LIVING MATHEMATICS

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LIVING MATHEMATICS

PREFACE TO THE SECOND EDITION

The reception accorded the first edition of *Living Mathematics* has been highly gratifying. However, now that certain strengths and weaknesses have become apparent, a rather thorough revision has been thought advisable in order to improve the book as a text while retaining and enhancing its general appeal.

Much as we hate to admit it, the first edition actually fell somewhat short of perfection as a text. There were not enough easy practice problems for the less sure-footed beginners, and there were also a few large gaps in the lower treads of the mathematical stairway that needed filling in for the sake of smooth climbing. Besides, the less adept among the readers, who probably found it necessary on occasion to retrace their steps between the "ifs" and the "therefores," may possibly have been less and less intrigued by a given facetious remark on each successive encounter, until finally, on some height of frustration, they were not intrigued at all, to put it mildly.

Now all this is part of the past. In the revised edition many early practice problems have been included in each exercise; inadequate treatments (as in the case of fractions) have been amplified; and some of the original flippancy has been eliminated to soothe the feelings of the earnest beginner. Yet the book no longer would be the *Living Mathematics* that found favor with many if we were to banish the light tone and treatment. The businessmen, lawyers, doctors, painters, and second-story men who plead guilty to a mathematical flair, and who naturally have some facility in the field (otherwise they never would have accepted our previous challenge to a lighthearted tilt) are hereby given permission to skip the insultingly easy problems included for drill purposes and to concentrate upon the amplified supply of potential posers that may prove worthy of their mettle.

In teachers colleges, especially, there is a very definite place, we believe, for a textbook of this type. The budding professor of literature should learn, in what may well be his terminal course in mathematics, that the subject is not so cut and dried and humorless as he had supposed; the prospective teacher of high-school mathematics most certainly should encounter fresh points of view, fresh approaches to old landmarks, and, above all, the feeling that his subject may be made interesting and stimulating as well as useful.

In Fun with Figures, which was previously the final chapter, we are frankly appealing to the interested amateur mathematician, whether or not (and the "nots" probably have it) he is currently working for a diploma. In other words, freshmen, this is not for you unless you consider yourself a little, shall we say, superior. And that goes for the final chapter too. It opens a new and largely unexplored field for those of you who like that sort of thing. But frankly, these last two chapters are not written for college students; they are offerings, pure and simple, to the dilettantes of the pencil and the throbbing brain, for whom all of living is not the pursuit of bread.

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LUBBOCK, TEXAS
March, 1949

PREFACE TO THE FIRST EDITION

In writing this book we had in mind two distinct but, we trust, compatible goals. The first was the production of a college textbook which would provide enough of the conventional subject matter to meet practical credit-transfer requirements. The second was the goal of highlighting for nonspecialists the interest that is inherent in mathematics itself and of fostering an appreciation of its place in modern life. Our aim, in short, is to answer in the text itself that perennial and petulant student query: "What's the good of all this?"

The book provides a one-year course for those who will theoretically pursue the subject no farther, but among whom there may possibly be salvaged a few devoted and surprised lifetime addicts. Part One specifically covers the ground of an orthodox three-semester-hour course in algebra. Part Two is a rounding-out survey of the mathematical highlights in trigonometry, analytic geometry, more advanced algebra, and calculus, with a seasoning touch of the theory of numbers. There is more than enough material for a full-bodied second course. And since those who leave the subject should carry with them a background of sympathy for and interest in the science which exalts the human reason, a somewhat unconventional treatment has seemed to be in order. The one adopted supplements the usual mathematical technique with an informal discussion of its role in life, so that the drudgery of the drill may be relieved somewhat by recurrent glimpses of the objective.

But whatever the surface novelty of treatment, difficulties have not been avoided, and reasonable rigor has been preserved. We are acutely aware of, and quite in sympathy with, the hostile criticism of any text from which the student "learns about mathematics but does not learn mathematics." While one who has mastered our book will not be an accomplished mathematician (how many sophomores are?), he will have done about as much plain thinking as is expected of most freshmen.

The question remains whether a light and jaunty treatment is suitable for a book with an essentially serious aim. We see no harm in it. If it helps to dispel the idea that mathematics is a painfully dull form of hard work which is to be highly recommended for the specialists who will need it, but which contains in itself nothing of drama, zest, humor, surprise, challenge, and general human interest—then, we think, this manner of writing will have been justified. We are writing, let us repeat, for the great battalion of those who fear the subject—not with the idea of removing their difficulties, but rather with the hope of adding interest and pleasure to their work.

Special acknowledgment and thanks are due to Professor C. V. Newsom of the University of New Mexico, whose valuable suggestions were for the most part incorporated into the text. Additional help was received from sources too numerous to mention, but including, of course, our colleagues at Texas Technological College.

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PART ONE

ALGEBRA

CHAPTER I

THE HAPHAZARD BEGINNING

1. Early counts. Man needed to count long before he learned to write, and hence we can never get the true story of his mathematical awakening. Our guesses, however, are not altogether unguided, for some of the fossils of prehistory are imbedded in words as well as in rocks. The fact that the words for *five* and *hand* are much alike in some languages[1] suggests that our ancestors may have found the fingers convenient to keep track of arrows, wives, sheep, and other things of that sort. Probably an early number sense developed similar to that of the legendary darky who, noticing an alligator making off with one of his brood, yelled triumphantly into the cabin: "See dere, Mandy, I done tole you somethin's been agittin dem chillun."

Such a vague and inadequate number sense, however, which is shared in some degree by animals and even birds, could not long serve satisfactorily the needs of human society. The basic idea of matching objects with fingers must have come long before history opened her first page. Then, when the grand inspiration of repeated matches came to some genius; that is, when five objects became a "hand," so that one could start a second series with the fingers again free to match, the useful art of counting may be said to have begun.

2. And how they kept track of them. The methodical cave-keeper of the "good old days" probably used three scratches on a bone or three stones in a pile to keep track of certain items in her budget before she bothered her frowzy head about a name for the abstract number "three." We might reasonably suspect, therefore, that the art of recording numbers developed along with, or at least not much behind, the ability to count, and that both of these

[1] For example, compare the Sanskrit *pantcha* (five) with the related Persian *pentcha* (hand), and the Russian *piat* (five) with *piast* (the outstretched hand). (After Dantzig.)

accomplishments stirred and flustered the brain beneath many a thick skull long before anyone thought about noting events on a brick or two for history. This surmise is given substance by the historical fact that the Sumerians, Egyptians, and Chinese had all developed systems of number writing before 3500 B.C.—a fact which, with allowance for development time, places the primitive beginning certainly much earlier. It is a highly significant fact, indicating the influence of finger counting, that numbers less than ten are represented by similar strokes, while a new symbol is used for ten.

The Phoenicians were probably the first people to use the letters of the alphabet to represent numbers, and their scheme was later adopted by the Greeks and Hebrews. Probably the most compact of the early number systems was that of the Romans—borrowed, perhaps, from the Etruscans. This system also shows the decided influence of finger counting, since new symbols were adopted for each multiple of five. Enumeration of fairly large numbers is relatively simple with it; but the absence of a symbol for zero and of an efficient place-value scheme made it cumbersome and in fact practically useless for computations. It is not surprising that mechanical computing machines like the abacus flourished in the days when 368 times 3233 was "CCCLXVIII times MMMCCXXXIII." Perhaps matters of that sort were responsible for some of the milder homicides of Nero, Caligula, and other impetuous Roman gentlemen.

3. First fumbles. As soon as integers were invented and given distinguishing marks for recording purposes, the problem of what happens when they are shuffled was certain to come to the front. Someone noticed for the first time that two stones combined with two more always made a pile of four, and suddenly arithmetic, with all its woes, descended upon little Willie's ancestors. A couple of stones were taken away experimentally, and lo, subtraction was born! Three piles of four each were found, upon investigation, to merge infallibly into twelve, starting the multiplication table on its painful and mangled course through the schoolrooms. And finally some ancient scholar, faced with a domestic insurrection over a pile of turtle eggs during a famine, went into a fervor of computation and came forth, in his extremity, with the art of division.

While the reliability of this unverified account of prehistoric events may be called into question by the sticklers for absolute accuracy, it nevertheless does give proper emphasis to one funda-

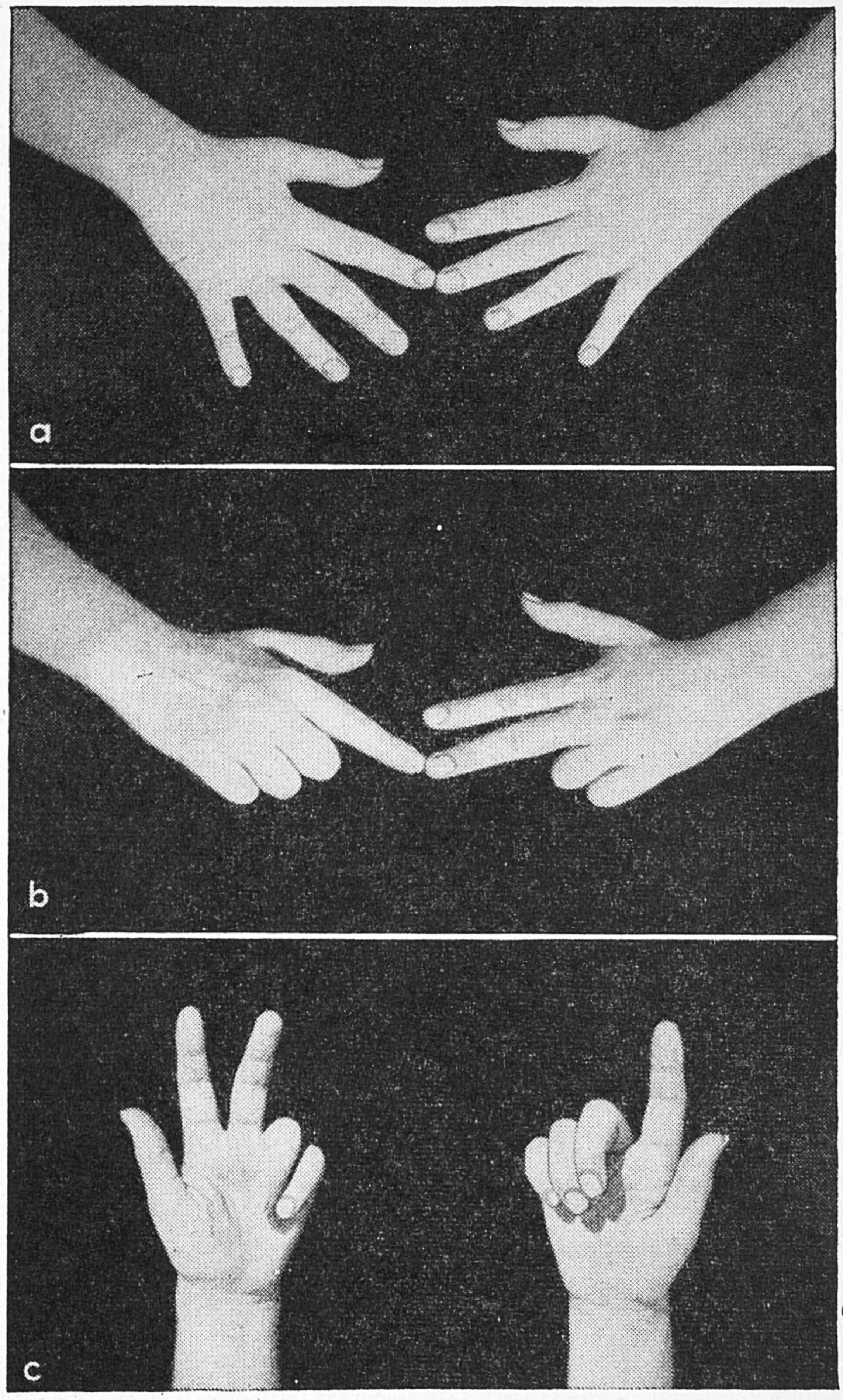

FIG. 1.

mental fact—the fact that early computations were almost certainly made with mechanical help of one sort or another. Again the fingers, and sometimes the toes, came to the aid of the early scholar. The art of finding the sum of two numbers less than 10 is known and surreptitiously practiced by present-day first graders struggling with their addition combinations. However, the fact that the fingers can be used to obtain the product of two numbers from 6 to 9 inclusive may be a surprise to most moderns, although the method is still used by the peasant class in some parts of the world today. In this process the thumbs in all cases represent 6, and the other fingers in turn stand for 7, 8, 9, and 10. To find the product of 7 and 8, for example, the finger on one hand representing 7 is placed in contact with 8 on the other (Fig. 1*a*), and all fingers beyond these are folded (Fig. 1*b*). Then the number of fingers extended (Fig. 1*c*) is the tens digit of the result, and the product of the folded fingers on each hand ($3 \times 2 = 6$) is the units digit. Thus the answer is 56. In finding the product of 6 and 7 by this method the number of extended fingers is 3, and the product of the folded fingers is 12, but $30 + 12 = 42$, the correct product.

Number representations themselves served to record the final results, but they did not come into the picture as part of the computing machinery until a surprisingly late date—until, in fact, after the Renaissance in Europe. The explanation of this strange backwardness in operations which seem easy for us is not hard to find. It lies in the unscientific nature of number representation before the Hindu place system came into use. We have already suggested the tussles with Roman numerals which would have taken place if the merchants of old Rome had multiplied with pencil and paper as we do. As a matter of fact, they didn't. They used the abacus, a mechanical contrivance as old as history, embodying in its multitudinous forms the simple principle of computing by moving beads along strings, much as we tally a billiard game. It was the principle of the stone pile, grown up and refined a bit. In some of its forms it even presaged the place system in the number writing of today. That is, units were represented on one string, tens on another, and so forth. The skilled manipulator could flip his beads and get quick results with fair-sized figures, and this was all that was necessary in the days of old-fashioned

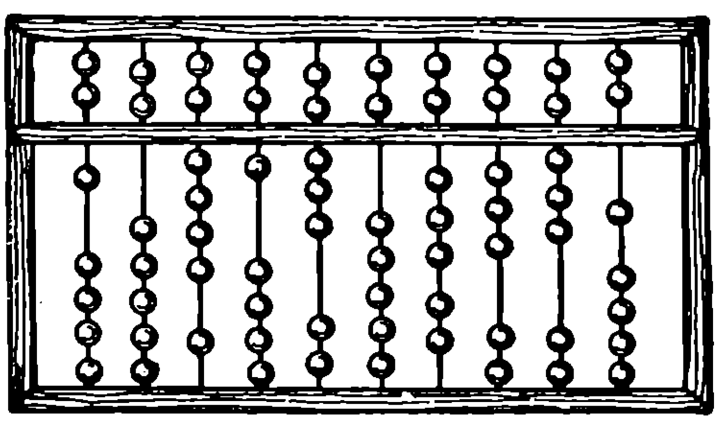
FIG. 2. Abacus, or Chinese swanpan.

astronomy and no national debts. Perhaps it was this very efficiency of the abacus which delayed the invention of a way of writing numbers serviceable in their manipulation too. At any rate these simple aids were still used by some European merchants as late as the eighteenth century; in fact, they are used expertly today in many oriental countries. Nor are they doomed, apparently, to early extinction outside of museums, for their elegant and talented modern descendants, the computing machines, still perform prodigies in our counting rooms, doing our multiplying and adding with a truly marvelous offhand infallibility.

4. Success at last! In spite of the great utility of the abacus and its modern cousins, a better system of recording was sorely needed for a long time. Not every farmer who sold thirteen pigs at twenty-four coins apiece could carry along a computing machine to keep from being cheated. The Hindus found a way out, and to them we owe a debt of gratitude. During or before the ninth century, they invented the *place system* of number writing, which, coming to Europe through the Arabs, had gradually overcome the inevitable human resistance to change and had won fairly widespread acceptance there by the year 1400. And any new idea in those days which could spread all the way from Asia to Europe in a scant five centuries had to be of more than passing merit.

But the place system really was a masterpiece, though in principle it merely poached upon the uses of the abacus. A number which would have been represented on the abacus by three, two, and six beads, respectively, on the three right-hand strings, became in the place system three hundreds, two tens, and six units, or 326. The idea was mathematical dynamite. Though apparently absurdly simple, it opened tremendous possibilities, both in written compactness and in computing power. Numbers ten times as large as those handled by the most bestringed abacus could be written down slapdash with just one more stroke of the pencil.

But why was this great boon to mankind, which the great mathematician Laplace places "in the first rank of useful inventions," overlooked for so long by the keenest thinkers of the not-too-keen dark ages? Maybe this is a clue: How would you represent, with marks for the digits "one" to "nine" inclusive, the quantity represented on the four right-hand strings of the abacus by "six, blank, blank, two"? Try it, and it comes out "6 2," which looks like nothing more than "62" with a printer's rupture. We might estimate the number of unbeaded strings in the abacus representation by measuring the width of the break, but who wants to read numbers with a ruler? We'd better go back to the abacus. But wait; let's try indicating the number of spaces left out. Perhaps with a digit—like this: "622." No, that's an obvious misrepresentation. We'll just have to mark the spaces somehow—say with a simple, handsome figure like a circle. All right—we have "6002." Eureka, it is done! We have invented a new symbol, a new digit, and a wonderful system of number representation!

Why didn't we do it before? Well, it seemed rather silly to place along with solid symbols like 6 and 8, whose meanings were plain to anyone, a mark representing nothing at all. The Babylonians did that 2,100 years ago; but it didn't seem to get them anywhere. That, upon reflection, is because they missed the really important thing about that hard-to-invent zero symbol, which is not the mark at all, but rather the space it fills up. That the Europeans were just beginning to learn about it in the fourteenth century merely shows how much they were behind the true Americans. For the Mayans of Central America used a scientific place-value "zero" in their sexagesimal system at the beginning of the Christian era, an accomplishment which certainly entitled them to laugh at our ancestors in a very superior way.

5. Growing pains.[1] Any schoolboy knows that when we multiply 826 by 725 we (1) write the numbers down, one above the other; (2) multiply the top number by the digits below taken in succession, beginning with the one at the right; (3) arrange the piecemeal results in a sort of stagger formation backsliding to the left as we go down; and (4) add the columns, getting the right answer (or anyway we should). It seems natural to assume that a trick as easy as all that must have been known always. But perhaps we'll be willing to give the honor rightfully due the inventors of the process when we try for ourselves a few of the complicated schemes which they were good enough to improve for us. Most of these methods were worked out by clever individuals dur-

[1] This article may be omitted with no loss of continuity.

ing the transition to the place system before the latter's full possibilities were realized; but others were still older.

One of the earlier methods, which could be carried out by use of the abacus, was known as *duplation.* The method is based on the fact that the product of two numbers is equal to the product of one-half of one of them by twice the other. For example

$$(16)(27) = (8)(54) = (4)(108) = (2)(216) = (1)(432) = 432$$

If the corresponding pairs of halves and doubles are arranged one above the other in a line, we have

16	8	4	2	1
27	54	108	216	432

in which we see that the only odd number in the line of halves is 1, under which the answer is found. But of course if the multiplier is odd to start with or if one of the halves turns out to be odd, a difficulty enters that requires a bit of dodging, as in the case of (19)(26). Here we start by writing 19 as 18 + 1, and then we have

$$\begin{aligned}(19)(26) &= (18 + 1)(26) = (18)(26) + 26 \\ &= (9)(52) + 26 = (8 + 1)(52) + 26 = (8)(52) + 52 + 26 \\ &= (4)(104) + 52 + 26 \\ &= (2)(208) + 52 + 26 \\ &= (1)(416) + 52 + 26 \\ &= 416 + 52 + 26 \\ &= 494\end{aligned}$$

The essential steps of this process can be shown by again using the double-line scheme, thus

19	9	4	2	1
26	52	104	208	416

in which the top row is obtained by successive halving with the leftover fractions discarded, and the bottom row by doubling. The numbers to be added for the product are then found below the odd numbers. Can you see why this will always work out?

Another scheme, known as the *Gelosia* or *grating* method, spread from India to China and Arabia, and thence to Italy. Though old, it seems to involve the place-system principle, at least as we reproduce it. Its use in the multiplication of 327 by 243 is shown in Fig. 3. The product, 12, of 4 by 3, for example, is in the second rectangle below 3. We get the final product by turning the page so that the diagonal lines are vertical and then adding the columns from right to left. The method is ingenious but it fell into disuse after the invention of printing (1454) because of the difficulty of drawing the grating.

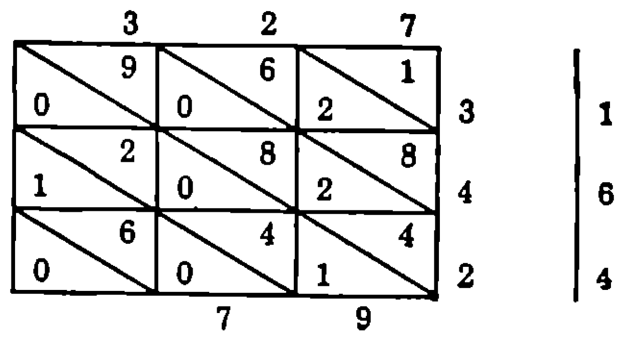

FIG. 3. Product 79,461.

A method that uses a mixture of the Roman and Hindu-Arabic numbers and represents a transitional stage between the abacus and a positional notation was found in a Paris manuscript. It is illustrated in Fig. 4 in the multiplication of 321 by 34.

This was probably the forerunner of the *chessboard* method shown in Fig. 5.

	CM	*XM*	*M*	*C*	*X*	*I*
				3	2	1
					8	4
			1 9	2 6	3	
		1	0	9	1	4
					3	4

FIG. 4.

This method needed only the elimination of the "boxes" to become the one of today. And certainly one would be rash to deny that still further improvements are possible.

In the matter of division we are forced to conclude, upon inspection of the older available methods, that the accomplished divider in the days of Charlemagne must have been a person of some importance. If he had had his social rights (which seems, on the whole, a bit unlikely), he should have been equal, roughly, to about two knights and perhaps an earl. He probably performed the simpler of his dividing chores by repeated subtraction on his abacus, as in the division of 29 by 8 thus: $29 - 8 = 21; 21 - 8 = 13; 13 - 8 = 5$; and hence $29 \div 8 = 3$, with 5 over. Since the quotients weren't always so small,

			3	2	6	7
			2	1	3	4
		1	3	0	6	8
		9	8	0	1	
	3	2	6	7		
6	5	3	4			
6	9	7	1	7	7	8

FIG. 5.

he worked out various improvements for the big jobs; but we'll pass on to other things and leave the complicated details for the larger histories.

6. The final achievement. In the place system we have a method of number representation which is both compact and tractable, making the ordinary operations of arithmetic fairly easily mastered even by a child. Certain finishing touches are needed, the chief of which is the decimal system. If we divide 25,300 by 10 and 100 successively we get 2,530 and 253, in which the obvious resemblance is that the order of the nonzero digits is not changed. Once it occurs to us to indicate the number 25,300/10 by 2,530.0, using a period at the right of the unit's place to signify the end of the number, the rest is easy. Then $\frac{2530}{10} = 253.0$, which suggests the convenience of writing $\frac{253}{10} = 25.3$ and $\frac{253}{100} = 2.53$. Obviously this is allowable if we define the symbols 25.3 and 2.53 as meaning the mixed numbers $25\frac{3}{10}$ and $2\frac{53}{100}$, which are the true quotients. The word "period" is no longer suitable to describe the marking point, however, since it is now not necessarily at the end of the number, so that it has been appropriately named the *decimal point* (derived from the Latin *decem*, or *ten*, which is the base of the place system). It is interesting that the comma replaces the decimal point in the continental European countries, and that, though the period is used in England, it is written above the line instead of on it.

The introduction of the decimal notation, first mentioned in a treatise by Stevin in 1585, was almost as important as the place system. For now a double infinity of digit places stretched to the right as well as to the left of the decimal point, and the ordered digits themselves represented both the infinitesimally small and the increasingly large according to the degree of the removal.

Admirable as it was, however, the completed system had limitations and defects which persist to this day. For one thing, such simple fractions as $\frac{1}{3}$ and $\frac{1}{7}$ cannot be represented exactly in the decimal notation without recourse to some device such as dots to indicate infinite repetition. Thus $\frac{1}{3}$ is written decimally as .333 . . . , meaning "$\frac{1}{3}$ equals $\frac{3}{10}$ plus $\frac{3}{100}$ plus similar fractions, each one tenth as large as the preceding one, added on without end." In addition, other definite numbers, such as the one represented geometrically by the diagonal of a square with a side one unit long, completely elude the system. It is impossible to represent this number in the decimal notation even with the added convention of dots representing repetitions, since there is not even a sequence of digits, as in the case $\frac{1}{7} = .142857142857\ldots$ which is repeated without end. Numbers of this kind, called *irrationals*, will be dealt with in more detail later; they are the incorrigibles, the outcasts, the nonconformists in a scheme of number representation which, though evidently short of perfection, is still one of the greatest and least appreciated achievements of the human race.

EXERCISE 1

1. Write the following numbers in a column and find their sum: three hundred and two, four thousand and twenty, five thousand and five, one thousand one hundred and one, three million two hundred thousand four hundred and thirty-seven.

Add the numbers in Probs. 2 to 4. Remember that in addition we keep the decimal points in a line.

2. .006, 3.217, 467.2, 21.38.

3. 1,026, 4.139, .0712, 600.1.

4. 426.781, 10,073.2, .03,761.

5. Find the sum of six hundred and one-tenth, four and twenty-one thousandths, three thousand twenty-one and twenty-two hundredths.

6. Change the following to decimal form and add: $12\frac{3}{4}$, $37\frac{5}{8}$, $426\frac{1}{8}$.

7. Change the following to mixed decimals and find the sum to three decimal places: $32\frac{1}{3}$, $471\frac{2}{15}$, $7\frac{5}{12}$, $577\frac{7}{18}$.

8. If the multiplicand is multiplied by 1,000 and the multiplier by 100, how is the product changed?

9. When we multiply two numbers containing decimal places, we treat the numbers as if they were integers and then point off as many places as there are in the multiplicand and multiplier together. Use the principle of Prob. 8 to show that this practice is correct.

10. Multiply 4,162 by 321. In the process of multiplication, why are the products obtained when we multiply by 2 and 3 set over one place and two places to the left respectively? Check your answer by dividing it by 321.

11. Multiply 4,261 by 405. How does the zero in the multiplier 405 affect the multiplication process?

Find the products of the numbers in Probs. 12 to 15.

12. 126.021 and 23.41.

13. 3.621 and .00627.

14. .5268 and .0782.

15. $37\frac{7}{8}$ and $.126\frac{3}{4}$.

16. A number contains four digits, with the second digit 7. How is the value of the number changed if 7 is replaced by 5? by 9? by 0?

17. The first two digits of a four-place number are 3 and 8 respectively. How is the value of the number changed if the 3 is erased? If the 3 is erased and 8 is replaced by 9? If the 3 and the 8 are interchanged?

18. If the first and last digit of a four-place number are 7 and 2, respectively, how is the value of the number changed if they are interchanged? If the 7 is erased and the 2 is replaced by 0?

19. If 35 is multiplied by a two-digit number ending in 6, how is the product changed if 6 is changed to 7? to 9? to 3?

20. Note that $\frac{1}{2} = .5$ is a *terminating* decimal fraction, whereas $\frac{1}{3} = .333 \ldots$ is *repeating*. Write $3/n$ in decimal form when $n = 4$, 5, 6, 7, 8, 9, 10, 11, and 12. How many of these are repeating decimal fractions? What can be said of the values of n that do not yield repeating decimals?

21. Write each of the fractions $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$, $\frac{1}{6}$, $\frac{1}{7}$, $\frac{1}{8}$, $\frac{1}{9}$, $\frac{1}{10}$, $\frac{1}{11}$, $\frac{1}{12}$ in decimal form. Which of these are repeating decimals?

22. If the dividend is multiplied by 1,000 and the divisor is multiplied by 10, how is the quotient affected?

23. When the dividend contains n more decimal places than the divisor we treat the two as whole numbers, and then, after dividing, we point off n places in the quotient. Use the principle of Prob. 22 to justify this procedure.

24. What is the procedure in division if the number of decimal places in the divisor exceeds the number in the dividend? How can it be justified?

25. How many times must 327 be successively subtracted from 69,978 in order to obtain the remainder zero?

Find the quotients obtained by dividing the first numbers of Probs. 26 to 29 by the second.

26. 462.17 by 32.3, answer to three decimal places.

27. 52.16 by 3.725, answer to four decimal places.

28. 137,067 by .00321.

29. .2373 by 99.9. Divide until the digits start repeating.

30. Without changing to the Hindu-Arabic notation, show that CXVI + XIII + VI + CCLXV = CD.

31. Without changing to the Hindu-Arabic notation, show that XLIX + XIX + XXIV + XC + XXVIII = CCX.

32. If people had four instead of five fingers on each hand, how might this have changed our place system of number representation? In the system which would then probably be used, if the digits 0, 1, 2, 3, 4, 5, 6, 7 were unchanged, how would 8 be written?

33. Multiply .142857 successively by 1, 2, 3, 4, 5, 6, 7, and 8. Do you notice anything interesting about the arrangement of the digits in the products?

34. To multiply a number by 25, we can divide it by 4 and then move the decimal point in the quotient two places to the right. Why does this work? Devise one more short cut in multiplication.

35. To divide a number by 250 we can first multiply by 4 and then move the decimal point three places to the left. Why? Devise another short cut in division.

36. Prove that if the sum of the digits in a number is divisible by 3 the number itself will be divisible by 3. HINT: Note that 426 (for example) $= 4(100) + 2(10) + 6 = 4(99) + 2(9) + 4 + 2 + 6$, in which the last three digits are those in 426.

37. How can you tell without dividing whether a number is divisible by 2? by 4? by 5? by 9?

CHAPTER II

ONWARD TO ALGEBRA

7. Two arts and some early artists. Just as the Hindu-Arabic notation, with its two key inventions of zero and the place system, gave us modern arithmetic, so the introduction of letters and symbols to represent quantities and operations was the fundamental invention of algebra. This somewhat brash statement seems to call for definitions of the two fields mentioned, so here they are:

Arithmetic *is the art of numerical representations and computing.*

Algebra *is the art of solving problems, or of finding unknown quantities, when the nature of the arithmetical computations leading to them is not at once evident.*

It will be noticed that both of these subjects are defined as arts rather than as sciences—a point of view which is in line with modern tendencies. Many arithmetic and algebraic results, such as the conclusion that two and two make four, are scientific discoveries of fact perhaps inherent in the nature of things; but many other mathematical verdicts, and certainly all the methods by which they were obtained, are creative inventions belonging to the realm of art in that the forms they took were designed rather than discovered by man.

Another point in connection with the second definition is that it does not mention the use of letters and symbols, which is the essential feature of modern algebra. The view that symbolic language is a powerful aid to algebra rather than a part of its basic content and purpose is supported inferentially by mathematical historians who list Ahmes of Egypt (about 1550 B.C.) as an early, if not the first, algebraist.

This man Ahmes obligingly provided his readers with "Rules for inquiring into nature and for knowing all that exists, every mystery, every secret." Since he undertook to meet this somewhat large order without an adequate number system or convenient symbols of operation, we should perhaps pardon a few lapses in the technique of the accomplishment. He tells us, for instance, that "a heap, its seventh, its whole makes 19" and then defends his claim to "all knowledge" by guessing, correcting, and struggling with that heap until he finally gets it hacked and patched to the right size. The schoolboy of today would let x represent the number, or "heap." It would follow that

$$x + \frac{x}{7} = 19 \qquad 7x + x = 7(19) \qquad 8x = 133 \qquad \text{and} \qquad x = 16\tfrac{5}{8}$$

The fact that Ahmes did not know how to invoke the mysterious Mr. X for help should not bar him from due honor as an indomitable pioneer on the frontier of algebra.

The game became more or less fashionable. The Chinese writer Sun Tsu[1] (first century A.D.) puts these words into the mouth of a woman washing dishes at a river: "I don't know how many guests there were, but every two used a dish for rice between them, every three a dish for broth, every four a dish for meat, and there were 65 dishes in all." He follows this with a rule for solving: Arrange the 65 dishes, multiply by 12, and then divide by 13. Though this method was perhaps one of trial and error, his result was the correct answer, which the aforesaid modern schoolboy would probably get about as follows: Let x = the number of guests. Then there would be $x/2$ dishes for rice, $x/3$ for broth, and $x/4$ for meat. The fact that there were 65 dishes altogether is expressed in the equation

$$\frac{x}{2} + \frac{x}{3} + \frac{x}{4} = 65$$

whence

$$12\left(\frac{x}{2} + \frac{x}{3} + \frac{x}{4}\right) = 12(65)$$

or

$$6x + 4x + 3x = 13x = 65(12)$$
$$x = 5(12) = 60$$

8. The oldest international language. These two solutions by the modern hypothetical schoolboy (who, we must admit, is a little surer of his x's than some of his flesh-and-blood contemporaries) illustrate the three essential elements in the up-to-date algebraic method. First there is the letter for the unknown,

[1] Florian Cajori, *A History of Mathematics*, p. 73.

usually x or, where several are involved, letters taken from the last of the alphabet in accordance with the original suggestion by the Frenchman Descartes. Then there are the symbols denoting operations, such as $+$ for *plus* and $=$ for *equals*. Finally there is the all-important statement of equality, or *equation*, which contains the desired unknown wrapped up in such form that its value can be found by anyone who knows the proper solving technique.

We'll go into that matter of technique more fully in the next chapter. For the present we wish merely to note that problems are solved expeditiously by use of a special compact language consisting of letters for quantities and symbols for operations, put together in groups to make statements. This useful mathematical language naturally developed slowly, being preceded by cumbersome written explanations from the pens of early investigators who lacked our short-hand equipment even when, as in the case of Ahmes, they did not noticeably lack confidence. It undoubtedly came about in response to definite needs.

For instance, in regard to operational symbols, the descriptions of the same processes over and over again must have grown very, very tiresome. It has been conjectured, therefore, in line with this happy thought, that the plus sign ($+$) comes from a corruption of the Latin *et* for *and*; the minus sign ($-$) descended from the word *minus* itself by way of hurried scrawlings of the abbreviation *m*; the division sign ($\div$) is due to the status of division as a kind of extended or glorified subtraction; and, finally, the multiplication sign ($\times$) came into being from the fact that in multiplying 45 by 23 in the column form we first take 3 times 5 and 4 and then 2 times 5 and 4 in a manner which may be shown diagrammatically thus: $|\times|$.

Again, the need for letters to represent unknown quantities is impressed upon us rather painfully in questions of this sort: "What is the number whose double, increased by 3, is 25 less than the number which is equal to 6 times the sum of 2 and the original number?" Our thinking is speeded up tremendously if we rewrite the question thus: "What is x such that $2x + 3 = 6(x + 2) - 25$?" When we put it that way our troubles are practically over. We can plunge into the problem with confidence and emerge after very little head-scratching with $x = 4$. Though it sounded depressingly hard it was really quite simple, after all, when translated into modern algebraic language.

Thus we certainly have something valuable here. Besides being a language which puts our problems into such compact form that their difficulties often prove to be merely a part of the old verbal trappings, the mathematician's written medium is international in its scope. His statement "$3 + 5 = 8$" is understood by many persons to whom the English sentence "Three plus five equals eight" would be unintelligible. As a medium of thought exchange which needs no translation for readers in many lands, the language of algebra is one of the binding cultural forces in the world.

EXERCISE 2

Illustrative Example

Restate the following in symbolic form:

Four times a number diminished by 8 is equal to three times the number increased by $\frac{1}{5}$ of itself.

Solution: If we represent the number by x, replace the phrase "diminished by" with the symbol $-$, and "is equal to" by $=$, the statement becomes

$$4x - 8 = 3x + (\tfrac{1}{5})x$$

Restate the following assertions in symbolic form, noting that the value of x is not to be found:

1. A number (x) increased by its double is 93.

2. A man lost one-third of his savings (x) and had $2,000 left.

3. The integer 21 is the sum of one-third and one-fourth of a certain number (x).

4. Twice a certain number (x) increased by 6 is equal to three times the number increased by one-half of itself.

5. The length and width of a rectangle whose dimensions are 10 yd. by 6 yd. are each increased by x yd. thereby increasing the area by 35 sq. yd.

6. Tom and Bill had x dollars apiece when they went to the county fair. Tom found a five-dollar bill at the race track and Bill lost $2 on the first race. Tom then had twice as much as Bill.

7. After a garden 20 yd. by 60 yd. is fenced off in one corner of a square plot of side x, there remains 800 sq. yd. outside the garden.

8. I have x dollars invested which yield an income of \$100 per year. If I save the earnings for 20 years I will have twice the amount of the original investment.

9. Dick has x dollars, which is half as much money as Harry has, but if Harry gives Dick \$10, they will have equal amounts.

10. Two towns, A and B, are 100 miles apart. A car leaves A at x miles per hour. A second car, traveling $\frac{2}{3}$ as fast, starts from B at the same time, and they meet in 1 hr.

11. Mr. Benson drove to town from his ranch in 2 hr. at x miles per hour. Coming back, he drove 10 miles an hour slower, and it took him 3 hr. HINT: Remember the formula, Distance = (rate) (time).

12. A freight train which traveled at the rate of 40 miles per hour left Jonesville at noon, 2 hr. ahead of a passenger train whose average speed was 60 miles per hour. The passenger train overtook the freight in x hr.

13. Ten years ago a father, whose age is now twice that of his son, was three times as old as the boy. (Let x be the boy's age.)

14. A stream flows at the rate of 4 miles per hour. A canoeing party, which can travel at x miles per hour in still water, rowed 10 miles downstream and back in 2 hr.

15. If x lb. of coffee worth 20 cents per pound is mixed with 20 lb. of 30-cent coffee, the mixture is worth 25 cents per pound.

16. Ten pounds of milk containing x per cent of butterfat, combined with 20 lb. of 3 per cent milk, yields a mixture which tests $4\frac{1}{4}$ per cent butterfat.

17. Three men trade for a business worth \$10,000. Mr. Smith puts up \$1,000 in cash and an automobile worth x dollars; Mr. Brown, a vendor's lien note worth twice as much as Mr. Smith's car; and Mr. Jones, a town lot worth as much as the car and the note.

18. A college student left the campus for his home 400 miles away on a truck which traveled for x hr. at 40 miles per hour. When it left the highway, he proceeded on foot for 2 hr. at the rate of 4 miles per hour, and then finished the journey in a passenger car whose speed was 60 miles per hour and which required half as much time as he spent in the truck.

19. It takes Tom 30 days, Dick 28 days, and Harry x days, each working alone, to paint a house. When working together, they can paint the house in 10 days.

20. In accordance with an early practice called "riding and tying," two men, having one horse between them, proceed as follows: Starting at noon, the first rides 1 hr., ties the horse, and proceeds on foot. The second walks until he reaches the horse, then mounts and overtakes the first at x o'clock in the afternoon. The horse travels at the rate of 8 miles per hour, and the men walk at the rate of 4 miles per hour.

9. Our own approach. Having seen how to state a problem or condition by use of symbols, we come next to the matter of their manipulation. In some fields, such as engineering, it is necessary, in view of the time allotted to mathematics, to load the students down with a lot of formal algebraic rules concerning allowable ways to juggle letters and exponents, in the hope that they will gradually come to see the usefulness of these rules later on, when they actually have to apply them to technical problems. For our purposes at least this method is not necessary. We shall consider only the simpler algebraic machinery necessary for our main business of solving practical problems met in everyday life. Those who have learned to handle well those basic tools will probably be able, if the need arises, to work out for themselves the refinements necessary for more difficult problems.

10. The first steps. Addition is such a rock-bottom necessity that we shall consider it first of all. The matter of finding the sum of two like objects and of three more of the same kind is simple; experience has taught us that this makes five like objects. That is, in the language of algebra, $2a + 3a = 5a$. Consider, however, the problem of adding $2a + b$ and $3a + 2b$. Is $2a + b + 3a + 2b$ the same as $2a + 3a + b + 2b$? If so, the answer is easily seen to be $5a + 3b$. But how do we know that we can change the order of the terms in an addition problem? Well, to be frank, we don't; we merely see that it works every time it is tried with numbers, as in the case $2 + 3 = 3 + 2 = 5$. It has been found helpful in practice to assume without proof the so called *commutative law of addition*, which says of any two terms a and b that $a + b = b + a$.

The usefulness of this law may be seen in the problem of finding the sum of the algebraic expressions: $2a + b + 3c + d$, $2d + c + a$, and $b + d + c$. The agreement that addition is commutative allows us to change the order of the letters and to write the various parts in columns of likes, thus:

$$\begin{array}{rcrcrcr} 2a & + & b & + & 3c & + & d \\ a & & & + & c & + & 2d \\ & & b & + & c & + & d \end{array}$$

The sum $3a + 2b + 5c + 4d$ is then readily found by adding the separate columns.

Another important question comes up in connection with the addition of three like terms, such as $5a + 2a + 8a$. Does this mean that we add together $5a$ and $2a$ and then combine their sum with $8a$, or does it mean that we add $5a$ to the sum of the last two terms? When we look at the results we see that the question is immaterial in this case, since the answer is $15a$ regardless of the preliminary grouping. Repeated trials lead to the same conclusion. Hence we find it convenient to assume, again without proof, that addition is *associative*. In symbolic language $(a + b) + c = a + (b + c)$.

Just as, in the words of an old adage, "it is a poor rule that does not work both ways," so in mathematics "it is a poor process that does not have an inverse." This is a technical way of saying that it should be possible to unwind that which is wound up, to undo whatever is done. What, then, should be the inverse of addition? The latter is the process of finding c in the equation $a + b = c$, when a and b are known. But if in the same equation the a and c are known first and we are to find b so that the sum $a + b$ shall be the indicated c, then our method must be a taking apart of the final addition to get the unknown portion of the sum. This inverse of addition is known as *subtraction* and is exhibited in the form $c - a = b$. It presents no new difficulty as long as a is less than c. Even the ancient computer with his abacus could flip out the answer with easy disdain, since all he had to do was to take a beads from c beads and note how many he had left.

11. Into the looking glass. But when a is larger than c, there we have something else again. We can't take six beads away from four and get anything that makes sense on the abacus. Nor on anything else, according to the ancient mathematicians. But our symbolic language is not so helpful after all if impossibilities keep popping up in it. Just as a matter of curiosity, is there any sensible way of representing x such that $5 + x = 3$? Not only does curiosity kill cats but it has also a way of resurrecting problems thought to be very dead by the baffled solvers of the past. So it kept on bringing up this one. Since there is no x among the known numbers such that $5 + x = 3$, and since $5 - 2 = 3$, why not make up a new number "-2" such that $5 + (-2) = 3$? Certainly there was nothing to stop mathematicians from inventing such numbers and thereby furnishing themselves with ready-made solutions of the hitherto unsolvable. The only practical limitation of a good mathematical invention is that it shall not get the inventor into contradictions. This condition was met easily by the new creation, and so *negative* numbers came into being, distinguished from the corresponding *positive* ones by the presence of the minus signs at the immediate left of their middles.

It's about time, then, for a formal definition. *A* ***negative*** *number* $-\mathbf{b}$ *is the solution* $\mathbf{x} = -\mathbf{b}$ *of an equation* $\mathbf{a} + \mathbf{x} = \mathbf{c}$ *where* $\mathbf{a}$, $\mathbf{b}$, *and* $\mathbf{c}$ *are positive and* $\mathbf{c} + \mathbf{b} = \mathbf{a}$.

Since the negative number is invented, we are free to make any rules about it which do not involve contradictions. The rule that $a + (-b) = a - (+b)$, however, follows from our definition; and the convention that the negative of a negative number is positive, or $-(-b) = +b$, can be justified, as we shall see, as a special case of a rule for multiplication to be developed later.

And what a help these numbers proved to be! With the new mathematical magic to bolster her calculations, Mrs. Housewife could now neatly subtract the cost of a dress (10 dollars) from the balance on hand (4 dollars) and enter the remainder (minus 6 dollars) in the list of family assets. Thus dead-end streets, financial and otherwise, were opened on every side. Little Jimmie's progress toward school at times might be logged with minus figures. The two eternities of before and after, split apart by the passing moment, could be calibrated with negative and positive numbers to measure the past and the future. Thermometers fluctuate; trees grow up and down. The notion of direction in time or place is fundamental in our thinking; and it was given an important place in mathematics through the great idea of Descartes. This we shall discuss in due time.

EXERCISE 3

Combine the numbers in Probs. 1 to 4.

1. $3 + 2 - 7$. **2.** $7 - (-6) - 2$.

3. $11 - 7 + (-2) - (-7)$. **4.** $-(-9) - (-5) + (-6) - 7$.

5. Noting your results in Probs. 1 to 4, formulate rules for adding quantities with like signs and with unlike signs.

Find the sum of the expressions in Probs. 6 to 13.

6. $3x + 2y + 2$, $x + 4y + 1$, $7x + 3y + 4$.

7. $5a + 2b + 4c$, $3a + b + 5c$, $a + 6b + 5c$.

8. $3x + 2y$, $3y + 5z$, $7x + 4y + 2z$, $x + y$.

9. $3a$, $4b$, $5a + c$, $7b + 4c$, $2a + 6b + 5c$.

10. $2x - 4y - 5z$, $5x + 8y - 2z$, $-3x - 4y + 7z$.

11. $-r - 4s - 3t$, $5r - 2s + 4t$, $-3r + 7s - 6t$.

12. $-3u + 4v - 2w$, $2u - 6v$, $8v + 2w$.

13. $3a - 2b + 4d$, $5a - 2c - 6d$, $-2a + 3b - 7d$, $7b - 5c + 4d$.

In Probs. 14 to 21, subtract the second expression from the first.

14. $8a$, $5a$. **15.** $a + 3b$, $2a + 5b$.

16. $-7x$, $5x$. **17.** $4x - 3y + 7z$, $2x + 5y + 3z$.

18. $-5r + 2s + 3t$, $6r - 4s - 2t$.

19. $8a - 2b - 3c$, $11a - 7b + 6c$.

20. $5x + z$, $2x - 3y + 2z$.

21. $4u + 7v + 7x$, $3v - 8w + 2x$.

22. Noting your results in Probs. 14 to 21, formulate a rule for subtracting like quantities.

Find x in each of Probs. 23 to 26.

23. $3 + x = 7$. **24.** $8 + x = 2$. **25.** $4 + x = 4$.

26. $-5 + x = -5$.

27. Define and illustrate (*a*) the commutative law of addition; (*b*) the associative law of addition.

28. The symbols ( ), [ ], and { } are called *symbols of grouping.* When a pair of the above symbols preceded by a minus sign in an expression is removed, the sign of every term inclosed in the pair must be changed. Why?

Remove the symbols of grouping in Probs. 29 to 36 and combine. Note that when one or more pairs are included in another, it is customary and as good a way as any, to remove the innermost symbols first.

29. $a - 3b + (2a - b)$.
30. $3x + 2y + (-x - 2y)$.
31. $3a + 4b - (2a - 3b)$.
32. $4x - 3y - (-2x + 4y)$.
33. $4a - [2a - (4a - 2a) + b]$.
34. $4x - [-3x - (2y - 4x)]$.
35. $2a - \{a - [3a - (2a + 3b) - b]\}$.
36. $3x - \{4x - [2a - (3a - b) + 2b] - 4a\}$.

37. If 0 indicates the normal height of a river on a river gauge, $+1$ and -1 would indicate stages 1 ft. above and below, respectively, the normal level. Give three other examples in which physical significance is given to negative numbers.

38. If $a + b = m$ and $a - b = n$, express $3a + b$ in terms of m and n.

39. A boy 17 years of age attends school in a building that was constructed 10 years before his birth. His home was built 5 years after that and was remodeled when he was 8 years of age. The family car was bought 6 years later. Which of the above numbers can be interpreted as negative? How old is the car? What arithmetical process did you use to get the last answer? How old is the school building? What rule dealing with negative numbers would enable you to get the last result by subtraction?

40. The lobby of an office building is 20 ft. high, the other floors are 10 ft. apart, and the basement floor is 12 ft. below the lobby floor. An elevator boy starts from the lobby and makes the following successive stops: third floor, second floor, eighth floor, third floor, lobby, basement, fourth floor, lobby. If the upward direction is positive, find the total positive and total negative distances traversed.

12. Speeding up. In the early, formative stages of the symbolic language, changes naturally came rather rapidly. For instance, the product of the two numbers a and b was shortened to $a \times b$ and finally to the form ab which we use today. Likewise the product of four a's was at first written $aaaa$; of five a's, $aaaaa$; of a thousand a's—well, you can see for yourself that an improvement was needed. Herigone used the form $a4$ for $aaaa$ (about 1634) and Descartes in 1637 suggested the present notation of a^n to represent the product of n factors a. The index n is known as the *exponent* and the number a as the *base*.

The product a^5a^7, for example, must contain five a's because of the first factor and seven a's on account of the second. Hence there are twelve a's in the product, and $a^5a^7 = a^{12}$. The more general case in which the exponents are letters, so that they can represent any positive integer whatever, yields the *first law of exponents,*

(**1**)[1] $$\boldsymbol{a^m a^n = a^{m+n}} \quad (m \text{ and } n \text{ positive integers})$$

When the exponents are equal and the bases different, as is the case a^4b^4, the early algebraist probably worried around with it somewhat like this: "Wonder if I can get another law out of that. Maybe I'd better write it out in the old form, thus: *aaaabbbb*. Now if I rearrange it like this, *abababab*, and put some parentheses in, so, $(ab)(ab)(ab)(ab)$, then I'll have—why, sure, I'll have $(ab)^4$. Or, just as easy, $a^{100}b^{100} = (ab)^{100}$. Looks like I've got something there! Of course I'll have to shift the letters around, but that ought to be allowable since we can do it with numbers. For instance, $4 \times 3 \times 2 = 4 \times (3 \times 2) = (2 \times 4) \times 3 = 24$. Evidently to prove this law I'll need a couple of postulates."

At this point we unquote and give you that second law,

(**2**) $$\boldsymbol{a^m b^m = (ab)^m} \qquad (m \text{ a positive integer})$$

The postulates which we have to assume in order to prove the law seem sensible enough:

1. *Multiplication is* **commutative:** that is, $ab = ba$.
2. *Multiplication is* **associative;** or, in symbols,

$$a(bc) = (ab)c = abc$$

A third postulate which should be mentioned here, though it is not needed above, is this:

3. *Multiplication is* **distributive,** or $a(b + c) = ab + ac$. Try it out with numbers and be convinced.

There is a useful *third law of exponents,* as follows:

(**3**) $$\boldsymbol{(a^m)^n = a^{mn}} \quad (m \text{ and } n \text{ positive integers})$$

This law follows rather easily from Descartes' definition of the exponential form, since by it $(a^m)^n$ contains n factors a^m, and each of these in turn has m factors a. Altogether, then, there are mn factors a, which proves the law.

[1] Equations whose numbers are printed in boldface are important reference formulas.

So far we have had an easy time following the old pioneers. The path of least resistance in algebraic progress lay practically dead ahead, broad and smooth and without forks. But trouble must have brought them up with a jerk when they tried multiplying with the newfangled negative numbers. What, come to think about it, ought to be the sign of the product of two minus quantities? Probably you've heard that it's positive, but is there any special reason why that makes sense? Before we look into the matter we'll give the umpire's final decision, set forth in such easily remembered form that all future headscratching can be reserved for weightier matters: *The product of two terms of like signs is positive; of unlike signs, negative.*

Fair enough, you say, but what is this? Is it one of those convenient postulates, or is it, maybe, something that can be proved? High-school algebras don't say; but if you'll grant us our premises, definitions, and postulates, we'll show that it belongs in the latter category.[1]

We need first to recall our definition of a negative number $-b$ as the solution of the equation $a + x = c$, where a is positive and larger than c and $c + b = a$. We also need, by way of premises, the following postulates, which are so in accord with our sense of the fitness of things that they are what we might call "postulates in the best of standing," or *axioms*. Formally, *an axiom is an unproved statement which seems to accord with experience.* As far as the mathematician is concerned, axioms and postulates are essentially the same, being merely bowstrings from which are shot the inevitable conclusions that make up his stock in trade. On the other hand, to the philosopher-mathematician the seeming harmony with experience gives the axiom a decided edge in any contest about usefulness. You may think of one informally, if you wish, as something which any idiot should see ought to be true, but which no idiot did until the Greek geometer Euclid made seeing it fashionable. But enough of this; let us pass on to the axioms themselves before the water gets too deep.

***Axiom* 1.** *If equals are multiplied by equals, the products are equal.*

[1] The ensuing discussion up to the last three paragraphs of this article will perhaps be of more interest to the teacher than to the student, and may be omitted at the teacher's discretion without loss of continuity.

Axiom **2.** *If equals are added to equals the sums are equal.*

Axiom **3.** *If* ***a, c,*** *and* ***d*** *represent three positive integers with* ***a*** *greater than* ***c,*** *then* ***da*** *is greater than* ***dc.***

Our rule for signs can then be deduced piecemeal in the following steps. In all cases the multiplicand, or number multiplied, will be represented by $+b$ or $-b$, the multiplier by $+d$ or $-d$, and all letters will stand for positive numbers. We shall also find it convenient to use the symbol $>$ for *is greater than* and $<$ for *is less than.*

i. Let $a + b = c$, with $c > a$.

Then

$$d(a + b) = dc \qquad \text{(Axiom 1)}$$

or

$$da + db = dc$$

(by the distributive postulate for multiplication)

Furthermore, $dc > da$, by Axiom 3. Hence, db is positive.

ii. Let $a + (-b) = a - b = c$, with $a > c$.

Then

$$da + d(-b) = dc \qquad \text{(Axiom 1)}$$

But $da > dc$ by Axiom 3. Hence, $d(-b)$ is negative.

iii. $(-d)(b) = b(-d)$

by the commutative postulate for multiplication. Therefore $(-d)(b)$ is negative by the preceding result.

iv. Let $a + (-b) = c$, with $a > c$.

$$(-d)[a + (-b)] = (-d)c \qquad \text{(Axiom 1)}$$

$$(-d)(a) + (-d)(-b) = (-d)c$$

(by the distributive postulate)

or

$$-da + (-d)(-b) = -dc \qquad \text{(by iii)}$$

Adding $da + dc$ to both sides, by Axiom 2,

$$dc + (-d)(-b) = da$$

But since $da > dc$ by Axiom 3, $(-d)(-b)$ must be positive.

Now that we have our rule of signs in multiplication as a consequence of certain postulates and definitions, we can justify some of the rules for addition and subtraction of algebraic terms, which most students use without knowing why. "When we subtract we change the sign of the quantity subtracted and add to the minuend"—so drones the placid unquestioning learner. In symbols, $a - b = a + (-b)$—a conclusion which follows from our definition of a negative number $-b$. But what about addition? What *is* this sum $a + (-b)$ when a and b are positive quantities, or in general what is the sum of terms of opposite signs? We can get this sum by the formal rule of "subtracting the numerically smaller from the numerically larger and prefixing the sign of the larger." Thus $5x - 2x = 3x$ obviously; but why should $(-5x) + 2x$ be $(-3x)$? Because $(-5x) + (2x) = (-1)(5x) + (-1)(-2x)$ (by the rule of signs) $= (-1)[5x + (-2x)]$ (by the distributive postulate of multiplication) $= (-1)(5x - 2x)$ (from the above rule of subtraction) $= (-1)(3x) = -3x$.

In the solution of many problems it is necessary to find the product of two algebraic sums. The limited machinery we have thus far worked out enables us to do this in some cases, as for instance in the multiplication of $x^2 - 2x + 1$ by $2x^2 - x - 1$. The process may be indicated thus:

$$
\begin{aligned}
(x^2 - 2x + 1)(2x^2 - x - 1) &= (x^2 - 2x + 1)[2x^2 + (-x) + (-1)] \\
&= (x^2 - 2x + 1)(2x^2) + (x^2 - 2x + 1)(-x) + (x^2 - 2x + 1)(-1) \\
&\quad \text{(by the distributive postulate)} \\
&= 2x^4 - 4x^3 + 2x^2 - x^3 + 2x^2 - x - x^2 + 2x - 1 \\
&\quad \text{(by the distributive postulate and by the laws of exponents and signs)} \\
&= 2x^4 + (-4x^3 - x^3) + (2x^2 + 2x^2 - x^2) + (-x + 2x) + (-1) \\
&\quad \text{(by the commutative and associative postulates of addition)} \\
&= 2x^4 - 5x^3 + 3x^2 + x - 1
\end{aligned}
$$

The work may be much more conveniently arranged in the form

$$\begin{array}{rrrrr}
2x^2 & - \ x & - \ 1 & & \\
x^2 & - \ 2x & + \ 1 & & \\
\hline
2x^4 & - \ x^3 & - \ x^2 & & \\
 & - \ 4x^3 & + \ 2x^2 & + \ 2x & \\
 & & 2x^2 & - \ x & - \ 1 \\
\hline
2x^4 & - \ 5x^3 & + \ 3x^2 & + \ x & - \ 1
\end{array}$$

When the two algebraic sums to be multiplied are binomials we can perform the operation mentally by noting the pattern suggested by the following typical multiplication:

$$\begin{array}{rrr}
3x & + \ 4y & \\
2x & - \ 5y & \\
\hline
6x^2 & + \ 8xy & \\
 & - \ 15xy & - \ 20y^2 \\
\hline
6x^2 & - \ 7xy & - \ 20y^2
\end{array}$$

Here the first and last terms in the product are obtained by multiplying the two first and the two last terms, respectively, of the binomials. The middle term is the sum of the products of the two "inside" and the "outside" terms when the indicated product is written in the form $(3x + 4y)(2x - 5y)$.

It may be noted that the sum of the inside and outside products can be combined, as in the example above, whenever the two left terms and also the two right terms are "like" or contain the same letters with the same corresponding exponents. But even when this is not the case, the product can be obtained mentally by following the indicated pattern. For example,

$$\begin{aligned}(a + b)(c + d) &= \text{“left”} + \text{“inside”} + \text{“outside”} + \text{“right”} \\ &= ac + bc + ad + bd\end{aligned}$$

Or again,

$$(3x^2 + y^2)(2x + y^3) = 6x^3 + 2xy^2 + 3x^2y^3 + y^5$$

Certain special cases of these products of binomials recur often enough to be worth memorizing by themselves, though we should not lose sight of the fact that they may be obtained almost as rapidly by following the one rule for multiplying binomials written side by side. They are

$$(a + b)^2 = a^2 + 2ab + b^2 \tag{4}$$
$$(a - b)^2 = a^2 - 2ab + b^2 \tag{5}$$
$$(a + b)(a - b) = a^2 - b^2 \tag{6}$$

The last formula may be stated in English thus: *The product of the sum and difference of two quantities is equal to the difference of their squares.* The corresponding statements for the other two are deferred to Prob. 55, Exercise 4.

EXERCISE 4

Perform the operations indicated in Probs. 1 to 8 and simplify.

1. $(3)(4) + (2)(-6) - (-5)(4)$.

2. $(-5)(4) - (-2)(-5) + (7)(-3)$.

3. $(3x)(-2y) + (-4x)(-2y) - (4x)(-3y)$.

4. $(2a)(-3b)(-c) - (-3a)(+2b)(-3c) - (4a)(-5b)(2c)$.

5. $(-2x)(-7y) - (4a)(-2b) - (-3a)(-2b) - (7x)(-2y)$.

6. $(-5r)(3s) - (7t)(-3u) + (2r)(-s) - (7t)(-4u)$.

7. $4(a - b) - 6(a + b) - 2a + 3b$.

8. $6[x - 2(x - y) + 3y] - 4(x - 2y) - 3x + 2y$.

Reduce and simplify the expressions in Probs. 9 to 22.

9. a^2a^3a. **10.** $(2b^2)(3b^4)$. **11.** $(a^2b)(3a^3b^2)$.

12. $(2a^{b-1})(3a)$. **13.** $(a^2b^3)^4$. **14.** $(2x^2y)^3$.

15. $(-4ab)^2(a^2b)$. **16.** $(2x^2y)^2(-xy^2)^3$. **17.** $(2a^2bc)^2(3ab)^4$.

18. $(2a^b)(-3a^{1-b})(4a^{b-1})$. **19.** $(2x^a)(4x^b)$.

20. $(x^{a-1})(x^{2a+3})(x^{a+1})^2$. **21.** $(a^{m-1})^2(b^3)(ab)^2$.

22. $(a^{2m+1})(a^{1-m})^2(a)^2$.

