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DENHAM LARRETT

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SCHOOL CERTIFICATE ALGEBRA

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SCHOOL CERTIFICATE ALGEBRA

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PREFACE

THIS *School Certificate Algebra* has been prepared in response to several requests for a text-book which shall contain in one volume a course of study from the beginning up to the standard of the examination. The contents have therefore been determined by the requirements of the eight examining bodies. Actually the book consists of the whole of my *Junior Algebra* bound up with the first four chapters of my *Senior Algebra*, with the additional feature of a large collection of examples from recent examination papers. The latter have been arranged in ten groups, according to their nature, so as to facilitate pre-examination revision.

I have not thought it necessary to alter the pagination of the material carried over from the two works named, and this, therefore, follows the original arrangement.

My thanks are due to the following examining bodies for permission to use questions set by them: The University of Bristol; the University of Cambridge Local Examinations Syndicate; the University of Durham Schools Examinations; the Senate of the University of London; the Joint Matriculation Board of the Universities of Manchester, Liverpool, Leeds, Sheffield, and Birmingham; the Delegates of the Oxford Local Examinations; the Central Welsh Board; and the Oxford and Cambridge Schools Examination Board.

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D. L.

PART I

PART II

PART III

PART I

CHAPTER I

THE FORMULA AND ITS CONSTRUCTION

1. Fig. 1 represents a piece of ribbon, part of which is coloured red and the remainder white. The length of the red part is 2 inches and the length of the white part

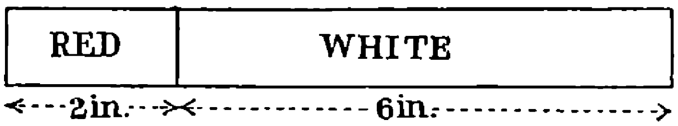

FIG. 1

is 6 inches. Then the total length of the piece of ribbon is the sum of these two lengths, and is 8 inches. To find this total length we have added the length of the red part to the length of the white part. If we use the plus sign (+) to represent this addition, as we do in arithmetic, then we can write:

Total length of ribbon
= length of red part + length of white part
= 2 inches + 6 inches
= 8 inches.

Fig. 2 represents a piece of ribbon coloured green and blue. Again, we can write:

Total length of ribbon
= length of green part + length of blue part.

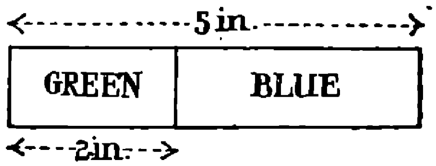

FIG. 2

In this case we know that the total length of the ribbon is 5 inches and the length of the green part is 2 inches, so that we can write:

5 inches = 2 inches + length of blue part.

From which we can see that since $2 + 3 = 5$, length of blue part $= 3$ inches.

Fig. 3 represents a number of pieces of ribbon, each of which is coloured black and yellow. No measurements are given, but in each case we can write:

Total length of ribbon
= length of black part + length of yellow part.

If the lengths of the coloured parts are known, then the

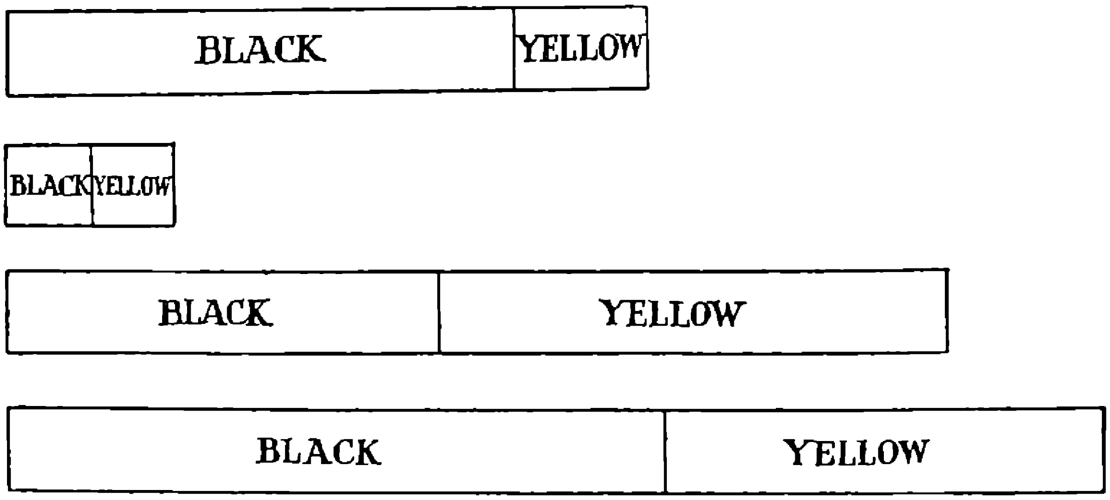

Fig. 3

total length of the ribbon can always be found by the simple addition indicated by this *general statement*:

Total length of ribbon
= length of black part + length of yellow part.

This applies to every case illustrated in Fig. 3 and to all similar cases which you can draw for yourself. A general statement such as this is called a *formula.*

The formula for the total length of a piece of ribbon part of which is black and the remainder of which is yellow is:

Total length of ribbon
= length of black part + length of yellow part.

And from this we can find the total length when the length of each of the coloured parts is known.

A formula written in this way is rather long, and can be shortened by using a letter to stand for each of the lengths we are considering. Thus we can let l stand for the total length of the ribbon in inches, m stand for the length in inches of the black part, and n for the length in inches of the yellow part; and then our formula can be written very shortly as

$$l = m + n.$$

If we are told in some particular case that the black part is 3 inches long, then $m = 3$, and

$$l = 3 + n.$$

If at the same time we are also told that the yellow part is 7 inches long, then $n = 7$. In this case

$$\begin{aligned} l &= 3 + 7 \\ &= 10. \end{aligned}$$

In other words, the total length of the ribbon is 10 inches.

EXERCISE I

1. A piece of ribbon is coloured red and white. Find the total length of the ribbon

(*a*) When the red part is 4 inches and the white part is 5 inches.

(*b*) When the red part is 7 inches and the white part is 2 inches.

(*c*) When the red part is 5 inches and the white part is 9 inches.

2. A stick 3 feet long is stuck in the ground.

(*a*) If 6 inches are in the ground, how many are above the ground?

(*b*) If 34 inches are above the ground, how many are in the ground?

3. A certain formula is $P = x + y$.

(*a*) If $x = 4$ yards and $y = 7$ yards, find P.

(*b*) If $x = 3$ yards and $y = 12$ yards, find P.

(*c*) If $x = 10$ yards and $P = 17$ yards, find y.

(*d*) If $P = 24$ yards and $x = 9$ yards, find y.

4. (*a*) Fig. 4 shows two boxes, one of which is 6 inches deep and the other 4 inches deep. What is the height of the two boxes?

(*b*) If one box is p inches deep and the other is q inches deep, write down a formula for the total height in inches (H) of the two boxes.

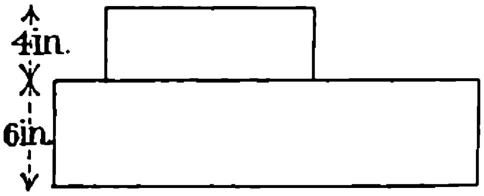

FIG. 4

(*c*) From your formula find H if $p = 5$ and $q = 4$.

5. (*a*) One bag contains 12 oranges and another bag contains 8 oranges. What is the total number of oranges?

(*b*) One bag contains c oranges and another bag contains k oranges. Write down a formula for the total number (n) of oranges.

(*c*) From your formula find n if $c = 12$ and $k = 7$.

(*d*) If $n = 20$ and $k = 4$, find c.

6. (*a*) One bag of coffee contains 4 lb. and another bag contains 2 lb. What is the total weight of the coffee?

(*b*) One bag of coffee contains m lb. and another bag contains n lb. Write down a formula for the total weight in lb. (W) of the coffee.

(*c*) Find W when $m = 6$ and $n = 8$.

(*d*) Find m when $W = 14$ and $n = 7$.

7. Fig. 5 shows a piece of ribbon coloured red, white, and blue.

(*a*) If the red part is 4 inches long, the white part is 5

FIG. 5

inches long, and the blue part is 6 inches long, what is the total length of the ribbon?

(*b*) If the red part is x inches long, the white part is y inches long, and the blue part is z inches long, write down a formula for the total length in inches (l) of the ribbon.

(*c*) Find l when $x = 5$, $y = 7$, and $z = 2$.
(*d*) Find l when $x = 10$, $y = 21$, and $z = 14$.

8. A postman is carrying three parcels.

(*a*) If the first parcel weighs 1 lb., the second 3 lb., and the third 5 lb., what is the total weight that the man is carrying?

(*b*) If the first parcel weighs p lb., the second q lb., and the third r lb., write down a formula for the total weight in lb. (w) that the man is carrying.

(*c*) Find w when $p = 1$, $q = 2$, and $r = 3$.
(*d*) Find w when $p = 1$, $q = 1$, and $r = 1$.

9. A certain formula is $K = c + e + g$.

(*a*) Find K when $c = 14$, $e = 72$, and $g = 29$.
(*b*) If $K = 100$ when $e = 21$ and $g = 47$, find c.

10. There are four big boys in a class. The first is w feet high, the second is x feet high, the third is a feet high, and the fourth is d feet high. Write down a formula for their total height in feet (H).

2. We shall now consider another type of formula. Suppose that a boy has a bag containing 80 marbles. He plays with a friend and loses 27. To find the number of marbles which he has left we must subtract 27 from 80. This leaves 53. Using the minus sign (–) to represent this subtraction, we can write:

Number of marbles left
= total number of marbles – marbles lost
= 80 – 27
= 53.

This boy may play many games, and whenever he loses he can *always* write at the end of each game:

Number of marbles left
= total number of marbles – marbles lost.

This is a formula, and we can use letters as before in order to write it more quickly. Thus, if the boy has a marbles and loses b marbles, and n is the number which he has left, then

$$n = a - b.$$

In the particular case that was considered above $a = 80$ and $b = 27$, so that

$$\begin{aligned} n &= a - b \\ &= 80 - 27 \\ &= 53. \end{aligned}$$

EXERCISE II

1. (*a*) A plank is 10 feet long. If 3 feet are sawn off, how much is left?

(*b*) A plank is 10 feet long. If x feet are sawn off, how much is left?

(*c*) A plank is a feet long and p feet are sawn off. Write down a formula for the length in feet (s) which remains.

(*d*) Find s when $a = 12$ and $p = 7$.

2. (*a*) A man is 40 years old and his son is 12 years old. What is the difference in their ages?

(*b*) A man is n years old and his son is k years old. Write down a formula for the difference in years of their ages (d).

(*c*) Find d when $n = 37$ and $k = 8$.

(*d*) Find n when $d = 30$ and $k = 17$.

3. (*a*) A jug contains 3 pints. If 2 pints are poured away, how much remains?

(*b*) A jug contains x pints. If y pints are poured away, write down a formula for the number of pints (p) which remain.

(*c*) Find p when $x = 6$ and $y = 2$.

(*d*) Find y when $p = 3$ and $x = 8$.

4. (*a*) A basket contains 20 eggs. If 5 get broken and have to be thrown away, how many are left?

(*b*) A basket contains m eggs. If n eggs are broken, write down a formula for the number of eggs (E) that remain.

(*c*) Find E when $m = 36$ and $n = 2$.

5. (*a*) A truck contains 10 tons of coal. If 3 tons are sold, how many tons are left?

(*b*) A truck contains t tons of coal. If s tons are sold, write down a formula for the number of tons (C) which remain.

(*c*) Find C when $t = 15$ and $s = 8$.

6. (*a*) A book contains 324 pages. If 137 pages have been read, how many pages remain to be read?

(*b*) A book contains P pages. If M pages have been read, write down a formula for the number of pages (K) which remain to be read.

(*c*) Find K when $P = 430$ and $M = 254$.

7. (*a*) There are 1200 passengers on a ship. If 212 of them leave at the first port, how many continue the journey?

(*b*) There are N passengers on a ship. If x of them leave at the first port, write down a formula for the number (Z) that continue the journey.

(*c*) Find Z when $N = 475$ and $x = 129$.

8. (*a*) A farmer has 300 sheep. If he sells 107, how many does he keep?

(*b*) A farmer has p sheep. If he sells t sheep, write down a formula for the number (N) he keeps.

(*c*) If $N = 280$ and $t = 74$, find p.

9. (*a*) John has 18 shillings and spends 6 shillings. How much has he left?

(*b*) John has m shillings and spends n shillings. Write down a formula for the number of shillings (S) he has left.

(*c*) Find S if $m = 20$ and $n = 12$.

10. (*a*) A box contains 56 lb. of tea. If 18 lb. are sold, how many lb. are left?

(*b*) A box contains P lb. of tea. If Q lb. are sold, write down a formula for the number of lb. (X) which remain.

(*c*) Find X when $P = 80$ and $Q = 35$.

3. The following is another simple example showing how some formulæ are constructed.

Suppose a boy starts with 5 marbles and plays two games, winning 3 marbles on his first game and then losing 2 marbles. Then

$$\begin{aligned}\textbf{Number of marbles left} &= \textbf{total number of marbles} \\ &\quad + \textbf{marbles won} - \textbf{marbles lost} \\ &= 5 + 3 - 2 \\ &= 6.\end{aligned}$$

In exactly the same way, if the boy starts with p marbles, wins q marbles, and loses r, then if N is the number of marbles he has left,

$$\mathrm{N} = p + q - r.$$

In the particular case just considered $p = 5$, $q = 3$, and $r = 2$. So that

$$\begin{aligned}\mathrm{N} &= 5 + 3 - 2 \\ &= 6.\end{aligned}$$

That is, the boy has 6 marbles left.

EXERCISE III

1. (*a*) A man has 100 oranges. He buys 50 more, and then gives away 86. How many has he left?

(*b*) A man has x oranges. He buys y more, and then gives away z. Write down a formula for the number of oranges (N) he has left.

(*c*) Find N when $x = 135$, $y = 27$, and $z = 92$.

2. (*a*) A grocer has 56 lb. of tea. He sells 37 lb., and then buys 28 lb. How many lb. of tea has he now?

(*b*) A grocer has m lb. of tea. He sells k lb., and then buys c lb. Write down a formula for the weight in lb. (w) of the tea he has left.

(*c*) Find w when $m = 112$, $k = 107$, and $c = 56$.

3. (*a*) A regiment consists of 1000 men. If 28 recruits join up and 500 men are drafted abroad, how many remain?

(*b*) A regiment consists of p men. If x recruits join up and n men are drafted abroad, write down a formula for the number of men (H) that remain.

(*c*) Find H if $p = 750$, $x = 104$, and $n = 435$.

4. (*a*) A man walks 3 miles and motors 10 miles more to his destination. He returns 8 miles by tram. How far is he from home?

(*b*) A man walks a miles and motors k miles more to his destination. He returns s miles by tram. Write down a formula for the distance (D) the man is from home.

(*c*) Find D when $a = 2$, $k = 34$, and $s = 12$.

5. (*a*) A man receives 35 shillings a week as wages, out of which he spends 32 shillings. He also receives 5 shillings a week in tips. How much does he save in a week?

(*b*) A man receives b shillings a week as wages, out of which he spends s shillings. He receives t shillings a week in tips. Write down a formula for the amount (A) he saves in a week.

(*c*) Find A when $b = 45$, $s = 40$, and $t = 7$.

4. There are three bags each containing 4 apples. Then

$$\begin{aligned}\text{Total number of apples} &= 4 + 4 + 4\\ &= 12.\end{aligned}$$

In other words, we have added the three 4's together so as to find the total number of apples.

Suppose there were 50 bags each containing 4 apples. Then

Total number of apples

$$= 4 + 4 + 4 \qquad \text{(there will be fifty 4's)}$$

Now, the sum of fifty 4's is best found by multiplying 50 by 4. So that

$$\begin{aligned}\text{Total number of apples} &= 4 + 4 + 4 \qquad . \text{(fifty 4's)}\\ &= 50 \times 4\\ &= 200.\end{aligned}$$

In just the same way $4 + 4 + 4 = 3 \times 4.$

$$7 + 7 + 7 = 3 \times 7.$$
$$a + a + a = 3 \times a.$$

Hence if T is the total number of apples contained in three bags, each of which holds a apples,

$$\begin{aligned} T &= a + a + a \\ &= 3 \times a. \end{aligned}$$

It is usual to write $3 \times a$ as $3a$. Our formula now becomes

$$T = 3a.$$

In the particular case considered at first $a = 4$.

$$\begin{aligned} T &= 3 \times 4 \\ &= 12. \end{aligned}$$

Notice that in the product of a number and a letter the number is written first. Thus we write $3a$, and not $a3$.

EXERCISE IV

1. If $a = 4$, find the value of $2a$, $4a$, $6a$, $8a$, $12a$, $20a$.
2. If $p = 12$, find the value of $3p$, $8p$, $11p$, $13p$.
3. Write the following more shortly:

(a) $x + x$.	(f) $k + k$.
(b) $x + x + x$.	(g) $T + T + T + T$.
(c) $x + x + x + x$.	(h) $d + d + d$.
(d) $x + x + x + x + x$.	(i) $h + h + h + h + h + h$.
(e) $m + m + m$.	(j) $P + P + P + P + P$.

4. If $X = 5y$, find X when $y = 15$.
5. If $P = 5a + 3b$, find P when $a = 2$ and $b = 3$.
6. If $A = 2r + 5p$, find A when $r = 7$ and $p = 4$.
7. If $M = 5K - 2N$, find M when $K = 7$ and $N = 4$.

5. The following example shows an important application of the work done in the preceding lesson.

$$
\begin{aligned}
\text{The number of pence in 1 shilling} &= 12. \\
\text{2 shillings} &= 2 \times 12. \\
\text{5 shillings} &= 5 \times 12. \\
\text{,, } k \text{ shillings} &= k \times 12 \\
&= 12k.
\end{aligned}
$$

Notice that $k \times 12$ is written $12k$, and not as $k12$.

MESTO

EXERCISE V

1. (*a*) How many pence are there in 1 shilling?
(*b*) How many pence are there in 7 shillings?
(*c*) How many pence are there in x shillings?

2. (*a*) How many ounces are there in 1 lb.?
(*b*) How many ounces are there in 2 lb.?
(*c*) How many ounces are there in m lb.?

3. (*a*) How many inches are there in 1 yard?
(*b*) How many inches are there in 3 yards?
(*c*) How many inches are there in p yards?

4. (*a*) Write down a formula for the number of seconds (T) in h hours.
(*b*) Use this formula to find T when $h = 3$.

5. (*a*) Write down a formula for the number of shillings (S) in g guineas.
(*b*) From your formula find S when $g = 4$.

6. (*a*) Write down a formula for the number of cwts. (C) in t tons.
(*b*) From your formula find C when $t = 12$.

7. (*a*) Write down a formula for the number of yards (n) in d miles.
(*b*) From your formula find n when $d = 8$.

6. We have seen that

$$5k = k + k + k + k + k$$

and

$$3k = k + k + k.$$

Then

$$5k + 3k = \underbrace{k + k + k + k + k}_{5k} + \underbrace{k + k + k}_{3k}$$
$$= 8k.$$

Similarly, $12m + 19m = 31m$, and so on.

EXERCISE VI

1. Simplify

(a) $3a + 8a$.
(b) $7q + 12q$.
(c) $10p + 13p$.
(d) $18c + 21c$.
(e) $35m + 17m$.
(f) $4a + 8a + 13a$.
(g) $17q + 14q + 13q$.
(h) $21p + 9p + 14p$.
(i) $14c + 23c + 18c$.
(j) $17m + 41m + 2m$.

2. If $x = 3$ find the value of

(a) $4x + 7x$.
(b) $6x + x$.
(c) $13x + 5x$.
(d) $3x + 8x + 2x$.
(e) $7x + 3x + 11x$.

3. If $A = 3m + 4m + 2m$, find A when $m = 7$.

4. If $Q = 2a + 12a + 3a$, find Q when $a = 10$.

5. If $w = p + 2p + 3p$, find w when $p = 4$.

7. Subtraction is performed in a similar manner.

Thus, $$5k - 3k = \overbrace{k + k + \underbrace{\not{k} + \not{k} + \not{k}}_{3k}}^{5k}$$
$$= 2k.$$

The subtraction is shown by crossing out the $k + k + k$ $(= 3k)$.

Thus, $5k - 3k = 2k$.

Similarly, $17m - 4m = 13m$, and so on.

EXERCISE VII

1. Simplify

(a) $7a - 2a$.
(b) $12q - 4q$.
(c) $10p - 3p$.
(d) $21c - 19c$.
(e) $11m - 2m$.
(f) $40x - 27x$.
(g) $37b - 19b$.
(h) $93e - 48e$.
(i) $112g - 74g$.
(j) $87t - 49t$.

2. If $d = 5$, find the value of

(a) $6d - 2d$.
(b) $12d - 5d$.
(c) $11d - 5d$.
(d) $38d - 23d$.
(e) $47d - 29d$.

3. If $X = 14x - 2x$, find X when $x = 3$.

4. If $h = 24c - 17c$, find h when $c = 5$.

5. If $w = 34m - 18m$, find w when $m = 10$.

6. Simplify

(a) $4a + 3a + 7a + 12a$.
(b) $15m + 13m - 4m$.
(c) $23c - 14c + 8c$.
(d) $12x + 7x + 4x - 5x$.
(e) $14w + 2w + 5w - 7w + 12w$.

8. We have seen that $3p = p + p + p$ and that $2q = q + q$, and so on. We must now consider how more complicated expressions can be simplified. Consider the following example.

Simplify $3p + 2q + 4p + 3q$.

$$3p = p + p + p.$$
$$2q = q + q.$$
$$4p = p + p + p + p.$$
$$3q = q + q + q.$$

Hence $3p + 2q + 4p + 3q$

$$= p + p + p + q + q + p + p + p + p + q + q + q$$
$$= p + p + p + p + p + p + p + q + q + q + q + q$$
$$= 7p + 5q.$$

We need not always write out in full the values of $3p$, $2q$, etc., as we have done here, because we can always rearrange the expression so as to gather similar letters together. Thus in this example

$$3p + 2q + 4p + 3q = \underbrace{3p + 4p} + \underbrace{2q + 3q}$$
$$= 7p + 5q.$$

Here is another example. Find the value of $2x + 4y + 6x + 8y$ when $x = 2$ and $y = 3$.

$$2x + 4y + 6x + 8y = \underbrace{2x + 6x} + \underbrace{4y + 8y}$$
$$= 8x + 12y.$$

If $x = 2$, then $8x = 16$.
If $y = 3$, then $12y = 36$.

$$\therefore 2x + 4y + 6x + 8y = 8x + 12y$$
$$= 16 + 36$$
$$= 52.$$

EXERCISE VIII

1. Simplify

(a) $2x + y + x$.
(b) $3a + 2b + 5a$.
(c) $4k + 3m + 2m + 5k$.
(d) $7c + 4w + 2c + 3w$.
(e) $8h + 4t + h + 3t$.
(f) $10p + 3r + 7r + 8p$.
(g) $4b + 7e + 3e + 5b$.
(h) $13m + 7w + 14m + 11w$.
(i) $15l + 16k + 3l + k$.
(j) $3t + 6x + 9t + 12x$.
(k) $x + 2y + 3z + 5x + 2y + 4z$.
(l) $2a + 3b + c + 4b + 3a + 3c$.
(m) $5k + 7m + 2n + 3k + 5n + 6m$.
(n) $7w + 12t + 4v + 6t + 5v + 10w$.
(o) $x + 2y + 3z + 4x + 5y + 6z + 7x + 8y + 9z$.
(p) $2a + 4b + 3c + 5a + 9c + 2a + 4b + 5c$.

(*q*) $5k + 7m + 2n + 3n + 5m + 7k + 6m + k + 3m$.
(*r*) $7t + 5u + 4t + 8v + 7u + 18v + 9t + 10u$.
(*s*) $4w + 3l + 6h + 7w + 2h + 5l + 4w$.
(*t*) $5a + 12c + 8b + 9c + 7a + 5c + 8b + 17a + 12b$.

2. Find the value of

(*a*) $2p + 3q + 4p + q$ when $p = 3$ and $q = 5$.
(*b*) $5x + 12y + 3x + 2y$ when $x = 8$ and $y = 3$.
(*c*) $3u + 5v + 4u + 7v$ when $u = 10$ and $v = 12$.
(*d*) $3u + 5v + 8w + 4v + 7u + 5w$ when $u = 1$, $v = 2$, and $w = 3$.
(*e*) $7a + 5c + 4b + 8c + 5a + 9b$ when $a = 3$, $b = 5$, and $c = 7$.

EXERCISE IX

1. (*a*) How many shillings are there in £1?
(*b*) How many shillings are there in £2?
(*c*) How many shillings are there in £x?
(*d*) How many shillings are there in £1 + 5 shillings?
(*e*) How many shillings are there in £2 + 5 shillings?
(*f*) How many shillings are there in £x + 5 shillings?
(*g*) How many shillings are there in £x + y shillings?

2. (*a*) How many lb. are there in T tons?
(*b*) How many lb. are there in T tons + 4 cwts.?
(*c*) How many lb. are there in T tons + C cwts.?

3. How many pence are there in £x + y shillings + z pence?

4. How many feet are there in p yards + q feet?

5. How many seconds are there in h hours + k minutes + m seconds?

6. How many shillings are there in £$2x$ + y shillings?

7. How many shillings are there in £$3x$ + $2y$ shillings?

8. How many feet are there in $3p$ yards + $2q$ feet?

9. How many inches are there in a yards + $2b$ feet + $3c$ inches?

10. How many minutes are there in $3t$ hours + $5k$ minutes?

11. How many pence are there in $5x$ florins?

12. How many ounces are there in $4m$ lb. + $15k$ ounces?

13. How many pence are there in $3x$ shillings + $5y$ pence?

14. How many yards are there in $4q$ miles?

15. How many pence are there in $5x$ half-crowns?

9. A basket contains 5 loaves of bread, each of which weighs 2 lb. Then the total weight of the bread is 5×2 lb.

Similarly, if the basket contains n loaves, each of which weighs 2 lb., then the total weight of the bread is $n \times 2$ lb.

If each loaf weighs a lb., then the total weight of n loaves is $n \times a$ lb.

This is written na, which stands for the product of n and a.

If P is the total weight in lb. of n loaves, each of which weighs a lb., we have the formula $P = na$.

If $n = 5$ and $a = 2$, then $P = 5 \times 2$.

$$= 10.$$

Hence the total weight in this case is 10 lb.

But $5 \times 2 = 2 \times 5.$

Similarly, $n \times a = a \times n.$

In other words, $na = an.$

EXERCISE X

1. (*a*) What is the cost in pence of 5 stamps, each costing $2d.$?

(*b*) Write down a formula for the cost in pence (x) of k stamps, each costing p pence.

(*c*) From this formula find x when $k = 12$ and $p = 3$.

2. (*a*) What is the weight of 7 cakes, each of which weighs 3 lb.?

(*b*) Write down a formula for the weight in lb. (w) of n cakes, each weighing c lb.

(*c*) From this formula find w when $n = 8$ and $c = 4$.

3. (*a*) What is the cost of 12 tickets for a concert if each ticket costs 5 shillings?

(*b*) Write down a formula for the cost in shillings (S) of x tickets, each costing a shillings.

(*c*) From this formula find S when $x = 50$ and $a = 3$.

4. (*a*) A man walks at 3 m.p.h. for 2 hours. How far does he travel?

(*b*) A man walks at a m.p.h. for t hours. Write down a formula for the distance in miles (D) that he walks.

(*c*) From this formula find D when $a = 4$ and $t = 3$.

5. If $P = xy + 5$, find P when $x = 1$ and $y = 4$.

6. If $K = 10 - pq$, find K when $p = 2$ and $q = 3$.

7. Find the value of $ab + bc$ when $a = 1$, $b = 2$, and $c = 3$.

8. Find the value of $pq + tq$ when $p = 5$, $q = 2$, and $t = 3$.

9. (*a*) Write down a formula for the total rent in shillings (R) received from n tenants, each of whom pays x shillings a week.

(*b*) Find R when $n = 10$ and $x = 12$.

10. (*a*) Write down a formula for the total weight in lb. (w) of a parcel that contains n packets, each weighing p lb., and m packets, each weighing q lb.

(*b*) Find w when $n = 2$, $m = 3$, $p = 2$, and $q = 4$.

10. We have seen that $x \times y$ is written xy.

Similarly, $x \times y \times z = xyz$.

Notice that $xy \times z = xz \times y = yz \times x$ and that $xyz = xzy = yzx$.

Study the following examples carefully.

EXAMPLE 1

Simplify $a \times 6b$.

$$\begin{aligned} a \times 6b &= a \times 6 \times b \\ &= 6 \times a \times b \\ &= 6ab. \end{aligned}$$

EXAMPLE 2

Simplify $3x \times 6y$.

$$\begin{aligned} 3x \times 6y &= 3 \times x \times 6 \times y \\ &= \underbrace{3 \times 6} \times \underbrace{x \times y} \\ &= 18xy. \end{aligned}$$

EXERCISE XI

1. Simplify

(a) $x \times 2y$.
(b) $2a \times c$.
(c) $4k \times 2l$.
(d) $8w \times 3t$.
(e) $7a \times 12x$.
(f) $12c \times 14f$.
(g) $a \times b \times c$.
(h) $2x \times y \times z$.
(i) $3w \times 2y \times 4z$.
(j) $5p \times 3q \times r$.
(k) $7k \times 4t \times 2w$.
(l) $m \times 2n \times 3p \times 4k$.

2. Find the value of

(a) $2g \times 3h$ when $g = 3$ and $h = 5$.
(b) $4k \times 2m$ when $k = 1$ and $m = 7$.
(c) $5m \times 12n$ when $m = 17$ and $n = 1$.
(d) $3a \times 7b$ when $a = 2$ and $b = 0$.
(e) $x \times y \times z$ when $x = 2$, $y = 5$, and $z = 3$.
(f) $w \times 2p \times q$ when $w = 1$, $p = 6$, and $q = 2$.
(g) $2a \times 3d \times 4h$ when $a = d = h = 2$.
(h) $3m \times 2l \times 4n$ when $m = 1$, $l = 5$, and $n = 5$.

11. Products of the same letters can be added and subtracted in the same way as the simpler quantities that we have already considered.

Thus, $5pq + 3pq = 8pq$
and $5pq - 3pq = 2pq$.

In a product the *order* of the letters does not affect the value of the product, and so when an addition is done care must be taken to see that the products are properly collected. The following examples illustrate this.

EXAMPLE 1

Simplify $5pq + 3qp$.

We have seen that $pq = qp$, so that

$$5pq + 3qp = 5pq + 3pq$$
$$= 8pq.$$

EXAMPLE 2

Simplify $5xyz + 3ab + 4yzx + 2ba$.

Since $xyz = yzx$ and $ab = ba$ we can write:

$$5xyz + 3ab + 4yzx + 2ba = 5xyz + 3ab + 4xyz + 2ab$$
$$= 5xyz + 4xyz + 3ab + 2ab$$
$$= 9xyz + 5ab.$$

EXERCISE XII

1. Simplify

(a) $xy + yx$.
(b) $2cg + 3gc$.
(c) $5km - mk$.
(d) $7ab + 3ba$.
(e) $10nf - 3fn$.
(f) $ab + bc + 2ab$.
(g) $5pqr - 2qpr$.
(h) $4dk + 5kd + 4ab$.
(i) $7abc + 8bca + 9cab$.
(j) $4ca + 2ab + 5ac + 3ba$.

2. Find the value of

(a) $2xy + 5yx$ when $x = 1$ and $y = 3$.
(b) $14km - 2mk$ when $k = 5$ and $m = 1$.
(c) $ab + cd + 2ba + 3dc$ when $a = c = 1$ and $b = d = 2$.
(d) $10pq + 4qp + 3rp + 2pr$ when $p = 1$, $q = 2$, and $r = 3$.
(e) $5abc$ when $a = 4$, $b = 5$, and $c = 6$.
(f) $12prq - 4rpq$ when $r = 3$, $p = 2$, and $q = 0$.
(g) $4cdf + 4fdc + 5dfc$ when $d = 3, f = 8$, and $c = 2$.
(h) $17xyz - 15xzy$ when $x = 25$, $y = 6$, and $z = 4$.
(i) $10fk + 3kf + 4fk$ when $f = 6$ and $k = 2$.
(j) $7wt + 4uv + 8tw + 3vu$ when $w = 1$, $u = 2$, $v = 3$, and $t = 4$.

12. If we divide 20 marbles into five equal groups there will be 4 marbles in each group. The number in each group is obtained by dividing 20 by 5, and we can write this process of division as either $20 \div 5 = 4$ or $\frac{20}{5} = 4$.

Similarly, if we divide n marbles into five groups the number in each group is obtained by dividing n by 5, and we write this quotient as $\frac{n}{5}$.

If we divide n marbles into x groups, then the number in each group is $\frac{n}{x}$. Hence we can write down a formula for the number of marbles in a group (K) when n marbles are divided into x groups as

$$K = \frac{n}{x}.$$

If $n = 150$ and $x = 25$, then

$$\begin{aligned} K &= \frac{150}{25} \\ &= 6. \end{aligned}$$

That is, there are 6 marbles in each group.

EXERCISE XIII

1. (*a*) If 40 oranges are divided equally among 8 boys, how many does each boy receive?

(*b*) If q oranges are divided equally among h boys, write down a formula for the number (n) of oranges each boy receives.

(*c*) From this formula find n when $q = 150$ and $h = 30$.

2. (*a*) If 12 yards of ribbon cost 1*s*. 6*d*., what is the cost in pence of 1 yard?

(*b*) If d yards of ribbon cost x pence, write down a formula for the cost in pence (c) of 1 yard.

(*c*) From this formula find c when $d = 20$ and $x = 50$.

3. (*a*) A motor travels 70 miles on 2 gallons of petrol. How far does it travel on 1 gallon?

(*b*) A motor travels m miles on t gallons of petrol. Write down a formula for the distance in miles (d) which it travels on 1 gallon.

(*c*) From this formula find d when $m = 100$ and $t = 5$.

4. (*a*) An aeroplane travels at 120 m.p.h. How many miles does it travel in 1 minute?

(*b*) An aeroplane travels at k m.p.h. Write down a formula for the distance in miles (M) that it travels in 1 minute.

(*c*) From this formula find M when $k = 180$.

5. (*a*) How many bags of sugar, each containing 2 lb., can be obtained from a box containing 56 lb.?

(*b*) Write down a formula for the number of bags of sugar (a) that can be obtained from a box containing w lb. if each bag weighs p lb.

(*c*) From this formula find a when $w = 112$ and $p = 4$.

6. Find the value of

(*a*) $\dfrac{ab}{c}$ when $a = 5$, $b = 3$, and $c = 1$.

(*b*) $\dfrac{3fg}{h}$ when $f = 4$, $g = 7$, and $h = 14$.

(*c*) $\dfrac{2xy}{3z}$ when $x = 5$, $y = 2$, and $z = 3$.

(*d*) $\dfrac{2p}{q} + \dfrac{3q}{2}$ when $p = 2$ and $q = 4$.

(*e*) $\dfrac{5a}{b} - \dfrac{4b}{a}$ when $a = 4$ and $b = 2$.

13. The following examples show an important application of the process of division. Study these examples carefully.

EXAMPLE 1

(*a*) How many pounds are there in 120 shillings?

(*b*) Write down a formula for the number of pounds (p) in S shillings.

(*a*) We know that 20 shillings = £1, and therefore, to reduce shillings to pounds, we must divide by 20.

Hence 120 shillings $= £\frac{120}{20}$.

(*b*) In an exactly similar way,

$$\text{S shillings} = £\frac{S}{20}.$$

Therefore our formula is

$$p = \frac{S}{20}.$$

EXAMPLE 2

Write down a formula for expressing x yards in miles (M).

Since 1760 yards = 1 mile, yards are reduced to miles by division by 1760. Hence our formula is

$$M = \frac{x}{1760}.$$

EXERCISE XIV

1. (*a*) How many hours are there in 240 minutes?
(*b*) How many hours are there in T minutes?

2. (*a*) How many tons are there in 4000 lb.?
(*b*) How many tons are there in p lb.?

3. (*a*) How many yards are there in 144 inches?
(*b*) How many yards are there in k inches?

4. (*a*) How many miles are there in 5280 feet?
(*b*) How many miles are there in m feet?

5. (*a*) How many cwts. are there in 1000 lb.?
(*b*) How many cwts. are there in d lb.?

6. (*a*) Write down a formula for expressing feet (f) in yards (y).
(*b*) Use your formula to find y when $f = 132$.

7. (*a*) Write down a formula for expressing h ounces in lb. (n).

(*b*) From this formula find n when $h = 176$.

8. (*a*) Write down a formula for expressing m hours in days (d).

(*b*) From this formula find d when $m = 192$.

9. (*a*) How many pounds are there in 250 florins?

(*b*) Write down a formula for the number of pounds (P) in t florins.

10. Express £$x + y$ shillings $+ z$ pence in (*a*) pounds, (*b*) shillings, and (*c*) pence.

11. Express a tons $+ b$ cwts. $+ c$ lb. in (*a*) tons, (*b*) cwts., and (*c*) lb.

12. Express p yards $+ q$ feet $+ r$ inches in (*a*) yards, (*b*) feet, and (*c*) inches.

14. We have seen how the familiar processes of arithmetic—addition, subtraction, multiplication, and division—are applied so as to construct a general statement, or formula, which is written in a 'shorthand' form by means of letters. The following exercises introduce no new ideas, but are a revision of what has been done before in this chapter and will provide further practice in the construction and use of formulæ.

EXERCISE XV (Revision Exercise)

(A)

1. If $x = 6$, what are the values of $x + 2$, $2x$, $x - 2$, and $\frac{x}{2}$?

2. If $p = 4$, $q = 7$, and $r = 6$, what are the values of $p + q + r$, $pq + qr + rp$, pqr, and $\frac{pq}{r}$?

3. (*a*) How many pence are there in £k?
(*b*) How many ounces are there in m lb.?
(*c*) How many yards are there in x miles?

4. Simplify the following:

$$4x + 5y + 2z + 3x + 4z + 8y + 5x.$$

5. (*a*) A train travels at 30 m.p.h. How far does it travel in 3 hours?

(*b*) A train travels at p m.p.h. Write down a formula for the distance in miles (D) that it travels in t hours.

(*c*) From this formula find D when $p = 25$ and $t = 4$.

(B)

6. (*a*) A parcel contains 2 bags of sugar, each weighing 1 lb., and 3 bags of flour, each weighing 2 lb. What is the total weight in pounds of the parcel?

(*b*) A parcel contains N bags of sugar, each weighing x lb., and M bags of flour, each weighing y lb. Write down a formula for the total weight in lb. (w) of this parcel.

(*c*) From this formula find w when $N = 3$, $x = 2$, $M = 4$, and $y = 2$.

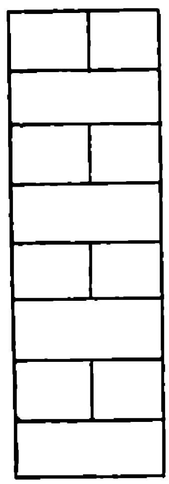

Fig. 6

7. Multiply $5x$ by $7y$, and find the value of the product when $x = 1$ and $y = 2$.

8. (*a*) How many gallons are there in c pints?
(*b*) How many seconds are there in k hours?
(*c*) How many shillings are there in m pence?

9. (*a*) Fig. 6 shows a part of a brick wall. Each brick is t inches thick (including the mortar). Write down a formula for the height of the wall in inches (H) when there are (i) 12 rows of bricks, (ii) 25 rows of bricks, and (iii) n rows of bricks.

(*b*) From this formula find H when $n = 40$.

10. From the formula $P = \frac{EV}{K} + 1$ find P when $E = 8$, $V = 13$, and $K = 2$.

(C)

11. (*a*) A box containing 64 oranges is divided among 16 boys. How many oranges does each boy receive?

(*b*) A box containing A oranges is divided among x boys. Write down a formula for the number of oranges (n) each boy receives.

(*c*) From this formula find

(i) n when $A = 132$ and $x = 12$.

(ii) A when $x = 27$ and $n = 3$.

12. (*a*) Reduce p half-crowns to pounds.

(*b*) Reduce L cwts. to tons.

13. Simplify the following:

(*a*) $2x + 7x - 3x$.

(*b*) $4a + 7b + 2c + 4b + 8a$.

(*c*) $2k \times 5m \times 6n$.

14. (*a*) A man buys a piano for £x and sells it for £y. Write down a formula for his gain (G).

(*b*) If he sold it at a loss, write down a formula for the loss (L).

15. Find the value of $\frac{xy}{z} + \frac{yz}{x} + \frac{zx}{y}$ when $x = 1$, $y = 2$, and $z = 4$.

(D)

16. (*a*) A basket which weighs 1 lb. contains 3 loaves of bread, each of which weighs 2 lb. What is the total weight of the basket and its contents?

(*b*) A basket which weighs w lb. contains n loaves of bread, each of which weighs x lb. Write down a formula for the total weight in lb. (W) of the basket and its contents.

(*c*) From this formula find W when $w = 2$, $n = 3$, and $x = 2$.

17. Simplify the following:

(*a*) $8cd - 5dc$.

(*b*) $17ab + 3ba + 2ab$.

(*c*) $8xyz + 7yzx - 2zxy$.

18. (*a*) Fig. 7 shows a number of equal steps. If the rise in each case is h inches, write down a formula for the total height in inches (H) of the steps when there are (i) 5 steps, (ii) 12 steps, and (iii) n steps.

(*b*) From this formula find H when $n = 7$ and $h = 6$.

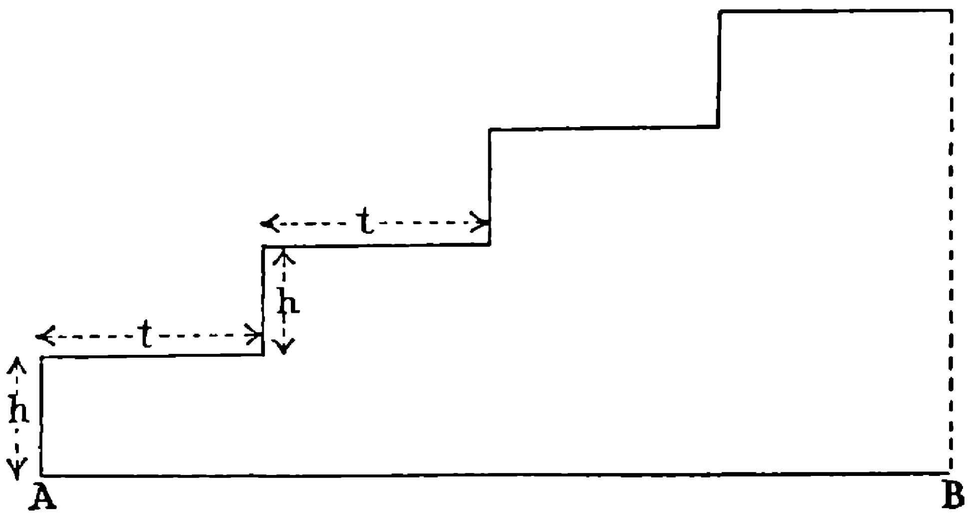

FIG. 7

19. If in the previous question the tread of each step is t inches and the distance AB is s inches, write down a formula for s when there are (*a*) 5 steps, (*b*) 12 steps, and (*c*) n steps.

20. If $g = 16$ find the value of

(*a*) $2g + 4g$. (*b*) $\frac{g}{2} + \frac{g}{4}$. (*c*) $\frac{g}{2} - \frac{g}{4}$.

CHAPTER II

POWERS AND ROOTS

15. When a number is multiplied by itself the product is called the *square* of the number. Thus the product 3×3 gives the square of 3, and so on. This is written for short 3^2, and $3^2 = 9$.

Similarly, $5 \times 5 = 5^2 = 25.$

$7 \times 7 = 7^2 = 49.$

And $x \times x = x^2.$

EXERCISE XVI

1. Find the value of 2^2, 5^2, 8^2, 11^2, and 13^2.

2. Find the value of

(*a*) $2^2 + 3^2$.
(*b*) $5^2 + 7^2 + 9^2$.
(*c*) $1^2 + 2^2 + 3^2$.
(*d*) $8^2 - 3^2$.
(*e*) $10^2 - 7^2$.

3. If $a = 3$ and $b = 5$ find the value of

(*a*) $a^2 + b^2$.
(*b*) $b^2 - a^2$.
(*c*) $a + 2b + a^2$.
(*d*) $a^2 + ab + b^2$.
(*e*) $a^2 - ab + b^2$.

4. If $x = 2$, $y = 4$, and $z = 6$ find the value of

(*a*) $x^2 + y^2 + z^2$.
(*b*) $x^2 + xy + z^2$.
(*c*) $xy + yz + x^2 + y^2$.
(*d*) $z^2 - yz$.
(*e*) $y^2 + x^2 - z$.

16. The small 2 that is used to show the process of squaring is called an *index*. We can multiply a number by itself as many times as we please, the index showing the number of terms in the product.

Thus, $4^3 = 4 \times 4 \times 4 = 64.$

$2^7 = 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 = 128.$

And $x^5 = x \times x \times x \times x \times x.$

And so on.

4^3 is called "four cubed," 2^7 is called "two to the seventh," x^5 is called "x to the fifth," or "x raised to the *power* five."

EXERCISE XVII

1. Find the value of 2^3, 3^5, 6^4, 7^3, and 4^4.

2. Find the value of

(a) $2^3 + 3^2$.
(b) $5^2 + 2^4$.
(c) $7^3 + 3^2 + 2^3$.
(d) $6^4 - 3^5$.
(e) $4^4 - 2^3$.

3. If $a = 3$ and $b = 5$ find the value of

(a) $a^3 + b^3$.
(b) $ab + b^4$.
(c) $a^4 - b^2$.
(d) $ab + a^4 + b^2$.
(e) $a + a^2 + a^3 + a^4 + a^5$.

4. If $x = 2$, $y = 4$, and $z = 6$ find the value of

(a) $x + xy + y^2$.
(b) $x^3 + y^3 + z^3$.
(c) $x^4 + y^3 + z^2$.
(d) $x + yz - x^4$.
(e) $z^3 - y^3$.

17. The addition and subtraction of quantities raised to some power are performed in a similar manner to the addition and subtraction of the quantities themselves.

Thus, as we have seen, $a + a + a = 3a$.

Similarly, $a^2 + a^2 + a^2 = 3a^2$.

Suppose $a = 2$, then

$$\begin{aligned} a^2 + a^2 + a^2 &= 2^2 + 2^2 + 2^2 \\ &= 4 + 4 + 4 \\ &= 3 \times 4 \\ &= 3 \times 2^2 \\ &= 3a^2. \end{aligned}$$

In exactly the same way,

$$p^5 + p^5 + p^5 + p^5 = 4p^5.$$

EXAMPLE 1

Simplify $4a^3 + 3a^3$.

$$4a^3 + 3a^3 = \underbrace{a^3 + a^3 + a^3 + a^3}_{4a^3} + \underbrace{a^3 + a^3 + a^3}_{3a^3}$$
$$= 7a^3.$$

EXAMPLE 2

Simplify $5k^2 + 7k^2$.

It is not necessary to write the values of $5k^2$ and $7k^2$ in full, as we have done for $4a^3$ and $3a^3$ in the previous example.

$$5k^2 + 7k^2 = 12k^2.$$

EXAMPLE 3

Simplify $12m^2 - 5m^2$.

$$12m^2 - 5m^2 = 7m^2.$$

EXERCISE XVIII

1. Simplify
   (a) $t^2 + t^2$.
   (b) $m^2 + 2m^2 + 3m^2$.
   (c) $5l^4 + 12l^4$.
   (d) $7a^5 + 4a^5 + 6a^5$.
   (e) $12b^9 + 5b^9 + 8b^9 + 2b^9$.
   (f) $17e^5 - 15e^5$.
   (g) $24g^7 - 9g^7$.
   (h) $3p^4 + 7p^4 - 2p^4$.
   (i) $10y^5 + 8y^5 - 7y^5$.
   (j) $5n^3 + 8n^3 - 3n^3 + 7n^3$.

2. Find the value of
   (a) $a^2 + 2b^3$ when $a = 2$ and $b = 3$.
   (b) $2m^3 + 3m^3$ when $m = 3$.
   (c) $4t^5$ when $t = 1$.
   (d) $5l^2 - 2l^2$ when $l = 5$.
   (e) $10c^3 + 5c^3 - 3c^3$ when $c = 4$.

(*f*) $a^2 + 2a^2 + 3a^2 + 4a^2$ when $a = 1$.
(*g*) $8g^5 - 4g^5$ when $g = 2$.
(*h*) $2k^3 + 5k^3 - 3k^3$ when $k = 3$.
(*i*) $17n^2 - 5n^2$ when $n = 10$.
(*j*) $4p^3 + 5p^3 + 6p^3$ when $p = 2$.
(*k*) $c + 2c^2 + 3c^3$ when $c = 2$.
(*l*) $6d^2 - 2d^3$ when $d = 3$.
(*m*) $5g + 2h^2$ when $g = 7$ and $h = 5$.
(*n*) $4w + 5w^3 - w^2$ when $w = 3$.
(*o*) $x + 2x^2 + 3x^3 + 4x^4 + 5x^5$ when $x = 2$.

18. More complicated quantities are raised to powers in a similar way.

Thus,
$$\begin{aligned}(2x)^2 &= 2x \times 2x \\ &= 2 \times x \times 2 \times x \\ &= 2 \times 2 \times x \times x \\ &= 4x^2.\end{aligned}$$

Hence $(2x)^2 = 4x^2$.

Notice the difference between $(2x)^2$ and $2x^2$.

Study the following examples carefully.

EXAMPLE 1

Simplify $(4p)^4$.

$$\begin{aligned}(4p)^4 &= 4p \times 4p \times 4p \times 4p \\ &= 4 \times 4 \times 4 \times 4 \times p \times p \times p \times p \\ &= 256p^4.\end{aligned}$$

Hence $(4p)^4 = 256p^4$.

Notice the difference between $(4p)^4$ and $4p^4$.

EXAMPLE 2

Simplify $(2pq)^2$.

$$\begin{aligned}(2pq)^2 &= 2pq \times 2pq \\ &= 2 \times p \times q \times 2 \times p \times q \\ &= 2 \times 2 \times p \times p \times q \times q \\ &= 4p^2q^2.\end{aligned}$$

Notice the difference between $(2pq)^2$ and $2pq^2$.

EXERCISE XIX

1. Simplify the following: $(3k)^4$, $(10p)^2$, $(5m)^3$, $(2pq)^3$, and $(4xy)^4$.
2. If $A = (kx)^2$, find A when $k = 2$ and $x = 3$.
3. If $P = (kx)^2 - kx^2$, find P when $k = 2$ and $x = 3$.
4. If $R = (3xy)^3$, find R when $x = y = 2$.
5. Simplify the following:
   (a) $(2k)^5 + 3k^5$.
   (b) $(3m)^6 + (2m)^6$.
   (c) $(2xy)^2 + (3xy)^2$.
   (d) $(5ab)^3 - (3ab)^3$.
   (e) $(2pqr)^2 - (pqr)^2$.
   (f) $(3xyz)^2 + 3x^2y^2z^2 - 2y^2x^2z^2$.

19. The multiplication of quantities which have been raised to a power is done in the following way.

Consider $x^2 \times x^3$.

$$\begin{aligned} x^2 \times x^3 &= x \times x \times x \times x \times x \\ &= x \times x \times x \times x \times x \\ &= x^5. \end{aligned}$$

Therefore $x^2 \times x^3 = x^5$.

Similarly, $p^2 \times p^4 = p \times p \times p \times p \times p \times p$
$= p^6$.

Notice that in the multiplication of two powers of the same quantity the index of the product is the sum of the indices of the quantities.

EXERCISE XX

Simplify

1. $a^3 \times a^4$.	6. $c^2 \times c^5 \times c^7$.
2. $p^2 \times p^{10}$.	7. $h^2 \times h \times h^5 \times h^8$.
3. $k^{12} \times k^5$.	8. $m^{13} \times m^6 \times m^2$.
4. $t^2 \times t^8$.	9. $x^2 \times x^a$.
5. $x \times x^2 \times x^3$.	10. $a^m \times a^n$.

20. When such quantities have numerical coefficients these must be considered when obtaining the final product. Thus,

$$\begin{aligned} 2p^4 \times 5p^2 &= 2 \times p^4 \times 5 \times p^2 \\ &= 2 \times 5 \times p^4 \times p^2 \\ &= 10p^6. \end{aligned}$$

Therefore $2p^4 \times 5p^2 = 10p^6$.

Similarly, $8x^3 \times 12x^5 = 96x^8$.

And so on.

EXERCISE XXI

Simplify

1. $2a^2 \times 3a^5$.	6. $6q^2 \times 7q^6 \times 4q$.
2. $4x^3 \times 7x^8$.	7. $10t \times 10t^2 \times 10t^3$.
3. $3p \times 5p^2 \times 2p^3$.	8. $w^2 \times 4w^4 \times 8w^8$.
4. $7g^2 \times 8g^4 \times 2g$.	9. $3y^2 \times 5y^4 \times 4y$.
5. $3m^3 \times 9m^7$.	10. $d \times 2d^2 \times 3d^3 \times 4d^4$.

21. The method of multiplication that has been described in the preceding pages can be extended to products which are more complex than those which have been considered in the last few exercises. The following examples illustrate this extension.

EXAMPLE 1

Simplify $p^2q \times pq^3$.

$$\begin{aligned} p^2q \times pq^3 &= p \times p \times q \times p \times q \times q \times q \\ &= p \times p \times p \times q \times q \times q \times q \\ &= p^3q^4. \end{aligned}$$

Therefore $p^2q \times pq^3 = p^3q^4$.

EXAMPLE 2

Simplify $2a^2b \times 5b^2c \times 3cd$.

$$\begin{aligned} 2a^2b \times 5b^2c \times 3cd &= 2 \times a^2 \times b \times 5 \times b^2 \times c \times 3 \times c \times d \\ &= 2 \times 5 \times 3 \times a^2 \times b \times b^2 \times c \times c \times d \\ &= 30a^2b^3c^2d. \end{aligned}$$

Therefore $2a^2b \times 5b^2c \times 3cd = 30a^2b^3c^2d$.

EXERCISE XXII

Simplify

1. $xy \times xy^2$.
2. $a^2b \times a^2b^3$.
3. $f^3g \times gf$.
4. $mn^2 \times ny$.
5. $p^2qr \times pq^2r$.
6. $4bc \times 2b^2c^3$.
7. $5km \times 2m^2$.
8. $m^2np \times np \times pm^2$.
9. $2xy \times 7xy^2$.
10. $3ab^3 \times 2a^2b \times 4ab$.
11. $3mn \times 6nm \times 7m$.
12. $p^2qr \times 2pq^2r$.
13. $5bc \times 8ca^2 \times 2abc$.
14. $3k^2m \times 7m^2n \times 2kmn$.
15. $3abc \times 2a^2bc \times 5abc^2$.
16. $18ef \times f^2e \times 2e^2f$.
17. $4gw^2 \times 5w \times 6g^2$.
18. $13h^3m \times 6m^2n \times mn^3$.
19. $(ab)^2$.
20. $(x^2y)^3$.
21. $(abc)^4$.
22. $(2p^2q)^3$.
23. $(k^2m^3)^2$.
24. $(ab^2c^3)^3$.
25. $(2a^2)^5$.
26. $(5m^2n^3)^2$.
27. $(a^2b)^3 + a^6b^3$.
28. $(2p^2)^3 + (2p^3)^2$.
29. $(4mn)^4 + (2m^2n^2)^2$.
30. $(3fgh)^2 + 5ghf$.

22. We have seen (on p. 20) that the quotient obtained by dividing x by y is written $\frac{x}{y}$. In the same way the quotient obtained by dividing x^3 by y^2 is written $\frac{x^3}{y^2}$. If we are told that numerical values can be given to the letters, then by substituting these a numerical answer can be obtained by arithmetical work, as the following examples will show

EXAMPLE 1

Find the value of $\frac{x^3}{y^2}$ when $x = 4$ and $y = 2$.

If $x = 4$, then
$$x^3 = x \times x \times x = 4 \times 4 \times 4 = 64.$$

If $y = 2$, then
$$y^2 = y \times y = 2 \times 2 = 4.$$

Then
$$\frac{x^3}{y^2} = \frac{64}{4} = \mathbf{16}.$$

EXAMPLE 2

If $P = \frac{2k^2}{3mn^3}$, find P when $k = 3$, $m = 2$, and $n = 1$.

$$\begin{aligned}
\text{If } k = 3, \text{ then } k^2 &= k \times k \\
&= 3 \times 3 \\
&= 9. \\
\therefore 2k^2 &= 18. \\
\text{If } n = 1, \text{ then } n^3 &= n \times n \times n \\
&= 1 \times 1 \times 1 \\
&= 1. \\
\therefore 3mn^3 &= 3 \times m \times n^3 \\
&= 3 \times 2 \times 1 \\
&= 6. \\
P &= \frac{2k^2}{3mn^3} \\
&= \frac{18}{6}. \\
\therefore P &= 3.
\end{aligned}$$

EXERCISE XXIII

1. Find the value of

(a) $\frac{a^4}{b^2}$ when $a = 3$ and $b = 1$.

(b) $\frac{2c^2k}{a}$ when $c = 5$, $k = 4$, and $a = 10$.

(c) $\left(\frac{ab}{cd}\right)^2$ when $a = 4$, $b = 3$, $c = 2$, and $d = 1$.

(d) $\left(\frac{x^2y}{z}\right)^3$ when $x = 3$, $y = 5$, and $z = 9$.

(e) $\frac{f^2g^3}{h^2}$ when $f = g = h = 1$.

2. If $P = \dfrac{3K^2}{MN^3}$, find P when $M = 1$, $N = 3$, and $K = 6$.

3. If $A = \dfrac{4pr^3}{3}$, find A when $p = 3$ and $r = 2$.

4. If $X = \dfrac{a}{b} + \dfrac{b}{c}$, find X when $a = 6$, $b = 3$, and $c = 1$.

5. Find the value of $\dfrac{x^2}{y^3z^3}$ when $x = 12$, $y = 3$, and $z = 2$.

6. Find the value of $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2}$ when $x = 1$, $y = 2$, $a = 3$, and $b = 4$.

23. Sometimes the quotients, such as those we have just considered, can be much simplified. This is so when powers of the same quantity are being divided. Thus $x^5 \div x^3$ is written $\dfrac{x^5}{x^3}$. This can be simplified in the following way:

$$\begin{aligned} x^5 \div x^3 &= \frac{x^5}{x^3} \\ &= \frac{x \times x \times \cancel{x} \times \cancel{x} \times \cancel{x}}{\cancel{x} \times \cancel{x} \times \cancel{x}} \\ &= x \times x \\ &= x^2. \end{aligned}$$

Similarly,
$$\begin{aligned} p^8 \div p^3 &= \frac{p^8}{p^3} \\ &= \frac{p \times p \times p \times p \times p \times \cancel{p} \times \cancel{p} \times \cancel{p}}{\cancel{p} \times \cancel{p} \times \cancel{p}} \\ &= p^5. \end{aligned}$$

Notice that in the division of two powers of the same quantity the index of the quotient is the difference between the indices of the quantities.

The same method is used when products of two or more quantities are involved, as in the following example.

Simplify $x^2y^5z^2 \div x^3y^2z^3$.

$$\frac{x^2y^5z^2}{x^3y^2z^3} = \frac{\not x \times \not x \times \not y \times \not y \times y \times y \times y \times \not z \times \not z}{\not x \times \not x \times x \times \not y \times \not y \times \not z \times \not z \times z}$$

$$= \frac{y^3}{xz}.$$

EXERCISE XXIV

Simplify

1. $a^5 \div a^2$.
2. $p^{10} \div p^2$.
3. $m^8 \div m$.
4. $g^{12} \div g^7$.
5. $x^{15} \div x^{14}$.
6. $w^9 \div w^3$.
7. $n^{12} \div n^8$.
8. $h^3 \div h$.
9. $xy \div x$.
10. $x \div xy$.
11. $a^2b \div ab^2$.
12. $a^3b^5 \div ab^2c$.
13. $xy^2 \div x^2z^3$.
14. $c^2fg \div f^2gc$.
15. $p^2qr \div r^2qp$.
16. $m^2n^2p^2 \div mn^2p^3$.
17. $abcd \div bcd^2$.
18. $g^7h^5 \div g^3h^9$.
19. $k^2lm^4 \div kl^2m$.
20. $p^3q^2r^5 \div p^4qr^6$.

24. We have seen that the product obtained by multiplying a number by itself gives the square of the number. Thus $5 \times 5 = 5^2 = 25$, and we say that 25 is the square of 5. We shall now consider the question of finding a number which when squared produces a given number. For example, suppose our given number is 36. Then we require a number which when multiplied by itself gives 36. This is clearly 6, because $6 \times 6 = 36$. In such a case 6 is called the *square root* of 36.

Notice that 6 squared is 36, and that the square root of 36 is 6.

We use the symbol $\sqrt{\ }$ to stand for the words "square root of," and write $\sqrt{36} = 6$.

Similarly, $\sqrt{64} = 8$, and so on.

Some square roots can be calculated exactly—*e.g.*, $\sqrt{\ }$,

$\sqrt{9}$—but others can be obtained only approximately. Thus $\sqrt{2} = 1{\cdot}414$ and $\sqrt{3} = 1{\cdot}732 \ldots$ Here is a complete list for the first ten numbers.

$\sqrt{1} = 1$	$\sqrt{6} = 2{\cdot}449$
$\sqrt{2} = 1{\cdot}414$.	$\sqrt{7} = 2{\cdot}646$.
$\sqrt{3} = 1{\cdot}732 \ldots$	$\sqrt{8} = 2{\cdot}828 \ldots$
$\sqrt{4} = 2$	$\sqrt{9} = 3$
$\sqrt{5} = 2{\cdot}236 \ldots$	$\sqrt{10} = 3{\cdot}162 \ldots$

EXERCISE XXV

1. Find the value of $\sqrt{16}$, $\sqrt{36}$, $\sqrt{25}$, $\sqrt{49}$, $\sqrt{64}$, and $\sqrt{144}$.

2. Find the value of

(*a*) $\sqrt{4} + \sqrt{25}$. (*f*) $\sqrt{196} - \sqrt{169}$.

(*b*) $\sqrt{16} + \sqrt{121}$. (*g*) $\sqrt{225} + \sqrt{100}$.

(*c*) $\sqrt{169} - \sqrt{49}$. (*h*) $\sqrt{289} - \sqrt{256}$.

(*d*) $\sqrt{64} - \sqrt{36}$. (*i*) $\sqrt{324} + \sqrt{121} - \sqrt{64}$.

(*e*) $\sqrt{4} + \sqrt{9} + \sqrt{16}$. (*j*) $\sqrt{256} + \sqrt{289} + \sqrt{169}$.

3. If $a = 36$, $b = 64$, and $c = 81$, find the value of

(*a*) $\sqrt{a} + \sqrt{b} + \sqrt{c}$. (*d*) $\sqrt{a} + \sqrt{b} - \sqrt{c}$.

(*b*) $\sqrt{b} - \sqrt{a}$. (*e*) $\sqrt{b} - \sqrt{a} + \sqrt{c}$.

(*c*) $\sqrt{c} - \sqrt{a}$.

4. If $x = 3$, $y = 7$, and $z = 6$, find to two places of decimals the value of

(*a*) $\sqrt{x} + \sqrt{y} + \sqrt{z}$. (*d*) $\sqrt{z} - \sqrt{x}$.

(*b*) $\sqrt{x} + \sqrt{y} - \sqrt{z}$. (*e*) $\sqrt{y} - \sqrt{z} + \sqrt{x}$.

(*c*) $\sqrt{y} - \sqrt{z}$.

25. If letters are used to represent numbers the square root is obtained in a similar way. Thus since $a^2 = a \times a$, the square root of a^2 is a, and we write $\sqrt{a^2} = a$.

Similarly, since $k^8 = k^4 \times k^4$

$$\sqrt{k^8} = k^4.$$

Again, since $9x^4y^2 = 3x^2y \times 3x^2y$

$$\sqrt{9x^4y^2} = 3x^2y.$$

Notice the effect of taking the square root of a number raised to a power. The index of the square root is one-half the index of the original number.

EXERCISE XXVI

1. Simplify $\sqrt{x^4}$, $\sqrt{a^8}$, $\sqrt{p^6}$, $\sqrt{s^{10}}$, and $\sqrt{T^{12}}$.

2. Simplify $\sqrt{4m^2}$, $\sqrt{16q^6}$, $\sqrt{49k^{10}}$, $\sqrt{100x^4}$, and $\sqrt{169w^8}$.

3. Simplify $\sqrt{a^4b^4}$, $\sqrt{c^{10}b^6}$, $\sqrt{a^4b^2c^2}$, $\sqrt{p^4q^8r^6}$, and $\sqrt{g^2h^4m^2}$.

4. Simplify $\sqrt{16x^4y^2}$, $\sqrt{25a^6b^2}$, $\sqrt{81p^4q^2}$, $\sqrt{144a^2h^4c^2}$, and $\sqrt{49x^2y^2z^2}$.

26. We have seen that a number when multiplied by itself three times is said to be 'cubed.' Thus we write:

$$5 \times 5 \times 5 = 5^3 = 125.$$

From this we see that the *cube root* of 125 is 5. This is written $\sqrt[3]{125} = 5$.

Similarly, since $3 \times 3 \times 3 \times 3 = 81$, the *fourth root* of 81 is 3. This is written $\sqrt[4]{81} = 3$.

Again, $2 \times 2 \times 2 \times 2 \times 2 = 32$, so that the *fifth root* of 32 is 2. This is written $\sqrt[5]{32} = 2$.

EXERCISE XXVII

Find the value of

1. $\sqrt[3]{8}$.	11. $\sqrt[5]{32} + \sqrt[3]{27}$.
2. $\sqrt[4]{16}$.	12. $\sqrt{121} + \sqrt[3]{1728}$.
3. $\sqrt[3]{64}$.	13. $\sqrt[3]{64} + \sqrt[4]{81}$.
4. $\sqrt[5]{243}$.	14. $\sqrt[3]{1000} - \sqrt{100}$.
5. $\sqrt[7]{128}$.	15. $\sqrt{169} - \sqrt[8]{256}$.
6. $\sqrt[4]{256}$.	16. $\sqrt[10]{1024} + \sqrt[7]{2187}$.
7. $\sqrt[3]{343}$.	17. $\sqrt[4]{625} + \sqrt[3]{216}$.
8. $\sqrt[3]{1000}$.	18. $\sqrt{64} - \sqrt[3]{343}$.
9. $\sqrt[4]{1296}$.	19. $\sqrt[3]{512} + \sqrt[3]{729}$.
10. $\sqrt[5]{3125}$.	20. $\sqrt[3]{1331} + \sqrt[3]{1728}$.

27. When letters are used to represent numbers cube and other roots are calculated in a similar way to that just described. Thus $a \times a \times a = a^3$; therefore $\sqrt[3]{a^3} = a$. Again, $k^6 = k^2 \times k^2 \times k^2$, so that $\sqrt[3]{k^6} = k^2$.

Notice that the index of the cube root is one-third of the index of the original quantity.

Study the following example carefully.

EXAMPLE 1

Find the cube root of $343x^6y^{12}z^3$.

The cube root of 343 is 7
The cube root of x^6 ,, x^2
The cube root of y^{12} ,, y^4
The cube root of z^3 ,, z

$$\sqrt[3]{343x^6y^{12}z^3} = 7x^2y^4z.$$

The fourth and higher roots are calculated in a similar way, as the following examples will show.

EXAMPLE 2

Find the fifth root of $32x^{10}y^{15}$.

$$\left.\begin{array}{l}\text{The fifth root of } 32 \text{ is } 2\\ \text{The fifth root of } x^{10} \text{ ,, } x^2\\ \text{The fifth root of } y^{15} \text{ ,, } y^3\end{array}\right\} \qquad \sqrt[5]{32x^{10}y^{15}} = 2x^2y^3.$$

EXAMPLE 3

Find the fourth root of $256a^4b^8$.

$$\left.\begin{array}{l}\text{The fourth root of } 256 \text{ is } 4\\ \text{The fourth root of } a^4 \text{ ,, } a\\ \text{The fourth root of } b^8 \text{ ,, } b^2\end{array}\right\} \qquad \sqrt[4]{256a^4b^8} = 4ab^2.$$

EXERCISE XXVIII

Simplify

1. $\sqrt[3]{a^6}$.
2. $\sqrt[4]{x^8}$.
3. $\sqrt[7]{c^{14}}$.
4. $\sqrt[6]{a^{12}}$.
5. $\sqrt[5]{p^{10}}$.
6. $\sqrt[8]{r^{16}}$.
7. $\sqrt[7]{m^{21}}$.
8. $\sqrt[3]{x^3y^3}$.
9. $\sqrt[3]{m^3n^6}$.
10. $\sqrt[4]{p^4q^8}$.
11. $\sqrt[3]{27x^3y^3}$.
12. $\sqrt[4]{256a^8b^8}$.
13. $\sqrt[5]{32m^5n^5}$.
14. $\sqrt{64m^8p^2}$.
15. $\sqrt[3]{125h^6g^9}$.
16. $\sqrt[3]{343f^{12}g^{12}h^{12}}$.
17. $\sqrt{4a^2b^4c^6d^8}$.
18. $\sqrt[6]{64x^6y^6}$.
19. $\sqrt[5]{243p^{10}t^{15}}$.
20. $\sqrt{49a^8m^{16}}$.

EXERCISE XXIX (REVISION EXERCISE)

(A)

1. If $a = 2$, $b = 3$, and $c = 4$, find the value of

(a) $a + b + c$.

(b) $a + 2b + 3c$.

(c) $a + b^2 + c^3$.

(d) $a^2 + b^3 + c^4$.

(e) $a^3 + 2b^2 + 3c$.

2. Simplify the following:

(a) $\sqrt{4a^2b^4}$.
(b) $(3x^2)^3 + (4x^3)^2$.
(c) $(3a^2)^3 + (4b^3)^2$.
(d) $\sqrt{16x^2y^2} - \sqrt[3]{8x^3y^3}$.
(e) $\sqrt{16a^2b^2} - \sqrt[3]{8x^3y^3}$.

3. From the following formula find P when $r = 5$, $l = 4$, and $m = 1$.

$$P = \frac{rl^2}{8m^3}.$$

4. Simplify the following:

(a) $f^3gh^2 \div fh$. (b) $a^4b^2c \div ab^3c^2$. (c) $x^5y \div y^5x$.

5. (a) A man buys 4 lb. of apples at 3*d.* per lb. and 12 oranges at 2*d.* each. How much did he spend altogether?

(b) A man buys m lb. of apples at x pence per lb. and n oranges at y pence each. Write down a formula for the total cost in pence (D) of this purchase.

(c) From the formula obtained in (b) find D when $m = 12$, $x = 4$, $n = 8$, and $y = 1$.

(B)

6. If $x = \sqrt{2}$ and $y = \sqrt{3}$, find the value of

(a) $x^2 + y^2$.
(b) $2x^2 + 3y^2$.
(c) $\dfrac{x^2 + y^2}{y^2 - x^2}$.
(d) $x + y$ } to three places of decimals.
(e) $y - x$ }

7. A bag contains 40 marbles. Write down a formula for the total weight (w) if each marble weighs x ounces and the bag itself weighs y ounces.

8. From the formula $R = \dfrac{ax^2}{10y^3}$ find R when $a = 5$, $x = 80$, and $y = 10$.

9. (a) The charge for admission to an exhibition is 2*s.* 6*d.*,

and at the end of one day it was found that £134 had been taken. How many persons went to the exhibition?

(*b*) The charge for admission to an exhibition is k shillings. If the takings on a certain day amount to £P, write down a formula for the number of persons (n) who were present.

(*c*) From the formula obtained in (*b*) find n when $P = 186$ and $k = 5$.

10. Simplify

(*a*) $2c + 5c + 7c + 8c$.
(*b*) $3x + 4y + 5z + 6y + x + 4z + 2x + 3z + y$.
(*c*) $\sqrt[4]{81a^4b^8c^4}$.

(C)

11. Express x tons $+\ y$ cwts. $+\ z$ lb. in (*a*) tons, (*b*) cwts., (*c*) pounds.

12. Find the value of each of the following to three places of decimals:

(*a*) $\sqrt{2} + \sqrt{3} + \sqrt{5}$.
(*b*) $\sqrt{3} + \sqrt{7} + \sqrt{9}$.
(*c*) $\sqrt{4} + \sqrt{6} + \sqrt{8}$.

13. Simplify the following:

(*a*) $4x \times 7y \times 8z$.
(*b*) $3ab \times 4bc \times 9ca$.
(*c*) $2fg^2 \times 7g^3hf$.
(*d*) $(7k^2)^3$.
(*e*) $2c + 5m + 7mc + 8m + 4cm$.

14. (*a*) A man's step is 30 inches long. How many steps does he make in one hour if he is walking at the rate of 3 miles an hour?

(*b*) A man's step is k inches long. Write down a formula for the number of steps (D) which he makes in an hour if he is walking at the rate of x miles an hour.

15. Find the value of $(2a)^3 + 2a^3$ when $a = 2$.

(D)

16. (*a*) Write down the area of each of the following rectangles:

(i) 3 inches long by 7 inches wide.
(ii) 4 cm. long by 19 cm. wide.
(iii) p feet long by 4 feet wide.
(iv) a yards long by b yards wide.

(*b*) Write down a formula expressing the area of a rectangle (A) in terms of its length (l) and width (w).

17. Fig. 8 is made up of two rectangles.

(*a*) What is the area of the rectangle ABCD if AB $= a$ inches, BC $=$ GH $= b$ inches?

(*b*) What is the area of the rectangle GEFH if GE $=$ c inches?

(*c*) Write down a formula for the total area (A) of Fig. 8.

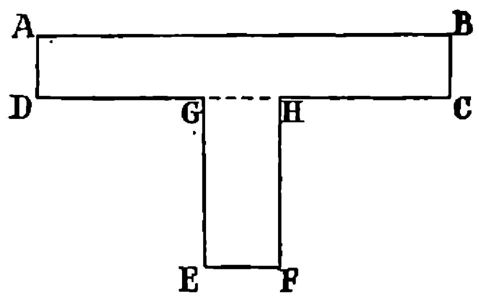

FIG. 8

18. Fig. 9 is the plan of a lawn x feet long, y feet wide,

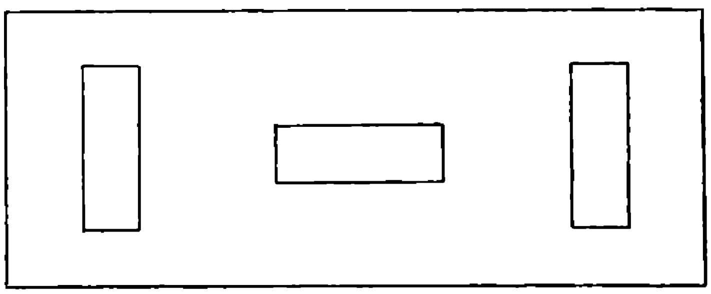
FIG. 9

and containing three equal flower-beds, each measuring m feet long and n feet wide.

(*a*) Write down a formula for the area of the lawn (A).

(*b*) From this formula find A when $x = 75$, $y = 42$, $m = 18$, and $n = 10$.

19. Fig. 10 is the plan of a square courtyard, in the centre of which is a square lawn. The courtyard is p feet square, and the lawn is q feet square.

(*a*) Write down a formula for the area (A) of the pathway round the lawn.

(*b*) From this formula calculate the area of the pathway if $p = 15$ and $q = 13$.

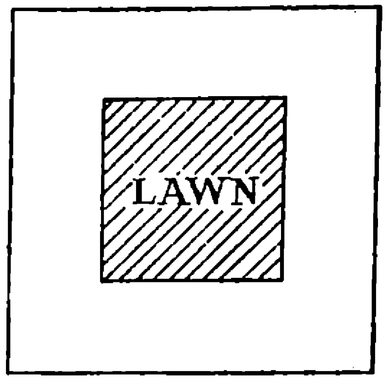

Fig. 10

20. A trapezium is a four-sided figure in which two of the sides are parallel. The area of a trapezium is the product of half the sum of the parallel sides and the perpendicular distance between them. Using the letters shown in Fig. 11, write down a formula for the area of a trapezium, and use it to find the area of a trapezium in which the parallel sides are 3 inches and 5 inches and are 2 inches apart.

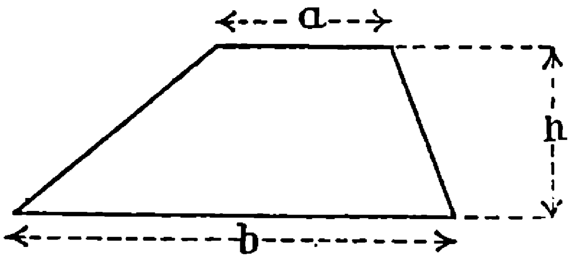

Fig. 11

CHAPTER III

H.C.F., L.C.M., AND FRACTIONS

28. When a number is expressed as a product of two or more numbers it is said to have been *factorized*. Thus we know that $12 = 4 \times 3$. In this case 4 and 3 are factors of 12. Notice that

$$12 = 4 \times 3.$$
$$12 = 6 \times 2.$$
$$12 = 12 \times 1.$$
$$12 = 2 \times 2 \times 3.$$

From which we can see at once that other factors of 12 are 6 and 2; 12 and 1; and 2, 2, and 3.

Algebraical quantities can be factorized in a similar way. Thus $3a^2 = 3 \times a \times a$. In other words, 3, a, and a are the three factors of $3a^2$.

Similarly $5p^3q^2 = 5 \times p \times p \times p \times q \times q$, showing the six factors of $5p^3q^2$.

EXERCISE XXX

1. Write down the factors of 28, 45, 108, and 121.
2. Express each of the following as a product of prime factors: 16, 18, 28, 45, and 72.
3. Write down the factors of $5t^3$, $7a^4$, $2x^2y$, $11pq^3r^2$, and $13abcd$.

29. We shall now apply this process of factorizing to the finding of the highest common factor (or H.C.F.) of two or more numbers.[1] This is the largest number which will divide exactly into the numbers we are considering. In such calculations we always use prime factors.

[1] The highest common factor is sometimes known as the greatest common measure (or G.C.M.).

Example 1

Find the H.C.F. of 60 and 84.

$$60 = \textcircled{2} \times \textcircled{2} \times \textcircled{3} \times 5.$$
$$84 = \textcircled{2} \times \textcircled{2} \times \textcircled{3} \times 7.$$

Factors which are to be found in both numbers are said to be *common* to them. In this example the common factors have been paired off. The product of the common factors is the *highest common factor* of the numbers. In this case,

The common factors of 60 and 84 are 2, 2, and 3.
The highest common factor is $2 \times 2 \times 3 = 12$.

Hence 12 is the largest number which will divide exactly into 60 and 84.

Example 2

Find the H.C.F. of 42, 63, and 84.

$$42 = 2 \times \textcircled{3} \times \textcircled{7}.$$
$$63 = 3 \times \textcircled{3} \times \textcircled{7}.$$
$$84 = 2 \times 2 \times \textcircled{3} \times \textcircled{7}.$$

(Notice that all the factors are prime.)

The common factors are 3 and 7.
The highest common factor is 21.

EXERCISE XXXI

Find the highest common factor in each of the following examples:

1. 6 and 9.
2. 24 and 84.
3. 22 and 110.
4. 45 and 60.
5. 32 and 48.
6. 15, 20, and 25.
7. 28, 42, and 56.
8. 36, 54, and 72.
9. 46, 69, and 92.
10. 48, 72, and 96.

30. The H.C.F. of two or more algebraical quantities is found in an exactly similar way, as the following examples will show.

EXAMPLE 1

Find the H.C.F. of a^2b and ab^2c.

$$a^2b = \textcircled{a} \times a \times \textcircled{b}.$$
$$ab^2c = \textcircled{a} \times b \times \textcircled{b} \times c.$$

The common factors are a and b.
The highest common factor is ab.

EXAMPLE 2

Find the H.C.F. of $12p^3q^2r$ and $18p^2qr^3$.

$$12p^3q^2r = \textcircled{2} \times 2 \times \textcircled{3} \times \textcircled{p} \times \textcircled{p} \times p \times \textcircled{q} \times q \times \textcircled{r}.$$
$$18p^2qr^3 = \textcircled{2} \times 3 \times \textcircled{3} \times \textcircled{p} \times \textcircled{p} \times \textcircled{q} \times \textcircled{r} \times r \times r.$$

The common factors are 2, 3, p, p, q, and r.
The highest common factor is $6p^2qr$.

EXERCISE XXXII

Find the highest common factor in each of the following examples:

1. x^3y and xy^3.
2. a^2bc^2 and a^3b^2c.
3. $3p^2q$ and $5pq^2$.
4. $7xy^2z$ and $11yz^2$.
5. $23m^3n^2$ and $7m^2n^2p$.
6. a^2b, ab^2, and a^2b^2.
7. $2x^5y^2z^3$, $3x^4y^3z$, and $5x^2yz^4$.
8. $15x^3$, $3x^4y$, and $5x^3y^2$.
9. $6p^3q^5$, $9pq^2$, and $12p^2q^3$.
10. $21a^2bc^3$, $35ab^2c$, and $21a^3bc$.

11. $4m^2l$, $20l^2n$, and $28lmn$.
12. a, $2ab$, $3abc$, and $4abcd$.
13. $2x^2$, $4x^4$, and $6x^6$.
14. p^3q^2, $7p^2q^3$, and $11p^3q$.
15. $15t^2wx$, $20tw^2x^3$, and $30t^3wx^2$.

31. The *lowest common multiple* (or L.C.M.) of some numbers is the smallest number that is exactly divisible by them all. Thus 12 is the L.C.M. of 2, 3, and 4, since 12 is the smallest number that can be divided by 2, 3, and 4 without leaving a remainder. The L.C.M. of two or more numbers is the product of the highest powers of their prime factors. Consider the following example.

Find the L.C.M. of 6, 24, and 36.

$$6 = 2 \times 3.$$
$$24 = 2^3 \times 3.$$
$$36 = 2^2 \times 3^2.$$

The L.C.M. of 6, 24, and 36 is $2^3 \times 3^2$
$$= 8 \times 9$$
$$= 72.$$

In other words, 72 is the smallest number which is exactly divisible by 6, 24, and 36.

EXERCISE XXXIII

Find the L.C.M. in each of the following examples:

1. 3, 5, and 7.
2. 10, 12, and 18.
3. 9, 12, and 18.
4. 14, 18, and 21.
5. 15, 25, and 30.
6. 12, 36, and 50.
7. 28, 45, and 50.
8. 2, 3, 4, and 5.
9. 4, 6, 9, and 12.
10. 12, 14, 21, and 28.

32. The L.C.M. of two or more algebraical quantities is obtained in an exactly similar way, as the following examples will show.

EXAMPLE 1

Find the L.C.M. of a^3b, ab^2c, and a^2c^2.

$$a^3b = \mathbf{a}^3 \times b.$$
$$ab^2c = a \times \mathbf{b}^2 \times c.$$
$$a^2c^2 = a^2 \times \mathbf{c}^2.$$

The L.C.M. is the product of the highest power of each factor. In this case

The L.C.M. of a^3b, ab^2c, and $a^2c^2 = a^3 \times b^2 \times c^2$
$= a^3b^2c^2$.

EXAMPLE 2

Find the L.C.M. of $9x^3y$, $16xy^3z$, and $18x^2yz^2$.

$$9x^3y = \mathbf{3}^2 \times \mathbf{x}^3 \times y.$$
$$16xy^3z = \mathbf{2}^4 \times x \times \mathbf{y}^3 \times z.$$
$$18x^2yz^2 = 2 \times 3^2 \times x^2 \times y \times \mathbf{z}^2.$$

Hence the required L.C.M. is $3^2 \times 2^4 \times x^3 \times y^3 \times z^2$
$= 144x^3y^3z^2$.

EXERCISE XXXIV

Find the L.C.M. in each of the following examples:

1. a^2b, ab^2, and a^2b^2.
2. x^3yz, xy^3z, and xyz^3.
3. m^2n^3, m^3n^2, and mn^2.
4. $3p^2q$, $6pq$, and $9pq^2$.
5. $4t^2$, $12t^2w$, and $9w^2t$.
6. $6a^2b$, $8ab^2c$, and $18a^2c^3$.
7. $2a$, $4b$, $6c$, and $8d$.
8. $45p^3r^2$ and $75r^5q$.
9. $3l^2m$, $5m^2n$, and $7n^2l$.
10. $9a^5x$, $12ax^4y$, and $18axy$.
11. $8t^2v$, $28tv^3w$, and $14v^2w^2$.
12. x, x^2y, x^3y^2, and x^4y^3.
13. $2a$, $3b$, $4c$, and $5d$.
14. $12p^2qr$, $14pq^2r$, and $16pqr^2$.
15. $27k^2m$, $36kmp$, and $12m^2p^3$.

33. Fractions in algebra have the same meaning and are manipulated by the same rules as fractions in arithmetic.

Thus we know that $\frac{2}{7}+\frac{3}{7}=\frac{5}{7}.$

Similarly, $\frac{2a}{7}+\frac{3a}{7}=\frac{5a}{7}$

and $\frac{4}{x}-\frac{1}{x}=\frac{3}{x}.$

EXERCISE XXXV

Simplify the following:

1. $\frac{2}{5}+\frac{1}{5}.$
2. $\frac{2}{m}+\frac{1}{m}.$
3. $\frac{2}{5}-\frac{1}{5}.$
4. $\frac{2}{m}-\frac{1}{m}.$
5. $\frac{7}{13}+\frac{4}{13}.$
6. $\frac{7a}{13}+\frac{4a}{13}.$
7. $\frac{7}{13}-\frac{4}{13}.$
8. $\frac{7a}{13}-\frac{4a}{13}.$
9. $\frac{5x}{11}+\frac{7x}{11}.$
10. $\frac{a}{13}+\frac{b}{13}.$
11. $\frac{5}{19}+\frac{3}{19}+\frac{4}{19}.$
12. $\frac{5}{p}+\frac{3}{p}+\frac{4}{p}.$
13. $\frac{a}{p}+\frac{b}{p}+\frac{c}{p}.$
14. $\frac{8}{15x}-\frac{2}{15x}.$
15. $\frac{8a}{15x}-\frac{2a}{15x}.$
16. $\frac{p}{15x}-\frac{q}{15x}.$
17. $\frac{p}{x}+\frac{q}{x}-\frac{r}{x}.$
18. $\frac{7}{a}+\frac{8}{a}-\frac{2}{a}.$
19. $\frac{7m}{a}+\frac{8m}{a}-\frac{2m}{a}.$
20. $\frac{7m}{5a}+\frac{8m}{5a}-\frac{2m}{5a}.$

34. We know from our arithmetic that the multiplication of the numerator and denominator of a fraction by the same number does not affect the value of the fraction.

Thus, $$\frac{5}{8}=\frac{5\times 2}{8\times 2}=\frac{10}{16}.$$

Similarly, $$\frac{5}{8}=\frac{5\times m}{8\times m}=\frac{5m}{8m}$$

and $$\frac{4}{7x^2y}=\frac{4\times xy^2}{7x^2y\times xy^2}=\frac{4xy^2}{7x^3y^3}.$$

EXERCISE XXXVI

Find the value of K in each of the following:

1. $\frac{4}{7}=\frac{K}{28}.$
2. $\frac{3}{11}=\frac{15}{K}.$
3. $\frac{4x}{7}=\frac{K}{28}.$
4. $\frac{3m}{11}=\frac{15m}{K}.$
5. $\frac{9}{13}=\frac{9a}{K}.$
6. $\frac{11xy}{17}=\frac{K}{17x^2y}.$
7. $\frac{5a}{7b}=\frac{15a^2m}{K}.$
8. $\frac{K}{a}=\frac{15b^2}{3a}.$
9. $\frac{12}{ab}=\frac{36a^2b^2}{K}.$
10. $\frac{3x}{5yz}=\frac{K}{5xyz}.$
11. $\frac{7}{2w^2}=\frac{49a}{K}.$
12. $\frac{ab}{pq}=\frac{K}{3p^2q^2}.$
13. $\frac{2}{5}=\frac{K}{15a^2m}.$
14. $\frac{K}{7x}=\frac{2axy}{14x^2y}.$
15. $\frac{8p}{K}=\frac{72pqr}{9mqr}.$

35. When the numerator and denominator of a fraction are divided by the same number the value of the fraction is unaltered, but it is said to have been reduced to its lowest terms. This process is sometimes described as *cancelling*.

Thus, $\frac{10}{16} = \frac{5 \times \cancel{2}}{8 \times \cancel{2}} = \frac{5}{8}$ (dividing numerator and denominator by 2).

Similarly, $\frac{10a^2}{16a^2} = \frac{5 \times \cancel{2} \times \cancel{a^2}}{8 \times \cancel{2} \times \cancel{a^2}} = \frac{5}{8}$ (dividing numerator and denominator by $2a^2$)

and $\frac{p^2qr}{pq^2r} = \frac{\cancel{p} \times p \times \cancel{q} \times \cancel{r}}{\cancel{p} \times \cancel{q} \times q \times \cancel{r}} = \frac{p}{q}$ (dividing numerator and denominator by pqr).

EXERCISE XXXVII

1. Reduce the following fractions to their lowest terms:

(a) $\frac{18}{30}$.

(b) $\frac{18m^2}{30m^2}$.

(c) $\frac{18m^2}{30m}$.

(d) $\frac{am^2}{bm}$.

(e) $\frac{5x^2y}{13xy^2}$.

(f) $\frac{4ab}{12bc}$.

(g) $\frac{3p^2q}{15q^2r}$.

(h) $\frac{ab^2c}{abc^2}$.

(i) $\frac{7wt^2v}{21w^2v^2}$.

(j) $\frac{6am^2p}{9a^2mp^2}$.

(k) $\frac{12p^2rt}{20prt^2}$.

(l) $\frac{10x^2y^3z}{35xy^2z^4}$.

(m) $\frac{m^2tu^2}{t^2uv^2}$.

(n) $\frac{12a^2bc^2d}{27ab^2cd^2}$.

(o) $\frac{12xy^2z^3}{14x^3y^2z}$.

2. Find the value of P in each of the following:

(a) $\dfrac{P}{5ab} = \dfrac{6ab}{15a^2b^2}$. (d) $\dfrac{ab}{P} = \dfrac{a^2bc}{ac^2d}$.

(b) $\dfrac{4x}{P} = \dfrac{12x^2y^3}{9xy^4}$. (e) $\dfrac{3x}{5y} = \dfrac{P}{5xy^2}$.

(c) $\dfrac{2k}{3m} = \dfrac{8k^2m}{P}$.

36. When it is necessary to add or subtract fractions whose denominators are unlike the first step is to express each of the fractions with a new denominator. This is usually the L.C.M. of the denominators of the fractions. The addition or subtraction can now be performed in the manner described in the previous pages.

Example 1

Simplify $\dfrac{2}{9} + \dfrac{1}{6}$.

The L.C.M. of 9 and 6 is 18.

$$\left.\begin{aligned} \frac{2}{9} &= \frac{2 \times 2}{9 \times 2} = \frac{4}{18}. \\ \frac{1}{6} &= \frac{1 \times 3}{6 \times 3} = \frac{3}{18}. \end{aligned}\right\}$$ This expresses each of the fractions in terms of the new denominator 18.

$$\frac{2}{9} + \frac{1}{6} = \frac{4}{18} + \frac{3}{18}$$
$$= \frac{7}{18}.$$

Example 2

Simplify $\dfrac{3}{ab^2} - \dfrac{2}{a^2b}$.

The L.C.M. of ab^2 and a^2b is a^2b^2.

$$\frac{3}{ab^2} = \frac{3 \times a}{ab^2 \times a} = \frac{3a}{a^2b^2}.$$

$$\frac{2}{a^2b} = \frac{2 \times b}{a^2b \times b} = \frac{2b}{a^2b^2}.$$

$$\frac{3}{ab^2} - \frac{2}{a^2b} = \frac{3a}{a^2b^2} - \frac{2b}{a^2b^2}$$

$$= \frac{3a - 2b}{a^2b^2}.$$

EXERCISE XXXVIII

Simplify the following:

1. $\frac{1}{5} + \frac{2}{3}$.
2. $\frac{3}{4} - \frac{1}{6}$.
3. $\frac{1}{x} + \frac{1}{y}$.
4. $\frac{5}{ab} - \frac{3}{a}$.
5. $\frac{m}{2} + \frac{m}{3}$.
6. $\frac{k}{5} - \frac{k}{7}$.
7. $\frac{2x}{5} - \frac{x}{6}$.
8. $\frac{3x}{2} + \frac{4x}{3} + \frac{5x}{4}$.
9. $\frac{a}{x} + \frac{a}{x^2}$.
10. $\frac{4}{3x} - \frac{3}{4x}$.
11. $\frac{4}{3x} - \frac{3}{4x^2}$.
12. $\frac{a}{b} + \frac{b}{a}$.
13. $\frac{a}{b} - \frac{b}{a}$.
14. $\frac{a}{2x} + \frac{a}{2y}$.
15. $\frac{2}{3x} + \frac{3}{4x} + \frac{4}{5x}$.
16. $\frac{5}{xy} - \frac{3}{y^2}$.

17. $\frac{5p}{3q} + \frac{5q}{3p}$.

18. $\frac{1}{x^2y} - \frac{1}{xy^2}$.

19. $\frac{2}{w^2v^3} - \frac{3v}{w}$.

20. $\frac{1}{a} + \frac{1}{b} + \frac{1}{c}$.

21. $\frac{2}{x} + \frac{3}{2y} + \frac{4}{3z}$.

22. $\frac{2a}{b} + \frac{3b}{c} + \frac{4c}{a}$.

37. Multiplication of fractions in algebra follows exactly the method used in arithmetic. Thus,

$$\frac{2}{3} \times \frac{4}{5} = \frac{2 \times 4}{3 \times 5} = \frac{8}{15}.$$

Similarly,
$$\frac{a}{b} \times \frac{x}{y} = \frac{a \times x}{b \times y} = \frac{ax}{by}$$

and
$$\frac{2p}{q^2} \times \frac{3q}{5p^2} = \frac{2p \times 3q}{q^2 \times 5p^2}$$
$$= \frac{6pq}{5p^2q^2}$$
$$= \frac{6}{5pq} \quad \text{(dividing numerator and denominator by } pq\text{)}.$$

EXERCISE XXXIX

Simplify the following:

1. $\frac{4}{7} \times \frac{2}{5}$.

2. $\frac{5}{6} \times \frac{2}{3}$.

3. $\frac{a}{b} \times \frac{a}{c}$.

4. $\frac{a}{b} \times \frac{b}{a}$.

5. $\frac{2x}{y} \times \frac{3y}{4x}$.

6. $\frac{5p}{q} \times \frac{rp}{q}$.

7. $\frac{5x}{2} \times \frac{7}{15}$.

8. $\frac{4}{9} \times \frac{2k}{3m}$.

9. $\frac{w}{8} \times \frac{5w}{v}$.

10. $\frac{a}{b} \times \frac{b}{c} \times \frac{c}{a}$.

11. $\frac{2x}{3} \times \frac{3x}{5y}$.

12. $\frac{a^2}{2} \times \frac{2a^2}{3}$.

13. $\frac{4k^2}{e} \times \frac{5k}{2m}$.

14. $\frac{2x^2y}{5} \times \frac{3xy^2}{7}$.

15. $\frac{7pq}{2r} \times \frac{p^2}{q^3}$.

16. $\frac{x^2}{y} \times \frac{y^2}{x}$.

17. $\frac{8p^2q}{3r} \times \frac{9r^2p}{2q}$.

18. $\frac{ax}{p^2} \times \frac{by}{q^2} \times \frac{cz}{r^2}$.

19. $\frac{5x}{a^2} \times \frac{3y}{b^2} \times \frac{z}{c^2}$.

20. $\frac{3a^2b^2}{c^2} \times \frac{2b^2c^2}{a^2} \times \frac{c^2a^2}{6b^2}$.

21. $\frac{1}{m^2} \times \frac{2}{n^2} \times \frac{3}{l^2}$.

22. $\frac{w^3v}{u^2} \times \frac{v^3u}{w^2} \times \frac{u^3w}{v^2}$.

23. $\frac{a^3}{2bc} \times \frac{b^3}{3ca} \times \frac{c^3}{4ab}$.

24. $\frac{2p^3q^2}{5r} \times \frac{7p^2q}{3r^2}$.

25. $\frac{9x^2y}{7m^2} \times \frac{11xy^3}{3n^3}$.

26. $\frac{a}{2bc} \times \frac{3b}{4ca} \times \frac{5c}{6ab}$.

38. The division of fractions in algebra also follows the method employed in arithmetic. Thus,

$$\frac{2}{3} \div \frac{5}{7} = \frac{2}{3} \times \frac{7}{5}$$
$$= \frac{14}{15}.$$

Similarly,
$$\frac{a}{b} \div \frac{x}{y} = \frac{a}{b} \times \frac{y}{x}$$
$$= \frac{ay}{bx}$$

and
$$\frac{2m}{p} \div \frac{p^2}{3m^3} = \frac{2m}{p} \times \frac{3m^3}{p^2}$$
$$= \frac{6m^4}{p^3}.$$

EXERCISE XL

Simplify the following:

1. $\frac{5}{9} \div \frac{2}{3}$.
2. $\frac{4}{7} \div \frac{5}{11}$.
3. $\frac{a}{5} \div \frac{3}{8}$.
4. $\frac{a}{b} \div \frac{3}{8}$.
5. $\frac{2x}{5y} \div \frac{4}{9}$.
6. $\frac{x}{y} \div \frac{y}{x}$.
7. $\frac{x}{y} \div \frac{x}{y}$.
8. $\frac{3}{5} \div \frac{m}{n}$.
9. $\frac{7}{12} \div \frac{3a^2b}{2c}$.
10. $\frac{a^2b}{c} \div \frac{ab}{c^2}$.
11. $\frac{2x^2y}{3} \div \frac{5x}{y}$.
12. $\frac{7a^3}{x} \div \frac{21x^3}{a}$.
13. $\frac{21x^3}{a} \div \frac{7a^3}{x}$.
14. $\frac{5m}{p} \div \frac{3m^2p}{7}$.
15. $\frac{9a^2}{2x} \div \frac{6a}{x^2}$.
16. $\frac{4pq}{r} \div \frac{4r}{pq}$.
17. $\frac{a^3b^2c}{xy} \div \frac{x^3y^2z}{ab}$.
18. $\frac{4w^2v^3}{m} \div \frac{3wv^2}{2m}$.
19. $\frac{7x^2y}{z} \div \frac{y^2}{14z}$.
20. $\frac{12a^2b}{7pq} \div \frac{8p^2q}{9ab^2}$.

EXERCISE XLI (Revision Exercise)

(A)

1. Find the H.C.F. and L.C.M. of the following:
   (*a*) 2, 4, 6, and 8.
   (*b*) a, b, c, and d.
   (*c*) a^3b^2 and b^3a^2.

2. Simplify the following:

(*a*) $\frac{a}{3}+\frac{a}{4}-\frac{a}{6}$.

(*b*) $\frac{3}{a}+\frac{4}{a}-\frac{6}{a}$.

(*c*) $\frac{x}{y}+\frac{y}{z}+\frac{z}{x}$.

3. Simplify the following:

(*a*) $\frac{x^2y}{m}\times\frac{m^2y}{x^3}$.

(*b*) $\frac{3p^2q}{2x}\div\frac{4x^2p^2}{q^2}$.

(*c*) $\frac{a^2b}{c}\times\frac{b^2a}{c}\div\frac{a^2b^2}{c^2}$.

4. If $p = 3$, $q = 5$, and $r = 7$, find the value of

(*a*) $p + 2q + 3r$.
(*b*) $p + q^2 + r^3$.
(*c*) $p + 2q^2 + 3r^3$.

5. Simplify the following:

(*a*) $\frac{16a^4b^2}{8ab^3}$. (*b*) $\frac{7x^7}{3x^3}$. (*c*) $(2a^2)^2$.

(B)

6. If $m = \sqrt{2}$, $n = \sqrt{3}$, and $p = \sqrt{4}$, find the value of

(*a*) $m + n + p$. (*b*) $m^2 + n^2 + p^2$. (*c*) $\frac{3m + 2n}{p}$.

7. (*a*) Express x tons $+ y$ cwts. $+ z$ lb. in (i) tons, (ii) cwts., (iii) lb.

(*b*) Express x days $+ y$ hours $+ z$ minutes in (i) days, (ii) hours, (iii) minutes.

8. Find the H.C.F. and L.C.M. of

(*a*) $4x^3y^2$, $6xy^3$, and $16x^2y$.
(*b*) $3m^3n$, $6m^2n^2$, and $9mn^3$.

9. Simplify the following:

(*a*) $144k^4m^2 \times 9k^6m^4$.

(*b*) $\sqrt{144k^4m^2} \times \sqrt{9k^6m^4}$.

(*c*) $\dfrac{2a^2}{3x} + \dfrac{3b^2}{4y}$.

10. Reduce the following fractions to their lowest terms:

(*a*) $\dfrac{14a^2b^3}{63a^3}$. (*b*) $\dfrac{16p^4r^2}{24p^2qr}$. (*c*) $\dfrac{5xyz.}{125zyx}$.

(C)

11. Find P if $\dfrac{\text{P}}{3a^2bc^2} = \dfrac{20abc}{12a^2b^2c^2}$.

12. Simplify the following:

(*a*) $\dfrac{1}{x} + \dfrac{1}{y} + \dfrac{1}{z}$.

(*b*) $\dfrac{1}{-} \times 1 \times \dfrac{1}{z}$.

(*c*) $\dfrac{5x^2y}{7xz^2} \times \dfrac{3x^2}{4z^2} \div \dfrac{5xy}{6yz}$.

13. (*a*) A man owns 24 houses, each of which pays 12 shillings a week rent. What is his annual income in pounds if the upkeep of the houses is £5?

(*b*) A man owns N houses, each of which pays k shillings a week rent. Write down a formula for his annual income in pounds (I) if the upkeep of the houses is £x.

(*c*) From the formula in (*b*) find I if $\text{N} = 42$, $k = 15$, and $x = 12$.

14. If $a = 1$, $b = 2$, $c = 3$, and $d = 4$, find the value of

(a) $\sqrt{a} + \sqrt{b} + \sqrt{c} + \sqrt{d}$ (to two places of decimals).

(b) $\sqrt{a + 4b}$.

(c) $\sqrt{2abd}$.

15. Simplify

(a) $5p + 7q + 3q + 8p + 9r + 2p + 3r + 5q$.

(b) $\sqrt{64a^3b^9c^6}$.

(D)

16. In the formula $s = ut + \frac{1}{2}ft^2$ find s when $u = 24$, $t = 3$, and $f = 18$.

17. Simplify

(a) $\frac{2x^2y}{5} + \frac{3xy^2}{7}$. (b) $\frac{2x^2y}{5} \div \frac{3xy^2}{7}$.

18. A formula for the area of a triangle (A) is

$$A = \sqrt{s(s-a)(s-b)(s-c)},$$

where $s = \frac{1}{2}(a + b + c)$ and a, b, and c are the lengths of the three sides.

Use this formula to find the area of a triangle whose sides are 6 inches, 8 inches, and 10 inches.

19. If $a = x^2 - 2y$ and $b = y^2 - 2x$, find the value of $a^2 + b^2$ when $x = 3$ and $y = 4$.

20. Say what value must be given to m in order that $\frac{5a}{7b^2} = \frac{20a^2b}{mb^3}$.

CHAPTER IV

BRACKETS AND FACTORS

39. Suppose that a boy has a bag containing 20 marbles, to which he adds first 8 and then 3 more. Altogether he has $20 + 8 + 3 = 31$ marbles. Instead of adding the 8 marbles and the 3 marbles separately he could have added them together, and put $8 + 3$ marbles into the bag at the same time. This would have made no difference to the final number of marbles in the bag. To show that the $8 + 3$ marbles were added at one time we write this as $(8 + 3)$, the brackets showing that the $8 + 3$ is to be considered as one quantity. Since the total number of marbles in the bag is the same whether the 8 marbles and the 3 marbles are put in separately or together, we can write:

$$20 + 8 + 3 = 20 + (8 + 3).$$

Similarly, if the bag contains m marbles and x marbles, and then y marbles are added, the total number is the same as if $(x + y)$ marbles had been put in.

Hence $\qquad m + x + y = m + (x + y).$

The presence of brackets shows that the quantities which they contain are to be considered together, and not separately.

Sometimes it is possible to simplify an expression containing brackets. Consider the following examples.

Example 1

Simplify $(3x + 7y) + (8x + 3y)$.

The position of the brackets shows that one quantity, $(8x + 3y)$, is being added to another quantity, $(3x + 7y)$. As the *order* in which this addition is done does not matter, we can write:

$$\begin{aligned}(3x + 7y) + (8x + 3y) &= 3x + 7y + 8x + 3y \\ &= \underbrace{3x + 8x} + \underbrace{7y + 3y} \text{ (rearranging)} \\ &= 11x + 10y.\end{aligned}$$

EXAMPLE 2

Simplify $p + (2p + 3q + r) + (p + 3r)$.

In this case

$$\begin{aligned} p + (2p + 3q + r) + (p + 3r) \\ &= p + 2p + 3q + r + p + 3r \\ &= p + 2p + p + 3q + r + 3r \\ &= 4p + 3q + 4r. \end{aligned}$$

EXERCISE XLII

Find the value of

1. $2 + (3 + 5)$.
2. $8 + (7 + 9)$.
3. $(6 + 10) + 4$.
4. $a + (2 + 3)$.
5. $(a + 2) + 3$.
6. $p + (p + q)$.
7. $(p + q) + p$.
8. $x + (2x + 3x)$.
9. $(x + 2x) + 3x$.
10. $3 + (2 + 3 + 4)$.
11. $(3 + 4 + 7) + (2 + 1)$.
12. $5 + (9 + 7 + 6) + (6 + 3)$.
13. $a + (2a + 4b + a)$.
14. $(x + 3x) + (5x + 9x + 11x)$.
15. $(2a + 3b) + (4a + 8b)$.
16. $(a + b + c) + (a + 2b + 3c)$.
17. $(x + 2y + 4z) + (8x + 16y + 32z)$.
18. $m + (3m + 2n) + n$.
19. $2w + 3v + (w + 5v)$.
20. $2w + (3v + w) + 5v$.

40. We have seen that $(8 + 3) = 8 + 3 = 11$.

Now,
$$\begin{aligned} 2(8 + 3) &= 2 \times (8 + 3) \\ &= 2 \times 11 \\ &= 22. \end{aligned}$$

But twice (8 + 3) must be the same as two 8's together with two 3's. In other words,

$$2(8 + 3) = 2 \times 8 + 2 \times 3$$
$$= 16 + 6$$
$$= 22.$$

Similarly, $p(x + y) = px + py.$

EXERCISE XLIII

1. Simplify the following:

(a) $3(2 + 5)$.
(b) $7(4 + 6)$.
(c) $10(6 + 12)$.
(d) $5(1 + 2 + 3)$.
(e) $9(2 + 6 + 10)$.
(f) $12(4 + 7)$.
(g) $a(p + q)$.
(h) $a(p + q + r)$.
(i) $x(5 + x)$.
(j) $2m(1 + p + q)$.
(k) $m(1 + m + m^2)$.
(l) $2k(a + bk + ck^2)$.

2. Simplify the following by removing the brackets and collecting together like terms:

(a) $5(x + 2) + 2(x + 5)$.
(b) $12(a + b) + 5(a + 2b)$.
(c) $3(2x + 5y) + 4(3x + 7y)$.
(d) $3(a + 4) + 7(2a + 3)$.
(e) $4(2x + 5) + 5(3 + 4x)$.
(f) $2(a + b + c) + 3(a + 2b + 3c)$.
(g) $2a(b + a) + 2b(a + b)$.
(h) $5a(2 + b) + 4b(2 + a) + 5c(1 + b)$.

41. From what has just been done we have seen that

$$p(x + y) = px + py.$$

In other words, we have a *product* of two quantities expressed as a *sum* of two algebraical quantities. Each of the quantities forming the product is called a *factor*, and the factors of $px + py$ are p and $(x + y)$. The process of expressing an algebraical expression as a product of its factors is called factorizing. Consider the following example.

Find the factors of (*a*) $2xy + 4yz$ and (*b*) $3k^3 + 9k^2 + 12k$.

(*a*) $2xy + 4yz = 2y(x + 2z)$.

(*b*) $3k^3 + 9k^2 + 12k = 3k(k^2 + 3k + 4)$.

