100. Thus $\frac{21}{100}$ is 21 per cent., or 21 out of a hundred.

$\frac{27}{200}$ is $13\frac{1}{2}$ per cent., or $13\frac{1}{2}$ out of a hundred. Express also as $\frac{13\frac{1}{2}}{100}$.

The children know that to compare fractions they must be expressed as equivalent fractions with the same denominators. If we wish to arrange a series of given fractions in order of size, we bring them all to the same denominator; from these the required order is at once attained.

Let the children arrange these groups of fractions in order of size:

(*a*) $\frac{3}{8}$ $\frac{1}{2}$ $\frac{3}{4}$ $\frac{2}{3}$ $\frac{5}{6}$ (*b*) $\frac{1}{4}$ $\frac{3}{10}$ $\frac{4}{5}$ $\frac{7}{8}$

Set (*a*) will have a different common denominator from set (*b*).

If afterwards we have to arrange another set of fractions in order of size, it is unlikely that the same denominators for (*a*) or (*b*) will serve. If all three sets of fractions are to be arranged in order in one list, the working already carried out is of little use, and a new denominator has to be found. Clearly it would be very convenient when a great many comparisons are being made to have a 'universal denominator' in terms of which any fraction can easily be expressed. The number 100 has been chosen for this purpose. Children find it interesting to express any given fraction as a percentage. Fractions with denominators which are factors of 100 present little difficulty. For example:

$\frac{1}{2} = \frac{50}{100} = 50$ per cent. $\frac{1}{4} = \frac{25}{100} = 25$ per cent.
$\frac{3}{10} = \frac{30}{100} = 30$ per cent. $\frac{6}{20} = \frac{30}{100} = 30$ per cent.

Now we must consider converting *any* fraction to a percentage.

Here is a general process to meet all cases. Let us convert $\frac{5}{8}$ to a percentage:

$$\frac{5}{8}=\frac{\frac{5}{8}}{1}=\frac{\frac{5}{8}\times 100}{100}=(\tfrac{5}{8}\times 100)\text{ per cent.}$$
$$=\tfrac{500}{8}\text{per cent.}=62\tfrac{1}{2}\text{ per cent.}$$

Examination of the above process shows at once a quick way of converting any vulgar fraction to a percentage—namely, multiply the given fraction by 100, thus:

$$\tfrac{11}{15}=(\tfrac{11}{15}\times 100)\text{ per cent.}$$
$$=73\tfrac{1}{3}\%.$$

Little beyond an introduction to percentage should be taken in the Junior School. Some Juniors are quite ready for it, and one must consider the intelligent children who are capable of working on their own, and have a real gift for mathematics. Children in the Junior School may be shown that a percentage is a fraction in which the denominator is always 100. In the business world $\frac{3}{100}$ is not called three hundredths (a difficult word to say) but 3 per cent., from the Latin words *per* (that is, for each), and *centum* (a *hundred*). Per cent. is often shown by the sign, %. Very simple examples may be given: What percentage is 6*d*. of one shilling? The children may reason that 6*d*. is $\frac{1}{2}$ of a shilling, and that $\frac{1}{2}$ is $\frac{50}{100}$, or 50%, or they can find the percentage thus: $\frac{1}{2}\times 100=50\%$.

Some practical examples, such as the following, should also be given. There are 45 names on the class register, and five pupils were absent. What part of the class is away? What percentage of the class is away?

$$(\tfrac{5}{45}\times 100)\text{ per cent.}=11\tfrac{1}{9}\%$$

Children by means of percentage compare the number of children absent in different classes. (For percentage and comparing fractions, see also Chapter XIV.)

CHAPTER XIV

# DECIMAL FRACTIONS

*Place-value again and the Use of Noughts. Introduction to Decimals. Multiplication and Division of Decimals by* 10, 100, 1,000, *etc. The Metric System. Addition and Subtraction of Decimal Fractions. Multiplication of Decimals: the Direct Method. Multiplication of Decimals by the Popular or Traditional Method. Multiplication of Decimals by Standard Form Method. Simple Division Exercises involving Decimals. Long Division with the Divisor Whole Numbers. Dividing a Decimal Fraction by a Decimal Fraction. Changing Decimal Fractions to Vulgar Fractions. Changing Vulgar Fractions to Decimal Fractions. Comparison of Decimal Fractions.*