Find the products indicated in Probs. 23 to 30.

23. $(x^2 - xy + y^2)(x + y)$.

24. $(a^2 + ab + b^2)(a - b)$.

25. $(x + y - z)(x - y + z)$.

26. $(2x^3 - 3x^2 + 2x - 1)(x - 1)$.

27. $(3a^2b - 2ab^2 - b^3)(a + b)$.

28. $(3a^3 - 4a^2b - 2ab^2 + b^3)(a - b)$.

29. $(x^4 + x^3y + x^2y^2 + xy^3 + y^4)(x - y)$.

30. $(a^2 + b^2 + c^2 - ab + ac - bc)(a + 2b - c)$.

Find the products indicated in Probs. 31 to 46 mentally.

31. $(2x + y)(x - 2y)$.

32. $(3a + b)(2a + 3b)$.

33. $(4a - 3b)(a + b)$.

34. $(3x - 2y)(4x + 3y)$.

35. $(3r + 2s)(5r - 3s)$.

36. $(x - 2y)(x + 2y)$.

37. $(3a - b)(3a + b)$.

38. $(5x + 7y)(5x - 7y)$.

39. $(6x - 5y)(x + y)$.

40. $(2a - 3b)(2a - 5b)$.

41. $(2a^2 + b)(a - b)$.

42. $(3x - 2)(2x^2 + 3)$.

43. $(2a + b)(c + 3d)$.

44. $(ax^2 + by)(ax + by^2)$.

45. $(5h + 7k)(2h^2 - k^2)$.

46. $(xz - 2v)(vz - x)$.

Given $(15)^2 = 225$, $(18)^2 = 324$, $(25)^2 = 625$, find the products in Probs. 47 to 54 mentally.

47. (14)(16).

48. (17)(19).

49. (24)(26).

50. (13)(23).

51. (22)(28).

52. (15)(35).

53. (11)(25).

54. (19)(31).

55. Put into English the rule stated by Formulas (4) and (5).

56. The square of any two-digit number ending in 5 may be obtained mentally by the following rule: Multiply the first digit by one more than itself and write 25 after the product. Prove that this is true by squaring $10t + 5$.

57. Using the process of Prob. 56, find the squares of all two-digit numbers ending in 5.

58. Using $(10t + u)^2$, devise a rule for squaring any two-digit number mentally.

13. Breaking up. Now that we have finished temporarily with multiplication, or "putting together," the inverse process of division, or "taking apart," comes up for attention.

The symbol a/b, read "a divided by b," is called an *algebraic fraction* and represents a number x such that

$$bx = a \tag{1}$$

In the fraction $\frac{a}{b}$, a is the *numerator*, b the *denominator*, and the horizontal bar (division sign) represents the operation of division, or of finding x when a and b are numbers. It also represents the operation of finding a simpler form for an algebraic fraction when the numerator and denominator are such combinations of letters and numbers that the simplification is possible.

To illustrate, just as $\frac{100}{4} = 25$, since $(4)(25) = 100$, so

$$\frac{x^3 + 3x^2 + 5x + 6}{x^2 + x + 3} = x + 2 \text{ since } (x + 2)(x^2 + x + 3) = x^3 + 3x^2 + 5x + 6$$

Evidently the indicated division is considerably shortened if we know in advance two factors of the dividend, one of which is the denominator. Thus the process of division may be divided into two parts, to wit: (1) the laborious kind, or *long* division; and (2) the prefabricated type of division known as *factoring.*

But before we can deal with long division it is necessary to have before us some postulates concerning operations with fractions and some additional laws of exponents. The first postulate is:

$$(2) \qquad \left(\frac{a}{b}\right)\left(\frac{c}{d}\right) = \frac{ac}{bd}$$

or, *the product of two or more fractions equals the product of the numerators divided by the product of the denominators.*

This rule is a good algebraic postulate in that it is in accord with the rules of arithmetic. For example, since $\frac{1}{3}(6)$, or "$\frac{1}{3}$ times 6," is also read "$\frac{1}{3}$ of 6," which is 2, we see that $\frac{1}{3}(6) = (\frac{1}{3})(\frac{6}{1}) = \frac{(1)(6)}{(3)(1)} = \frac{6}{3} = 2$. Note that the denominator 1 is understood when none is written, so that in general $(a)\left(\frac{b}{c}\right) = \left(\frac{a}{1}\right)\left(\frac{b}{c}\right) = ab/c$.

Another law of exponents is needed here, as follows:

$$(3) \qquad \frac{a^m}{a^n} = a^{m-n} \quad (m > n;\ m \text{ and } n \text{ positive integers})$$

Proof: By our definition of a fraction, a^m/a^n is a number x such that $a^n x = a^m$. But $a^n(a^{m-n}) = a^{n+m-n} = a^m$ by Law 1, Art. 12, and hence $x = a^{m-n}$. The symbol a^{m-n} as yet has no meaning when $m \leqq n$.

Thus a fraction such as ax^m/bx^n, which often comes up in long division, may be simplified as follows:

$$\frac{ax^m}{bx^n} = \left(\frac{a}{b}\right)\left(\frac{x^m}{x^n}\right) \text{ by (2) } = \frac{a}{b}x^{m-n} \text{ by (3)}$$

14. Long division. The details of a long division problem are perhaps best explained by means of an example.

Problem: Find the quotient and remainder when $6x^3 - 2y^3 - 6xy^2 - 7x^2y$ is divided by $3x + y$.

NOTE. If we call the first expression the *dividend* and the second one the *divisor* (replacing in long division the words "numerator" and "denominator" for the parts of a fraction) we have

$$\frac{\text{dividend}}{\text{divisor}} = \text{quotient} + \frac{\text{remainder}}{\text{divisor}}$$

Thus the word "quotient," as used here, does not mean the total result of the division unless the remainder is zero. The quotient of 17 divided by 5, for example, is 3, while the remainder is 2.

Solution: We'll write the mechanical steps first, numbering the lines, and explain the details afterward.

$$\begin{array}{lrl}
1 & & 2x^2 - 3xy - y^2 \text{ (quotient)} \\
2 & 3x + y \,/ & 6x^3 - 7x^2y - 6xy^2 - 2y^3 \text{ (dividend)} \\
3 & & \underline{6x^3 + 2x^2y} \\
4 & & \quad - 9x^2y - 6xy^2 - 2y^3 \\
5 & & \quad \underline{- 9x^2y - 3xy^2} \\
6 & & \qquad - 3xy^2 - 2y^3 \\
7 & & \qquad \underline{- 3xy^2 - y^3} \\
 & & \qquad\qquad - y^3 \text{ (remainder)}
\end{array}$$

On line 2 the divisor appears at the left of the dividend. In both the terms are arranged in the order of descending powers of x (we could have used y).

The first term of the dividend was divided by the first term of the divisor, and the result, $2x^2$, was written above in line 1 as the first term in the quotient.

Next, the divisor was multiplied by $2x^2$, and the product terms were placed on line 3 to be subtracted from corresponding terms in line 2.

The process was repeated until we finally reached the remainder, $-y^3$.

15. Prefabricated division, or factoring. The easiest part of factoring is that which can be disposed of in some formulas to be memorized, such as the following:

(1) $$ax + ay = a(x + y)$$

(2) $$x^3 - y^3 = (x - y)(x^2 + xy + y^2)$$

(3) $$x^3 + y^3 = (x + y)(x^2 - xy + y^2)$$

These formulas may of course be checked by simple multiplication of the indicated factors on the right. Equations (4) to (6) of Art. 12 may also be considered as factoring formulas when read from right to left. As for trinomials in general, probably the best method of factoring them is that of "trial and error." We start with any pair of trial factors yielding the correct first and last terms in the product, and then we test the choice by noting mentally the middle term in the product. For example, given $6x^2 - 7xy - 3y^2$, we might try $(6x - y)(x + 3y)$. This is rejected at once since the product's middle term is $17xy$. Incidentally this disposes of the trial factors $(6x + y)(x - 3y)$ at the same time. Next, $(2x + 3y)(3x - y)$ yields $7xy$ as the middle term, so that we reverse the second signs and get $(2x - 3y)(3x + y)$ as the factors.

Illustrative Examples

(1) $12a^3b^6 - 6a^2b^3 + 18a^4b^5 = 6a^2b^3(2ab^3 - 1 + 3a^2b^2)$. Here we have divided each of the terms by their greatest common divisor, writing the latter as a factor, according to Rule 1

(2) $64a^4 - 9b^2 = (8a^2 - 3b)(8a^2 + 3b)$ (Rule 6, Art. 12)

(3) $(x + y)^2 - 4z^2 = (x + y - 2z)(x + y + 2z)$ (Rule 6, Art. 12)

(4) $9a^2 - (b + c)^2 = [3a - (b + c)]\,[3a + (b + c)]$
$= (3a - b - c)(3a + b + c)$ (Rule 6, Art. 12)

(5) $1 - 27x^3 = 1^3 - (3x)^3 = (1 - 3x)\,[1 + 3x + (3x)^2]$ (Rule 2)
$= (1 - 3x)\,(1 + 3x + 9x^2)$

(6) $8a^3 + 27b^3 = (2a)^3 + (3b)^3 = (2a + 3b)(4a^2 - 6ab + 9b^2)$ (Rule 3)

EXERCISE 5

Find the quotients indicated in Probs. 1 to 10.

1. $(a^3 - b^3) \div (a - b)$. **2.** $(a^3 + b^3) \div (a + b)$.

3. $(x^3 - 8) \div (x - 2)$. **4.** $(27x^3 + 8y^3) \div (3x + 2y)$.

5. $(x^3 - x^2y - xy^2 + y^3) \div (x + y)$.

6. $(x^3 + x^2y - xy^2 - y^3) \div (x^2 + 2xy + y^2)$.

7. $(6a^4 - 7a^3b + a^2b^2 + 5ab^3 - 2b^4) \div (3a^2 + ab - b^2)$.

8. $(3x^4 - 4x^3y - 3x^2y^2 - 4y^4) \div (x^2 - xy - 2y^2)$.

9. $(x^5 - y^5) \div (x - y)$. **10.** $(x^4 + 4y^4) \div (x^2 + 2xy + 2y^2)$.

Find the quotients and remainders in Probs. 11 to 16.

11. $(3x^2 - 4xy + y^2) \div (x - 2y)$.

12. $(4a^2 - 5ab + 3b^2) \div (2a - b)$.

13. $(3x^3 - 2xy^2 + y^3) \div (x - 2y)$.
14. $(6a^4 - 4a^3b - 3ab^3 - 5b^4) \div (3a - 2b)$.
15. $(x^4 + y^4) \div (x + y)$.
16. $(x^6 - y^6) \div (x + y)$.

Factor the expressions in Probs. 17 to 54.

17. $2a - 4b + 6c$.
18. $3x - 9y + 12z$.
19. $4x^3 - 8x^2 + 12xy$.
20. $3a^2b - 45ab + 18ab^2$.
21. $2c(a + b) + 3d(a + b)$.
22. $(x + y + z)(2u) - (x + y + z)(3v)$.
23. $a^2 - 4$.
24. $4x^2 - y^2$.
25. $9x^4 - 4y^2$.
26. $25a^6 - 16b^4$.
27. $16x^8 - 25y^6$.
28. $9a^4 - 144b^8$.
29. $(x + y)^2 - 4$.
30. $(a - b)^2 - c^2$.
31. $(2x - 3y)^2 - 9z^2$.
32. $(4a - 2b)^2 - a^2$.
33. $4 - (x + y)^2$.
34. $x^2 - (y + z)^2$.
35. $4a^2 - (2b - c)^2$.
36. $9z^2 - (2x - 3y)^2$.
37. $a^2 + 4ab + 4b^2$.
38. $4x^2 + 4xy + y^2$.
39. $a^2 + 6ab + 9b^2$.
40. $9x^2 - 6xy + y^2$.
41. $4a^2 - 12ab + 9b^2$.
42. $9x^2 - 24xy + 16y^2$.
43. $a^2 - ab - 2b^2$.
44. $3x^2 - 2xy - y^2$.
45. $2a^2 + 3ab - 2b^2$.
46. $3x^2 + 8xy - 3y^2$.
47. $6a^2 - 13ab + 6b^2$.
48. $8x^2 + 10xy - 12y^2$.
49. $12a^2 + 95ab - 8b^2$.
50. $24x^2 - 145xy + 6y^2$.
51. $ab - 2b - 3a + 6$.
52. $x^2 - 2xy + 2x - 4y$.
53. $ac - 2bc + 2ab - 4b^2$.
54. $3xz - 6yz - wx + 2wy$.

16. Fractions, and what to do with them. In all operations with fractions it is important to keep in mind the following

Basic rule: *The value of a fraction is not changed when the numerator and denominator are multiplied or divided by the same quantity.*

For $a/b = (a/b)(1) = (a/b)(c/c) = ac/bc$ by (2), Art. 13, and we may also follow this argument from right to left.

The basic rule permits the dangerous practice of *cancellation*, illustrated herewith in correct form:

$$\frac{abc}{ade} \qquad \frac{\not{a}bc}{\not{a}de} = \frac{bc}{de}$$

A very solemn warning, however, is in order here. Many students think that this official word "cancellation" is a heaven-sent license to easy-chair algebraists permitting, and in fact recommending, the gleeful abolishment of all letters loitering in pairs above and below some horizontal line, or even some different lines in the same general neighborhood. For example, consider the fraction $\frac{a+b}{a+c}$, which might be operated upon like this: $\frac{\cancel{a}+b}{\cancel{a}+c} = \frac{b}{c}$. This type of mathematical sinning is not indulged in by clear thinkers because the value of a fraction is usually changed when the same quantity is *subtracted* from numerator and denominator. If we could do it with numbers we would get, for instance, $\frac{2}{3} = \frac{1+1}{1+2} = \frac{\cancel{1}+1}{\cancel{1}+2} = \frac{1}{2}$. To summarize, cancellation brings about a permissible and helpful simplification when the thing canceled, say a, can be shown to be *a factor of both the numerator and the denominator,* thus:

$$\frac{\cancel{a}(\text{rest of numerator})}{\cancel{a}(\text{rest of denominator})} = \frac{\text{rest of numerator}}{\text{rest of denominator}}.$$

Frequently, in the multiplication of fractions by Rule 2, Art. 13, we can save time if we look ahead and see what common factors will occur in the numerator and denominator of the product. We can then cancel them before multiplying, thus avoiding unnecessary copying.

Example 1. $$\left(\frac{8a^3b}{3c}\right)\left(\frac{6c}{4a^2}\right) = \left(\frac{\overset{2a}{\cancel{8a^3b}}}{\cancel{3c}}\right)\left(\frac{\overset{2}{\cancel{6c}}}{\cancel{4a^2}}\right) = 4ab.$$

Example 2. $$\left(\frac{a^2-b^2}{ab}\right)\left(\frac{a^2}{a^2-ab+2b^2}\right)\left(\frac{a-2b}{a-b}\right)$$

$$= \left[\frac{\cancel{(a-b)}\cancel{(a+b)}}{\cancel{a}b}\right]\left[\frac{\cancel{a}a}{\cancel{(a-2b)}\cancel{(a+b)}}\right]\left[\frac{\cancel{a-2b}}{\cancel{a-b}}\right] = \frac{a}{b}.$$

Note that if, in Example 1, we had started with $\frac{8a^3b}{3c} + \frac{6c}{4a^2}$, the canceling would have been ruled out by the presence of the plus sign. We'll take up shortly the correct procedure in this case.

The rule for division of fractions now follows logically from that for multiplication. By our definition of an algebraic fraction, $\frac{a}{b} \div \frac{c}{d}$, or $\frac{a/b}{c/d}$, means a number x such that $x\left(\frac{c}{d}\right) = \frac{a}{b}$ Since

$$\left(\frac{ad}{bc}\right)\left(\frac{c}{d}\right) = \frac{a}{b}, x = \frac{ad}{bc} = \left(\frac{a}{b}\right)\left(\frac{d}{c}\right)$$

Therefore we get the formula

$$(1) \qquad \frac{a}{b} \div \frac{c}{d} = \left(\frac{a}{b}\right)\left(\frac{d}{c}\right)$$

which, stated as a directive, tells us that *to simplify the quotient of two fractions we invert the denominator fraction and multiply it by the numerator.*

Example: $$\frac{6x^2 - 6y^2}{x + 2y} \div \frac{3(x + y)^2}{x-2y} = \frac{\overset{2}{\cancel{6}}(x - y)\cancel{(x + y)}}{x + 2y} \cdot \frac{x - 2y}{\cancel{3}(x + y)\cancel{(x + y)}}$$
$$= \frac{2(x - 2y)(x - y)}{(x + 2y)(x + y)}$$

For addition of fractions we need the formula

$$(2) \qquad \frac{a}{c} + \frac{b}{c} = \frac{a + b}{c}$$

which states in symbolic form the rule that *to add two or more fractions with the same denominator, we place the sum of the numerators over the common denominator.*

This formula follows readily from the distributive law of multiplication and the foregoing rules concerning fractions, thus:

$$\frac{a}{c} + \frac{b}{c} = a\left(\frac{1}{c}\right) + b\left(\frac{1}{c}\right) = (a + b)\left(\frac{1}{c}\right) = \left(\frac{a + b}{1}\right)\left(\frac{1}{c}\right) = \frac{a + b}{c}$$

It applies to *subtraction* as well as addition of fractions, since

$$\frac{a}{c} - \frac{b}{c} = \frac{a}{c} + \frac{(-b)}{c} = \frac{a - b}{c}$$

At this point we need to learn, or recall, the meaning of the phrase "least common multiple."

A *common multiple* of a set of algebraic expressions[1] (including numbers as special cases) is an expression which is divisible by each member of the set. It is the *least common multiple* (**L.C.M.**) of the set if its degree and its numerical coefficients are smaller than those of any other common multiple.[2] For example, the L.C.M. of 24, 36, and 48, or of $2^3 \times 3$, $2^2 \times 3^2$, and $2^4 \times 3$, is $2^4 \times 3^2$, or 144; while the L.C.M. of $(y - x)(y + x)$ and $(x - y)^3$ is either $(x - y)^3(x + y)$ or $(y - x)^3(x + y)$.

To apply the principle indicated by (2) to all fractions, we find first the L.C.M. of the denominators involved, which we call the *least common denominator* (**L.C.D.**). Then we replace each fraction by an equivalent one containing the L.C.D. as its denominator.

Example 1. Find the sum of a/bc, c/bd, and e/c^2d.

Solution: Here the L.C.D. is bc^2d. Using the basic rule stated at the beginning of this article, we have

$$\frac{a}{bc} + \frac{c}{bd} + \frac{e}{c^2d} = \frac{acd}{bc^2d} + \frac{c^3}{bc^2d} + \frac{eb}{bc^2d} = \frac{acd + c^3 + eb}{bc^2d}$$

Example 2. $\dfrac{x}{(x - y)^2} + \dfrac{y}{x^2 - y^2} - \dfrac{2xy}{(x^2 - y^2)(x - y)}$

$$= \frac{x(x + y)}{(x - y)^2(x + y)} + \frac{y(x - y)}{(x^2 - y^2)(x - y)} - \frac{2xy}{(x^2 - y^2)(x - y)}$$

$$= \frac{x^2 - y^2}{(x^2 - y^2)(x - y)} = \frac{1}{x - y}$$

EXERCISE 6

Perform the operations indicated in Probs. 1 to 14 and express the answer in the simplest form.

1. $\left(\frac{1}{2}\right)\left(\frac{3}{4}\right)\left(\frac{16}{9}\right)$.

2. $\left(\frac{5}{8}\right)\left(\frac{16}{25}\right)\left(\frac{15}{4}\right)$.

3. $\left(\frac{2}{3}\right)\left(\frac{9}{4}\right) \div \left(\frac{15}{8}\right)$.

4. $\left(\frac{7}{8}\right)\left(\frac{16}{21}\right) \div \left(\frac{4}{9}\right)$.

5. $\left(\frac{4x^2}{3y^2}\right)\left(\frac{x}{8y}\right)\left(\frac{12y^4}{x^3}\right)$.

6. $\left(\frac{5a}{2b}\right)\left(\frac{6b^2}{5a^3}\right)\left(\frac{7a}{9b}\right)$.

[1] The expressions are here understood to be integral and rational, which means that the exponents of all letters involved are positive integers (see Art. 22).

[2] We here consider an L.C.M. and its negative as essentially the same.

7. $\left(\frac{2a^2}{5b^2}\right)\left(\frac{25ab}{12c^2}\right) \div \left(\frac{20b^2}{9c}\right)$.

8. $\left(\frac{4x^3}{7y^2}\right)\left(\frac{21y^3}{16x}\right) \div \left(\frac{9y}{4x^2}\right)$.

9. $\left(\frac{xy}{x+y}\right)\left(\frac{x^2-y^2}{x^2}\right)$.

10. $\left(\frac{a+b}{a}\right)\left(\frac{a-b}{b}\right)\left(\frac{a^2b^2}{a^2-b^2}\right)$.

11. $\left(\frac{2x-y}{4xy}\right)\left(\frac{2x^2}{4x^2-y^2}\right)\left(\frac{2x+y}{x}\right)$.

12. $\left(\frac{9a^2-4b^2}{12ab}\right)\left(\frac{4a^2b^2}{3a+2b}\right)\left(\frac{3}{3a-2b}\right)$.

13. $\left(\frac{4x^2-y^2}{x-y}\right)\left(\frac{x-y}{2x+y}\right)\left(\frac{x}{2x-y}\right)$.

14. $\left(\frac{a^2-ab-6b^2}{a-b}\right)\left(\frac{a^2-b^2}{a-3b}\right)\left(\frac{b}{a+2b}\right)$.

Combine the fractions in Probs. 15 to 38 into a single fraction in the simplest form.

15. $\frac{9}{4}+\frac{5}{12}-\frac{3}{2}$.

16. $\frac{7}{18}+\frac{1}{2}-\frac{2}{9}$.

17. $\frac{1}{8}+\frac{17}{12}-\frac{19}{24}$.

18. $\frac{7}{6}-\frac{2}{3}+\frac{1}{10}$.

19. $\frac{7}{3a}+\frac{1}{2a}-\frac{5}{6a}$.

20. $\frac{3}{x}+\frac{2}{3x}-\frac{1}{2x}$.

21. $\frac{3}{5y}+\frac{7}{4y}-\frac{17}{20y}$.

22. $\frac{5}{2x}+\frac{7}{12x}-\frac{11}{4x}$.

23. $\frac{2}{b}+\frac{2+a}{2a}-\frac{b+2a}{ab}$.

24. $\frac{3x+4y}{2xy}+\frac{2x-6}{3x}-\frac{3}{2y}$.

25. $\frac{b+ac}{bc}-\frac{a-1}{b}-\frac{1}{c}$.

26. $\frac{2+3k}{2hk}+\frac{3h-6}{4h}+\frac{2h-1}{hk}-\frac{2}{k}$

27. $\frac{5x+2y}{3x+3y}-\frac{x}{x+y}$.

28. $\frac{a}{a+b}-\frac{a-2b}{3a+3b}$.

29. $\frac{2x^2+6xy}{x^2-4y^2}-\frac{2x+y}{x-2y}$.

30. $\frac{3a+b}{a^2-b^2}-\frac{2}{a-b}$.

31. $\frac{6x+4y}{x^2-y^2}-\frac{1}{x+y}-\frac{3}{x-y}$.

32. $\frac{1}{a+b}+\frac{2}{a-b}-\frac{4b}{a^2-b^2}$.

33. $\dfrac{a-2b}{a+b}+\dfrac{2a}{a-b}-\dfrac{2a^2+2b^2}{a^2-b^2}.$

34. $\dfrac{2zy-x+y}{x^2-y^2}+\dfrac{2z+1}{x+y}-\dfrac{z}{x+y}.$

35. $\dfrac{a+b}{a^2-2ab-3b^2}+\dfrac{a-8b}{a^2-ab-6b^2}.$

36. $\dfrac{3}{x+y}+\dfrac{4x+3y}{2x^2+3xy-2y^2}-\dfrac{4x+7y}{x^2+3xy+2y^2}.$

37. $\dfrac{4a-2b}{a^2-b^2}+\dfrac{b}{2a^2-5ab+3b^2}-\dfrac{5a-10b}{2a^2-ab-3b^2}.$

38. $\dfrac{x+5y}{x^2-y^2}+\dfrac{7x+9y}{3x^2+7xy+4y^2}-\dfrac{8x+13y}{3x^2+xy-4y^2}.$

17. More Stories. Frequently we meet in mathematics a fraction which is *complex*, or "several stories high." Technically a fraction is complex if its numerator or denominator contains a fraction, or if there are fractions in both. In the sample: $\dfrac{\dfrac{a}{2}+b}{\dfrac{c}{4e}+d}$, 2 and $4e$ are called *minor denominators.*

To simplify a complex fraction we employ our basic rule and multiply numerator and denominator by the L.C.M. of the minor denominators. In the example above we get

$$\frac{\left(\dfrac{a}{2}+b\right)4e}{\left(\dfrac{c}{4e}+d\right)4e}=\frac{2ae+4be}{c+4ed}$$

In case the minor denominators themselves consist of two or more terms, it is often simpler to employ the principle involved in (1), Art. 16.

Example:
$$\frac{1-\dfrac{1}{x^2-y^2}}{1-\dfrac{1}{x^3-y^3}}=\frac{x^2-y^2-1}{x^2-y^2}\div\frac{x^3-y^3-1}{x^3-y^3}$$
$$=\frac{x^2-y^2-1}{(x-y)(x+y)}\cdot\frac{(x-y)(x^2+xy+y^2)}{x^3-y^3-1}$$
$$=\frac{(x^2-y^2-1)(x^2+xy+y^2)}{(x+y)(x^3-y^3-1)}$$

EXERCISE 7

Simplify the following complex fractions.

1. $\dfrac{1+\frac{1}{2}}{\frac{1}{2}}$.

2. $\dfrac{2}{1+\dfrac{2}{3}}$.

3. $\dfrac{2+\dfrac{1}{2}}{3-\dfrac{1}{3}}$.

4. $\dfrac{\dfrac{3}{4}-\dfrac{2}{3}}{1+\dfrac{1}{2}}$.

5. $\dfrac{1+\dfrac{1}{a}}{1-\dfrac{1}{a^2}}$.

6. $\dfrac{x-\dfrac{1}{x}}{1+\dfrac{1}{x}}$.

7. $\dfrac{\dfrac{3}{b}-\dfrac{3}{a}}{\dfrac{a}{b}-1}$.

8. $\dfrac{\dfrac{x}{y}-1}{1-\dfrac{y}{x}}$.

9. $\dfrac{a-\dfrac{a}{a-1}}{1-\dfrac{1}{a-1}}$.

10. $\dfrac{x-\dfrac{xy}{x-y}}{y-\dfrac{y^2}{x-y}}$.

11. $\dfrac{x+\dfrac{x}{x-1}}{x-\dfrac{x}{x+1}}$.

12. $\dfrac{1-\dfrac{b}{a+b}}{1+\dfrac{b}{a-b}}$.

13. $\dfrac{x+y+\dfrac{y^2}{x-y}}{x-\dfrac{xy}{x+y}}$.

14. $\dfrac{1+\dfrac{a+4}{a^2-4}}{1+\dfrac{3}{a-2}}$.

15. $\dfrac{1+1/x}{1-\dfrac{1}{1-1/x}}$.

16. $\dfrac{1+\dfrac{b^2}{a^2-b^2}}{a+b+\dfrac{1}{\dfrac{a}{b^2}-\dfrac{1}{b}}}$.

18. Some useful laws. The five so-called *laws of exponents*, some of which have been given already, are so important that they should be considered as a group in a form easily memorized, such as the following:

Group one (repeated base)

(1) $$a^m a^n = a^{m+n}$$

(2) $$\frac{a^m}{a^n} = a^{m-n}*$$

Note that we *hold the base*

Group two (repeated exponent)

(3) $$a^m b^m = (ab)^m$$

(4) $$\frac{a^m}{b^m} = \left(\frac{a}{b}\right)^m$$

Here we *hold the exponent*

Extra (single base)

(5) $$(a^m)^n = a^{mn}$$

When the laws are learned in the form above the student is somewhat less likely than he otherwise would be to make the mistake of claiming that $2^3 5^4$ is 10^7, or 10^{12}, or 7^7, or 7^{12}, or some other equally absurd figure, because he can see that *neither* the base nor the exponent is repeated and that there are *two* bases. Thus he knows at once (or should know) that none of the five laws of exponents applies in this case. On the other hand, if a law does apply, as in the case of $5^{14}2^{14}$, he should remember that *whatever is repeated is held*, so that $5^{14}2^{14} = (5 \times 2)^{14} = 10^{14}$, or 1 followed by 14 zeros, or 100 trillion. In this case the use of the law is much quicker than multiplication.

For the cases in which the exponents are positive integers the proofs of the laws are simple, and those not already given will be left to the reader.

19. We define with discretion. The foregoing laws of exponents may be said to give us a glimpse of one phase of absolute truth in the sense that they are necessary and eternal consequences of our agreed-upon definition of a^n, where n is a positive integer, $a \neq 0$, and $m > n$ in (2), Art. 18. As distinguished from such "laws," mathematical postulates in general are merely provisional and convenient agreements which may or may not represent underlying verities.

* The qualification given in (3), Art. 13, will be shown in the next article to be unnecessary.

When we examine the laws more closely, however, we find that much remains to be desired. For as yet they have meaning only when the exponents are positive integers and, in the case of the second law, when $m > n$. Now mathematicians look upon an exception much as nature is said to regard a vacuum. Their unfriendly attitude is due to the fact that exceptions are injurious if not fatal to each of the two most desirable attributes of a mathematical result—simplicity and generality. Every qualifying phrase works havoc with its simplicity; every exception hamstrings its generality.

Therefore, with a unanimity seldom matched in other branches of science, mathematicians all agreed that it would be highly convenient if the five laws could stand grandly intact and unqualified in all cases. Of course the wish might have to be given up, though with reluctance, if it led to contradictions. But, as it turned out, the wish could be and was fulfilled almost better than one had any right to expect in advance. The following definitions did the job.

(1) $$a^{-n} = \frac{1}{a^n}$$

(2) $$a^0 = 1$$

(3) $$a^{\frac{p}{q}} = \left(a^{\frac{1}{q}}\right)^p = \left(\sqrt[q]{a}\right)^p = \sqrt[q]{a^p} \quad \text{when } a > 0$$

(Unfortunately, it will be seen, we end with one final qualification after all; but one can't have everything. The even roots of negative numbers pose special problems which will be treated in the next chapter.)

As soon as it was realized that the process of broadening the five basic laws of exponents was merely a matter of defining the meaning of a^n, when n is not a positive integer, in such a way that the laws still hold without contradictions, the formulation of the proper definitions was easy. For example, since $a^3/a^5 = 1/a^2$ actually, and since $a^3/a^5 = a^{3-5} = a^{-2}$ by Law 2, it is incumbent upon us to decree that a^{-2} shall mean $1/a^2$. Again, since $a^m/a^m = 1$, and $a^m/a^m = a^{m-m} = a^0$ by Law 2, it seems wise, if we want to keep out of trouble, to insist that a^0 represents 1, whatever one might have been inclined to guess from its appearance. Note that then $a^0a^m = 1 \cdot a^m = a^m$, as it should by Law 1, so that everything is still lovely. And finally $(a^{2/3})(a^{2/3})(a^{2/3})$ ought to be, if Law 1 is to hold, the same as $a^{2/3 + 2/3 + 2/3} = a^2$, whereas, if the usual meaning of cube root is upheld, $(\sqrt[3]{a^2})(\sqrt[3]{a^2})(\sqrt[3]{a^2}) = a^2$ for sure. Hence we define $a^{2/3}$ as $\sqrt[3]{a^2}$, or more generally, $a^{\frac{p}{q}}$ as $\sqrt[q]{a^p}$. But since either of the symbols $(a^p)^{\frac{1}{q}}$ or $(a^{\frac{1}{q}})^p$ should represent $a^{\frac{p}{q}}$ if Law 3 is to hold, we decide that $a^{\frac{p}{q}} = (a^{\frac{1}{q}})^p = (\sqrt[q]{a})^p$ as well as $\sqrt[q]{a^p}$. For instance, $8^{2/3} = \sqrt[3]{8^2} = \sqrt[3]{64} = 4$, and $8^{2/3} = (\sqrt[3]{8})^2 = 2^2 = 4$. Since there is no contradiction in this, or any other arithmetic example of the two definitions (with the given restriction), they are consistent and may be used interchangeably, as convenient. The second form involves less "figuring" in the above example, while the first would be better in a case such as $7^{2/3} = \sqrt[3]{7^2} = \sqrt[3]{49}$.

20. How to treat radicals. The expression $\sqrt[n]{a}$ is called a *radical* of order n. If the *radicand* a is not a perfect nth power, $\sqrt[n]{a}$ is called a *surd*. Though radicals, as we have seen, may always be expressed by means of fractional exponents and combined by use of the five basic laws, it is sometimes convenient to pass directly to the desired result without taking the exponential detour. We therefore list below some of the laws governing radicals:

(1) $$\sqrt[n]{a}\sqrt[n]{b} = \sqrt[n]{ab}$$

and

(2) $$\frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\frac{a}{b}} \quad \text{when } a \text{ and } b \text{ are not both negative}$$

(3) $$\sqrt[m]{\sqrt[n]{a}} = \sqrt[mn]{a}$$

(4) $$\sqrt[n]{a^{kn}b} = a^k\sqrt[n]{b}$$

Laws 1, 2, and 4 may be proved by raising both sides to the nth power, applying the proper laws of exponents, and remembering that $(\sqrt[n]{a})^n = a$ by definition of $\sqrt[n]{a}$. To prove (3), we note that $\sqrt[m]{\sqrt[n]{a}} = (a^{\frac{1}{n}})^{\frac{1}{m}} = a^{\frac{1}{mn}}$ [by Law 5, Art. 18] $= \sqrt[mn]{a}$. Or, $(\sqrt[m]{\sqrt[n]{a}})^{mn} = [(\sqrt[m]{\sqrt[n]{a}})^m]^n = (\sqrt[n]{a})^n = a$, and also $(\sqrt[mn]{a})^{mn} = a$.

Illustrative Examples

(1) $\sqrt[3]{2x}\sqrt[3]{4x^2} = \sqrt[3]{(2x)(4x^2)} = \sqrt[3]{8x^3} = \sqrt[3]{(2x)^3} = 2x.$

(2) $\dfrac{\sqrt[5]{64x^{-3}}}{\sqrt[5]{-2x^2}} = \sqrt[5]{\dfrac{64x^{-3}}{-2x^2}} = \sqrt[5]{\dfrac{-32}{x^3x^2}} = \sqrt[5]{\left(\dfrac{-2}{x}\right)^5} = -\dfrac{2}{x}.$

(3) $\sqrt{\sqrt[3]{25x^2y^4}} = \sqrt[3]{\sqrt{25x^2y^4}} = \sqrt[3]{\sqrt{(5xy^2)^2}} = \sqrt[3]{5xy^2}.$

(4) $\sqrt[3]{16x^5y^7z^0} = \sqrt[3]{16x^5y^7} = \sqrt[3]{(8x^3y^6)(2x^2y)} = 2xy^2\sqrt[3]{2x^2y}.$

Here we have sketched briefly the considerations beneath the formal definitions, and we have seen that they are both simple and logical. But since the human race took, as nearly always, a rather devious path winding and looping toward this goal, you should appreciate that goal, now that it is reached, more than perhaps you do. For instance, going back along the zigzag trail a little, Chuquet (1484) used 12^0 for 12; 12^1 for 12 times a *nombre linear*, 12^2 for $12x^2$, 12^3 for $12x^3$, and so on. Though he shows that he had some ideas about negative exponents, it was more than two centuries before the troublesome details were cleared up. The mathematical historian D. E. Smith states that "the first of the writers to explain with any completeness the significance of negative and fractional exponents was John Wallis (1655)."

EXERCISE 8

Reduce and simplify the expressions in Probs. 1 to 14.

1. $\dfrac{2^5}{2^2}$

2. $\dfrac{2^{1000}}{2^{995}}.$

3. $\dfrac{x^{10}}{x^3}.$

4. $\dfrac{x^{3a}}{x^{2a}} \times \dfrac{x^{2b}}{x^b}.$

5. $\dfrac{(m^{25})^a}{(m^a)^{12}}.$

6. $\left(\dfrac{2a^2b^3}{3a^5b}\right)^2.$

7. $\left(\dfrac{4x^3y^4}{12x^2y^2}\right)^3.$

8. $\left(\dfrac{3a^2}{2b}\right)^3\left(\dfrac{8b^3}{9a^4}\right)^2.$

9. $(3x)^2(6y^3)^2.$

10. $(4a^2)^3(2a)^3.$

11. $(27x)^a(2y)^a.$

12. $(3a^2b)^{2c}(2ab^3)^{2c}.$

13. $[(3a^2b)^x(4a^4b^x)][(6a^3b^2)^y(2a^3)^y].$

14. $(2x^3)^{a+1}(3x)^{a+1}(6x^2)^{1-a}(x^2)^{1-a}.$

Prove the statements in Probs. 15 to 17.

15. $(24)^2(36)^2 = (2^5 3^3)^2$.

16. $\left[\dfrac{9(28)(49)}{2(3)^2(7)}\right]^2 = 2^2 7^4$.

17. $\dfrac{(625)(72)}{100} = 2(5^2)(3^2)$.

Reduce the expressions in Probs. 18 to 23 to common fractions without negative exponents:

18. 2^{-2}.

19. $\dfrac{3^{-2}}{2^{-3}}$.

20. $16^{-1/2}$.

21. $[(64)^{1/3}]^{-2}$.

22. $\left[\left(\dfrac{27}{8}\right)^{1/3}\right]^{-1}$.

23. $\left(\dfrac{8^{1/3}}{16^{1/4}}\right)^{-1}$

Find the numerical value of each of the expressions in Probs. 24 to 29.

24. $8^{2/3} + 4^{3/2} + 2^0$.

25. $(7^0 - 1)^5$.

26. $4^{-1/2} + 8^{-1/3} + 7(2)^{-4}$.

27. $\sqrt{32}\sqrt{2}$.

28. $\dfrac{\sqrt[3]{-24}}{\sqrt[3]{3}}$.

29. $\sqrt{16 + 9}$.

Express each of Probs. 30 to 33 as a simple fraction or integer without negative or fractional exponents:

30. $\left(\dfrac{2^{3a}}{2^{4a}}\right)^{-1/a}$

31. $\left(\dfrac{a^{3b}}{a^b}\right)^{-1/b}$.

32. $\left(\dfrac{x^{m+n}}{x^{m+2n}}\right)^{-2/n}$

33. $\left[\dfrac{x^{(a+b)/2}}{x^{(a-b)/2}}\right]^{2b}$.

Simplify Probs. 34 to 35, canceling where it is allowable.

34. $\dfrac{(m - n)x^{-2}y^3z^{-4}}{(m - n)x^4y^{-2}z^6}$.

35. $\dfrac{3^{-1}(2 + 4^{-1/2})}{3^{-2}(2 - 4^{-1/2})}$.

Change the radical expression in each of Probs. 36 to 47 to the simplest form.

36. $\sqrt{3x^3y}\sqrt{6xy^3}$.

37. $\sqrt{12a^5b^3}\sqrt{3a^2b}$.

38. $\sqrt{5x^7y^3}\sqrt{15xy^4}$.

39. $\sqrt[3]{16a^5b^2}\sqrt[3]{4ab^3}$.

40. $\sqrt[5]{2a^4b^4}\sqrt[5]{16ab^6}$.

41. $\sqrt[4]{8x^3y^2}\sqrt[4]{12x^5y^3}$.

42. $\dfrac{\sqrt{8a^3b^5}}{\sqrt{2ab^2}}$.

43. $\dfrac{\sqrt[3]{54x^5y^7}}{\sqrt[3]{2x^2y^2}}$.

44. $\dfrac{\sqrt[4]{128a^{15}b^{13}}}{\sqrt[4]{4a^3b^4}}$.

45. $\dfrac{\sqrt{288x^7y^{11}}}{\sqrt{2x^5y^5}}$.

46. $\dfrac{\sqrt{3a^2b}\,\sqrt{6ab^4}}{\sqrt{2ab}}$.

47. $\dfrac{\sqrt[3]{9x^4y^4}\,\sqrt[3]{12xy^2}}{\sqrt[4]{4x^2y}}$.

48. Prove that $\sqrt[m]{\sqrt[n]{a}} = \sqrt[n]{\sqrt[m]{a}} = \sqrt[mn]{a}$.

49. (*a*) Prove that $\sqrt{x^2 + y^2} = x + y$ is false.

(*b*) Can you give special values to x and y for which the statement in (*a*) is true?

Simplify each of the expressions in Probs. 50 to 53.

Illustrative Example

In order to apply the above instructions to

(1) $$4^{1/2}\sqrt{18} + \sqrt[3]{27}(50)^{1/2} - (16)^{1/4}\left(\frac{1}{\sqrt{32}}\right)^{-1}$$

we first simplify each term separately thus:

$$4^{1/2}\sqrt{18} = (2^2)^{1/2}\sqrt{(9)(2)} = (2)(3)(\sqrt{2}) = 6\sqrt{2}$$

$$\sqrt[3]{27}(50)^{1/2} = 3[(25)(2)]^{1/2} = 3(25)^{1/2}(2)^{1/2} = (3)(5)\sqrt{2} = 15\sqrt{2}$$

$$(16)^{1/4}\left(\frac{1}{\sqrt{32}}\right)^{-1} = 2\left(\frac{1}{\sqrt{(16)(2)}}\right)^{-1} = 2\left[\frac{1}{(4\sqrt{2})}\right]^{-1} = (2)(4)(\sqrt{2}) = 8\sqrt{2}.$$

Therefore, the expression (1) becomes

$$6\sqrt{2} + 15\sqrt{2} - 8\sqrt{2} = 13\sqrt{2}$$

50. $(50)^{1/2} + (16)^{1/2} - (\frac{1}{8})^{-1/2}$.

51. $\sqrt{48} - (27)^{1/2} + 4^{1/2}\sqrt{8}$.

52. $(4a)^{1/2} - (16a)^{1/2} + \left(\dfrac{1}{25a}\right)^{-1/2}$.

53. $(27)^{1/3}\sqrt{12} + \sqrt[3]{8}(27)^{1/2} - \dfrac{9^{1/2}}{(3)^{-1/3}}$.

CHAPTER III

A SENTENCE AND WHAT COMES OF IT

21. Questions and answers. An equation is a sentence in the international language of mathematics. It is a compact symbolic statement which records a result or proposes a question, and in the latter form it is the detective tool which lies at the very heart of algebra.

The working or ***conditional equation*** *is a statement that two expressions will be equal if and only if certain values, as yet unknown, are substituted for particular letters used in the statement.* Such an equation, for instance, is $x + \frac{x}{7} = 19$, which appeared when we reduced to symbolic language the problem which worried Ahmes of Egypt. Here the statement is true *on condition* that $x = \frac{133}{8}$ and not otherwise. This number $\frac{133}{8}$ is called the *root* of the equation.

Much of the world's knowledge is packed in *formulas*, or ex-working equations which have been solved and, with the permanent answer replacing the one-time unknown, honorably retired subject to call. For example, if we represent by the Greek letter π the constant ratio of the circumference to the diameter of a circle (this fraction, incidentally, turns out to be nearly $\frac{22}{7}$), then the final answer to the question, "What is the area A of a circle with a given radius r?" is neatly stored in the formula $A = \pi r^2$.

A second form of the equation which records a result or answers an implied question is the *identity*, sometimes written with the sign $\equiv$ replacing the equality sign. An identity is a statement that a given mathematical expression is convertible by the laws of mathematics to a second one also given. Or to put it in another way, *an identity is an equation which is true for all values of the letters involved.* For example

$$(a + b)^2 \equiv a^2 + 2ab + b^2$$

as one may discover for himself by straight multiplication. If the identity isn't trivial and obvious, such as $x \equiv x$, it may serve as a useful reference formula. Many formulas, however, such as Boyle's law for gas, $pv = c$, meaning "pressure times volume is constant," are not in any sense identities, since the equivalence of the two expressions is shown by other than mathematical means.

Having disposed of the formula and identity as types of the mathematical sentence which are much too important to be ignored completely, we'll get back to the first type discussed—sentences strewed intriguingly with the scrawled signature of the illiterate Mr. X. These conditional equations are the main business in hand for this chapter. Each one is a question, and we propose to show when to accept the implied challenge with confidence, as well as when to wait cannily for the reserves. The latter, by the way, are coming up in Chap. IX.

Conditional equation sentences are *simple* or *compound* according to the number of unknowns. The single unknown is usually represented by x (as a matter of custom, not necessity), and several unknowns by letters near the end of the alphabet. Examples of the simple sentence are $2x - x = 4$ or $2x^2 - 3x + 1 = 0$. The compound sentence is illustrated by the symbolic translation of the question: "What are two numbers whose sum is ten and whose difference is four?" We translate thus: "$x + y = 10$; $x - y = 4$." The two parts of this statement are usually called *simultaneous equations* and are given as separate assertions bracketed together, but we prefer to think of them as the constituent parts of a compound sentence which gives us the necessary clue to the required pair of numbers. Such sentences will be dealt with later.

22. We take in less territory. First on our program is the simple sentence. This comes up in the solution of many everyday problems—such, for instance, as those which the student has already met. In Exercise 2 his task was to reduce the English statement of each problem to a mathematical sentence or equation. It will be noticed that in each case x appeared unencumbered with exponents, or that, in other words, the exponent *one* was understood. In the next article we'll consider equations containing x^2 with or without an accompanying x. Those containing $x^{1/2}$, x^{-2}, etc., will be dodged consistently this side of our chapter with the promised reserves. The symbolic statements of most practical problems one meets will be found to involve x with exponents which are positive integers only, and since the study of such equations is especially useful we'll restrict the rest of this chapter to their consideration. And now, for the sake of clarity, we'll define our restricted subject in formidably precise terms.

A ***rational integral equation*** *is one of the type*

$$a_0x^n + a_1x^{n-1} + \cdots + a_n = 0$$

in which the ***a's*** *are fixed numbers free of* **x** *and* **n** *is a positive integer.*

The a's are called *coefficients*, and represent integers usually, though not necessarily.

If $n = 1$, the sentence is a *first degree* or *linear* equation (connected, as we shall see, with a line). If $n = 2$, it is *second degree*, or *quadratic*. The adjectives "rational" and "integral" signify respectively that the exponents of x are whole numbers and are positive. The origin of this usage is suggested by numerical examples such as $3^{1/2}$, or $\sqrt{3}$, which is not rational (see Art. 25), and 3^{-2}, or $\frac{1}{9}$, which is not an integer.

Another formal definition is needed about here, to wit: *Equations which are satisfied by the same values of the variable are called* ***equivalent.***

Now, with the necessary formalities out of the way, we can proceed.

23. The linear equation and how to subdue it. Please stay on the side lines while we dispose of the following sample assertion:

$$\frac{9x}{4} - \frac{3}{4} = \frac{x}{2} + \frac{11}{4}$$

(1) Multiply both sides of the equation by 4, the least common multiple of the denominators:

$$9x - 3 = 2x + 11$$

(2) Subtract $2x$ from both sides:

$$9x - 3 - 2x = 2x + 11 - 2x$$

or

$$7x - 3 = 11$$

(3) Add 3 to both sides:

$$7x - 3 + 3 = 11 + 3$$

or

$$7x = 14$$

(Evidently if it had been convenient we could have subtracted just as easily.)

(4) Divide both sides by 7:

$$x = \tfrac{14}{7} = 2 \qquad \text{(answer)}$$

and we know we're right when we substitute $x = 2$ back in the original equation, getting an identity. The reader should check this for himself.

Notice that the various steps in the solution were allowable under Axioms 1 and 2 already given (about equals added to and multiplied by equals, etc.), and by those below:

Axiom 4. *If equals are subtracted from equals, the remainders are equal.*

Axiom 5. *If equals are divided by nonzero equals the quotients are equal.*

Those statements certainly look plausible enough, though you should remember that we have not proved them. We can see the reason for the adjective "nonzero" in Axiom 5 when we try dividing two numbers by zero, thus:

$0 \times 2 = 0 \times 5$ (true, since zero times any number is zero). Then

$$\frac{\cancel{0} \times 2}{\cancel{0}} = \frac{\cancel{0} \times 5}{\cancel{0}} \text{ or } 2 = 5. \quad \text{(Can you see where we slipped?)}$$

To avoid such difficulties, it is agreed by algebraists that *division by zero is barred*, since any fraction in the form $a/0$ does not represent a number for any choice whatever of a.

Another point worth noticing is that the successive equations we got in our solution were not the same at all. For instance, the two sides of $7x = 14$ are different from the sides of $x = 2$. How, then, could the final one tell us what value of x will work in the first one? The answer is that any number which, when it is substituted for x, makes the two sides equal in the first sentence will keep them equal in the sentences that follow. In other words, the process of solution consists of getting a succession of *equivalent* equations in which the final one ($x = 2$ in our example) makes perfectly clear the value of x necessary to balance the two sides of all equations above it, including the first one.

We should be missing a golden opportunity if we did not, at this strategic spot, point out with all the emphasis at our command the one infallible guide to the solution of equations. To this guide the faltering student should by habit retreat in moments of doubt:

Golden rule of algebra. *Always do to the left side what you do to the right side, and vice versa.*

And by that temporarily vague "do" we mean: add to it or subtract from it anything you please, and multiply it or divide it by any number you please except zero. One who observes this practical rule may not get to his destination (the solution) by the shortest route, but he will at least be staying on correct ground where light may break through at any time, and he will not do this awful thing that follows: "$2x = 3$, hence $x = 3 - 2$ or 1" (this is a sample of the more pitiful "reasoning" by the gropers without a chart). Students who were equally weak but were more devout followers of the algebraist's golden rule would *never* have divided the left side by two without doing *just that* to the right side.

We have dwelt at some length on this seemingly trivial point because it is the essential guiding principle in the mechanical solution of algebra's fundamental problem—the tagging of unknowns tied up in equations. Experience has shown that time spent upon this principle may be subtracted from that frittered away otherwise in the wastelands of mathematical error.

EXERCISE 9

1. Give an example of a rational integral equation which is (*a*) linear, (*b*) quadratic, (*c*) of the sixth degree.

2. Give three equations which are different from but equivalent to each of your three equations in Prob. 1.

3. (*a*) Solve the equation $3(x - 4) + x = 3 - x$.

(*b*) How many different equivalent equations did you get in your solution of (*a*)?

(*c*) Prove that your solution of (*a*) is correct.

Solve the equations in Probs. 4 to 23 and check your answers.

4. $3x - 2(x - 2) = 5x.$

5. $2x - 7 - 3(2x - 5) = 4 - 2x.$

6. $5x = 4(3x - 6) - 3(2x - 7).$

7. $x(2x - 3) - 3(4x - 15) = 2x(x - 3).$

8. $2x - \frac{3}{4} = \frac{x}{4} + 6.$

9. $\frac{1}{2} + \frac{2}{3} = \frac{1}{x}.$

10. $\frac{1}{2}(4x - 8) = 4\left(\frac{x}{4} - \frac{3}{4}\right) - 2(x + 5).$

11. $\frac{x}{2} + \frac{x}{3} + \frac{1}{4} = \frac{3}{2} - \frac{2}{3} + \frac{x}{4}.$

12. $3\left(x - \frac{4}{3}\right) = 2(x - 1) - 3x.$

13. $\frac{2}{3}\left(3x - \frac{3}{2}\right) = \frac{3}{2}(4x + 1) - 2(x + 2).$

14. $\frac{3}{4}(2x - 3) - \frac{2}{3}(3x + 4) = -\frac{37}{12} - \frac{1}{2}(5x + 1).$

15. $\frac{7x - 5}{8} + \frac{11x - 12}{6} - \frac{3x - 2}{2} = \frac{5x - 27}{24}.$

16. $\frac{2}{x + 1} - \frac{1}{x - 1} + \frac{1}{x^2 - 1} = 0.$

17. $\frac{1}{2x - 1} = \frac{10}{2x^2 + 3x - 2} - \frac{1}{x + 2}.$

18. $\frac{4}{x - 1} + \frac{2}{x + 2} - \frac{6}{x - 2} = 0.$

19. $\frac{3}{2x - 1} - \frac{2}{2x + 1} = \frac{1}{2x + 3}.$

20. $ax + a^2 = bx - a^2.$

21. $\frac{x}{a} + \frac{x}{b} + \frac{x}{c} = \frac{ab + bc + ac}{abc}.$

22. $a^2x - a^2 = b^2x - 2ab + b^2.$

23. $\frac{x}{a - b} - \frac{x}{a + b} = \frac{2ab}{a^2 - b^2}.$

24. When a boy pedals his bicycle at the rate of 8 miles per hour it requires 6 min. longer for him to get to school than it does for him to return to his home at the rate of 10 miles per hour. How far is the school from his home?

25. Tom, Dick, and Harry have 100 marbles between them. Tom has $\frac{2}{3}$ as many as Dick, and Harry has as many as both of the others. How many does each have?

26. Sam and Jim were operating lemonade stands in the same block, and Sam was getting 6 cents per glass for his product. When Jim had disposed of all but 30 glasses of his supply, he sold out to Sam for 3 cents per glass. After a bit of figuring, Sam saw that he could mix the purchased stock with his own and sell the mixture for 5 cents per glass with no change in his total profit. How many glasses did Sam have when the trade was made?

27. A farmer paid two laborers a total of $117 for their services. If one of them worked $1\frac{1}{2}$ days longer than the other and the daily wage was $6, how long did each work?

28. A salesman left Briggs at 8 A.M. for Atherton, where he had an appointment for 4 P.M. He figured that if he averaged 40 miles per hour, he would just make it. At a highway intersection he misread the sign and traveled 30 miles in the wrong direction. Upon discovering his error, he increased his speed to 50 miles per hour, continued to Atherton by way of the intersection, and was just in time for his appointment. How far was the intersection from Briggs?

29. If Jim can hoe a garden in 2 hr. and Peter can hoe it in 3 hr., how many hours will be required by the two of them working together?

30. A does a piece of work in a days and B in b days. (a) How many days does it take the two of them? (b) Substitute in the formula obtained in (a) to get the result when $a = 8$ and $b = 10$.

31. If 2 qt. of water and 1 gal. of milk containing 3 per cent butterfat are added to 2 gal. of 4 per cent milk, what is the percentage of butterfat in the resulting mixture?

32. Point out the error in the following reasoning:

Let

$$a = b$$

Then

$$a^2 = ab \qquad \text{(Axiom 1)}$$

Also,

$$a^2 - b^2 = ab - b^2 \qquad \text{(Axiom 4)}$$

$$(a - b)(a + b) = b(a - b) \quad \text{(factoring each side)}$$

Therefore,

$$(a + b) = b \qquad \text{(Axiom 5)}$$

Hence,

$$b + b = b \qquad (\text{since } a = b)$$

or

$$2b = b$$

Therefore,

$$2 = 1 \qquad \text{(Axiom 5)}$$

33. (*a*) Solve the equation

$$\frac{4}{(x-1)(x+1)} + \frac{1}{x+1} - \frac{2}{x-1} = 0$$

(*b*) Substitute your solution in (*a*) and show that it is not satisfied. Explain.

24. The "quadratic" and how to tame it. Consider now our formally defined rational integral equation with $n = 2$. The general equation then becomes

$$a_0x^2 + a_1x + a_2 = 0$$

of which a special case is, for example,

$$2x^2 + 9x + 5 = 0$$

Our problem is to find a numerical value for x which will make the left side of the equation equal to zero. The first solvers probably "beat around the bush" for some time, but we'll give you the benefit of their experience and make only motions that count. Watch that guiding principle.

(1) $$2x^2 + 9x + 5 = 0$$

We take off here.

(2) $$2x^2 + 9x = -5$$

This is called *transposing* or, in this case, bringing 5 across the equality symbol and changing its sign. But notice that it really amounts to subtracting 5 from each side.

(3) $$x^2 + (\tfrac{9}{2})x = -\tfrac{5}{2}$$

(Divide both sides by 2, the coefficient of x^2.)

Now comes the step which few students would find easily for themselves. If the left side were a perfect square, as in $x^2 = 4$, we could solve the equation easily by extracting roots, thus: $x = +\sqrt{4} = 2$ or $x = -\sqrt{4} = -2$ (or, more briefly, $x = \pm\sqrt{4} = \pm 2$). Since we can add any number we please to both sides, let's try to find one which will make the left side of (3) a perfect square. The difficulty is overcome when we notice that the third term, a^2, of the perfect square $x^2 + 2ax + a^2 = (x + a)^2$ is merely the square of one-half the coefficient of x in the second term of $x^2 + 2ax$. Comparing this with $x^2 + \frac{9}{2}x$, we see that a^2 is in this case $[\frac{1}{2}(\frac{9}{2})]^2 = (\frac{9}{4})^2 = \frac{81}{16}$. Hence we fall back upon Axiom 2 and add $\frac{81}{16}$ to both sides of (3), thus:

(4) $$x^2 + \frac{9}{2}x + \frac{81}{16} = -\frac{5}{2} + \frac{81}{16}$$

or

(5) $$\left(x + \frac{9}{4}\right)^2 = -\frac{40}{16} + \frac{81}{16} = \frac{41}{16}$$

Finally,

(6) $$x + \frac{9}{4} = \pm\sqrt{\frac{41}{16}} = \pm\frac{\sqrt{41}}{4}$$

so that

$$x = -\frac{9}{4} + \frac{\sqrt{41}}{4}$$

or

$$x = -\frac{9}{4} - \frac{\sqrt{41}}{4}$$

For the present we'll think of $\sqrt{41}$ as the number between 6 and 7 whose square is 41. We shall soon have more to say about such numbers, but until further notice they should be left as radicals and not changed to their approximate decimal forms (even if, as we optimistically assume, you know how to do this).

At this point you should hitch up your mental suspenders and prove to yourself that you're getting a certain amount of skill in algebraic manipulation. Try substituting

$$x = -\frac{9}{4} + \frac{\sqrt{41}}{4} = \frac{-9 + \sqrt{41}}{4}$$

back into equation (1), remembering, of course, that $(\sqrt{41})^2$ may be replaced by 41. Perhaps something will go wrong at first, but the sky will clear at last if you'll try as the copybook advises. The bothersome $\sqrt{41}$ will conveniently expire at the hands of an obliging negative counterpart. After convincing yourself that the two sides of the equation really do balance, you'll probably have less trouble with the second root, $x = \frac{-9 - \sqrt{41}}{4}$. Then you'll feel a pleasant glow of satisfaction and will realize that the proper question to ask yourself about *any* answer is simply "Does it work?" The answer of a fellow student (or, for that matter, the one in the back of the book) may occasionally, at least, be incorrect; what you should try to develop is the "let me see for myself" attitude. When you have made this attitude your habit you will have passed the first milestone on the road to mathematical self-reliance.

Now that we have untangled our first quadratic, we'll have an intermission, and use it to see how the Greeks, who liked geometric pictures of things, looked at the important unsnarling operation of "completing the square." Diophantus suggested

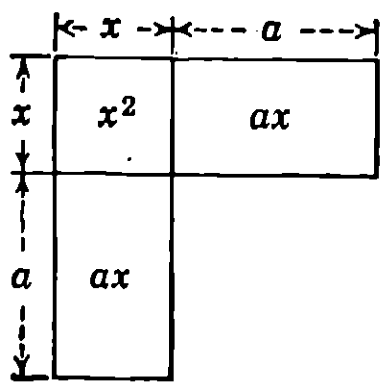

FIG. 6.

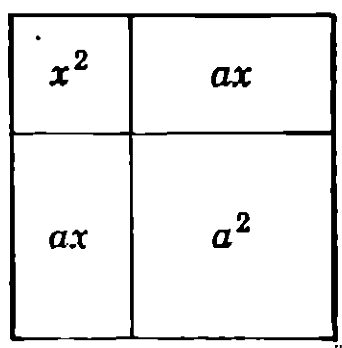

FIG. 7.

that if we think of x and a as lengths, then the expression $x^2 + 2ax$ represents the area of Fig. 6. It is easy to see that we "complete the square" by filling in the "southeast" section, as in Fig. 7. Obviously, the big square has a side equal to $x + a$, so that its area is $(x + a)^2$.