EXERCISE XLIV

Factorize the following:

1. $ax + ay$.
2. $2p + 2q$.
3. $2a + 4b$.
4. $mk + kn$.
5. $k^2 + k$.
6. $3k^2p + kp$.
7. $3k^2p + 6kp$.
8. $ax + ay + az$.
9. $a^2x + a^2y + a^2z$.
10. $2p + 4q + 6r$.
11. $3pq^2 + 6p^2q$.
12. $m^3 + m^2n + mn^2$.
13. $k^5 + k^2$.
14. $2a^2x + 4a^2y + 6a^2z$.
15. $5w^3v^2 + 25w^2v^3$.

42. Just as $(8 + 3)$ is considered as one quantity, so $(8 - 3)$ must be treated in the same way. In order to understand this, we will return to the illustration used earlier in the chapter. Suppose a boy has 20 marbles in his bag. If he adds 8 and then takes away 3, the number that remains is $20 + 8 - 3 = 25$ marbles. In other words, the effect of adding 8 and then taking away 3 is the same as an addition of 5 at one time. Consequently we can write:

$$20 + (8 - 3) = 20 + 8 - 3$$
$$= 25.$$

Similarly, $m + (x - y) = m + x - y$.

A multiplying factor outside the brackets multiplies each term inside when the brackets are removed. Thus

$$5(8 - 3) = 5 \times 8 - 5 \times 3$$
$$= 40 - 15$$
$$= 25.$$

Similarly, $2x(p - q) = 2px - 2qx$.

EXERCISE XLV

1. Find the value of

(a) $15 + (7 - 2)$.
(b) $(15 - 7) + 2$.
(c) $(4 - 1) + (3 - 2)$.
(d) $10 + (a - 6)$.
(e) $(10 - a) + 6$.
(f) $p + (p - q)$.
(g) $(p - q) + p$.
(h) $x + (3x - 2x)$.
(i) $(5x - 3x) + 7x$.
(j) $(a + 1) + (5a - 3a + 1)$.
(k) $3(5 - 4)$.
(l) $15(4 - 1)$.
(m) $a(b - c)$.
(n) $2x(p - x + y)$.
(o) $7m(a + m - n)$.

2. Simplify the following by removing the brackets and collecting together like terms:

(a) $3(x + 2) + 2(x - 1)$.
(b) $12(a + b) + 7(2a - b)$.
(c) $3(2x + 5y) + 2(x - y)$.
(d) $4a(p - q) + 2ap$.
(e) $5m(m + n) + 3m(2m - n)$.

3. Factorize the following:

(a) $ax - ay$.
(b) $2p - 2q$.
(c) $2a - 4b$.
(d) $mk - kn$.
(e) $k^2 - k$.
(f) $3k^2p - kp$.
(g) $3k^2p - 6kp$.
(h) $3pq^2 - 6p^2q$.
(i) $k^5 - k^2$.
(j) $5w^3v^2 - 25w^2v^3$.
(k) $xy - xz + xw$.
(l) $2x + 4y - 6z$.
(m) $10m + 15l - 5n$.
(n) $3x^2 - 2x + 3xy$.
(o) $7a^2 - 28ab + 14a^2$.
(p) $5mp + 10m^2 + 5m$.
(q) $3cd - 2df - 5d^2$.
(r) $7ab - 8a^2 + 3ab^2$.
(s) $p - p^2q - p^3q^2$.
(t) $4n^2 + 2nk - 8n^2k$.

43. We must now consider the effect of taking away $(8 + 3)$ marbles from the bag. Once again suppose there are 20 marbles to start with. If $(8 + 3)$ marbles are taken out the number that remain must be the same as if the 8 marbles and the 3 marbles were removed separately. In other words, it makes no difference to the final result if

$(8 + 3) = 11$ marbles are taken out at one time, or if 8 marbles and then 3 marbles are taken in two successive operations. In either case 9 marbles remain. Consequently we can write:

$$20 - (8 + 3) = 20 - 8 - 3.$$

Look at this very carefully. Notice, in particular, that when the brackets are removed the minus sign outside the bracket has changed the sign inside the bracket from + to −.

Similarly we can write:

$$m - (x + y) = m - x - y$$

and

$$2a - (3b + a) = 2a - 3b - a$$
$$= a - 3b \text{ (on collecting like terms).}$$

EXERCISE XLVI

1. Find the value of

(a) $5 - (1 + 2)$.
(b) $(7 - 3) - 2$.
(c) $8 - (3 + 4)$.
(d) $10 - 2(1 + 2)$.
(e) $3(1 + 2) - 2(2 + 1)$.
(f) $18 - (1 + 3 + 5)$.
(g) $(2 + 4) - (1 + 2 + 3)$.
(h) $20 - 2(2 + 3 + 4)$.

2. Simplify the following:

(a) $a - (b + c)$.
(b) $a - 2(b + c)$.
(c) $(3a + b) - (a + c)$.
(d) $p - q(x + y + z)$.
(e) $a(b + c) - a(b + d)$.

3. Simplify the following by removing the brackets and collecting together like terms:

(a) $10 - (a + 4)$.
(b) $14 - 2(3x + 5)$.
(c) $5(x + 2) - 3(x + 3)$.
(d) $12(a + b) - 5(a + 2b)$.
(e) $3(2x + 5y) - 2(3x + 5y)$.
(f) $7(a + 4) - 3(2a + 5)$.
(g) $4(2x + 5) - 2(3x + 8)$.

4. Write down the contents of the brackets in each of the following:

(a) $a + b + c = a + (\qquad)$.
(b) $a - b - c = a - (\qquad)$.
(c) $x + 2y + 4z = x + 2(\qquad)$.
(d) $x - 2y - 4z = x - 2(\qquad)$.
(e) $ab + bc - xy - yz = b(\qquad)\ \ y(\qquad)$.
(f) $pq - qr - tq = q(\qquad)$.
(g) $ax - ay + m = a(\qquad) + m$.
(h) $mv + mv^2 = mv(\qquad)$.
(i) $xy + xz - pq - pr = x(\qquad) - p(\qquad)$.
(j) $a + a^2 + a^3 + a^4 = a(\qquad)$.

44. At this stage in our study of the effect of brackets there is one more case to consider—namely, the subtraction of a quantity such as (8 – 3). If we suppose that a boy has 20 marbles to start with, then we have to find the value of 20 – (8 – 3). Since the contents of the brackets have to be considered as *one* quantity,

$$\begin{aligned} 20 - (8 - 3) &= 20 - 5 \\ &= 15. \end{aligned}$$

This is the same as subtracting 8 from 20 and then **adding** 3. In other words,

$$\begin{aligned} 20 - (8 - 3) &= 20 - 8 + 3 \\ &= 15. \end{aligned}$$

Similarly, $m - (x - y) = m - x + y.$

Notice that the – sign *immediately* before the bracket changes the sign inside the bracket from – to +. In the preceding section we saw that the minus sign outside the bracket changed the + to – when the brackets were removed. This change of sign takes place in just the same way when there is a multiplying factor outside the brackets. Consider the following example.

Simplify $18a + 10b - 2(3a + b) - 4(2a - 3b)$.

$$18a + 10b - 2(3a + b) - 4(2a - 3b)$$
$$= 18a + 10b - 6a - 2b - 8a + 12b \text{ (removing the brackets)}$$
$$= 18a - 6a - 8a + 10b - 2b + 12b$$
$$= 4a + 20b \text{ (collecting like terms).}$$

EXERCISE XLVII

1. Find the value of the following:

(*a*) $8 - (5 - 3)$.
(*b*) $(8 - 5) - 3$.
(*c*) $10 - 2(4 - 1)$.
(*d*) $3(1 + 2) - 2(3 - 1)$.
(*e*) $10 - (1 + 2 + 3)$.

2. Simplify the following by removing the brackets:

(*a*) $a - (b - c)$.
(*b*) $(a - b) - c$.
(*c*) $a - 2(b - c)$.
(*d*) $(2a + b) - (a - b)$.
(*e*) $p - q(x + y + z)$.
(*f*) $a(b + c) - b(a - c)$.

3. Simplify the following by removing the brackets and collecting together like terms:

(*a*) $10 - (a - 4)$.
(*b*) $15 - 2(3x - 5)$.
(*c*) $7(x + 1) - 3(x - 2)$.
(*d*) $9(2a + 3b) - 5(3a - b)$.
(*e*) $8(x + 2y + z) - 5(x - 2y + z)$.

4. Write down the contents of the brackets in each of the following:

(*a*) $a - b - c = a - (\qquad)$.
(*b*) $a - b + c = a - (\qquad)$.
(*c*) $x - 2y + 4z = x - 2(\qquad)$.
(*d*) $x - 2y - 4z = x - 2(\qquad)$.
(*e*) $ab - bc - xy + xz = b(\qquad) - x(\qquad)$.
(*f*) $pq + pr - ab + bc = p(\qquad) - b(\qquad)$.
(*g*) $ab - ab^2 = ab(\qquad)$.
(*h*) $x^2 - xy + y^2 - y^3 = x(\qquad) + y^2(\qquad)$.
(*i*) $a - a^2 + a^3 - a^4 = a(\qquad)$.

45. The brackets and the process of factorization which have just been described find many applications in all branches of mathematics. They are particularly useful in assisting numerical calculation, and frequently sums that appear to be long and difficult can, by means of careful factorizing, be worked out very easily. Study the following examples.

EXAMPLE 1

Find the value of $8{\cdot}6 \times 2{\cdot}35 + 8{\cdot}6 \times 6{\cdot}43 + 7{\cdot}22 \times 8{\cdot}6$.

Notice that we have to find the sum of three products, each of which contains the factor 8·6. Consequently we can write :

$$\begin{aligned} 8{\cdot}6 \times 2{\cdot}35 + 8{\cdot}6 \times 6{\cdot}43 + 7{\cdot}22 \times 8{\cdot}6 \\ &= 8{\cdot}6(2{\cdot}35 + 6{\cdot}43 + 7{\cdot}22) \\ &= 8{\cdot}6 \times 16 \\ &= 137{\cdot}6. \end{aligned}$$

EXAMPLE 2

Express the formula $A = 6m^2 - 9mn$ in a form more suitable for calculation. Find A when $m = 12{\cdot}5$ and $n = 8$.

$$\begin{aligned} A &= 6m^2 - 9mn \\ &= 3m(2m - 3n) \\ &= 37{\cdot}5(25 - 24) \quad \text{(on substituting the values of } m \text{ and } n) \\ &= 37{\cdot}5. \end{aligned}$$

EXERCISE XLVIII

1. Find the value of each of the following by the shortest method :

(a) $3{\cdot}4 \times 1{\cdot}7 + 1{\cdot}3 \times 3{\cdot}4$.
(b) $100 \times 18 + 27 \times 100 + 100 \times 53$.
(c) $23 \times 49 + 37 \times 23 + 23 \times 14$.
(d) $32 \times 57 - 22 \times 32$.
(e) $24 \times 25 \times 26 - 22 \times 26 \times 25$.

2. Express each of the following in a form more suitable for calculation:

(*a*) $p^2q + p^2qr$.
(*b*) $ar^3 - br^2 + abr^5$.
(*c*) $x^3 - xy$.
(*d*) $mn^2p + m^2np^2 - m^2n^2$.
(*e*) $6a^3 - 18a^2b$.

3. Express each of the following formulæ in a form more suitable for calculation:

(*a*) $A = \frac{2}{3}p^2 - p^2q$. Find A when $p = 5$ and $q = \frac{2}{3}$.
(*b*) $P = 12x^2 - 8x + 16$. Find P when $x = 7$.
(*c*) $K = 4a^2b - 4ab^2$. Find K when $a = 25$ and $b = 4$.
(*d*) $V = 5r^3h - 7r^2h$. Find V when $r = 7$, $h = 5$.
(*e*) $S = \dfrac{na + nl}{2}$. Find S when $n = 50$, $a = 1$, and $l = 50$.

46. Fig. 12 shows a rectangle 5 inches long and 4 inches wide divided into squares each of side 1 inch. There are twenty of these squares, and each has an area of 1 square inch, so that the area of the rectangle in Fig. 12 is 20 square inches. A rectangle of length *a* inches and width *b* inches has an area of *ab* square inches. If A denotes this area, we have the formula $A = ab$.

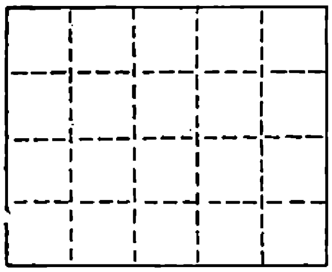

Fig. 12

In Fig. 13 is shown the plan of a large dining-room, the measurements being made in feet. The dotted line divides this into two rectangular parts, the first of which has an area of *ab* square feet and the second an area of *ac* square feet. If the complete area is A, then we have the formula

$$A = ab + ac.$$

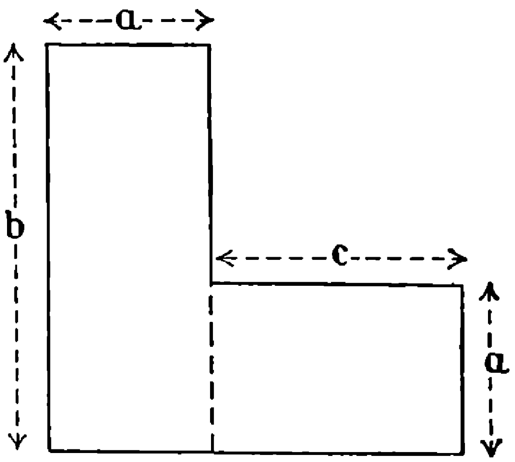

Fig. 13

For purposes of calculation we should write this:

$$A = a(b + c).$$

If $a = 20$, $b = 60$, and $c = 40$, then

$$\begin{aligned} A &= 20(60 + 40) \\ &= 20 \times 100 \\ &= 2000. \end{aligned}$$

Hence the area of the room is 2000 square feet.

EXERCISE XLIX

1. Find the areas of the following rectangles:
   (*a*) 5 inches long by 7 inches wide.
   (*b*) 14 cm. long by 12 cm. wide.
   (*c*) 17 inches long by x inches wide.
   (*d*) p cm. long by q cm. wide.
   (*e*) a feet long by a feet wide.

2. Fill in the blank spaces in the following table:

	Length of rectangle	*Width of rectangle*	*Area of rectangle*
(*a*)	$3x$ feet	$7y$ feet	
(*b*)	$8m$ yards	$8m$ yards	
(*c*)	5 inches		30 square inches
(*d*)	10 feet		120 square feet
(*e*)		$4b$ yards	$28bc$ square yards
(*f*)		$13p$ feet	$169p^2$ square feet

3. (*a*) Write down a formula for the area (A) of the plan in Fig. 14, arranging it in the form most suitable for calculation.
(*b*) Find A when $b = 25$ feet and $a = 13$ feet.

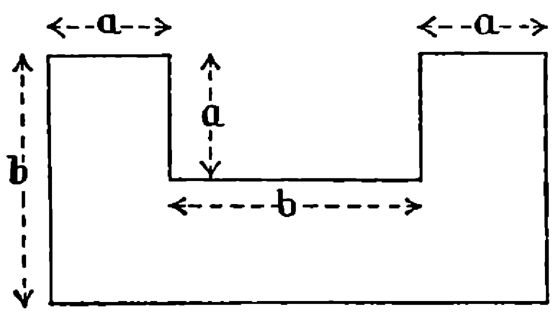

FIG. 14

4. (*a*) Write down a formula for the area (A) of the design in Fig. 15.

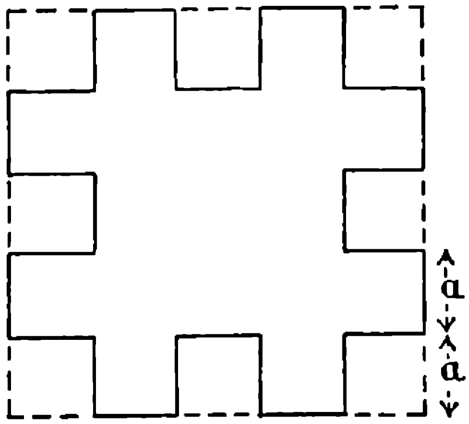

FIG. 15

(*b*) Find A if $a = 2$ inches.

5. (*a*) Write down a formula in a form suitable for calculation for the area shown in Fig. 16.

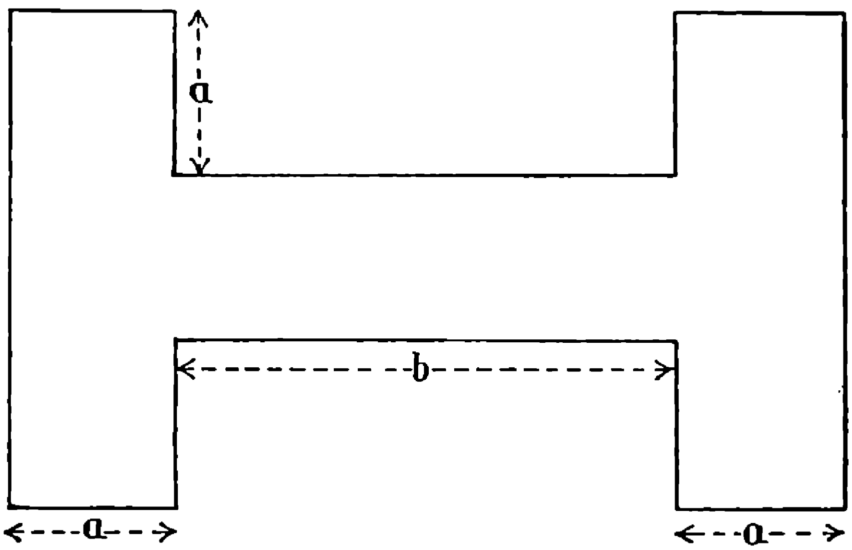

FIG. 16

(*b*) From this formula find A when $a = 5$ inches and $b = 8$ inches.

6. (*a*) A room is *a* feet long, *b* feet wide, and *c* feet high. Write down a formula for the area of its walls (A) in square feet.

(*b*) Use the formula found in (*a*) to calculate the area of the walls of a room (i) length 15 feet, width 13 feet, height 10 feet, (ii) length 20 feet, width 15 feet, height 12 feet.

47. In Fig. 17 we have a square of side a units out of which has been removed a smaller square of side b units. Then the area of this design is $a^2 - b^2$ square units. The

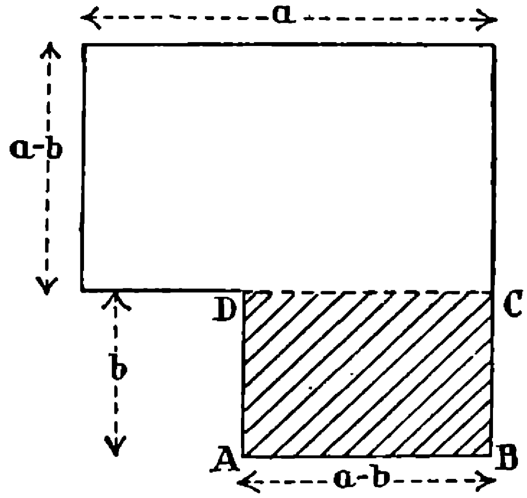

FIG. 17

lower part of this figure, the rectangle ABCD, is of length $(a - b)$ units and of width b units, and has been cut off and put in another position, so as to form the large rectangle shown in Fig. 18. The length of this new rectangle is

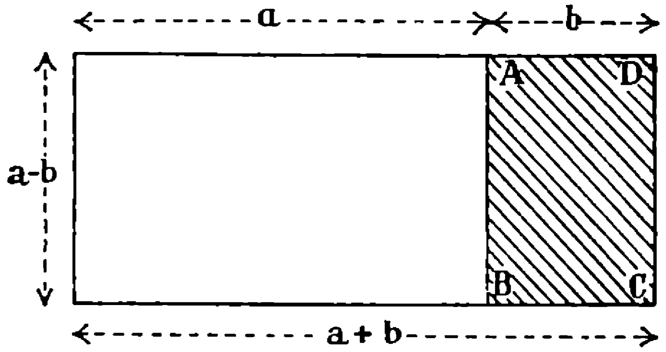

FIG. 18

$(a + b)$ units and its width $(a - b)$ units, so that its area will be $(a + b)(a - b)$ square units. Clearly both figures must have the same area, so that we have the relationship

$$a^2 - b^2 = (a + b)(a - b),$$

from which we see that the *factors* of $a^2 - b^2$ are $(a + b)$ and $(a - b)$. This method of factorizing the *difference of two squares* has many applications, some of which we will now proceed to consider.

Find the value of $389^2 - 387^2$.

Of course, it would be possible to evaluate 389^2 and 387^2 by the ordinary process of multiplication, but notice how much easier it is to use the factors which have been discovered above.

Let $a = 389$ and $b = 387$.

$$\begin{aligned}\text{Then}\quad 389^2 - 387^2 &= a^2 - b^2\\ &= (a + b)(a - b)\\ &= (389 + 387)(389 - 387)\\ &= 776 \times 2\\ &= 1552.\end{aligned}$$

EXERCISE L

Find the value of

1. $25^2 - 24^2$.
2. $38^2 - 33^2$.
3. $79^2 - 72^2$.
4. $121^2 - 119^2$.
5. $246^2 - 242^2$.
6. $1000^2 - 999^2$.
7. $999^2 - 1$.
8. $2 \times 19^2 - 2 \times 17^2$.
9. $7 \times 15^2 - 13^2 \times 7$.
10. $29^2 \times 11 - 11 \times 27^2$.

48. The relationship $a^2 - b^2 = (a + b)(a - b)$ expresses the difference of any two squares as a product. Hence it can be used to find the factors of any algebraical expression consisting of the difference of two squares. Let us consider the following example.

Find the factors of $16x^2 - 9y^2$.

Remember that $a^2 - b^2 = (a + b)(a - b)$.

$$\begin{aligned}16x^2 - 9y^2 &= (4x)^2 - (3y)^2\\ &= (4x + 3y)(4x - 3y).\end{aligned}$$

EXERCISE LI

Find the factors of

1. $a^2 - 16$.
2. $p^2 - 49$.
3. $4p^2 - q^2$.
4. $4p^2 - 25q^2$.
5. $49x^2 - 64y^2$.
6. $c^2 - 16m^2$.
7. $144 - 16m^2$.
8. $36k^2 - 81c^2$.
9. $100w^2 - 121v^2$.
10. $9g^2 - 1$.
11. $81h^2 - 64f^2$.
12. $169m^2 - 144n^2$.
13. $a^2b^2 - 1$.
14. $a^2b^2 - x^2y^2$.
15. $a^2b^2 - x^4y^4$.
16. $25p^2q^2 - 16m^2n^2$.
17. $36p^2q^4 - 49m^4n^2$.
18. $1 - 64x^4$.
19. $25a^4 - 49b^4$.
20. $81p^2 - 144q^6$.

49. Sometimes the fact that an algebraical expression can be factorized by the methods just described is not seen at once because of the presence of some other factor, which if put outside some brackets reveals the difference of two squares. The two examples which follow illustrate how we proceed in such cases.

EXAMPLE 1

Find the factors of $4mp^2 - mq^2$.

The expression $4mp^2 - mq^2$ is not the difference of two squares. Notice, however, that we can factorize this expression as follows:

$$4mp^2 - mq^2 = m(4p^2 - q^2).$$

The quantity inside the brackets is readily recognized as $(2p)^2 - q^2$, which we can factorize as $(2p + q)(2p - q)$. Consequently we write:

$$\begin{aligned}4mp^2 - mq^2 &= m(4p^2 - q^2)\\ &= m\{(2p)^2 - q^2\}\\ &= m(2p + q)(2p - q).\end{aligned}$$

EXAMPLE 2

Find the factors of $8x^4y - 18x^2y^3$.

Proceeding as in the last example, we write:

$$\begin{aligned}8x^4y - 18x^2y^3 &= 2x^2y(4x^2 - 9y^2)\\ &= 2x^2y(2x + 3y)(2x - 3y).\end{aligned}$$

EXERCISE LII

Find the factors of

1. $ax^2 - ay^2$.
2. $8x^2 - 2y^2$.
3. $12p^2 - 3q^2$.
4. $20a^2 - 45b^2$.
5. $63p^2q^2 - 7k^2$.
6. $x^2 - x^2y^2$.
7. $x^3 - xy^2$.
8. $4x^2y^2 - 9x^2$.
9. $16ab^2 - c^2a$.
10. $49ab^2 - a^3$.
11. $a^4b^3 - a^2b$.
12. $16p^2q^2 - p^4$.
13. $9a - ax^4$.
14. $16ab - 25ab^3$.
15. $x^3y^3 - xy$.
16. $c^2d - f^4d$.
17. $2a^4k - 18kg^4$.
18. $3p^4q - 12p^2q^5$.
19. $12a^4b - 27a^2b^5$.
20. $64x^2y^4 - 49x^4y^2$.

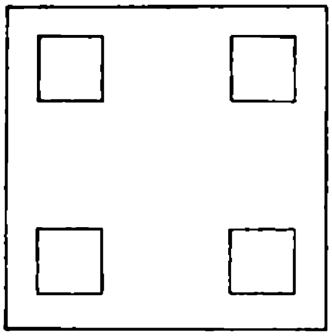

FIG. 19

50. In Fig. 19 we have a lawn x feet square containing four flower-beds, each a feet square. Then the area of the lawn is x^2 square feet and the area of the flower-beds is $4a^2$ square feet.

If A is the area of the grass in square feet we have the formula

$$A = x^2 - 4a^2.$$

If $x = 60$ and $a = 15$, then in order to calculate the area of the grass

$$\begin{aligned} A &= x^2 - 4a^2 \\ &= (x + 2a)(x - 2a) \\ &= (60 + 30)(60 - 30) \\ &= 90 \times 30 \\ &= 2700. \end{aligned}$$

Hence the area of the grass is 2700 square feet. Notice how the formula has been factorized and has thus simplified the calculation which followed.

EXERCISE LIII

1. (*a*) In Fig. 20 we have a square court-yard containing a square lawn. The court-yard is x feet square and the lawn y feet square. What is the width of the path round the lawn?

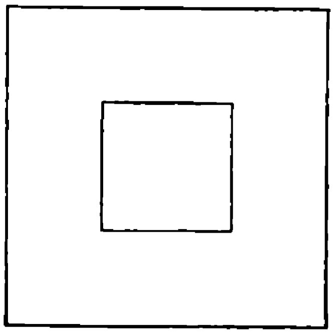

FIG. 20

(*b*) Write down a formula for the area of the path (A).

(*c*) Arrange the formula in a form suitable for calculation, and then find A when $x = 74$ and $y = 26$.

(*d*) Find A when $x = 100$ and $y = 84$.

2. (*a*) In Fig. 21 we have a piece of cardboard p inches square, and out of each corner a small square of side q inches has been removed, so that by folding the cardboard along the dotted lines a shallow box will be formed. How large will the box be? What will be its volume?

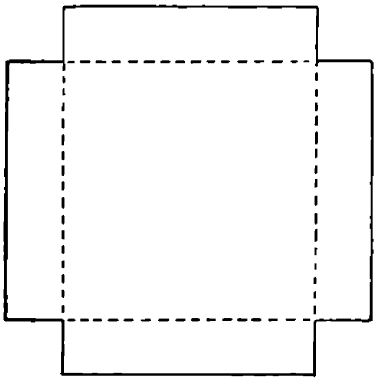

FIG. 21

(*b*) Write down a formula for the area of the bottom of the box (A).

(*c*) Arrange the formula in a form suitable for calculation, and find the area of the bottom of the box when $p = 21$ and $q = 5$.

(*d*) Write down a formula for the area (S) of the four sides of this box.

(*e*) Find S when $p = 21$ and $q = 5$.

3. (*a*) Write down a formula for the area (T) of the brass plate shown in Fig. 22.

(*b*) Find T when $k = 14$ inches and $m = 3$ inches.

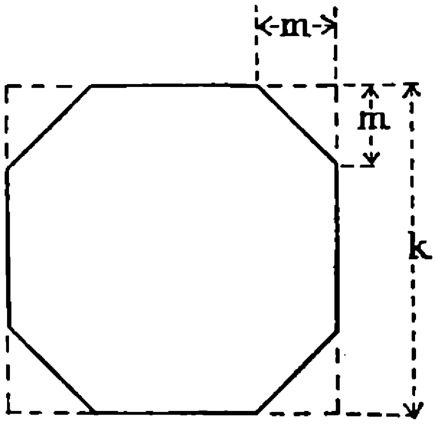

FIG. 22

4. (*a*) Write down a formula for the area (B) of the plate shown in Fig. 23.

(*b*) Find B when $x = 15$ inches and $y = 1$ inch.

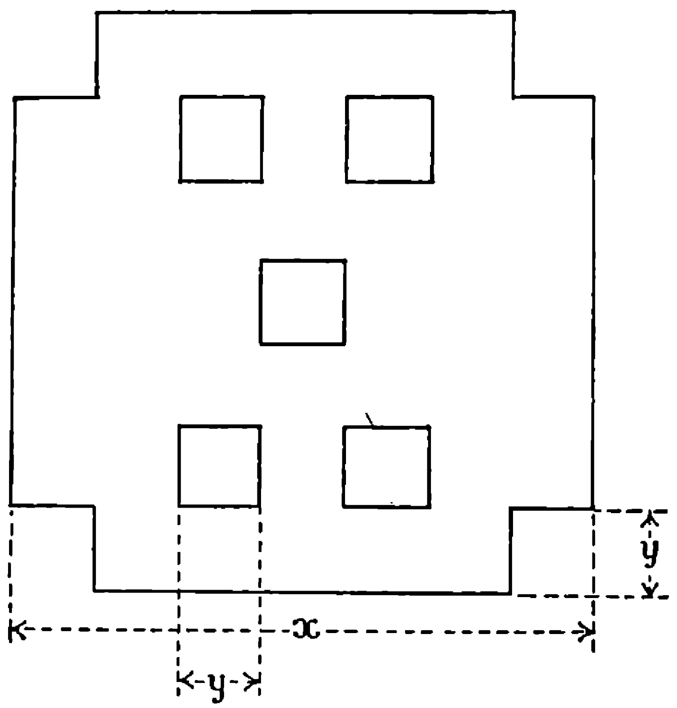

FIG. 23

5. (*a*) Write down a formula for the area of the star (S) shown in Fig. 24.

(*b*) Find S when $x = 5$ inches and $y = \frac{1}{2}$ inch.

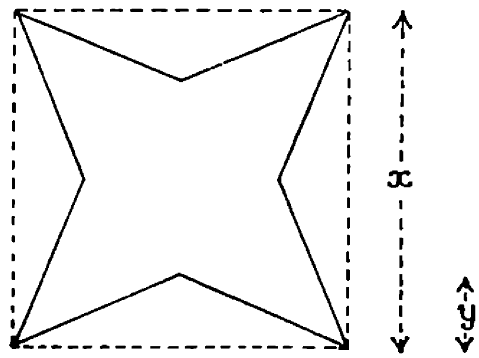

FIG. 24

EXERCISE LIV (Revision Exercise)

(A)

1. Simplify
   (a) $2a + 3(a + 2b)$.
   (b) $7x - 3(2x + y)$.
   (c) $12p - 7(p - 2q)$.

2. If $m = 7$, $n = 5$, and $p = 3$, find the value of
   (a) $m^2 + n^2 + p^2$.
   (b) $3m - 2(p + n)$.
   (c) $3m - 2(n - p)$.

3. Find the factors of
   (a) $14ax + 6xy$.
   (b) $64 - 9m^2$.
   (c) $ab^2 - c^2a$.

4. Find the value of $12{\cdot}4 \times 7{\cdot}19 + 7{\cdot}19 \times 7{\cdot}6$.

5. State what value must be given to P if

$$\frac{4a}{5b} = \frac{12a^2b}{Pb^2}.$$

(B)

6. Simplify the following:
   (a) $2(1 + a + b) + 3(2 + 4a + 6b)$.
   (b) $7x + 8y - 5(x + y)$.

7. Find the factors of
   (a) $14p^2 + 7pq$.
   (b) $8x^2 - 2$.
   (c) $5p^2q + 10pq^2 + 15pqr$.

8. Write down the contents of the brackets in each of the following:
   (a) $2p - 3q + 6 = 2p - 3(\qquad)$.
   (b) $cd + cf - xy - yz = c(\qquad) - y(\qquad)$.
   (c) $18x^2 - 2 = 2(\qquad)(\qquad)$.

9. Find the H.C.F. and L.C.M. of

(*a*) $5m^3n^2$, $25mn^3$, and $15m^2n^2$.

(*b*) x, $2x^2$, $3x^3$, and $4x^4$.

10. Simplify the following:

(*a*) $24a^5b^2 \times 8a^3b$.

(*b*) $24a^5b^2 \div 8a^3b$.

(*c*) $\frac{2a}{5x} + \frac{3b}{7y}$.

(C)

11. (*a*) Write down a formula for the area (A) in Fig. 25.

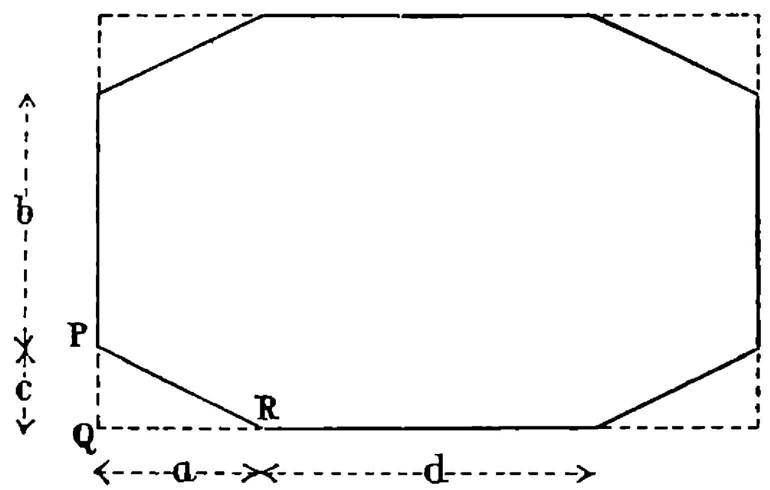

FIG. 25

(*b*) Write down a formula for the area (T) of the triangle PQR.

(*c*) Write down a formula in terms of A and T for the area (X) of the rectangle that contains the area A.

12. Simplify

(*a*) $7a + 5a$, $7a - 5a$, $7a \times 5a$, and $7a \div 5a$.

(*b*) $\frac{2p^3q}{4pq^3}$ and $\frac{5x^2y}{7x} \times \frac{21x^3}{25y^3}$.

13. If $a = 4$, $b = 3$, and $c = 2$ find the value of

(*a*) $a + 2b + 3c$.

(*b*) $a^2 + b^2 + c^2$.

(*c*) $a + b^2 + c^3$.

(*d*) $\sqrt{a} + \sqrt{b} + \sqrt{c}$.

14. Write down the contents of the brackets in each of the following:

(a) $121p^2q^2 - 1 = (\qquad)(\qquad)$.

(b) $m^3p - m^2p^2 + m^2p = m^2p(\qquad)$.

(c) $2a + 6b - 7c - 21d = 2(\qquad) - 7(\qquad)$.

15. Write down a formula for the area (A) of the design in Fig. 26.

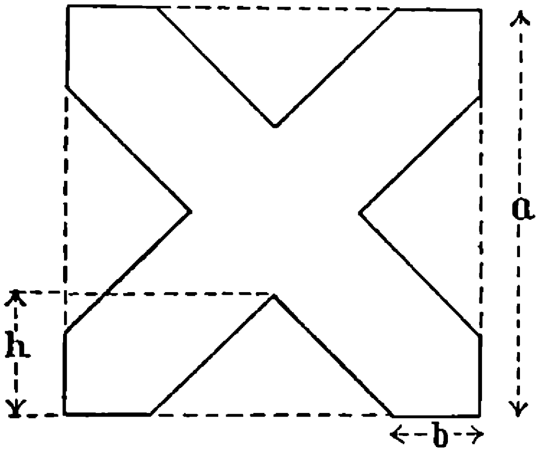

FIG. 26

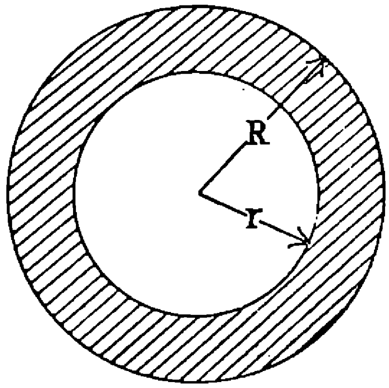

FIG. 27

(D)

16. (a) Find a formula for the area (A) of the shaded part between the two circles in Fig. 27.

(b) Calculate this area when R = 7 inches and r = 3 inches.

(c) What does the formula become if $R = 2r$?

17. If $P = a^2 - 2b$ and $Q = b^2 - 2a$, find the value of $Q^2 - P^2$ when $a = 4$ and $b = 5$.

18. (a) Fig. 28 shows the cross-section of an iron girder.

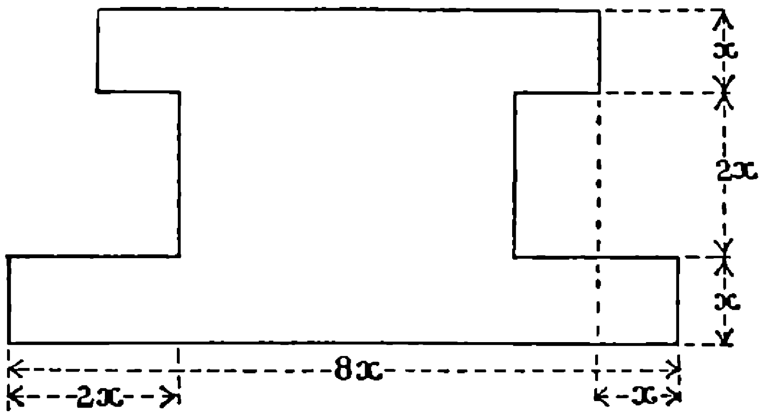

FIG. 28

Write down a formula for the area (E) of this cross-section.

(*b*) Calculate the area when $x = 2$ inches.

19. (*a*) A clock loses m minutes a day. Write down a formula for the number of *hours* (H) lost in a week.

(*b*) If the clock is put right at 9 A.M. on Monday morning, use your formula to calculate the time it indicates at 9 A.M. (real time) on the following Monday when $m = 2$.

20. (*a*) A tradesman buys goods at p pence per dozen and sells them at p farthings each. Write down a formula for the profit in pence (G) he makes on each article.

(*b*) Find G when $p = 3$.

CHAPTER V

CONSECUTIVE NUMBERS

51. If we take the marbles out of a bag one by one and count as we do it, we say 1, 2, 3, 4 until the required number of marbles is reached. These numbers are called *consecutive* numbers.

7, 8, 9, 10, are consecutive numbers; so are 12, 13, 14, 15, 16, 17. Notice that starting with 7

$$8 = 7 + 1$$
$$9 = 8 + 1$$
$$10 = 9 + 1, \text{ and so on.}$$

Again,

$$13 = 12 + 1$$
$$14 = 13 + 1$$
$$15 = 14 + 1$$
$$16 = 15 + 1$$
$$17 = 16 + 1$$

This shows how a series of consecutive numbers is built up. Notice also that a series of consecutive numbers can be built up from any starting number. Thus, starting with 14, we have:

$$15 = \mathbf{14 + 1}$$
$$16 = 15 + 1 = 14 + 1 + 1 = \mathbf{14 + 2}$$
$$17 = 16 + 1 = 14 + 2 + 1 = \mathbf{14 + 3}$$
$$18 = 17 + 1 = 14 + 3 + 1 = \mathbf{14 + 4}$$

Thus the consecutive numbers starting with 14 are 14, 15, 16, 17, 18, and can be written:

$$14$$
$$14 + 1$$
$$14 + 2$$
$$14 + 3$$
$$14 + 4, \text{ and so on.}$$

In exactly the same way the series of consecutive numbers starting with n is

$$n$$
$$n + 1$$
$$n + 2$$
$$n + 3$$
$$n + 4, \text{ and so on.}$$

EXERCISE LV

1. Write down three consecutive numbers, starting with 20.
2. Write down five consecutive numbers, starting with 17.
3. Write down four consecutive numbers, starting with 100.
4. (*a*) Are 3, 4, 5, 6, 7, consecutive numbers?
(*b*) Are 3^2, 4^2, 5^2, 6^2, 7^2, consecutive numbers?
(*c*) Are 3^3, 4^3, 5^3, 6^3, 7^3, consecutive numbers?
5. Write down five consecutive numbers of which 17 is the middle one.
6. How many consecutive numbers are there *between* 12 and 21?
7. What is the sum of the seven consecutive numbers starting with 6?
8. Write down four consecutive numbers, beginning with n.
9. Write down six consecutive numbers, beginning with $2x$.
10. What is the sum of the consecutive numbers in Question 8?

52. In the four consecutive numbers starting with 7—namely, 7, 8, 9, 10—the starting number, 7, is the smallest of the four. We can, however, write down four consecutive numbers starting with 7 in which 7 is the largest of the four—namely, 7, 6, 5, 4.

Notice that

$$6 = 7 - 1$$
$$5 = 7 - 2$$
$$4 = 7 - 3$$

In just the same way, if we start with n, which we make the largest of four consecutive numbers, then the others are

$$\begin{gathered}(n)\\ n-1\\ n-2\\ n-3\end{gathered}$$

EXERCISE LVI

1. Write down four consecutive numbers in which 10 is the largest.
2. Write down six consecutive numbers in which 15 is the largest.
3. (*a*) Are 14, 13, 12, 11, 10, 9, consecutive numbers?
   (*b*) Are 14^2, 13^2, 12^2, 11^2, 10^2, 9^2, consecutive numbers?
4. Write down five consecutive numbers in which n is the largest.
5. Write down seven consecutive numbers in which $2x$ is the largest.
6. What is the sum of the numbers in Question 4?
7. What is the sum of the numbers in Question 5?

53. We have seen how to write down a series of consecutive numbers in which our starting number is either the largest or the smallest of the series. We can, however, take our starting number anywhere in the series. Thus five consecutive numbers of which 7 is the middle one would be

$$5 \quad 6 \quad \mathbf{7} \quad 8 \quad 9$$

In just the same way five consecutive numbers of which n is the middle one would be

$$n-2 \quad n-1 \quad \mathrm{n} \quad n+1 \quad n+2$$

EXERCISE LVII

1. Provide the missing number in each of the following series of consecutive numbers:

   (*a*) 12, 11, 10, —, 8, 7.
   (*b*) 57, 56, —, —, 53, —, 51.

(*c*) —, $n - 1$, $n - 2$.
(*d*) n, $n + 1$, —, $n + 3$, $n + 4$.
(*e*) $n + 2$, —, n, $n - 1$.
(*f*) $n + 3$, $n + 2$, —, —, —, $n - 2$.
(*g*) $n + 5$, $n + 4$, —, —.

2. Write down five consecutive numbers in which

(*a*) The largest number is 12.
(*b*) The smallest number is 12.
(*c*) The middle number is 12.

3. Write down seven consecutive numbers in which

(*a*) The largest number is n.
(*b*) The smallest number is n.
(*c*) The middle number is n.

4. Write down five consecutive numbers in which

(*a*) The largest number is $5x$.
(*b*) The smallest number is $5x$.
(*c*) The middle number is $5x$.

5. Find the sum of five consecutive numbers whose middle term is n.

6. Find the sum of seven consecutive numbers whose middle term is $3x$.

7. Write down three consecutive numbers whose sum is 24.

8. Supply five consecutive numbers whose sum is 45.

9. Write down a formula for the sum (S) of seven consecutive numbers whose middle term is n.

10. Give three consecutive numbers whose sum is $18x$.

54. Numbers which are exactly divisible by 2 are called *even* numbers—*e.g.*, 4, 12, 62, 108, are even numbers.

Numbers which are not exactly divisible by 2 are called *odd* numbers—*e.g.*, 3, 7, 87, 121, are odd numbers.

Consecutive even numbers are even numbers which follow one another—*e.g.*, 6, 8, 10, 12, are four consecutive even numbers. Notice how consecutive even numbers are built up.

$$\begin{aligned} &\mathbf{6} \\ 8 &= \mathbf{6+2} \\ 10 &= 8+2 = 6+2+2 = \mathbf{6+4} \\ 12 &= 10+2 = 6+4+2 = \mathbf{6+6} \end{aligned}$$

In the same way, if n is an even number the three consecutive even numbers after n are

$$\begin{aligned} &n+2 \\ &n+4 \\ &n+6 \end{aligned}$$

Consecutive odd numbers are constructed in a similar way. Consider the consecutive odd numbers 3, 5, 7, 9.

$$\begin{aligned} 5 &= \mathbf{3+2} \\ 7 &= 5+2 = 3+2+2 = \mathbf{3+4} \\ 9 &= 7+2 = 3+4+2 = \mathbf{3+6} \end{aligned}$$

If n is an odd number, then the consecutive odd numbers are

$$\begin{aligned} &n+2 \\ &n+4 \\ &n+6, \text{ and so on.} \end{aligned}$$

EXERCISE LVIII

1. Write down four consecutive even numbers beginning with 8.
2. Supply four consecutive even numbers of which the largest is 8.
3. Give five consecutive odd numbers, starting with 17.
4. Write down five consecutive odd numbers of which 17 is the middle one.
5. Give four consecutive even numbers, starting with the even number n.

6. Write down four consecutive even numbers of which the largest is the even number n.

7. Write down four consecutive even numbers, beginning with $2n$.

8. Verify for six different values of n that $2n$ is always an even number.

9. Verify for six different values of n that $2n + 1$ is always an odd number.

10. By taking several different values of n discover whether $2n - 1$ is always an odd number or always an even number.

EXERCISE LIX (Revision Exercise)

(A)

1. If $x = 3$ and $y = 5$, find the value of

   (a) $(x + y)(x^2 - xy + y^2)$.

   (b) $(x^2 + xy + y^2)(x^2 - xy + y^2)$.

2. Simplify

   (a) $3x^5y^2 \times 6x^2y$. (b) $72a^3b^2c \div 24abc^2$.

3. Write down the value of

   (a) $(7x^2)^3$ when $x = 1$.

   (b) $(2abc)^2$ when $a = b = c = 3$.

4. Find the factors of

   (a) $12ax^2 - 27a$. (b) $4p^2q - 8pq^2 + 12pq$.

5. Write down seven consecutive numbers of which

   (a) $2p + 1$ is the largest.

   (b) $2p + 1$ is the smallest.

   (c) $2p + 1$ is the middle number.

(B)

6. Simplify

   (a) $\frac{2}{x^2y} + \frac{4x}{y} - \frac{3y}{x}$. (b) $3(x + 4) - 2(x - 1)$.

7. (*a*) If 5 men do a piece of work in 14 days, how long would 7 men take to do it?

(*b*) If n men do a piece of work in k days, how long would x men take to do it?

8. (*a*) Find a formula for the area (A) of the white cross on the flag shown in Fig. 29.

(*b*) From the formula calculate A when $x = 7$ inches, $y = 5$ inches, and $a = 1\frac{1}{2}$ inches.

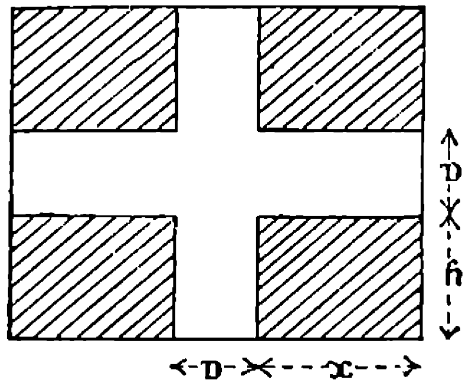

FIG. 29

9. Simplify

(*a*) $\sqrt{49x^2y^4} - \sqrt{64x^4y^2}$.

(*b*) $\sqrt{121a^6} + \sqrt{169b^8}$.

10. What is the sum of the five consecutive even numbers of which $2x$ is the middle number?

(C)

11. Find the H.C.F. and L.C.M. of $12a^2b$, $14ab^2$, and $18a^2b^2$.

12. An article is bought for £$a + b$ shillings and sold for £$x + y$ shillings. How much profit was made?

13. Simplify

(*a*) $7(3x + 2y) - 5(2x - 3y) + 6x$.

(*b*) $\dfrac{12a^2b}{4x^2} \times \dfrac{5x^2y}{15a^2}$.

14. (*a*) Write down four consecutive odd numbers, starting with $2n - 1$.

(*b*) Write down four consecutive odd numbers, ending with $2n - 1$.

15. Factorize the following:

(*a*) $a^2x^2y - 2a^2xy^2$. (*b*) $18x^3y - 8xy^3$.

(D)

16. Write down the contents of the brackets in each of the following:

(*a*) $5(x + 5) - 3(x + 3) = 2(\qquad)$.

(*b*) $7(2x + 3) - 3(\qquad) = 11x + 18$.

17. (*a*) Write down a formula for the area (A) of the square metal plate shown in Fig. 30.

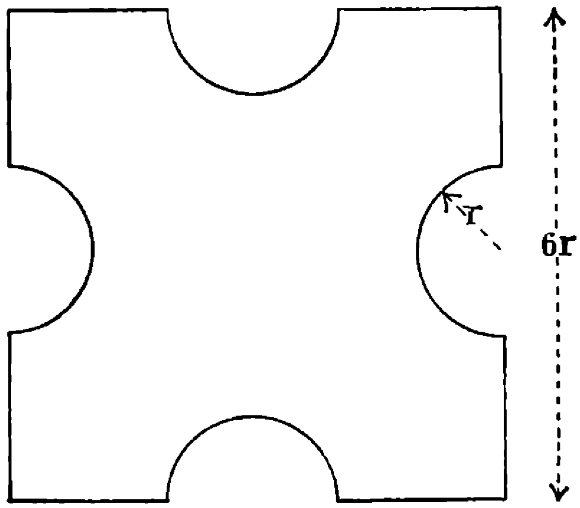

FIG. 30

(*b*) Arrange the formula in a form suitable for numerical work, and then from it calculate A when $r = 1\frac{1}{2}$ inches.

18. (*a*) If 5 books cost 12*s*. 6*d*., what will be the cost of 9 books?

(*b*) If *n* books cost *x* shillings, write down a formula for the cost in shillings (C) of *m* books.

19. Simplify the following:

(*a*) $a + 2b + 3c + 4(a + 2b + 3c) - 2(a - 2b - 3c)$.

(*b*) $\dfrac{64a^2bc}{18b^2ca} \times \dfrac{9c^2ba}{16abc}$.

20. Find the sum of seven consecutive numbers whose middle term is $7x$.

CHAPTER VI

NUMBER PUZZLES AND SIMPLE EQUATIONS

55. *If* 8 *is added to a certain number the result is* 15. By using a letter of the alphabet to stand for the words "a certain number" we can write this statement in the short form

$$8 + n = 15,$$

where n stands for the words "a certain number." It is easy to see that the certain number must be 7, since $8 + 7 = 15$.

We may write, therefore, $n = 7$.

Here is another puzzle.

If 12 *is subtracted from twice a certain number the result is* 16.

If we let x represent the unknown number, then $2x$ will be twice the number. Hence we can write the puzzle very shortly as $2x - 12 = 16$. It is not difficult to verify that the number in question must be 14, because if

$$x = 14$$

then $$2x = 28$$

and $$2x - 12 = 28 - 12$$
$$= 16.$$

The following is another puzzle which can be dealt with in a similar way.

The sum of two consecutive numbers is 13.

We can let k represent the smaller of the two numbers.

Then $(k + 1)$ will be the other number.

(Notice that $(k + 1)$ is enclosed in brackets so as to indicate that it is to be considered as one quantity.)

Hence our two consecutive numbers are k and $(k + 1)$. The sum of these two numbers is $k + (k + 1)$, and

$$k + (k + 1) = 13.$$

This can be simplified by removing the brackets according to the rules discovered on p. 61.

Then $$k + k + 1 = 13.$$

I.e., $$2k + 1 = 13.$$

Hence the original statement can be written very briefly as

$$2k + 1 = 13.$$

It is not difficult to verify that $k = 6$ and that the two consecutive numbers are 6 and 7, because if

$$k = 6$$

then $$(k + 1) = 7$$

and $$\begin{aligned} k + (k + 1) &= 6 + 7 \\ &= 13. \end{aligned}$$

EXERCISE LX

1. Write the following statements in a short form:

(*a*) A certain number is 5.
(*b*) Three times a certain number is 24.
(*c*) Eight times a certain number is 96.
(*d*) If 2 is added to a certain number the result is 14.
(*e*) If 23 is added to a certain number the result is 57.
(*f*) If 8 is subtracted from a certain number the result is 10.
(*g*) If 35 is subtracted from a certain number the result is 23.
(*h*) If 5 is added to twice a certain number the result is 11.
(*i*) If 42 is added to four times a certain number the result is 70.
(*j*) If 13 is subtracted from twice a certain number the result is 17.
(*k*) If 34 is subtracted from seven times a certain number the result is 8.
(*l*) Half a certain number is 18.
(*m*) Two-thirds of a certain number is 56.
(*n*) Seven-eighths of a certain number is 49.

(*o*) The sum of two consecutive even numbers is 26.
(*p*) The sum of two consecutive odd numbers is 28.
(*q*) The sum of three consecutive numbers is 21.
(*r*) The square of a certain number is 81.
(*s*) The cube of a certain number is 64.
(*t*) The difference between the square and the cube of a number is 18.

2. Write out in full the statements represented by the following:

(*a*) $x = 19$.	(*k*) $(t + 2) - t = 2$.
(*b*) $3p = 96$.	(*l*) $\frac{1}{2}v = 19$.
(*c*) $12k = 132$.	(*m*) $\frac{4}{7}w = 32$.
(*d*) $2a + 1 = 15$.	(*n*) $\frac{1}{3}x + 1 = 22$.
(*e*) $8c + 5 = 37$.	(*o*) $y^2 = 144$.
(*f*) $17 - f = 13$.	(*p*) $a^4 = 81$.
(*g*) $26 - 3g = 14$.	(*q*) $2b^3 = 54$.
(*h*) $5m - 1 = 24$.	(*r*) $5(h + 2) = 35$.
(*i*) $3n - 7 = 50$.	(*s*) $20 = 18 + g$.
(*j*) $p + (p + 1) = 21$.	(*t*) $38 = 2m - 1$.

56. When a statement is written in a short form and consists of two equal parts it is called an *equation*. Questions 2 (*a*)–2 (*t*) in the previous exercise are equations. The process by which we find out the number which the letter in the equation represents is called "solving the equation."

EXAMPLE 1

Solve $2x + 1 = 7$.

In this case we have to find out what number x represents. The full statement which this equation represents is: If 1 is added to twice a certain number the result is 7.

Notice that 6 must be added to 1 to make 7, so that

$$\boxed{2x} + 1 = 7.$$
$$\boxed{6} + 1 = 7.$$

Hence $$2x = 6.$$
$$x = 3.$$

The solution of $2x + 1 = 7$ is $x = 3$. In other words, the number which this statement describes is 3.

EXAMPLE 2

Solve $47 - 6x = 5$.

Since 42 must be subtracted from 47 to leave 5, we write:

$$47 - \boxed{6x} = 5.$$
$$47 - \boxed{42} = 5.$$

Hence
$$6x = 42.$$
$$x = 7.$$

EXERCISE LXI

1. Solve the following equations:

(*a*) $x + 1 = 6$.
(*b*) $m + 5 = 12$.
(*c*) $a + 19 = 37$.
(*d*) $2p + 1 = 7$.
(*e*) $3k + 4 = 16$.
(*f*) $4 + 3x = 16$.
(*g*) $5 - y = 4$.
(*h*) $7 - z = 5$.
(*i*) $12 - 2c = 4$.
(*j*) $27 - 3g = 12$.
(*k*) $2(x + 1) = 6$.
(*l*) $5(p + 1) = 35$.
(*m*) $4(t - 1) = 12$.
(*n*) $7(k + 3) = 42$.
(*o*) $12(2m + 3) = 84$.
(*p*) $9(5g + 2) = 153$.
(*q*) $(x + 8) - 2 = 17$.
(*r*) $\frac{1}{2}a = 4$.
(*s*) $\frac{1}{3}c = 8$.
(*t*) $\frac{1}{7}k = 5$.
(*u*) $\frac{2}{3}m = 4$. (What is $\frac{1}{3}m$? What is m?)
(*v*) $\frac{3}{5}p = 9$. (What is $\frac{1}{5}p$? What is p?)
(*w*) $\frac{1}{2}x + 1 = 7$. (What is $\frac{1}{2}x$? What is x?)
(*x*) $\frac{1}{3}y + 2 = 10$.
(*y*) $\frac{3}{4}a - 5 = 7$.

2. Write the following statements as equations, and then, by solving the equation, find the number referred to in each case:

(*a*) If 4 is added to a certain number the result is 15.
(*b*) If 9 is subtracted from a certain number the result is 6.
(*c*) Twice a certain number is 18.

(*d*) Three times a certain number is 42.
(*e*) Half of a certain number is 13.
(*f*) One-quarter of a certain number is 11.
(*g*) Two-thirds of a certain number is 16.
(*h*) Five-ninths of a certain number is 10.
(*i*) The sum of twice a certain number and 3 is 11.
(*j*) If 4 is subtracted from three times a certain number the result is 17.
(*k*) The sum of two consecutive numbers is 21.
(*l*) The sum of three consecutive numbers is 24.
(*m*) The sum of two consecutive even numbers is 26.
(*n*) When five times a certain number is subtracted from 28 the remainder is 13.
(*o*) When two-thirds of a certain number is subtracted from 50 the remainder is 29.

57. Consider the equation $2n + 5 = 19$.

We have seen that we can solve this equation, for if

$$\boxed{2n} + 5 = 19$$

and

$$\boxed{14} + 5 = 19$$

then $2n = 14$ and $n = 7$.

Notice that $14 = 19 - 5$.

Therefore $2n = 19 - 5$.

Compare this result with the original equation. You will see that the 5 which was originally on the *left-hand side* of the equals sign, and which was **added** to $2n$, is now on the *right-hand side* of the equals sign and **subtracted** from 19.

Now consider the equation $15 - 4x = 3$. Solving this by the methods already described, we write:

$$15 - \boxed{4x} = 3.$$
$$15 - \boxed{12} = 3.$$

Therefore $4x = 12$ and $x = 3$.
But $15 = 3 + 12$.
Therefore $15 = 3 + 4x$.

Compare this result with the original equation. You will see that the $4x$ which was on the left-hand side of the equals sign and which was *subtracted* from 15 is now on the right-hand side and *added* to 3.

These two results suggest a very important property of equations which we may briefly summarize by saying: If one side of an equation consists of the sum of two numbers, one of them may be moved to the other side and subtracted from the number already there. If one side of an equation consists of the difference of two numbers, the number which is subtracted may be moved to the other side and added to the number already there.

EXERCISE LXII

1. Rewrite the following equations so that the a terms are on the left-hand side:

(*a*) $12 = 15 - a$. (*c*) $17 = 19 - 2a$.
(*b*) $17 = 14 + a$. (*d*) $24 = 9 + 3a$.

2. Solve the following equations by arranging the letters on the left-hand side and the numbers on the right-hand side:

(*a*) $2x + 1 = 4 + x$.
(*b*) $4m - 3 = 2 + 3m$.
(*c*) $10 - 2a = 16 - 5a$.
(*d*) $k = 40 - 4k$.
(*e*) $7p = 3p + 20$.
(*f*) $3t + 4 = 13 + 2t$.
(*g*) $5w - 4 = 4w + 2$.
(*h*) $3x - (x - 4) = 16$.
(*i*) $5m - (4m + 1) = 8$.
(*j*) $(a + 1) + 3(2a + 1) = 18$.
(*k*) $2(k + 1) + 7(2k + 3) = 71$.
(*l*) $2(5p + 1) - 3(3p + 2) = 1$.
(*m*) $2(x + 1) + 3(2x + 3) = 3(2x + 5)$.

(*n*) $3(2y + 5) - (y - 1) = 4y + 19.$
(*o*) $5(2a - 11) - 7(a - 6) = a - 1.$
(*p*) $9(k + 1) - 2(3k + 4) = k + 9.$
(*q*) $3(m + 2) = (2m + 1) + 17.$
(*r*) $p + 3(2p - 7) = 4(p - 1) + 13.$
(*s*) $12 + 5(2t - 7) = 7(t - 2) + (t + 5).$
(*t*) $5v + 8 - (3v - 7) = 23.$
(*u*) $9(x + 1) - 6(x - 2) = 2(x + 9) + 8.$

58. We have seen how the use of a letter enables us to write a statement in a shortened form and sometimes as an equation which can be solved by the methods just described. Consider the following.

EXAMPLE 1

There were 500 people at a concert. The number of men present exceeded the number of women by 4, and there were no children. How many men were there?

Let n be the *number* of men at the concert. Since there were 500 people altogether, the *number* of women present must have been $500 - n$. We are told that there were 4 more men than women, so that the result of subtracting the *number* of women from the *number* of men is the *number* 4. (Remember that you cannot subtract women from men, but only the NUMBER of women from the NUMBER of men. That is why you must choose the letter to represent the number of men at the concert.)

We can write this statement in equation form.

Number of men − number of women = 4.

$$n - (500 - n) = 4.$$

Removing the brackets, $n - 500 + n = 4.$

Collecting terms together, $2n - 500 = 4.$

$2n = 4 + 500$ (moving 500 to the other side of the equation and *adding* it instead of *subtracting*).

$2n = 504.$

$n = 252.$

Hence there must have been 252 men present at the concert. The number of women present

$$= 500 - n$$
$$= 500 - 252$$
$$= 248.$$

EXAMPLE 2

A house and grounds cost £2100. The house cost five times as much as the ground. Find the cost of each.

Let x be the number of pounds which the ground cost.

Then $5x$ is the number of pounds which the house cost.

Since the total cost was £2100, we have the equation

$$5x + x = 2100.$$
$$6x = 2100.$$
$$x = 350.$$

Hence the cost of the ground was £350, and the cost of the house = £2100 - £350 = £1750.

EXERCISE LXIII

Use the methods just described to answer the following questions:

1. Divide £20 between two persons so that one has £3 more than the other.
2. Divide £20 between two persons so that one has three times as much as the other.
3. A bag contains two kinds of marbles. Some are red, and the others are white. If there are 47 marbles altogether and the number of red ones exceeds the number of white ones by 15, how many of each kind are there?
4. The perimeter of a square field is 156 yards. What is the length of one side?
5. Find the area of the field described in the previous question.
6. The total receipts for a concert amounted to £15. There were 250 people present, some of whom paid 1 shilling and the remainder 2 shillings. If the number of people in

the cheaper seats was four times the number in the dearer, find how many were in each kind of seat.

7. I bought 43 eggs, some at 4 a shilling and some at 5 a shilling. If I spent 10 shillings altogether, find how many of each kind of egg were bought.

8. Divide £35 between A, B, and C so that B receives twice as much as A, and C twice as much as B.

9. Divide £35 between A, B, and C so that B receives £5 more than A and C receives £15 more than A.

10. A money-box contains twice as many shillings as florins. If the value of the money is £2 8*s*., find how many of each kind of coin are in the box.

11. A silver collection was made at the end of a performance, with the result that there was a mixture of shillings and sixpences. There were 34 more shillings than sixpences, and the total value of the collection was £5 9*s*. How many of each kind of coin were there?

12. Two boys have 54 marbles between them. If one gives the other 9 they will both have the same number. How many has each?

13. When the cost of postage of letters was raised from 1*d*. to $1\frac{1}{2}d$. the monthly expenditure of a tradesman increased from 4*s*. 2*d*. to 6*s*. 3*d*. How many letters a month did he send?

14. An express train travelling at 60 m.p.h. completes a certain journey 2 hours quicker than a slow train travelling at 30 m.p.h. How long is the journey?

15. A man and his wife earn together £3 per week. The man earns three times as much as his wife. How much does the man earn in a week?

16. Divide a line 25 inches long into two parts, one of which is four times as long as the other.

17. At a party there were 5 more women than men, and the number of children was equal to the total number of grown-ups. If there were 50 persons altogether, how many children were there?

18. A house cost four times as much as the furniture it

contained. If the whole was valued at £1400, what was the value of the furniture?

19. A man is three times as old as his son. If their combined ages amount to 48 years, how old is the man?

20. A man is 25 years older than his son. Five years ago his age was six times the age of the son. How old is each now?

59. In all the problems that have been considered so far in this chapter we have used a letter to stand for an unknown number. In the previous exercise the letter stood for the *number* of things under consideration in the particular question, but in every case the letter represented some number the determination of which was the object of our work. The equation which we developed was simply a relationship between the various numbers in the question. In our study of equations we discovered two rules by the application of which we were able to solve them—namely:

1. If the expression on one side of an equation is the sum of two numbers, then the number which is added can be removed and subtracted from the number on the other side of the equals sign.
2. If the expression on one side of an equation is the difference of two numbers, then the number which is subtracted can be removed and added to the number on the other side of the equals sign.

It is not difficult to see that these two rules may have a wider application than merely solving number puzzles.

Consider the equation

$$2x - 1 = 7.$$

Then $$2x = 7 + 1 \text{ (applying Rule 2).}$$

$$\therefore x = \frac{7+1}{2} = 4.$$

Now compare this with the general formula $ax - m = n$.

Then $$ax = n + m.$$

$$\therefore x = \frac{n + m}{a}.$$

Remember that $ax - m = n$ is a short way of writing the statement: The result of subtracting the number m from a times a certain number is n.

The certain number which fulfils this condition is given by the formula $x = \dfrac{n + m}{a}$.

EXERCISE LXIV

In each of the following x stands for a certain number. Write out in full the statement that each question represents, and also find a formula for x in each case.

1. $px + q = t$.
2. $mx - k = r$.
3. $fx = p$.
4. $\dfrac{x}{y} = a$.
5. $\dfrac{x}{a} + \dfrac{x}{b} = m$.
6. $(p + q)x = t$.
7. $(p^2 - q^2)x = m + n$.
8. $\dfrac{a}{x} = b$.
9. $5(x + c) = d$.
10. $x(a + b) = c - d$.

60. Fig. 31 represents the plan of a room.

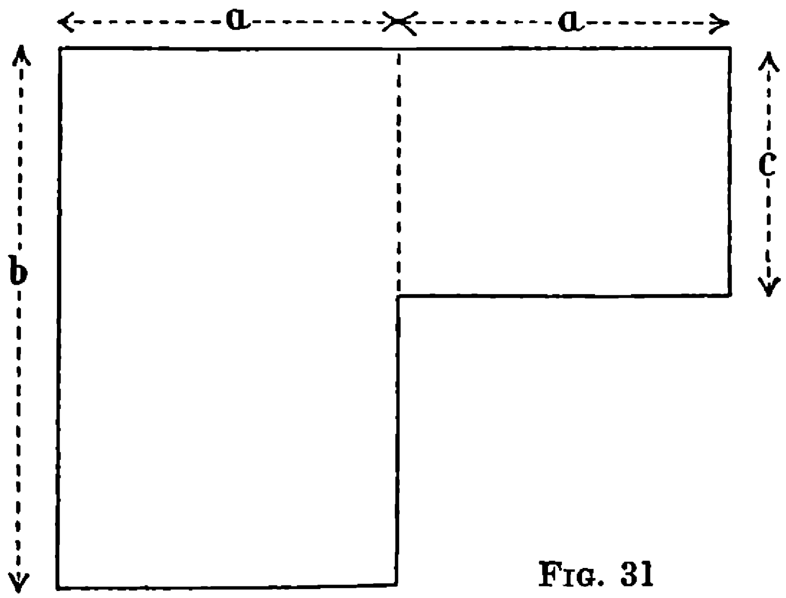

FIG. 31

If A is the area of the room, then we have the formula

$$A = ab + ac.$$
$$A = a(b + c).$$

This is a formula which can be used for calculating the area of the room, for if $a = 15$ feet, $b = 20$ feet, and $c = 8$ feet, then

$$\begin{aligned} A &= a(b + c) \\ &= 15(20 + 8) \\ &= 15 \times 28 \\ &= 420 \text{ square feet.} \end{aligned}$$

The formula written in this form has as its object the calculation of the area of the room the plan of which is shown in Fig. 31. For this reason A (the area of the room) is called the *subject of the formula.*

Suppose we knew the area of this room, and also the lengths of a and b. We could then use the formula to calculate the length c. It would be easier, however, to obtain a new formula, in which *c is the subject.* We can do this by applying the two rules on p. 100 to the formula $A = ab + ac$.

We have $\qquad A = ab + ac.$

$$\therefore A - ab = ac \text{ (applying Rule 1).}$$

This is the same as

$$ac = A - ab.$$
$$\therefore c = \frac{A - ab}{a}.$$

This is a formula in which c is the subject. This process of obtaining from a given formula a new formula with a different subject is called "changing the subject of the formula."

Study the following examples carefully.

Example 1

From the formula $v = u + ft$ find a formula for t.

We have $\qquad v = u + ft.$

$$v - u = ft.$$

This is the same as

$$ft = v - u.$$

$$\therefore t = \frac{v-u}{f}.$$

EXAMPLE 2

If $E = \frac{1}{2}mv^2$, make v the subject of the formula.

We have $$E = \tfrac{1}{2}mv^2.$$

$$\therefore \tfrac{1}{2}mv^2 = E.$$

$$\therefore v^2 = \frac{E}{\frac{1}{2}m}$$

$$= \frac{E}{1} \times \frac{2}{m}.$$

$$\therefore v^2 = \frac{2E}{m}.$$

$$\therefore v = \sqrt{\frac{2E}{m}}.$$

EXERCISE LXV

1. If $x + y + z = K$, make y the subject of this formula.
2. If $D = 2r$, express r in terms of D.
3. If $C = 2\pi r$, express r in terms of C and π.
4. If $A = \pi r^2$, express r in terms of A and π.
5. If $s = \frac{1}{2}gt^2$, express t in terms of s and g.
6. If $PV = RT$, write down a formula in which the subject is (*a*) T, (*b*) P, and (*c*) V.
7. If $v^2 = u^2 + 2fs$, construct a formula in which the subject is (*a*) u, (*b*) f, and (*c*) s.
8. If $C = \frac{E}{R}$, write down a formula in which the subject is R.
9. If $S = \frac{n}{2}(a + l)$, give a formula in which the subject is n.
10. If $nP = W + w$, find a formula for W.

11. If $P = \dfrac{E^2}{R}$, find a formula for E.

12. If $E = \dfrac{wv^2}{2g}$, find a formula for (*a*) v, (*b*) w.

13. If $R = 2 + \dfrac{5D}{d}$, find a formula for D.

14. If $xy = K$, find a formula in which y is the subject.

15. If $I = \dfrac{PLAN}{33{,}000}$, find a formula in which P is the subject

EXERCISE LXVI (Revision Exercise)

(A)

1. Solve the following equations:

    (*a*) $x + 7 = 19$.
    (*b*) $15 - 2p = 3$.
    (*c*) $5(p - 1) = 60$.

2. Divide 54 into two parts, one of which is 8 times the other.

3. Find three consecutive numbers whose sum is 39.

4. Find the factors of

    (*a*) $8p^2q - 24pq^2$.
    (*b*) $x^2 - 64y^2$.
    (*c*) $a^2bc + ab^2c + abc^2$.

5. (*a*) If 8 yards of cloth cost 32 shillings, what will be the cost of 10 yards?

(*b*) If p yards of cloth cost m shillings, write down a formula for the cost in shillings (C) of y yards.

(B)

6. Write down five consecutive odd numbers, of which

    (*a*) the smallest is $2m + 1$.
    (*b*) the largest is $2m + 1$.
    (*c*) the middle number is $2m + 1$.

7. Simplify the following:

(a) $\frac{7}{x} + \frac{8}{2x} + \frac{9}{5x}$.

(b) $\sqrt{121a^4x^2} + \sqrt[3]{343a^6x^3}$.

8. Fill in the contents of the brackets in each of the following:

(a) $4(a + 2b) - 3(a - 2b) = 2(\qquad\qquad)$.

(b) $9(2x + y) - 3(\qquad\qquad) = 3(5x + 2y)$.

9. One side of a rectangle is twice as long as the other. If the area of the rectangle is 162 square inches, find the length of each side.

10. Divide £100 among A, B, and C so that B receives £15 more than A and C receives £5 less than B.

(C)

11. Solve the equations

(a) $4(2x - 1) = 6(x + 2)$.

(b) $5(1 + 2x) - 3(1 + x) = 37$.

12 If V is the volume of a sphere of radius r, then $V = \frac{4}{3}\pi r^3$. Make r the subject of this formula.

13. A bag contains an equal number of shillings and florins. If the total value of the money is £4 10s., how many of each coin are there?

14. Find three consecutive odd numbers whose sum is 45.

15. Simplify the following:

(a) $3x^2y \times 5xy^2 \times 9xy$.

(b) $\frac{3a}{5x} + \frac{2b}{7y} + \frac{1}{xy}$.

(c) $\sqrt{169x^4y^6}$.

(D)

16. Say what value must be given to P so that

$$\frac{4a^2b}{9c} = \frac{28a^3b^2c}{P}.$$

17. Calculate as quickly as you can the value of $999^2 - 1$.

18. A man is 25 years older than his son. Five years ago the man was six times as old as the son. Find their present ages.

19. A bill of £50 is paid with one-pound notes and ten-shilling notes. If there were 14 more pound notes than ten-shilling notes, how many of each were used?

20. Find a number such that if I divide it by 3 and add 6 to the answer the result is 12 less than the original number.

ADDITIONAL EXERCISES

A 1

1. If $x = 8$, $y = 6$, and $z = 4$, find the value of

(a) $x + y + z$.

(b) $x + 2y + 3z$.

(c) $x^2 + y^2 + z^2$.

(d) $\sqrt{x} + \sqrt{z}$.

(e) $\dfrac{x + y}{z}$.

2. Find the factors of

(a) $3pq + 5p^2$.

(b) $xy + yz + xyz$.

(c) $a^2 - b^2$.

(d) $4k^2 - 9m^2$.

(e) $12a^3 - 3ab^2$.

3. (a) Find a formula for the area (A) of the design in Fig. 32.

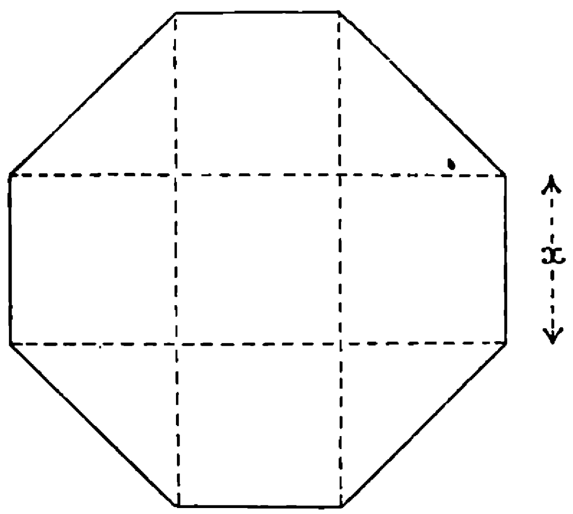

FIG. 32

(b) Use this formula to calculate the area when $x = 3$ inches.

4. Solve the equations

(a) $7x = 91$. (b) $5p + 8 = 53$. (c) $4(a + 2) = 36$.

5. The sum of two numbers is 32. If one of them is three times as large as the other find the numbers.

6. Simplify

$$(a)\ \frac{2p}{7}+\frac{8p}{7}.\quad (b)\ \frac{9k}{5}-\frac{2k}{7}.\quad (c)\ p^2k \div pk^2.$$

7. Divide £20 between A, B, and C so that B has £7 more than A and C has £2 less than A.

8. Make h the subject of the formula $V = \pi r^2 h$.

9. The sum of the angles of any triangle is 180°. If one angle is 46° and the other two angles are equal, how large is each of the equal angles?

10. Find the sum of four consecutive numbers of which k is the largest.

A 2

1. Simplify the following:

$$(a)\ (4p)^2.\qquad (b)\ \sqrt{169a^4} \div \sqrt{225a^6}.$$

2. Express £p + q shillings + r pence in pence.

3. Find the L.C.M. and H.C.F. of $15a^2b$, $25ab^2$, and $40a^2b^2$.

4. In 20 years' time a man will be twice as old as he is now. How old is he now?

5. Solve the equations

$$(a)\ \tfrac{2}{3}x + 1 = 19.\quad (b)\ 5(2x+1) - 2(x-1) = 39.$$

6. Two consecutive odd numbers have a sum of 56. Find the numbers.

7. Express the price of £x per ton in pence per lb.

8. (*a*) A train is travelling at 60 m.p.h. How many feet does it travel in one second?

(*b*) A train is travelling at k m.p.h. How many feet does it travel in one second?

(*c*) How many feet does the train in (*b*) travel in m seconds?

9. (*a*) A clerk is appointed at a salary of £100 per annum, increasing by £10 a year to £250 per annum. What is his

salary at the end of the tenth year? How long does it take him to reach his maximum salary?

(*b*) A clerk is appointed at a salary of £P per annum, increasing by £x a year to £Q per annum. Write down a formula for his salary (S) at the end of the nth year. How long does it take him to reach his maximum salary?

10. A bag contains shillings and sixpences, there being twice as many sixpences as shillings. If the value of the money is £2, how many of each coin are there?

A 3

1. Solve the following equations:

(*a*) $\frac{2x}{3} - \frac{3x}{8} = 7.$

(*b*) $5(2x + 1) - 4(x - 1) = 4(x + 4).$

2. (*a*) How long will a train 500 feet in length and travelling at 30 m.p.h. take to pass over a bridge 380 feet in length?

(*b*) How long will a train k feet in length and travelling at v m.p.h. take to pass over a bridge l feet in length?

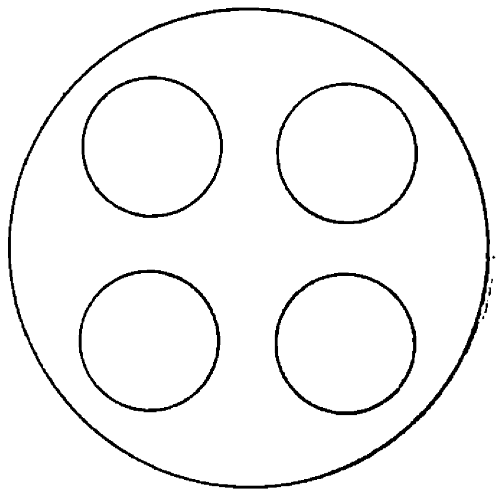

FIG. 33

3. (*a*) The area of a circle is πr^2, where r is its radius. Find a formula for the area (A) of the metal disk shown in Fig. 33 if the radius of the disk is R and the radius of each of the small holes is r.

(*b*) Arrange the formula so as to make it suitable for use in calculations.

(*c*) Find A when R = 8 cm. and $r = 0{\cdot}5$ cm.

4. Complete the brackets in each of the following:

(*a*) $4x^2 + 14xy + 28xy^2 = 2x(\qquad)$.

(*b*) $7x - 5y - 6(\qquad) = x + y$.

(*c*) $64 - m^2 = (\qquad)(\qquad)$.

5. Calculate as quickly as possible the value of

(*a*) 999^2. (*b*) $185 \times 123 + 123 \times 161$.

6. Express x guineas in pounds.

7. If $p = 3$ and $q = 5$, find the value of

(*a*) $p^3 + q^3$. (*b*) $p^2 + 2pq + q^2$. (*c*) $(p + q)^2$.

8. (*a*) If p yards of cloth cost q shillings, what will be the cost (C) of x yards?

(*b*) Find C when $x = 10\frac{1}{2}$, $q = 3\frac{1}{2}$, and $p = 3$.

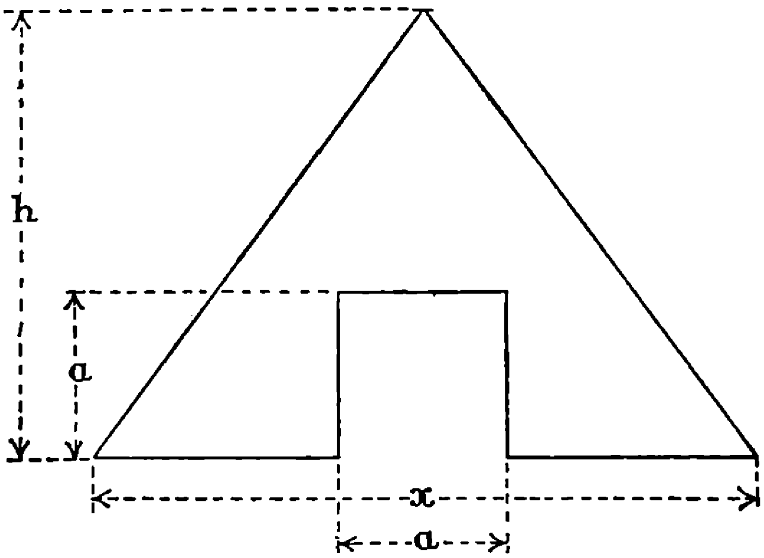

FIG. 34

9. If $A = x + 2y$, $B = y + 2z$, and $C = z + 2x$, find the value of $A + B + C$.

Why will $A + B + C$ be always divisible by 3?

10. A boy receives x shillings a week pocket-money, and spends on an average y pence a week. What do his savings amount to in a year? (Give your answer in shillings.)

A 4

1. The angles of a triangle are $x°$, $5x°$, and $9x°$. What is x?

2. Simplify the following:

(*a*) $x^2y^3 \div 3xy^2$. (*b*) $p\left(\frac{1}{x} + \frac{2}{y}\right)$.

3. If I add 6 to a certain number and divide the answer by 2 the result is 24. What is the number?

4. A party consists of 8 adults and 5 children. When travelling by train the adults pay full fare and the children half-fare, with the result that the railway tickets cost in all £1 6*s*. 3*d*. What was the full fare?

5. (*a*) Find a formula for the area (A) of the diagram in Fig. 34.

(*b*) Find A when $x = 7$ inches, $h = 12$ inches, and $a = 1$ inch.

6. Solve the equation

$$\frac{x+5}{4} + \frac{x+1}{8} = 4.$$

7. If $A = 5x + 7y$ and $B = 3x - 2y$, find the value of (*a*) A + B and (*b*) A − B.

8. Simplify the following by removing the brackets and collecting like terms:

(*a*) $a + 2(a + 3b) + b$.

(*b*) $5(3a + 5b) - 3(5a - 3b)$.

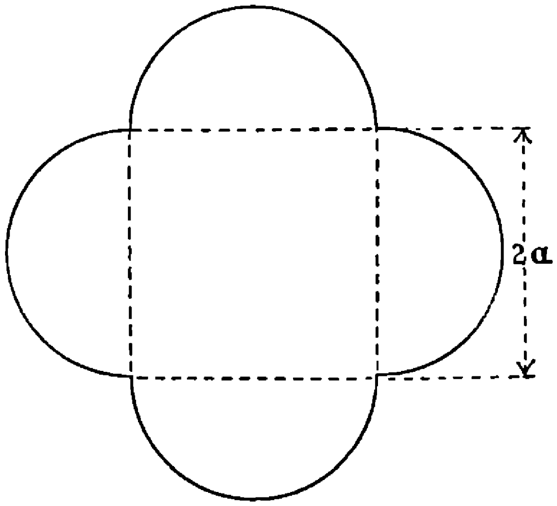

FIG. 35

9. If V is the volume of a sphere of diameter *d*, then $V = \frac{\pi}{6}d^3$. Make *d* the subject of this formula.

10. Divide £30 between A, B, and C so that B has twice as much as A and C has as much as A and B together.

A 5

1. (*a*) Write down a formula for the area (A) of the design (called a *quatrefoil*) shown in Fig. 35.

(*b*) Calculate A when $a = 6$ inches.

2. (*a*) Write down a formula for the perimeter (C) of the design in Fig. 35.

(*b*) Calculate C when $a = 6$ inches.

3. If $X = ax + by$ and $Y = bx + ay$, prove that $X + Y = x(a + b) + y(a + b)$.

4. Solve the following equations:

(a) $\dfrac{x + 3}{2} = \dfrac{4x - 3}{3}$.

(b) $5(x + 4) - 3(x + 1) = 31$.

5. A border x inches wide is left all the way round the inside of a room 20 feet long and 15 feet wide, and the remainder of the room is carpeted. What are the dimensions of the carpet?

6. Find the value of

(a) $\sqrt{3} + \sqrt{3}$. (c) $\sqrt{3} \times \sqrt{3}$.

(b) $\sqrt{3} - \sqrt{3}$. (d) $\sqrt{3} \div \sqrt{3}$.

7. From the formula $\dfrac{1}{f} = \dfrac{1}{u} + \dfrac{1}{v}$ find f in terms of u and v.

8. Write down five consecutive odd numbers of which the middle one is $2m + 1$. What is the sum of these numbers?

9. If 9 is added to 4 times a certain number the result is 109. What is the number?

10. If $S = ut + \frac{1}{2}ft^2$, find S when $u = 0$, $t = 1$, and $f = 24$.

A 6

1. The area of a sphere is given by the formula $A = 4\pi r^2$, where r is the radius. If d is the diameter of the sphere, then $d = 2r$. Prove that $A = \pi d^2$.

2. If $X = \sqrt{a} + \sqrt{2a} + \sqrt{3a}$, find X when $a = 3$. (Give your answer to two places of decimals.)

3. Solve the equations

(a) $\dfrac{1}{x} = \dfrac{3}{7}$. (b) $\dfrac{2}{x} + \dfrac{3}{2x} = \dfrac{7}{6}$.

4. If 6 is added to a certain number and the result is divided by 7 the answer is 5. What is the number?

5. Eggs are bought at x shillings a dozen and sold at a

profit of $\frac{1}{2}x$ pence each. At what price per dozen were the eggs sold?

6. Multiply $3ab + 4c$ by $2a^2b$.

7. Divide $3x^3 + 6x^2y + 9x^2z$ by $3x$.

8. (*a*) If 5 men dig a trench in 8 hours, how long would 4 men have taken to do this?

(*b*) If p men dig a trench in k hours, how long would q men have taken to do this?

9. If a, b, and c are three consecutive numbers, prove that $ac = a(b + 1)$.

10. If x is a proper fraction, will x^2 be a fraction or a whole number?

A 7

1. For what value of m does the sum of $5(2m + 3)$ and $3(m + 1)$ amount to 31?

2. A bag contains 20 coins, some of which are florins and the remainder sixpences. The value of the coins is £1 8*s*. How many florins are there?

3. (*a*) Find a formula for the area (A) of the quatrefoil design in Fig. 36.

(*b*) Calculate the area of the design when $x = 6$ inches.

(*c*) Find a formula for the perimeter (C) of this design.

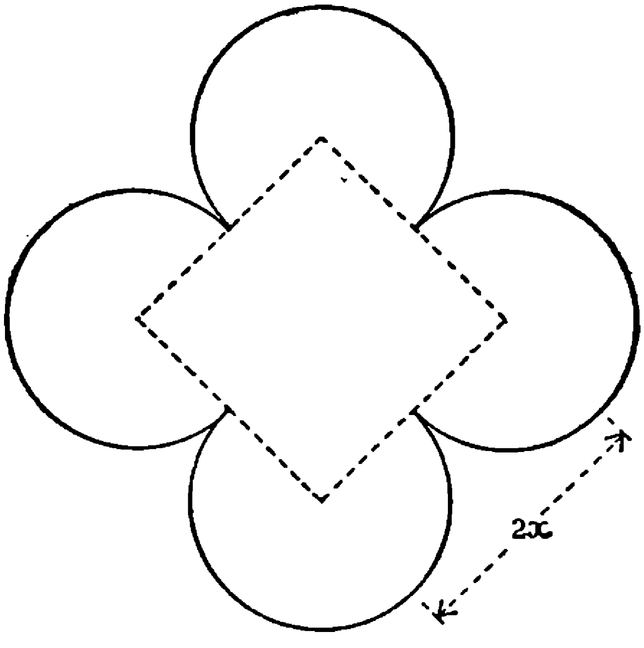

FIG. 36

4. Find a number such that one-seventh of it exceeds one-eighth by one.

5. If $\frac{1}{R} = \frac{2}{p} + \frac{3}{q}$, express R in terms of p and q.

6. If $\frac{a}{x} + \frac{b}{2x} + \frac{c}{3x} = a + \frac{b}{2} + \frac{c}{3}$, find x.

7. Simplify

$$\frac{3p^2q}{7qr^3} \times \frac{5pqr}{9r^2q} \times \frac{21p}{20qr}.$$

8. A runs v feet a second and B runs w feet a second. In a certain race A wins by x feet. By how many seconds does A win?

9. The perimeter of a square is $28x$ inches. What is the area of the square?

10. Simplify $5(3x + 4) - 2(x - 1) - (x + 6)$.

For what value of x is this expression equal to 40?

A 8

1. A map is drawn to a scale in which $2x$ inches represent 1 mile. How many square inches on the map will represent a square mile?

2. If $px + q = 5$ when $p = 1$ and $q = 3$, what will be its value when $p = 2$ and $q = 7$?

3. If a, b, c, and d are four consecutive numbers of which a is the smallest, prove that their average is $b + \frac{1}{2}$.

4. The length of a rectangular garden is 10 yards longer than its width. If the perimeter of the garden is 80 yards, find (a) the length of the garden, (b) its area.

5. If $A = 3x + 5y$ and $B = 5x + 3y$, prove that $A + B$ is divisible by 8.

6. The angles of a triangle are $2x°$, $2x°$, and $x°$. What is x?

7. Using the data of question 5, find the value of $2A + 3B$ when $x = 2$ and $y = 3$.

8. Solve the equations

$$(a)\ \frac{2}{x} + \frac{5}{3x} = \frac{11}{12}. \qquad (b)\ \frac{x}{2} + \frac{3x}{5} = 3\tfrac{3}{10}.$$

9. If $2s = a + b + c$, prove that
$$s + (s - a) + (s - b) + (s - c) = a + b + c.$$

10. For what values of x will the two fractions $\frac{x+5}{3}$ and $\frac{2x+7}{5}$ be equal?

A 9

1. Find two consecutive numbers whose sum is 291.

2. If $ax + b = 17$ when $a = 2$ and $b = 3$, what will be the value of the expression when $a = 5$ and $b = 18$?

3. Each side of a regular hexagon is x inches in length. If the sides are produced so as to form a six-pointed star, find the perimeter of this design.

4. Find the factors of

$$(a)\ 225 - 169m^2. \quad (b)\ 5ax + 25ay. \quad (c)\ 36a^3 - 64ab^2.$$

5. If $\frac{1}{a} + \frac{1}{b} = \frac{1}{c}$, find b when $a = 2$ and $c = 5$.

6. The length and width of a rectangle are both doubled. By how many times is the area increased?

7. A runs 100 yards in x seconds, and thus beats B by one second. How many yards start can be given to B so as to make the race a dead heat?

8. What is the simplest value that can be given to t in order to make $t + 1$ exactly divisible by $2x + 3$?

9. One tap fills a bath in x seconds and another in y seconds. If both taps are turned on how long will it take to fill the bath?

10. In the previous question if the waste tap empties the bath in z seconds and is left open when the other taps are running, how long will it take to fill the bath then?

A 10

1. If $a = \sqrt{b}$ and $b = \sqrt{c}$, express a in terms of c. Find c when $a = 2$.

2. If $x = 3$ and $y = 5$, find the value of

(a) x^x. (c) $(x + y)^x$.
(b) y^y. (d) $x^y + y^x$.

3. The length of a garden is 5 yards more than its width. If its perimeter is 50 yards, find its area.

4. Subtract $\frac{a}{3}$ from $\frac{3}{a}$ and multiply the result by 3.

5. A piece of wire 1 foot in length is bent into the form of a triangle whose sides are in the proportion 1 : 2 : 5. Find the lengths of the sides.

6. A motor-car travels x miles on a gallon of petrol which costs p pence. What will be the cost of a journey of k miles?

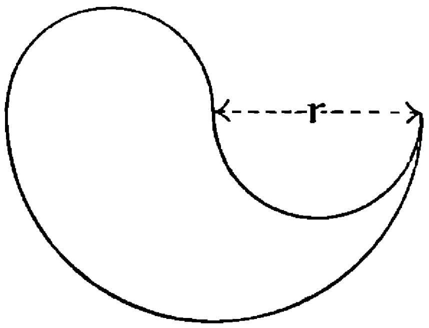

FIG. 37

7. Find a formula for the area (A) of the design in Fig. 37.

8. The sum of three consecutive odd numbers is 249. Find the numbers.

9. What value of x will make $7(2x + 1) - 4(x - 6)$ equal to 91?

10. How many seconds will it take a train x yards long and travelling at v m.p.h. to pass completely through a station y yards in length?

A 11

1. Find what value must be given to x in each of the following:

(a) $2^x = 4$. (c) $5^x = 125$.
(b) $3^x = 81$. (d) $2^x = 64$.

2. Simplify

(a) $\sqrt{100a^2b^4} + \sqrt{169a^4b^2}$. (b) $\frac{3x + 5}{7} - \frac{x + 1}{6}$.

3. If I walk at $x - 1$ miles an hour a journey takes p hours. If I walk at $x + 1$ miles an hour the journey takes q hours. Prove that $x = \dfrac{p + q}{p - q}$.

4. What is the total surface area of a cube of edge x inches? What is the volume of the cube?

5. Find two numbers in the ratio of 7 : 4 whose sum is 55.

6. Divide

(a) $xy + yz$ by y. (c) $x^2 - a^2$ by $x - a$.
(b) $5x + 15xy$ by $5x$. (d) $x^2 - a^2$ by $x + a$.

7. Multiply

(a) $p + 3q$ by $2p$. (c) $x + a$ by $x - a$.
(b) $2p + q$ by $7q$. (d) $x - a$ by $x + a$.

8. (a) What must be added to 5 to make 9?

(b) What must be added to $\dfrac{1}{a}$ to make $\dfrac{1}{b}$.

9. A map is drawn on the scale of 3 inches to the mile. How many square miles are represented by a square of side x inches?

10. Express the cost of p shillings a ton in pence per pound.

A 12

1. A farmer has a number of hurdles each x yards in length. How many will he require to enclose a rectangular piece of ground m yards long and n yards wide?

2. A slow train travelling at v miles an hour leaves King's Cross at 9 A.M. An hour later a fast train travelling at w miles an hour leaves for the same destination. How long will it take the fast train to overtake the slow one?

3. Simplify

$$\frac{5a^2}{7b} \times \frac{8ab}{10b^2a} \times \frac{14ab^2}{24a^2b^2}.$$

4. Divide $(3x)^2 - (3y)^2$ by $9(x + y)$.

5. The sum of two consecutive numbers is $2n + 1$. What are the numbers?

6. If $ax + b = 19$ when $a = 4$ and $b = 3$, find the value of the expression when $a = 7$ and $b = 2$.

7. Solve the equation

$$\frac{1}{2x} - \frac{1}{5x} = \frac{3}{40}.$$

8. If $xy = c$, express y in terms of c and x and calculate y when $c = 150$ and $x = 4$.

9. Find the contents of the bracket in the following:

$$7(5x + 4) - 3(x + 1) = 5(\qquad\qquad) + 2x.$$

10. Solve the equation $0{\cdot}5x - 0{\cdot}25x = 1{\cdot}2$.

A 13

1. At an election 17,816 people voted for one of the two candidates. If the successful candidate had a majority of 3126, how many votes did he receive?

2. Express as a formula the radius of a circle in terms of its circumference.

Use your formula to calculate the radius required to construct a circular running-track a quarter of a mile in length.

3. Solve the equations

(a) $0{\cdot}75x - 1{\cdot}37 = 5{\cdot}63$.

(b) $5(2x - 3) = 7(x + 1) - 1$.

4. Make A the subject of the formula $Y = \dfrac{FL}{Ae}$.

5. A train 300 feet in length is travelling at 60 m.p.h. How long would it take to pass a workman walking along the side of the line in the opposite direction at a rate of x miles an hour?

6. Find two consecutive numbers whose sum is 55.

7. Show that $(2a)^2 + (3a)^2 + (4a)^2 = (5a)^2 + (2a)^2$.

8. If the three angles of a triangle are equal, how large is each?

9. Find the cost in pence of x lb. if the cost of x tons is £p.

10. A square has an area of $256a^2$ square inches. How long is each side?

A 14

1. Find a formula for the diagram in Fig. 38, and use it to calculate the area when $r = 2$ inches.

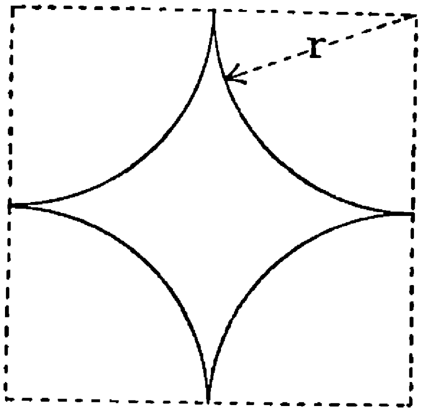

FIG. 38

2. Express a speed of x miles an hour in feet per second.

3. A cube has a volume of $2197p^3$ cubic feet. What is the length of each edge?

4. Simplify

(a) $\frac{4x^3y}{5y^2z} \times \frac{3x}{7y} \times \frac{15xy^2z^3}{12z}$. (b) $\sqrt[4]{81a^8b^{12}c^{16}}$.

5. Multiply

(a) $2x - y$ by x. (b) $2y - 3x$ by $2y$.

6. Multiply

(a) $4x^2$ by $x + y$. (b) $3x - 2y$ by $3y^3$.

7. Solve the equations

(a) $4{\cdot}83 + 5{\cdot}5x = 8{\cdot}98$.

(b) $3(x - 1) + 4(2x - 1) = 37$.

8. Divide three guineas between A, B, and C so that B has $\frac{2}{3}$ of A's share and C has half of B's share.

9. For what value of k will $7(2k - 3)$ and $3(k - 5)$ have the same value?

10. Children are admitted to an entertainment at half-price. If a party of 20 children and 2 adults paid £2 8*s*. in all, what was the price of admission?

A 15

1. A man is 12 times as old as his son. In ten years' time the difference between their ages will be 22. Find their ages now.

2. Divide

(*a*) $5x^2 + xy$ by x.
(*b*) $9x^2 - 16a^2$ by $3x + 4a$.
(*c*) $9x^2 - 16a^2$ by $3x - 4a$.

3. Solve the following equations:

(*a*) $3^x = 729$.
(*b*) $17{\cdot}5x + 3{\cdot}85 = 4{\cdot}5x + 5{\cdot}15$.
(*c*) $\frac{1}{2x} + \frac{1}{3x} = \frac{5}{6}$.

4. If $\frac{1}{v} + \frac{1}{u} = \frac{1}{f}$, express f in terms of v and u.

5. The numerator of a fraction is one less than the denominator. If the reciprocal of the fraction is $1\frac{1}{2}$, find the original fraction.

6. If $7(2x + 4) - 3(2x - 5) = A(x + 5) + 3$, find A.

7. Calculate as quickly as possible the value of

(*a*) $45^2 - 43^2$.
(*b*) $17 \times 19 - 19 \times 13$.
(*c*) $\frac{14 \times 15 \times 16 + 15 \times 16 \times 17}{15 \times 16}$.

8. The perimeter of a card is 26 cm. and it is 3 cm. longer than it is wide. Find its area.

9. A clerk is appointed at a salary of £a, which is increased by £b for each completed year of service. What salary will he receive (*a*) at the beginning of, (*b*) and the end of, the nth year?

10. Find a formula for (*a*) all numbers which are multiples of 7, (*b*) all numbers ending in 7.

PART II

CHAPTER VII

DIRECTED NUMBERS

61. We are all familiar with the idea of direction. The traveller from London goes *north* when he journeys to Edinburgh, and *south* when he returns. The words 'north' and 'south' indicate the directions in which he has travelled, and we know that they are opposite directions. This sense of 'oppositeness' occurs in other cases. A boy may *win* 16 marbles, and next time he plays may *lose* 5 marbles. Winning and losing are two opposites. Again, a tradesman may *gain* £5 on a transaction one day and make a *loss* of £5 another day. Here gain and loss are two opposites. We use the familiar plus (+) and minus (–) signs to indicate this oppositeness. Thus we may write (+ 10) to stand for a journey of 10 miles *north* of a certain town. Then (– 10) will represent a journey of 10 miles *south* of the same town. We call (+ 10) and (– 10) *directed numbers*. In the same way, the boy who wins 16 marbles can describe his winnings as (+ 16). His loss of 5 marbles would be written as (– 5). The tradesman would write his gain of £5 as (+ 5) and his loss of £5 as (– 5).

EXERCISE LXVII

1. If (+ 7) stands for a gain of £7, write down (*a*) a gain of £10, (*b*) a loss of £7, and (*c*) a loss of £10.

2. If (+ 1000) stands for the height in feet of a town above sea-level, write down, in similar form, a height of (*a*) 1500 feet above sea-level, (*b*) 12 feet below sea-level.

3. If (– 30) represents a fall in temperature of 30°, write down (*a*) a fall of 40°, (*b*) a rise of 38°, and (*c*) a rise of 15°.

4. If (– 24) stands for a journey of 24 miles west of a certain town, write down a journey of (*a*) 5 miles west, (*b*) 34 miles east, of this town.

5. A clock gains 10 seconds a day. If this is written (+ 10), how would you write down a loss of 10 seconds a day?

6. A boy's weight increases by x lb. If this is written (+ x), how would you write down (*a*) a gain of p lb., (*b*) a loss of p lb.?

7. A soldier takes 2 paces forward. If this is written (+ 2), how would you write down (*a*) 7 paces forward, (*b*) m paces forward, (*c*) 5 paces backward, and (*d*) k paces backward?

8. If (+ 12) represents the number of steps up to my bedroom from the ground floor, how would you write down the fact that there are 7 steps downward from the ground floor to the cellar?

9. The speed of a train is increased from 30 m.p.h. to 35 m.p.h. If this increase in speed is written as (+ 5), say how you would write down the following changes of speed:

(*a*) 20 m.p.h. to 38 m.p.h.
(*b*) 40 m.p.h. to 38 m.p.h.
(*c*) From rest to 20 m.p.h.
(*d*) From 20 m.p.h. to rest.

10. If (+ x) stands for the payment into the bank of £x and (– x) stands for the withdrawal of £x, say how you would write down the following transactions:

(*a*) A payment in of £24.
(*b*) A payment in of £a.
(*c*) A withdrawal of 10 shillings.
(*d*) A withdrawal of p shillings.

11. If (+ 4) represents a journey of 4 miles in the direction north-east, what does (– 4) represent?

12. If (+ 1930) represents the year A.D. 1930, what does (– 100) represent?

How would you write 55 B.C.?

62. Fig. 39 shows a method for representing directed numbers in a diagram. Numbers of the form (+ 2), (+ 3), and so on are called *positive* numbers, and are represented above the horizontal line. Numbers of the form (– 2), (– 3), and so on are called *negative* numbers, and are represented

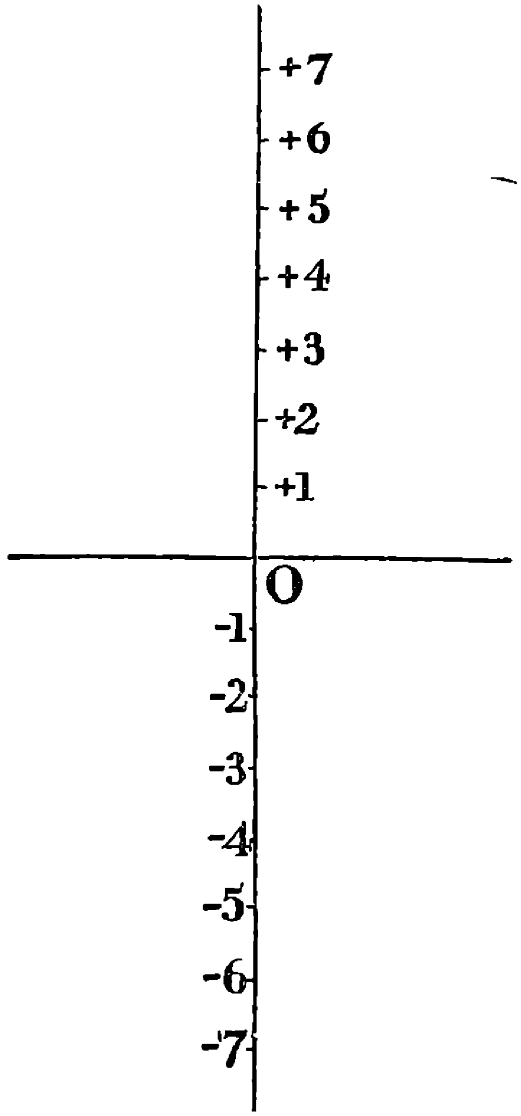

FIG. 39

in a corresponding manner below the line. The point where the horizontal line crosses the vertical is called the *origin*, and is always represented by the letter O. Notice that (+ 1) and (– 1) are equally spaced above and below the origin respectively. Similarly, (+ 2) is as far above O as (– 2) is below it, and so on for the other numbers.

Sometimes this diagram is drawn horizontally instead of vertically, as in Fig. 40. In this case positive and negative

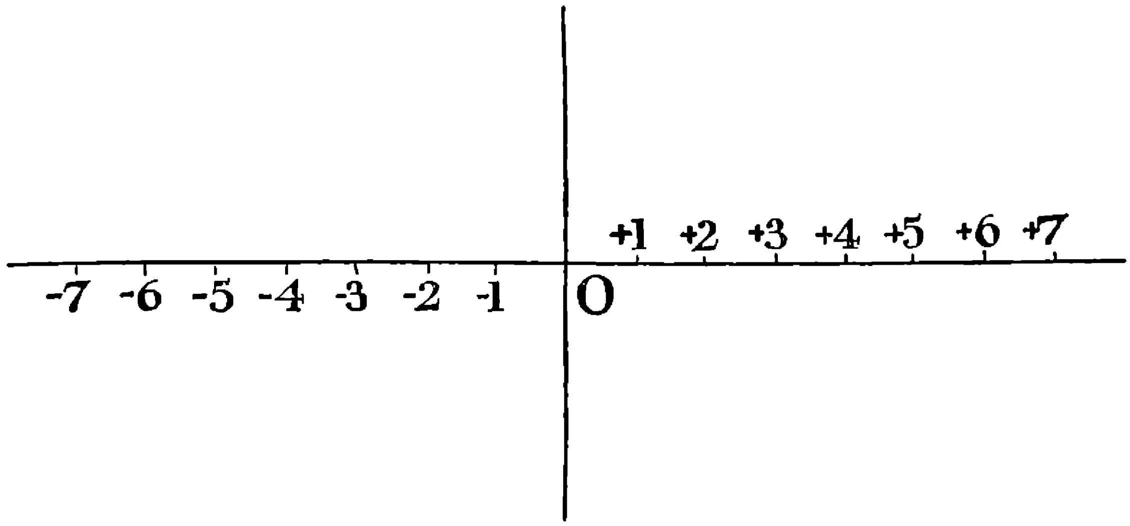

FIG. 40

numbers are represented on the right and on the left of the origin respectively.

EXERCISE LXVIII

1. Draw a diagram similar to Fig. 39, and on it mark by a cross each of the following points: (+ 1), (+ 4), (+ 7), (– 2), (– 5), (– 6).

2. Draw a diagram similar to Fig. 40, and on it mark by a cross each of the following points: (+ 2), (+ 5), (+ 6), (– 1), (– 3), (+ O), (– O).

63. We must now consider how the addition of directed numbers can be performed—for example, (+ 2) + (+ 5).

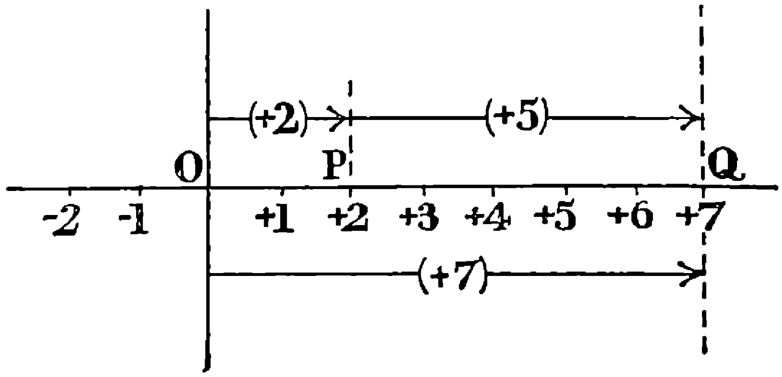

FIG. 41

In our diagram we shall measure positive numbers to the right of the origin and negative numbers to the left. Hence in Fig. 41 (+ 2) will be represented by the point P, which is 2 spaces to the right. We have to add on to this (+ 5), which means moving toward the right a further 5 spaces. This brings us to Q (Fig. 41). But Q represents the number (+ 7), because it is 7 spaces to the right of the origin. Therefore (+ 2) + (+ 5) = (+ 7).

EXERCISE LXIX

1. By means of diagrams similar to Fig. 41 prove the following:

(*a*) (+ 1) + (+ 2) = (+ 3).
(*b*) (+ 2) + (+ 3) = (+ 5).
(*c*) (+ 1) + (+ 2) + (+ 3) = (+ 6).