## Place-value again and the Use of Noughts

If children understand place-value, and if the numbers are kept simple, they should find little difficulty in learning 'decimal fractions.' They should know well that every digit or figure has a meaning or value according **to its position.** They study the diagrams below.

| H | T | U | t | h |
|---|---|---|---|---|
| 4 | 4 | 4 | 4 | 4 |

(*a*)

| H | T | U | t | h |
|---|---|---|---|---|
| 4 | 0 | 0 | 4 | 0 |

(*b*)

A figure in any of the above compartments has **ten times** the value of the figure in the compartment immediately on the right. Thus the 4 to the left of the units place is 4 tens. The 4 in the next place to the left is 4 hundreds—see (*a*) and (*b*). In general a figure in any compartment has 10 times the value of the same digit occupying the place immediately to the **right**. We cannot decide on the value of a figure unless the **unit figure** is clearly marked. Now, if a figure is put to the **right** of the unit place, it must be 10 times smaller than the unit figure—that is, it must be 4 tenths or $\frac{4}{10}$. The diagram below may help to explain this:

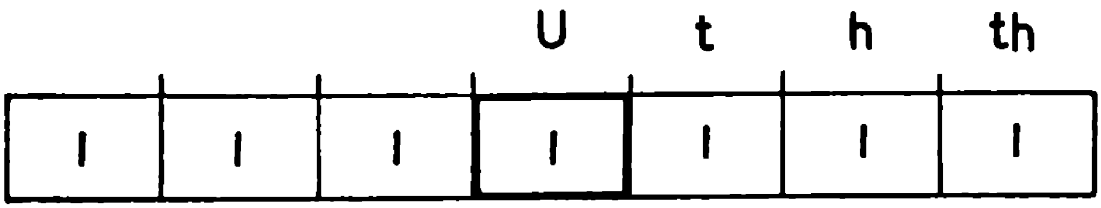



FIG. 58

The 1 in the compartment to the **right** of the units compartment is a tenth of the 1 in the units place, hence it represents 1 tenth. In the same way the 1 in the compartment marked *h* to the right of the compartment *t* is a tenth of a tenth, or 1 hundredth. Similarly the 1 in the compartment *th* represents a tenth of a hundredth, or 1 thousandth, and so on. The children should understand clearly that the places to the **right** of the units are kept for **tenths**, **hundredths**, **thousandths**, etc., just as those to the left are kept for tens, hundreds, thousands. So far children have been concerned with the value of figures to the **left** of the unit place, so they need practice in reading figures to the right of the units place.

One needs to stress the importance of the **unit place** because when the 'point' is introduced they tend to think there is a big gap between units and **tenths** and **no gap** between units and **tens**. It must be emphasized that there is no discontinuity between whole numbers and fractions: $2\frac{3}{10}$ is $\overset{U}{2}3$. Let them read some numbers written as below:

| | | | |
|---|---|---|---|
| U t h | T U t h | U t h th | H T U t |
| 2 3 4 | 3 0 1 2 | 0 1 5 4 | 2 4 3 4 |

## Introduction to Decimals

In this way we have just described children who are really familiar with the system of decimal notation as applied to whole numbers will readily understand the same system extended to include fractions. The fractions denoted by the figures to the right of the units are called "decimal fractions" because they are based on ten, and **decimal** means tenths. Point out again to the children the function of zero as a **place-holder** only. Thus the 0's in $\overset{\mathrm{U}}{0}023$ have no value in themselves, but they are *essential* to show the units place. Although the U on top to show the units place is extremely helpful, it is obvious to the children that some smaller, simpler mark is needed. The custom is to put a dot on the right-hand side of the unit to point it out; thus 28·1, where 8 is the unit. Obviously the point must not be separated from the unit, and must be as near to it as possible. It is the **unit place** that is important, not the point. Mathematicians always write 0·23 instead of just ·23, because, as the real use of the point is to *indicate* which is the unit figure or place, there must be some figure or 0 to the left of the point. Children are generally very keen to put the 0 if necessary in the units place when they hear that "real mathematicians do."