The next question that comes up is whether we can improve upon the above method for solving quadratics. When a process is repeated so frequently that the worker can carry it through with little mental effort, it is said to become "mechanical." In industry a machine is invented for such an operation; in mathematics, we have the formula. Certainly a formula is needed here, where, as a matter of routine, we always

(1) transpose the absolute term (the one not involving x) to the right side;
(2) divide each side by the coefficient of x^2;

(3) complete the square;
(4) extract the square root of each side;
(5) solve for x, *i.e.*, get it off by itself on the left.

These operations, being repeated with monotonous regularity, set the stage perfectly for the triumphant entry of the formula. To get it the mathematician resorts to an old but dependably effective trick. Instead of special numbers connected with particular problems, he uses letters to stand for any numbers whatever. Then the infinite array of equations, such as $3x^2 - 2x + 5 = 0$ and $9{,}286x^2 + 11{,}000{,}001x - \frac{231}{385} = 0$ (customary samples, like the first one, seldom give a fair idea of the extensive ground we're being equipped to cover), are *all* represented by the simple and innocent looking equation.

$$ax^2 + bx + c = 0 \tag{7}$$

Just picture them in your mind's eye—quadratic equations by the millions and billions, covering row after row of fine script on a mile-wide page stretching endlessly beyond the sun and stars, with coefficients as big or little as you please—all to be solved neatly in one fell swoop when the roots of (7) are torn out and laid away in a formula! You should not leave your imagination behind if you wish to appreciate the wonders of mathematics.

Watch closely, then, while we dispose of an infinite horde with a few simple twists of our axioms. The technique is ready-made for us with our very first conquest of the particular case. We need only apply it for the last time with the literal coefficients replacing the numerical ones, and the thing is done. All right then—begin.

$$ax^2 + bx + c = 0 \tag{8}$$

We're off! Now transpose:

$$ax^2 + bx = -c \tag{9}$$

Now divide:

$$x^2 + \left(\frac{b}{a}\right)x = -\frac{c}{a} \tag{10}$$

Next, of course,

$$x^2 + \left(\frac{b}{a}\right)x + \left(\frac{b}{2a}\right)^2 = -\frac{c}{a} + \left(\frac{b}{2a}\right)^2 \tag{11}$$

or, simplifying,

$$(12)\qquad \left(x + \frac{b}{2a}\right)^2 = -\frac{c}{a} + \frac{b^2}{4a^2}$$
$$= -\frac{4ac}{4a^2} + \frac{b^2}{4a^2}$$
$$= \frac{b^2 - 4ac}{4a^2}$$

(The road is easy, you see, if the bumps are taken slowly.)

$$(13)\qquad x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}}$$
$$= \pm\frac{\sqrt{b^2 - 4ac}}{2a}$$

and finally

$$(14)\qquad x = \frac{-b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

There. The second degree equation is solved, once and for all. It has two answers, which we'll call r and s for convenience, thus:

$$r = \frac{-b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a}$$

and

$$s = \frac{-b}{2a} - \frac{\sqrt{b^2 - 4ac}}{2a}$$

The shorter form (14) is called the *quadratic formula.* To get two solutions of a particular quadratic we simply replace the three letters in the answers by the numbers they represent in the given case. For example

$$2x^2 - 3x - 4 = 0$$

is the same as

$$2x^2 + (-3)x + (-4) = 0$$

so that $a = 2$, $b = -3$ and $c = -4$

Then

$$r = \frac{-(-3) + \sqrt{(-3)^2 - (4)(2)(-4)}}{2(2)} = \frac{3 + \sqrt{9 + 32}}{4}$$
$$= \frac{3 + \sqrt{41}}{4}$$

Similarly, s turns out to be $\frac{3 - \sqrt{41}}{4}$.

But while we have now found a formula which will work in all cases, it does not follow that all quadratic equations should be solved by use of it. An elephant gun is somewhat ludicrous when used on a chipmunk. On the same principle, given a quadratic equation of the form $ax^2 + bx + c = 0$ in which (a) $b = 0$, (b) $c = 0$, or (c) two linear factors of the left side may be seen easily by inspection, it is best to leave heavy equipment in the rear and do the job without fuss and ceremony. For instance:

Example 1. Solve $9x^2 - 4 = 0$.
Solution: $x^2 = \frac{4}{9}$; $x = \pm \frac{2}{3}$.

Example 2. Solve $3x^2 + 5x = 0$.
Solution: Factoring the left side we have $x(3x + 5) = 0$. This is satisfied when either factor is set equal to zero, since 0(any number) = 0. Thus $x = 0$ and $3x + 5 = 0$ or $x = -\frac{5}{3}$.

Example 3. Solve $2x^2 - x - 3 = 0$.
Solution: Here $(2x - 3)(x + 1) = 0$, so that $x = \frac{3}{2}$ or -1.

In the following exercise the student should apply these shorter methods wherever possible. However, he should not spend much time hunting for factors which are not readily apparent to him.

EXERCISE 10

Find the constants which, when added to the binomials in the respective parts of Probs. 1 to 10, make the resulting trinomials perfect squares. Show the squares obtained.

Example: $x^2 + \frac{2}{3}x$. The constant is $(\frac{1}{2} \text{ of } \frac{2}{3})^2 = \frac{1}{9}$. Then

$$x^2 + \tfrac{2}{3}x + \tfrac{1}{9} = (x + \tfrac{1}{3})^2.$$

1. $x^2 + 6x$.

2. $x^2 + 4ax$.

3. $x^2 - \frac{4x}{5}$.

4. $x^2 + \frac{6bx}{7}$.

5. $x^2 - \frac{3x}{4}$.

6. $x^2 - \frac{5mx}{6}$.

7. $x^2 + \frac{ax}{b}$.

8. $x^2 - 2(m + n)x$.

9. $x^2 - \left(\frac{a - b}{c}\right)x$.

10. $x^2 + \frac{2(r + s)}{(u - v)}x$.

11 to 20. Write "= 0" after the given function in Probs. 1 to 10 and solve the resulting equation by the shortest method.

Solve the equations in Probs. 21 to 30.

21. $x^2 - 2 = 0.$

22. $9x^2 - 25 = 0.$

23. $x^2 - x - 2 = 0.$

24. $x^2 + 2x - 3 = 0.$

25. $2x^2 - x - 3 = 0.$

26. $4x^2 + 4x - 3 = 0.$

27. $6x^2 - 13x + 6 = 0.$

28. $15x^2 + 11x - 12 = 0.$

29. $8x^2 - 10x + 3 = 0.$

30. $2a^2x^2 + abx - b^2 = 0.$

Solve for x the equations in Probs. 31 to 40.

31. $3x^2 + 9 = x^2 + 25.$

32. $x^2 - \frac{x^2 + 5}{2} = \frac{x^2 - 3}{3}.$

33. $\frac{3}{4x^2} - \frac{7}{3} + \frac{x}{2} + 1 = \frac{1}{6x^2} + \frac{x + 2}{2}.$

34. $x + \frac{x^2 - 9}{4} - \frac{1}{2} = \frac{x^2 + 1}{5} + \frac{2x - 1}{2}.$

35. $mx^2 + n = nx^2 + m.$

36. $x^2 + 2ax + b = a\left(2x + \frac{b}{a}\right).$

37. $\frac{1}{x - 1} - \frac{1}{x + 1} = 1.$

38. $\frac{1}{x - a} - \frac{1}{x + a} = \frac{x^2 + 2a}{x^2 - a^2}.$

39. $\frac{a}{x} + \frac{x}{a} = \frac{9a^2 - x^2}{ax}.$

40. $(x + 4)(x + 5) - 5 = 3(x + 1)(x + 2) + 1.$

Solve Probs. 41 to 54 by completing the square:

41. $x^2 + 4x = 21.$

42. $x^2 - 6 = -x.$

43. $8t = 4t^2 - 3.$

44. $3r^2 = 3r + 6.$

45. $\frac{7v}{5} - \frac{5}{3v} = \frac{20}{3}.$

46. $\frac{2w^2}{3} + \frac{3w}{2} = 15.$

47. $6x^2 + \frac{x-5}{2} = \frac{3x-1}{2}$.

48. $3x^2 + x + \frac{1}{2} = \frac{8x+5}{2}$.

49. $\frac{3x}{4} + \frac{4}{3x} + x + 1 = \frac{19+6x}{6}$.

50. $x^2 - \frac{x}{3} = \frac{4x-1}{12}$.

51. $5ax - 2x^2 = 2a^2$.

52. $x^2 - \frac{x}{m} = \frac{3}{4m^2}$.

53. $x^2 + ax - ab = bx$.

54. $\frac{3ax^2}{4} - \frac{13}{3a} = -\frac{2x}{3}$.

Solve Probs. 55 to 64 by use of the quadratic formula:

55. $x^2 + 2x - 3 = 0$.

56. $3x^2 - 4x + 1 = 0$.

57. $2x^2 + 5x + 2 = 0$.

58. $x - 2x^2 + 1 = 0$.

59. $3 + x = x^2$.

60. $4y = 2y^2 - 7$.

61. $3t^2 - 5t = 2$.

62. $2x^2 - 7x + 4 = 0$.

63. $6x^2 + x - 2 = 0$.

64. $5z^2 - 7z + 1 = 0$.

Solve Probs. 65 to 70 by formula:

Example: $2x^2 - 5rx + d = mx^2 - ex + g + x$

Grouping in powers of x:

$$2x^2 - mx^2 + ex - 5rx - x + d - g = 0$$

or

$$(2 - m)x^2 + (e - 5r - 1)x + (d - g) = 0$$

Evidently

$$a = 2 - m,\ b = e - 5r - 1,\ c = d - g$$

and

$$x = \frac{-(e - 5r - 1) \pm \sqrt{(e - 5r - 1)^2 - 4(2 - m)(d - g)}}{2(2 - m)}$$

65. $ax^2 - bx - c = 0$.

66. $mx^2 + nx + r = 0$.

67. $ax - x^2 = 2 - bx + c$.

68. $mx^2 + nx + r = x^2 + 2x + 3$.

69. $x^2 + 2xy - 3y^2 = 1$.

70. $ax^2 + bx + c = x^2 + x + 1$.

Solve the quadratics in Probs. 71 to 76 and check your solutions by substituting each answer in the left side of the equation.

71. $x^2 + 4x - 5 = 0$.

72. $3x^2 - 7 = 0$.

73. $3x^2 + 2x - 1 = 0$.

74. $2x^2 + x - 5 = 0$.

75. $3x^2 + 6x + 2 = 0$.

76. $x^2 + 3x - 5 = 0$.

77. Find the side of a square whose diagonal is one foot longer than its side.

78. A tinsmith wishes to cut a square piece of tin such that the number of feet in the perimeter is equal to the number of square feet in its area. Find the length of the side.

79. Two boys with only one lawn mower contract to mow a rectangular lawn 50 by 120 ft. How wide a strip must the first boy mow around the outer edge in order to cut half of it?

80. A college boy hitchhiking home made 100 miles in 2 hr. He rode 45 miles in one car and 55 in another which traveled 10 miles per hour faster than the first. Find the speed of the cars.

81. A man and boy, working together, can hoe a field in 12 hr. The man working alone could have hoed it in 10 hr. less time than would have been required for the boy. How long would it take each?

82. When a man became violently ill on a barge floating down the Missouri, a power boat put out from the barge, went downstream to a city 30 miles away, and returned with serum, meeting the barge when it had floated 10 miles downstream. If the current flowed 6 miles per hour, what was the speed of the boat?

83. The theory of relativity was formulated following an experiment which was based on the simple principle that a boat requires more time for making a round trip up and down a river than it needs to go the same distance in still water. If the current flows 10 miles per hour, at what speed would the time required for a round trip on the river be twice that required for an equal round trip on a lake?

25. We meet a stranger. It is about time to do something about those numbers like $\sqrt{31}$ and $\sqrt{41}$, in which the radicand is not a perfect square. A simple one to begin with is $\sqrt{2}$, which comes up in the solution of $x^2 + x - \frac{1}{4} = 0$. We define it as the number whose square is two, and find with little trouble that it is between 1.4 and 1.5 since $(1.4)^2 = 1.96 < 2$, and $(1.5)^2 = 2.25 > 2$. The ancients did a lot of "scratch work" on their sand tables trying to find the fraction between $\frac{14}{10}$ and $\frac{15}{10}$ which would exactly do the trick. It was evident that $\sqrt{2}$ should be called a number of some kind, since it was represented geometrically by the diagonal of a square with unit sides, and it seemed desirable that the general term "number" should include all words used to represent particular points in the line below:

0 $\frac{1}{2}$ 1 2 and so on

There was surely a point between 1 and 2 which was distant from the zero point by the length of the unit square's diagonal; and therefore someone sufficiently bright should be able to find the proper "fraction," or quotient of two integers, to represent it. The idea seems plausible enough, and some bright Greek boy might have been expected, under ordinary circumstances, to turn up with the necessary quotient. The fact that no one did so need not be set down in disparagement of the Greeks because, as a matter of fact, the long-sought fraction simply does not exist. And anyway it was a Greek—Euclid—who pointed out this surprising fact. He did it about as follows:

Suppose $\sqrt{2} = p/q$, where p and q are integers and the fraction is reduced to lowest terms. This means that all common factors in numerator and denominator are divided out, as in the example:

$$\frac{30}{42} = \frac{(2)(3)(5)}{(2)(3)(7)} = \frac{5}{7}$$

Then either p or q must be odd, since if both are even we may still divide out the common factor 2. Resorting again to our indispensable axioms, we have

$$\sqrt{2}q = p$$

after multiplying both sides by q. Hence

$$2q^2 = p^2$$

since squaring both sides amounts to multiplying equals by equals. Therefore p^2 is even, since it is twice an integer. But if p^2 is even so must be p, since an odd number may be written in the form $2n + 1$, and $(2n + 1)^2 = 4n^2 + 4n + 1 = 2(2n^2 + 2n) + 1$, proving that the square of any odd number is odd. Hence $p = 2s$, with s an integer, and we have

$$2q^2 = (2s)^2 = 4s^2$$

or

$$q^2 = 2s^2$$

This makes q as well as p necessarily even, in spite of our original premise that at least one of them was odd. Where's the hitch? When the conclusion of a mathematical argument is as absurd or impossible as the reasoning is flawless, the trained mathematician automatically turns his critical gaze backward to the premise which started the argument on its way. He knows that the trouble must be in the "if" part, and he uses the impressive Latin phrase *reductio ad absurdum,* whose meaning should be obvious, as a sort of a death chant over the ill-fated premise.

And now, if you'll look back a bit, you'll see that we said "Suppose $\sqrt{2} = p/q$," etc. Evidently our "supposing" enthusiasm got us into a pardonable error from which we were ultimately saved only by the *reductio* technique. Having hatched up a premise which collided mathematically with its conclusion, we can announce with confidence the eternal falsity of the premise and thus get a neat headlock on reality from the rear. We know something for certain now, and that is that $\sqrt{2}$ cannot be expressed as the quotient of two integers.

But there it is. Not an integer, and not the quotient of two integers, $\sqrt{2}$ nevertheless has a faithful geometric representation as the diagonal of a unit square, and hence must be admitted into the fold of numbers. Here is a whole extensive class, including $\sqrt{7}$ and $\sqrt[3]{9}$ (as well as outlanders like π, the ratio of the circumference of a circle to the diameter), which calls for a new adjective and a new definition—such as this:

An ***irrational number*** *is one which cannot be expressed as the quotient of two integers.*

It follows, incidentally, that an irrational number cannot be expressed exactly in decimal form. If, for instance, $\sqrt{2}$ were exactly 1.4142, it would be 14,142/10,000, or the quotient of two integers. By the same token, any number representable in decimal notation with a finite number of digits must be rational.

The account of the struggle to develop irrational numbers is one of the most interesting chapters in the history of mathematics. The combined genius of Dedekind and Cantor (working in the latter part of the nineteenth century) brought about its complete solution. Unfortunately, space demands that we leave the fascinating sidetrack incompletely explored, and turn back to the main highway.

26. Final touches on the real number system. Just what, then, do we mean by a number? The concept seems to have altered no little since we graduated from the finger-counting, or positive-integer, stage and began to deal with fractions, negative numbers, and slippery irrationals. A formal definition is next on our docket; but it will be given much more meaning by a picture of the so-called *axis of real numbers*, looking like this:

No end here -3 -2 -1 0 1 2 3 or here either.

←•——•——•——•——•——•——•→

(On the side: That adjective *real* is a technical word to distinguish the numbers we have met up to date from some others, called *imaginary*, which will turn up in the next article.)

It should be noted that there is a place on the line, though not on the paper, for any positive or negative integer whatever. Evidently there are many points in between those representing integers. If we agree that each of these points shall represent a number, then we can say:

A ***real number*** *is a positive or negative quantity which corresponds to one and only one point on the scaled axis of reals.*

A ***rational number*** *is a real one which can be expressed as the quotient of two integers.* We reserve the privilege of calling it just a fraction when we feel like it. For instance, $\frac{2}{3}$, $\frac{3}{4}$, and 7 (or $\frac{7}{1}$) are both rational numbers and fractions; but $\sqrt{2}/3$ and $1/\sqrt{5}$ cannot qualify.

It is a simple matter, of course, to find the points on the axis of reals corresponding to any given fractions, and by choosing increasingly large denominators we can get two such points in any part of the line as close together as we please. The point representing $\sqrt{2}$, for example, can be boxed in by the closely paired dots standing for 1.4142 and 1.4143—two rational numbers differing by one ten-thousandth of a unit. Tightening the snare still more, we find an unending series of other rational pairs (obtainable by the root-extraction process as well as in other ways) representing closer and closer dot-pairs with the $\sqrt{2}$ point still enmeshed between them but never, as we have shown, coinciding with either one of its guards. In fact, though the points standing for fractions are clustered in unlimited number in any minute segment of the line, there remains an infinity of other dots in this same segment which, like the $\sqrt{2}$ one, represent the irrationals. The latter are convenient "hole-fillers," so when we combine them with the row of densely packed points standing for rational numbers we can say that any point whatever on the axis of reals is representable by one and only one number.

This completes the list of the kinds of real numbers, but for the sake of the record we'll mention here that by another classification, which includes the to-be-described imaginaries in its scope, numbers are separated into two groups, the *algebraic* and the *transcendental*. The former are the roots of rational integral equations (for instance, $\sqrt{2}$ is one root of $x^2 - 2 = 0$) and the latter, including the famous π in their ranks, are not. That is to say, there does not exist a single equation, among the infinite number of the type described, which contains π among its roots. We realize that this unbolstered statement may seem a bit sweeping, but it is just one of those things which have been proved by methods too advanced for consideration here.

27. We are forced to invent. One would think that mathematicians would be satisfied by the endless array of real numbers at their disposal. But, as luck would have it, that formidable set failed to provide them with the kind of numbers they needed for roots of certain quadratics—even such a simple, innocent-looking one as $x^2 = -1$. Solving this equation they got $x = \pm\sqrt{-1}$ *formally*, but what actually? Certainly not $+1$ nor -1, since $(+1)^2 = (-1)^2 = +1$ and not -1. At first it seemed the sensible thing to say that equations whose formal solutions involved square roots of negative numbers had no actual solutions, which was true enough if roots had to be numbers of the kind known. Then the algebraists perhaps got to thinking that it would be better for their prestige as solvers if they could produce solutions in every case. Since they already had them in a sort of written mathematical nonsense such as $\sqrt{-17}$ and the like, it was necessary only to assign meaning to the gibberish, or, in other words, invent a new kind of number, in order to increase tremendously their solving range.

The idea does not seem startlingly brilliant now, and we can appreciate its ingenuity more when we learn how long it was incubating. One of the mathematicians who first faced the problem was Chuquet, who in 1484 took the "no root" position.

Bombelli suggested, along about 1550, that the mathematical outlanders be admitted to good standing. Girard, Wallis, and Newton in the sixteenth and seventeenth centuries began to use $\sqrt{-2}$, $\sqrt{-3}$, etc., without compunction. In 1777 Euler suggested the use of i for $\sqrt{-1}$, and the thing was done when, in 1801, Gauss began a systematic use of the symbol. We now say that the quantity $1 \pm \frac{\sqrt{-16}}{2}$, which comes up as the originally meaningless root of $x^2 - 2x + 5 = 0$, is a number, albeit an *imaginary* one. The part $\sqrt{-16}$ is simplified thus: $\sqrt{-16} = \sqrt{16(-1)} = \sqrt{16}\sqrt{-1} = 4i$.

That symbol "i" turns out to be unexpectedly potent, with astonishing adaptability. With it we can express not only square roots, but also fourth roots, sixth roots, and in fact any even roots whatever of negative numbers. For example, remembering that $i^2 = -1$ by definition, we find that one number whose fourth power is -1 is $(1/\sqrt{2})(1 + i)$, since $[(1/\sqrt{2})(1 + i)]^2 = \frac{1}{2}(1 + 2i + i^2) = \frac{1}{2}(1 + 2i - 1) = i$, and hence $[(1/\sqrt{2})(1 + i)]^4 = i^2 = -1$.

If we seek the point of intersection of two curves which really do not cross, the usual algebraic method (to be explained later) comes through tractably enough but leaves us with an i on our hands. In fact, whenever there are conditions in a problem which cannot be satisfied by any members of the real number system, the ubiquitous i usually comes gallantly to the rescue and, by its very appearance, warns us of the physical impossibility of the conditions. We can then explain learnedly that there is no real solution, and yet show that we are equal to any task by producing an *imaginary* one.

If, however, one gains the impression that i is a device for saving the face of mathematicians, he is as wrong as it is possible for a freshman to be, and this is very strong language. The development of mathematics through the use of the imaginary concept has been of inestimable *practical* service in many fields, as in mechanics, physics, and engineering—particularly electrical engineering. To be sure, the perverse engineers, having adopted i as their symbol for electrical current, callously replace the mathematicians' sacred letter by a plebian j, but it's the same idea in a different dress.

Numbers of the type $a + bi$, where $i = \sqrt{-1}$ and a and b are real, are called *complex*. When b is not equal to zero (written $b \neq 0$) they are *imaginary;* when $b = 0$, they are *real*; and when $a = 0$ and $b \neq 0$ they are *pure imaginaries*. Thus they provide a comprehensive kind of number system, since *all* real numbers are included as special cases. The wide field of mathematics dealing with them is called the *complex variable field*. Geometrically, they can all be represented in the plane containing

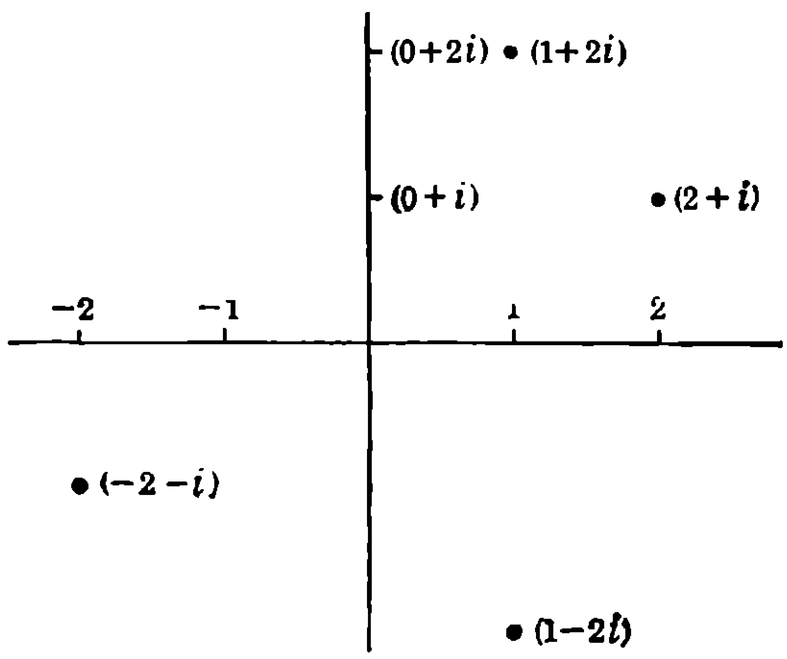

FIG. 8.

the real axis (Fig. 8). We'll close the subject for our purposes here, though we have barely opened it.

EXERCISE 11

1. A theorem in plane geometry states that a line segment drawn from any point on the circumference of a circle perpendicular to a diameter is the mean proportional between the segments of the diameter. Show how this theorem can be used to construct a line of length $\sqrt{3}$; $\sqrt{5}$; $\sqrt{8}$; $\sqrt{n}$ (n an integer).

2. By use of a series of right triangles show how to construct a line of length $\sqrt{5}$ units. (HINT: start with a right triangle whose legs are 1 unit in length.)

3. In the equation $x^2 + bx - 1 = 0$, which of the integral values of b, from 0 to 5 inclusive, give irrational roots?

4. In the equation $x^2 + x + c = 0$, which of the integral values of c, from 0 to 5 inclusive, give imaginary roots?

Solve Probs. 5 to 10, expressing the answers in terms of i.

5. $x^2 + x + 1 = 0$. **6.** $2x^2 - x + 3 = 0$.

7. $5 + x^2 = 0$. **8.** $5 = x - x^2$.

9. $x^2 - 2x + k = 0$. $(k > 1)$. **10.** $x^2 + 4x - k = 0$. $(k < -4)$.

11. (*a*) Consider the two following contradictory results:

$$\sqrt{-4}\sqrt{-4} = \sqrt{(-4)(-4)} = \sqrt{+16} = +4$$
$$\sqrt{-4}\sqrt{-4} = (2i)(2i) = 4i^2 = 4(-1) = -4.$$

(*b*) Which of the two answers above is consistent with our definition of the square root?

(*c*) In view of your decision in (*b*), formulate a rule for multiplying square roots of negative quantities and use this rule to find:

(1) $\sqrt{-9}\sqrt{-16}$; (2) $\sqrt{-3}\sqrt{-12}$;
(3) $\sqrt{-2}\sqrt{-5}$; (4) $\sqrt{-10}\sqrt{-1}$.

Find the products indicated in Probs. 12 to 17.

12. $(3 + \sqrt{-3})(2 - \sqrt{-12})$. **13.** $(2i + 4)(3 + i)$.

14. $(4 - \sqrt{-2})(1 + i\sqrt{3})$. **15.** $(2i + 3)(4 + 5i)$.

16. $\sqrt{2}i(5 + \sqrt{-3})$. **17.** $\sqrt{3}\sqrt{-4}(1 + \sqrt{-5})$.

Find the quotients indicated in Probs. 18 to 22, expressing the answer in the form $a + bi$.

Example:
$$\frac{2 + \sqrt{-3}}{3 - \sqrt{-5}} = \frac{2 + i\sqrt{3}}{3 - i\sqrt{5}} = \frac{(2 + i\sqrt{3})(3 + i\sqrt{5})}{(3 - i\sqrt{5})(3 + i\sqrt{5})}$$
$$= \frac{6 + i^2\sqrt{15} + (3\sqrt{3} + 2\sqrt{5})i}{3^2 - (i\sqrt{5})^2} = \frac{6 - \sqrt{15} + (3\sqrt{3} + 2\sqrt{5})i}{9 - (-5)}$$
$$= \frac{6 - \sqrt{15}}{14} + \left(\frac{3\sqrt{3} + 2\sqrt{5}}{14}\right)i$$

18. $\dfrac{2}{i}$. **19.** $\dfrac{i}{i + 1}$. **20.** $\dfrac{1 - \sqrt{-4}}{1 + \sqrt{-5}}$.

21. $\dfrac{3 - \sqrt{-3}}{3 + \sqrt{-12}}$. **22.** $\dfrac{2}{1 + \sqrt{-3}}$.

23. Noting that $i = i$, $i^2 = -1$, $i^3 = i(i^2) = i(-1) = -i$, $i^4 = i(i^3) = i(-i) = -(i^2) = -(-1) = +1$, find similar simple forms for i^5, i^6, i^7, i^8, i^9, i^{10}, i^{11}, i^{12}. What is the general rule which will enable you to express i^n in simple form when n is any positive integer?

24. Demonstrate algebraically that the conditions of the following problems are impossible.

(*a*) Find the side of a square such that its perimeter in inches is 5 units more than its area in square inches.

(*b*) Find the side of an equilateral triangle such that its altitude in inches is $\sqrt{3}$ more than its area in square inches.

25. In (3), Art. 19, we specified $a > 0$. In $(\sqrt{-4})^3$, replace $\sqrt{-4}$ by $2i$ and show that $(\sqrt{-4})^3 \neq \sqrt{(-4)^3}$.

26. Show that $\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)^3 = \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)^3 = 1^3 = 1.$

27. Use the method of Euclid to prove that $\sqrt{3}$ cannot be expressed as the quotient of two integers.

28. Side lights on quadratics. The amateur detective may find ample opportunity for practice in hunting and interpreting clues in connection with the unknowns snarled up in rational integral equations. It may at first seem useless to draw inferences about the nature of the roots of a given quadratic when the actual roots themselves can be found with little more effort. Our justification for playing the Sherlock role soon appears, however. Aside from the fact that the information obtained in the detective process is interesting in itself as well as suggestive of profitable lines of attack on the higher degree equations to be met later, we find, surprisingly enough, that there are at least two applications which expedite the solution of the quadratic itself. These we shall mention after setting forth the pertinent information.

Consider our formula portrait

$$x = \frac{-b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

of the elusive Mr. X buried in the equation

$$ax^2 + bx + c = 0$$

The portrait shows him to be a two-faced individual; and if we employ a letter alias for each of his aspects, we have

$$r = \frac{-b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a}$$

and

$$s = \frac{-b}{2a} - \frac{\sqrt{b^2 - 4ac}}{2a}$$

If a, b, and c are integers or fractions, we can classify at once the number-type of r and s by examining the quantity $b^2 - 4ac$. Since this last expression is an important clue in our analysis, it is honored with the special name of *discriminant*, and is often represented by the letter D.

Evidently, if $D = 0$, $r = s = -b/2a$ are identical twins; but if $D \neq 0$, the two roots are necessarily unequal, since the condition that $r = s$, or that

$$\frac{-b}{2a} + \frac{\sqrt{D}}{2a} = \frac{-b}{2a} - \frac{\sqrt{D}}{2a}$$

requires that $D = 0$. If D is negative, its square root is imaginary, and so, therefore, are r and s. Finally, if D is positive, r and s are seen to be rational or irrational according as D is or is not a perfect square.

Another interesting and sometimes helpful result is that the sum of the two roots of any quadratic is the negative of the quotient of the coefficients of x and x^2, and that the product of the roots is the constant term divided by the coefficient of x^2. In symbolic language, for the quadratic $ax^2 + bx + c = 0$, $r + s$ turns out to be $-b/a$, and rs to be c/a. To prove this, we need only do the necessary adding and multiplying of the foregoing values for r and s. With the help of this formula we can tell almost instantly that the sum of the roots of $x^2 - x + \frac{298}{732} = 0$ (for example) is just plain *one*, though the roots themselves are such cumbersome imaginaries that the feat is impressive for one not in on the secret.

And now we are ready for the applications. By way of one example, since we find that, for the quadratic $6x^2 - x - 3 = 0$, $D = 73$ and not a perfect square, we know in advance that it is a waste of time to seek factors of the left side. On the other hand, the time spent in finding the value of D is not lost in this case, since D appears in the quadratic formula.

The second use is in a quick check of the supposed answers as soon as we get them (and a quadratic is not really solved if the so-called answers are incorrect). For instance, suppose that in solving $2x^2 - 3x - 4 = 0$, we get $\frac{3}{4} + \frac{\sqrt{41}}{4}$ and $\frac{3}{4} - \frac{\sqrt{41}}{4}$. The sum of these two numbers is $\frac{3}{2}$ and their product is $\left(\frac{3}{4}\right)^2 - \left(\frac{\sqrt{41}}{4}\right)^2 = \frac{9}{16} - \frac{41}{16} = -\frac{32}{16} = -2$. Upon inspection of the equation we see that these results are exactly what we should get according to the respective formulas $r + s = -b/a$ and $rs = -c/a$. Actually we have here a check which is often considerably faster and more convenient than that obtained by substitution of the roots as found in the given equation.

EXERCISE 12

Find the discriminants of the quadratics in Probs. 1 to 10, and then in each case determine whether the roots are (*a*) real or imaginary, (*b*) equal or unequal, and (*c*) rational or irrational. NOTE: If the roots are imaginary, omit parts (*b*) and (*c*).

1. $x^2 + 3x + 1 = 0$.
2. $2x^2 + 5x + 2 = 0$.
3. $x^2 + 2x + 1 = 0$.
4. $3x^2 + 5 = 0$.
5. $x^2 + x + 1 = 0$.
6. $2x^2 - 3x = 0$.
7. $2x^2 - 3x + 4 = 0$.
8. $4 = x^2$.
9. $2x^2 + 3x + 1 = 0$.
10. $100x^2 + 200x + 300 = 0$.

11 to **20**. Still without solving, find the sum and the product of each pair of roots in the equations of Probs. 1 to 10.

Find the equations whose roots are the pairs of numbers in Probs. 21 to 28.

Examples: (1) 2 and -3.

The equation is $(x - 2)[x - (-3)] = 0$ or $(x - 2)(x + 3) = 0$, since $(2 - 2)(2 + 3) = (0)(5) = 0$ and also $(-3 - 2)(-3 + 3) = (-5)(0) = 0$.

(2) $2 + \sqrt{3}$ and $2 - \sqrt{3}$ (or $2 \pm \sqrt{3}$).

The equation is $[x - (2 + \sqrt{3})][x - (2 - \sqrt{3})] = 0$ or $[(x - 2) - \sqrt{3}][(x - 2) + \sqrt{3}] = 0$, or $(x - 2)^2 - (\sqrt{3})^2 = 0$, or $x^2 - 4x + 1 = 0$.

21. $-2, -3$.
22. $\pm \sqrt{5}$.
23. $\pm i\sqrt{3}$.
24. $3 \pm \sqrt{2}$.
25. $4 \pm \sqrt{-9}$.
26. $5 \pm i\sqrt{7}$.
27. $a \pm \sqrt{b}$.
28. $a \pm i\sqrt{b}$.

Given $2x^2 + kx + k = 2$, find the value of k such that the conditions in Probs. 29 to 34 are satisfied.

29. The sum of the roots shall be 4.

30. The product of the roots shall be -2.

31. One root shall be zero.

32. The roots shall be equal numerically but of opposite signs.

33. One root shall be twice the other (two answers).

34. One root shall be one more than the other (two answers).

Extend the method of Probs. 21 to 28 to find the equation whose roots are the numbers in Probs. 35 to 38.

35. 1, 2, 3, 4.

36. $1, -2, -3, 4$.

37. $1 \pm \sqrt{2}, 1 \pm i$.

38. a, b, c, d.

39. From your result in 38, draw some conclusions in regard to the roots of a general fourth-degree equation similar to the results in the text concerning the sum and product of the roots of a quadratic.

40. Show that the roots of $x^2 + 2x + c = 0$ are rational if c has the form $(1 + n)(1 - n)$.

41. Show that the roots of $x^2 + bx - 1 = 0$ are rational if b^2 has the form $(n + 2)(n - 2)$.

29. A look ahead. Since the thorough subduing of mere quadratics involved so much mathematical spadework, including the development of the irrational and imaginary number concepts, we need not be surprised that equations of higher degree baffled the algebraists for a long time. The story of various attacks on the problem carries us back to the ancients—to Greece, Persia, India, and China. During the Middle Ages the effort was popular in Italy, where, in the sixteenth century, the cubic or third-degree equation was solved by Cardan and Tartaglia, and the biquadratic or fourth-degree by Ferrari. The assault on fifth- and higher degree equations continued until the nineteenth century, when it was finally proved that in the general case they could not be solved, as were the lower degree equations, in terms of radicals involving the coefficients. The two brilliant mathematicians responsible for the proof were Abel of Norway, who, incidentally, was twenty-seven when he died, and Galois of France, killed in a duel at the age of twenty-one.

It should not be inferred, however, that the results of Abel and Galois mean that we cannot get the roots of an equation such as $x^5 - 4x^3 + 7x - 1 = 0$, or even $3x^{975} + 9x^4 + 2 = 0$ as precisely as may be required for any practical purpose whatever. As a matter of fact, a method has been devised by which the roots of all rational integral equations (and those last four words cover a lot of ground) can be approximated decimally to as many places as desired. This, after all, is the best we can do *decimally* with even such "found" roots as $\sqrt{2}$ or $\sqrt{3}$. Since this general method of attack (to be discussed in Chap. IX) is the most practical method for solving any equation of degree higher than two, we shall omit here the usual explanations of the special method applying only to cubics and biquadratics. These latter methods have considerable mathematical and historical interest, to be sure, and should be looked into by students who wish to pursue the subject further; but they are rather too involved to be thrust upon a first explorer, especially in view of the fact that there is an easier way.

30. A couple of flank attacks. We have met, then, and as yet have not conquered, the problem which, more than any other, perhaps, lies at the very heart of algebra. That problem is the working out of a practical technique for solving the general rational integral equation. We have rather thoroughly disposed of the first- and second-degree cases, and will attack the others later, after we've brought up several chapters-full of mathematical ammunition. Before we turn back, however, we should notice that there are many higher degree equations which we can solve even with our present equipment. One of these, for instance, is

$$2x^5 - x^4 - 3x^3 + 16x^2 - 8x - 24 = 0$$

whose formidable front conceals a weakness that could easily bring about its downfall and solution at the hands of a bright grade-school student. The solid front is more apparent than real; it can be cracked by anyone who knows the first principles of factoring. The discussion of this mathematical art appeared in Chap. II, but it seems advisable at this point to carry it a bit further.

While the technique of factoring, or the breaking up of an algebraic expression into simpler parts whose product it is, has not been reduced to an infallible routine system even by skilled algebraists, there are a few suggestions as to procedure which often get results and are therefore worth remembering. One of these is the experimental blocking of the expression to be factored into groups of one or more terms each. The "three-three" grouping of the left side of the equation above yields the following result:

$$(2x^5 - x^4 - 3x^3) + (16x^2 - 8x - 24)$$

For brevity, suppose we designate this whole expression as $f(x)$. If now we take out the obvious common factors which show up in each of the two groups, we have

$$f(x) = x^3(2x^2 - x - 3) + 8(2x^2 - x - 3)$$

Obviously we're getting somewhere now, since the same expression appears in two places. It will be correct to divide this out and place it on the left, thus,

$$f(x) = (2x^2 - x - 3)(x^3 + 8)$$

as may be verified by referring to the distributive law of multiplication or by performing the indicated multiplication. Since $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$, we have $x^3 + 8 = (x + 2)(x^2 - 2x + 4)$, and our discouraging first equation becomes

$$f(x) = (2x^2 - x - 3)(x + 2)(x^2 - 2x + 4) = 0$$

Now, as we have already seen in the case of the quadratic equation, any value of x which makes one of the factors of $f(x)$ equal to zero will satisfy the whole equation. The three factors, set separately equal to zero, yield the five roots $\frac{3}{2}$, -1, -2, $1 + i\sqrt{3}$ and $1 - i\sqrt{3}$; and that, as a matter of fact, completes the list. (It is easy to remember, as a matter of incidental information not proved here, that an equation of the fifth degree always has five roots, of the sixth degree, six roots, and in general of the nth degree, n roots, where n is any integer whatever.) Whenever, then, an equation has zero at the right of the equality sign (note this detail; it is highly important) and an expression at the left which has been broken up into factors of the first or second degree, the quadratic-and-linear-equation-solver can look upon it scornfully as conquerable territory, no matter how high and imposing may be its total degree.

In addition to our general suggestion of experimental blocking, a few more hints on factoring may prove useful here. First of all, an expression is *not* factored which looks mechanically like this: ( ) + ( ), or maybe like this: ( ) − ( ). Misguided students trying to factor $x^2 - a^2 - 3x + 3a$ often think they have the job done when they get it looking like this: $(x + a)(x - a) - 3(x - a)$. This is a correct step *on the way* to the desired result, which is $(x + a - 3)(x - a)$, but every upright grader will recognize it as worth one emphatic and chastizing zero when it is innocently exhibited as "the answer."

The first rule, then, is to learn how a factored result appears to the eye, leaving the mind out of consideration. After that it will help to memorize a few formulas such as the following. (Some of these appeared in Chap. II, but will be repeated here for convenience.)

(1) $x^2 - y^2 = (x + y)(x - y)$

(2) $x^2 + 2xy + y^2 = (x + y)^2$

(3) $x^2 - 2xy + y^2 = (x - y)^2$

(4) $x^2 + (a + b)x + ab = (x + a)(x + b)$

(5) $x^3 - y^3 = (x - y)(x^2 + xy + y^2)$

(6) $x^3 + y^3 = (x + y)(x^2 - xy + y^2)$

The first equation above often helps in the "three-one" trial blocking of a four term expression where the "two-two" attempt fails. Thus the left side of $x^2 + 2ax + a^2 - b^2 = 0$ may be attacked as follows: $(x^2 + 2ax) + (a^2 - b^2) = x(x + 2a) + (a + b)(a - b)$. This is correct and may look promising to the novice, but it certainly does not show up the factors, if there are any. Trying next the grouping: $(x^2 + 2ax + a^2) - b^2$, we see that the expression becomes $(x + a)^2 - b^2$, which is obviously the difference of two squares, so that (1) applies. This shows that the factors are $(x + a + b)(x + a - b)$ and that the roots of the equation are accordingly $x = -a - b$ and $x = -a + b$.

Another piece of mathematical strategy, which sometimes works on specially selected equations of high degree, is illustrated in the solution of the equation

$$(7) \qquad x^4 + 2x^3 - 7x^2 - 8x + 12 = 0$$

If we write $-7x^2$ as $x^2 - 8x^2$, it becomes

$$x^4 + 2x^3 + x^2 - 8x^2 - 8x + 12 = 0$$

The left side now breaks up into blocks which turn out to be related like this:

$$x^2(x^2 + 2x + 1) - 8x(x + 1) + 12 = 0$$

or

$$(8) \qquad [x(x + 1)]^2 - 8[x(x + 1)] + 12 = 0$$

This shows that the equation is a quadratic—not, to be sure, in x, but rather in the bracketed expression $[x(x + 1)]$. If we call this z, (8) becomes

$$(9) \qquad z^2 - 8z + 12 = 0$$

from which we see, either by factoring the left side of (9) or by use of the formula, that $z = 6$ or 2. A common pitfall at this point is the restful assumption that "$z = 6$ or 2" constitutes the answer. A quick glance at (7) should expose this lazy man's error at once. The job is evidently to find the x there involved, not z or any other letter turning up incidentally along the way. Remembering now that $z = x(x + 1)$, we have the two cases $x(x + 1) = 6$ and $x(x + 1) = 2$ corresponding with the two possible values for z. These last two quadratics yield the respective results $x = 2$ or -3 and $x = 1$ or -2. Hence, in line with our previous announcement, the roots of the *fourth*-degree equation (7) are the *four* numbers 2, -3, 1, and -2.

Sentences of the above type, called "equations in quadratic form," evidently increase our solving range in a rather important way.

EXERCISE 13

Solve the equations in Probs. 1 to 15 by factoring.

1. $x^2 - 3x + 2 = 0$.

2. $4x^2 = 3 - x$.

3. $2x^2 + 6 = 7x$.

4. $x^2 - ax = 2bx - 2ab$.

5. $x^2 - ax - ba - b^2 = 0.$

6. $x^2 - 2ax + a^2 - b^2 = 0.$

7. $x^6 + 3x^3 - 4 = 0.$

8. $x^4 + 4 = 5x^2.$

9. $x^4 + x^3 + x^2 + x = 0.$

10. $x^3 - x^2 + x - 1 = 0.$

11. $x^4 + 5x^2 + 4 = 0.$

12. $x^3 + x = 2x^2 + 2.$

13. $2x^3 - 2x^2 - 4x = 0.$

14. $x^4 - 2x^3 + x^2 - 2x = 0.$

15. $x^3 - a^2x + bx^2 - a^2b = 0.$

Solve the equations in quadratic form in Probs. 16 to 25.

16. $x^4 - 3x^2 - 4 = 0.$

17. $x^6 - 7x^3 - 8 = 0.$

18. $2(x^2 + x + 1)^2 + 3(x^2 + x + 1) - 2 = 0.$

19. $x^{-4} + 4x^{-2} - 5 = 0.$

20. $3(x^2 + 2)^2 - 5(x^2 + 2) + 2 = 0.$

21. $3(x^2 + 2x)^2 - 5(x^2 + 2x) - 2 = 0.$

22. $4(2x^2 - x - 1)^2 + 12(2x^2 - x - 1) + 9 = 0.$

23. $(2x^2 + 3x)^2 - 6(2x^2 + 3x) + 9 = 0.$

24. $x^{-6} - 19x^{-3} - 216 = 0.$

25. $3(x^2 - x + 1)^2 + x^2 - x + 1 = 0.$

Change the equations of Probs. 26 to 30 to quadratic form and solve.

26. $(2x^2 + x + 2)^2 + 4x^2 + 2x = 4.$

27. $(3x^2 - 9x)^2 + 4(x^2 - 3x + 3) = 40.$

28. $x^2 - 1 - \dfrac{3}{x^2 - 1} = 2.$

29. $\dfrac{2x^2 - 3x + 4}{x + 3} + \dfrac{3x + 9}{2x^2 - 3x + 4} = 4.$

30. $2(x^2 + 3x - 1)^2 + 9x = 8 - 3x^2.$

Solve the equations in Probs. 31 to 35, getting four roots for the fourth-degree equations, six roots for the sixth, etc.

31. $x^4 = 1.$

32. $x^6 = 1.$

33. $3x + 2ax + 6x^2 - 2ax^3 + 4ax^2 - 3x^3 = 0.$

34. $a(2x + 3)^3 + 2bx - 6x - 9 + 3b = 0.$

35. $(x^3 + 1)^2 = 9 - 5x^3.$

31. We work equations together. *Two at a time.* Back in Art. 21 we promised to show you how to deal with problems involving two unknowns, such as a pair of numbers whose sum is ten and whose difference is four. In symbolic language this gives the compound sentence $x + y = 10$; $x - y = 4$, or, as we say more frequently, a *pair of simultaneous equations*. For convenience consider them separately thus:

(1) $$x + y = 10$$

(2) $$x - y = 4$$

Adding the left and right sides separately, we get $2x = 14$, whence $x = 7$. Then (1) becomes $7 + y = 10$, or $y = 10 - 7 = 3$. Substituting this pair of values in place of x and y in (1) and (2), we get $7 + 3 = 10$ and $7 - 3 = 4$, which are evidently true. Clearly we have somehow or other found a pair of values which meet the required condition. Most students do this sort of thing mechanically and let the matter go at that. Delighted and somewhat astonished when they find a mathematical process in which it is easy to follow directions, they proceed happily on their way, proud of the ability to produce "the answer" with little or no agitation of their brains. It rarely occurs to them to worry about how it happens that the process gets results. And the pity is all the greater because it really isn't hard at all to get a better insight into what the innocents are doing.

In the first place, as the reader may or may not have noticed, we might have found any number of pairs of values, such as $x = 2$, $y = 8$, or $x = 11$, $y = -1$, which will satisfy equation (1). Likewise an unlimited number of pairs will work in (2). Our task in solving them "simultaneously," however, is to find a single pair which will satisfy both (1) and (2). If we really think our way through, our mental operations will be indicated roughly in the following steps:

(a) Let's assume, to begin with, that there really is one such pair at least (not necessarily *only* one). If we find it, our assumption will have been proved correct; but if we get contradictory results we must remember the possibility that the required pair does not exist. Such cases will come up in due time.

(b) If, then, the two sought-for numbers exist, they may be considered as already there, incognito, satisfying (1) and (2) while disguised as x and y.

(c) If (b) is accepted, we know that the right and left sides of

(1) and (2) are equal respectively, although the actual numerical values of x and y are as yet unknown. We can therefore apply Axiom 2 about equals added to equals, and get

$$(3) \qquad 2x = 14$$

This equation will be true, it should be noted, for the number-pair we are seeking, but not for the many other pairs designated separately by (1) and (2), since our assumption that $(x + y) + (x - y) = 2x$ requires that the two x's on the left stand for the same number, as must likewise the two y's. Equation (3) holds good, then, just as do (1) and (2), for the hunted number-pair; but since y has now been *eliminated*, we can see that x is restricted to a single value 7. From (1) and (2) it appears that the corresponding y value must be 3.

It should now not be hard to apply the same sort of reasoning to justify our mechanical solution in a somewhat more difficult case, such as the following:

$$(4) \qquad 2x + 3y = 8$$
$$(5) \qquad 3x + 4y = 11$$

Remembering that our steps apply only to a particular pair of numbers whose existence we tentatively assume, and noticing that the least common multiple of the coefficients of y is 12, we get, upon multiplying both sides of (4) and (5) by 4 and 3, respectively, (see Axiom 1):

$$(6) \qquad 8x + 12y = 32$$
$$(7) \qquad 9x + 12y = 33$$

Hence, $-x = -1$ by Axiom 4, and therefore $x = 1$. Substituting this value in (4), we have

$$3y = 8 - 2 = 6$$
$$y = 2$$

Checking in (5), $3(1) + 4(2) = 11$. Evidently our quarry, now bagged, is the pair $(x = 1, y = 2)$ (answer). We must always be sure to get both values together in the final result to make it clear that we know the answer to be a *pair* of values. The following, for instance, would be not merely a sloppy way of recording the result; it would be definitely incorrect: "$x = 1$ (answer)" (space here for figuring) . . . "$y = 2$ (answer)."

We have already warned that our optimistic assumption about the existence of a pair of numbers satisfying two linear equations in two unknowns might not always prove warranted. For instance, such a pair would not exist for the equations:

(8) $$2x + 4y = 5$$
and
(9) $$x + 2y = 6$$

since obviously any pair satisfying (8) would also satisfy $x + 2y = \frac{5}{2}$ and therefore would not make $x + 2y$ equal to 6 as required by (9). Such equation-pairs are called *inconsistent.*

Again, we did not guarantee in advance that our number-pair solution would be unique; and furthermore, our argument did not require this condition. It applied, as a matter of fact, to any pair which would satisfy both of the initial equations, however many such pairs there might be. Though in each of the examples given just one answer-pair actually turned up, such is not the case for the equations

(10) $$x + 2y = 6$$
(11) $$3x + 6y = 18$$

There *all* pairs satisfying (10) do the same for (11), so that we have an unlimited number of answers, such as (0,3), $(1,\frac{5}{2})$, (2,2), $(3,\frac{3}{2})$, etc. Equations (10) and (11) are called *dependent.*

When we start picturing equations in the next chapter, we'll get a "pictorial review" of each of the cases discussed (*ordinary, inconsistent,* and *dependent* equation pairs). The geometric interpretation will help to give more meaning and significance to the adjectives.

Three at a time. The compound sentences we've met thus far involved two variables, and two statements about them were required before we could fix the values of the pair of numbers thus indicated in mathematical code language. We now come to problems involving three equations and three unknowns. Here the principle of the attack is still easy. Notice that we say "principle." The details get many a student hopelessly bogged down in a mass of contradictions, not because he doesn't know how to start, but rather because he can't finish. Since the course of the computer is somewhat longer than usual between start and checkup it is hard to avoid at least one tiny but ruinous bungle somewhere short of the finish. The moral, of course, is that a little extra caution is needed here, though there should be no real difficulty for one who understands the plan of action. Consider, for instance, the equations

(12) $$3x - 2y + 2z = 5$$
(13) $$4x + 5y - 4z = 2$$
(14) $$2x + 3y - z = 5$$

If, as we don't know but cheerfully assume, there is a trio of actual numbers enmeshed by implication in that threefold statement, awaiting only the skilled touch of an algebraist to bring it to light, our unveiling operations must be conducted with the same understanding as the one adopted in the two-variable case. By that we mean that the operations and equations obtained along the way are not supposed to be necessarily true for any of the multitudinous number-trios assôciated exclusively with (12) or (13) or (14). They must be democratic triplets agreeing to *all three* of the conditions before we will admit them into our solving equations. If that point is clear we'll announce our plan of action, which is simplicity itself. *Oust one letter at a time from the equations involving the unknowns.* Eliminate z from the first two by doubling both sides of (12) and adding it to (13). This gives

(15) $$10x + y = 12$$

Here there are only two unknowns, so that we need just one more equation to find both of them. Continue the assault on z, then, by expelling it from (12) and (14), leaving among the remains the information that

(16) $$7x + 4y = 15$$

Making short work now of (15) and (16) by the two-at-a-time method, we learn that $x = 1$ and $y = 2$. Putting these values back in (12), (13), and (14), we find that in each case $z = 3$, so that our checked solution is $(x = 1,\ y = 2,\ z = 3)$. Furthermore, it is the only solution in this case, since the equations forced this one trio of numbers upon us.

We'll admit, if you pin us down, that we haven't settled all phases of the matter. Sometimes the three original equations allow many sets of answers, in which case they are *dependent.* Again, they may be self-contradictory, permitting no solution at all (*inconsistent*). An example of equations which are obviously so badly matched that not even two of them can get along is the following:

$$\begin{aligned} x + 2y + 3z &= 1 \\ x + 2y + 3z &= 2 \\ x + 2y + 3z &= 3 \end{aligned}$$

Or the incompatibility may be a shade more subtle, as in this case:

$$\begin{aligned} x + y + z &= 3 \\ 2x + y + z &= 4 \\ 3x + 2y + 2z &= 8 \end{aligned}$$

Likewise the examples of dependence may be delightfully naive and obvious, as when one equation, such as $x + y + z = 1$, masquerades also as a second one in the form $2x + 2y + 2z = 2$; or not so easily seen, as in the example $x + y + z = 3$; $2x + y + z = 4$; $3x + 2y + 2z = 7$. The fact of the matter is, that this is a big field here which, as is true in so many cases, we can point out but not explain in its entirety.

EXERCISE 14

Solve the simultaneous equations in Probs. 1 to 34.

1. $x + 2y = 0$; $x - 3y = 5$.

2. $2x - 3y = 12$; $x + 3y = -3$.

3. $x - 3y = 5$; $x - 5y = 9$.

4. $x + y = 2$; $2x - y = 1$.

5. $3x - 4y = 18$; $5x + 8y = -14$.

6. $2x - 3y = 4$; $5x - 6y = 13$.

7. $3x - 2y = 4$; $2x + 3y = 7$.

8. $4x - 3y = 6$; $3x + 5y = -10$.

9. $2x + 3y = 3$; $4x + 9y = 8$.

10. $4x - 3y = 8$; $8x + 9y = -9$.

11. $\frac{r}{2} + s = 1;$
$\frac{r}{4} + \frac{s}{2} = \frac{7}{8}.$

12. $\frac{x}{2} - \frac{y}{3} = 0;$
$\frac{x}{3} + 2y = 20.$

13. $\frac{3x}{2} + \frac{2y}{3} = 2;$
$\frac{x}{4} + \frac{y}{3} = \frac{5}{12}.$

14. $\frac{2x}{3} - \frac{3y}{4} = -\frac{5}{6};$
$\frac{x}{9} + \frac{3y}{2} = \frac{28}{9}.$

15. $\frac{1}{x} + \frac{2}{y} = 1;$
$\frac{2}{x} - \frac{4}{y} = 0.$

16. $\frac{3}{x} - \frac{2}{y} = 1;$
$\frac{6}{x} + \frac{5}{y} = 11.$

17. $ax - by = ab;$
$x + y = b.$

18. $abx + b^2y = 2b;$
$a^2x + aby = 2a.$

19. $ax - by = a^2;$
$2bx + 3by = 5ab + 2b^2.$

20. $bx + 2ay = 3;$
$\frac{x}{a} + \frac{y}{b} = \frac{a + b}{ab}.$

21. $x + y + z = 3;$
$x - 2y + z = 0;$
$3x - y + 2z = 4.$

22. $x + 2y + z = 3;$
$3x - 4y - 4z = 2;$
$2x + y + z = 5.$

23. $2x + 4y + z = -3;$
$x - 3y - 2z = 1;$
$3x + 2y + 2z = 2.$

24. $2x + 3y - 4z = -3;$
$3x - y + 2z = 9;$
$4x - 2y + 3z = 13.$

25. $ax - by = a^2 - b^2;$
$bx + az = -a^2 + b^2;$
$y + z = 0.$

26. $\frac{x}{b} - \frac{y}{a} = 0;$
$ax + cz = 2;$
$by + 2cz = 3.$

Determine, in each of Probs. 27 to 34, whether the equation is consistent, inconsistent, or dependent.

27. $3x - 2y = 1;$
$6x - 4y = 3.$

28. $2x + y = 3;$
$3x - 2y = 1.$

29. $x + 2y = 3;$
$\frac{x}{2} + y = \frac{3}{2}.$

30. $2x + y + z = 7;$
$x + 2y + 2z = 5;$
$x - y - z = 1.$

31. $3x + 9y = 7;$
$4x + 12y = 1.$

32. $2x + 4y = 6;$
$7x + 14y = 2.$

33. $4x + 2y + 3z = 7;$
$x + y - 2z = 2;$
$3x + y + 5z = 5.$

34. $x - y = 1;$
$x + z = 5;$
$y - z = 2.$

35. Two years ago John was three times as old as Marvin. Eight years hence he will be twice as old. How old is each?

36. The ages of Tom, Dick, and Harry total 112 years. Eight years ago Dick was three times as old as Harry. Eight years hence Tom's age will be 80 per cent greater than Dick's. How old is each?

37. A police car, giving chase to a stolen car which had a start of 1 hr. 6 min., reached an intersecting highway in 3 hr. The police car turned to the left, whereas the thief had turned to the right. Ten minutes later the police learned their mistake through a radio call, reversed their direction, and caught the thief $7\frac{1}{2}$ hr. after the start of the chase. If the cars were 52 miles apart when the mistake was discovered, what was the speed of each car?

38. A football coach was asked about the prospects for winning his next game. He replied, "Some of my best men are disabled, and the Dean's eligibility report has disqualified one-fourth of the remainder of the squad. The number of men who cannot play at present would make two teams. If the disabled recover and all but 3 of the disqualified make up their conditions, I will have 76 available men." How many men were disabled and how many were disqualified?

39. The seats in a football stadium are priced at \$3 and \$2, and a "sell-out" game yields \$46,000. If the gate receipts when only $\frac{1}{4}$ of the better seats and $\frac{1}{2}$ of the others are sold are \$15,500, what is the capacity of the stadium?

40. A man paid each of his two children \$1.35 in dimes and quarters. The second child received $\frac{1}{5}$ as many quarters as the first but twice as many coins altogether. How many coins of each denomination did each receive?

41. A man and his son painted one side of a chicken house in 40 min. He was joined by another son, and the three painted another side in 30 min. Then the boys ran off to the circus, and the father painted the third side alone. How long did it take him if the three sides were the same size and the two boys painted at the same speed?

42. A grocer found that neither his 35-cent coffee nor his 20-cent coffee was selling well. When he mixed the two, he had 100 lb. of the blend which sold readily at 29 cents per pound. How many pounds of each grade did he have if he did not lose on the deal?

32. Linear and quadratic teams. This matter of deducing the full implications of a mathematician's compound sentence, or, as he would say, of *solving simultaneous equations*, can often lead us into sudden and surprising algebraic entanglements, even when the way ahead looks clear. For instance, consider the simultaneous quadratics

(1) $$2x^2 - 3xy - 4y^2 + x + y - 2 = 0$$

(2) $$3x^2 - 5xy + y^2 - 2x - 3y - 1 = 0$$

Treating (1) as a quadratic equation in y, we have $a = -4$, $b = 1 - 3x$, and $c = 2x^2 + x - 2$. The solution formula yields this monstrosity:

$$y = \frac{3x - 1 \pm \sqrt{(1 - 3x)^2 - 4(-4)(2x^2 + x - 2)}}{2(-4)}$$

$$= \frac{3x - 1 \pm \sqrt{41x^2 + 10x - 31}}{-8}$$

Here are two unpromising values of y which must be substituted separately in (2), giving in each case (after we have segregated the radical on one side of the equation in order to exterminate it by squaring) an equation in x of the fourth degree which we would rather attack some other time, and which shouldn't be handled at this stage anyway. Hence we shall relieve the tension and the reader simultaneously by proclaiming a halt right here. We got this far, as it happens, merely to indicate a solving method which can always be carried to the painful and tedious end by anyone who knows how to deal with rational integral equations of any degree. Nevertheless, we can use the method successfully even now in the case of a mixed equation-pair—one linear and one quadratic. Consider this pair:

(3) $$4x + 3y - 4 = 0$$

(4) $$4x^2 + xy - y^2 + 4 = 0$$

Solving (3) for y we get

$$3y = 4 - 4x$$

or

(5) $$y = \frac{4 - 4x}{3}$$

Substituting in (4),

(6) $$4x^2 + x\left(\frac{4 - 4x}{3}\right) - \left(\frac{4 - 4x}{3}\right)^2 + 4 = 0$$

Here the typical tyro makes a blunder which the teacher can practically depend upon. He (the beginner) forgets that in $[(4 - 4x)/3]^2$ the denominator as well as the numerator must be squared. Remembering this point as we complete the indicated square, and simplifying, we get

$$2x^2 + 11x + 5 = 0$$

so that

$$x = -\tfrac{1}{2} \text{ or } x = -5 \qquad \text{(not answers)}$$

To get an answer, which of course is a pair of numbers, we solve (3) or, even more conveniently (5) for y when $x = -\frac{1}{2}$, and also when $x = -5$. This gives the two answers ($x = -\frac{1}{2}$, $y = 2$) and ($x = -5$, $y = 8$). Notice that we always go back to the first- and *not* the second-degree equation to find the second value of an answer-pair when the first one is known. If we should use (4) above, certain additional answer-pairs would appear which would contribute nothing to the solution except a possible headache, since they would have to be rejected in the end as failing to meet requirement (3).