2. From a diagram (or otherwise) find the value of

(*a*) (+ 4) + (+ 3).
(*b*) (+ 6) + (+ 2).
(*c*) (+ 1) + (+ 3) + (+ 5).
(*d*) (+ 4) + (+ 10) + (+ 1).
(*e*) (+ 2) + (+ 9) + (+ 3).
(*f*) (+ 7) + (+ 5) + (+ 4).
(*g*) (+ 3) + (+ 8) + (+ 7).

64. The addition of two negative numbers is performed in a similar manner. Thus, if we wish to find the value of (– 2) + (– 5) we choose P 2 spaces to the left of our origin (Fig. 42). The addition of (– 5) means a further 5 spaces in

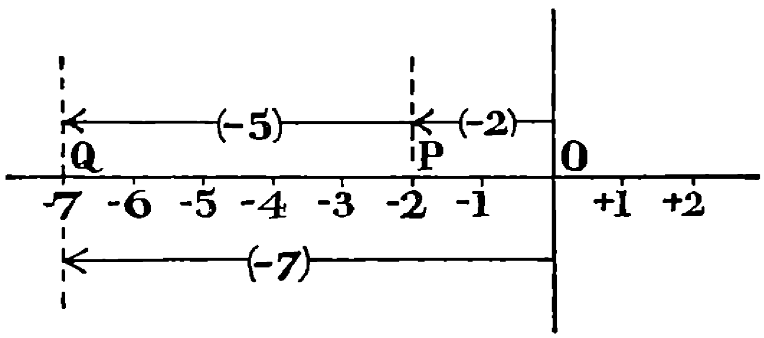

FIG. 42

the negative direction—*i.e.*, to Q. It is easy to see that Q represents (– 7).

Hence (– 2) + (– 5) = (– 7).

EXERCISE LXX

1. By means of diagrams similar to Fig. 41 prove the following:

   (*a*) (– 1) + (– 2) = (– 3).
   (*b*) (– 2) + (– 3) = (– 5).
   (*c*) (– 1) + (– 2) + (– 3) = (– 6).

2. From a diagram (or otherwise) find the value of

   (*a*) (– 3) + (– 4).
   (*b*) (– 8) + (– 3).
   (*c*) (– 1) + (– 3) + (– 5).
   (*d*) (– 4) + (– 8) + (– 2).
   (*e*) (– 1) + (– 10) + (– 8).
   (*f*) (– 7*a*) + (– 10*a*) + (– 8*a*).
   (*g*) (– 5*x*) + (– 7*x*) + (– 9*x*) + (– 11*x*).

65. We must now consider the addition of two directed numbers, one of which is positive and the other negative —*e.g.*, (– 2) + (+ 5).

Following the method adopted in the preceding pages, (– 2) will be represented in Fig. 43 by P, which is 2 spaces

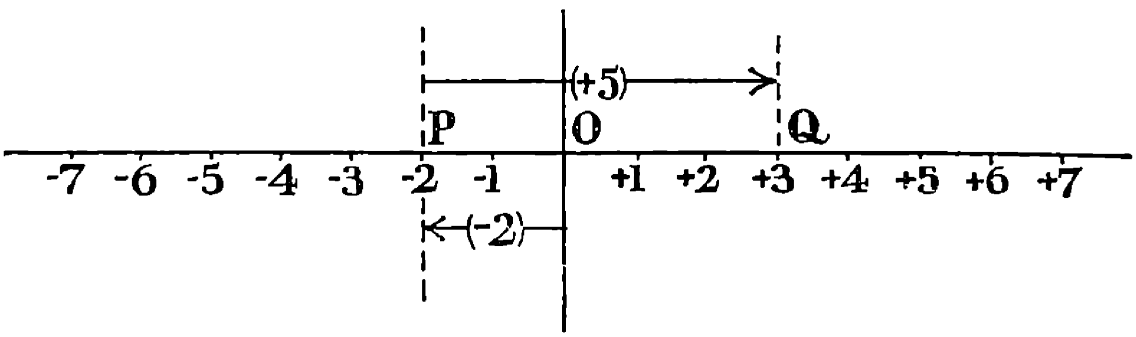

FIG. 43

on the negative side—*i.e.*, the left side—of the origin, O. The addition to this of (+ 5) means moving 5 spaces in the positive direction—*i.e.*, toward the right—from P. This takes us to Q, which is 3 spaces on the right of O.

Hence $(-2) + (+5) = (+3)$.

As a second example, consider the sum $(-7) + (+4)$. This is illustrated in Fig. 44. The point P, which is 7 spaces

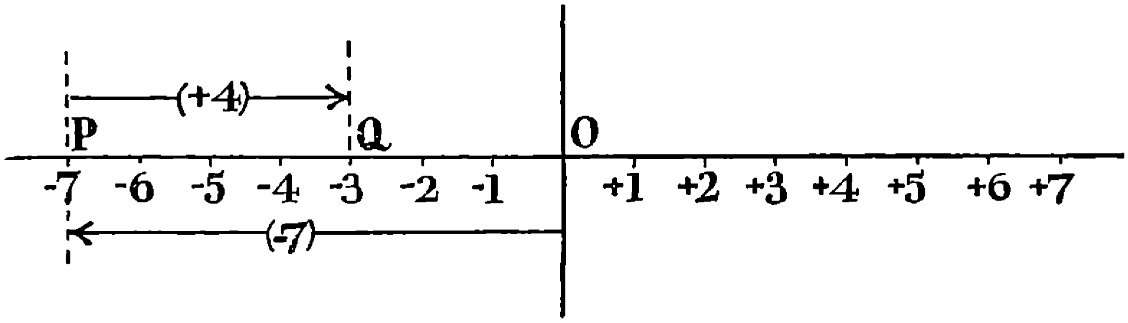

FIG. 44

on the negative side of the origin, represents (-7). The addition of $(+4)$ is represented by a movement of 4 spaces in the positive direction—*i.e.*, to Q. This is 3 spaces to the left of O.

Hence $(-7) + (+4) = (-3)$.

EXERCISE LXXI

1. By means of diagrams similar to Figs. 43 and 44 prove the following:

(*a*) $(-3) + (+2) = (-1)$.
(*b*) $(+3) + (-2) = (+1)$.
(*c*) $(+4) + (-7) = (-3)$.
(*d*) $(-1) + (-2) + (+5) = (+2)$.

2. From a diagram (or otherwise) find the value of

(*a*) $(+10) + (-11) + (-3)$.
(*b*) $(-8) + (-10) + (+18)$.
(*c*) $(+4) + (-7) + (-8)$.
(*d*) $(-7) + (-9) + (+11) + (+13)$.
(*e*) $(-1) + (+2) + (-3) + (+4)$.
(*f*) $(-5) + (+7) + (+9) + (-11)$.

(*g*) $(+a)+(+3a)+(-5a)$.
(*h*) $(-4k)+(-3k)+(+k)+(+2k)$.
(*i*) $(+3x)+(+12x)+(-9x)$.
(*j*) $(-12x)+(+21x)+(-4x)$.
(*k*) $(-14m)+(+8m)+(-10m)$.

3. The lift in a hotel went from the ground floor to the seventh floor, on to the eighth floor, and then down to the second floor. Illustrate these movements by means of directed numbers.

4. A boy's diary revealed the following financial transactions: August 1, received 10*s.*; August 2, paid out 2*s.* 6*d.*; August 17, spent 5*s.* 6*d.*; August 29, received 15*s.* from father and 7*s.* 6*d.* from aunt; September 1, paid out 8*s.* 6*d.*

Represent these receipts and payments by directed numbers, and thus calculate how much money the boy had left.

66. The subtraction of directed numbers is performed by means of complementary addition—that is to say, instead of *subtracting* 7 from 13 we find what must be *added* to 7 in order to make 13. The following examples will show the application of the method to directed numbers.

Example 1

Find the value of $(+5)-(+2)$.

In this case we have to ask ourselves what must be *added* to $(+2)$ in order to make $(+5)$. In Fig. 45 $(+2)$ is represented by P and $(+5)$ by Q. In order to go from P

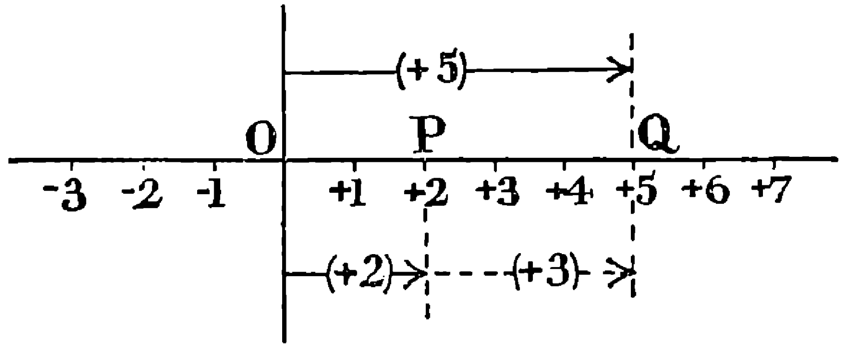

Fig. 45

to Q we must move $(+3)$, or, in other words, we must add $(+3)$ to $(+2)$ in order to get $(+5)$. Hence

$$(+5) - (+2) = (+3).$$

EXAMPLE 2

Find the value of $(-5) - (-2)$.

Here we have to find what must be added to (-2) in order to make (-5). In Fig. 46 P represents (-2) and Q

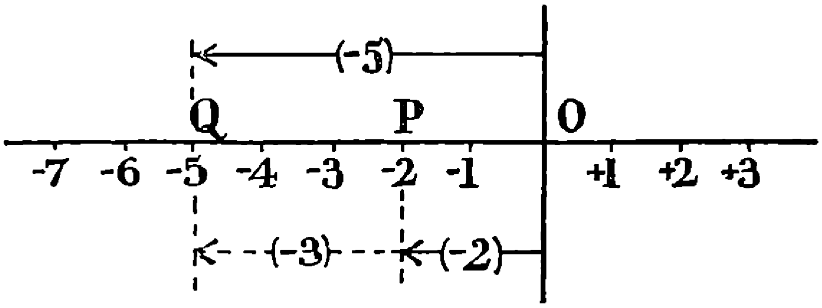

FIG. 46

represents (-5). To go from P to Q means an addition of (-3). Hence

$$(-5) - (-2) = (-3).$$

EXERCISE LXXII

1. Show by means of diagrams similar to Figs. 45 and 46 that

(*a*) $(+4) - (+1) = (+3)$.
(*b*) $(-4) - (-1) = (-3)$.
(*c*) $(+5) - (+4) = (+1)$.
(*d*) $(-5) - (-4) = (-1)$.

2. From a diagram (or otherwise) find the value of

(*a*) $(+10) - (+7)$.
(*b*) $(+14) - (+8)$.
(*c*) $(-10) - (-7)$.
(*d*) $(-14) - (-8)$.
(*e*) $(-10a) - (-4a)$.
(*f*) $(+24x) - (+17x)$.
(*g*) $(+14m^2) - (+11m^2)$.

(*h*) $(+29xy) - (+15xy)$.
(*i*) $(+5) + (+7) - (+8)$.
(*j*) $(+6) + (+10) - (+12)$.
(*k*) $(-8) - (-6) + (-5)$.
(*l*) $(+14a) + (-8a) - (+4a)$.
(*m*) $(-15x) - (-8x) + (+5x)$.
(*n*) $(+21m) - (+15m) + (-6m)$.

67. We have seen from Fig. 45 that

$$(+5) - (+2) = (+3).$$

This gives the same result as $(+5) + (-2)$, for

$$(+5) + (-2) = (+3)$$

according to the rules we obtained for the addition of directed numbers. Compare these two results and notice that the effect of *subtracting* $(+2)$ is the same as *adding* (-2).

From Fig. 46 we have

$$(-5) - (-2) = (-3).$$

This gives the same result as $(-5) + (+2)$, for

$$(-5) + (+2) = (-3)$$

according to our rules for the addition of two directed numbers. Hence the *subtraction* of (-2) gives the same result as the *addition* of $(+2)$.

These suggest that *we can change the process of subtraction into one of addition provided we change the sign of the number we subtract.*

EXERCISE LXXIII

1. Insert the proper directed number in the empty brackets in each of the following examples:

(*a*) $(+5) - (+1) = (+5) + (\quad\quad)$.
(*b*) $(+7) - (+4) = (+7) + (\quad\quad)$.
(*c*) $(-8) - (-5) = (-8) + (\quad\quad)$.
(*d*) $(-10a) - (-8a) = (-10a) + (\quad\quad)$.
(*e*) $(-15xy) - (-8xy) = (-15xy) + (\quad\quad)$.

2. Write down the answers to each of Nos. 1 (*a*)–1 (*e*).

3. Verify the following from a diagram similar to Figs. 44 and 45:

(*a*) $(+5) - (+7) = (-2)$.
(*b* $(+8) - (+15) = (-7)$.
(*c*) $(-5) - (-7) = (+2)$.
(*d*) $(-8) - (-15) = (+7)$.

4. Complete each of the following:

(*a*) $(-12x) - (-14x) = (-12x) + (\qquad)$.
(*b*) $(-10ab) - (-24ab) = (-10ab) + (\qquad)$.
(*c*) $(+14xyz) - (+15xyz) = (+14xyz) + (\qquad)$.
(*d*) $(+12m^2) - (+18m^2) = (+12m^2) + (\qquad)$.

5. Write down the answers to each of Nos. 4 (*a*)–4 (*d*).

68. The subtraction of directed numbers of opposite kinds follows the rules just established.

Example 1

Find the value of $(+5) - (-3)$.

Here we have to find what must be *added* to (3) in order to make $(+5)$. This is $(+8)$, since

$$(+8) + (-3) = (+5).$$

Hence $$(+5) - (-3) = (+8).$$

We see at once from this that our rule of signs (p. 130), by which we change a subtraction into an addition, still holds good, for

$$(+5) - (-3) = (+5) + (+3) \quad \text{(from the rule of signs)}$$
$$= (+8).$$

We can illustrate this subtraction by means of a diagram, as in Fig. 47. Here P represents (-3). In order to obtain $(+5)$ we must move 8 spaces to the right—*i.e.*, in the positive direction—to Q. But Q represents $(+5)$. Hence

$$(+5) - (-3) = (+8).$$

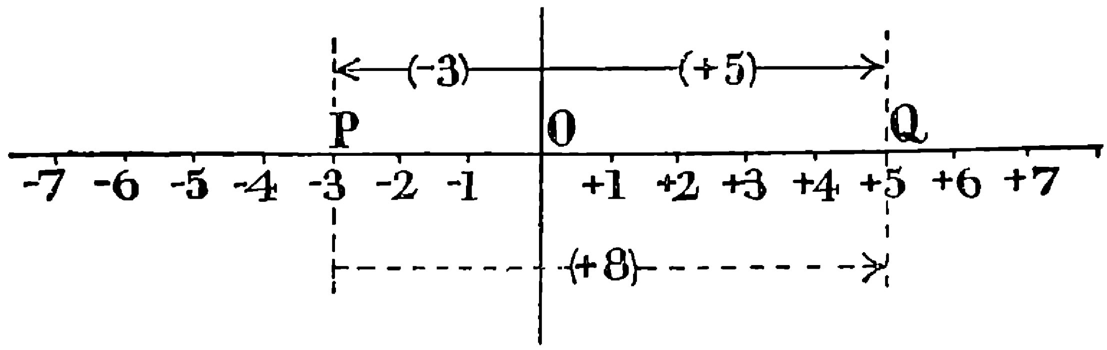

FIG. 47

EXAMPLE 2

Find the value of (– 3) – (+ 5).

Here again we ask ourselves what must be added to (+ 5) in order to make (– 3). Clearly (– 8). Notice that the rule of signs still holds, for

$$(-3)-(+5) = (-3)+(-5) \quad \text{(according to the rule of signs)}$$
$$= (-8).$$

EXERCISE LXXIV

1. Draw a diagram similar to Fig. 47 to illustrate each of the following:

(*a*) (+ 7) – (– 2) = (+ 9).
(*b*) (– 7) – (+ 5) = (– 12).
(*c*) (– 5) – (+ 4) = (– 9).

2. Rewrite each of the following subtractions as additions, and hence find the required answer:

(*a*) (– 18) – (+ 4).
(*b*) (+ 18) – (– 4).
(*c*) $(-12x)-(+7x)$.
(*d*) $(-14ab)-(+21ab)$.
(*e*) $(+12x^2)-(-4x^2)$.
(*f*) $(+41xyz)-(-32xyz)$.
(*g*) $(-32xyz)-(+41xyz)$.
(*h*) $(+20a^2)-(-47a^2)$.
(*i*) $(-35mn)-(+14mn)$.
(*j*) $(-\frac{1}{2}k^3)-(+\frac{1}{3}k^3)$.
(*k*) $(+3{\cdot}5p)-(-2{\cdot}7p)$.

3. If $a = (-7)$, $b = (+8)$, and $c = (-9)$, find the value of

(a) $a + b$.
(b) $a + b + c$.
(c) $a - b$.
(d) $b - a$.
(e) $a - b + c$.
(f) $a + b - c$.
(g) $a - b - c$.
(h) $b - a$.
(i) $b - c$.
(j) $c - a$.
(k) $c - b + a$.
(l) $c - b - a$.
(m) $c + b - a$.
(n) $c - a + b$.

4. If $x = (-4)$, $y = (-6)$, and $z = (+8)$, find the value of

(a) $x + y$.
(b) $x + y + z$.
(c) $x + (y + z)$.
(d) $x - (y + z)$.
(e) $x - y - z$.
(f) $(x + y) - z$.
(g) $z - (x + y)$.
(h) $(x + y) + (y + z)$.
(i) $(x - y) - (y + z)$.
(j) $(x - y) + (y - z) + (z - x)$.

69. We have already seen that multiplication is nothing more than repeated addition. Thus,

$$4 \times 3 = 4 + 4 + 4$$
$$= 12.$$

We shall deal with directed numbers in the same way, for

$$(+4) \times (+3) = (+4) + (+4) + (+4)$$
$$= (+12)$$

and

$$(-4) \times (+3) = (-4) + (-4) + (-4)$$
$$= (-12$$

EXERCISE LXXV

By the process of repeated addition show that

1. $(+2) \times (+3) = (+6)$.
2. $(-2) \times (+3) = (-6)$.
3. $(+2) \times (-3) = (-6)$.
4. $(+1) \times (+2) \times (+3) = (+6)$
5. $(-1) \times (+2) \times (+3) = (-6)$.
6. $(-4) \times (+7) = (-28$

7. $(+3) \times (-5a) = (-15a)$.
8. $(-2x) \times (+5) = (-10x)$.
9. $(+5) \times (+5) = (+25)$.
10. $-5) \times (+5) = (-25)$.

70. Just as we define the product $(+4)$ by $(+3)$ as a repeated addition thus:

$$\begin{aligned}(+4) \times (+3) &= (+4) + (+4) + (+4)\\ &= (+12)\end{aligned}$$

so we shall define the product (-4) by (-3) as a repeated subtraction in this manner:

$$\begin{aligned}(-4) \times (-3) &= -(-4) - (-4) - (-4)\\ &= -(-12)\\ &= (+12).\end{aligned}$$

We thus obtain the following rule for the product of two directed numbers:

When two directed numbers are multiplied together the product is positive if the signs of the two numbers are the same, or negative if the signs of the two numbers are unlike.

EXERCISE LXXVI

1. Find the value of

(*a*) $(+5) \times (-4)$.
(*b*) $(+7) \times (+8)$.
(*c*) $(-7) \times (-9)$.
(*d*) $(+4) \times (-9)$.
(*e*) $(-1) \times (+2) \times (-3)$.
(*f*) $(-8) \times (+7) \times (+6)$.
(*g*) $(-1) \times (-2) \times (-3)$.
(*h*) $(+4) \times (+12)$.
(*i*) $(-4) \times (-12)$.
(*j*) $(-3) \times (-5) \times (-7)$.
(*k*) $(-2x) \times (+3y)$.
(*l*) $(+2x) \times (-3y)$.
(*m*) $(-3a^2) \times (-4b)$.
(*n*) $(+4b) \times (-3ab)$.
(*o*) $(-5m) \times (-6p)$.
(*p*) $(-8k) \times (+7k)$.
(*q*) $(-x) \times (-y) \times (-z)$.
(*r*) $(-2x) \times (-3y) \times (-4z)$.
(*s*) $(+5mn) \times (-6np)$.
(*t*) $(-3ab) \times (-10bc)$.

2. If $a = (+2)$, $b = (-3)$, and $c = (-4)$, find the value of

(*a*) ab.	(*e*) $b(b + c)$.
(*b*) bc.	(*f*) abc.
(*c*) ac.	(*g*) $c(b + a)$.
(*d*) $a(a + b)$.	(*h*) $a(b - c) + b(c - a) + c(a - b)$.

3. If $x = (-5)$, $y = (-4)$, and $z = (-3)$, verify by substitution that

(*a*) $x(y + z) = xy + xz$.
(*b*) $x(y - z) = xy - xz$.

71. We have seen that

$$(+12) = (+3) \times (+4).$$

If we divide both sides of this equation by $(+4)$ we get

$$\frac{(+12)}{(+4)} = (+3) \qquad (1)$$

Compare this with

$$(-12) = (+3) \times (-4).$$

Dividing both sides of this equation by (-4), we get

$$\frac{(-12)}{(-4)} = (+3) \qquad (2)$$

The results (1) and (2) show that if two directed numbers of the *same* sign—*i.e.*, both positive or both negative—are divided the quotient is positive. Now consider the equation

$$(-12) = (-3) \times (+4)$$

Dividing both sides by $(+4)$, we get

$$\frac{(-12)}{(+4)} = (-3).$$

This shows that the quotient of two directed numbers of *opposite* signs is negative.

EXERCISE LXXVII

Find the quotient in each of the following:

1. $(-8) \div (+2)$, $(+16) \div (-4)$.
2. $(-8) \div (-2)$, $(-16) \div (+4)$.
3. $(+2) \div (-8)$, $(-3) \div (+9)$.
4. $(+16) \div (+4)$, $(-9) \div (+3)$.
5. $(-16) \div (-4)$, $(+64) \div (-8)$.
6. $(+10) \div (-5)$, $(-64) \div (+8)$.
7. $(-49) \div (+7)$, $(+56) \div (-7)$.
8. $(-108) \div (+12)$, $(+144) \div (+9)$.
9. $(-12a) \div (-4a)$.
10. $(+4a) \div (-12a)$.
11. $(+35x^2) \div (-7x)$.
12. $(-15xy) \div (-3x)$.
13. $(+4pq) \div (+20pq)$.
14. $(-18m^2n) \div (-3mn^2)$.
15. $(+32p^2q) \div (+4pq^2)$.

EXERCISE LXXVIII

(General Revision Exercise on Directed Numbers)

If $a = (+2)$, $b = (-7)$, and $c = (-8)$, find the value of

1. $a + b$.
2. $a - b$.
3. $a + b + c$.
4. $a - b + c$.
5. $a - b - c$.
6. $a + (b + c)$.
7. $a - (b + c)$.
8. $a - (b - c)$.
9. $(a - b) - c$.
10. $a(b + c)$.
11. $c(b + a)$.
12. $b(a + c)$.
13. $c(a + b) + b(c + a) + a(b + c)$.
14. $\frac{a}{b}$.
15. $\frac{b}{c}$.
16. $\frac{a}{b} + \frac{b}{c}$.
17. $\frac{ac}{b}$.
18. $\frac{bc}{a}$.
19. $\frac{ab}{c} + \frac{bc}{a} + \frac{ca}{b}$.

20. $\frac{a}{b}(c + a)$.

21. $4a$.

22. $7b$.

23. $4a + 7b$.

24. $a + 2(b + c)$.

25. $a - 2(b + c)$.

26. $2a - 3(b - c)$.

27. $\frac{2a}{b}$.

28. $\frac{2b}{a}$.

29. $\frac{2}{a} + \frac{4}{c}$.

30. $5(a + 2b + 3c)$.

31. $\frac{a + b}{a - b}$.

32. $\frac{2a + 3b}{4a - 5b}$.

33. $a + 2b + 3c + 4a + 5b + 6c$.

34. $7(a + b) + 8(b + c) - 9(c + a)$.

35. $4ab - 5bc$.

72. Powers of directed numbers are calculated in the same way as powers of non-directed numbers. Thus,

$$(+2)^2 = (+2) \times (+2) = (+4)$$
$$(-2)^2 = (-2) \times (-2) = (+4)$$
$$(+2)^3 = (+2) \times (+2) \times (+2) = (+8)$$
$$(-2)^3 = (-2) \times (-2) \times (-2) = (-8)$$

and so on.

These four cases are sufficient to show what will be the proper rules for signs.

1. All powers of a positive directed number are positive.
2. Even powers of a negative directed number are positive.
3. Odd powers of a negative directed number are negative.

EXERCISE LXXIX

1. Find the value of

(a) $(+3)^2$.

(b) $(-3)^3$.

(c) $(+2)^4$.

(d) $(-2)^4$.

(e) $(-4)^2$.

(f) $(+5)^2$.

(g) $(-1)^2 + (-2)^2$.

(h) $(-2)^3 + (+3)^3$.

(i) $(+1)^2 + (+2)^2 + (-3)^2$.

(j) $(-1)^3 + (+2)^3 + (-3)^3$.

2. If $x = (-4)$ and $y = (+2)$, verify the following identities:

(*a*) $(x + y)^2 = x^2 + 2xy + y^2$.
(*b*) $(x - y)^2 = x^2 - 2xy + y^2$.
(*c*) $(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$.
(*d*) $(x - y)^3 = x^3 - 3x^2y + 3xy^2 - y^3$.
(*e*) $x^2 - y^2 = (x + y)(x - y)$.
(*f*) $x^3 - y^3 = (x - y)(x^2 + xy + y^2)$.
(*g*) $x^3 + y^3 = (x + y)(x^2 - xy + y^2)$.

3. Verify each of the identities in Nos. 2 (*a*)–2 (*g*) for the values $x = (+2)$ and $y = (-4)$.

73. Since $(+2) \times (+2) = (+4)$, therefore $\sqrt{(+4)} = (+2)$.

Similarly, since $(+2) \times (+2) \times (+2) = (+8)$, therefore $\sqrt[3]{(+8)} = (+2)$.

And so on for other roots.

Notice that not only does

$$(+2) \times (+2) = (+4)$$

but also $$(-2) \times (-2) = (+4).$$

Therefore *there are two square roots of* $(+4)$—namely, $(+2)$ and (-2). We write this fact as follows:

$$\sqrt{(+4)} = (\pm 2).$$

Since we cannot find any number which when squared will give as the product (-4), we say that the square root of (-4) is imaginary.

Thus, $$\sqrt{(+16)} = (\pm 4)$$

$$\sqrt{(+144)} = (\pm 12)$$

and so on, but $\left.\begin{matrix}\sqrt{(-16)} \\ \sqrt{(-144)}\end{matrix}\right\}$ are imaginary.

EXERCISE LXXX

Find the value of

1. $(+3)^2$.
2. $(-3)^2$.
3. $(-5)^3$.
4. $(+5)^3$.
5. $(-6)^2$.
6. $(+1)^4 - (+2)^3$.
7. $(-1)^3 - (-2)^3$.
8. $(+5)^2 - (-3)^3$.
9. $(-2)^7 - (+2)^7$.
10. $(+5)^2 - (-5)^2$.
11. $\sqrt{(+25)}$.
12. $\sqrt{(-25)}$.
13. $\sqrt{(+64)}$.
14. $\sqrt{(+169)}$.
15. $\sqrt{(+121)}$.
16. $\sqrt{\left(+\frac{4}{9}\right)}$.
17. $\sqrt{\left(-\frac{4}{9}\right)}$.
18. $\sqrt{\left(+\frac{16}{49}\right)}$.
19. $\sqrt{\left(+\frac{100}{121}\right)}$.
20. $\sqrt{\left(+\frac{81}{16}\right)}$.
21. $(+x)^2$.
22. $(-x)^2$.
23. $(+3x)^2$.
24. $(-3x)^2$.
25. $(+4k)^3$.
26. $(-4k)^3$.
27. $(+2p)^2 - (-2p)^2$.
28. $(-4m)^2 + (+4m)^2$.
29. $(+2bc)^3 + (+3cb)^3$.
30. $(-2bc)^3 - (-3cb)^3$.
31. $\sqrt{(+4a^2)}$.
32. $\sqrt{(+169k^2)}$.
33. $\sqrt{(+121m^4n^2)}$.
34. $\sqrt{(+25p^4)}$.
35. $\sqrt{(+64a^4b^2)}$.

74. The two square roots of a number are equal in magnitude but opposite in sign. This can be extended to any *even* root. *E.g.*,

$$\sqrt[4]{(+16)} = (\pm 2)$$

since $(+2) \times (+2) \times (+2) \times (+2) = (+16)$

and $(-2) \times (-2) \times (-2) \times (-2) = (+16)$.

There is no corresponding ambiguity, however, in the case of *odd* roots. Thus,

$$\sqrt[3]{(+8)} = (+2)$$

since $(+2) \times (+2) \times (+2) = (+8)$.

But $(-2) \times (-2) \times (-2) = (-8)$.

So that $\sqrt[3]{(-8)} = (-2)$.

Similarly, $\sqrt[4]{(+256)} = (\pm 4)$

and $\sqrt[4]{(-256)}$ is imaginary.

But $\sqrt[5]{(+243)} = (+3)$

and $\sqrt[5]{(-243)} = (-3)$.

EXERCISE LXXXI

Find the value of

1. $\sqrt[3]{(+64)}$.
2. $\sqrt[3]{(-64)}$.
3. $\sqrt{(+64)}$.
4. $\sqrt{(-64)}$.
5. $\sqrt[4]{(+81)}$.
6. $\sqrt[5]{(+1024)}$.
7. $\sqrt[5]{(-1024)}$.
8. $\sqrt[3]{(+27x^3)}$.
9. $\sqrt[3]{(-125m^6)}$.
10. $\sqrt[6]{(+64k^{12})}$.
11. $\sqrt[3]{(-216a^3b^6)}$.
12. $\sqrt[4]{(+625x^4)}$.
13. $\sqrt[5]{(-32t^{10})}$.
14. $\sqrt[3]{(+343p^3)}$.
15. $\sqrt[7]{-2187x^{14}}$.

75. In practice we consider all quantities with which we deal in algebra as directed quantities unless otherwise stated, and manipulated by the rules we have just considered. Consequently we can write them without the small brackets and omit the addition sign.

Thus $(+5x) + (-3x)$

would be written $5x - 3x$.

Similarly $(-8a) + (-5a)$ would be written $-8a - 5a$, and this is $-13a$.

EXERCISE LXXXII

1. Simplify

(a) $2a + 3a$.
(b) $2x - 3x$.
(c) $3p - 2p$.
(d) $2p - 3p$.
(e) $7m + 4m + 2m$.
(f) $7m - 4m + 2m$.
(g) $-7m + 4m - 2m$.
(h) $10a^2 - 8a^2$.
(i) $8a^2 - 10a^2$.
(j) $12pq + 8pq - 20pq$.
(k) $a + b + a - b$.
(l) $2a - 3b + 4a - 5b$.
(m) $7x - 9y + 8y - 10x$.
(n) $1 - 5x + 7x - 3$.
(o) $9 - 10mn + 7mn - 8$.
(p) $x^2 + 5x - 6 - 2x^2 + 8x - 12$.
(q) $3p^2 + 4pq - 5q^2 - 9pq + q^2$.
(r) $9x^2y - 8y + 10x^2y - 12y$.
(s) $-12ab + 14b^2 + 8ba - 7b^2$.
(t) $10pm - 9m^2 + 8mp^2 + 3m^2$.

2. Simplify the following by removing the brackets and collecting together like terms:

(a) $(x - y) + (y - z) + (z - x)$.
(b) $a + 4(b - a)$.
(c) $a - 4(b - a)$.
(d) $(2m - p) - (p - 2m)$.
(e) $7(x - 1) - 8(2x + 3)$.
(f) $8(3k - 5) + 5(7k - 4)$.
(g) $x(y - z) + y(z - x) + z(x - y)$.
(h) $a(2b - c) + c(2a - 1)$.
(i) $3x(4x + 3) - 5x(3 - 4x)$.
(j) $2p(p + 1) - 3 + 4p(3 - 2p) + 5$.

76. It is sometimes an advantage to arrange the addition of terms as is done in arithmetic, by putting like terms in columns. Consider the following example.

Find the sum of $4x^2 + 5xy - y^2$, $3xy + 4y^2$, $-7x^2 - 2xy + 3y^2$, $2x^2 - 7y^2$.

Arranging the terms in columns, we have

$$\begin{array}{rrr} 4x^2 & + 5xy & - y^2 \\ & 3xy & + 4y^2 \\ -7x^2 & - 2xy & + 3y^2 \\ 2x^2 & & - 7y^2 \\ \hline -x^2 & + 6xy & - y^2 \end{array}$$

The sum is $-x^2 + 6xy - y^2$

EXERCISE LXXXIII

1. Find the sum of

(*a*) $7a + 3b - c$
$4a - 2b + 5c$
$-3a + 5b - 2c$

(*b*) $10m - 4mn - 6p$
$-5m - 3mn + 10p$
$3m + 5mn - p$

(*c*) $5k + 6l - 4m$
$-7k - 8l + 9m$
$10k + 12l - 21m$

(*d*) $7p - 8q + 3r$
$-10p + 4q - 9r$
$-3p - 5q + r$

(*e*) $x^2 + 5x - 4$
$10x^2 - 11x + 9$
$-3x^2 - 2x - 8$

(*f*) $3x^2 - 5xy - 7y^2$
$-8x^2 + 10xy - 12y^2$
$-11x^2 - 9xy + 10y^2$

(*g*) $4ab - 9bc + 8abc$
$7ab - 9abc$
$-10bc + 4abc$

(*h*) $x + y - z$
$x - y + z$
$-x + y + z$

(*i*) $12a^2 - 15ab + 9b^2$
$-3a^2 + ab$
$-8ab - 10b^2$

(*j*) $5m^2 - 6n^2 + 12t^2$
$-m^2 - t^2$
$9n^2 + 5t^2$

(*k*) $a^2 + b^2, 3ab - 9b^2, 7a^2 - 4ab.$

(*l*) $3x^2 - y^2 + yz, 2(x^2 - xy), 3y(y - z).$

(*m*) $7 + p^2 + 9p(p - 5), 8p^2 + 10, 31 - 4p^2 + 3p.$

(*n*) $a(7b - c), 5b(4a - 1), 7c - 3b.$

(*o*) $5p^2 - 6q^2 - 9pq, 8q^2 - pq, 27p^2 + 8pq.$

2. If $A = 4a^2 + 7a - b$, $B = 8a^2 - 18a + 3b$, and $C = a - 5b$, find the value of

(*a*) $A + B + C.$
(*b*) $A + 2B + 3C.$
(*c*) $A + B.$
(*d*) $B + C.$
(*e*) $C + A.$

77. We have seen in our consideration of the subtraction of directed numbers on p. 130 that we can change the process of subtraction into one of addition *provided we change the sign of the number we subtract.* We can do this mentally and write down our result as in the following examples.

EXAMPLE 1

$$\begin{array}{r} 7x^2 - 8xy + y^2 \\ 3x^2 - 4xy + 3y^2 \\ \hline 4x^2 - 4xy - 2y^2 \end{array}$$

Note. In order to perform this subtraction we say

$$7x^2 - 3x^2 = 4x^2.$$
$$-8xy + 4xy = -4xy.$$
$$y^2 - 3y^2 = -2y^2.$$

In each case we have changed (mentally) the sign of the bottom line of the subtraction and then added.

EXAMPLE 2

From $7a - b + 3c$ subtract $8b - 5c$.

Writing this as a subtraction, we have

$$\begin{array}{r} 7a - b + 3c \\ 8b - 5c \\ \hline 7a - 9b + 8c \end{array}$$

Difference is $7a - 9b + 8c$

EXERCISE LXXXIV

1. Perform the following subtractions:

(a) $\begin{array}{r} 8x + 9y - 5 \\ 3x - 8y + 6 \end{array}$

(b) $\begin{array}{r} 7a^2 - 8ab + b^2 \\ -a^2 + 5ab + 6b^2 \end{array}$

(c) $\begin{array}{r} -8x^2 - 9xy - y^2 \\ 5x^2 + 10xy - 7y^2 \end{array}$

(d) $\begin{array}{r} 10p + 16pq - q + 5 \\ 8p - 9pq + 10q + 7 \end{array}$

(e) $\begin{array}{r} 12ab - 9bc + 14ca \\ 8ab + 10bc + 5ca \end{array}$

(f) $\begin{array}{r} -14m^2 + 20m - 8 \\ 5m^2 + 37m - 16 \end{array}$

(g) $\begin{array}{r} 4a + 10a^2 + 5a^3 \\ 6a + 7a^2 + 8a^3 \end{array}$

(h) $\begin{array}{r} 1 + 2m - 3mn + 6n^2 \\ -5 - 3m + 4mn - 10n^2 \end{array}$

(i) $\begin{array}{r} a + b - abc \\ -5a + 6b - 3abc \end{array}$

(j) $\begin{array}{r} xy + 7y^2 - 10yz \\ -5xy + 9y^2 + 15yz \end{array}$

2. Subtract

(a) $9a + b$ from $9b + a$.

(b) $x^2 + 2x + 3$ from $x^2 - 2x + 3$.

(c) $8p^2 - q^2$ from $8q^2 - p^2$.

(d) $12xy + y^2 - x^2$ from $18x^2 - 6xy + 5y^2$.

(e) $10mn - 9n^2$ from $8m^2 + 7mn - 6n^2$.

3. If $A = 3x^2 + 5x - 6$, $B = 8x^2 - 7x + 9$, and $C = 10x^2 - 8y^2$, find the value of

(a) $A - B$.
(b) $B - A$.
(c) $C - B$.
(d) $A + B - C$.
(e) $A - B + C$.

78. We have seen (p. 134) that when two directed numbers of *like* signs—*i.e.*, *both* positive or *both* negative—are multiplied together the product is *positive*. If the signs are unlike, then the product is negative.

EXAMPLE 1

Multiply $7x - 8y$ by $9x$.

$$7x \times 9x = 63x^2.$$
$$-8y \times 9x = -72xy.$$
$$\therefore (7x - 8y) \times 9x = 63x^2 - 72xy.$$

The following example shows a convenient method for arranging the multiplication.

EXAMPLE 2

Multiply $10ab - 9a^2 + 8a$ by $-3ab$.

$$\begin{array}{l} 10ab - 9a^2 + 8a \\ -3ab \\ \hline -30a^2b^2 + 27a^3b - 24a^2b \end{array}$$

Note. The various steps in this multiplication are

(i) $10ab \times -3ab = -30a^2b^2$.
(ii) $-9a^2 \times -3ab = +27a^3b$.
(iii) $8a \times -3ab = -24a^2b$.

EXERCISE LXXXV

1. Multiply

(a) $5a - 3b$ by $2a$.
(b) $12x^2 + 5x$ by $-3x$.
(c) $10m - n$ by $4mn$.
(d) $x + y + z$ by $2x$.
(e) $3p - 5q + r$ by p^2q.
(f) $3mn - 5n^2$ by $-3n$.
(g) $7a^2 - 3ab + b^2$ by -4.
(h) $10bc - 8c$ by $5c$.
(i) $2p - 5q + 3pq$ by $-3q$.
(j) $-14k + 3m$ by $-2mk$.

2. Simplify the following by removing the brackets and collecting like terms:

(a) $4x(5x + 7) - 3x(8 - 2x)$.
(b) $5a(7a^2 - ab + b) - 8a^2(a + b)$.
(c) $8p(4 - p^2) - 3(p^3 + 2p - 1)$.
(d) $-3bc(a - b - c) - 4a(bc - 1)$.
(e) $10x^2(3x^2 + x - 1) - 3x^2(x^2 - x + 4)$.

79. If we multiply $4a - 3b$ by $2a$ we get

$$\begin{array}{l} 4a - 3b \\ 2a \\ \hline 8a^2 - 6ab \end{array}$$

If we multiply $4a - 3b$ by $-5b$ we get

$$\begin{array}{r} 4a - 3b \\ -5b \quad\;\; \\ \hline -20ab + 15b^2 \end{array}$$

The sum of these two results will give the product of $4a - 3b$ and $2a - 5b$.

Hence we have

$$\begin{array}{lr} & 8a^2 - 6ab \qquad\quad \\ & -20ab + 15b^2 \\ \cline{2-2} \text{The product is} & 8a^2 - 26ab + 15b^2 \end{array}$$

This work can be arranged in a more satisfactory manner by following the methods of long multiplication in arithmetic. Thus in this case we should write:

$$\begin{array}{rl} 4a - 3b \qquad\quad & \\ 2a - 5b \qquad\quad & \\ \hline 8a^2 - \;\; 6ab \qquad\quad & \text{(multiplying by } 2a\text{)} \\ -20ab + 15b^2 & \text{(multiplying by } -5b\text{)} \\ \hline 8a^2 - 26ab + 15b^2 & \end{array}$$

•

EXERCISE LXXXVI

1. Multiply

(*a*) $2x - y$ by $3x + 2y$.
(*b*) $4p + 3q$ by $3p - 4q$.
(*c*) $5m - 6n$ by $7m + 8n$.
(*d*) $x + 3$ by $4x - 5$.
(*e*) $10a - 2b$ by $-3a + b$.
(*f*) $x + y + 1$ by $2x - 3y$.
(*g*) $3p - 2q + 4$ by $2p + q$.
(*h*) $13w - 12t$ by $2w - t + 1$.
(*i*) $4c - 5d$ by $2a - c$.
(*j*) $10m - n$ by $10n - m$.

2. Simplify the following:

(*a*) $(a + b)^2$.
(*b*) $(a + b)^2 + (a - b)^2$.
(*c*) $(2x - y)^2$.
(*d*) $(3y - 2x)^2$.
(*e*) $(4p - 3q)^2$.
(*f*) $(4x - 3y)(5x + 6y)$.
(*g*) $(3a - 2b)(4a - b)$.
(*h*) $(xy + 1)(2x - y)$.
(*i*) $(1 + x^2)(1 - x)$.
(*j*) $(a + b - c)(a - b + c)$.
(*k*) $(p - qr)(q - pr)$.
(*l*) $(2x^2 - 3)(4 - x)$.
(*m*) $(ab - c)(bc - a)$.
(*n*) $(wx + a)(wx + 5a)$.
(*o*) $(1 - abc)(1 - a)$.

80. It is frequently an advantage to arrange algebraical expressions in *ascending* or *descending* order of the letters. Thus the expression $4x^4 - 6x^3 + 8x^2 + 9x - 10$ is arranged in descending order, the powers of x decreasing as we move from left to right. Similarly $3 + 5a^2 + 7a^4 + 8a^5$ is an expression arranged in ascending order. It sometimes happens that by arranging an expression in ascending order of one letter the expression is at the same time arranged in descending order of another letter. Thus $5p^4q - 7p^3q^2 + 8p^2q^3 - 10pq^4$ is in descending order of the p's but in ascending order of the q's.

EXERCISE LXXXVII

Arrange the following expressions (*a*) in ascending, (*b*) in descending, order:

1. $5a^4 + 6a - 4 + 3a^3 - a^2$.
2. $4p^4 - 5 + 7p^2 - p + 3p^3$.
3. $x + 4 + 5x^2 - 9x^4 + 3x^3$.
4. $5 - 9c^2 + 10c + 8c^5 - c^3$.
5. $10 - k^4 + 2k - 3k^3 + k^2$.

81. It is always advisable in the multiplication of algebraical expressions to arrange the terms in order before starting. It does not matter whether it is an ascending or a descending order which is selected, but usually the question itself suggests which order should be chosen. Consider the following example.

Multiply $3 + 4x^4 - x^2$ by $3x^2 - x + 1$.

Rearranging so that the terms are arranged in descending order, we have

$$\begin{array}{rrrrrrr}
4x^4 & & -x^2 & & +3 & & \\
3x^2 - x & +1 & & & & & \\
\hline
12x^6 & & -3x^4 & & +9x^2 & & \\
 & -4x^5 & & +x^3 & & -3x & \\
 & & 4x^4 & & -x^2 & & +3 \\
\hline
12x^6 & -4x^5 & +x^4 & +x^3 & +8x^2 & -3x & +3
\end{array}$$

Notice the spaces left for the missing powers of x. These are gradually filled up as the working progresses. The final answer arranges itself in descending order of the powers of x.

EXERCISE LXXXVIII

Multiply

1. $3x^2 - 4x + 1$ by $2x^2 + x - 1$.
2. $5p^2 - 3pq + q^2$ by $2p^2 - 2pq - 3q^2$.
3. $4 - 3x + x^2$ by $2 + x + x^2$.
4. $10a^2 - ab + b^2$ by $a^2 - ab + 2b^2$.
5. $3x^2 + 2$ by $x^2 + x - 1$.
6. $4m^2 + 3m - 1$ by $m^2 + 2$.
7. $p^2 - pq + q^2$ by $p + q$.
8. $p^2 + pq + q^2$ by $p - q$.
9. $1 + 3x + 3x^2 + x^3$ by $1 + x$.
10. $4 - 2k^2 + k$ by $3k^2 + k - 1$.
11. $5a^3 - 2a^2 + 3a + 1$ by $a^2 + 3a - 2$.
12. $4t^2 - 5t + 4$ by $2 + t + t^2$.
13. $2k^3 - 5k + 8$ by $k^2 - 2k$.
14. $yz + zx$ by $xy + 1$.
15. $x^2 + xy + y^2$ by $x^2 - xy + y^2$.

82. Division in algebra follows the method of long division in arithmetic, with the difference that at each stage of the work the difference of the first two terms is zero. The following examples will illustrate this.

EXAMPLE 1

Divide $6x^3 - x^2 - 11x + 6$ by $3x - 2$.

$$
\begin{array}{l}
3x - 2)6x^3 - x^2 - 11x + 6(2x^2 + x - 3 \\
\quad\quad\; \underline{6x^3 - 4x^2} \\
\quad\quad\quad\quad\; 3x^2 - 11x \\
\quad\quad\quad\quad\; \underline{3x^2 - 2x} \\
\quad\quad\quad\quad\quad\; -9x + 6 \\
\quad\quad\quad\quad\quad\; \underline{-9x + 6}
\end{array}
$$

Note

(i) $6x^3 \div 3x = 2x^2$.

(ii) $(3x - 2) \times 2x^2 = 6x^3 - 4x^2$.

(iii) Subtract. Notice that there are no terms in x^3 in the remainder.

(iv) $3x^2 \div 3x = x$.

(v) $(3x - 2) \times x = 3x^2 - 2x$.

(vi) Subtract. Notice that there are no terms in x^2 in the remainder.

(vii) $-9x \div 3x = -3$.

(viii) Proceed as before.

The required quotient is $2x^2 + x - 3$.

EXAMPLE 2

Divide $27x^3 - 125y^3$ by $3x - 5y$.

$$
\begin{array}{l}
3x - 5y)27x^3 \quad\quad\quad\quad\quad - 125y^3(9x^2 + 15xy + 25y^2 \\
\quad\quad\quad\; \underline{27x^3 - 45x^2y} \\
\quad\quad\quad\quad\quad\quad 45x^2y \\
\quad\quad\quad\quad\quad\quad \underline{45x^2y - 75xy^2} \\
\quad\quad\quad\quad\quad\quad\quad\quad\quad 75xy^2 - 125y^3 \\
\quad\quad\quad\quad\quad\quad\quad\quad\quad \underline{75xy^2 - 125y^3}
\end{array}
$$

The required quotient is $9x^2 + 15xy + 25y^2$.

EXERCISE LXXXIX

Divide

1. $x^3 - 2x^2 + x + 4$ by $x + 1$.
2. $6p^3 - p^2 + 7p - 6$ by $3p - 2$.
3. $8a^3 - 26a^2 + 5a + 25$ by $2a - 5$.
4. $10m^3 - 33m^2 + 5m - 42$ by $2m - 7$.
5. $16p^3 - 10p^2q + 7pq^2 - 3q^3$ by $2p - q$.
6. $9x^3 - 21x^2y + 4xy^2 + 4y^3$ by $3x - 2y$.
7. $3a^4 + a^3 + 2a^2 - a + 1$ by $a^2 + a + 1$.
8. $2k^4 - 3k^3 + 2k^2 - 2k - 3$ by $2k^2 - k + 3$.
9. $10m^4 + 11m^3 - 10m^2 - 7m + 4$ by $2m^2 + m - 1$.

10. $2p^5 + 6p^3 - 5p^2 - 15$ by $p^2 + 3$.
11. $8x^3 - 26x^2y + 19xy^2 - 10y^3$ by $4x^2 - 3xy + 2y^2$.
12. $a^4 - a^2 + 2a - 1$ by $a^2 - a + 1$.
13. $3x^4 + x^3 - 4x^2 + 3x - 1$ by $3x^2 - 2x + 1$.
14. $2k^4 + k^3 - 12k^2 - 2k + 15$ by $2k^2 - k - 5$.
15. $6m^4 - m^3n - 3m^2n^2 + 3mn^3 - n^4$ by $2m^2 + mn - n^2$.

83. Mention has been made of the advantages of arranging algebraical expressions in ascending or descending order of powers of the letters. This should always be done in multiplication and division, not only because it simplifies the calculation, but also because the answer obtained will then be arranged in proper order. Since the nature of the answer depends upon the coefficients, we can perform the operations of multiplication and division by writing down only the coefficients at each stage of the work. The following example will illustrate this method of 'detached coefficients.'

Multiply $7x^2 - 3x + 2$ by $2x^2 - 3$.

Ordinary method:

$$\begin{array}{rrrrr} & & 7x^2 & -\ 3x & +\ 2 \\ & & 2x^2 & & -\ 3 \\ \hline & & -\ 21x^2 & +\ 9x & -\ 6 \\ 14x^4 & -\ 6x^3 & +\ 4x^2 & & \\ \hline 14x^4 & -\ 6x^3 & -\ 17x^2 & +\ 9x & -\ 6 \end{array}$$

Method of detached coefficients:

$$\begin{array}{rrrrr} & & 7 & -\ 3 & +\ 2 \\ & & 2 & +\ 0 & -\ 3 \\ \hline & & -\ 21 & +\ 9 & -\ 6 \\ 14 & -\ 6 & +\ 4 & & \\ \hline 14x^4 & -\ 6x^3 & -\ 17x^2 & +\ 9x & -\ 6 \end{array}$$

Note
(i) The signs in front of the coefficients are determined by the ordinary rules.
(ii) Missing term is shown by + 0.

EXERCISE XC

Perform the multiplications of Exercise LXXXVIII by the method of detached coefficients.

84. The following example shows the application of the method of detached coefficients to division.

Divide $10x^4 - 9x^3 - 26x^2 + 38x - 15$ by $2x^2 + x - 5$.

Ordinary method:

$$\begin{array}{l}
2x^2 + x - 5)10x^4 - 9x^3 - 26x^2 + 38x - 15(5x^2 - 7x + 3 \\
\qquad\qquad\quad \underline{10x^4 + 5x^3 - 25x^2} \\
\qquad\qquad\qquad\quad - 14x^3 - x^2 + 38x \\
\qquad\qquad\qquad\quad \underline{- 14x^3 - 7x^2 + 35x} \\
\qquad\qquad\qquad\qquad\qquad 6x^2 + 3x - 15 \\
\qquad\qquad\qquad\qquad\qquad \underline{6x^2 + 3x - 15}
\end{array}$$

The required quotient is $5x^2 - 7x + 3$.

Method of detached coefficients:

$$\begin{array}{l}
2 + 1 - 5)10 - 9 - 26 + 38 - 15(5 - 7 + 3 \\
\qquad\qquad\ \underline{10 + 5 - 25} \\
\qquad\qquad\qquad - 14 - 1 + 38 \\
\qquad\qquad\qquad \underline{- 14 - 7 + 35} \\
\qquad\qquad\qquad\qquad\quad 6 + 3 - 15 \\
\qquad\qquad\qquad\qquad\quad \underline{6 + 3 - 15}
\end{array}$$

The required quotient is $5x^2 - 7x + 3$.

EXERCISE XCI

Perform the divisions of Exercise LXXXIX by the method of detached coefficients.

EXERCISE XCII (Revision Exercise)

(A)

1. If $x = 5$, $y = -7$, and $z = -9$, find the value of

(*a*) $x + y + z$. (*d*) xyz.

(*b*) $x^2 + y^2 + z^2$. (*e*) $x - y - z$.

(*c*) $xy + yz + zx$.

2. Multiply $2a^2 + 5ab - 6b^2$ by $3a - 2b$.

3. For what value of x is $17x = -153$?

4. Divide $10x^4 - 11x^3y - 4x^2y^2 + 4xy^3 + y^4$ by $2x^2 - xy - y^2$.

5. If $A = 3p - 5q$ and $B = 5p - 3q$, find the value of

(a) $A + B$. (b) $A - B$. (c) AB.

(B)

6. Solve the following equations:

(a) $2x + 5 = 1$. (b) $3(x - 4) = 2(x - 10)$.

7. If $a^3 - 3a^2 - 6a + K$ is exactly divisible by $a + 2$, find K.

8. Find two numbers whose sum is 70 and whose difference is 24.

9. If $X = 4a - 3b$ and $Y = 3b - 4a$, verify that

(a) $(X + Y)(X - Y) = X^2 - Y^2$.

(b) $(X + Y)(X^2 - XY + Y^2) = X^3 + Y^3$.

(c) $(X - Y)(X^2 + XY + Y^2) = X^3 - Y^3$.

10. For what values of n is $(-1)^n$ positive?

(C)

11. Change the subject of the formula $T = 2\pi\sqrt{\frac{l}{g}}$ from T to l.

12. A sum of £4 14*s.* 6*d.* is made up of an equal number of sixpences, shillings, and florins. How many of each kind of coin are there?

13. Solve the following equations:

(a) $\frac{3x + 1}{4} - \frac{4x - 3}{5} = \frac{3}{4}$.

(b) $(3x + 2)(x + 1) = 3x(x + 4) - 11$.

14. Simplify

(a) $\frac{5p + 3q}{4} - \frac{4q - 3p}{5}$.

(b) $\frac{m + 2n + p}{6} - \frac{4m - 3n + 2p}{8}$.

15. Find the factors of

(*a*) $1 - 81m^4$.

(*b*) $p^2q - q^2p + p^2q^2$.

(*c*) $x - \frac{b^2x}{4}$.

(D)

16. If $10x^3 - 27x^2 + 38x - K$ is exactly divisible by $2x^2 - 3x + 4$, find K.

17. After A has spent one-eleventh of his money he has seven times as much as B. How much had each at first?

18. Find by using the method of detached coefficients the coefficient of a^3 in $(1 + 2a - 3a^2)^3$.

19. If $x = -2a$ and $y = -3b$, find the value of

$$\frac{x^2 - xy + y^2}{x - y}$$

in terms of a and b.

20. Solve the following equations:

(*a*) $\frac{x+1}{2} + \frac{3x-5}{3} - \frac{3x-13}{6} = 6$.

(*b*) $(4x + 1)(2x - 3)$
$= (5x - 2)(x + 4) + (3x - 4)(x + 1)$.

CHAPTER VIII

SIMULTANEOUS EQUATIONS

85. If $y = 2x + 3$, then as x is given a succession of values a corresponding set of values for y can be calculated. Thus,

$$\begin{aligned} \text{if } x &= 1, \text{ then } y = 2.1 + 3 &&= 5; \\ \text{if } x &= 3, \text{ then } y = 2.3 + 3 &&= 9; \\ \text{if } x &= -5, \text{ then } y = 2.(-5) + 3 &&= -7; \end{aligned}$$

and so on.

We may represent these corresponding values on a diagram by the method illustrated in Fig. 48. Two straight lines called *axes* are drawn at right angles to each other, O being their point of intersection, called the *origin*. The corresponding values $x = 1$ and $y = 5$ are represented on the diagram by the point A, which is said to have these numbers as *co-ordinates*. For this reason A is described as the point (1, 5), the x co-ordinate being the first of this pair of numbers. Similarly B is the point (3, 9) and C is the point (– 5, – 7). In the particular case of Fig. 48 these three points lie on a straight line.

The straight line BAC in Fig. 48 has some important properties. The co-ordinates of every point on it are connected by the relationship $y = 2x + 3$. For example, consider the point P. Its co-ordinates are 2, 7.

$$\begin{aligned} \text{If } x = 2, \text{ then } y &= 2.2 + 3 \\ &= 4 + 3 \\ &= 7, \text{ which is the } y \text{ co-ordinate of P.} \end{aligned}$$

And so on for every other point on the line. For this reason the straight line can be regarded as a *locus* of all points whose co-ordinates satisfy the equation $y = 2x + 3$. Such a locus is called the *graph* of $y = 2x + 3$. The graph of an equation is not always a straight line. We shall use several

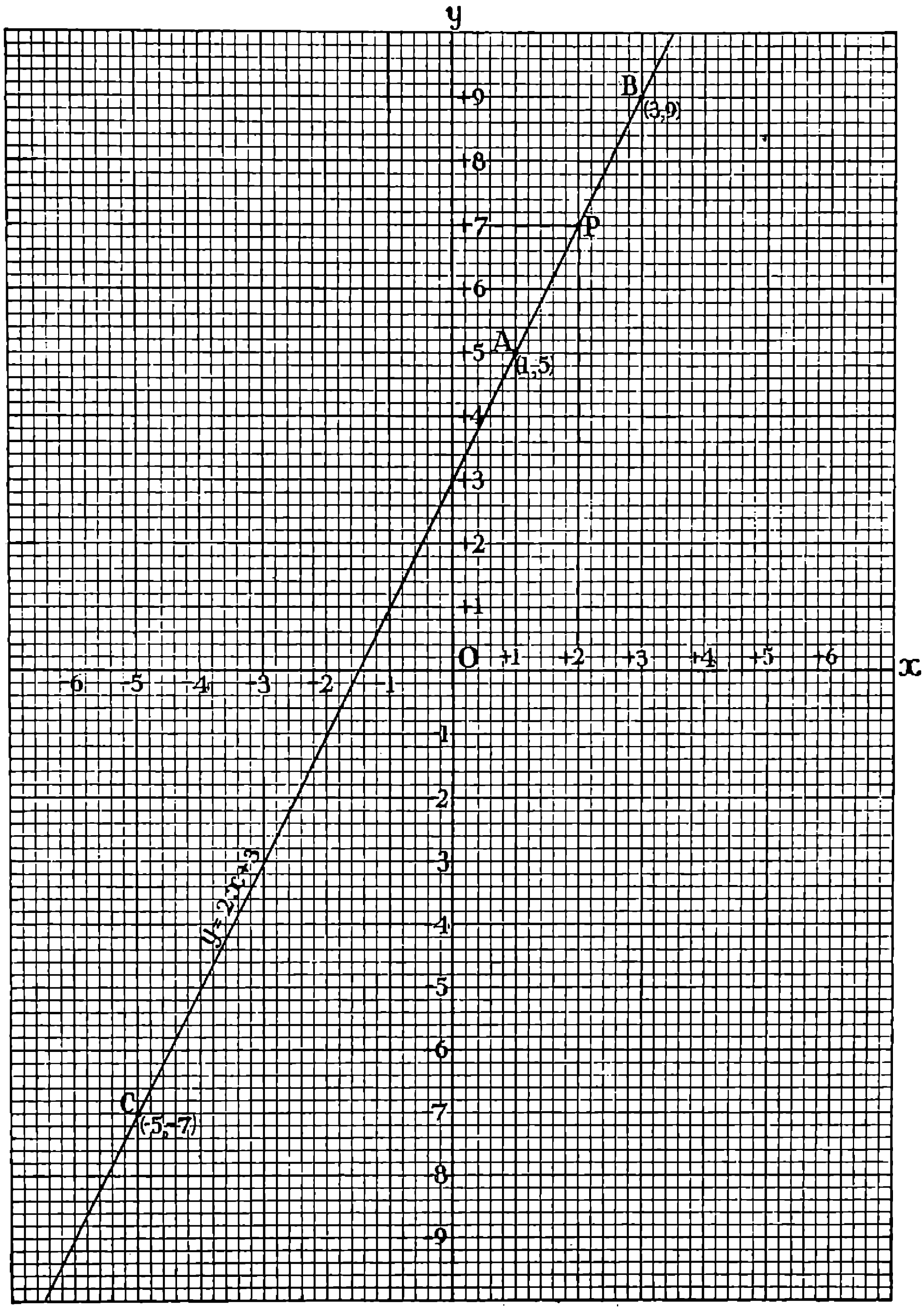

FIG. 48

curved graphs in later chapters of this book, but the graph of an equation of the first degree—*i.e.*, an equation containing no squares or higher powers—is always a straight line, and for this reason equations of the first degree are sometimes called *linear* equations.

EXERCISE XCIII

(It will be an advantage to use squared paper for this exercise.)

1. Using the same scale on both axes, plot the following points: $(3, -1)$, $(3, 3)$, $(-1, 3)$, $(-1, -1)$.

Verify that these four points are the corners of a square.

2. Plot the points $(-2, 2)$, $(2, -2)$, and $(3, 3)$. Verify that they are the vertices of an isosceles triangle.

3. Show that the points $(-3, -2)$, $(0, -1)$, and $(6, 1)$ lie on a straight line. Draw this line.

4. Join the points $(-2, 5)$ and $(0, -1)$, and show that the straight line on which they lie is perpendicular to the line drawn in Question 3.

5. Join each of the points $(1, 3)$ and $(3, 1)$ to the origin and measure the angle between these two lines.

6. Draw the graphs of $y = x$, $y = 2x$, and $y = 3x$. What is the smallest number of points necessary in order to draw each of these graphs?

7. Draw the graph of $y = -x$. This is sometimes called the "mirror graph" of $y = x$. Why is this?

8. Draw the graph of $y = x - 5$. On the same figure draw in dotted lines the mirror graph (*a*) in the x axis, (*b*) in the y axis. Write down the equation of the mirror graph in each case.

9. Plot the graphs of $y = x + 3$, $y = x - 1$, and $y = x - 7$ on the same axes, and verify that they are three parallel straight lines.

10. Draw the graph of $y = -3x + 1$. Say which of the following points lie on the graph: $(0, 0)$, $(1, -2)$, $(-2, 7)$, $(-3, 4)$.

11. Using the graph drawn in Question 10, find the values of y corresponding to the following values of x: -3, -1, 1, 3. Verify by calculation in each case.

12. Using the graph drawn in Question 10, find the values of x corresponding to the following values of y: $-3, -1, 1, 3$. Verify by calculation in each case.

13. State which of the following graphs will be a straight line:

$$(a)\ y = x. \quad (b)\ y = x^2. \quad (c)\ y = \frac{1}{x}.$$

14. The Fahrenheit and Centigrade thermometer scales are connected by the equation $F = \frac{9}{5}C + 32$. Draw this graph for values of C between 0° and 10°, and from it obtain the Fahrenheit temperatures corresponding to 5° C., 8° C., and $-$ 10° C.

15. Use the graph drawn in the previous question to obtain the Centigrade temperatures corresponding to 5° F., 8° F., and $-$ 10° F.

16. What is the graph of $y = 0$?

17. What is the graph of $x = 0$?

86. In Fig. 49 we have again the graph of $y = 2x + 3$. The graph cuts the x axis in the point $(-\frac{3}{2}, 0)$, which shows that the corresponding value of x for $y = 0$ is $x = -\frac{3}{2}$.

If $y = 0$, then $\quad 2x + 3 = 0.$

$\therefore 2x = -3$

and $\quad x = -\frac{3}{2}.$

Hence the x co-ordinate of the point where the graph cuts the x axis is the solution of the equation $2x + 3 = 0$.

The graph $y = 2x + 3$ cuts the graph $y = 5$ at the point P, whose co-ordinates are 1, 5.

If $\quad 2x + 3 = 5$

then $\quad 2x = 5 - 3$

$= 2$

and $\quad x = 1.$

Hence the x co-ordinate of the point P $(x = 1)$ is the solution of the equation $2x + 3 = 5$.

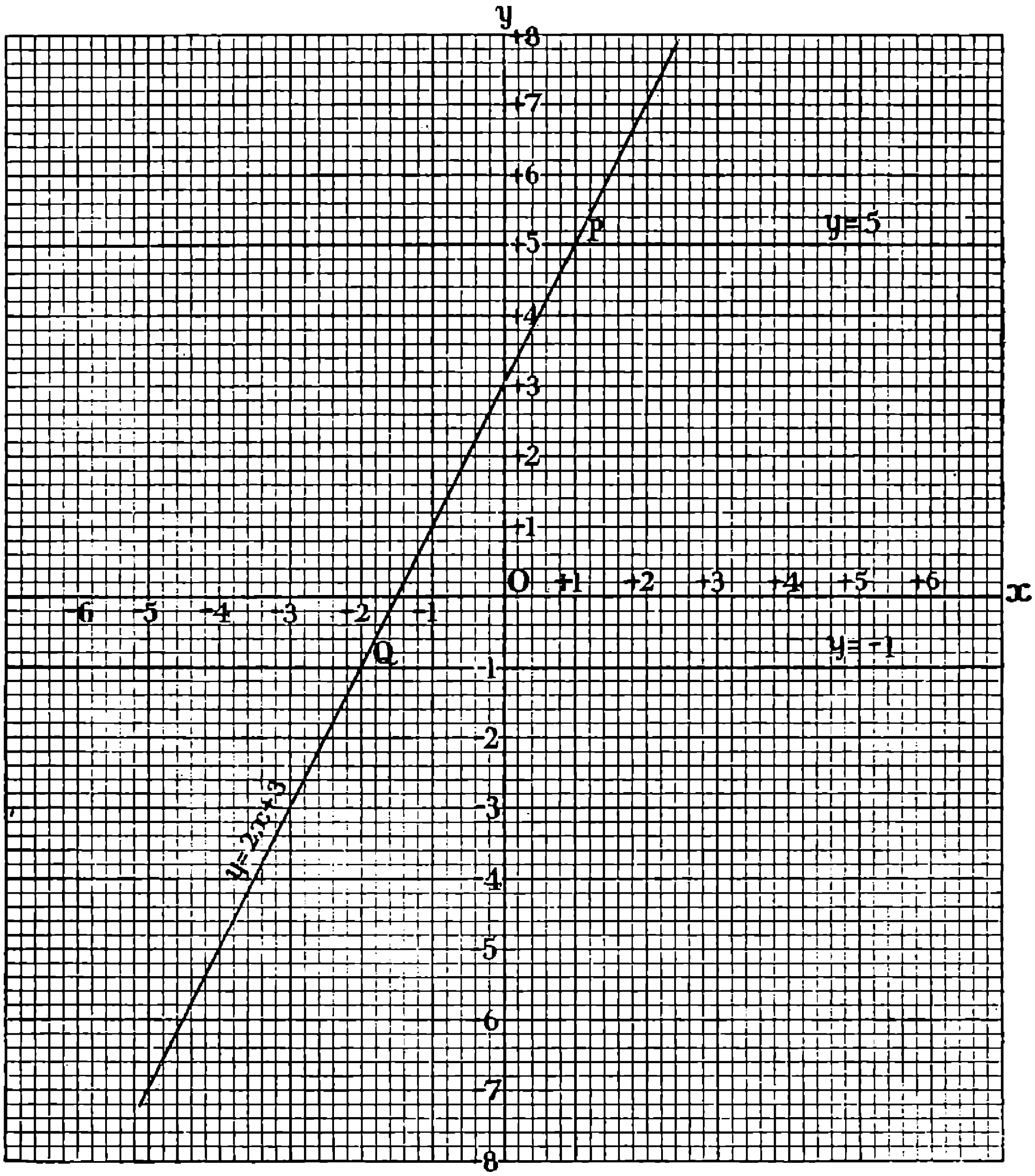

FIG. 49

Similarly, the graph $y = 2x + 3$ cuts the graph $y = -1$ at the point Q, whose co-ordinates are $-2, -1$. The solution of the equation $2x + 3 = -1$ is $x = -2$.

This is the x co-ordinate of Q, the point of intersection of the graphs $y = 2x + 3$ and $y = -1$.

EXERCISE XCIV

(It will be an advantage to use squared paper for this exercise.)

1. Draw the graph of $y = x - 2$, and from it read off solutions of the following equations:

(*a*) $x - 2 = 0$. (*b*) $x - 2 = 3$. (*c*) $x - 2 = -5$.

2. Draw the graph of $y = 3x - 4$, and from it read off solutions of the following equations:

(*a*) $3x - 4 = 0$. (*b*) $3x - 4 = 1$. (*c*) $3x - 4 = -10$.

3. Draw the graph of $y = 5 - 2x$, and from it read off solutions of the following equations:

(*a*) $5 - 2x = 0$. (*b*) $5 - 2x = 3$. (*c*) $5 - 2x = -5$.

87. Fig. 50 shows the graphs of $y = 2x + 3$ and $y = 5 + x$. These intersect at the point P, whose co-ordinates are 2, 7. As x is given a succession of values, the corresponding values of y when $y = 2x + 3$ and when $y = 5 + x$ are shown in the following table:

	$x = -1$	$x = 0$	$x = 1$	$x = 2$	$x = 3$	$x = 4$
$y = 2x + 3$	1	3	5	7	9	11
$y = 5 + x$	4	5	6	7	8	9

The table shows that when $x = 2$ both $y = 2x + 3$ and $y = 5 + x$ have the same values—namely, $y = 7$; and this is the value of the y co-ordinate of the point P in Fig. 50.

The equation $y = 2x + 3$ can be rewritten in the form

$$2x - y + 3 = 0$$

and this equation is true when $x = 2$ and $y = 7$. This can be seen by substituting these values of x and y.

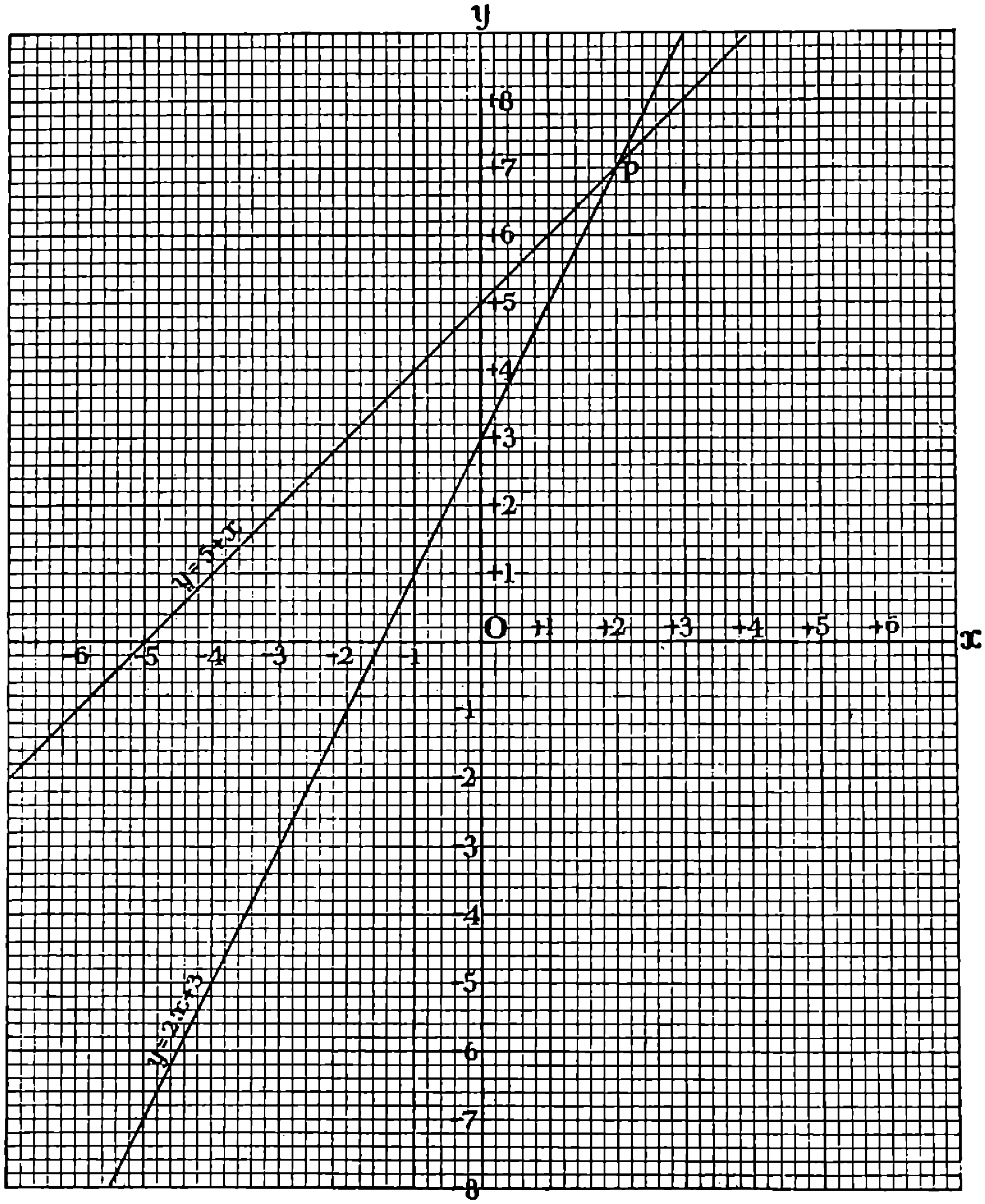

FIG. 50

Similarly, the equation $y = 5 + x$ can be rewritten in the form

$$x - y + 5 = 0$$

and this equation is also true when $x = 2$ and $y = 7$. This can be seen by substituting these values of x and y.

Hence for the values $x = 2$ and $y = 7$ the equations $2x - y + 3 = 0$ and $x - y + 5 = 0$ are true *at the same time*. Other values of x and y can be chosen to satisfy the equations individually. Some of these are shown in the above table, but the only pair of values which will satisfy both equations *simultaneously*—*i.e.*, at the same time—are $x = 2$ and $y = 7$. For this reason we say that the solutions of the two simultaneous equations $2x - y + 3 = 0$ and $x - y + 5 = 0$ are $x = 2$ and $y = 7$.

Fig. 50 shows that the solutions of two simultaneous equations of the first degree are given by the x and y co-ordinates of the point of intersection of the two graphs of the equations.

EXERCISE XCV

(It will be an advantage to use squared paper for this exercise.)

1. Solve by means of graphs the following simultaneous equations:

(*a*) $y = x + 1$. $y = 3x - 1$.

(*b*) $y = x + 5$. $y = 1 - x$.

(*c*) $x + y + 1 = 0$. $x - y - 3 = 0$.

(*d*) $3x - y = 4$. $x + 2y = 6$.

(*e*) $x - y = 2$. $x + y = -2$.

(*f*) $y = x + 3$. $2x + y + 6 = 0$.

(*g*) $x = y$. $x + y = 4$.

(*h*) $x - y - 2 = 0$. $y - 2x + 1 = 0$.

(*i*) $2x - y = 1$. $2y - x = 4$.

2. Draw on the same diagram the graphs of (*a*) $y = x - 1$, (*b*) $2y - x = 1$, and (*c*) $x - y = 2$, and hence find the solutions of the following pairs of simultaneous equations:

(*a*) $y = x - 1$. $2y - x = 1$.

(*b*) $2y - x = 1$. $x - y = 2$.

88. Although all simultaneous equations with two unknowns can be solved by the graphical method just described, it is quite clear that except in the simplest cases it is not the best or the quickest method of obtaining the required solution. The following is the method usually adopted.

EXAMPLE 1

Solve the simultaneous equations

$$2x - y = -3.$$
$$x - y = -5.$$

The graphs of these two equations will be found in Fig. 50. The point P, where the two graphs intersect, is common to both straight lines, and therefore at this point

$$2x + 3 = 5 + x.$$

I.e., $x = 2.$

Since $y = 2x + 3$, if $x = 2$, then by substitution we have

$$y = 2.2 + 3$$
$$= 4 + 3.$$
$$\therefore y = 7.$$

Hence the solutions of the two simultaneous equations are $x = 2$ and $y = 7$.

This suggests the method to employ—namely, from the *two* original equations containing *two* unknowns we should obtain *one* equation containing *one* unknown. This is done by a process called *elimination.*

The following system of numbering equations is usually adopted.

$$2x - y = -3 \quad (1)$$
$$x - y = -5 \quad (2)$$

Subtract equation (2) from equation (1).

$$x = -3 + 5 \quad \text{(This has eliminated } y.\text{)}$$
$$x = 2.$$

Substituting the value $x = 2$ in equation (2),

$$2 - y = -5.$$
$$-y = -7.$$
$$y = 7.$$

Hence the solutions of the equations are $x = 2$ and $y = 7$.

EXAMPLE 2

Solve the simultaneous equations

$$2x - y = 1.$$
$$3x + 2y = 12.$$

Numbering the equations as we did in the last example,

$$2x - y = 1 \qquad (1)$$
$$3x + 2y = 12 \qquad (2)$$

Multiplying equation (1) by 2, we have

$$4x - 2y = 2 \qquad (3)$$
$$3x + 2y = 12 \qquad (2)$$

Adding equations (3) and (2), we have

$$7x = 14. \quad \text{(This has eliminated } y.\text{)}$$
$$x = 2.$$

Substituting this value of x in (1), we have

$$4 - y = 1 \qquad (4)$$
$$-y = 1 - 4$$
$$= -3.$$
$$y = 3.$$

Hence the solutions of the equations are $x = 2$ and $y = 3$.

EXERCISE XCVI

Solve the following simultaneous equations:

1. $x + y = 5.$ $x - y = -1.$
2. $x - y = -1.$ $3x - y = 1.$
3. $y - x = 5.$ $y + x = 1.$
4. $x + y + 1 = 0.$ $x - y - 3 = 0.$
5. $3x - y = 4.$ $x + 2y = 6.$
6. $x - y = -3.$ $2x + y + 6 = 0.$
7. $x = y.$ $x + y = 4.$
8. $x - y - 2 = 0.$ $y - 2x + 1 = 0.$
9. $2x - y = 1.$ $2y - x = 4.$
10. $2m + 3n = 2.$ $6m + 8n = 5.$
11. $7p - q + 21 = 0.$ $q - p = 5.$
12. $4a + b = 5.$ $4b - a = 3.$

13. $x + y = 4x - y = 5$.

14. $m + n + 1 = 2m + 3n - 3 = 2$.

15. $2p + q + 4 = 3p + q + 1 = 7$.

16. $3x + y = 7x - y = 5$.

17. $m + n + 4 = 3m - 2n = 2m - 3n - 3$.

18. $\frac{1}{x} + \frac{2}{y} = 3.$

$\frac{2}{x} + \frac{6}{y} = 5.$

Hint. Put $\frac{1}{x} = a$ and $\frac{1}{y} = b$, then the equations become

$$a + 2b = 3.$$
$$2a + 6b = 5.$$

Solve for a and b and then obtain the values of x and y from $\frac{1}{x} = a$ and $\frac{1}{y} = b$.

19. $\frac{2}{x} + \frac{3}{y} = 2.$

$\frac{3}{x} + \frac{7}{y} = \frac{5}{6}.$

20. $\frac{2}{m} + \frac{5}{n} = 7.$

$\frac{3}{m} - \frac{2}{n} = 1.$

21. $\frac{4}{p} + \frac{2}{q} = -\frac{3}{2}.$

$\frac{6}{p} + \frac{4}{q} = -2.$

22. $\frac{x}{2} + y = 1.$

$\frac{3x}{4} - y = 2.$

89. The solution of simultaneous equations with three (or more) unknowns is performed in a similar way to the case where there are only two unknowns. The number of equations is gradually reduced by a process of elimination until a simple equation is obtained. The following example will illustrate this, and also show how the solution of such a set of equations should be written out.

Solve the simultaneous equations

$$x + y + z = 6.$$
$$2x - 3y + z = -1.$$
$$3x + y + 2z = 11.$$

Numbering the equations as before, we have

$$x + y + z = 6 \quad (1)$$
$$2x - 3y + z = -1 \quad (2)$$
$$3x + y + 2z = 11 \quad (3)$$

Multiplying equation (1) by 2, we have

$$2x + 2y + 2z = 12 \quad (4)$$
$$2x - 3y + z = -1 \quad (2)$$

Subtracting equation (2) from equation (4), we have

$$5y + z = 13 \quad (5)$$

Multiplying equation (2) by 3 and equation (3) by 2, we have

$$6x - 9y + 3z = -3 \quad (6)$$
$$6x + 2y + 4z = 22 \quad (7)$$

Subtracting equation (7) from equation (6), we have

$$-11y - z = -25 \quad (8)$$
$$5y + z = 13 \quad (5)$$

Adding equation (5) and equation (8), we have

$$-6y = -12.$$
$$y = 2.$$

Substituting the value $y = 2$ in equation (5), we have

$$z = 3.$$

Substituting the values $y = 2$ and $z = 3$ in equation (1), we have

$$x = 1.$$

Hence the solutions of the three simultaneous equations are $x = 1$, $y = 2$, and $z = 3$.

EXERCISE XCVII

Solve the following simultaneous equations:

1. $x + 2y + z = 6.$
   $x - y + z = 3.$
   $2x + y + 3z = 12.$

2. $x + y + 2z = 7.$
   $3x - y - z = 6.$
   $2x + 3y - 4z = 8.$

3. $4x + y + z = 6.$
$3x - y + z = 6.$
$2x - 3y + 4z = 3.$

4. $p + q + r = 6.$
$3p + q - r = -8.$
$p - 3q - 2r = -21.$

5. $3a - b + c = -13.$
$a - 3b + 2c = -16.$
$-2a + b + c = 12.$

6. $l - m + n = -2.$
$l + m - n = 0.$
$-l + m + n = -4.$

7. $2x - y + 3z = 12.$
$3x + 2y - z = 26.$
$x - 3y + 4z = -4.$

8. $a - b - c = 6.$
$3b + 2c = 7.$
$3a - 4c = 19.$

9. $7p - 3q + 2r = 12.$
$5p + q - 4r = 4.$
$3p + 2q + r = 12.$

10. $x + y - 4z = -2.$
$5x + 2y + z = 17.$
$3x - 5y + 4z = 26.$

90. Problems involving simultaneous equations are of frequent occurrence, but their solution introduces no fresh rules. When the equations which describe the problem have been written down they are solved in the manner already explained.

EXAMPLE 1

Find two numbers whose sum is 54 and whose difference is 8.

Let x be the larger number and y the smaller number.

Then $$x + y = 54 \quad (1)$$
and $$x - y = 8 \quad (2)$$

Adding equation (1) and equation (2), we have
$$2x = 62.$$
$$x = 31.$$

Substituting this value of x in equation (1), we have
$$y = 23.$$

Therefore the two numbers are 31 and 23.

EXAMPLE 2

One table and five chairs cost £23. Two tables and six chairs cost £36. Find the cost of a table and of a chair.

Let £x be the cost of a table and £y the cost of a chair.

Then $$x + 5y = 23 \quad (1)$$
and $$2x + 6y = 36 \quad (2)$$

Multiplying equation (1) by 2, we have

$$2x + 10y = 46 \quad (3)$$
$$2x + 6y = 36 \quad (2)$$

Subtracting equation (2) from equation (3), we have

$$4y = 10.$$
$$y = 2\tfrac{1}{2}.$$

Substituting this value of y in equation (1), we have

$$x = 10\tfrac{1}{2}.$$

Hence the table cost £10 10*s*. and each chair cost £2 10*s*.

EXERCISE XCVIII

1. Find two numbers whose sum is 43 and whose difference is 13.

2. Find two numbers whose sum is 212 and whose difference is 34.

3. Find two numbers such that the sum of the first and twice the second is 42, and the sum of the second and twice the first is 39.

4. Divide 210 into two parts in the ratio of 9 : 12.

5. A bag contains 23 coins, some of which are florins and the remainder shillings. The total value of the coins is £1 18*s*. How many of each kind of coin are there?

6. For £2 5*s*. it is possible to buy either 7 lb. of tea and 6 lb. of coffee, or 11 lb. of tea and 3 lb. of coffee. Find the cost per lb. of each.

7. One lb. of tea and 4 lb. of sugar can be bought for 4 shillings. If the price of the tea is increased $33\frac{1}{3}$ per cent. and the price of the sugar is increased 25 per cent. they would cost 5*s*. 3*d*. Find the price per pound of the tea and the sugar.

8. A gramophone and 8 records can be bought for £13 12*s*. The same gramophone and 20 records can be bought for £15 2*s*. Find the cost of the gramophone.

9. A is 7 years older than B. In 10 years' time the sum of their ages will be 63. How old is each now?

10. A sum of £2 17*s*. is made up of florins and half-crowns. If the numbers of florins and half-crowns were interchanged the value of the money would be £3. How many of each kind of coin are there?

11. Each of three numbers being left out in turn, the sums of the remaining two are 26, 21, and 23 respectively. Find the three numbers.

12. A number consists of two digits, one of which is twice the other. If the digits are interchanged the number is diminished by 36. Find the original number.

13. A number consists of two digits whose sum is 8. If the digits are reversed the number is diminished by 18. Find the original number.

14. A factory employs both skilled and unskilled workers. The weekly wages bill was £235 for 80 skilled men and 20 unskilled: it was £247 for 75 skilled men and 34 unskilled. Find the weekly wages of the skilled and the unskilled worker.

15. A man bought two kinds of tea for £16 6*s*. Some of the tea cost 2*s*. 6*d*. per lb. and the remainder cost 3*s*. 6*d*. per lb. He sells the mixture at 3*s*. per lb. and makes a total profit of £1 2*s*. How many lb. of each kind of tea did he buy?

EXERCISE XCIX (Revision Exercise)

(A)

1. Multiply $3x^2 - 4x - 8$ by $x^2 + x + 1$.

2. If $20x^4 - 7x^3 + 10x^2 + Ax + B$ is exactly divisible by $4x^2 + x + 1$, what values must be given to A and B?

3. Solve the equations

(*a*) $\dfrac{2x+7}{3} - \dfrac{x+1}{5} = 4.$

(*b*) $5x + y = 7.$
$3x - 4y = 18.$

4. Find the factors of

(*a*) $7x^2 - 35x^3y + 49x^2y^2$.

(*b*) $\dfrac{144x^4}{y^4} - 49$.

5. If $y = ax + b$ and $y = 7$ when $x = 1$, and $y = 9$ when $x = 2$, find a and b.

Hint. Substitute in turn $x = 1$, $y = 7$, and $x = 2$, $y = 9$, in the equation $y = ax + b$, and thus obtain two simultaneous equations for a and b.

(B)

6. If $x = -5$, $y = 7$, and $z = -3$, find the value of

(*a*) $x(y - z) - z(y - x)$.

(*b*) $(x - y) + (y - z) + (z - x)$.

7. A number consists of two digits of which the ten's digit exceeds the unit's digit by 3. The number itself exceeds the sum of its digits by 72. Find the number.

8. Solve the equations

(*a*) $\dfrac{2}{x} + \dfrac{3}{y} = 2.$

$\dfrac{8}{x} - \dfrac{9}{y} = 1.$

(*b*) $\dfrac{2x + y}{3} = 5.$

$\dfrac{3x - y}{5} = 1.$

9. Divide $6m^4 - m^3 - 15m^2 + 11m - 5$ by $3m^2 - 2m + 1$, using the method of detached coefficients.

10. Find what numerical values must be given to A, B, and C in $y = Ax^2 + Bx + C$ so that when $x = 1$, $y = 6$, when $x = 2$, $y = 11$, and when $x = 3$, $y = 18$.

(C)

11. The perimeter of a rectangular garden is 70 yards. If the length is increased by 5 yards and the width is decreased

by 3 yards, the area remains unaltered. Find the length and width of the garden.

12. Solve the equations

(*a*) $(2x+1)(x-3)-28=2(x+1)(x-5)$.

(*b*) $2(3x-5y)+25=3(2x+5y)$.

$7x-2y=12$.

13. At what time between 10 o'clock and 11 o'clock will the hands of a clock be in a straight line?

14. If $A=3x^2-x-2$, $B=2x^2+5x+3$, and $C=x^2-1$, find the value of $\frac{AB}{C}$ in terms of x.

15. A man pays the keeper of a shooting-range 1*d.* for a miss and receives 3*d.* for a hit. After firing a dozen shots he has to pay the keeper 4*d.* How many hits did he make?

(D)

16. Say by how much the expression $\frac{3(a+2b+3c)}{4}$ exceeds the expression $\frac{4(3a+2b+3c)}{3}$.

17. Solve the equations

(*a*) $\frac{2}{x}+\frac{1}{y}=-\frac{3}{5}$.

$\frac{2}{y}+\frac{1}{x}=-1\frac{4}{5}$.

(*b*) $5(2x+3y+1)-4(3x-2y+5)=25$.

$5x-2y=11$.

18. A boy has two bags, the first of which contains x marbles and the second y marbles. How many must he take out of the first bag and put in the second so that both bags contain the same number?

19. A father is 6 times as old as his son. In 10 years' time their combined ages will be 62. How old is each now?

20. Show that $\frac{1}{1-a}=1+a+a^2+\frac{a^3}{1-a}$.

CHAPTER IX

HARDER FACTORS AND SQUARE ROOTS

91. We have already used the relationship

$$\mathbf{x^2 - y^2 = (x - y)\,(x + y)}$$

by which the difference of two squares can be expressed as a product of two factors. Although it was originally discovered in the course of our investigations of non-directed numbers, it can be proved to hold good for directed numbers as well. Further, it also holds good when x and y represent algebraic expressions. The following examples will illustrate this.

EXAMPLE 1

Find the factors of $(a + 2b)^2 - (2a + b)^2$.

In this case $x = (a + 2b)$ and $y = (2a + b)$. So that

$$\begin{aligned}(a + 2b)^2 - (2a + b)^2 \\ &= x^2 - y^2 \\ &= (x - y)\,(x + y) \\ &= \{(a + 2b) - (2a + b)\}\ \{(a + 2b) + (2a + b)\} \\ &= \{a + 2b - 2a - b\}\ \{a + 2b + 2a + b\} \\ &= (-a + b)\,(3a + 3b).\end{aligned}$$

Notice the necessity for the brackets and how they are gradually cleared. Once the process is understood it is not usual to show the substitution in full, as in this example. The next example illustrates the way in which the work is generally set out.

EXAMPLE 2

Find the factors of $9(p + q)^2 - 25r^2$.

$$\begin{aligned}9(p + q)^2 - 25r^2 &= \{3(p + q)\}^2 - \{5r\}^2 \\ &= \{3(p + q) - 5r\}\ \{3(p + q) + 5r\} \\ &= (3p + 3q - 5r)\,(3p + 3q + 5r).\end{aligned}$$

EXERCISE C

1. Verify that $x^2 - y^2 = (x - y)(x + y)$ for the values

(i) $x = -1, y = 5$. (iii) $x = -2, y = -4$.
(ii) $x = 4, y = -3$.

2. Find the factors of the following:

(i) $(a + b)^2 - (a - b)^2$.
(ii) $(a - b)^2 - (a + b)^2$.
(iii) $(x - 2y)^2 - (2x + y)^2$.
(iv) $(x + y)^2 - z^2$.
(v) $z^2 - (x + y)^2$.
(vi) $9(p + q)^2 - 64q^2$.
(vii) $16a^2 - 36(a + b)^2$.
(viii) $25(x + y)^2 - 100(x - y)^2$.
(ix) $81(a + b)^2 - 9b^2$.
(x) $(m - n)^2 - 49$.
(xi) $(x + y + z)^2 - (x - y - z)^2$.
(xii) $25p^{10} - 9q^{10}$.
(xiii) $(p + p^2q^2)^2 - (q - q^2p^2)^2$.
(xiv) $2(u + v)^2 - 18w^2$.

Hint. Notice that 2 is a common factor, and then deal with the difference of two squares.

(xv) $3a^2 - 75(a + b)^2$.
(xvi) $7x^2y^2 - 63(x + y)^2$.
(xvii) $4(c + d)^2 - 121$.
(xviii) $p^2(x - y)^2 - p^2y^2$.
(xix) $x^2y^2(a + b)^2 - a^2b^2(x + y)^2$.
(xx) $(ax + b)^2 - 49$.
(xxi) $(mx + c)^2 - (c - mx)^2$.
(xxii) $144 - (3y - x)^2$.
(xxiii) $(a + 2b + 3c)^2 - (3a + 2b + c)^2$.
(xxiv) $(x - 3y + z)^2 - (x + 2y - z)^2$.
(xxv) $32a^2 - 18(x - y)^2$.
(xxvi) $(2l - m)^2 - (2m - l)^2$.
(xxvii) $(5p + q - r)^2 - (5q - r + p)^2$.
(xxviii) $100 - 16(p - q - r)^2$.
(xxix) $p(a - b)^2 - pc^2$.

3. Show that $29^2 - 15^2$ is divisible by 11.
4. Show that $(1 + x + x^2)^2 - (1 - x - x^2)^2$ is divisible by 4.

92. It can be proved by actual multiplication that

$$(\mathbf{x} + \mathbf{y})^2 = \mathbf{x}^2 + 2\mathbf{xy} + \mathbf{y}^2.$$
$$(\mathbf{x} - \mathbf{y})^2 = \mathbf{x}^2 - 2\mathbf{xy} + \mathbf{y}^2.$$

Notice that in each case the result of squaring $(x + y)$ and $(x - y)$ consists of *three* terms arranged in the following order.

1. The square of the first term.
2. Plus (or minus) *twice* the product of the terms.
3. The square of the second term.

By this means we are able to write down at sight the square of any expression of the form $(x + y)$ or $(x - y)$.

EXAMPLE 1

Write down the square of $(3a + 2b)$.

$$(3a + 2b)^2 = (3a)^2 + 2.(3a)(2b) + (2b)^2$$
$$= 9a^2 + 12ab + 4b^2.$$

EXAMPLE 2

Write down the value of $(4p - 3q)^2$.

$$(4p - 3q)^2 = (4p)^2 - 2.(4p)(3q) + (3q)^2$$
$$= 16p^2 - 24pq + 9q^2.$$

EXERCISE CI

1. Write down the squares of

(a) $a + 1$.
(b) $a - 2$.
(c) $m + 3$.
(d) $m - 4$.
(e) $2p + q$.
(f) $2p - q$.
(g) $3u + 2v$.
(h) $3u - 2v$.
(i) $4b + 3c$.
(j) $x + \frac{1}{x}$.
(k) $a - \frac{2}{b}$.
(l) $xy + p$.
(m) $2p - 3qr$.
(n) $\frac{4}{x} + 1$.
(o) $5k - 2m$.

2. Write down the square roots of

(*a*) $x^2 - 4x + 4$.
(*b*) $a^2 + 4ab + 4b^2$.
(*c*) $4m^2 + 4mn + n^2$.
(*d*) $m^2 - 4mn + 4n^2$.
(*e*) $4x^2 + 12xy + 9y^2$.
(*f*) $9x^2 - 12xy + 4y^2$.
(*g*) $1 + 2x + x^2$.
(*h*) $1 - 6p + 9p^2$.
(*i*) $q^2 - 10qr + 25r^2$.
(*j*) $4x^2 - 12xy + 9y^2$.

3. Fill in the missing terms in the following squares:

(*a*) $(x - 3)^2 = x^2 - \quad + 9$.
(*b*) $(2x - 1)^2 = \quad - 4x + 1$.
(*c*) $(3p - q)^2 = 9p^2 - \quad +$
(*d*) $(q - \quad)^2 = q^2 - 4qr + 4r^2$.
(*e*) $(\quad - 5n)^2 = m^2 - 10mn +$

4. Add a third term to each of the following, so as to make the expression a perfect square. Write down its square root.

(*a*) $a^2 + 2ab$.
(*b*) $a^2 - 4ab$.
(*c*) $4a^2 - 4ab$.
(*d*) $9x^2 + 12xy$.
(*e*) $4x^2 - 12xy$.
(*f*) $1 - 2x$.
(*g*) $1 - 6p$.
(*h*) $q^2 - 4qr$.
(*i*) $4 + 4k$.
(*j*) $9m^2 + 30mn$.

93. It is very easy to show by actual multiplication that the product

$$(x + 5)(x - 7) = x^2 - 2x - 35.$$

It is possible, however, to write down such products at sight instead of performing the multiplication each time.

$$(x + 5)(x - 7)$$

The products of the end terms (shown in the line above by the arrows) clearly give the end terms x^2 and -35 in the answer.

$$(x + 5)(x - 7)$$

The algebraic sum of the two products of the unlike terms (shown in the line above by the arrows) clearly gives

$$5x - 7x = -2x.$$

This is the middle term of the answer.

Hence altogether we have

$$(x + 5)(x - 7) = x^2 - 2x - 35.$$

EXERCISE CII

Write down—*i.e.*, without actual multiplication—the answers of the following products:

1. $(x + 2)(x + 3)$.
2. $(x - 2)(x - 3)$.
3. $(x - 2)(x + 3)$.
4. $(x + 2)(x - 3)$.
5. $(p - 3q)(p + 2q)$.
6. $(p + 3q)(p - 2q)$.
7. $(2p - q)(p + 2q)$.
8. $(m + 2n)(2m - 3n)$.
9. $(3k + t)(5k - 2t)$.
10. $(2a - 3b)(4a + 5b)$.
11. $(6a + 7b)(8a - 9b)$.
12. $(3x - 7y)(5x + 2y)$.
13. $(8p - 3q)(4p + q)$.
14. $(1 - 5x)(1 + 2x)$.
15. $(1 - pq)(2 - 3pq)$.
16. $(x - ay)(a - xy)$.
17. $(2 - k)(5 - 7k)$.
18. $(3p + 4q)(7p - q)$.
19. $(2x - 5y)(6x - y)$.
20. $(3a + 5b)(a - 7b)$.

94. We must now consider the converse process—namely, finding the factors (if there are any) of an algebraic expression of three terms. We will first of all consider the case in which the coefficient of the first of the three terms is unity.

EXAMPLE 1

Find the factors of $x^2 - x - 6$.

It is clear that we can write down immediately

$$x^2 - x - 6 = (x \quad *)(x \quad *)$$

and our only difficulty will be to find the numbers represented by the stars, and of course the proper signs to place before them. Since the product of the two numbers is 6, the two numbers themselves must be factors of 6. The only factors of 6 which can give 1 as their difference are 2 and 3. Hence we can write

$$x^2 - x - 6 = (x \quad 2)(x \quad 3).$$

The question of signs must now be settled. Since the final product (– 6) is negative, the signs before the 2 and the 3 must be *unlike*. Hence we shall have either

$$(x + 2)(x - 3)$$

or

$$(x - 2)(x + 3).$$

Now, from what we have just done we can write down these products at sight.

$$(x + 2)(x - 3) = x^2 - x - 6.$$
$$(x - 2)(x + 3) = x^2 + x - 6.$$

The second result does not give the set of factors required. Hence we have

$$x^2 - x - 6 = (x + 2)(x - 3).$$

EXAMPLE 2

Write down the factors of $x^2 + 2x - 15$.

Proceeding by the stages described in the last example, we have

(i) $x^2 + 2x - 15 = (x \quad)(x \quad)$.

(ii) $x^2 + 2x - 15 = (x \quad 5)(x \quad 3)$.

(iii) $x^2 + 2x - 15 = (x + 5)(x - 3)$ or $(x - 5)(x + 3)$.

But $(x - 5)(x + 3)$ does not give the required product.

$$x^2 + 2x - 15 = (x + 5)(x - 3).$$

EXAMPLE 3

Write down the factors of $x^2 - 9x + 20$.

Proceeding by the stages described in Example 1, we have

(i) $x^2 - 9x + 20$
$= (x \quad)(x \quad)$.

(ii) $x^2 - 9x + 20$
$= (x \quad 4)(x \quad 5)$ or $(x \quad 2)(x \quad 10)$.

Since the final product is positive, the signs in both pairs of brackets must be alike. Hence

(iii) $x^2 - 9x + 20$

$$= \begin{matrix}(x + 4)(x + 5) \\ (x - 4)(x - 5)\end{matrix} \text{ or } \begin{matrix}(x + 2)(x + 10). \\ (x - 2)(x - 10).\end{matrix}$$

The second set of factors can be rejected at once, since they will not give a product of $-9x$ as the middle term. It is then not difficult to make the proper selection:

$$x^2 - 9x + 20 = (x - 4)(x - 5).$$

EXERCISE CIII

Find the factors of the following. In each case check your answer by actual multiplication.

1. $x^2 + 3x + 2$.
2. $x^2 + 5x + 6$.
3. $x^2 + 6x + 8$.
4. $x^2 - 3x + 2$.
5. $x^2 - 5x + 6$.
6. $x^2 - 6x + 8$.
7. $a^2 + 8a + 15$.
8. $m^2 - 6m - 7$.
9. $k^2 - k - 2$.
10. $p^2 + 5p + 6$.
11. $p^2 + 5pq + 6q^2$.
12. $x^2 + 2x - 15$.
13. $x^2 - 18x + 80$.
14. $a^2 + a - 56$.
15. $a^2 + ab - 56b^2$.
16. $p^2 + 4pq - 5q^2$.
17. $x^2 + 2x - 99$.
18. $x^2 - 8x - 9$.
19. $a^2 - 2ab - 35b^2$.
20. $x^2 - xy - 6y^2$.
21. $c^2 - 3cd - 10d^2$.
22. $x^2 - 7xy - 8y^2$.
23. $a^2 - 2ab - 3b^2$.
24. $m^2 - mn - 6n^2$.
25. $k^2 - 2km - 3m^2$.
26. $x^2 + 7xy + 10y^2$.
27. $p^2 - 12pq + 27q^2$.
28. $a^2 - 13ab + 30b^2$.
29. $b^2 + 4bd - 5d^2$.
30. $x^2 + 9xy + 8y^2$.

95. Where the coefficient of the first of the three terms of an algebraic expression is not unity the process of finding the factors has to be slightly modified.

EXAMPLE 1

Find the factors of $6x^2 + 7xy - 5y^2$.

The two factors which give $6x^2$ are either $3x$ and $2x$ or $6x$ and x. Consequently we may write:

$$\text{(i) } 6x^2 + 7xy - 5y^2 = (3x \qquad)(2x \qquad) \text{ or } (6x \qquad)(x \qquad).$$

The factors of $5y^2$ are $5y$ and y, so that our next stage of the work will be:

$$\text{(ii)}\quad 6x^2 + 7xy - 5y^2$$

$$= \begin{matrix}(3x \quad 5y)\,(2x \quad y) \\ (3x \quad y)\,(2x \quad 5y)\end{matrix} \text{ or } \begin{matrix}(6x \quad 5y)\,(x \quad y). \\ (6x \quad y)\,(x \quad 5y).\end{matrix}$$

Now, the middle term ($7xy$) cannot be obtained by either of the two combinations starting with the factors $6x$ and x, so that we need consider only the first pair of alternatives. Of these the second can be rejected at once, because by no method of arrangement of signs will it be possible to obtain $7xy$ as the middle term. Hence we must write:

$$\text{(iii)}\quad 6x^2 + 7xy - 5y^2 = (3x \quad 5y)\,(2x \quad y).$$

The signs which give $+ 7xy$ produce

$$6x^2 + 7xy - 5y^2 = (3x + 5y)\,(2x - y).$$

After some practice much of the work shown in this example can be done mentally. The next example is not worked out in such detail, but all the steps shown in Example 1 have been performed.

EXAMPLE 2

Find the factors of $10x^2 - 33xy + 27y^2$.

$$\text{(i)}\quad 10x^2 - 33xy + 27y^2$$

$$= (5x \quad)\,(2x \quad) \text{ or } (10x \quad)\,(x \quad).$$

$$\text{(ii)}\quad 10x^2 - 33xy + 27y^2$$

$$= \left(5x \quad \begin{matrix}27y\\ \text{or}\\ 3y\\ \text{or}\\ 9y\\ \text{or}\\ y\end{matrix}\right)\left(2x \quad \begin{matrix}y\\ \text{or}\\ 9y\\ \text{or}\\ 3y\\ \text{or}\\ 27y\end{matrix}\right)$$

or

$$\left(10x \quad \begin{matrix}27y\\ \text{or}\\ 3y\\ \text{or}\\ 9y\\ \text{or}\\ y\end{matrix}\right)\left(x \quad \begin{matrix}y\\ \text{or}\\ 9y\\ \text{or}\\ 3y\\ \text{or}\\ 27y\end{matrix}\right)$$

In order to obtain the term of $33xy$ the only possible choice is $(5x \quad 9y)(2x \quad 3y)$. To obtain the proper signs, notice that the third term is positive, and that consequently both signs must be alike. Since the second term is negative, the two signs must be minus. Hence

$$\text{(iii)} \quad 10x^2 - 33xy + 27y^2 = (5x - 9y)(2x - 3y).$$

EXERCISE CIV

Find the factors of

1. $2x^2 + 5x + 3$.
2. $2x^2 - 5x + 3$.
3. $2x^2 + 7x + 3$.
4. $2x^2 - 7x + 3$.
5. $6a^2 - ab - b^2$.
6. $6a^2 + 5ab + b^2$.
7. $8x^2 + 10xy - 3y^2$.
8. $3x^2 - xy - 2y^2$.
9. $10q^2 + 13qp - 3p^2$.
10. $12q^2 + 5qp - 2p^2$.
11. $6x^2 + 7xy + y^2$.
12. $5m^2 - 6mn + n^2$.
13. $4k^2 - 7kl - 2l^2$.
14. $5a^2 - 3ab - 2b^2$.
15. $10a^2 + 19ab - 2b^2$.
16. $27g^2 - 3gf - 2f^2$.
17. $25p^2 - 25pq + 6q^2$.
18. $2x^2 - 23xy - 12y^2$.
19. $2m^2 + 7mn + 6n^2$.
20. $3a^2 - 19ab - 14b^2$.
21. $2x^2 - 19x + 9$.
22. $8x^2 + 2xy - 15y^2$.
23. $3a^2 + 16ab - 35b^2$.
24. $10a^2 - 13ab - 3b^2$.
25. $42x^2 - 16x - 10$.
26. $9k^2 - 14km + 5m^2$.
27. $6p^2 - 31pq + 33q^2$.
28. $54a^2 + 57ab + 10b^2$.
29. $7m^2 + 50mn + 7n^2$.
30. $4p^2 - 8pq + 3q^2$.
31. $2a^2 - amn - m^2n^2$.
32. $3a^2 - 23ab - 8b^2$.
33. $2x^2 + 5xy + 3y^2$.
34. $6x^2 - 5xy - y^2$.
35. $18p^2 - 77pq - 18q^2$.

96. It is easy to prove by actual multiplication that

$$\mathbf{x^3 + y^3 = (x + y)(x^2 - xy + y^2)}$$

and that

$$\mathbf{x^3 - y^3 = (x - y)(x^2 + xy + y^2)}.$$

These two results enable us to factorize the *sum* and the *difference* of two cubes.

We have seen that the *difference* of two squares can be factorized—e.g., $x^2 - y^2 = (x - y)(x + y)$—but there are *no* factors for the sum of two squares.

EXAMPLE 1

Find the factors of $8a^3 + b^3$.

$$8a^3 + b^3 = (2a)^3 + b^3$$
$$= (2a + b)(4a^2 - 2ab + b^2).$$

EXAMPLE 2

Find the factors of $64 - \frac{x^3}{27}$.

$$64 - \frac{x^3}{27} = (4)^3 - \left(\frac{x}{3}\right)^3$$
$$= \left(4 - \frac{x}{3}\right)\left\{4^2 + 4 \cdot \frac{x}{3} + \left(\frac{x}{3}\right)^2\right\}$$
$$= \left(4 - \frac{x}{3}\right)\left(16 + \frac{4x}{3} + \frac{x^2}{9}\right).$$

EXERCISE CV

Find the factors of

1. $x^3 + 1$.
2. $x^3 - 1$.
3. $1 - 8p^3$.
4. $1 + 8p^3$.
5. $27a^3 - b^3$.
6. $b^3 + 27a^3$.
7. $125m^3 + 64n^3$.
8. $1 - \frac{b^3}{27}$.
9. $1000x^3 - y^3$.
10. $v^3k^3 + k^3$.
11. $a^3 - (x + y)^3$.
12. $a^3 + (a - b)^3$.
13. $(a + b)^3 + (a - b)^3$.
14. $(a + b)^3 - (a - b)^3$.
15. $a^6 - b^6$.
16. $(x + y)^3 - (y + z)^3$.
17. $1 - p^3q^6$.
18. $27k^3 + 343m^3$.
19. $x^5 - x^2y^3$.
20. $5x^3 + 625y^3$.

97. It is difficult to give any general rules for factorizing algebraic expressions of four or more terms. Frequently they are found to be more complicated illustrations of the types of factors already dealt with. Later on (p. 182), however, we shall consider a method by which the linear factors of an expression can always be discovered. In this exercise no fresh ideas are introduced, but familiar processes are applied to more complicated expressions.