Let the children measure lines of different lengths with rulers marked in inches and tenths. They write down their measurements—for example, $4\frac{3}{10}$, $2\frac{5}{10}$, $11\frac{8}{10}$, $7\frac{7}{10}$, $\frac{6}{10}$, etc., in the new and simple notation—4·3, 2·5, 11·8, 7·7, 0·6, etc., inches.

## The Metric System

Link up decimals with the Metric System of measurement. The **metre** is the standard unit; compared with our units it is about $38\frac{3}{8}$ inches, or rather more than 1 yard. It is used in general where we should use the yard. The metre is divided into tenths (decimetres), hundreds (centimetres), thousandths (millimetres). The children see how easy it is to change one unit to another. Then there are larger multiples of the metre; 10 metres are 1 dekametre, 100 metres are 1 hectometre, and 1,000 metres 1 kilometre. Children need not be troubled with the higher multiples, but they should be able to compare a kilometre with a mile. If metre rulers cannot be had, strips of stiff paper a metre long can be pinned up in the classroom (as suggested for the foot and yard). Good 12-inch rulers usually show graduations for the smaller metric units. Let them prove that 1 inch is approximately equal to 2·54 centimetres. They read different lengths—2·34 metres is 2 metres, 3 decimetres, 4 centimetres; 23·4 metres, etc. Change 234 decimetres to (*a*) metres (23·4 m.) (*b*) centimetres (2,340 cm), and so on. All reduction exercises are simply multiplying by 10, 100, 1,000.

Explain that, although 4·3 really means 4 and 3 tenths, it is usually read as "4 point 3." Mixed numbers like the above, and fractions as 0·6, are usually called "decimals."

Intelligent children may see that all numbers, whether wholes or fractions, may be called decimals, because each number is ten times bigger or ten times smaller than its neighbour on the right or left respectively. The figures to the right of the units are decimal fractions.

Let the children write some fractions, as below:

$$4\tfrac{23}{100},\quad 12\tfrac{3}{100},\quad \tfrac{4}{100},\quad \tfrac{34}{100}$$

Then rewrite them in the new notation:

$$4{\cdot}23,\quad 12{\cdot}03,\quad 0{\cdot}04,\quad 0{\cdot}34$$

The first decimal is 4 units, 2 tenths, 3 hundredths, and is read "four point two three," and *never* "four point twenty-three." Let the children write the above decimal fractions thus:

$$4{\cdot}23 = 4 + \tfrac{2}{10} + \tfrac{3}{100} = 4\tfrac{23}{100}$$

Lead them to see from the above that any 'two-place' decimal fraction may be expressed as a number of hundredths, a fact which will be of great value to them later on. Examples are:

$$
\begin{aligned}
23{\cdot}34 &= 2334 \text{ hundredths}\\
3{\cdot}54 &= 354 \text{ hundredths}\\
0{\cdot}54 &= 54 \text{ hundredths}\\
0{\cdot}04 &= 4 \text{ hundredths}
\end{aligned}
$$

If they are doubtful about 23·34 being equal to 2334 hundredths, let them prove it:

$$2334 \text{ hundredths} = \tfrac{2334}{100} = 23\tfrac{34}{100} = 23{\cdot}34$$

Some teachers may like to link up percentage with the work being done with decimal fractions. This may well be done with bright children. Let them find the connexion between the three types of fractions, using simple examples as:

$$5\% = \tfrac{5}{100} = 0{\cdot}05; \qquad 30\% = \tfrac{30}{100} = \tfrac{3}{10} = 0{\cdot}3$$

30 hundredths could equally well be written ·30. The affixing of a 0 at the end of a decimal fraction, because it does not affect the place of any of the other digits, leaves the value of the decimal unchanged. Compare affixing a 0 or 0's to the last or end figures of a whole number, thus: $00\overset{U}{3}{\cdot}30$, $03\overset{U}{4}{\cdot}300$.