Simultaneous equations illustrate very well one of the best common-sense rules for checking answers, or making sure that no arithmetical error has been made. Maybe you can see for yourself how the rule applies to them. It may be stated as follows:

Checking principle. Check the answer where possible by testing directly the final condition required of it, rather than by going back over the devious route of the first calculations and making, as likely as not, the same mistakes over again.

EXERCISE 15

Solve the simultaneous equations in Probs. 1 to 14.

1. $x + y = 3;$
$x^2 + y^2 = 5.$

2. $x - y = 2;$
$x^2 - y^2 = 8.$

3. $2x - y = 5;$
$x^2 + y^2 = 5.$

4. $x + 3y - 1 = 0;$
$x^2 + xy - 2y^2 = 0.$

5. $2x + 3y + 3 = 0;$
$x^2 + 9y^2 + 3xy = 3.$

6. $2x = 3y - 2;$
$2x = 3y^2 - 2xy - 1.$

7. $x^2 + xy + y^2 = \frac{1}{4}$;
$3x - y = 2$.

8. $x + 2y = 0$;
$3x^2 + 2y^2 - 4x + \frac{y}{2} = 3$.

9. $x - y = 2b$;
$x^2 - y^2 = 4ab$.

10. $ax + by = 2a$;
$a^2bx^2 + b^3y^2 = 2a^2b$.

11. $b^2x^2 + a^2y^2 = a^2b^2$;
$bx + ay = ab$.

12. $x - y = 3b$;
$2x^2 + 3xy + y^2 = 6a^2 - 3ab$.

13. $ax + by = a + 1$;
$a^2bx^2 + ab^2y^2 = b + a^3$.

14. $x - y = a - b$;
$x^2 - y^2 = 4ab - 3b^2 - a^2$.

In which of Probs. 15 to 20 are the solutions real? imaginary? equal?

15. $x^2 + y^2 = 4$;
$x + y = 4$.

16. $x^2 + y^2 = 25$;
$4x + 3y = 25$.

17. $x^2 - y^2 = 16$;
$5x - 3y = 16$.

18. $x^2 - y^2 = 16$;
$3x - 5y = 0$.

19. $x + y = 2$;
$x^2 - 4(x + y) + 9 = 0$.

20. $y^2 - 4x^2 = 5$;
$x - y + 1 = 0$.

Devise a short method for solving the equations in Probs. 21 and 22.

21. $2x - 3y = 1$;
$(x + y - 1)(x - y + 1) = 0$.

22. $x^2 + xy + y^2 = 7$;
$xy + 6 = 0$.

Extend the method used in solving Prob. 21 to the solution of Probs. 23 to 26.

23. $(x - y - 1)(x + y - 2) = 0$;
$(2x + y)(x - 2y) = 0$.

24. $(2x - y - 1)(x - y + 2) = 0$;
$(x - 2y + 3)(2x - 3y) = 0$.

25. $(x - y)^2 = 1$;
$(x + y)^2 = 4$.

26. $(x + y - a)^2 = a^2$;
$(x - y - b)^2 = b^2$.

NOTE: In some of the following problems the conditions are impossible. In others, one of the solutions must be rejected. The algebraic results should indicate these facts.

27. If the perimeter of a rectangular plot is 40 rods and its area is 144 sq. rods, what are its dimensions?

28. The side of a square is increased by a certain amount and an adjacent side by twice that amount. The perimeter of the resulting rectangle is 16 ft., and its area is 8 sq. ft. greater than that of the square. Find the dimensions of the square and of the rectangle.

29. A tinsmith received an order for an air duct 4 ft. in length with a rectangular cross section having an area of 384 sq. in. He used a rectangular piece of sheet metal 48 by 80 in. for the purpose. What were the dimensions of the cross section?

30. A man who owned a rectangular piece of ground containing 2,400 sq. yd. built a stone fence around a square plot containing 100 sq. yd. in one corner. He fenced the remainder with wire netting and used 180 yd. Find the dimensions of his property.

31. Nine times the area of one square plus four times the area of another is equal to 72 sq. ft. The sum of four times the side of the first and three times the side of the second is 17 ft. Find the side of each.

32. A cowboy left the ranch on a pony at 7 A.M. for a town 30 miles away. At 8:30 A.M. he met his boss, who had left town in a car at 8 A.M. and who reached the ranch 1 hr. before the cowboy reached town. Find the average speed at which each traveled.

33. A man on a ranch, bitten by a rattlesnake, can choose between two doctors, A and B, in a town 100 miles away, and can get help (a) from A, who will come by plane or (b) by riding his pony to meet B, who will come by car or (c) by waiting for B to reach the ranch. Method (b) will require 40 min. more than (a) and method (c) one-fourth more time than (b). If the plane travels 90 miles per hour faster than the car, how fast do plane, car, and pony travel?

34. An airplane flew from Minneapolis to Detroit against a head wind, and then back to Minneapolis, making the round trip in 8 hr. On the next day, when the wind was blowing twice as hard but still in the same direction, the round trip took 10 hr. Assuming that the distance between the cities is 500 miles (in round numbers), what were the two wind speeds involved, as well as the speed of the plane in still air?

35. A ranch owner intended to drive his car to a town 120 miles away to catch a streamliner. When he found that his battery was dead he drove a team which averaged one-quarter the speed of the car to a bus station, took a bus which averaged 45 miles per hour and reached the town in time to catch a train which left 3 hr. after the streamliner. If the latter route was 80 miles longer than the former, find the speed of the car and the distances he traveled by team and by bus.

33. We make comparisons. In the equation-sentences we have met thus far the verb has invariably been the equality sign, and the sentences have stated that one thing *equals* another. In many of our scientific laws one finds mathematical sentences in which the verb is *varies as*. Charles's law states, for instance, that when the pressure is constant the volume of a given mass of gas varies as the temperature. Newton's famous *law of gravitation* states that the gravitational attraction between two spherical bodies varies inversely as the square of the distance between their centers, and directly as the product of their masses. So many things, in fact, vary as certain other things that mathematicians decided upon the symbol $\propto$ to take care of the idea. Accordingly $a \propto b$ is read "a varies as b" and means that a is always equal to the product of b by some constant, say k. This k is called the *constant of variation*. Its judicious use is tending to make the symbol $\propto$ unnecessary and probably due for extinction, even though the variation idea is still highly important and useful. For $a \propto b$ is now rewritten $a = kb$, becoming more manageable mathematically in that form.

The statement that "y varies inversely as x" means that the size of y is proportional to, or k times, the reciprocal of x; that is, $y = k/x$. Again, "y varies jointly as x and z" leads to the equation $y = kxz$. Thus if we let m and M stand for the masses to two spheres, F for the mutual gravitational attraction, and d for the distance between their centers, Newton's law becomes

$$F = \frac{kmM}{d^2}$$

Such statements involving the constant k are called *equations of variation*. To get an idea of what can be done with them, consider this problem:

The distance s passed over by a ball rolling down an inclined plane varies as the square of the time t. If a ball rolls 24 ft. in 2 sec., how far will it roll in 5 sec.?

Here the old-fashioned notation $s \propto t^2$ is translated to the working equation

(1) $$s = kt^2$$

Now when $t = 2$, $s = 24$; hence

$$24 = k(2)^2$$

and

$$k = 6$$

Inserting this value of k in (1) we have

$$s = 6t^2$$

which gives us much more than the information required in the problem. In addition to telling us that when $t = 5$, $s = 6(5)^2$ ft. $= 150$ ft., it tells us the computation to make to calculate s when t has any positive value whatever.

Evidently, in order to solve a problem involving a variation, we must

(1) *know the law operating in the given case;*

(2) *know a set of data from which we can obtain the constant of variation;*

(3) *have another set of data in which one of the quantities is unknown.*

EXERCISE 16

Express as equations the statements in Probs. 1 to 6.

1. y varies as the reciprocal of x.

2. a varies jointly as b and c.

3. w varies jointly as x and y and inversely as z.

4. m varies as the square of n.

5. g varies inversely as the product of p and q.

6. P varies jointly as the product of M and the square of n and inversely as the square of d.

7. If $y \propto xz$, and $y = 12$ when $x = 3$ and $z = 2$, find y if $x = 4$ and $z = 2$.

8. Given $a \propto bc/d^2$, with $a = 27$, $b = 18$, $c = 2$, $d = 2$, find b when $a = 243$, $c = 18$, $d = 18$.

9. Given $p \propto qr^2/st$, $p = 108$, $q = 9$, $r = 12$, $s = 16$, $t = 3$, find s when $p = 48$, $q = 7$, $r = 12$, $t = 14$.

10. Given $y \propto w^2\sqrt{x}/\sqrt{z}$, with $y = 6$, $x = 12$, $w = 2$, $z = 72$, find x if $y = 9$, $w = 3$, $z = 6$.

11. If $x = k\sqrt{yz}/tw$, compare the value of x when $y = 8$, $z = 9$, $t = 15$, $w = 12$ with the value when $y = 6$, $z = 12$, $t = 5$, $w = 3$.

12. The length of exposure in seconds necessary to photograph a given subject varies as the square of the distance in feet of the subject from the source of light. If an exposure of 2 sec. is required when the subject is 6 ft. from the light, how much time is required when the subject is 9 ft. from the light?

13. The fuel consumption (in pounds) of an aircraft motor varies jointly as the power delivered (expressed in horsepower) and the number of hours of operation. If an 800-hp. motor consumes 2,374 lb. of fuel in 2 hr., how many pounds will a 750-hp. motor consume in 3 hr.?

14. The volumes of two like bodies vary as the cubes of their corresponding dimensions, and the surfaces as the squares of the dimensions. The earth's surface area is roughly sixteen times that of the moon. How do their radii and volumes compare?

15. By the law of gravitation the attractive force between the earth and the moon varies inversely as the square of their distance apart. Compare the attractions when they are separated by the distances of 225,000 and 250,000 miles.

16. The time in seconds necessary to make an enlargement from a photographic negative varies as the area of the proposed enlargement. If 10 sec. are required to make a 3- by 5-in. enlargement, how much time would be required to make an enlargement 6 by 10 in.?

17. The lift of an airplane with a wing area of 500 sq. ft. flying at the rate of 100 miles per hour is 7,300 lb. If the lift varies jointly as the wing area and the square of the velocity, find the lift of a plane with wing area of 170 sq. ft. flying with a velocity of 80 miles per hour.

18. The squares of the revolution periods of the planets vary as the cubes of their mean distances to the sun. Assuming the unit of time to be one year and the unit of distance one astronomical unit (93 million miles) find the approximate revolution periods of the planets listed below. The approximate mean distance of each from the sun, in astronomical units, is given after the planet's name: Mercury (0.4); Venus (0.7); Earth (1); Mars (1.6); Jupiter (5); Saturn (10); Uranus (20); Neptune (30); Pluto (40).

19. The amount of light received by a star varies inversely as the square of the distance to the star. How many times fainter would our sun be if it were at the distance of the nearest fixed star, or 280,000 times as far away as it is?

20. The relative speeds of any two camera lenses vary inversely as the squares of their *f/numbers.* Compare the speeds of an $f/4$ and an $f/8$ lens.

21. Objects in the sky with like apparent diameters vary in actual diameters directly with the distance. Granting that the sun is about 400 times as far away as the moon and that their apparent diameters are the same, how do their volumes compare?

34. We balance some ratios. Before we can compare quantitatively two such things as, for instance, an oyster and a grand opera singer, we must first reduce them somehow to the same units. Measuring them in terms of something they have in common, such as their gravitational affinity to the earth, the oyster is to Madame X as one-twentieth pound is to, say, 230 lb. (1890) or 120 lb. (today). The indicated fraction y/x, when y and x are expressed in terms of the same unit, is called the "ratio of y to x" and leads to a form of equation much used by the ancients. This is the *proportion*, or the *statement of equality between two ratios.* The nineteenth-century teachers, however, did not write a simple little proportion such as $x/y = u/v$ in that way; they dressed it up more impressively like this:

$$x:y::v:w$$

or later, when they had weakened a little and begun to suspect that it was nothing more than an ordinary equation, like this:

$$x:y = v:w$$

They worked out a lot of rules such as "the product of the means" (y and v in our example) "equals the product of the extremes" (x and w). In fact, our grandfathers and great-grandfathers had a rather elaborate technique for handling proportions which they called the "rule of three" and honored with a prominent place in the early arithmetics. When, however, the pompous trappings of symbolic dots were seen through and the unwrapped thing proved to be just an equation, the rule of three became as unfashionable as the bustle, and the proportion, shorn of its glamorous means and extremes, stayed in active service subject merely to the ordinary rules of algebra. You knew three of the quantities x, y, v, and w, and you solved for the other one. For instance, if $2/x = \frac{3}{5}$, then $10 = 3x$ and $x = \frac{10}{3}$. As simple as that.

A slightly more complicated problem is the following:

If

$$\frac{x}{y} = \frac{v}{w} \tag{1}$$

find w when $x = 2$, $y = 3$, and $v + w = 30$.

In this case we need an equation involving $v + w$, which we may get by adding 1 to both sides of (1), thus

$$\frac{x}{y} + 1 = \frac{v}{w} + 1$$

or

$$\frac{x + y}{y} = \frac{v + w}{w} \tag{2}$$

Substituting the given values,

$$\frac{2 + 3}{3} = \frac{30}{w}$$

we have

$$w = 18$$

Similarly, subtracting 1 from both sides of (1), we get

$$\frac{x - y}{y} = \frac{v - w}{w} \tag{3}$$

Dividing corresponding sides of (2) and (3) (Axiom 5), we have

$$\frac{x - y}{x + y} = \frac{v - w}{v + w} \tag{4}$$

or, for that matter,

$$\frac{x + y}{v + w} = \frac{x - y}{v - w} \tag{5}$$

By this time, one who has the hang of it and the proper respect for the laws of algebra, can add just as many new members to the congregation (2) to (5) as he pleases, all of which follow in an interesting and not too obvious way from (1). While various ones in the group prove useful at times, there's no sense in cluttering the memory with a lot of results which can be hacked as needed from the bedrock of algebra. In dealing with equations our technique will be sufficient for most purposes if we just remember that one of the axioms lies somewhere behind each valid operation.

A rule-of-three problem from Davie's *University Arithmetic* (published about 1863) carries a certain amount of historical as well as mathematical illumination. To quote:

"A grocer bought a hogshead of rum for 80 cents a gallon, and, after adding water, sold it for 60 cents a gallon, when he found the selling and buying prices were proportional to the original quantity and the mixture. How much water did he add?"

Solution: (Authors' note: Anyone bright enough to read an arithmetic in 1863 was probably expected to know that a hogshead contains 63 gal.) Letting x represent the number of gallons in the new mixture, we have

$$\frac{x}{63} = \frac{80}{60}$$

whence

$$x = 84$$

Therefore the amount of water added was $84 - 63 = 21$ gal., and perhaps, after all, the rum was improved.

EXERCISE 17

Find the ratio of the first to the second quantity in each of Probs. 1 to 10.

1. $2\frac{1}{2}$ lb.; 12 oz.

2. 1,320 ft.; 2 miles.

3. 1.2 cc.; .24 cc.

4. \$.50; \$350.

5. 3 quarters; a ten-dollar bill.

6. 2 weeks; 2 years.

7. 11 basketball teams; 4 football teams.

8. The area of a square whose side is 4 ft.; the area of a square whose side is 2 yd.

9. The volume of a cube whose side is 3 in.; the volume of a cube whose side is 1 ft.

10. The volumes of two boxes whose dimensions are 8 by 6 in. by 1 ft., and 2 by 3 by 4 ft., respectively.

Find the values of x in the proportions of Probs. 11 to 20.

11. $4:12 = 18:x$.

12. $x/2 = \frac{5}{12}$.

13. $2:x = x:18$. (In this case, x is called the *mean proportional,* or the *geometric mean,* between 2 and 18.)

14. $3:4 = 7:x$. (Here x is the *fourth proportional* to 3, 4, and 7.)

15. $4:5 = 5:x$. (Here x is the *third proportional* to 4 and 5.)

16. Find the mean proportional between 6 and 24.

17. Find the third proportional to 5 and 7.

18. Find the fourth proportional to 4, 9, 32, and the mean proportional to 4 and 9.

19. $a:\frac{1}{x} = x^2:(a - x)$.

20. $\frac{2 - x}{2 + x} = \frac{3x}{x + 8}$.

21. Two towns 60 miles apart are represented by points on a map $1\frac{1}{2}$ in. apart. What is the scale on the map?

22. The areas of two similar geometrical figures have the same ratio as the squares of two corresponding sides. Find the area of a triangular plot with one side 240 rods long, if the corresponding side of a similar plot containing $1\frac{1}{2}$ acres is 80 rods in length.

23. The amounts of water delivered by two pipes have the same ratio as the areas of their cross sections. If a pipe of diameter 4 in. fills a tank in 1 hr., find the time required by a 6-in. pipe to fill the same tank.

24. A man working on a "skyscraper" dropped a steel ball from a window 64 ft. from the ground and found that it struck the ground in 2 sec. He ascended to the top and dropped a second ball and found that it struck the ground in 6 sec. If the distance a body falls is proportional to the square of the time of the fall, find the height of the building.

25. The volume of a mass of gas at a fixed temperature is inversely proportional to the pressure to which it is subjected. If a mass of gas under 20 lb. pressure contains 100 cu. in., what is its volume if the pressure is increased to 30 lb.?

26. Two girls weighing 80 and 100 lb., respectively, are balanced on a teeter board. If their weights vary inversely as their distances from the point of support, find the distance of the heavier girl, if the lighter is 8 ft. from the pivot.

27. If $\frac{x}{y} = \frac{u}{v}$, find

(*a*) y when $u = 4$, $v = 5$ and $x - y = 3$;
(*b*) x when $u = -2$, $v = 4$ and $x + y = 6$;
(*c*) v when $x = 3$, $y = -4$ and $u - v - x = 2$;
(*d*) u when $x = 1$, $y = 2$ and $x + y + u + v = 4$.

28. By the law of gravitation, the weight of a body on the surface of a globe varies directly as the mass of the globe and inversely as the square of its radius. Find the weight of a 200-lb. man on each of the globes designated below. The two figures after each name represent the globe's approximate mass and radius, respectively, on the scale in which those of the earth are unity. The solution can, of course, be accomplished by the variation as well as the proportion method.

(*a*) Moon $(\frac{1}{80}, \frac{1}{4})$; (*b*) Mars $(\frac{1}{9}, \frac{1}{2})$;
(*c*) Jupiter (300, 11); (*d*) Saturn (100, 9);
(*e*) Neptune (17, 4); (*f*) Sun (1,000,000/3, 108).

35. We lose our balance. Up to this point our statements have been the balanced assertions known as equations. One thing has always been definitely equal to another. This ideal mathematical situation makes possible an impressively precise answer. Unfortunately, however, the equation method cannot be applied to all problems. Sometimes we need a mathematical expression for the statement that one thing is *different* from another. Such a statement is an *inequality*, which replaces the equality sign by the symbol $<$ (less than) or $>$ (greater than). For instance, $3 < 4$ and $-10 < -2$ (algebraically).

Inequalities are occasionally the forerunners of equations. For example, before the area of the circle was known in terms of the radius, the searchers for a precise formula noted that the area was certainly less than that of a circumscribing polygon, however many sides the latter might have. Since it was possible to compute exactly the area of each such polygon when the number of its sides was known, this method of approach led to a more and more accurate approximation to the desired area.

The axioms applying to inequalities are much like those for equations, but the points of dissimilarity are important enough to warrant the separate following statements:

Axiom 6. *If equals are added to or subtracted from unequals, the sums or differences are unequal in the same order.*

Axiom 7. *If equals are multiplied or divided by unequals, the products or quotients are unequal in the same order or in the reverse order, according as the multipliers or divisors are positive or negative.*

Consider, then, the inequality

$$4x + 3 < x + 8$$

Axiom 6 evidently allows transposition as in ordinary equations. Thus

$$4x - x < 8 - 3$$

or

$$3x < 5$$

The final solution again resembles that of the equation when we divide by 3, thus:

$$x < \tfrac{5}{3}$$

Or again, given

$$-x < -2$$

we divide both sides by -1 and note that by Axiom 7 we have

$$x > 2$$

A word should be said about the meaning of the answer in the case of an inequality. Unlike the root of a conditional equation, it designates by implication an infinite number of specific solutions. The answer $x > -2$, for instance, tells us that $-\frac{3}{2}$, 0, 10, and 1,000, as well as an endless array of other numbers, are all solutions of the inequality $x + 2 > 0$; whereas $-\frac{5}{2}$, -3, etc., are barred from the elect. The information thereby given is often specific enough for practical purposes. For instance, if we lower a rope to a man stranded somewhere on the side of a bluff known to be 100 ft. high, we could use any strong enough rope which is more than 95 ft. long.

EXERCISE 18

Solve the inequalities in Probs. 1 to 10.

1. $3x - 2 > x + 7$.

2. $x + 1 < -2x - 2$.

3. $4x + 2 > 5x - 3$.

4. $2x - 1 < 3x - 3$.

5. $2x - 1 > 3x - 3$.

6. $3x - a + 2b < x + 2a - b.$

7. $-\frac{3x}{2} < \frac{2x}{3} - \frac{4}{5}.$

8. $\frac{x}{a+b} - 2 < \frac{x}{a-b} - 1,\ a > b > 0.$

9. $\frac{3ax + bx}{b} > \frac{2ax}{b} - b,\ a > 0,\ b > 0.$

10. $\frac{3ax + bx}{b} - a > \frac{2ax}{b^2} + x - \frac{ab - 1}{b},\ a < 0,\ b < 0.$

11. If $x^2 > 4$, solve for x.

Incorrect solution: $x > \pm 2$, or $x > 2,\ x > -2$.

Correct solution: $x^2 - 4 > 0$

or

$$(x + 2)(x - 2) > 0$$

Hence, the factors $(x + 2)$ and $(x - 2)$ must have like signs. Now $x + 2 = 0$ when $x = -2$, and $x - 2 = 0$ when $x = 2$, and hence we must examine three intervals.

(*a*) $x < -2$. Here both factors are negative, and the inequality is satisfied.

(*b*) $-2 < x < 2$. Here $x + 2$ is positive and $x - 2$ is negative, hence the inequality is not satisfied.

(*c*) $x > 2$. Here both factors are positive, satisfying the inequality.

The solution, then, is $x < -2$ and $x > 2$; and the two inequalities *may not* be combined thus: $2 < x < -2$, since the same x cannot meet the requirements of both inequality signs—as it may, for instance, in $-2 < x < 2$.

Solve for x the inequalities in Probs. 12 to 17.

12. $x^2 < 9.$

13. $(x + 2)(x - 3) < 0.$

14. $x(x + 3)(x - 1) < 0.$

15. $x^2(x + 3)(x - 1) < 0.$

16. $(x - 1)(x - 2)(x - 3)(x - 4) > 0.$

17. $(x - 1)(x - 2)^2(x - 3)^3(x - 4)^4 > 0.$

18. In a sailboat race the boats are allowed to take any course which will make best use of the wind, subject to the condition that they must round a buoy which is 10 miles from the start-and-finish line. State mathematically a condition applying to the distance covered in the race by any boat which completes the race.

19. What can be said mathematically about the area enclosed by a fence surrounding a house 40 by 30 ft. if the fence is inside a rectangular lot 50 by 150 ft.?

CHAPTER IV

NOW WE PICTURE IT

36. A very bright idea. The kindergarten hopeful, beginning his education with a book of pictures, works up through the alphabet, the reading stage, and the "movies" to the goal of the comic supplement or the great work of art, according to instinct or training. In any case, a large part of his store of information comes to him through his eyes, and this applies to his mathematical development no less than to his artistic appreciation. The use of diagrams and figures is a big help to most of us in grasping difficult ideas; for the human mind learns more easily by "seeing" physically as well as mentally. Obviously, then, if the processes of algebra can be put into a readily interpreted pictorial form, it will be easier for all of us to get along with them. Any pioneering and workable scheme to this end belongs automatically in the category of a very bright idea.

The ancients, however, did not know, or at least did not apply, this principle. To them algebra was a manipulation of symbols for the purpose of problem solving, and if one talked about picturing a mathematical subject he got willy-nilly into geometry. Nothing in the science which dealt with triangles, areas, volumes, etc., seemed likely to help the algebraist, and only the simplest of algebra was employed by the geometer. The two sciences, in fact, flourished for the most part independently and on different soils, Arabia for algebra and Greece for geometry—a useful fact for alliterative purposes but bad for mathematical progress.

Even then, however, the bright idea was stirring feebly on the intellectual horizon. The Egyptian surveyors divided their country into districts by means of boundaries parallel to two fixed lines, or axes. The Romans used this principle for the plan of their cities, which were usually laid off with respect to two axes—the *decimanos*, running in most cases north and south, and the *cardo*, meeting the other at right angles. And in a related realm of thought, points in the sky and on the surface of the earth were located by means of a system similar to modern latitude and longitude, which are measured in degrees from the equator and the prime meridian, respectively, as reference lines.

A Roman lad who lived in the town whose "skeletonized" map is shown in Fig. 9 might describe the lines AB and CD as paths on which one is always twice as far from the decimanos as from the cardo. With our superior advantages, however, we can now outclass the Roman lad in briefness and elegance by describing AB and CD in the sentences $x - 2y = 0$ and $x + 2y = 0$. We can do this because the principle of two perpendicular reference lines finally culminated in the birth of the full-fledged Bright Idea in the active brain of one René Descartes, a Frenchman who lived in the seventeenth century. With this he married the hitherto unrelated subjects of algebra and geometry, bringing forth out of the union the wonder child of *analytic geometry*.

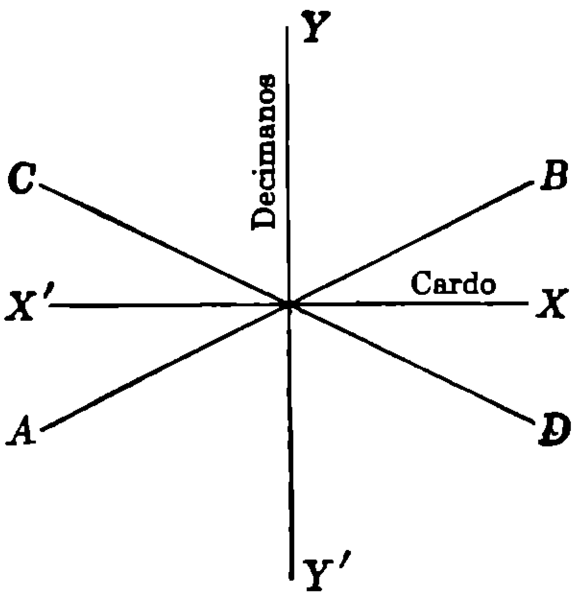

FIG. 9.

37. The essence of the scheme. Descartes's plan, like many great ideas, was simplicity itself. He merely appropriated for mathematics the principle of the decimanos and the cardo, or of latitude circles and meridians on large-scale local maps. The horizontal and vertical reference lines became the X *axis* and Y *axis*, respectively ($X'X$ and $Y'Y$ in Fig. 10). The two lines are referred to collectively as the *coordinate axes*, and their important intersection is the *origin*. The distance AP from the Y axis to the point P (Fig. 10) is called the *abscissa* of P, and is positive, negative, or zero according as P is to the right of, to the left of, or on the Y axis. Similarly, the distance BP, called the *ordinate* of P, is positive for points above the X axis, negative for those below, and zero for all points on the axis.[1]

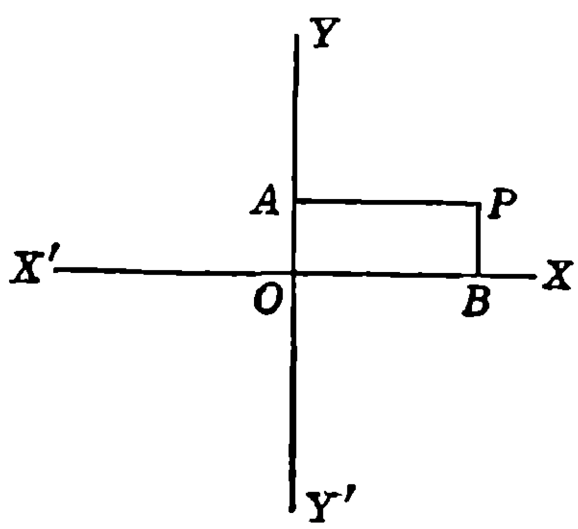

FIG. 10.

[1] The words *abscissa* and *ordinate* were first used by Leibnitz about 1692.

The abscissa and ordinate of a point are called the *coordinates*, and are usually written as a pair of numbers enclosed in parentheses and separated by a comma, with the abscissa invariably on the left. Thus, in Fig. 11, the coordinates of the points A, B, and C are the respective pairs (4,6), (3,2), and (−2, −4). A simple way to locate a point whose coordinates are given is to begin at the origin and measure to the right or left the distance indicated by the positive or negative abscissa, and then, from the point on the X axis thus reached, lay off the vertical distance upward or downward according to the sign of the ordinate. The downright simplicity of the thing is likely to make the beginner so confident and supercilious about the matter that he won't practice enough and then he'll be surprised and probably chagrined to find how easy it is even for great intellects to get the order mixed. It helps to remember that the order is "horizontal and then vertical," or "abscissa and then ordinate," exactly as we would have it from the alphabetical standpoint.

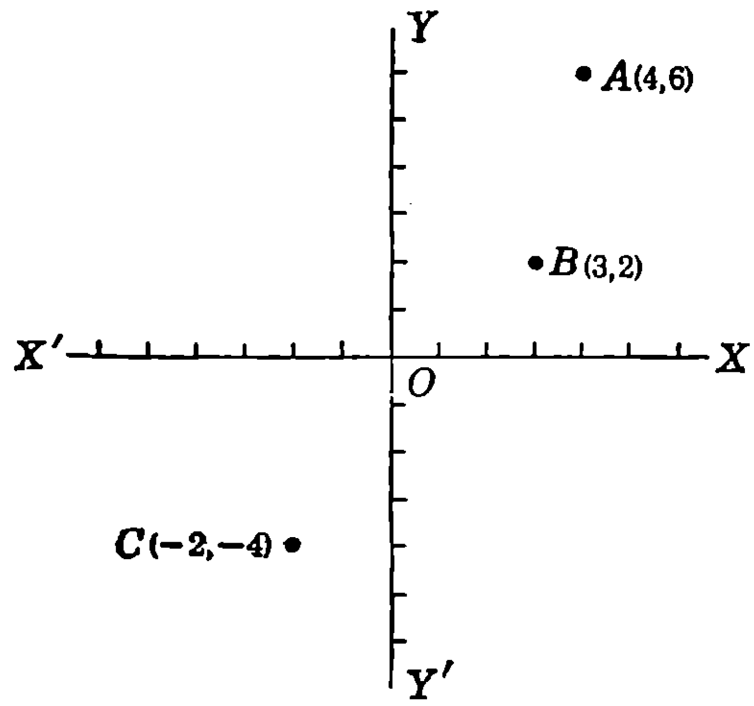

FIG. 11.

The plane is divided into four *quadrants*, or quarters, as shown in Fig. 10. Quadrants I, II, III, and IV are numbered in counterclockwise order beginning with the one above the X axis and to the right of the Y axis.

EXERCISE 19

1. Plot the following points: (1,2), (−2,4), (−3,−2), (−4,0), (0,0), (0,−2), (6,−1), (−5,4), (−3,2), (4,0), (2,−5).

2. In what quadrant is a point located if
(*a*) both of the coordinates are positive;
(*b*) both of the coordinates are negative;
(*c*) the abscissa is positive and the ordinate is negative;
(*d*) the ordinate is positive and the abscissa is negative?

3. If a point moves so that its ordinate is zero and its abscissa is positive, what line does it trace? Answer the same question if the abscissa is zero and the ordinate is negative.

4. What line is traced by a point that moves so that its coordinates are equal?

5. What can be said of one or both of the coordinates of

(*a*) all points on the X axis;
(*b*) all points on the Y axis;
(*c*) all points on a line 2 units below the X axis;
(*d*) all points to the right of the Y axis and below the X axis?

6. A rectangle whose sides are 6 and 4 units in length is placed so that the two sides lie along the coordinate axes. Find the four possible sets of coordinates for the vertices if the long side is on the Y axis.

7. A famous reply to a surrender demand in the Second World War can be deciphered if the following sets of points are plotted and the points in each set connected in the order indicated by lines. What was the reply?

(*a*) $(-16,0)$, $(-16,6)$, $(-11,0)$, $(-11,6)$.
(*b*) $(-6,6)$, $(-6,2)$, $(-3,0)$, $(0,2)$, $(0,6)$.
(*c*) $(4,6)$, $(10,6)$; then $(7,6)$, $(7,0)$.
(*d*) $(20,6)$, $(17,6)$, $(14,4)$, $(17,3)$, $(20,2)$, $(17,0)$, $(14,0)$.

38. The picture of a sentence. The coordinate-axes scheme enables us to find a unique pair of numbers to designate any point on a plane, and also to associate one and only one point with any given number-pair. This is the essential tie-up between algebraic sentences involving two variables, such as x and y, and the picture method of analytic geometry. For every equation in x and y lays down a rule by which each number automatically gets itself a mate, and the home addresses of these mating pairs on Mr. Descartes' convenient blackboard are points which are related to each other in a most surprising and interesting way. Instead of forming a helter-skelter mess of dots, as might be expected, they are frequently ranged neatly and thickly in tandem formations which turn out to be geometric lines and curves. Thus the ordinary algebraic sentence in two variables has a straight or curved picture, and the unlimited field for investigation opened by the bright idea stretches still unconquered before the mathematical pioneers of the future.

Consider the sentence $x + 2y = 0$, which we told you describes one of the paths of our hypothetical Roman boy through his city. To get a few of the countless number-pairs thus designated, we'll first transpose the x and divide by 2, thus getting $y = -x/2$. This solving for y makes it easier to name the numerical mate of any value which we assign to x. Taking a few numbers more or less at random, except that they should be conveniently small, we'll find the values of y which correspond with the following values for x: $-6, -2, 0, 2, 4, 6$. When $x = -6$, $y = -(-\frac{6}{2}) = 3$. There's our first pair, designating at once the point $(-6,3)$—above and to the left of the origin. The other pairs, found similarly, are $(-2,1)$, $(0,0)$, $(2,-1)$, $(4,-2)$, and $(6,-3)$. If we plot them, as in Fig. 12, we find that they lie along a straight line. The fact that the ordinate of any point in this line is half as large as the corresponding abscissa, as well as opposite it in sign, is expressed neatly in the sentence $y = -x/2$.

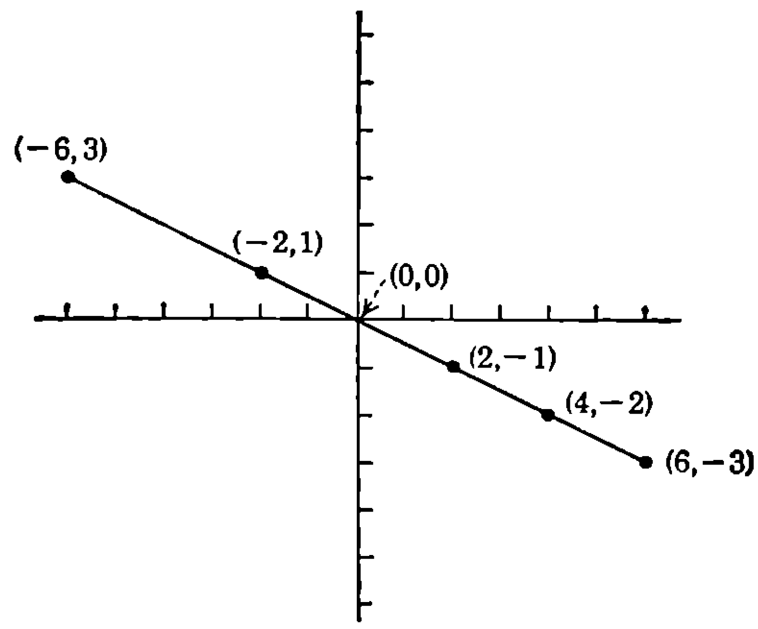

FIG. 12.

Suppose next that we get a few of the number-pairs designated by the equation $3x + 2y = 12$ and find its geometric picture, or *locus*. Solving for y, we get

$$y = \frac{12 - 3x}{2}$$

In order to set a good precedent, we'll arrange the pairs in vertical columns thus:

x	y
-2	9
0	6
2	3
4	0
6	-3

Again the picture turns out to be a straight line. In fact this is always true with reference to a first-degree, or *linear*, equation.

Evidently, then, only two points are necessary to determine it. Nevertheless, it will help greatly to keep down errors if a third point is used for checking purposes, at least temporarily.

The picture of a sentence, or, as the technical mathematician would say, the *locus* or *graph* of an equation, is important enough to merit a definition all by itself:

The ***locus*** *of an equation in not more than two variables is the geometric picture made by all of the points whose coordinates satisfy the equation.*

Usually a locus consists of some straight or curved lines, though it may also contain one or more isolated points.

EXERCISE 20

Plot the loci of the equations in Probs. 1 to 10.

1. $y = 4x$.
2. $y = -6x$.
3. $2x - 3y = 6$.
4. $4x = 12 - 3y$.
5. $x = 4$.
6. $x + 2y - 3 = 0$.
7. $2x + 5y = 8$.
8. $3y + 7 = 5x$.
9. $y = 0$.
10. $\frac{x}{3} - \frac{y}{4} = 1$.

11. Plot the graph of the equation $x + y = b$ when
(*a*) $b = 1$; (*b*) $b = 3$; (*c*) $b = 5$; (*d*) $b = -1$.

12. From the loci obtained in Prob. 11, what can you say about the effect on the graph of $x + y = b$ when we increase or decrease the value of b?

13. Plot $y = ax$ when (*a*) $a = -2$, (*b*) $a = 0$, and (*c*) $a = 3$. What fact is true of all lines $y = ax$ with a a constant, and what is the effect of an increase in the value of a?

14. Find the equation of the locus of
(*a*) all points 2 units to the right of the Y axis (answer: $x = 2$);
(*b*) all points on the Y axis;
(*c*) all points equidistant from the two axes whose coordinates are either both positive or both negative;
(*d*) all points twice as far from the X axis as from the Y axis, and having coordinates with unlike signs.

15. The points $(-2,-3)$, $(0,-1)$, $(6,5)$, and $(8,7)$ lie on a line. Note a relation which holds good for the two coordinates of each point and express this relation as an equation.

16. Three times the abscissa of any point on a line is two less than its ordinate. What is its equation?

17. Under normal braking conditions the distance a car travels after brakes are applied is given by the formula $d = 0.045r^2 + 1.1r$. Plot the graph of this equation in which d (the ordinate) is expressed in feet and r (the abscissa) in miles per hour. Answer the following questions: (*a*) Assuming the headlights enable the driver to see 150 ft. ahead, what speed at night is at the danger point? (*b*) If a car is moving at 30 miles per hour, when should a driver signal his intention to stop? (*c*) If two cars going in the same direction are 50 ft. apart, what is the maximum safe speed?

18. The relation between the Fahrenheit and centigrade thermometer scales is given by the equation $F = 9C/5 + 32$. Plot the graph of this equation, letting $C = 0°$ and $20°$, and extend the graph in each direction until the temperatures from $-50°C$ to $50°C$ will show. Then use the graph to answer the following questions: (*a*) For what centigrade reading is the Fahrenheit reading zero? (*b*) For what Fahrenheit reading is the centigrade reading zero? (*c*) What temperature gives the same reading on both scales? (*d*) What Fahrenheit reading is exactly five times the corresponding centigrade reading?

39. A composite sketch. Two straight paths on a level plain, unless they are parallel, will eventually cross. Though the spot marked with an X may have had little or nothing to recommend it previously, the crossing at once changes its status. Great cities have grown up at the meeting of dim trails which widened into highways—a fact which at last gives dignity and consequence to the intellectual life of the carefree pioneering cattle whose offhand decisions made history. In brief, crossings are often very, very important.

If this be true, we should be able to attach some algebraic significance to the point at which two lines intersect. And we can. It is obviously the only point which lies in both lines, and therefore it is the only one whose coordinates satisfy both equations at the same time. Its coordinates, in other words, make up the number-pair which we called the "answer" when we solved the compound sentence algebraically, and its location gives us a graphical or pictorial check on the algebraic processes.

For instance, if we solve the compound sentence

(1) $$x - 2y = 1$$
(2) $$3x + 2y = 7$$

by the method of the last chapter, we get the answer-pair ($x = 2$, $y = \frac{1}{2}$). To give a geometrical meaning to this operation we need only draw the lines (1) and (2) on the same set of axes. The picture then looks like that in Fig. 13. The reader should check this for himself. Note that the two lines cross at the point which, if we draw the figure with care, proves to be exactly the one indicated by the algebraic solution. Thus we have found a method not only of illustrating our algebra, but also of checking up on it. Furthermore, the geometrical relations of the figures are now available to suggest conclusions which do not follow readily from the algebra alone. For example, we can now see that two first-degree equations, whose pictures are straight lines, will always have a number-pair in common if the lines intersect, and also that the solution, like the intersection, will be unique. Likewise, the interpretation of inconsistent equations such as $2x + 3y = 4$ and $2x + 3y = 5$ becomes obvious—their graphs are parallel lines.

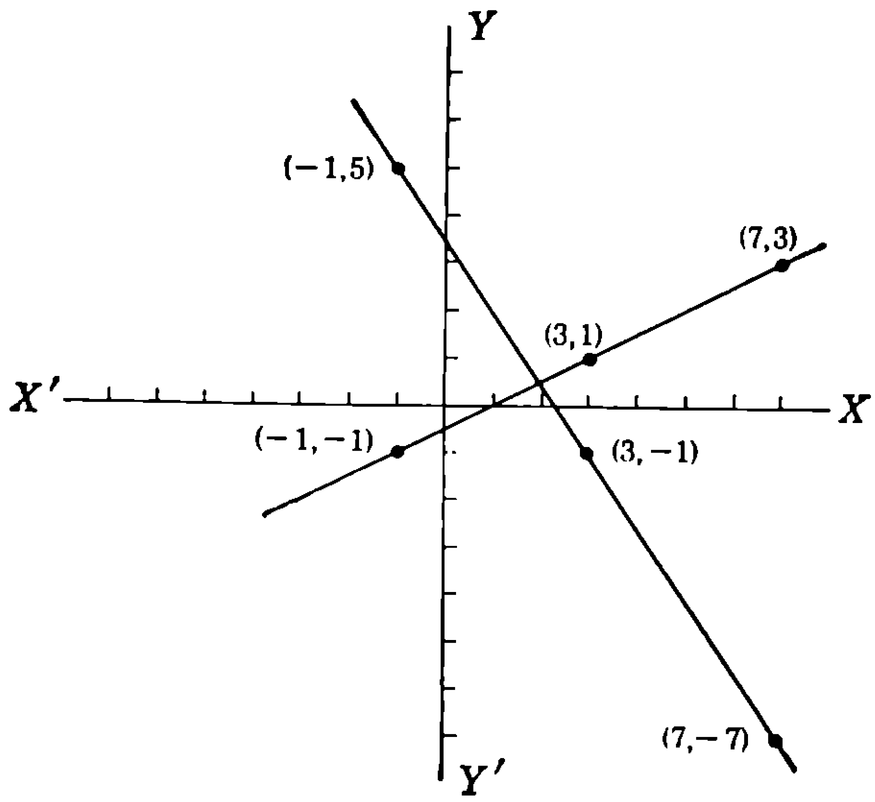

FIG. 13.

Here is an intriguing question. Is it possible to find easily, by some wholesale method, a whole group of lines which unanimously pass through a given point? It is, as we shall show in the case of the point $(2,\frac{1}{2})$. After transposing the absolute terms 1 and 7 in equations (1) and (2) and getting

$$x - 2y - 1 = 0 \tag{3}$$

and

$$3x + 2y - 7 = 0 \tag{4}$$

we multiply one of them, say (4), by k, and put them together thus:

$$x - 2y - 1 + k(3x + 2y - 7) = 0 \tag{5}$$

We can now make the sweeping statement that (5) represents an infinite number of lines, all passing through the intersection of

(3) and (4). Each separate value of k gives us a particular line of the "family" which can be absolutely depended upon to head straight for the crossroad point. For example, if $k = 2$, (5) becomes $7x + 2y - 15 = 0$, which, as we predicted, goes right through $(2,\frac{1}{2})$ since $7(2) + 2(\frac{1}{2}) - 15 = 0$. You can try other values of k for yourself.

The rather simple explanation of this surprising result is that the crossroad point is naturally on both of the lines (1) and (2), and that therefore its coordinates, substituted for x and y, make the left sides of (3) and of (4) both equal to zero. If we will note how these left sides are involved in (5) we will see that this particular point will make (5) become

$$0 + k(0) = 0$$

so that (5) is obviously satisfied for any value of k whatever.

If now we let $k = 1$, (5) becomes

$$4x - 8 = 0$$

or

$$x = 2$$

This is the line on which the abscissa of every point is 2, so that it is parallel to the Y axis and two units to the right. Similarly, setting $k = -\frac{1}{3}$, we get the line

$$y = \frac{1}{2}$$

parallel to and one-half a unit above the X axis. But we already know in advance that both of these lines go through the intersection of (3) and (4), since they are special cases of (5). Hence we see that the algebraic method of solving two simultaneous linear equations by eliminating one of the variables amounts geometrically to getting two lines, one vertical and one horizontal, which go through the same point as the original equations and thereby show the coordinates of the previously unknown junction point.

EXERCISE 21

Draw each of the line-pairs indicated in Probs. 1 to 10, and estimate the coordinates of the points of intersection when they exist.

1. $2x - 3y = 4$;
$3x + 4y = 6$.

2. $y = 3x$;
$x = 2y$.

3. $y = 3 - 2x;$
$x = 3 - 2y.$

4. $4x - 3y = 6;$
$x + 2 = 0.$

5. $5x + 2y = 10;$
$x + y = 2.$

6. $4x + 3y = 12;$
$3x - 4y = 6.$

7. $4 - 3x - y = 0;$
$4x - 3y - 1 = 0.$

8. $4x - 3y = 6;$
$6y = 8x - 2.$

9. $y = x;$
$5x = 6 + 5y.$

10. $x + y = 2;$
$x + y + 2 = 0.$

11. (*a*) Through what point do all the lines of the family $x + y - 7 + k(4x - 3y) = 0$ pass?
(*b*) Find k so that the line will pass through $(3,-2)$.

12. Using k as in Prob. 11, make up an equation of a family of lines all of which go through (*a*) the origin, (*b*) the point (2,3), and (*c*) the point $(-1,-2)$.

40. Curves ahead. The pictures of first-degree equations, as we have seen, are remarkably simple and easy to indicate on paper as long as full-length portraits are not required; but it must be admitted that they have a tendency to pall on one. The total lack of surprise in the quality of straightness, whether in geometry or in roads, is at best a bit monotonous in the long run. But curves are different. There lies the beauty of the unexpected, of rhythmic change, new vistas, and the artist Nature in many of her best lines.

We hope, therefore, that the reader will be greatly pleased to learn that the mathematical portraits of second- and higher degree equations usually forsake the paths laid down by the ruler and sweep along the pleasing arcs of rainbows, comet trails, and meandering brooks. And since the lines of nature are seldom straight, while her laws are mathematical, it is a striking sidelight upon the aptness of our scheme of illustration that most mathematical pictures turn out to be curves!

Consider, for instance, the equation which expresses the relation between the horizontal—or x—and vertical—or y—distances covered by a stone thrown over a house. This relation, depends of course upon the initial speed of the stone and the slant of its path. Without at present going into the method of derivation, we find that it comes out under one set of conditions to be $x^2 = 18 - 9y$.

Now if we graph this relation on a paper-size scale, the part of the curve near the origin turns out to be a replica in miniature of the actual path of the stone. To see this we first solve the equation for y, getting

$$y = \frac{18 - x^2}{9}$$

and then compute y when x has each of several not-too-large values which are easy to work with, such as 9, 6, 3, 0, -3, -6, -9. The corresponding pairs of values are given in the table below.

x	—12	−9	—6	—3	0	3	6	9	12
y	—14	−7	—2	1	2	1	−2	−7	—14

When we plot these points and connect them with a smooth curve, we get the graph shown in Fig. 14. This type of curve, called a *parabola,* is a favorite in the works of nature and of man alike, being used independently, for example, by thrown stones, waterfalls, and manufacturers of headlights. However, since it is our purpose in this chapter to explain the translation-to-picture machinery rather than to study its results in detail, we shall merely greet the parabola, in passing, as an interesting curve turned up along the way.

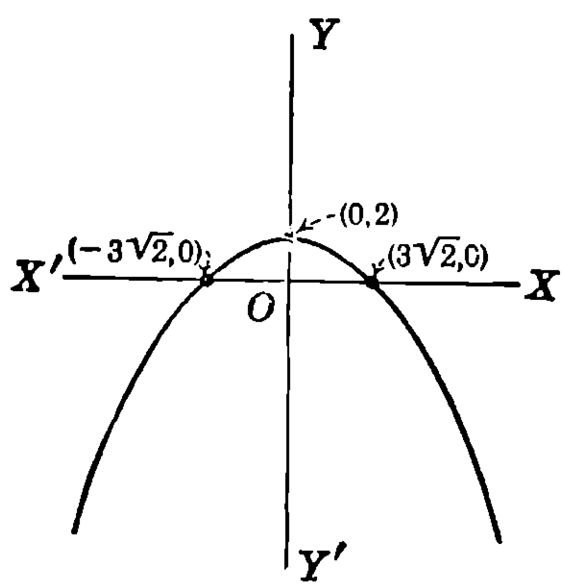

Fig. 14.

Again, if we carefully plot the graph of $y = x^3 - 3x + 1$, we find that the sample points

x	− 3	—2	−1	0	1	2	3
y	−17	−1	3	1	−1	3	19

fail to indicate clearly just how the curve runs. In such a case, it is advisable to get some in-between points, such as $x = -\frac{1}{2}$, $y = \frac{19}{8}$; $x = \frac{3}{2}$, $y = -\frac{1}{8}$, etc. The picture, as shown in Fig. 15, has, you notice, more twists than the parabola. Generally speaking, the higher the exponents on x and y in an equation, the more intricate is its curve, and when we admit fractional exponents we get

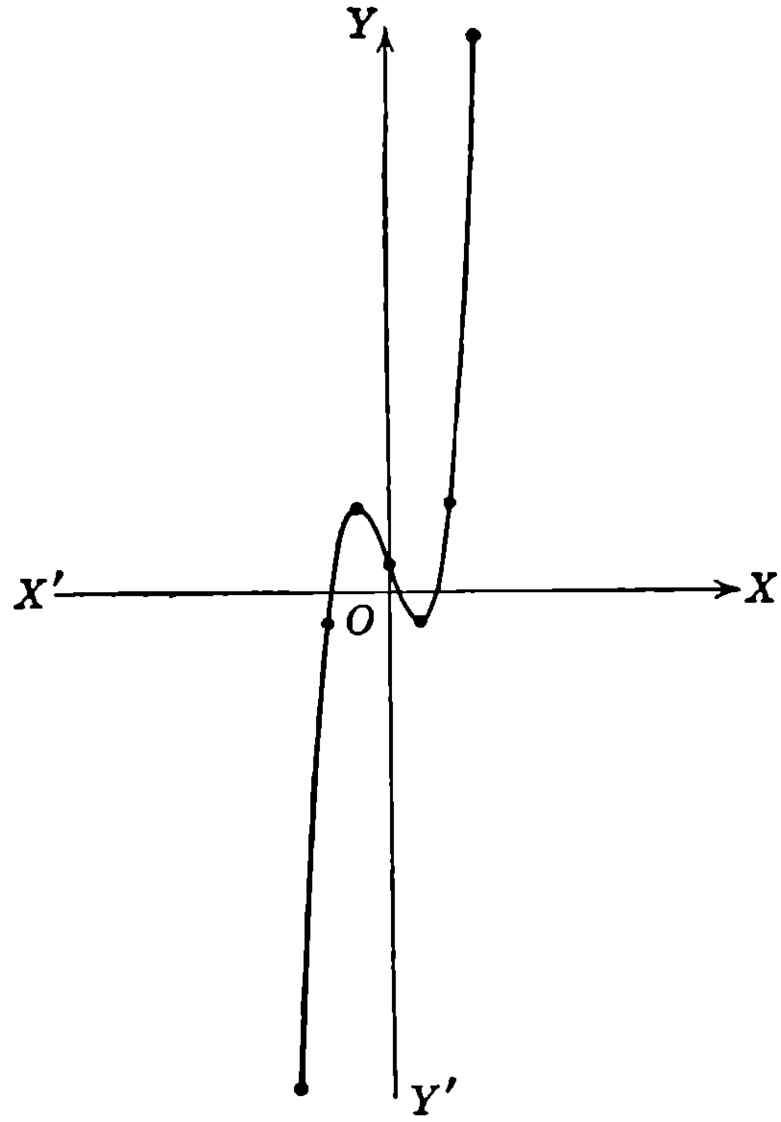

FIG. 15.

some things which look like smoke-trails of stunting airplanes. Just as matter of interest we'll show in Fig. 16 a few samples of the not-so-easy-to-plot curves, with their equations.

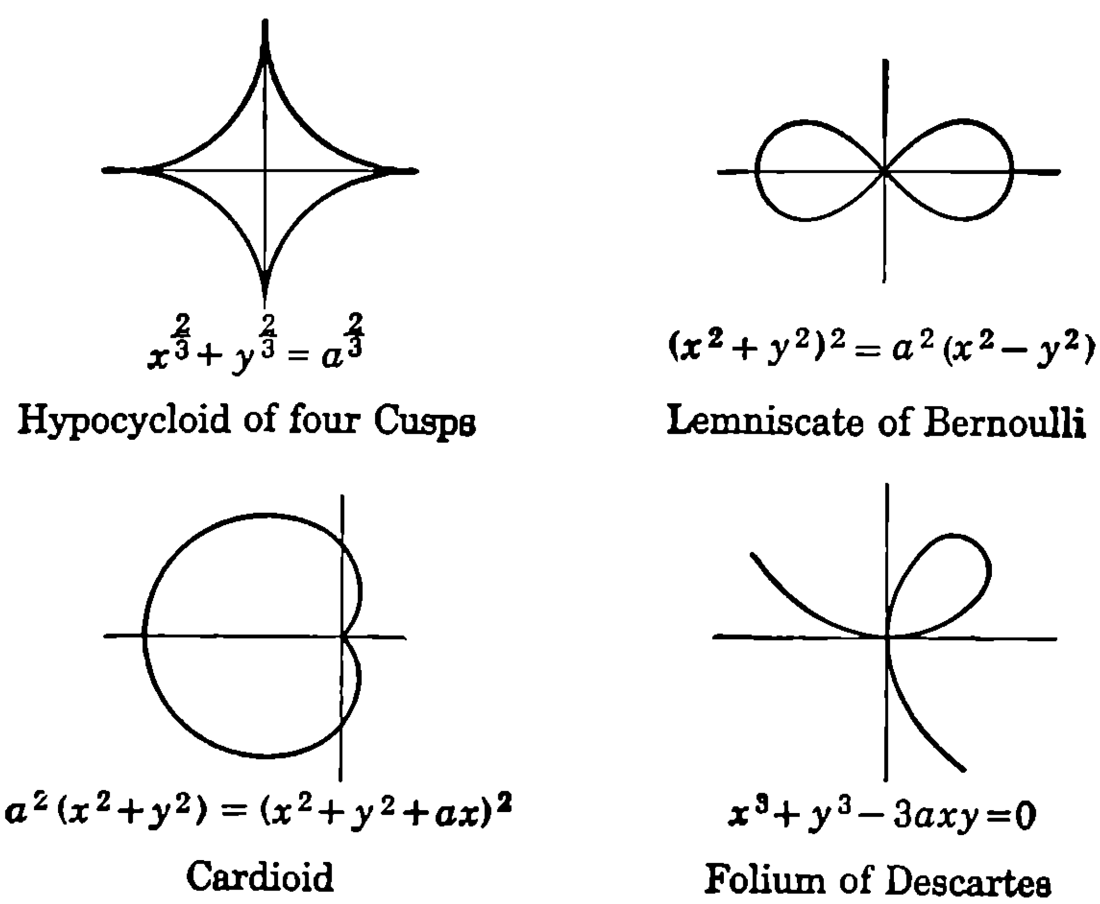

FIG. 16.

EXERCISE 22

Plot the curves indicated in Probs. 1 to 10, choosing about 10 reasonably small positive and negative values of x and adjusting the vertical scale so that the points found can be placed in the available space.

1. $y = x^2$.

2. $y = x^2 - 2x - 3$.

3. $y = 4 - 3x - x^2$.

4. $x^2 = 4y$.

5. $y = x^3$.

6. $x^2 + y^2 = 25$.

7. $y = x^4$.

8. $y = x^3 - 4x$.

9. $x^2 - xy = 0$.

10. $y = x - \frac{1}{x}$.

Make a graphic representation of the well-known laws of nature stated in Probs. 11 to 15. Use values for the first and second variables given as ordinates and abscissas, respectively. In each case get about 10 points fairly close to the origin.

11. $p = 1/v$. (This is a special case of Boyle's law applying to a gas under constant temperature. The variables p and v represent pressure and volume.)

12. $s = 16t^2$. (Here s equals the distance in ft. traversed by a falling body in time t, expressed in sec.)

13. $v = 32t$. (v is the falling velocity attained in time t.)

14. $F = 1/d^2$. (This is one version of the famous law of gravitation, with a proper choice of units. Here F = force of attraction and d = distance between objects.)

15. $A = a^t$, for the case $a = 2$. (This is the law of growth applying to anything, from a colony of bacteria to a city, whose size is multiplied by a factor a in one unit of time. Here A is the growing quantity and t is the time.)

41. Using our charts. Pictures of all kinds are becoming more and more indispensable in modern civilization. In the form of photographs they come to the aid of astronomy, geology, and crime detection, for example—and certainly they are a big help to the motion-picture industry. In the form of drawings they are used by the architect, engineer, and cartoonist, to say nothing of the artist. In mathematics we have already seen how the two-line image of a compound sentence helps us to understand the meaning of its solution. This is merely a start in the matter of using the illustrative scheme which enables us to work problems in algebra and many fields of higher mathematics much more easily because we can visualize them. A second and very important application lies in the solution of higher degree equations.

Consider, for example, the sentence

$$(1) \qquad x^3 - 3x + 1 = 0$$

which is a sample of the third-degree or *cubic* equation in one variable—only a degree, to be sure, higher than the quadratic but about a dozen degrees more resistant to a purely algebraic attack. The picture method immediately suggests an easier and a more practical approach. If we designate the left side of (1) by y we get the equation

$$(2) \qquad y = x^3 - 3x + 1$$

whose graph is shown in Fig. 15. Turning back to this, we notice that it crosses the X axis at points whose abscissas are roughly -1.9, .4, and 1.5. For about those values of x, then, the y of (2) will be zero, and therefore (1) will be satisfied. Look at it closely and note that we have here a solving method which does not apply merely to cubics. It works just as well for higher degree equations in one variable. We simply move all terms of the given equation to the left of the equality sign, replace the zero on the right by y, and plot the curve indicated. The abscissas of its intersections with the X axis, or technically the *x-intercepts,* will be the roots of the first equation.