EXAMPLE 1

Find the factors of $ax + bx + ay + by$.

$$ax + bx + ay + by = x(a + b) + y(a + b)$$

$$= (a + b)\left\{x + y\right\}$$ (taking the common factor $(a + b)$ outside the curled brackets).

$$= (a + b)(x + y).$$

EXAMPLE 2

Find the factors of $x^2 + xy - ax - ay$.

$$x^2 + xy - ax - ay = x(x + y) - a(x + y)$$ (Notice this + sign.)

$$= (x + y)\left\{x - a\right\}$$ (taking the common factor $(x + y)$ outside the curled brackets).

$$= (x + y)(x - a).$$

EXERCISE CVI

Find the factors of

1. $x^2 + x + xy + y$.
2. $x^2 - x + xy - y$.
3. $x^2 - x - xy + y$.
4. $ax + 2bx - ay - 2by$.
5. $2ax - 2bx + ay - by$.
6. $2ax + 4bx - 3ay - 6by$.
7. $mf - nf + 3mg - 3ng$.
8. $2ak + 6am - kb - 3mb$.
9. $6ax + 2bx - 9ay - 3by$.
10. $am - bm - an + bn$.
11. $ax + bx + cx + ay + by + cy$.
12. $ax - bx - cx - ay + by + cy$.
13. $x + ax + a^2x + y + ay + a^2y$.
14. $x - ax - a^2x - y + ay + a^2y$.
15. $a - y + ax - xy$.
16. $a + y - ax - xy$.

17. $2ax + 3by - 3bx - 2ay$.
18. $2mp + 4mn - 5pn - 10n^2$.
19. $3f - gx + 3fx - g$.
20. $3mx - 2my + 12kx - 8ky$.

98. If $3x^3 - 5x^2 + 4x - 7$ is divided by $x - 2$ there is a remainder of $+ 5$. This can be shown by performing the actual division as follows:

$$\begin{array}{r l}
x - 2)3x^3 - 5x^2 + 4x - 7(3x^2 + x + 6 & \\
\underline{3x^3 - 6x^2} & \\
x^2 + 4x & \\
\underline{x^2 - 2x} & \\
6x - 7 & \\
\underline{6x - 12} & \\
+ 5 \text{ remainder} &
\end{array}$$

If we substitute the value $+ 2$ for x in the original expression the result is $+ 5$, for if $x = + 2$, then

$$\begin{aligned}
3x^3 - 5x^2 + 4x - 7 &= 3(+ 2)^3 - 5(+ 2)^2 + 4(+ 2) - 7 \\
&= 24 - 20 + 8 - 7 \\
&= + 5.
\end{aligned}$$

This is an illustration of a general rule called the "Remainder Theorem," which states that the remainder obtained by dividing an algebraic expression containing powers of x by $x - a$ is equal to the result of substituting $x = + a$. Similarly, if the expression is divided by $x + a$, then the remainder is equal to the result of substituting $x = - a$.

EXAMPLE 1

Find the remainder when $3x^3 + 4x^2 - 7x + 1$ is divided by $x - 3$.

If $x = + 3$, then

$$\begin{aligned}
3x^3 + 4x^2 - 7x + 1 &= 3(+ 3)^3 + 4(+ 3)^2 - 7(+ 3) + 1 \\
&= 81 + 36 - 21 + 1 \\
&= + 97.
\end{aligned}$$

EXAMPLE 2

Find the remainder when $3x^3 + 4x^2 - 7x + 1$ is divided by $x + 3$.

If $x = -3$, then

$$\begin{aligned} 3x^3 + 4x^2 - 7x + 1 &= 3(-3)^3 + 4(-3)^2 - 7(-3) + 1 \\ &= -81 + 36 + 21 + 1 \\ &= -23. \end{aligned}$$

EXERCISE CVII

Find the remainder in each of the following divisions:

1. $3x^2 - x + 1$ by $x - 3$.
2. $7x^2 + x - 5$ by $x + 3$.
3. $3x^3 - 7x^2 + 5x + 4$ by $x - 5$.
4. $3x^3 - 7x^2 + 5x + 4$ by $x + 5$.
5. $7x^2 + 8x - 9$ by $x - 2$.
6. $9x^4 - 3x^3 + x + 4$ by $x + 1$.
7. $5x^3 - 8x^2 + 13x - 5$ by $x - 4$.
8. $7p^2 - 5p + 3$ by $p + 4$.
9. $10m^2 - 8m + 7$ by $p - 3$.
10. $8a^3 + 7a^2 - 9a + 5$ by $a - 8$.

99. If we proceed to find the remainder when $2x^3 - 9x^2 + 7x + 6$ is divided by $x - 2$, we shall find that it is zero, for substituting the value $x = 2$ we have

$$\begin{aligned} 2x^3 - 9x^2 + 7x + 6 &= 2(2^3) - 9(2^2) + 7(2) + 6 \\ &= 16 - 36 + 14 + 6 \\ &= 0. \end{aligned}$$

Since there is no remainder when $2x^3 - 9x^2 + 7x + 6$ is divided by $x - 2$, therefore $x - 2$ is a factor of this expression. In this particular case $x - 3$ is another factor, for substituting the value $x = 3$ we have

$$\begin{aligned} 2x^3 - 9x^2 + 7x + 6 &= 2(3)^3 - 9(3)^2 + 7(3) + 6 \\ &= 54 - 81 + 21 + 6 \\ &= 0. \end{aligned}$$

Since the original expression is of the third degree and two of its factors are $(x-2)$ and $(x-3)$, there is a third linear factor. This can be found by dividing by the product of the two known factors, and we then have

$$2x^3-9x^2+7x+6=(x-2)(x-3)(2x+1).$$

EXERCISE CVIII

1. Show that $x-1$ is a factor of x^3-7x+6, and then completely factorize this expression.

2. Show that $x+1$ and $x-2$ are factors of x^3-4x^2+x+6, and then completely factorize this expression.

3. Show that $x+3$ is a factor of $2x^3+5x^2-4x-3$. Find the other factors.

4. Find the factors of

(*a*) $2x^3-7x^2-5x+4$.
(*b*) $x^3-9x^2+15x-7$.
(*c*) $4x^3-12x^2-x+3$.
(*d*) $2x^3-17x^2+38x-15$.
(*e*) $4x^3+4x^2-23x-30$.
(*f*) $x^3-6x^2-7x+60$.
(*g*) $3x^3-4x^2-27x-20$.

5. If $6x^3-17x^2-5x+\mathrm{K}$ is exactly divisible by $x-3$, find K.

Hint. If $x-3$ is a factor of $6x^3-17x^2-5x+\mathrm{K}$, then on substituting $x=3$ in $6x^3-17x^2-5x+\mathrm{K}$ there will be no remainder, and we shall have

$$6(3)^3-17(3)^2-5(3)+\mathrm{K}=0.$$

This is a simple equation for K.

6. Find K if $3x^3+7x^2+\mathrm{K}x+4$ is exactly divisible by $x+4$.

7. What value of K will make $\mathrm{K}x^3+11x^2+20x+12$ exactly divisible by $x+2$?

8. What values must be given to p and q in order that x^3-6x^2+px+q may be divisible by $x-1$ and $x-2$?

9. What values must be given to p and q in order that $px^3+qx^2-13x-6$ may be divisible by $x-3$ and $x+2$?

10. Find the factors of $3x^3 - 4x^2 - 35x + 12$.

11. If $ax^3 + bx^2 + cx - 6$ is exactly divisible by $x - 3$, $x + 2$, and $2x + 1$, find the values of a, b, and c.

12. Show that $2x + 1$ is a factor of $6x^3 + 5x^2 - 3x - 2$, and hence factorize this expression.

13. The expressions $3x^2 - 11x + 10$ and $2x^3 - 3x^2 - 5x + 6$ have a common factor. Find it.

14. Find what values must be given to p and q in order that $px^3 - 46x^2 + qx + 6$ may be divisible by $2x - 3$ and $3x - 2$.

EXERCISE CIX

(General Revision Exercise on Factors)

Find the factors of the following:

1. $1 - 81p^2$.
2. $1 + 27p^3$.
3. $20x^2 - 9x + 1$.
4. $8 - \frac{a^3}{27}$.
5. $3x^2 - x - 10$.
6. $abx^2 + (a^2 + b^2)xy + aby^2$.
7. $6x^2 + 5x - 4$.
8. $1 + 2x + 2x^2 + x^3$.
9. $ax - ay + by - bx$.
10. $28x^2 + 23x - 15$.
11. $x^2 - a^2 - 2ab - b^2$.
12. $x^3 - (a - b)^3$.
13. $14x^2 - x - 30$.
14. $1000p^3 - q^3$.
15. $x^3 + 3x^2 + 3x + 1$.
16. $x^3 - 3x^2 + 3x - 1$.
17. $16x^2 + 2x - 3$.
18. $5x^2 - 27x - 18$.
19. $ax - 6by - 3ay + 2bx$.
20. $1 + x^3$.
21. $21ax - 35bx - 6ay + 10by$.
22. $36p^2 + 81pq - 40q^2$.
23. $72m^2 - 31mn - 28n^2$.
24. $3x^3 - 25x^2 + 42x + 40$.
25. $6x^3 - 7x^2 - 7x + 6$.

100. We shall conclude this chapter with a brief account of the process used for finding the square root of an algebraic expression. Mention has already been made of one method (see p. 172), when the square root is written down by inspection. When the algebraic expression is complicated the square root cannot be written down in this manner, and consequently some other method must be used. The following is the one usually adopted. It is modelled on the corresponding process in arithmetic, the only difference being that the square root of the first term at each stage of the process is evaluated exactly. Consider the following example.

Find the square root of $4x^4 - 4x^3 + 13x^2 - 6x + 9$.

$$
\begin{array}{r|l}
 & 2x^2 - x + 3 \\
\hline
 & 4x^4 - 4x^3 + 13x^2 - 6x + 9 \\
 & 4x^4 \\
\hline
 & -4x^3 + 13x^2 \\
4x^2 - x & -4x^3 + x^2 \\
\hline
4x^2 - 2x + 3 & 12x^2 - 6x + 9 \\
 & 12x^2 - 6x + 9 \\
\hline
\end{array}
$$

Note

(i) The square root of $4x^4$ is $2x^2$. Note that the square root of the first term is evaluated exactly.

(ii) Following the arithmetical process, double $2x^2$ and divide it into the next term — namely, $-4x^3$. This gives $-x$.

(iii) Multiply $4x^2 - x$ by $-x$ and subtract, so as to get the remainder $12x^2$.

(iv) Double each term in the answer and continue the process as before.

EXERCISE CX

Find the square roots of the following:

1. $x^4 + 4x^3 + 10x^2 + 12x + 9$.
2. $4a^4 - 12a^3 + 25a^2 - 24a + 16$.
3. $9a^2 + 12ab + 18a + 4b^2 + 12b + 9$.
4. $25p^4 - 10p^3 - 19p^2 + 4p + 4$.
5. $16x^4 - 56x^3 + 73x^2 - 42x + 9$.
6. $9p^2 + 30pq + 6pr + 25q^2 + 10qr + r^2$.
7. $9x^2 + 12xy - 24x + 4y^2 - 16y + 16$.
8. $x^6 - 4x^5 + 6x^4 - 6x^3 + 5x^2 - 2x + 1$.
9. $16a^4 - 8a^3 - 23a^2 + 6a + 9$.
10. $b^2c^2 + 2abc^2 + 2ab^2c + a^2c^2 + 2a^2bc + a^2b^2$.
11. $1 + 2x - 2xy + x^2 - 2x^2y + x^2y^2$.
12. $1 - 4x + 10x^2 - 12x^3 + 9x^4$.
13. $a^2 + 2a - 1 - \dfrac{2}{a} + \dfrac{1}{a^2}$.
14. $4x^4 - 4x^3 + x^2 + 12x - 6 + \dfrac{9}{x^2}$.
15. $16a^2 - 24a + 25 - \dfrac{12}{a} + \dfrac{4}{a^2}$.

EXERCISE CXI (Revision Exercise)

(A)

1. Find the factors of

(*a*) $25k^2 - 49$. (*b*) $125k^3 - 343$. (*c*) $6x^2 - 13x + 6$.

2. Solve the equations

(*a*) $5x + y = 8.$
$3x - 2y = 23.$

(*b*) $\frac{x+4}{4} - \frac{x-4}{3} = 2.$

3. Show without actual division that $x + 3$ is a factor of $x^3 - 14x - 15$.

4. Find two numbers whose sum is 48 and whose difference is 14.

5. Find the common factor of $6x^2 + 5x - 4$, $10x^2 - 11x + 3$, and $2x^2 + x - 1$.

(B)

6. Solve the equations

(*a*) $\frac{3}{x} + \frac{6}{y} = -\frac{1}{2}.$

$\frac{5}{x} - \frac{9}{y} = 5\frac{1}{2}.$

(*b*) $2x + 3y = 2y + 3z = 2z + 3x = 5.$

7. What values must be given to p and q in order that $4x^3 - 8x^2 + px + q$ may be exactly divisible by $x - 3$ and $2x - 1$?

8. What value of x will make $(2x + 5)^2$ equal to $(2x - 4)^2$?

9. A number consists of two digits. If the digits are reversed the number is diminished by 27. The sum of the digits is 5. Find the original number.

10. If $2n$ is the smallest of four consecutive numbers, write down the four numbers.

Find the numbers if their sum is 62. What will be the value of n in this case?

(C)

11. Find the square root of $9x^4 + 6x^3 - 23x^2 - 8x + 16$.

12. A man spent £4 on two kinds of coffee, some of which cost 3 shillings per lb. and the remainder 4 shillings per lb. He sells the mixture at 3*s*. 6*d*. per lb. and makes a profit of 7*s*. 6*d*. How much of each kind did he buy?

13. Say what values must be given to A, B, and C if

$$(2x + 3)(3x - 2) = Ax^2 + Bx + C.$$

14. The sum of two numbers is 8, and the difference of their squares is 16. Find the numbers.

15. If $x = -5a$ and $y = -5b$, find the value of $\frac{x^2 + xy + y^2}{x - y}$ in terms of a and b.

(D)

16. The perimeter of a garden is 190 feet, and the length exceeds the width by 25 feet. Calculate its area.

17. Find the factors of

(*a*) $6ax - 4ay + 9bx - 6by$.
(*b*) $3x^3 - 11x^2 - 6x + 8$.

18. Find the square root of $4m^4 - 12m^3 - 11m^2 + 30m + 25$.

19. What values must be given to a and b so that $ax + by + 1$ may be the square root of $4x^2 - 12xy + 4x + 9y^2 - 6y + 1$?

20. A sum of £1 5*s*. 6*d*. is made up of sixpences and shillings. If the numbers of sixpences and shillings were interchanged the value of the money would be £1 10*s*. How many of each kind of coin are there?

CHAPTER X

COMPLEX FRACTIONS AND PARTIAL FRACTIONS

101. Although this chapter will be concerned mainly with fractions of a more complex nature than those considered up to the present, it is necessary, first of all, to revise and extend our knowledge of the highest common factor (H.C.F.) and the lowest common multiple (L.C.M.) of a number of quantities.

The H.C.F. of a number of quantities is the product of their common factors.

EXAMPLE 1

Find the H.C.F. of $x^2 - 1$, $2x^2 + x - 3$, and $3x^2 - 5x + 2$.
Factorizing each of these three, we have

$$x^2 - 1 = (\mathbf{x-1})(x+1).$$
$$2x^2 + x - 3 = (2x+3)(\mathbf{x-1}).$$
$$3x^2 - 5x + 2 = (3x-2)(\mathbf{x-1}).$$

The common factor is obviously $x - 1$, and therefore in this case the H.C.F. is $x - 1$.

EXAMPLE 2

Find the H.C.F. of $ax^3 - ax$, $x^4 - x$, and $x^3 - 2x^2 + x$.

$$\begin{aligned} ax^3 - ax &= ax(x^2 - 1) \\ &= a\mathbf{x}(\mathbf{x-1})(x+1). \\ x^4 - x &= x(x^3 - 1) \\ &= \mathbf{x}(\mathbf{x-1})(x^2 + x + 1). \\ x^3 - 2x^2 + x &= x(x^2 - 2x + 1) \\ &= \mathbf{x}(\mathbf{x-1})(x-1). \end{aligned}$$

The common factors are x and $x - 1$. Therefore the H.C.F. is $x(x - 1)$.

EXERCISE CXII

Find the highest common factor of

1. $x^2 - 1$ and $x^2 - 2x + 1$.
2. $x^3 + 1$ and $x^2 + 2x + 1$.
3. $2x^2 - 3x + 1$ and $4x^2 + 4x - 3$.
4. $3x^2 - x - 2$ and $3x^2 + 8x + 4$.
5. $8x^2 - 2x - 3$ and $12x^2 - 17x + 6$.
6. $x^2 + 2x + 1$, $x^2 - 1$, and $2x^2 + 3x + 1$.
7. $6x^2 - 4x$, $3x^2 + x - 2$, and $3x^2 - 5x + 2$.
8. $4x^2 + 12x + 9$, $6x^2 + 7x - 3$, and $8x^2 + 14x + 3$.
9. $3x^2 - 8x + 5$, $9x^2 - 30x + 25$, and $6x^2 - x - 15$.
10. $x^2 - 1$, $x^3 - x^2 - x + 1$, and $x^3 + x^2 - x - 1$.

102. The L.C.M. of a number of quantities is the product of the highest powers of each kind of factor.

Example 1

Find the L.C.M. of $x^2 - 1$, $x^2 - 2x + 1$, and $x^2 + 2x + 1$.
Factorizing each of these, we have

$$x^2 - 1 = (x - 1)(x + 1).$$
$$x^2 - 2x + 1 = (x - 1)^2.$$
$$x^2 + 2x + 1 = (x + 1)^2.$$

The L.C.M. is $(x - 1)^2 (x + 1)^2$.

Example 2

Find the L.C.M. of $2x^2 + x$, $4x^2 + 4x + 1$, and $2x^2 - x - 1$.

$$2x^2 + x = x(2x + 1).$$
$$4x^2 + 4x + 1 = (2x + 1)^2.$$
$$2x^2 - x - 1 = (x - 1)(2x + 1).$$

The L.C.M. is $x(2x + 1)^2 (x - 1)$.

EXERCISE CXIII

Find the lowest common multiple of each of the questions given in Exercise CXII.

103. The addition and subtraction of fractions follow exactly the corresponding operations in arithmetic. Care must be taken to use brackets, so as to avoid mistakes of sign.

EXAMPLE 1

Simplify $\dfrac{5}{x+3} + \dfrac{4}{x+5}$.

$$\frac{5}{x+3} + \frac{4}{x+5}$$

$$= \frac{5(x+5) + 4(x+3)}{(x+3)(x+5)}$$ (The L.C.M. is $(x + 3)(x + 5)$.)

$$= \frac{5x + 25 + 4x + 12}{(x+3)(x+5)}$$

$$= \frac{9x + 37}{(x+3)(x+5)}.$$

EXAMPLE 2

Simplify $\dfrac{5}{x+3} - \dfrac{4}{x+5}$.

$$\frac{5}{x+3} - \frac{4}{x+5}$$

$$= \frac{5(x+5) - 4(x+3)}{(x+3)(x+5)}$$

$$= \frac{5x + 25 - 4x - 12}{(x+3)(x+5)}$$ (Notice the importance of inserting brackets so as to obtain the proper signs.)

$$= \frac{x + 13}{(x+3)(x+5)}.$$

EXERCISE CXIV

Simplify the following:

1. $\frac{1}{x+2}+\frac{2}{x+1}$.

2. $\frac{2}{x+2}-\frac{1}{x+1}$.

3. $\frac{8}{x+4}+\frac{3}{x+2}$.

4. $\frac{8}{x+4}-\frac{3}{x+2}$.

5. $\frac{3}{x+1}+\frac{2}{x+2}+\frac{1}{x+1}$.

6. $\frac{2}{x+3}+\frac{3}{x+1}$.

7. $\frac{2}{x+3}-\frac{3}{x+1}$.

8. $\frac{a}{x+1}+\frac{a}{x+2}$.

9. $\frac{a}{x+1}-\frac{a}{x+2}$.

10. $\frac{x+1}{x+2}+\frac{x+2}{x+1}$.

11. $\frac{x+1}{x+2}-\frac{x+2}{x+1}$.

12. $\frac{1}{x^2-y^2}+\frac{2}{x+y}$.

13. $\frac{a}{a+b}+\frac{b}{a-b}+\frac{a+b}{a^2-b^2}$.

14. $\frac{m-n}{m}-\frac{m}{m-n}$.

15. $\frac{1}{3x^2-8x+5}+\frac{2}{6x^2-x-15}$.

16. $\frac{1}{3x^2-8x+5}-\frac{2}{6x^2-x-15}$.

17. $\frac{x-y}{x+y}-\frac{x+y}{x-y}+\frac{x}{x^2-y^2}$.

18. $\frac{5}{2x^2-5x-3}+\frac{4}{6x^2-x-2}$.

19. $\frac{5}{2x^2-5x-3}-\frac{4}{6x^2-x-2}$.

20. $\frac{1}{b-c}+\frac{1}{c-a}+\frac{1}{a-b}$.

104. Multiplication and division of fractions follow exactly the corresponding operations in arithmetic. Again it is important to use brackets and to remember that the contents of a pair of brackets must be considered as *one* term, and consequently must be treated as a single term if any 'cancelling' is done.

EXAMPLE 1

Simplify $\dfrac{6}{x^2-1} \times \dfrac{x+1}{15}$.

$$\frac{6}{x^2-1} \times \frac{x+1}{15}$$

$$= \frac{\overset{2}{\cancel{6}}}{\cancel{(x+1)}(x-1)} \times \frac{\cancel{(x+1)}}{\underset{5}{\cancel{15}}}$$

(Note that $(x+1)$ is a factor in both numerator and denominator and is cancelled in exactly the same way as the 6 and 15 are cancelled by 3.)

$$= \frac{2}{5(x-1)}.$$

EXAMPLE 2

Simplify $\dfrac{x^2+y^2}{x^2-y^2} \div \dfrac{x-y}{x+y}$.

$$\frac{x^2+y^2}{x^2-y^2} \div \frac{x-y}{x+y}$$

$$= \frac{x^2+y^2}{x^2-y^2} \times \frac{x+y}{x-y}$$

(Note that ÷ sign is replaced by × sign and that the following fraction is inverted.)

$$= \frac{x^2+y^2}{\cancel{(x+y)}(x-y)} \times \frac{\cancel{(x+y)}}{x-y}$$

(Note that $x^2-y^2 = (x+y)(x-y)$, and that $(x+y)$ is cancelled in both numerator and denominator.)

$$= \frac{x^2+y^2}{(x-y)(x-y)}$$

$$= \frac{x^2+y^2}{(x-y)^2}.$$

EXERCISE CXV

1. Simplify the following:

(a) $\dfrac{10}{x^2 + 2x + 1} \times \dfrac{x + 1}{15}$.

(b) $\dfrac{a + x}{x^2 - 2x + 1} \div \dfrac{a - x}{x - 1}$.

(c) $\dfrac{x^2 - 1}{x^2 + 1} \div \dfrac{x + 1}{x - 1}$.

(d) $\dfrac{x^2 + x}{15x} \times \dfrac{5x^2}{x + 1}$.

(e) $\dfrac{a^3 - 1}{a^2 + a + 1} \times \dfrac{a}{a^2 - 1}$.

(f) $\dfrac{m^3 + n^3}{m^3 - n^3} \div \dfrac{m + n}{m - n}$.

(g) $\dfrac{3x^2 - 8x + 5}{6x^2 - x - 15}$.

(h) $\dfrac{4x^2 - x - 3}{8x^2 + 2x - 3}$.

(i) $\dfrac{14x^2 - 17x - 6}{35x^2 + 17x + 2}$.

(j) $\dfrac{a^3 + 1}{a^2 + 1} \div \dfrac{a^2 - a + 1}{a - 1}$.

(k) $\dfrac{a^2 + 3a + 1}{a^2 - 2a + 1} \times \dfrac{a - 1}{a + 3} \div \dfrac{a + 1}{a + 3}$.

(l) $\dfrac{x^2 + x - 6}{x^2 - x - 6} \times \dfrac{x^2 - 4}{x^2 - 9}$.

(m) $\dfrac{3x + 5}{x + 1} \div \dfrac{12x^2 + 23x + 5}{5x^2 + 2x - 3}$.

(n) $\dfrac{\dfrac{1}{x} + \dfrac{1}{y}}{\dfrac{1}{x} - \dfrac{1}{y}}$.

(o) $\dfrac{\dfrac{x - a}{x + a}}{\dfrac{x + a}{x^2 - a^2}}$.

(p) $\dfrac{1 + \dfrac{1}{x}}{1 + \dfrac{1}{y}} \times \dfrac{1 - \dfrac{1}{x}}{1 - \dfrac{1}{y}}$.

2. If $a = \dfrac{1}{1-x}$ and $b = \dfrac{1}{1-y}$, find the value of $\dfrac{a+b}{a-b}$ in terms of x and y.

3. Find the value of $\dfrac{x^2+1}{x^2-1}$ if $\dfrac{1}{1-a}$ is substituted for x.

4. Simplify

(a) $\dfrac{\dfrac{1}{a^3} - \dfrac{1}{b^3}}{\dfrac{1}{a^3} + \dfrac{1}{b^3}} \div \dfrac{\dfrac{1}{a} - \dfrac{1}{b}}{\dfrac{1}{a} + \dfrac{1}{b}}$.

(b) $\dfrac{x^6 - y^6}{x^4 - y^4}$.

105. It is a well-known property of fractions that the numerator and denominator can be multiplied by the *same* number without altering the value of the fraction. Thus

$$\frac{3}{4} = \frac{3 \times 7}{4 \times 7} = \frac{21}{28}.$$

More generally we can write:

$$\frac{x}{y} = \frac{kx}{ky}.$$

We shall now consider an important application of this principle. The process of calculating the value of $\dfrac{2}{\sqrt{3}}$ involves a laborious division of decimals, and the accuracy of the answer depends upon the number of places of decimals which are taken as the value of $\sqrt{3}$. For a high degree of accuracy it is necessary to take $\sqrt{3}$ to several places of decimals, but this will cause the division sum of $\dfrac{2}{\sqrt{3}}$ to be a lengthy and difficult task. This calculation is greatly simplified by multiplying the numerator and denominator of $\frac{2}{\sqrt{3}}$ by $\sqrt{3}$, as follows:

$$\begin{aligned} \frac{2}{\sqrt{3}} &= \frac{2 \times \sqrt{3}}{\sqrt{3} \times \sqrt{3}} \\ &= \frac{2 \times \sqrt{3}}{3} \\ &= \frac{2 \times 1{\cdot}732 \ldots}{3} \\ &= 1{\cdot}154 \end{aligned}$$

Division by 3 is much simpler than division by $\sqrt{3}$. This process, by which the denominator is cleared of awkward square roots, is called *rationalizing the denominator.* The following examples are some further illustrations.

EXAMPLE 1

Find the value of $\frac{3}{\sqrt{5}}$.

$$\begin{aligned} \frac{3}{\sqrt{5}} &= \frac{3 \times \sqrt{5}}{\sqrt{5} \times \sqrt{5}} \\ &= \frac{3\sqrt{5}}{5} \\ &= \frac{3 \times 2{\cdot}236}{5} \\ &= 1{\cdot}34. \end{aligned}$$

EXAMPLE 2

Find the value of $\frac{5}{\sqrt{7}}$.

$$\frac{5}{\sqrt{7}} = \frac{5 \times \sqrt{7}}{\sqrt{7} \times \sqrt{7}}$$

$$= \frac{5\sqrt{7}}{7}$$

$$= \frac{5 \times 2{\cdot}646}{7}$$

$$= 1{\cdot}89.$$

EXERCISE CXVI

Find the value of

1. $\frac{1}{\sqrt{2}}$.
2. $\frac{3}{\sqrt{7}}$.
3. $\frac{5}{\sqrt{8}}$.
4. $\frac{2}{\sqrt{3}} + \frac{3}{\sqrt{5}}$.
5. $\frac{3}{\sqrt{7}} + \frac{5}{\sqrt{3}}$.
6. $\frac{1}{\sqrt{2}} + \frac{2}{\sqrt{3}} + \frac{3}{\sqrt{4}}$.
7. $\frac{3}{\sqrt{6}} + \frac{4}{\sqrt{8}}$.
8. $\frac{a}{\sqrt{x}}$ if $x = 2$.
9. $k\sqrt{\frac{m}{n}}$ if $k = 3$, $m = 1$, and $n = 5$.
10. $\frac{7}{\sqrt{6}} - \frac{6}{\sqrt{5}}$.

106. When the denominator of a fraction consists of two (or more) terms the process of rationalization depends upon the relationship.

$$(x + y)(x - y) = x^2 - y^2.$$

Suppose $x = \sqrt{5}$ and $y = \sqrt{3}$.

Then $(x + y)(x - y) = (\sqrt{5} + \sqrt{3})(\sqrt{5} - \sqrt{3})$

$= (\sqrt{5})^2 - (\sqrt{3})^2$

$= 5 - 3$, which is free from square-root signs.

The following examples illustrate the process of rationalization of the denominator in such cases:

EXAMPLE 1

Find the value of $\frac{3}{5 - \sqrt{3}}$.

$$\frac{3}{5-\sqrt{3}} = \frac{3(5+\sqrt{3})}{(5-\sqrt{3})(5+\sqrt{3})}$$

$$= \frac{3(5+\sqrt{3})}{25-3}$$

$$= \frac{3(5+\sqrt{3})}{22}$$

$$= \frac{3 \times 6{\cdot}732}{22}$$

$$= 0{\cdot}918.$$

(Both numerator and denominator are multiplied by $5 + \sqrt{3}$. We thus obtain a denominator of the form $(x + y)(x - y)$, in which $x = 5$ and $y = \sqrt{3}$. This product is $x^2 - y^2$, which must be free from square-root signs.)

EXAMPLE 2

Find the value of $\frac{\sqrt{5}}{\sqrt{7} + \sqrt{3}}$.

$$\frac{\sqrt{5}}{\sqrt{7}+\sqrt{3}} = \frac{\sqrt{5}(\sqrt{7}-\sqrt{3})}{(\sqrt{7}+\sqrt{3})(\sqrt{7}-\sqrt{3})}$$

$$= \frac{\sqrt{5}(\sqrt{7}-\sqrt{3})}{7-3}$$

$$= \frac{\sqrt{35}-\sqrt{15}}{4}$$

$$= \frac{5{\cdot}916 - 3{\cdot}873}{4}$$

$$= 0{\cdot}51.$$

(In this case we have to multiply numerator and denominator by $\sqrt{7} - \sqrt{3}$ in order to obtain a product of the form $(x + y)(x - y)$. The rest of the calculation follows as before.)

EXERCISE CXVII

Find the value of the following to two places of decimals:

1. $\dfrac{1}{1+\sqrt{2}}$.
2. $\dfrac{3}{2-\sqrt{3}}$.
3. $\dfrac{1}{\sqrt{2}+\sqrt{3}}$.
4. $\dfrac{\sqrt{2}}{\sqrt{3}+\sqrt{5}}$.
5. $\dfrac{\sqrt{3}}{\sqrt{5}-\sqrt{3}}$.
6. $\dfrac{\sqrt{2}+1}{\sqrt{2}-1}$.
7. $\dfrac{\sqrt{2}-1}{\sqrt{2}+1}$.
8. $\dfrac{\sqrt{3}+\sqrt{2}}{\sqrt{3}-\sqrt{2}}$.
9. $\dfrac{\sqrt{3}-\sqrt{2}}{\sqrt{3}+\sqrt{2}}$.
10. $\dfrac{\sqrt{x}}{\sqrt{x}-\sqrt{a}}$.

107. Before concluding this chapter on the study of fractions we must digress for a moment and consider two kinds of statement which appear in algebra.

$$2x = 10.$$
$$(x + y)(x - y) = x^2 - y^2.$$

The first of these statements—namely, $2x = 10$—*is true only when* $x = 5$. For other values of x, $2x$ is *not* equal to 10. Hence this statement is true only in the *one* case in which $x = 5$. Now consider the second statement—namely, $(x + y)(x - y) = x^2 - y^2$. This is true for *all* values of x and y.

The first of these two statements is called an *equation*; the second is called an *identity*, and is sometimes written

$$(x + y)(x - y) \equiv x^2 - y^2,$$

the sign $\equiv$ being used to show that the two expressions which it separates are not only equal, but are equal for *all* values of the letters concerned. Now, this can only be so when the two expressions are really the same although written in different forms. Hence to establish the truth of an identity it is necessary to show that either one side can be transformed into the other, or that both sides can be transformed into the same expression.

EXAMPLE 1

Prove that $(x + y)^2 - (x - y)^2 \equiv 4xy$.

$$\begin{aligned}(x + y)^2 - (x - y)^2 &\equiv (x^2 + 2xy + y^2) - (x^2 - 2xy + y^2)\\ &\equiv x^2 + 2xy + y^2 - x^2 + 2xy - y^2\\ &\equiv 4xy.\end{aligned}$$

EXAMPLE 2

Prove that $(x - y)(a - b) \equiv (y - x)(b - a)$.

Left-hand side

$$= (x - y)(a - b) \equiv xa - ya - xb + yb.$$

Right-hand side

$$= (y - x)(b - a) \equiv yb - xb - ya + xa.$$

left-hand side $\equiv$ right-hand side

EXERCISE CXVIII

Establish the following identities:

1. $a - b \equiv -(b - a)$.
2. $\dfrac{-1}{1 - x} \equiv \dfrac{1}{x - 1}$.
3. $(x + y)^2 + (x - y)^2 \equiv 2(x^2 + y^2)$.
4. $x(y - z) + y(z - x) + z(x - y) \equiv 0$.
5. $x^2(y - z) + y^2(z - x) + z^2(x - y) \equiv yz(y - z) + zx(z - x) + xy(x - y)$.
6. $\dfrac{b + c}{(a - b)(a - c)} + \dfrac{c + a}{(b - c)(b - a)} + \dfrac{a + b}{(c - a)(c - b)} \equiv 0$.
7. $s + (s - a) + (s - b) + (s - c) \equiv a + b + c$ where $s = \frac{1}{2}(a + b + c)$.
8. $(x - y)(x^2 + xy + y^2) \equiv x^3 - y^3$.
9. $(x + y)(x^2 - xy + y^2) \equiv x^3 + y^3$.

10. $a^3 + 3a^2b + 3ab^2 + b^3 \equiv (a + b)^3$.

11. $a^3 - 3a^2b + 3ab^2 - b^3 \equiv (a - b)^3$.

12. $a^2 + b^2 + c^2 + 2ab + 2bc + 2ca \equiv (a + b + c)^2$.

13. $a^3 + b^3 + c^3 - 3abc \equiv (a + b + c)(a^2 + b^2 + c^2 - bc - ca - ab)$.

14. $(y - z)^2 + (z - x)^2 + (x - y)^2 \equiv 2(x^2 + y^2 + z^2) - 2(yz + zx + xy)$.

15. $s^2 + (s - a)^2 + (s - b)^2 + (s - c)^2 \equiv a^2 + b^2 + c^2$ where $s = \frac{1}{2}(a + b + c)$.

16. $(x-1)(3x-4) \equiv (2x + 1)(x-7) + 6(x + 1) + 5 + x^2$.

17. $(3a + b)(4a - 3b) + (2a - b)(a - 2b) \equiv (2a - 3b)(4a + b) + 2(3a^2 + b^2)$.

18. $(x - y)(y - z)(z - x) \equiv x(y^2 - z^2) + y(z^2 - x^2) + z(x^2 - y^2)$.

19. $(x + y)(y + z)(z + x) \equiv (x + y + z)(xy + yz + zx) - xyz$.

20. $(s - a)^3 + (s - b)^3 + (s - c)^3 \equiv s^3 - 3abc$ where $s = \frac{1}{2}(a + b + c)$.

108. Since both sides of an identity must be the same, even if expressed differently, it follows that we can compare coefficients of corresponding terms. Thus, if

$$(x - 3)(x - 4) \equiv Ax^2 + Bx + C$$

then, since $(x - 3)(x - 4) \equiv x^2 - 7x + 12$, it follows that

$$A = 1, B = -7, \text{ and } C = 12.$$

Study the following example carefully.

If $6x - 11 \equiv A(3x - 2) + B(2x + 1)$, find A and B.

$$\begin{aligned} 6x - 11 &\equiv A(3x - 2) + B(2x + 1) \\ &\equiv 3Ax - 2A + 2Bx + B \\ &\equiv 3Ax + 2Bx - 2A + B \\ &\equiv (3A + 2B)x - (2A - B). \end{aligned}$$ (Notice the minus sign inside the brackets.)

Comparing coefficients,

$$3A + 2B = 6 \quad (1)$$
$$2A - B = 11 \quad (2)$$

Here are two simultaneous equations for A and B.
Multiplying equation (2) by 2, we have

$$4A - 2B = 22 \qquad (3)$$

Adding equations (1) and (3), we have

$$7A = 28.$$
$$\therefore A = 4.$$

By substitution $\qquad B = -3.$
Hence $6x - 11 \equiv 4(3x - 2) - 3(2x + 1)$.

Alternative Method

$6x - 11 \equiv A(3x - 2) + B(2x + 1)$.

Since this is an identity, it is true for *all* values of x. Therefore it is true if $x = 1$.

If $x = 1$ we have, on substitution of this value,

$$6 - 11 = A(3 - 2) + B(2 + 1).$$
$$A + 3B = -5.$$

Since the identity is true for all values of x, it is true if $x = 2$. If $x = 2$ we have, on substitution of this value,

$$12 - 11 = A(6 - 2) + B(4 + 1).$$
$$4A + 5B = 1.$$

Hence we have the two simultaneous equations

$$A + 3B = -5 \qquad (1)$$
$$4A + 5B = 1 \qquad (2)$$

Multiplying equation (1) by 4, we have

$$4A + 12B = -20 \qquad (3)$$
$$4A + 5B = 1 \qquad (2)$$

Subtracting equation (2) from equation (3),

$$7B = -21.$$
$$B = -3.$$

Substituting $B = -3$ in equation (1), we have

$$A = 4.$$

EXERCISE CXIX

1. Find A and B if

(*a*) $5x + 1 \equiv A(x - 1) + B(x + 1)$.
(*b*) $2x - 13 \equiv A(x - 2) + B(x + 1)$.
(*c*) $12 - 5x \equiv A(2x + 3) + B(x - 5)$.
(*d*) $7a - b - c \equiv A(a + b + c) + B(a - b - c)$.
(*e*) $22x - 23 \equiv A(3x - 7) + B(2x + 1)$.
(*f*) $3x - 46 \equiv A(5 - 2x) + B(8 - x)$.
(*g*) $-15x \equiv A(18 + x) + B(3 + x)$.
(*h*) $9(x - 2y) \equiv A(2x - y) + B(x + 4y)$.
(*i*) $8x + 21y \equiv A(x - 3y) + B(2x + 3y)$.
(*j*) $2x - 5y \equiv A(3x - 2y) + B(x + 3y)$.

2. Find A, B, C, and D if

(*a*) $(x - 7)(x + 2) \equiv Ax^2 + Bx + C$.
(*b*) $(2x + 7)(3x - 2) \equiv Ax^2 + Bx + C$.
(*c*) $(x + 1)(x + 2)(x + 3) \equiv Ax^3 + Bx^3 + Cx + D$.
(*d*) $(x - 1)(x - 2)(x - 3) \equiv Ax^3 + Bx^2 + Cx + D$.
(*e*) $(2x + 1)(2x - 1)(x + 1) \equiv Ax^3 + Bx^2 + Cx + D$.

109. We can now continue our study of fractions by a consideration of the following example.

If $\dfrac{5x + 12}{(x + 2)(x + 3)} \equiv \dfrac{A}{x + 2} + \dfrac{B}{x + 3}$, find A and B.

$$\frac{5x + 12}{(x + 2)(x + 3)} \equiv \frac{A}{x + 2} + \frac{B}{x + 3}$$
$$\equiv \frac{A(x + 3) + B(x + 2)}{(x + 2)(x + 3)}.$$

$$5x + 12 \equiv A(x + 3) + B(x + 2)$$
$$\equiv (A + B)x + (3A + 2B).$$

Comparing coefficients,

$$A + B = 5 \qquad (1)$$
$$3A + 2B = 12 \qquad (2)$$

Multiplying equation (1) by 2, we have

$$2A + 2B = 10 \qquad (3)$$

Subtracting equation (3) from equation (2), we have

$$A = 2.$$

By substitution in equation (1)

$$B = 3.$$

Therefore $$\frac{5x + 12}{(x + 2)(x + 3)} \equiv \frac{2}{x + 2} + \frac{3}{x + 3}.$$

We have thus expressed a fraction as the sum of two simpler fractions, each of which has as its denominator a factor of the denominator of the original fraction. These fractions are called *partial fractions.*

EXERCISE CXX

1. Complete the following partial fractions:

(*a*) $$\frac{3x + 4}{(x + 1)(x + 2)} \equiv \frac{A}{x + 1} + \frac{B}{x + 2}.$$

(*b*) $$\frac{3(x + 1)}{(x - 2)(x + 7)} \equiv \frac{A}{x - 2} + \frac{B}{x + 7}.$$

(*c*) $$\frac{7x - 23}{(x - 3)(x - 4)} \equiv \frac{A}{x - 3} + \frac{B}{x - 4}.$$

(*d*) $$\frac{11x - 2}{(2x + 1)(x - 2)} \equiv \frac{A}{2x + 1} + \frac{B}{x - 2}.$$

(*e*) $$\frac{16(x - 1)}{(3x - 5)(5x - 3)} \equiv \frac{A}{3x - 5} + \frac{B}{5x - 3}.$$

(*f*) $$\frac{x + 9}{(x - 1)(2x + 3)} \equiv \frac{A}{x - 1} + \frac{B}{2x + 3}.$$

(*g*) $$\frac{7(x - 1)}{(2x + 1)(3x - 2)} \equiv \frac{A}{2x + 1} + \frac{B}{3x - 2}.$$

(*h*) $\dfrac{17x + 11}{(5x + 2)(x - 1)} \equiv \dfrac{A}{5x + 2} + \dfrac{B}{x - 1}.$

(*i*) $\dfrac{5x^2 - x - 2}{x(x - 1)(x + 1)} \equiv \dfrac{A}{x} + \dfrac{B}{x - 1} + \dfrac{C}{x + 1}.$

(*j*) $\dfrac{2x + 1}{(x - 1)^2} \equiv \dfrac{A}{x - 1} + \dfrac{B}{(x - 1)^2}.$

2. Find A, B, and C so that the following statements may be identities:

(*a*) $\dfrac{2x + 5}{x + 1} \equiv A + \dfrac{B}{x + 1}.$

(*b*) $\dfrac{5x - 16}{x - 3} \equiv A + \dfrac{B}{x - 3}.$

(*c*) $\dfrac{2(7x - 6)}{2x - 3} \equiv A + \dfrac{B}{2x - 3}.$

(*d*) $\dfrac{3x^2 + 2x - 2}{x^3 - 1} \equiv \dfrac{A}{x - 1} + \dfrac{Bx + C}{x^2 + x + 1}.$

(*e*) $\dfrac{2x^2 + 2x - 3}{(x + 1)^2 (x - 2)} \equiv \dfrac{A}{x + 1} + \dfrac{B}{(x + 1)^2} + \dfrac{C}{x - 2}.$

EXERCISE CXXI (Revision Exercise)

(A)

1. Simplify

(*a*) $\dfrac{3}{2x - 5} - \dfrac{4}{x - 1} + \dfrac{7}{3x + 4}.$

(*b*) $\dfrac{x + 1}{x + 2} + \dfrac{x + 3}{x + 4}.$

2. Calculate the value of

$$\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{5}}.$$

3. Solve the following equations:

(a) $(x + 3)(2x - 3) - 12 = (x - 3)(2x + 3)$.

(b) $7x + 3y = -5$.
$3x - 7y = 31$.

4. One bag contains x marbles and another bag contains y marbles. How many must be taken out of the second bag and put in the first so that there will be twice as many in the first bag as there are in the second?

5. Find three consecutive even numbers whose sum is 54.

(B)

6. Find by rationalizing the denominator the values of

$$(a)\ \frac{2}{2+\sqrt{3}}. \quad (b)\ \frac{2}{2-\sqrt{3}}.$$

7. A number consists of two digits. If the digits are reversed a second number is formed. This is 27 more than the first number, and the sum of the two numbers is 55. Find the first number.

8. Say what numerical values must be given to A and B in the following identities:

(a) $5(1 - x) \equiv A(2 - x) + B(1 + 2x)$.

$$(b)\ \frac{2(x+7)}{(x-3)(3x+1)} \equiv \frac{A}{x-3} + \frac{B}{3x+1}.$$

9. Using the Remainder Theorem (or otherwise), find the factors of $x^3 + 3x^2 - 4x - 12$.

10. The sum of two numbers is 11. The difference of their squares is 33. Find the numbers.

(C)

11. Simplify

$$\frac{p^2 - q^2}{p^3 + q^3} \times \left(\frac{p+q}{p-q}\right)^2 \div \frac{p+q}{p^2 - pq + q^2}.$$

12. From the following simultaneous equations find x and y in terms of a and b:

$$ax + by = a^2 + b^2.$$
$$bx + ay = 2ab.$$

13. A fraction becomes $\frac{5}{7}$ if 1 is added to the numerator and $\frac{2}{3}$ if 1 is subtracted from the denominator. Find the fraction.

14. Find the square root of $4x^4 + 4x^3 - 11x^2 - 6x + 9$.

15. Prove that if $s \equiv \frac{1}{2}(a + b + c)$

(a) $(s - a)(2s - a) + (s - b)(2s - b) + (s - c)(2s - c)$
$\equiv a^2 + b^2 + c^2$.

(b) $$\frac{1}{(s-b)(s-c)} + \frac{1}{(s-c)(s-a)} + \frac{1}{(s-a)(s-b)}$$
$$\equiv \frac{s}{(s-a)(s-b)(s-c)}.$$

(D)

16. Find by rationalizing the denominator the value of

$$\frac{\sqrt{3}}{\sqrt{7} - \sqrt{2}}.$$

17. Find the factors of

(a) $(x + y)^3 + (x - y)^3$. (b) $(x + y)^3 - (x - y)^3$.

18. At what times between 6 and 7 o'clock are the hands of a clock coincident?

19. Find A, B, and C if

(a) $(x + 2)(3x - 2) \equiv Ax^2 + Bx + C$.

(b) $\dfrac{3x + 17}{x + 7} \equiv A + \dfrac{B}{x + 7}$.

20. Find the square root of $x^4 - 2x^3 + x^2 - 2x + 2 + \dfrac{1}{x^2}$.

CHAPTER XI

QUADRATIC EQUATIONS

110. In Fig. 51 we have the graph of $y = x^2$ drawn over the range $x = -3$ to $x = +3$. The table of values from which the graph is drawn is as follows:

x	-3	-2	-1	0	$+1$	$+2$	$+3$
y	$+9$	$+4$	$+1$	0	$+1$	$+4$	$+9$

Notice that y is always positive, and consequently its graph is always above the x axis in the first and second quadrants. This curve is called a *parabola*.

EXERCISE CXXII

(*It will be an advantage to use squared paper for this exercise.*)

1. Draw the graphs of $y = x^2$, $y = 2x^2$, and $y = 5x^2$ on the same figure and using the same scale. Describe the graph $y = mx^2$ when (*a*) m is small, (*b*) m is large.

2. Draw the graphs $y = x^2 + 2$, $y = x^2 + 5$, and $y = x^2 - 3$ on the same figure and using the same scale. Describe the graph $y = x^2 + c$ when (*a*) c is positive, (*b*) c is negative, (*c*) c is zero.

3. Draw the graph of $y = -2x^2$. (This is the mirror graph of $y = 2x^2$.)

4. Draw the graph of $x = y^2$.

5. Compare the graph of Question 4 with the graph of $y = x^2$ in Fig. 51. Both these graphs may be used for

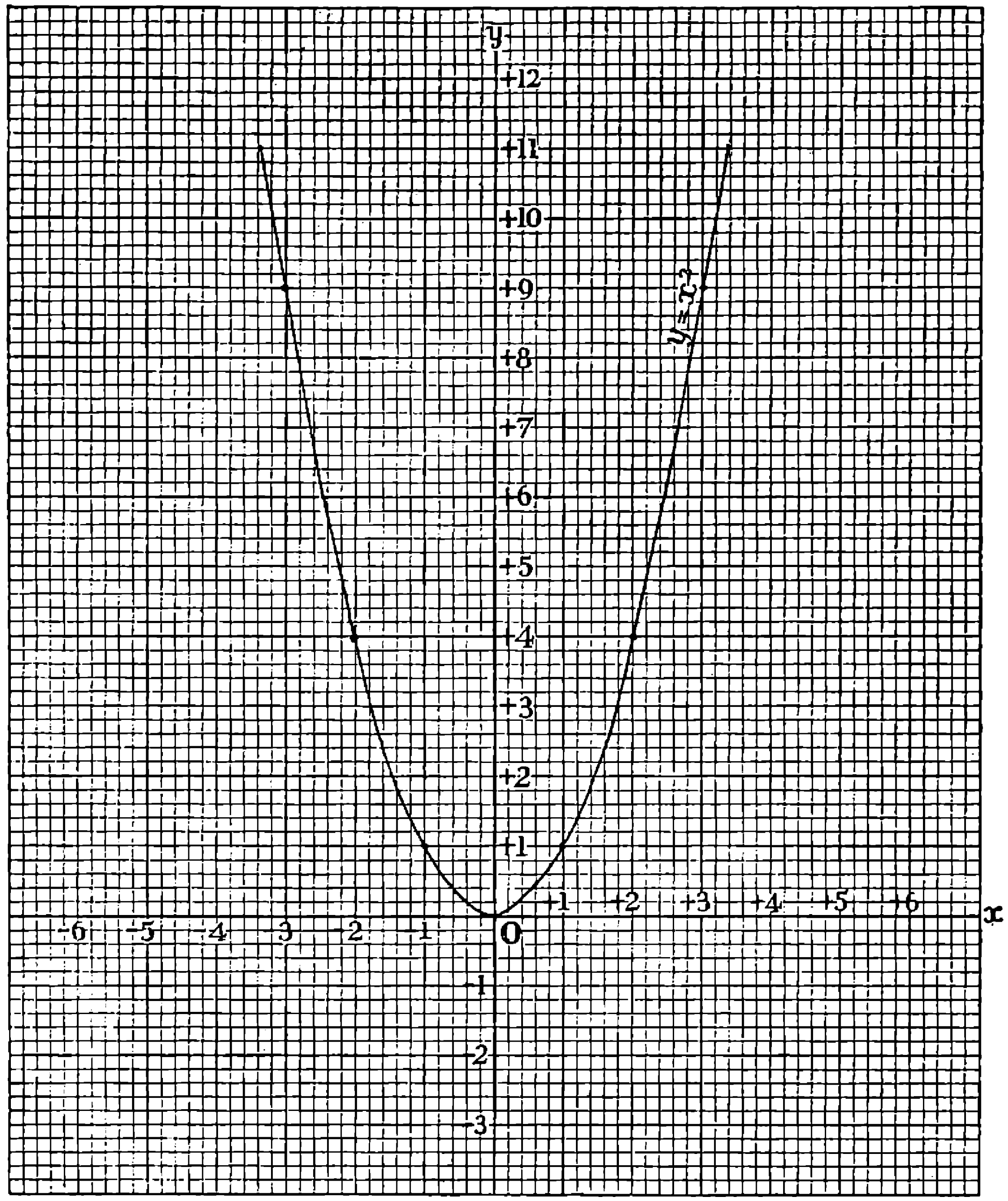

FIG. 51

finding either the squares or the square roots of numbers. Use them to find the values of

(a) $(1{\cdot}5)^2$, $({\cdot}75)^2$, and $(2{\cdot}5)^2$.

(b) $\sqrt{1{\cdot}5}$, $\sqrt{{\cdot}75}$, and $\sqrt{2{\cdot}5}$.

Verify by calculation in each case.

6. From the graph of $y = x^2 - 3$ (see Question 2) find the

minimum value that y can have. For what value of x does y have this minimum value?

7. Draw the graph of $y = 3 - x^2$. Find the maximum value that y can have. For what value of x does y have this maximum value?

8. By studying the graph of $y = x^2 - 3$ (Question 6) state if y can have a maximum value.

9. If $y = 3 - x^2$ (Question 7), can y have a minimum value?

10. Verify from the graphs you have drawn in this exercise that the *shape* of the curves at a minimum is concave *upward*, and in the case of a maximum that it is *concave* downward.

111. If we draw on the same figure (and using the same scale) the graphs of $y = x^2$ and $y = 5x - 6$ we shall find that the graph of $y = 5x - 6$ is a straight line cutting the parabola $y = x^2$ in two points, A and B (Fig. 52). At each of these two points we can write

$$x^2 = 5x - 6.$$

The corresponding values of x at these points of intersection are $x = 2$, for A, and $x = 3$, for B.

If $x = 2$, then $x^2 = 4$

and $5x - 6 = 10 - 6 = 4.$

Therefore $\mathbf{x = 2}$ is a solution of the equation $\mathbf{x^2 = 5x - 6}$.

If $x = 3$, then $x^2 = 9$

and $5x - 6 = 15 - 6 = 9.$

Therefore $\mathbf{x = 3}$ is a solution of the equation $\mathbf{x^2 = 5x - 6}$.

Hence the equation $x^2 = 5x - 6$ has **two** solutions—namely, $x = 2$ and $x = 3$; and these are given by the x co-ordinates of the two points of intersection of the parabola $y = x^2$ and the straight line $y = 5x - 6$.

An equation in which the unknown appears as a square is called an equation of the *second degree*, or a *quadratic equation*. Hence $x^2 = 5x - 6$ is a quadratic equation.

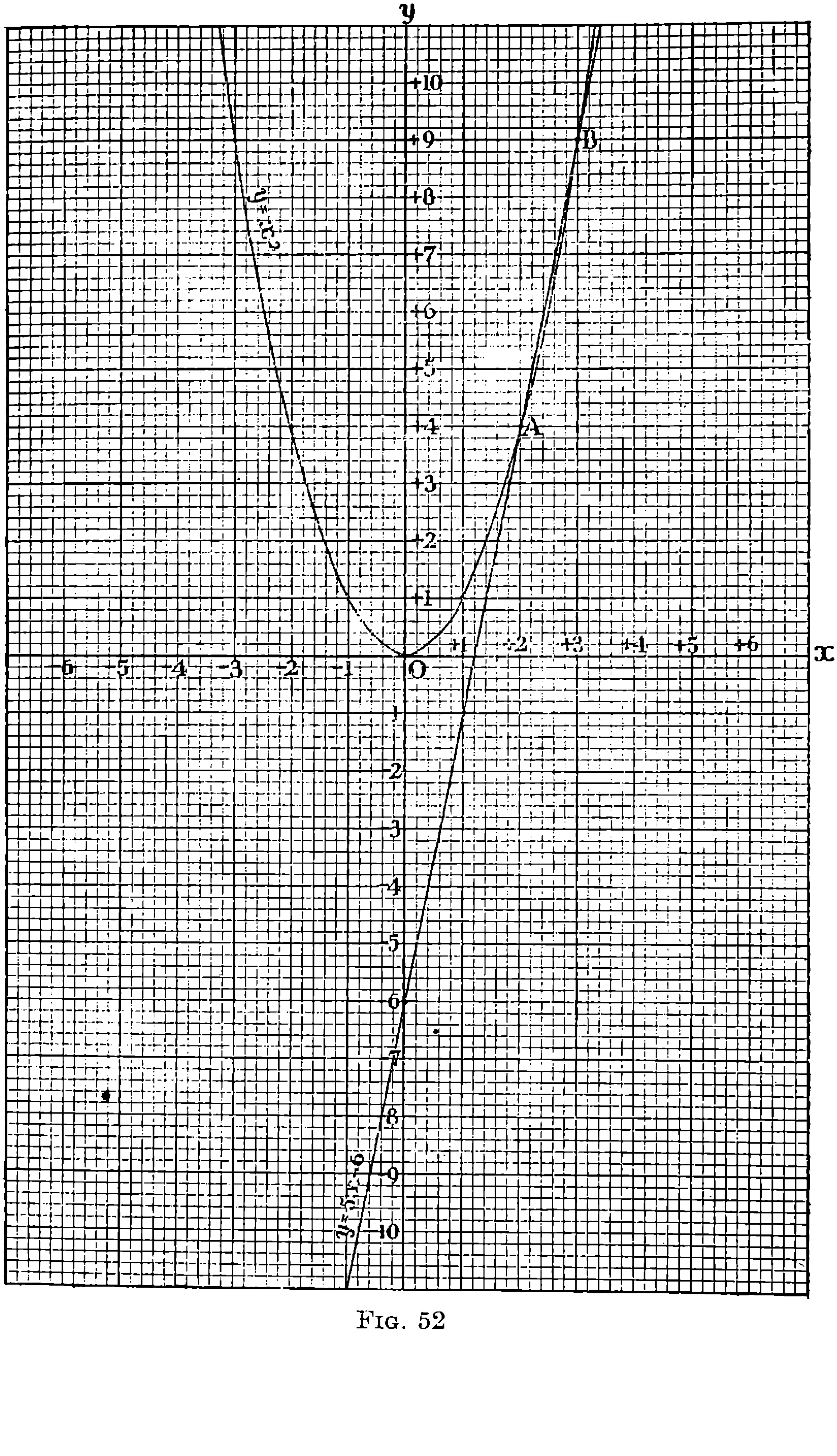

Fig. 52

EXERCISE CXXIII

(It will be an advantage to use squared paper for this exercise.)

1. Solve graphically, as in Fig. 52, the following quadratic equations and *verify by substitution that the x co-ordinates of the two points of intersection in each case satisfy the original equation*:

(*a*) $x^2 = 3x - 2$. (*d*) $x^2 = 8 - 2x$.
(*b*) $x^2 = 6x - 8$. (*e*) $x^2 = 15 - 2x$.
(*c*) $x^2 = x + 6$.

2. Can a straight line cut a parabola in more than two points?

3. Can a straight line cut a parabola in less than two points? Give illustrations.

4. Why cannot the equation $x^2 = 2x - 3$ be solved graphically by the method described above?

112. We have considered the quadratic equation $x^2 = 5x - 6$, and found that its two solutions are $x = 2$ and $x = 3$. We may rewrite this equation in the form

$$x^2 - 5x + 6 = 0.$$

If we draw the graph of $y = x^2 - 5x + 6$, then the co-ordinates of the points where it cuts the x axis are the solutions of the equation $x^2 - 5x + 6 = 0$. The following is the table of values:

Values of x	-2	-1	0	$+1$	$+2$	$+3$	$+4$	$+5$
$y = x^2 - 5x + 6$	20	12	6	2	0	0	2	6

Notice that the graph is parabolic in shape (Fig. 53) and cuts the x axis in the two points $+2$ and $+3$, which are the two solutions of the equation $x^2 - 5x + 6 = 0$. Following the method we adopted in Chapter VIII, we can read off from this graph the solutions of other quadratic equations. Thus the straight line $y = 2$ cuts this parabola in two points, P and Q, whose x co-ordinates are $+1$ and $+4$. These are the solutions of the equation

$$x^2 - 5x + 6 = 2.$$

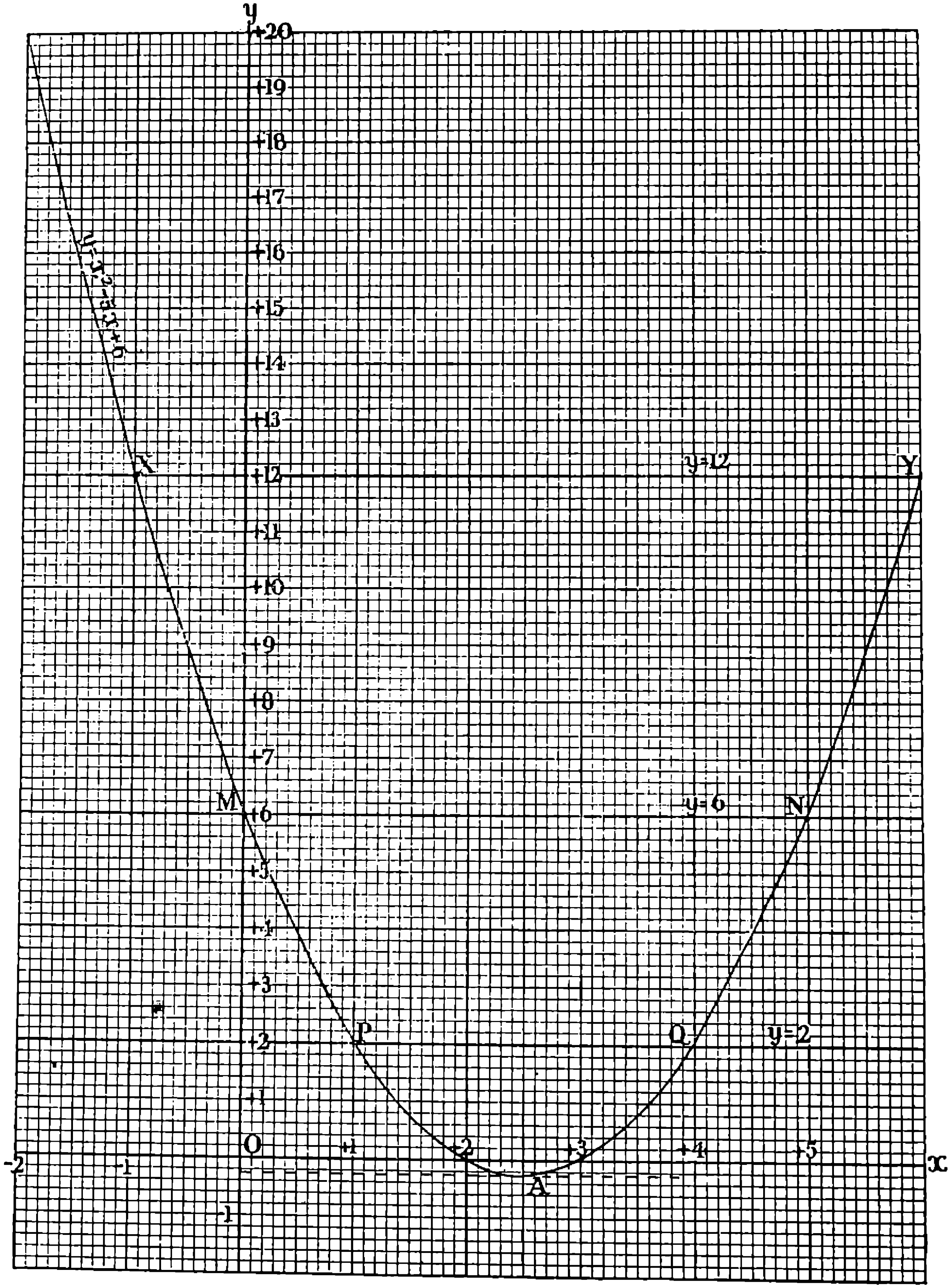

FIG. 53

Similarly, the straight line $y = 6$ cuts the parabola in M and N, whose x co-ordinates are 0 and $+ 5$. These are the solutions of the equation

$$x^2 - 5x + 6 = 6.$$

Again, the straight line $y = 12$ cuts the parabola in X and Y, whose x co-ordinates are $- 1$ and $+ 6$. These are the solutions of the equation

$$x^2 - 5x + 6 = 12.$$

Notice the point A. This is called the *vertex* of the parabola. Its y co-ordinate is $- \frac{1}{4}$, and this is the smallest value that the expression $x^2 - 5x + 6$ can have. For this reason A is sometimes called a *minimum* turning-point, and we say that $x^2 - 5x + 6$ has a minimum value when $x = + 2\frac{1}{2}$, for when $x = 2\frac{1}{2}$ then $y = \quad \frac{1}{4}$.

EXERCISE CXXIV

(It will be an advantage to use squared paper for this exercise.)

1. Draw the graph of $y = x^2 - 3x + 2$, and from it read off the solutions of the following equations:

(a) $x^2 - 3x + 2 = 0.$
(b) $x^2 - 3x + 2 = 2.$
(c) $x^2 - 3x + 2 = 6.$

What is the minimum value of $x^2 - 3x + 2$? What value of x gives this minimum?

2. Draw the graph of $y = x^2 - 6x + 8$, and from it read off the solutions of the following equations:

(a) $x^2 - 6x + 8 = 0.$
(b) $x^2 - 6x + 8 = 3.$
(c) $x^2 - 6x + 8 = 8.$

For what value of x is $x^2 - 6x + 8$ a minimum? What is this minimum?

3. Draw the graph of $y = x^2 - 2x - 3$, and from it read off the solutions of the following equations:

(a) $x^2 - 2x - 3 = 0$.
(b) $x^2 - 2x - 3 = 5$.
(c) $x^2 - 2x - 3 = -3$.

For what value of x is $x^2 - 2x - 3$ a minimum? What is this minimum?

4. Draw the graph of $y = 2 - x - x^2$, and hence solve the equation $2 - x - x^2 = 0$.

What is the maximum value of $2 - x - x^2$? What value of x gives this maximum?

5. Draw the graph of $y = 8 - 2x - x^2$, and hence solve the equation $8 - 2x - x^2 = 0$.

For what value of x is $8 - 2x - x^2$ a maximum? What is this maximum?

6. For what range of values of x is $x^2 - 3x + 2$ negative? (See Question 1.)

7. For what range of values of x is $x^2 - 6x + 8$ negative? (See Question 2.)

8. For what range of values of x is $8 - 2x - x^2$ positive? (See Question 5.)

113. We have seen that the two solutions of the equation $x^2 - 5x + 6 = 0$ are $x = 2$ and $x = 3$.

We can obtain this result by another method.

If $$x^2 - 5x + 6 = 0$$
then $$(x - 2)(x - 3) = 0.$$

Here we have the product of two factors equal to zero. Consequently one of them must be zero—*i.e.*, either $x - 2 = 0$ or $x - 3 = 0$.

If $x - 2 = 0$, then $x = 2$.

If $x - 3 = 0$, then $x = 3$.

As we have seen by our graphical methods, these are the two solutions of the quadratic equation $x^2 - 5x + 6 = 0$. This method can be extended to equations of higher degrees, as will be shown in the following examples.

EXAMPLE 1

Solve the equation $2x^2 + x - 6 = 0$.

$$2x^2 + x - 6 = 0.$$
$$(2x - 3)(x + 2) = 0.$$
$$\therefore \text{ either } 2x - 3 = 0$$
$$\text{or } x + 2 = 0.$$

If $2x - 3 = 0$, then $2x = 3$ and $x = \frac{3}{2}$.
If $x + 2 = 0$, then $x = -2$.

EXAMPLE 2

Solve the equation $x^3 + 4x^2 + x - 6 = 0$.

By the Remainder Theorem we can factorize the left-hand side and write $(x - 1)(x + 2)(x + 3) = 0$.

Hence if $x - 1 = 0$, then $x = 1$;
if $x + 2 = 0$, then $x = -2$;
if $x + 3 = 0$, then $x = -3$.

The *three* solutions of the *cubic* equation $x^3 + 4x^2 + x - 6 = 0$ are $x = 1$, -2, and -3. These can be verified by substitution, and also demonstrated by the graphical method already described. The graph of $y = x^3 + 4x^2 + x - 6$ is shown in Fig. 54. The points where it cuts the x axis give the three solutions of the cubic equation

$$x^3 + 4x^2 + x - 6 = 0.$$

EXAMPLE 3

Construct a quadratic equation whose two solutions are $x = +2$ and $x = -\frac{5}{3}$.

If $x = +2$, then $x - 2 = 0$.
If $x = -\frac{5}{3}$, then $x + \frac{5}{3} = 0$.
To obtain the quadratic equation we must write

$$(x - 2)\left(x + \tfrac{5}{3}\right) = 0.$$

I.e., $x^2 - \frac{1}{3}x - \frac{10}{3} = 0$ (working out the product).

$\therefore 3x^2 - x - 10 = 0$ (multiplying by 3, so as to clear the fractions).

This is the required quadratic equation.

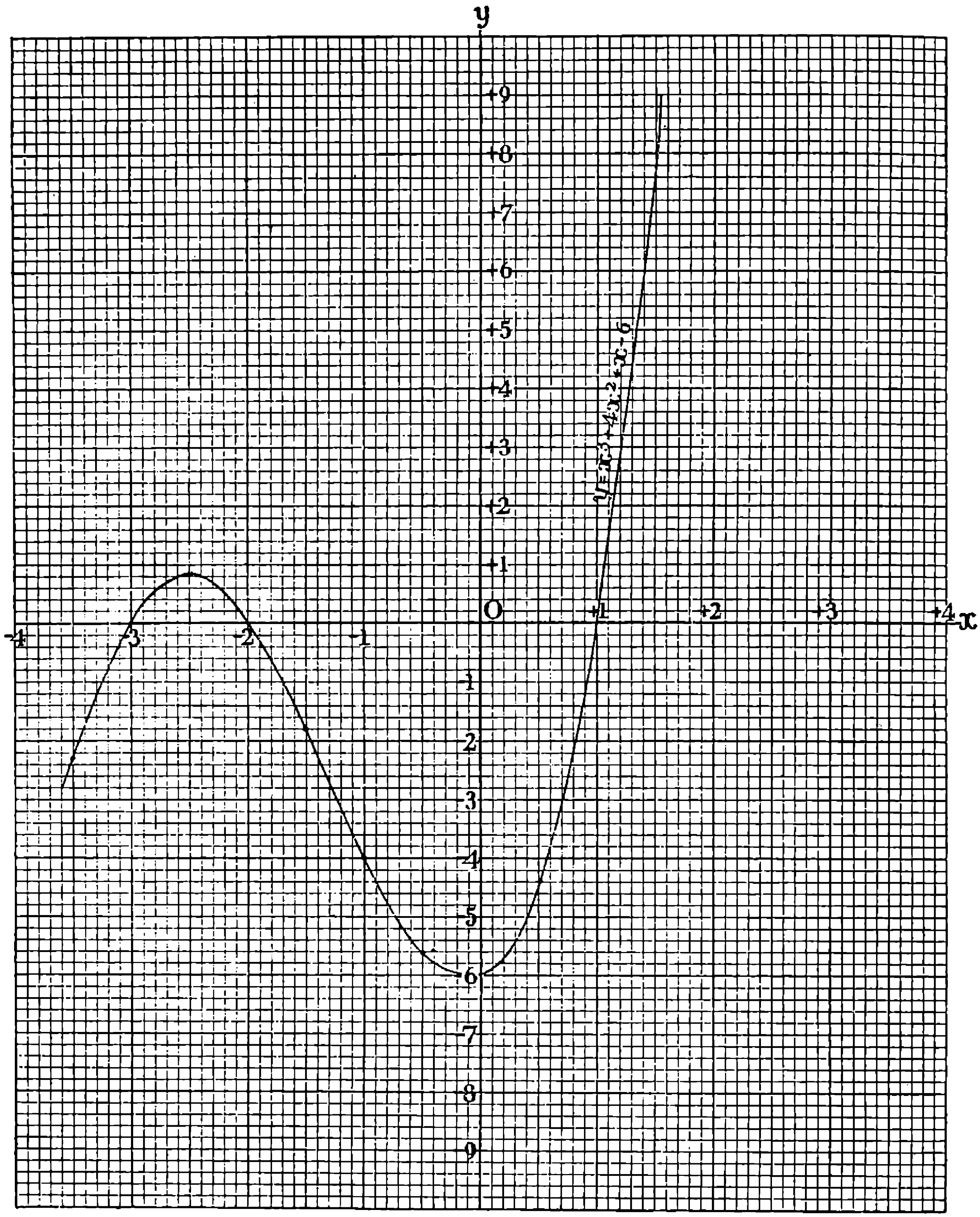

Fig. 54

EXERCISE CXXV

1. Find by factorizing the solutions of the following equations:

(a) $x^2 - 3x + 2 = 0$.
(b) $x^2 - x - 6 = 0$.
(c) $x^2 + 4x - 21 = 0$.
(d) $2x^2 - 5x + 2 = 0$.
(e) $2x^2 + x - 6 = 0$.
(f) $3x^2 - 2x - 21 = 0$.
(g) $x^2 - 16x + 63 = 0$.
(h) $10x^2 - 13x - 3 = 0$.
(i) $3x^2 + 10x - 25 = 0$.
(j) $2x^2 + 5x + 3 = 0$.
(k) $2x^2 - 11x - 40 = 0$.
(l) $2x^2 - 21x + 40 = 0$.
(m) $7x^2 + 6x - 1 = 0$.
(n) $4x^2 - 5x - 6 = 0$.
(o) $12x^2 - 17x + 6 = 0$.
(p) $x^3 - 3x^2 + 2x = 0$.
(q) $x^3 + 4x^2 - 21x = 0$.
(r) $x^3 + 6x^2 + 11x + 6 = 0$.
(s) $x^3 - x^2 - x + 1 = 0$.
(t) $6x^3 - 3x^2 - 10x + 5 = 0$.

2. Construct in their simplest form equations whose solutions are:

(a) $x = -2, x = 7$.
(b) $x = 1, x = -5$.
(c) $x = \frac{1}{2}, x = \frac{1}{3}$.
(d) $x = a, x = b$.
(e) $x = \frac{1}{7}, x = -\frac{1}{6}$.
(f) $x = \dfrac{1}{p}, x = \dfrac{1}{q}$.
(g) $x = 2, x = 14$.
(h) $x = -1\frac{2}{3}, x = 5$.
(i) $x = 1\frac{1}{2}, x = 4\frac{1}{4}$.
(j) $x = -3, x = -13$.
(k) $x = 1, x = 2, x = 3$.
(l) $x = -4, x = 1, x = -2$.
(m) $x = a, x = b, x = c$.
(n) $x = \frac{1}{2}, x = -\frac{1}{3}, x = \frac{1}{4}$.

3. Solve the following equations:

(a) $x^2 + 8x + 15 = 0$.
(b) $(x - 1)^2 - 5(x - 1) + 6 = 0$.
(c) $(2x - 1)^2 - 5(2x - 1) + 4 = 0$.
(d) $(x + 3)^2 + (x + 3) - 2 = 0$.
(e) $(x - 5)^2 + 2(x - 5) = 8$.
(f) $(3x + 5)^2 - 5(3x + 5) = 6$.
(g) $(2x + 1)^2 - 5(2x + 1) + 6 = 0$.
(h) $(3x + 2)^2 - 3(3x + 2) = 4$.
(i) $(3x + 2)^2 - 5(3x + 2) + 4 = 0$.
(j) $(4x + 1)^2 - 6(4x + 1) + 8 = 0$.

EXERCISE CXXVI (Revision Exercise)

(A)

1. Solve the following equations:

(a) $3x - 2y = 31.$
$8x - 5y = 81.$

(b) $2x^2 - 13x - 24 = 0.$

2. Form a quadratic equation whose solutions are (a) 2 and 3, (b) $\sqrt{2}$ and $\sqrt{3}$.

3. A sum of £2 10s. is made up of an equal number of florins, half-crowns, and sixpences. How many of each kind of coin are there?

4. What values must be given to P and Q so that $3x^4 - 2x^3 - 10x^2 + Px + Q$ may be divisible by $x^2 + x - 3$? What will be the quotient?

5. State what values must be given to A and B so that

(a) $5(x - 4) \equiv A(3x - 4) + B(2x - 1).$

(b) $$\frac{7x - 19}{(x - 3)(2x - 5)} \equiv \frac{A}{x - 3} + \frac{B}{2x - 5}.$$

(B)

6. Find the square root of $9x^4 - 6x^3 + 19x^2 - 6x + 9$.

7. Calculate by rationalizing the denominator the value of

$$\frac{2}{\sqrt{3} - \sqrt{2}} + \frac{3}{\sqrt{5} - \sqrt{3}}.$$

8. Find the factors of

(a) $1 - 729a^6.$

(b) $45x^2 + 117x - 8.$

(c) $10x^3 - 35x^2 - 6x + 21.$

9. Construct a quadratic equation whose two solutions are $x = a + b$ and $x = a - b$.

10. One of the solutions of the cubic equation $6x^3 - 11x^2 - 19x = 6$ is $x = 3$. Find the other two solutions.

(C)

11. Simplify $(3\sqrt{2} + \sqrt{3})(\sqrt{2} - \sqrt{3})$.

12. Find the factors of $(2x - 5)^2 + (2x - 5) - 6$, and hence solve the equation

$$(2x - 5)^2 + (2x - 5) = 6.$$

13. If $2p - 1$ is the middle one of five consecutive odd numbers, write down the others in terms of p.

14. In the previous question, if the sum of the numbers is 85 find them.

15. A certain number of persons at a concert are seated on benches of equal size. If there were 5 fewer benches it would be necessary for each bench to accommodate 2 more persons. If there were 15 more benches the number of persons on each bench could be reduced by 3. Find the number of people at the concert and the number of persons on each bench.

(D)

16. Prove that if $x + y + z = 0$, then $x^3 + y^3 + z^3 = 3xyz$.

17. Find p and q so that the solutions of the equation $x^2 + px + q = 0$ may be $x = 7$ and $x = 6$.

18. Construct a quadratic equation whose solutions are three times those of the equation $4x^2 - 8x = 5$.

19. Make f the subject of the formula $s = ut + \frac{1}{2}ft^2$.

20. Find two consecutive numbers such that the sum of their squares is 145.

CHAPTER XII

QUADRATIC EQUATIONS—*continued*

114. We must now consider the question of solving a quadratic equation when it cannot be factorized—e.g., $x^2 + 8x + 2 = 0$. As there are no factors of $x^2 + 8x + 2$, we write

$$x^2 + 8x = -2.$$

To both sides of this equation we add the number which will make the left-hand side a perfect square. This will be *the square of half the coefficient of x*. The coefficient of x is 8. Hence half this coefficient is 4, and the addition of $(4)^2$ to $x^2 + 8x$ makes a perfect square.

$$\begin{aligned} x^2 + 8x + (4)^2 &= x^2 + 8x + 16 \\ &= (x + 4)^2. \end{aligned}$$

Now, adding $(4)^2$ to both sides of the original equation, we have

$$x^2 + 8x + (4)^2 = -2 + (4)^2.$$

I.e.,

$$\begin{aligned} (x + 4)^2 &= -2 + 16 \\ &= 14. \end{aligned}$$

Taking the square root of both sides, we have

$$x + 4 = \pm \sqrt{14}. \text{ (Notice the } \pm \text{ signs.)}$$

$$\therefore x = -4 \pm \sqrt{14}.$$

Hence the two roots of the quadratic equation are $x = -4 + \sqrt{14}$ and $x = -4 - \sqrt{14}$—i.e., $x = -\cdot 258$ and $-7\cdot 742$.

In the next exercise by completing the squares in question 1 the corresponding quadratic equations in Question 2 can be solved by the method just described.

EXERCISE CXXVII

1. Say what must be added to each of the following in order to make a perfect square:

(a) $x^2 + 2x$.
(b) $x^2 + 4x$.
(c) $x^2 + 6x$.
(d) $p^2 + 8p$.
(e) $m^2 - 10m$.
(f) $k^2 + 11k$.
(g) $y^2 - 13y$.
(h) $a^2 + 5a$.
(i) $g^2 - 7g$.
(j) $k^2 - 12k$.

2. Solve the following equations by the process of "completing the square":

(a) $x^2 + 2x = 1$.
(b) $x^2 + 4x = 3$.
(c) $x^2 + 6x = 5$.
(d) $p^2 + 8p + 2 = 0$.
(e) $m^2 - 10m + 9 = 0$.
(f) $k^2 + 11k - 5 = 0$.
(g) $y^2 - 13y + 8 = 0$.
(h) $a^2 + 5a + 6 = 0$.
(i) $g^2 - 7g - 9 = 0$.
(j) $k^2 - 12k + 20 = 0$.

115. In the previous exercise the coefficient of the squared term in each of the equations is unity. It is only in such cases that the process of "completing the square" can be applied. Consequently the first step in solving a quadratic equation such as $3x^2 + 9x - 5 = 0$ is to divide throughout by 3, so as to reduce it to this form. The solution is then effected by the same process as before. Consider the following example.

Solve the equation $3x^2 + 9x - 5 = 0$.

$$3x^2 + 9x = 5.$$

$$x^2 + 3x = \frac{5}{3}$$ (dividing by 3, so as to make the coefficient of x^2 unity).

$$x^2 + 3x + \left(\frac{3}{2}\right)^2 = \frac{5}{3} + \left(\frac{3}{2}\right)^2$$ ("completing the square" on the left-hand side by adding the square of half the coefficient of x—this must be added to the other side of the equation as well).

I.e., $$\left(x + \frac{3}{2}\right)^2 = \frac{5}{3} + \frac{9}{4}$$

$$= \frac{47}{12}.$$

$$\therefore x + \frac{3}{2} = \pm\sqrt{\frac{47}{12}}$$ (taking the square root of both sides of the equation).

$$\therefore x = -\frac{3}{2} \pm \sqrt{\frac{47}{12}}.$$

Hence the two solutions of the equation $3x^2 + 9x - 5 = 0$ are

$$x = -\frac{3}{2} + \sqrt{\frac{47}{12}} = +0{\cdot}479$$

and $$x = -\frac{3}{2} - \sqrt{\frac{47}{12}} = -3{\cdot}479.$$

EXERCISE CXXVIII

1. Solve the following quadratic equations:

(*a*) $2x^2 + 5x - 3 = 0$.
(*b*) $5x^2 + x - 1 = 0$.
(*c*) $2x^2 - 5x = 6$.
(*d*) $3x^2 - x - 4 = 0$.
(*e*) $2x^2 - 7x = 15$.
(*f*) $4x^2 + 7x - 1 = 0$.
(*g*) $6x^2 + 5x = 1$.
(*h*) $2x^2 + 5x - 2 = 0$.
(*i*) $10x^2 + 19x = 2$.
(*j*) $3x^2 - 4x - 5 = 0$.
(*k*) $7x^2 + x - 3 = 0$.
(*l*) $2x^2 + 6x - 1 = 0$.
(*m*) $4x^2 + 7x + 2 = 0$.
(*n*) $2x^2 - x - 1 = 0$.
(*o*) $3x^2 - 2x - 1 = 0$.

2. Prove that $x^2 - 6x + 11 = (x - 3)^2 + 2$, and hence find its minimum value.

Hint. $(x - 3)^2$ must be *always* positive. (Why?) Therefore the smallest value it can have is zero, and it has this value when $x = 3$. Hence the minimum value of $x^2 - 6x + 11$ occurs when $x = 3$, and when $x = 3$ then $x^2 - 6x + 11 = 2$, which is its smallest value. This can be verified graphically.

3. Find the minimum value of $x^2 + 4x + 1$ and verify the result graphically.

4. For what value of x does $2x^2 - 8x + 3$ have its minimum value? What is this value?

Hint. $2x^2 - 8x + 3 = 2(x - 2)^2 - 5$.

5. Find the minimum value of $3x^2 + 24x + 38$ and the value of x which produces it.

6. Find the maximum value of $1 - x^2 + 4x$.

Hint. $1 - x^2 + 4x = 1 - (x-2)^2 + 4 = 5 - (x-2)^2$.
Hence the maximum value of $1 - x^2 + 4x$ occurs when $x = 2$. (Why?) Verify this graphically.

7. Find the maximum value of $-6 - x^2 - 6x$ and the value of x which produces it.

8. Find the maximum value of $6 - x^2 + 4x$ and the value of x which produces it.

9. Find the maximum value of $5 - 3x^2 - 6x$ and the value of x which produces it.

Hint. $5 - 3x^2 - 6x = 8 - 3(x+1)^2$.

10. Find the maximum value of $7 - 18x^2 + 24x$.

11. Find the minimum value of $18x^2 - 24x + 7$.

116. We are now in a position to consider the quadratic equation $ax^2 + bx + c = 0$, which we shall proceed to solve by the method of "completing the square" already described.

$$ax^2 + bx + c = 0.$$

$$x^2 + \frac{b}{a}x = -\frac{c}{a}$$ (dividing throughout by a, so as to make the coefficient of x^2 unity, and also putting $\frac{c}{a}$ on the right-hand side of the equation).

$$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = \left(\frac{b}{2a}\right)^2 - \frac{c}{a}$$ (adding $\left(\frac{b}{2a}\right)^2$ to both sides of the equation).

I.e., $$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a}.$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}.$$

$$\left(x + \frac{b}{2a}\right) = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}}$$ (taking the square root of both sides of the equation).

$$\left(x + \frac{b}{2a}\right) = \frac{\pm\sqrt{b^2 - 4ac}}{2a}.$$

$$\therefore x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}.$$

$$\therefore \mathbf{x} = \frac{-\mathbf{b} \pm \sqrt{\mathbf{b}^2 - 4\mathbf{ac}}}{2\mathbf{a}}.$$

We have thus found the two solutions of the quadratic equation in terms of the coefficients a, b, and c. Since any values can be given to these letters, this result is a formula which can be used to solve any quadratic.

The formula $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ *is most important, and must be remembered.* The following examples show how it can be used to solve quadratic equations.

EXAMPLE 1

Solve the equation $2x^2 - 5x - 7 = 0$.

In this case $a = 2$, $b = -5$, and $c = -7$.

$$\therefore x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-(-5) \pm \sqrt{(-5)^2 - 4.2.(-7)}}{2.2}$$

$$= \frac{5 \pm \sqrt{25 + 56}}{4}$$

$$= \frac{5 \pm \sqrt{81}}{4}$$

$$= \frac{5 \pm 9}{4}$$

Taking the + sign, $x = \frac{5 + 9}{4} = \frac{14}{4} = 3\frac{1}{2}$.

Taking the − sign, $x = \frac{5 - 9}{4} = -\frac{4}{4} = -1$.

These are the two solutions of the equation $2x^2-5x-7=0$. They can be verified, either by direct substitution or by solving the equation by the factor method.

EXAMPLE 2

Solve the equation $2x^2-10x+1=0$.

In this case $a=2$, $b=-10$, and $c=1$.

$$\therefore\ x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$=\frac{-(-10)\pm\sqrt{(-10)^2-4.2.1}}{2.2}$$

$$=\frac{10\pm\sqrt{100-8}}{4}$$

$$=\frac{10\pm\sqrt{92}}{4}.$$

Taking the + sign, $x=\dfrac{10+9{\cdot}592}{4}=\dfrac{19{\cdot}592}{4}=4{\cdot}898.$

Taking the − sign, $x=\dfrac{10-9{\cdot}592}{4}=\dfrac{0{\cdot}408}{4}=0{\cdot}102.$

Hence the two solutions of the quadratic equation $2x^2-10x+1=0$ are $x=4{\cdot}898$ and $x={\cdot}102$, correct to three places of decimals.

EXERCISE CXXIX

Solve the following quadratic equations by the formula method:

1. $x^2+2x-1=0$.
2. $3x^2+x-4=0$.
3. $2x^2-13x+15=0$.
4. $6x^2+5x=1$.
5. $2x^2+x-4=0$.
6. $3x^2-5x+2=0$.
7. $4x^2-9x-3=0$.
8. $2x^2+x-1=0$.
9. $3x^2=5x+7$.
10. $4x^2=x+1$.

11. $3x^2 - x = 4$.
12. $2x^2 - 7x = 15$.
13. $x^2 + 2x - 3 = 0$.
14. $2x^2 - 3x - 4 = 0$.
15. $4x^2 - 3x = 0$.
16. $3a^2 - a - 1 = 0$.
17. $2p^2 - 2p - 12 = 0$.
18. $m^2 + m - 1 = 0$.
19. $2k^2 - k - 3 = 0$.
20. $3t^2 + 2t - 1 = 0$.

117. Consider the equation $x^2 - x + 2 = 0$. Solving this by the formula method, we have

$$a = 1,\ b = -1, \text{ and } c = 2.$$

Hence
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$= \frac{-(-1) \pm \sqrt{(-1)^2 - 4.1.2}}{2.1}$$
$$= \frac{1 \pm \sqrt{1-8}}{2}$$
$$= \frac{1 \pm \sqrt{-7}}{2}.$$

We cannot find $\sqrt{-7}$. There is no number which will give -7 when it is squared, and consequently we describe $\sqrt{-7}$ as an *imaginary number*, and say that the solutions of the equation $x^2 - x + 2 = 0$ are imaginary. Fig. 55 shows the graph of $y = x^2 - x + 2$. Notice that it does not cut the x axis like the graph described in Fig. 53. Thus the solutions of a quadratic equation may be either *real* or *imaginary*. They are imaginary when they contain the square root of a negative number, as in the example described above, and it is easy to see that this will be so when $b^2 < 4ac$.[1] If $b^2 > 4ac$ then the solutions of the quadratic equation are real. Turn again to the solutions of Exercise CXXVIII. All of them are real, but in some cases the square root could be evaluated exactly, while in other cases it could not, and the answer had to be left with the root sign still in it or

[1] The sign $<$ stands for the words "is less than," and the sign $>$ stands for the words "is greater than."

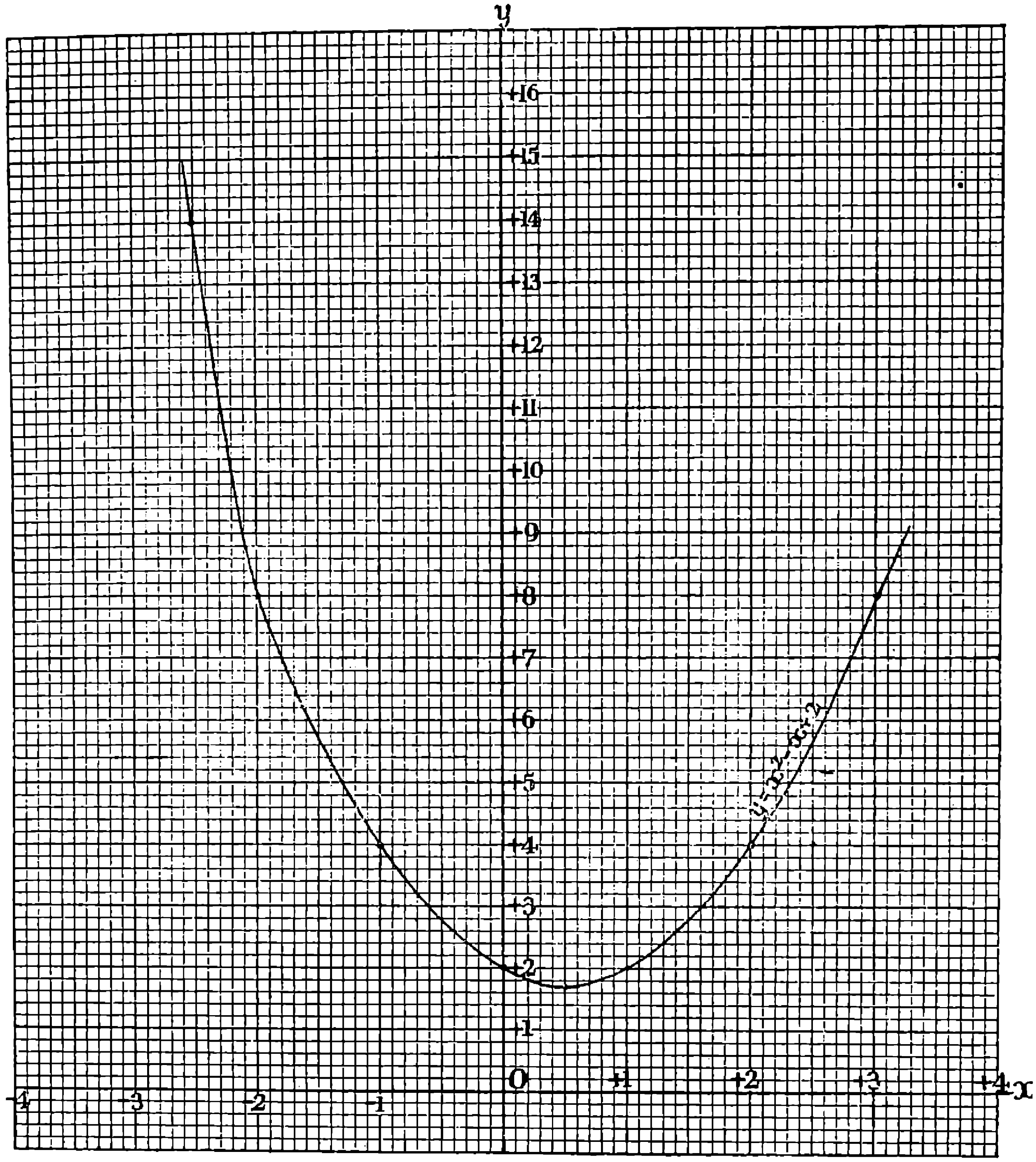

FIG. 55

worked out approximately to a number of places of decimals. Solutions of a quadratic equation in which the square root can be evaluated exactly—*i.e.*, in which $b^2 - 4ac$ is a perfect square—are said to be *rational*, while solutions in which $b^2 - 4ac$ is *not* a perfect square are called *irrational*. We may summarize these facts as follows:

SOLUTIONS OF QUADRATIC EQUATIONS

Real ($b^2 > 4ac$)		Imaginary ($b^2 < 4ac$)
Rational ($b^2 - 4ac$ a perfect square)	Irrational ($b^2 - 4ac$ not a perfect square)	

It is quite clear from this that the expression $b^2 - 4ac$, which occurs in the formula $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, tells us the kind of roots the quadratic equation has. For this reason it is called the *discriminant.*

EXERCISE CXXX

By calculating the discriminant $b^2 - 4ac$ describe the solutions of the following quadratic equations—*i.e.*, say if they are real or imaginary; and if they are real state also whether they are rational or irrational.

1. $x^2 + 2x + 3 = 0.$
2. $2x^2 - x - 1 = 0.$
3. $3x^2 + 5x - 2 = 0.$
4. $2x^2 - 3x + 4 = 0.$
5. $5x^2 - x - 1 = 0.$
6. $2x^2 = 3x + 1.$
7. $2x^2 = 3x - 1.$
8. $3x^2 - 7x + 2 = 0.$
9. $x^2 - 5x + 2 = 0.$
10. $3x^2 + 5x - 3 = 0.$
11. $2p^2 - p = 4.$
12. $p = 5p^2 + 2.$
13. $m^2 + m + 1 = 0.$
14. $2q^2 - 5q - 1 = 0.$
15. $a^2 - 2a + 3 = 0.$

118. Before considering problems which involve quadratic equations it is necessary to devote some space to equations which involve fractions. The manipulation of algebraical fractions has already been considered, on pp. 190–194, and so few new ideas will arise. As a start, consider the equation

$$\frac{x+1}{x+2} = \frac{2x-1}{3x-2}.$$

The L.C.M. of the denominators of these two fractions is $(x+2)(3x-2)$. Multiplying both sides of the equation by this L.C.M., we have

$$\frac{x+1}{x+2} \times (x+2)(3x-2) = \frac{2x-1}{3x-2} \times (x+2)(3x-2).$$

This simplifies to

$$(x+1)(3x-2) = (2x-1)(x+2).$$

I.e., $$3x^2 + x - 2 = 2x^2 + 3x - 2.$$

I.e., $$x^2 - 2x = 0.$$

$$x(x-2) = 0.$$

either $x = 0$ or $x - 2 = 0$.

$x = 0$ and $x = 2$ are the solutions of the equation.

Notice that the multiplication by the L.C.M. of the denominators frees the equation from fractions. Study the following examples carefully.

EXAMPLE 1

Solve the equation $\frac{9}{2x+1} - \frac{2}{3x-8} = \frac{1}{2}$.

The L.C.M. of the denominators is $2(2x+1)(3x-8)$. Multiplying each term by this, we have

$$\frac{9}{2x+1} \times 2(2x+1)(3x-8) - \frac{2}{3x-8} \times 2(2x+1)(3x-8)$$
$$= \tfrac{1}{2} \times 2(2x+1)(3x-8).$$

I.e., $$18(3x-8) - 4(2x+1) = (2x+1)(3x-8).$$

$$54x - 144 - 8x - 4 = 6x^2 - 13x - 8.$$ (Note the minus sign in -4.)

$$6x^2 - 59x + 140 = 0.$$

I.e., $$(x-4)(6x-35) = 0.$$

$\therefore$ either $x - 4 = 0$ or $6x - 35 = 0$.

Hence $x = 4$ and $\frac{35}{6}$.

EXAMPLE 2

Solve the equation $\frac{x+4}{x+3} + \frac{x-2}{x-1} = 1\frac{2}{3}$.

$$\frac{x+4}{x+3} + \frac{x-2}{x-1} = \frac{5}{3}.$$

(Note the mixed number $1\frac{2}{3}$ is written as the improper fraction $\frac{5}{3}$.)

Multiplying each term by the L.C.M. of the denominators, we have

$$\frac{x+4}{x+3} \times 3(x+3)(x-1) + \frac{x-2}{x-1} \times 3(x+3)(x-1)$$
$$= \frac{5}{3} \times 3(x+3)(x-1).$$

I.e., $3(x+4)(x-1) + 3(x-2)(x+3) = 5(x+3)(x-1)$.

I.e., $3(x^2+3x-4) + 3(x^2+x-6) = 5(x^2+2x-3)$.

$$3x^2 + 9x - 12 + 3x^2 + 3x - 18 = 5x^2 + 10x - 15.$$
$$\therefore x^2 + 2x - 15 = 0.$$
$$(x+5)(x-3) = 0.$$

Hence either $x + 5 = 0$ or $x - 3 = 0$.

I.e., $x = -5$ and $x = 3$ are the solutions of the equation.

EXERCISE CXXXI

Solve the following equations:

1. $\frac{3}{2x-1} = 1.$
2. $\frac{2x-1}{3x-5} = \frac{9}{10}.$
3. $\frac{1-4x}{2-3x} = 1\frac{1}{8}.$
4. $\frac{3}{2x-5} = \frac{4}{3x+1}.$
5. $\frac{7}{5x+3} = \frac{2}{3x-4}.$
6. $\frac{x+2}{x-3} = \frac{x-5}{x+4}.$
7. $\frac{4x+5}{2x-4} = \frac{2x-3}{x+4}.$
8. $\frac{x+1}{x+2} + \frac{x+3}{x+4} = 2.$
9. $\frac{3}{x-5} + \frac{4}{x-6} = \frac{7}{x+1}.$
10. $\frac{5x-2}{x+1} = \frac{4x-3}{2x-1}.$

11. $x + \frac{4}{x} = 5.$

12. $2x + \frac{5}{x} = 11.$

13. $\frac{4}{x-3} - \frac{3}{x-2} = \frac{1}{2x+1}.$

14. $\frac{5}{2x-1} + \frac{1}{x-2} = \frac{3}{x+4}.$

15. $\frac{x+7}{x+5} - \frac{2}{x} = \frac{x+2}{x+1}.$

16. $\frac{3x+4}{4x-1} - \frac{2x+3}{5x+1} = \frac{1}{3}.$

17. $\frac{x+1}{x} - \frac{x}{x+1} = \frac{2x+1}{8x}.$

18. $\frac{1-2x}{1-3x} - \frac{1-3x}{1-2x} = \frac{2-5x}{2x}.$

19. $\frac{5}{x-5} + \frac{6}{x-6} = \frac{6}{x^2-11x+30}.$

20. $\frac{3(2x-1)}{x+1} - \frac{2(x-1)}{x-1} = 1.$

21. $\frac{1-x}{1-3x} = 0{\cdot}4.$

22. $\frac{x-a}{x-b} - \frac{x-b}{x-a} = 0.$

23. $\frac{x}{2(3x-2)} = \frac{1}{x+2}.$

24. $\frac{4}{x^2-5x+6} - \frac{5}{x^2-7x+12} = \frac{3}{x^2-6x+8}.$

25. $\frac{1}{(x+1)(x+2)} + \frac{1}{3(x-1)(x+2)} = \frac{1}{2(x^2-1)}.$

119. Problems which involve the solution of quadratic equations are of frequent occurrence. When once the equation which describes them has been written down the solution can be effected by one of the methods described.

EXAMPLE 1

Find two positive consecutive numbers such that the sum of their squares is 145.

Let x be the smaller of the two numbers.

Then $x + 1$ is the second number.

Since the sum of their squares is 145, we have the equation

$$x^2 + (x + 1)^2 = 145.$$

I.e., $$x^2 + x^2 + 2x + 1 = 145.$$

I.e., $$2x^2 + 2x - 144 = 0.$$

$$x^2 + x - 72 = 0 \quad \text{(dividing throughout by 2).}$$

I.e., $$(x + 9)(x - 8) = 0.$$

$\therefore$ either $x + 9 = 0$ and $x = -9$,

or $x - 8 = 0$ and $x = 8$.

Since the problem restricts us to positive numbers only, we must select the solution $x = 8$. Hence the two consecutive numbers required are 8 and 9.

EXAMPLE 2

A tradesman bought a number of rings for £120. If he had bought 10 more for the same sum the cost of each ring would have been £1 less. Find the number of rings and the cost of each.

Let x be the number of rings.

Then £$\frac{120}{x}$ is the cost of each.

If there had been $x + 10$ rings the cost of each would have been £$\frac{120}{x + 10}$.

The difference of these two costs is £1. Hence the equation is

$$\frac{120}{x} - \frac{120}{x + 10} = 1.$$

I.e., $$\frac{120(x + 10) - 120x}{x(x + 10)} = 1.$$

$$120x + 1200 - 120x = x(x + 10).$$

I.e., $$x^2 + 10x - 1200 = 0.$$

$$(x - 30)(x + 40) = 0.$$

$$x = 30 \text{ or } -40.$$

The positive solution is the one required. Hence the tradesman purchased 30 rings, each of which cost him £4.

EXERCISE CXXXII

1. Find two consecutive even numbers, the sum of whose squares is 340.

2. The sum of two numbers is 20. The sum of their squares is 208. Find the numbers.

3. Divide 25 into two parts, whose product is 136.

4. The length of a rectangle exceeds its width by 7 inches. If the area is 228 square inches, find the dimensions of the rectangle.

5. A rectangular lawn has a perimeter of 40 yards. If the length is increased by 4 yards and the width is decreased by 2 yards, the area remains unaltered. Find the dimensions of the lawn.

6. A man bought a certain number of eggs for £5. If 50 of them were broken and the remainder sold at $\frac{1}{2}d.$ more than they cost, the profit made would be £1 5*s*. How many eggs were bought in the first place?

7. The perimeter of a right-angled triangle is 30 inches. If one of the sides is 5 inches, find the length of the hypotenuse.

8. The sum of the squares of three consecutive even numbers is 116. Find the numbers.

9. Three lines are of lengths 20 inches, 18 inches, and 11 inches. Equal lengths are cut off each, and the remainders form the sides of a right-angled triangle. What length was cut off?

10. One man can complete a piece of work in 2 days less

time than another. If the two men work together, they can do this work in $5\frac{5}{11}$ days. How long would each take to do it alone?

11. A passenger train travelling at 10 m.p.h. more than a goods train does a journey of 240 miles in two hours less time. Find the average speed of each.

12. If eggs are reduced by $\frac{1}{2}d.$ each, the number which can be purchased for a shilling is increased by 4. What was the original price of the eggs?

13. A sum of £1 10*s*. is divided equally among a number of boys. If the number of boys is reduced by 5, then each could receive one shilling more. How many boys were there?

14. The perimeter of a rectangular garden is 86 yards. If its area is 450 square yards find the length of its sides.

15. A man bought a number of shares for £1250. He kept 150 of them and sold the remainder for £1275, thereby making a profit of 5 shillings a share. How many shares did he buy, and what did each cost?

16. The sum of the reciprocals of two consecutive numbers is $\frac{29}{210}$. Find the numbers.

17. The product of the last two of five consecutive numbers exceeds the product of the first two by 36. Find the five numbers.

18. Two pipes fill a cistern in 5 minutes 50 seconds. If one of them takes 4 minutes less than the other to fill the cistern, find how long it takes each pipe singly.

19. A man rows downstream for 3 miles, turns round, and returns to his starting-point, the return journey taking an hour longer than the outer one. If the stream flows at 2 m.p.h. find the rate of rowing in still water.

20. At an auction sale one plot of land was sold for £880. Another plot, smaller than the first by 5 acres, was sold for £1050, with the result that the price per acre in the second case was £8 more than in the first. How large was the first plot?

120. Simultaneous equations in which one or more of them are quadratics can be solved, but the problem of eliminating the letters until one equation containing only one unknown is obtained is more difficult than when the equations are all linear. For this reason we shall confine our attention to the case of two simultaneous equations, *one* of which is quadratic, and leave the consideration of more complicated cases for a later stage in the study of the subject.

The general method of solution which is applicable to all such simultaneous equations is as follows:

(i) Express one unknown in terms of the other by means of the *linear* equation.

(ii) Substitute this value in the *quadratic* equation, thus obtaining a quadratic equation with *one* unknown.

The following examples illustrate the operation.

EXAMPLE 1

Solve the following equations:

$$x^2 + y^2 = 29 \text{ and } 2x + y = 9.$$

The linear equation is $2x + y = 9$.

$$\therefore y = 9 - 2x.$$

Substituting this value of y in the quadratic equation $x^2 + y^2 = 29$, we have

$$x^2 + (9 - 2x)^2 = 29.$$

I.e.,

$$x^2 + 81 - 36x + 4x^2 = 29.$$

$$\therefore 5x^2 - 36x + 52 = 0.$$

$$(5x - 26)(x - 2) = 0.$$

$$\text{either } 5x - 26 = 0 \text{ and } x = \frac{26}{5} = 5\tfrac{1}{5},$$

$$\text{or } x - 2 = 0 \text{ and } x = 2.$$

These *two* values of x have *two corresponding* values of y, which can be obtained by substituting them in turn in the linear equation $2x + y = 9$.

Since $y = 9 - 2x$, we have

(i) If $x = 5\frac{1}{5}$, then

$$y = 9 - 2 \times 5\tfrac{1}{5}$$
$$= -1\tfrac{2}{5}.$$

(ii) If $x = 2$, then

$$y = 9 - 2 \times 2$$
$$= 5.$$

Hence the solutions of the simultaneous equations are

$$x = 5\tfrac{1}{5} \text{ and } 2,$$
$$y = -1\tfrac{2}{5} \text{ and } 5.$$

These solutions are shown graphically in Fig. 56. The graph of $x^2 + y^2 = 29$ is a circle whose centre is at the origin and whose radius is $\sqrt{29}$. (Why?)

The graph of $2x + y = 9$ is a straight line. This cuts the circle in two points, A and B. The co-ordinates of A are 2, 5, and of B are $5\frac{1}{5}$, $-1\frac{2}{5}$, which are the two sets of solutions of the simultaneous equations.

Example 2

The perimeter of a right-angled triangle is 30 inches. If the hypotenuse is 13 inches find the other two sides.

Let the lengths of the other two sides be x inches and y inches respectively.

Then, by the theorem of Pythagoras, $x^2 + y^2 = 13^2 = 169$, and $x + y + 13 = 30$.

Hence the two simultaneous equations are

$$x^2 + y^2 = 169 \qquad (1)$$
$$x + y = 17 \qquad (2)$$

From the linear equation (2) we have

$$x = 17 - y \qquad (3)$$

Substituting this value of x in the quadratic equation (1),

we have $$(17 - y)^2 + y^2 = 169.$$

I.e., $$289 - 34y + y^2 + y^2 = 169.$$

$$\therefore 2y^2 - 34y + 120 = 0.$$

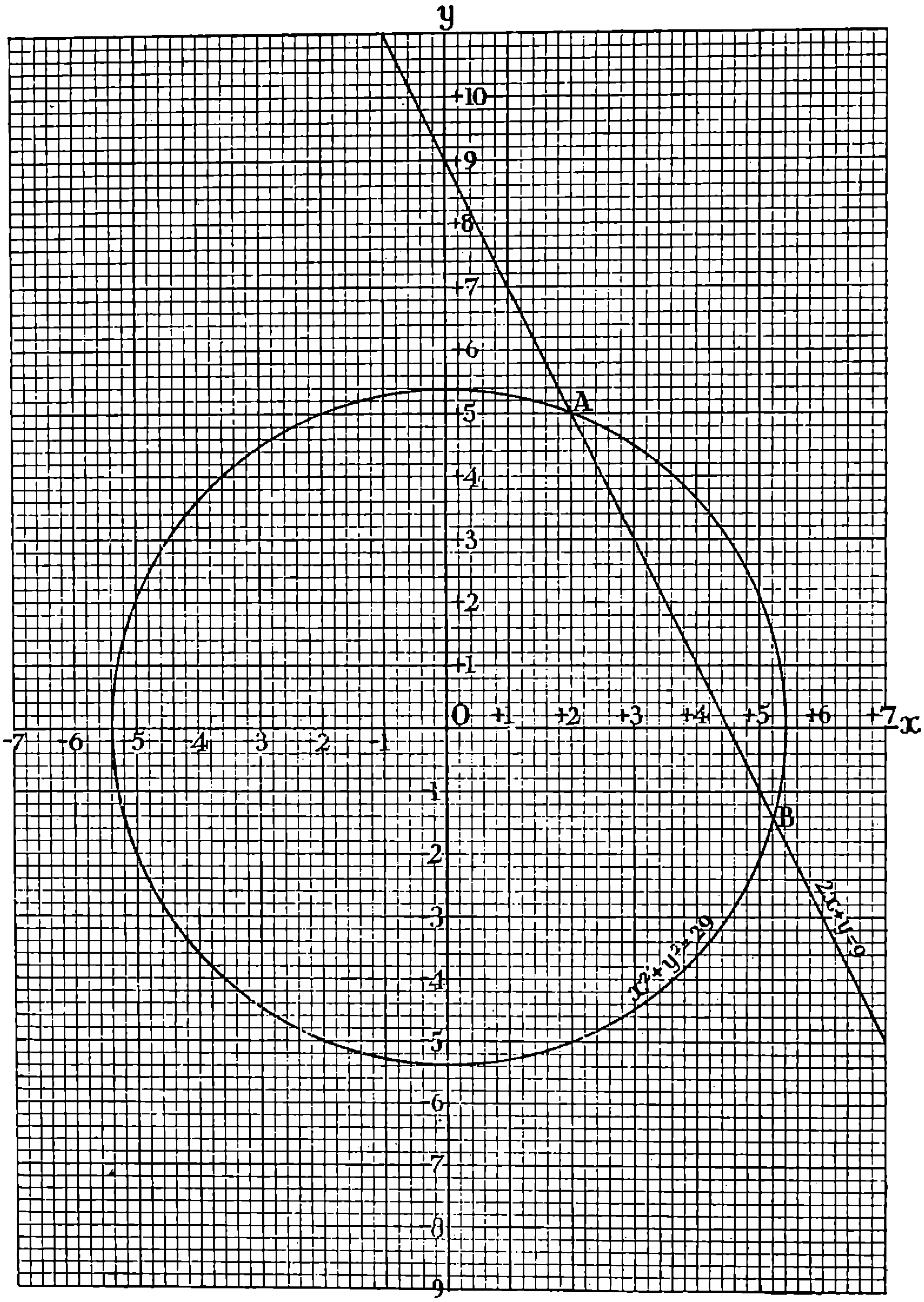

FIG. 56

$$y^2 - 17y + 60 = 0.$$
$$(y - 5)(y - 12) = 0.$$
$$\therefore y = 5 \text{ and } y = 12.$$

Substituting these values of y in equation (3), we obtain the corresponding values of x—namely, $x = 12$ and $x = 5$.