## Addition and Subtraction of Decimal Fractions

These exercises conform to all the rules of procedure that the children have learnt in connexion with whole numbers. Give the children examples as in (*a*), (*b*), (*c*):

| (*a*) | (*b*) | (*c*) |
|---|---|---|
| 13·7 | 22·82 | TUt |
| 6·5 | 6·06 | 324·3 |
| 0·4 | 17·7 | 176·74 |
| 7·5 | 8· | —— |
| —— | —— | |
| | | —— |
| —— | —— | |

There may be, in the case of the first sum (*a*), the danger of the children making two sums, one for the 'decimals' and one for the whole numbers. This is because of the 'point' column. It helps the children if they name the columns T U t at first. They begin by extending the points to the answer space below, as shown. With whole-number addition the children are familiar with the irregular left-hand edge of the column as in (*a*), but when there is irregularity on both edges, as (*b*), some children may be a little disconcerted. In this case it may be suggested to them that 0 can be inserted in the gaps without altering the value of the decimal fraction. After a few examples, they will soon see the 0's can be dispensed with, as they are not **place-holders.** In the subtraction exercises (*c*), a pupil may be puzzled by having *literally* 'nothing' from which to subtract 4. Here again let him insert a 0. He soon learns that this device is unnecessary.

### Multiplication and Division of Decimals by 10, 100, 1,000, etc.

If children understand place-value, they will find no difficulty in multiplying by 10, 100, etc. Remind them that, if a figure is moved one place to the left, it becomes ten times bigger—that is, it is multiplied by ten, as in the example below, where the whole set of figures has been moved one place to the left.

U
23·42 × 10
=234·2

In the movement the 2 tens have become 2 hundreds, the 3 units have become 3 tens, the 4 tenths have become 4 units, and the 2 hundredths have become 2 tenths. Hence we have the rule for multiplying by 10. Move the figures of the number one place to the left. It follows that in multiplying by 100 we move the digits two places, and by 1,000, three places, and so on. (See Chapter VIII.)

As multiplying by ten means **promoting** the figures one place to the **left**, so dividing by ten means reducing the figures by moving them one place to the right. In the example below multiplication and division are compared:

| | |
|---|---|
| 32·54 × 10 | 32·54 ÷ 10 |
| =325·4 | =3·254 |

Examples such as the following may be taken:

| | | | |
|---|---|---|---|
| (*a*) | 3·276 ÷ 10<br>=0·3276 | (*b*) | 0·023 ÷ 10<br>=0·0023 |
| (*c*) | 562 ÷ 10<br>=56·2 | (*d*) | 6·0 ÷ 10<br>=0·6 |

(*c*) and (*d*) above are examples of whole numbers divided by 10. It may help children if the 'point' is placed to the right of the unit figure 'to point it out'; thus 562 is written 562·. Let the children revise division of whole numbers by 10, 100, etc., in Chapter IX. The simple rule for dividing by 10 explained above could not be given in that chapter because there was no 'place' into which the amount over could go. Now the children have learnt new places, tenths and hundredths, a want is satisfied. Instead of putting 2 as a remainder in 562 ÷ 10 and not dividing or sharing it, the dividing and sharing can be completed, and 2 goes into the tenths column, as 56·2.

## Multiplication of Decimals: the Direct Method

If children understand place-value, they can multiply right away without altering the position of the point. Indeed, it is a good test of their understanding of place-value. Here are some examples:

(1) *Multiplying a Decimal by the Numbers from* 2 *to* 9, *inclusive.* This is easy, but the children must be careful to put the unit multiplier under the unit figure in the multiplicand. All the figures remain in the same places.

```
 31·24
  4·
------
124·96
```

(2) *Multiplying a Decimal by* 20, 30, *etc.* The children already know how to multiply by 10, 100, etc., but they should practise multiplying by 20. 30, 40, 200, 300, etc.

```
 31·24      31·24
 20·       200·
------    -------
624·8     6248·00
```

To multiply by 20, they multiply by 2, at the same time moving each figure one place to the left to multiply by 10. In the case of 200 the figures of the result are moved 2 places to the left. It may help the children if two 0's are put in to mark the tenth and hundredth place as above.

(3) *Multiplying a Decimal by a Decimal Fraction, such as* 0·3 *and* 0·03. They notice the multiplier 3 is in the **tenth place,** it is $\frac{3}{10}$, so they multiply by 3, at the same time moving each figure one place to the right to reduce it. This the children often find difficult at first, because they are so used to moving figures to the left.

```
31·24
 0·300
------
 9·372
```

It may help them to put the figures in the right places if 0's are inserted after ·3 in the hundredths and thousandths places. The children say, "3 times 4 are 12, the 2 goes one place to the right—that is, into the thousandths place." The other figures fit into their correct places if the first one is right.