Assuming that we can get these revealing crossing points with any required degree of accuracy (our Chap. IX will explain the method in greater detail), we have certainly made more than just a stride forward in the matter of solving equations. In fact, only moving-picture language can do justice to the simple truth; our leap has been stupendous, gigantic, and supercolossal, for we have here hit upon a way of solving equations of *any degree whatever.* A better illustration of the helpfulness of pictures in our study of the symbolic language of algebra would be very, very hard to find.

42. Introducing the function. When, as in the above discussion of a powerful solving method, we wish to talk about the left side of a particular equation in x, and at the same time to emphasize the fact that our remarks apply to countless other expressions involving x, of which the one used is merely a sample, we clearly need some word or phrase to describe all such expressions. The word customarily used is *function.* The idea back of it is so basic and important in mathematics that nothing will do here except a definition:

*A **function of** **x** is a quantity so related to **x** that its value is automatically fixed whenever a value is named for **x**.*

Naturally, the definition would still be good if we should replace x all the way through by any other letter. This the reader should do in several ways, using w in place of x—then z, etc., until the important idea gets pretty well fixed in his mind.

It should be clear from the definition that we can use our discretion about values to be assigned to x. When that is done, however, we have no choice whatever about the function's value. For instance, one sample function of x is x^2, and if x is 12, then x^2 has to be 144. Another function is $7x^2 - 3x + 2$, which is necessarily 6 in case x is 1, or 12 in case x is -1. With the symbol $f(x)$ (read "function of x" or "f of x") we can indicate all such quantities dependent upon x for their values, and we can cut down the territory covered to one particular function by means of a simple little "if." The usual trick is to put it like this: "If $f(x) = x^2 + 2$, then $f(3) = 3^2 + 2 = 11$; but if $f(x) = 1/x^3$, then $f(3) = \frac{1}{3^3} = \frac{1}{27}$." Here $f(3)$ is read "f of 3" and means "the value of $f(x)$ when $x = 3$." Since x can have various values in the problem, it is called a *variable* as distinguished from a *constant,* such as 2, or π, or numbers usually represented by letters at the first of the alphabet. These constants, when once assigned values, have to keep them from there on to the end of the problem, come what will. Variables like x which can run wild are called *independent,* while those like $f(x)$ which must tag along with x and change only as directed are, naturally enough, dubbed *dependent.*

Now these subservient functions themselves are of two main types: mathematical and otherwise. The first are those, such as

$$\left(\frac{1}{2x + 3}\right)^3$$

or the samples mentioned above, which can be expressed in such form with mathematical symbols that the form gives all the information necessary to compute the functional value corresponding with a given value for x. The second type, which is met in real life perhaps even more often than the first, is illustrated by this statement: "The temperature at a given spot on the earth's surface is a function of the time, since that spot has had or will have a certain degree of warmth at any given time, past or future." The pairing of one with the other is all that is needed to make a functional relation; but we cannot express it in explicit mathematical form allowing computation. As a result we can look it up, perhaps, for some instant in the past; but we cannot figure it out, either before or after the event. We'll discuss this class of function at more length in the next article.

To show that functional relations of one kind or another press about us from all sides, so that much can be learned from a mathematical treatment of them, consider these simple truths: The temperature and atmospheric pressure about any given person vary from midnight to midday, from day to day, from year to year, and from one place to another. Each step that he takes affects not only his total walking record but in some degree his health and his life. The distance traveled by a falling body and its rate of fall both increase with the time of descent. The area of a circle depends upon its radius. Since the latter two functional relations can today be expressed in precise mathematical form, we can *predict* the distance when we know the time, or the area of a circle when given the radius. On the other hand, lacking the necessary formulas, we cannot anticipate temperature in terms of time, or a man's health according to his mileage, although we know that a relationship is there. Evidently each new translation of a functional relation into mathematical language means one more turn in the wheel of progress.

In the formal algebraic treatment of functions, any equation in two variables (which we may as well call x and y for convenience) expresses implicitly a functional relation between them, since either one is determined when the other is assigned a value. If we solve the equation for y in terms of x, the resulting picture is said to be a graph of y or of $f(x)$, and is a straight line, as we have seen, if $f(x)$ contains only first-degree terms in x. Otherwise it is usually a curve of some sort.

EXERCISE 23

Plot the graphs of the functions in Probs. 1 to 5; then, using your graph, estimate to one decimal place the value of x for which the function is zero. (In plotting the graphs of the functions, use corresponding values of the variable and the function as abscissas and ordinates of the points.)

1. $\frac{1}{x} - 2.$ **2.** $x^2 - 2x + 2.$ **3.** $x^3 - 2x^2 + x - 1.$

4. $x^2 - 3.$ **5.** $x^3 - 4.$

6. If $f(x) = x^3 + x^2 + 1$, find (*a*) $f(2)$; (*b*) $f(0)$; (*c*) $f(1)$.

7. If $f(x) = 3x^2 - x + 1$ and $g(x) = x^2 - 1$, find

(*a*) $f(2)$; (*b*) $f(\frac{1}{2})$;
(*c*) $f(z + 1)$; (*d*) $f(1) + g(2)$;
(*e*) $\frac{f(3)}{g(2)}$; (*f*) $f(z) + g(z - 1)$.

Express y as an explicit function of x in the equations of Probs. 8 to 11.

Example: Given $y + 2xy = x + 1$, then $y = \frac{x + 1}{1 + 2x}$.

8. $x^2 + xy + x + y = 0.$ **9.** $x^2 + xy + y^2 = 0.$

10. $3y^2 - 25x^{10} = 0.$ **11.** $\frac{1}{x} + \frac{1}{y} - \frac{2}{xy} = 0.$

Find y as an explicit function of x in each of Probs. 12 to 16.

12. $y = u^2 - 1;\ u = x + 1.$
13. $y = u^2 + 5;\ u = x + 3.$
14. $u - uy = v;\ ux + x^2 = 0;\ x^2v = 1.$
15. $y = u + 1;\ u = v + 1;\ v = x + 1.$
16. $y = u^2 + 1;\ u = v^2 + 1;\ v = x^2 + 1.$

17. Think of 10 functional relations met in life in addition to the following example: The price of wheat is a function of the supply and demand. (In a general way, neglecting other factors involved, it varies directly with the demand and inversely with the supply.)

18. Which, if any, of the relations you cited in Prob. 17 can be expressed, at the present stage of our progress, in precise mathematical form? Are there any of the others which hold promise of such expression in the near future?

43. Maps for everything. Since the functions which have as yet no mathematical expression are so numerous and important in real life, the application of our picture method to them not only looks promising but has turned out to be, in actual fact, extremely useful. The whole field of *statistics*, or the tabulation of figures encountered in any investigation or occupation, is lighted up, so to speak, by easy-to-grasp messages to the eye. These pictures usually adopt in principle our mathematical framework. Some

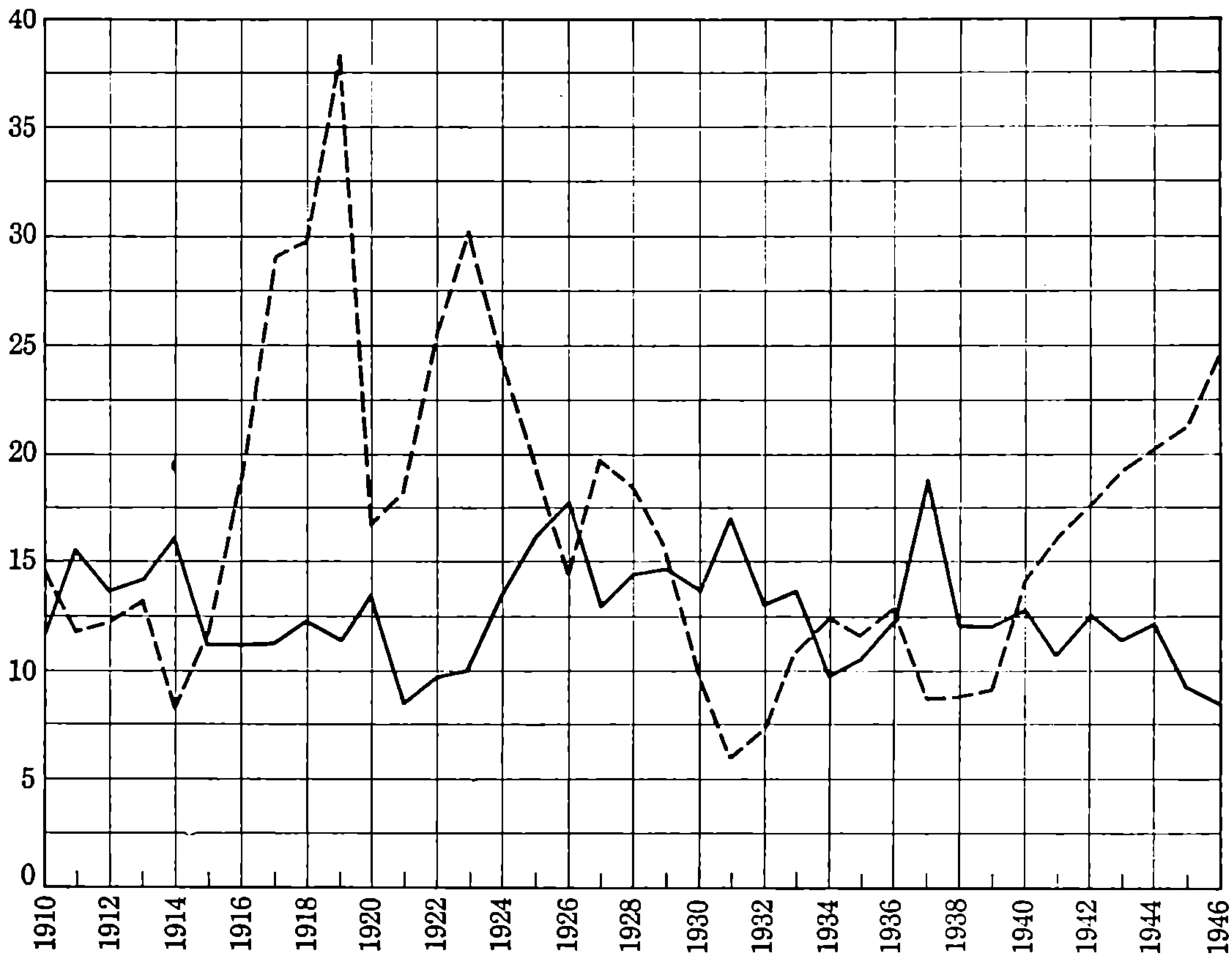

FIG. 17.

independent variable, such as time, for instance, is represented, like the serviceable x, along a horizontal line, while another variable related to, if not dependent upon, the first is represented vertically like the nearly as serviceable y. The second variable might be the world production of cotton, or again it might be the price. To show the sort of picture which results, we present in Fig. 17 the graphs of the cotton production (heavy line) in multiples of 1,000,000 bales, and the cotton price (dotted line) in cents per pound, for the years 1910–1946 inclusive. Certain trends and conclusions are at once apparent. We expect a rise in price with a drop in production, and vice versa—an expectation which is borne out by the graph for the years 1911, 1914, 1915, 1923–1927, 1931, 1937. We note also an exceptional price level from 1915 to 1919, and from 1939 to 1946, as well as an unusual drop in 1920, and it requires no great mental effort to connect these line waves with the First and Second World Wars.

Such are the pictures which are pored over daily by economists and businessmen. They are the next best thing to mathematical curves when the latter are not available. For while the information which they carry to the brain through the eye is sometimes useful in itself, it is more frequently valued only as it suggests that which lies *off the page*. The important question for the economist is seldom, "What has happened?" Rather it is, "What is going to happen?" or, "Is that price curve likely to go up or down?" Now if his curve were a mathematical one he could answer the question for any date in the near or distant future with the easy assurance of the kindergarten teacher. As it is, his curves have enough independence and surprise bends to keep him worried and occupied with the business of making forecasts, while at the same time they do enough near-repeating in response to conditions such as have been met before to make his expert predictions much more valuable than mere guesswork. As a consequence, the study of statistics is being recognized more and more as an essential part of the training of the economist, businessman, lawyer, scientist, or in fact the worker in any field involving tabulated data (and few if any fields are left out by that description). The full-time professional statisticians employed by many large firms give the final touch of respectability to this baby science offshooting from mathematics.

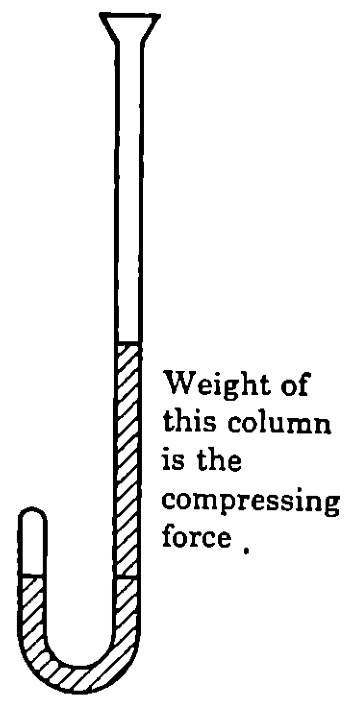

FIG. 18.

Since we have called the wavy line plotted from a set of figures (or from so-called "empirical" data) the "next best thing to mathematical curves," the finding that any particular empirical curve is really a mathematical one can obviously amount in some instances to a very important discovery. A case in point is the simple laboratory experiment in physics which connects the pressure on a given amount of gas with its volume. The essential apparatus is a bent glass tube closed at one end and fixed in a vertical position (see Fig. 18).

The liquid metal mercury, when poured into the open end of the tube, traps a certain amount of air in the closed part, and the volume of this imprisoned gas gets smaller as more mercury is added. In the experiment the various volumes of air, together with the corresponding heights of the compressing column, are measured carefully. The following results were obtained in one case:

Height of Mercury Column	C. C. of Air
10	150
12	124
15	100
18	84
25	60
50	29.5
75	19.6
100	15.1
125	11.8
150	10

When these number-pairs are plotted on the coordinate system with the recorded heights of the mercury column and the corresponding air volumes as abscissas and ordinates, respectively, and when the points are connected by a smooth curve, we get Fig. 19.

Here we have a curve with such beautiful and symmetrical contours that a mathematical relation is suggested. Following up this clue, we examine the pairs of numbers more closely, and note that the product of the pairs is in each case exactly or very nearly 1,500. This leads us to suspect that if the errors in the experiment could be eliminated, the product of the corresponding readings would always be a constant in the neighborhood of 1,500. Since the height of the mercury column is directly proportional to its weight and hence to the pressure exerted, we seem to have discovered a relation in the field of physics which is expressed by the statement: "Pressure times volume remains constant in the case of a small bit of air," or more compactly by the formula $pv = c$.

Actually such a mathematical relation, applied not only to air but to any gas, was announced in 1662 and called *Boyle's law* after the discoverer. It seems probable that an analysis of his experimental results in the form of a curve may have led him to the result. In any case, we know that the scientist is often led to new discoveries from the revealing antics of his curves. The so-called physical "laws" are usually merely relations between quantities which obey, in the measured cases, the rules laid down by a mathematical equation. Such is Boyle's law, and such, indeed, is Newton's law of gravitation—the best known, perhaps, of all generalizations about nature.

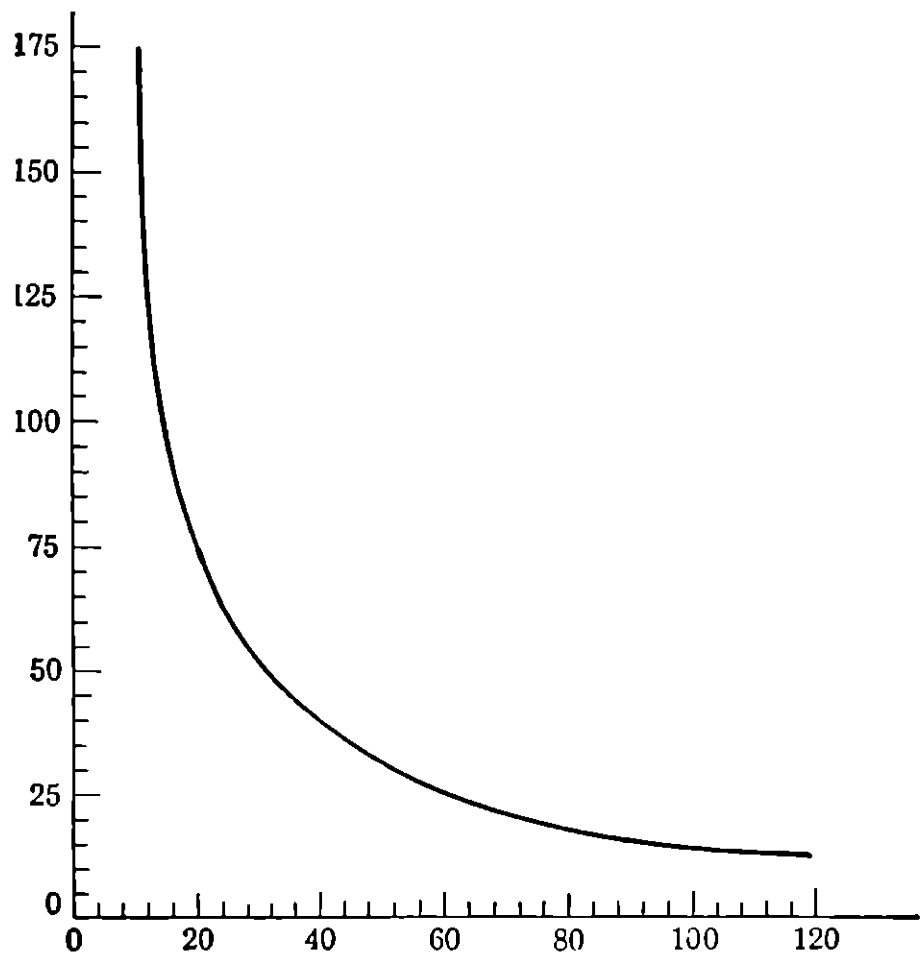

FIG. 19.

EXERCISE 24

1. The temperature readings in a certain city during one 24-hr. day were as follows: 2 (A.M.), 63°; 4, 60°; 6, 62°; 8, 65°; 10, 77°; 12 (noon), 84°; 2 (P.M.), 89°; 4, 93°; 6, 92°; 8, 81°; 10, 68°; 12, 64°. Plot the curve and explain its trend.

In each of the following problems draw graphs representing the given data. Official sources are here used, but often the last figure is a preliminary estimate. Each set of figures tells a story in the light of history; and if you are bright enough perhaps you can see "the picture behind the picture."

2. (Two graphs) The total imports and exports of the United States for the years 1932–46 inclusive, in hundreds of millions dollars, were as follows (the first figure in each pair representing imports): 13.2, 16.1; 14.5, 16.8; 16.6, 21.3; 20.5, 22.8; 24.2, 25.6; 30.8, 33.5; 19.6, 30.9; 23.2, 31.8; 26.2, 40.2; 33.5, 51.5; 27.5, 81.0; 33.8, 130; 39.2, 143; 41.4, 98.1; 50.0, 97.8.

3. The population of the United States, in millions, over the indicated period was as follows:

Population	July 1940	1941		1942		1943		1944		1945		1946	
		Jan.	July	Jan.	July	Jan.	July	Jan.	July	Jan.	July	Jan.	July
Total...	132	133	133	134	135	135	136	137	138	139	140	140	141
De facto.	132	133	133	134	134	134	134	134	133	132	132	137	140
Civilian.	132	132	132	132	132	129	127	127	126	127	127	133	138
Military.	0.3	0.7	1.6	2.0	3.8	6.9	9.1	10.3	11.5	11.9	12.2	6.8	3.0

4. Birth and death rates in the United States, per 1,000 of the population, for the years 1920–1945 inclusive, were as follows (the first figure of each pair representing births): 23.7, 13.0; 24.2, 11.5; 22.3, 11.7; 22.1, 12.1; 22.2, 11.6; 21.3, 11.7; 20.5, 12.1; 20.5, 11.3; 19.7, 12.0; 18.8, 11.9; 18.9, 11.3; 18.0, 11.1; 17.4, 10.9; 16.6, 10.7; 17.2, 11.1; 16.9, 10.9; 16.7, 11.6; 17.1, 11.3; 17.6, 10.6; 17.3, 10.6; 17.9, 10.7; 18.9, 10.5; 20.9, 10.4; 21.5, 10.9; 20.2, 10.6; 19.6, 10.6. Why do births fluctuate more than deaths? What general trends are indicated?

5. The per capita income (dollars) in the United States for the years 1919–1946 was as follows: 455, 414, 377, 451, 508, 499, 518, 521, 524, 541, 566, 471, 411, 334, 364, 400, 440, 509, 536, 494, 549, 591, 696, 796, 906, 927, 899, 837.

6. The number of individuals in the United States with net incomes of $1,000,000 or more (before taxation) for the years 1925–1938 inclusive, was 207, 231, 290, 511, 513, 150, 77, 20, 50, 33, 41, 61, 49, 48.

7. The following table shows the expenditure for education in the United States, expressed in terms of millions of dollars, for the indicated years:

1870	83	1910	450
1880	125	1920	1000
1890	185	1925	1800
1900	250		

8. The following table shows the population of the United States, expressed in millions of individuals, for the years indicated:

1850	23.1	1890	62.9
1860	31.4	1900	76.0
1870	38.6	1910	92.0
1880	50.1	1920	105.7

9. Compare the graph of Prob. 7 to that of Prob. 8 and then decide whether the cost of education per capita increased or decreased during the interval from 1870 to 1920.

10. The mean temperature during each month for two cities, the first in a low altitude and the second in a high altitude, was as follows: January, 10°, 0°; February, 6°, 4°; March, 31°, 25°; April, 38°, 42°; May, 50°, 69°; June, 63°, 79°; July, 75°, 81°; August, 82°, 77°; September, 74°, 68°; October, 60°, 44°; November, 34°, 20°; December, 17°, 3°. Draw the two graphs. Can you propose a theory to account for the general shift to the left of the graph representing temperatures in a high altitude?

11. The number of hours of sunlight during a midsummer day at the indicated latitudes is approximately as follows: 0° (on equator), 12; 17°, 13; 33°, 14; 40°, 15; 50°, 16; 60°, 18; 67°, 24. Plot the hours of sunlight as a function of the latitude. How would the tendency indicated be likely to affect the highest midsummer temperatures of Minneapolis and New Orleans?

CHAPTER V

PUTTING EXPONENTS TO WORK

44. Help wanted. Back in Chap. II we introduced exponents of various kinds—positive, negative, and fractional. We worked out the pertinent rules of the game, known as the "laws of exponents," which took care of the situation admirably even when those upraised numbers (as in the case of $8^{-2/3} = \frac{1}{4}$) departed no little from the plain positive integer of the first definition. At that point we discretely let the matter stand, confining ourselves in Chap. III to the positive integral exponents which indicate the degrees of equations.

Now comes the time to justify this strategy. If asked to show the practical uses of exponents, we can give examples enough to suit anybody, but most of them require computations which are discouragingly tiresome without the modern short cuts. For example, suppose a proud father, on the birth of a son, deposits to the latter's account in a savings bank the sum of $100, which is to draw interest at 3 per cent per year compounded semi-annually. This means that every six months the accrued interest is to be added to the principal. Now if the father is weak in arithmetic, doesn't trust the bank's statement, and wishes to figure out just how much his son will have coming to him on his twenty-first birthday, he may be hurried to complete the calculation of the correct amount before the son finds it out for himself by the simple process of growing up and getting it. The first steps in the computation would run about as follows:

(1) Interest on $100 for 6 months at 3 per cent per year.	$ 1.50
(2) Amount due at the end of 6 months.	101.50
(3) Interest on $101.50, etc., (101.50)(.015)	1.5225
(4) Amount due after one year.	103.0225

Somewhere in the process the fond computer might learn that he could have saved some time by simply figuring the value of $100(1.015)^{42}$ to get the amount due at the end of 42 periods, since $100 + 100(.015) = 100(1 + .015) = 100(1.015)$, and $100(1.015) + (100)(1.015)(.015) = 100(1.015)^2$, etc. Even then he would find the task so annoyingly drawn out that he would probably get disgusted and quit. And if he were to try to figure the amount of \$100 for (say) 5 years and 72 days, or 10.4 years (which *should* be an easy decimal number to work with) the quantity $(1.015)^{10.4}$ or $(1.015)^{10\frac{4}{10}}$ or $(1.015)^{5\frac{2}{5}}$ or $\sqrt[5]{(1.015)^{52}}$ would rise from the scratch-page to haunt him. His frame of mind would by that time make him thoroughly appreciate the blessed computing device coming out of the past which makes the calculation of such vexers as $(1.015)^{52}$ or $(1.015)^{10.4}$ a mere matter of minutes at the most, instead of hours or days. This would be one form of mathematical help, at least, whose usefulness he would not be inclined to deny.

45. Briggs to the rescue. In the early part of the seventeenth century help came dramatically. It took the form of a mathematical invention given the formidable name of *logarithms.* The latter, according to Lord Moulton,[1] "came on the world as a bolt from the blue. No previous work had led up to it, foreshadowed it, or heralded its arrival." The timesaver, fittingly enough, was the brain child of a Scotchman, one John Napier (1550–1617). Since, however, the application to rapid computing was due chiefly to the Englishman Henry Briggs (1556–1631), we shall honor him with the official role of rescuer. Indeed, the special tables of numbers which we shall explain are known as the "Briggs" or "common" logarithms. They are nothing more abstruse nor difficult than plain and simple *exponents of ten.* But to be a little more explicit, we'll give a

Definition: *The* ***common logarithm*** *of a number* ***A*** *is the exponent which, applied to the base* 10, *gives* ***A***.

For instance, the logarithm of 100 is 2, since $10^2 = 100$. This same statement in symbolic language would be this: $\log 100 = 2$ (read "logarithm of 100 is 2"). Also, the logarithm of 1,000 is 3, since $10^3 = 1{,}000$. Hence the logarithm of 567, which is between 100 and 1,000, should be between 2 and 3, and

[1] "Inaugural Address: The Invention of Logarithms." *Napier Tercentenary Memorial Volume,* 1914, p. 1.

will be the value of x in the equation $10^x = 567$, provided that equation is solvable. As a matter of fact, no rational exponent applied to 10 will give 567, but the equation is nevertheless solvable if we admit irrational exponents. You should note that this involves an extension of our previous definitions, which covered exponents that were at first positive and integral, and then fractional, negative, and zero. We'll omit the precise statement of this extension, and say only that $x = 2.7536$ is the four-decimal rational approximation to the exact irrational solution of $10^x = 567$. This means that $10^{2.7536}$, or $10^{(27,536)/(10,000)}$, or the ten-thousandth root of the super-huge number written with the digit 1 followed by 27,536 zeros, will come out very nearly equal to 567.

Granting, then, that $567 = 10^{2.7536}$, to a degree of accuracy sufficient for our purposes, it follows that $56.7 = \frac{567}{10} = 10^{2.7536}/10 = 10^{1.7536}$, so that log 56.7 = 1.7536. There we see illustrated the extremely striking property of logarithms that enables us to shorten tremendously the labor of constructing working tables, as well as to telescope the space necessary for them. But before we can describe that property we need to break up a logarithm, for the purpose of convenient description, into the integral part to the left of the decimal point, or the *characteristic*, and the fractional or decimal part, called the *mantissa*. With these definitions taken care of, we are ready to note that the mantissas of the logarithms of the two numbers 567 and 56.7 are exactly alike. Furthermore, since $5.67 = 56.7/10 = (10^{1.7536})/10 = 10^{0.7536}$, it follows that log 5.67 = 0.7536, with the mantissa still unchanged. By a similar argument, log .567 turns out to be $0.7536 - 1$, and by the trick of writing this $9.7536 - 10$ instead of -0.2464 we can avoid changing the part following the decimal point even in this case. In general, the mantissa of the logarithm of a number, if we define it as necessarily positive, depends only upon the sequence of digits in the number and not at all upon where the decimal point comes in that sequence. And when to this item is added the helpful observation that the characteristic of the logarithm can be determined by inspection, as explained later, so that it need not be included in the recorded values, we see why a table that needs only the one entry 7536 for the logarithms of such widely different numbers as 0.0000567 and 56,700,000 can be fairly compact. It is an interesting fact, however, that the characteristics were printed in the tables until well into the eighteenth century.

A good working rule for finding the characteristic of the logarithm of a number may be stated after a couple of preliminary definitions are disposed of, as follows:

Definition 1. *The **significant digits** in a number are all of its digits except the zeros which precede the left-most nonzero digit.*

Definition 2. *The **key position** of the decimal point in a number is the position following the first significant digit.*

Rule. *If the decimal point in a number lies **n** digits to the* $\begin{pmatrix} \textit{right} \\ \textit{left} \end{pmatrix}$ *of the key position, the characteristic of the logarithm is* $\begin{pmatrix} n \\ -n \end{pmatrix}$.

Illustrative Examples

(1) log 0056700 = 4.7536
(2) log 005.6700 = 0.7536
(3) log 00.56700 = $-1 + .7536$, or $9.7536 - 10$
(4) log .0056700 = $-3 + .7536$, or $7.7536 - 10$

EXERCISE 25

1. What are the significant digits in the numbers (*a*) 0.021040, (*b*) 000.009, (*c*) 100,000?

2. In Prob. 1, the key position follows what digit?

3. Give the characteristics of the logarithms of the following numbers:

(*a*) 235;
(*b*) 3,670,000;
(*c*) 0.831;
(*d*) 0.000462;
(*e*) 41.00;
(*f*) 008.12;
(*g*) 32.80000;
(*h*) 2.70000;
(*i*) 0.03000;
(*j*) 0.0050.

4. Given $10^{.6542} = 4.51$, how much is (*a*) log 4.51; (*b*) log 451,000; (*c*) log 0.00451; (*d*) log 0.04510; (*e*) log 4,510.000?

5. How much is A if log A equals -3? 0? $+2$? $+7$?

6. (*a*) Why is it that log 300 is 2 more than log 3?
(*b*) Why is log 12.347 less by 3 than log 12,347?

7. In the so-called *scientific notation* a number is expressed as a product of two factors of which one is a power of 10 and the other is a number less than 10 but not less than 1.

Examples: $1,360,000 = 1.36(10)^6$; $0.0000327 = 3.27(10)^{-5}$. Note that the exponent of 10 is the characteristic of the logarithm of the number.

Write the numbers of Prob. 3 in scientific notation.

8. If log N is the number indicated in the following problems, where is the decimal point located in N?

(*a*) 2.1623;	(*b*) 0.1724;
(*c*) 9.1763 − 10;	(*d*) 5.3241;
(*e*) 8.1728 − 10;	(*f*) 3.4217;
(*g*) 6.1124 − 10;	(*h*) 4.2167.

46. Table technique. This seems to be the proper place to explain the use of the condensed table we've been telling about. After this somewhat tedious detail is disposed of we'll be ready to demonstrate the almost magical efficiency of the logarithm in the prosaic but highly necessary art of computation. Turn, therefore, to Table I and get out the pencil which should always be used along with the brain in the reading of mathematical statements. (The impression is widespread that the pencil should rest close to the feet on top of the desk until the official list of problems looms through the smoke-haze. The student thus deluded usually reports that he "can't understand the explanation.")

The first lesson in table technique has to do with finding the mantissa of the logarithm of a given number. Observe that in the column below N at the upper left of the page are found the numbers from 10 to 99, inclusive; while across the top of the page at the head of each column are the integers 0 to 9. Suppose now that we want the logarithm of 3.67. In line horizontally with the figure 36 under N, and in the vertical column headed by 7, we find the digits 5647, which, when preceded by a decimal point, give us the mantissa. Since the characteristic is zero by the rule of the preceding article, we have log 3.67 = 0.5647. To find log 0.031 we look in the row with 31 at the left and in the column under 0 for the mantissa, which is .4914. The characteristic is −2 by our rule, and hence

$\log 0.031 = -2 + .4914$ or $8.4914 - 10$. Similarly, $\log 3 = \log 3.00 = 0.4771$.

Next, we need to know how to find a number whose logarithm is given, or, if we must be technical, how to get the "antilogarithm." To be specific, let's find N if $\log N = 1.6684$. Looking in the "body" of the tables, we note that these four-digit figures, or the mantissa minus their preceding decimal points, increase from the upper left corner to the lower right, so that we can scan them in the way anyone but a Chinese reads and find 6,684 in short order. This is in line with 46 on the left and in the column with 6 above, so that the digits in N are 466. Since the characteristic of $\log N$ is 1, there are two digits preceding the decimal point in N and hence $N = 46.6$. We could give a rule for pointing off the antilogarithm, but the most practical course is to learn the rule of Art. 45 thoroughly and then shift the decimal point back and forth experimentally until the corresponding characteristic is the one required. After a few shifts one gets the knack and can drive the decimal point home on the first try. Now, for practice, try doing this one: $\log N = 8.9212 - 10$. The mantissa 0.9212 shows that N has the digits 834, and the characteristic $8 - 10$ or -2 indicates that the decimal point is to be placed two digits to the left of the key position. Accordingly $N = 0.0834$.

To get from our tables the mantissas of the logarithms of four-place numbers, we use a process called "interpolation." For the number 21.63 the suggested arrangement below will show the essence of the scheme.

$$\left.\begin{aligned} \log 21.70 &= 1.3365 \\ \log 21.63 &= \text{———} \\ \log 21.60 &= 1.3345 \end{aligned}\right\} 20$$

Here the dash represents the quantity to be found. Since 21.63 lies $\frac{3}{10}$ of the way from 21.60 to 21.70, we seek a number which is $\frac{3}{10}$ of the way from 3,345 to 3,365—the digits in the respective mantissas. Since the difference of the last two figures is 20, $(\frac{3}{10})(20) = 6$, and $3{,}345 + 6 = 3{,}351$, the desired logarithm is 1.3351.

Theoretically this method is not quite accurate. However, when the interpolation is between two adjacent table entries, as in our example, the error due to the "rounding off" of the mantissa, or the dropping of all digits after the fourth, is more important than the flaw in the theory, so that the practical result is satisfactory.

Finally, to complete our table technique, we must know how to interpolate for N when log N is not in the tables. Suppose, for example, log N = 1.3472. The mantissa digits 3,464 and 3,483, between which 3,472 lies, are readily found, and the corresponding sequences of digits turn out to be 222 and 223. After we insert the decimal point in the proper place and add a zero to make the desired four-digit antilogs on the left, the preliminary situation looks like this:

$$\left.\begin{array}{l} \log 22.30 = 1.3483 \\ \left.\begin{array}{l}\log \qquad\;\; = 1.3472 \\ \log 22.20 = 1.3464\end{array}\right\}8\end{array}\right\}19$$

Since the indicated differences on the right, disregarding the decimal point, are 8 and 19 respectively, and hence the mantissa 0.3472 is $\frac{8}{19}$ of the way from 0.3464 to 0.3483, we shall assume that N is $\frac{8}{19}$ of the way from 22.20 to 22.30. But $(\frac{8}{19})$ (.10) = 0.80/19 = 0.04(+), so that N = 22.20 + 0.04 = 22.24.

Now don't get the impression that the experienced computer has to go through all this written rigmarole every time he needs to read between the lines in the tables. Actually, after you've practiced awhile, you'll probably find that you yourself can get the correct result mentally without writing a scratch. It's a good idea, however, to use a paper-and-pencil scheme like that above until the interpolation process becomes more or less automatic, so that you can carry it through mentally without much fear of bungling.

EXERCISE 26

1. Using the table, find the logarithms of the numbers in Prob. 3, Exercise 25.

2. Find the antilogarithms of the following logarithms:

(*a*) 0.8414;	(*b*) 8.3766 − 10;
(*c*) 3.9350;	(*d*) 4.5729;
(*e*) 7.4487 − 10;	(*f*) 6.8306;

(*g*) 3.9624 − 10; (*h*) 2.5821;
(*i*) 9.2014 − 10; (*j*) 1.7466.

3. Find by interpolation the logarithms of the following numbers:
(*a*) 23.45; (*b*) 32,280;
(*c*) 0.002981; (*d*) 00.2897;
(*e*) 316.800; (*f*) 2,796,000.00;
(*g*) 000.038620; (*h*) 41.78;
(*i*) 3,295; (*j*) 0.6342.

4. Find by interpolation the antilogs of the following logarithms:
(*a*) 2.3965; (*b*) 0.2178;
(*c*) 6.8241 − 10; (*d*) 3.7496;
(*e*) 9.8258 − 10; (*f*) 3.2168 − 10;
(*g*) 3.3471; (*h*) 1.2345;
(*i*) 7.8163; (*j*) 2.4169.

47. The potent logarithm. Now that we've disposed of the mechanical details of digging out logarithms and antilogarithms from the tables, let's observe these exponents of ten (and don't forget that common logarithms are just that) at their main job of cutting down work and saving time. They do this by virtue of certain so-called "properties" which not only enable the user to replace multiplication by addition and division by subtraction, but also (and here the timesaving is much more marked) permit him to substitute for the tedious and long-drawn-out steps in *involution* (raising to powers) and *evolution* (extraction of roots) the simple processes of multiplication and division. Before we can see the laborsaver in operation, then, we must consider these marvelous and useful properties. As we list them here below they perhaps look like nothing to get excited about; but their usefulness grows upon one as he reflects.

Property I. *log* $\boldsymbol{MN}$ = *log* $\boldsymbol{M}$ + *log* $\boldsymbol{N}$. *More generally, the logarithm of the product of two or more numbers is the sum of their logarithms.*

Proof: Let $M = 10^x$ and $N = 10^y$. Then $x = \log M$, and $y = \log N$, so that we may write

$$\begin{aligned} \log (MN) &= \log (10^x 10^y) \\ &= \log (10^{x+y}) && \text{[by (1) of Art. 12]} \\ &= x + y && \text{(by the definition of logarithms)} \\ &= \log M + \log N \end{aligned}$$

Similarly, using (3) of Art. 13,

$$\log \frac{M}{N} = \log \left(\frac{10^x}{10^y}\right)$$
$$= \log 10^{x-y}$$
$$= x - y$$
$$= \log M - \log N$$

and we have

Property II. $\log \frac{M}{N} = \log M - \log N$, or, *the logarithm of a quotient is the logarithm of the dividend minus the logarithm of the divisor.*

Finally, since

$$\log M^p = \log (10^x)^p$$
$$= \log 10^{px} \qquad \text{[by (3), Art. 12]}$$
$$= px$$
$$= p \log M$$

we can add

Property III. $\log M^p = p \log M$, or, *the logarithm of a power of a given number is the product of the exponent of the power by the logarithm of the number.*

This third property takes care of the logarithm of an indicated root, since $\sqrt[r]{M}$ may be written $M^{1/r}$.

Coming now to the application of these surprisingly useful properties, let's find first the value of that quantity $x = 100 (1.025)^{42}$ (dollars) which our hypothetical boy is growing up to inherit. By Property I, $\log x = \log 100 + \log (1.025)^{42}$, so that, changing the latter term by Property III, we have

$$\log x = \log 100 + 42 \log 1.025$$

We'll now make an outline of the work in column form, thus:

$$42(\log 1.025) = 42(0. \qquad) =$$
$$\log 100 = \underline{2. \qquad\qquad}$$
$$\log x =$$
$$x = \underline{\qquad\qquad}$$

We just can't recommend this preliminary outline too highly. It illustrates several points which must be heeded before one can get the full timesaving benefit of this invention. Notice in particular that

1. the column form with the precise vertical array of decimal points facilitates the necessary addition and subtraction;

2. the characteristics are filled in so far as possible while the outline is being made, since otherwise they might be forgotten; and finally

3. every figure in the column *is labeled or described at the left of the equality sign*, so that the work will be easy to follow and check even after it grows "cold."

Turning to the tables, we fill in the values, interpolating where necessary.

$$
\begin{aligned}
42(\log 1.025) = 42(0.0107) &= 0.4494 \\
\log 100 &= \underline{2.0000} \quad \text{(We knew this mantissa without looking it up.)} \\
\log x &= 2.4494 \\
x &= 281.50
\end{aligned}
$$

Evidently the limited number of digits in the mantissas of our table makes the figures representing cents in our answer only approximately correct. We could get the result accurate to the cent by means of logarithms, but this would require more extensive tables than it is practicable for us to supply.

Consider next the value of Q in the equation

$$Q = \left[\frac{(4.21)(2.631)(1.32)}{46.32}\right]^2$$

To get log Q we must use all three properties of logarithms, thus:

$$
\begin{aligned}
\log 4.21 &= 0. \\
\log 2.631 &= 0. \\
\log 1.32 &= \underline{0.\qquad} \quad \text{Add these. (Property I)} \\
\log \text{numerator} &= \\
\log 46.32 &= \underline{1.\qquad} \quad \text{Subtract. (Property II)} \\
\log \sqrt{Q} &= \qquad\qquad \text{Multiply by 2 (Property III)} \\
& \quad \underline{\qquad 2} \\
\log Q &= \\
Q &=
\end{aligned}
$$

Please remember that in actual practice the mantissas should always be inserted in the *original outline.* There should be two separate stages in the work, but *only one* finished result. Trusting that this point is sufficiently clear, we now indicate the mantissas which should go into the *first* outline.

$$\begin{aligned}
\log 4.21 &= \quad 0.6243 \\
\log 2.631 &= \quad 0.4202 \\
\log 1.32 &= \quad \underline{0.1206} \\
\log \text{numerator} &= \quad 1.1651 \\
\log 46.32 &= \quad \underline{1.6658} \\
\log \sqrt{Q} &= -0.5007 \qquad \text{(Here, here—this won't do!)}
\end{aligned}$$

The trouble just encountered is that negative mantissas aren't in the tables. Of course we can rewrite the last logarithm in the form $10 - 10 - 0.5007 = (10 - 0.5007) - 10 = 9.4993 - 10$; but all this bother could have been avoided if we had changed 1.1651 to $11.1651 - 10$ in the first place. Doing this, then, and carrying on from there, we have

$$\begin{aligned}
\log \text{numerator} &= 11.1651 - 10 \\
\log 46.32 &= \underline{\ \ 1.6658 \qquad\ } \\
\log \sqrt{Q} &= \ \ 9.4993 - 10. \qquad \text{(Multiply by 2)} \\
& \qquad\qquad\quad \underline{\ \ 2} \\
\log Q &= 18.9986 - 20 \\
Q &= \ \ 0.09968
\end{aligned}$$

Notice that the characteristic of $\log Q$ is $18 - 20 = -2$, so that Q is a decimal fraction with one zero to the right of the decimal point.

Another neat device for abolishing inconvenient forms of the logarithm is illustrated in the process of finding $R = \sqrt[3]{.0621}$. Of course $\log R$ is $\frac{1}{3}$ of $\log 0.0621$ (by Property III, with $p = \frac{1}{3}$). The division yields $\log R = (\frac{1}{3})(8.7931 - 10) = 2.9310 - 3.3333$. Again the troublesome form could be converted into a normal logarithm, but again it could have been prevented from coming on the scene by a little artful dodging at the strategic point, as illustrated herewith:

$$\begin{aligned}
\log 0.0621 &= \ \ 8.7931 - 10 \\
&= 28.7931 - 30 \qquad (\textit{Now} \text{ we can divide by 3})
\end{aligned}$$

Then

$$\log R = 9.5977 - 10$$

Hence,

$$R = 0.3960$$

Naturally, to get $\sqrt[4]{.0621}$ we would write log 0.0621 in the form 38.7931 − 40, etc.

48. Caution. The critical reader may be somewhat disturbed by the fact that the above results contain no more than four nonzero digits, while the actual computations by the methods of arithmetic would yield much longer numbers. While it is true that our tables yield answers which are dependable in general to only four places, this limitation in most practical problems is not so serious as it may at first appear. For example, if one were to compute the volume in cubic feet of a room whose dimensions are 15.2 by 11.1 by 8.3 ft. by actual measurement, he would get 1,400.376 by straight multiplication. Assuming, however, (and this is usually the case) that each measured figure is not exactly correct in the last place, the pretended accuracy of the arithmetical figure is seen to be thoroughly misleading. To demonstrate this, suppose that the more accurately determined height of the room is 8.29 ft. The corresponding altered product is 1,398.6888, and the two contrasting figures show up in its true light the worthlessness of those right-end digits. The practical man would say that the room contained 1,400 cu. ft., retaining the same number of nonzero digits—two—as found in the number with fewest digits used in the computation. The multiplication by logarithms, incidentally, gives exactly 1,400.

In financial problems, of course, the results should be accurate to the cent. For amounts of any consequence this usually requires logarithm tables with seven or more digits in the mantissas. Such tables are available in libraries, offices of accountants, and elsewhere, and they are used in principle exactly like the shorter ones we have explained. In general we may obtain results from a logarithm table which yield the same number of dependable significant digits as the number of digits in a sample mantissa of the table.

A second point worth mentioning is that in finding log 26.1762 in our tables, we replace the longer figure by 26.18. This practice not only shortens the work of interpolation, but it yields results which are just as satisfactory and dependable as the more laboriously arrived at figures, since the latter have to be rounded off to four places anyway. In finding log 26.17421, the replaced figure would of course be 26.17. A problem of procedure comes up when we try to curtail 26.17500, for example, to four significant figures. Many writers (and we'll join this group) make the arbitrary ruling that the rounded-off number shall be the even one nearest to the two theoretically tied candidates. The latter are 26.17 and 26.18 in our example, and the honor falls upon 26.18 according to our rule. Curtailing 26.165, on the other hand, would lead by agreement to the smaller number 26.16.

These, then, are the rules of the game in using our four-place table: Round off to *four significant figures* both antilogarithms and numbers whose logarithms are found. Drop all but four digits in mantissas interpolated between table values, and use the above even-number rule in case of ties.

EXERCISE 27

Using logarithms, make the computations indicated in Probs. 1 to 22.

1. $(267)(35.1)(.086)$.

2. $\dfrac{(0.035)(614)}{(21.7)(14.9)}$.

3. $(2.13)^{10}$.

4. $\sqrt[5]{8.14}$.

5. $\dfrac{14.6}{38(9.62)}$.

6. $(\sqrt{285})(\sqrt[3]{16.4})$.

7. $\dfrac{(.647)^4}{\sqrt{4.64}}$.

8. $\sqrt[3]{0.000324}$.

9. $(12.35)^4$.

10. $\sqrt[5]{0.00026783}$.

11. $\sqrt[4]{83.796}$.

12. $(0.02485)^5$.

13. $\sqrt[10]{326,999}$.

14. $\sqrt{25.2455}$.

15. $(0.3165)^4$.

16. $(2.2222)^{10}$.

17. $\dfrac{(3.624)(5.761)}{(4.259)^3}$.

18. $\dfrac{(4.3786218)^{1/4}}{(0.016892)(0.0023)}$.

19. $\sqrt[5]{\dfrac{0.216847}{(1.369)(2.47)}}$.

20. $\dfrac{(0.4189)^2(87.52)^3}{\sqrt{0.0679}}$.

21. $\left[\dfrac{(0.6934012)(0.2825)}{4.8735}\right]^4$.

22. $\sqrt[5]{\dfrac{3(762.18)}{4(3.1416)}}$.

23. Work Prob. 1 by the arithmetic method and compare with the previous results. How many significant figures are dependable in the answer if the given figures represent measured quantities?

24. Why is there no real logarithm of a negative number?

25. Find x if

$$x = \frac{(-3.417)(2.162)(-0.621)}{(9.2176)(-4.321)}$$

HINT: By inspection, the quotient is negative, having three negative signs involved. Find the corresponding positive number and prefix the negative sign.

26. Compute $\frac{(2.387)^2 - (3.916)^2}{(3.816)^2 - (0.0024)^2}$ by the most convenient method.

SUGGESTION: Factor algebraically before using logarithms.

27. Using logarithms, express $(2.416)^{1000}$ in the form $A(10^n)$ where A is carried to four significant figures and n is an integer. Under what condition would this result be as dependable as that obtained by carrying through the 999 indicated arithmetic multiplications and retaining all figures?

49. A new conquest. With our logarithmic equipment as a tool, we are now prepared to solve problems which would have given us much trouble before.

An *exponential equation* is one in which the unknown is perched up inaccessibly on one or more constants, as in the example

$$2^{x-1} = 3^{2x}$$

Now we can proceed blithely to the attack and cut down the figurative trees by the simple process of taking the logarithms of both sides, thus

$$(x - 1) \log 2 = 2x \log 3$$

or

$$(x - 1)0.3010 = 2x(0.4771)$$

There lies exposed the pitiful relic of the once-proud poser—now a mere first-degree equation scarcely worth our notice. Continuing the process, we get

$$\begin{aligned} 0.3010x - 0.3010 &= 0.9542x \\ -0.6532x &= 0.3010 \\ x &= -\frac{0.3010}{0.6532} = -.4608 \end{aligned}$$

Since we used four-place logarithms we are not justified in carrying the division in the last line to more than four places; but we could evidently get a more accurate value of x by using tables with more places in the mantissas.

Though the power of logarithms is obvious in the case cited, they are not, it must be admitted, equal to the task of solving all exponential equations. For example, if we try to do likewise with the equation

$$2^x + 3^x = 4$$

we get

$$\log (2^x + 3^x) = \log 4$$

and there we pause strategically to look over the situation. The trouble is that we have no way of simplifying the expression for the logarithm of a sum. So again we'll leave some unfinished business for our versatile Chap. IX, in which we'll show how to solve any equation in one variable whenever one of these shorter methods fails. There is no need for this last resort, however, in the case of exponential equations which contain only products, quotients, powers, and roots rather than sums and differences, for the potent logarithm is then perfectly equal to the task of bringing its x to the ground.

EXERCISE 28

Solve the equations in Probs. 1 to 15 by means of logarithms.

1. $2^x = 128$.

2. $16^x = 4{,}096$.

3. $81^x = 27$.

4. $(625)^x = 5$.

5. $16(27)^x = 9(64)^x$.

6. $16^{4/x} = 2^x$.

7. $\sqrt[x]{256} = (4)^x$.

8. $8^{1/x} = 3\sqrt{(512)^{1/x}}$.

9. $(2)(3^x) = (2^x)(3^4)$.

10. $6^x - \dfrac{2}{3^x} = 0$.

11. $416 = 10^{x^3}$.

12. $2^x = (3)(4^x)(5^x)$.

13. $5^{x-2} - 3^{4x} = 0$.

14. $5^{x^2-2x} = 4$.

15. $2^7 - 3^{x^2} = 0$.

16. Observe that the exact solution of each of Probs. 1 to 7 is a rational number. Find the numbers each of which will be nearly or exactly equal to the value found by use of logarithms.

50. Napier's supposed contribution. It is just too bad that common logarithms, which work so well in numerical computations, are not so well suited for the more advanced branches of mathematics such as calculus. It turns out, for reasons we'll not try to explain here, that the "natural" or "Napierian" logarithms are there much more convenient. Instead of being exponents of ten, like common logarithms, they are exponents of an irrational constant equal to 2.71828(+)—a surprising kind of rival for the simple and elegant 10, to be sure, but one which is nevertheless of considerable importance throughout the mathematical world. This constant is generally designated by the letter e. Along with its brother curiosity π, it plays a very important role in physics and mechanics, and they both are connected with 1, 0, and i (the remaining aristocrats of the constant world) through the impressively simple relation

$$e^{\pi i} + 1 = 0$$

Incidentally, e, like π, has been proved to be transcendental—which means, you may remember, that it cannot be the root of a rational integral equation.

It is an interesting sidelight on the quirks of history that Napier, who is credited with the invention of logarithms and who gave the name to this Napierian or natural base e, did not think of logarithms as exponents at all. Natural logarithms first appeared in 1618, in the appendix of Edward Wright's[1] translation of Napier's *Descriptio*. In 1620 John Speidell[2] used e for the first time as a base when he published his *New Logarithms*.

51. Backstage. The complete story of the computation and construction of a table of common logarithms belongs in more advanced textbooks than this. We can, however, explain a few things which may make the logarithm table a little less mysterious.

In the first place, it will be convenient to make a distinction between prime and composite numbers. A *prime number* n is a number greater than unity which has no divisors except ± 1 or $\pm n$. Examples are 2, 3, 5, 7, 13, 17, 19, 23, etc. The number 1 is customarily excluded from the list and put in a class by itself.

[1] D. E. Smith, *History of Mathematics*, Vol. II, p. 516.
[2] *Ibid.*, p. 517.

A *composite number* is one which has two or more prime factors. Now it is evident that the properties of logarithms allow us to express the logarithm of a composite number in terms of the logarithm of its prime factors. Thus

$$\log 6 = \log[(2)(3)] = \log 2 + \log 3$$
$$\log 24 = \log[(2^3)(3)] = 3 \log 2 + \log 3$$

There remain only the logarithms of primes to puzzle one. To complete the removal of the veil of mystery, suppose we find log 323, not by the efficient method actually used in the construction of tables, but by a cumbersome device which *could* be used and which has the merit of being easily understood. It depends upon the extraction of square roots, which one can certainly do for himself, in case he has forgotten the tedious high-school method, merely by squaring various trial approximations. Consider, then, the following:[1]

$$\sqrt{10} = 3.1623, \text{ hence } \log 3.1623 = \tfrac{1}{2} = 0.5000$$
$$\sqrt[4]{10} = \sqrt{3.1623} = 1.7783, \text{ hence } \log 1.7783 = \tfrac{1}{4} = 0.2500$$

and finally, continuing by this method,

$$\sqrt[64]{10} = 1.0366, \text{ hence } \log 1.0366 = \tfrac{1}{64} = 0.0156$$
$$\sqrt[128]{10} = 1.0181, \text{ hence } \log 1.0181 = \tfrac{1}{128} = 0.0078$$

Now $323 = (100)(3.23) = (100)(3.1623)(1.0214)$ (nearly).

Hence, $\log 323 = \log 100 + \log 3.1623 + \log 1.0214$

But $\log 1.0214 = 0.0092$, by interpolation between log 1.0181 and log 1.0366.

Therefore, $\log 323 = 2 + 0.500 + 0.0092 = 2.5092$ (nearly).

It should be understood here that we are merely removing a mystery and not exhibiting the table-construction technique in actual use. The latter depends upon the use of *series*—a mathematical tool of great value which we hope to bring into the picture in pleasant installments, beginning here below.

In calculus it is proved that

$$\log(1 + x) = .4343\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots\right)$$

[1] Since the method here illustrated is impractical, the uninterested student should not be required to use it or explain it.

The *infinite series* on the right never terminates, but when x is small the terms decrease very rapidly as we consider them from left to right, and hence we need use only a limited number of them to get $\log(1 + x)$ to a given number of decimal places. To get log 1.12, for instance, we take $x = .12$ and find that

$$\begin{aligned}\log 1.12 &= 0.4343\left(0.12 - \frac{0.12^2}{2} + \frac{0.12^3}{3} - \cdots\right)\\ &= 0.4343(0.1200 - 0.0072 + 0.0006 - 0.0001\\ &\qquad + 0.0000 - \cdots)\\ &= 0.4343(0.1133) = 0.0492 \text{ (nearly) to four places}\\ &= 0.049 \text{ to three places}\end{aligned}$$

This, we see, is much shorter than the other method, and in actual practice still other series are used which are even more effective in their places. It should be noticed that we almost always find the mantissas to a certain number of decimal places, and not exactly. This is a necessary consequence of the fact that the logarithms of most integers are irrational, and therefore cannot be expressed exactly in decimal form. It gives a rough idea, moreover, of how very frequently the number which bobs up before the mathematician turns out to be one of those strange irrationals.

EXERCISE 29

Given log 2 = 0.3010, log 3 = 0.4771, and log 10 = 1, make use of these values to find the quantities indicated in Probs. 1 to 8.

1. log 6.
2. log 1.5.
3. log $\frac{2}{3}$.
4. log 12.
5. log 72.
6. log 0.75.
7. log 750.
8. log 5.

Using the series in Art. 51, find the three decimal places the numbers indicated in Probs. 9 to 12.

9. log 1.1.
10. log 1.2.
11. log 1.15.
12. log 0.995.

CHAPTER VI

A TOY BECOMES A TOOL

52. Meet the "series" family. Just a few pages ago we demonstrated the magical efficiency of a string of x's which we brought in from the tool house for the purpose of calculating a logarithm. We called this thing a *series* and promised to introduce more or less painlessly the large mathematical tribe from which it hailed. Comes the time.

A chain of numbers or symbols obtained in succession according to some law is called a *sequence*. Thus 2, 4, 6, . . . , $2n$ is a sequence, since each term is two more than the preceding one. The indicated sum of a sequence, such as $2 + 4 + 6 + \ldots + 2n$, is a *series*. The idea is so simple that one is not likely to anticipate the scope or variety of the services performed by the versatile mathematical family which grew out of it.

Whose idea was it? We can't say, exactly, any more than we can name the fellow who cooked the first meal and thus broke ground for the long trail leading to the electric range. It goes back at least to the period in history when people began playing with the so-called *figurate* numbers, or numbers which can be represented by geometric arrangements of dots. This was the time when magical properties were assigned to various integers, particularly those whose dot representations made symmetrical and interesting patterns. While the study of these numbers did not lead at first to particularly important discoveries, it fostered an interest in the matter of their relation to each other. This, it turned out, was the fruitful angle of approach, as opposed to the study of the numbers themselves, for this was the basic idea which flowered eventually into the highly practical and useful series of today.

Three classes of figurate numbers which interested the ancients were the following:

1. Triangular numbers:

1 3 6 10

2. Square numbers:

1 4 9 16

3. Pentagonal numbers:

1 5 12 22

It is evidently a simple matter, in the case of any one of these sequences, to construct the next succeeding arrangement of dots and thus determine the next number in the sequence. It is even a comparatively easy matter to make any pattern in the row, however far removed to the right. The thousandth figure in the pentagonal sequence, for instance, will evidently be a square with one thousand dots on a side, surmounted by a triangle in which the number of dots decreases by one in successive rows from bottom to top. We could make the figure and then get the thousandth pentagonal number by counting the dots. This method, however, seems a trifle childish. Anyone with a grain of intellectual curiosity would naturally feel challenged before long, if he dabbled with this thing, to figure out in advance how many dots there would be in any particular preassigned pattern. The fourth one, for instance, can be rearranged like this:

Evidently the dots in the triangle surmounting the square could be supplemented by those in another triangle like this:

to form a rectangle three dots by four containing $(3)(4) = 12$ dots. The square below has $(4)(4) = 16$ dots. Evidently, then, the fourth pentagonal number is $(4)(4) + \frac{1}{2}(4)(3) = 16 + 6 = 22$. Similarly, the thousandth one must be $(1{,}000)(1{,}000) + \frac{1}{2}(1{,}000)(999) = 1{,}000(1{,}000 + \frac{999}{2}) = (\frac{1000}{2})(2{,}999) = 1{,}499{,}500$.

Now we are ready for the final step, which leads us to a general algebraic result derived from a geometric picture. What is the nth pentagonal number, when n is *any integer whatever?* (Naturally the mathematician *would* have to know that.) But now it is easy. Of course it is $(n)(n) + \frac{1}{2}n(n - 1)$, which we can simplify to $(n/2)(3n - 1)$. This gives us the nth term of our sequence and we can write our result elegantly like this, discarding the now useless dot patterns:

$$1, 5, 12, 22, \cdots, \left(\frac{n}{2}\right)(3n - 1)$$

where the last term shown stands for the nth number in the sequence. We can calculate the seventeenth, or the nine hundred and eighty-first, or the billionth term, offhand, thus showing that our algebra has already far exceeded in power our geometric stepping stones.

Probably, moreover, the reader could construct for himself numerous and increasingly complicated geometric patterns which might lead to algebraic results of interest. In order not to lead him too far afield, however, we'll point out in the following articles some members of the series family which have proved to be particularly simple, manageable, and helpful in everyday life.

EXERCISE 30

1. Find in terms of n the expression for the nth term of the series of triangular numbers; of square numbers.

2. Find the nth term of the series of figurate numbers in which a sample pattern consists of a square with four triangles on its respective sides similar to the one triangle in the pentagonal pattern.