Hence the other sides of the right-angled triangle are 5 inches and 12 inches.

EXERCISE CXXXIII

1. Solve the following simultaneous equations:

(a) $x^2 + y^2 = 13.$
$x + y = 5.$

(b) $x^2 + y^2 = 13.$
$x - y = 1.$

(c) $x^2 - y^2 = 45.$
$x + y = 9.$

(d) $x^2 - y^2 = 45.$
$x - y = 5.$

(e) $x^2 - xy + y^2 = 9.$
$x + y = 6.$

(f) $x^2 + y^2 + 3xy = 4.$
$x + y = -4.$

(g) $xy + y^2 = 4.$
$x + y = 1.$

(h) $3x + 2y = 19.$
$\frac{x^2}{3} + \frac{y^2}{5} = 8.$

(i) $xy = 4.$
$x + 2y = 9.$

(j) $\frac{1}{x} + \frac{1}{y} = \frac{9}{20}.$
$y - x = 1.$

(k) $\frac{x}{y} + \frac{y}{x} = 2\frac{1}{6}.$
$x + y = 10.$

(l) $x^2 + 2y^2 - 3 = 0 = x - y.$

(m) $xy = -3.$
$4x - y = -13.$

(n) $x(x + 1) + y(y + 1) = 36.$
$3x - 5y = 5.$

(o) $x^2 + 2y^2 - 22 = 0 = 3x + 2y.$

2. The hypotenuse of a right-angled triangle is 10 inches. If the perimeter is 24 inches find the lengths of the other two sides.

3. In a right-angled triangle the difference between the

sides containing the right angle is 31 inches. If the hypotenuse is 41 inches find the lengths of the other two sides.

4. The sum of two numbers is 10. The sum of the squares of the numbers is 58. Find the numbers.

5. The sum of two numbers is 12. The sum of their squares exceeds their product by 39. Find the numbers.

6. The perimeter of a rectangular garden is 80 yards. If its area is 375 square yards find its length and breadth.

7. A man can go from A to B either by car or by train. The average speed of the train is 10 m.p.h. more than that of the car, and consequently it does the journey one hour quicker. If the distance from A to B is 120 miles find the average speed of the car.

(*It will be an advantage to use squared paper for the following questions.*)

8. Solve Question 1 (*a*) graphically.
9. Solve Question 1 (*b*) graphically.
10. Solve Question 1 (*i*) graphically.

Note. Study the graph of $xy = 4$ carefully. Notice in particular that it consists of two parts situated in the first and third quadrants respectively, and that these branches of the curve approach the axes without actually cutting them. This curve is called the *rectangular hyperbola*.

11. Solve graphically the equations $xy = 8$ and $x^2 + y^2 = 20$.

Note. The solutions of these two equations are given by the points of intersection of the rectangular hyperbola $xy = 8$ and the circle $x^2 + y^2 = 20$.

EXERCISE CXXXIV (Revision Exercise)

(A)

1. Find the factors of

(*a*) $8x^2 + 14xy - 15y^2$.
(*b*) $8(p-2q)^2 + 14(p-2q)(2p-q) - 15(2p-q)^2$.

2. Solve the equations

(a) $15x^2 - x - 5 = 1$.

(b) $\dfrac{x+4}{x+7} + \dfrac{x+3}{x+1} = \dfrac{7}{3}$.

3. A man walks 10 miles, part of which is uphill, in 3 hours. If his rate of walking uphill is 3 m.p.h. and his other walking speed is $3\frac{1}{2}$ m.p.h., what length of the distance was uphill?

4. Find the square root of $4x^4 + 4x^3 - 11x^2 - 6x + 9$.

5. Show that $x^2 + 3x - 4$ can be written in the form $\left(x + \frac{3}{2}\right)^2 - \left(\frac{25}{4}\right)$.

Hence find for what value of x the expression $x^2 + 3x - 4$ has a minimum value. What is this minimum value?

(B)

6. Solve the equation $\dfrac{1}{x-3} + \dfrac{2}{x-4} = \dfrac{3}{x-5}$.

7. Find without actually solving them which of the following quadratic equations have real solutions. In each case state whether the solution is rational or irrational.

(a) $x^2 - 5x = 4$. (d) $15x^2 = 7x + 2$.
(b) $3x^2 + 2x - 1 = 0$. (e) $2m^2 - m + 1 = 0$.
(c) $5p^2 - p + 2 = 0$.

8. Solve the equations in the previous question.

9. A motor-cyclist saves an hour in a journey of 60 miles by increasing his average speed by 3 m.p.h. Find the slower speed.

10. For what values of p and q will $2x^4 - 9x^3 + x^2 + px + q$ be exactly divisible by $x^2 - 5x + 4$?

(C)

11. Prove that $a^2(b^3 - c^3) + b^2(c^3 - a^3) + c^2(a^3 - b^3)$
$\equiv a^2(b-c)^3 + b^2(c-a)^3 + c^2(a-b)^3$.

12. Find the factors of

(*a*) $(x + 2y)^3 + (x - 2y)^3$.

(*b*) $(x + 2y)^2 - (x - 2y)^2$.

One of the factors of $6a^3 - 13a^2b + 4b^3$ is $a - 2b$. Find the other factors.

13. Solve the equation $3x^2 - 7x - 6 = 0$. Find graphically (or otherwise) for what value of x the expression $3x^2 - 7x - 6$ is a minimum. What is this minimum value?

14. A garden is surrounded by a path 3 feet wide and having an area of 198 square feet. What is the perimeter of the garden?

15. A certain number between 10 and 100 is the square of the sum of its digits. If 63 is subtracted from the number the digits are reversed. Find the number.

(D)

16. Find a quadratic equation whose solutions are three times as large as those of $2x^2 - 3x - 5 = 0$.

17. The difference between the cubes of two consecutive numbers is 169. Find the numbers.

18. Find a fraction such that if 1 is subtracted from the numerator it reduces to $\frac{1}{2}$, and if 1 is added to the numerator it reduces to $\frac{3}{4}$.

19. A sum of £18 is divided equally among a number of boys. If there had been 3 more boys each would have received 6 shillings less. How many boys were there?

20. The perimeter of a right-angled triangle is 5 feet and one of the sides is 2 feet. Find the hypotenuse and other side.

ADDITIONAL EXERCISES

A 16

1. If $p = 2$, $q = -3$, and $r = -5$, find the value of

(*a*) $p + q + r$. (*f*) $p^2 + q^2 + r^2$.
(*b*) $p + 2q + 3r$. (*g*) $p^2 - q^2$.
(*c*) $p - q$. (*h*) $p^3 + q^3 - r^3$.
(*d*) $q - r$. (*i*) $(p + q)^3$.
(*e*) $(p - q) + (q - r) + (r - p)$. (*j*) $(q - r)^3$.

2. Find the sum of $2x - 3y$, $3y - 4z$, and $4z - 2x$.

3. Subtract $5a - 4b + 7c$ from $7a + 4b - 5c$.

4. (*a*) Multiply $a^2 + ab + b^2$ by $a - b$.

(*b*) Divide $a^3 + b^3$ by $a + b$.

5. Solve the equations

(*a*) $3x + 2 = 3 - 2x$.

(*b*) $\frac{x+4}{4} - \frac{x+5}{5} = \frac{1}{20}$.

6. Write down five consecutive numbers beginning with $2n$. If the sum of these numbers is 40, find them.

7. A man is six times as old as his son. In ten years' time the difference between their ages will be 25 years. Find their present ages.

8. If A has £x and B has £y, how much must B give A so that they will both have equal amounts?

How much will each have then?

9. If $px + qy = 12$ and $px - qy = 8$ when $x = 2$ and $y = 1$, find p and q.

10. Simplify the following expressions:

(*a*) $5(a + 2b) - 3(a - 3b)$.

(*b*) $\left(\frac{a}{b}\right)^2 - \left(\frac{b}{a}\right)^2$

A 17

1. If $p = 2$, $q = -3$, and $r = -5$, find the value of

(a) pq.
(b) $pq + qr + rp$.
(c) $\frac{p}{q}$.
(d) $\frac{q}{r}$.
(e) $\frac{p}{q} + \frac{q}{r} + \frac{r}{p}$.
(f) p^2q.
(g) pq^2.
(h) $(pq)^2$.
(i) $(-p)(-q)$.
(j) $(2p^2q)^3$.

2. Eggs are advertised as costing 6*d.* for x eggs. What is the price (a) per dozen, (b) per score?

3. Multiply $3x^2 - 2xy - y^2$ by $2x^2 + xy - y^2$ and divide the answer by $x^2 - y^2$.

4. For what value of x is $5(2x - 3)$ equal to $3(x + 2)$? Check your answer.

5. Find three consecutive odd numbers such that the difference between the product of the first two and the product of the last two is 36.

6. If $3x + 2y = 12$ and $3y + 2x = 13$, find the value of (a) $x + y$, (b) $x - y$, and (c) $x^2 - y^2$.

7. Simplify

(a) $\frac{x + 3}{4} + \frac{x - 5}{7} - \frac{2x - 8}{14}$.

(b) $\frac{p^2 - q^2}{pq} \times \frac{p}{p + q} \times \frac{q}{p - q}$

8. Solve the equations

(a) $\frac{x + 4}{x - 3} = \frac{x + 5}{x - 4}$.

(b) $5x - 2y = 7x + y = 19$.

9. At a cricket-match 1040 persons were present, some of whom paid a shilling and the remainder paid half a crown for admission. The total takings amounted to £70. How many paid a shilling for their entrance?

10. A rectangle is a feet long and b feet wide. If the length is increased by c feet and the width decreased by c feet, what is the increase in area?

A 18

1. If $x = a + 2b$ and $y = 2a + b$, prove that

(a) $x^2 + 2xy + y^2 = 9(a + b)^2$.
(b) $x^2 - y^2 = 3(b^2 - a^2)$.

2. Solve the equations

(a) $5(3x - 4) - 3(x + 2) = 7(2x - 1) - (7x - 1)$.
(b) $3(x + y) - 2(x - y) = 7$.
$5(2x - y) + 3(x - 2y) = 15$.

3. Multiply $1 + 3a - 2a^2$ by $1 - a + a^2$. Check your answer for the value $a = -2$.

4. A number consists of two digits, one of which is twice the other. If the digits are reversed the number is reduced by 18. Find the number.

5. If a, b, c, and d are four consecutive numbers, prove that

(a) $(c + d) - (a + b) = 4$.
(b) $cd - ab = 2(a + d)$.

6. Simplify

(a) $\dfrac{a}{x + a} - \dfrac{a}{x - a} + \dfrac{a^2}{x^2 - a^2}$.

(b) $\dfrac{x(1 - x)(1 + x + x^2)}{1 - x^3}$.

7. (a) What is the square of $(3p - 2q)$?

(b) What is the square root of $16m^2 - 8m + 1$?

8. One tap fills a cistern in x minutes and another fills it in y minutes. If both taps are running, how long will it take to fill the cistern?

Verify your answer for $x = 3$ and $y = 5$.

9. If $x = \dfrac{1}{a}$ and $y = \dfrac{x^2}{1 - x}$, express y in terms of a.

10. A bill of 18*s*. 6*d*. is paid in florins and sixpences, the total number of coins being 13. How many of each kind of coin are used?

A 19

1. If $3x + 2y = 31$ and $4x - 3y = 13$, find the value of $x^2 - y^2$.

2. Two concentric circles are of radii $2x$ and x respectively. If A is the area of the ring formed between them, prove that $A = 3\pi x^2$.

Calculate A if $x = 1{\cdot}5$ and $\pi = 3{\cdot}14$.

3. Divide $3x^4 + 4x^3 - 3x - 4$ by $1 + x + x^2$.

4. Simplify

(*a*) $\frac{1}{x} + \frac{2}{x+1} + \frac{3}{x+2}$.

(*b*) $\frac{x}{1} + \frac{x+1}{2} + \frac{x+2}{3}$.

5. Solve the equations

(*a*) $\frac{x-1}{3} + \frac{x+1}{4} = \frac{2x+1}{3} - \frac{x-2}{5}$.

(*b*) $x + y = 4$.
$y + z = 8$.
$z + x = 6$.

6. A man is 45 years old and his son is 10 years old. In how many years will the father be twice as old as his son? What will be their ages then?

7. Find five consecutive odd numbers whose sum is 55.

8. Find the H.C.F. and L.C.M. of $24a^3b^3$, $32b^2c$, and $56a^2b^2c^2$.

9. Find the factors of

(*a*) $8a + 12ab - 16abc$. (*b*) $p^2 - 64q^2$.

10. The wages of 8 men and 8 boys amount to £26. The wages of 12 men and 6 boys amount to £34 10*s*. Find the wages of a man and of a boy.

A 20

1. For what values of p and q will $2x^4 + x^3 + px + q$ be exactly divisible by $x^2 + x + 1$?

2. Find the factors of

(a) $x^3 + x^2 + 7x + 7$.
(b) $p^3 - 8q^3$.
(c) $2x^2 + xy - y^2$.

3. Solve the equations

(a) $\dfrac{x+a}{x+b} = \dfrac{x+b}{x+c}$.

(b) $$\begin{aligned} x + 2y + z &= 1. \\ 2x + y + z &= 6. \\ 5x - y - 2z &= 3. \end{aligned}$$

4. If the numerator of a fraction is diminished by 2 the fraction reduces to $\frac{1}{3}$. If the numerator is increased by 3 the fraction reduces to $\frac{2}{3}$. Find the fraction.

5. What must be added to $4x^2 + 12x + 5$ to make it a perfect square?

6. A quantity of oranges was to be distributed among a number of children. If each child received 8 oranges, then there would be 10 left over, but in order to give each child 9 oranges 15 more were wanted. How many children were there?

7. Find A and B in each of the following:

(a) $22x - 15 \equiv A(2x + 1) + B(3x - 5)$.
(b) $6x - 17 \equiv A(3x - 2) + B(2x + 3)$.

8. Simplify

(a) $xyz\left(\dfrac{1}{x} + \dfrac{1}{y} + \dfrac{1}{z}\right)$.

(b) $xy\left(\dfrac{1}{x} + \dfrac{1}{y}\right) + yz\left(\dfrac{1}{y} + \dfrac{1}{z}\right) + zx\left(\dfrac{1}{z} + \dfrac{1}{x}\right)$.

9. If $5x + 3$, $8x - 1$, and $10x - 3$ are three consecutive odd numbers in ascending order of magnitude, find x.

10. If $ax + by$ is equal to 5 when $x = 1$ and $y = 1$, and is equal to 45 when $x = 7$ and $y = 2$, find a and b.

A 21

1. Find the factors of

   (a) $12x^2 - 27y^2$.
   (b) $6x^2 - xy - 15y^2$.
   (c) $x^3 + 2x^2 - 5x - 6$.

2. Prove by actual division that

$$\frac{1}{1-x} \equiv 1 + x + x^2 + \frac{x^3}{1-x}.$$

3. If $y = 7x$, find the value of $\dfrac{3x + 2y}{2x + 3y}$.

4. Solve the equations

   (a) $x^2 + x = 6$.
   (b) $2(2x - y) = \frac{1}{2}(y - x)$.
   $y = 2x - 1$.

5. Two pounds of tea and 3 pounds of sugar cost 5*s*. 9*d*., and 5 pounds of tea and 8 pounds of sugar cost 14*s*. 6*d*. Find the price per pound of the tea and of the sugar.

6. Find three positive consecutive numbers such that the sum of their squares is 110.

7. Each edge of a cube is x inches in length. If it expands until each edge becomes $(x + h)$ inches find (a) the change in volume, (b) the change in surface area.

8. Simplify

   (a) $\dfrac{2}{3}x^2yz^2 \times \dfrac{27x}{y^2} \div \left(\dfrac{20x}{yz} \times \dfrac{x^2}{5}\right)^2$

   (b) $\dfrac{\dfrac{1}{x} + \dfrac{1}{y} + \dfrac{1}{z}}{\dfrac{1}{xy} + \dfrac{1}{yz} + \dfrac{1}{zx}}$.

9. A slow train leaves London and is followed half an hour later by a train travelling on the average 10 m.p.h. faster. The fast train catches the slow one after $1\frac{1}{2}$ hours. Find the speed of each train.

10. A number consists of two digits, the difference of which is 3. If the digits are reversed a new number is formed, which is 25. Find the original number.

A 22

1. Find the factors of

(a) $2x^3 - 2x - x^2 + 1.$

(b) $1 - \dfrac{x^2}{9}.$

(c) $3x^4 + x^2 - 2.$

2. Solve the equations

(a) $2(2x + y) + 3(x - y) = 2(3x - 2y) - 9.$

$$\frac{2x - y}{2} = \frac{x - 2y}{3} + 1\tfrac{1}{3}.$$

(b) $16x^2 + 8x - 15 = 0.$

3. Find what values must be given to A and B in the following identities:

(a) $\dfrac{\quad - \quad + 1}{(x + \quad (2x - 1)} \equiv \dfrac{A}{x + 1} + \dfrac{B}{2x - 1}.$

(b) $3(x + 10) \equiv A(2x - 1) + B(x - 5).$

4. Find the square root of $4x^4 - 4x^3 + 13x^2 - 6x + 9.$

5. Construct a quadratic equation whose solutions are three times as large as those of $6x^2 - 7x + 2 = 0.$

6. A line 15 inches long is divided into two parts such that 3 times one part is equal to twice the other. What are the lengths of the two parts?

7. Simplify $\dfrac{1}{6x^2 - 7x + 2} - \dfrac{1}{12x^2 + x - 6} + \dfrac{1}{8x^2 + 2x - 3}.$

8. Evaluate $\dfrac{2\sqrt{3}}{\sqrt{2}-1}+\dfrac{3\sqrt{2}}{\sqrt{3}-1}$.

9 A bag contains 32 coins, some of which are half-crowns and the remainder florins. The value of the coins is $3\frac{1}{2}$ guineas. How many of each kind of coin are there?

10. A is 24 years older than B, and 12 years ago A was twice as old as B. What are their present ages?

A 23

1. (*a*) Divide $3x^4-11x^3+17x^2-13x+4$ by x^2-2x+1.

(*b*) Show that $\dfrac{1}{(1-x)^2}\equiv 1+2x+3x^2+4x^3+\dfrac{R}{(1-x)^2}$. Find R.

2. Solve the equations

(*a*) $\dfrac{1}{x}+\dfrac{1}{y}=\dfrac{7}{12}$.

$\dfrac{3}{x}+\dfrac{4}{y}=2$.

(*b*) $\dfrac{3x-2}{4}+\dfrac{5x-6}{5}=\dfrac{25x+6}{10}$.

3. (*a*) For what values of p and q will $9x^4+30x^3+13x^2+px+q$ have an exact square root?

(*b*) What must be added to $16x^2+40x+8$ in order to make it a perfect square?

4. Find by rationalizing the denominators the value of

$$\frac{\sqrt{5}}{\sqrt{3}+\sqrt{2}}+\frac{\sqrt{3}}{\sqrt{5}-\sqrt{2}}.$$

5. Find A and B in each of the following:

(*a*) $15x-22\equiv A(5x-3)+B(x+2)$.

(*b*) $\dfrac{x-10}{(2x+1)(3x-2)}\equiv\dfrac{A}{2x+1}+\dfrac{B}{3x-2}$.

6. The cost of a picnic was £1 8*s*. Two members of the

party did not contribute to the cost, with the result that each of the remainder had fourpence more to pay than his share. How many were there at the picnic?

7. A path 2 feet 6 inches wide surrounds a rectangular garden. The area of the path is 185 square feet. Find the perimeter of the garden.

8. Prove that $a^2(b^3 - c^3) + b^2(c^3 - a^3) + c^2(a^3 - b^3)$

$$\equiv a^2(b - c)^3 + b^2(c - a)^3 + c^2(a - b)^3.$$

9. At an examination the maximum number of marks was 150 and the top candidate obtained 115 and the bottom one 5. By means of a graph scale these marks down so that the top candidate obtains 100. How many marks would the candidate who received 60 get under the new scale?

10. Draw the graphs of $y = x^2$ and $y = 5x - 4$, and thus solve the equation $x^2 - 5x + 4 = 0$.

If $x^2 - 5x + k = 0$, for what value of k will the solutions of this equation be equal?

For what values of k will the solutions be imaginary?

A 24

1. Solve the equations

(a) $$\frac{1}{x+1} + \frac{1}{x+2} = \frac{1}{x+3} + \frac{1}{x+4}.$$

(b) $$x^2 + y^2 = 53.$$
$$3x + y = 13.$$

2. Draw a graph to illustrate the equations in (b) of the preceding question.

3. Express $x^2 - 4x - 5$ as the difference of two squares, and hence (or otherwise) find for what value of x the expression $x^2 - 4x - 5$ has its minimum value. What is this minimum value?

4. Illustrate the preceding question graphically. From your graph *read off* solutions of the following equations:

(a) $x^2 - 4x - 5 = 0$.

(b) $x^2 - 4x - 7 = 0$ (i.e., $x^2 - 4x - 5 = 2$).

(c) $x^2 - 4x + 4 = 0$.

If $x^2 - 4x - 5 = k$, for what values of k will this equation have solutions which are (i) real, (ii) imaginary ?

5. Find a fraction such that if 1 is added to the denominator the fraction reduces to $\frac{2}{3}$, and if 3 is subtracted from the denominator it reduces to $\frac{6}{7}$.

6. Find two consecutive numbers such that the difference of their cubes exceeds the sum of their squares by 56.

7. Write down the square of $(\sqrt{a} + \sqrt{b} + \sqrt{c})$.

8. Four pounds of tea and 8 pounds of sugar cost 12 shillings. If the price of tea is increased 25 per cent. and the price of sugar is increased 50 per cent., they would cost 15*s*. 6*d*. Find the original price per pound of the tea and of the sugar.

9. Find the H.C.F. and L.C.M. of $2x^3 - 4x^2$, $4x^2 - 8x$, and $8x - 16$.

10. Find A, B, C, and D in each of the following:

(*a*) $(x + 1)(x + 2) \equiv x^2 + Ax + B$.
(*b*) $(x + 1)(x + 2)(x + 3) \equiv x^3 + Ax^2 + Bx + C$.
(*c*) $(x + 1)(x + 2)(x + 3)(x + 4)$
$\equiv x^4 + Ax^3 + Bx^2 + Cx + D$.

A 25

1. Find the factors of

(*a*) $5x^2 + 7xy - 6y^2$.
(*b*) $5(a + 2b)^2 + 7(a + 2b)(2a - 3b) - 6(2a - 3b)^2$.

2. Find the square root of $16x^2 + 40x + 17 - \frac{10}{x} + \frac{1}{x^2}$.

3. Show without actual division that $6x^3 - 25x^2 + 34x - 15$ is divisible by $3x - 5$.

Hence (or otherwise) find the factors of this expression.

4. If, a, b, and c are three consecutive numbers, prove that

(*a*) $ac + 1 = b^2$.
(*b*) $\frac{1}{2}(ab + bc) = b^2$.
(*c*) $ab + bc - 2ca = 2$.

5. There were 350 seats in a concert-hall, some of which were priced 1*s*. 6*d*. and the remainder 1 shilling. The total takings amounted to £21 5*s*. How many of each kind of seat were there?

6. Solve the equations

(*a*) $\dfrac{3{\cdot}75}{x} + \dfrac{x}{0{\cdot}5} = 10{\cdot}75.$

(*b*) $0{\cdot}25(3x + y) = 5x - 4y.$
$0{\cdot}5x + 0{\cdot}75y = 1{\cdot}25.$

7. For what value of x is $3 + 4x - x^2$ a maximum? What is this maximum value?

Verify your calculations by drawing the graph of $y = 3 + 4x - x^2$ and from it reading the value of x which gives the maximum value of $3 + 4x - x^2$.

8. (*a*) Find A and B such that $\dfrac{3x - 5}{x - 1} \equiv A + \dfrac{B}{x - 1}.$

(*b*) Prove that $(b - c)^3 + (c - a)^3 + (a - b)^3$
$\equiv 3(b - c)(c - a)(a - b).$

9. If $v = u + ft$, express t in terms of v, u, and f. By substituting this value of t in the formula $s = ut + \frac{1}{2}ft^2$, prove that $v^2 = u^2 + 2fs$.

10. The average speed of one train is 5 m.p.h. more than that of another train, with the result that it finishes a journey of 280 miles an hour sooner than the second. Find the speeds of the two trains.

A 26

1. If $5x^2 - x + 20 \equiv A(x - 1)(x + 2) + B(x - 2)(x + 1) + C(x^2 - 1)$, find A, B, and C.

2. (*a*) Construct a quadratic equation whose two solutions are $x = \frac{1}{2}$ and $x = -\frac{3}{5}$.

(*b*) Solve the equation $(3x + 7)(x - 4)(2x - 9) = 0$.

3. (*a*) Show that $3896 = 3 \times 10^3 + 8 \times 10^2 + 9 \times 10 + 6 = 3t^3 + 8t^2 + 9t + 6$ (where t is written instead of 10).

(*b*) Write in a similar manner (using t for 10) the following numbers:

187 347 1896 2756 3978.

4. Using the notation of the preceding question, say what numbers are represented by the following expressions:

(*a*) $2t^2 + 1$. (*d*) $2t - 1$.
(*b*) $5t^3 + 3t^2 + 7$. (*e*) $8t^3 - 3t^2 + 4t - 7$.
(*c*) $12t^3 + 8t^2 + 7t + 5$.

5. Solve the equations

(*a*) $2^x = 64$. (*b*) $2x^2 - x = 1$.

6. Prove that $\frac{1}{(1+x)^2} \equiv \frac{1}{1+2x+x^2}$

$$\equiv 1 - 2x + 3x^2 - 4x^3 + \frac{R}{(1+x)^2}.$$

Find R.

7. A grass plot is 18 feet long and 12 feet wide, and is surrounded by a gravel path of uniform width. If the area of the path is 136 square feet, find its width.

8. Three lines are of lengths 2 cm., 9 cm., and 10 cm. What equal lengths must be added to these so that the lines may form a right-angled triangle?

9. The sum of two numbers is 20. The sum of their squares is 202. Find the numbers.

10. Find the gain per cent. in selling eggs at x shillings a score if they were bought at the rate of a eggs for p pence.

A 27

1. Find the factors of

(*a*) $x^3 - \frac{27}{64}y^6$.

(*b*) $x^3 - 4x^2 + x + 6$.

2. Solve the equations

$$(a)\ \frac{2x-3}{5} - \frac{x+3}{3} = \frac{x}{6} - \tfrac{1}{2}.$$

$$(b)\ \frac{2x^2-3}{5} - \frac{x^2+3}{3} = \frac{x^2}{6} - \tfrac{1}{2}.$$

3. Find the square root of $4x^4 - 32x^3 + 108x^2 - 176x + 121$.

4. For what value of x has $4x^4 - 32x^3 + 108x^2 - 176x + 121$ a minimum value? Show that this minimum value is 9. (See the preceding question.)

5. The difference between a number and its reciprocal is $3\frac{3}{4}$. Find the number.

6. A does a piece of work in x days, B in y days, and C can do it in the same time as A and B working together. If all three work together, in how many days will they finish the work?

7. Without actually solving the following equations state in each case whether the solutions will be real or imaginary, and if real whether rational or irrational.

(*a*) $2x^2 - 5x - 7 = 0$.
(*b*) $2x^2 - 5x + 7 = 0$.
(*c*) $2x^2 - 5x + 2 = 0$.

8. Illustrate your answers in the preceding question by graphs.

9. If a, b, c, and d are four consecutive numbers, prove that

$$4(a^2 + b^2 + c^2 + d^2) - (a + b + c + d)^2 = 20.$$

10. A manufacturer sells an article to a wholesale merchant, thereby making a profit of 10 per cent. The wholesale merchant sells it to the retailer at a profit of 10 per cent. The retailer sells it to the customer, and he also makes a profit of 10 per cent. If the customer pays £1, what was the original cost of the article to the manufacturer?

A 28

1. Solve the equations

(*a*) $0{\cdot}5(2x - 3) = 2(0{\cdot}5 + 7{\cdot}5x)$.

(*b*) $1{\cdot}75x^2 - 3{\cdot}5x = 2{\cdot}25$ (answer to two places of decimals).

2. A sum of money is divided between A, B, and C. A and B together have 14 shillings more than C; B and C together have 30 shillings more than A; C and A together have 20 shillings more than B. How much does each receive?

3. If a, b, and c are three consecutive numbers, prove that

$$\tfrac{1}{3}(a^3 + b^3 + c^3) = b(b^2 + 2).$$

4. Express $\dfrac{x - 3}{(x + 3)(x + 1)}$ as the sum of two partial fractions.

5. Owing to a fall in price of a farthing per pound, it is possible to purchase two more pounds of sugar than before for 5*s*. 6*d*. What was the original price per pound?

6. Find the coefficient of x^3 in $(1 - 2x)^4$.

7. If $x = 2 - \sqrt{3}$, find the value of

$$(a)\ x + \frac{1}{x}. \quad (b)\ x^2 + \frac{1}{x^2}.$$

8. Construct quadratic equations whose solutions are (*a*) $x = 3$ and $x = -5\tfrac{1}{2}$, (*b*) $x = m + 1$ and $x = m - 1$.

9. One of the three solutions of the equation $2x^3 + x^2 - 5x + 2 = 0$ is $x = \tfrac{1}{2}$. Find the other two solutions.

10. A number consists of two digits, one of which is twice the other. If the digits are reversed the new number is 18 more than the old one. Find the original number.

A 29

1. Find the value of $\dfrac{x(y - z) + y(z - x) + z(x - y)}{x^2 + y^2 + z^2 + 2xy + 2yz + 2zx}$ when $x = 1$, $y = -2$, and $z = -3$.

2. Divide $(x - 3)^3 - (x - 3)(x - 2)^2$ by $2x^2 - 11x + 15$.

3. Solve the equations

(a) $\frac{1}{5}(2x + y) + \frac{1}{2}(x + 2y) = 3x + 4y.$
$3(2x - y) - 2(x - 2y) = 13.$

(b) $x^2 + 2y^2 = 9.$
$3x + 2y = 7.$

4. Find the value of $\dfrac{\sqrt{5} + 1}{\sqrt{5} - 1}$.

5. Find the square root of

$$(2x^2 - 7x + 3)(2x^2 - 9x + 4)(x^2 - 7x + 12).$$

6. Construct a quadratic equation whose solutions are reciprocals of those of $6x^2 - x - 2 = 0$.

7. The area of a rectangle is unaltered if its length is increased by 10 feet and its width diminished by 4 feet or if its length is diminished by 5 feet and its width increased by 5 feet. Find its area.

8. When $x - 2$ is multiplied by a certain number and $2x + 1$ is multiplied by another number the sum of the products is $20x$ for all values of x. Find the two numbers.

9. The perimeter of a rectangle is 142 feet and its area is 660 square feet. Find the length of a diagonal.

10. The hypotenuse of a right-angled triangle is 13 inches and of the other two sides one of them is 7 inches more than the other. Find the sides of the triangle.

A 30

1. Find the square root of $4x^4 + 4x^3 + 5x^2 + 2x + 1$.

2. Solve the equations

$$2x + y + z = 6.$$
$$3x + 2y - 3z = -15.$$
$$x - y + 2z = 15.$$

3. If a, b, and c are three consecutive odd numbers, prove that

$$a^2 + b^2 + c^2 = 3b^2 + 8.$$

Would this result hold if a, b, and c were three consecutive *even* numbers?

4. There are twice as many boys in one form as there are in another. In the smaller form the average age is 14 years 6 months and in the other it is 15 years exactly. If the two forms are combined, what will be the average age?

5. Find x if $4^x = 2^6$.

6. Solve the following equations to two places of decimals:

(a) $6x^2 + 3x - 15 = 0$.
(b) $6x^2 + 3x - 14 = 0$.

7. Simplify $\dfrac{(5x^2 + 52x + 20)(x^2 - 1)}{(x^2 + 11x + 10)(x - 1)}$.

8. The postage for a letter is $1\frac{1}{2}d$. and for a postcard it is $1d$. A man sent 44 communications—some of them letters and the remainder postcards—for $4s. 8d$. How many of each did he send?

9. At what times between 7 o'clock and 8 o'clock are the hands of a clock (a) coincident, (b) in a straight line, (c) at right angles?

10. What must be added to $9x^2 - 30x + 21$ to make it a perfect square?

Hence (or otherwise) find for what value of x the expression $9x^2 - 30x + 1$ is a minimum.

What is this minimum value?

PART III

CHAPTER I

THE THEORY OF QUADRATIC EQUATIONS

1. Our study of quadratic equations[1] has shown that two solutions are always obtained. It is not difficult to prove that a quadratic equation cannot have more than two solutions. Suppose that the equation $ax^2 + bx + c = 0$ has *three* different solutions—namely, α, β, and γ.

Then $a\alpha^2 + b\alpha + c = 0$, because α is a solution.
$a\beta^2 + b\beta + c = 0$, $\quad\beta$
$a\gamma^2 + b\gamma + c = 0$, $\quad\gamma$

Subtracting the second of these equations from the first,

$$a(\alpha^2 - \beta^2) + b(\alpha - \beta) = 0.$$

Since α, β, and γ are all different, $\alpha - \beta$ is not zero. Hence by division by $\alpha - \beta$ we have

$$a(\alpha + \beta) + b = 0 \qquad (1)$$

Similarly, by subtracting the third of the above equations from the second we have

$$a(\beta^2 - \gamma^2) + b(\beta - \gamma) = 0,$$

which on division by $\beta - \gamma$ reduces to

$$a(\beta + \gamma) + b = 0 \qquad (2)$$

By subtracting (2) from (1) we have

$$a(\alpha - \gamma) = 0,$$

which shows that either $a = 0$ or $\alpha - \gamma = 0$.

If $a = 0$, then $ax^2 + bx + c = 0$ ceases to be a quadratic equation.

[1] See *A Junior Algebra*, pp. 207–241.

If $\alpha - \gamma = 0$, then $\alpha = \gamma$, which means that the three solutions α, β, and γ are not all different. Hence a quadratic equation has only two solutions.

As we have seen,[1] the equation $ax^2 + bx + c = 0$ has two solutions, given by the formula $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

If we let α and β be these two solutions, then

$$\alpha = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$$

and
$$\beta = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

The Value of $\alpha + \beta$

$$\alpha + \beta = \frac{-b + \sqrt{b^2 - 4ac}}{2a} + \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

$$= -\frac{2b}{2a}.$$

$$= -\frac{b}{a}.$$

$$\boldsymbol{\alpha + \beta = -\frac{b}{a}.}$$

The Value of $\alpha\beta$

$$\alpha\beta = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \times \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{(-b)^2 - (\sqrt{b^2 - 4ac})^2}{4a^2}$$

$$= \frac{4ac}{4a^2}$$

$$= \frac{c}{a}.$$

$$\boldsymbol{\alpha\beta = \frac{c}{a}.}$$

[1] *A Junior Algebra*, p. 224.

ALTERNATIVE METHOD

Suppose that α and β are two solutions of $ax^2 + bx + c = 0$. Then $x - \alpha$ and $x - \beta$ are factors of $ax^2 + bx + c$, by the Remainder Theorem.

$$(x - \alpha)(x - \beta) \text{ is a factor of } ax^2 + bx + c.$$

By comparing $ax^2 + bx + c$ and $(x - \alpha)(x - \beta)$ it is clear that the only other factor is a.

$$ax^2 + bx + c \equiv a(x - \alpha)(x - \beta).$$

Suppose that γ is a solution of $ax^2 + bx + c = 0$.

Then $\quad 0 = a\gamma^2 + b\gamma + c \equiv a(\gamma - \alpha)(\gamma - \beta),$

from which either $\gamma - \alpha = 0$ (*i.e.*, $\alpha = \gamma$)

or $\quad \gamma - \beta = 0$ (*i.e.*, $\beta = \gamma$).

Hence there cannot be three distinct solutions of the equation $ax^2 + bx + c = 0$.

Since α and β are the two solutions, we have

$$\begin{aligned} ax^2 + bx + c &\equiv a(x - \alpha)(x - \beta) \\ &= ax^2 - a(\alpha + \beta)x + a\alpha\beta. \end{aligned}$$

Comparing coefficients,

$$b = -a(\alpha + \beta),$$

from which $\quad \alpha + \beta = -\dfrac{b}{a}.$

$$c = a\alpha\beta,$$

from which $\quad \alpha\beta = \dfrac{c}{a}.$

By dividing each term of the quadratic equation $ax^2 + bx + c = 0$ by a we have

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0.$$

This can be written $x^2 - (\alpha + \beta)x + \alpha\beta = 0.$

In other words, *provided that the coefficient of x^2 in a quadratic equation is unity*, the coefficient of x with its sign changed is equal to the sum of the solutions of the equation, and the third term is equal to their product.

The Nature of α and β

The actual values of α and β are calculated from the formula $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, from which it can be seen at once that the nature of α and β—*i.e.*, whether they are real or imaginary, rational or irrational—depends upon the value of the 'discriminant,' $b^2 - 4ac$.

1. If $b^2 - 4ac$ is positive α and β are real and unequal.
2. If $b^2 - 4ac$ is negative α and β are imaginary.
3. If $b^2 - 4ac$ is a perfect square α and β are real and rational.
4. If $b^2 - 4ac$ is positive, but not a perfect square, α and β are real and irrational.
5. If $b^2 - 4ac = 0$, then $\alpha = \beta = -\dfrac{b}{2a}$.

Note. The nature of α and β is sometimes revealed by the absence of one of the coefficients a, b, or c.

Thus if $a = 0$ the equation $ax^2 + bx + c = 0$ ceases to be a quadratic and reduces to a linear equation with one solution.

If $b = 0$ the equation $ax^2 + bx + c = 0$ reduces to

$$ax^2 + c = 0,$$

$$x = \pm\sqrt{-\frac{c}{a}}.$$

In this case if α and β are to be real, then *either* c or a must be negative, otherwise the square root cannot be evaluated. If α and β are real they will be equal in magnitude, but opposite in sign.

If $c = 0$ the equation $ax^2 + bx + c = 0$ reduces to

$$ax^2 + bx = 0.$$

I.e.,

$$x(ax + b) = 0.$$

$$x = 0 \text{ or } x = -\frac{b}{a}.$$

Hence if $c = 0$ one of the two solutions is zero.

EXERCISE I

1. From an examination of the discriminant $b^2 - 4ac$ describe the solutions of the following quadratic equations:

(*a*) $x^2 - 5x - 6 = 0$.
(*b*) $x^2 - 5x + 6 = 0$.
(*c*) $2x^2 + 7x + 1 = 0$.
(*d*) $3x^2 - 5x = 0$.
(*e*) $3x^2 - 7 = 0$.
(*f*) $3 - 4k = k^2$.
(*g*) $p^2 = 6p + 5$.
(*h*) $m = 3m^2 + 2$.
(*i*) $a^2 - 2a + 3 = 0$.
(*j*) $1 + n + n^2 = 0$.

2. Write down the sum of the solutions in each of the quadratic equations in the preceding question.

3. Write down the product of the solutions in each of the quadratic equations in Question 1.

4. Construct quadratic equations whose solutions are

(*a*) 3 and 2.
(*b*) 3 and -2.
(*c*) $-1\frac{1}{2}$ and $2\frac{3}{4}$.
(*d*) $-3k$ and $7k$.
(*e*) $-5m$ and $-2m$.

5. One solution of $x^2 - px + 18 = 0$ is 3. Find the other solution and the value of p.

6. One solution of $8x^2 - kx + 1 = 0$ is double the other. Find k.

7. If the solutions of $x^2 - 3x + m = 0$ are equal, find m.

8. If one solution of $x^2 - px + q = 0$ is four times the other, prove that $4p^2 = 25q$.

9. Find the values of $\alpha + \beta$ and $\alpha\beta$ for the equation $px^2 + 2qx + r = 0$.

10. If one of the solutions of $ax^2 + bx + c = 0$ is three times the other, prove that $3b^2 = 16ac$.

2. The values of $\alpha + \beta$ and $\alpha\beta$ which have just been obtained are used for evaluating all other functions of α and β. In such problems the *actual values* of α and β should NOT be used, but the function should be rearranged so that it is expressed in terms of $\alpha + \beta$ and $\alpha\beta$. The following examples illustrate the method.

EXAMPLE 1

If $ax^2 + bx + c = 0$, find the value of $\alpha^2 + \beta^2$.

Since $ax^2 + bx + c = 0$, we have $\alpha + \beta = -\frac{b}{a}$ and $\alpha\beta = \frac{c}{a}$.

Hence
$$\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$$
$$= \left(-\frac{b}{a}\right)^2 - 2\frac{c}{a}$$
$$= \frac{b^2}{a^2} - \frac{2c}{a}$$
$$= \frac{b^2 - 2ac}{a^2}.$$

EXAMPLE 2

If $x^2 + px + q = 0$, find the value of $\alpha^3 + \beta^3$.

Since $x^2 + px + q = 0$, we have $\alpha + \beta = -p$ and $\alpha\beta = q$.

Hence
$$\alpha^3 + \beta^3 = (\alpha + \beta)(\alpha^2 - \alpha\beta + \beta^2)$$
$$= (\alpha + \beta)\{(\alpha + \beta)^2 - 3\alpha\beta\}$$
$$= -p\{(-p)^2 - 3q\}$$
$$= -p(p^2 - 3q)$$
$$= 3pq - p^3.$$

EXERCISE II

1. If $ax^2 + bx + c = 0$, find the values of:

(a) $\alpha - \beta$.

(b) $\alpha^2 + \beta^2$.

(c) $\alpha^3 + \beta^3$.

(d) $\alpha^2 - \beta^2$.

(e) $\alpha^3 - \beta^3$.

(f) $\frac{\alpha}{\beta} + \frac{\beta}{\alpha}$.

(g) $\frac{\alpha + \beta}{\alpha - \beta}$.

(h) $\alpha^2\beta + \alpha\beta^2$.

(i) $\alpha^4 - \beta^4$.

(j) $(1 + \alpha)(1 + \beta)$.

2. If α and β are the solutions of $x^2 - px + q = 0$, find the values of:

(a) $\alpha + \beta$.
(b) $\alpha - \beta$.
(c) $\alpha\beta$.
(d) $\alpha^2 - \beta^2$.
(e) $\alpha^3 - \beta^3$.
(f) $\alpha^3 + \beta^3$.
(g) $\dfrac{\alpha}{\beta} + \dfrac{\beta}{\alpha}$.
(h) $\alpha^2\beta + \alpha\beta^2$.

3. If α and β are the solutions of the equation $ax^2 + bx + c = 0$, find equations whose solutions are

(a) $\dfrac{1}{\alpha}$ and $\dfrac{1}{\beta}$.
(b) $\dfrac{1}{\alpha^2}$ and $\dfrac{1}{\beta^2}$.
(c) $1 + \alpha$ and $1 + \beta$.
(d) $\dfrac{2\alpha}{\beta}$ and $\dfrac{2\beta}{\alpha}$.
(e) α^3 and β^3.

4. If $2x^2 + 3x - 4 = 0$, prove that $\alpha^2 + \beta^2 = 6\frac{1}{4}$, where α and β are the solutions of the equation.

5. If $x^2 - 2x - 3 = 0$, prove that $(1 + \alpha + \alpha^2)(1 + \beta + \beta^2) = 13$, where α and β are the solutions of the equation.

3. The theory of quadratic equations is used for determining the range of values of the variable which will make a given quadratic expression positive or negative. As a very simple example consider $x^2 + 6x - 7$. This can be written $(x + 7)(x - 1)$, and reference to its graph (Fig. 1) shows that this expression is positive for all values of x *except those which lie between the two solutions of the equation* $x^2 + 6x - 7 = 0$. This method can be extended to more complicated quadratic expressions.

EXAMPLE

If x is real, prove that the expression $\dfrac{x^2 + 5x + 8}{x + 4}$ can have all numerical values except those which lie between -7 and 1.

Let $$\frac{x^2 + 5x + 8}{x + 4} = y.$$

Then $$x^2 + 5x + 8 = xy + 4y.$$

I.e., $$x^2 + (5 - y)x + (8 - 4y) = 0.$$

Since x is real, the discriminant of this quadratic equation must be positive.

Hence $(5 - y)^2 - 4(8 - 4y)$ must be positive.

$25 - 10y + y^2 - 32 + 16y$ must be positive.

$\therefore\ y^2 + 6y - 7$ must be positive.

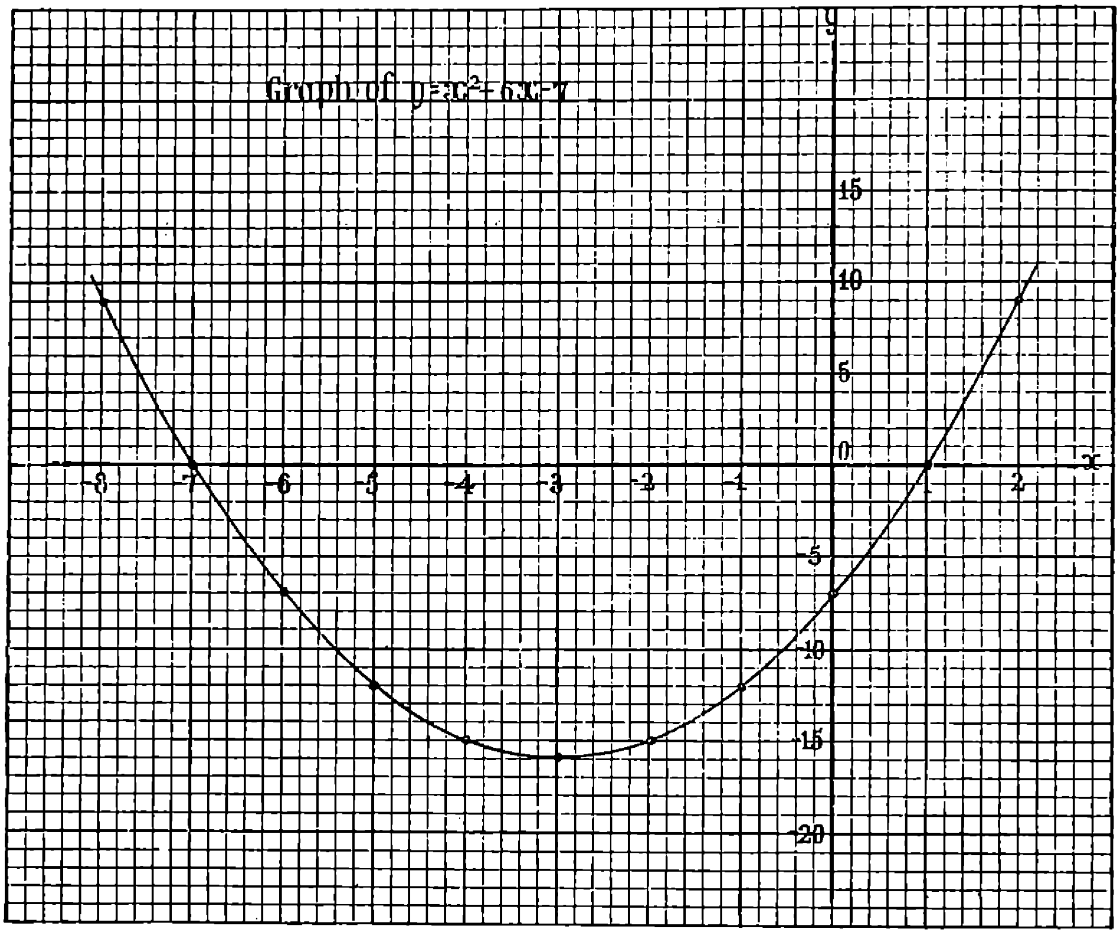

FIG. 1

From what we have just seen by reference to the graph in Fig. 1 $y^2 + 6y - 7$ will be *positive* for *all values of* y *except those which lie between the solutions of* $y^2 + 6y - 7 = 0$.

Since

$$y^2 + 6y - 7 = 0$$

$$(y + 7)(y - 1) = 0.$$

$$y = -7 \text{ and } 1.$$

Hence y must not lie between -7 and 1.

Therefore $\dfrac{x^2 + 5x + 8}{x + 4}$ cannot lie between -7 and 1.

Note on the Sign of $ax^2 + bx + c$.

Let α and β be the two solutions of $ax^2 + bx + c = 0$.

Then $\quad y = ax^2 + bx + c \equiv a(x - \alpha)(x - \beta)$.

(i) If $a > 0$, then for y to be positive $x - \alpha$ and $x - \beta$ must both have the same sign.

$\therefore$ x CANNOT lie between α and β.

(ii) If $a < 0$, then for y to be positive $x - \alpha$ and $x - \beta$ must have opposite signs.

x MUST lie between α and β.

(iii) If α and β are imaginary, then

$$y = ax^2 + bx + c \equiv a\left\{\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a^2}\right\}.$$

But $\left(x + \frac{b}{2a}\right)^2$ must be positive and $\frac{4ac - b^2}{4a^2}$ must also be positive if α and β are imaginary.

y has the same sign as a.

EXERCISE III

1. Determine for what range of values of x the following quadratic expressions will be positive:

(*a*) $x^2 - 5x + 6$. (*d*) $6 + 7x - 3x^2$.
(*b*) $x^2 + 4x + 4$. (*e*) $-3 + 5x + 2x^2$.
(*c*) $2 - 3x - 2x^2$.

2. Prove that if x is real the expression $\frac{x^2 + 4x - 3}{x + 6}$ cannot have a numerical value between -14 and -2.

3. Prove that if x is real the expression $\frac{2x^2 + 3x - 1}{2x^2 + 3x + 1}$ has no value between 1 and 17.

4. For what values of x is $(3x + 2)(2x - 1)(x + 1)$ positive?

5. Show that the expression $3x^2 - 5x + 7$ is positive for all real values of x.

6. For what range of values is $\frac{1 - 2x}{3x + 4}$ positive?

7. Prove that for real values of x the expression $\frac{x^2 - 2x + 1}{x^2 + x + 1}$ will always lie between zero and 4.

8. Prove that $(x-3)(x-4) = 16$ has real solutions. Hence (or otherwise) show that the solutions of $(x-a)(x-b) = k^2$ are always real.

9. For what values of x will $\dfrac{x^2-3x+2}{x-3}$ be positive?

10. What restriction must be placed on the value of k so that the solutions of the equation $\dfrac{1}{x^2+x+1} = k$ may be real?

EXERCISE IV (Revision Exercise)

(A)

1. Solve the equations

   (a) $2x + y = 1$ and $5x - 2y = 34$.
   (b) $3x^2 + x - 14 = 0$.

2. What values must be given to p and q so that $12a^3 + pa^2 + qa - 20$ may be exactly divisible by $a + 2$ and $4a - 5$? Hence find the factors of this expression.

3. A car will travel n miles on a gallon of petrol costing p pence per gallon, and m miles on a pint of oil costing q pence per pint. Calculate the cost in pence per mile of running the car.

4. Find the factors of

   (a) $20x^2 + 3x - 9$.
   (b) $20(p + 2q)^2 + 3(p + 2q) - 9$.

5. Find a quadratic equation whose solutions are four times as large as those of $x^2 - 7x - 8 = 0$.

(B)

6. Calculate the values which must be given to A, B, and C if

$$\frac{2(2x+5)}{(x+1)(x+3)(x+4)} \equiv \frac{A}{x+1} + \frac{B}{x+3} + \frac{C}{x+4}.$$

7. Prove that if $a + b + c = 0$, then $a^3 + b^3 + c^3 = 3abc$.

8. A certain number of children share £1 equally among themselves. If there had been two more children each would have received sixpence less. How much more would each have received if there had been two fewer children?

9. If α and β are the two solutions of $x^2 + 2px + q = 0$, find the values of

$$(a)\ \alpha + \beta. \quad (b)\ \alpha\beta. \quad (c)\ \alpha^2 + \beta^2.$$

Hence (or otherwise) find the equation whose solutions are $\frac{1}{\alpha}$ and $\frac{1}{\beta}$.

10. For what range of values is $x^2 + 7x + 6$ negative? Show that $x^2 + 7x + 6 = \left(x + \frac{7}{2}\right)^2 - 6\frac{1}{4}$, and hence find the minimum value of $x^2 + 7x + 6$ and the value of x which makes it a minimum.

(C)

11. Find the highest common factor of $x^3 - 7x - 6$ and $x^3 - 3x^2 - 4x + 12$.

12. If a, b, and c are three consecutive odd numbers, prove that $a^2 + b^2 + c^2 = 3(ab + 2c) - 4$.

13. If $2x + 3y = 26$ and $14x - y = 17$, find the value of $\frac{x + y}{x - y}$.

14. A man invests £3000 partly at 4 per cent. and partly at 5 per cent. If his total income is £142, how is his investment divided?

15. A rectangular lawn is surrounded by a path 5 feet wide, having an area of 500 square feet. Find the perimeter of the lawn.

(D)

16. Solve the equations

$$2x + 3y = 17.$$
$$2x^2 + y^2 - xy = 22.$$

17. What value must be given to p so that one of the solutions of the equation $x^2 - (p + 2)x + (p + 10) = 0$ may be five times the other?

18. A man travels a certain distance in $2\frac{1}{2}$ hours by walking part of it at 3 m.p.h. and cycling the remainder at 8 m.p.h. If the speeds with which the respective parts were covered had been interchanged he would have completed the journey in 25 minutes less time. Find the length of the journey.

19. Show that for real values of x the expression $\frac{x^2+x+2}{x-1}$ cannot lie between -1 and 7.

20. The perimeter of a rectangle is 46 cm. and its diagonal is 17 cm. Find the length of the sides.

CHAPTER II

INDICES AND LOGARITHMS

4. We have already made some acquaintance with the index notation and its use. Thus $x \times x \times x \times x$ is written x^4, the number 4 being described as the index which shows the power to which x has been raised. Similarly $x \times x \times x \times x \times x \times x \times x$ is written x^7.

The product $x^7 \times x^4 = x \times x \times x \times x \times x \times x \times x \times x \times x \times x \times x$
$= x^{11}$.

Thus $\qquad x^7 \times x^4 = x^{11}$.

Similarly, in general $\mathbf{x^m \times x^n = x^{m+n}}$, where m and n are positive whole numbers.

$$\text{The quotient } x^7 \div x^4 = \frac{x \times x \times x \times x \times x \times x \times x}{x \times x \times x \times x}$$
$$= x^3.$$

In general, therefore, $\mathbf{x^m \div x^n = x^{m-n}}$ $(m > n)$, where m and n are positive whole numbers.

The value of $\qquad (x^2)^3 = x^2 \times x^2 \times x^2$
$= x^6$.

Notice also that $(x^3)^2 = x^6$.

Similarly, in general $\mathbf{(x^m)^n = (x^n)^m = x^{mn}}$, where m and n are positive whole numbers.

The following exercise provides further illustrations of these *three index laws*.

EXERCISE V

1. Write down the values of

(*a*) $x^2 \times x^3$.	(*f*) $b^{16} \div b^5$.
(*b*) $p^4 \times p^{10}$.	(*g*) $(n^2)^4$.
(*c*) $k^7 \times k^2$.	(*h*) $(r^5)^3$.
(*d*) $a^9 \div a$.	(*i*) $(t^3)^5$.
(*e*) $m^{17} \div m^{12}$.	(*j*) $(h^4)^2$.

2. Simplify the following:

(a) $\frac{a^4 \times a^6}{a^3}$.

(b) $(x^2)^3 \div x^4$.

(c) $\frac{(m^2)^5 \times (m^5)^2}{2m^5}$.

(d) $(3k)^2 + (2k)^3$.

(e) $(3d^2)^2 \times (2d)^3$.

(f) $(3d^2)^2 \div (2d)^3$.

(g) $\frac{3g^4 \times 5g^6}{(3g)^2}$.

(h) $\frac{10h^2}{15h^3 \times 2h^7}$.

(i) $\frac{(5l^2)^3}{(2l^5)^2}$.

(j) $\frac{4p^2 \times 5p^3 \times 6p^4}{7p^5 \times 8p^6}$.

3. Find x in each of the following equations:

(a) $2^x = 256$.

(b) $4^x = 1024$.

(c) $3^x = 243$.

(d) $m^3 \times x = m^5$.

(e) $p^5 \times x = p^9$.

(f) $ab^3 \times x = a^2b^4$.

(g) $h^7 \div x = h^2$.

(h) $k^{10} \div x = k$.

(i) $2m^{12} \div x = \frac{m^7}{16}$.

(j) $x \times 10^5 = 250000$.

4. Use the index notation to write the following large numbers in a more compact form (*e.g.*, $3650000 = 3{\cdot}65 \times 10^6$):

(a) 478600.

(b) 5700000000.

(c) 634700.

(d) 8750000.

(e) 98700000000.

5. So far the indices which we have considered have all been positive integers, and it is clear that the meaning we have given to x^m no longer holds if m is fractional or negative. Thus, if $m = \frac{3}{5}$, the definition of x^m as meaning $x \times x \times x$ for m factors is useless. If we decide that the three index laws—viz.,

(i) $x^m \times x^n = x^{m+n}$,

(ii) $x^m \div x^n = x^{m-}$

(iii) $(x^m)^n = x^{mn}$,

shall be good for **all** values of m and n, we can then assign meanings to x^m when m is fractional or negative.

EXAMPLE 1

To find the meaning of $x^{\frac{1}{2}}$.

Since $x^m \times x^n = x^{m+n}$

let $m = n = \frac{1}{2}$.

Then
$$x^{\frac{1}{2}} \times x^{\frac{1}{2}} = x^{\frac{1}{2}+\frac{1}{2}}$$
$$= x.$$
$$x^{\frac{1}{2}} = \sqrt{x}.$$

In general, $\mathbf{x^{\frac{1}{p}} = \sqrt[p]{x}}.$

EXAMPLE 2

To find the meaning of $x^{\frac{3}{5}}$.

Since
$$(x^m)^n = x^{mn}$$
$$(x^{\frac{3}{5}})^5 = x^3.$$
$$x^{\frac{3}{5}} = \sqrt[5]{x^3}$$ (taking the fifth root of both sides of the equation).

In general, $\mathbf{x^{\frac{p}{q}} = \sqrt[q]{x^p}}.$

EXERCISE VI

Find the value of

1. $4^{\frac{1}{2}}$.
2. $8^{\frac{1}{3}}$.
3. $16^{\frac{1}{2}}$.
4. $16^{\frac{1}{4}}$.
5. $27^{\frac{1}{3}}$.
6. $49^{\frac{1}{2}}$.
7. $32^{\frac{1}{5}}$.
8. $125^{\frac{1}{3}}$.
9. $256^{\frac{1}{4}}$.
10. $1024^{\frac{1}{10}}$.
11. $8^{\frac{2}{3}}$.
12. $16^{\frac{3}{4}}$.
13. $27^{\frac{2}{3}}$.
14. $32^{\frac{4}{5}}$.
15. $125^{\frac{2}{3}}$.
16. $256^{\frac{3}{4}}$.
17. $81^{\frac{3}{4}}$.
18. $64^{\frac{2}{3}}$.
19. $216^{\frac{2}{3}}$.
20. $343^{\frac{2}{3}}$.
21. $10{,}000^{\frac{3}{4}}$.
22. $2401^{\frac{1}{4}}$.
23. $15{,}625^{\frac{5}{6}}$.
24. $1024^{\frac{3}{5}}$.
25. $6561^{\frac{1}{8}}$.
26. $2048^{\frac{9}{11}}$.
27. $2401^{\frac{3}{4}}$.
28. $1296^{\frac{3}{4}}$.
29. $625^{\frac{1}{4}}$.
30. $2187^{\frac{5}{7}}$.

6. In a similar manner we can assign a meaning to x^m when m is negative.

EXAMPLE 1

To find the meaning of x^{-4}.

Since $x^m \div x^n = x^{m-n}$, then, if $m = 3$ and $n = 7$, we shall have

$$x^3 \div x^7 = x^{3-7} = x^{-4}.$$

But $$x^3 \div x^7 = \frac{x^3}{x^7} = \frac{1}{x^4}.$$

$$x^{-4} = \frac{1}{x^4}.$$

In general, $$\mathbf{x^{-n} = \frac{1}{x^n}}.$$

EXAMPLE 2

To find the meaning of x^0.

Since $x^m \div x^n = x^{m-n}$, then, if $m = n$, we shall have

$$x^m \div x^m = x^{m-m}$$
$$= x^0.$$

But $$x^m \div x^m = 1.$$

Hence $\mathbf{x^0 = 1}$ for all values of x except zero. In this case 0^0 has no meaning.

EXERCISE VII

Find the value of

1. 3^{-2}.
2. 5^{-3}.
3. 3^{-3}.
4. $\frac{1}{2^{-5}}$.
5. $\frac{4}{2^{-7}}$.
6. $(-k)^{-3}$.
7. $(2m)^{-2}$.
8. $\left(\frac{2}{m}\right)^{-2}$
9. $16^{-\frac{1}{2}}$.
10. $27^{-\frac{1}{3}}$.
11. $\frac{1}{36^{-\frac{1}{2}}}$.
12. 8^0.

EXERCISE VIII

(General Exercise on Indices)

1. Express with positive indices

(a) $3p^{-\frac{1}{2}}$.
(b) $7m^{-\frac{4}{5}}$.
(c) $5a^{-3}b^2$.
(d) $7xy^{-5}$.
(e) $(h^{-4})^2$.
(f) $(h^2)^{-4}$.
(g) $(3d^{-2})^{\frac{1}{2}}$.
(h) $\sqrt[3]{x^{-6}}$.
(i) $\sqrt{\dfrac{a^2}{b^{-2}}}$.
(j) $(27p^{-3}q^3)^{\frac{1}{3}}$.

2. Find the values of 2^n when $n = -3, -2, -1, 0, 1, 1\frac{1}{2}, 2, 2\frac{1}{2}, 3$.

3. If $y = 2^x$, use the calculations of the preceding question to draw the graph of 2^x over this range of values.

4. Solve $x^{\frac{1}{2}} + 2x^{-\frac{1}{2}} = 3$.

Hint. Substitute $x^{\frac{1}{2}} = a$, and solve for a.

5. Solve $2(3x - 2x^{-1}) = 5$.

6. Using the results of Question 2, draw the graph of $y = 2^x + 2^{-x}$. What is the minimum value of y?

7. Find the product of

(a) $x^{\frac{1}{2}} + 1$ and x^{-2}.
(b) $3m^{-5} + 7$ and m^2.
(c) $1 + x^{\frac{1}{2}} + x^{\frac{1}{4}}$ and $1 + x^{\frac{1}{2}}$.
(d) $3x^{\frac{2}{3}} + 5$ and $2x^{\frac{1}{3}} + 1$.
(e) $x + 2 + x^{-1}$ and $x^{\frac{1}{2}} - 3$.
(f) $x + x^{-1}y^{-1} + y$ and $x^{-1}y^{-1}$.
(g) $5p^{\frac{3}{4}} + q^{\frac{1}{2}}$ and $5p^{\frac{1}{2}} + q^{\frac{3}{4}}$.
(h) $x^{\frac{1}{2}} + 1$ and $x^{\frac{1}{2}} - 1$.
(i) $1 + a$ and $a^{-1} + a^{-2}$.

8. Divide

(a) $x^2 + x^{\frac{3}{2}} - x^{\frac{1}{2}} - 1$ by $x - 1$.
(b) $3x + 8x^{\frac{3}{4}} - 3x^{\frac{1}{2}}$ by $x^{\frac{1}{2}} + 3x^{\frac{1}{4}}$.
(c) $10x^{-\frac{2}{3}} + 14x^{-\frac{1}{3}} + 4 + 2x^{\frac{2}{3}} + 2x$ by $2x^{-\frac{1}{3}} + 2$.
(d) $3a - 9a^{\frac{3}{5}}b^{\frac{3}{5}} + a^{\frac{2}{5}}b^{\frac{2}{5}} - 3b$ by $3a^{\frac{3}{5}} + b^{\frac{2}{5}}$.
(e) $a^{\frac{2}{3}} + 2a^{\frac{1}{3}}b^{\frac{2}{3}} - 3b^{\frac{4}{3}}$ by $a^{\frac{1}{3}} + 3b^{\frac{2}{3}}$.

9. Find the square root of

(*a*) $a^{\frac{2}{3}} + 4a^{\frac{1}{3}}b^{\frac{1}{3}} + 4b^{\frac{2}{3}}$.
(*b*) $1 - 6x^{\frac{1}{5}} + 11x^{\frac{2}{5}} - 6x^{\frac{3}{5}} + x^{\frac{4}{5}}$.
(*c*) $a^{\frac{8}{7}} + 6a - a^{\frac{6}{7}} - 30a^{\frac{5}{7}} + 25a^{\frac{4}{7}}$.
(*d*) $4a^{\frac{2}{3}} + 12a + 5a^{\frac{4}{3}} - 6a^{\frac{5}{3}} + a^2$.
(*e*) $9x^{-\frac{2}{5}} + 6x^{-\frac{1}{5}}y^{-\frac{1}{3}} + y^{-\frac{2}{3}}$.

LOGARITHMS

7. Look at the following powers of 3:

$3^0 = 1$.	$3^4 = 81$.
$3^1 = 3$.	$3^5 = 243$.
$3^2 = 9$.	$3^6 = 729$.
$3^3 = 27$.	

Each of the numbers 1, 3, 9, 27, 81, 243, and 729 is expressed as a power of 3. In such a case 3 is described as the *base*, and its index is called the *logarithm*, of the number.

Thus 4 is the logarithm of 81 to the base 3, and 6 is the logarithm of 729 to the base 3, and so on. Notice in particular that a **logarithm is an index.**

Similarly, $2^7 = 128$, so that the logarithm of 128 to the base 2 is 7. We write this $\log_2 128 = 7$. In the same way, using the results above, we have $\log_3 81 = 4$ and $\log_3 729 = 6$.

EXERCISE IX

1. Say what are the logarithms of the following numbers if 2 is the base used: 8, 32, 256, 1024.

2. Say what are the logarithms of the following numbers if 5 is the base used: 1, 5, 125.

3. Give the logarithms of the following numbers to the base 10: 10,000, 100, 10, 1, 0·1, 0·01.

4. Using the base 4, what are the logarithms of 16, 64, and 1024? Verify that $\log_4 16 + \log_4 64 = \log_4 1024$.

5. Prove that $\log_2 8 + \log_2 32 = \log_2 256$.

Hint. Since $2^3 = 8$
$\therefore \log_2 8 = 3.$
Similarly, $2^5 = 32.$
$\therefore \log_2 32 = 5$
and $2^8 = 256.$
$\therefore \log_2 256 = 8.$

6. Prove the following:

(*a*) $\text{Log}_3 9 + \log_3 81 = \log_3 729.$
(*b*) $\text{Log}_2 16 + \log_2 64 = \log_2 1024.$
(*c*) $\text{Log}_4 16 = 2 \log_4 4.$
(*d*) $\text{Log}_2 8^2 = 2 \log_2 8.$

7. Prove the following:

(*a*) $\text{Log}_2 64 - \log_2 16 = \log_2 4.$
(*b*) $\text{Log}_5 125 - \log_5 5 = \log_5 25.$
(*c*) $\text{Log}_3 729 - \log_3 9 = \log_3 81.$
(*d*) $\text{Log}_3 \sqrt{81} = \frac{1}{2} \log_3 81.$

8. Prove that the logarithm of 1 is zero for any base.
9. Solve the following equations:

(*a*) $\text{Log}_x 125 = 3.$
(*b*) $\text{Log}_x 512 = 9.$
(*c*) $\text{Log}_x 2187 = 7.$

10. Find the value of $3^{\log_3 7}$.

Hint. Let $3^{\log_3 7} = y$; then $\log_3 7 = \log_3 y$. Therefore $y = 7$.

Write down the values of

(*a*) $5^{\log_5 2}$. (*b*) $3^{\log_3 9}$. (*c*) $7^{\log_7 5}$.

8. We can now investigate some of the important properties of logarithms in general—*i.e.*, logarithms to any base.
If $a^x = n$, then $\log_a n = x$.
Similarly, if $a^y = m$, then $\log_a m = y$.

The product $mn = a^x \times a^y$
$= a^{x+y}$ (first law of indices).
$\therefore \log_a mn = x + y.$

$$\mathbf{\log_a mn = \log_a m + \log_a n.}$$

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In other words, the logarithm of the product of two numbers is equal to the sum of their logarithms. Of course the same base must be used throughout. Numerical examples illustrating this rule in particular cases occur in Exercise IX, Nos. 4, 5, and 6.

The quotient $$\frac{m}{n} = \frac{a^x}{a^y}$$

$$= a^{x-y} \quad \text{(second law of indices).}$$

$$\log_a \frac{m}{n} = x - y.$$

$$\mathbf{\log_a \frac{m}{n} = \log_a m - \log_a n.}$$

The logarithm of the quotient of two numbers is equal to the difference between the logarithms of the dividend and divisor. Some further examples of the properties of logarithms in general will be found in the following exercise.

EXERCISE X

1. Prove that $\mathbf{\log_a m^k = k \log_a m}$.

2. Prove that $\mathbf{\log_a \sqrt[k]{m} = \frac{1}{k} \log_a m}$.

3. Express as single logarithms

   (*a*) $\text{Log}_a l + \log_a m + \log_a n$. (*d*) $\frac{1}{2}(\log_a m + \log_a n)$.
   (*b*) $\text{Log}_a l + \log_a m - \log_a n$. (*e*) $\frac{1}{2}\log_a(m + n)$.
   (*c*) $\text{Log}_a l - \log_a m - \log_a n$.

4. Prove that $\log_a m = -\log_a \frac{1}{m}$.

5. Find an equation connecting m and n in each of the following:

   (*a*) $\text{Log}_a m = 3 \log_a n$. (*d*) $\text{Log}_a m - \log_a n = 1$.
   (*b*) $\text{Log}_a m = -3 \log_a n$. (*e*) $\text{Log}_a m + 2 = 2 \log_a n + 3$.
   (*c*) $\text{Log}_a m + \log_a n = 1$.

6. Write down the values of

(a) $a^{\log_a m}$. (b) $a^{-\log_a m}$. (c) $a^{\log_a a}$.

7. If $a^x = N$ and $b^y = N$, prove that $\mathbf{\log_a N = \log_b N \,.\, \log_a b}$.

Hint. $\text{Log}_a (b^y) = y \log_a b$
$= \log_b N \,.\, \log_a b.$

But $\log_a (b^y) = \log_a N.$

Hence the result follows. This result is useful for transforming logarithms from one base to another.

8. Prove that $\mathbf{\log_a b \times \log_b a = 1}$.

9. For numerical work the base used is always 10. Thus $\log_{10} 100 = 2$, and so on. This practice is so universal that it is usual to dispense with the suffix 10 and write simply $\log 100 = 2$. When any other base is used it must always be indicated. With 10 as base we have

$$
\begin{array}{rlrl}
1000 &= 10^3. & \log 1000 &= 3. \\
100 &= 10^2. & \log 100 &= 2. \\
10 &= 10^1. & \log 10 &= 1. \\
1 &= 10^0. & \log 1 &= 0. \\
\cdot 1 &= \frac{1}{10} = 10^{-1}. & \therefore \log \cdot 1 &= -1. \\
\cdot 01 &= \frac{1}{100} = 10^{-2}. & \therefore \log \cdot 01 &= -2. \\
\cdot 001 &= \frac{1}{1000} = 10^{-3}. & \log \cdot 001 &= -3.
\end{array}
$$

Hence the logarithms of numbers between 100 and 1000 lie between 2 and 3. Thus $\log 849 = 2\cdot 9289$, so that $10^{2\cdot 9289} = 849$. Similarly, the logarithms of numbers between 1 and 10 lie between 0 and 1. Thus $\log 6\cdot 54 = \cdot 8156$, so that $10^{\cdot 8156} = 6\cdot 54$. The decimal part of a logarithm is called the *mantissa*, and the whole number is called the *characteristic*. The characteristic of a logarithm can always be written down at sight, and tables are provided from which the mantissa can be obtained.

Once the mantissa of a number between 1 and 10 is known the logarithm of any number having the same significant figures can be written down.

Thus,

$$654 = 10^2 \times 6{\cdot}54.$$
$$65{\cdot}4 = 10^1 \times 6{\cdot}54.$$
$$6{\cdot}54 = 10^0 \times 6{\cdot}54. \quad \text{(Note } 10^0 = 1.)$$
$$\cdot 654 = 10^{-1} \times 6{\cdot}54.$$
$$\cdot 0654 = 10^{-2} \times 6{\cdot}54, \text{ and so on.}$$

Hence

$$\begin{aligned}\log 654 &= \log(10^2 \times 6{\cdot}54)\\ &= \log 10^2 + \log 6{\cdot}54\\ &= 2 \log 10 + \log 6{\cdot}54\\ &= 2 + \cdot 8156. \quad (\text{Log } 6{\cdot}54 = \cdot 8156.)\end{aligned}$$

Thus we have

log 654 =	$2 + \cdot 8156$,	which is written	$2{\cdot}8156$.
65·4 =	$1 + \cdot 8156$		$1{\cdot}8156$.
6·54 =	$0 + \cdot 8156$		$\cdot 8156$.
·654 =	$-1 + \cdot 8156$		$\bar{1}{\cdot}8156$.
·0654 =	$-2 + \cdot 8156$		$\bar{2}{\cdot}8156$.

Notice that in the case of decimals—*e.g.*, ·645—we write log ·654 = $\bar{1}{\cdot}8156$, and *not* $-\cdot 1844$; similarly, log ·0654 = $\bar{2}{\cdot}8156$, and *not* $-1{\cdot}1844$. By so doing the mantissa is the same for all numbers having the same significant figures—a great advantage in the construction of tables of logarithms.

The following is a further example to illustrate this important principle.

Example

If log 2·374 = ·3754, write down the logarithms of 2374, 237·4, 23·74, 2·374, ·2374, ·02374, ·002374, and ·0002374.

We have

$2374 = 10^3 \times 2{\cdot}374.$	log 2374 = 3·3754.
$237{\cdot}4 = 10^2 \times 2{\cdot}374.$	log 237·4 = 2·3754.
$23{\cdot}74 = 10^1 \times 2{\cdot}374.$	log 23·74 = 1·3754.
$2{\cdot}374 = 10^0 \times 2{\cdot}374.$	log 2·374 = ·3754.
$\cdot 2374 = 10^{-1} \times 2{\cdot}374.$	log ·2374 = $\bar{1}{\cdot}3754$.
$\cdot 02374 = 10^{-2} \times 2{\cdot}374.$	log ·02374 = $\bar{2}{\cdot}3754$.
$\cdot 002374 = 10^{-3} \times 2{\cdot}374.$	log ·002374 = $\bar{3}{\cdot}3754$.
$\cdot 0002374 = 10^{-4} \times 2{\cdot}374.$	log ·0002374 = $\bar{4}{\cdot}3754$.

EXERCISE XI

1. If log 4·361 = ·6396, write down the logarithms of 4361, 436·1, 43·61, ·4361, ·04361, and ·004361.

2. If log 15·62 = 1·1937, write down the logarithms of 156·2, 1·562, ·1562, and ·00001562.

3. If log 2·459 = ·3908, say what numbers have the following logarithms: 1·3908, 2·3908, 4·3908, $\bar{1}$·3908, and $\bar{3}$·3908.

4. If log 684·2 = 2·8352, say what numbers have the following logarithms: 4·8352, 1·8352, $\bar{1}$·8352, and $\bar{3}$·8352.

5. If log 7·932 = ·8994, write down the logarithms of 793·2, 79·32, ·007932, $(7{\cdot}932)^2$, $\sqrt{7{\cdot}932}$, and $(793{\cdot}2)^{\frac{1}{3}}$.

6. If log 8·254 = ·9167, write down the logarithm of $\frac{1}{(8{\cdot}254)^2}$.

Hint.

$$\frac{1}{(8{\cdot}254)^2} = (8{\cdot}254)^{-2}.$$

$$\begin{aligned} \log \frac{1}{(8{\cdot}254)^2} &= \log (8{\cdot}254)^{-2} \\ &= -2 \log 8{\cdot}254 \\ &= -2 \times {\cdot}9167 \\ &= -1{\cdot}8334. \end{aligned}$$

Remembering that the mantissa of a logarithm is always written as a positive quality, we must write $-1{\cdot}8334 = -2 + {\cdot}1666$.

$$\therefore \log \frac{1}{(8{\cdot}254)^2} = \bar{2}{\cdot}1666.$$

7. If log 7·564 = ·8787, write down the logarithms of $\frac{1}{7{\cdot}564}$, $\frac{1}{(7{\cdot}564)^2}$, $\frac{1}{75{\cdot}64}$, $\frac{1}{\sqrt{756{\cdot}4}}$, and $\frac{1}{7564}$.

8. If log 95·41 = 1·9795, write down the numbers whose logarithms are 4·9795, 2·9795, $\bar{3}$·9795, and $\bar{5}$·9795.

9. If log 2 = ·3010 and log 3 = ·4771, write down the values of log 6, log 12 (*i.e.*, $2^2 \times 3$), log 18, log 36, and log 24.

10. Write down the values of log $\sqrt{12}$, log $\sqrt[3]{18}$, log ·036, and log 24^2.

10. From what we have just done it is easy to see that the only tabulation of logarithms required is for numbers between 0 and 10. This has been done to varying degrees of accuracy. Tables to four places of decimals will be found on pages 166–167, and these will give answers (not necessarily correct to the fourth significant figure) sufficiently accurate for most purposes. Five, six, seven, and in some cases more places of decimals are used in some computations, and specially prepared tables for the higher degree of accuracy are available.

The following is an extract from the tables on page 166:

LOGARITHMS

	0	1	2	3	4	5 ↓	6	7	8	9	1	2	3	4	5	6	7 ↓	8	9
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201	2	4	6	8	11	13	15	17	19
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404	2	4	6	8	10	12	14	16	18
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598	2	4	6	8	10	12	14	15	17
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784	2	4	6	7	9	11	13	15	17
→ 24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962	2	4	5	7	9	11	12	14	16
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133	2	3	5	7	9	10	12	14	15
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298	2	3	5	7	8	10	11	13	15
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456	2	3	5	6	8	9	11	13	14
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609	2	3	5	6	8	9	11	12	14
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757	1	3	4	6	7	9	10	12	13

Notice that no decimal points are printed. These are unnecessary, as the mantissæ only are tabulated, the characteristic being added by inspection, using the methods just described. Employing the above extract to find the logarithm of 2·457, we find the horizontal line beginning 24 and move along it to the column headed 5, where we find 3892. The 'difference' for the fourth digit, 7, is found in the appropriate 'difference' column, and is 12. This is added to 3892, and we have $3892 + 12 = 3904$.

Then $\log\ 2{\cdot}457 = {\cdot}3904$

and consequently $\log\ 24{\cdot}57 = 1{\cdot}3904$

$\log\ 245{\cdot}7 = 2{\cdot}3904$

$\log\ {\cdot}2457 = \bar{1}{\cdot}3904$

$\log\ {\cdot}002457 = \bar{3}{\cdot}3904$, and so on.

These tables can be used for the converse problem of finding a number when its logarithm is known, but it is easier to use antilogarithm tables for this purpose. The extract of the antilogarithms table corresponding to the extract shown on page 32 is as follows:

ANTILOGARITHMS

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
·34	2188	2193	2198	2203	2208	2213	2218	2223	2228	2234	1	1	2	2	3	3	4	4	5
·35	2239	2244	2249	2254	2259	2265	2270	2275	2280	2286	1	1	2	2	3	3	4	4	5
·36	2291	2296	2301	2307	2312	2317	2323	2328	2333	2339	1	1	2	2	3	3	4	4	5
·37	2344	2350	2355	2360	2366	2371	2377	2382	2388	2393	1	1	2	2	3	3	4	4	5
·38	2399	2404	2410	2415	2421	2427	2432	2438	2443	2449	1	1	2	2	3	3	4	4	5
·39	2455	2460	2466	2472	2477	2483	2489	2495	2500	2506	1	1	2	2	3	3	4	5	5
·40	2512	2518	2523	2529	2535	2541	2547	2553	2559	2564	1	1	2	2	3	4	4	5	5
·41	2570	2576	2582	2588	2594	2600	2606	2612	2618	2624	1	1	2	2	3	4	4	5	5
·42	2630	2636	2642	2649	2655	2661	2667	2673	2679	2685	1	1	2	2	3	4	4	5	6
·43	2692	2698	2704	2710	2716	2723	2729	2735	2742	2748	1	1	2	3	3	4	4	5	6

Using this to find the number whose logarithm is ·3904, we start with the horizontal line commencing ·39 (notice that the decimal point is inserted) and move along to the column headed 0, where we find 2455. The 'difference' for the fourth digit, 4, is found in the appropriate 'difference' column, and is 2. This is added, and we have 2455 + 2 = 2457. Hence the number whose logarithm is ·3904 has 2457 as significant figures. Since the logarithm lies between 0 and 1 the number must lie between 1 and 10. Hence we have

the number whose logarithm is ·3904 is 2·457
,, 1·3904 is 24·57
,, 2·3904 is 245·7
,, $\bar{3}$·3904 is ·002457, and so on.

In using antilogarithm tables remember that only the mantissa is used, and that the characteristic indicates where the decimal point must be placed when the digits have been found.

EXERCISE XII

1. From the tables find the logarithms of the following numbers:

(*a*) 25·62. (*b*) 74·81. (*c*) 849·2. (*d*) ·7654. (*e*) 914·8. (*f*) 817600. (*g*) 8910. (*h*) 7·325. (*i*) ·00614. (*j*) 82·12.

2. Find the numbers whose logarithms are

(*a*) 1·8946. (*b*) $\bar{1}$·8763. (*c*) 2·3814. (*d*) 3·1234. (*e*) $\bar{2}$·1106. (*f*) $\bar{1}$·4611. (*g*) ·5753. (*h*) 1·6413. (*i*) $\bar{1}$·2498. (*j*) $\bar{2}$·4132.

3. Use logarithms to find the value of

(*a*) $(47{\cdot}13)^2$. (*b*) $(845{\cdot}9)^{\frac{1}{2}}$. (*c*) $\sqrt[3]{238{\cdot}4}$. (*d*) $\frac{1}{76{\cdot}11}$. (*e*) $\frac{1}{(76{\cdot}11)^2}$. (*f*) $\frac{1}{\sqrt{76{\cdot}11}}$.

4. Solve the equation $3^x = 7$.

Hint. Take logarithms of both sides, then $x \log 3 = \log 7$—*i.e.*, $\cdot 4771x = \cdot 8451$, which is a simple equation for x.

5. Find x to three significant figures in each of the following:

(*a*) $4^x = 9$. (*b*) $5^{x+2} = 8^3$. (*c*) $2^{x-1} = 3^{x-2}$.

6. Find the value of 24·28 × 17·65.

Hint. The following is a suitable method for arranging the work. A modification of this will be given later, when greater facility in the use of logarithms and logarithm tables has been acquired.

$$\begin{aligned} 24{\cdot}28 \times 17{\cdot}65 &= 10^{1{\cdot}3852} \times 10^{1{\cdot}2467} \\ &= 10^{2{\cdot}6319} \\ &= 428{\cdot}5. \end{aligned}$$

7. Find the value of

(a) $29{\cdot}47 \times 63{\cdot}21$.

(b) $\cdot 3516 \times 4{\cdot}718$.

(c) $765{\cdot}2 \times \cdot 01764 \times 3{\cdot}214$.

(d) $8{\cdot}125 \times 3{\cdot}142 \times \cdot 007613$.

(e) $\dfrac{37{\cdot}85}{23{\cdot}19}$.

(f) $\dfrac{146{\cdot}2}{1{\cdot}015}$.

(g) $\dfrac{725{\cdot}1 \times 8{\cdot}341}{2{\cdot}149}$.

(h) $\dfrac{3{\cdot}004 \times 23{\cdot}05}{7{\cdot}635}$.

(i) $\dfrac{847{\cdot}6}{19{\cdot}68 \times 9{\cdot}014}$.

(j) $\dfrac{23{\cdot}79 \times 13{\cdot}14}{3{\cdot}256 \times 2{\cdot}635}$.

(k) $(8{\cdot}761)^2 \times (4{\cdot}891)^2$.

(l) $\dfrac{(14{\cdot}38)^3}{(7{\cdot}64)^2}$.

(m) $3{\cdot}142 \times (4{\cdot}35)^2$.

(n) $\dfrac{2914 \times 3{\cdot}015}{(14{\cdot}61)^3}$.

(o) $15{\cdot}76 \div (1{\cdot}098)^2$.

(p) $\sqrt{175{\cdot}6} \div \sqrt{2{\cdot}846}$.

(q) $(49{\cdot}75)^2 \times \sqrt{3{\cdot}064}$.

(r) $(14{\cdot}21)^3 + (13{\cdot}15)^3$.

(s) $\dfrac{(33{\cdot}52)^2 + (451{\cdot}1)^{\frac{1}{2}}}{19{\cdot}81}$

11. The preceding exercise will have afforded some practice in calculation by means of logarithms, and will have indicated their value in complicated arithmetical work. By writing each number as a power of 10 (see Question 6, Exercise XII) the whole operation is shown very clearly, and the fundamental fact, that *a logarithm is an index*, is well illustrated. Now that more skill in manipulation has been acquired this method of setting out the work can be shortened considerably. Although all logarithms are arranged in columns at the side, and only the sum and its answer appear on the left-hand side, it must be remembered that the actual operations that are being done are still additions or subtractions of indices. For this reason we shall repeat Question 6, Exercise XII, showing how the calculation may now be set out.

EXAMPLE 1

Find the value of

$$24{\cdot}28 \times 17{\cdot}65.$$

$$24{\cdot}28 \times 17{\cdot}65 = 428{\cdot}5.$$

	NUMBER	LOGARITHM
	24·28 17·65	1·3852 1·2467
Add		2·6319
	428·5	

Study the following examples carefully, and in particular the notes following Examples 3 and 4.

EXAMPLE 2

Find the value of

$$\frac{137{\cdot}8 \times {\cdot}7615}{\sqrt{4{\cdot}372}}.$$

$$\frac{137{\cdot}8 \times {\cdot}7615}{\sqrt{4{\cdot}372}} = 50{\cdot}20.$$

	NUMBER	LOGARITHM
	137·8 ·7615	2·1393 $\bar{1}$·8817
Add		2·0210
Divide by 2	4·372	·6407 ·3203
	Numerator Denominator	2·0210 ·3203
Subtract		1·7007
	50·20	

EXAMPLE 3

Find the value of

·01764 ÷ ·0003915.

	Number	Logarithm
	·01764 ·0003915	$\bar{2}$·2465 $\bar{4}$·5927
Subtract		1·6538
·01764 ÷ ·0003915 = 45·06.	45·06	

Note this subtraction of logarithms carefully. Remember that the mantissa is always positive, and that consequently the 'carrying figure' when passing the decimal point is also positive. The subtraction of the characteristics follows the ordinary rule of signs. In this case $\bar{2} - (-3) = +1$. As work with negative characteristics is apt to be confusing, it is a good plan to check the subtraction by adding the difference to the bottom line. In this case 1·6538 + $\bar{4}$·5927 = $\bar{2}$·2465.

EXAMPLE 4

Find the value of $\sqrt[3]{\cdot 04865}$.

	Number	Logarithm
	·04865	$\bar{2}$·6870 = $\bar{3}$ + 1·6870
Divide by 3		$\bar{1}$·5623
$\sqrt[3]{\cdot 04865} = \cdot 3651$.	·3651	

Note the division of $\bar{2}$·6870 by 3. Remember that the mantissa (·6870) is positive, while the characteristic ($\bar{2}$) is negative. In order to facilitate the division 1 is added to the mantissa (making 1·6870) and subtracted from the characteristic ($\bar{2} - 1 = \bar{3}$). This gives $\bar{3}$ + 1·6870. The division by 3 is now straightforward.

EXERCISE XIII

1. Find by logarithms the value of

(a) $\dfrac{241\cdot 6 \times 3\cdot 142 \times \cdot 8463}{(2\cdot 768)^2}$.

(b) $\sqrt{\dfrac{48\cdot 31 \times 29\cdot 56}{134\cdot 5}}$.

(c) 347·2 × ·0006392.

(d) $(\cdot 0786)^2$.

(e) $\sqrt[3]{\cdot 0786}$.

(f) $\dfrac{(29\cdot 27)^2 \times \cdot 004613}{1\cdot 576}$.

(g) $\dfrac{(\cdot 7632)^3}{\sqrt{15\cdot 15}}$.

(h) $\dfrac{\sqrt{765\cdot 2 \times 4\cdot 375}}{(1\cdot 04)^2}$.

(i) $\dfrac{328\cdot 7 \times \sqrt{5\cdot 914} \times (22\cdot 14)^2}{\cdot 0315 \times (84\cdot 3)^3}$.

(j) $\dfrac{(\cdot 0706)^5 \times \sqrt{(1\cdot 543)}}{\sqrt[3]{1\cdot 543} \times \cdot 0245}$.

2. The formula $V = \frac{4}{3}\pi r^3$ gives the volume of a sphere of radius r. If $r = 2\cdot 25$ inches and $\pi = 3\cdot 142$, find V.

3. The formula $T = 2\pi\sqrt{\dfrac{l}{g}}$ gives the time in seconds for a complete swing of a pendulum of length l feet. Find T when $l = 2\cdot 85$ and $g = 32\cdot 2$.

4. The formula $V = \frac{1}{3}\pi r^2 h$ gives the volume of a cone. Find V when $r = 3\cdot 75$ inches and $h = 10$ inches.

5. The formula $V = \dfrac{\pi h}{6}(3R^2 + h^2)$ gives the volume of a segment of a sphere of height h. Find V when $R = 8$ cm. and $h = 3\cdot 746$ cm.

6. The formula $C = \pi\left(\dfrac{a+b}{2} + \sqrt{\dfrac{a^2+b^2}{2}}\right)$ gives the approximate circumference of an ellipse of semi-axes a and b. Find C when $a = 2\cdot 34$ inches and $b = 3\cdot 87$ inches.

7. Find the radius of a circle having the same circumference as the ellipse in the previous question.

8. The formula $V = (39B^2 + 25H^2 + 26BH)L \times \cdot 00003147$ gives the volume in imperial gallons of any cask, where

B = the bung diameter, H = the head diameter, and L = length of the cask, all measured in inches.

Find V when B = 30, H = 24, and L = 42.

9. The formula $d = \sqrt{\frac{3l(L - l)}{8}}$ gives the sag of a telegraph wire, where l = the span between two supports and L = actual length of the wire, all measurements being made in feet. Find d when $l = 60$ and L = 62·3.

10. The formula $H = \frac{PLAN}{33,000}$ gives the indicated horsepower of an engine, where P = the mean effective pressure in lb. per square inch, L = the length of the stroke in feet, A = the area in square inches of the cross-section of the cylinder, and N = the number of strokes per minute. Find H when P = 34, L = 12·25, A = 35·38, N = 235.

11. From the formula $K = \sqrt{\frac{v^2}{3 \cdot 4h}}$ find K when $v = 132$ and $h = \cdot 001346$.

12. Find the value of $\frac{e^x - e^{-x}}{e^x + e^x}$ when $e = 2 \cdot 7183$ and $x = 1 \cdot 6$.

13. The formula $R = 2 \cdot 5 + \frac{v^{\frac{5}{3}}}{65 \cdot 82}$ gives the resistance to motion of a train, where v = velocity in miles per hour and R = resistance in lb.

Find R when $v = 60$.

14. The formula $\frac{T}{T_o} = e^{\mu\theta}$ gives the ratio of the tensions in a rope passing round an axle. T and T_o are the tensions at the two ends, θ (measured in radians) is the angle subtended at the centre of the axle by the part of the rope in contact with it, and μ = the coefficient of friction.

Find T when $T_o = 54$ lb. weight, $\mu = \cdot 57$, $e = 2 \cdot 7183$, and $\theta = 1 \cdot 5$.

15. The temperatures and pressures of a perfect gas are connected by the equation $\frac{T_1}{T_2} = \left(\frac{p_1}{p_2}\right)^{\frac{\gamma - 1}{\gamma}}$

Find T_1 when $T_2 = 421$, $p_1 = 3{\cdot}27$, $p_2 = 4{\cdot}81$, and $\gamma = 1{\cdot}2$.

16. From the formula $E = 2 - \left(\frac{2}{k}\right)^{n-1}$ find E when $k = 2{\cdot}47$ and $n = 3{\cdot}5$.

17. The formula $A = P\left(1 + \frac{r}{100}\right)^n$ gives the amount (A) of £P in n years at r per cent. compound interest. Find A when $P = 1$, $r = 4{\cdot}5$, and $n = 25$.

18. The formula $\mu = {\cdot}2 + {\cdot}004\sqrt{v}$ connects the coefficient of friction between a belt and pulley with the velocity of the belt in feet per minute.

Find μ when $v = 1500$.

19. From the formula $H = 1{\cdot}37M^{4{\cdot}18}$ find H when $M = 835{\cdot}3$.

EXERCISE XIV (Revision Exercise)

(A)

1. Solve the following equations:

(a) $\frac{x+1}{x+3} = \frac{2x-1}{2x+1}$.

(b) $x^2 + xy + y^2 = 4$.

$2x - y = 6$.

2. At what times between 9 o'clock and 10 o'clock are the hands of a clock (a) coincident, (b) in a straight line?

3. What must be added to $9x^2 - 24x + 10$ in order to make it a perfect square?

Hence (or otherwise) find for what value of x the expression $9x^2 - 24x + 10$ is a minimum. What is this minimum value?

4. Find what values must be given to p and q so that $4x^4 - 4x^3 - 11x^2 + px + q$ may be a perfect square.

5. The difference between a number and its reciprocal is $7\frac{11}{30}$. Find the number.

(B)

6. The sum of the solutions of a quadratic equation is 6, and the sum of their reciprocals is $\frac{3}{4}$. Find the equation.

7. If $p = 2$, $q = 3$, and $r = 4$, find the values of

(*a*) pq^r. (*b*) p^rq. (*c*) $(pq)^r$.

8. Find *without using tables* the values of

(*a*) Log $7{\cdot}6 + \log 1{\cdot}5 - \log 1{\cdot}14$.

(*b*) $5^{\log_5 5}$.

9. Two pipes can fill a tank in 2 minutes 55 seconds if both are turned on together. If turned on separately one fills the tank in 2 minutes less time than the other. How long does each take separately?

10. If α and β are the two solutions of the equation $ax^2 + 2bx + c = 0$, find the quadratic equation whose solutions are $\frac{1}{\alpha}$ and $\frac{1}{\beta}$.

(C)

11. Simplify $\dfrac{(3x^2 + 2x - 21)(4x^2 - 9x - 9)}{12x^2 - 19x - 21}$.

12. Express $\dfrac{5x + 7}{(2x + 3)(x + 1)}$ as the sum of two partial fractions.

13. Find the value of

(*a*) $\dfrac{437{\cdot}1 \times 0{\cdot}0128}{1{\cdot}356}$. (*b*) $\sqrt{\dfrac{235{\cdot}1}{81{\cdot}46}}$.

14. The sum of two numbers is 11. The sum of their squares is 65. Find the numbers.

15. Find the value of

$$\frac{1 + \sqrt{3}}{\sqrt{3} + \sqrt{5}} + \frac{1 + \sqrt{5}}{\sqrt{3} - \sqrt{5}}$$

correct to two places of decimals.

(D)

16. Prove that for real values of x the value of the expression $\dfrac{x-1}{x^2+x+2}$ lies between -1 and $\frac{1}{7}$.

17. Simplify $\dfrac{(27)^{\frac{1}{3}} + (64)^{\frac{2}{3}}}{(\frac{1}{2})^{-2}}$.

18. Find the values of

(*a*) $(1{\cdot}764)^{2{\cdot}5}$. (*b*) $(1{\cdot}764)^{-2{\cdot}5}$.

19. Three lines are of lengths 6 cm., 13 cm., and 15 cm. What equal lengths must be added to each of these so that they may form the sides of a right-angled triangle?

20. By drawing the graphs of $y = x^2$ and $y = 6 - x$ solve the equation $x^2 + x - 6 = 0$.

Hence determine for what value of k the equation $x^2 + x + k = 0$ will have (*a*) equal, (*b*) imaginary solutions.

CHAPTER III

SERIES

12. A number of quantities arranged in a definite order form a series. Thus we may select the even numbers between 0 and 10. If these are arranged in the order 2, 4, 6, 8, they form a series. The following are some other examples of series:

$$2, 6, 18, 54, 162, 486.$$

It is not difficult to see that these numbers are not arranged in haphazard fashion, but that with the initial number 2 each succeeding number is three times its predecessor.

$$\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{7}, \frac{1}{8}, \frac{1}{9}, \frac{1}{10}.$$

This series consists of the reciprocals of the first ten numbers.

$$2, 6, 12, 20, 30, 42$$

This series of numbers consists of the products, 1×2, 2×3, 3×4, . and so on.

EXERCISE XV

Study the following series and describe how each is constructed:

1. 1, 3, 5, 7, 9,
2. 1, 3, 9, 27, 81,
3. $1, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \frac{1}{9},$
4. $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6},$
5. 2, 2, 2, 2, 2,

Thus in the series 2, 7, 12, 17, 22,

$$\begin{aligned}
7 &= 2 + 5 & &= 2 + 1 \times 5\\
12 &= 7 + 5 &= 2 + 5 + 5 &= 2 + 2 \times 5\\
17 &= 12 + 5 &= 2 + 2 \times 5 + 5 &= 2 + 3 \times 5\\
22 &= 17 + 5 &= 2 + 3 \times 5 + 5 &= 2 + 4 \times 5
\end{aligned}$$

so that if 2 is the first term of the series, then the

second term is $2 + 1 \times 5$
third term is $2 + 2 \times 5$
fourth term is $2 + 3 \times 5$

nth term is $2 + (n - 1)5$

EXAMPLE

Find the common difference, the nth term, and the 14th term of the following series: (*a*) 3, 7, 11, and (*b*) 8, 2, – 4,

(*a*) The series is 3, 7, 11,

The common difference $= \text{2nd term} - \text{1st term}$
$= 7 - 3$
$= 4.$

The nth term $= 3 + (n - 1)4$
$= 3 + 4n - 4$
$= 4n - 1.$

If $n = 14$, then $4n - 1 = 56 - 1$
$= 55.$

the 14th term of the series is 55.

(*b*) The series is 8, 2, – 4,

The common difference $= \text{2nd term} - \text{1st term}$
$= 2 - 8$
$= -6.$

The nth term $= 8 + (n - 1)(-6)$
$= 8 - 6n + 6$
$= 14 - 6n.$

If $n = 14$, then $14 - 6n = 14 - 84$
$= -70.$

the 14th term of the series is – 70.

EXERCISE XVI

1. Find the common difference, the nth term, and the 8th term in each of the following arithmetical progressions:

(*a*) 2, 4, 6,
(*b*) 2, 5, 8,
(*c*) 1, 8, 15,
(*d*) 2, − 1, − 4,
(*e*) 3, − 4, − 11,
(*f*) a, $2a$, $3a$,
(*g*) $5x^2$, $4x^2$, $3x^2$,
(*h*) 2, 2, 2,

2. Find the missing terms in each of the following arithmetical progressions:

(*a*) 2, —, 6,
(*b*) 3, 7, —,
(*c*) —, 10, 15,
(*d*) 3, —, − 5,
(*e*) 5, − 2, —,

3. Insert the missing terms in the following arithmetical progressions (these are called *arithmetic means*):

(*a*) 2, —, —, —, 14.

Hint. The three arithmetic means, *together* with the first term, 2, and the last term, 14, form an arithmetical progression of *five* terms in which 14 is the fifth term.

∴ if d is the common difference we have

$$14 = 2 + 4d,$$

from which

$$d = 3.$$

∴ the series is 2, 5, 8, 11, 14, so that the three arithmetic means which can be inserted between 2 and 14 are 5, 8, and 11.

(*b*) 3, —, —, —, —, 23.
(*c*) 7, —, —, —, 15.
(*d*) 4, —, —, —, —, − 16.
(*e*) − 8, —, —, —, 12.

4. The first term of an arithmetical progression is 8 and the eighth term is 57. What is the common difference?

5. The nth term of an arithmetical progression is given by the formula $7n - 2$. Write down the first three terms of the series. What is the 400th term?

6. Write down the first three terms of the series whose nth terms are

(a) $3n$.
(b) $3n + 5$.
(c) $3n - 5$.
(d) $3n^2$.
(e) $\frac{3}{n}$.
(f) $\frac{n}{3}$.
(g) 3^n.
(h) $1 + \frac{1}{n}$.
(i) $n^2 + 1$.
(j) $1 - 2n^2$.

7. What is the 9th term of the arithmetical progression whose nth term is $4n - 7$? What is the common difference?

14. In order to find the sum of a given number of terms of an arithmetical progression it is not necessary to add them up, as would be the case with a collection of numbers selected in a haphazard manner. The calculation can be simplified considerably by a device which will be illustrated in the following example.

EXAMPLE 1

Find the sum of the first 18 terms of the series 2, 5, 8,

The 18th term of this series is $2 + 17 \times 3 = 2 + 51 = 53$.
,, 17th ,, ,, ,, $2 + 16 \times 3 = 2 + 48 = 50$.
If S is the sum of the 18 terms, then

$$S = 2 + 5 + 8 + \quad + 50 + 53. \qquad (1)$$

As the order in which a sequence is added makes no difference to the sum, we can reverse the above order and write

$$S = 53 + 50 + 47 + \quad + 5 + 2 \qquad (2)$$

Adding equations (1) and (2), we have

$$2S = 55 + 55 + 55 + \quad + 55 + 55$$

Since there are 18 terms to be added,

$$\therefore 2S = 55 \times 18.$$

$$S = \frac{55 \times 18}{2}$$

$$= 495.$$

EXAMPLE 2

Find a formula for the sum (S_n) of the first n terms of the series 2, 5, 8,

The nth term of this series is

$$2 + (n-1)3 = 2 + 3n - 3 = 3n - 1.$$

The $(n-1)$th term of this series is

$$2 + (n-2)3 = 2 + 3n - 6 = 3n - 4.$$

Hence we have

$$S_n = 2 + 5 + 8 + \quad + (3n-7) + (3n-4) + (3n-1).$$

Reversing the order of the terms,

$$S_n = (3n-1) + (3n-4) + (3n-7) + . \quad . + 8 + 5 + 2.$$

By addition we have

$$2S_n = (3n+1) + (3n+1) + (3n+1) + \quad . \text{ to } n \text{ terms}$$
$$= n(3n+1).$$
$$S_n = \tfrac{1}{2}n(3n+1).$$

Note. As a check, let $n = 18$. Substituting this value of n in the above formula, we have $S_{18} = \frac{1}{2}.18.55 = 495$, which agrees with the result obtained in Example 1.

EXERCISE XVII

1. Find the sum of the first 12 terms in each of the following arithmetical progressions:

(*a*) 1, 2, 3,
(*b*) 3, 7, 11,
(*c*) 5, 15, 25,
(*d*) 4, 13, 22,

(e) 3, – 1, – 5,
(f) – 3, – 5, – 7, .
(g) $1\frac{1}{2}$, $2\frac{1}{4}$, 3,
(h) 5a, 11a, 17a,

2. Find a formula for the sum (S_n) of n terms of each of the following arithmetical progressions:

(a) 2, 4, 6,
(b) 3, 7, 11,
(c) 6, 4, 2,
(d) x, $5x$, $9x$,
(e) 2, $2\frac{1}{4}$, $2\frac{1}{2}$,
(f) a, $a + d$, $a + 2d$,

15. It is a great advantage to have formulæ to facilitate the calculations involved in the study of arithmetical progressions. In these it is usual to let a represent the first term, d the common difference, and n the number of terms of the progression.

Then the nth term of an arithmetical progression is given by the formula a + (n – 1)d.

If S_n denotes the sum of n terms of an arithmetical progression, then

$$S_n = a + (a + d) + (a + 2d) + \quad . \\ + \{a + (n-2)d\} + \{a + (n-1)d\}.$$

Reversing the order, as in the example on page 48,

$$S_n = \{a + (n-1)d\} + \{a + (n-2)d\} + \{a + (n-3)d\} \\ + \quad + (a + d) + a.$$

By addition we have

$$2S_n = \{2a + (n-1)d\} + \{2a + (n-1)d\} \\ + \{2a + (n-1)d\} + \quad + \{2a + (n-1)d\}$$
$$= n\{2a + (n-1)d\}, \text{ since there are } n \text{ terms.}$$

$$\mathbf{S_n = \frac{n}{2}\{2a + (n-1)d\}.}$$

The nth term of the progression—viz., $a+(n-1)d$—is also the last term. If we denote this by l we have

$$S_n = \frac{n}{2}\{2a + (n-1)d\}$$

$$= \frac{n}{2}\{a + a + (n-1)d\}.$$

$$\mathbf{S_n = \frac{n}{2}(a + l).}$$

This variation of the formula for the sum of a number of terms of an arithmetical progression is used when the first and last terms are known. The following examples illustrate the application of some of these formulæ.

EXAMPLE 1

The 7th term of an arithmetical progression is 27 and the 10th term is 39. Find the first term and the common difference.

Let a be the first term and d the common difference.

Then the 7th term is $a + 6d$ and the 10th term is $a + 9d$.

Hence we have $a + 6d = 27$

and $a + 9d = 39.$

By subtraction $3d = 12.$

$d = 4.$

By substitution $a = 3.$

Hence the first term of the series is 3 and the common difference is 4.

EXAMPLE 2

Find the sum of 18 terms of the series $5, -1, -7,$

In this case $a = 5$

$d = \text{2nd term} - \text{1st term}$

$= -1 - 5$

$= -6$

and $n = 18.$

If S is the sum of 18 terms

$$\begin{aligned} S &= \frac{18}{2}\{2.5 + (18-1)(-6)\} \\ &= 9\{10 + 17(-6)\} \\ &= 9(10-102) \\ &= 9 \times (-92) \\ &= -828. \end{aligned}$$

EXERCISE XVIII

1. Find the sum of the first 15 terms of each of the following arithmetical progressions:

(*a*) 2, 7, 12,
(*b*) 1, 9, 17,
(*c*) 5, 4, 3,
(*d*) 5, – 1, – 7,
(*e*) 34, 29, 24,
(*f*) $1\frac{1}{2}$, $2\frac{3}{4}$, 4,
(*g*) 10, $11\frac{1}{2}$, 13,
(*h*) 2·5, 1·4, 0·3,

2. Find the sum of n terms of each of the progressions in the preceding question.

3. The ninth term of an arithmetical progression (A.P.) is 35 and the twentieth term is 79. Find the first term and the tenth term.

4. The fourth term of an A.P. is 23 and the sixth term is 37. Find the sum of the first ten terms.

5. Find the sum of all the multiples of 7 between 1 and 100.

6. How many terms of the series 2, 6, 10, will be required to make a sum of 200?

7. The nth term of an A.P. is $3n + 7$. What is the common difference? What is the sum of n terms?

8. The third term of an A.P. is 19 and the sum of twenty terms is 1580. Find the series.

9. Insert four arithmetic means between 3 and 23.

10. A clerk is appointed at a commencing salary of £150 per annum and receives an increment of £10 at the end of

each complete year of service. What will be his salary during the eleventh year?