In the case of the next example, where the figures have to be moved two places to the right, it is again a help if the two places to the right of ·03 are filled with 0's.

```
31·24
 0·0300
-------
 0·9372
```

It is essential that the children keep the point column straight and the figures exactly under each other.

(4) *Long Multiplication of a Decimal by a Decimal.* To help the children to keep the figures in their right places, let them

carry the 'point column' down to the answer space; as there will be three lines of partial products, this will be some distance.

```
  31·44
  23·4
 -------
 628·8
  94·32
  12·576
 -------
 735·696
```

As with ordinary long multiplication, they begin with the digit of highest order, the 2; they note it is in the tens place, and, as they multiply by 2, they promote the digits of the result one place. The next figure, 3, is in the units' place, and there is neither promotion nor reduction. The next figure, 4, is in the tenths place, so they multiply by 4 and reduce the digits of the product one place by moving them one place to the right. Having worked several exercises, the children see how important it is to have the first line correct. The first figure of the first line of working is the critical one, 8 in the last example. If this figure is placed correctly, the rest of the work is almost automatic, as in the multiplication of whole numbers. Give children plenty of examples to work, such as the following:

1·74 × 2·3, 2·83 × 1·35, 5·99 × 25·4, 17·83 × 15·9.

## Multiplication of Decimals by the Popular or Traditional Method

This is the method most used in schools, and the easiest method of all. The children rewrite the multiplicand and the multiplier as though they were whole numbers. Then multiply in the ordinary way. Lastly they place the decimal point in the answer so that there are as many digits to the right of it as there are in the original multiplicand and multiplier. See example below:

$$31{\cdot}44 \times 23{\cdot}4 \qquad \begin{array}{r} 3144 \\ 234 \\ \hline 628800 \\ 94320 \\ 12576 \\ \hline 735696 \end{array} \div 1000 = 735{\cdot}696$$

In this method the decimal point is suppressed altogether, and only reappears in the answer. The partial products have not much meaning as they stand. Exercises such as the above give the children plenty of practice in **long multiplication**, but they do not help them to understand place-value or decimal fractions.

However, the method can be justified thus:

$$31{\cdot}44 \times 23{\cdot}4 = 31\tfrac{44}{100} \times 23\tfrac{4}{10} = \tfrac{3144}{100} \times \tfrac{234}{10}$$

Here we have to multiply a whole number by a second whole number, 3,144 × 234, and divide it by 1,000. This is exactly what was done in the example worked above. The numbers were multiplied together **as whole numbers**, then divided by 1000. It is most important that the children write the numbers down as whole numbers without the points; otherwise dividing by a thousand makes no sense.

## Multiplication of Decimals by Standard Form Method

This is perhaps best left for the Senior School, but if children have a real grasp of place-value they will be interested. It is a useful method for finding an approximate answer. The easiest decimals by which to multiply are those with one figure in the units place, as 3·012. If the multiplier is a unit, each digit of the product is placed exactly under the figure being multiplied, as in (*a*) below.

$$(a) \quad \begin{array}{l} 0{\cdot}00327 \\ 3{\cdot}23 \quad \downarrow \\ \hline 0{\cdot}00981 \\ 0{\cdot}000654 \end{array}$$

The rest of the working causes practically no difficulty, as all the figures fall into their right places with the help of the first line. The 4 of ·2 times 7 comes one place to the right of the line. A number with a figure in the units column followed by decimal fractions is said to be in "standard form," a convenient and recognized form. In an exercise such as 22·42 × 35·4, the multiplier may be put in standard form by dividing it by 10. Thus 35·4 ÷ 10 becomes 3·54, but it is now ten times smaller, and will make the answer ten times smaller. We must remedy this by multiplying the other number, 22·42, by 10, so that it becomes 224·2. The exercise becomes 224·2 × 3·54, as below.

```
224·2
  3·54
------
672·6
112·10
```

Either of the two numbers can be chosen as multiplier and put in standard form, for the children know that 3 × 4 = 4 × 3, so that we can exchange multiplier and multiplicand. It is best to put the number that needs *less* alteration in the required standard form, and to alter the other accordingly. Thus in the exercise 72·3 × ·0345 it is best to put 72·3 in standard form.