53. A simple member. In the early part of the seventeenth century, Galileo, the brilliant Italian pioneer in the realm of experimental physics, created a storm of controversy by announcing that light and heavy bodies fall at rates which would be identical if it were not for air resistance. Since the latter is unimportant for fairly large stones moving at relatively slow speeds, he proposed to show that if two different-sized stones were dropped at the same time from the top of the now famous Leaning Tower of Pisa, they would reach the ground simultaneously. Before a large crowd containing many scoffers, the miracle was performed according to prediction, although in the light of the contrary assertion by the long-dead Greek philosopher Aristotle, whose word had hitherto been final and conclusive, these blasphemous stones revealed a rank heresy in nature herself. In such spectacular and dramatic fashion the news broke upon a doubting world that falling bodies near the earth travel in successive seconds distances which increase by a constant and dependable increment. That increment is 32.2 feet—the acceleration of gravity at the earth's surface—and the distances traversed in successive seconds are 16.1, 48.3, 80.5, 112.7, etc., feet.

Here we have one of the earliest of the sequences recognized in nature itself, as opposed to the man-made figurate-number playthings. It is important enough to merit a

Definition: *An* ***arithmetic progression*** *is a sequence in which each term after the first may be obtained from the preceding one by adding to it a fixed quantity.* This quantity, which may be either positive or negative, is called the *common difference.*

To illustrate the type of problem which involves such a progression, consider what would be the height of a tower from whose top a dropped stone would continue to fall for 10 sec. Of course we *could* figure the distances traveled in each of the 10 sec. and add those results, but somehow the mathematician in us rebels. Certainly if we were to use this strong-arm method on a similar problem with *one hundred* substituted for ten, our glow of achievement might at least be tempered by annoying misgivings about efficiency. In any case, we want to work out formulas here applying to many problems and not just one, as we always do when faced with a mathematical situation involving factors likely to recur over and over again.

Remembering, then, our formula-deriving technique as applied, for instance, in connection with quadratic equations, we need to designate by letters the various recurring elements in an arithmetic progression. Suppose we let a represent the first term, l the nth term, and d the common difference. Then

$$\begin{aligned} \text{first term} &= a \\ \text{second term} &= a + d \\ \text{third term} &= a + d + d = a + 2d \\ \text{fourth term} &= a + 3d \end{aligned}$$

and so on. We now notice that in each case the coefficient of d is one less than the number of the term in the series. Hence the nth term is $a + (n - 1)d$, and we have

$$\mathbf{(1)} \qquad \boldsymbol{l = a + (n - 1)d}$$

Next we need a formula for s, where

$$(2) \quad s = a + (a + d) + (a + 2d) + \cdots + [a + (n - 1)d]$$

Of course the last equation as it stands gives such a formula, but it is not in satisfactory form. We can get a much more convenient result by reversing the order of the terms in (2), getting [after noting that the last term in (2) equals l and that succeeding terms now decrease by d]:

$$(3) \quad s = l + (l - d) + (l - 2d) + \cdots + l - (n - 1)d$$

If now we add the terms in (2) and (3) with attention to the fact that the d's in corresponding terms drop out, we have

$$2s = (a + l) + (a + l) + (a + l) + \cdots + (a + l) = n(a + l)$$

since there are n terms in each of (2) and (3). Hence

$$s = \left(\frac{n}{2}\right)(a + l) \tag{4}$$

Note that the five elements involved in an arithmetic progression are represented by the letters in the word "lands." Evidently if any three of them are known, formulas (1) and (4) will then involve just two unknowns and hence will be sufficient for the solution. Since a progression is not uniquely specified unless at least three elements are given, these two formulas are all that we need.

Going back to our tower problem, we have $a = 16.1$, $d = 32.2$, and $n = 10$. Substituting these known values in (1) and (4) we have

$$l = 16.1 + 9(32.2) \tag{5}$$

$$s = (\tfrac{10}{2})(16.1 + l) \tag{6}$$

From (5) we find that $l = 305.9$, which value substituted in (6) gives $s = 1{,}610$. Hence our hypothetical tower must be much higher than the Empire State Building.

Frequently, as in the example just cited, only one unknown appears initially in one or the other of equations (1) and (4). Sometimes, however, the two unknowns appear in both equations, forcing the solver to brush up on his simultaneous-equations technique. Suppose, for instance, we are given $s = 56$, $d = 3$, $l = 17$. Then (1) and (4) yield

$$17 = a + (n - 1)3 \tag{7}$$

$$56 = \left(\frac{n}{2}\right)(a + 17) \tag{8}$$

From (7) $a = 17 - (n - 1)3 = 20 - 3n$. Substituting this value in (8) we have

$$56 = \left(\frac{n}{2}\right)(20 - 3n + 17) = \left(\frac{n}{2}\right)(37 - 3n)$$

which reduces, after simplification, to

$$3n^2 - 37n + 112 = 0$$

Applying our quadratic-equation formula, we have

$$n = \frac{37 \pm \sqrt{1{,}369 - 1{,}344}}{6} = \frac{37 \pm 5}{6} = 7 \text{ or } \frac{16}{3}$$

Remembering that n must be an integer, we can immediately discard the latter root, leaving as the only possibility $n = 7$, and hence, from (7), $a = -1$.

When an arithmetic progression contains three terms, the middle one is called the *arithmetic mean* of the other two. It occurs so frequently that it deserves a formula, and hence we'll get b in terms of a and c in the arithmetic progression a, b, c. Evidently, from the definition

$$b - a = c - b$$

Hence

$$2b = a + c$$

and

$$\text{(9)} \qquad \mathbf{b} = \frac{a + c}{2}$$

so the arithmetic mean of two quantities is evidently nothing more than their average value dressed up in somewhat more elegant and impressive language.

EXERCISE 31

In Probs. 1 to 8 add two more terms to each and find the 20th term.

1. 1, 2, 3.

2. 3, 7, 11.

3. 14, 6, -2.

4. 1, $\frac{3}{2}$, 2.

5. 2, $\sqrt{2}$, $2\sqrt{2} - 2$.

6. m, n, $2n - m$.

7. 6, 5.14, 4.28.

8. $x + y$, $2x$, $3x - y$.

In each of Probs. 9 to 18 the values of three of the five letters l, a, n, d, s, are given. Find the other two.

9. $n = 5$, $d = -2$, $s = -10$.

10. $a = \frac{1}{2}$, $d = \frac{1}{2}$, $s = 18$.

11. $a = 3$, $n = 10$, $s = 120$.

12. $a = \frac{3}{4}$, $d = -\frac{1}{4}$, $n = 8$.

13. $l = 14$, $d = 2$, $s = 50$.

14. $l = -43$, $n = 10$, $s = -205$.

15. $l = -33$, $n = 9$, $d = -10$.

16. $l = 26$, $a = \frac{3}{2}$, $s = 110$.

17. $l = \frac{23}{2}$, $a = \frac{3}{2}$, $d = \frac{5}{3}$.

18. $l = 204$, $a = 4$, $n = 11$.

19. Find the arithmetic mean of

(*a*) $7, -4$; (*b*) $x + y, x - y$;
(*c*) $\sqrt{11}, \sqrt{5}$; (*d*) $\sqrt{12}, -2\sqrt{3}$.

20. Insert four arithmetic means between

(*a*) $8, -3$; (*b*) r, t;
(*c*) $10, \frac{1}{2}$; (*d*) $2, \sqrt{2}$.

21. Work out formulas for s involving all of the letters l, a, n, d, s, except

(*a*) l; (*b*) a; (*c*) n.

22. Find the sum of all positive multiples of 3 between 1,000 and 5,000.

23. In a charity lottery, tickets were numbered in multiples of 5 from 100 to 1,000, and were sold for a price which equaled in cents the numbers on the tickets. The prizes amounted to $100. How much of the proceeds went to charity?

24. In a pile of logs there is one less log in any row not on the ground than in the row below. There are 10 and 20 logs respectively in the top and bottom rows. How many logs are there in the pile?

25. Neglecting air resistance, how far does a body fall from rest in 10 sec.?

26. If a body, starting from rest, falls 660.1 ft. in a given second, how long has it been falling at the end of that second?

27. If the terms of an arithmetic progression are multiplied by a constant, show that the products form an arithmetic progression.

28. Show that if the corresponding terms of two arithmetic progressions are added, the sums form an arithmetic progression.

29. Prove that the sum of the first n odd positive integers is a perfect square.

54. A useful one. In accord with a convenient custom, the geometric progression now trails along on schedule right on the heels of its arithmetic predecessor. We'll find that the new progression parallels the arithmetic one so closely in many of its features that the two are learned most easily in conjunction. Also both progressions have immediate and direct applications that we shall use in our articles dealing with workaday business.

A ***geometric progression*** *is a sequence of terms in which each one after the first may be derived from the preceding one by multiplying it by a fixed quantity called the* **common ratio.**

Examples are 2, 6, 18, 54, and 5, $-\frac{10}{3}$, $\frac{20}{9}$, $-\frac{40}{27}$, in which the common ratios are 3 and $-\frac{2}{3}$ respectively.

Designating the first and nth terms by a and l respectively, the sum by s, and the number of terms by n, as in the arithmetic progression, we have only to replace d by r, the common ratio, and the key word "lands" by the key word "snarl," to exhibit our new equipment. Then the

$$\begin{aligned} \text{first term} &= a \\ \text{second term} &= ar \\ \text{third term} &= ar^2 \end{aligned}$$

and in general, the

$$n\text{th term} = ar^{n-1}$$

giving us the formula

$$l = ar^{n-1} \tag{1}$$

Adding the first n terms of the series, we have

$$s = a + ar + ar^2 + ar^3 + \cdots + ar^{n-1} \tag{2}$$

If we now multiply each member of (2) by r, we get

$$sr = ar + ar^2 + ar^3 + \cdots + ar^{n-1} + ar^n \tag{3}$$

Notice here that the next-to-last term in (3) is obtained from the next-to-last term in (2), namely, ar^{n-2}, which it is convenient to omit. Notice also that the right members of (2) and (3) are alike except for the first term in (2) and the last one in (3). Hence, if we subtract corresponding sides of the equations, most of the terms drop out and we have

$$s - sr = a - ar^n \tag{4}$$

or

$$s(1 - r) = a - ar^n$$

yielding the important formula

$$s = \frac{a - ar^n}{1 - r} \tag{5}$$

Again we have two equations, (1) and (5) in this case, dealing with five quantities, so that if any three of them are given the other two may be found. Again, also, one of these equations sometimes contains only one unknown when the three given quantities are inserted, and sometimes the two equations must be solved simultaneously. The essential added element of difficulty is the fact that the degree of the equation to be solved may now be n, whereas in the arithmetic progression it is never higher than two. The fact that n *must be* an integer is a helpful thought which it is here particularly important to keep in mind. For instance, if $a = \frac{32}{81}$, $r = \frac{3}{2}$ and $l = \frac{9}{2}$, we have by (1)

$$\frac{9}{2} = \left(\frac{32}{81}\right)\left(\frac{3}{2}\right)^{n-1}$$

or

$$\left(\frac{3}{2}\right)^{n-1} = \frac{729}{64}$$

Knowing that there must be an integral n which works here if the given conditions are possible, we can scorn the use of logarithms and simply try out successively higher powers of $\frac{3}{2}$ until we find that

$$\left(\frac{3}{2}\right)^{6} = \frac{729}{64}$$

so that

$$n - 1 = 6 \text{ and } n = 7$$

Again, if $a = 10$, $l = \frac{5}{512}$, $s = \frac{3415}{512}$, formulas (1) and (5) yield, respectively,

(6) $$\frac{5}{512} = 10r^{n-1}$$

and

(7) $$\frac{3{,}415}{512} = \frac{10 - 10r^{n}}{1 - r}$$

To eliminate n, we multiply (6) by r, getting

$$\frac{5r}{512} = 10r^{n}$$

Substituting this value of $10r^{n}$ in (7), we have

$$\frac{3{,}415}{512} = \frac{10 - \dfrac{5r}{512}}{1 - r} = \frac{5{,}120 - 5r}{512(1 - r)}$$

Hence,

$$3{,}415(1 - r) = 5{,}120 - 5r$$

and, after solving, we have

$$r = -\tfrac{1}{2}$$

(Fill in the details for yourself.)

Now suppose, by way of application, that mamma and papa rabbit have four little rabbits each year, and that these four are responsible in the very next season for eight new entries into rabbit society. What can be said of the hundredth generation in this very moderate-appearing program from the rabbit standpoint? Let's see.

We have to use (1) with $a = 2$, $r = 2$, $n = 100$. This gives

$$l = 2(2)^{99} = 2^{100}$$

Calling in our trusty logarithms, we find that

$$100 \log 2 = 100(0.3010) = 30.10$$

or roughly, 30. This means that l is about equal to the antilog of 30, or 1 followed by 30 ciphers, so that the innocent-looking program produces more than a trillion rabbits for every square inch of land or sea on the surface of the earth! And perhaps the moral of this pleasant little calculation is that we shouldn't blame nature too much for her cruel and exterminating propensities, geometric progressions being what they are.

Another pertinent moral lesson to be learned from this phase of mathematics is that one should not dwell too long or emphatically in the presence of others upon some more or less reliable genealogical information he may have which proves him to be a descendant of, say, William the Conqueror. For, considering that the number of our ancestors increases in geometric ratio as we go back through the generations, with a certain loss in numbers, of course, due to the intermarrying in backward-spreading lines, each of us has perhaps a million ancestors of a generation as comparatively recent as that of Columbus. In our own case we are confident that a thorough investigation of our family tree would carry us sooner or later through ducal halls and royal premises, but we fear that some of the sideswipings of our research would lead us now and then into the haunts of the socially unacceptable, to say nothing of embarrassing ancestral residences such as pirate ships. For our part, we are willing to let the matter rest, secure in the belief that too much family pride may invariably be traced to faulty training in the implications of the geometric progression.

EXERCISE 32

In Probs. 1 to 8, add two more terms to each sequence and find the twentieth term.

1. 3, 9, 27.

2. 8, 4, 2.

3. 2, 4, 8.

4. $\frac{1}{3}$, $-\frac{1}{9}$, $\frac{1}{27}$.

5. 2, $2\sqrt{2}$, 4.

6. a, $-a$, a.

7. 1, 0.1, 0.01.

8. a, a^2b, a^3b^2.

State whether the terms in each of Probs. 9 to 16 form an arithmetic progression, a geometric progression, or neither:

9. 2, 6, 10.

10. 1, 4, 9.

11. 3, $\sqrt{3}$, 1.

12. 5, $\sqrt{5}$, $2\sqrt{5} - 5$.

13. 2, 6, 18.

14. 16, 9, 4.

15. a, b, b^2/a.

16. x, x, x.

In each of Probs. 17 to 26, the values of three of the five letters s, n, a, r, l are given. Find the other two.

17. $a = 3$, $r = -\frac{1}{3}$, $l = -\frac{1}{81}$.

18. $n = 4$, $r = 0.01$, $l = 0.00000012$.

19. $n = 7$, $a = \frac{1}{16}$, $r = 4$.

20. $s = \frac{63}{144}$, $r = -\frac{1}{2}$, $l = -\frac{1}{48}$.

21. $n = 7$, $a = 3$, $l = \frac{1}{243}$. (Two cases)

22. $a = 3$, $r = \frac{2}{3}$, $l = \frac{32}{81}$.

23. $n = 5$, $r = -\frac{1}{2}$, $l = 4$.

24. $n = 4$, $a = 10$, $l = 0.00001$.

25. $a = 500$, $n = 6$, $r = 0.04$.

26. $s = \frac{3}{4}$, $a = \frac{1}{4}$, $l = 1$.

27. Insert four geometric means between

(*a*) 4 and -1;

(*b*) 10 and $\frac{2}{625}$;

(*c*) m and n;

(*d*) a and b^5/a^4.

28. Work out a formula for the geometric mean between a and b. (There will be two of them.) Use this formula to find the geometric means of

(*a*) 5, 6; (*b*) -3, -12;
(*c*) $x + y$, $x - y$; (*d*) $\sqrt{8}$, $\sqrt{32}$.

29. Work out formulas involving all of the letters s,n,a,r,l except

(*a*) n; (*b*) a; (*c*) r.

Find the values of the two letters not given in each of Probs. 30 to 33.

30. $n = 6$, $r = -\frac{1}{2}$, $l = -\frac{1}{48}$.

31. $s = -\frac{11}{2}$, $a = -8$, $r = -\frac{1}{2}$.

32. $s = \frac{7}{3}$, $n = 3$, $l = \frac{1}{3}$.

33. $s = -\frac{3}{2}$, $n = 3$, $a = -2$.

34. Assuming that the population of a city was 100,000 on May 1, 1930, and that it increases 10 per cent a year, what will it be on May 1, 1951? (Use logarithms.)

35. A boy intends to save 1 cent on June 1, 2 cents on June 2, 4 cents on June 3, etc. How much would he have to save on June 30?

36. Starting a poultry farm with one hen and a rooster, an optimist hopes to have 10 more hens at the end of the first year. Each of these hens is to produce 10 more during the second year, and so on. Excluding the roosters, how many hens will have sojourned on this dream farm by the end of the tenth year? (The figures are easy to handle without logarithms.)

37. If the numbers 0, -1, -1, and $-\frac{1}{2}$ are added to the four terms of an arithmetic progression, the resulting terms form a geometric progression. Find the two progressions.

38. Prove that if the terms of a geometric progression are squared, the resulting terms form a geometric progression.

55. This thing called "infinity." Recalling the series we used some time back to calculate a logarithm, you may remember that the new and interesting thing about this series was the fact that it *had no end.* Let anyone name a real positive number as large as he pleases, and still we can say confidently, regardless of the strength of his imagination, that the number of terms in the series will be larger than *that.* In fact, it is larger than any positive number whatever. Another way of saying the same thing is that this number is *infinite* or without end. We designate it sometimes by the symbol ∞, which looks like the digit 8 capsized and requires a bit of explaining.

The point is that this symbol represents, not a number at all, but an idea—*the idea of endlessness.* A long way back we said that division by zero was barred by algebraists, so $\frac{1}{0}$, for instance, is not a number. But occasionally one runs across this in print:

$$\frac{1}{0} = \infty$$

This may be considered allowable (though of questionable propriety, at that) as long as we do not think of it as a true equation involving definite numbers, but rather as a symbolic statement of an idea which goes into somewhat tedious English about like this: "If we divide 1 by x and let x get closer and closer to zero, the quotient $1/x$ can be made larger than any number named in advance, however large the latter may be." Certainly the symbolic statement outclasses the English one, for brevity at least;

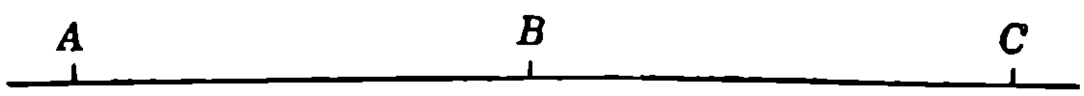

FIG. 20.

but one should keep in mind the lengthy translation of its meaning.

The adjective *finite* is applied to all *specific* numbers, however huge, to bring out the fact that they are definite and surpassable in size.

Now when we get to dealing with things of which there are an infinite number, we run into all sorts of strange situations. We can't even depend upon such seemingly obvious axioms as this: "The whole is greater than any of its parts." For instance, if we consider the line segment AC shown in Fig. 20, it seems safe to assert that there are more points in this than in the smaller segment AB. But are there? Let's place them in parallel position and draw lines connecting the two ends and intersecting at D, Fig. 21. Now every point on AC, as G, has a corresponding point F on AB lying on the line through G and D. Furthermore, any two points on AC which are actually distinct, however close together they may

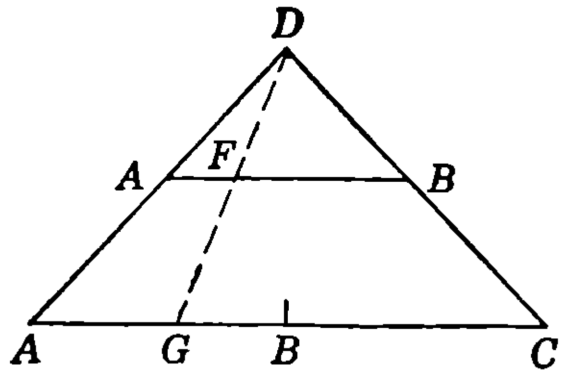

FIG. 21.

be, will be matched on AB by two more points which will likewise not coincide. It follows that we cannot assert the existence of more distinct points in AC than in AB. As a matter of fact, there is an endless number of them in each case, and that is what causes the difficulty. It shows us, however, that we must think carefully and cautiously when we deal with the infinite.

56. Series without end. Remembering, then, that we are treading on new thought ground when we deal with the endless in mathematics, let's have a look at a particular infinite series, such as this:

$$1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \cdots$$

and so on forever. It will be observed that the successive terms decrease rapidly in size, each one being one-half of its predecessor. In the calculation of log 1.12, we assumed without proof in a similar instance that the decimal value obtained by adding a few terms of the series would not have been changed with regard to the three digits next to the decimal had we added as many additional terms as we pleased. But if we have an infinite number of them to draw from, isn't that conclusion a little dubious?

Certainly the Greeks thought it was. In fact, it seemed so undeniable to them that a series of things, each greater than zero, would add up to a sum which would "go out of bounds" when enough terms were taken that Zeno of Elea[1] used this "obvious" conclusion to prove that a runner could not reach his goal. For, in order to do so, he must first reach the mid-point, which would require a certain positive time interval. A second interval would elapse before he would reach the point halfway between the mid-point and the end, and so on. Evidently there would be an infinite number of such intervals, growing smaller and smaller, to be sure, but all positive and all increasing the total time, so that this time would never end and the runner could not finish, according to Zeno.

Don't get the impression, however, that the Greeks failed to see that their logic was undependable in a spot or two. They

[1] See Smith's *History of Mathematics*, Vol. I, p. 18, and Dantzig's *Number*, p. 122.

were far from fools and knew something was wrong, but they just couldn't see what it was.

Actually the mistake of the Greeks was their failure to see that the numbers in certain series simply will not add up to a sum exceeding a certain fixed quantity, no matter how many of them we may take. For instance, if Zeno's runner covers half of the distance in one minute, say, he will go half the remaining distance in one-half a minute, the next prescribed interval in one-fourth a minute, and so on. Adding these time intervals in minutes we get the series

$$1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \cdots$$

which turns out to be the one given at the start of this article. But now we know that, since the runner covered one-half the distance in a minute, he must have gone the whole way in two minutes, so that the sum of any given number of terms in the series, no matter how large it may be, cannot exceed two. That is the first important point to notice. The second one is that this sum of the first n terms which we'll call S_n, comes as close to two as we please if we take n large enough, as we can show by the geometric-progression formula for the sum. In this case, $a = 1$, $n = n$, $r = \frac{1}{2}$, so

$$S_n = \frac{1 - (\frac{1}{2})^n}{1 - \frac{1}{2}} = 2 - \left(\frac{1}{2}\right)^{n-1}$$

which, as we saw was necessary, is less than two, and which also differs from two by $(\frac{1}{2})^{n-1}$—a quantity that approaches zero as n increases.

The number 2, then, has a special relation to the given series and is said by definition to be its *sum.* In general the *sum of an infinite series, when it exists, is the number which differs from the sum of the first* **n** *terms in the series by a quantity which we can make as small as we please by increasing* **n** *sufficiently.*

We haven't shown that such a number does exist in every case, and as a matter of fact we can't show it. There are cases in which the sum of the first n terms increases without limit as n increases, even though the successive terms get nearer and nearer to zero, and others in which this sum dodges back and forth endlessly, getting nowhere in particular. An example of the first kind, illustrating the going-out-of-bounds situation which the Greeks thought would always hold, is the series[1]

$$\tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \tfrac{1}{5} + \cdots$$

of the second, the series

$$1 - 2 + 3 - 4 + \cdots$$

If the sum exists the series is called *convergent;* otherwise it is *divergent.*

There is one type of convergent series for which we are now prepared to find the sum. This is the geometric progression with infinitely many terms in which $-1 < r < 1$. As before, let S_n be the sum of the first n terms. By the usual formula.

$$S_n = \frac{a - ar^n}{1 - r} = \frac{a}{1 - r} - \frac{ar^n}{1 - r}$$

Now as n increases the quantity ar^n approaches zero, since it may be written in the form

$$a\left(\frac{1}{R}\right)^n = a\frac{1^n}{R^n} = \frac{a}{R^n}$$

where R $(= 1/r)$ is numerically greater than one and hence R^n increases indefinitely in size with n. Remembering that r is some fixed quantity between -1 and $+1$, we see that $1 - r$ is a constant different from zero and hence $(ar^n)/(1 - r)$ comes as close to zero as we please as n increases. This means that the sum of the infinite geometric progression is $a/(1 - r)$

[1] To show that the sum of the first n terms of this series increases beyond any given constant as n increases, we may write it thus: $\frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \cdots + \frac{1}{8}\right) + \left(\frac{1}{9} + \cdots + \frac{1}{16}\right) + \cdots + \left[\frac{1}{(2^{n-1} + 1)} + \cdots + \frac{1}{2^n}\right] + \cdots$. Each quantity within parentheses is greater than $\frac{1}{2}$. (Why?) Hence, since there are an unlimited number of such quantities available, we can add enough of them together to get a sum exceeding any given number, no matter how large it may be.

according to the above definition, and we can add the important new formula

$$(1) \qquad S = \frac{a}{1 - r} \text{ when } n = \infty \text{ and } -1 < r < 1$$

An interesting but not especially important application appears in connection with *repeating decimal* numbers, such as $N = 1.2353535 \cdots$, which we can now show to be rational. Writing N as a summation we have

$$N = 1.2 + 0.035 + 0.00035 + 0.0000035 + \cdots$$

in which the series beginning with the second term is seen to be an infinite series with $a = 0.035$ and $r = 0.01$.
Hence

$$\begin{aligned} N &= 1.2 + \frac{0.035}{1 - .01} \\ &= 1.2 + \frac{0.035}{0.99} = \frac{12}{10} + \frac{35}{990} \\ &= \frac{1188 + 35}{990} \\ &= \frac{1223}{990} \end{aligned}$$

It is a simple matter to check the result by division.

EXERCISE 33

In Probs. 1 to 8, find the sum of the series with infinitely many terms:

1. $2 + \frac{1}{3} + \frac{1}{18} + \frac{1}{108} + \cdots$
2. $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \cdots$
3. $2.1 + 0.021 + 0.00021 + \cdots$
4. $4 + 0.04 + 0.004 + \cdots$
5. $11 + 0.011 + 0.000011 + \cdots$
6. $1 + \frac{1}{3} + \frac{1}{9} + \cdots$
7. $a + 1/a + 1/a^3 + \cdots \qquad (a > 1)$.
8. $1 + r/a + (r/a)^2 + \cdots \qquad (0 < r < a)$.

Express the repeating decimals in Probs. 9 to 16 as rational fractions:

9. 0.3333 · · · **10.** 0.5555 · · ·

11. 1.2333 · · · **12.** 0.312312312 · ·

13. 2.140140140 · · · **14.** 1.2010101

15. 0.99999 · · · **16.** 1.000111

17. A ship which is 105 miles from shore sails 10 miles the first hour and decreases its rate by $\frac{1}{10}$ on each succeeding hour. Does it reach the shore? If so, when?

18. A boy in a swing travels through an arc of 10 ft. from north to south, then three-quarters as far from south to north, and so on until he comes to a stop. How far does he travel altogether?

19. A ton of metal is extracted from a mining shaft in one day. The next day an equal excavation yields one-quarter less metal. If this decreasing yield continues indefinitely, how much metal was probably in the shaft originally?

57. Business connections: Simple interest. This is the article in which we literally get down to business. While the relation of the arithmetic progression to the physical laws governing falling stones, and the tendency of the geometric progression to complicate the lives of rabbits as well as school children, add a certain amount of zest and flavor to the subject, perhaps, they of course do not show very clearly why the wage earner should know about progressions. As a matter of fact, he can use them in the simplest items of his workaday life. They enter most emphatically into problems concerning his private collection of dollars and cents.

Take, for instance, the matter of buying a used car on the installment plan. Suppose the price is \$1,200 and the seller agrees to accept payment in 12 monthly installments consisting of \$100.00 each plus total accrued interest at 6 per cent, the first payment being due one month after the deal. Since 6 per cent interest per year is equivalent to 0.5 per cent per month, and since the debt is reduced by \$100.00 each month, the accrued interest will decrease monthly by (0.005)(\$100.00) = \$0.50. The first payment will be \$100.00 + (0.005)(\$1,200) = \$106.00. To get the total amount paid, then, we have an arithmetic progression with $a = 106.00$, $d = -0.50$, and $n = 12$. Hence

$$l = 106.00 - 11(0.50) = 100.50$$

and

$$s = 6(106.00 + 100.50) = 1{,}239.00$$

so that the buyer pays $1,239.00 altogether.

That extra $39 paid in the above problem represents *interest* or *rent* for the use of money, payment of which is a fundamental principle of business as conducted in the so-called "capitalistic" nations. We get interest (presumably) when we buy bonds or deposit money in a savings bank, and we pay interest (usually) in one way or another for money we have borrowed as cash or in the form of services. It is the calculation of that interest under various conditions that constitutes the central technical problem of most financial transactions; and it is the progressions, particularly the geometric ones, which furnish the key to nearly all the answers. So well, indeed, do they do the job that bankers seldom tear their hair, even when it is available, over the question, "How much will it work out to be at such and such a rate?" (They push a button and ask a clerk, who looks it up in a table.) *Their* central problem is, "What rate *should I get* for money rented out to that man or group, considering the human and business factors involved?"

If this point is sufficiently clear, we shall proceed to the subject treated at length in some whole textbooks as "the mathematics of finance." It deals, not with the difficult human factors involving credit and fixing rates, but rather with the purely technical computations to be made after the haggling is over. Evidently every educated person should know, for his own protection, how to make and check these computations.

In that which follows we shall let P represent the *principal*, or the sum on which the interest is paid; i, the rate; and n, the number of conversion periods. Other letters used will be defined as they are introduced. A *conversion period*, by the way, is the interval during which the principal draws interest at the given rate and at the end of which the accrued interest is *com-*

pounded or combined with the former principal to make a new one. If the conversion period is as long as the time involved in the transaction, the interest is *simple;* otherwise it is *compound.*

The important formulas involving simple interest are

$$(1) \qquad I = Pit$$

and

$$(2) \qquad A = P + Pit = P(1 + it)$$

where I represents the interest, t the time in years (not necessarily integral), and A the principal plus interest, or the *amount.*

Evidently if a debt is paid off in installments which include in each case a fixed amount plus the simple interest thus far accrued (as is frequently the case when the installments are completed within a year), they (the installments) form an arithmetic progression, as in our "used car" problem. In general, if P is the original debt, t the interval between payments in years, and R the fixed amount included in each payment, the first term of the progression, or the first payment, is $R + Pit$, and the common difference is $-Rit$ since there are R less dollars each month in the unpaid principal.

EXERCISE 34

1. Find the simple interest at 5 per cent on $1,000 for 9 months.

2. Find the amount of $500 for 15 months at 4 per cent, simple interest.

3. A debt of $120, due January 1, is paid in monthly installments, beginning February 1, of $10 plus accrued interest at 6 per cent, each being made at the first of a month. Find the total amount repaid on the debt.

4. Work Prob. 3 if the debt is $360, the installments $30.00 each, and the interest rate 4 per cent.

5. Work Prob. 3 if the debt is $540, the installments $45.00 each, and the interest rate 5 per cent.

58. Business connections: Compound interest. When the conversion period is shorter than the transaction time, so that the interest is compounded, we come to the formula that is the mathematical keystone of the whole vast world of finance. So important is it, in fact, that a large proportion, if not practically all, of the multitudinous other formulas dealing with money, insurance, and business are merely variations of, and deductions from, this particular mathematical sentence. They are convenient timesavers, in other words, but only this one is essential. Here, then, is the key sentence of all finance:

$$A = P(1 + i)^n \tag{1}$$

where A is the compound amount of the principal P, which draws interest at rate i per period for n conversion periods.

The derivation is simple. The amount at the end of the first period is

$$P + Pi = P(1 + i) \tag{2}$$

Remember that our i here is the rate per conversion period, corresponding with the it of the simple interest formula when t is the time of that period in years. Thus, if the conversion period is 6 months and the *nominal* annual rate is 6 per cent, $i = 0.06(\frac{1}{2}) = 0.03$. (We say "nominal" rate because the actual or *effective* rate per year is in this case slightly greater than 6 per cent.) This accumulated amount $P(1 + i)$ then becomes the new principal, so that the amount at the end of the second period is

$$P(1 + i) + P(1 + i)i = P(1 + i)(1 + i) = P(1 + i)^2 \tag{3}$$

Continuing in this manner, we get

$$P(1 + i)^3,\ P(1 + i)^4,\ \cdots,\ P(1 + i)^n \tag{4}$$

as the amounts of the original principal P at the end of the third, fourth, and finally the nth periods. That's all there is to it, and there we have the mathematics of finance in a nutshell. To be sure, we'll add some of the usual frills at the proper places, such as the timesaving annuity equations derived from (1) by means of geometric progression formulas, but (1) is after all the one essential and basic result.

To take a specific case, suppose we find the amount of $100 at 4 per cent converted annually for 15 years. Formula (1) gives us

$$A = 100(1.04)^{15}$$

The computation could of course be carried through rapidly by means of logarithms; but a six-place table would be necessary to give the result correct to the cent. In actual practice, tables are used giving the values of $(1 + i)^n$ for the values of n and i most frequently encountered in business deals. We have provided a table of this nature (Table II) extensive enough to give some practice, though not sufficiently complete to serve the needs of business. Here, in the horizontal line with 15 on the left and in the column headed by 4 per cent, appears the number 1.8009. This is the value of $(1.04)^{15}$ to five significant figures. Hence

$$A = 100(1.8009) = 180.09$$

so that the required amount is $180.09.

A second type of problem which is likely to confront almost anyone is this: "Now that I've inherited $1,500 from Uncle Peter and will probably fritter away unaccountably any part of it under my control, how much of it should I tie up in a trust fund so that my son John can have $2,000 in 20 years, if I can get, as they assure me, 3 per cent compounded annually?"

The problem looks simple enough, but when we put the given quantities in equation (1) the unknown P appears on the right side and has to be solved for, thus:

$$2{,}000 = P(1.03)^{20}$$

$$P = \frac{2{,}000}{(1.03)^{20}} = 2{,}000(1.03)^{-20}$$

The corresponding form of (1) in the general case is

$$\textbf{(5)} \qquad \boldsymbol{P = A(1 + i)^{-n}}$$

Evidently we need another table, and this is obligingly provided as Table III. There we find that $(1.03)^{-20} = 0.55368$, so that

$$P = 2{,}000(0.55368) = 1{,}107.36$$

Again, we might want to find out how long it will take a certain amount of money to accumulate to a specified sum at a given rate. For example, in what time would \$100.00 double itself at 5 per cent nominal rate converted semiannually?

Here the figure 100, come to think of it, has little bearing on the problem. The time is the same as that required for the doubling of a single dollar, and hence may be found, approximately, by inspection of Table II. In 28 half-year periods the dollar amounts to \$1.9965, and in 29 to \$2.0464. The doubling is effected, then, in just a little more than 14 years.

There remains one more letter in (1), namely, i, which might be the unknown. Again referring to Table II, suppose we find the interest rate, convertable annually, which would exactly double a dollar in a period of 20 years. Reading along the line for $n = 20$, we find that the figure 2 is missing, as might be expected. However, it will lie between the given figures 1.9898 and 2.1911, which correspond with interest rates of 3.5 per cent and 4 per cent, respectively. Interpolating to get a fair approximation, we find that 2 is $\frac{102}{2013}$ of the way from the first to the second figure, so that the desired percentage is $3.5 + (\frac{102}{2013})(0.5) = 3.525$.

In every one of the four cases involving formula (1), then, the problem may be worked most simply by reference to Tables II and III, provided the given data are there covered. Even when this is not the case, however, we needn't be stumped, provided ordinary logarithm tables are available. Taking the logarithm of both sides of (1) we have

$$\log A = \log P + n \log (1 + i) \tag{6}$$

from which any of the four quantities A, P, n, or i may be computed when the other three are given. To consider just one example, let's find how long it will take one dollar to triple itself at 10 per cent interest compounded annually.

The solution of (6) for n yields

$$n = \frac{\log A - \log P}{\log (1 + i)}$$

Inserting $A = 3$, $P = 1$, $i = 0.10$, we have

$$n = \frac{\log 3}{\log 1.1} = \frac{0.4771}{0.0414} = 11.5$$

so that the time is between 11 and 12 years.

In some installment and savings accounts, payments are made monthly while interest is converted at longer periods. But though there are tables and methods to take care of this situation, no essential change in principle is involved. It suits our purpose to deal only with those problems in which interest and conversion periods coincide.

EXERCISE 35

1. By use of Table II, find

(*a*) the amount of \$1,000 for 10 years with interest at 6 per cent compounded annually;
(*b*) the amount of \$100 for 30 years at 5 per cent, compounded annually;
(*c*) the approximate time required for money to double itself at 4 per cent compounded annually;
(*d*) the time required for money to triple itself at 6 per cent compounded twice a year;
(*e*) the approximate effective rate at which money will double itself in 15 years;
(*f*) the approximate effective rate at which it will triple itself in 20 years.

2. By use of Table III, find

(*a*) the sum which will amount to \$1,000 in 10 years at 4 per cent converted semiannually;
(*b*) the present value of a payment of \$10,000 to be made 15 years hence with interest at 6 per cent converted semiannually;
(*c*) the sum which will amount to \$600 in 25 years at 5 per cent converted semiannually.

3. By use of logarithms, find

(*a*) the amount of \$500 for 60 years at 4 per cent converted annually;
(*b*) the amount of \$1,000 for 20 years at 6 per cent converted quarterly;

(*c*) the approximate time in years required for money to triple itself at $6\frac{1}{2}$ per cent interest compounded semiannually;
(*d*) the effective annual rate at which money will triple itself in 40 years.

59. Business connections: Annuities. In the last article we discussed the key sentence of finance, and sampled a few of the problems which are direct and simple applications. Probably the most important and widely used of business formulas, however, are the somewhat more complicated applications, or rather extensions, applying to annuities.

America is the land of installment buying, and an *annuity* is merely an installment gone technical. It is any periodic payment made to liquidate a debt or accumulate a reserve, whether paid to a business concern for goods delivered, to a savings bank for future credit, or to a beneficiary by an insurance company. The important feature is that it is a payment of equal amounts at equal intervals of time.

Suppose, for example, that the sum of $100 is paid at the end of each 6-month period for 20 periods, and that interest is allowed for money paid at the nominal annual rate of 4 per cent, to be compounded semiannually at the time each payment is made. Two problems are immediately suggested: (1) If the payments are made to a loan company or savings bank, what will be the amount accumulated to the credit of the investor at the end of the twentieth period? (2) If the payments are for the purpose of clearing up a debt already incurred, what was the original debt?

Evidently we need some more formulas. In line with the usual custom, we shall let R be the amount of the payment, or periodic rent, n the number of payments, i the rate per period, A_n the *present value*, and S_n the *amount* or *future value* of the annuity. In the problem above, $R = 100$, $n = 20$, and $i = 2$.

Now let's see what can be done by way of short cuts. If we seek first a formula for S_n, we note that S_n can be obtained by the simple but long and tedious process of finding the accumulated amount of each payment and then adding these amounts. Since the first payment was made at the end of the first period, it will draw interest for $(n-1)$ periods. The second payment draws interest for $(n-2)$ intervals, and so on until the last, or nth, payment—made at the time of the final reckoning—draws no interest whatever. Applying our indispensable formula (1), Art. 58, we find that the various compound amounts will be as follows:

first, $R(1+i)^{n-1}$
second, $R(1+i)^{n-2}$
third, $R(1+i)^{n-3}$

nth, $R(1+i)^0 = R$

Here the geometric progression comes into its own and gives us the sum with neatness and dispatch. Reversing the order for convenience, we have $a = R$, $r = 1 + i$, $n = n$; so that, by (5) of Art. 54, rewriting $(a - ar^n)/(1 - r)$ in the form $\dfrac{a(r^n - 1)}{(r - 1)}$, we have

$$S_n = R\left[\frac{(1+i)^n - 1}{i}\right]$$

If we place $R = 1$, $S_n = \dfrac{(1+i)^n - 1}{i}$ in (1), which accordingly represents the amount of an annuity of one dollar per period for n periods at rate i per period. Designating this useful quantity by $s_{\overline{n}|i}$, we have

$$\textbf{(2)} \qquad S_n = Rs_{\overline{n}|i}$$

the first important annuity formula.

Evidently $s_{\overline{n}|i}$ can be calculated by means of logarithms. This has been done for values of n and i likely to occur in practice, and the results have been tabulated in extensive tables, of which an abridged sample is Table IV.

In our problem,

$$\begin{aligned} S_{20} &= 100s_{\overline{20}|0.02} \\ &= 100(24.2974) \\ &= 2{,}429.74 \end{aligned}$$

the accumulated amount of the deposits.

To find the formula for A_n, or the so-called *present or cash value* of the annuity, we note that A_n should be the present value, as the annuity starts, of its maturing value. Hence by (5), Art. 58

$$\begin{aligned} A_n &= S_n(1+i)^{-n} = R[s_{\overline{n}|i}(1+i)^{-n}] \\ &= R(1+i)^{-n}\left[\frac{(1+i)^n - 1}{i}\right] = R\left[\frac{1-(1+i)^{-n}}{i}\right] \end{aligned}$$

This latter quantity, which is the present value of an annuity of one dollar per period for n periods at rate i, we shall call, according to the very satisfactory custom, $a_{\overline{n}|i}$, so that our second important annuity formula may be written

$$\textbf{(3)} \qquad A_n = Ra_{\overline{n}|i}$$

In Table V we're again supplying merely an illustrative sample, but it should be enough to give the idea. At least it's enough to make short work of finding A_n, or the cash value of the annuity in our sample problem, thus:

$$A_{20} = 100a_{\overline{20}|.02} = 100(16.3514) = 1{,}635.14$$

By way of an independent check, this sum should amount to S_{20} in the 20 half-year periods of the annuity. Trying it out, we find, by Table II, that

$$1{,}635.14(1.4859) = 2{,}429.65$$

missing the otherwise-determined S_n by 9 cents. The discrepancy need not distress us at all in view of the fact that the 1.4859 used in the last computation gives accuracy only to five significant figures, so that the first figure for S_n is the correct one.

Thus far we have shown how to find the first two of the five quantities S_n, A_n, R, i, and n involved in an ordinary annuity. Evidently we can make up problems in which each of the letters R, i, and n is the unknown. There is a point to this, however, only if it becomes necessary in practice to do this very thing.

Certainly it is in the case of R. For when bonds are issued by a corporation, a sinking fund is usually created into which equal sums are deposited periodically in order that the fund will be sufficient to retire the bonds at maturity. In this case we know S_n, n, and i. Hence the desired formula is that obtained by solving (2) for R, thus:

$$R = S_n\left(\frac{1}{s_{\overline{n}|i}}\right) \tag{4}$$

Or again, solving (3), we have

$$R = A_n\left(\frac{1}{a_{\overline{n}|i}}\right) \tag{5}$$

a formula to be used, evidently, when it is desired to find the amount of the periodic payment which will pay a certain debt by installments in a specified time and at a given rate. This is the situation which frequently occurs when an automobile is bought on the installment plan, since the payments are usually so adjusted that they will end in 6, 12, or 18 months. Clearly the formulas for R are among those most needed in practice.

Granting the last statement, then, it seems unlikely that businessmen and financiers would be satisfied with Tables IV and V for the determination of the quantities $1/s_{\overline{n}|i}$ and $1/a_{\overline{n}|i}$. Our usual abridged sample of the new data thus required is Table VI. A fortunate algebraic result makes a separate table for $1/s_{\overline{n}|i}$ unnecessary, since $1/s_{\overline{n}|i} = 1/a_{\overline{n}|i} - i$, as may be deduced from the definitions of $s_{\overline{n}|i}$ and $a_{\overline{n}|i}$. We can thus get $1/s_{\overline{n}|i}$ from the value of $1/a_{\overline{n}|i}$ as altered by an easy subtraction to be made mentally.

A couple of illustrations will do no harm here.

Illustrative Examples

1. Mr. Jones buys a house for \$12,000, paying \$2,000 cash and agreeing to pay the balance in 20 equal semiannual installments with interest at 4 per cent convertible semiannually, the first payment to be made in 6 months. Find the amount of each payment.

Here $A = \$10{,}000$, $n = 20$, $i = .02$. Then

$$R = \$10{,}000\left(\frac{1}{a_{\overline{20}|0.02}}\right) = \$10{,}000(0.0611567) = \$611.57$$

2. A father wishes to provide \$4,000 for his son on his twenty-first birthday. How much should he deposit every 6 months in a savings

bank which pays 3 per cent converted semiannually, if the first deposit is made when the son is $3\frac{1}{2}$ years old?

In this case $S_n = \$4{,}000$, $i = 0.015$, and $n = 36$. (CAUTION: Always be sure to check the value of n closely; if you don't you'll often miss the correct figure by one. Use care and common sense, noting Prob. 1 in the following exercises.) Then

$$\begin{aligned} R &= \$4{,}000\left(\frac{1}{s_{\overline{36}|0.015}}\right) \\ &= \$4{,}000(0.0361524 - 0.015) \\ &= \$4{,}000(0.0211524) \\ &= \$84.61 \end{aligned}$$

We could take up the case involving n as the unknown if we wanted to; but fortunately we don't. There's no sense in manufacturing business problems which will never worry anyone except us.

The last remark, however, is not at all apropos when i is the unknown. Suppose by way of an illustration which may be called fictitious (though parallel cases are not altogether missing in the marts of trade) a benevolent-looking gentleman wishes to lend you \$100, saying that you can return it in 12 monthly installments at \$10 each, thus paying him \$20 dollars or 20 per cent on his money. You may think that his interest rate, while rather high, is not altogether unreasonable in view of the smallness of the loan and your urgent need. Remember, however, that you start paying back the loan one month after it is made, so that the total interest of \$20 is charged on an outstanding debt averaging approximately \$50 during the year. The actual interest you pay is then in the neighborhood of 40 per cent, so that i, or the rate per month, is roughly $\frac{40}{12} = 3\frac{1}{3}$ per cent or 0.033. In the given case, $n = 12$, $R = 10$, $A_n = 100$, S_n is *not* \$120 (Why not?) and i, the unknown, is near 0.03. Substituting this trial value in the right side of Equation (3), we have

$$A_n = 10(a_{\overline{12}|0.03}) = 10(9.9540) = 99.54$$

This shows that 0.03 is slightly too large, since A_n decreases as i increases. Trying $i = 0.025$,

$$A_n = 10(a_{\overline{12}|0.025}) = 10(10.2578) = 102.58$$

Hence $i = 0.0293$ (by interpolation), and the nominal annual rate is 12(2.93 per cent) = 35.16 per cent. Now suppose you wish to find the actual or effective annual rate charged by the kind gentleman. If we let this be r, we have

$$1 + r = (1.0293)^{12}$$

or

$$r = (1.0293)^{12} - 1$$

which turns out to be 0.4145 (using five-place logarithms), so the rate has climbed, upon more careful examination, to nearly 41.5 per cent. The moral is—but on second thought, we'll leave that to you. After all, this is a book on mathematics.

EXERCISE 36

1. The first of a series of annual payments was due Jan. 1, 1900, and the final payment was due Jan. 1, 1920. How many payments were there?

2. Find the amounts and the present values of the following annuities:

(*a*) \$100 at the end of each 6-month period for 15 years with interest at 5 per cent compounded semiannually;

(*b*) \$1,000 payments made at the end of each year over a 20-year period with interest at 6 per cent effective.

3. Find the periodic rent, given

(*a*) A_n = \$1,000, $n = 15$, $i = 0.02$;

(*b*) A_n = \$10,000, $n = 30$, $i = 0.015$;

(*c*) S_n = \$1,500, $n = 20$, $i = 0.0125$;

(*d*) S_n = \$2,000, $n = 12$, $i = 0.01$.

4. Find the approximate value of i, given

(*a*) A_n = \$1,000, $n = 16$, R = \$70;

(*b*) A_n = \$10,000, $n = 8$, R = \$1,300;

(*c*) S_n = \$1,000, $n = 30$, R = \$25;

(*d*) S_n = \$10,000, $n = 40$, R = \$200.

5. A man buys a house, paying \$1,000 cash and agreeing to pay \$500 at the end of each 6-month period for 5 years. If money is worth 5 per cent converted semiannually, what was the price of the house?

6. A boy deposits \$10 every 6 months in a savings bank which pays 4 per cent converted semiannually. How much has he to his credit after the twenty-fifth deposit?

7. How much should be deposited annually in a sinking fund in order to retire a debt of $100,000 due in 10 years, if money is worth 2 per cent converted annually? The first deposit is made at the end of the first year.

8. A man buys an automobile for $1,000, agreeing to pay for it in six equal payments made at the end of 3-month intervals. How much are the payments if the interest charged is 8 per cent converted quarterly?

9. A "loan shark" lends $50 to be repaid in 6 monthly installments of $10, the first occurring one month after the loan was made. Find the approximate effective interest rate if interest is converted monthly.

10. At maturity an insurance policy offers the option of $3,000 in cash or an annuity of $150 semiannually for life, the first payment being due 6 months after the policy matured. If the insured accepted the annuity, with which payment would he begin to win if money is worth 4 per cent compounded semiannually?

11. Mr. Brown bought land for $15,000, paying $5,000 in cash and agreeing to pay the remainder in 20 semiannual installments. On the day that he paid the tenth installment, he sold the house and figured that he made $1,715 on the trade. Assuming that money was worth 4 per cent compounded semiannually, and that the land yielded no income during the 10 years, find the amount of his installments, and, to the nearest dollar, the amount he received for the land.

12. On his fortieth birthday a man invested a sum sufficient to yield an annuity of $600 semiannually for 20 years, starting 6 months after his sixtieth birthday. If money will yield 4 per cent compounded semiannually throughout the 20 years, what was the amount of his investment?

PART TWO

A LOOK OVER THE FIELD

CHAPTER VII

MEASURING THE INACCESSIBLE

60. We reach out. In the last two chapters we showed the importance and application of a number of highly useful tables. In this select company there belongs unquestionably the table of trigonometric values, without which people who measure things would be lost and helpless. And this brings us to *trigonometry*, or the art of measuring lengths and angles, together with the surprising array of mathematical tools and relations springing from that simple problem.

The measurements involved require a new technique and new theory, since often the rule or the tapeline cannot be applied directly to the subject being investigated. No earth-man as yet has stood on the moon with the end of a long string in his hand, and yet we know accurately the distance to our satellite. No aviator has zoomed to the lofty spot five hundred miles above the earth's polar regions where the northern lights are playing strange electrical pranks, nor has he climbed even to the minor heights one hundred miles or so above us where the incoming meteors first glow white-hot from air friction. The height of mountains, the width of rivers, the depth of oceans, the size of farms or cities or townsites—even the staggering distance to the sun and the unimaginable stretches of emptiness between us and the nearer stars—all of these things and many more are measured by the indirect and yet dependably accurate methods of trigonometry. It has told us how big, far, wide, and high are all the main features of the earth as well as the inaccessible objects in the sky, and has shown us our place, so to speak, in the universe. Important? Well, judge for yourself.

We shall not try to be meticulously accurate and specific about the beginnings of trigonometry, since the history books themselves are of necessity somewhat hazy on the matter. Some of its underlying principles began to be recognized, perhaps, about the second or third millennium B.C.[1] One of its earliest products was the sundial, an instrument for telling time that was widely used. It consisted essentially of an upright staff called the *gnomon*, whose shadow gave the most accurate information then available regarding the time of day and also of the season. It seems only natural that the observed relation between the height of the gnomon, the length of its shadow, and the angular distance of the sun above the horizon should eventually suggest the possibility of using the shadow of a building or even of a mountain to determine its height. In fact, Thales of Greece did that very thing in connection with one of the pyramids. The ancient Chinese must have had the same general idea, since we find in one of their early manuscripts[2] the statement that "The knowledge comes from the shadow, and the shadow comes from the gnomon."

61. The principle of the thing. The essentials of this measuring art were, in fact, exemplified in the simple sundial. One would think that a device powerful enough to put intellectual calipers on remote and inaccessible objects must be the mental product of a supreme genius, and must be so profound and hard to understand that the average mortal had better not waste his time on it. It should be a pleasant surprise, therefore, to learn that the heart of trigonometry is so ingloriously easy to understand that it can be put, by way of anticlimax, into a simple concept familiar to most schoolboys—that of *similiar triangles*. To find the distance to the moon all we need is a couple of similar triangles, one with a base of 8,000 miles stretching, say, from Quito in Ecuador, South America, to Singapore in Asia, and with an altitude so great that its sharp vertex can prod the moon; while the other, small enough to be put on paper, is made similar to the big one by use of the angles at the extremities of the base which are measured at a given time by observers in the two cities. Figure 22 will show the principle. Then the distance from Singapore to the moon is to the diameter of the earth as the distance SM in the small triangle of the figure is to the distance SQ, or about thirty to one. This

[1] Smith, *History of Mathematics*, Vol. II, p. 600.
[2] *Ibid.*, p. 602.

makes the distance to the moon about 240,000 miles, which is close enough for our purpose, since the moon actually keeps

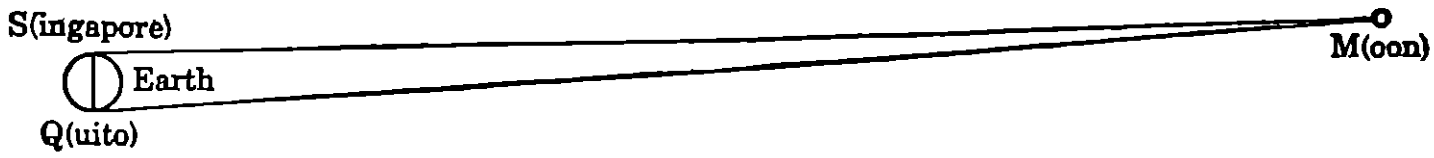

FIG. 22.

changing its distance in a disconcertingly complicated manner. We have thus already made our first step off the earth. Less ambitious problems, such as the height of a tree 20 ft. away whose top makes a measured angle with the ground, become now mere child's play, since we can obviously make a small right triangle similar to the big one with the tree for its side, and can then measure the ratio of the height to the base of the small triangle.

But, although the essential principles of trigonometry are employed in this method depending upon measurements, it is much too crude for a modern, well developed science. The mathematician prefers to do less measuring and more figuring, particularly when, as in this case, most of the figuring can be done once for all and recorded in tables. Evidently certain precise definitions are needed with regard to the ratios and terms used, so that the necessary comparisons can be made more efficiently. To that end we present in the next article some of the customary technical tools which even the most casual student of trigonometry must learn to recognize at sight in order to get a passing acquaintance with the subject.

We'll take time out here for two definitions which will be needed in the following exercise.

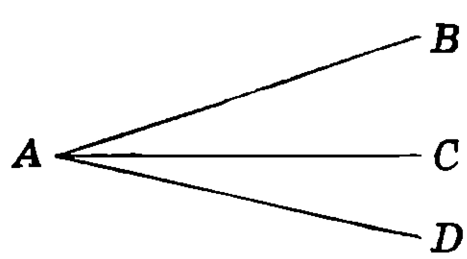

FIG. 23.

The ***angle of elevation*** *of a point **B** from a point **A** is the angle made with the horizontal plane by the line **AB**.* (*Angle **BAC** in Fig.* 23.)

The ***angle of depression*** *of a point **D** from a point **A** is the angle made with the horizontal plane by the line **AD**.* (***CAD*** *in Fig.* 23.)

EXERCISE 37

In this exercise the results should be, in most cases, approximations only, obtained by construction of triangles and rough measurements such as would have been made under the circumstances cited.

1. A man 6 ft. tall is standing at the foot of a windmill tower. If the shadow of the tower and that of the man are 40 ft. and 4 ft. in length, respectively, find the height of the tower.

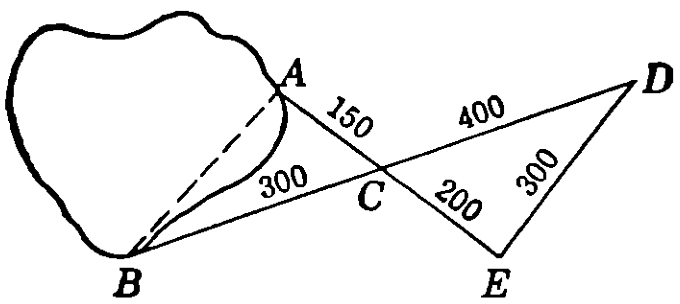

Fig. 24.

2. Two boys, wishing to obtain the distance between two points A and B on the shore of a lake, set stakes in positions indicated by each letter in Fig. 24, and then obtained the indicated distances by measuring. What is the distance from A to B?

3. At a given time an observer A in Fort Worth saw a meteor flash at a point M in the sky, which seemed to be directly over Dallas, making with the horizon an angle of 60 degrees (written 60°; a right angle contains 90°). Observer B in Dallas saw the same flash, apparently over Fort Worth, but 70° above the horizon. Assuming that the observers were 40 miles apart, draw a triangle to scale with the points A, B, and M as vertices. From this, *estimate* the height of the meteor above the ground when seen.

4. A party of pioneers, coming to a river, wished to estimate its width. At A (see Fig. 25) they obtained a direction AD at right angles to the line AB toward a tree on the far side. At C on AD, 200 ft. from A, they sighted toward points A and B and then drew a small triangle on the ground like the one shown. About how wide was the river?

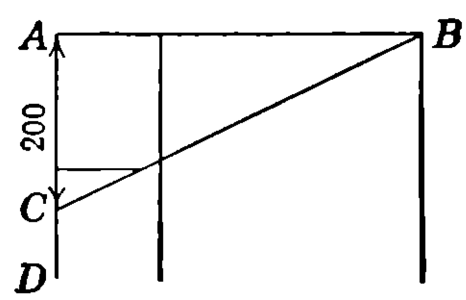

Fig. 25.

5. Approaching a mountain beyond a great plain, a mounted scout noted that its top made an angle of 10° with the horizon. An hour later, when he knew that he had covered about 12 miles, the angle was 30°. About how high was the mountain?

6. A tourist who had reached the top of a mountain noted that the angle of depression of city A from his position was 40°, while that of city B directly beyond A was 15°. He knew that A and B were 10 miles apart. Estimate the height of the mountain above the plain of the cities.

7. A man on the left curb of a street faces a fireplug some distance away on his own side of the street. Holding a yardstick against his right eye and in line with the fireplug, he finds that the far end of the stick is between his left eye and a telephone post directly across the street from the fireplug. How far away is the latter, if his eyes are $2\frac{1}{2}$ in. apart and the street is 40 ft. wide?

8. A geometry student, standing 6 ft. from a telephone post which was between him and a tower, had his assistant mark the points on the pole in line with the top and bottom of the tower. If these points were 4.5 ft. apart and the tower was 100 ft. away, what was its height?

9. A man finds that when he holds a ruler before him horizontally and at right angles to his line of sight, the two front corners of a building facing him span 10 and 12 in. on the ruler respectively when the middle of the ruler is 24 and 28 in. from his eye. How wide and how far away is the building?

10. When the hand is held at arm's length, with fingers outspread, the ends of the thumb and little finger are about 20° apart. An observer riding in a car going at 30 miles per hour found by this means that when he was opposite a signboard paralleling the highway it subtended a horizontal angle of 20° and that 2 sec. later the angle was 10°. About how wide was the signboard and about how close to it did he get?

62. The tools for the job. To get properly started, we need a figure showing an acute angle, such as AOB in Fig. 26. An *angle*, by the way, is one of those simple things which most of us can picture satisfactorily without a formal definition, but just to get something down we'll say that *it is the figure made by two half-lines with a common end point.* If the two half-lines are perpendicular, the smaller, or *right*, angle contains 90 *degrees* (written 90°). This statement will also serve to define

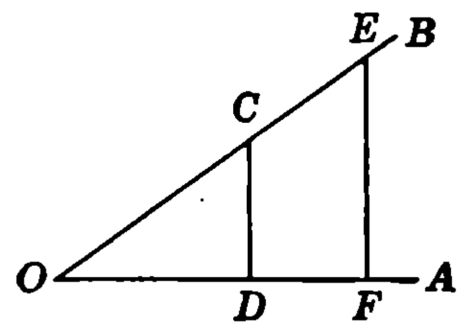

FIG. 26.

one of the common units, the *degree*, which in turn consists of sixty minutes (60′). An angle containing less than 90° is called *acute*, while one between 90° and 180° in size is *obtuse*. The angles dealt with in trigonometry range up to any size whatever (consider the one turned through each minute by the spoke of a rapidly revolving wheel), but the acute and obtuse ones will occupy all of our attention for the present.