How much will he receive altogether for his first complete ten years' service?

11. The sum of four numbers in arithmetical progression is 26 and the sum of their squares is 214. Find the numbers.

12. The ratio of the 5th term of an A.P. to the 6th term is 22   27. If the first term is 2, find the common difference.

13. The sum of the first four terms of an A.P. is 36. The sum of the next four is 100. Find the first term and the common difference.

14. Find the sum of 25 consecutive odd numbers of which the last is 99.

15. The nth term of an A.P. is $3n + 5$. Find the sum of (a) 8 terms and (b) n terms.

16. If 5, x, y, 26, form an A.P., find x and y.

17. The first two terms of an A.P. are a and b. Find the nth term and the sum of n terms.

18. An arithmetical progression consists of $2n$ terms whose sum is S_{2n}. If S_n is the sum of the first n terms, prove that $S_{2n} - 2S_n = n^2d$.

19. The sum of four numbers in arithmetical progression is 24. The sum of the first two is double the sum of the last two. Find the numbers.

20. Four numbers are in arithmetical progression. The ratio of the first to the second is 4   7. Find the numbers. What is the ratio of the third term to the fourth term?

21. Insert four arithmetic means between 19 and 54.

22. If $S = 1 + 3 + 5 + \quad + (2n - 1)$, prove that $S = n^2$.

23. If $S = 2 + 4 + 6 + \quad + 2n$, prove that $S = n(n + 1)$.

24. How many terms of the A.P. 4, 9, 14,   must be taken in order that their sum may be 1476?

25. Show that if each term of an A.P. is multiplied by the same quantity, x, the products form an A.P. in which the common difference is x times the common difference of the original series.

GEOMETRICAL PROGRESSION

16. A series in which each term (after the first) is obtained by multiplying its predecessor by some constant factor is called a *geometrical progression*. Thus 3, 6, 12, 24, 48, is a geometrical progression in which the first term is 3 and the *common ratio* is 2. The series 3, $-2\frac{1}{4}$, $1\frac{11}{16}$, is a geometrical progression in which the first term is 3 and the common ratio is $-\frac{3}{4}$. Notice that the terms of the series are alternatively positive and negative. The common ratio is calculated by the rule

$$\text{Common ratio} = \frac{\text{2nd term}}{\text{1st term}}$$

$$= \frac{\text{3rd term}}{\text{2nd term}}$$

$$= \frac{n\text{th term}}{(n-1)\text{th term}}.$$

Applying this to the series 3, 6, 12, 24, . , we have

$$\text{Common ratio} = \frac{\text{2nd term}}{\text{1st term}}$$

$$= \frac{6}{3}$$

$$= 2.$$

Applying the rule to the series 3, $-2\frac{1}{4}$, $1\frac{11}{16}$, . , we have

$$\text{Common ratio} = \frac{\text{2nd term}}{\text{1st term}}$$

$$= \frac{-2\frac{1}{4}}{3}$$

$$= -\frac{9}{4} \times \frac{1}{3}$$

$$= -\frac{3}{4}.$$

Every term of a geometrical progression can be calculated from the first term and the common ratio.

Thus, in the series 3, 6, 12,

$$\begin{aligned} 6 &= 3.2 \\ 12 &= 6.2 = 3.2^2 \\ 24 &= 12.2 = 3.2^3. \end{aligned}$$

So that if 3 is the first term of the series, then the

second term is 3.2
third term is 3.2^2
fourth term is 3.2^3

nth term is 3.2^{n-1}.

EXAMPLE

Find the common ratio, the nth term, and the 5th term of the series (*a*) 1, − 2, 4, and (*b*) 3, $1\frac{1}{2}$, $\frac{3}{4}$,

(*a*) The series is 1, − 2, 4,

$$\text{The common ratio} = \frac{\text{2nd term}}{\text{1st term}} = -\frac{2}{1} = -2.$$

$$\text{The } n\text{th term} = 1.(-2)^{n-1} = (-2)^{n-1}.$$

Note. The value of the nth term is positive or negative according to whether n is odd or even.

$$\text{The 5th term} = 1.(-2)^4 = 16.$$

(*b*) The series is $3,\ 1\frac{1}{2},\ \frac{3}{4},\ .$

$$\text{The common ratio} = \frac{\text{2nd term}}{\text{1st term}}$$

$$= \frac{1\frac{1}{2}}{3}$$

$$= \frac{1}{2}.$$

$$\text{The } n\text{th term} = 3(\tfrac{1}{2})^{n-1}$$

$$= \frac{3}{2^{n-1}}$$

$$= 3.2^{1-n}.$$

(These alternative forms are obtained by applying the Laws of Indices.)

$$\text{The 5th term} = 3.(\tfrac{1}{2})^4$$

$$= \frac{3}{16}.$$

EXERCISE XIX

1. Find the common ratio, the nth term, and the 8th term in each of the following geometrical progressions:

(*a*) $1,\ 2,\ 4,$
(*b*) $2,\ 6,\ 18,$
(*c*) $2,\ -6,\ 18,$
(*d*) $3,\ 2,\ 1\frac{1}{3},$
(*e*) $2,\ -3,\ 4\frac{1}{2},$
(*f*) $4,\ 1,\ \frac{1}{4},$
(*g*) $1,\ x,\ x^2,$
(*h*) $a,\ ar,\ ar^2,$
(*i*) $a,\ -\frac{a}{r},\ \frac{a}{r^2},$

2. Find the missing terms in each of the following geometrical progressions:

(*a*) 2, —, 50,
(*b*) 3, —, 48,
(*c*) —, 28, 196,
(*d*) —, 18, 108,
(*e*) 9, – 36, —,

3. Insert the missing terms in each of the following geometrical progressions (these are called *geometric means*):

(*a*) 2, —, —, —, 162.

Hint. The three geometric means, *together* with the first term, 2, and the last term, 162, form a geometrical progression of *five* terms, in which the fifth term is 162. Therefore if r is the common ratio, we have

$$162 = 2.r^4.$$
$$r^4 = 81.$$
$$r = \pm 3. \quad \text{(Notice that both signs are admissible.)}$$

If $r = +3$, the series is 2, 6, 18, 54, 162, and the three geometric means are 6, 18, and 54.

If $r = -3$, the series is 2, -6, 18, -54, 162, and the three geometric means are -6, 18, and -54.

(*b*) 1, —, —, —, —, 243.

(*c*) 3, —, —, —, 243.

(*d*) 4, —, —, —, —, $\frac{1}{8}$.

(*e*) 3, —, —, —, $\frac{16}{27}$.

4. The fourth term of a geometrical progression (G.P.) is 27 and the seventh term is 729. Find the first term and the common ratio.

Hint. Let a be the first term and r the common ratio, then the fourth term is ar^3 and the seventh term is ar^6.

$$ar^3 = 27.$$
$$ar^6 = 729.$$
$$\frac{ar^3}{ar^6} = \frac{27}{729}.$$
$$\frac{1}{r^3} = \frac{1}{27}.$$
$$r = 3.$$

The value of a can be found by substitution in $ar^3 = 27$.

5. The third term of a G.P. is 18 and the sixth term is 486. Find the series.

6. The second term of a G.P. is $1\frac{1}{2}$ and the fifth term is $\frac{3}{16}$. Find the series.

7. Find the geometric mean between 4 and 64.

8. If a, b, and c are in geometrical progression, prove that $b^2 = ac$.

Note. b is called the geometric mean between a and c.

9. The ratio of the second term of a G.P. to the fifth term is 1 27. Find the ratio of the third term to the seventh term.

10. Write down (a) the nth term and (b) the fifth term of the series $\frac{a^2}{b^2}, \frac{a}{b}, 1,$

11. The first term of a G.P. is a and the second term is b. What is the third term? Write down the nth term.

12. If a series of numbers are in G.P., prove that their logarithms are in A.P.

17. For finding the sum of a number of terms in geometrical progression the following device is used to avoid the lengthy process of calculating each term of the series and then performing the addition.

EXAMPLE

Find the sum of 12 terms of the series 3, 6, 12,

The common ratio of the series is $\frac{6}{3} = 2$.

Hence the twelfth term is $3.2^{11} = 3 \times 2048$

$= 6144$

and the eleventh term is 3072.

If S is the required sum, then

$$S = 3 + 6 + 12 + \quad\quad + 3072 + 6144.$$

Multiply throughout by the common ratio.

$$2S = \quad 6 + 12 + \quad\quad\quad + 6144 + 12288.$$

Each term on the right-hand side is moved one place to the right, so as to bring terms of equal value beneath one another.

By subtraction,

$$2S - S = 12288 - 3.$$

$$S = 12285.$$

EXERCISE XX

Find the sums of the following series:

1. 2, 4, 8, to 8 terms.
2. 1, 3, 9, to 7 terms.
3. 4, 2, 1, to 10 terms.
4. 3, $1\frac{1}{2}$, $\frac{3}{4}$, to 6 terms.
5. 5, $3\frac{1}{3}$, $2\frac{2}{9}$, to 5 terms.

18. The calculations involved in the study of geometrical progression are simplified by the use of formulæ. We shall now establish for geometrical progressions a set of formulæ similar to those obtained on page 49 for arithmetical progressions. It is usual to let a be the first term of the series, r the common ratio, and n the number of terms. Then the G.P. is

$$a, ar, ar^2,$$

from which it can be seen at once that **the nth term of the G.P. is given by the formula ar^{n-1}.**

If S_n denotes the sum of n terms of the geometrical progression $a + ar + ar^2 + \quad . ar^{n-1}$, then

$$S_n = a + ar + ar^2 + \quad + ar^{n-1}. \qquad (1)$$

Multiplying by r and moving each term of the series one place to the right,

$$rS_n = \quad ar + ar^2 + \quad + ar^{n-1} + ar^n \qquad (2)$$

Subtracting equation (2) from equation (1), we have

$$S_n - rS_n = a - ar^n.$$

$$S_n(1 - r) = a(1 - r^n).$$

$$\mathbf{S_n = \frac{a(1 - r^n)}{1 - r}.}$$

By subtracting equation (1) from equation (2) we have

$$rS_n - S_n = -a + ar^n.$$

$$S_n(r-1) = a(r^n - 1).$$

$$\mathbf{S_n = \frac{a(r^n - 1)}{r - 1}.}$$

There are thus two forms for the sum of a G.P., although it is not difficult to see that both will give the same result when used in any particular case. It is an advantage, though it is not essential, to use the form [1]

$$S_n = \frac{a(1 - r^n)}{1 - r} \text{ when } |r| < 1$$

and the form

$$S_n = \frac{a(r^n - 1)}{r - 1} \text{ when } |r| > 1.$$

EXAMPLE 1

Find the sum of 8 terms of the series 2, 6, 18,
In this case $a = 2$, $r = 3$, $n = 8$.
Since $|r| > 1$, we use the form

$$S = \frac{a(r^n - 1)}{r - 1}$$

$$= \frac{2\{(3)^8 - 1)\}}{3 - 1}$$

$$= \frac{2(6561 - 1)}{2}$$

$$= 6560.$$

EXAMPLE 2

Find the sum of 9 terms of the series 3, $-1\frac{1}{2}$, $\frac{3}{4}$,
In this case $a = 3$, $r = -\frac{1}{2}$, $n = 9$.

[1] $|r|$ means the numerical value of r—for example, $|3| = 3$ and $|-3| = 3$.

Since $|r| < 1$, we use the form

$$S = \frac{a(1 - r^n)}{1 - r}$$

$$= \frac{3\{1 - (-\frac{1}{2})^9\}}{1 - (-\frac{1}{2})}$$

$$= \frac{3\{1 - (-\frac{1}{512})\}}{1 + \frac{1}{2}}$$

$$= \frac{3(1 + \frac{1}{512})}{1\frac{1}{2}}$$

$$= 2\tfrac{1}{256}.$$

(Notice the care which must be used with regard to signs. The value of r^n should always be calculated separately and shown as in this example.)

EXERCISE XXI

1. Find the sum of the first 9 terms of each of the following geometrical progressions:

(*a*) 1, 2, 4, (*d*) $4, -1\frac{1}{3}, \frac{4}{9}$,
(*b*) $2, 1, \frac{1}{2}$, (*e*) $1, x, x^2$,
(*c*) 36, 24, 16,

2. Find the sum of n terms of each of the series in Question 1.

3. How many terms of the series 1, 3, 9, must be taken to make 3280?

4. The fourth term of a G.P. is 54 and the seventh term is 1458. Find the series and the sum of 10 terms.

5. Find the sum of 6 terms of the series $1, -\sqrt{3}, 3,$

6. How many terms of the series 2, 6, 18, must be taken to have a sum of 2186?

7. Find the sum of all the powers of 3 between 2 and 500.

8. The average of the first 2 terms of a G.P. is 4. The average of the third and fourth terms is 36. Find the series.

9. The sum of the first 4 terms of a G.P. whose common ratio is 3 is 40. Prove that the sum of the next 4 terms is 3240.

10. If the common ratio of a G.P. is r, and it contains $2n$

terms, prove that the sum of the n even terms is r times the sum of the n odd terms.

11. If 2, p, q, $182\frac{1}{4}$, are in G.P., find p and q.

12. The first term of a G.P. is a and the second term is b. What is the nth term and the sum of n terms?

13. The first term of a G.P. is 4 and the second term is 20. What is the nth term? What is the smallest number of terms which will make a sum of more than 5000?

14. If p, q, r, s, are in geometrical progression and $q - p = 18$ and $s - r = 162$, find p.

15. What must be added to 11, 47, and 155 to make three numbers in geometrical progression?

16. Find the sum of 6 terms of the series 7, $5\frac{1}{4}$, $3\frac{15}{16}$,

17. Three numbers, whose sum is 15, are in arithmetical progression. If 1, 3, and 23 are added to them respectively the results will be in geometrical progression. Find the original numbers.

18. If p, q, r, and s are in G.P., prove that $q(p + r) = p(q + s)$.

19. A ball is dropped from a height of 6 feet and bounces to a height of 4 feet. After each bounce it rises two-thirds of the distance which it has just fallen. How high will it rise after the fourth bounce?

20. A geometrical progression consists of 10 terms, the first of which is 8, and the common ratio is $\frac{1}{4}$. Write down the series and prove that the first term is greater than the sum of all the others.

19. We shall now consider the series 1, $\frac{1}{2}$, $\frac{1}{4}$, This is a geometrical progression in which the first term is 1 and the common ratio is $\frac{1}{2}$. If S_2 denotes the sum of the first two terms, S_3 the sum of the first three terms, and so on, we have

$$S_2 = 1\tfrac{1}{2}$$
$$S_3 = 1\tfrac{3}{4}$$
$$S_4 = 1\tfrac{7}{8}$$
$$S_5 = 1\tfrac{15}{16}, \text{ and so on.}$$

These results show that as the number of terms is increased their sum approaches 2. We can write

$$S_2 = 2 - \tfrac{1}{2}$$
$$S_3 = 2 - \tfrac{1}{4}$$
$$S_4 = 2 - \tfrac{1}{8}$$
$$S_5 = 2 - \tfrac{1}{16}, \text{ and so on.}$$

Hence the amount by which S_n differs from 2 can be made as small as we please by taking n sufficiently large. This can be written in another way: $S_n \to 2$ as $n \to \infty$, using the arrow to stand for the word 'approaches.' In this case we say that the series *converges* to the limit 2.

Now consider the series 1, 2, 4, ... This is a geometrical progression in which the common ratio is 2, and

$$S_2 = 3$$
$$S_3 = 7$$
$$S_4 = 15$$
$$S_5 = 31.$$

It is clear that $S_n \to \infty$ as $n \to \infty$, or, in other words, the sum of n terms increases indefinitely as the number of terms is increased. Such a series is said to *diverge.*

Some series neither converge nor diverge. Consider the geometrical progression 1, -1, 1, ... The common ratio is -1, and

$$S_2 = 0$$
$$S_3 = 1$$
$$S_4 = 0$$
$$S_5 = 1, \text{ and so on.}$$

In this case S_n is either 0 or 1, according to whether n is even or odd; it does not converge to a limit, as in the first series we considered, nor does it diverge to infinity, as did the second series. The value of S_n is said to *oscillate* between 0 and 1, and the series 1, -1, 1, ... is called an oscillating series.

It is our purpose to consider the convergent geometric series. The number to which the sum of n terms converges as n is increased is called the 'limiting sum,' or more commonly the 'sum to infinity.' The general geometrical progression $a, ar, ar^2, \ldots$ has for the sum of n terms

$$S_n = \frac{a(1-r^n)}{1-r}$$

$$= \frac{a}{1-r} - \frac{ar^n}{1-r}.$$

In the series considered first of all—namely, $1, \frac{1}{2}, \frac{1}{4}$,

$$\frac{a}{1-r} = \frac{1}{1-\frac{1}{2}} = 2$$

and when $n = 2$ $\qquad \frac{ar^n}{1-r} = \frac{(\frac{1}{2})^2}{1-\frac{1}{2}} = \frac{1}{2}.$

If $n = 3$ $\qquad \frac{ar^n}{1-r} = \frac{(\frac{1}{2})^3}{1-\frac{1}{2}} = \frac{1}{4}.$

If $n = 4$ $\qquad \frac{ar^n}{1-r} = \frac{(\frac{1}{2})^4}{1-\frac{1}{2}} = \frac{1}{8}$, and so on.

From which we see at once that as n increases $\frac{ar^n}{1-r}$ decreases. We can write this $\frac{ar^n}{1-r} \to 0$ as $n \to \infty$

It is clearly necessary for $\frac{ar^n}{1-r} \to 0$ *that* $|r| < 1$. *If* $|r| > 1$ *the series diverges, and if* $|r| = 1$ *the series will either diverge or oscillate.*

Hence in $\qquad S_n = \frac{a}{1-r} - \frac{ar^n}{1-r}$

we have $\qquad \frac{ar^n}{1-r} \to 0$ as $n \to \infty$ (if $|r| < 1$).

Therefore as $n \to \infty$, $S_n \to \frac{a}{1-r}$.

If we write S_∞ as the sum to infinity we have

$$\mathbf{S_\infty = \frac{a}{1-r}}.$$

A geometrical progression only converges to a limiting sum or sum to infinity when its common ratio is numerically less than unity.

EXERCISE XXII

1. Say which of the following geometrical progressions have a sum to infinity:

(*a*) $3, 2\frac{1}{4}, 1\frac{11}{16},$
(*b*) $3, 4, 5\frac{1}{3},$
(*c*) $1, -2, 4,$
(*d*) $5, 4, 3\frac{1}{5}, .$
(*e*) $5, -4, 3\frac{1}{5},$

2. In the cases where it is possible calculate the sum to infinity of the progressions in the preceding question.

3. In the G.P. $1, \frac{1}{2}, \frac{1}{4},$ what is the difference between the sum to infinity and the sum of the first 10 terms?

4. The first two terms of a G.P. are 4 and 3 respectively. What is the sum to infinity?

5. The first two terms of a G.P. are a and b respectively. What is the sum to infinity?

6. Say for what values of x will the following geometrical series converge:

(*a*) $1, 3x, 9x^2,$ (*b*) $1, -3x, 9x^2,$

7. Assuming that the series in the preceding question do converge, what is the sum to infinity in each case?

8. What values of x will make the series in Question 6 oscillate?

20. An interesting application of the sum to infinity of a convergent geometrical progression occurs in the theory of recurring decimals. It is well known that in the calculation of decimals from vulgar fractions it is impossible in many cases to express the fraction as a terminating decimal —*i.e.*, to express it exactly in a definite number of places of decimals. In certain non-terminating decimals the digits repeat themselves in the same order. Thus,

$$\frac{5}{7} = 0{\cdot}714285714285714285$$

It will be seen from this that the digits 714285 repeat themselves in this order. These constitute a *period*, and it is a usual abbreviation to place a dot over the first and last digit of the period and write $\frac{5}{7} = 0{\cdot}\dot{7}1428\dot{5}$.

Notice that a period can be made to commence at any point—*e.g.*, $\frac{5}{7} = 0{\cdot}71\dot{4}2857\dot{1}$, which is clearly the same as $0{\cdot}\dot{7}1428\dot{5}$.

The following example illustrates the recurring decimal as an infinite series and establishes the rule for converting recurring decimals to vulgar fractions.

EXAMPLE

Express $0{\cdot}6\dot{7}$ as a vulgar fraction.

$$0{\cdot}6\dot{7} = {\cdot}67777$$
$$= \frac{6}{10} + \underbrace{\frac{7}{10^2} + \frac{7}{10^3} + \frac{7}{10^4}}$$

(This is an infinite G.P., in which $a = \frac{7}{10^2}$ and $r = \frac{1}{10}$.)

If we find the sum to infinity of this G.P. we have

$$S_\infty = \frac{\frac{7}{10^2}}{1 - \frac{1}{10}}$$
$$= \frac{7}{100} \times \frac{10}{9}$$
$$= \frac{7}{90}.$$

Hence $$0{\cdot}6\dot{7} = \frac{6}{10} + \frac{7}{90}$$
$$= \frac{61}{90}.$$

It is easy to verify by the usual arithmetical process that $0{\cdot}6\dot{7} = \frac{61}{90}$. The method employed above can be used in general terms to prove the rule for converting recurring decimals to vulgar fractions that is sometimes found in text-books on arithmetic—viz.,

> *The numerator of the fraction is the difference between the given decimal and the non-recurring part. The denominator of the fraction contains as many* 9's *as there are figures in the period followed by as many* 0's *as there are non-recurring digits in the decimal.*

EXERCISE XXIII

1. Express each of the following recurring decimals as an infinite G.P. By finding the value of S_∞ in each case, convert them into vulgar fractions.

(*a*) $0{\cdot}\dot{5}$. (*b*) $3{\cdot}\dot{1}\dot{4}$. (*c*) $5{\cdot}7\dot{1}\dot{4}$. (*d*) $5{\cdot}\dot{7}1\dot{4}$. (*e*) $8{\cdot}4\dot{2}\dot{7}$.

2. Prove that $0{\cdot}\dot{9} = 1$ and $0{\cdot}0\dot{9} = 0{\cdot}1$.

21. An application of the theory of geometrical progression to be considered arises in connexion with compound interest.

If £P are invested at r per cent. per annum at the end of the first year the amount—*i.e.*, principal + interest—will be A_1, where

$$A_1 = P + \frac{Pr}{100}$$
$$= P\left(1 + \frac{r}{100}\right).$$

During the second year the sum £A_1 is earning interest at the rate of r per cent. per annum. At the end of the second year this will amount to A_2, where

$$A_2 = A_1\left(1 + \frac{r}{100}\right)$$
$$= P\left(1 + \frac{r}{100}\right)^2$$

Similarly, $A_3 = P\left(1 + \frac{r}{100}\right)^3$, and so on. If, therefore, a sum of £P remains at compound interest, in n years it will amount to £A, where $A = P\left(1 + \frac{r}{100}\right)^n$ Regarded from another point of view, £P is the *present value* of £A due in n years' time.

If we denote $1 + \frac{r}{100}$ by R we have $A = PR^n$, from which $P = \frac{A}{R^n} = AR^{-n}$. The difference between the amount and the present value is called the *discount*. Denoting this by D, we have

$$\begin{aligned} D &= A - P \\ &= A - AR^{-n} \\ &= A(1 - R^{-n}). \end{aligned}$$

We shall now consider the effect of allowing a number of equal annual instalments to accumulate at compound interest. The following example illustrates the principles used in the calculation.

EXAMPLE

At the beginning of each of ten successive years a sum of £10 is invested at 3 per cent. per annum. How much money will have accumulated by the end of the tenth year?

In this example we have ten sums of money, each of which is allowed to accumulate at compound interest.

The first payment of £10 remains at compound interest for 10 years, and therefore amounts to $£10(1{\cdot}03)^{10}$ at the end of this period. The second payment of £10 remains at compound interest for 9 years, and therefore amounts to $£10(1{\cdot}03)^9$. The third payment of £10 amounts to $£10(1{\cdot}03)^8$. The last payment of all is made at the beginning of the 10th year and by the end of that year has amounted to

£10(1·03). Hence at the end of the 10th year the sum accumulated will be

$$\begin{aligned}
&\pounds 10(1{\cdot}03)^{10} + 10(1{\cdot}03)^{9} + 10(1{\cdot}03)^{8} + \quad . + 10(1{\cdot}03) \\
&= \pounds 10[(1{\cdot}03)^{10} + (1{\cdot}03)^{9} + (1{\cdot}03)^{8} + \quad + 1{\cdot}03] \\
&= \pounds 10 \, \frac{1{\cdot}03\{(1{\cdot}03)^{10} - 1\}}{1{\cdot}03 - 1} \\
&= \pounds 10 \, \frac{1{\cdot}03(1{\cdot}3439 - 1)}{{\cdot}03} \\
&= \pounds 118 \ 1s.\ 5d.
\end{aligned}$$

If instead of making these ten annual payments the investor had placed £87 17*s.* 2*d.* at compound interest at 3 per cent. per annum for ten years this sum would have accumulated to £118 1*s.* 5*d.* Consequently a loan of £87 17*s.* 2*d.*, allowed to accumulate at this rate of interest would amount to exactly the same sum in 10 years as the annual payments of £10. The latter are called an *annuity*, and in this particular case the present value or cost of the annuity is £87 17*s.* 2*d.*

This method of repaying a loan both as to capital and interest by means of equal annual instalments finds a wide application in house purchase schemes through building societies. The society advances a proportion of the purchase money on mortgage in order to help the borrower pay for the house. The loan is repaid by equal annual instalments spread over 10, 15, or 20 years, according to arrangement. Suppose the advance was £500, which was to be cleared in 15 years, the rate of interest agreed on being 4 per cent. per annum. This sum at compound interest would accumulate to £900 in 15 years.[1] If the purchaser makes fifteen annual payments of £45, these, invested at 4 per cent., would amount to £900 at the end of the fifteenth

[1] These figures are only approximate. The determination of the exact annual payments which the borrower would have to make is left as an exercise to the reader.

year, and would therefore clear the debt. This method of repayment is frequently adopted by local authorities. For example, suppose that a town council embarked on an improvement scheme which is to cost £50,000. This sum would be raised by means of a loan and a definite interest paid on the debt. If no plan of repayment is adopted the interest due and the amount which can be repaid will have to be determined each year. This is unsatisfactory both from the point of view of the investor and of the town council. The investor likes to feel that his money need not be disturbed for a definite number of years and the town council find it simplest to make a fixed annual charge on the rates. The plan frequently adopted is to raise the loan on the understanding that it will be repaid in (say) twenty years. A sinking fund is created to which twenty equal annual instalments are added and allowed to accumulate at compound interest. At the end of this time they will repay the original loan.

EXERCISE XXIV

1. Calculate the amount of £100 in ten years at compound interest at 5 per cent. per annum. ($(1{\cdot}05)^{10}=1{\cdot}6289$.)

2. A National Savings Certificate (third issue) costs 16*s*., and is worth £1 in six years. At what rate per cent. per annum is the interest added?

3. A man raises a loan of £600 from a building society for which he has to pay interest at 5 per cent. per annum. What annual repayment must be made if he is to repay the debt in ten years? ($(1{\cdot}05)^{10} = 1{\cdot}6289$.)

4. What is the present value of £1 in 5 years' time, interest being calculated at 3 per cent. per annum?

5. The sum of £10 is deposited in the bank at the beginning of each year for eight years. If interest at 3 per cent. per annum is paid, how much will have accumulated by the end of the eighth year? ($(1{\cdot}03)^8 = 1{\cdot}2668$.)

6. In ten years the population of a city increased from

50,000 to 75,000. Find the rate of increase (supposed uniform) per cent. per annum.

7. How long will it take a sum of money to double itself at compound interest at 4 per cent. per annum?

8. What is the present value of £100 payable in 8 years' time, allowing compound interest at 4 per cent. per annum?

9. What is the present value (or purchase price) of an annuity of £100 to be paid for five successive years commencing in 8 years' time if the rate of interest is 5 per cent. per annum?

Hint. The purchase price of the annuity will be the sum of the present values of £100 in 8, 9, 10, 11, and 12 years' time.

10. A city borrows £100,000 for an improvement scheme. What annual charge must be made on the rates if this debt is to be cleared in thirty years? Allow interest at 5 per cent. per annum.

HARMONICAL PROGRESSION

22. The series 1, 4, 7, ... is an arithmetical progression. The reciprocals of the terms, taken in the same order, form the series $\frac{1}{1}, \frac{1}{4}, \frac{1}{7}, \ldots$ This series is called an *harmonical progression*. Similarly, $1\frac{1}{2}, 1\frac{3}{4}, 2, \ldots$ is an A.P., and $\frac{2}{3}, \frac{4}{7}, \frac{1}{2}, \ldots$ is the corresponding H.P. We thus define an harmonical progression as a series of terms which are the reciprocals of a corresponding arithmetical progression. Most problems which arise in connexion with harmonical progressions are solved by means of the corresponding arithmetical progressions; thus the nth term is obtained by inverting the nth term of the corresponding A.P., and the insertion of harmonic means is performed by inverting the corresponding arithmetic means. There is no formula for the sum of a given number of terms of a harmonical progression. Should the sum ever be required, then it has to be obtained by actual addition.

EXAMPLE

Insert four harmonic means between $\frac{1}{2}$ and $\frac{2}{9}$.

Representing the means by dashes, we shall have the harmonical progression

$$\frac{1}{2}, \text{—}, \text{—}, \text{—}, \text{—}, \frac{2}{9}.$$

The corresponding arithmetical progression is

$$2, \frac{1}{\text{—}}, \frac{1}{\text{—}}, \frac{1}{\text{—}}, \frac{1}{\text{—}}, \frac{9}{2}.$$

Here $a = 2$, and if d is the common difference the sixth term is

$$a + 5d = \frac{9}{2}.$$

$$\therefore 5d = \frac{9}{2} - 2$$

$$= \frac{5}{2}.$$

$$d = \frac{1}{2}.$$

Hence the arithmetical progression is

$$2, 2\tfrac{1}{2}, 3, 3\tfrac{1}{2}, 4, 4\tfrac{1}{2}.$$

So that the harmonical progression is

$$\frac{1}{2}, \frac{2}{5}, \frac{1}{3}, \frac{2}{7}, \frac{1}{4}, \frac{2}{9}.$$

Hence the four harmonic means between $\frac{1}{2}$ and $\frac{2}{9}$ are

$$\frac{2}{5}, \frac{1}{3}, \frac{2}{7}, \frac{1}{4}.$$

EXERCISE XXV

1. Find the fifth term and the nth term of each of the following harmonical progressions:

(a) $\frac{1}{2}, \frac{1}{3}, \frac{1}{4},$ (c) $\frac{3}{10}, \frac{3}{8}, \frac{1}{2},$

(b) $\frac{2}{3}, \frac{2}{5}, \frac{2}{7},$ (d) $\frac{6}{19}, \frac{3}{8}, \frac{6}{13},$

2. Insert four harmonic means between

(a) $\frac{2}{3}$ and $\frac{2}{13}$.

(b) $\frac{3}{10}$ and ∞.

(c) $\frac{6}{11}$ and 1.

3. If a, b, and c are in harmonical progression, prove that

$$\frac{a}{c} = \frac{a-b}{b-c}.$$

4. Prove that if $y + z$, $z + x$, and $x + y$ are in harmonical progression, then x^2, y^2, and z^2 are in arithmetical progression.

5. If p, q, r, and s are in harmonical progression, prove that $rs(p-q) = pq(r-s)$.

6. If a, H, and b are in harmonical progression, prove that

$$\text{H} = \frac{2ab}{a+b}.$$

23. *Note on the Three Progressions.* If A, H, and G are respectively the arithmetic, harmonic, and geometric means between a and b, we have

$$\text{A} = \tfrac{1}{2}(a+b).$$

$$\text{H} = \frac{2ab}{a+b}.$$

$$\text{G} = \sqrt{ab}.$$

From which we have

$$AH = \frac{a + b}{2} \times \frac{2ab}{a + b}$$
$$= ab$$
$$= G^2.$$

The relationship $AH = G^2$ connects the three kinds of means which can be inserted between two quantities.

The three means have a further property, which we may consider here:

$$A - G = \tfrac{1}{2}(a + b) - \sqrt{ab}$$
$$= \tfrac{1}{2}(a - 2\sqrt{ab} + b)$$
$$= \tfrac{1}{2}(\sqrt{a} - \sqrt{b})^2, \text{ which must always be positive.}$$

(Why?)

$$\therefore A > G.$$

Since $G^2 = AH$ the value of G must lie between A and H, and since $A > G$ we have $A > G > H$. In other words, the three means, A, G, and H, are in descending order of magnitude. Further, $G^2 = AH$ may be written $\frac{G}{A} = \frac{H}{G}$, so that A, G, and H form a descending geometrical progression, in which $\frac{G}{A}$ is the common ratio.

MISCELLANEOUS SERIES

24. The numbers 1, 2, 3, form an arithmetical progression in which the first term is 1 and the common difference is 1. These numbers are sometimes described as the *natural numbers*.

I. The sum of the first n natural numbers.

Let $S = 1 + 2 + 3 + \quad + n$.

Then we have to find the sum of n terms of an A.P. in which $a = 1$ and $l = n$.

Hence $$S = \frac{n}{2}(1 + n).$$

$$\therefore S = \frac{n(n + 1)}{2}.$$

II. The sum of the squares of the first n natural numbers.

Let $S = 1^2 + 2^2 + 3^2 + \quad\quad + n^2$.

We shall use the identity (which can be easily verified)

$$(x + 1)^3 - x^3 \equiv 3x^2 + 3x + 1.$$

If $x = 1$, then $2^3 - 1^3 = 3.1^2 + 3.1 + 1.$

If $x = 2$, then $3^3 - 2^3 = 3.2^2 + 3.2 + 1.$

If $x = 3$, then $4^3 - 3^3 = 3.3^2 + 3.3 + 1.$

If $x = n$, then $(n + 1)^3 - n^3 = 3.n^2 + 3.n + 1.$

If for all the values of x from $x = 1$ to $x = n$ we add these equations, we shall get on the *left-hand side*

$$(n + 1)^3 - 1^3,$$

all the other terms having cancelled out, and on the *right-hand side*

$$3(1^2 + 2^2 + 3^2 + \quad\quad + n^2) + 3(1 + 2 + 3 + \quad\quad + n) + n.$$

But $1^2 + 2^2 + 3^2 + \quad\quad + n^2 = S$

and $$1 + 2 + 3 + \quad\quad + n = \frac{n(n + 1)}{2}.$$

Hence we have

$$(n + 1)^3 - 1^3 = 3S + 3\,\frac{n(n + 1)}{2} + n.$$

$$3S = (n + 1)^3 - 1^3 - \frac{3n(n + 1)}{2} - n$$

$$= n^3 + 3n^2 + 3n + 1 - 1 - \frac{3n(n + 1)}{2} - n$$

$$= \frac{2n^3 + 3n^2 + n}{2}$$

$$= \frac{n(n+1)(2n+1)}{2}.$$

$$S = \frac{n(n+1)(2n+1)}{6}.$$

III. The sum of the cubes of the first n natural numbers.

Let $S = 1^3 + 2^3 + 3^3 + \quad + n^3$.

We shall use the identity

$$(x+1)^4 - x^4 \equiv 4x^3 + 6x^2 + 4x + 1.$$

If $x = 1$, then $2^4 - 1^4 = 4.1^3 + 6.1^2 + 4.1 + 1$.
If $x = 2$, then $3^4 - 2^4 = 4.2^3 + 6.2^2 + 4.2 + 1$.
If $x = 3$, then $4^4 - 3^4 = 4.3^3 + 6.3^2 + 4.3 + 1$.

If $x = n$, then $(n+1)^4 - n^4 = 4.n^3 + 6.n^2 + 4.n + 1$.

By addition we have on the *left-hand side*

$$(n+1)^4 - 1^4,$$

all other terms having cancelled out, and on the *right-hand side*

$$4(1^3 + 2^3 + 3^3 + \quad + n^3) + 6(1^2 + 2^2 + 3^2 + \quad . n^2) + 4(1 + 2 + 3 + \quad + n) + n.$$

But $1^3 + 2^3 + 3^3 + \quad + n^3 = S$.

$$1^2 + 2^2 + 3^2 + \quad + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

$$1 + 2 + 3 + \quad + n = \frac{n(n+1)}{2}.$$

Hence we have

$$(n+1)^4 - 1^4 = 4S + 6 . \frac{n(n+1)(2n+1)}{6} + 4 . \frac{n(n+1)}{2} + n$$

$$= 4S + 2n^3 + 3n^2 + n + 2n^2 + 2n + n$$

$$= 4S + 2n^3 + 5n^2 + 4n.$$

$$4S = (n+1)^4 - 1^4 - 2n^3 - 5n^2 - 4n$$

$$= n^4 + 4n^3 + 6n^2 + 4n + 1 - 1 - 2n^3 - 5n^2 - 4n$$

$$= n^4 + 2n^3 + n^2$$

$$= n^2(n+1)^2.$$

$$\therefore S = \frac{n^2(n+1)^2}{4}$$

$$= \left[\frac{n(n+1)}{2}\right]^2$$

The Sigma Notation. We shall now introduce a notation for the sum of a series of terms which is frequently met in various branches of mathematics. The Greek letter Σ placed before a term means the sum of all terms like the one in question. Thus, applied to the three series just considered, we have

$$\Sigma n = 1 + 2 + 3 + \quad + n = \frac{n(n+1)}{2}.$$

$$\Sigma n^2 = 1^2 + 2^2 + 3^2 + \quad + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

$$\Sigma n^3 = 1^3 + 2^3 + 3^3 + \quad + n^3 = \left(\frac{n(n+1)}{2}\right)^2$$

Applications of Σn, Σn^2, and Σn^3. When the nth term of a series is an expression in n, n^2, and n^3 a formula for the sum of n terms can sometimes be found by application of the results just obtained for Σn, Σn^2, and Σn^3. The method is illustrated in the following examples.

EXAMPLE 1

Find a formula for the sum of n terms of the series $1.2 + 2.3 + 3.4 +$

The nth term of the series is clearly $n(n + 1) = n^2 + n$.

If $n = 1$ the first term is $1^2 + 1$.

If $n = 2$ the second term is $2^2 + 2$.

If $n = 3$ the third term is $3^2 + 3$.

If $n = n$ the nth term is $n^2 + n$.

If S is the required sum we have

$$S = (1^2+2^2+3^2+. \quad .+n^2) + (1+2+3+. \quad .+n)$$

$$= \frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2}$$

$$= \frac{n(n+1)(2n+1+3)}{6}$$

$$= \frac{n(n+1)(n+2)}{3}.$$

EXAMPLE 2

Find the number of spherical shot in a pyramid the base of which is square and contains 10 shot.

Consider the base alone. Since each side contains 10 shot there will be 10^2 or 100 shot required to form the bottom layer.

The 10 shot have nine spaces between them, so that the second layer will contain 9^2 or 81 shot.

The third layer will contain 8^2 or 64, and so on, until finally there will be one shot left to form the apex of the pyramid.

Consequently the total number of shot will be

$$10^2 + 9^2 + 8^2 + \quad + 1^2.$$

Applying the formula $\Sigma n^2 = \frac{n(n+1)(2n+1)}{6}$, we shall obtain as the total number of shot 385.

EXERCISE XXVI

1. Write down the first three terms of the series whose nth terms are

(*a*) $4n - 3$.	(*f*) $n(n + 1)$.
(*b*) $5n + 7$.	(*g*) $n^2 + n$.
(*c*) n^2.	(*h*) $n^3 + n$.
(*d*) $(2n)^2$.	(*i*) $3n^2 + n$.
(*e*) $(2n + 1)^2$.	(*j*) $n^3 + 2n^2 + n$.

2. Write down the nth term of each of the following series:

(a) $1.3 + 2.5 + 3.7 +$
(b) $1.1 + 2.3 + 3.5 +$
(c) $1.2.3 + 2.3.4 + 3.4.5 +$
(d) $1^2 + 3^2 + 5^2 +$
(e) $2^2 + 4^2 + 6^2 +$
(f) $1^2.2 + 2^2.3 + 3^2.4 +$
(g) $1.3.5 + 3.5.7 + 5.7.9 +$
(h) $2^3 + 4^3 + 6^3 +$

3. Find a formula for the sum of n terms of each of the series in Question 2.

4. Find the value of $11^2 + 12^2 + 13^2 + \qquad + 20^2$.

5. Prove that $\Sigma n^3 = (\Sigma n)^2$.

6. Calculate the number of shot in a complete pyramid with a square base, each side of which contains 6 shot.

7. Calculate the number of shot in a pyramid with an equilateral triangular base, each side of which contains 6 shot.

8. If each side of the equilateral triangular base in Question 7 contains n shot, prove that the total number of shot in the pyramid is $\frac{1}{6}n(n + 1)(n + 2)$.

9. Find the number of shot in an incomplete square pile of 5 courses, having 10 shot on each side of the top.

10. The number of shot in the top layer of a square pile is 100 and in the bottom layer 900. How many shot does the pile contain?

EXERCISE XXVII

(General Revision Exercise on Series)

1. Find the sum of (i) n terms and (ii) 10 terms of each of the following series:

(a) $1 + 2\frac{1}{2} + 4 +$
(b) $5\frac{1}{4}, 2\frac{3}{4}, \frac{1}{4},$
(c) $3, 4\frac{1}{2}, 6\frac{3}{4},$

(*d*) $3, -4\frac{1}{2}, 6\frac{3}{4},$

(*e*) $\frac{1}{n}, \frac{2}{n}, \frac{3}{n},$

(*f*) 1·05, 1·5, 1·95,

2. Write down the *n*th term of each of the following series:

(*a*) $5 + 9 + 13 + .$

(*b*) $1.5 + 2.9 + 3.13 +$

3. Find the sum of (*a*) *n* terms and (*b*) 10 terms of each of the series in Question 2.

4. The fourth term of an A.P. is 17 and the ninth term is 42. Find the series and the sum of the first 20 terms.

5. What is the twentieth term of the series in Question 4?

6. Insert four geometric means between 3 and $\frac{32}{81}$.

7. The first term of an A.P. is 2 and the second term is $3\frac{1}{2}$. How many terms are required to give a sum of at least 500?

8. Insert three harmonic means between $\frac{2}{3}$ and $\frac{2}{13}$.

9. If the *p*th, *q*th, and *r*th terms of an A.P. are P, Q, and R respectively, prove that

$$P(q-r) + Q(r-p) + R(p-q) = 0.$$

10. An electric car starts from rest and increases its speed uniformly for five minutes, when it is travelling at 30 m.p.h. How far has it travelled?

11. A clerk is appointed at an annual salary of £250, and receives an annual increment of £10 for each completed year of service. How much will he receive altogether in twenty years?

12. The sum of the first six terms of an A.P. whose first term is 1 is 51. What is the sum of the next six terms?

13. The sum of the first, third, and fifth terms of a G.P. is $26\frac{1}{4}$. The sum of the second, fourth, and sixth terms is $52\frac{1}{2}$. Find the series.

14. How many terms of the series 3, 8, 13, ... must be taken to have a sum of 1010?

15. How many terms of the series $1\frac{1}{4}$, $2\frac{1}{2}$, 5, ... must be taken to have a sum of $78\frac{3}{4}$?

16. For what values of x will the series x, x^2, x^3, ... converge? What will be the sum to infinity of the series?

17. The first three of the four numbers 10, p, q, $16\frac{1}{3}$, are in A.P., and the last three are in G.P. Find p and q.

18. How many terms of the series 1, 2, 4, ... must be taken in order that their sum may exceed 10,000?

19. In how many years will £1 become £2 at compound interest at 4 per cent.?

20. A sum of £600 is borrowed from a building society in order to buy a house. Repayment is made by fifteen annual instalments, the first of which is made on the first anniversary of the loan. If interest is charged at 4 per cent., calculate what each annual payment will be. ($(1{\cdot}04)^{15} = 1{\cdot}8009$.)

21. If $S = 1 + \frac{1}{2} + \frac{1}{4} + \dots$ to n terms, prove that lim. $S = 2$ when $n \to \infty$.

22. If $S = 1 - \frac{1}{2} + \frac{1}{4} - \dots$ to n terms, prove that lim. $S = \frac{2}{3}$ when $n \to \infty$.

23. Solve the equation $1 + x + x^2 + \dots$ to $\infty = 8$.

24. Prove that $\frac{1}{2} - \frac{1}{3} = \frac{1}{2.3}$, $\frac{1}{3} - \frac{1}{4} = \frac{1}{3.4}$, and hence prove that

$$\frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \dots + \frac{1}{n(n+1)} = 1 - \frac{1}{n+1}.$$

25. Prove that

$$1.2 + 2.3 + 3.4 + \dots + n(n+1) = \tfrac{1}{3}n(n+1)(n+2).$$

26. What is the nth term of the series 1, 11, 111, ...?

27. Find the sum of 10 terms of $1 + 11 + 111 + 1111 + \dots$.

28. If $S = 1 + 2x + 3x^2 + \dots$ to n terms, what is the nth term? Find the value of $S - Sx$, and hence find an expression for S.

29. For what values of x will the series $1 + 2x + 3x^2 + \ldots$ converge? If the value of x is such that the series is convergent, what will be its sum to infinity?

30. Express the recurring decimal $0{\cdot}3\dot{8}\dot{5}$ as an infinite G.P., and by calculating S_∞ find to what vulgar fraction it is equivalent.

EXERCISE XXVIII

(Revision Exercise)

(A)

1. Solve the following equations:

(a) $$\frac{x+1}{2x-3} - \frac{3x-5}{6(x-2)} = 0.$$

(b) $$x^2 + xy - y^2 = 19.$$
$$3x - 2y = 10.$$

2. Prove that $\log \frac{1}{2} + \log \frac{2}{3} + \log \frac{3}{4} + \ldots + \log \frac{9}{10} = -1.$

3. If p eggs are bought for x pence and q eggs are bought at y pence per dozen, find the average cost of an egg.

4. Find the value of (a) $(1{\cdot}8742)^{3{\cdot}5}$ and (b) $(1{\cdot}8742)^{-3{\cdot}5}$.

5. If one solution of the equation $a^2x^2 - 3a(b+c)x + (b+2c)(2b+c) = 0$ is $\frac{b+2c}{a}$, find the other solution.

(B)

6. Find the quadratic equation whose solutions are 1 more than those of $6x^2 - 7x + 2 = 0$.

7. Simplify $\frac{1}{\sqrt{3}+\sqrt{5}} + \frac{2}{\sqrt{5}+\sqrt{8}}$, and find its value correct to two places of decimals.

8. Find the factors of

(a) $x^{\frac{1}{3}} - y^{\frac{2}{3}}$. (b) $x^{\frac{9}{5}} - y^{\frac{3}{5}}$.

9. The fifth term of an A.P. is -1 and the twelfth term is -15. Find the sum of the first 16 terms.

10. Find by logs the value of $\sqrt{\frac{12 \cdot 4}{8 \cdot 7} - \frac{8 \cdot 7}{12 \cdot 4}}$.

(C)

11. The first term of a G.P. is 15 and the second term is -30. Find the sum of the first 15 terms.

12. A man buys two houses for £1550. He sells one of them at a gain of 10 per cent. and the other at a loss of 10 per cent. and receives altogether £1555. How much did each house cost?

13. How many balls would be required to make a complete pyramid on a square base, each side of which contains 15 balls?

14. Prove that if $|x| < 1$, then $\frac{1}{1+x} = 1 - x + x^2 - x^3 + \ldots$ to ∞.

Hence solve the equation $1 - x + x^2 - \ldots = 1\frac{1}{2}$.

15. How many 0's after the decimal point will there be in $(\cdot 02)^{100}$?

(D)

16. Find a formula for the sum of n terms of the series $1 + 3x + 5x^2 + \ldots$ For what values of x will the series converge, and what will be its sum to infinity?

17. If $x^2 - px + q = 0$ has two solutions—namely, $x = \alpha$ and $x = \beta$—find the value of $\frac{\alpha}{\beta} + \frac{\beta}{\alpha}$ in terms of p and q.

Hence find the equation whose two solutions are $\frac{\alpha}{\beta}$ and $\frac{\beta}{\alpha}$.

18. The middle points of the sides of a square are joined so as to form a second square. By joining the middle points of the second square a third square is formed, and so on. What is the ratio of the area of the first square to the area of the sixth square?

19. The nth term of the series 5, 22, 63, . . . is given by

$pn^3 + qn$. Find p and q and a formula for the sum of n terms.

20. Prove the following identities:

(*a*) $x(y-z)^3 + y(z-x)^3 + z(x-y)^3$
$$\equiv (y-z)(z-x)(x-y)(x+y+z).$$

(*b*) $x^2(y-z)^3 + y^2(z-x)^3 + z^2(x-y)^3$
$$\equiv (y-z)(z-x)(x-y)(yz+zx+xy).$$

CHAPTER IV

RATIO, PROPORTION, AND VARIATION

25. We compare two quantities when we say that one is three times as large as the other, or that they are in the *ratio* of three to one. Thus two pieces of string, one of which is 6 feet and the other 2 feet in length, are in the ratio of 3 to 1, for the larger is three times as long as the smaller. This ratio is written either in fractional form, as $\frac{3}{1}$, or in the symbolic form, as $3:1$. Since a ratio is only another way of writing a comparison, it is necessary to express all the quantities which are being compared in the same units. Notice in particular that since a ratio describes the number of *times* one quantity is greater than another the ratio itself is a *number*, and not a quantity.

EXAMPLE 1

Find the ratio of 4 feet to 1 foot 6 inches.

The required ratio is $\dfrac{4 \text{ feet}}{1 \text{ foot } 6 \text{ inches}} = \dfrac{48 \text{ inches}}{18 \text{ inches}} = \dfrac{8}{3}.$

This may also be written $8:3$.

EXAMPLE 2

If $x : y = 2 \quad 3$, find the value of the ratio

$$2x + y : 5x - 2y.$$

Since
$$\frac{x}{y} = \frac{2}{3},$$

$$\therefore x = \frac{2y}{3}.$$

Hence $$\frac{2x+y}{5x-2y} = \frac{2\cdot\frac{2y}{3}+y}{5\cdot\frac{2y}{3}-2y} \qquad \left(\text{substituting } x=\frac{2y}{3}\right)$$

$$= \frac{\frac{4y}{3}+y}{\frac{10y}{3}-2y}$$

$$= \frac{\frac{7y}{3}}{\frac{4y}{3}}$$

$$= \frac{7y}{3} \times \frac{3}{4y}$$

$$= \frac{7}{4}.$$

Hence we have $2x + y : 5x - 2y = 7 : 4$.

EXERCISE XXIX

1. Find the ratios between the following quantities:

(*a*) £5 and £1 2*s*. 6*d*.
(*b*) 4 tons 3 cwt. and 2 tons 14 cwt.
(*c*) 8 yds. 1 ft. 8 in. and 3 yds. 2 ft. 6 in.
(*d*) 2 hrs. 12 min. and 5 hrs. 42 min.
(*e*) 1 sq. yd. and 1 acre.
(*f*) 45 m.p.h. and 21 m.p.h.
(*g*) 5 guineas and 5 shillings.
(*h*) 4 litres and 40 c.c.
(*i*) 3 gal. 1 qt. 1 pt. and 2 gal. 2 qts.
(*j*) 12 dollars and 18 cents.

2. Simplify the following ratios:

(a) $1\frac{1}{2} : 2\frac{1}{2}$.
(b) $2\frac{3}{4} : 4\frac{1}{4}$.
(c) $1\frac{1}{5} : 3\frac{3}{4}$.
(d) $4\frac{1}{2} : 1\frac{1}{4}$.
(e) $\frac{1}{9} : 1\frac{1}{3}$.
(f) $\cdot 5 : \cdot 25$.
(g) $1\cdot 85 : 2\cdot 15$.
(h) $7\frac{1}{2} : \frac{5}{12}$.

3. If $\dfrac{x}{y} = \dfrac{2}{3}$, find the value of $\dfrac{Kx}{Ky}$.

4. If $a : b = 1 : 2$, find the value of

(a) $a + b : a - b$.
(b) $2a + 3b : 3a - b$.
(c) $a^2 : b^2$.
(d) $a^2 + 3b^2 : 3a^2 - b^2$.
(e) $(a + b)^2 : (a - b)^2$.
(f) $(a + b)^3 : (a - b)^3$.
(g) $7a^2b^3 : 9a^3b^2$.
(h) $a^2 + b^2 : a^2 - b^2$.

5. Find the ratio $x : y$ if

(a) $x + y : x - y = 3 : 2$.
(b) $7x + 3y : 3x + 2y = 5 : 3$.
(c) $4x - y : 3x + 2y = 2 : 5$.
(d) $3x^2 + y^2 : x^2 + 3y^2 = 1 : 2$.

6. Divide 42 into two parts in the ratio 2 : 5.

Hint. Let x be one of the parts. Then $42 - x$ is the other part.

$$\frac{x}{42 - x} = \frac{2}{5}.$$

This is an equation for x.

7. Divide 140 into two parts in the ratio 3 : 7.

8. Two numbers are in the ratio of 3 : 4. If 2 is added to each of them the ratio is then 4 : 5. Find the numbers.

9. The ratio of a man's age to that of his son is 8 : 3. Five years ago this ratio was 7 : 2. Find their present ages.

10. What is the ratio of the perimeter of a square to its diagonal?

11. The ratio of the radii of two circles is 7 : 13. What is the ratio of (a) their circumferences, (b) their areas?

12. The ratio of the radii of two spheres is 7 : 13. What is the ratio of (a) their areas, (b) their volumes?

13. The perimeter of a triangle is 36 cm. The sides are in the ratio 2 : 3 : 4. Find the length of each side.

14. Two numbers are in the ratio 8 : 3. What is the ratio of their sum to their difference?

15. The sides of a rectangle are in the ratio of 7 : 13. If its area is 364 sq. cm., find the lengths of the sides.

16. Two numbers are in the ratio 2 : 3. What fraction of the first must be added to them both so as to make them in the ratio 5 : 7?

17. If x is positive, show that $(10 + x) : (7 + x) < 10 : 7$.

18. If x is positive, show that $(8 + x) : (13 + x) > 8 : 13$.

19. If x and b are positive and $\frac{a}{b}$ is some ratio, prove that

(*a*) If $a > b$, then $\frac{a}{b} > \frac{a+x}{b+x}$.

(*b*) If $a < b$, then $\frac{a}{b} < \frac{a+x}{b+x}$.

Hint.

$$\frac{a}{b} - \frac{a+x}{b+x} = \frac{ax - bx}{b(b+x)} = \frac{x(a-b)}{b(b+x)}.$$

If $a > b$ this is positive, and therefore

$$\frac{a}{b} > \frac{a+x}{b+x}.$$

Case (*b*) is proved in a similar way.

20. If x is positive and $\left(\frac{a+x}{b+x}\right)^2 = \frac{a}{b}$, prove that $x^2 = ab$. $(a \neq b.)$

26. If two ratios are equal the four quantities involved are said to be in *proportion.*

Thus, if $\frac{a}{b} = \frac{c}{d}$, the four quantities a, b, c, and d are in proportion, and d is said to be the *fourth proportional* to a, b, and c.

If $\frac{a}{b} = \frac{b}{c}$, then c is said to be the *third proportional* to a and b, and b is called the *mean proportional* between a and c.

If $\frac{a}{b} = \frac{b}{c} = \frac{c}{d} = \frac{d}{e} = \frac{e}{f} = \quad$, then the quantities a, b, c, d, e, f, are said to be in *continued proportion*.

EXERCISE XXX

1. Complete the following:

(a) $\frac{2}{3} = \frac{10}{}$. (e) $\frac{7}{} = \frac{12}{17}$.

(b) $\frac{3}{5} = \frac{}{35}$. (f) $\frac{8}{3} = \frac{}{10}$.

(c) $\frac{4}{} = \frac{36}{63}$. (g) $\frac{5}{} = \frac{8}{9}$.

(d) $\frac{3}{8} = \frac{4}{}$. (h) $\frac{x}{y} = \frac{k}{}$.

2. Find the fourth proportional to

(a) 3, 5, and 7. (b) 1, 2, and 3. (c) x, y, and z.

3. Find the mean proportional between

(a) 8 and 2. (b) 4 and 9. (c) x and y.

4. If $\frac{a}{b} = \frac{c}{d}$, prove that

(a) $\frac{a+b}{b} = \frac{c+d}{d}$.

Hint. Add 1 to both sides of the equation $\frac{a}{b} = \frac{c}{d}$.

(b) $\frac{a-b}{b} = \frac{c-d}{d}$.

(c) $\frac{a+b}{a-b} = \frac{c+d}{c-d}$.

5. If $a : b = c : d$, find the value of the ratio $\frac{1}{a} : \frac{1}{b}$.

6. If $\frac{1}{a} \quad \frac{2}{b} = \frac{3}{c} : \frac{4}{d}$, find the value of the ratio $a : b$.

7. If $ab + cd$ is a mean proportional between $a^2 + c^2$ and $b^2 + d^2$, prove that $\frac{a}{b} = \frac{c}{d}$.

8. If a, b, and c are in continued proportion, prove that

$$(a)\quad \frac{a+b}{b} = \frac{b+c}{c}.$$

$$(b)\quad \frac{a-b}{b} = \frac{b-c}{c}.$$

$$(c)\quad \frac{a-b}{a+b} = \frac{b-c}{b+c}.$$

27. The numerous relationships which connect quantities in proportion are best investigated by the "k method." The following examples illustrate this.

Example 1

If $\frac{a}{b} = \frac{c}{d} = \frac{e}{f}$, prove that each ratio is equal to $\frac{a+2c+3e}{b+2d+3f}$.

Let each one of the equal ratios be equal to k, so that we have

$$\frac{a}{b} = \frac{c}{d} = \frac{e}{f} = k,$$

from which $a = bk$, $c = dk$, and $e = fk$.

$$\frac{a+2c+3e}{b+2d+3f} = \frac{bk+2dk+3fk}{b+2d+3f} \quad \text{(Substituting } a = bk\text{, etc.)}$$

$$= \frac{k(b+2d+3f)}{b+2d+3f}$$

$$= k$$

$$= \frac{a}{b} = \frac{c}{d} = \frac{e}{f}.$$

EXAMPLE 2

If $\dfrac{a}{x+y-z} = \dfrac{b}{y+z-x} = \dfrac{c}{z+x-y}$, prove that $\dfrac{x}{c+a} = \dfrac{y}{a+b} = \dfrac{z}{b+c}$.

As before, let $\dfrac{a}{x+y-z} = \dfrac{b}{y+z-x} = \dfrac{c}{z+x-y} = k.$

Then
$$a = kx + ky - kz$$
$$b = -kx + ky + kz$$
$$c = kx - ky + kz,$$

from which
$$b + c = 2kz$$
$$c + a = 2kx$$
$$a + b = 2ky.$$

$$\frac{x}{c+a} = \frac{x}{2kx} = \frac{1}{2k}.$$

Similarly, $\dfrac{y}{a+b} = \dfrac{1}{2k}$ and $\dfrac{z}{b+c} = \dfrac{1}{2k}$.

Hence
$$\frac{x}{c+a} = \frac{y}{a+b} = \frac{z}{b+c}.$$

EXERCISE XXXI

1. If $\dfrac{a}{b} = \dfrac{c}{d}$, prove the following results:

(a) $\dfrac{a^2+b^2}{c^2+d^2} = \dfrac{b^2}{d^2}$.

(b) $\dfrac{2a^2+5c^2}{2b^2+5d^2} = \dfrac{ac}{bd}$.

(c) $\dfrac{5a-3b}{8a-5b} = \dfrac{5c-3d}{8c-5d}$.

(d) $\dfrac{a^n + b^n}{c^n + d^n} = \dfrac{b^n}{d^n}$.

(e) $\dfrac{a + c}{b + d} = \dfrac{a - c}{b - d}$.

2. If a, b, and c are in continued proportion, prove the following results:

(a) $\dfrac{b^2 + c^2}{a^2 + b^2} = \dfrac{c}{a}$.

(b) $\dfrac{4a - 3b}{5a - 7b} = \dfrac{4b - 3c}{5b - 7c}$.

3. If $\dfrac{a}{b} = \dfrac{c}{d} = \dfrac{e}{f}$, prove that $\dfrac{8a + 7c}{8b + 7d} = \dfrac{3c - 5e}{3d - 5f}$.

4. If $\dfrac{a}{b} = \dfrac{c}{d} = \dfrac{e}{f}$, prove that

(a) $\dfrac{a}{b} = \dfrac{a + c + e}{b + d + f}$.

(b) $\dfrac{c^2}{d^2} = \dfrac{a^2 + c^2 + e^2}{b^2 + d^2 + f^2}$.

5. If $\dfrac{a}{b} = \dfrac{c}{d} = \dfrac{e}{f} =$, prove that

$$\left(\frac{a}{b}\right)^n = \frac{p_1a^n + p_2c^n + p_3e^n + \cdots}{p_1b^n + p_2d^n + p_3f^n +}$$

where $p_1, p_2, p_3, \ldots$ are any numbers.

6. Solve the equations

$$\frac{x + y + 3}{4} = \frac{3x + y + 1}{6} = \frac{2(3x + y)}{11}.$$

7. If $\dfrac{a}{x - y} = \dfrac{b}{y - z} = \dfrac{c}{z - x}$, prove that

(a) $a + b + c = 0$.

(b) $(x + y)a + (y + z)b + (z + x)c = 0$.

8. What number must be subtracted from each of 6, 7, 10, and 12 so that the remainders are in proportion?

9. If $\dfrac{2x + y}{4} = \dfrac{y + 2z}{8} = \dfrac{3z - 2x}{7}$, find $x : y \quad z$.

10. If $x = \dfrac{a}{b - c}$, $y = \dfrac{b}{c - a}$, $z = \dfrac{c}{a - b}$, prove that

$$xy + yz + zx = -1.$$

11. If $x : y : z = 3 : 5 : 7$ and $2x - 3y + 5z = 78$, find the values of x, y, and z.

12. If $\dfrac{a + c}{a + b + c + d} = \dfrac{a}{a + b}$, prove that $\dfrac{a}{b} = \dfrac{c}{d}$.

13. Solve the equations

$$\frac{5x + y + 7}{4} = \frac{2x - 3y + 1}{6} = \frac{3x - 2y - 1}{5}.$$

14. If $\dfrac{a}{b} = \dfrac{x}{y}$, prove that

$$a^2 + b^2 : \frac{a^3}{a + b} = x^2 + y^2 : \frac{x^3}{x + y}.$$

15. If $x^2 + y^2 = 1$ and $\dfrac{x}{a} = \dfrac{y}{b} = \dfrac{1}{k}$, find k.

28. If a cyclist is travelling at 10 m.p.h. we know that in

1 hour he travels	10 miles	
2 hours	20	
3	30	and so on.

It is quite clear from this that as the time he spends travelling is *increased* the distance covered is *increased* also; further, it is not difficult to see that as the time is increased the distance is increased in the same ratio, for if the time is doubled the distance covered is doubled also, and so on. In such a case we say that the distance *varies directly* as the time, and we write this as $D \propto t$, where D represents the distance and t the time.

Notice in particular that

$$\frac{\text{any distance travelled by the cyclist}}{\text{the time taken to do this distance}} = 10,$$

a fact which we may write as $\frac{D}{t} = 10$, or $D = 10t$.

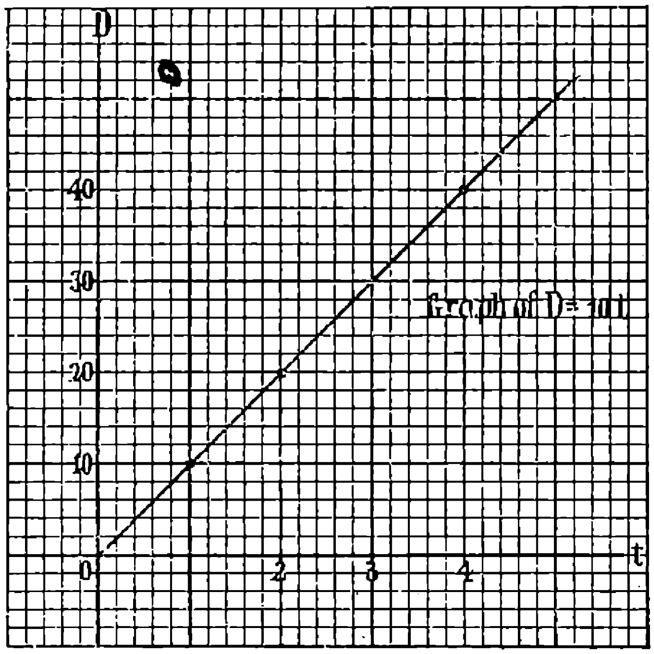


FIG. 2

This can be extended to any case of direct variation, and thus we have $x \propto y$; then $x = ky$, where k is some constant whose value is determined by the numerical data of the question. The graph of the equation $x = ky$ is a straight line passing through the origin. This is a particular feature of the direct variation of two quantities. In Fig. 2 the two quantities are the distance covered and the time taken by the cyclist, and the travel graph which has thus been constructed has the equation $D = 10t$. Study the two following examples carefully The first shows how a particular value of one of two varying quantities is calculated from numerical data, and the second extends the method to concrete problems.

EXAMPLE 1

If $y \propto x$ and $y = 15$ when $x = 3$, find the value of y when $x = 8$.

We have $y \propto x$.

Therefore $y = kx$, where k is some constant.

If $y = 15$, $x = 3$, so that on substitution we have

$$15 = k.3.$$

I.e., $15 = 3k$, or $k = 5$.

Hence $y = 5x$.

Therefore if $x = 8$, $y = 40$.

EXAMPLE 2

The area of a circle varies as the square of the radius. If the area of a circle of radius 2 in. is $12\frac{4}{7}$ sq. in., find the area of a circle of radius 3 in.

If we let A denote the area and r the radius of the circle, we have $A \propto r^2$.

$$\therefore A = kr^2, \text{ where } k \text{ is a constant.}$$

If $r = 2$, then $A = 12\frac{4}{7}$,

so that we have $12\frac{4}{7} = k.4.$

$$k = \frac{12\frac{4}{7}}{4}$$

$$= 3\frac{1}{7}.$$

Hence our equation is $A = 3\frac{1}{7}r^2$.

Therefore, if $r = 3$, then $A = 3\frac{1}{7}.3^2$

$$= \frac{22}{7} \times \frac{9}{1}$$

$$= \frac{198}{7}$$

$$= 28\frac{2}{7}.$$

Hence the area of a circle of radius 3 in. is $28\frac{2}{7}$ sq. in.

EXERCISE XXXII

1. If $x \propto y$ and $y = 28$ when $x = 7$, find the value of y when $x = 3$.

2. If $y \propto x^2$ and $y = 27$ when $x = 3$, find the value of y when $x = 4$.

3. If $x \propto y$, prove that

$$(a)\ x^n \propto y^n. \quad (b)\ x^2 - y^2 \propto xy. \quad (c)\ x + y \propto x - y.$$

4. The circumference of a circle varies as the radius. If the radius is 7 in. the circumference is 3 ft. 8 in. Find the radius of a circle whose circumference is 10 ft.

5. If $y \propto x^2 - 2$ and $y = 21$ when $x = 3$, find the value of y when $x = 2\frac{1}{2}$.

6. The volume of a sphere is proportional to the cube of its radius. If the volume of a sphere of radius one inch is $4\frac{4}{21}$ cu. in., find the volume of a sphere of radius $3\frac{1}{4}$ in.

7. The radii of three spheres are in the ratio of 1 : 3 : 5. What is the ratio of (*a*) their areas, (*b*) their volumes?

8. If $y \propto x^3$, complete the following table:

x	1	2	3	4	5	6
y				32		

9. It costs p pence to whiten a ceiling of area 200 sq. ft. What will it cost to do a similar ceiling with an area of 235 sq. ft.?

10. The sag of a beam supported at the ends is proportional to the cube of its length. For a beam 10 ft. in length the sag is observed to be 2·5 in. What would be the sag in the case of a similar beam 12 ft. in length?

11. The pressure on the windscreen of a car varies as the square of its speed. When moving at 15 m.p.h. this pressure is 7·5 lb. What would it be when the speed of the car is increased to 20 m.p.h.?

12. Check your answer to the preceding question by graphical construction. Plot P (pressure on the windscreen) and v^2 (square of speed of car).

13. A cube of edge 2 in. weighs 1 lb. What will be the weight of a cube of edge 4 in.?

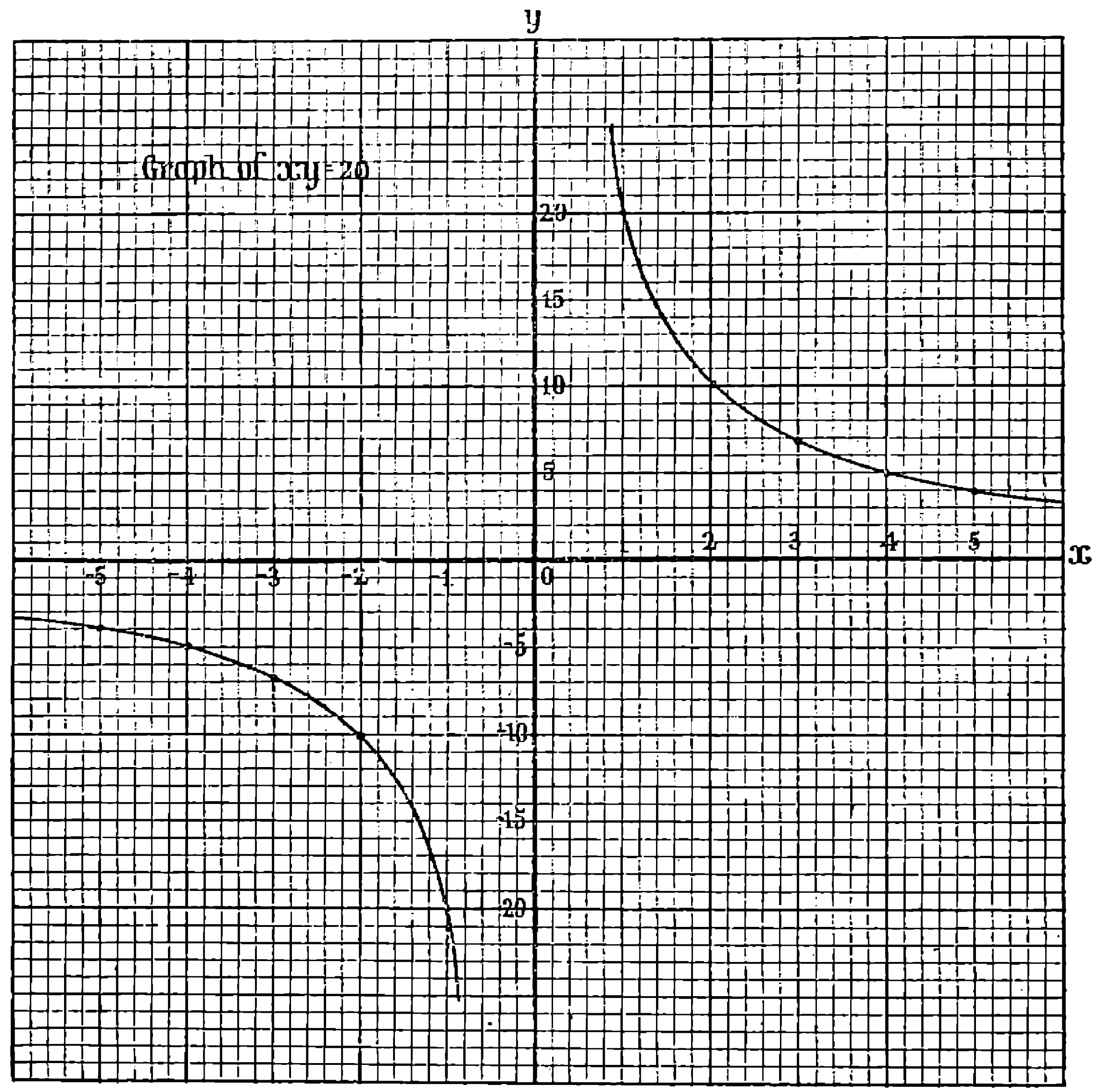


FIG. 3

14. What is the ratio of the surfaces of the two cubes in the preceding question?

15. The time of oscillation of a pendulum from rest to rest varies as the square root of its length. A seconds pendulum does this in one second, and is 39·12 in. long. What length of pendulum would do this in two seconds?

29. If y varies directly as the reciprocal of x—*i.e.*, $y \propto \frac{1}{x}$—then y is said to vary *inversely* as x.

If $y \propto \frac{1}{x}$, then $y = k \cdot \frac{1}{x}$, where k is a constant.

$$\therefore xy = k.$$

The graph of $xy = k$ is a rectangular hyperbola situated in the first and third quadrants. The extremities of the curves approach the axes, but the curves never actually cut them. In such a case the axes are called *asymptotes*.

As an example of inverse variation, consider a journey of 20 miles, which can be accomplished in

1 hour	if the speed is 20 m.p.h.	
2 hours	10	
4	5	
5	4	, and so on.

Notice that an INCREASE in time is associated with a DECREASE in speed. In general, if y varies inversely as x, then an increase in y causes a decrease in x. The graph of $xy = 20$ (illustrating the journey of 20 miles accomplished at different speeds) is shown in Fig. 3. Problems involving inverse variation are solved by methods which follow closely those just described in our consideration of direct variation.

EXAMPLE

If y varies inversely as x and $y = 10$ when $x = 2$, find y when $x = 5$.

Since $y \propto \frac{1}{x}$, then $y = \frac{k}{x}$, where k is a constant.

But $y = 10$ when $x = 2$, so that

$$10 = \frac{k}{2}.$$

$$\therefore k = 20.$$

$$\therefore y = \frac{20}{x}.$$

If $x = 5$, $\quad \therefore y = 4.$

EXERCISE XXXIII

1. If y varies inversely as x and $y = 7$ when $x = 2$, find the value of y when $x = 5$.

2. If y varies inversely as $\sqrt{x}$ and $y = 5$ when $x = 9$, find the value of y when $x = 4$.

3. The following table shows a set of corresponding values of x and y.

x	2	3	4	5	6	7	8
y	7·5	5	3·75	3	2·5	2·1	1·8

(*a*) Why does y vary inversely as x?

(*b*) Plot y and x and obtain a rectangular hyperbola.

(*c*) Plot y and $\frac{1}{x}$ and obtain a straight line.

(*d*) In experimental work, what advantages has (*c*) over (*b*)?

4. The force between two magnetic poles varies inversely as the square of the distance between them. If the two poles are 5 cm. apart this force is 40 dynes. What will it be if this distance is increased to 8 cm.?

5. If y is inversely proportional to $x + a$, and when $x = 4$ $y = 7$, and when $x = 5$ $y = 12$, find an equation connecting x and y.

6. If $(x + 2)(y + 4) = 16$, write this as an inverse proportion.

7. The pressure of a gas at constant temperature varies inversely as its volume. If a certain quantity of gas occupies a volume of 5 cu. ft. when subjected to a pressure of one atmosphere, what will be the volume when this pressure is increased to four atmospheres?

8. The force of the earth's attraction varies inversely as the square of the distance of the body from the centre of the earth. Assuming that the earth is a sphere of radius

4000 miles, find the difference in weight when a body weighing 1 ton is raised 100 miles above the earth's surface.

9. The time of vibration of a loaded beam is inversely proportional to the square root of the deflection caused by the load. When the deflection was ·05 in. the time was ·25 sec. What would be the time when the deflection was ·04 in.?

10. If $x \propto \frac{1}{y^2}$ and $x \propto z^2$, what is the effect on y of trebling z?

11. Find the values of x and y which satisfy simultaneously the relations $xy = 12$ and $x - 2y = 2$.

12. Illustrate your answer to the preceding question by means of two graphs drawn to the same axes.

30. One quantity is said to *vary jointly* as two other quantities if it varies as their product. Thus, if y varies jointly as x and z, then $y \propto xz$, and consequently $y = kxz$, where k is a constant. Such variation is very common. The following are a few examples:

1. The volume of a cone varies jointly as the height and area of its base. ($V \propto hA$.)
2. The weight of a metal rod varies jointly as the length and the square of its diameter. ($W \propto ld^2$.)
3. The electrical resistance of a wire varies directly as its length and inversely as the square of its diameter. $\left(R \propto \frac{l}{d^2}.\right)$

31. An Important Theorem.

If $y \propto x$ when z is constant,
and $y \propto z$ when x is constant,
then $y \propto xz$ when both x and z are varying.

Since $y \propto z$ when x is constant, therefore $y = Kz$, where K is independent of y and z, and therefore is dependent only on x.

Since $$y = Kz,$$

$$\frac{y}{x} = \frac{Kz}{x}.$$

$$\therefore \frac{K}{x} = \frac{1}{z} \cdot \frac{y}{x}.$$

If, now, x varies and z remains constant, then y will vary, but $\frac{y}{x}$ will remain constant.

Therefore $\frac{K}{x}$ will remain constant when x varies.

But K depends only on x.

$$\therefore K \propto x.$$

$$\therefore K = kx,$$

where k is independent of ALL the variables.

But $$y = Kz.$$

$$\therefore y = kxz.$$

I.e., $$y \propto xz.$$

EXAMPLE

The electrical resistance of a wire varies directly as its length and inversely as the square of its diameter. If the resistance of 50 cm. of wire of diameter ·08 cm. is 2 ohms, what will be the resistance of 85 cm. of wire having a diameter of ·1 cm.?

Using the letters R, l, and d for resistance, length, and diameter, we have

$$R \propto \frac{l}{d^2}.$$

$$R = k\frac{l}{d^2}.$$

$R = 2$ when $l = 50$ and $d = \cdot 08$.

$$\therefore 2 = k\frac{50}{(\cdot 08)^2}.$$

$$k = \cdot 000256.$$

$$\therefore R = \frac{\cdot 000256 l}{d^2}.$$

If $l = 85$ and $d = \cdot 1$,

$$\therefore R = \frac{\cdot 000256 \times 85}{(\cdot 1)^2}$$

$$= 2\cdot 176.$$

Hence the required resistance is 2·176 ohms.

EXERCISE XXXIV

1. Write down formulæ connecting the variables in the following:

(*a*) y varies jointly as p and q.
(*b*) y varies jointly as p and $\sqrt{q}$.
(*c*) y varies as p and inversely as q.
(*d*) y varies as p^2 and inversely as $\sqrt{q}$.
(*e*) y varies jointly as p and q and inversely as r.

2. The volume of a cone varies jointly as the height and the square of the radius. For a cone of height 7 in. and radius 1 in. the volume is $7\frac{1}{3}$ cu. in. What will be the volume of a cone of height 8 in. and with a radius $1\frac{1}{2}$ in.?

3. The energy stored in a fly-wheel varies as the fifth power of the diameter, and also as the square of its speed of rotation. For a wheel of diameter 6 ft. running at 50 revolutions per min. this energy amounts to 12,000 ft.-lb. What would it amount to for a wheel half this size, but running at twice the speed?

4. The weight of a sphere varies jointly as the cube of the radius and the density of the material. A sphere of radius 2 cm. made of zinc, whose density is 7 grm. per c.c.,

weighs 235 grm. What would be the weight of a sphere of hard wood, 3·5 cm. radius, and having a density of 1·2 grm. per c.c.?

5. If p varies directly as q and inversely as r^2, and $p = 1$ when $q = 2$ and $r = 3$, find the value of p when $q = 4$ and $r = 5$.

6. The greatest weight which can be hung from one end of a beam, the other end of which is securely fastened, is directly proportional to its breadth and the square of its thickness, and inversely proportional to its length. For a beam 12 ft. long, $2\frac{1}{2}$ in. wide, and 2 in. thick this weight is 5 cwt. What would it be for a beam 8 ft. long and 3 in. square?

7. The weight of a metal rod varies directly as its length and also as the square of the diameter. A rod 18 in. long and of 1 in. diameter weighs $3\frac{1}{2}$ lb. What will be the weight of a rod 10 ft. long and of $1\frac{1}{2}$ in. diameter?

8. The cost of conducting a school is partly constant and partly proportional to the number of scholars, all of whom pay the same fee. If there are 30 scholars the expenses are just covered. If there are 60 scholars there is a profit of £450. Find a formula for the profits when there are n scholars.

9. Find the profit for the preceding question when there are 150 scholars.

10. If y varies jointly as x and z and $x + z = 10$, express y in terms of x.

If $y = 9$ when $x = 1$, find the two values of x when $y = -11$. What are the corresponding values of z?

11. If q varies directly as x and p varies inversely as x, and $p + q = 6$ when $x = 1$, and $p + q = 10\frac{1}{2}$ when $x = 2$, find the value of $p + q$ when $x = -2$.

12. The densities of two spheres are in the ratio 3 : 4. The weights of the spheres are in the ratio 16 : 9. What is the ratio of their radii?

13. The distance which a body falls from rest varies as the square of the time it has been falling. If a body falls

144 ft. in 3 sec., how far will it fall in 6 sec.? How far will it fall in the sixth second?

14. The volume of a cylinder varies jointly as the square of the diameter and the height. The volume of a cylinder of diameter 2 ft. and height 14 ft. is 44 cu. ft. What will be the volume of a cylinder of the same height, but with a diameter twice as great?

15. The loss of head of water flowing through a pipe due to pipe friction is proportional to the length and inversely proportional to the diameter of the pipe. For 60 ft. of 3-in.-diameter piping this is $4\frac{1}{2}$ ft. What would it be for 75 ft. of $2\frac{1}{2}$-in.-diameter piping?

16. If p, v, and t are the pressure, volume, and absolute temperature respectively of a gas, Boyle's Law states that $p \propto \frac{1}{v}$ when t is constant, and Charles' Law states that $v \propto t$ when p is constant. Prove that $pv = \mathrm{R}t$, where R is a constant independent of all the variables.

17. If $y \propto x^{\frac{1}{2}}$ when z is constant and $y \propto \frac{1}{z^3}$ when x is constant, prove that $y \propto \frac{x^{\frac{1}{2}}}{z^3}$.

EXERCISE XXXV (Revision Exercise)

(A)

1. Simplify $\left[\frac{1+a}{1-a} - \frac{1-a}{1+a}\right]^2 \div \left[\frac{1-a}{1+a} + \frac{1+a}{1-a}\right]^2$

2. Solve the following equations:

(*a*) $3x^2 - x - 2 = 0$.
(*b*) $3(2x - 1)^2 - (2x - 1) - 2 = 0$.

3. Prove that

$$\frac{21 + 23 + 25 + \ldots + 49}{2 + 4 + 6 + \ldots + 20} = \frac{105}{22}.$$

4. A cubic foot of lead is melted down and recast into small spherical pellets, each $\frac{1}{4}$ in. in radius. Find by using logarithms how many pellets would be obtained, assuming that there is no loss of lead during the process.

5. A number of men can form a solid square, or, if one of them remains out, a hollow square of four ranks, the front rank of which has 18 more men than before. How many men formed the solid square?

(B)

6. Solve the following equations:

(*a*) $x^2 + xy + y^2 = 7.$
$2x + y = 4.$

(*b*) $3^x = 21.$

7. The ratio of a man's age to that of his son is 9 : 1. In six years the ratio will be 21 : 5. In how many years from now will it be 3 : 1?

8. In a potato race 10 potatoes are placed at equal intervals of 2 yards, and the basket into which they have to be placed is 5 yards from the nearest potato. How far will each competitor run if he travels from the basket to each potato and back again, gathering in the potatoes one at a time?

9. If $p, q, r, s, t, w,$ are in geometrical progression, prove that rs is a mean proportional between pq and tw.

10. What number must be added to 4, 14, 25, and 70 so that the sums may be in proportion?

(C)

11. (*a*) Prove that $x^{-1} + 2x^{-2} + x^{-3} \equiv (x^{-\frac{1}{2}} + x^{-\frac{3}{2}})^2.$

(*b*) Simplify $\dfrac{a}{(b-c)(c-a)} + \dfrac{b}{(c-a)(a-b)} + \dfrac{c}{(a-b)(b-c)}.$

12. The population of a town increased from 120,000 to 175,000 in ten years. If the same rate of progress is

maintained, what will be the population at the end of the next ten years?

13. If $\frac{a}{b} = \frac{c}{d} = \frac{e}{f}$, prove that

$$(ab + bc + cd)(cd + de + ef)(b^2 + d^2 + f^2) \\ = (b^2 + bd + d^2)(d^2 + fd + f^2)(a^2 + c^2 + e^2).$$

14. If $6(2x^2 - y^2) = xy$, find the ratio of $x : y$.

15. The sum of the numerator and the denominator of a proper fraction is 11. The difference between the fraction and its reciprocal is $1\frac{5}{28}$. Find the fraction.

(D)

16. If $\frac{3x - 1}{(3x + 1)(9x + 2)} = \frac{A}{3x + 1} + \frac{B}{9x + 2}$, find A and B.

17. If $a^2 = 17b^3$ and $b^2 = 17c^3$, express a in terms of c, and calculate by logarithms the value of a when $c = 4{\cdot}372$.

18. A sequence of natural numbers is arranged in groups as follows: [1], [2, 3], [4, 5, 6], [7, 8, 9, 10],

How many numbers will there be in the nth bracket, and what will be their sum?

19. Find the square root of $(6x^2 - 7x - 3)(12x^2 + x - 1)(8x^2 - 14x + 3)$.

20. In what ratio must the edge of a cube be increased so that (a) its surface, (b) its volume, may be doubled?

EXAMINATION QUESTIONS

THE following questions are taken from past examination papers of various examining bodies, the source in each case being indicated by the letters in brackets. The authorities referred to are:

The University of Bristol (B.).
The University of Cambridge (C.).
The University of Durham (D.).
The University of London (L.).
The Joint Matriculation Board (N.).
The University of Oxford (O.).
The Central Welsh Board (C.W.B.).
The Oxford and Cambridge Schools Examination Board (O. and C.).

The questions are arranged in groups:

	Nos.
Formulæ	1–14
Factors	15–28
Simplification	29–46
Equations (Simple and Quadratic)	47–67
Equations (Simultaneous)	68–89
Indices and Logarithms	90–105
Series	106–121
Ratio, Proportion, and Variation	122–134
Graphs	135–153
Problems	154–175

FORMULÆ

1. You are given the formula

$$x = y\left(\frac{2t}{p+q} - 1\right).$$

(*a*) Change the subject of this formula to t—*i.e.*, express t in terms of the other letters.

(*b*) Change the subject to p. (O. and C.)

2. If T, V, and m are connected by the formula $T = \dfrac{76\,V}{12V + 10{\cdot}6m}$, calculate the value of T, correct to two places of decimals, when $V = 13{\cdot}8$ and $m = 6{\cdot}25$. Rearrange this formula so as to give V in terms of T and m, and calculate the value of V when $T = 5{\cdot}5$ and $m = 7{\cdot}8$. (C.W.B.)

3. A cyclist rides d miles out at x miles an hour, and returns home without stopping at y miles an hour; find the total time taken, and prove that his average speed for the whole journey is $\dfrac{2xy}{x + y}$ miles per hour. (B.)

4. If $y = \dfrac{x - 2}{x + 2}$, find both x and $\dfrac{x^2 - 4}{x^2 + 4}$ in terms of y. (O.)

5. (*a*) If $a = 1 - \dfrac{2b}{ct - b}$, express t in terms of the other letters.

(*b*) Find what the value of t reduces to if $a = 4$ and $b = 2c$. (O. and C.)

6. Given that $V = \pi h^2\left(r - \dfrac{h}{3}\right)$, find V, correct to three significant figures, if $\pi = 3{\cdot}142$, $r = 6{\cdot}87$, and $h = 3{\cdot}94$.

Express r in terms of V, h, and π. (C.W.B.)

7. A sheet of paper is l inches by b inches; n of these sheets are cut up to make envelopes. If x per cent. of the paper is wasted, and the area of paper in each finished envelope is a square inches, write down an expression for the number of envelopes made. (O.)

8. A man buys x tons of potatoes at £y per ton; 10 per cent. are wasted, and he sells the remainder at z pence per stone (14 lb.). How much does he receive? Prove that he makes no profit unless $3z$ is greater than $5y$, and that if $z = 2y$ his profit is 20 per cent. (D.)

9. A batsman's average is found by dividing the total number of runs he has made by the number of times he is 'out.' A Test-match cricketer has an average of r runs for six completed innings before the last match is played. In the next match he scores altogether l runs, and is 'out' in both innings. Show that his average is now $\frac{3}{4}r + \frac{1}{8}l$. If he doubles his average as a result of the last match show that $l = 10r$. (D.)

10. How many eggs can be purchased for one shilling if x eggs cost u shillings and v pence? (B.)

11. A horse which cost £a is sold for £b. What is the gain per cent.? If sold for £c the loss per cent. would be half the previous gain per cent. Prove that $a = \frac{1}{3}(b + 2c)$. (D.)

12. A cyclist travels a miles in b hours. Find how many miles he would travel in c hours at half this speed. (C.)

13. (*a*) A man's gross income is £b. Find in pounds his net income after payment of tax at the rate of c shillings in the pound.

(*b*) Given that $pv = cd\left(1 + \frac{t}{273}\right)$, express t in terms of the other quantities in the formula. (N.)

14. (*a*) At a concert the charge for admission was 1*s*. 6*d*., but a further charge of 2*s*. was made for a seat. If the number of people admitted was N and the total receipts were £P, find a formula for the number n of people that were seated.

(*b*) If $v = a\left(1 - \frac{T}{t}\right)\sqrt{h}$, express T in terms of the other quantities. (N.)

FACTORS

15. Factorize

(*a*) $12x^2 + 5x - 72$.
(*b*) $(2x - y)^2 - 4(3x - 2y)^2$.
(*c*) $x^3 + 2x^2 - 11x - 12$.

16. Given that $x - 2$ is a factor of the expression $6x^3 - a(5x^2 - 3x - 5) - 3x^2$, a being a constant, determine whether $3x + 2$ is a factor of the same expression or not. (C.W.B.)

17. Find all the factors of

(a) $6x^2 - 13x + 6$.
(b) $2a^2 - 18(b - c)^2$.

One of the three factors of $x^3 + x^2 - ax + 8$ is $x + 4$. Find the value of a and the other two factors. (B.)

18. Factorize

(a) $30(a^2b^2 - 1) + 32ab$.
(b) $(x - 2)^3 + (x + 2)^3$.
(c) $(3a - 1)bc + ac^2 - 3b^2$. (O.)

19. (a) Factorize

(i) $6x^2 + 7x - 10$.
(ii) $a^2 - 2ab - 2a + 4b$.
(iii) $x^4 - (x^2 - x - 1)^2$.

(b) $x - 2$ and $2x + 1$ are factors of $4x^3 - 4x^2 + ax + b$. Find the values of a and b and factorize the expression. (C.W.B.)

20. Factorize $3y^2 + 32y + 20$, and hence or otherwise prove that if $y = 6x^2 - 17x + 2$, then

$$3y^2 + 32y + 20 = (2x - 3)(3x - 4)(3x - 8)(6x - 1).$$

(O.)

21. Factorize

(a) $x^2 - 4x - 21$.
(b) $18x^2 + 9xy - 20y^2$.

For what values of c will $x^2 + x - 6$ and $x^2 + 5x + c$ have a common factor? (L.)

22. Find all the factors of

(a) $3x^2 - 7x - 6$.
(b) $ax^2 - bxy - bx^2 + axy$.
(c) $(a^2 - b^2)x^2 - 4abx - (a^2 - b^2)$. (B.)

23. Factorize the following:

$x^2 - 4$, $x^3 + 4x^2 + 3x$, $x^2 + 4x + 4$, and $x^2 - x - 2$
and find the value of

$$\frac{x^2 - 4}{x^2 + 4x + 4} \div \frac{x^2 - x - 2}{x^3 + 4x^2 + 3x}. \qquad \text{(D.)}$$

24. By the Remainder Theorem or otherwise find

(*a*) The value of a such that $12x^3 - ax + 60$ and $8x^3 + 27$ may have a common linear factor.

(*b*) The factors of $3x^3 - 4x^2 - 17x + 6$. (B.)

25. Find the factors of the expression $3x^2 + 16x + 21$, and hence find the two prime factors of the number 31,621. (C.)

26. Resolve into their simplest factors

(*a*) $al - 6bm - 3am + 2bl$.

(*b*) $6p^2 - 7pq - 5q^2$.

(*c*) $2a^2 - 3ab + b^2 + c(a - b)$. (N.)

27. Resolve into their simplest factors

(*a*) $x^2 - y^2 - 2x + 1$.

(*b*) $ab - 3ad + bk - 3dk$.

For what value of p is $3x^3 + 13x^2 + px - 9$ exactly divisible by $x + 3$? (N.)

28. (*a*) Find the H.C.F. of $6x^2 + 19x - 7$ and $6x^3 + 17x^2 - 16x - 7$.

(*b*) Find the value of c if the L.C.M. of $x^2 + 3x - 10$, $x^2 - 2x - 35$, and $x^3 - 5x^2 - 29x + c$ is $(x - 2)(x + 5)(x - 3)(x - 7)$. (N.)

SIMPLIFICATION

29. $\dfrac{2x}{(x-2)^2} - \dfrac{1}{2x-1} - \dfrac{3x}{2x^2 - 5x + 2}$. (L.)

30. Express as one fraction in its lowest terms

$$\frac{19x - 3}{6x^2 - x - 12} - \frac{5}{3x + 4} + \frac{3}{2x - 3}. \qquad \text{(C.W.B.)}$$

31. Without working out any square roots show that

$$\frac{3\sqrt{2}+2\sqrt{3}}{3\sqrt{2}-2\sqrt{3}} - \frac{3\sqrt{2}-2\sqrt{3}}{3\sqrt{2}+2\sqrt{3}} \doteq 4\sqrt{6}.$$ (B.)

32. (*a*) State a simple value of x which makes $4a^2 + x + 9b^2$ and $4a^2 - x + 9b^2$ perfect squares; and work out the product of the two expressions when x is given this value.

(*b*) Simplify

$$\frac{1-\frac{1}{2x}}{2(1+2x)} + \frac{1+\frac{1}{2x}}{2(1-2x)}.$$ (O. and C.)

33. (*a*) Simplify $\frac{p+q}{1-pq}$, where $p = \frac{2x}{1-x}$ and $q = \frac{1-x}{1+x}$.

(*b*) If $5x = \frac{15-3y}{2}$ and $3y = \frac{15-20x}{3}$, find the numerical value of $5x^2 - 3y^2$. (O. and C.)

34. Simplify

$$\left(\frac{p}{q} - \frac{p-q}{p+q}\right) \div \left(\frac{q}{p} + \frac{p-q}{p+q}\right).$$ (C.W.B.)

35. In the expression $3K^2 + 2KL - 4L^2 - 3K - 7L - 16$ substitute $(x + a)$ for K and $(y + b)$ for L, and find the coefficients of x and y in the result. If both these coefficients are zero, what are the numerical values of a and b? (O.)

36. Prove that

$$(a^2 + 4b^2)^4 - 64a^2b^2(a^2 - 4b^2)^2 = (a^4 - 24a^2b^2 + 16b^4)^2.$$ (O.)

37. (*a*) When $a = 1, b = 2$, and $c = -1$, find the value of

$$\frac{a^2}{b-c} + \frac{b^2}{c-a} + \frac{c^2}{a-b}.$$

(*b*) Show that $(x - 2)(6x^2 - 7x - 20)$ is identically equal to $(3x + 4)(2x^2 - 9x + 10)$. (C.W.B.)

38. Simplify

(a) $\frac{3}{1-2x} - \frac{7}{1+2x} - \frac{4-20x}{4x^2-1}$.

(b) $(a^4 - 81b^4) \div (a^3 - 3a^2b + 9ab^2 - 27b^3)$. (O.)

39. Simplify

(a) $\frac{3}{(x-2)^2} - \frac{2}{x^2-5x+6} - \frac{1}{x^2-4}$.

(b) $a^{\frac{1}{2}}b^{\frac{1}{3}}(ab)^{\frac{1}{4}} \div \frac{a^{\frac{5}{12}}}{b^{\frac{5}{12}}}$. (L.)

40. Simplify

(a) $\left(x^2 + 1 - \frac{3}{x^2-1}\right) \div \left(x + 1 - \frac{1}{x-1}\right)$.

(b) $2^{n-3} \times 4^{-n} \times 16^{n+1} \div 8^{n-1}$. (D.)

41. If $X = 5a + 2b$ and $Y = 5a - 7b$, find the difference between $X^2 + XY + Y^2$ and $3XY$ in terms of a and b. (D.)

42. (a) Without using tables, find the value of $(0{\cdot}16)^{-\frac{1}{2}} \div (0{\cdot}125)^{\frac{2}{3}}$.

(b) Express in its simplest form

$$(a-b)\left\{\frac{1}{(\sqrt{a}-\sqrt{b})^2} \div \frac{1}{(\sqrt{a}+\sqrt{b})^2}\right\}.$$ (B.)

43. (a) Show that

$$\frac{x(x-1)(x-2)}{1.2.3} + \frac{x(x-1)}{1.2} = \frac{(x+1)\,x\,(x-1)}{1.2.3}$$

(b) Simplify

$$\left(2x - 7 + \frac{6}{x}\right)\left(2x + 7 + \frac{6}{x}\right) \div \left(x - \frac{4}{x}\right).$$ (D.)

44. (a) Find the value of $\frac{n(n-1)(n-2)(n-3)}{24}$ when $n = -3$.

(*b*) Express $\frac{5}{12(x-5)} - \frac{17}{12(x-17)}$ as a single fraction in its simplest form. (C.)

45. Express in their simplest forms

(*a*) $\frac{a^4 - b^4}{(a-b)^2} \div \frac{a^2 + b^2}{a^2b - ab^2}$.

(*b*) $\frac{3}{1+x} + \frac{7}{1-x} + \frac{x}{4}$. (N.)

46. (*a*) Find, as a single fraction, the expression by which unity exceeds $\frac{a-b}{b} + \frac{2a}{a-b} - \frac{a^2}{b(a-b)}$.

(*b*) If $2K = m(u+v)^2 + m(u-v)^2$ and $2L = m(u+v)^2 - m(u-v)^2$, where $u = vt$, express, as simply as possible, $\frac{L}{K}$ in terms of t. (N.)

EQUATIONS (SIMPLE AND QUADRATIC)

47. $\frac{1}{x-3} + \frac{1}{x-4} = \frac{2}{x-5}$. (O. and C.)

48. $3x^2 - 10x = 3$. (Give the result correct to two decimal places.) (C.W.B.)

49. $11 - \frac{x-2}{10} = \frac{x+8}{6} - 10$. (O.)

50. $\frac{5}{x-5} - \frac{3}{x-3} = \frac{2}{x+2}$. (O. and C.)

51. $\frac{1}{x-2} + \frac{1}{3x+2} = 1$. (Give the result correct to two decimal places.) (C.W.B.)

52. $\frac{x-5}{2} - \frac{3-2x}{5} = 6\frac{4}{5}$. (O.)

53. $\dfrac{7(x-3)}{4} - \dfrac{5(x-7)}{2} + 1 = 0.$ (O.)

54. $3 - \dfrac{x-7}{5} \doteq 5 - \dfrac{8x-4(x-3)}{3}.$ (C.W.B.)

55. $\dfrac{3}{x-2} - \dfrac{2}{x+3} = \dfrac{3}{4}.$ (L.)

56. $\dfrac{1}{x-1} - \dfrac{2}{x+5} = \dfrac{1}{4}.$ (L.)

57. $\dfrac{2x-3}{x-3} = 1 - \dfrac{2x}{x-4}.$ (D.)

58. $\dfrac{a+x}{a} - \dfrac{b-x}{b} - 2 = 0.$ (B.)

59. $2(x-1)^2 + 2(x-1) - 1 = 0.$ (Give the roots correct to two significant figures.) (B.)

60. $\dfrac{2x-1}{3x-5} = \frac{1}{7}(x+3).$ (D.)

61. $\dfrac{3x+5}{x+2} = \dfrac{5}{(x+2)(2x-3)} + \dfrac{3}{2(2x-3)}.$ (D.)

62. $\dfrac{2}{3(x+2)} - \dfrac{3}{2x+7} = \dfrac{1}{15}.$ (C.)

63. If $x^2 + bx + c \equiv (x-\alpha)(x-\beta)$, find b and c in terms of α and β. Hence obtain in terms of b and c the value of $\dfrac{1}{\alpha^2} + \dfrac{1}{\beta^2}$ and the equation whose roots are $\dfrac{1}{\alpha^2}$ and $\dfrac{1}{\beta^2}$. (N.)

64. (*a*) The roots of the equation $5x^2 - 20x + 12 = 0$ are α and β. Factorize the expression $\alpha^3 + \beta^3$, and hence find its numerical value without solving the equation.

(*b*) The roots of $x^2 + px + q = 0$ are double the roots of $x^2 - (b+c)x + bc = 0$. Express p and q in terms of b and c.

(*c*) Find the value of p and the roots of the equation

$2x^2 - 33x + p = 0$, given that one root is ten times the other. (N.)

65. (*a*) If a, p are the roots of $x^2 - 100x + 2491 = 0$, and a, q are the roots of $x^2 + 50x - 4559 = 0$, find *without solving* these equations the values of $p - q$ and $\frac{p}{q}$.

(*b*) Find in its simplest form the equation whose roots are $\frac{3 \pm \sqrt{5}}{2}$. (N.)

66. (*a*) Find, without solving the second equation, the values of p and q so that the roots of the equation $9x^2 + px + q = 0$ may be the squares of the roots of the equation $3x^2 - 4x - 2 = 0$.

(*b*) If α and β are the roots of the equation $7x^2 - 8x + 2 = 0$, find, without solving this equation, the quadratic equation whose roots are $2\alpha + 1$ and $2\beta + 1$. (N.)

67. (*a*) If one of the roots of the equation $x^2 + px + 32 = 0$ is -8, while the equation $x^2 + px + q = 0$ has equal roots, find the value of q.

(*b*) If α and β are the roots of the equation $5x^2 - 7x - 19 = 0$, find, without solving this equation, the quadratic equation whose roots are $\alpha + 2$ and $\beta + 2$. (N.)

EQUATIONS (SIMULTANEOUS)

68. $\frac{x}{3} + \frac{y - x}{5} + 1\frac{1}{2} = 0.$

$$7x + 3y = 0.$$ (O. and C.)

69. $\frac{x + y}{3} = \frac{x - y}{2} + 9.$

$$\frac{x}{2} = 5 - \frac{x + y}{9}.$$ (C.W.B.)

70. $2x + 3y = 1.$
$3x^2 - xy + 2y^2 = 16.$ (C.W.B.)

71. $\dfrac{x^2 - 3y^2}{x^2 + 3y^2} = \dfrac{1}{2} = \dfrac{2x - y}{10}.$ (O.)

72. $0{\cdot}5x + 1{\cdot}2y = 1{\cdot}4.$
$0{\cdot}6x - 7y = 5{\cdot}9.$ (O. and C.)

73. $\dfrac{x + 7}{3} = \dfrac{y + 1}{2}.$
$\dfrac{2y - x}{3x - 2y} = \dfrac{5}{3}.$ (O.)

74. $\dfrac{x^2}{9} + \dfrac{y^2}{4} = 8.$
$x + 2y = 14.$ (O.)

75. $\dfrac{x - y}{1} = 2(2x - y) = y - 13.$ (O.)

76. $(x + 5)^2 + (y - 6)^2 = 2(xy - 24).$
$y - x = 1.$ (O.)

77. $9x^2 - 2xy - 4y^2 = 3.$
$3x - 2y = 1.$ (L.)

78. $2x - y = 3.$
$(x + y)^2 = xy + 1.$ (L.)

79. $0{\cdot}5x + 0{\cdot}2y = 0{\cdot}55.$
$0{\cdot}7x - 0{\cdot}5y = 0{\cdot}185.$ (D.)

80. $2x + y = 5.$
$4x^2 + y^2 = 17.$ (D.)

81. $3x = 4y.$
$10x - 7y = 1{\cdot}9.$ (D.)

82. $5x - 2y = 4.$
$x(x + y) = 10.$ (D.)

83. $2x + 3y + 4xy = 0.$
$4x + 5y = 3.$ (B.)

84. $\frac{1}{7}(x+y)-\frac{1}{4}(x-y)=1.$
$\frac{1}{4}(x+y)-\frac{1}{3}(x-y)=3.$ (D.)

85. $\dfrac{2x+3y+1}{3}-\dfrac{x-y+2}{5}+\dfrac{5}{7}=0.$

$\dfrac{7(x+y)}{5}-\dfrac{2x+y+1}{2}+1=0.$ (C.)

86. $\dfrac{7x+9}{6}-(y-2)=4.$

$\dfrac{7y+10}{6}-(x+3)=-3.$ (C.)

87. $x^2-5xy+25y^2=21.$
$2x-5y=3.$ (C.)

88. $2x+y=3a+b.$
$x+2y=3a-b.$ (N.)

89. $x+y=6.$
$3y-2x=3.$ (N.)

INDICES AND LOGARITHMS

90. Evaluate by logarithms

(*a*) $\dfrac{0{\cdot}654\times\sqrt{4729}}{6{\cdot}892}.$

(*b*) $\dfrac{(9{\cdot}576)^2+(8{\cdot}642)^2}{\sqrt{0{\cdot}884}}.$ (C.W.B.)

91. (*a*) If $-2=\log x$, $y=\log\dfrac{1}{\sqrt[3]{1000}}$, and $3\log 2+\log z=0$, find x, y, and z without using tables.

(*b*) Use tables to find the value of $\dfrac{(0{\cdot}0125)^{0{\cdot}35}}{\sqrt{15{\cdot}69}}.$ (B).

92. Reduce to its simplest form $(2^{-1}x^{\frac{1}{2}}y^{-\frac{3}{2}})^2 \div \left(\frac{xy}{2}\right)^{-2}$ (B.)

93. If $x = a^2y + b^2y^{-1}$, use logarithms to find the value of x when $a = 32{\cdot}4$, $b = 2{\cdot}31$, and $y = 0{\cdot}056$. (O.)

94. If $r^3 = (5{\cdot}43)^3 + (7{\cdot}56)^3$, use logarithms to find r and the value of $\frac{4}{3} \times 3{\cdot}1416 \times r^3$. (O.)

95. (*a*) By using tables of logarithms calculate $(1{\cdot}44)^2$ and $\sqrt{0{\cdot}144}$.

(*b*) The area S sq. cm. of the curved surface of a cone is given by $S = \pi r\sqrt{r^2 + h^2}$, where h cm. is the height of the cone and r cm. is its base radius. Making use of tables of logarithms, calculate the curved surface area of a cone of height 17·6 cm. and base radius 13·4 cm., assuming π to be 3·142.

(*c*) If $S = \pi r\sqrt{h^2 + r^2}$, express h in terms of S, r, and π. (C.W.B.)

96. Use logarithms to find the approximate values of

(*a*) $\dfrac{1{\cdot}081}{2{\cdot}592 \times 3{\cdot}673}$.

(*b*) $\dfrac{12\sqrt{2}}{0{\cdot}017}$.

If four numbers are in geometrical progression, prove that their logarithms are in arithmetical progression. (O.)

97. Without using tables of logarithms, find the value of $(729)^{-\frac{2}{3}}$. Use your tables to find (*a*) the number of digits in 19^{20}, (*b*) the value of $\sqrt[3]{(17{\cdot}02)} \times (0{\cdot}853)^{10}$. (L.)

98. Find by means of logarithm tables

(*a*) The square root of $(0{\cdot}7582)^2 - (0{\cdot}6928)^2$.

(*b*) The value of x when $253\left[\left(1+\frac{x}{100}\right)^{20}-1\right]=357{\cdot}2$.

(D.)

99. (*a*) The volume V of a cylinder of radius r and height h is given by the formula $V=\pi r^2 h$. Taking $\pi = 3{\cdot}142$, find to three significant figures the radius of a cylinder whose volume is 1000 c.c. and whose height is equal to its diameter.

(*b*) If $x = ab^{1\cdot25}c^{-0\cdot5}$ express x in terms of a, b, and c, using only positive whole-number indices and root signs. Express also c in terms of x, a, and b. (D.)

100. Find without using tables the values of (*a*) $27^{-\frac{2}{3}} \times 4$, (*b*) log 37·5 – log 0·21 + log 5·6.

Using tables, calculate the value of $(23{\cdot}46)^2 \times \sqrt[3]{0{\cdot}0257} \div 6{\cdot}821$. (B.)

101. Calculate the values of

(*a*) $(0{\cdot}093)^{\frac{1}{3}}$.

(*b*) $\sqrt[5]{\dfrac{13{\cdot}9 \times (0{\cdot}34)^3}{(8{\cdot}145)^2}}$. (D.)

102. (*a*) Evaluate correct to five decimal places $\sqrt{0{\cdot}2035} \div 100$.

(*b*) Find the positive whole number n such that 10,000,000 lies between 3^n and 3^{n+1}. (C.)

103. (*a*) Given that log 2 = 0·3010 and log 3 = 0·4771, find without use of the tables log 18 and log 0·012.