## Simple Division Exercises involving Decimals

First let the children apply their knowledge of decimal fractions to ordinary division exercises—*e.g.*, 26 ÷ 4. If they keep to **whole** numbers only, the answer is "6 and 3 over." If vulgar fractions are used, the answer may be given as $6\frac{2}{4}$, or $6\frac{1}{2}$. In order to find the corresponding decimal fraction, the pupil may proceed like this, saying, "4's in 26, 6 and 2 over," writing down the 6 as in ordinary division.

```
     6·5
4 ) 26·0
```

He now affixes a 0 after the point in the dividend; the remaining 2 units are changed to 20 tenths. Then he proceeds, "4's in 20, 5," writing down the 5 in the tenths place.

Here are some further examples of short division:

(1) 352·8 ÷ 9

$$\begin{array}{r} 39{\cdot}2 \\ 9\,\overline{)\,352{\cdot}8} \end{array}$$

The division is worked in the usual way as far as the units columns. Here there is a remainder of 1 unit. This remainder is reduced to 10 tenths and added to the 8 tenths to make 18 tenths in all. 9's in 18, 2. The 2 goes in the tenths place.

(2) 69 ÷ 8

In this case the dividend does not divide exactly by 8 until after the tenth place. "8's in 69, 8 and 5 over." Change or reduce the 5 units to 50 tenths. 8's in 50, 6 and 2 over. Reduce or change the 2 tenths to 20 hundredths. 8's in 20, twice and 4 hundredths over, reduce to 40 thousandths. 8's in 40, 5 exactly.

$$\begin{array}{r} 8\ \ \ 6\ 2\ 5 \\ 8\,\overline{)\,6\ 9\ {\cdot}\,{}^{5}0\,{}^{2}0\,{}^{4}0} \end{array}$$

Noughts are perhaps best affixed to the dividend before the division begins. It makes it easier for children. The point must be placed in the quotient vertically above its position in the dividend. The children soon see the process is exactly the same as for ordinary short division except that 0's to any extent can be affixed to the dividend to the right of the decimal point.

(3) 60·12 ÷ 3

This example is easy, except that children tend to write 4 instead of ·04.

$$\begin{array}{r} 20{\cdot}04 \\ 3\,\overline{)\,60{\cdot}12} \end{array}$$

Varied examples are important to test understanding.

## Long Division with the Divisors Whole Numbers

The process is as above, but the long division form is necessary as in (*a*).

```
(a)          13·61
       21 ) 285·81
            21
            ---
             75·
             63
            -----
             12·8 tenths
             12·6
            -----
               ·21 hundredths
               ·21
              ----
```

Strictly speaking, the point should be brought down throughout all the workings, but in practice it is enough if it is shown clearly in the dividend and quotient. It is essential that the point be put at once in the quotient, otherwise the children may not keep the figures in their right places in the quotient.

```
(b)          0·67
       35 ) 23·485
            21·0
            -----
             2·48
```

In the case of (*b*) the **divisor** is **greater** than the whole numbers in the dividend. A 0 is put in the units place in the quotient, the remaining 23 units are reduced to 230 tenths + 4 tenths—that is, to 234 tenths. 35's in 234, 6, and the work proceeds in the usual way.

So far the children have worked exercises in which the division comes to an end after a few steps. They soon find there are exercises where "the division goes on for ever." When they are working exercises for practice they may be told to continue the division for a certain number of decimal places only—say three. The children may see for themselves that a stage can be reached where the remainder is so small as to be negligible. This leads to approximate quotients.

### Dividing a Decimal Fraction by a Decimal Fraction

So far all the divisors have been whole numbers. If the divisor is a decimal, the best method to use is to make the divisor a whole number. Take, for example, $3{\cdot}25 \div 0{\cdot}4$. This equals $3{\cdot}25 \div \frac{4}{10} = 3{\cdot}25 \times \frac{10}{4} = \frac{32{\cdot}5}{4} = 8{\cdot}125$. In practice it is simplest for children to rewrite the exercise in fractional form, $\frac{3{\cdot}25}{0{\cdot}4}$. They know that the value of a fraction is not altered if its numerator and denominator are each multiplied by the same number, such as 10. Thus $\frac{3{\cdot}25}{0{\cdot}4}$ becomes $\frac{32{\cdot}5}{4{\cdot}}$. The division is then carried out.