Getting back to the angle AOB in Fig. 26, we note that if, from a couple of sample points on OB such as C and E, we drop perpendiculars (CD and EF in this case) to the side OA, then we have two similar triangles ODC and OFE, so that the ratios of corresponding sides in the two triangles are equal. Thus $DC/OC = FE/OE$, for instance, so that this particular ratio is a *number* (about three-fifths in our figure) associated with this particular angle. Since the value of the ratio depends not at all upon the position of the chosen point C or E, but solely upon the size of the angle O, it is, according to a previous definition,[1] a function of that angle. Evidently there are six of these ratios, or functions, involved for any acute angle. They are named respectively the *sine*, *cosine*, *tangent*, *cotangent*, *secant*, and *cosecant* of the angle, and are abbreviated, without the usual periods, as *sin*, *cos*, *tan*, *cot*, *sec*, and *csc*. Note that, as defined below, they apply only to acute angles. A meaning is also assigned to each function of an angle of any size whatever, but never mind that now. For the present, here are the six essential definitions as applying to the angle A in Fig. 27. Note that the side BC opposite the angle A is designated by the small letter a, the side opposite B by b, etc. This is a convenient practice which we shall follow in our treatment of both right and oblique triangles.

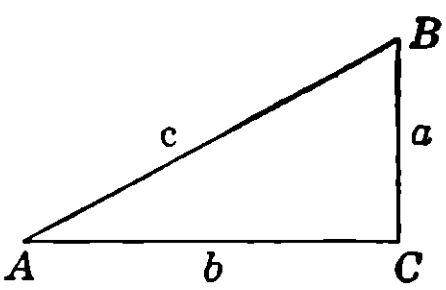

FIG. 27.

In the following abbreviations "opp", "adj", and "hyp" stand for the side opposite the angle A, the side adjacent to the angle A, and the hypotenuse, respectively.

$$\sin A = \frac{\text{opp}}{\text{hyp}} = \frac{a}{c}$$

$$\cos A = \frac{\text{adj}}{\text{hyp}} = \frac{b}{c}$$

$$\tan A = \frac{\text{opp}}{\text{adj}} = \frac{a}{b}$$

$$\cot A = \frac{\text{adj}}{\text{opp}} = \frac{b}{a}$$

$$\sec A = \frac{\text{hyp}}{\text{adj}} = \frac{c}{b}$$

$$\csc A = \frac{\text{hyp}}{\text{opp}} = \frac{c}{a}$$

[1] See Art. 42.

These definitions may be applied to the angle B also, but we must remember that the sides opposite and adjacent to B are now b and a, respectively. Hence $\sin B = b/c$. Since $B = 90° - A$, or the *complement* of A, we note that $\sin A = \cos(90° - A)$; or in words, *the sine of any acute angle is the cosine of its complement.* A good practice problem for you at this point is the following: Write the other five functions of B and find other relations similar to the one given.

It happens that there are three acute angles, namely 30°, 45°, and 60°, for which we can easily figure the *exact* values of their functions. If we bisect one of the angles of an equilateral triangle we get a right triangle with acute angles of 30° and 60° respectively, in which the side opposite the 30° angle is half the length of the hypotenuse. Choosing our scale so that this short side is 1 and the hypotenuse 2, we get $\sqrt{3}$ for the side b in Fig. 28, since $b^2 + 1 = 2^2$ and $b^2 = 4 - 1 = 3$, according to the Pythagorean theorem. Looking at the triangle and remembering the definitions above, we can now find six functions of both 30° and 60°. The corresponding functions of 45° may be obtained from the triangle in Fig. 29 involving this angle, in which the two sides are obviously equal and may be considered one unit long. This is an excellent place for the reader to stop and fix more firmly in mind the six fundamental definitions by writing down, with the help of the two figures, the 18 numerical values for the functions of 30°, 45°, and 60°. Having done that, he can check his results by comparison with the following table:

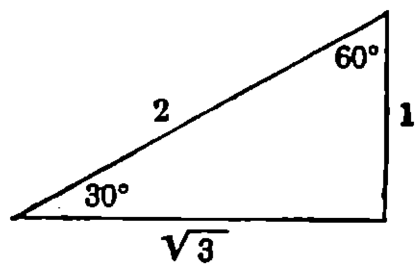

FIG. 28.

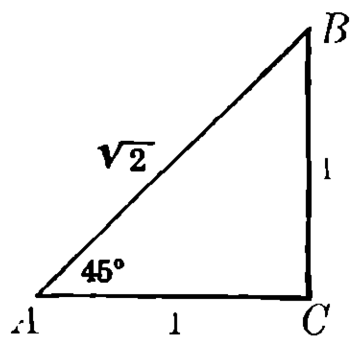

FIG. 29.

	sin	cos	tan	cot	sec	csc
30°	$\frac{1}{2}$	$\sqrt{3}/2$	$1/\sqrt{3}$	$\sqrt{3}/1$	$2/\sqrt{3}$	2
45°	$1/\sqrt{2}$	$1/\sqrt{2}$	1	1	$\sqrt{2}$	$\sqrt{2}$
60°	$\sqrt{3}/2$	$\frac{1}{2}$	$\sqrt{3}$	$1/\sqrt{3}$	2	$2/\sqrt{3}$

To get the approximate decimal values of such numbers as $1/\sqrt{2}$, and $2/\sqrt{3}$, which have radicals in the denominators, it is best to *rationalize* the denominator thus: $1/\sqrt{2} = (1/\sqrt{2})(1) = (1/\sqrt{2})(\sqrt{2}/\sqrt{2}) = \sqrt{2}/2 = 1.4142/2 = .7071$. Such a form as $(\frac{2}{3})(\sqrt{3})$ should be recognized as merely another way of writing the number $2/\sqrt{3}$; but, contrary to a fairly widespread impression, it is not a particularly better way *unless* one wishes to evaluate it decimally.

It is worth noticing that twelve of the eighteen numbers in the small table are irrational, which means, it may be recalled, that they cannot be expressed exactly in decimal form. In fact, of all the functions of multiples of 1° from 1° to 89°, inclusive, only sin 30°, csc 30°, tan 45°, cot 45°, cos 60°, and sec 60° are rational. It follows that almost all of the values found in trigonometric tables such as Table VII are decimal approximations rather than exact values. For instance, sin 1° is given there as 0.0175; but it would appear as .01745 in five-place tables.

Looking at Table VII, we note that we can read from the top downward to get the sine, cosine, tangent, or cotangent of selected angles from 0° to 45°, and from the bottom upward to get the same functions of angles from 45° to 90° inclusive. One should have little difficulty in finding, for instance, that sin 13°10′ is 0.2278, and that cot 64°40′ is 0.4734, without tedious and elaborate instruction on our part. Try it and see. Intermediate values such as cos 13°27′ = 0.9726 can be found by interpolation, which we explained in connection with logarithms.

If, now, the reader is one of those rare and admirable characters who is not satisfied with a mere set of data, and if he burns with curiosity about how the figures were actually obtained, we'll dispose of the matter in a somewhat peremptory manner, for reasons that will be evident in the answer. Most of these values were obtained by use of special infinite series whose discussion, unfortunately, belongs in more advanced mathematics.

Since, however, our definitions of the functions of an acute angle do not apply to 0° or to 90°, one may legitimately ask here what is meant by the table values for these angles. They are simply definitions decided upon in this way: If in a right triangle containing the acute angle A we hold the hypotenuse fixed in size and let A get as small as we please, evidently the side opposite A, and hence the ratio named sin A, gets as near zero as we please, while the side next to A approaches equality with the hypotenuse, so that cos A comes closer and closer to one. Evidently, then, we can avoid sudden "jumps" or changes in the values of these two functions for two angles close together, one of which is 0°, by *defining* sin 0° as 0, and cos 0° as 1. For the same reason we let tan 0° = 0, sec 0° = 1, sin 90° = csc 90° = 1 and cos 90° = cot 90° = 0. The symbol ∞ for cot 0° and tan 90° means that these two functions are *infinite*. More precisely, in one sample case, the tangent of an angle increases without limit as the angle approaches 90°, so that tan 90° does not exist.

EXERCISE 38

Using Table VII, find the sine, cosine, tangent, and cotangent of each of the following angles, using interpolation where necessary.

1. 20°30′. **2.** 41°23′. **3.** 67°10′.
4. 81°46′. **5.** 47°28′. **6.** 38°14′.
7. 19°40′. **8.** 63°2′. **9.** 89°42′.
10. 17′.

Find the angles when the functions are as indicated.

11. $\sin A = 0.2391$. **12.** $\cos B = 0.0640$.
13. $\tan \theta = 2.1445$. **14.** $\cot \theta = 0.0035$.
15. $\sin \theta = 0.9981$. **16.** $\cot A = 4.1628$.
17. $\sin B = 0.0003$. **18.** $\cos B = 0.1111$.
19. $\cot A = 3.1491$. **20.** $\tan A = 0.9988$.

Assuming that a and b are the sides of a right triangle while c is the hypotenuse, use the Pythagorean theorem to find the values indicated by dashes below. Leave the result in radical form *if* it is irrational.

	a	b	c
21.	9	12	—
22.	8	—	17
23.	—	12	13
24.	7	8	—
25.	7	—	9
26.	—	$\frac{1}{2}$	$\frac{3}{4}$
27.	—	24	25
28.	2	5	—
29.	2	—	5
30.	2	$\frac{2}{3}$	—

31. If $A = 30°$, and $c = 6$, what are the values of a and b?

32. Answer the same question if $c = 10$.

33. Find the values of the sine, cosine, tangent and cotangent of 25°40′; then, without reference to the table, get the values of the same functions of 64°20′.

34. Using Figs. 28 and 29, find the values of $\sin^2 30° + \cos^2 30°$, $\sin^2 45° + \cos^2 45°$, and $\sin^2 60° + \cos^2 60°$.

35. Using Table VII, find the value of $\sin^2 27° + \cos^2 27°$.

36. What general conclusion is suggested by Prob. 34? Why is this conclusion not borne out exactly in Prob. 35?

63. How we do it. Right triangles. With a table of values of the trigonometric functions before us, we may smile complacently at the passé methods of the similar-triangle-measurers, and may demonstrate our own efficiency by calculating the unknown parts of a triangle without moving from the easy chair. Since, however, the process is simpler in the case of right triangles, we'll dispose of that problem first.

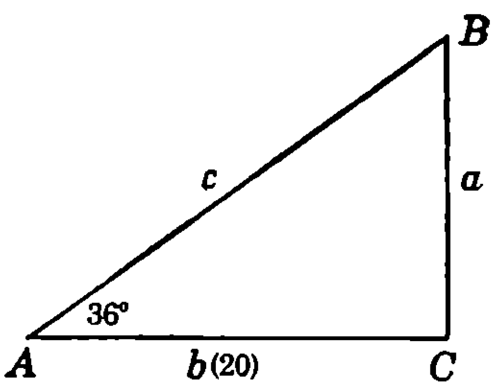

FIG. 30.

Suppose, for example, in a right triangle with sides and angles labeled according to our previously agreed-upon convention (side a opposite angle A, etc.), $A = 36°$ and $b = 20$ feet *exactly* (Fig. 30). Our problem is to find the unknowns B, a, and C. Since our tables contain only the sine, cosine, tangent, and cotangent values, we'll consider first the eight equations:

(1) $$\sin A = \frac{a}{c}$$

(2) $$\cos A = \frac{b}{c}$$

(3) $$\tan A = \frac{a}{b}$$

(4) $$\cot A = \frac{b}{a}$$

$$(5) \qquad \sin B = \frac{b}{c}$$

$$(6) \qquad \cos B = \frac{a}{c}$$

$$(7) \qquad \tan B = \frac{b}{a}$$

$$(8) \qquad \cot B = \frac{a}{b}$$

Seeking among these equations those which contain the two knowns A and b together with one unknown, we find that only (2), (3), and (4) meet these conditions. This will suggest a systematic mode of attack, though with very little experience we can find immediately the equations which can be used. Applying (3), we have $\tan 36° = a/20$. From our table we find that $\tan 36° = 0.7265$, so that $0.7265 = a/20$, or $a = 20(0.7265) = 14.53$. Similarly, using (2), we have $\cos 36° = 20/c$, or $0.8090 = 20/c$, so that $0.8090c = 20$, and $c = 20/0.8090 = 24.72$.

In the problem above, since b was assumed to be exactly 20, it could be written 20.00, and hence our results were accurate to four significant digits, as were the numbers in the tables used. In practice the given values are usually *measured* ones (as in a surveying problem), so that we can consider them correct only to the number of places indicated. In subsequent problems the number of significant digits in the answers should be the same as in the given data. Angles are to be found in each case to the nearest minute.

It is further worth noticing here that the "parts" of a triangle consist of three sides and three angles. When three of these parts are known, including at least one side, the triangle is solvable. In a right triangle, then, a side and an angle or two sides must be known in addition to the right angle.

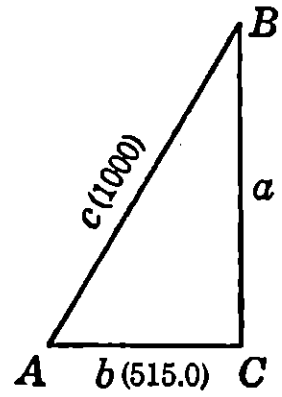

FIG. 31.

A second type of solvable right-triangle problem is that in which neither acute angle is known, as in the case represented by Fig. 31. Running through the same eight equations involving A and B as were used in the preceding problem, we find that there are two equations which contain only one unknown each. They are

$$\cos A = \frac{b}{c}$$

and

$$\sin B = \frac{b}{c}$$

Inserting known values in the first equation, we have

$$\cos A = \frac{515.0}{1{,}000} = .5150$$

Turning to the table, we find .5150 in the cosine column opposite 59°, so that $A = 59°$. Of course, then, $B = 90° - 59° = 31°$, and the other available equation need not be used. The value of a remains to be found, but now any of our eight original equations which contains a will have only that one unknown. A convenient one to apply is the following:

$$\sin A = \sin 59° = \frac{a}{1{,}000}$$

or, using the table,

$$.8572 = \frac{a}{1{,}000}$$

so that

$$a = 1{,}000(0.8572) = 857.2$$

Where, as in the above problem, two or more different equations can be used to find the same quantity, the correctness of the value obtained in one way may be checked by comparing it with that found by another method. If the two results are exactly alike or differ only slightly, there is probably no bad arithmetic or algebraic blunder. A small difference can occur when the work is correct because the tabular values are irrational, as explained before, so that the final digits are rounded off. Thus, if we use the equation

$$\tan 59° = \frac{a}{515.0}$$

we get

$$a = 857.1$$

instead of 857.2, but feel reassured, nevertheless, about our calculations. Of course the *first* check the solver should apply, if he can possibly get the habit, is that which catches the bigger errors immediately by inspection. He should always draw the figure approximately to scale, so that he can look at it as soon as he gets a required value and decide whether or not it is reasonable. If, for example, on the problem of Fig. 31 he should get by one method $a = 8.571$, his common sense and his eyesight should combine to veto that answer almost before it is written down.

EXERCISE 39[1]

In Probs. 1 to 10 dealing with right triangles, c will stand for the hypotenuse opposite the right angle C, while A and B are the acute angles opposite sides a and b respectively. *Be sure* to draw each triangle about to scale before completing the computations. Then solve for the values not given.

	a	b	c	A	B
1.	21	37	—	—	—
2.	133	—	214	—	—
3.	216.3	—	—	37°14′	—
4.	—	123.4	312.6	—	—
5.	—	—	43,720	—	41°10′
6.	—	0.2389	—	81°49′	—
7.	0.00314	—	—	—	77°11′
8.	—	86.31	—	9°59′	—
9.	0.6578	0.8641	—	—	—
10.	—	214,000	423,000		—

11. A rectangle is 11.22 in. long and 10.09 in. wide. Find the length of its diagonal and the angle it makes with the long side.

12. When the sun's angle of elevation is 20°, a tree casts a shadow 159.3 ft. long. How high is the tree?

13. From a point 100 ft. from the base of a tower, the angle of elevation of its top is 26°40′. Find the height of the tower.

[1] For the convenience of teachers who wish their classes to use logarithms of trigonometric functions in triangle solutions, we have included these, without explanation, in Table VII.

14. From a lighthouse 75 ft. high, the angle of depression of a boat is 31°20′. How far is the boat from the base of the lighthouse?

15. In aerial navigation a *course* is a direction designated by an angle measured clockwise from due north. For example, 270° means due west.

A pilot flies his plane at a course 0° to an airport 500 miles away. Refueling, he follows course 132°30′ to a point due east of his home field. How many miles did he travel?

16. A carpenter is employed to build a stairway that rises 10 ft. in a hall 25 ft. in length. If the lower end of the stairs is 4 ft. from the wall, what is the inclination of the stairs, and the length of the handrail?

17. A waterpipe runs under a house, emerging at two points—one 21 ft. north and one 18 ft. east of the southwest corner. Find the acute angle that the pipe makes with the west wall, and the length of the pipe.

18. Centerville and Cross Cut are located on east-west and north-south highways, respectively, with the former 48 miles west of, and the latter 72 miles north of, the intersection. The two towns are also connected directly by a dirt road. A salesman in Centerville can average 50 miles per hour on the highways and 40 miles per hour on the road. Which road is the better for him to take to Cross Cut?

19. If, at a spot designated as A, the sun rises in the east at 6 A.M. and reaches the point straight overhead at noon, at what time will it go behind a mountain range whose rim is 10 miles west of A and 2.5 miles higher than A?

20. How big an angle in the sky would be made by the earth, 8,000 miles in diameter, to an observer on the moon, 240,000 miles away from the earth's center?

21. The orbit of Venus is inside that of the earth, in nearly the same plane. When the angle between the lines of sight from the earth to Venus and to the sun respectively is greatest, it is 46°. Given that the earth is 93 million miles from the sun, how far from the latter is Venus?

22. A room's dimensions are 20.52 by 13.65 by 8.33 ft. Find the greatest distance between two points in the room.

23. Work Prob. 3, Exercise 37, by the methods of this exercise.

24. Work Prob. 6, Exercise 37, by the methods of this exercise.

25. Two buildings face each other across a span of 12 ft. In the course of a criminal trial a witness claimed to have seen, from a window in the lower floor of house A and through a window in the second floor of house B, the face of the defendant, C. If C was standing back of a desk 3 ft. wide, if the top of the glass in A was 7 ft. below the bottom of the glass in B, the latter being 3.5 ft. above the floor on which the accused man was standing, and if the latter was 5 ft. 3 in. tall, prove by trigonometry that the witness was a liar.

64. How we do it. Oblique triangles. The plot thickens when we come to the triangles which do not contain right angles. These so-called *oblique* triangles may be solved by breaking them up into right triangles, but that method is tedious and long. Fortunately, a couple of short cuts known respectively as *the law of sines* and *the law of cosines* enable us to telescope the solutions very neatly. Before we can add them to our mathematical repertoire, however, we'll need definitions for the sine and cosine of an obtuse angle.

Consider, then, the obtuse angle DAB (Fig. 32). We first produce the side AD through the vertex and drop the perpendicular BC from a point B upon it. The lengths AD, AB, and CB are taken as positive, and the length AC as negative (a natural choice, since AD and AC are oppositely directed). Then we make the following definitions:

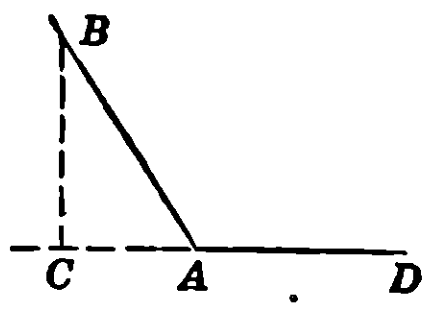

FIG. 32.

$$\sin DAB = \frac{CB}{AB}$$

and

$$\cos DAB = \frac{AC}{AB} \quad \text{where } AC\text{, and hence } \cos DAB\text{, is negative}$$

If we designate the angle DAB by θ, the acute angle CAB will evidently be $180° - \theta$, and we see that

$$(1) \qquad \sin \theta = \sin (180 - \theta)$$
$$(2) \qquad \cos \theta = -\cos (180 - \theta)$$

Relations (1) and (2) then enable us to express the sine and cosine of an angle less than 180° in terms of the corresponding functions of its supplement, so that they are true whether θ itself is obtuse or acute.

Now we're ready for the all-conquering laws of sines and cosines, which, between them, can subdue any solvable triangle. Consider, then, the triangles in Fig. 33.

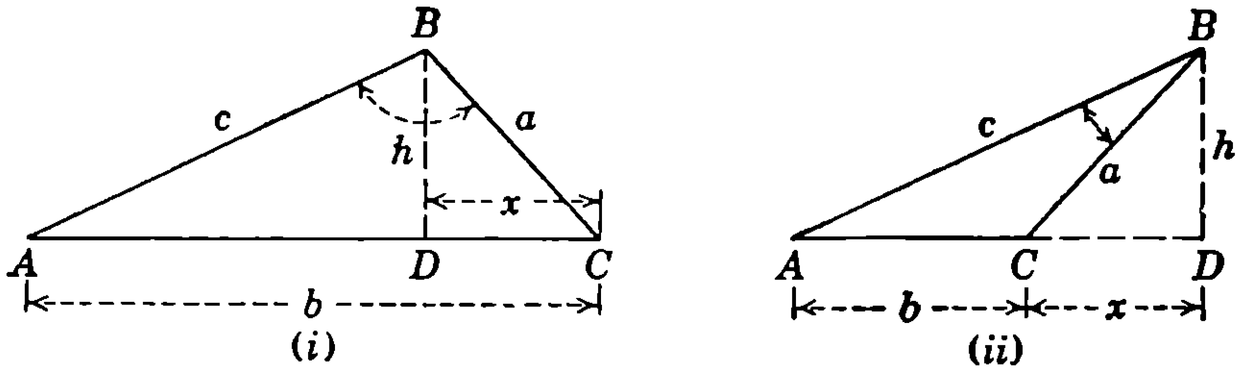

FIG. 33.

The important difference between triangles (*i*) and (*ii*) is that C is obtuse in the latter. In both cases the constructed line h represents the altitude from B to side b and, in the right triangle ABD, $\sin A = h/c$, so that $h = c \sin A$. Furthermore, in (*i*) $h = a \sin C$ and in (*ii*) $h = a \sin (180° - C)$, and since $\sin (180° - C) = \sin C$ by equation (1), $h = a \sin C$ in both (*i*) and (*ii*). Equating the two values of h, we have

$$c \sin A = a \sin C$$

Dividing both sides of the latter equation by $\sin A \sin C$, we get

$$\frac{c}{\sin C} = \frac{a}{\sin A}$$

Similarly, we can prove that

$$\frac{c}{\sin C} = \frac{b}{\sin B}$$

so that finally we arrive at the equation which states the law of sines in brief but adequate form

$$\text{(3)} \qquad \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

In words, the law may be stated thus:

The lengths of the sides of any triangle are to each other as the sines of the corresponding opposite angles.

The law of cosines can also be obtained by use of Fig. 33. In (*i*)

$$c^2 = h^2 + (AD)^2 = h^2 + (b - x)^2$$
$$= h^2 + b^2 - 2bx + x^2$$

But since $h^2 + x^2 = a^2$ and $x = a \cos C$, we have

$$(4)_1 \qquad c^2 = a^2 + b^2 - 2ab \cos C$$

which is the desired law as proved for triangle (*i*). From (*ii*) we get

$$\begin{aligned} c^2 &= h^2 + (AD)^2 = h^2 + (b + x)^2 \\ &= (h^2 + x^2) + b^2 + 2bx \\ &= a^2 + b^2 + 2ba \cos (180° - C) \end{aligned}$$

Now by (2), $\cos (180 - c) = - \cos C$, so that in this case also equation $(4)_1$ is obtained.

Evidently, since the side c is not essentially different from the other two sides, we can also conclude that

$$(4)_2 \qquad a^2 = b^2 + c^2 - 2bc \cos A$$

and

$$(4)_3 \qquad b^2 = a^2 + c^2 - 2ac \cos B$$

Taking the law of cosines out of this compact symbolic form and restating it in ordinary English, we find that:

The square of any side of a triangle is equal to the sum of the squares of the other two sides minus twice their product times the cosine of their included angle.

Sometimes it is more easily remembered in this form.

An interesting special application of the law of cosines is that in which the included angle is 90°, so that its cosine is zero. Then the square of the side opposite the right angle equals the sum of the squares of the other two sides, and this, come to think of it, is nothing more than the old familiar Pythagorean relation. But there is an objection to the use of the law of cosines in the proof of that gem from geometry. Can you see what it is?

Having added these two admirable laws to our triangle-solving equipment, let's see how well they meet the test. Consider, then, the different cases which will arise. Excluding the case of three known angles (a condition which defines the triangle's shape but not its size), there remain the following sets of conditions, each of which involves three known parts and is sufficient for a complete solution: (1) two angles and a side; (2) two sides and an angle opposite one of them; (3) two sides and their included angle; and (4) three sides. To save you from this terrible suspense, we'll state flatly and at once that the law of sines takes care of (1) and (2), while the law of cosines disposes of (3) and (4). What more could be asked of them? As a pair they meet all comers in the triangle world, and all we have to do now is to demonstrate their prowess in some sample numerical cases.

Case 1. *Two angles and a side.* Suppose $A = 26°$, $B = 46°$, and $a = 125.0$. Since the three angles of a triangle add up to 180°, we know that the angle $C = 180° - (26° + 46°) = 108°$. And to encourage a rock-bound habit which we cannot too strongly recommend—that of making a figure in each case to show what the letters represent—we'll include one here (Fig. 34). Note our adherence to several convenient practices already mentioned but worth repeating, to wit:

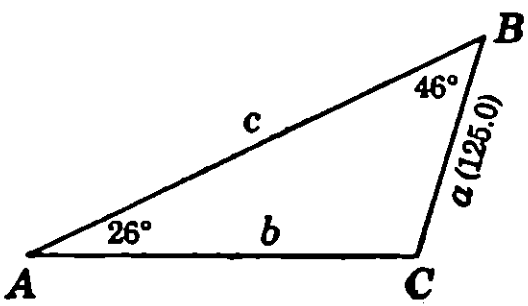

Fig. 34.

1. The figures are drawn *about to scale,* so that the eye can catch all the big bobbles and bungles in the answers.

2. Sides a, b, and c are opposite angles A, B, and C, respectively.

3. Only the given values are shown in the figure, so that it is not necessary to write: "Given, so and so; to find, thus and so." The figure takes care of that.

Next, we'll write down the law of sines and fill in the known values:

$$\frac{125}{\sin 26°} = \frac{b}{\sin 46°} = \frac{c}{\sin 108°}$$

If in the equations above we can find one which contains only one unknown, the rest is algebra, table using, and arithmetic. Two such equations are before our face, namely:

$$\frac{125}{\sin 26°} = \frac{b}{\sin 46°} \tag{5}$$

and

$$\frac{125}{\sin 26°} = \frac{c}{\sin 108°} \tag{6}$$

From (5) we get

$$b = \frac{125 \sin 46°}{\sin 26°} = \frac{(125)(0.7193)}{0.4384} = 205.1$$

Note that the quantity

$$\frac{125(0.7193)}{0.4384}$$

can be computed easily by means of logarithms. It will be understood henceforth that this method should be employed in the indicated computations wherever it is most convenient.

Similarly, from (6), since sin 108° = sin (180° − 108°) by (1),

$$c = \frac{125 \sin 108°}{\sin 26°} = \frac{125 \sin 72°}{\sin 26°} = \frac{125(0.9511)}{0.4384} = 271.2 \text{ (reasonable)}$$

Just to show how foolproof is this matter of choosing the right one from the redoubtable pair of triangle-solving laws, suppose we write down all three forms of the law of cosines and fill in the values given in the above problem. Each equation will turn out to have *more than one unknown*, so that it cannot be used. In other words, if in his blind groping among the available equations supplied by (3) and (4) the beginner comes across one which contains just one unknown, he may solve it with perfect assurance. He should understand, of course, that it will not be necessary to use this blundering hit-and-miss method if he learns how to diagnose each case and meet it in the direct and efficient way.

EXERCISE 40

In Probs. 1 to 10 find the value of each part not given.[1]

	A	B	C	a	b	c
1.	26°10′	35°	—	200	—	—
2.	42°30′	—	17°10′	—	12.5	—
3.	—	38°20′	61°40′	—	—	250
4.	56°20′	41°30′	—	11.20	—	—
5.	—	47°	81°13′	—	—	133.2
6.	32°11′	—	67°20′	—	21.21	—
7.	—	28°11′	49°40′	4.812	—	—
8.	17°48′	—	68°42′	—	48.91	—
9.	13°42′	62°13′	—	18.31	—	—
10.	53°11′	61°17′	—	—	23.46	—

[1] Exercises 40 to 43 involve purely formal triangle problems. Applications will appear in Exercise 44. It seems best to demand technical proficiency before introducing distracting elements.

Case 2. Two sides and an angle opposite one of them. Let the data be as follows: $a = 1{,}002$; $b = 2{,}240$; $A = 20°$. Drawing the figure (Fig. 35) conscientiously (and by that we mean with an at-

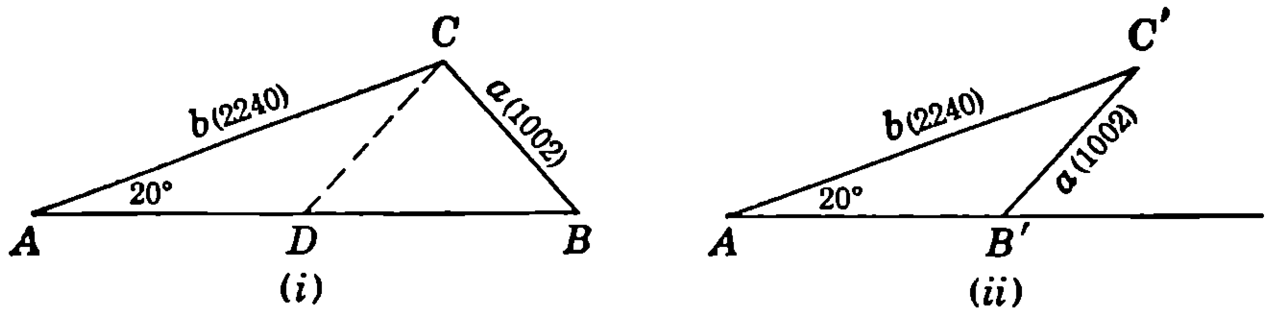

Fig. 35.

tempt to get the sides and angles at least roughly to the proper scale), we are at once struck with the fact that there are two triangles which meet all of the required conditions. In triangle (*i*) a second possible position for the side a is indicated by the dotted line CD, while (*ii*) is the triangle which places a in the position CD of (*i*). The changed values of C, B, and c thereby resulting are indicated by use of the letters C', B', and c'. Again referring to the law of sines, which we promised would work in the first two cases, we have

$$\frac{1{,}002}{\sin 20°} = \frac{2{,}240}{\sin B} = \frac{c}{\sin C}.$$

Evidently

$$\begin{aligned}\sin B &= \tfrac{2240}{1002} \sin 20° \\ &= \tfrac{2240}{1002} (0.3420) \\ &= 0.7645\end{aligned}$$

By interpolating in the tables we find that $B = 49°52'$ to the nearest minute—a result which applies by inspection to the B of (*i*) rather than the B' or (*ii*). Completing the solution of (*i*) we find that

$$C = 180° - (20° + 49°52') = 110°8'$$

so that

$$\frac{c}{\sin 110°8'} = \frac{1{,}002}{\sin 20°}$$

and

$$\begin{aligned}c &= \frac{1{,}002 \sin 110°8'}{\sin 20°} \\ &= \frac{1{,}002 \sin 69°52'}{\sin 20°} \\ &= \frac{1{,}002(0.9389)}{0.3420} \\ &= 2{,}751 \qquad \text{(reasonable?)}\end{aligned}$$

To get the angle B' of (*ii*) we note that, by (1),

$$\sin 49°52' = \sin (180° - 49°52') = \sin 130°8'$$

so that the value 130°8′ for B also meets the conditions of the equation. Clearly this is the B' of triangle (*ii*), which is also seen by the geometry of the situation in (*i*) to be the supplement of the angle BDC and hence of the equal angle DBC (or B). We then have $B' = 130°8'$, $C' = 180° - (20° + 130°8') = 29°52'$, and

$$\frac{c'}{\sin 29°52'} = \frac{1{,}002}{\sin 20°}$$

or

$$c' = \frac{1{,}002(0.4998)}{0.3420} = 1{,}464 \qquad \text{(reasonable?)}$$

Thus we see that in the case cited there are two different triangles which meet the original or given conditions. To show that this is not always the case when two sides and an angle opposite one are given we need only draw the triangles illustrating the following conditions:

(I) $b = 2{,}960,\ c = 2{,}240,\ B = 20°$
(II) $a = 1{,}000,\ b = 500,\ B = 70°$
(III) $c = 2{,}000,\ b = 3{,}000,\ B = 150°$

In each case it will be found convenient to place the given angle on the left with the unknown side horizontal and with the given side which is not opposite the given angle above and on the left. The triangles are shown in Fig. 36. The mere rough construc-

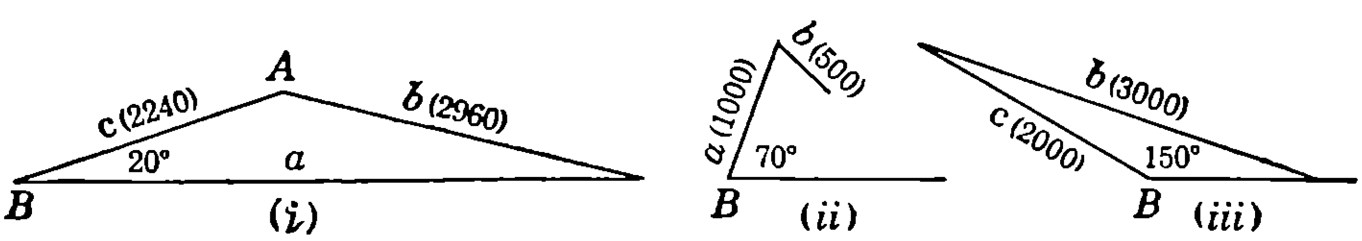

FIG. 36.

tion of the figures shows that there is one solution for each of the triangles *i* and *iii* and no solution of *ii*, since b is not long enough to reach the horizontal side. Nor would there be a solution of *iii* if b were not longer than c. Evidently a more precise mathematical test will be necessary *only* in those cases in which, after the figure is drawn with moderate care, with the given angle at the left of the base, it still remains doubtful whether the slanting side on the right will or will not reach down to horizontal line.

This doubt exists only when the given angle is acute, when the side opposite it is shorter than the other given side, and when it is neither obviously too short nor clearly long enough to reach down to the horizontal side. As an example of this somewhat rare case consider the conditions $A = 40°$, $b = 100$, $a = 640$ (Fig. 37). Applying the law of sines,

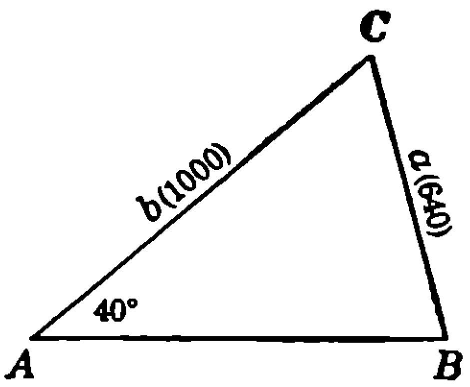

FIG. 37.

$$\sin B = \frac{1{,}000 \sin 40°}{640}$$
$$= \frac{1{,}000(0.6428)}{640}$$
$$= 1.004$$

Now a hasty inspection of the tables shows that there is no angle whose sine is greater than one (in case you did not already know it), so that there is no solution in this case. If, however, a had been given as 642.8, sin B would be 1 and B would be 90°, so that there would be a solution (a right triangle). If a were 643.0, sin B would turn out to be a value in the table (either directly or by interpolation) and the solution would be like that in our first example.

Because of the *possibility* of more than one solution, Case 2 is called the *ambiguous case.* But, as we have shown, the practical way to attack the problem is to *draw the figure first* before worrying about the number of solutions, and then, on the rare occasions when the conclusion seems in doubt, *don't worry even then* since the results will show up willy-nilly in the equations. This is one of the pleasant surprises of mathematics.

EXERCISE 41

In the following triangles, find the values of the parts not given, including those for both solutions when there are two. In some cases there are no solutions.

	A	*B*	*C*	*a*	*b*	*c*
1.	42°40′	—	—	296.0	358.0	—
2.	116°20′	—	—	0.0381	—	0.0562

	A	B	C	a	b	c
3.	—	63°11′	—	2,978	2,031	—
4.	—	—	58°28′	—	7,384	6,931
5.	123°14′	—	—	8,964	9,221	—
6.	—	—	43°30′	1,000	—	688.4
7.	—	67°55′	—	—	843.1	704.2
8.	20°47′	—	—	132.0	—	303.0
9.	45°	—	—	82	85	—
10.	—	—	47°20′	205.4	—	150.0

Case 3. *Two sides and their included angle.* Let the data be as shown in Fig. 38. Here the law of sines is not immediately useful, and the law of cosines comes into its own. It should not be hard to see that the most promising one of its three forms is (4_1), since it involves the known angle C. Applying this and substituting the known values, we have

$$c^2 = (0.3940)^2 + (0.4373)^2 - 2(0.3940)(0.4373)(0.3907)$$

This is as good a place as any to make a somewhat painful admission. In our previous gloating over the infallible ability of the sine-and-cosine-laws team to meet all triangle comers, we neglected to mention the fact that the law of cosines has objectionable features. Its chief defect lies in its contrariness in the matter of logarithms. For instance, it would be easy to compute c from the above equation by means of logarithms if the plus and minus signs were omitted, and only multiplication, division, and the extraction of roots were indicated. As it is, we must return to the antilogarithm in each of the three terms which make up the value of c^2, and the computation is tedious. Because of this, alternative methods of solution have been devised which use logarithms more efficiently. However, in a short survey of the subject it seems advisable to omit these refined methods, since the law of cosines *will* do the job in its own awkward and cumbersome way.

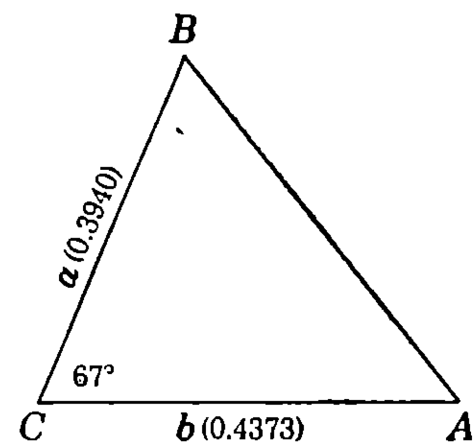

FIG. 38.

Calculating by means of logarithms the values of each of the three parts of c^2, and retaining only as many significant figures our table allows, we get

$$\begin{aligned} c^2 &= 0.1552 + 0.1912 - 0.1346 \\ &= 0.2118 \end{aligned}$$

and

$$c = 0.4602$$

again found by the use of logarithms.

The stigma of logarithmic inefficiency does not rest upon the law of sines, so that we frequently turn to it to finish a problem just as soon as it can be used. In the present case, and by this admirably compact law, we find that

$$\begin{aligned} \sin A &= \frac{0.3940 \sin 67^\circ}{0.4602} \\ &= \frac{(0.3940)(0.9205)}{0.4602} \\ &= 0.7880 \end{aligned}$$

and

$$A = 52^\circ 0' \quad \text{(to the nearest minute)}$$

Hence,

$$\begin{aligned} B &= 180^\circ - (52^\circ 0' + 67^\circ) \\ &= 61^\circ 0'. \end{aligned}$$

EXERCISE 42

Find the unknown side in each triangle in Probs. 1 to 10.

1. $A = 20^\circ$, $b = 25$, $c = 36$.

2. $B = 49^\circ 10'$, $a = 202$, $c = 305$.

3. $C = 79^\circ 22'$, $a = 201.4$, $b = 328.6$.

4. $B = 129^\circ 31'$, $a = 0.2914$, $c = 0.3528$.

5. $A = 69^\circ 14'$, $b = 1{,}800$, $c = 2{,}002$.

6. $C = 160^\circ 40'$, $a = 301$, $b = 402$.

7. $A = 90^\circ 42'$, $b = 0.0028$, $c = 0.0037$.

8. $C = 157^\circ$, $a = 222$, $b = 333$.

9. $B = 35'$, $a = 10.0$, $c = 11.0$.

10. $C = 120^\circ$, $a = 3{,}000$, $b = 5{,}000$.

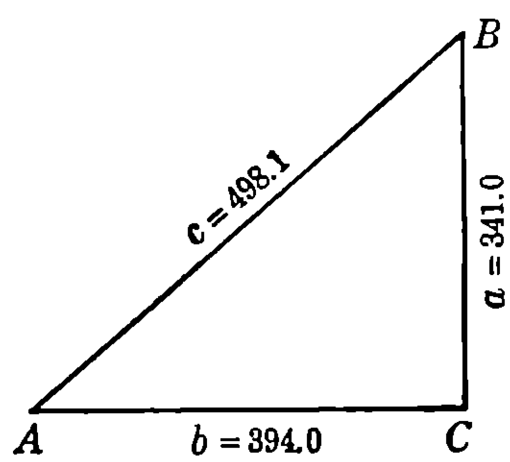

FIG. 39

Case 4. *Three sides.* Let $a = 341.0$, $b = 394.0$, and $c = 498.1$, as in Fig. 39.

Again the law of cosines works in its tediously computed way. Starting with

$$a^2 = b^2 + c^2 - 2bc \cos A$$

and solving for cos A, we get

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$
$$= \frac{(394.0)^2 + (498.1)^2 - (341.0)^2}{2(394.0)(498.1)}$$

which turns out, with the help of logarithms and a little side figuring, to be

$$\frac{155{,}200 + 248{,}100 - 116{,}300}{392{,}500} = \frac{287{,}000}{392{,}500}$$
$$= 0.7313$$

Hence

$$A = 43°0'$$

Similarly, from the equations

$$\cos B = \frac{a^2 + c^2 - b^2}{2ac}$$

and

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

we find that $B = 52°0'$ and $C = 85°0'$. Adding, we get

$$A + B + C = 43° + 52° + 85° = 180°$$

and this check makes us confident that our results are fairly accurate.

Take special note of that adverb "fairly." We cannot of course expect accuracy to within a fraction of a minute without more extensive trigonometric tables. Our emphasis here is on principles—not the technical refinements requiring better tools. And never forget that those principles are far-reaching in their application, ranging from the petty problems of local surveying to the measurements of astronomical distances too vast for human comprehension. And remember also that these unthinkable distances, even though they defy the grand sweep of the imagination, have already yielded on paper to the simple magic of trigonometry.

EXERCISE 43

Assume that the sides have *exactly* the values indicated in Probs. 1 to 10, and get the angles to the nearest minute.

	a	*b*	*c*
1.	3	4	5
2.	4	5	6
3.	5	6	7
4.	6	7	8
5.	7	8	9
6.	5	12	13
7.	8	15	17
8.	7	24	25
9.	25	35	50
10.	2	4	5

EXERCISE 44

Miscellaneous Problems

1. A farmer's house A and his windmill B are separated by his barn. In order to determine the length of a straight pipe line connecting A and B, the farmer locates a point C 150 ft. from A and 125 ft. from B from which both are visible, and finds that the angle ACB is 52°. Using the above data, find the length of the pipe line.

2. Two observers, A and B, are 14.38 miles apart. A balloon is directly over the line AB. From A, the angle of elevation of the balloon is 22°35′ and from B it is 19°14′. How high is the balloon above the ground?

3. Two cars which left a highway junction simultaneously were 83.0 miles apart at the end of 3 hr. If they traveled at 50.0 and 60.0 miles per hour, respectively, at what angle did the highways intersect?

4. Two highways designated as AB and CD intersect at O so that angle AOC is 60° and angle COB is 120°. Two cars leave the intersection O simultaneously at 40 and 50 miles per hour respectively. How far apart are they at the end of an hour if their respective directions are OD and OB?

5. Find the answer to Prob. 4 if the respective directions are OC and OB.

6. Aviator A left an air field at 2 P.M., traveling in a straight line at 115 miles per hour. B left the field at 3 P.M. at 200 miles per hour. At 4 P.M. A learns by radio that B has just reached a large city whose lights he can see. If the line from A to this city makes an angle of 20° with the direction from which A started, how far is he from the city? If there are two solutions, which is the reasonable one?

7. Two points on the earth's equator are at longitudes 0° and 85° respectively. If the equatorial radius of the earth is 3,963 miles, how far apart are the points?

8. If two points on the equator are 2,140 miles apart, what is their difference in longitude?

9. An observer on a mountaintop finds that the angle AOB, in a vertical plane, made by lines from his position O, to two airplane beacons, A and B, is equal to 38°41′. He knows that his distance from A is 27 miles and that A and B are 82 miles apart. How far is he from B?

10. A meteor flash is observed simultaneously by A and B, who are in a north-south line 27 miles apart. To both observers the flash appears over the due-south point, but to A its angle of elevation is 47°, while to B the corresponding angle is 58°. How high was the meteor when seen? (Neglect the spherical shape of the earth.)

11. Three towns A, B, and C are located at the vertices of a triangle whose angles at A and B are 32° and 46°, respectively, and whose sides are straight highways. The sheriff at B is notified that an escaped convict has just been seen leaving A for C in a car capable of traveling 65 miles per hour. If A is 75 miles from B, how fast must the sheriff travel in order to intercept the convict at C?

12. Referring to Prob. 11, a crossroad leaves the highway connecting B and C at a point D 15 miles from B, and intersects AC at the point E, 12 miles from C. If the sheriff takes this road, how fast must he travel in order to intercept the convict at E?

13. A was 30 miles due north of B. They both observed the same meteor, which appeared at point C in the sky and disappeared at point D, farther south. The line CD was directly over the north-south line which passed through A and B. The angles of elevation of C and D were respectively 73° and 21° from A and 90° and 26° from B. If they both estimated the duration of the flash at 3 sec., how fast was the meteor traveling?

14. In order to fit a business building to the available lot space it is necessary to build it in the form of a quadrilateral with inside corner angles of 100°, 51°, 80°, and 129° in the clockwise order from above. The sides are 98 ft., 87 ft., a, and b, in the same order, the one 98 ft. long being between the 100° and 51° angles. Find a and b.

15. Work out a formula for the area of a triangle in terms of two sides a and b and their included angle C.

16. Work out a formula for the area of a triangle in terms of two angles A and B and their included side c.

17. To get the area of a large field in the form of a quadrilateral with vertices at A, B, C, and D, respectively, a surveyor placed his transit at a point O near the center. An assistant held a rod at each vertex in succession, and by measuring the angular distance between marks on the rod the surveyor was able to determine the distance to the rod directly. He thus found the lengths OA, OB, OC, and OD to be 347, 298, 402, and 351 yd., respectively. At the same place he found the angles AOB, BOC, COD, and DOA to be 97°, 103°, 81°, and 79°, respectively. Find the area, using the result obtained in Prob. 15.

65. The field broadens. Important as is the problem of the solution of triangles, with all its implications, it does not by any means exhaust the human and intellectual service possibilities of trigonometry. In fact, the triangle problem merely started this amazing mathematical creation on its useful career as the handmaiden of engineering and the physical sciences. There were discovered a great many relations among the so-called *trigonometric functions* (sine, cosine, tangent, etc.) which proved to have unexpected applications to the problems of astronomy, physics, chemistry, and in fact of almost every exact science known today. A fairly complete discussion of these other uses, however, would automatically expand this chapter into a book. To avoid this dire possibility, and to stick to our original purpose, we'll have to indicate rather than explore the rich field of trigonometric relations.

We should, nevertheless, be playing a mean trick upon the inquiring student not to explain to him briefly the more general definitions of the trigonometric functions. For these are the definitions which apply to angles of any size whatever, and not just the pigmies having less than 180° to their names such as we have met up to date. Furthermore, all the interesting relations which turn up in connection with the definitions applying to a restricted angular range still hold when we get in an expansive mood and deal with, say, an angle having one million degrees. And don't

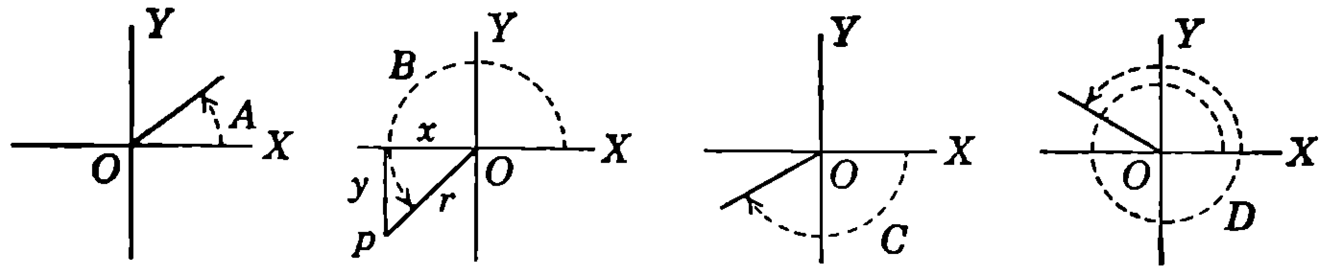

FIG. 40.

ever think that such an angle is never met outside of a book. How many degrees do you suppose the blade of an airplane propeller could reel off in a minute? This should keep you busy while we set the stage for a final scene.

Definition. *An angle is in* ***standard position*** *with respect to a pair of rectangular coordinate axes if its initial side is on the positive X axis and its vertex is at the origin.*

In Fig. 40, for instance, angles A, B, C, and D are all in standard position.

Assuming, now, that the meaning of "standard position" is clear, we hereby formally introduce to you trigonometric functions which apply to angles of any and all sizes.

Designate as θ *a general angle placed in standard position. Let* P *be a point chosen at random on its terminal side, and let* x, y, *and* r *be respectively the abscissa, ordinate, and radius vector of* P, *where the* ***radius vector*** *is the distance from the origin to* P. *Let the signs of the numbers represented by* x *and* y *be governed by the usual rules for the coordinate system, while* r *is positive* ***always.*** *Then*

$$\sin\theta = \frac{\text{ord.}}{\text{rad.}}\left(\frac{\text{ordinate of } P}{\text{radius vector of } P}\right) = \frac{y}{r}$$

$$\cos\theta = \frac{\text{abs.}}{\text{rad.}} = \frac{x}{r}$$

$$\tan\theta = \frac{\text{ord.}}{\text{abs.}} = \frac{y}{x}$$

$$\cot\theta = \frac{\text{abs.}}{\text{ord.}} = \frac{x}{y}$$

$$\sec\theta = \frac{\text{rad.}}{\text{abs.}} = \frac{r}{x}$$

$$\csc\theta = \frac{\text{rad.}}{\text{ord.}} = \frac{r}{y}$$

With attention, then, to these definitions, and recalling the all-important theorem from geometry to the effect that

$$x^2 + y^2 = r^2$$

the reader should have no trouble whatever in proving for himself, by mere substitution of the letter ratios for the functions, the eight so-called *fundamental relations of trigonometry*:

$$\sin\theta \csc\theta = 1 \quad (1)$$
$$\cos\theta \sec\theta = 1 \quad (2)$$
$$\tan\theta \cot\theta = 1 \quad (3)$$
$$\tan\theta = \frac{\sin\theta}{\cos\theta} \quad (4)$$
$$\cot\theta = \frac{\cos\theta}{\sin\theta} \quad (5)$$
$$\sin^2\theta + \cos^2\theta = 1 \quad (6)$$
$$1 + \tan^2\theta = \sec^2\theta \quad (7)$$
$$1 + \cot^2\theta = \csc^2\theta \quad (8)$$

It should not be difficult to imagine, moreover, that the deductions from these simple statements carry us into a maze of further relations which is practically endless. We can sometimes take a complicated-looking mess of trigonometric functions which appears to be unmanageable by anyone, and, by virtue of relations (1) to (8), strip it and trim it down until the overdressed fraud collapses into a mere mathematical nobody. For example,

$$\cos A \sqrt{1 - \sin^2 A} - (1 - \sec^2 A) \cos^2 A \tan A \cot A$$

is simply a mystifying and high-brow way of writing the digit 1. Not that it would be fair to expect the beginner to get that information offhand out of equations (1) to (8), even though it really is tied up there. The problem of showing that two utterly unlike-looking trigonometric expressions are "brothers under the skin" is known as *proving identities*, and is regarded by many freshmen as the worst form of indoor sport. The reason for inflicting it is that it is supposed to help fix the fundamental relations in mind and thus to give more facility in dealing with the trigonometric expressions which really do come up in engineering practice and scientific investigations. Since, however, we are not trying to prepare the reader for those occupations, we'll let him off lightly in our forthcoming exercise. As we said before, anyway, we merely wished to point out in this article the first pathway leading away from the simple but important triangle problem to the wider field whose borders we have barely touched.

EXERCISE 45

Prove the following identities:

1. $\tan \theta + \cot \theta = \sec \theta \csc \theta$.

Proof: Left side $= \dfrac{\sin \theta}{\cos \theta} + \dfrac{\cos \theta}{\sin \theta}$

$$= \frac{\sin^2 \theta + \cos^2 \theta}{\cos \theta \sin \theta} = \frac{1}{\cos \theta \sin \theta}$$

$$= \frac{1}{\cos \theta} \cdot \frac{1}{\sin \theta}$$

$= \sec \theta \csc \theta =$ right side.

2. $\cos \theta = \pm \sqrt{1 - \sin^2 \theta}$.

3. $\tan \theta = \pm \sqrt{\sec^2 \theta - 1}$.

4. $\sec \theta \sin \theta = \tan \theta$.

5. $\csc \theta \cos \theta = \cot \theta$.

6. $\sec \theta \csc \theta (\sin \theta + \cos \theta) = \sec \theta + \csc \theta$.

7. $\sin \theta \cos \theta (\sec \theta + \csc \theta) = \sin \theta + \cos \theta$.

8. $(\tan \theta + \cot \theta)^2 = \sec^2 \theta + \csc^2 \theta$.

9. $(\sin \theta + \cos \theta)^2 = 1 + 2 \sin \theta \cos \theta$.

10. $(\sec \theta + \csc \theta)^2 = (\tan \theta + \cot \theta)^2 + 2 \sec \theta \csc \theta$.

11. $\sin \theta \cos \theta \tan \theta \cot \theta \sec \theta \csc \theta = 1$.

12. $\sec^2 \theta + \csc^2 \theta = \sec^2 \theta \csc^2 \theta$.

13. $\cos^2 \theta - \sin^2 \theta = 2 \cos^2 \theta - 1$.

14. $(1 + \tan^2 \theta) \cos^2 \theta = 1$.

15. $\sec \theta - \cos \theta = \tan \theta \sin \theta$.

16. $\dfrac{\cos^2 \theta}{1 - \sin \theta} = 1 + \sin \theta$.

17. $(\cos \theta + \sin \theta)^2 + (\cos \theta - \sin \theta)^2 = 2$.

18. Using Prob. 12, find three pairs of rational numbers whose products and sums are equal.

19. Referring to the fundamental relations, find six trigonometric expressions which can be replaced by 1.

20. Show that the identities in Probs. 1 to 10 hold when $\theta = 30°$.

CHAPTER VIII

NEW LIFE IN OLD FIGURES

66. Into geometry with algebra. In Chap. IV we explained the bright idea of René Descartes which, by means of a simple device for picturing the equation, united the two great fields of algebra and geometry. In that chapter, in addition to the development of the picture machinery itself, the chief emphasis was placed upon the use of that machinery as an aid to algebra. For instance, the graphs proved to be helpful in the solution of simultaneous equations, and also in finding the roots of third- and higher-degree equations. Furthermore, the essential scheme involved served to make an effective pictorial representation of statistical data in a form which is highly useful in a thousand fields, and which sometimes leads to discoveries of the greatest importance.

In *analytic geometry*, the natural child of Descartes' simple but revolutionary idea, the emphasis is largely reversed. Instead of using the picture to study the equation (an algebraic end) we use the equation to study the picture (a geometric problem). For, fundamentally, analytic geometry *is* geometry, or a study of the properties of figures in space. Its difference from the old love of Euclid and the Greeks lies in its utterly new, or algebraic, method. The conclusions of geometry are likely to be intuitive guesses followed by proofs which are easy to understand but hard to produce for the first time. The conclusions of analytic geometry, on the other hand, are forced upon us by the nature of routine operations, and fall out in great and unearned batches. For instance, the pure geometer must study separately each of an endless array of figures to determine whether certain lines in them are *concurrent*, or intersect in one point. The new geometry furnishes one simple method for *all* such cases, thus cutting down the amount of necessary *original* thinking from infinity to schoolboy dimensions. While it may seem a shame to wipe out in one mechanical operation a million or more chances for individual brilliance, it really is a big help to science, since there is always plenty of work for the released brain power. And the gigantic lift to pure geometry illustrated in this case is just one sample of the work which can be done by the new method. For the old familiar triangles, circles, and curves made by falling stones and circling moon (for example), when made subject to attack by the powerful tools of algebra, almost immediately gave up secrets which were never suspected by the keen but mathematically unprepared Greeks.

67. Before the fusion. When the human race developed sufficiently to derive most of its living from the land, surface measurements became necessary and geometry was born. Probably it grew most vigorously in the lower valley of the Nile, where annual floods, obliterating all landmarks, made necessary the yearly survey. The geometry thus developed consisted largely (and naturally enough) of formulas for finding areas. These formulas, probably guessed at intuitively and then cut and patched into shape by trial and error, were valued less for scientific accuracy than for practical usefulness. Probably the value $3r^2$ for the area of a circle of radius r was sufficiently accurate for their purpose, and anyone who announced that this area was more nearly equal to $22r^2/7$ would undoubtedly have been considered, in the earlier days, a rather troublesome hairsplitter.

According to Smith,[1] the Greek Thales seems to have been the first to use the demonstrative method for establishing the truth of a proposition. About two hundred years later, Euclid, starting with a set of axioms, or unproved but plausible statements, worked out a systematized demonstrative geometry which in some respects has not been improved upon to this day. The whole imposing array of his theorems was deduced from a few elementary axioms. So elegant and logical were his methods that they strongly appealed to the Greek mind, and the geometry of Euclid flourished mightily in the land. Nevertheless, the Greeks' unfortunate lack of interest in algebra and arithmetic caused them to miss completely some of the more powerful geometric aids. Though there is a somewhat cumbersome and confusing suggestion of the use of a coordinate system, the effective use of this

[1] D. E. Smith, *History of Mathematics*, Vol. II, p. 271.

revolutionary tool was reserved for Descartes and his followers.

Some of the most elegant and useful results of the old geometry had to do, as might be expected, with areas and volumes. The fact that the surface area of a sphere is *exactly* four times the area inside a circle with the same diameter is an example of the surprising and intriguing simplicity of these results. Another conclusion of the same type is the fact that the volume and surface of a sphere are two-thirds of the volume and surface, respectively, of a cylinder circumscribed about the sphere. The discovery of this truly remarkable relation is said to have so elated Archimedes that he had the figure relating to it inscribed upon his tomb.

68. Formulas concerning distances. At this point the reader should take time out to restudy Art. 37. As a test of his preparation, he might draw the rectangular axes and locate on the coordinate plane the points (2,3), (−1,4), (−3,−5), and (4,−6), which should be found to lie in the first, second, third, and fourth quadrants, respectively.

As the first step in the main business of this chapter, we shall derive formulas for: (*a*) the distance between two fixed points; (*b*) the distance from a point to a line; (c) the coordinates of a point which lies any given fraction of the way from one point to another; and (*d*) some definitions and relations concerning the slant of a line with the horizontal (technically called the *slope*). With these relatively simple tools we'll show how to obtain easily and directly results which are difficult to prove by the methods of deductive geometry.

First we need a formula for the distance between two points, say P_1 and P_2, whose coordinates it will be convenient to call (x_1,y_1) and (x_2,y_2). (The subscripts indicate that the letters represent fixed distances, while plain x and y always stand for the coordinates of a point which is movable in the problem.) Three cases then appear.