(*b*) Find by the use of the tables the logarithm of $\sqrt{28{\cdot}5}$ and the number whose logarithm is $\bar{1}{\cdot}7441$. (N.)

104. (*a*) If $W = C^2R + \frac{t^2}{R}$, find by logarithms the value of W when R = 0·044, C = 41·1, and t = 1·8.

(*b*) Show, without the use of tables, that

$$2(\log \sqrt{125} + \log 27 - \log \sqrt{1000}) = 3(\log 9 - \log 2).$$

(N.)

105. (*a*) Solve the equation

$$(3{\cdot}981)^{2x\cdot 5} = (7{\cdot}943)^{x}.$$

(*b*) Find by logarithms the value of

$$\sqrt[3]{\left(\frac{59{\cdot}26 \times (1{\cdot}414)^2}{0{\cdot}022 \times 365}\right)}.$$ (N.)

SERIES

106. Find the sum of the series $1 + 2 + 3 + \quad + r$. A man's house is the xth house in a row of n houses, numbered consecutively 1, 2, 3, $\quad n$. The sum of the numbers of the houses preceding his house is equal to the sum of the numbers following it. Prove that $n^2 + n - 2x^2 = 0$, and find n when x has the value 35. (L.)

107. (*a*) Find the 22nd term and the sum of the first twenty-two terms of the progression $2\frac{2}{3} + 3\frac{1}{12} + 3\frac{1}{2} +$

(*b*) The 2nd and 3rd terms of a G.P. are 24 and $12(b + 1)$ respectively. What is the first term?

If these three terms have a sum of 76, what are the possible numerical values of the third term? (C.W.B.)

108. (*a*) The 4th and 7th terms of an arithmetical progression are 11 and 20; find the sum of the first fifteen terms.

(*b*) The 1st and 2nd terms of a geometrical progression are a and b; prove that the sum of the first eight terms is

$$\frac{(a + b)(a^2 + b^2)(a^4 + b^4)}{a^6}.$$ (B.)

109. The sum of the 5th and 13th terms of an arithmetical progression is 58, and the 21st term is $2\frac{1}{2}$ times the 8th term. Find the sum of the first fifteen terms. (O.)

110. (*a*) The 1st term of an arithmetical progression is $(1 - x)^2$ and the 2nd term is $1 + x^2$. Find the possible values of x if the sum of ten terms is 310.

(*b*) The 2nd term of a geometrical progression is -6 and the 5th term is $20\frac{1}{4}$. Find the 4th term and the sum of five terms. What is the nth term? (C.W.B.)

111. Use logarithms to find the value of the sum of twelve terms of the geometrical progression $27 + 18 + 12 + \ldots$ (O.)

112. Prove that the sum of n terms of the series $a + ar + ar^2 + \ldots$ is $\frac{a(1 - r^n)}{1 - r}$.

If £1 is invested at 5 per cent. per annum compound interest, prove that the sum due after n years is £$(1{\cdot}05)^n$. A man invests £10 at the beginning of each year at 5 per cent. per annum compound interest. What is the total sum due to him, to the nearest £, at the end of 10 years? (C.W.B.)

113. If the 5th and 10th terms of an arithmetical progression are 20 and 75 respectively, find the sum of twenty terms.

A party of climbers starts to ascend a mountain 10,000 feet high. They climb 2000 feet the first hour, and in each succeeding hour the height ascended is 20 per cent. less than in the preceding hour. Estimate their height at the end of 6 hours, and show that, if their rate of ascent continues to decrease in this ratio, they will never reach the summit. (D.)

114. (*a*) If a, b, and c are a series in arithmetical progression, prove that $x - a$, $x - b$, and $x - c$ are also a series in arithmetical progression. If the sum of these two series is the same, find x in terms of b only.

(*b*) Sum to ten terms the series $1\frac{1}{2} + 2\frac{1}{4} + 3\frac{1}{8} + 4\frac{1}{16} + \ldots$ (B.)

115. A man pays an insurance company £50 each year on January 1 for 40 successive years, and receives in turn at the end of the 40th year a lump sum of £4512. Find

how much the insurance company will have gained on the transaction at the end of the 40th year, reckoning compound interest at 4 per cent. per annum. ($1{\cdot}04^{40} = 4{\cdot}8010$.)

Show that the man would have taken 47 years to save £4512 if he had saved £50 the first year and increased this amount by £2 per year (reckoning no interest on the amounts saved). (D.)

116. Three concentric circles have the lengths of their radii in arithmetical progression, and the area between the second and third is $1\frac{1}{2}$ times the area between the first and second. If the radius of the first, or least, circle is $1\frac{1}{2}$ inches, find the radii of the others. (O.)

117. A trapezium is such that its two parallel sides and the distance between them are in arithmetical progression, with the last-named as the first term of the A.P. Give a formula for the area of the trapezium in terms of the lengths of the parallel sides. If a, b, and c are in arithmetical progression, $a = \frac{2}{3}b$, and $a(b + c) = 896$, determine the values of a, b, and c.

If these values of b and c are the lengths of the parallel sides of the trapezium mentioned above, what is its area? (L.)

118. (*a*) Find the sum of n terms of the series of which the rth term is $2r - 1$.

(*b*) If £P is allowed to accumulate at compound interest at the rate of 5 per cent. per annum, prove that the amount accumulated at the end of n years is £P$(1{\cdot}05)^n$.

At the beginning of every year a man invests £120 at 5 per cent. per annum compound interest. Find the total amount standing to his credit at the end of 10 years as accurately as your tables allow. (L.)

119. (*a*) Write out the first five terms of the progression whose nth term is $10 - 4n$.

(*b*) A man who has to pay £61 7*s*. 6*d*. income tax on January 1, 1933, puts a certain sum of money into the bank on deposit on the last day of each month in 1932. If the bank allows simple interest at the rate of 5 per cent. per annum on each amount for the length of time it is on deposit, find the monthly payment necessary to pay the tax. (Reckon a calendar month as $\frac{1}{12}$ of a year.) (D.)

120. A contractor knows that a certain piece of work would be completed by 15 men in 19 weeks; he employs 15 men in the first week, and at the beginning of each subsequent week he increases by x the number of men employed on this work, and it is completed in 10 weeks. Assuming that the amount of work done in any week is proportional to the number of men employed throughout that week, find the number x. (C.)

121. (*a*) The sum of five numbers in arithmetical progression is 55, and the product of the first, third, and fifth is 935. Find the numbers.

(*b*) The sum of the first four terms of a geometrical progression is 30, and the sum of the next four terms is 480; find two real values of the 1st term and the common ratio. (C.)

RATIO, PROPORTION, AND VARIATION

122. (*a*) Two numbers are in the ratio of 9 16. When each number is increased by 15 the ratio becomes 2 3. What does the ratio become if each of the original numbers is decreased by 12?

(*b*) The length of wire that can be coiled on a cylinder varies as the radius of the cylinder and inversely as the square of the diameter of the wire. On a cylinder $2\frac{1}{4}$ inches in radius 75 yards of wire 0·2 inch thick can be wound. What length of wire, to the nearest tenth of a yard, 0·3 inch thick can be coiled on a cylinder 8 inches in radius?

(C.W.B.)

123. In a party travelling by rail the number of people requiring whole tickets was to the number requiring half-tickets as 2 to 3. Find the number of people in the party if all the tickets together were equivalent to 21 whole tickets. (O. and C.)

124. (*a*) If $y = 2x^2$, find the value of $\dfrac{2y^3 - x^2y^2}{2x^4y}$.

(*b*) The height of a cylinder of given material varies directly as the weight and inversely as the square of the radius. A certain cylinder has a height of 10 cm. It is required to make another cylinder of one-quarter the weight and with a radius which is 0·3 that of the first cylinder. Find the height of the new cylinder to the nearest millimetre. (C.W.B.)

125. On a council the ratio of the number of men to the number of women was 7 3. After five men and one woman had been added the ratio was 12 5. Find the number of people originally on the council. (O.)

126. (*a*) You are given that y and x are connected by the equation $y = mx + c$. When $x = 2$ the value of y is $3\frac{1}{2}$, and when $x = -1$ the value of y is $-2\frac{1}{2}$. Find the values of m and c, and hence find the value of x when $y = -2$.

(*b*) The weight of a sphere varies as the cube of its radius and also as the specific gravity of the material of which it is made. The specific gravity of gold is 19·25 and of silver 10·5. Find the radius of a sphere of silver equal in weight to three times that of a sphere of gold of radius 2 cm. (C.W.B.)

127. (*a*) If $\dfrac{a}{b} = \dfrac{c}{d} = \dfrac{e}{f}$, prove that each ratio

$$= \sqrt{\frac{2a^2 - 3c^2 + 5e^2}{2b^2 - 3d^2 + 5f^2}}.$$

(*b*) The number of lb. of an explosive required to destroy a given length of wall varies jointly as the height and the square of the thickness of the wall. If a charge of 60 lb. destroys a wall 12 feet high and 4 feet thick, what charge would destroy a wall 8 feet high and 6 feet thick? (B.)

128. The distance of the horizon at sea varies as the square root of the height of the observer above sea-level. At a height of 20 feet one can see 5·5 miles. To what distance could one see from a height of 45 feet, and to what height must one ascend in order to be able to see 30 miles? (D.)

129. If y is the algebraic sum of two terms, one of which is constant and the other varies as the product of x and y, and if $y = 3$ when $x = 1$, and $y = \frac{1}{3}$ when $x = 5$, find the value of y when $x = 11$. (B.)

130. The annual incomes of two persons are in the ratio $\frac{a}{b}$, and their annual expenditures are in the ratio $\frac{p}{q}$; if each person saves an amount £C per annum, find their annual incomes. (C.)

131. (*a*) If $\frac{a}{b} = \frac{c}{d} = \frac{e}{f}$, prove that $\left(\frac{a-c}{b-d}\right)^2 = \frac{c^2+e^2}{d^2+f^2}$.

(*b*) The distance of the horizon at sea varies as the square root of the height of the eye above sea-level. When the distance is 9 miles the height of the eye is 54 feet. Find in miles the distance when the height of the eye is 6 feet, and find in feet the height of the eye when the distance is 4 miles. (C.)

132. (*a*) If x varies directly as s and inversely as the square of t, and if $x = 6$ when $s = 8$ and $t = 2$, find s when $t = 9$ and $x = \frac{1}{27}$.

(*b*) A clock keeps accurate time at 60° F., but gains as the temperature falls and loses as it rises, the rate of gain or loss varying as the square of the number of degrees between the actual temperature and 60° F. If it gains 2 seconds per day when the temperature is 47° F., how much does it lose in 3 days when the temperature is 75° F.? Give your answer to the nearest second. (N.)

133. (*a*) If z varies directly as x and inversely as the square of y, find the percentage increase in z due to an increase in x of 12 per cent. and a decrease in y of 20 per cent.

(*b*) The weight, W, of a body varies jointly as its height h and the square of the diameter, d, of its base. Find suitable numbers to fill in the blanks in the following table of values:

W	25		7·2
h	2·5	4	2
d	2	0·6	

(N.)

134. (*a*) If $\frac{a}{b} = \frac{c}{d} = \frac{e}{f}$, prove that each fraction is equal to $\sqrt{\left(\frac{a^2 - 3c^2 + 5e^2}{b^2 - 3d^2 + 5f^2}\right)}$.

(*b*) The cost of making an overcoat is assumed to consist of a fixed sum, together with an additional sum, which varies inversely as the number of coats made at the factory in a day. When the number made is 40 the cost of each is £3 3*s*. 0*d*., and when the number made is 100 the cost is £3. Find the cost of an overcoat when the daily production is 80. (N.)

GRAPHS

135. (*a*) Taking one inch as unit on each axis, draw the graph of $5y = 10 + 5x - 2x^2$ for values of x from -2 to $+4$.

(*b*) Find the co-ordinates of the highest point of the curve, and give the equation of the straight line which touches the curve at this point. (O. and C.)

136. Draw the graph of $2x^2 - 3x - 5$, plotting points at intervals of one unit from $x = -2$ to $x = 3$. By plotting one or two extra points, which must be shown clearly, determine the minimum value of $2x^2 - 3x - 5$ and the value of x that gives this minimum. From your graph find the values of x for which $2x^2 - 3x - 4 = 0$. With the same scales and the same axes draw the graph of $2x - 2$, and write down the values of x where the two graphs cut. These values can be found from an equation, obtained without reference to the graph. Obtain this equation, and write it in the form $ax^2 + bx + c = 0$. (C.W.B.)

137. Draw the graph of $y = (x + 1)(2x - 3)$ for values of x between -2 and $+2$, and determine from your graph (*a*) the roots of the equation $2x^2 - x - 3 = 0$, (*b*) the least value of $2x^2 - x - 3$, and (*c*), by drawing a straight line parallel to the axis of x, the roots of the equation $2x^2 - x - 5 = 0$. (B.)

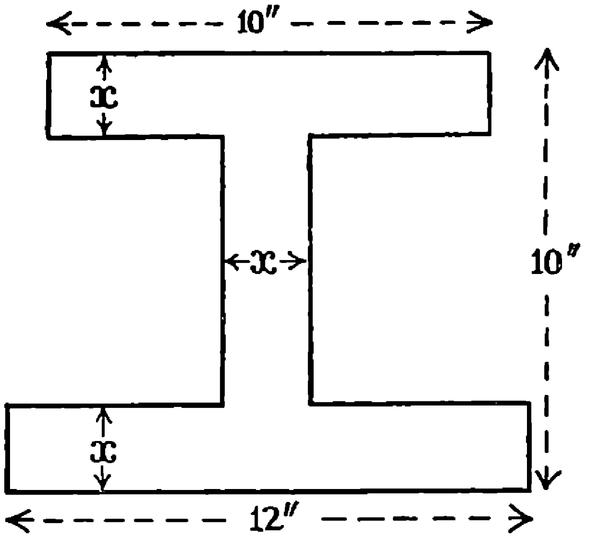
138. Plot the graph of $\dfrac{12}{x}$ for the following values of x 0·5, 1, 1·5, 2, 2·5, 3, 3·5, 4, and with the same origin and axes plot the graph of x.

By considering the intersection of the two graphs determine the square root of 12. (B.)

139. Find the values of $\dfrac{(x + 1)(x - 3)}{x + 2}$ when x has the values -1, 0, 3, 5. What would the sign of this expression

be when x lies between -1 and $+3$? Give reasons for your answer. (B.)

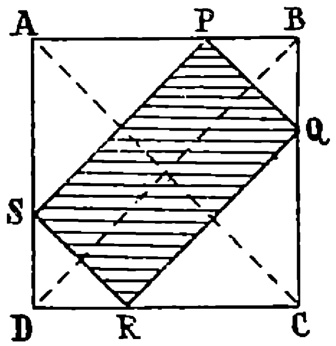
140. The diagram represents a section of a girder in which the metal is everywhere x inches thick. Prove that the area of the section is $2x(16-x)$ square inches.



Draw a graph of this from $x = \frac{1}{2}$ to $x = 3$, and find the thickness if the area is 50 square inches. (D.)

141. A rectangle PQRS is obtained from a square ABCD by cutting along lines parallel to the diagonals, as shown in the diagram. If the side of the square is 12 inches and AP is x inches, show that the area of the rectangle is $2x(12-x)$ square inches.

Plot the values of this from $x = 0$ to $x = 12$ on squared paper, and find (*a*) the greatest possible area of the rectangle, (*b*) the value of x when the rectangle is one-quarter of the square. (D.)



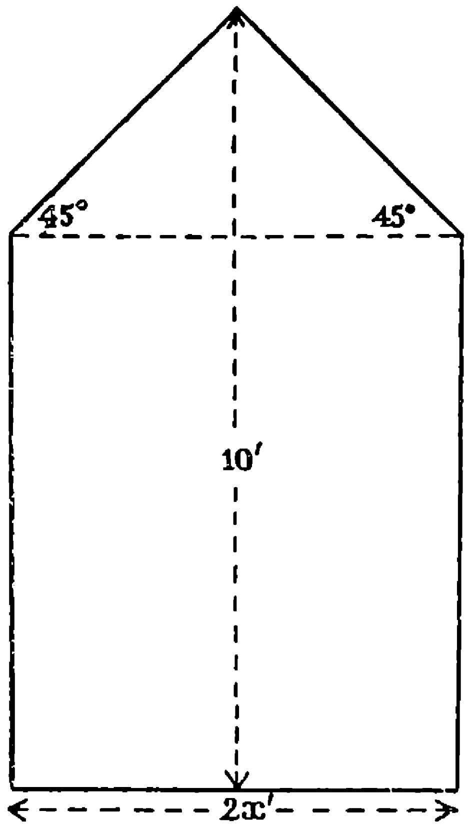
142. Write down the values of 3^x when $x = 2, 1\frac{1}{2}, 1, \frac{1}{2}, 0, -\frac{1}{2}, -1, -1\frac{1}{2}$, each to two places of decimals. ($3^{\frac{1}{2}} = 1{\cdot}73$.)

Use these results to draw the graph of 3^x. From your graph find the value of $3^{1\cdot75}$ and the two values of x which satisfy the equation $3^x = 2x + 1$. (O.)

143. Draw a graph of the function $(1-2x)(1-x)$ for values of x from -1 to $2\frac{1}{2}$.

(*a*) Make use of your graph to solve the equation $2x^2 - 3x - 3 = 0$.

(*b*) $1-2x$, $1-x$, and 1 are three numbers in arithmetical progression. What is the common difference? Find from your graph the possible values of the common difference when the product of the three numbers is 2. (C.W.B.)



144. Trace the graph of the function $x^3 - 6x^2 + 9x + 1$ between $x = 0$ and $x = 4$, taking one inch as unit. Use your graph to find the roots of the equation $x^3 - 6x^2 + 9x = 2$. (L.)

145. A gable window is in the form of a rectangle surmounted by an isosceles triangle whose sloping sides make 45° with its base, and the total height of the window is 10 feet. If $2x$ feet be the width of the window, show that its area is $20x - x^2$ square feet. Represent this function graphically from $x = 1$ to $x = 5$. Find from the graph the width of the window when the area is 40 square feet. (D.)

146. (*a*) Taking one inch as the unit on each axis, draw the graph of $10y = 3x^2 - 4x + 5$ for values of x from -2 to $+4$.

(*b*) On the same diagram draw the graph of $4x - 7y + 2 = 0$, and find from the diagram the co-ordinates of the points of intersection of the two graphs. (O. and C.)

147. Draw the graph of $6 + x - 2x^2$ from $x = -2$ to $x = 3$. Make use of your graph to determine

(*a*) The maximum value of $6 + x - 2x^2$.

(*b*) The roots of the equation $2x^2 - x - 7 = 0$.

(*c*) The range of values of x for which $6 + x - 2x^2$ is greater than 3. (C.W.B.)

148. Draw the graph of the equation $y = 4x - \frac{15}{2x}$, plotting for the values of $x, -2, -1\frac{1}{2}, -1, 1, 1\frac{1}{2}, 2$, and, using the same axes of reference, draw also that of $x + y = 2$.

Use your graph to find two values of x which satisfy $10x^2 - 4x - 15 = 0$. Verify your answers by solving this equation algebraically to two places of decimals. (O.)

149. Taking one inch as the unit for both x and y, plot the points whose co-ordinates are given by

$x =$	1	1·5	2	2·5	3	3·5
$y =$	- 0·3	0	0·2	0·1	- 0·1	- 0·4

and draw a smooth curve through them.

Using the same units and the same axes, plot the graph of $y = 7 - 4x$, and write down the approximate values of x and y for the common point of the two graphs. (C.)

150. Taking one inch as the unit for both x and y, and choosing the usual directions for positive and negative values along each axis, draw the graph of $10y = 15 + 4x - 4x^2$ from $x = -3$ to $x = +3$.

On the same diagram draw the graph of $5y = 2x + 3$, and state the co-ordinates of the points common to the two graphs. (C.)

151. Draw the graph of $y = 5 + 2x - x^2$ for values of x from -2 to $+4$, using your graph to find

(*a*) The value of y when $x = 1{\cdot}8$.

(*b*) The greatest value of y.

(c) The values of x between which the expression $5 + 2x - x^2$ is always positive. (N.)

152. Draw the graphs of $y = (x + 1)(x + 2)$ for values of x from -3 to $+1{\cdot}5$. Between what values of x within this range does the expression $(x + 1)(x + 2)$ decrease in value as x increases? Use your graph to find for what values of x the expression $x^2 + 3x + 2$ is equal to $1{\cdot}6$. Draw (and label) the line whose points of intersection with the curve would give solutions of the equation $x^2 + 3x + \frac{3}{2} \doteq 0$. (N.)

153. Draw the graph of $y = x^2 - 4x + 5$ for values of x from -1 to $+5$. Use your graph to solve the equation $2x^2 - 8x + 1 = 0$. By drawing a line through the origin find the range of values of x within which the values of x are less than the corresponding values of y. (N.)

PROBLEMS

154. I buy a certain number of books at 2*s*. 6*d*. each, keep 3, and sell the rest at 2*s*. 9*d*. each, gaining 5 per cent. on my outlay. How many books did I buy? (L.)

155. A tradesman who had bought some articles of equal value for £60, and had marked them at 25 per cent. above cost price, offered them in his sale week at 2*s*. each less than the price he had put on them. When he had got back his initial outlay he found that he had five of the articles left.

Find the cost price of each, being given that it is an exact number of shillings. (B.)

156. A motor-cyclist went in pursuit of a motor-car which had 20 minutes' start, and caught it when he had gone 80 miles. If the average speed of the cycle was 8 miles an hour greater than that of the car find the speed of each. (O.)

157. A sum of money was divided equally among 50

children. If each boy among them had received 2*s*. more and each girl 6*d*. less there would have been 10*s*. left over. How many of the children were boys? (O.)

158. The distance between two towns is 22 miles; a motor-bus which runs from one to the other travelled on a misty day at $2\frac{1}{2}$ miles per hour less than its usual average speed, and arrived 4 minutes late. What is the usual time taken for the journey? (O.)

159. When the price of petrol rose from 1*s*. 2*d*. to 1*s*. 5*d*. a gallon a motorist got a new carburettor, which enabled him to run his car 5 miles more to the gallon than before. His petrol bill for 3000 miles was 1*s*. 8*d*. more than before the rise in price. How many miles to the gallon did his car run with the old carburettor? (O.)

160. The perimeter of a rectangular plot of ground is 240 yards. Had the length been diminished by 8 yards and the width increased by 6 yards the area would have been unaltered. Find the dimensions of the plot of ground. (L.)

161. A man walking on a moor finds that at 11.25 A.M. he is $\frac{1}{2}$ mile from the nearest point, A, of a straight road, and he wishes to reach a house situated by the road at a distance of 1 mile 7 furlongs from A at noon. If he walks at 3 miles an hour on the moor and 4 miles an hour on the road, find the position of the point on the road for which he must make. (D.)

162. When the price of a certain kind of tea was reduced by 2*d*. a pound it was found that at the lower price one pound more could be bought for £3 3*s*. 4*d*. than at the higher price. Find the original price of the tea per pound. (O. and C.)

163. Eight solid spheres, each of 1 inch radius, are fitted into a hollow cube whose internal side length is 4 inches. Show that the ratio of the total volume of the spheres to the volume of the cube is a little greater than one-half.

What side length of cube is necessary in order that the cube may just contain n^3 solid spheres each of 1 inch radius when the lines joining the centres of any two spheres in contact are parallel to an edge of the cube? Show that the ratio of the total volume of the n^3 spheres to the volume of the containing cube is the same whatever value n has. (O. and C.)

164. The area of each of two rooms is 240 square feet. One is 2 feet longer than the other, but its breadth is $1\frac{2}{3}$ feet less. Find the length and breadth of each room. (C.W.B.)

165. A boy was given 14*s*. 7*d*. to buy 4 lb. of tea and $3\frac{1}{2}$ lb. of bacon, but he bought $3\frac{1}{2}$ lb. of tea and 4 lb. of bacon, and consequently brought back 5*d*. change. Find the prices of the tea and bacon per pound. (O.)

166. A total sum of £5000 is invested, part at $3\frac{1}{2}$ per cent. per annum, twice as much at 4 per cent., and the remainder at $4\frac{1}{2}$ per cent. If the total income is £213, how is the money invested? (D.)

167. The area of a rectangular grass plot is 375 square feet. If the length was diminished by 20 per cent. the distance round the plot would be diminished by 4 feet. Find the original length and breadth. (B.)

168. A motorist sets out for a place 80 miles distant, and does the first part of the journey at 30 miles per hour. Then he has a slight accident, and can only continue at 18 miles per hour. If his total running time is 3 hrs. 50 min., where does the accident occur? (D.)

169. A sum of money was divided equally among a number of people. If the sum had been 10*s*. more each would have received 15*s*.; and if the number of recipients had been 8 less each would have received 17*s*. 6*d*. What was the sum of money? (B.)

170. If x is a number of 3 digits, which, taken in order,

are in arithmetical progression, and y is the number obtained by reversing the digits, prove that $x + y$ is divisible by 222 and that $x^2 - y^2$ is divisible by 43956. (D.)

171. A man buys some £100 shares when they are below par. He sells all but 10 of them when they are as much above par as they were below par when he bought them. If he invested £4500, and the proceeds of the sale were £4400, find the number of shares he bought and the price paid per share. (N.)

172. A man sold his stock of eggs for £2. If he had had two dozen more he would have got the same money by selling the whole a penny a dozen cheaper. How many dozen eggs had he? (C.)

173. In a race of 100 yards when A and B start level A passes the winning-post $\frac{1}{5}$ second before B, but if A gives B a start of 3 yards B can win by $\frac{1}{10}$ second. Find the number of seconds each takes to run 100 yards. (N.)

174. The length of a table is twice its width. If the table were 2 feet shorter and 3 feet wider it would be square. Find its width. (N.)

175. A bag contains 30 silver coins, each of which is either a florin or a half-crown. If the total value of the coins is £3 8*s*. 6*d*., find the number of half-crowns. (L.)

TABLES OF LOGARITHMS AND ANTILOGARITHMS

LOGARITHMS

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374	4	8	12	17	21	25	29	33	37
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755	4	8	11	15	19	23	26	30	34
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106	3	7	10	14	17	21	24	28	31
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430	3	6	10	13	16	19	23	26	29
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732	3	6	9	12	15	18	21	24	27
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014	3	6	8	11	14	17	20	22	25
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279	3	5	8	11	13	16	18	21	24
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529	2	5	7	10	12	15	17	20	22
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765	2	5	7	9	12	14	16	19	21
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989	2	4	7	9	11	13	16	18	20
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201	2	4	6	8	11	13	15	17	19
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404	2	4	6	8	10	12	14	16	18
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598	2	4	6	8	10	12	14	15	17
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784	2	4	6	7	9	11	13	15	17
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962	2	4	5	7	9	11	12	14	16
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133	2	3	5	7	9	10	12	14	15
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298	2	3	5	7	8	10	11	13	15
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456	2	3	5	6	8	9	11	13	14
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609	2	3	5	6	8	9	11	12	14
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757	1	3	4	6	7	9	10	12	13
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900	1	3	4	6	7	9	10	11	13
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038	1	3	4	6	7	8	10	11	12
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172	1	3	4	5	7	8	9	11	12
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302	1	3	4	5	6	8	9	10	12
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428	1	3	4	5	6	8	9	10	11
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551	1	2	4	5	6	7	9	10	11
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670	1	2	4	5	6	7	8	10	11
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786	1	2	3	5	6	7	8	9	10
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899	1	2	3	5	6	7	8	9	10
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010	1	2	3	4	5	7	8	9	10
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117	1	2	3	4	5	6	8	9	10
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222	1	2	3	4	5	6	7	8	9
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325	1	2	3	4	5	6	7	8	9
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425	1	2	3	4	5	6	7	8	9
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522	1	2	3	4	5	6	7	8	9
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618	1	2	3	4	5	6	7	8	9
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712	1	2	3	4	5	6	7	7	8
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803	1	2	3	4	5	5	6	7	8
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893	1	2	3	4	4	5	6	7	8
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981	1	2	3	4	4	5	6	7	8
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067	1	2	3	3	4	5	6	7	8
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152	1	2	3	3	4	5	6	7	8
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235	1	2	2	3	4	5	6	7	7
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316	1	2	2	3	4	5	6	6	7
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396	1	2	2	3	4	5	6	6	7

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474	1	2	2	3	4	5	5	6	7
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551	1	2	2	3	4	5	5	6	7
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627	1	2	2	3	4	5	5	6	7
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701	1	1	2	3	4	4	5	6	7
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774	1	1	2	3	4	4	5	6	7
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846	1	1	2	3	4	4	5	6	6
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917	1	1	2	3	4	4	5	6	6
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987	1	1	2	3	3	4	5	6	6
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055	1	1	2	3	3	4	5	5	6
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122	1	1	2	3	3	4	5	5	6
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189	1	1	2	3	3	4	5	5	6
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254	1	1	2	3	3	4	5	5	6
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319	1	1	2	3	3	4	5	5	6
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382	1	1	2	3	3	4	4	5	6
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445	1	1	2	2	3	4	4	5	6
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506	1	1	2	2	3	4	4	5	6
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567	1	1	2	2	3	4	4	5	5
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627	1	1	2	2	3	4	4	5	5
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686	1	1	2	2	3	4	4	5	5
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745	1	1	2	2	3	4	4	5	5
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802	1	1	2	2	3	3	4	5	5
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859	1	1	2	2	3	3	4	5	5
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915	1	1	2	2	3	3	4	4	5
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971	1	1	2	2	3	3	4	4	5
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025	1	1	2	2	3	3	4	4	5
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079	1	1	2	2	3	3	4	4	5
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133	1	1	2	2	3	3	4	4	5
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186	1	1	2	2	3	3	4	4	5
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238	1	1	2	2	3	3	4	4	5
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289	1	1	2	2	3	3	4	4	5
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340	1	1	2	2	3	3	4	4	5
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390	1	1	2	2	3	3	4	4	5
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440	0	1	1	2	2	3	3	4	4
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489	0	1	1	2	2	3	3	4	4
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538	0	1	1	2	2	3	3	4	4
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586	0	1	1	2	2	3	3	4	4
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633	0	1	1	2	2	3	3	4	4
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680	0	1	1	2	2	3	3	4	4
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727	0	1	1	2	2	3	3	4	4
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773	0	1	1	2	2	3	3	4	4
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818	0	1	1	2	2	3	3	4	4
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863	0	1	1	2	2	3	3	4	4
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908	0	1	1	2	2	3	3	4	4
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952	0	1	1	2	2	3	3	4	4
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996	0	1	1	2	2	3	3	3	4

ANTILOGARITHMS

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
·00	1000	1002	1005	1007	1009	1012	1014	1016	1019	1021	0	0	1	1	1	1	2	2	2
·01	1023	1026	1028	1030	1033	1035	1038	1040	1042	1045	0	0	1	1	1	1	2	2	2
·02	1047	1050	1052	1054	1057	1059	1062	1064	1067	1069	0	0	1	1	1	1	2	2	2
·03	1072	1074	1076	1079	1081	1084	1086	1089	1091	1094	0	0	1	1	1	1	2	2	2
·04	1096	1099	1102	1104	1107	1109	1112	1114	1117	1119	0	1	1	1	1	2	2	2	2
·05	1122	1125	1127	1130	1132	1135	1138	1140	1143	1146	0	1	1	1	1	2	2	2	2
·06	1148	1151	1153	1156	1159	1161	1164	1167	1169	1172	0	1	1	1	1	2	2	2	2
·07	1175	1178	1180	1183	1186	1189	1191	1194	1197	1199	0	1	1	1	1	2	2	2	2
·08	1202	1205	1208	1211	1213	1216	1219	1222	1225	1227	0	1	1	1	1	2	2	2	3
·09	1230	1233	1236	1239	1242	1245	1247	1250	1253	1256	0	1	1	1	1	2	2	2	3
·10	1259	1262	1265	1268	1271	1274	1276	1279	1282	1285	0	1	1	1	1	2	2	2	3
·11	1288	1291	1294	1297	1300	1303	1306	1309	1312	1315	0	1	1	1	2	2	2	2	3
·12	1318	1321	1324	1327	1330	1334	1337	1340	1343	1346	0	1	1	1	2	2	2	2	3
·13	1349	1352	1355	1358	1361	1365	1368	1371	1374	1377	0	1	1	1	2	2	2	3	3
·14	1380	1384	1387	1390	1393	1396	1400	1403	1406	1409	0	1	1	1	2	2	2	3	3
·15	1413	1416	1419	1422	1426	1429	1432	1435	1439	1442	0	1	1	1	2	2	2	3	3
·16	1445	1449	1452	1455	1459	1462	1466	1469	1472	1476	0	1	1	1	2	2	2	3	3
·17	1479	1483	1486	1489	1493	1496	1500	1503	1507	1510	0	1	1	1	2	2	2	3	3
·18	1514	1517	1521	1524	1528	1531	1535	1538	1542	1545	0	1	1	1	2	2	2	3	3
·19	1549	1552	1556	1560	1563	1567	1570	1574	1578	1581	0	1	1	1	2	2	3	3	3
·20	1585	1589	1592	1596	1600	1603	1607	1611	1614	1618	0	1	1	1	2	2	3	3	3
·21	1622	1626	1629	1633	1637	1641	1644	1648	1652	1656	0	1	1	2	2	2	3	3	3
·22	1660	1663	1667	1671	1675	1679	1683	1687	1690	1694	0	1	1	2	2	2	3	3	3
·23	1698	1702	1706	1710	1714	1718	1722	1726	1730	1734	0	1	1	2	2	2	3	3	4
·24	1738	1742	1746	1750	1754	1758	1762	1766	1770	1774	0	1	1	2	2	2	3	3	4
·25	1778	1782	1786	1791	1795	1799	1803	1807	1811	1816	0	1	1	2	2	2	3	3	4
·26	1820	1824	1828	1832	1837	1841	1845	1849	1854	1858	0	1	1	2	2	3	3	3	4
·27	1862	1866	1871	1875	1879	1884	1888	1892	1897	1901	0	1	1	2	2	3	3	3	4
·28	1905	1910	1914	1919	1923	1928	1932	1936	1941	1945	0	1	1	2	2	3	3	4	4
·29	1950	1954	1959	1963	1968	1972	1977	1982	1986	1991	0	1	1	2	2	3	3	4	4
·30	1995	2000	2004	2009	2014	2018	2023	2028	2032	2037	0	1	1	2	2	3	3	4	4
·31	2042	2046	2051	2056	2061	2065	2070	2075	2080	2084	0	1	1	2	2	3	3	4	4
·32	2089	2094	2099	2104	2109	2113	2118	2123	2128	2133	0	1	1	2	2	3	3	4	4
·33	2138	2143	2148	2153	2158	2163	2168	2173	2178	2183	0	1	1	2	2	3	3	4	4
·34	2188	2193	2198	2203	2208	2213	2218	2223	2228	2234	1	1	2	2	3	3	4	4	5
·35	2239	2244	2249	2254	2259	2265	2270	2275	2280	2286	1	1	2	2	3	3	4	4	5
·36	2291	2296	2301	2307	2312	2317	2323	2328	2333	2339	1	1	2	2	3	3	4	4	5
·37	2344	2350	2355	2360	2366	2371	2377	2382	2388	2393	1	1	2	2	3	3	4	4	5
·38	2399	2404	2410	2415	2421	2427	2432	2438	2443	2449	1	1	2	2	3	3	4	4	5
·39	2455	2460	2466	2472	2477	2483	2489	2495	2500	2506	1	1	2	2	3	3	4	5	5
·40	2512	2518	2523	2529	2535	2541	2547	2553	2559	2564	1	1	2	2	3	4	4	5	5
·41	2570	2576	2582	2588	2594	2600	2606	2612	2618	2624	1	1	2	2	3	4	4	5	5
·42	2630	2636	2642	2649	2655	2661	2667	2673	2679	2685	1	1	2	2	3	4	4	5	6
·43	2692	2698	2704	2710	2716	2723	2729	2735	2742	2748	1	1	2	3	3	4	4	5	6
·44	2754	2761	2767	2773	2780	2786	2793	2799	2805	2812	1	1	2	3	3	4	4	5	6
·45	2818	2825	2831	2838	2844	2851	2858	2864	2871	2877	1	1	2	3	3	4	5	5	6
·46	2884	2891	2897	2904	2911	2917	2924	2931	2938	2944	1	1	2	3	3	4	5	5	6
·47	2951	2958	2965	2972	2979	2985	2992	2999	3006	3013	1	1	2	3	3	4	5	5	6
·48	3020	3027	3034	3041	3048	3055	3062	3069	3076	3083	1	1	2	3	4	4	5	6	6
·49	3090	3097	3105	3112	3119	3126	3133	3141	3148	3155	1	1	2	3	4	4	5	6	6

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
·50	3162	3170	3177	3184	3192	3199	3206	3214	3221	3228	1	1	2	3	4	4	5	6	7
·51	3236	3243	3251	3258	3266	3273	3281	3289	3296	3304	1	2	2	3	4	5	5	6	7
·52	3311	3319	3327	3334	3342	3350	3357	3365	3373	3381	1	2	2	3	4	5	5	6	7
·53	3388	3396	3404	3412	3420	3428	3436	3443	3451	3459	1	2	2	3	4	5	6	6	7
·54	3467	3475	3483	3491	3499	3508	3516	3524	3532	3540	1	2	2	3	4	5	6	6	7
·55	3548	3556	3565	3573	3581	3589	3597	3606	3614	3622	1	2	2	3	4	5	6	7	7
·56	3631	3639	3648	3656	3664	3673	3681	3690	3698	3707	1	2	3	3	4	5	6	7	8
·57	3715	2724	3733	3741	3750	3758	3767	3776	3784	3793	1	2	3	3	4	5	6	7	8
·58	3802	3811	3819	3828	3837	3846	3855	3864	3873	3882	1	2	3	4	4	5	6	7	8
·59	3890	3899	3008	3917	3926	3936	3945	3954	3963	3972	1	2	3	4	5	5	6	7	8
·60	3981	3990	3999	4009	4018	4027	4036	4046	4055	4064	1	2	3	4	5	6	6	7	8
·61	4074	4083	4093	4102	4111	4121	4130	4140	4150	4159	1	2	3	4	5	6	7	8	9
·62	4169	4178	4188	4198	4207	4217	4227	4236	4246	4256	1	2	3	4	5	6	7	8	9
·63	4266	4276	4285	4295	4305	4315	4325	4335	4345	4355	1	2	3	4	5	6	7	8	9
·64	4365	4375	4385	4395	4406	4416	4426	4436	4446	4457	1	2	3	4	5	6	7	8	9
·65	4467	4477	4487	4498	4508	4519	4529	4539	4550	4560	1	2	3	4	5	6	7	8	9
·66	4571	4581	4592	4603	4613	4624	4634	4645	4656	4667	1	2	3	4	5	6	7	9	10
·67	4677	4688	4699	4710	4721	4732	4742	4753	4764	4775	1	2	3	4	5	7	8	9	10
·68	4786	4797	4808	4819	4831	4842	4853	4864	4875	4887	1	2	3	4	6	7	8	9	10
·69	4898	4909	4920	4932	4943	4955	4966	4977	4989	5000	1	2	3	5	6	7	8	9	10
·70	5012	5023	5035	5047	5058	5070	5082	5093	5105	5117	1	2	4	5	6	7	8	9	11
·71	5129	5140	5152	5164	5176	5188	5200	5212	5224	5236	1	2	4	5	6	7	8	10	11
·72	5248	5260	5272	5284	5297	5309	5321	5333	5346	5358	1	2	4	5	6	7	9	10	11
·73	5370	5383	5395	5408	5420	5433	5445	5458	5470	5483	1	3	4	5	6	8	9	10	11
·74	5495	5508	5521	5534	5546	5559	5572	5585	5598	5610	1	3	4	5	6	8	9	10	12
·75	5623	5636	5649	5662	5675	5689	5702	5715	5728	5741	1	3	4	5	7	8	9	10	12
·76	5754	5768	5781	5794	5808	5821	5834	5848	5861	5875	1	3	4	5	7	8	9	11	12
·77	5888	5902	5916	5929	5943	5957	5970	5984	5998	6012	1	3	4	5	7	8	10	11	12
·78	6026	6039	6053	6067	6081	6095	6109	6124	6138	6152	1	3	4	6	7	8	10	11	13
·79	6166	6180	6194	6209	6223	6237	6252	6266	6281	6295	1	3	4	6	7	9	10	11	13
·80	6310	6324	6339	6353	6368	6383	6397	6412	6427	6442	1	3	4	6	7	9	10	12	13
·81	6457	6471	6486	6501	6516	6531	6546	6561	6577	6592	2	3	5	6	8	9	11	12	14
·82	6607	6622	6637	6653	6668	6683	6699	6714	6730	6745	2	3	5	6	8	9	11	12	14
·83	6761	6776	6792	6808	6823	6839	6855	6871	6887	6902	2	3	5	6	8	9	11	13	14
·84	6918	6934	6950	6966	6982	6998	7015	7031	7047	7063	2	3	5	6	8	10	11	13	15
·85	7079	7096	7112	7129	7145	7161	7178	7194	7211	7228	2	3	5	7	8	10	12	13	15
·86	7244	7261	7278	7295	7311	7328	7345	7362	7370	7396	2	3	5	7	8	10	12	13	15
·87	7413	7430	7447	7464	7482	7499	7516	7534	7551	7568	2	3	5	7	9	10	12	14	16
·88	7586	7603	7621	7638	7656	7674	7691	7709	7727	7745	2	4	5	7	9	11	12	14	16
·89	7762	7780	7798	7816	7834	7852	7870	7889	7907	7925	2	4	5	7	9	11	13	14	16
·90	7943	7962	7980	7998	8017	8035	8054	8072	8091	8110	2	4	6	7	9	11	13	15	17
·91	8128	8147	8166	8185	8204	8222	8241	8260	8279	8299	2	4	6	8	9	11	13	15	17
·92	8318	8337	8356	8375	8395	8414	8433	8453	8472	8492	2	4	6	8	10	12	14	15	17
·93	8511	8531	8551	8570	8590	8610	8630	8650	8670	8690	2	4	6	8	10	12	14	16	18
·94	8710	8730	8750	8770	8790	8810	8831	8851	8872	8892	2	4	6	8	10	12	14	16	18
·95	8913	8933	8954	8974	8995	9016	9036	9057	9078	9099	2	4	6	8	10	12	15	17	19
·96	9120	9141	9162	9183	9204	9226	9247	9268	9290	9311	2	4	6	8	11	13	15	17	19
·97	9333	9354	9376	9397	9419	9441	9462	9484	9506	9528	2	4	7	9	11	13	15	17	20
·98	9550	9572	9594	9616	9638	9661	9683	9705	9727	9750	2	4	7	9	11	13	16	18	20
·99	9772	9795	9817	9840	9863	9886	9008	9931	9954	0977	2	5	7	9	11	14	16	18	20

ANSWERS

JUNIOR: PART I

EXERCISE I

1. (*a*) 9 inches.
(*b*) 9 inches.
(*c*) 14 inches.

2. (*a*) 2 feet 6 inches.
(*b*) 2 inches.

3. (*a*) 11 yards.
(*b*) 15 yards.
(*c*) 7 yards.
(*d*) 15 yards.

4. (*a*) 10 inches.
(*b*) $H = p + q$.
(*c*) 9 inches.

5. (*a*) 20 oranges.
(*b*) $n = c + k$.
(*c*) 19.
(*d*) 16.

6. (*a*) 6 lb.
(*b*) $W = m + n$.
(*c*) 14.
(*d*) 7.

7. (*a*) 15 inches.
(*b*) $l = x + y + z$.
(*c*) 14.
(*d*) 45.

8. (*a*) 9 lb.
(*b*) $w = p + q + r$.
(*c*) 6.
(*d*) 3.

9. (*a*) 115.
(*b*) 32.

10. $H = w + x + a + d$.

EXERCISE II

1. (*a*) 7 feet.
(*b*) $10 - x$ feet.
(*c*) $s = a - p$.
(*d*) 5.

2. (*a*) 28 years.
(*b*) $d = n - k$.
(*c*) 29.
(*d*) 47.

3. (*a*) 1 pint.
(*b*) $p = x - y$.
(*c*) 4.
(*d*) 5.

4. (*a*) 15.
(*b*) $E = m - n$.
(*c*) 34.

5. (*a*) 7 tons.
(*b*) $C = t - s$.
(*c*) 7.

6. (*a*) 187.
(*b*) $K = P - M$.
(*c*) 176.

7. (*a*) 988 passengers.
(*b*) $Z = N - x$.
(*c*) 346.

8. (*a*) 193 sheep.
(*b*) $N = p - t$.
(*c*) 354.

9. (*a*) 12 shillings.
(*b*) $S = m - n$.
(*c*) 8.

10. (*a*) 38 lb.
(*b*) $X = P - Q$.
(*c*) $X = 45$.

EXERCISE III

1. (*a*) 64 oranges.
(*b*) $N = x + y - z$.
(*c*) 70.

2. (*a*) 47 lb.
(*b*) $w = m - k + c$.
(*c*) 61.

3. (*a*) 528 men.
(*b*) $H = p + x - n$.
(*c*) 419.

4. (*a*) 5 miles.
(*b*) $D = a + k - s$.
(*c*) 24.

5. (*a*) 8 shillings.
(*b*) $A = b - s + t$.
(*c*) 12.

EXERCISE IV

1. 8, 16, 24, 32, 48, 80.

2. 36, 96, 132, 156.

3. (*a*) $2x$. (*b*) $3x$. (*c*) $4x$. (*d*) $5x$. (*e*) $3m$. (*f*) $2k$. (*g*) 4T. (*h*) $3d$. (*i*) $6h$. (*j*) 5P.

4. 75.

5. 19.

6. 34.

7. 27.

EXERCISE V

1. (*a*) 12.
(*b*) 84.
(*c*) $12x$.

2. (*a*) 16.
(*b*) 32.
(*c*) $16m$.

3. (*a*) 36.
(*b*) 108.
(*c*) $36p$.

4. (*a*) $T = 3600h$.
(*b*) 10,800.

5. (*a*) $S = 21g$.
(*b*) 84.

6. (*a*) $C = 20t$.
(*b*) 240.

7. (*a*) $n = 1760d$.
(*b*) 14,080.

EXERCISE VI

1. (*a*) $11a$. (*b*) $19q$. (*c*) $23p$. (*d*) $39c$. (*e*) $52m$. (*f*) $25a$. (*g*) $44q$. (*h*) $44p$. (*i*) $55c$. (*j*) $60m$.

2. (*a*) 33. (*b*) 21. (*c*) 54. (*d*) 39. (*e*) 63.

3. 63.

4. 170.

5. 24.

EXERCISE VII

1. (*a*) $5a$. (*b*) $8q$. (*c*) $7p$. (*d*) $2c$. (*e*) $9m$. (*f*) $13x$. (*g*) $18b$. (*h*) $45e$. (*i*) $38g$. (*j*) $38t$.

2. (*a*) 20. (*b*) 35. (*c*) 30. (*d*) 75. (*e*) 90.

3. 36.

4. 35.

5. 160.

6. (*a*) $26a$. (*b*) $24m$. (*c*) $17c$. (*d*) $18x$. (*e*) $26w$.

EXERCISE VIII

1. (a) $3x + y$.
(b) $8a + 2b$.
(c) $9k + 5m$.
(d) $9c + 7w$.
(e) $9h + 7t$.
(f) $18p + 10r$.
(g) $9b + 10e$.
(h) $27m + 18w$.
(i) $18l + 17k$.
(j) $12t + 18x$.
(k) $6x + 4y + 7z$.
(l) $5a + 7b + 4c$.
(m) $8k + 13m + 7n$.
(n) $17w + 18t + 9v$.
(o) $12x + 15y + 18z$.
(p) $9a + 8b + 17c$.
(q) $13k + 21m + 5n$.
(r) $20t + 22u + 26v$.
(s) $15w + 8l + 8h$.
(t) $29a + 28b + 26c$.

2. (a) 38.
(b) 106.
(c) 214.
(d) 67.
(e) 192.

EXERCISE IX

1. (a) 20.
(b) 40.
(c) $20x$.
(d) 25.
(e) 45.
(f) $20x + 5$.
(g) $20x + y$.

2. (a) 2240T.
(b) 2240T + 448.
(c) 2240T + 112C.

3. $240x + 12y + z$.

4. $3p + q$.

5. $3600h + 60k + m$.

6. $40x + y$.

7. $60x + 2y$.

8. $9p + 2q$.

9. $36a + 24b + 3c$.

10. $180t + 5k$.

11. $120x$.

12. $64m + 15k$.

13. $36x + 5y$.

14. $7040q$.

15. $150x$.

EXERCISE X

1. (a) 10 pence.
(b) $x = kp$.
(c) 36.

2. (a) 21 lb.
(b) $w = nc$.
(c) 32.

3. (a) 60 shillings.
(b) $S = xa$.
(c) 150.

4. (a) 6 miles.
(b) $D = at$.
(c) 12.

5. 9.

6. 4.

7. 8.

8. 16.

9. (a) $R = nx$.
(b) 120.

10. (a) $w = np + mq$.
(b) 16.

EXERCISE XI

1. (*a*) $2xy$. (*b*) $2ac$. (*c*) $8kl$. (*d*) $24wt$. (*e*) $84ax$. (*f*) $168cf$. (*g*) abc. (*h*) $2xyz$. (*i*) $24wyz$. (*j*) $15pqr$. (*k*) $56ktw$. (*l*) $24\,mnpk$.

2. (*a*) 90. (*b*) 56. (*c*) 1020. (*d*) 0. (*e*) 30. (*f*) 24. (*g*) 192. (*h*) 600.

EXERCISE XII

1. (*a*) $2xy$. (*b*) $5gc$. (*c*) $4mk$. (*d*) $10ab$. (*e*) $7fn$. (*f*) $3ab + bc$. (*g*) $3pqr$. (*h*) $9kd + 4ab$. (*i*) $24abc$. (*j*) $5ab + 9ac$.

2. (*a*) 21. (*b*) 60. (*c*) 14. (*d*) 43. (*e*) 600. (*f*) 0. (*g*) 624. (*h*) 1200. (*i*) 204. (*j*) 102.

EXERCISE XIII

1. (*a*) 5 oranges. (*b*) $n = \dfrac{q}{h}$. (*c*) 5.

2. (*a*) $1\frac{1}{2}$ pence. (*b*) $c = \dfrac{x}{d}$. (*c*) $2\frac{1}{2}$.

3. (*a*) 35 miles. (*b*) $d = \dfrac{m}{t}$. (*c*) 20.

4. (*a*) 2 miles. (*b*) $M = \dfrac{k}{60}$. (*c*) 3.

5. (*a*) 28 bags. (*b*) $a = \dfrac{w}{p}$. (*c*) 28.

6. (*a*) 15. (*b*) 6. (*c*) $2\frac{1}{5}$. (*d*) 7. (*e*) 8.

EXERCISE XIV

1. (*a*) 4 hours. (*b*) $\dfrac{T}{60}$.

2. (*a*) $1\frac{11}{14}$ tons. (*b*) $\dfrac{p}{2240}$.

3. (*a*) 4 yards. (*b*) $\dfrac{k}{36}$.

4. (*a*) 1 mile. (*b*) $\dfrac{m}{5280}$.

5. (*a*) $8\frac{13}{14}$ cwts. (*b*) $\dfrac{d}{112}$.

6. (*a*) $y = \dfrac{f}{3}$. (*b*) 44.

7. (*a*) $n = \dfrac{h}{16}$. (*b*) 11.

8. (*a*) $d = \dfrac{m}{24}$. (*b*) 8.

9. (*a*) £25. (*b*) $P = \dfrac{t}{10}$.

10. (*a*) $x + \dfrac{y}{20} + \dfrac{z}{240}$. (*b*) $20x + y + \dfrac{z}{12}$. (*c*) $240x + 12y + z$.

11. (*a*) $a + \dfrac{b}{20} + \dfrac{c}{2240}$. (*b*) $20a + b + \dfrac{c}{112}$. (*c*) $2240a + 112b + c$.

12. (*a*) $p + \dfrac{q}{3} + \dfrac{r}{36}$. (*b*) $3p + q + \dfrac{r}{12}$. (*c*) $36p + 12q + r$.

EXERCISE XV (Revision Exercise)

(A)

1. 8, 12, 4, 3.

2. 17, 94, 168, $4\frac{2}{3}$.

3. (*a*) $240k$.
(*b*) $16m$.
(*c*) $1760x$.

4. $12x + 13y + 6z$.

5. (*a*) 90 miles.
(*b*) $D = pt$.
(*c*) 100.

(B)

6. (*a*) 8 lb.
(*b*) $w = Nx + My$.
(*c*) 14.

7. $35xy$, 70.

8. (*a*) $\frac{c}{8}$. (*b*) $3600k$. (*c*) $\frac{m}{12}$.

9. (*a*) $H = 12t$.
(*b*) $H = 25t$.
(*c*) $H = nt$.

10. 53.

(C)

11. (*a*) 4 oranges.
(*b*) $n = \frac{A}{x}$.
(*c*) (i) 11. (ii) 81.

12. (*a*) $\frac{p}{8}$.
(*b*) $\frac{L}{20}$.

13. (*a*) $6x$.
(*b*) $12a + 11b + 2c$.
(*c*) $60kmn$.

14. (*a*) $G = y - x$.
(*b*) $L = x - y$.

15. $10\frac{1}{2}$.

(D)

16. (*a*) 7 lb.
(*b*) $W = w + nx$.
(*c*) 8.

17. (*a*) $3cd$.
(*b*) $22ab$.
(*c*) $13xyz$.

18. (*a*) (i) $H = 5h$.
(ii) $H = 12h$.
(iii) $H = nh$.
(*b*) 42.

19. (*a*) $s = 5t$.
(*b*) $s = 12t$.
(*c*) $s = nt$.

20. (*a*) 96.
(*b*) 12.
(*c*) 4.

EXERCISE XVI

1. 4, 25, 64, 121, 169.
2. (*a*) 13. (*b*) 155. (*c*) 14. (*d*) 55. (*e*) 51.
3. (*a*) 34. (*b*) 16. (*c*) 22. (*d*) 49. (*e*) 19.
4. (*a*) 56. (*b*) 48. (*c*) 52. (*d*) 12. (*e*) 14.

EXERCISE XVII

1. 8, 243, 1296, 343, 256.
2. (*a*) 17. (*b*) 41. (*c*) 360. (*d*) 1053. (*e*) 248.
3. (*a*) 152. (*b*) 640. (*c*) 56. (*d*) 121. (*e*) 363.
4. (*a*) 26. (*b*) 288. (*c*) 116. (*d*) 10. (*e*) 152.

EXERCISE XVIII

1. (*a*) $2t^2$. (*b*) $6m^2$. (*c*) $17l^4$. (*d*) $17a^5$. (*e*) $27b^9$. (*f*) $2e^5$. (*g*) $15g^7$. (*h*) $8p^4$. (*i*) $11y^5$. (*j*) $17n^3$.
2. (*a*) 58. (*b*) 135. (*c*) 4. (*d*) 75. (*e*) 768. (*f*) 10. (*g*) 128. (*h*) 108. (*i*) 1200. (*j*) 120. (*k*) 34. (*l*) 0. (*m*) 85. (*n*) 138. (*o*) 258.

EXERCISE XIX

1. $81k^4$, $100p^2$, $125m^3$, $8p^3q^3$, $256x^4y^4$.
2. 36.
3. 18.
4. 1728.
5. (*a*) $35k^5$. (*b*) $793m^6$. (*c*) $13x^2y^2$. (*d*) $98a^3b^3$. (*e*) $3p^2q^2r^2$. (*f*) $10x^2y^2z^3$.

EXERCISE XX

1. a^7.
2. p^{12}.
3. k^{17}.
4. t^{10}.
5. x^6.
6. c^{14}.
7. h^{16}.
8. m^{21}.
9. x^{2+a}.
10. x^{m+n}.

EXERCISE XXI

1. $6a^7$.
2. $28x^{11}$.
3. $30p^6$.
4. $112g^7$.
5. $27m^{10}$.
6. $168q^9$.
7. $1000t^6$.
8. $32w^{14}$.
9. $60y^7$.
10. $24d^{10}$.

EXERCISE XXII

1. x^2y^3.
2. a^4b^4.
3. f^4g^2.
4. mn^3y.
5. $p^3q^3r^3$.
6. $8b^3c^4$.
7. $10km^3$.
8. $m^4n^2p^3$.
9. $14x^2y^3$.
10. $24a^4b^5$.
11. $126m^3n^2$.
12. $2p^3q^3r^3$.
13. $80a^3b^2c^3$.
14. $42\,k^3m^4n^2$.
15. $30a^4b^3c^4$.
16. $36e^4f^4$.
17. $120g^3w^3$.
18. $78h^3m^4n^4$.
19. a^2b^2.
20. x^6y^3.
21. $a^4b^4c^4$.
22. $8p^6q^3$.
23. k^4m^6.
24. $a^3b^6c^9$.
25. $32a^{10}$.
26. $25m^4n^6$.
27. $2a^6b^3$.
28. $12p^6$.
29. $260m^4n^4$.
30. $9f^2g^2h^2 + 5fgh$.

EXERCISE XXIII

1. (*a*) 81.
   (*b*) 20.
   (*c*) 36.
   (*d*) 125.
   (*e*) 1.
2. 4.
3. 32.
4. 5.
5. $\frac{2}{3}$.
6. $\frac{13}{36}$.

EXERCISE XXIV

1. a^3.
2. p^8.
3. m^7.
4. g^5.
5. x.
6. w^6.
7. n^4.
8. h^2.
9. y.
10. $\frac{1}{y}$.
11. $\frac{a}{b}$.
12. $\frac{a^2b^3}{c}$.
13. $\frac{y^2}{xz^3}$.
14. $\frac{c}{f}$.
15. $\frac{p}{r}$.
16. $\frac{m}{p}$.
17. $\frac{a}{d}$.
18. $\frac{g^4}{h^4}$.
19. $\frac{km^3}{e}$.
20. $\frac{q}{pr}$.

EXERCISE XXV

1. 4, 6, 5, 7, 8, 12.
2. (*a*) 7. (*b*) 15. (*c*) 6. (*d*) 2. (*e*) 9. (*f*) 1. (*g*) 25. (*h*) 1. (*i*) 21. (*j*) 46.
3. (*a*) 23. (*b*) 2. (*c*) 3. (*d*) 5. (*e*) 11.
4. (*a*) 6·827. (*b*) 1·929. (*c*) 0·197. (*d*) 0·717. (*e*) 1·929.

EXERCISE XXVI

1. x^2, a^4, p^3, s^5, T^6.
2. $2m$, $4q^3$, $7k^5$, $10x^2$, $13w^4$.
3. a^2b^2, c^5b^3, a^2bc, $p^2q^4r^3$, gh^2m.
4. $4x^2y$, $5a^3b$, $9p^2q$, $12ah^2c$, $7xyz$.

EXERCISE XXVII

1. 2.	**6.** 4.	**11.** 5.	**16.** 5.
2. 2.	**7.** 7.	**12.** 23.	**17.** 11.
3. 4.	**8.** 10.	**13.** 7.	**18.** 1.
4. 3.	**9.** 6.	**14.** 0.	**19.** 17.
5. 2.	**10.** 5.	**15.** 11.	**20.** 23.

EXERCISE XXVIII

1. a^2.	**6.** r^2.	**11.** $3xy$.	**16.** $7f^4g^4h^4$.
2. x^2.	**7.** m^3.	**12.** $4a^2b^2$.	**17.** $2ab^2c^3d^4$.
3. c^2.	**8.** xy.	**13.** $2mn$.	**18.** $2xy$.
4. a^2.	**9.** mn^2.	**14.** $8m^4p$.	**19.** $3p^2t^3$.
5. p^2.	**10.** pq^2.	**15.** $5h^2g^3$.	**20.** $7a^4m^8$.

EXERCISE XXIX (Revision Exercise)

(A)

1. (*a*) 9.
(*b*) 20.
(*c*) 75.
(*d*) 287.
(*e*) 38.

2. (*a*) $2ab^2$.
(*b*) $43x^6$.
(*c*) $27a^6 + 16b^6$.
(*d*) $2xy$.
(*e*) $4ab - 2xy$.

3. 10.

4. (*a*) f^2gh. (*b*) $\frac{a^3}{bc}$. (*c*) $\frac{x^4}{y^4}$.

5. (*a*) 36 pence.
(*b*) $D = mx + ny$.
(*c*) 56.

(B)

6. (*a*) 5.
(*b*) 10.
(*c*) 5.
(*d*) 3·146.
(*e*) 0·318.

7. $w = 40x + y$.

8. 3·2.

9. (*a*) 1072.
(*b*) $n = \frac{20P}{k}$.
(*c*) 744.

10. (*a*) $22c$.
(*b*) $6x + 11y + 12z$.
(*c*) $3ab^2c$.

(C)

11. (*a*) $x + \frac{y}{20} + \frac{z}{2240}$.

(*b*) $20x + y + \frac{z}{112}$.

(*c*) $2240x + 112y + z$.

12. (*a*) 5·382.
(*b*) 7·37[illegible].
(*c*) 7·277.

13. (*a*) $224xyz$.
(*b*) $108a^2b^2c^2$.
(*c*) $14f^2g^5h$.
(*d*) $343k^6$.
(*e*) $2c + 13m + 11mc$.

14. (*a*) 6336.
(*b*) $D = \frac{63,360x}{k}$.

15. 80.

(D)

16. (*a*) (i) 21 square inches.
(ii) 76 square cm.
(iii) $4p$ square feet.
(iv) ab square yards.
(*b*) $A = lw$.

17. (*a*) ab square inches.
(*b*) bc square inches.
(*c*) $A = ab + bc$.

18. (*a*) $A = xy - 3mn$.
(*b*) 2610.

19. (*a*) $A = p^2 - q^2$.
(*b*) 56 square feet.

20. $A = \frac{1}{2}h(a + b)$, 8 square inches.

EXERCISE XXX

1. 2×14, 4×7; 3×15, 5×9; 2×54, 4×27, 12×9, 36×3; 11×11.

2. 2^4, 2×3^2, $2^2 \times 7$, $3^2 \times 5$, $2^3 \times 3^2$.

3. $5 \times t \times t \times t$, $7 \times a \times a \times a \times a$, $2 \times x \times x \times y$, $11 \times p \times q \times q \times q \times r \times r$, $13 \times a \times b \times c \times d$.

EXERCISE XXXI

1. 3.
2. 12.
3. 22.
4. 15.
5. 16.
6. 5.
7. 14.
8. 18.
9. 23.
10. 24.

EXERCISE XXXII

1. xy.
2. a^2bc.
3. pq.
4. yz.
5. m^2n^2.
6. ab.
7. x^2yz.
8. x^3.
9. $3pq^2$.
10. $7abc$.
11. $4l$.
12. a.
13. $2x^2$.
14. p^2q.
15. $5twx$.

EXERCISE XXXIII

1. 105.
2. 180.
3. 36.
4. 126.
5. 150.
6. 900.
7. 6300.
8. 60.
9. 36.
10. 84.

EXERCISE XXXIV

1. a^2b^2.
2. $x^3y^3z^3$.
3. m^3n^3.
4. $18p^2q^2$.
5. $36w^2t^2$.
6. $72a^2b^2c^3$.
7. $24abcd$.
8. $225p^3r^5q$.
9. $105l^2m^2n^2$.
10. $36a^5x^4y$.
11. $56t^2v^3w^3$.
12. x^4y^3.
13. $60abcd$.
14. $336p^2q^2r^2$.
15. $108k^2m^2p^3$.

EXERCISE XXXV

1. $\frac{3}{5}$.
2. $\frac{3}{m}$.
3. $\frac{1}{5}$.
4. $\frac{1}{m}$.
5. $\frac{11}{13}$.
6. $\frac{11a}{13}$.
7. $\frac{3}{13}$.
8. $\frac{3a}{13}$.
9. $\frac{12x}{11}$.
10. $\frac{a+b}{13}$.
11. $\frac{12}{19}$.
12. $\frac{12}{p}$.
13. $\frac{a+b+c}{p}$.
14. $\frac{6}{15x}$.
15. $\frac{6a}{15x}$.
16. $\frac{p-q}{15x}$.
17. $\frac{p+q-r}{x}$.
18. $\frac{13}{a}$.
19. $\frac{13m}{a}$.
20. $\frac{13m}{5a}$.

EXERCISE XXXVI

1. 16.
2. 55.
3. $16x$.
4. 55.
5. $13a$.
6. $11x^3y^2$.
7. $21amb$.
8. $5b^2$.
9. $3a^3b^3$.
10. $3x^2$.
11. $14aw^2$.
12. $3abpq$.
13. $6a^2m$.
14. a.
15. m.

EXERCISE XXXVII

1. (a) $\frac{3}{5}$. (b) $\frac{3}{5}$. (c) $\frac{3m}{5}$. (d) $\frac{am}{b}$. (e) $\frac{5x}{13y}$. (f) $\frac{a}{3c}$. (g) $\frac{p^2}{5qr}$. (h) $\frac{b}{c}$. (i) $\frac{t^2}{3wv}$. (j) $\frac{2m}{3ap}$. (k) $\frac{3p}{5t}$. (l) $\frac{2xy}{7z^3}$. (m) $\frac{m^2u}{tv^2}$. (n) $\frac{4ac}{9bd}$. (o) $\frac{6z^2}{7x^2}$.

2. (a) 2. (b) $3y$. (c) $12km^2$. (d) cd. (e) $3x^2y$.

EXERCISE XXXVIII

1. $\frac{13}{15}$.
2. $\frac{7}{12}$.
3. $\frac{x+y}{xy}$.
4. $\frac{5-3b}{ab}$.
5. $\frac{5m}{6}$.
6. $\frac{2k}{35}$.
7. $\frac{7x}{30}$.
8. $\frac{49x}{12}$.
9. $\frac{ax+a}{x^2}$.
10. $\frac{7}{12x}$.
11. $\frac{16x-9}{12x^2}$.
12. $\frac{a^2+b^2}{ab}$.
13. $\frac{a^2-b^2}{ab}$.
14. $\frac{ax+ay}{2xy}$.
15. $\frac{133}{60x}$.
16. $\frac{5y-3x}{xy^2}$.
17. $\frac{5p^2+5q^2}{3pq}$.
18. $\frac{y-x}{x^2y^2}$.
19. $\frac{2-3wv^4}{w^2v^3}$.
20. $\frac{bc+ca+ab}{abc}$.
21. $\frac{12yz+9zx+8xy}{6xyz}$.
22. $\frac{2ca^2+3ab^2+4bc^2}{abc}$.

EXERCISE XXXIX

1. $\frac{8}{35}$.
2. $\frac{5}{9}$.
3. $\frac{a^2}{bc}$.
4. 1.
5. $\frac{3}{2}$.
6. $\frac{5rp^2}{q^2}$.
7. $\frac{7x}{6}$.
8. $\frac{8k}{27m}$.
9. $\frac{5w^2}{8v}$.
10. 1.
11. $\frac{2x^2}{5y}$.
12. $\frac{a^4}{3}$.
13. $\frac{10k^3}{em}$.
14. $\frac{6x^3y^3}{35}$.
15. $\frac{7p^3}{2rq^2}$.
16. xy.
17. $12p^3r$.
18. $\frac{abcxyz}{p^2q^2r^2}$.
19. $\frac{15xyz}{a^2b^2c^2}$.
20. $a^2b^2c^2$.
21. $\frac{6}{m^2n^2l^2}$.
22. $u^2v^2w^2$.
23. $\frac{abc}{24}$.
24. $\frac{14p^5q^3}{15r^3}$.
25. $\frac{33x^3y^4}{7m^2n^3}$.
26. $\frac{5}{16abc}$.

EXERCISE XL

1. $\frac{5}{6}$.
2. $\frac{44}{35}$.
3. $\frac{8a}{15}$.
4. $\frac{8a}{3b}$.
5. $\frac{9x}{10y}$.
6. $\frac{x^2}{y^2}$.
7. 1.
8. $\frac{3n}{5m}$.
9. $\frac{7c}{18a^2b}$.
10. ac.
11. $\frac{2xy^2}{15}$.
12. $\frac{a^4}{3x^4}$.
13. $\frac{3x^4}{a^4}$.
14. $\frac{35}{3mp^2}$.
15. $\frac{3ax}{4}$.
16. $\frac{p^2q^2}{r^2}$.
17. $\frac{a^4b^3c}{x^4y^3z}$.
18. $\frac{8wv}{3}$.
19. $\frac{98x^2}{y}$.
20. $\frac{27a^3b^3}{14p^3q^2}$.

EXERCISE XLI (Revision Exercise)

(A)

1. H.C.F.: (*a*) 2. (*b*) 1. (*c*) a^2b^2.
   L.C.M.: (*a*) 24. (*b*) $abcd$. (*c*) a^3b^3.

2. (*a*) $\frac{5a}{12}$. (*b*) $\frac{1}{a}$. (*c*) $\frac{zx^2 + xy^2 + yz^2}{xyz}$.

3. (*a*) $\frac{my^2}{x}$. (*b*) $\frac{3q^3}{8x^3}$. (*c*) ab.

4. (*a*) 34. (*b*) 371. (*c*) 1082.

5. (*a*) $\frac{2a^3}{b}$. (*b*) $\frac{7x^4}{3}$. (*c*) $4a^4$.

(B)

6. (*a*) 5·146. (*b*) 9. (*c*) 3·853.

7. (*a*) (i) $x + \frac{y}{20} + \frac{z}{2240}$.
   (ii) $20x + y + \frac{z}{112}$.
   (iii) $2240x + 112y + z$.
   (*b*) (i) $x + \frac{y}{24} + \frac{z}{1440}$.
   (ii) $24x + y + \frac{z}{60}$.
   (iii) $1440x + 60y + z$.

8. H.C.F.: (*a*) $2xy$. (*b*) $3mn$.
   L.C.M.: (*a*) $48x^3y^3$. (*b*) $18m^3n^3$.

9. (*a*) $1296k^{10}m^6$. (*b*) $36k^5m^3$. (*c*) $\frac{8a^2y + 9b^2x}{12xy}$.

10. (*a*) $\frac{2b^3}{9a}$. (*b*) $\frac{2p^2r}{3q}$. (*c*) $\frac{1}{25}$.

(C)

11. $5ac$.

12. (*a*) $\dfrac{yz + zx + xy}{xyz}$.

(*b*) $\dfrac{1}{xyz}$.

(*c*) $\dfrac{9x^2y}{14z^3}$.

13. (*a*) £743 16*s*. 0*d*.

(*b*) $I = \dfrac{13Nk}{5} - x$.

(*c*) 1626.

14. (*a*) 6·15.
(*b*) 3.
(*c*) 4.

15. (*a*) $15p + 15q + 12r$.
(*b*) $4ab^3c^2$.

(D)

16. 153.

17. (*a*) $\dfrac{14x^2y + 15xy^2}{35}$.

(*b*) $\dfrac{14x}{15y}$.

18. 24 square inches.

19. 101.

20. $28a$.

EXERCISE XLII

1. 10.
2. 24.
3. 20.
4. $a + 5$.
5. $a + 5$.
6. $2p + q$.
7. $2p + q$.
8. $6x$.
9. $6x$.
10. 12.
11. 17.
12. 36.
13. $4a + 4b$.
14. $29x$.
15. $6a + 11b$.
16. $2a + 3b + 4c$.
17. $9x + 18y + 36z$.
18. $4m + 3n$.
19. $3w + 8v$.
20. $3w + 8v$.

EXERCISE XLIII

1. (*a*) 21.
(*b*) 70.
(*c*) 180.
(*d*) 30.
(*e*) 162.
(*f*) 132.
(*g*) $ap + aq$.
(*h*) $ap + aq + ar$.
(*i*) $5x + x^2$.
(*j*) $2m + 2mp + 2mq$.
(*k*) $m + m^2 + m^3$.
(*l*) $2ak + 2bk^2 + 2ck^3$.

2. (*a*) $7x + 20$.
(*b*) $17a + 22b$.
(*c*) $18x + 43y$.
(*d*) $17a + 33$.
(*e*) $28x + 35$.
(*f*) $5a + 8b + 11c$.
(*g*) $4ab + 2a^2 + 2b^2$.
(*h*) $9ab + 10a + 8b + 5c + 5bc$.

EXERCISE XLIV

1. $a(x + y)$.
2. $2(p + q)$.
3. $2(a + 2b)$.
4. $k(m + n)$.
5. $k(k + 1)$.
6. $kp(3k + 1)$.
7. $3kp(k + 2)$.
8. $a(x + y + z)$.
9. $a^2(x + y + z)$.
10. $2(p + 2q + 3r)$.
11. $3pq(q + 2p)$.
12. $m(m^2 + mn + n^2)$.
13. $k^2(k^3 + 1)$.
14. $2a^2(x + 2y + 3z)$.
15. $5w^2v^2(w + 5v)$.

EXERCISE XLV

1. (a) 20.
   (b) 10.
   (c) 4.
   (d) $4 + a$.
   (e) $16 - a$.
   (f) $2p - q$.
   (g) $2p - q$.
   (h) $2x$.
   (i) $9x$.
   (j) $3x + 2$.
   (k) 3.
   (l) 45.
   (m) $ab - ac$.
   (n) $2px - 2x^2 + 2xy$.
   (o) $7am + 7m^2 - 7mn$.

2. (a) $5x + 4$.
   (b) $26a + 5b$.
   (c) $8x + 13y$.
   (d) $6ap - 4aq$.
   (e) $11m^2 + 2mn$.

3. (a) $a(x - y)$.
   (b) $2(p - q)$.
   (c) $2(a - 2b)$.
   (d) $k(m - n)$.
   (e) $k(k - 1)$.
   (f) $kp(3k - 1)$.
   (g) $3kp(k - 2)$.
   (h) $3pq(q - 2p)$.
   (i) $k^2(k^3 - 1)$.
   (j) $5w^2v^2(w - 5v)$.
   (k) $x(y - z + w)$.
   (l) $2(x + 2y - 3z)$.
   (m) $5(2m + 3l - n)$.
   (n) $x(3x - 2 + 3y)$.
   (o) $7a(3a - 4b)$.
   (p) $5m(p + 2m + 1)$.
   (q) $d(3c - 2f - 5d)$.
   (r) $a(7b - 8a + 3b^2)$.
   (s) $p(1 - pq - p^2q^2)$.
   (t) $-n(2n + k - 4nk)$.

EXERCISE XLVI

1. (a) 2.
   (b) 2.
   (c) 1.
   (d) 4.
   (e) 3.
   (f) 9.
   (g) 0.
   (h) 2.

2. (a) $a - b - c$.
   (b) $a - 2b - 2c$.
   (c) $2a + b - c$.
   (d) $p - qx - qy - qz$.
   (e) $ac - ad$.

3. (a) $6 - a$.
   (b) $4 - 6x$.
   (c) $2x + 1$.
   (d) $7a + 2b$.
   (e) $5y$.
   (f) $a + 13$.
   (g) $2x + 4$.

EXERCISE XLVI—*continued*

4. (*a*) $b + c$.
(*b*) $b + c$.
(*c*) $y + 2z$.
(*d*) $y + 2z$.
(*e*) $a + c$, $x + z$.
(*f*) $p - r - t$.
(*g*) $x - y$.
(*h*) $1 + v$.
(*i*) $y + z$, $q + r$.
(*j*) $1 + a + a^2 + a^3$.

EXERCISE XLVII

1. (*a*) 6.
(*b*) 0.
(*c*) 4.
(*d*) 5.
(*e*) 4.

2. (*a*) $a - b + c$.
(*b*) $a - b - c$.
(*c*) $a - 2b + 2c$.
(*d*) $a + 2b$.
(*e*) $p - qx - qy - qz$.
(*f*) $ac + bc$.

3. (*a*) $14 - a$.
(*b*) $25 - 6x$.
(*c*) $4x + 13$.
(*d*) $3a + 32b$.
(*e*) $3x + 26y + 3z$.

4. (*a*) $b + c$.
(*b*) $b - c$.
(*c*) $y - 2z$.
(*d*) $y + 2z$.
(*e*) $a - c$, $y - z$.
(*f*) $q + r$, $a - c$.
(*g*) $1 - b$.
(*h*) $x - y$, $1 - y$.
(*i*) $1 - a + a^2 - a^3$.

EXERCISE XLVIII

1. (*a*) 10·2.
(*b*) 9800.
(*c*) 2300.
(*d*) 1120.
(*e*) 1300.

2. (*a*) $p^2q(1 + r)$.
(*b*) $r^2(ar - b + abr^3)$.
(*c*) $x(x^2 - y)$.
(*d*) $mn(np + mp^2 - mn)$.
(*e*) $6a^2(a - 3b)$.

3. (*a*) $A = p^2(\frac{2}{3} - q)$, $A = 0$.
(*b*) $P = 4(3x^2 - 2x + 4)$,
$P = 548$.
(*c*) $K = 4ab(a - b)$,
$K = 8400$.
(*d*) $V = r^2h(5r - 7)$,
$V = 6860$.
(*e*) $S = \frac{n(a + l)}{2}$, $S = 1275$.

EXERCISE XLIX

1. (*a*) 35 square inches.
(*b*) 168 square cm.
(*c*) $17x$ square inches.
(*d*) pq square cm.
(*e*) a^2 square feet.

2. (*a*) $21xy$ square feet.
(*b*) $64m^2$ square yards.
(*c*) 6 inches.
(*d*) 12 feet.
(*e*) $7c$ yards.
(*f*) $13p$ feet.

3. (*a*) $A = b(a + b)$.
(*b*) 950.

4. (*a*) $A = 17a^2$.
(*b*) 68 square inches.

5. (*a*) $A = a(6a + b)$.
(*b*) 190 square inches.

6. (*a*) $A = 2c(a + b)$.
(*b*) (i) 560 square feet.
(ii) 840 square feet.

EXERCISE L

1. 49.
2. 355.
3. 1057.
4. 480.
5. 1952.
6. 1999.
7. 998,000.
8. 144.
9. 392.
10. 1232.

EXERCISE LI

1. $(a + 4)(a - 4)$.
2. $(p + 7)(p - 7)$.
3. $(2p + q)(2p - q)$.
4. $(2p + 5q)(2p - 5q)$.
5. $(7x + 8y)(7x - 8y)$.
6. $(c + 4m)(c - 4m)$.
7. $(12 + 4m)(12 - 4m)$.
8. $(6k + 9c)(6k - 9c)$.
9. $(10w + 11v)(10w - 11v)$.
10. $(3g + 1)(3g - 1)$.
11. $(9h + 8f)(9h - 8f)$.
12. $(13m + 12n)(13m - 12n)$.
13. $(ab + 1)(ab - 1)$.
14. $(ab + xy)(ab - xy)$.
15. $(ab + x^2y^2)(ab - x^2y^2)$.
16. $(5pq + 4mn)(5pq - 4mn)$.
17. $(6pq^2 + 7m^2n)(6pq^2 - 7m^2n)$.
18. $(1 + 8x^2)(1 - 8x^2)$.
19. $(5a^2 + 7b^2)(5a^2 - 7b^2)$.
20. $(9p^2 + 12q^3)(9p^2 - 12q^3)$.

EXERCISE LII

1. $a(x + y)(x - y)$.
2. $2(2x + y)(2x - y)$.
3. $3(2p + q)(2p - q)$.
4. $5(2a + 3b)(2a - 3b)$.
5. $7(3pq + k)(3pq - k)$.
6. $x^2(1 + y)(1 - y)$.
7. $x(x + y)(x - y)$.
8. $x^2(2y + 3)(2y - 3)$.
9. $a(4b + c)(4b - c)$.
10. $a(7b + a)(7b - a)$.
11. $a^2b(ab + 1)(ab - 1)$.
12. $p^2(4q + p)(4q - p)$.
13. $a(3 + x^2)(3 - x^2)$.
14. $ab(4 + 5b)(4 - 5b)$.
15. $xy(xy + 1)(xy - 1)$.
16. $d(c + f^2)(c - f^2)$.
17. $2k(a^2 + 3g^2)(a^2 - 3g^2)$.
18. $3p^2q(p + 2q^2)(p - 2q^2)$.
19. $3a^2b(2a + 3b^2)(2a - 3b^2)$.
20. $x^2y^2(8y + 7x)(8y - 7x)$.

EXERCISE LIII

1. (*a*) $\frac{1}{2}(x - y)$ feet.
   (*b*) $A = x^2 - y^2$.
   (*c*) 4800 square feet.
   (*d*) 2944 square feet.
2. (*a*) $(p - 2q)$ inches square; volume $= q(p - 2q)^2$ cu. inches.
   (*b*) $A = (p - 2q)^2$.
   (*c*) 121.
   (*d*) $S = 4q(p - 2q)$.
   (*e*) 220.
3. (*a*) $T = k^2 - 2m^2$.
   (*b*) 178 square inches.
4. (*a*) $B = x^2 - 9y^2$.
   (*b*) 216 square inches.
5. (*a*) $S = x^2 - 2xy$.
   (*b*) 20 square inches.

EXERCISE LIV (Revision Exercise)

(A)

1. (a) $5a + 6b$.
(b) $x - 3y$.
(c) $5p + 14q$.

2. (a) 83.
(b) 5.
(c) 17.

3. (a) $2x(7a + 3y)$.
(b) $(8 + 3m)(8 - 3m)$.
(c) $a(b + c)(b - c)$.

4. 143·8.

5. $15a$.

(B)

6. (a) $8 + 14a + 20b$.
(b) $2x + 3y$.

7. (a) $7p(2p + q)$.
(b) $2(2x + 1)(2x - 1)$.
(c) $5pq(p + 2q + 3r)$.

8. (a) $q - 2$.
(b) $d + f$, $x + z$.
(c) $3x + 1$, $3x - 1$.

9. (a) H.C.F. is $5mn^2$, L.C.M. is $75m^3n^3$.
(b) H.C.F. is x, L.C.M. is $12x^4$.

10. (a) $192a^8b^3$.
(b) $3a^2b$.
(c) $\frac{14ay + 15bx}{35xy}$.

(C)

11. (a) $A = db + 2(dc + ab + ac)$.
(b) $T = \frac{1}{2}ac$.
(c) $X = A + 4T$.

12. (a) $12a$, $2a$, $35a^2$, $1\frac{2}{5}$.
(b) $\frac{p^2}{2q^2}$, $\frac{3x^4}{5y^2}$.

13. (a) 16.
(b) 29.
(c) 21.
(d) 5·146.

14. (a) $11pq + 1$, $11pq - 1$.
(b) $m - p + 1$.
(c) $a + 3b$, $c + 3d$.

15. $A = a^2 - 2h(\quad - 2b)$.

(D)

16. (a) $A = \pi(R^2 - r^2)$.
(b) 40π square inches.
(c) $A = 3\pi r^2$.

17. 253.

18. (a) $E = 22x^2$.
(b) 88 square inches.

19. (a) $H = \frac{7m}{60}$.
(b) 8.46 A.M.

20. (a) $G = \frac{p}{6}$.
(b) $\frac{1}{2}$.

EXERCISE LV

1. 20, 21, 22.

2. 17, 18, 19, 20, 21.

3. 100, 101, 102, 103.

4. (a) Yes.
(b) No.
(c) No.

5. 15, 16, 17, 18, 19.

6. 8.

7. 63.

8. n, $n + 1$, $n + 2$, $n + 3$.

9. $2x$, $2x + 1$, $2x + 2$, $2x + 3$, $2x + 4$, $2x + 5$.

10. $4n + 6$.

EXERCISE LVI

1. 10, 9, 8, 7.

2. 15, 14, 13, 12, 11, 10.

3. (*a*) Yes.
(*b*) No.

4. n, $n-1$, $n-2$, $n-3$, $n-4$.

5. $2x$, $2x-1$, $2x-2$, $2x-3$, $2x-4$, $2x-5$, $2x-6$.

6. $5n-10$.

7. $14x-21$.

EXERCISE LVII

1. (*a*) 9.
(*b*) 55, 54, 52.
(*c*) n.
(*d*) $n+2$.
(*e*) $n+1$.
(*f*) $n+1$, n, $n-1$.
(*g*) $n+3$, $n+2$.

2. (*a*) 12, 11, 10, 9, 8.
(*b*) 12, 13, 14, 15, 16.
(*c*) 10, 11, 12, 13, 14.

3. (*a*) n, $n-1$, $n-2$, $n-3$, $n-4$, $n-5$, $n-6$.
(*b*) n, $n+1$, $n+2$, $n+3$, $n+4$, $n+5$, $n+6$.
(*c*) $n-3$, $n-2$, $n-1$, n, $n+1$, $n+2$, $n+3$.

4. (*a*) $5x$, $5x-1$, $5x-2$, $5x-3$, $5x-4$.
(*b*) $5x$, $5x+1$, $5x+2$, $5x+3$, $5x+4$.
(*c*) $5x-2$, $5x-1$, $5x$, $5x+1$, $5x+2$.

5. $5n$.

6. $21x$.

7. 7, 8, 9.

8. 7, 8, 9, 10, 11.

9. $S = 7n$.

10. $6x-1$, $6x$, $6x+1$.

EXERCISE LVIII

1. 8, 10, 12, 14.

2. 8, 6, 4, 2.

3. 17, 19, 21, 23, 25.

4. 13, 15, 17, 19, 21.

5. n, $n+2$, $n+4$, $n+6$.

6. n, $n-2$, $n-4$, $n-6$.

7. $2n$, $2n+2$, $2n+4$, $2n+6$.

EXERCISE LIX (Revision Exercise)

(A)

1. (*a*) 152.
(*b*) 931.

2. (*a*) $18x^7y^3$.
(*b*) $\frac{3a^2b}{c}$.

3. (*a*) 343.
(*b*) 2916.

4. (*a*) $3a(2x+3)(2x-3)$.
(*b*) $4pq(p-2q+3)$.

5. (*a*) $2p+1$, $2p$, $2p-1$, $2p-2$, $2p-3$, $2p-4$, $2p-5$.
(*b*) $2p+1$, $2p+2$, $2p+3$, $2p+4$, $2p+5$, $2p+6$, $2p+7$.
(*c*) $2p-2$, $2p-1$, $2p$, $2p+1$, $2p+2$, $2p+3$, $2p+4$.

(B)

6. (a) $\dfrac{2 + 4x^3 - 3xy^2}{x^2y}$.

(b) $x + 14$.

7. (a) 10 days.

(b) $\dfrac{kn}{x}$ days.

8. (a) $A = a^2 + 2a(x + y)$.
(b) $38\frac{1}{4}$ square inches.

9. (a) $7xy^2 - 8x^2y$.
(b) $11a^3 + 13b^4$.

10. $10x$.

(C)

11. H.C.F. is $2ab$,
L.C.M. is $252\, a^2b^2$.

12. £$(x - a) + (y - b)$ shillings.

13. (a) $17x + 29y$.
(b) by.

14. (a) $2n - 1,\ 2n + 1,\ 2n + 3,\ 2n + 5$.
(b) $2n - 7,\ 2n - 5,\ 2n - 3,\ 2n - 1$.

15. (a) $a^2xy(x - 2y)$.
(b) $2xy(3x + 2y)(3x - 2y)$.

(D)

16. (a) $x + 8$.
(b) $x + 1$.

17. (a) $A = 2r^2(18 - \pi)$.
(b) $66\frac{6}{7}$ square inches.

18. (a) £1 2*s*. 6*d*.

(b) $C = \dfrac{xm}{n}$.

19. (a) $3a + 14b + 21c$.

(b) $\dfrac{2ac}{b}$.

20. $49x$.

EXERCISE LX

1. (a) $n = 5$.
(b) $3n = 24$.
(c) $8n = 96$.
(d) $n + 2 = 14$.
(e) $n + 23 = 57$.
(f) $n - 8 = 10$.
(g) $n - 35 = 23$.
(h) $2n + 5 = 11$.
(i) $4n + 42 = 70$.
(j) $2n - 13 = 17$.
(k) $7n - 34 = 8$.
(l) $\frac{1}{2}n = 19$.
(m) $\frac{2}{3}n = 56$.
(n) $\frac{7}{8}n = 49$.
(o) $n + (n + 2) = 26$.
(p) $n + (n + 2) = 28$.
(q) $n + (n + 1) + (n + 2) = 21$.
(r) $n^2 = 81$.
(s) $n^3 = 64$.
(t) $n^3 - n^2 = 18$.

EXERCISE LXI

1. (a) 5.
(b) 7.
(c) 18.
(d) 3.
(e) 4.
(f) 4.
(g) 1.
(h) 2.
(i) 4.
(j) 5.
(k) 2.
(l) 6.
(m) 4.
(n) 3.
(o) 2.
(p) 3.
(q) 11.
(r) 8.
(s) 24.
(t) 35.
(u) 2, 6.
(v) 3, 15.
(w) 6, 12.
(x) 24.
(y) 16.

EXERCISE LXI—*continued*

2. (*a*) $n + 4 = 15$; number is 11.
(*b*) $n - 9 = 6$; number is 15.
(*c*) $2n = 18$; number is 9.
(*d*) $3n = 42$; number is 14.
(*e*) $\frac{1}{2}n = 13$; number is 26.
(*f*) $\frac{1}{4}n = 11$; number is 44.
(*g*) $\frac{2}{3}n = 16$; number is 24.
(*h*) $\frac{5}{9}n = 10$; number is 18.
(*i*) $2n + 3 = 11$; number is 4.
(*j*) $3n - 4 = 17$; number is 7.
(*k*) $n + (n + 1) = 21$; numbers are 10 and 11.
(*l*) $n + (n + 1) + (n + 2) = 24$; numbers are 7, 8, and 9.
(*m*) $n + (n + 2) = 26$; numbers are 12 and 14.
(*n*) $28 - 5n = 13$; number is 3.
(*o*) $50 - \frac{2}{3}n = 29$; number is $31\frac{1}{2}$.

EXERCISE LXII

1. (*a*) $a = 15 - 12$.
(*b*) $17 - a = 14$.
(*c*) $17 + 2a = 19$.
(*d*) $24 - 3a = 9$.

2. (*a*) 3. (*b*) 5. (*c*) 2. (*d*) 8. (*e*) 5. (*f*) 9. (*g*) 6. (*h*) 6. (*i*) 9. (*j*) 2. (*k*) 3. (*l*) 5. (*m*) 2. (*n*) 3. (*o*) 6. (*p*) 4. (*q*) 12. (*r*) 10. (*s*) 7. (*t*) 4. (*u*) 5.

EXERCISE LXIII

1. £11 10*s*., £8 10*s*.
2. £15, £5.
3. 31 red and 16 white.
4. 39 yards.
5. 1521 square yards.
6. 200 and 50.
7. 28 at 4 a shilling and 15 at 5 a shilling.
8. A £5, B £10, C £20.
9. A £5, B £10, C £20.
10. 12 florins and 24 shillings.
11. 84 shillings and 50 sixpences.
12. 36 and 18.
13. 50.
14. 120 miles.
15. £2 5*s*.
16. 20 inches and 5 inches.
17. 10 men, 15 women, 25 children.
18. £280.
19. 36 years.
20. 35 years and 10 years.

EXERCISE LXIV

1. $x = \frac{t - q}{p}$.
2. $x = \frac{r + k}{m}$.
3. $x = \frac{p}{f}$.
4. $x = ay$.
5. $x = \frac{mab}{a + b}$.
6. $x = \frac{t}{p + q}$.
7. $x = \frac{m + n}{p^2 - q^2}$.
8. $x = \frac{a}{b}$.
9. $x = \frac{d - 5c}{5}$.
10. $x = \frac{c - d}{a + b}$.

EXERCISE LXV

1. $y = K - (x + z)$.

2. $r = \frac{D}{2}$.

3. $r = \frac{c}{2\pi}$.

4. $r = \sqrt{\frac{A}{\pi}}$.

5. $t = \sqrt{\frac{2s}{g}}$.

6. (a) $T = \frac{PV}{R}$.

(b) $P = \frac{RT}{V}$.

(c) $V = \frac{RT}{P}$.

7. (a) $u = \sqrt{v^2 - 2fs}$.

(b) $f = \frac{v^2 - u^2}{2s}$.

(c) $s = \frac{v^2 - u^2}{2f}$.

8. $R = \frac{E}{c}$.

9. $n = \frac{2S}{a + l}$.

10. $W = nP - w$.

11. $E = \sqrt{PR}$.

12. (a) $v = \sqrt{\frac{2gE}{w}}$.

(b) $w = \frac{2gE}{v^2}$.

13. $D = \frac{d(R - 2)}{5}$.

14. $y = \frac{K}{x}$.

15. $P = \frac{33{,}000\ I}{LAN}$.

EXERCISE LXVI (Revision Exercise)

(A)

1. (a) $x = 12$.
(b) $p = 6$.
(c) $p = 13$.

2. 48 and 6.

3. 12, 13, and 14.

4. (a) $8pq(p - 3q)$.
(b) $(x + 8y)(x - 8y)$.
(c) $abc(a + b + c)$.

5. (a) 40 shillings.

(b) $C = \frac{my}{p}$.

(B)

6. (a) $2m + 1, 2m + 3, 2m + 5, 2m + 7, 2m + 9$.
(b) $2m - 7, 2m - 5, 2m - 3, 2m - 1, 2m + 1$.
(c) $2m - 3, 2m - 1, 2m + 1, 2m + 3, 2m + 5$.

7. (a) $\frac{128}{10x}$.

(b) $18a^2x$.

8. (a) $\frac{1}{2}a + 7b$.

(b) $x + y$.

9. 18 inches and 9 inches.

10. A £25, B £40, C £35.

(C)

11. (*a*) $x = 8$.
(*b*) $x = 5$.

12. $r = \sqrt[3]{\frac{3V}{4\pi}}$.

13. 30.

14. 13, 15, and 17.

15. (*a*) $135x^4y^4$.
(*b*) $\frac{21ay + 10bx + 35}{35xy}$.
(*c*) $13x^2y^3$.

(D)

16. $63abc^2$.

17. 998,000.

18. Man 35 years, son 10 years.

19. 38 £1 notes and 24 10*s*. notes.

20. 27.

ADDITIONAL EXERCISES

A 1

1. (*a*) 18.
(*b*) 32.
(*c*) 116.
(*d*) 4·828.
(*e*) $3\frac{1}{2}$.

2. (*a*) $p(3q + 5p)$.
(*b*) $y(x + z + xz)$.
(*c*) $(a - b)(a + b)$.
(*d*) $(2k + 3m)(2k - 3m)$.
(*e*) $3a(2a + b)(2a - b)$.

3. (*a*) $A = 7x^2$.
(*b*) 63 square inches.

4. (*a*) $x = 13$.
(*b*) $p = 9$.
(*c*) $a = 7$.

5. 24 and 8.

6. (*a*) $\frac{10p}{7}$.
(*b*) $\frac{53k}{35}$.
(*c*) $\frac{p}{k}$.

7. A £5, B £12, C £3.

8. $h = \frac{V}{\pi r^2}$.

9. 67°.

10. $4k - 6$.

A 2

1. (*a*) $16p^2$.
(*b*) $\frac{13}{15a}$.

2. $240p + 12q + r$.

3. L.C.M. is $600\,a^2b^2$, H.C.F. is $5ab$.

4. 20 years.

5. (*a*) $x = 27$.
(*b*) $x = 4$.

6. 27 and 29.

7. $\frac{3x}{28}$ pence.

8. (*a*) 88.
(*b*) $\frac{22k}{15}$.
(*c*) $\frac{22km}{15}$.

9. (*a*) £190, 16 years.
(*b*) $S = P + (n - 1)x$,
$\frac{Q - P + x}{x}$ years.

10. 20 shillings and 40 sixpences.

A 3

1. (a) $x = 24$.
 (b) $x = 3\frac{1}{2}$.
2. (a) 20 seconds.
 (b) $\frac{15(l + k)}{22v}$.
3. (a) $A = \pi(R^2 - 4r^2)$.
 (b) $A = \pi(R + 2r)(R - 2r)$.
 (c) 198 square cm.
4. (a) $2x + 7y + 14y^2$.
 (b) $x - y$.
 (c) $8 + m$ and $8 - m$.
5. (a) 998,001.
 (b) 42,558.
6. $£\frac{21x}{20}$.
7. (a) 152.
 (b) 64.
 (c) 64.
8. (a) $C = \frac{qx}{p}$.
 (b) $12\frac{1}{4}$.
9. $3(x + y + z)$.
10. $\frac{13(12x - y)}{3}$ shillings.

A 4

1. $x = 12$.
2. (a) $\frac{xy}{3}$.
 (b) $\frac{p(y + 2x)}{xy}$.
3. 42.
4. 2*s*. 6*d*.
5. (a) $A = \frac{1}{2}xh - a^2$.
 (b) 41 square inches.
6. $x = 7$.
7. (a) $8x + 5y$.
 (b) $2x + 9y$.
8. (a) $3a + 7b$.
 (b) $34b$.
9. $d = \sqrt[3]{\frac{6V}{\pi}}$.
10. A £5, B £10, C £15.

A 5

1. (a) $A = a^2(4 + 2\pi)$.
 (b) $370\frac{2}{7}$ square inches.
2. (a) $C = 4\pi a$.
 (b) $75\frac{3}{7}$ inches.
4. (a) $x = 3$.
 (b) $x = 7$.
5. $\left(20 - \frac{x}{6}\right)$ feet long and $\left(15 - \frac{x}{6}\right)$ feet wide.
6. (a) 3·464.
 (b) 0.
 (c) 3.
 (d) 1.
7. $f = \frac{uv}{u + v}$.
8. $2m - 3, 2m - 1, 2m + 1, 2m + 3, 2m + 5; 10m + 5$.
9. 25.
10. $S = 12$.

A 6

2. 7·18.
3. (a) $x = 2\frac{1}{3}$.
   (b) $x = 3$.
4. 29.
5. $\frac{3x}{2}$ shillings.
6. $6a^3b^2 + 8a^2bc$.
7. $x^2 + 2xy + 3xz$.
8. (a) 10 hours.
   (b) $\frac{kp}{q}$ hours.
10. A fraction.

A 7

1. $m = 1$.
2. 12.
3. (a) $A = x^2(3\pi + 4)$.
   (b) $483\frac{3}{7}$ square inches.
   (c) $C = 6\pi x$.
4. 56.
5. $R = \frac{pq}{3p + 2q}$.
6. $x = 1$.
7. $\frac{p^4}{4qr^5}$.
8. $\frac{x}{w}$ seconds.
9. $49x^2$ square inches.
10. $x = 2$.

A 8

1. $4x^2$.
2. 11.
4. (a) Length 25 yards, width 15 yards.
   (b) 375 square yards.
6. $x = 36$.
7. 99.
8. (a) $x = 4$.
   (b) $x = 3$.
10. $x = 4$.

A 9

1. 145 and 146.
2. 53.
3. $12x$ inches.
4. (a) $(15 - 13m)(15 + 13m)$.
   (b) $5a(x + 5y)$.
   (c) $4a(3a - 4b)(3a + 4b)$.
5. $b = 3\frac{1}{3}$.
6. Four times.
7. $\frac{100}{x + 1}$ yards.
8. $t = 2(x + 1)$.
9. $\frac{xy}{x + y}$ seconds.
10. $\frac{xyz}{yz + zx - xy}$.

A 10

1. $a = \sqrt[4]{c}$, $c = 16$.
2. (a) 27. (c) 512.
   (b) 3125. (d) 368.
3. 150 square yards.
4. $\dfrac{9 - a^2}{a}$.
5. $1\frac{1}{2}$ inches, 3 inches, $7\frac{1}{2}$ inches.
6. $\dfrac{pk}{x}$ pence.
7. $A = \frac{1}{2}\pi r^2$.
8. 81, 83, and 85.
9. $x = 6$.
10. $\dfrac{45(x + y)}{22v}$ seconds.

A 11

1. (a) $x = 2$. (c) $x = 3$.
   (b) $x = 4$. (d) $x = 6$.
2. (a) $10ab^2 + 13a^2b$.
   (b) $\dfrac{11x + 23}{42}$.
4. Total surface area $6x^2$ square inches; volume x^3 cubic inches.
5. 35 and 20.
6. (a) $x + z$.
   (b) $1 + 3y$.
   (c) $x + a$.
   (d) $x - a$.
7. (a) $2p^2 + 6pq$.
   (b) $14pq + 7q^2$.
   (c) $x^2 - a^2$.
   (d) $x^2 - a^2$.
8. (a) 4.
   (b) $\dfrac{a - b}{ab}$.
9. $\dfrac{x^2}{9}$ square miles.
10. $\dfrac{3p}{560}$ pence per lb.

A 12

1. $\dfrac{2(m + n)}{x}$ hurdles.
2. $\dfrac{v}{w - v}$ hours.
3. $\dfrac{a}{3b^2}$.
4. $x - y$.
5. n and $n + 1$.
6. 30.
7. $x = 4$.
8. $y = \dfrac{c}{x}$, $37\frac{1}{2}$.
9. $6x + 5$.
10. $x = 4{\cdot}8$.

A 13

1. 10,471 votes.
2. $r = \dfrac{C}{2\pi}$, 70 yards.
3. (a) $x = 9\frac{1}{3}$. (b) $x = 7$.
4. $A = \dfrac{FL}{Ye}$.
5. $\dfrac{4500}{1320 + 22x}$ seconds.
6. 27 and 28.
8. 60°.
9. $\dfrac{3p}{28}$ pence.
10. $16a$ inches.

A 14

1. $A = r^2(4 - \pi)$, $3\frac{3}{7}$ square inches.

2. $\frac{22x}{15}$ feet per second.

3. $13p$ feet.

4. (*a*) $\frac{3x^5z}{7}$.

(*b*) $3a^2b^3c^4$.

5. (*a*) $2x^2 - xy$.
(*b*) $4y^2 - 6xy$.

6. (*a*) $4x^3 + 4x^2y$.
(*b*) $9xy^3 - 6y^4$.

7. (*a*) $x = 0{\cdot}754 \ldots$
(*b*) $x = 4$.

8. A £1 11*s*. 6*d*.; B £1 1*s*. 0*d*.; C 10*s*. 6*d*.

9. $k = \frac{6}{11}$.

10. 4 shillings.

A 15

1. Man 24 years, son 2 years.

2. (*a*) $5x + y$.
(*b*) $3x - 4a$.
(*c*) $3x + 4a$.

3. (*a*) $x = 6$.
(*b*) $x = 0{\cdot}1$.
(*c*) $x = 1$.

4. $f = \frac{uv}{u + v}$.

5. $\frac{2}{3}$.

6. $A = 8$.

7. (*a*) 176.
(*b*) 76.
(*c*) 31.

8. 40 square cm.

9. (*a*) £$a + (n - 1)b$.
(*b*) The same as at the beginning of the year. The next year he will receive £$a + nb$.

10. (*a*) $N = 7x$.
(*b*) $N = 10x + 7$.

JUNIOR: PART II

EXERCISE LXVII

1. (*a*) (+ 10).
(*b*) (− 7).
(*c*) (− 10).

2. (*a*) (+ 1500).
(*b*) (− 12).

3. (*a*) (− 40).
(*b*) (+ 38).
(*c*) (+ 15).

4. (*a*) (− 5).
(*b*) (+ 34).

5. (− 10).

6. (*a*) (+ p).
(*b*) (− p).

7. (*a*) (+ 7).
(*b*) (+ m).
(*c*) (− 5).
(*d*) (− k).

8. (− 7).

9. (*a*) (+ 18).
(*b*) (− 2).
(*c*) (+ 20).
(*d*) (− 20).

10. (*a*) (+ 24).
(*b*) (+ a).
(*c*) $\left(-\frac{10}{20}\right)$.
(*d*) $\left(-\frac{p}{20}\right)$.

11. 4 miles south-west.

12. 100 B.C.; (− 55).

EXERCISE LXIX

2. (*a*) (+ 7).
(*b*) (+ 8).
(*c*) (+ 9).
(*d*) (+ 15).
(*e*) (+ 14).
(*f*) (+ 16).
(*g*) (+ 18).

EXERCISE LXX

2. (*a*) (− 7).
(*b*) (− 11).
(*c*) (− 9).
(*d*) (− 14).
(*e*) (− 19).
(*f*) (− 25a).
(*g*) (− 32x).

EXERCISE LXXI

2. (*a*) (− 4).
(*b*) 0.
(*c*) (− 11).
(*d*) (+ 8).
(*e*) (+ 2).
(*f*) 0.
(*g*) (− a).
(*h*) (− 4k).
(*i*) (+ 6x).
(*j*) (+ 5x).
(*k*) (− 16m).

3. (+ 7) + (+ 1) + (− 6).

4. (+ 10) + (− $2\frac{1}{2}$) + (− $5\frac{1}{2}$) + (+ 15) + (+ $7\frac{1}{2}$) + (− $8\frac{1}{2}$).
Amount left 16*s*.

EXERCISE LXXII

2. (*a*) $(+3)$.
(*b*) $(+6)$.
(*c*) (-3).
(*d*) (-6).
(*e*) $(-6a)$.
(*f*) $(+7x)$.
(*g*) $(+3m^2)$.
(*h*) $(+14xy)$.
(*i*) $(+4)$.
(*j*) $(+4)$.
(*k*) (-7).
(*l*) $(+2a)$.
(*m*) $(-2x)$.
(*n*) 0.

EXERCISE LXXIII

1. (*a*) -1.
(*b*) -4.
(*c*) $+5$.
(*d*) $+8a$.
(*e*) $+8xy$.

2. (*a*) $(+4)$.
(*b*) $(+3)$.
(*c*) (-3).
(*d*) $(-2a)$.
(*e*) $(-7xy)$.

4. (*a*) $(+14x)$.
(*b*) $(+24ab)$.
(*c*) $(-15xyz)$.
(*d*) $(-18m^2)$.

5. (*a*) $(+2x)$.
(*b*) $(+14ab)$.
(*c*) $(-xyz)$.
(*d*) $(-6m^2)$.

EXERCISE LXXIV

2. (*a*) $(-18)+(-4)=(-22)$.
(*b*) $(+18)+(+4) = (+22)$.
(*c*) $(-12x)+(-7x) = (-19x)$.
(*d*) $(-14ab)+(-21ab) = (-35ab)$.
(*e*) $(+12x^2)+(+4x^2) = (+16x^2)$.
(*f*) $(+41xyz)+(+32xyz) = (+73xyz)$.
(*g*) $(-32xyz)+(-41xyz) = (-73xyz)$.
(*h*) $(+20a^2)+(+47a^2) = (+67a^2)$.
(*i*) $(-35mn)+(-14mn) = (-49mn)$.
(*j*) $(-\frac{1}{2}k^3)+(-\frac{1}{3}k^3) = (-\frac{5}{6}k^3)$.
(*k*) $(+3{\cdot}5p)+(+2{\cdot}7p) = (+6{\cdot}2p)$.

3. (*a*) $(+1)$.
(*b*) (-8).
(*c*) (-15).
(*d*) $(+15)$.
(*e*) (-24).
(*f*) $(+10)$.
(*g*) (-6).
(*h*) $(+15)$.
(*i*) $(+17)$.
(*j*) (-2).
(*k*) (-24).
(*l*) (-10).
(*m*) $(+6)$.
(*n*) $(+6)$.

4. (*a*) (-10).
(*b*) (-2).
(*c*) (-2).
(*d*) (-6).
(*e*) (-6).
(*f*) (-18).
(*g*) $(+18)$.
(*h*) (-8).
(*i*) 0.
(*j*) 0.

EXERCISE LXXVI

1. (a) (20).
(b) (+ 56).
(c) (+ 63).
(d) (– 36).
(e) (+ 6).
(f) (– 336).
(g) (– 6).
(h) (+ 48).
(i) (+ 48).
(j) (– 105).
(k) (– $6xy$).
(l) (– $6xy$).
(m) (+ $12a^2b$).
(n) (– $12ab^2$).
(o) (+ $30mp$).
(p) (– $56k^2$).
(q) (– xyz).
(r) (– $24xyz$).
(s) (– $30mn^2p$).
(t) (+ $30ab^2c$).

2. (a) (– 6).
(b) (+ 12).
(c) (– 8).
(d) (– 2).
(e) (+ 21).
(f) (+ 24).
(g) (+ 4).
(h) 0.

EXERCISE LXXVII

1. (– 4), (– 4).
2. (+ 4), (– 4).
3. $\left(-\frac{1}{4}\right)$, $\left(-\frac{1}{3}\right)$.
4. (+ 4), (– 3).
5. (+ 4), (– 8).
6. (– 2), (– 8).
7. (– 7), (– 8).
8. (– 9), (+ 16).
9. (+ $3a$).
10. $\left(-\frac{1}{3}a\right)$.
11. (– $5x$).
12. (+ $5y$).
13. $\left(+\frac{1}{5}\right)$.
14. $\left(+\frac{6m}{n}\right)$.
15. $\left(+\frac{8p}{q}\right)$.