In the case of $0{\cdot}562 \div 3{\cdot}42$, or $\frac{0{\cdot}562}{3{\cdot}42}$, to make the divisor a whole number the numerator and denominator must each be multiplied by 100 and become $\frac{56{\cdot}2}{342{\cdot}}$. The division proceeds in the same way as before, correct to three decimal places, as below.

```
       00·164
342 ) 56·2ØØ
      34·2
      ------
      22·00
      20·52
      ------
       1·480
       1·368
      ------
```

Here is the rule of procedure: Write the exercise in fractional form, 'point' over 'point.' Multiply the divisor by the power of 10 that will make it a whole number. Multiply the dividend by the same power of ten. Then divide, keeping point under point.

## Changing Decimal Fractions to Vulgar Fractions

There is little difficulty in changing decimal fractions to vulgar fractions.

$0{\cdot}5 = 5$ tenths $= \frac{5}{10} = \frac{1}{2}$

$0{\cdot}75 = 75$ hundredths $= \frac{75}{100} = \frac{15}{20} = \frac{3}{4}$

$0{\cdot}135 = 135$ thousandths $= \frac{135}{1000} = \frac{27}{200}$

$0{\cdot}005 = 5$ thousandths $= \frac{5}{1000} = \frac{1}{200}$

Notice there are as many 0's in the denominators as there are decimal places.

## Changing Vulgar Fractions to Decimal Fractions

This is easy if the denominator is 10 or a multiple of 10, as

$$\tfrac{2}{10} = 0{\cdot}2,\ \tfrac{9}{100} = 0{\cdot}09.$$

There is no difficulty also if the fraction can be easily expressed as an equivalent fraction with a denominator which is 10 or a multiple of 10, as

$$\tfrac{1}{2} = \tfrac{5}{10} = 0{\cdot}5;\quad \tfrac{7}{20} = \tfrac{35}{100} = 0{\cdot}35;\quad \tfrac{2}{5} = \tfrac{4}{10} = 0{\cdot}4.$$

In the case of a fraction that cannot be expressed as an equivalent fraction with a denominator that is a power of 10 —for example, $\frac{3}{7}$—let the pupil imagine he has divided a number by 7 and has a remainder 3. He can express this as $\frac{3}{7}$, 3 still to be divided by 7, or $\frac{1}{7}$ of 3, or complete the division by changing $\frac{3}{7}$ into a decimal fraction thus:

```
7 ) 3·000
   ------
   0·428 . . .
```

Therefore $\frac{3}{7}$ equals approximately 0·428....

Here is an example where the divisor works out exactly $\frac{13}{40} = 0{\cdot}325$.

```
         0·325
40 ) 13·000
     12·0
     ----
      1·00
      0·80
      -----
      0·200
      0·200
```

## Comparison of Decimal Fractions

Remind the children that ordinary vulgar fractions are compared by bringing them to the same name or denominator. This is often tedious, and if another fraction has to be compared with them a new common denominator has to be found. Give them some vulgar fractions to compare. Decimal fractions are not only easy to compare but they give a definite impression of size. Suppose these fractions have to be compared: 0·023, 0·35, 0·142, 0·2. This can be done by arranging them in descending order of size. First write the fraction down in a column, so that the decimal points will be in a vertical line. Fill up any gaps with 0's, so that each has the same number of figures after the point.

0·023
0·350
0·142
0·200

Then read off the thousandths, thus: 23 thousandths, 350 thousandths, 142 thousandths, 200 thousandths. Clearly in descending order of size they read: ·35, ·2, ·142, ·023. With just a little practice in decimals, comparisons become almost as simple as the comparison of whole numbers.

It is worth while spending time over decimals in the light of the future, for decimals tend to be used more and more in measurements of all kinds. The children should find the decimal equivalent of the more familiar vulgar fractions, and in some cases know them by heart, for example:

$\frac{1}{2}$=0·5, $\frac{1}{4}$=0·25, $\frac{1}{8}$=0·125, $\frac{3}{8}$=0·375, $\frac{3}{4}$=0·75, $\frac{1}{5}$=0·2, $\frac{1}{3}$= 0·33

# SOME GENERAL SUGGESTIONS

## Practical Work, Geometry, Spatial Work

This type of work appeals to all children, but specially to children who cannot get on with abstract arithmetic. Children should have much practice with the use of the ruler, set-square, and compass, and be familiar with straight lines of various lengths and directions: horizontal lines, vertical lines, slanting lines, parallel lines. Some angles should be known, such as the right angle, and also the angle of 45° or any other angles on their set-squares. They should know these shapes: the square, rectangle, oblong, parallelogram, hexagon, and the difference between square and rectangle, as well as different kinds of triangles. The circle should be familiar, together with the terms radius, diameter, circumference.