EXERCISE LXXVIII
(General Revision Exercise on Directed Numbers)

1. (– 5).
2. (+ 9).
3. (– 13).
4. (+ 1).
5. (+ 17).
6. (– 13).
7. (+ 17).
8. (+ 1).
9. (+ 17).
10. (– 30).
11. (+ 40).
12. (+ 42).
13. (+ 52).
14. $\left(-\frac{2}{7}\right)$.
15. $\left(+\frac{7}{8}\right)$.
16. $\left(+\frac{33}{56}\right)$.
17. $\left(+\frac{16}{7}\right)$.
18. (+ 28).
19. (+ $32\frac{1}{28}$).
20. (+ $1\frac{5}{7}$).
21. (+ 8).
22. (– 49).
23. (– 41).
24. (– 28).
25. (+ 32).
26. (+ 1).
27. $\left(-\frac{4}{7}\right)$.
28. (– 7).
29. $\left(+\frac{1}{2}\right)$.
30. (– 180).
31. $\left(-\frac{5}{9}\right)$.
32. $\left(-\frac{17}{43}\right)$.
33. (– 111).
34. (– 101).
35. (– 336).

EXERCISE LXXIX

1. (*a*) (+ 9).
(*b*) (– 27).
(*c*) (+ 16).
(*d*) (+ 16).
(*e*) (+ 16).
(*f*) (+ 25).
(*g*) (+ 5).
(*h*) (+ 19).
(*i*) (+ 14).
(*j*) (– 20).

EXERCISE LXXX

1. (+ 9).
2. (+ 9).
3. (– 125).
4. (+ 125).
5. (+ 36).
6. (– 7).
7. (+ 7).
8. (+ 52).
9. (– 256).
10. 0.
11. (± 5).
12. Imaginary.
13. (± 8).
14. (± 13).
15. (± 11).
16. $\left(\pm \frac{2}{3}\right)$.
17. Imaginary.
18. $\left(\pm \frac{4}{7}\right)$.
19. $\left(\pm \frac{10}{11}\right)$.
20. $\left(\pm \frac{9}{4}\right)$.
21. $(+ x^2)$.
22. $(+ x^2)$.
23. $(+ 9x^2)$.
24. $(+ 9x^2)$.
25. $(+ 64k^3)$.
26. $(- 64k^3)$.
27. 0.
28. $(+ 32m^2)$.
29. $(+ 35b^3c^3)$.
30. $(+ 19b^3c^3)$.
31. $(\pm 2a)$.
32. $(\pm 13k)$.
33. $(\pm 11m^2n)$.
34. $(\pm 5p^2)$.
35. $(\pm 8a^2b)$.

EXERCISE LXXXI

1. (+ 4).
2. (– 4).
3. (± 8).
4. Imaginary.
5. (± 3).
6. (+ 4).
7. (– 4).
8. $(+ 3x)$.
9. $(- 5m^2)$.
10. $(\pm 2k^2)$.
11. $(- 6ab^2)$.
12. $(\pm 5x)$.
13. $(- 2t^2)$.
14. $(+ 7p)$.
15. $(- 3x^2)$.

EXERCISE LXXXII

1. (*a*) $5a$.
(*b*) $- x$.
(*c*) p.
(*d*) $- p$.
(*e*) $13m$.
(*f*) $5m$.
(*g*) $- 5m$.
(*h*) $2a^2$.
(*i*) $- 2a^2$.
(*j*) 0.
(*k*) $2a$.
(*l*) $6a - 8b$.
(*m*) $- 3x - y$.
(*n*) $2x - 2$.
(*o*) $1 - 3mn$.
(*p*) $- x^2 + 13x - 18$.
(*q*) $3p^2 - 5pq - 4q^2$.
(*r*) $19x^2y - 20y$.
(*s*) $7b^2 - 4ab$.
(*t*) $- 6m^2 + 8mp^2 + 10m^2p$.

EXERCISE LXXXII—*continued*

2. (*a*) 0.
(*b*) $4b - 3a$.
(*c*) $5a - 4b$.
(*d*) $4m - 2p$.
(*e*) $-9x - 31$.
(*f*) $59k - 60$.
(*g*) 0.
(*h*) $2ab + ac - c$.
(*i*) $32x^2 - 6x$.
(*j*) $-6p^2 + 14p + 2$.

EXERCISE LXXXIII

1. (*a*) $8a + 6b + 2c$.
(*b*) $8m - 2mn + 3p$.
(*c*) $8k + 10l - 16m$.
(*d*) $-6p - 9q - 5r$.
(*e*) $8x^2 - 8x - 3$.
(*f*) $-16x^2 - 4xy - 9y^2$.
(*g*) $11ab - 19bc + 3abc$.
(*h*) $x + y + z$.
(*i*) $9a^2 - 22ab - b^2$.
(*j*) $4m^2 + 3n^2 + 16t^2$.
(*k*) $8a^2 - ab - 8b^2$.
(*l*) $5x^2 - 2yz + 2y^2 - 2xy$.
(*m*) $48 - 42p + 14p^2$.
(*n*) $27ab - ac - 8b + 7c$.
(*o*) $32p^2 - 2pq + 2q^2$.

2. (*a*) $12a^2 - 10a - 3b$.
(*b*) $20a^2 - 26a - 10b$.
(*c*) $12a^2 - 11a + 2b$.
(*d*) $8a^2 - 17a - 2b$.
(*e*) $4a^2 + 8a - 6b$.

EXERCISE LXXXIV

1. (*a*) $5x + 17y - 11$.
(*b*) $8a^2 - 13ab - 5b^2$.
(*c*) $-13x^2 - 19xy + 6y^2$.
(*d*) $2p + 25pq - 11q - 2$.
(*e*) $4ab - 19bc + 9ca$.
(*f*) $-19m^2 - 17m + 8$.
(*g*) $-2a + 3a^2 - 3a^3$.
(*h*) $6 + 5m - 7mn + 16n^2$.
(*i*) $6a - 5b + 2abc$.
(*j*) $6xy - 2y^2 - 25yz$.

2. (*a*) $8b - 8a$.
(*b*) $-4x$.
(*c*) $9q^2 - 9p^2$.
(*d*) $19x^2 - 18xy + 4y^2$.
(*e*) $8m^2 - 3mn + 3n^2$.

3. (*a*) $-5x^2 + 12x - 15$.
(*b*) $5x^2 - 12x + 15$.
(*c*) $2x^2 + 7x - 9 - 8y^2$.
(*d*) $x^2 - 2x + 3 + 8y^2$.
(*e*) $5x^2 + 12x - 15 - 8y^2$.

EXERCISE LXXXV

1. (*a*) $10a^2 - 6ab$.
(*b*) $-36x^3 - 15x^2$.
(*c*) $40m^2n - 4mn^2$.
(*d*) $2x^2 + 2xy + 2xz$.
(*e*) $3p^3q - 5p^2q^2 + p^2qr$.
(*f*) $-3mn^2 + 15n^3$.
(*g*) $-28a^2 + 12ab - 4b^2$.
(*h*) $50bc^2 - 40c^2$.
(*i*) $-6pq + 15q^2 - 9pq^2$.
(*j*) $28\,mk^2 - 6m^2k$.

2. (*a*) $26x^2 + 4x$.
(*b*) $27a^3 - 13a^2b + 5ab$.
(*c*) $-11p^3 + 26p + 3$.
(*d*) $-7abc + 3b^2c + 3bc^2 + 4a$.
(*e*) $27x^4 + 13x^3 - 22x^2$.

EXERCISE LXXXVI

1. (*a*) $6x^2 + xy - 2y^2$.
(*b*) $12p^2 - 7pq - 12q^2$.
(*c*) $35m^2 - 2mn - 48n^2$.
(*d*) $4x^2 + 7x - 15$.
(*e*) $-30a^2 + 16ab - 2b^2$.
(*f*) $2x^2 - xy + 2x - 3y^2 - 3y$.
(*g*) $6p^2 - pq + 8p - 2q^2 + 4q$.
(*h*) $26w^2 - 37wt + 13w + 12t^2 - 12t$.
(*i*) $8ac - 10ad - 4c^2 + 5cd$.
(*j*) $-10m^2 + 101mn - 10n^2$.

2. (*a*) $a^2 + 2ab + b^2$.
(*b*) $2a^2 + 2b^2$.
(*c*) $4x^2 - 4xy + y^2$.
(*d*) $9y^2 - 12xy + 4x^2$.
(*e*) $16p^2 - 24pq + 9q^2$.
(*f*) $20x^2 + 9xy - 18y^2$.
(*g*) $12a^2 - 11ab + 2b^2$.
(*h*) $2x^2y + 2x - xy^2 - y$.
(*i*) $1 - x + x^2 - x^3$.
(*j*) $a^2 - b^2 + 2bc - c^2$.
(*k*) $pq - q^2r - p^2r + pqr^2$.
(*l*) $-2x^3 + 8x^2 + 3x - 12$.
(*m*) $ab^2c - bc^2 - a^2b + ac$.
(*n*) $w^2x^2 + 6awx + 5a^2$.
(*o*) $1 - abc - a + a^2bc$.

EXERCISE LXXXVII

1. (*a*) $-4 + 6a - a^2 + 3a^3 + 5a^4$.
(*b*) $5a^4 + 3a^3 - a^2 + 6a - 4$.

2. (*a*) $-5 - p + 7p^2 + 3p^3 + 4p^4$.
(*b*) $4p^4 + 3p^3 + 7p^2 - p - 5$.

3. (*a*) $4 + x + 5x^2 + 3x^3 - 9x^4$.
(*b*) $-9x^4 + 3x^3 + 5x^2 + x + 4$.

4. (*a*) $5 + 10c - 9c^2 - c^3 + 8c^5$.
(*b*) $8c^5 - c^3 - 9c^2 + 10c + 5$.

5. (*a*) $10 + 2k + k^2 - 3k^3 - k^4$.
(*b*) $-k^4 - 3k^3 + k^2 + 2k + 10$.

EXERCISE LXXXVIII

1. $6x^4 - 5x^3 - 5x^2 + 5x - 1$.

2. $10p^4 - 16p^3q - 7p^2q^2 + 7pq^3 - 3q^4$.

3. $8 - 2x + 3x^2 - 2x^3 + x^4$.

4. $10a^4 - 11a^3b + 22a^2b^2 - 3ab^3 + 2b^4$.

5. $3x^4 + 3x^3 - x^2 + 2x - 2$.

6. $4m^4 + 3m^3 + 7m^2 + 6m - 2$.

7. $p^3 + q^3$.

8. $p^3 - q^3$.

9. $1 + 4x + 6x^2 + 4x^3 + x^4$.

10. $-6k^4 + k^3 + 15k^2 + 3k - 4$.

11. $5a^5 + 13a^4 - 13a^3 + 14a^2 - 3a - 2$.

12. $4t^4 - t^3 + 7t^2 - 6t + 8$.

13. $2k^5 - 4k^4 - 5k^3 + 18k^2 - 16k$.

14. $xy^2z + x^2yz + yz + zx$.

15. $x^4 + x^2y^2 + y^4$.

EXERCISE LXXXIX

1. $x^2 - 3x + 4$.
2. $2p^2 + p + 3$.
3. $4a^2 - 3a - 5$.
4. $5m^2 + m + 6$.
5. $8p^2 - pq + 3q^2$.
6. $3x^2 - 5xy - 2y^2$.
7. $3a^2 - 2a + 1$.
8. $k^2 - k - 1$.
9. $5m^2 + 3m - 4$.
10. $2p^3 - 5$.
11. $2x - 5y$.
12. $a^2 + a - 1$.
13. $x^2 + x - 1$.
14. $k^2 + k - 3$.
15. $3m^2 - 2mn + n^2$.

EXERCISE XCII (Revision Exercise)

(A)

1. (a) -11.
   (b) 155.
   (c) -17.
   (d) 315.
   (e) 21.
2. $6a^3 + 11a^2b - 28ab^2 + 12b^3$.
3. $x = -9$.
4. $5x^2 - 3xy - y^2$.
5. (a) $8p - 8q$.
   (b) $-2p - 2q$.
   (c) $15p^2 - 34pq + 15q^2$.

(B)

6. (a) $x = -2$.
   (b) $x = -8$.
7. K = 8.
8. 47 and 23.
10. All even values.

(C)

11. $l = \frac{T^2g}{4\pi^2}$.
12. 27.
13. (a) $x = 2$.
    (b) $x = 1\frac{6}{7}$.
14. (a) $\frac{37p - q}{20}$.
    (b) $\frac{-8m + 17n - 2p}{24}$.
15. (a) $(1 - 3m)(1 + 3m)(1 + 9m^2)$.
    (b) $pq(p - q + pq)$.
    (c) $x\left(1 - \frac{b}{2}\right)\left(1 + \frac{b}{2}\right)$.

(D)

16. K = 24.
17. If A has £x, then B has £$\frac{10}{77}x$.
18. -28.
19. $\frac{4a^2 - 6ab + 9b^2}{-2a + 3b}$.
20. (a) $x = 5$.
    (b) $x = \frac{1}{3}$.

EXERCISE XCIII

16. The x axis.

17. The y axis.

EXERCISE XCV

1. (a) $x = 1, y = 2$.
   (b) $x = -2, y = 3$.
   (c) $x = 1, y = -2$.
   (d) $x = 2, y = 2$.
   (e) $x = 0, y = -2$.
   (f) $x = -3, y = 0$.
   (g) $x = 2, y = 2$.
   (h) $x = -1, y = -3$.
   (i) $x = 2, y = 3$.

2. (a) $x = 3, y = 2$.
   (b) $x = 5, y = 3$.

EXERCISE XCVI

1. $x = 2, y = 3$.
2. $x = 1, y = 2$.
3. $x = -2, y = 3$.
4. $x = 1, y = -2$.
5. $x = 2, y = 2$.
6. $x = -3, y = 0$.
7. $x = 2, y = 2$.
8. $x = -1, y = -3$.
9. $x = 2, y = 3$.
10. $m = -\frac{1}{2}, n = 1$.
11. $p = -2\frac{2}{3}, q = 2\frac{1}{3}$.
12. $a = 1, b = 1$.
13. $x = 2, y = 3$.
14. $m = -2, n = 3$.
15. $p = 3, q = -3$.
16. $x = 1, y = 2$.
17. $m = -1, n = -2$.
18. $x = \frac{1}{4}, y = -2$.
19. $x = \frac{10}{23}, y = -\frac{15}{13}$.
20. $m = 1, n = 1$.
21. $p = -2, q = 4$.
22. $x = 2\frac{2}{5}, y = -\frac{1}{5}$.

EXERCISE XCVII

1. $x = 1, y = 1, z = 3$.
2. $x = 3, y = 2, z = 1$.
3. $x = 2, y = -1, z = -1$.
4. $p = -2, q = 3, r = 5$.
5. $a = -3, b = 5, c = 1$.
6. $l = -1, m = -2, n = -3$.
7. $x = 5, y = 7, z = 3$.
8. $a = 9, b = 1, c = 2$.
9. $p = 2, q = 2, r = 2$.
10. $x = 4, y = -2, z = 1$.

EXERCISE XCVIII

1. 28 and 15.
2. 123 and 89.
3. 12 and 15.
4. 90 and 120.
5. 15 florins, 8 shillings.
6. Tea 3*s*., coffee 4*s*.
7. Tea 3*s*., sugar 3*d*.
8. Gramophone £12 12*s*., records 2*s*. 6*d*.
9. A 25 years, B 18 years.
10. 16 florins, 10 half-crowns.
11. 14, 12, 9.
12. 84.
13. 53.
14. Skilled £2 10*s*., unskilled £1 15*s*.
15. 80 at 2*s*. 6*d*., 36 at 3*s*. 6*d*.

EXERCISE XCIX (Revision Exercise)

(A)

1. $3x^4 - x^3 - 9x^2 - 12x - 8$.
2. A = – 1, B = 2.
3. (*a*) $x = 4$.
   (*b*) $x = 2, y = -3$.
4. (*a*) $7x^2(1 - 5xy + 7y^2)$.
   (*b*) $\left(\frac{12x^2}{y^2} - 7\right)\left(\frac{12x^2}{y^2} + 7\right)$.
5. $a = 2$ and $b = 5$.

(B)

6. (*a*) – 14.
   (*b*) 0.
7. 85.
8. (*a*) $x = 2, y = 3$.
   (*b*) $x = 4, y = 7$.
9. $2m^2 + m - 5$.
10. A = 1, B = 2, C = 3.

(C)

11. Length 20 yards, width 15 yards.
12. (*a*) $x = 7$.
    (*b*) $x = 2, y = 1$.
13. $21\frac{9}{11}$ minutes past 10.
14. $6x^2 + 13x + 6$.
15. 2 hits, 10 misses.

(D)

16. $\frac{1}{12}(-39a - 14b - 21c)$.
17. (*a*) $x = 5, y = -1$.
    (*b*) $x = 3, y = 2$.
18. $\frac{1}{2}(x - y)$.
19. Father 36 years, son 6 years.

EXERCISE C

2. (i) $4ab$.
(ii) $-4ab$.
(iii) $(3x-y)(-x-3y)$.
(iv) $(x+y+z)(x+y-z)$.
(v) $(z-x-y)(z+x+y)$.
(vi) $(3p+11q)(3p-5q)$.
(vii) $4(5a+3b)(-a-3b)$.
(viii) $25(3x-y)(-x+3y)$.
(ix) $9(3a+4b)(3a+2b)$.
(x) $(m-n+7)(m-n-7)$.
(xi) $4x(y+z)$.
(xii) $(5p^5-3q^5)(5p^5+3q^5)$.
(xiii) $(p+q)(p-q+2p^2q^2)$.
(xiv) $2(u+v+3w)(u+v-3w)$.
(xv) $3(6a+5b)(-4a-5b)$.
(xvi) $7(xy+3x+3y)(xy-3x-3y)$.
(xvii) $(2c+2d+11)(2c+2d-11)$.
(xviii) $p^2x(x-2y)$.
(xix) $\{xy(a+b)+ab(x+y)\}\{xy(a+b)-ab(x+y)\}$.
(xx) $(ax+b+7)(ax+b-7)$.
(xxi) $4cmx$.
(xxii) $(12+3y-x)(12-3y+x)$.
(xxiii) $8(a+b+c)(-a+c)$.
(xxiv) $(2x-y)(-5y+2z)$.
(xxv) $2(4a+3x-3y)(4a-3x+3y)$.
(xxvi) $3(l+m)(l-m)$.
(xxvii) $8(3p+3q-\mathrm{r})(p-q)$.
(xxviii) $4(5+2p-2q-2r)(5-2p+2q+2r)$.
(xxix) $p(a-b+c)(a-b-c)$.

EXERCISE CI

1. (a) a^2+2a+1.
(b) a^2-4a+4.
(c) m^2+6m+9.
(d) $m^2-8m+16$.
(e) $4p^2+4pq+q^2$.
(f) $4p^2-4pq+q^2$.
(g) $9u^2+12uv+4v^2$.
(h) $9u^2-12uv+4v^2$.
(i) $16b^2+24bc+9c^2$.
(j) $x^2+2+\frac{1}{x^2}$.
(k) $a^2-\frac{4a}{b}+\frac{4}{b^2}$.
(l) $x^2y^2+2pxy+p^2$.
(m) $4p^2-12pqr+9q^2r^2$.
(n) $\frac{16}{x^2}+\frac{8}{x}+1$.
(o) $25k^2-20km+4m^2$.

2. (a) $x-2$.
(b) $a+2b$.
(c) $2m+n$.
(d) $m-2n$.
(e) $2x+3y$.
(f) $3x-2y$.
(g) $1+x$.
(h) $1-3p$.
(i) $q-5r$.
(j) $2x-3y$.

3. (a) $6x$.
(b) $4x^2$.
(c) $6pq$, q^2.
(d) $2r$.
(e) m, $25n^2$.

4. (a) b^2, $(a+b)$.
(b) $4b^2$, $(a-2b)$.
(c) b^2, $(2a-b)$.
(d) $4y^2$, $(3x+2y)$.
(e) $9y^2$, $(2x-3y)$.
(f) x^2, $(1-x)$.
(g) $9p^2$, $(1-3p)$.
(h) $4r^2$, $(q-2r)$.
(i) k^2, $(2+k)$.
(j) $25n^2$, $(3m+5n)$.

EXERCISE CII

1. $x^2 + 5x + 6$.
2. $x^2 - 5x + 6$.
3. $x^2 + x - 6$.
4. $x^2 - x - 6$.
5. $p^2 - pq - 6q^2$.
6. $p^2 + pq - 6q^2$.
7. $2p^2 + 3pq - 2q^2$.
8. $2m^2 + mn - 6n^2$.
9. $15k^2 - kt - 2t^2$.
10. $8a^2 - 2ab - 15b^2$.
11. $48a^2 + 2ab - 63b^2$.
12. $15x^2 - 29xy - 14y^2$.
13. $32p^2 - 4pq - 3q^2$.
14. $1 - 3x - 10x^2$.
15. $2 - 5pq + 3p^2q^2$.
16. $ax - a^2y - x^2y + axy^2$.
17. $10 - 19k + 7k^2$.
18. $21p^2 + 25pq - 4q^2$.
19. $12x^2 - 32xy + 5y^2$.
20. $3a^2 - 16ab - 35b^2$.

EXERCISE CIII

1. $(x + 1)(x + 2)$.
2. $(x + 2)(x + 3)$.
3. $(x + 2)(x + 4)$.
4. $(x - 1)(x - 2)$.
5. $(x - 3)(x - 2)$.
6. $(x - 2)(x - 4)$.
7. $(a + 3)(a + 5)$.
8. $(m + 1)(m - 7)$.
9. $(k + 1)(k - 2)$.
10. $(p + 2)(p + 3)$.
11. $(p + 2q)(p + 3q)$.
12. $(x + 5)(x - 3)$.
13. $(x - 8)(x - 10)$.
14. $(a + 8)(a - 7)$.
15. $(a + 8b)(a - 7b)$.
16. $(p + 5q)(p - q)$.
17. $(x + 11)(x - 9)$.
18. $(x - 9)(x + 1)$.
19. $(a - 7b)(a + 5b)$.
20. $(x - 3y)(x + 2y)$.
21. $(c - 5d)(c + 2d)$.
22. $(x - 8y)(x + y)$.
23. $(a - 3b)(a + b)$.
24. $(m - 3n)(m + 2n)$.
25. $(k - 3m)(k + m)$.
26. $(x + 2y)(x + 5y)$.
27. $(p - 3q)(p - 9q)$.
28. $(a - 3b)(a - 10b)$.
29. $(b + 5d)(b - d)$.
30. $(x + 8y)(x + y)$.

EXERCISE CIV

1. $(x + 1)(2x + 3)$.
2. $(x - 1)(2x - 3)$.
3. $(2x + 1)(x + 3)$.
4. $(2x - 1)(x - 3)$.
5. $(2a - b)(3a + b)$.
6. $(2a + b)(3a + b)$.
7. $(4x - y)(2x + 3y)$.
8. $(3x + 2y)(x - y)$.
9. $(5q - p)(2q + 3p)$.
10. $(3q + 2p)(4q - p)$.
11. $(6x + y)(x + y)$.
12. $(5m - n)(m - n)$.

EXERCISE CIV—*continued*

13. $(4k + l)(k - 2l)$.
14. $(5a + 2b)(a - b)$.
15. $(10a - b)(a + 2b)$.
16. $(3g - f)(9g + 2f)$.
17. $(5p - 2q)(5p - 3q)$.
18. $(x - 12y)(2x + y)$.
19. $(m + 2n)(2m + 3n)$.
20. $(a - 7b)(3a + 2b)$.
21. $(2x - 1)(x - 9)$.
22. $(2x + 3y)(4x - 5y)$.
23. $(a + 7b)(3a - 5b)$.
24. $(2a - 3b)(5a + b)$.
25. $(6x + 2)(7x - 5)$.
26. $(9k - 5m)(k - m)$.
27. $(3p - 11q)(2p - 3q)$.
28. $(9a + 2b)(6a + 5b)$.
29. $(m + 7n)(7m + n)$.
30. $(2p - q)(2p - 3q)$.
31. $(a - mn)(2a + mn)$.
32. $(a - 8b)(3a + b)$.
33. $(2x + 3y)(x + y)$.
34. $(6x + y)(x - y)$.
35. $(9p + 2q)(2p - 9q)$.

EXERCISE CV

1. $(x + 1)(x^2 - x + 1)$.
2. $(x - 1)(x^2 + x + 1)$.
3. $(1 - 2p)(1 + 2p + 4p^2)$.
4. $(1 + 2p)(1 - 2p + 4p^2)$.
5. $(3a - b)(9a^2 + 3ab + b^2)$.
6. $(b + 3a)(b^2 - 3ab + 9a^2)$.
7. $(5m + 4n)(25m^2 - 20mn + 16n^2)$.
8. $\left(1 - \frac{b}{3}\right)\left(1 + \frac{b}{3} + \frac{b^2}{9}\right)$.
9. $(10x - y)(100x^2 + 10xy + y^2)$.
10. $k^3(v + 1)(v^2 - v + 1)$.
11. $(a - x - y)\{a^2 + a(x + y) + (x + y)^2\}$.
12. $(2a - b)(a^2 - ab + b^2)$.
13. $2a(a^2 + 3b^2)$.
14. $2b(3a^2 + b^2)$.
15. $(a - b)(a + b)(a^4 + a^2b^2 + b^4)$.
16. $(x - z)(x^2 + 3xy + 3y^2 + xy + 3yz + z^2)$.
17. $(1 - pq^2)(1 + pq^2 + p^2q^4)$.
18. $(3k + 7m)(9k^2 - 21km + 49m^2)$.
19. $x^2(x - y)(x^2 + xy + y^2)$.
20. $5(x + 5y)(x^2 - 5xy + 25y^2)$.

EXERCISE CVI

1. $(x + 1)(x + y)$.
2. $(x - 1)(x + y)$.
3. $(x - 1)(x - y)$.
4. $(a + 2b)(x - y)$.
5. $(a - b)(2x + y)$.
6. $(a + 2b)(2x - 3y)$.
7. $(m - n)(f + 3g)$.
8. $(k + 3m)(2a - b)$.
9. $(3a + b)(2x - 3y)$.
10. $(a - b)(m - n)$.
11. $(a + b + c)(x + y)$.
12. $(a - b - c)(x - y)$.
13. $(1 + a + a^2)(x + y)$.
14. $(1 - a - a^2)(x - y)$.
15. $(1 + x)(a - y)$.
16. $(1 - x)(a + y)$.
17. $(2a - 3b)(x - y)$.
18. $(2m - 5n)(p + 2n)$.
19. $(3f - g)(1 + x)$.
20. $(4k + m)(3x - 2y)$.

EXERCISE CVII

1. 25. **2.** 55. **3.** 229. **4.** – 571. **5.** 35. **6.** 15. **7.** 239. **8.** 135. **9.** 73. **10.** 4477.

EXERCISE CVIII

1. $(x-1)(x-2)(x+3)$.
2. $(x+1)(x-2)(x-3)$.
3. $(x+3)(x-1)(2x+1)$.
4. (*a*) $(x-4)(x+1)(2x-1)$.
(*b*) $(x-7)(x-1)(x-1)$.
(*c*) $(x-3)(2x+1)(2x-1)$.
(*d*) $(x-5)(2x-1)(x-3)$.
(*e*) $(x+2)(2x+3)(2x-5)$.
(*f*) $(x+3)(x-4)(x-5)$.
(*g*) $(x-4)(x+1)(3x+5)$.
5. $K = 6$.
6. $K = -19$.
7. $K = 2$.
8. $p = 11, q = -6$.
9. $p = 2, q = -1$.
10. $(x+3)(x-4)(3x-1)$.
11. $a = 2, b = -1, c = -13$.
12. $(2x+1)(3x-2)(x+1)$.
13. $x-2$.
14. $p = 24, q = 11$.

EXERCISE CIX

(General Revision Exercise on Factors)

1. $(1-9p)(1+9p)$.
2. $(1+3p)(1-3p+9p^2)$.
3. $(5x-1)(4x-1)$.
4. $\left(2-\frac{a}{3}\right)\left(4+\frac{2a}{3}+\frac{a^2}{9}\right)$.
5. $(3x+5)(x-2)$.
6. $(ax+by)(bx+ay)$.
7. $(2x-1)(3x+4)$.
8. $(1+x)(1+x+x^2)$.
9. $(a-b)(x-y)$.
10. $(7x-3)(4x+5)$.
11. $(x+a+b)(x-a-b)$.
12. $(x-a+b)\{x^2+x(a-b)+(a-b)^2\}$.
13. $(7x+10)(2x-3)$.
14. $(10p-q)(100p^2+10pq+q^2)$.
15. $(x+1)^3$.
16. $(x-1)^3$.
17. $(8x-3)(2x+1)$.
18. $(5x+3)(x-6)$.
19. $(a+2b)(x-3y)$.
20. $(1+x)(1-x+x^2)$.
21. $(3a-5b)(7x-2y)$.
22. $(12p-5q)(3p+8q)$.
23. $(9m+4n)(8m-7n)$.
24. $(3x+2)(x-5)(x-4)$.
25. $(2x-3)(3x-2)(x+1)$.

EXERCISE CX

1. x^2+2x+3.
2. $2a^2-3a+4$.
3. $3a+2b+3$.
4. $5p^2-p-2$.
5. $4x^2-7x+3$.
6. $3p+5q+r$.
7. $3x+2y-4$.
8. x^3-2x^2+x-1.
9. $4a^2-a-3$.
10. $bc+ca+ab$.
11. $1+x-xy$.
12. $1-2x+3x^2$.
13. $a+1-\frac{1}{a}$.
14. $2x^2-x+\frac{3}{x}$.
15. $4a-3+\frac{2}{a}$.

EXERCISE CXI (Revision Exercise)

(A)

1. (*a*) $(5k + 7)(5k - 7)$.
(*b*) $(5k - 7)(25k^2 + 35k + 49)$.
(*c*) $(3x - 2)(2x - 3)$.

2. (*a*) $x = 3, y = -7$.
(*b*) $x = 4$.

4. 31 and 17.

5. $2x - 1$.

(B)

6. (*a*) $x = 2, y = -3$.
(*b*) $x = y = z = 1$.

7. $p = -15, q = 9$.

8. $x = -\frac{1}{4}$.

9. 41.

10. $2n, 2n + 1, 2n + 2, 2n + 3$; $n = 7$.

(C)

11. $3x^2 + x - 4$.

12. 20 lb. at 3*s*. per lb. and 5 lb. at 4*s*. per lb.

13. A = 6, B = 5, C = − 6.

14. 5 and 3.

15. $\frac{5(a^2 + ab + b^2)}{b - a}$.

(D)

16. 2100 square feet.

17. (*a*) $(3x - 2y)(2a + 3b)$.
(*b*) $(3x - 2)(x - 4)(x + 1)$.

18. $2m^2 - 3m - 5$.

19. $a = 2, b = -3$.

20. 23 sixpences, 14 shillings.

EXERCISE CXII

1. $x - 1$.
2. $x + 1$.
3. $2x - 1$.
4. $3x + 2$.
5. $4x - 3$.
6. $x + 1$.
7. $3x - 2$.
8. $2x + 3$.
9. $3x - 5$.
10. $(x - 1)(x + 1)$.

EXERCISE CXIII

1. $(x - 1)^2 (x + 1)$.
2. $(x + 1)^2 (x^2 - x + 1)$.
3. $(2x - 1)(2x + 3)(x - 1)$.
4. $(3x + 2)(x - 1)(x + 2)$.
5. $(4x - 3)(2x + 1)(3x - 2)$.
6. $(x + 1)^2 (x - 1)(2x + 1)$.
7. $2x(3x - 2)(x + 1)(x - 1)$.
8. $(2x + 3)^2 (3x - 1)(4x + 1)$.
9. $(3x - 5)^2 (2x + 3)(x - 1)$.
10. $(x - 1)^2 (x + 1)^2$.

EXERCISE CXIV

1. $\frac{3x+5}{(x+2)(x+1)}$.
2. $\frac{x}{(x+2)(x+1)}$.
3. $\frac{11x+28}{(x+4)(x+2)}$.
4. $\frac{5x+4}{(x+4)(x+2)}$.
5. $\frac{6x+10}{(x+1)(x+2)}$.
6. $\frac{5x+11}{(x+1)(x+3)}$.
7. $\frac{-x-7}{(x+1)(x+3)}$.
8. $\frac{2ax+3a}{(x+1)(x+2)}$.
9. $\frac{a}{(x+1)(x+2)}$.
10. $\frac{2x^2+6x+5}{(x+1)(x+2)}$.
11. $\frac{-2x-3}{(x+1)(x+2)}$.
12. $\frac{1+2x-2y}{x^2-y^2}$.
13. $\frac{a^2+b^2+a+b}{a^2-b^2}$.
14. $\frac{n^2-2mn}{m(m-n)}$.
15. $\frac{4x+1}{(3x-5)(x-1)(2x+3)}$.
16. $\frac{5}{(3x-5)(x-1)(2x+3)}$.
17. $\frac{x-4xy}{x^2-y^2}$.
18. $\frac{19x-22}{(2x+1)(x-3)(3x-2)}$.
19. $\frac{11x+2}{(2x+1)(x-3)(3x-2)}$.
20. $\frac{ab+bc+ca-a^2-b^2-c^2}{(b-c)(c-a)(a-b)}$.

EXERCISE CXV

1. (*a*) $\frac{2}{3(x+1)}$.
(*b*) $\frac{a+x}{(a-x)(x-1)}$.
(*c*) $\frac{(x-1)^2}{x^2+1}$.
(*d*) $\frac{x^2}{3}$.
(*e*) $\frac{a}{a+1}$.
(*f*) $\frac{m^2-mn+n^2}{m^2+mn+n^2}$.
(*g*) $\frac{x-1}{2x+3}$.
(*h*) $\frac{x-1}{2x-1}$.
(*i*) $\frac{2x-3}{5x+1}$.
(*j*) $\frac{a^2-1}{a^2+1}$.
(*k*) $\frac{a^2+3a+1}{a^2-1}$.
(*l*) $\left(\frac{x-2}{x-3}\right)^2$.
(*m*) $\frac{5x-3}{4x+1}$.
(*n*) $\frac{y+x}{y-x}$.
(*o*) $\frac{(x-a)^2}{x+a}$.
(*p*) $\frac{y^2(x^2-1)}{x^2(y^2-1)}$.

EXERCISE CXV—*continued*

2. $\dfrac{2-y-x}{x-y}$.

3. $\dfrac{2-2a+a^2}{2a-a^2}$.

4. (a) $\dfrac{b^2+ba+a^2}{b^2-ba+a^2}$.

(b) $\dfrac{x^4+x^2y^2+y^4}{x^2+y^2}$.

EXERCISE CXVI

1. 0·707.
2. 1·134.
3. 1·767.
4. 2.495.
5. 4·020.
6. 3·361.
7. 2·638.
8. $\frac{1}{2}a\sqrt{2}$.
9. 1·341.
10. 0·174.

EXERCISE CXVII

1. 0·414.
2. 11·196.
3. 0·318.
4. 0·356.
5. 3·436.
6. 5·828.
7. 0·172.
8. 9·898.
9. 0·102.
10. $\dfrac{x+\sqrt{ax}}{x-a}$.

EXERCISE CXIX

1. (a) A = 2, B = 3.
(b) A = 5, B = − 3.
(c) A = − 1, B = − 3.
(d) A = 3, B = 4.
(e) A = 4, B = 5.
(f) A = 2, B = − 7.
(g) A = 3, B = − 18.
(h) A = 6, B = − 3.
(i) A = − 2, B = 5.
(j) A = 1, B = − 1.

2. (a) A = 1, B = − 5, C = − 14.
(b) A = 6, B = 17, C = − 14.
(c) A = 1, B = 6, C = 11, D = 6.
(d) A = 1, B = − 6, C = 11, D = − 6.
(e) A = 4, B = 4, C = − 1, D = − 1.

EXERCISE CXX

1. (a) A = 1, B = 2.
(b) A = 1, B = 2.
(c) A = 2, B = 5.
(d) A = 3, B = 4.
(e) A = 2, B = 2.
(f) A = 2, B = − 3.
(g) A = 3, B = − 1.
(h) A = − 3, B = 4.
(i) A = 2, B = 1, C = 2.
(j) A = 2, B = 3.

2. (a) A = 2, B = 3.
(b) A = 5, B = − 1.
(c) A = 7, B = 9.
(d) A = 1, B = 2, C = 3.
(e) A = 1, B = 1, C = 1.

EXERCISE CXXI (Revision Exercise)

(A)

1. (a) $\dfrac{-x^2 - 18x + 103}{(2x - 5)(x - 1)(3x + 4)}$.
   (b) $\dfrac{2(x^2 + 5x + 5)}{(x + 2)(x + 4)}$.
2. 1·731.
3. (a) $x = 2$.
   (b) $x = 1, y = -4$.
4. $\frac{1}{3}(2y - x)$.
5. 16. 18, 20.

(B)

6. (a) 0·536.
   (b) 7·464.
7. 14.
8. (a) A = 3, B = − 1.
   (b) A = 2, B = − 4.
9. $(x + 2)(x - 2)(x + 3)$.
10. 7 and 4.

(C)

11. $\dfrac{p + q}{p - q}$.
12. $x = a$. $y = b$.
13. $\frac{4}{7}$.
14. $2x^2 + x - 3$.

(D)

16. 1·406.
17. (a) $2x(x^2 + 3y^2)$.
    (b) $2y(3x^2 + y^2)$.
18. $32\frac{8}{11}$ minutes past 6.
19. (a) A = 3, B = 4, C = − 4.
    (b) A = 3, B = − 4.
20. $x^2 - x - \dfrac{1}{x}$.

EXERCISE CXXIII

1. (a) $x = 1$ and 2.
   (b) $x = 2$ and 4
   (c) $x = 3$ and − 2.
   (d) $x = 2$ and − 4.
   (e) $x = 3$ and − 5.

EXERCISE CXXIV

1. (a) $x = 1$ and 2.
   (b) $x = 0$ and 3.
   (c) $x = 4$ and − 1.
   Minimum value of $-\frac{1}{4}$ given by $x = 1\frac{1}{2}$.
2. (a) $x = 2$ and 4.
   (b) $x = 5$ and 1.
   (c) $x = 0$ and 6.
   Minimum value of − 1 given by $x = 3$.
3. (a) $x = 3$ and − 1.
   (b) x 4 and − 2.
   (c) $x = 0$ and 2.
   Minimum value of − 4 given by $x = 1$.
4. $x = 1$ and − 2.
   Maximum value of $2\frac{1}{4}$ given by $x = -\frac{1}{2}$.
5. $x = 2$ and − 4.
   Maximum value of 9 given by $x = -1$.
6. Between $x = 1$ and $x = 2$.
7. Between $x = 2$ and $x = 4$.
8. Between $x = 2$ and $x = -4$.

EXERCISE CXXV

1. (*a*) 2 and 1.
(*b*) 3 and – 2.
(*c*) 3 and – 7.
(*d*) 2 and $\frac{1}{2}$.
(*e*) $\frac{3}{2}$ and – 2.
(*f*) 3 and $-\frac{7}{3}$.
(*g*) 7 and 9.
(*h*) $\frac{3}{2}$ and $-\frac{1}{5}$.
(*i*) $\frac{5}{3}$ and – 5.
(*j*) $-\frac{3}{2}$ and – 1.
(*k*) 8 and $-\frac{5}{2}$.
(*l*) 8 and $\frac{5}{2}$.
(*m*) $\frac{1}{7}$ and – 1.
(*n*) 2 and $-\frac{3}{4}$.
(*o*) $\frac{2}{3}$ and $\frac{3}{4}$.
(*p*) 0, 1, and 2.
(*q*) 0, 3, and – 7.
(*r*) – 1, – 2, and – 3.
(*s*) 1, 1, and – 1.
(*t*) $x = \pm\sqrt{\frac{5}{3}}$ or $\frac{1}{2}$.

2. (*a*) $x^2 - 5x - 14 = 0$.
(*b*) $x^2 + 4x - 5 = 0$.
(*c*) $6x^2 - 5x + 1 = 0$.
(*d*) $x^2 - (a + b)x + ab = 0$.
(*e*) $42x^2 + x - 1 = 0$.
(*f*) $pqx^2 - (p + q)x + 1 = 0$.
(*g*) $x^2 - 16x + 28 = 0$.
(*h*) $3x^2 - 10x - 25 = 0$.
(*i*) $8x^2 - 46x + 51 = 0$.
(*j*) $x^2 + 16x + 39 = 0$.
(*k*) $x^3 - 6x^2 + 11x - 6 = 0$.
(*l*) $x^3 + 5x^2 + 2x - 8 = 0$.
(*m*) $(x - a)(x - b)(x - c) = 0$.
(*n*) $24x^3 - 10x^2 - 3x + 1 = 0$.

3. (*a*) – 3 and – 5.
(*b*) 3 and 4.
(*c*) 1 and $\frac{5}{2}$.
(*d*) – 2 and – 5.
(*e*) 7 and 1.
(*f*) – 2 and $\frac{1}{3}$.
(*g*) 1 and $\frac{1}{2}$.
(*h*) – 1 and $\frac{2}{3}$.
(*i*) $\frac{2}{3}$ and $-\frac{1}{3}$.
(*j*) $\frac{1}{4}$ and $\frac{3}{4}$.

EXERCISE CXXVI (Revision Exercise)

(A)

1. (*a*) $x = 7$ and $y = -5$.
(*b*) $x = 8$ and $-\frac{3}{2}$.

2. (*a*) $x^2 - 5x + 6 = 0$.
(*b*) $x^2 - (\sqrt{2} + \sqrt{3})x + \sqrt{6} = 0$.

3. 10.

4. P = 19, Q = – 12; $3x^2 - 5x + 4$.

5. (*a*) A = 7, B = – 8.
(*b*) A = 2, B = 3.

(B)

6. $3x^2 - x + 3$.

7. 12·244.

8. (*a*) $(1 + 3a)(1 - 3a)(1 + 9a^2 + 81a^4)$.
(*b*) $(15x - 1)(3x + 8)$.
(*c*) $(5x^2 - 3)(2x - 7)$.

9. $x^2 - 2ax + a^2 - b^2 = 0$.

10. $-\frac{1}{2}$ and $-\frac{2}{3}$.

EXERCISE CXXVI—*continued*

(C)

11. $3 - 2\sqrt{6}$.

12. $2(2x-7)(x-1)$; $x = 1$ and $3\frac{1}{2}$.

13. $2p-5$, $2p-3$, $2p-1$, $2p+1$, $2p+3$.

14. 13, 15, 17, 19, 21.

15. 200; 8.

(D)

17. $p = -13$, $q = 42$.

18. $4x^2 - 24x - 45 = 0$.

19. $f = \dfrac{2(s-ut)}{t^2}$.

20. 8 and 9.

EXERCISE CXXVII

1. (*a*) 1.
(*b*) 4.
(*c*) 9.
(*d*) 16.
(*e*) 25.
(*f*) $30\frac{1}{4}$.
(*g*) $42\frac{1}{4}$.
(*h*) $6\frac{1}{4}$.
(*i*) $12\frac{1}{4}$.
(*j*) 36.

2. (*a*) $-1 \pm \sqrt{2}$.
(*b*) $-2 \pm \sqrt{7}$.
(*c*) $-3 \pm \sqrt{14}$.
(*d*) $-4 \pm \sqrt{14}$.
(*e*) 9 and 1.
(*f*) $\dfrac{-11 \pm \sqrt{141}}{2}$.
(*g*) $\dfrac{13 \pm \sqrt{137}}{2}$.
(*h*) -3 and -2.
(*i*) $\dfrac{7 \pm \sqrt{85}}{2}$.
(*j*) 10 and 2.

EXERCISE CXXVIII

1. (*a*) $\frac{1}{2}$ and -3.
(*b*) $\dfrac{-1 \pm \sqrt{21}}{10}$.
(*c*) $\dfrac{5 \pm \sqrt{73}}{4}$.
(*d*) $\frac{4}{3}$ and -1.
(*e*) 5 and $-\frac{3}{2}$.
(*f*) $\dfrac{-7 \pm \sqrt{65}}{8}$.
(*g*) $\frac{1}{6}$ and -1.
(*h*) $\dfrac{-5 \pm \sqrt{41}}{4}$.
(*i*) -2 and $\frac{1}{10}$.
(*j*) $\dfrac{2 \pm \sqrt{19}}{3}$.
(*k*) $\dfrac{-1 \pm \sqrt{85}}{14}$.
(*l*) $\dfrac{-3 \pm \sqrt{11}}{2}$.
(*m*) $\dfrac{-7 \pm \sqrt{17}}{8}$.
(*n*) 1 and $-\frac{1}{2}$.
(*o*) 1 and $-\frac{1}{-}$.

EXERCISE CXXVIII—*continued*

3. Minimum value -3.

4. Minimum value -5 when $x = 2$.

5. Minimum value -10 when $x = -4$.

7. Maximum value 3 when $x = -3$.

8. Maximum value 10 when $x = 2$.

9. Maximum value 8 when $x = -1$.

10. Maximum value 15 when $x = \frac{2}{3}$.

11. Minimum value -1 when $x = \frac{2}{3}$.

EXERCISE CXXIX

1. $-1 \pm \sqrt{2}$.

2. 1 and $-\frac{4}{3}$.

3. 5 and $\frac{3}{2}$.

4. -1 and $\frac{1}{6}$.

5. $\frac{-1 \pm \sqrt{33}}{4}$.

6. 1 and $\frac{2}{3}$.

7. $\frac{9 \pm \sqrt{129}}{8}$.

8. -1 and $\frac{1}{2}$.

9. $\frac{5 \pm \sqrt{109}}{6}$.

10. $\frac{1 \pm \sqrt{17}}{8}$.

11. -1 and $\frac{4}{3}$.

12. 5 and $-\frac{3}{2}$.

13. 1 and -3.

14. $\frac{3 \pm \sqrt{41}}{4}$.

15. 0 and $\frac{3}{4}$.

16. $\frac{1 \pm \sqrt{13}}{6}$.

17. 3 and -2.

18. $\frac{-1 \pm \sqrt{5}}{2}$.

19. -1 and $\frac{3}{2}$.

20. -1 and $\frac{1}{3}$.

EXERCISE CXXX

1. Imaginary.

2. Real and rational.

3. Real and rational.

4. Imaginary.

5. Real and irrational.

6. Real and irrational.

7. Real and rational.

8. Real and rational.

9. Real and irrational.

10. Real and irrational.

11. Real and irrational.

12. Real and irrational.

13. Imaginary.

14. Real and irrational.

15. Imaginary.

EXERCISE CXXXI

1. 2.

2. 5.

3. – 2.

4. – 23.

5. $3\frac{1}{11}$.

6. $\frac{1}{2}$.

7. $-\frac{8}{35}$.

8. – 3.

9. $5\frac{9}{23}$.

10. $\frac{5 \pm \sqrt{-5}}{6}$.

11. 1 and 4.

12. 5 and $\frac{1}{2}$.

13. $-4 \pm \sqrt{21}$.

14. $-16 \pm 3\sqrt{34}$.

15. $\frac{-15 \pm \sqrt{185}}{2}$.

16. $-20 \pm 3\sqrt{42}$.

17. 7 and $-\frac{1}{2}$.

18. $\frac{2}{5}$, 1, and $\frac{1}{4}$.

19. 6.

20. 2.

21. – 3.

22. $\frac{1}{2}(a + b)$.

23. 2.

24. $\frac{3}{4}$.

25. 2.

EXERCISE CXXXII

1. 12 and 14.

2. 8 and 12.

3. 17 and 8.

4. 19 inches and 12 inches.

5. 12 yards and 8 yards.

6. 800.

7. 13 inches.

8. 4, 6, and 8.

9. 3 inches.

10. 10 days and 12 days.

11. 30 m.p.h.; 40 m.p.h.

12. $1\frac{1}{2}d.$ each.

13. 15 boys.

14. 25 yards, 18 yards.

15. 1000 shares; £1 5*s.*

16. 14 and 15.

17. 4, 5, 6, 7, 8.

18. 10 minutes, 14 minutes.

19. 4 m.p.h.

20. 40 acres.

EXERCISE CXXXIII

1. (*a*) $x = 2, 3;\ y = 3, 2.$
(*b*) $x = 3, -2;\ y = 2, -3.$
(*c*) $x = 7, y = 2.$
(*d*) $x = 7, y = 2.$
(*e*) $x = 3, y = 3.$
(*f*) $x = 2, -6;\ y = -6, 2.$
(*g*) $x = -3, y = 4.$
(*h*) $x = 3, 4\tfrac{13}{17};\ y = 5, 3\tfrac{4}{47}.$
(*i*) $x = 8, 1;\ y = \frac{1}{2}, 4.$
(*j*) $x = 4, -\frac{5}{9};\ y = 5, \frac{4}{9}.$
(*k*) $x = 4, 6;\ y = 6, 4.$
(*l*) $x = \pm 1, y = \pm 1.$
(*m*) $x = -\frac{1}{4}, -3;\ y = 12, 1.$
(*n*) $x = 5, -5\tfrac{5}{17};\ y = 2, -4\tfrac{3}{17}.$
(*o*) $x = 2, -2;\ y = -3, 3.$

2. 6 inches and 8 inches.

3. 9 inches and 40 inches.

4. 7 and 3.

5. 7 and 5.

6. 15 yards and 25 yards.

7. 30 m.p.h.

11. $x = \pm 4, \pm 2;\ y = \pm 2, \pm 4.$

EXERCISE CXXXIV (Revision Exercise)

(A)

1. (*a*) $(4x - 3y)(2x + 5y).$
(*b*) $-3(2p + 5q)(4p - 3q).$

2. (*a*) $\frac{2}{3}$ and $-\frac{3}{5}.$
(*b*) 2 and $-13.$

3. 3 miles.

4. $2x^2 + x - 3.$

5. A minimum value of $-6\tfrac{1}{4}$ when $x = -1\tfrac{1}{2}.$

(B)

6. $3\tfrac{1}{2}.$

7. (*a*) Real and irrational.
(*b*) Real and rational.
(*c*) Imaginary.
(*d*) Real and rational.
(*e*) Imaginary.

8. (*a*) $\frac{5 \pm \sqrt{41}}{2}.$
(*b*) -1 and $\frac{1}{3}.$
(*c*) $\frac{1 \pm \sqrt{-39}}{10}.$
(*d*) $\frac{2}{3}$ and $-\frac{1}{5}.$
(*e*) $\frac{1 \pm \sqrt{-7}}{4}.$

9. 12 m.p.h.

10. $p = 14,\ q = -8.$

(C)

12. (*a*) $2x(x^2 + 12y^2).$
(*b*) $4y(3x^2 + 4y^2)$
$(3a - 2b)(2a + b).$

13. $x = -\frac{2}{3}$ and 3.
Minimum value $-10\tfrac{1}{12}$ for the value $x = 1\tfrac{1}{6}.$

14. 54 feet.

15. 81.

EXERCISE CXXXIV—*continued*

(D)

16. $2x^2 - 9x - 45 = 0$.

17. 7 and 8.

18. $\frac{5}{8}$.

19. 12.

20. 2 feet 2 inches, 10 inches.

ADDITIONAL EXERCISES

A 16

1. (*a*) -6.
(*b*) -19.
(*c*) 5.
(*d*) 2.
(*e*) 0.
(*f*) 38.
(*g*) -5.
(*h*) 106.
(*i*) -1.
(*j*) 8.

2. 0.

3. $2a + 8b - 12c$.

4. (*a*) $a^3 - b^3$.
(*b*) $a^2 - ab + b^2$.

5. (*a*) $x = \frac{1}{5}$.
(*b*) $x = 1$.

6. $2n, 2n + 1, 2n + 2, 2n + 3, 2n + 4$.
6, 7, 8, 9, 10.

7. Man 30 years, son 5 years.

8. £$\frac{1}{2}(y - x)$.
They will have £$\frac{1}{2}(x + y)$ each.

9. $p = 5, q = 2$.

10. (*a*) $2a + 19b$.
(*b*) $\frac{a^4 - b^4}{a^2b^2}$.

A 17

1. (*a*) -6.
(*b*) -1.
(*c*) $-\frac{2}{3}$.
(*d*) $\frac{3}{5}$.
(*e*) $-2\frac{17}{30}$.
(*f*) -12.
(*g*) 18.
(*h*) 36.
(*i*) -6.
(*j*) $-13{,}824$.

2. (*a*) $\frac{72}{x}$.
(*b*) $\frac{120}{x}$.

3. $6x^2 - xy - y^2$.

4. 3.

5. 7, 9, 11.

6. (*a*) 5.
(*b*) -1.
(*c*) -5.

7. (*a*) $\frac{7x + 17}{28}$.
(*b*) 1.

8. (*a*) $-\frac{1}{2}$.
(*b*) $x = 3, y = -2$.

9. 800.

10. $c(b - a - c)$.

ADDITIONAL EXERCISES—*continued*

A 18

2. (*a*) 4.
(*b*) $x = 2, y = 1$.

3. $1 + 2a - 4a^2 + 5a^3 - 2a^4$.

4. 42.

6. (*a*) $-\dfrac{a^2}{x^2 - a^2}$.
(*b*) x.

7. (*a*) $9p^2 - 12pq + 4q^2$.
(*b*) $4m - 1$.

8. $\dfrac{xy}{x + y}$ minutes.

9. $y = \dfrac{1}{a(a - 1)}$.

10. 8 florins and 5 sixpences.

A 19

1. 24.

2. 21·195.

3. $3x^2 + x - 4$.

4. (*a*) $\dfrac{2(3x^2 + 5x + 1)}{x(x + 1)(x + 2)}$.
(*b*) $\dfrac{11x + 7}{6}$.

5. (*a*) $x = 7$.
(*b*) $x = 1, y = 3, z = 5$.

6. 25 years; 70 years, 35 years.

7. 7, 9, 11, 13, 15.

8. H.C.F. is $8b^2$; L.C.M. is $672\,a^3b^3c^2$.

9. (*a*) $4a(2 + 3b - 4bc)$.
(*b*) $(p + 8q)(p - 8q)$.

10. Man £2 10*s*., boy 15*s*.

A 20

1. $p = -2, q = -1$.

2. (*a*) $(x^2 + 7)(x + 1)$.
(*b*) $(p - 2q)(p^2 + 2pq + 4q^2)$.
(*c*) $(2x - y)(x + y)$.

3. (*a*) $x = \dfrac{b^2 - ac}{a + c - 2b}$.
(*b*) $x = 2, y = -3, z = 5$.

4. $\dfrac{7}{15}$.

5. 4.

6. 25.

7. (*a*) A = 5, B = 4.
(*b*) A = 4, B = − 3.

8. (*a*) $yz + zx + xy$.
(*b*) $2(x + y + z)$.

9. $x = 2$.

10. $a = 7, b = -2$.

A 21

1. (*a*) $3(2x + 3y)(2x - 3y)$.
(*b*) $(3x - 5y)(2x + 3y)$.
(*c*) $(x + 1)(x - 2)(x + 3)$.

3. $\dfrac{17}{23}$.

4. (*a*) 2 and − 3.
(*b*) $x = 5, y = 9$.

5. Tea 2*s*. 6*d*., sugar 3*d*.

6. 5, 6, and 7.

7. (*a*) $3x^2h + 3xh^2 + h^3$.
(*b*) $12xh + 6h^2$.

8. (*a*) $\dfrac{9yz^4}{8x^3}$.
(*b*) $\dfrac{yz + zx + xy}{x + y + z}$.

9. 30 m.p.h. and 40 m.p.h.

10. 52.

ADDITIONAL EXERCISES—*continued*

A 22

1. (*a*) $(x-1)(x+1)(2x-1)$.
   (*b*) $\left(1+\frac{x}{3}\right)\left(1-\frac{x}{3}\right)$.
   (*c*) $(3x^2-2)(x^2+1)$.
2. (*a*) $x=3, y=-4$.
   (*b*) $x=\frac{3}{4}$ and $-\frac{5}{4}$.
3. (*a*) A = 2, B = 3.
   (*b*) A = 5, B = − 7.
4. $2x^2-x+3$.
5. $2x^2-7x+6=0$.
6. 6 inches and 9 inches.
7. $\frac{5x+2}{(3x-2)(2x-1)(4x+3)}$.
8. $\frac{1}{2}(7\sqrt{6}+4\sqrt{3}+3\sqrt{2})$
   $=14{\cdot}156$.
9. 19 half-crowns and 13 florins.
10. A 60 years, B 36 years.

A 23

1. (*a*) $3x^2-5x+4$.
   (*b*) $R=5x^4-4x^5$.
2. (*a*) $x=3, y=4$.
   (*b*) $-3\frac{1}{15}$.
3. (*a*) $p=-20, q=4$.
   (*b*) 17.
4. $\frac{1}{3}(4\sqrt{15}+\sqrt{6}-3\sqrt{10})$
   $=2{\cdot}818$.
5. (*a*) A = 4, B = − 5.
   (*b*) A = 3, B = − 4.
6. 14.
7. 64 feet.
10. If $k=6\frac{1}{4}$ the solutions are equal. If $k>6\frac{1}{4}$ the solutions will be imaginary.

A 24

1. (*a*) $\frac{1}{2}(-5\pm\sqrt{3})$.
   (*b*) $x=2, 5\frac{1}{3}$; $y=7, -4\frac{2}{3}$.
3. The expression has a minimum value of − 9 for $x=2$.
4. For real solutions $k>-9$. For imaginary solutions $k<-9$.
5. $\frac{12}{17}$.
6. 7 and 8.
7. $a+b+c+2\sqrt{ab}+2\sqrt{bc}+2\sqrt{ca}$.
8. Tea 2*s*. 6*d*. per lb., sugar 3*d*. per lb.
9. H.C.F. is $2(x-2)$; L.C.M. is $8x^2(x-2)$.
10. (*a*) A = 3, B = 2.
    (*b*) A = 6, B = 11, C = 6.
    (*c*) A = 10, B = 35, C = 50, D = 24.

ADDITIONAL EXERCISES—*continued*

A 25

1. (*a*) $(5x - 3y)(x + 2y)$.
(*b*) $(-a + 19b)(5a - 4b)$.

2. $4x + 5 - \frac{1}{x}$.

3. $(2x - 3)(x - 1)(3x - 5)$.

5. 150 seats at 1*s*. 6*d*. each and 200 at 1*s*.

6. (*a*) $x = 5$ and 0·375.
(*b*) $x = 1, y = 1$.

7. Maximum value of 7 for $x = 2$.

8. (*a*) $A = 3, B = -2$.

9. $t = \frac{v - u}{f}$.

10. 35 m.p.h. and 40 m.p.h.

A 26

1. $A = -13, B = -12, C = 30$.

2. (*a*) $10x^2 + x - 3 = 0$.
(*b*) $x = -\frac{7}{3}$, 4 and $\frac{9}{2}$.

3. (*b*) $t^2 + 8t + 7, 3t^2 + 4t + 7,$
$t^3 + 8t^2 + 9t + 6,$
$2t^3 + 7t^2 + 5t + 6,$
$3t^3 + 9t^2 + 7t + 8$.

4. (*a*) 201.
(*b*) 5307.
(*c*) 12,875.
(*d*) 19.
(*e*) 7733.

5. (*a*) $x = 6$.
(*b*) $x = 1$ and $-\frac{1}{2}$.

6. $R = 5x^4 + 4x^5$.

7. 2 feet.

8. 3 cm.

9. 11 and 9.

10. $\frac{20(3ax - 5p)}{p}$.

A 27

1. (*a*) $\left(x - \frac{3}{4}y^2\right)\left(x^2 + \frac{3}{4}xy^2 + \frac{9}{16}y^4\right)$.
(*b*) $(x + 1)(x - 2)(x - 3)$.

2. (*a*) -11.
(*b*) $\pm\sqrt{-11}$.

3. $2x^2 - 8x + 11$.

4. A minimum value for $x = 2$ of 9.

5. 4.

6. $\frac{xy}{2(x + y)}$ days.

7. (*a*) Real and rational.
(*b*) Imaginary.
(*c*) Real and rational.

10. $£\frac{1000}{1331} = 15s.$ (approx.).

ADDITIONAL EXERCISES—*continued*

A 28

1. (*a*) $-\frac{5}{28}$.
(*b*) 2·51 and − 0·51.

2. A has 17*s*., B has 22*s*., C has 25*s*.

4. $\frac{3}{x+3} - \frac{2}{x+1}$.

5. 3*d*. per lb.

6. − 32.

7. (*a*) 4.
(*b*) 14.

8. (*a*) $2x^2 + 5x - 33 = 0$.
(*b*) $x^2 - 2mx + m^2 - 1 = 0$.

9. $x = 1$ and − 2.

10. 24.

A 29

1. 0.

2. − 1.

3. (*a*) $x = 4$, $y = -3$.
(*b*) $x = 1, 2\frac{9}{11}$; $y = 2, -\frac{8}{11}$.

4. 2·618.

5. $(2x - 1)(x - 3)(x - 4)$.

6. $2x^2 + x - 6 = 0$.

7. 150 square feet.

8. 4 and 8.

9. 61 feet.

10. 5 inches, 12 inches, 13 inches.

A 30

1. $2x^2 + x + 1$.

2. $x = 2$, $y = -3$, $z = 5$.

3. Yes.

4. 14 years, 10 months.

5. $x = 3$.

6. (*a*) $x = -1·85$ and 1·35.
(*b*) $x = -1·79$ and 1·29.

7. $5x + 2$.

8. 24 letters and 20 postcards.

9. (*a*) $38\frac{2}{11}$ minutes past 7.
(*b*) $5\frac{5}{11}$ minutes past 7.
(*c*) $21\frac{9}{11}$ minutes past 7 and $54\frac{6}{11}$ minutes past 7.

10. 4; minimum value of − 24 for $x = \frac{5}{3}$.

SENIOR

EXERCISE I

1. (*a*) Real and rational.
   (*b*) Real and rational.
   (*c*) Real and irrational.
   (*d*) One solution is zero.
   (*e*) Equal in magnitude, opposite in sign, and irrational.
   (*f*) Real and irrational.
   (*g*) Real and irrational.
   (*h*) Imaginary.
   (*i*) Imaginary.
   (*j*) Imaginary.

2. (*a*) 5.
   (*b*) 5.
   (*c*) $-3\frac{1}{2}$.
   (*d*) $1\frac{2}{3}$.
   (*e*) 0.
   (*f*) -4.
   (*g*) 6.
   (*h*) $\frac{1}{3}$.
   (*i*) 2.
   (*j*) -1.

3. (*a*) -6.
   (*b*) 6.
   (*c*) $\frac{1}{2}$.
   (*d*) 0.
   (*e*) $-2\frac{1}{3}$.
   (*f*) -3.
   (*g*) -5.
   (*h*) $\frac{2}{3}$.
   (*i*) 3.
   (*j*) 1.

4. (*a*) $x^2 - 5x + 6 = 0$.
   (*b*) $x^2 - x - 6 = 0$.
   (*c*) $8x^2 - 10x - 33 = 0$.
   (*d*) $x^2 - 4kx - 21k^2 = 0$.
   (*e*) $x^2 + 7mx + 10m^2 = 0$.

5. 6, 9.

6. 6 or -6.

7. $2\frac{1}{4}$.

9. $-\frac{2q}{p}, \frac{r}{p}$.

EXERCISE II

1. (*a*) $\frac{\sqrt{b^2 - 4ac}}{a}$.
   (*b*) $\frac{b^2 - 2ac}{a^2}$.
   (*c*) $-\frac{b}{a}\left(\frac{b^2 - 3ac}{a^2}\right)$.
   (*d*) $-\frac{b\sqrt{b^2 - 4ac}}{a^2}$.
   (*e*) $\frac{\sqrt{b^2 - 4ac}(b^2 - ac)}{a^3}$.
   (*f*) $\frac{b^2 - 2ac}{ac}$.
   (*g*) $-\frac{b}{\sqrt{b^2 - 4ac}}$.
   (*h*) $-\frac{bc}{a^2}$.
   (*i*) $-\frac{b\sqrt{b^2 - 4ac}(b^2 - 2ac)}{a^4}$.
   (*j*) $\frac{a - b + c}{a}$.

EXERCISE II—*continued*

2. (*a*) p.
   (*b*) $\sqrt{p^2 - 4q}$.
   (*c*) q.
   (*d*) $p\sqrt{p^2 - 4q}$.
   (*e*) $\sqrt{p^2 - 4q}(p^2 - q)$.
   (*f*) $p(p^2 - 3q)$.
   (*g*) $\dfrac{p^2 - 2q}{q}$.
   (*h*) pq.

3. (*a*) $cx^2 + bx + a = 0$.
   (*b*) $c^2x^2 - (b^2 - 2ac)x + a^2 = 0$.
   (*c*) $ax^2 - (2a - b)x + (a - b + c) = 0$.
   (*d*) $acx^2 - 2(b^2 - 2ac)x + 4ac = 0$.
   (*e*) $a^3x^2 + b(b^2 - 3ac)x + c^3 = 0$.

EXERCISE III

1. (*a*) All values except those between 2 and 3.
   (*b*) All values.
   (*c*) Values between $\frac{1}{2}$ and -2.
   (*d*) Values between 3 and $-\frac{2}{3}$.
   (*e*) All values except those between $\frac{1}{2}$ and -3.

4. Values of x between -1 and $-\frac{2}{3}$ and values greater than $\frac{1}{2}$.

6. Values of x between $\frac{1}{2}$ and $-1\frac{1}{3}$.

9. Values of x between 1 and 2 and values greater than 3.

10. $0 < k < \frac{4}{3}$.

EXERCISE IV (Revision Exercise)

(A)

1. (*a*) $x = 4, y = -7$.
   (*b*) $x = 2, -2\frac{1}{3}$.

2. $p = 17, q = -24$; $(3a + 2)(4a - 5)(a + 2)$.

3. $\dfrac{pm + qn}{mn}$.

4. (*a*) $(4x + 3)(5x - 3)$.
   (*b*) $(4p + 8q + 3)(5p + 10q - 3)$.

5. $x^2 - 28x - 128 = 0$.

(B)

6. A = 1, B = 1, C = -2.

8. 10*d*.

9. (*a*) $-2p$.
   (*b*) q.
   (*c*) $4p^2 - 2q$. $qx^2 + 2px + 1 = 0$.

10. For values of x between -1 and -6. Minimum value $-6\frac{1}{4}$ when $x = -3\frac{1}{2}$.

EXERCISE IV (Revision Exercise)—*continued*

(C)

11. $(x + 2)(x - 3)$.

13. $-1\frac{14}{23}$.

14. £800 at 4 per cent. and £2200 at 5 per cent.

15. 80 feet.

(D)

16. $x = 1, 3\frac{1}{4}$; $y = 5, 3\frac{1}{2}$.

17. $p = 10$ or $p = -6\frac{4}{5}$.

18. 10 miles.

20. 15 cm. and 8 cm.

EXERCISE V

1. (*a*) x^5.
(*b*) p^{14}.
(*c*) k^9.
(*d*) a^8.
(*e*) m^5.
(*f*) b^{11}.
(*g*) n^8.
(*h*) r^{15}.
(*i*) t^{15}.
(*j*) h^8.

2. (*a*) a^7.
(*b*) x^2.
(*c*) $\frac{1}{2}m^{15}$.
(*d*) $9k^2 + 8k^3$.
(*e*) $72d^7$.
(*f*) $\frac{9}{8}d$.
(*g*) $\frac{5}{3}g^8$.
(*h*) $\frac{1}{3h^8}$.
(*i*) $\frac{125}{4l^4}$.
(*j*) $\frac{15}{7p^2}$.

3. (*a*) 8.
(*b*) 5.
(*c*) 5.
(*d*) m^2.
(*e*) p^4.
(*f*) ab.
(*g*) h^5.
(*h*) k^9.
(*i*) $32m^5$.
(*j*) 2·5.

4. (*a*) $4{\cdot}786 \times 10^5$.
(*b*) $5{\cdot}7 \times 10^9$.
(*c*) $6{\cdot}347 \times 10^5$.
(*d*) $8{\cdot}75 \times 10^6$.
(*e*) $9{\cdot}87 \times 10^{10}$.

EXERCISE VI

1. 2.
2. 2.
3. 4.
4. 2.
5. 3.
6. 7.
7. 2.
8. 5.
9. 4.
10. 2.
11. 4.
12. 8.
13. 9.
14. 16.
15. 25.
16. 64.
17. 27.
18. 16.
19. 36.
20. 49.
21. 1000.
22. 7.
23. 3125.
24. 64.
25. 3.
26. 512.
27. 343.
28. 216.
29. 5.
30. 243.

EXERCISE VII

1. $\frac{1}{9}$.
2. $\frac{1}{125}$.
3. $\frac{1}{27}$.
4. 32.
5. 512.
6. $-\frac{1}{k^3}$.
7. $\frac{1}{4m^2}$.
8. $\frac{m^2}{4}$.
9. $\frac{1}{4}$.
10. $\frac{1}{3}$.
11. 6.
12. 1.

EXERCISE VIII

(General Exercise on Indices)

1. (*a*) $\frac{3}{p^{\frac{1}{2}}}$.

(*b*) $\frac{7}{m^{\frac{4}{5}}}$.

(*c*) $\frac{5b^2}{a^3}$.

(*d*) $\frac{7x}{y^5}$.

(*e*) $\frac{1}{h^8}$.

(*f*) $\frac{1}{h^8}$.

(*g*) $\frac{\sqrt{3}}{d}$.

(*h*) $\frac{1}{x^2}$.

(*i*) ab.

(*j*) $\frac{3q}{p}$.

2. $\frac{1}{8}, \frac{1}{4}, \frac{1}{2}, 1, 2, 2\sqrt{2}, 4, 4\sqrt{2}, 8$.

4. $x = 1, 4$.

5. $x = 1\frac{1}{3}$ or $-\frac{1}{2}$.

6. 2.

7. (*a*) $\frac{1}{x^{\frac{3}{2}}} + \frac{1}{x^2}$.

(*b*) $\frac{3}{m^3} + 7m^2$.

(*c*) $x + x^{\frac{3}{4}} + 2x^{\frac{1}{2}} + x^{\frac{1}{4}} + 1$.

(*d*) $6x + 3x^{\frac{2}{3}} + 10x^{\frac{1}{3}} + 5$.

(*e*) $x^{\frac{3}{2}} + 2x^{\frac{1}{2}} + x^{-\frac{1}{2}} - 3x - 6 - 3x^{-1}$.

(*f*) $y^{-1} + x^{-2}y^{-2} + x^{-1}$.

(*g*) $25p^{\frac{5}{4}} + 5p^{\frac{1}{2}}q^{\frac{1}{2}}(1 + p^{\frac{1}{4}}q^{\frac{1}{4}}) + q^{\frac{5}{4}}$.

(*h*) $x - 1$.

(*i*) $a^{-2} + 2a^{-1} + 1$.

8. (*a*) $x + x^{\frac{1}{2}} + 1$.

(*b*) $3x^{\frac{1}{2}} - x^{\frac{1}{4}}$.

(*c*) $5x^{-\frac{1}{3}} + 2 + x$.

(*d*) $a^{\frac{2}{5}} - 3b^{\frac{3}{5}}$.

(*e*) $a^{\frac{1}{3}} - b^{\frac{2}{3}}$.

9. (*a*) $a^{\frac{1}{3}} + 2b^{\frac{1}{3}}$.

(*b*) $1 - 3x^{\frac{1}{5}} + x^{\frac{2}{5}}$.

(*c*) $a^{\frac{4}{7}} + 3a^{\frac{3}{7}} - 5a^{\frac{2}{7}}$.

(*d*) $2a^{\frac{1}{3}} + 3a^{\frac{2}{3}} - a$.

(*e*) $3x^{-\frac{1}{5}} + y^{-\frac{1}{3}}$.

EXERCISE IX

1. 3, 5, 8, 10.

2. 0, 1, 3.

3. 4, 2, 1, 0, − 1, − 2.

4. 2, 3, 5.

9. (*a*) 5.
(*b*) 2.
(*c*) 3.

10. (*a*) 2.
(*b*) 9.
(*c*) 5.

EXERCISE X

3. (*a*) $\log_a lmn$.
(*b*) $\log_a \frac{lm}{n}$.
(*c*) $\log_a \frac{l}{mn}$.
(*d*) $\log_a \sqrt{mn}$.
(*e*) $\log_a \sqrt{m+n}$.

5. (*a*) $m = n^3$.
(*b*) $m = \frac{1}{n^3}$.
(*c*) $mn = a$.
(*d*) $m = an$.
(*e*) $m = an^2$.

6. (*a*) m. (*b*) $\frac{1}{m}$. (*c*) a.

EXERCISE XI

1. 3·6396, 2·6396, 1·6396, $\bar{1}$·6396, $\bar{2}$·6396, $\bar{3}$·6396.

2. 2·1937, ·1937, $\bar{1}$·1937, $\bar{5}$·1937.

3. 24·59, 245·9, 24590, ·2459, ·002459.

4. 68420, 68·42, ·6842, ·006842.

5. 2·8994, 1·8994, $\bar{3}$·8994, 1·7988, ·4497, $\bar{1}$·4497.

6. $\bar{2}$·1666.

7. $\bar{1}$·1213, $\bar{2}$·2426, $\bar{2}$·1213, $\bar{2}$·5606, $\bar{4}$·1213.

8. 95410, 954·1, ·009541, ·00009541.

9. ·7781, 1·0791, 1·2552, 1·5562, 1·3801.

10. ·5395, ·4184, $\bar{2}$·5562, 2·7602.

EXERCISE XII

1. (*a*) 1·4085.
(*b*) 1·8740.
(*c*) 2·9290.
(*d*) $\bar{1}$·8839.
(*e*) 2·9613.
(*f*) 5·9125.
(*g*) 3·9499.
(*h*) ·8648.
(*i*) $\bar{3}$·7882.
(*j*) 1·9144.

2. (*a*) 78·45.
(*b*) ·7521.
(*c*) 240·6.
(*d*) 1328.
(*e*) ·01290.
(*f*) ·2892.
(*g*) 3·761.
(*h*) 43·78.
(*i*) ·1777.
(*j*) ·02589.

3. (*a*) 2221.
(*b*) 29·09.
(*c*) 6·200.
(*d*) ·01314.
(*e*) ·0001726.
(*f*) ·1147.

EXERCISE XII—*continued*

4. 1·772.

5. (*a*) 1·58.
(*b*) 1·88.
(*c*) 3·71.

6. 428·5.

7. (*a*) 1862.
(*b*) 1·659.
(*c*) 43·38.
(*d*) ·1944.
(*e*) 1·632.
(*f*) 144·1.
(*g*) 2815.
(*h*) 9·067.
(*i*) 4·777.
(*j*) 36·43.
(*k*) 1835.
(*l*) 50·93.
(*m*) 59·46.
(*n*) 2·816.
(*o*) 13·07.
(*p*) 7·854.
(*q*) 4332.
(*r*) 5143.
(*s*) 57·8.

EXERCISE XIII

1. (*a*) 83·85.
(*b*) 3·258.
(*c*) ·2219.
(*d*) ·006177.
(*e*) ·4284.
(*f*) 2·507.
(*g*) ·1142.
(*h*) 53·51.
(*i*) 20·77.
(*j*) $7{\cdot}695 \times 10^{-5}$.

2. 47·73 cu. in.

3. 1·869 sec.

4. 147·2 cu. in.

5. 404·1 c.c.

6. 19·80 in.

7. 3·151 in.

8. 90·16 gal.

9. 7·191 ft.

10. 105 h.p.

11. 1951.

12. ·9217.

13. 16·47 lb.

14. 127 lb. weight.

15. 394·9.

16. 1·4102.

17. £3.

18. ·3549.

19. $2{\cdot}240 \times 10^{12}$.

EXERCISE XIV (Revision Exercise)

(A)

1. (*a*) $x = 2$.
(*b*) $x = 2, 2\frac{2}{7}$; $y = -2, -1\frac{3}{7}$.

2. (*a*) $49\frac{1}{11}$ minutes past nine.
(*b*) $16\frac{4}{11}$ minutes past nine.

3. 6; minimum value -6 when $x = 1\frac{1}{3}$.

4. $p = 6$, $q = 9$.

5. $7\frac{1}{2}$.

(B)

6. $x^2 - 6x + 8 = 0$.

7. 162, 48, 1296.

8. (*a*) 1.
(*b*) 5.

9. 5 min., 7 min.

10. $cx^2 + 2bx + a = 0$.

EXERCISE XIV (Revision Exercise)—*continued*

(C)

11. $x^2 - 9$.

12. $\dfrac{1}{2x + 3} + \dfrac{2}{x + 1}$.

13. (*a*) 4·127.
(*b*) 1·699.

14. 4, 7.

15. $-(4 + \sqrt{3}) = -5{\cdot}73$.

(D)

17. $4\frac{3}{4}$.

18. (*a*) 4·132.
(*b*) ·2419.

19. 2 cm.

20. (*a*) $\frac{1}{4}$.
(*b*) $k > \frac{1}{4}$.

EXERCISE XVI

1. (*a*) 2, $2n$, 16.
(*b*) 3, $3n - 1$, 23.
(*c*) 7, $7n - 6$, 50.
(*d*) -3, $5 - 3n$, -19.
(*e*) -7, $10 - 7n$, -46.
(*f*) a, na, $8a$.
(*g*) $-x^2$, $(6 - n)x^2$, $-2x^2$.
(*h*) 0, 2, 2.

2. (*a*) 4.
(*b*) 11.
(*c*) 5.
(*d*) -1.
(*e*) -9.

3. (*a*) 5, 8, 11.
(*b*) 7, 11, 15, 19.
(*c*) 9, 11, 13.
(*d*) 0, -4, -8, -12.
(*e*) -3, 2, 7.

4. 7.

5. 5, 12, 19, 2798.

6. (*a*) 3, 6, 9.
(*b*) 8, 11, 14.
(*c*) -2, 1, 4.
(*d*) 3, 12, 27.
(*e*) 3, $1\frac{1}{2}$, 1.
(*f*) $\frac{1}{3}$, $\frac{2}{3}$, 1.
(*g*) 3, 9, 27.
(*h*) 2, $1\frac{1}{2}$, $1\frac{1}{3}$.
(*i*) 2, 5, 10.
(*j*) -1, -7, -17.

7. 29, 4.

EXERCISE XVII

1. (*a*) 78.
(*b*) 300.
(*c*) 720.
(*d*) 642.
(*e*) -228.
(*f*) -168.
(*g*) $67\frac{1}{2}$.
(*h*) $456a$.

2. (*a*) $n(n + 1)$.
(*b*) $n(2n + 1)$.
(*c*) $n(7 - n)$.
(*d*) $nx(2n - 1)$.
(*e*) $\dfrac{n(n + 15)}{8}$.
(*f*) $\dfrac{n}{2}\left\{2a + (n - 1)d\right\}$.

EXERCISE XVIII

1. (a) 555:
(b) 855.
(c) – 30.
(d) – 555.
(e) – 15.
(f) $153\frac{3}{4}$.
(g) $307\frac{1}{2}$.
(h) – 78.

2. (a) $\frac{n}{2}(5n-1)$.
(b) $n(4n-3)$.
(c) $\frac{n}{2}(11-n)$.
(d) $n(8-3n)$.
(e) $\frac{n}{2}(73-5n)$.
(f) $\frac{n}{8}(7+5n)$.
(g) $\frac{n}{4}(37+3n)$.
(h) $\frac{n}{20}(61-11n)$.

3. 3, 39.

4. 335.

5. 735.

6. 10.

7. 3, $\frac{n}{2}(3n+17)$.

8. 3, 11, 19, 27, . . .

9. 7, 11, 15, 19.

10. £250, £1950.

11. 2, 5, 8, 11.

12. 5.

13. 3, 4.

14. 1875.

15. 148, $\frac{n}{2}(3n+13)$.

16. $x = 12$, $y = 19$.

17. $n(b-a)+2a-b$,
$\frac{n}{2}\left\{n(b-a)+3a-b\right\}$.

19. 3, 5, 7, 9.

20. $a, \frac{7}{4}a, \frac{10}{4}a, \frac{13}{4}a$; 10: 13.

21. 26, 33, 40, 47.

24. 24.

EXERCISE XIX

1. (a) 2, 2^{n-1}, 128.
(b) 3, $2 \times 3^{n-1}$, 4374.
(c) – 3, $2 \times (-3)^{n-1}$, – 4374.
(d) $\frac{2}{3}, \frac{2^{n-1}}{3^{n-2}}, \frac{128}{729}$.
(e) $-\frac{3}{2}, \frac{(-3)^{n-1}}{2^{n-2}}, \frac{-2187}{64}$.
(f) $\frac{1}{4}, \frac{1}{4^{n-2}}, \frac{1}{4096}$.
(g) x, x^{n-1}, x^7.
(h) r, ar^{n-1}, ar^7.
(i) $-\frac{1}{r}, \frac{a}{(-r)^{n-1}}, -\frac{a}{r^7}$.

2. (a) ± 10.
(b) ± 12.
(c) 4.
(d) 3.
(e) 144.

3. (a) 6, 18, 54 or – 6, 18, – 54.
(b) 3, 9, 27, 81.
(c) 9, 27, 81 or – 9, 27, – 81.
(d) 2, 1, $\frac{1}{2}$, $\frac{1}{4}$.
(e) 2, $\frac{4}{3}$, $\frac{8}{9}$ or – 2, $\frac{4}{3}$, $-\frac{8}{9}$.

4. 1, 3.

5. 2, 6, 18,

EXERCISE XIX—*continued*

6. $3, 1\frac{1}{2}, \frac{3}{4}, .$

7. 16.

9. 1 : 81.

10. $\left(\frac{b}{a}\right)^{n-3}, \frac{b^2}{a^2}.$

11. $\frac{b^2}{a}, \frac{b^{n-1}}{a^{n-2}}.$

EXERCISE XX

1. 510.

2. 1093.

3. $7\frac{127}{128}$.

4. $5\frac{29}{32}$.

5. $13\frac{2}{81}$.

EXERCISE XXI

1. (*a*) 511.
(*b*) $3\frac{127}{128}$.
(*c*) $105\frac{139}{729}$.
(*d*) $3\frac{1}{6561}$.
(*e*) $\frac{1-x^9}{1-x}$.

2. (*a*) $2^n - 1$.
(*b*) $4 - \frac{1}{2^{n-2}}$.
(*c*) $108 - \frac{2^{n+2}}{3^{n-3}}$.
(*d*) $3 + \frac{1}{(-3)^{n-1}}$.
(*e*) $\frac{1-x^n}{1-x}$.

3. 8.

4. 2, 6, 18, 59,048

5. $-13(\sqrt{3}-1)$.

6. 7.

7. 363.

8. 2, 6, 18, or -4, 12, -36, .

11. 9, $40\frac{1}{2}$.

12. $\frac{b^{n-1}}{a^{n-2}}, \frac{a^n - b^n}{a^{n-2}(a-b)}.$

13. $4 \times 5^{n-1}$, 6.

14. 9, 27, 81, . or $-4\frac{1}{2}$, $13\frac{1}{2}$, $-40\frac{1}{2}$, .

15. 7.

16. $23\frac{17}{1024}$.

17 1, 5, 9 . . or 31, 5, -21 .

19. $1\frac{5}{27}$ ft.

20. 8, 2, $\frac{1}{2}$, .

EXERCISE XXII

1. (*a*), (*d*), (*e*).

2. 12, 25, $2\frac{7}{9}$.

3. $\frac{1}{512}$.

4. 16.

5. $\frac{a^2}{a-b}$.

6. (*a*) $|x| < \frac{1}{3}$.
(*b*) $|x| < \frac{1}{3}$.

7. (*a*) $\frac{1}{1-3x}$.
(*b*) $\frac{1}{1+3x}$.

8. (*a*) $x = -\frac{1}{3}$.
(*b*) $x = \frac{1}{3}$.

EXERCISE XXIII

1. (*a*) $\frac{5}{10} + \frac{5}{10^2} + \qquad \frac{5}{9}.$

   (*b*) $3 + \frac{14}{10^2} + \frac{14}{10^4} +$

   $3\frac{14}{99}.$

   (*c*) $5\frac{7}{10} + \frac{14}{10^3} + \frac{14}{10^5} + \ldots;$

   $5\frac{707}{990}.$

   (*d*) $5 + \frac{714}{10^3} + \frac{714}{10^6} +$

   $5\frac{238}{333}.$

   (*e*) $8\frac{4}{10} + \frac{27}{10^3} + \frac{27}{10^5}$

   $+ . \qquad 8\frac{47}{110}.$

EXERCISE XXIV

1. £162 17*s*. 10*d*.
2. 3·8 per cent.
3. £77 14*s*.
4. 17*s*. 3*d*.
5. £91 12*s*.
6. 4·1 per cent.
7. 18 years.
8. £73.
9. £307.
10. £6500.

EXERCISE XXV

1. (*a*) $\frac{1}{6}, \frac{1}{n+1}.$

   (*b*) $\frac{2}{11}, \frac{2}{2n+1}.$

   (*c*) $\frac{3}{2}, \frac{3}{12-2n}.$

   (*d*) $\frac{6}{7}, \frac{6}{22-3n}.$

2. (*a*) $\frac{2}{5}, \frac{2}{7}, \frac{2}{9}, \frac{2}{11}.$

   (*b*) $\frac{3}{8}, \frac{1}{2}, \frac{3}{4}, 1\frac{1}{2}.$

   (*c*) $\frac{3}{5}, \frac{2}{3}, \frac{3}{4}, \frac{6}{7}.$

EXERCISE XXVI

1. (*a*) 1, 5, 9.
   (*b*) 12, 17, 22.
   (*c*) 1, 4, 9.
   (*d*) 4, 16, 36.
   (*e*) 9, 25, 49.
   (*f*) 2, 6, 12.
   (*g*) 2, 6, 12.
   (*h*) 2, 10, 30.
   (*i*) 4, 14, 30.
   (*j*) 4, 18, 48.

2. (*a*) $n(2n+1).$
   (*b*) $n(2n-1).$
   (*c*) $n(n+1)(n+2).$
   (*d*) $(2n-1)^2.$
   (*e*) $4n^2.$
   (*f*) $n^2(n+1).$
   (*g*) $(2n-1)(2n+1)(2n+3).$
   (*h*) $8n^3.$

EXERCISE XXVI—*continued*

3. (a) $\dfrac{n(n+1)(4n+5)}{6}$.

(b) $\dfrac{n(n+1)(4n-1)}{6}$.

(c) $\dfrac{n(n+1)(n+2)(n+3)}{4}$.

(d) $\dfrac{n(2n-1)(2n+1)}{3}$.

(e) $\dfrac{2n(n+1)(2n+1)}{3}$.

(f) $\dfrac{n(n+1)(n+2)(3n+1)}{12}$.

(g) $n(n+2)(2n^2+4n-1)$.

(h) $2n^2(n+1)^2$.

4. 2485.

6. 91.

7. 56.

9. 730.

10. 9170.

EXERCISE XXVII

(General Revision Exercise on Series)

1. (a) $\dfrac{n(3n+1)}{4}$, $77\frac{1}{2}$.

(b) $\dfrac{n(26-5n)}{4}$, -60.

(c) $\dfrac{3^{n+1}}{2^{n-1}}-6$, $339\frac{507}{512}$.

(d) $\dfrac{(-3)^{n+1}}{5.2^{n-1}}+\dfrac{6}{5}$, $-67\frac{511}{512}$.

(e) $\dfrac{n+1}{2}$, $\dfrac{55}{n}$.

(f) $\dfrac{3n(3n+11)}{40}$, 30·75.

2. (a) $4n+1$.
(b) $n(4n+1)$.

3. (a) $n(2n+3)$, 230.

(b) $\dfrac{n(n+1)(8n+7)}{6}$, 1595.

4. 2, 7, 12, . 990.

5. 97.

6. 2, $1\frac{1}{3}$, $\dfrac{8}{9}$, $\dfrac{16}{27}$.

7. 25.

8. $\dfrac{4}{11}$, $\dfrac{1}{4}$, $\dfrac{4}{21}$.

10. $1\frac{1}{4}$ miles.

11. £6900.

12. 159.

13. $1\frac{1}{4}$, $2\frac{1}{2}$, 5, 10, .

14. 20.

15. 6.

16. $|x| < 1$, $\dfrac{x}{(1-x)}$.

17. $p = 12$, $q = 14$ or $p = 2\frac{1}{12}$, $q = -5\frac{5}{6}$.

18. 14.

19. 18 years.

20. £53 19*s*.

23. $x = \dfrac{7}{8}$.

26. $\dfrac{1}{81}\left\{10(10^n-1)-9n\right\}$.

27. 1,234,567,900.

28. nx^{n-1}; $\dfrac{1-x^n}{(1-x)^2}-\dfrac{nx^n}{1-x}$.

29. $|x| < 1$; $\dfrac{1}{(1-x)^2}$.

30. $\dfrac{3}{10}+\dfrac{85}{10^3}+\dfrac{85}{10^5}+\ldots$; $\dfrac{191}{495}$.

EXERCISE XXVIII (Revision Exercise)

(A)

1. (*a*) $x = 2\frac{1}{13}$.
   (*b*) $x = 4, -44$; $y = 1, -71$

3. $\dfrac{12x + qy}{12(p + q)}$ pence.

4. 9·003, ·1111.

5. $\dfrac{2b + c}{a}$.

(B)

6. $6x^2 - 19x + 15 = 0$.

7. 0·65.

8. (*a*) $(x^{\frac{1}{4}} - y^{\frac{1}{3}})(x^{\frac{1}{4}} + y^{\frac{1}{3}})$.
   (*b*) $(x^{\frac{1}{4}} - y^{\frac{1}{5}})(x^{\frac{1}{2}} + x^{\frac{1}{4}}y^{\frac{1}{5}} + y^{\frac{2}{5}})$.

9. -128.

10. ·8507.

(C)

11. 163,845.

12. £800, £750.

13. 1240.

14. $x = -\frac{1}{3}$.

15. 169.

(D)

16. $\dfrac{1 - (2n - 1)x^n}{1 - x} + \dfrac{2x - 2x^n}{(1 - x)^2}$;
    $|x| < 1$, $\dfrac{1 + x}{(1 - x)^2}$.

17. $\dfrac{p^2 - 2q}{q}$; $qx^2 - (p^2 - 2q)x + q = 0$.

18. 32 : 1.

19. $p = 2$, $q = 3$;
    $\dfrac{n(n + 1)(n^2 + n + 3)}{2}$.

EXERCISE XXIX

1 (*a*) 40 : 9.
  (*b*) 83 : 54.
  (*c*) 154 : 69.
  (*d*) 22 : 57.
  (*e*) 1 : 4840.
  (*f*) 15 : 7.
  (*g*) 21 : 1.
  (*h*) 100 : 1.
  (*i*) 27 : 20.
  (*j*) 200 : 3.

2. (*a*) 3 : 5.
   (*b*) 11 : 17.
   (*c*) 8 : 25.
   (*d*) 18 : 5.
   (*e*) 1 : 12.
   (*f*) 2 : 1.
   (*g*) 37 : 43.
   (*h*) 18 : 1.

3. $\frac{2}{3}$.

4. (*a*) $-3 : 1$.
   (*b*) 8 : 1.
   (*c*) 1 : 4.
   (*d*) $-13 : 1$.
   (*e*) 9 : 1.
   (*f*) $-27 : 1$.
   (*g*) 14 : 9.
   (*h*) $-5 : 3$.

EXERCISE XXIX—*continued*

5. (*a*) 5 : 1.
(*b*) 1 : 6.
(*c*) 9 : 14.
(*d*) $1 : \sqrt{5}$ or $-1 \quad \sqrt{5}$.

6. 12, 30.

7. 42, 98.

8. 6, 8.

9. 40, 15.

10. $2\sqrt{2} : 1$.

11. (*a*) 7 : 13.
(*b*) 49 : 169.

12. (*a*) 49 : 169.
(*b*) 343 : 2197.

13. 8 cm., 12 cm., 16 cm.

14. 11 : 5.

15. 14 cm., 26 cm.

16. $\frac{1}{4}$.

EXERCISE XXX

1. (*a*) $\frac{2}{3} = \frac{10}{15}$.

(*b*) $\frac{3}{5} = \frac{21}{35}$.

(*c*) $\frac{4}{7} = \frac{36}{63}$.

(*d*) $\frac{3}{8} = \frac{4}{10\frac{2}{3}}$.

(*e*) $\frac{7}{9\frac{11}{12}} = \frac{12}{17}$.

(*f*) $\frac{8}{3} = \frac{26\frac{2}{3}}{10}$.

(*g*) $\frac{5}{5\frac{5}{8}} = \frac{8}{9}$.

(*h*) $\frac{x}{y} = \frac{k}{\frac{ky}{x}}$.

2. (*a*) $11\frac{2}{3}$.
(*b*) 6.
(*c*) $\frac{yz}{x}$.

3. (*a*) ± 4.
(*b*) ± 6.
(*c*) $\sqrt{xy}$.

5. $d : c$.

6. $2c : 3d$.

EXERCISE XXXI

6. $x = 2, y = 3$.

8. 2.

9. 1 : 2 : 3.

11. 9, 15, 21.

13. $x = 4, y = -7$.

15. $k = \sqrt{a^2 + b^2}$.

EXERCISE XXXII

1. 12.
2. 48.
4. $19\frac{1}{11}$ in.
5. $12\frac{3}{4}$.
6. $143\frac{148}{168}$ cu. in.
7. (*a*) 1 : 9 : 25.
   (*b*) 1 : 27 : 125.
8. Missing values: $\frac{1}{2}$, 4, $13\frac{1}{2}$, $62\frac{1}{2}$, 108.
9. $\dfrac{47p}{40}$ pence.
10. 4·32 in.
11. $13\frac{1}{3}$ lb.
13. 8 lb.
14. 1 : 4.
15. 156·48 in.

EXERCISE XXXIII

1. $2\frac{4}{5}$.
2. $7\frac{1}{2}$.
3. (*a*) $xy = 15$.
4. 15·875 dynes.
5. $y(6{\cdot}4 - x) = 16{\cdot}8$.
6. $y + 4 \propto \dfrac{1}{x+2}$.
7. $1\frac{1}{4}$ cu. ft.
8. 108 lb. approx.
9. ·28 sec.
10. Since yz is constant, y will be diminished to a third of its value.
11. $x = 6, -4$; $y = 2, -3$.

EXERCISE XXXIV

1. (*a*) $y = kpq$.
   (*b*) $y = kp\sqrt{q}$.
   (*c*) $y = \dfrac{kp}{q}$.
   (*d*) $y = \dfrac{kp^2}{\sqrt{q}}$.
   (*e*) $y = \dfrac{kpq}{r}$.
2. $18\frac{6}{7}$ cu. in.
3. 1500 ft.-lb.
4. 215·9 grammes.
5 ·72.
6. $20\frac{1}{4}$ cwt.
7. $52\frac{1}{2}$ lb.
8. $P = 15n - 450$ (where £P is the profit).
9. £1800.
10. $y = kx(10 - x)$; $x = 11, -1$; $z = -1, 11$.
11. $-10\frac{1}{2}$.
12. 4 : 3.
13. 576 ft., 176 ft.
14. 176 cu. ft.
15. $6\frac{3}{4}$ ft.

EXERCISE XXXV (Revision Exercise)

(A)

1. $\dfrac{4a^2}{(1 + a^2)^2}$.

2. (*a*) $x = -\frac{2}{3}, 1$.
(*b*) $x = \frac{1}{6}, 1$.

4. 26,400.

5. 625.

(B)

6. (*a*) $x = 3, 1;\ y = -2, 2$.
(*b*) 2·771.

7 12.

8. 280 yards.

10. 2.

(C)

11 (*b*) $\dfrac{a^2 + b^2 + c^2 - ab - bc - ca}{(a - b)(b - c)(c - a)}$.

12. 255,200.

14. $x : y = 3 : 4$ or $-2 : 3$.

15. $\dfrac{4}{7}$.

(D)

16. A = 2, B = – 5.

17. $a^4 = 17^5c^9$; 953·9.

18. $n;\ \dfrac{n(n^2 + 1)}{2}$.

19. $(3x + 1)(4x - 1)(2x - 3)$.

20. $\sqrt{2} : 1;\ \sqrt[3]{2} : 1$.

EXAMINATION QUESTIONS

1. (*a*) $t = \dfrac{(p+q)(x+y)}{2y}$.

(*b*) $p = \dfrac{2yt - q(x+y)}{x+y}$.

2. 4·52; $V = -\dfrac{10{\cdot}6mT}{12T - 76}$; 45·47.

3. $\dfrac{d(x+y)}{xy}$ hours.

4. $x = -\dfrac{2(y+1)}{y-1}$;

$\dfrac{x^2 - 4}{x^2 + 4} = \dfrac{2y}{y^2+1}$.

5. (*a*) $t = \dfrac{b(a-3)}{c(a-1)}$.

(*b*)

6. 271; $r = \dfrac{3V + \pi h^3}{3\pi h^2}$.

7. $\dfrac{nlb}{a}\left(\dfrac{100-x}{100}\right)$.

8. £$\dfrac{3xz}{5}$.

10. $\dfrac{12x}{12u+v}$.

11. $\dfrac{100(b-a)}{a}$.

12. $\dfrac{ac}{2b}$.

13. (*a*) $\dfrac{b(20-c)}{20}$.

(*b*) $t = \dfrac{273(pv - cd)}{cd}$.

14. (*a*) $n = \dfrac{40P - 3N}{4}$.

(*b*) $T = t\left(1 + \dfrac{v}{a\sqrt{h}}\right)$.

15. (*a*) $(3x+8)(4x-9)$.
(*b*) $(8x-5y)(3y-4x)$.
(*c*) $(x+1)(x+4)(x-3)$.

16. Yes.

17. (*a*) $(3x-2)(2x-3)$.
(*b*) $2(a+3b-3c)(a-3b+3c)$.
$a = 10$; $(x-1)(x-2)$.

18. (*a*) $2(3ab+5)(5ab-3)$.
(*b*) $2x(x^2+12)$.
(*c*) $(ac-b)(3b+c)$.

19. (*a*) (i) $(6x-5)(x+2)$,
(ii) $(a-2b)(a-2)$,
(iii) $(x+1)(x-1)(2x+1)$.
(*b*) $a = -7$, $b = -2$;
$(x-2)(2x+1)^2$.

20. $(3y+2)(y+10)$.

21. (*a*) $(x-7)(x+3)$.
(*b*) $(6x-5y)(3x+4y)$.
(*c*) $c = 6$ or -14.

22. (*a*) $(3x+2)(x-3)$.
(*b*) $x(x+y)(a-b)$.
(*c*) $\{(a+b)x + (a-b)\}\{(a-b)x - (a+b)\}$.

23. $(x+2)(x-2)$,
$x(x+1)(x+3)$, $(x+2)^2$,
$(x-2)(x+1)$, $\dfrac{x(x+3)}{x+2}$.

24. (*a*) $a = -13$.
(*b*) $(x+2)(x-3)(3x-1)$.

25. $(3x+7)(x+3)$; 307, 103.

26. (*a*) $(a+2b)(l-3m)$.
(*b*) $(3p-5q)(2p+q)$.
(*c*) $(a-b)(2a-b+c)$.

27. (*a*) $(x-1+y)(x-1-y)$.
(*b*) $(b-3d)(a+k)$.
$p = 9$.

28. (a) $2x + 7$.
(b) $c = 105$.

29. $\dfrac{4}{(x-2)^2}$.

30. $\dfrac{6}{2x-3}$.

32. (a) $x = 12ab$; $(4a^2 - 9b^2)^2$.
(b) $\dfrac{2}{1-4x^2}$.

33. (a) $\dfrac{1+3x^2}{(1-x)^2}$.
(b) -30.

34. $\dfrac{p}{q}$.

35. $6a + 2b - 3$; $2a - 8b - 7$; $a = \dfrac{19}{26}$, $b = -\dfrac{18}{26}$.

37. (a) $-2\frac{2}{3}$.

38. (a) 0.
(b) $a + 3b$.

39. (a) $\dfrac{2(x-8)}{(x-2)^2(x-3)(x+2)}$.
(b) $a^{\frac{1}{3}}b$.

40. (a) $\dfrac{x^2+2}{x+1}$.
(b) 16.

41. $81b^2$.

42. (a) 10.
(b) $\dfrac{(\sqrt{a}+\sqrt{b})^4}{a-b}$.

43. (b) $\dfrac{4x^2-9}{x}$.

44. (a) 15.
(b) $-\dfrac{x}{(x-5)(x-17)}$.

45. (a) $ab(a+b)$.
(b) $\dfrac{(5-x)(8+5x+x^2)}{4(1-x^2)}$.

46. (a) $\dfrac{b}{a-b}$.
(b) $\dfrac{2t}{t+1}$.

47. $3\frac{2}{3}$.

48. $3{\cdot}61, -0{\cdot}28$.

49. $74\frac{1}{2}$.

50. $1\frac{1}{2}$.

51. $3{\cdot}09, -0{\cdot}43$.

52. 11.

53. $17\frac{2}{3}$.

54. -3.

55. $5, -\dfrac{14}{3}$.

56. $3, -11$.

57. $0, 3\frac{1}{3}$.

58. $\dfrac{2ab}{a+b}$.

59. $1{\cdot}4, -0{\cdot}4$.

60. $-\frac{2}{3}, 4$.

61. $2, -\dfrac{23}{12}$.

62. $-1, -17$.

63. $b = -(\alpha+\beta)$, $c = \alpha\beta$; $\dfrac{b^2-2c}{c^2}$; $c^2x^2 - (b^2-2c)x + 1 = 0$.

64. (a) $35\frac{1}{5}$.
(b) $p = -2(b+c)$, $q = 4bc$.
(c) $p = 45$, $x = 1\frac{1}{2}, 15$.

65. (a) $150, -\dfrac{53}{97}$.
(b) $x^2 - 3x + 1 = 0$.

66. (a) $p = -28$, $q = 4$.
(b) $7x^2 - 30x + 31 = 0$.

67. (a) $q = 36$.
(b) $5x^2 - 27x + 15 = 0$.

68. $x = 4\frac{1}{2}$; $y = -10\frac{1}{2}$.

69. $x = 6$; $y = 12$.

70. $x = 2, -\frac{71}{41}$; $y = -1, \frac{61}{41}$.

71. $x = 3, \frac{15}{7}$; $y = 1, -\frac{5}{7}$.

72. $x = 4$; $y = -0{\cdot}5$.

73. $x = 8$; $y = 9$.

74. $x = 6, 4\frac{2}{25}$; $y = 4, 4\frac{24}{25}$.

75. $x = 2\frac{3}{5}$; $y = 7\frac{4}{5}$.

76. $x = 49$; $y = 50$.

77. $x = 1, \frac{4}{3}$; $y = 1, \frac{3}{2}$.

78. $x = 1, \frac{8}{7}$; $y = -1, -\frac{5}{7}$.

79. $x = 0{\cdot}8$; $y = 0{\cdot}75$.

80. $x = 2, \frac{1}{2}$; $y = 1, 4$.

81. $x = \frac{2}{5}$; $y = \frac{3}{10}$.

82. $x = 2, -\frac{10}{7}$; $y = 3, -\frac{39}{7}$.

83. $x = 1\frac{1}{8}, -2$; $y = -\frac{3}{10}, 1$.

84. $x = 20$; $y = 8$.

85. $x = -\frac{2}{7}$; $y = -\frac{3}{7}$.

86. $x = -3$; $y = -4$.

87. $x = -1, 4$; $y = 1, 1$.

88. $x = a + b$; $y = a - b$.

89. $x = 3$; $y = 3$.

90. (*a*) 6·525.
(*b*) 177.

91. (*a*) $x = 0{\cdot}01, y = -1, z = \frac{1}{8}$.
(*b*) 0·05446.

92. $\frac{x^3}{16y}$.

93. 154·06.

94. 8·397, 2480.

95. (*a*) 2·074, 0·3795.
(*b*) 931·1 sq. cm.
(*c*) $\frac{\sqrt{s^2 - \pi^2 r^4}}{\pi r}$.

96. (*a*) 0·1135.
(*b*) 998·4.

97. $\frac{1}{81}$.
(*a*) 26 digits.
(*b*) 0·52.

98. (*a*) 0·308.
(*b*) 4·5.

99. (*a*) 5·42.
(*b*) $x = \frac{a\sqrt[4]{b^5}}{c}, c = \frac{a^2\sqrt{b^5}}{x^2}$.

100. (*a*) $\frac{2}{9}$.
(*b*) 3.
23·81.

101. (*a*) 0·4531.
(*b*) 0·3829.

102. (*a*) 0·00451.
(*b*) 14.

103. (*a*) 1·2552, $\bar{2}{\cdot}0792$.
(*b*) 0·7279, 0·5547.

104. (*a*) 147·96.

105. (*a*) $x = 10$.
(*b*) 2·452.

106. $\frac{1}{2}r(r + 1)$; 49.

107. (*a*) $11\frac{5}{12}$, $154\frac{11}{12}$.
(*b*) $\frac{48}{b + 1}$.
36 and 16.

108. (*a*) 345.

109. 390.

110. (*a*) 3 and -10.
(*b*) $-13\frac{1}{2}$, $13\frac{3}{4}$, $\frac{(-3)^n}{2^{n-2}}$.

111. 80·37.

112. £132 1*s.* 10*d.*

113. 1610; 7378 feet.

114. (*a*) $2b$.
(*b*) $55\frac{1023}{1024}$.

115. £429.

116. $2\frac{1}{2}$ inches, $3\frac{1}{2}$ inches.

117. Area $= \frac{1}{2}(2b^2 + bc - c^2)$; $a = 16$, $b = 24$, $c = 32$; area 448.

118. (*a*) n^2.
(*b*) £1585.

119. (*a*) 6, 2, − 2, − 6, − 10.
(*b*) £4 17*s*.

120. $x = 3$.

121. (*a*) 5, 8, 11, 14, 17.
(*b*) 1st term 2 or − 6, common ratio 2 or − 2.

122. (*a*) 5 : 12.
(*b*) 118·5 yards.

123. 12 adults, 18 children.

124. (*a*) 3.
(*b*) 27·8 cm.

125. 130.

126. (*a*) $m = 2, c = -\frac{1}{2}, x = -\frac{3}{4}$.
(*b*) $\sqrt[3]{44} = 3{\cdot}53$.

127. (*b*) 135 lb.

128. 8·25 miles, 594·9 feet.

129. $\frac{1}{7}$.

130. $\frac{Ca(p-q)}{bp-aq}, \frac{Cb(p-q)}{bp-aq}$.

131. (*b*) 3 miles, $10\frac{2}{3}$ feet.

132. (*a*) $s = 1$.
(*b*) 6 seconds.

133. (*a*) 75 per cent.
(*b*) $W = 3{\cdot}6$, $d = 1{\cdot}2$.

134. (*b*) £3 0*s*. 6*d*.

135. (*b*) $(1\frac{1}{4}, 2\frac{5}{8})$; $y = 2\frac{5}{8}$.

136. Minimum value $-6\frac{1}{8}$ when $x = \frac{3}{4}$; $x = 2{\cdot}35, -0{\cdot}85$; $2x^2 - 5x - 3 = 0$.

137. (*a*) $x = 1\frac{1}{2}, -1$.
(*b*) Minimum value $-3\frac{1}{8}$.
(*c*) $x = 1{\cdot}85, -1{\cdot}35$.

138. 3·46.

140. 1·75 inches.

141. 72 square inches; $x = 10{\cdot}24, 1{\cdot}76$.

142. 9, 5·19, 3, 1·73, 1, 0·578, 0·333, 0·193; 0·6837; $x = 0, 1$.

143. (*a*) $x = 2{\cdot}19, -0{\cdot}69$.
(*b*) Common difference x; $x = 1{\cdot}78, -0{\cdot}28$.

144. 0·27, 2, 3·73.

145. 4 feet 6 inches.

146. (*b*) $(3, 2), \left(\frac{5}{21}, \frac{62}{147}\right)$.

147. (*a*) Maximum value $6\frac{1}{8}$.
(*b*) $x = 2{\cdot}14, -1{\cdot}64$.
(*c*) $-1 < x < 2$.

148. $x = 1{\cdot}44, -1{\cdot}04$.

149. (1·7, 0·1).

150. (− 1·5, 0) and (1·5, 1·2).

151. (*a*) 5·36.
(*b*) $y = 6$.
(*c*) $3{\cdot}45 < x < -1{\cdot}5$.

152. $x = 2{\cdot}8, -5{\cdot}8$.

153. $x = 3{\cdot}87, 0{\cdot}13$.

154. 66.

155. £2.

156. 40 m.p.h. and 48 m.p.h.

157. 6 boys.

158. 44 minutes.

159. 25 miles.

160. 72 yards and 48 yards.

161. 3 furlongs from A.

162. 3*s*. 4*d*.

163. $2n$ inches.

164. 16 feet by 15 feet or 18 feet by 13 feet 4 inches.

165. Tea 2*s*. 4*d*., bacon 1*s*. 6*d*.

166. £600 at $3\frac{1}{2}$ per cent., £1200 at 4 per cent., and £3200 at $4\frac{1}{2}$ per cent.

167. 37 feet 6 inches, and 10 feet.

168. $27\frac{1}{2}$ miles after the start.

169. £38 10*s*.

171. 50 shares, each costing £90.

172. 30.

173. A takes $9\frac{4}{5}$ seconds, B takes 10 seconds.

174. 5 feet.

175. 17.