This knowledge begins in the Infant School, when little ones match shapes, draw round shapes, colour and cut them out. Diagrams that the children can draw are suggested throughout the book. Chapter XI on length and area gives suggestions for drawing squares, and rectangles, as well as gardens and playgrounds of different shapes and sizes. Language work is most important; the actual drawing of horizontal and vertical lines, **finding** them in the **classroom**, as well as right angles, increases their vocabulary in such a way that the words have a **real** meaning for them. Some pattern-making may be done in the arithmetic lesson but most in the art and handwork lessons when all the above words will be freely used. In the geography lessons there will be drawing to scale which often helps children to understand ratio. Here they find new uses for the words **horizontal** (horizon) and **vertical**. The interrelation of subjects is most important. Whenever possible encourage children, especially backward children, to draw diagrams to illustrate their problems. Spatial work is dealt with in Chapters X and XI.

## Corrections

Children from the very first should check, verify, or prove in some way that their answers are correct. The establishment of this habit is too often neglected at school, partly through the mania for speed. Too many wrong answers may mean a child is practising his mistakes! There are two ways of avoiding this. (1) Let the child prove each example before he does the next. Little ones prove their answers by the use of counters; older children may also sometimes use counters, for the method of proof depends on the exercise. Ways of proving exercises have been given in the different chapters, but here are some additional ways that interest children. An addition sum is checked by adding each column separately and then adding the sums so obtained, or by making two shorter sums and adding the two answers together. 'Short' multiplication may be verified by addition; 'long' multiplication may be checked in several ways by reversing multiplier and multiplicand, by dividing the answer by one of the multipliers to get the other. Short and long division should be verified by not only multiplying divisor and quotient to obtain the dividend, but dividing the dividend by the quotient to get the divisor. These habits of testing help the pupil to find his own errors and his standard of accuracy for himself. He sees better the relations of the processes and the reasons why the *right* way of adding, subtracting, multiplying or dividing **is the right way**. The teacher's explanations in *words* are clear to a bright child, but the slow child finds out the reasons for the processes by **his own work**. Children see, for example, how multiplication can be used for checking, and gain a greater respect for this process. Moreover, when proving exercises the children are setting themselves **new exercises** that have a purpose, as well as being useful for revision. (2) Another way, commonly used, to prevent children from working too many exercises incorrectly is to go round and mark one or two exercises from time to time as they work.

## Oral Work and Written Work

We have no section or chapter on mental work because all arithmetic is mental. Text-books and teachers talk about *mental* arithmetic when they mean work done without pencils or paper—that is, *oral work. Oral* work is a less ambiguous word than *mental. Oral work* is valuable when children are learning their tables and additions facts, and as suggested in the various chapters. Easy problems and exercises are often taken in oral work. These are excellent for bright children, but for the dull or slow children they may be a waste of time; there is nothing intellectually criminal about using a pencil as well as inner thought. On the whole it is more valuable to give drill in easy problems as written rather than oral work. The gifted children do *all* the exercises and thus get many times as much practice as they could get in oral work. Slow children who rarely get a 'thought' answer at the rate demanded by oral work are sure to have time to do as many **as they are able.** Written answers are also of more value because children are less inclined to guess, and can read the question several times. The advantages of oral work are perhaps freedom from eye-strain, sociability, less trouble for the teacher. Needless to say, for oral work groups should be carefully graded.

The making of arithmetic notebooks, sample books of exercises, booklets for spatial work, etc., are of great value. The children enjoy making them—indeed, they make their own arithmetic books. One gets surprisingly good results. Some suggestions for booklets and notebooks have been given.

Finally, in all mathematical work it must be kept in mind that *understanding* is more important than speed, or even in some cases accuracy. Arithmetic should not mean to a child only tables learnt by heart and rules of thumb; if it does, then intelligent children will be bored, and advanced work made impossible.