1. Let the points be in a vertical line, so that $x_1 = x_2$. For convenience, let y_2 be the ordinate of the upper point. Then the difference

(1) $\boldsymbol{y_2 - y_1}$, **or upper ordinate minus lower ordinate**

will give as a positive number the distance between the two points. This is shown in the three parts of Fig. 41. Remembering that the ordinate of a point below the X axis is negative, we see that $-y_1$ is positive in Figs. (*ii*) and (*iii*) of Fig. 41, while y_2 is negative in (*iii*), so that in each case the difference $y_2 - y_1$ will give the positive length P_1P_2.

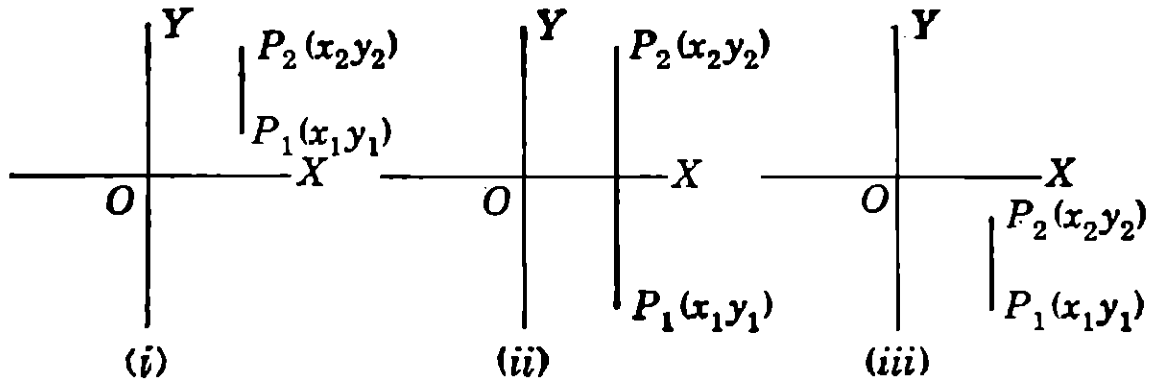

FIG. 41.

2. Let the points be in a horizontal line, with $y_1 = y_2$. Then, by an argument similar to that in (1), the positive length P_1P_2 is given by the formula

(2) $\quad x_2 - x_1$, **or right abscissa minus left abscissa**

regardless of the position of the points with reference to the axes.

3. Let the points be in a line oblique to the axes, as shown in Fig. 42. Drawing the right triangle in which P_1P_2 is the hypotenuse and the sides are parallel to the coordinate axes, noting that $DP_2 = y_2 - y_1$ and $P_1D = x_2 - x_1$ by the results in (1) and (2) above, and using the Pythagorean relation, we have $(P_1P_2)^2 = (P_1D)^2 + (DP_2)^2$, or

(3) $$P_1P_2 = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Evidently, since $(x_2 - x_1)^2 = (x_1 - x_2)^2$, it makes no difference which point we honor with the designation P_1. Hence we can say in words that *the square of the distance between two points is equal to the square of the difference of the abscissas plus the square of the difference of the ordinates.* But for heaven's sake *do not* use this formula when the points are in a vertical or horizontal line. Remember under such conditions those simple but useful phrases, *upper ordinate minus lower ordinate,* and *right abscissa minus left abscissa.*

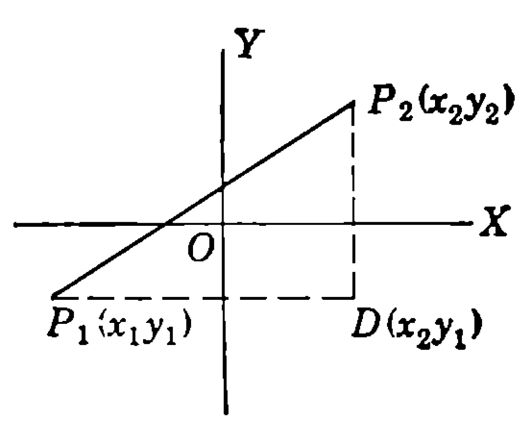

FIG. 42.

The same phrases prove their versatility and usefulness when we speak of the distance from a point to a vertical or horizontal line. Consider, for example, the various distances shown in Fig. 43. There the vertical line has the equation $x = 3$ since the abscissa (or x) of every point on it is 3, while the ordinate (or y) is not mentioned and hence not restricted. Similarly, the equation of the horizontal line is $y = -2$. Then one horizontal distance is shown as $x_1 - 3$ instead of $3 - x_1$ because x_1 is the right ab-

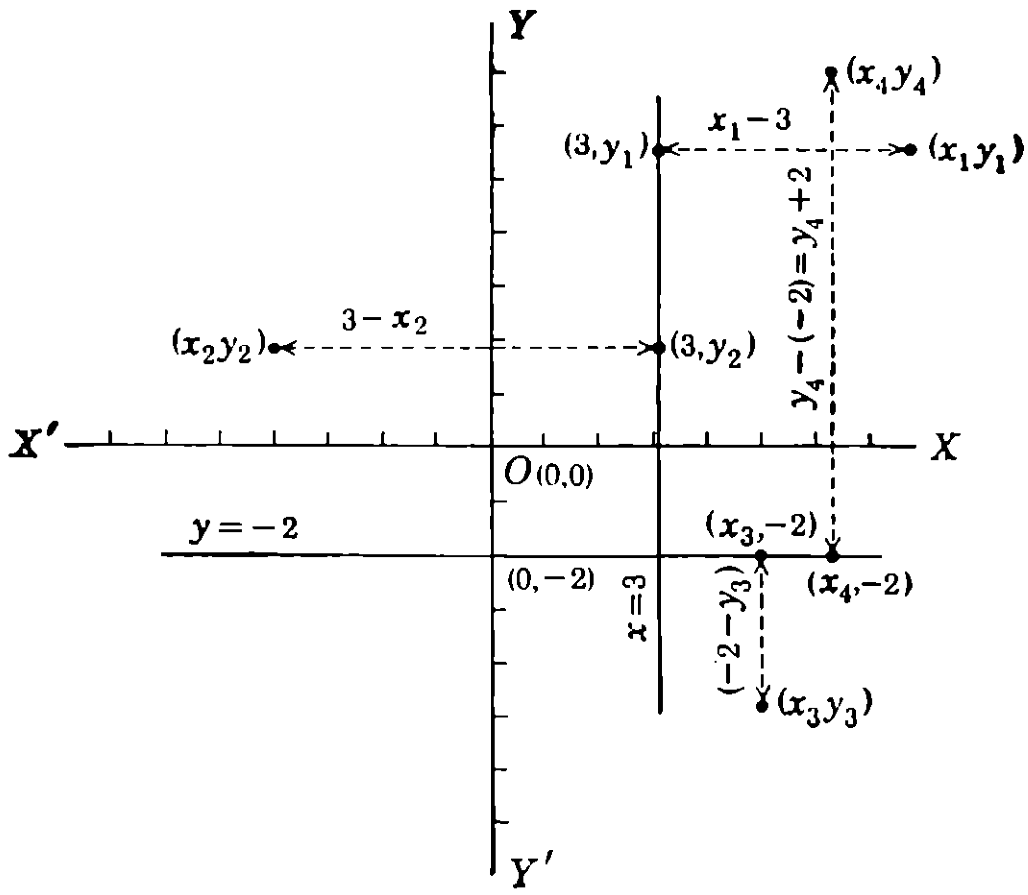

FIG. 43

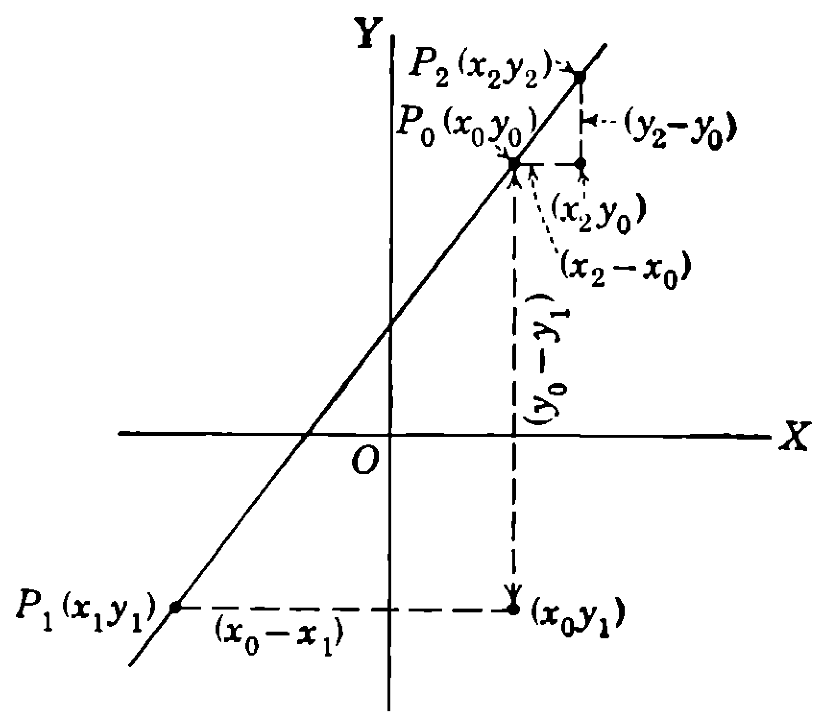

FIG. 44.

scissa, while another distance is $3 - x_2$ because here 3 is at the right. The point is, of course, that with the indicated order of subtraction the values for the distances will always come out as positive numbers.

The obvious generalization of the preceding results is a formula for determining the distance from a fixed point to an oblique line.

Since, however, such a formula involves complications for which we're not yet prepared, it will be strategic to get along temporarily without it.

Next on the program comes the matter of a point dividing a line segment in a given ratio. Suppose, for instance, we need the coordinates (x_0, y_0) of the point P_0 dividing the segment P_1P_2 so that $P_1P_0/P_0P_2 = r$, where r is a given number (see Fig. 44). From the similar triangles, we get the proportion

$$\frac{P_1P_0}{P_0P_2} = \frac{r}{1} = \frac{x_0 - x_1}{x_2 - x_0}$$

Solving the latter equation, we find that

$$x_0 = \frac{x_1 + rx_2}{1 + r}$$

Similarly, we find that

$$y_0 = \frac{y_1 + ry_2}{1 + r}$$

These results are probably not worth remembering as formulas, since on the rare occasions when they are needed one should be able to deduce the coordinates of the required point by means of similar triangles. For the special case in which P_0 is halfway between P_1 and P_2, however, they are important and often used. Then $r = 1$, and we get the useful *mid-point* formulas:

$$(4) \qquad x_0 = \frac{x_1 + x_2}{2} \qquad y_0 = \frac{y_1 + y_2}{2}$$

EXERCISE 46

Find the distances between the following points, or points and lines.

1. $(-2,3)$ and $(11,3)$.

2. $(-2,-7)$ and $(-2,-1)$.

3. $(3,-2)$ and $x = 3$.

4. $(-5,4)$ and $x = 10$.

5. $(2,-4)$ and $y = -7$.

6. The point $P(x,y)$ in (quadrant) I and $y = -5$.

7. The point $P(x,y)$ in II and $x = 6$.

8. The Y axis and point $P(x,y)$ in I.

[If you have used Formula (3) in any of the above problems, you should throw away your paper and work them again, using (1) and (2) *only*.]

Find the length of the line connecting the pair of points indicated in each of Probs. 9 to 16.

9. (3,2), (3,12). **10.** (2,4), $(-12,4)$.

11. $(-8,-1)$, $(-2,-1)$. **12.** $(-3,-10)$, $(-3,-1)$.

13. (2,1), (8,9). **14.** $(-8,-6)$, (4,10).

15. $(-3,-4)$, $(-15,1)$. **16.** $(4,-4)$, $(-4,8)$.

Find the distance between the point and line indicated in each of Probs. 17 to 24.

17. (12,7), $x = 3$. **18.** $(3,-4)$, $x = -2$.

19. $(-7,2)$, $x = 5$. **20.** $(-3,4)$, $x = -10$.

21. (3,4), $y = 16$. **22.** $(5,-2)$, $y = -8$.

23. $(-2,-5)$ $y = 8$. **24.** $(0,-12)$, $y = -3$.

Find the area of the triangle whose vertices are indicated in each of Probs. 25 to 28. NOTE: The area of a triangle is $\frac{1}{2}$ (base)(altitude).

25. (0,0), (12,0), (4,7). **26.** $(-2,3)$, (4,3), (0,9).

27. (3,2), (3,10), $(-6,1)$. **28.** $(-2,-3)$, $(-2,13)$, (7,7).

In each of Problems 29 to 34, determine whether the triangle whose vertices are as indicated is (*a*) right, (*b*) isosceles, (*c*) right and isosceles, or (*d*) scalene.

29. $(-1,-2)$, (3,1), $(-4,2)$. **30.** $(1,-2)$, (5,1), $(-3,1)$.

31. (2,3), $(-1,4)$, $(3,-2)$. **32.** $(3,-2)$, (4,3), $(-2,-1)$.

33. (5,2), $(3,-4)$, $(12,-7)$. **34.** $(-6,3)$, $(-1,-3)$, (5,2).

35-42. Find the midpoints of the segments designated in Probs. 9 to 16.

43. Find the coordinates of a point which is three-fifths of the way from (7,2) to $(3,-6)$.

44. Find the coordinates of a point on a line through the points $A(2,0)$ and $B(5,-3)$ which is not between A and B and is three times as far from B as from A.

69. Formulas concerning slopes. Still another tool needed in analytic geometry is a device to indicate in some manner the angle which a line drawn in the coordinate plane makes with the X axis. To this end we shall define the *inclination* of a line that crosses the X axis as the *particular one of the four angles it makes with the X axis which is above the axis and to the right of the line.* Thus in Fig. 45, the inclination of AB is the angle a, and of MN, the angle b. *If a line is parallel to the X axis its inclination is zero.*

Connected with every line there is a numerical constant which has proved to be very useful. It is called the *slope* of the line and is defined as the *tangent of the line's inclination.* Its convenience in analytic geometry arises largely from the fact that it can be expressed in terms of the coordinates of two points on the line. For example, in Fig. 45 it is obvious that the angle DP_1P_2 equals the angle a, since they are exterior-interior angles of parallel lines. Hence, if we let m be the slope of AB, then $m = \tan a = \tan (DP_1P_2)$ = (side opposite DP_1P_2)/(side adjacent to DP_1P_2) = $DP_2/P_1D = (y_2 - y_1)/(x_2 - x_1)$. In words:

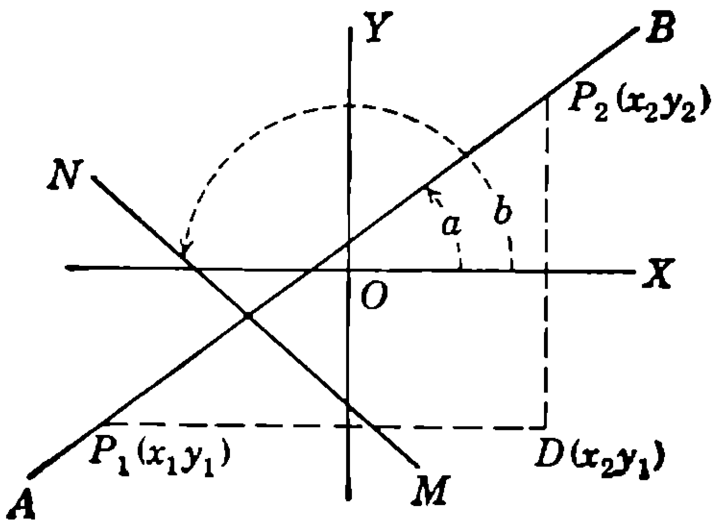

FIG. 45.

The slope of a line through two given points is the ratio of the difference of their ordinates to the difference of their abscissas, where both differences are taken in the same order.

Thus

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{y_1 - y_2}{x_1 - x_2} \tag{1}$$

Those who remember, probably not too fondly, their encounters with parallel and perpendicular lines in the high-school geometry, and who recollect in particular how the author, time and again, would haul out his hard working "supposer" and blithely suppose a couple of lines to be parallel or perpendicular when they obviously were anything but that in the figure, only to admit later that he had made a mistake, or else, on the other hand, had just deliberately drawn the thing all wrong—in short, those who have studied geometry and haven't got lost in the grammar of this sentence, will certainly appreciate analytic geometry's direct and simple approach to this usually troublesome matter. For it furnishes us with short and easily applied tests to tell whether two lines with given slopes are parallel, perpendicular, or askew to each other. In the corresponding cases the slopes are equal, negative reciprocals, or neither, and that's all there is to it.

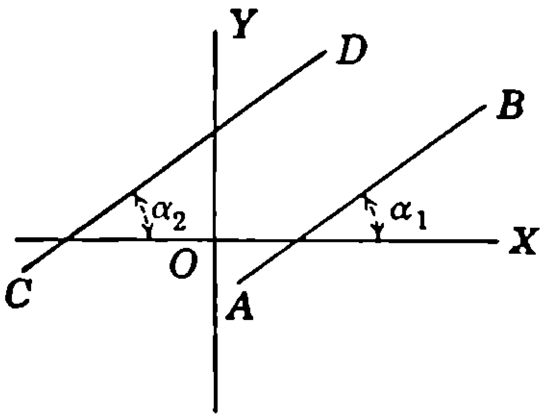

FIG. 46.

Suppose, for instance, that the lines AB and CD in Fig. 46 are parallel (as we have really tried to draw them). Then their inclinations, α_1 and α_2, are equal, being exterior-interior angles of parallel lines, and hence their slopes—the tangents of equal angles—are equal. On the other hand, if m_1, the slope of AB, equals m_2, the slope of CD, then $\tan \alpha_1 = \tan \alpha_2$ and hence $\alpha_1 = \alpha_2$, since the inclination of a line cannot exceed 180°. Therefore the lines are parallel, by the theorem that *if the exterior-interior angles of two lines crossed by a transversal are equal, the lines are parallel.* Here you may detect an indirect admission that we are considerably indebted to the old geometry after all—which is just as well, in view of some mistaken inferences you may have drawn from the remarks of the preceding paragraph. Our point is that, building on the hard labors of the geometer, we have reached a conclusion which may be stated concisely thus:

(2) **Two lines are parallel if and only if their slopes are equal.**

We come next to the interesting and less obvious proposition about the slopes of two perpendicular lines. It will be recalled (maybe) that we claimed these slopes to be *negative reciprocals* (and by that we mean a pair of numbers, such as $\frac{3}{2}$ and $-\frac{2}{3}$ or -7 and $\frac{1}{7}$, whose product is -1). To make good our assertion, we refer to the perpendicular lines AB and CD in Fig. 47. (Note that we can always move the X axis downward, without changing slopes, until the point E is above it.) Let the inclinations of the two lines be α_1 and α_2, respectively, and let the corresponding slopes be m_1 and m_2. Now if we refer to the general definitions of the trigonometric functions in Art. 65, and draw an

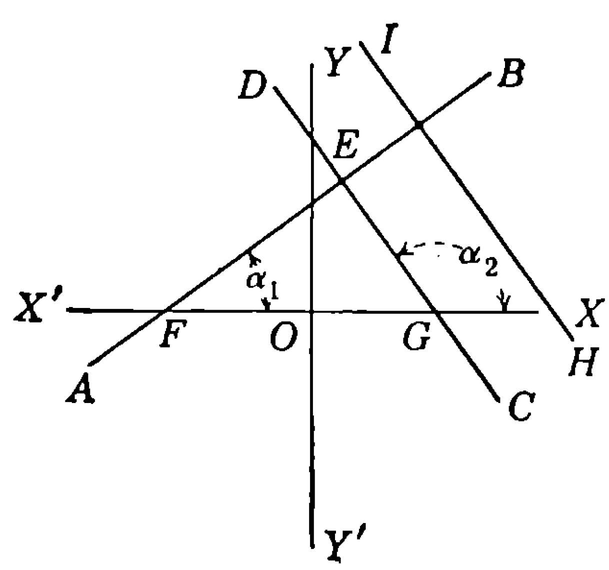

FIG. 47.

obtuse angle in standard position, we see that its tangent is negative and numerically equal to the tangent of the supplementary acute angle. Hence $\tan \alpha_2 = -\tan (FGE) = -FE/GE = -1/(GE/FE) = -1/\tan \alpha_1$. But $\tan \alpha_1 = m_1$ and $\tan \alpha_2 = m_2$, and hence $m_2 = -1/m_1$, or $m_1 m_2 = -1$.

We have proved that if two lines are perpendicular, the product of their slopes is -1. To make our test satisfactory we must prove the converse theorem. We'll suppose that the line HI in Fig. 47 has the slope m_3 and that $m_3 m_1 = -1$. Now since $m_1 m_2 = -1$, it follows that $m_1 m_2 = m_1 m_3$ and that therefore $m_2 = m_3$ (by what axiom?). Hence, by conclusion (2), lines HI and CD are parallel. Therefore, since CD is perpendicular to AB, so also is HI. This gives us the second important test, namely

(3) Two straight lines are perpendicular if and only if the product of their slopes is -1.

Having our tools, formulas (1) to (4) of Art. 68, and (1) to (3) of this article, sharpened and ready, we'll now demonstrate their power by dispatching neatly and briefly several sample theorems of plane geometry. First on the docket is that tried and true proposition, *the diagonals of a parallelogram bisect each other*, whose proof by the orthodox method of Euclid requires the preliminary

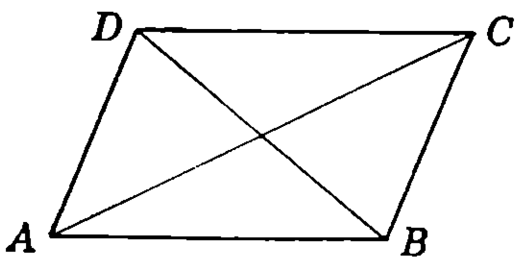

FIG. 48.

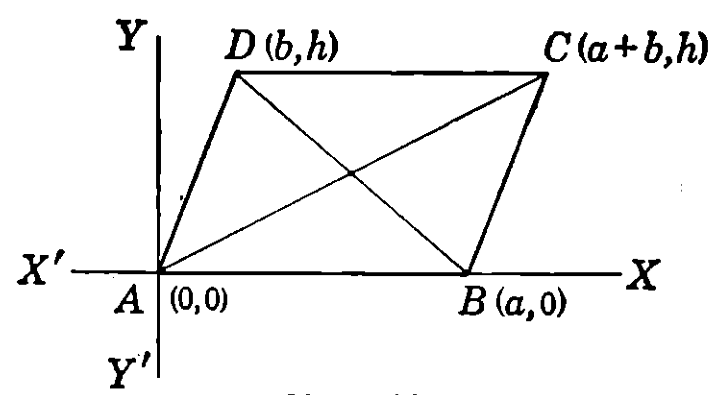

FIG. 49.

mastery of theorems on angles, triangles, and parallel lines. We first lay out our parallelogram thus (Fig. 48). Next we draw the coordinate axes, placing them *anywhere we please* so long as they are conveniently related to the figure. Obviously it will be helpful to draw them as in Fig. 49. Then, if the base and altitude of the parallelogram are a and h respectively and if we let the abscissa of D be b, the coordinates of the four corners are automatically fixed as shown in the figure. Designating the mid-points of

AC and of BD by (r,s) and (R,S), respectively, we have, by the mid-point formulas,

$$r = \frac{0 + a + b}{2} = \frac{a + b}{2}$$

$$s = \frac{0 + h}{2} = \frac{h}{2}$$

$$R = \frac{b + a}{2}$$

and

$$S = \frac{h + 0}{2} = \frac{h}{2}$$

Thus we see that AC and BD have the same mid-point, and suddenly we realize that the theorem is proved.

Our next attempted conquest is the well-known theorem: *The line joining the midpoints of two sides of a triangle is parallel to the third side and equal to half of it.* To prove it, we'll place the base of the triangle on the X axis, with the left vertex at the origin, as in Fig. 50. Of course, we are free to use any letters we please for the coordinates, except in the cases where they are necessarily zero. We shall choose the old reliables a, b, and c. Then the coordinates of E and F turn out to be $(b/2, c/2)$ and $[(a + b)/2,$

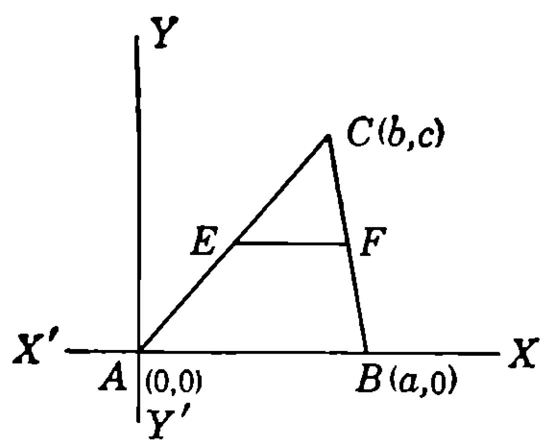

FIG. 50.

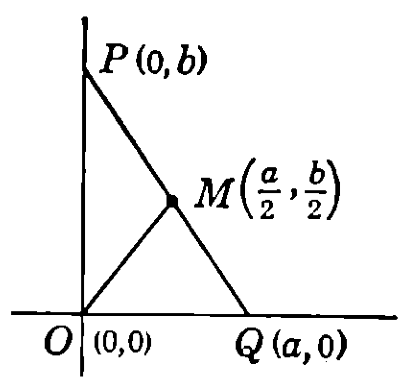

FIG. 51.

$c/2)]$ respectively, which shows already that EF is parallel to AB, since the two ordinates are equal. Furthermore, the length EF, by Formula 2 of Art. 68, is $(a + b)/2 - b/2 = a/2$, so that EF is one-half as long as AB. There you are.

To show that *the mid-point of the hypotenuse of a right triangle is equidistant from the three vertices,* consider a right triangle backed up against both axes as in Fig. 51. The mid-point M of PQ has, we find, the coordinates $(a/2, b/2)$. Applying the distance formula 3 of Art. 68, we get

$$OM = \sqrt{\left(\frac{a}{2} - 0\right)^2 + \left(\frac{b}{2} - 0\right)^2} = \frac{\sqrt{a^2 + b^2}}{2}$$

$$MP = \sqrt{\left(\frac{a}{2} - 0\right)^2 + \left(\frac{b}{2} - b\right)^2} = \frac{\sqrt{a^2 + b^2}}{2}$$

Also, of course, $MP = MQ$.

Finally, to show that the diagonals of a rhombus are perpendicular to each other, we call upon our tool labeled (3), to the effect that the product of the slopes of two lines is -1 if and only if the lines are perpendicular. A rhombus, by the way, is a parallelogram whose sides are equal. In Fig. 52 we find one of them placed in a convenient position with respect to the axes. The condition that $OA = OC$, making the parallelogram a rhombus, is not used in the figure, but we'll come to that. By (1), the slope of OB is

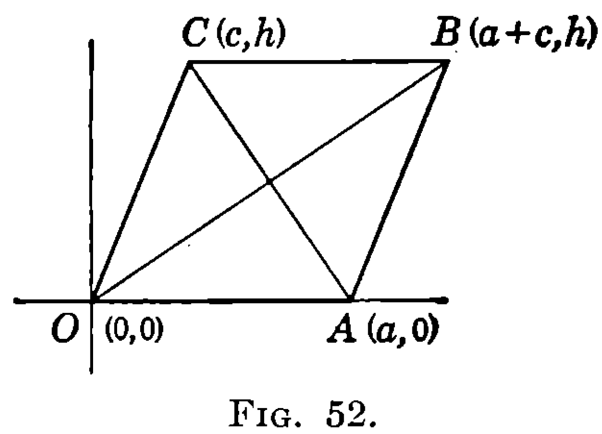

FIG. 52.

$$\frac{h - 0}{a + c - 0} = \frac{h}{a + c}$$

and that of AC is

$$\frac{h - 0}{c - a} = \frac{h}{c - a}$$

We wish to prove that the product of these slopes, or

$$\left(\frac{h}{a + c}\right)\left(\frac{h}{c - a}\right) = \frac{h^2}{c^2 - a^2}$$

is really -1. And so it turns out when we remember that $OC = OA = a$, since then $a^2 = c^2 + h^2$ by the distance formula, so that $c^2 - a^2 = -h^2$, and

$$\frac{h^2}{c^2 - a^2} = \frac{h^2}{-h^2} = -1$$

EXERCISE 47

1. Find the slopes of all the line segments in Probs. 9 to 16, Exercise 46.

2. Prove that the line joining the mid-points of two nonparallel sides of a trapezoid is parallel to the other two sides.

3. Prove that the diagonals of a square are perpendicular to each other.

4. Prove that the medians of any triangle intersect at a point which is two-thirds of the way from any vertex to the mid-point of the opposite side.

5. Prove that the segments joining the mid-points of the sides of any quadrilateral form a parallelogram.

6. Prove that the lines joining the mid-points of the sides of a square form another square.

7. Prove that if the side AB of an equilateral triangle with vertices at A, B, and C is produced its own length through B to D, then DC is perpendicular to AC.

8. Prove that if the opposite sides of a quadrilateral are parallel, they are also equal.

9. Prove that the lines joining the mid-points of the sides of a rhombus form a rectangle.

10. Prove that if the diagonals of a parallelogram are perpendicular to each other, the parallelogram is a rhombus.

70. Changing the picture into a sentence. Perhaps the reader has noticed already that the tools described in the preceding articles, as well as the proofs which use those tools, are essentially algebraic rather than geometric. It is this diversion of the resources of algebra into the domain of the old geometry that gives freshness, power, and vigor to the new approach. Before one can go very far into this land of new discoveries, however, he must master the first two fundamental problems of analytic geometry.

The first of these problems is that of changing the algebraic sentence into a picture; and this we dealt with in Chap. IV. It is true that we used there the laborious method of point-by-point plotting, and that the expert can often draw a more accurate and complete curve by mere inspection of the equation. Nevertheless, most of us must walk before we can fly, and the beginners naturally should take at first the slower and easier way.

The second main problem is that of changing the picture into an algebraic sentence. We'll give some hints about how to do it in this article and the succeeding ones, and the reader will find, incidentally, that practice in changing a graph to an equation will add considerably to his skill in backtracking the steps.

Fortunately, as in the case of the solution of an equation (where the "Golden Rule of Algebra" dictated each step), there is one simple principle that gives direction and sureness to each approach to the problem. We might call this the *principle of the sample point*, and state it thus.

To get the equation of any curve, take on it a sample point ***P****, with coordinates* (x,y)*, and write the relation between* x *and* y *which holds true because the point is on the curve. This relation will be the desired equation.*

Confidentially, the big obstacle in the application of this simple rule is the practical impossibility of finding the desired relation in many cases, as when the curves are drawn to fit statistical data. Fortunately, however, in many other important cases the point which "generates" the curve moves according to prescribed conditions. We'll now consider cases of the latter type.

71. The straight line. Suppose we seek the equation of a straight line through the fixed point $P_1(x_1,y_1)$ and with slope $m = v/h$. (This notation will serve as a reminder that the slope of a line is *always* a *vertical* distance divided by a *horizontal* distance.) The line is shown in Fig. 53, with the sample point $P(x,y)$ chosen on it at random, so that the resulting equation holds good for any point on the line. Then, using the slope formula, we have

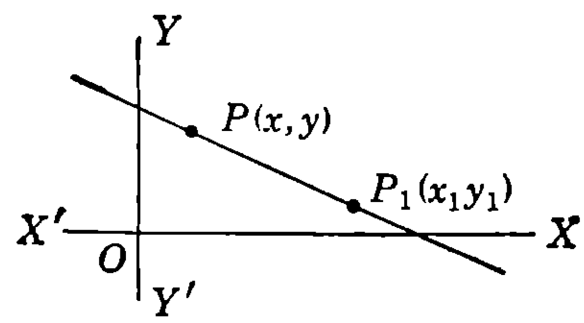

FIG. 53.

$$\frac{y - y_1}{x - x_1} = \frac{v}{h}$$

or, upon clearing the equation of fractions and transposing terms,

(1) $$vx - hy = vx_1 - hy_1$$

This result immediately makes evident a property of the equation of a line with a given slope whose usefulness has been grossly underestimated in most textbooks. The property is this:

The slope of a line with a given equation is the ratio of the coefficient of **x** *to the coefficient of* **y** *with the sign changed, when the variable terms are on the same side of the equality sign.*

Thus, in (1) the slope is $-(v/-h) = v/h$; and similarly those of $2x + 3y = 6$ and $3x - 5y = 7$ are $-(\frac{2}{3}) = -\frac{2}{3}$ and $-(3/-5) = \frac{3}{5}$, respectively. Also, we may immediately write the equations of *all* lines with any given slope, thus:

$$\text{Slope } -\tfrac{3}{8}: \quad 3x + 8y = \text{any number}$$
$$\text{Slope } \tfrac{2}{5}: \quad 2x - 5y = \text{any number}$$

A second point is brought out by (1) when we note that its right side is obtained from the left side by merely replacing x and y by x_1 and y_1. That is, we can obtain the constant at the right of the equality sign by substituting the coordinates of any known point on the line in the expression at the left. For instance, to get the equation of a line through $(2,-3)$ and with slope $-\frac{4}{5}$, we start immediately and confidently with the left side thus:

$$4x + 5y =$$

and then determine the constant at the right by using the point $(2,-3)$ and noting that $4(2) + 5(-3) = 8 - 15 = -7$, so that the equation is

$$4x + 5y = -7$$

The same method very successfully takes the place of the usual complicated and error-causing *two-point formula* for finding the equation of a line through two given points. For example, to get the equation of the line through $(-3,-4)$ and $(2,5)$ we first use the slope formula and get

$$m = \frac{v}{h} = \frac{5 - (-4)}{2 - (-3)} = \frac{9}{5}$$

so that our equation is

$$9x - 5y = \text{some constant}$$

Using $(-3,-4)$ we find that constant to be $9(-3) - 5(-4) = -7$. This may be written down as soon as found in place of our phrase "some constant," and the equation $9x - 5y = -7$ is checked immediately by using the *other* point $(2,5)$ and noting that $9(2) - 5(5)$ is also -7.

EXERCISE 48

Derive the equations of the lines determined by the data given in Probs. 1 to 5.

1. Its slope is $\frac{1}{3}$, and it passes through (2,3).

2. It passes through ($-3,-4$) and (2,7).

3. It is parallel to $2x - 7y = 1$, and passes through (-3,1).

4. It is perpendicular to $x + 4y = 7$ and passes through (2,-3).

5. It intersects the coordinate axes at points which are, respectively, three units above and four units to the right of the origin.

6. The extremities of the hypotenuse of a right triangle are (0,0) and (10,2). The slope of the leg through (10,2) is -1. Find the equations of the three sides.

In Probs. 7 to 15, assume that A, B, C are the vertices of a triangle, with coordinates (-4,5), (3,1) and (4,-6), respectively. Find the equations of the following lines:

7. The sides AB, AC, and BC.

8. The lines through the vertices parallel to the opposite sides.

9. The lines through the vertices perpendicular to the opposite sides.

10. The perpendicular bisectors of the sides.

11. The three medians of the triangle.

12. By solving the three equations of Prob. 9 simultaneously (see Art. 31), prove that the three lines are *concurrent* (pass through a common point).

13. Show that the three lines of Prob. 10 are concurrent.

14. Show that the lines of Prob. 11 are concurrent.

15. Show that the three points in Probs. 12 to 14 lie in a straight line.

16. Write the equation of the line through (3,2) with slope m.

17. Using your result for Prob. 16, find the equation of the line through (3,2) which makes with the coordinate axes a triangle with area 12 sq. units.

18. Find the equation of the line through (3,2) whose y intercept is three times its x intercept. (The *intercepts* are the nonzero coordinates of the points at which the line crosses the axes.)

19. Find the equation of the line through (3,2) which forms one side of a trapezoid whose other sides are the coordinate axes and the line $y = 2$, if the area of the trapezoid is 10 square units.

72. The circle. The *circle* is defined as a closed curve on a plane which contains all the points at a given distance from a fixed point called the *center*. The given distance is called the *radius* of the circle and is usually denoted by r. Fig. 54 shows a circle with radius r and center at C, whose coordinates are (h, k), and it also shows the inevitable sample point $P(x,y)$ placed on it at random. From the definition of the circle it is easily seen that the condition upon P is that its distance from C be r. Thus we obviously need the distance formula, which obligingly furnishes us the equation of the circle as follows:

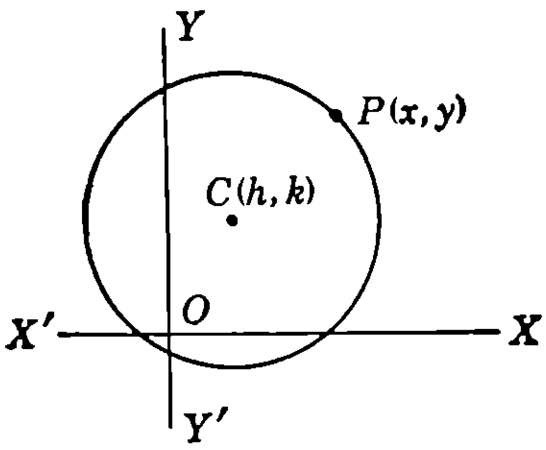

FIG. 54.

$$CP = \sqrt{(x - h)^2 + (y - k)^2} = r$$

or

$$(1) \qquad (x - h)^2 + (y - k)^2 = r^2$$

If we do the indicated squaring and shift the terms around a little, we get

$$(2) \qquad x^2 + y^2 - 2hx - 2ky = r^2 - h^2 - k^2$$

This shows that the equation of any circle is of the second degree but without an xy term, and that it can be written with the coefficients of x^2 and y^2 each equal to one. Evidently any equation in x and y containing as its only second-degree terms kx^2 and ky^2, where k is any integer, can be written in form (1). Thus $-2x^2 - 2y^2 + 4x + 10y = -6$ becomes, upon dividing both sides of the equation by -2,

$$x^2 + y^2 - 2x - 5y = 3$$

To show that this represents a circle we *complete the square,* or add a convenient sum to both sides of the equation thus:

$$x^2 - 2x + (-1)^2 + y^2 - 5y + (-\tfrac{5}{2})^2 = 3 + (-1)^2 + (-\tfrac{5}{2})^2$$

Here the quantities whose squares are added are evidently $\frac{1}{2}(-2)$

and $\frac{1}{2}(-5)$, respectively, as suggested by our experience with quadratic equations. Then

$$x^2 - 2x + 1 + y^2 - 5y + \tfrac{25}{4} = 3 + 1 + \tfrac{25}{4} = \tfrac{41}{4}$$

or

$$(x - 1)^2 + (y - \tfrac{5}{2})^2 = (\sqrt{\tfrac{41}{4}})^2$$

so that we have here a circle with center at $(1,\frac{5}{2})$ and with radius of $\sqrt{41/4} = \frac{1}{2}\sqrt{41}$, or 3.2 (to one decimal place). We can now draw it easily with a compass, and this certainly beats point-by-point plotting.

An equation of the type attacked above in which r^2 turns out to be zero is called a *point circle*, and is satisfied by the single pair of coordinates of the center. If r^2 is negative, the radius is imaginary and the curve may be called an *imaginary* circle. In that case it cannot be drawn on the plane. With these points understood, we can then say that any equation of the form

$$Ax^2 + Ay^2 + Bx + Cy + D = 0$$

where $A \neq 0$ and B, C, and D are any constants, positive, negative, or zero, will represent a circle, real or imaginary.

The most convenient simplification of all occurs when the center of the circle is placed at the origin, since, then, $h = k = 0$ and (1) reduces to the simple, elegant, and well-known form

$$x^2 + y^2 = r^2 \qquad (3)$$

EXERCISE 49

Write the equations of the circles which meet the conditions in Probs. 1 to 10.

1. Its center is at $(-3,4)$ and its radius is 5.
2. Its center is at $(0,0)$ and its radius is 3.
3. Its center is at $(4,0)$ and its radius is 0.
4. Its center is at $(\frac{3}{2},-\frac{2}{3})$ and its radius is $\frac{3}{2}$.
5. The ends of its diameter are at $(0,0)$ and $(6,-8)$.
6. Its center is $(-6,5)$ and it touches the Y axis.
7. Its center is at the origin, and it passes through $(-3,-4)$.
8. It passes through the points $(0,0)$, $(4,0)$, and $(0,2)$.
9. It passes through $(2,1)$, $(-2,1)$, and $(0,-4)$.
10. Its center is at $(-5,1)$, and it passes through the origin.

Find the radii and coordinates of the centers of the following circles; and draw the nonimaginary ones.

11. $x^2 + y^2 - 4x + 6y = 0.$

12. $2x^2 + 2y^2 - 3x + 4y - 2 = 0.$

13. $3x^2 + 3y^2 + 6x - y + 4 = 0.$

14. $4x^2 + 4y^2 - 4x + 16y + 17 = 0.$

15. $9x^2 + 9y^2 - 12x + 6y + 1 = 0.$

16. $4x^2 + 4y^2 + 8x - 4y + 13 = 0.$

17. Compare the two forms of the equation of the circle listed as (*a*) and (*b*) below, and determine the values of h, k, and r in terms of D, E, and F.

(*a*) $(x - h)^2 + (y - k)^2 = r^2$; (*b*) $x^2 + y^2 + Dx + Ey + F = 0.$

18. Put the equations of Probs. 12 to 16 in the form of 17(*b*) and find the centers and radii by the formulas developed in 17.

19. If three points on a circle are given, why is the form 17(*b*) preferable, in general, to 17(*a*)?

Express each of the following conditions upon the circle $(x - h)^2 + (y - k)^2 = r^2$ in the form of an equation involving *only* h, k, and r, or some of these letters.

20. The circle is below the X axis and tangent to it.

21. The circle is at the right of the line $x = 3$ and tangent to it.

22. The center of the circle is on the line $5x - 3y = 4$.

23. Using the technique developed in the last three problems, find the equation of the circle which is tangent to both axes and which passes through the point $(-1,2)$.

24. A point moves so that it is always twice as far from the point (2,1) as from the point $(6,-1)$. Find the equation of its path (or *locus*).

73. The conic sections. One of the most important of the discoveries which initiated modern astronomy was that of Kepler (1571–1630), who announced in 1609 that the planets of our solar system, including the earth, move about the sun in *ellipses*, or closed oval orbits of a special kind well known to the ancient Greeks. A striking illustration of what often happens in science is the fact, as stated by the historian Cajori,[1] that "the Greeks

[1] Florian Cajori, *A History of Mathematics*, p. 161.

never dreamed that these curves would be of practical use." And now, though they studied them as interesting playthings only, their results were to prove highly useful to the scientists of a later day. Facts such as these always comfort the mathematician with a practical flair. In our own case they bolster the incidental hope that the playthings we shall mention in our final chapters will in their turn prove to be useful as well as intellectually ornamental.

The special curves mentioned above which the Greeks studied so intensively were called by them *conic sections,* because they are the curves made by the trace, or section, of a plane on the surface of a cone (see Fig. 55). The mathematician's first task is to describe these curves in a precise way which will set them apart from all others. The awesome formal definition which accomplishes this rather remarkable feat is apparently so artificial that the student would never suspect the applications which have appeared from time to time. It (the definition) is as follows:

A **conic section** *is the path traced by a point which so moves on a plane that its distance from a fixed point is in a constant ratio to its distance from a fixed line.*

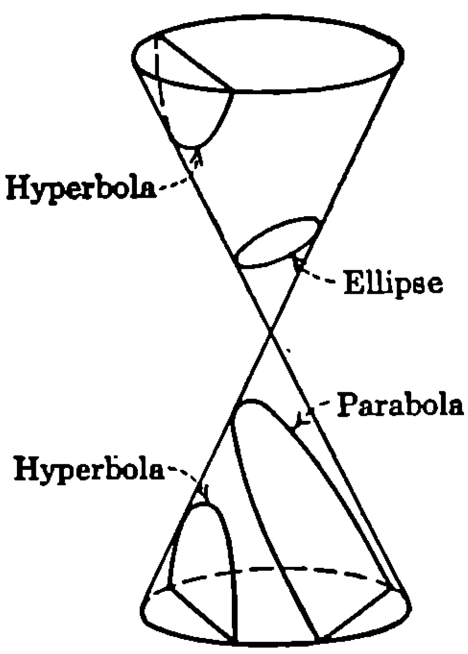

FIG. 55.

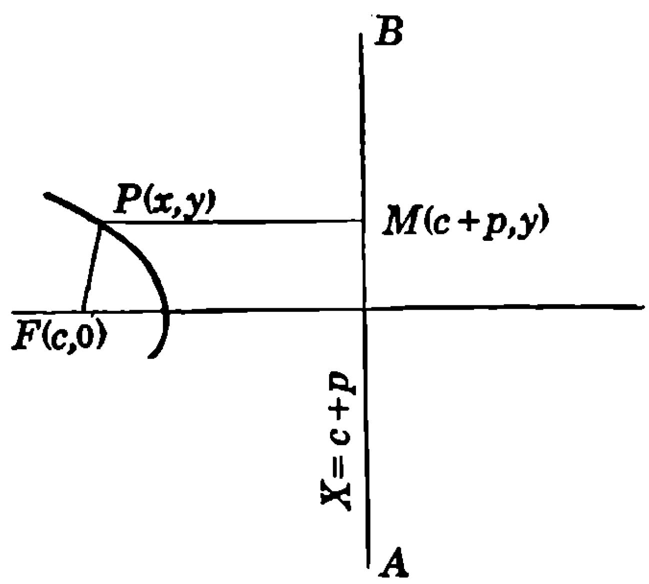

FIG. 56.

We start, then, in any given case, with merely a fixed point (called the *focus*), a fixed line (called the *directrix*), and a given constant (called the *eccentricity* and designated by the letter e). In order to derive the equation, we shall place the focus F and directrix AB in a convenient position with reference to the X axis, as shown in Fig. 56, and we'll show a small part of the curve as it would appear if the constant e were about $\frac{1}{2}$. The line through the focus perpendicular to the directrix is here made to coincide with the X axis. This fixes the ordinate of the focus as 0. The choice of its abscissa c will then fix the position of the Y axis. Since we are free to place that where we wish, we'll wait cannily until we find how to choose c in order to make the equation of the curve as simple as possible. Designating by p the fixed distance between F and AB, we find the equation of the latter to be $x = c + p$.

We're ready, now, to take our sample point $P(x,y)$ and apply to it the condition of our definition. If we let M be the point at the foot of the perpendicular from P to AB, this condition is that

$$\frac{PF}{PM} = e$$

or

$$PF = ePM$$

Using our distance-formula tools, we have

$$\sqrt{(x - c)^2 + (y - 0)^2} = e(c + p - x)$$

Squaring both sides to eliminate the radical and simplifying, we get

$$(1) \quad x^2 - 2cx + c^2 + y^2 = e^2[(c + p)^2 - 2(c + p)x + x^2]$$

or

$$(2) \quad x^2(1 - e^2) + y^2 + 2x[e^2(c + p) - c] = e^2(c + p)^2 - c^2$$

At this point we'll make our deferred choice of c so that

$$(3) \qquad e^2(c + p) - c = 0$$

[Remember that e and p were fixed in advance, so that (3) defines c.]

Then, noting that the bracketed part of (2) is zero and that $(c + p)^2$ can be replaced by $(c/e^2)^2$ according to (3), we find that (2) simplifies to

$$(4) \qquad x^2(1 - e^2) + y^2 = c^2\left(\frac{1 - e^2}{e^2}\right)$$

If now we let

$$(5) \qquad c = ae$$

(so that a new constant $a = c/e$ is introduced), and then divide both sides of (4) by $a^2(1 - e^2)$, we get

$$\frac{x^2}{a^2} + \frac{y^2}{a^2(1 - e^2)} = 1 \tag{6}$$

There are now three cases to examine.

A. If $e < 1$, the quantity $a^2(1 - e^2)$ is positive, so that it is safe to define a new positive constant b thus:

$$b^2 = a^2(1 - e^2) \tag{7}$$

Eq. (5) then takes the form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \tag{8}$$

This is the equation of a closed curve called an *ellipse* (of which the circle is a special case). Fig. 57 shows a typical ellipse and represents the graph of (8) when $a = 5$ and $b = 4$. One of the special properties of this curve is that, since $(-x)^2 = x^2$, it follows that if the coordinates (x,y) satisfy (8), then so do the coordinates $(-x,y)$ at the same distance from the Y axis on the other side. In other words, the curve is *symmetric* with respect to the Y axis—and similarly with respect to the X axis.

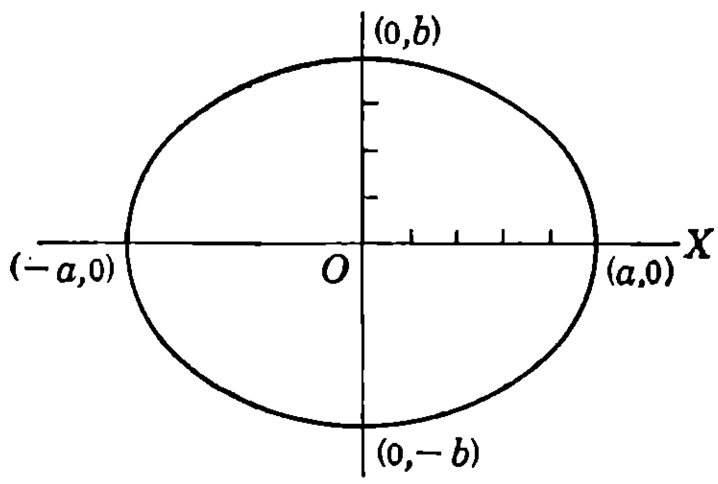

FIG. 57.

It is reasonable, then, to say that it has a *center*. The symmetry of the curve also shows us that it has *two* foci. If we define as *vertices* the points $V(a,0)$ and $V'(-a,0)$ and as *covertices* the points $E(0,b)$ and $E'(0,-b)$, we see that our arbitrarily defined constants a and b now have simple geometric meanings. Adding to these the value of c as we defined it originally, we have

(9)
a is the distance from the center to a vertex
b is the distance from the center to a covertex
c is the distance from the center to a focus

By use of (5), Eq. (7) may be rewritten thus:

$$b^2 = a^2 - (ae)^2 = a^2 - c^2$$

so that the relations between the four constants a, b, c, and e may be summarized thus:

$$(10) \qquad c = ae \qquad a^2 = b^2 + c^2$$

In case the longer axis of the ellipse is vertical instead of horizontal, equation (8) is replaced by

$$(11) \qquad \frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$$

to which the definitions and relations (9) and (10) still apply. When numbers are given instead of the letters a and b we can identify the equation as that of an *X ellipse* (8) or *Y ellipse* (11) from the fact that $a > b$ except when $a = b$ (the case of the circle).

B. If $e > 1$, the quantity $a^2(1 - e^2)$ in (6) is negative. Hence, since b^2 must be positive, we let

$$(12) \qquad b^2 = a^2(e^2 - 1)$$

and (6) takes the form

$$(13) \qquad \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

This curve, which also has two foci, two directrices, and is symmetric to both axes, is known as a *hyperbola*. A sample one, with $a = 5$ and $b = 3$, is shown in Fig. 58. It will be noted that the

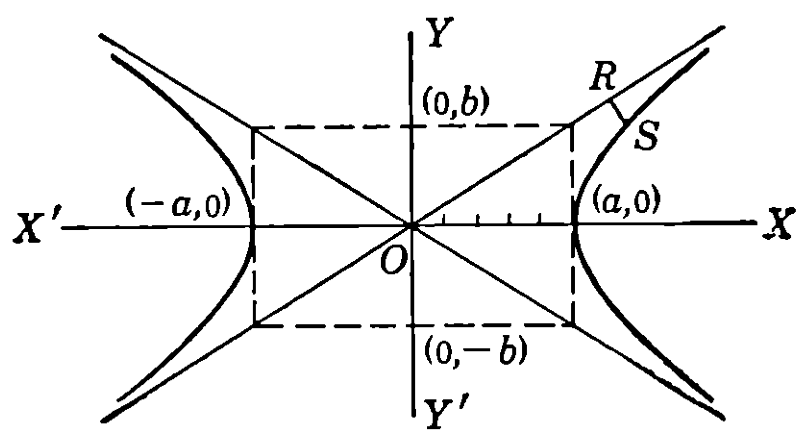

FIG. 58.

covertices $(0,b)$ and $(0,-b)$ are not on the curve; but that $2a$ and $2b$ are the dimensions of a rectangle (shown in dotted lines) whose preliminary construction has helped in the drawing of the curve. For it can be shown that the diagonals of the rectangle, called the *asymptotes* of the hyperbola, enclose its two branches completely and *are tangent to them at infinity*, which means that the distance RS perpendicular to OR shown in Fig. 58 comes as close to zero as we please if the segment RS is moved far enough outward from the center.

For this curve, equations (9) still hold good, but by virtue of the change from (7) to (12) in defining b^2, the relation between the constants for the case of the hyperbola is as follows:

$$c = ae \qquad c^2 = a^2 + b^2 \tag{14}$$

For the *Y hyperbola* whose vertices are on the Y axis, equation (**13**) is replaced by

$$\frac{x^2}{b^2} - \frac{y^2}{a^2} = -1 \tag{15}$$

to which (9) and (14) still apply. Here there is no restriction on the relative sizes of a and b, but we tell which is which in a given equation by putting it in standard form and comparing with (**13**) and (**15**). Thus

$$\frac{y^2}{4} - \frac{x^2}{25} = 1$$

may be rewritten as

$$\frac{x^2}{25} - \frac{y^2}{4} = -1$$

and we see that it represents a Y hyperbola with $a = 2$ and $b = 5$.

C. If $e = 1$, Eq. (6) becomes unmanageable because of the factor $1 - e^2 = 0$ in the denominator. Hence the previous choice of c does not work in this case. Going back to (2) and letting $c = -p/2$, we get (remembering that $e = 1$)

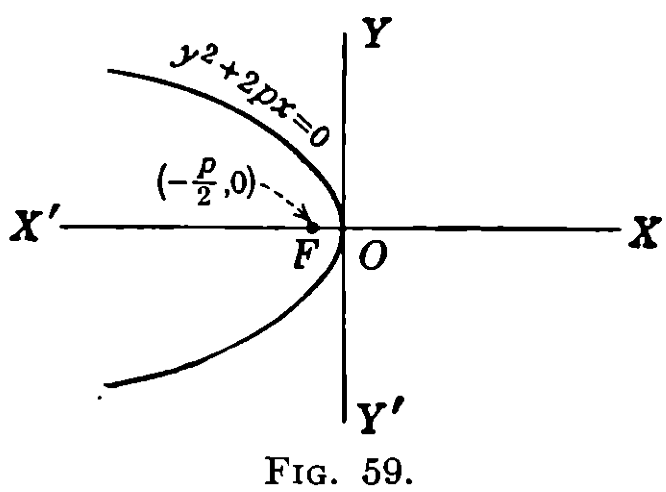

FIG. 59.

$$y^2 + 2px = 0 \tag{16}$$

which is the equation of the *parabola*—a highly important and much used curve, as we shall see. This curve is shown in Fig. 59. The constant

(17) **p is the distance from focus to directrix**

This, you perhaps remember, was our original definition of p when we started the discussion of the conic sections. Since the *vertex,* with coordinates (0,0), must be halfway between the focus and directrix by virtue of the definition of a conic and the fact that $e = 1$, it follows that the directrix is in this case the line $x = p/2$, and that the focus is the point $F(-p/2,0)$. The modification of the equation for the other directions of opening is shown in the more general forms (22) and (23).

The equations thus far obtained were simplified by a choice of axes such that the origin was at the *center* of the ellipse and of the hyperbola, and at the *vertex* of the parabola. If, now, the axes are moved to a new position parallel to the former one in such a way that the strategic point in the curve which was formerly at (0,0) is now at the point (h,k), then the distance on the unchanged curve which was formerly designated by x becomes $x - h$, and similarly the former y becomes $y - k$, as shown in Fig. 60.

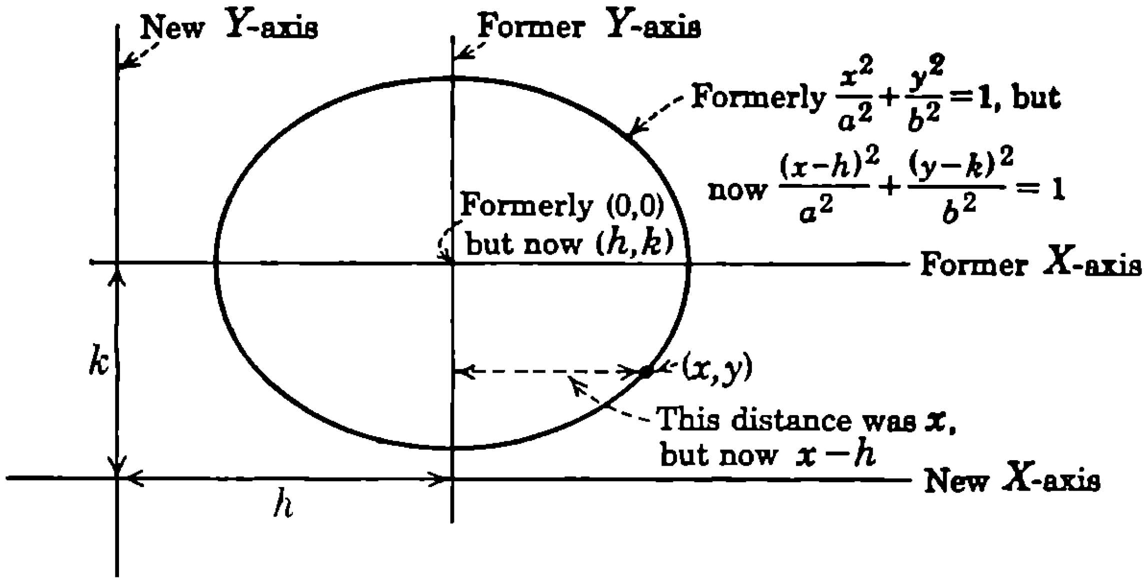

FIG. 60.

The various forms of the conic-section equations obtained when a vertex and focus are in a horizontal line (the *X curves*) or a vertical line (the *Y curves*), and when the axes are moved as in the preceding paragraph, are called *standard forms* and are summarized herewith:

(18) $$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$ *X ellipse*

(19) $$\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1$$ *Y ellipse*

(20) $$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$ *X hyperbola*

(21) $$\frac{(x-h)^2}{b^2} - \frac{(y-k)^2}{a^2} = -1$$ *Y hyperbola*

(22) $(y - k)^2 = \pm 2p(x - h)$ **X parabola** *(opens right with + sign; opens left with − sign)*

(23) $(x - h)^2 = \pm 2p(y - k)$ **Y parabola** *(opens upward with + sign; opens downward with − sign)*

The constants are defined and related by equations (9), (10), (14), and (17). The forms above are more useful than the simpler equations such as (8), (11), (13), (15), and (16), since we can complete the squares in any second-degree equation in x and y (provided an xy term is not present) and then, by comparison with the groups above, we can (or should) promptly recognize the curve. Thus, if we start with

$$25x^2 - 4y^2 - 100x + 8y + 196 = 0$$

we should proceed as follows:

$$25(x^2 - 4x) - 4(y^2 - 2y) = -196$$
$$25(x^2 - 4x + 4) - 4(y^2 - 2y + 1) = -196 + 25(4) - 4(1)$$
$$25(x - 2)^2 - 4(y - 1)^2 = -100$$
$$\frac{25(x - 2)^2}{100} - \frac{4(y - 1)^2}{100} = -1$$
$$\frac{(x - 2)^2}{4} - \frac{(y - 1)^2}{25} = -1$$

We thus see that the curve is a Y hyperbola, with center at (2,1) and with $a = 5$, $b = 2$, so that we can draw it immediately. Since second-degree relations between variables occur in nearly all fields of investigation, and since it is often helpful to exhibit these relations graphically, this method deserves careful study. It certainly leads point-by-point plotting by a wide margin of efficiency.

EXERCISE 50

A. The Ellipse

Sketch the graphs of the ellipses in Probs. 1 to 6.

1. $\dfrac{(x - 2)^2}{9} + \dfrac{(y + 4)^2}{4} = 1.$

Solution: By comparing this equation with (18) we see that $h = 2$, $k = -4$, $a = 3$, and $b = 2$. Hence we obtain the sketch in Fig. 61.

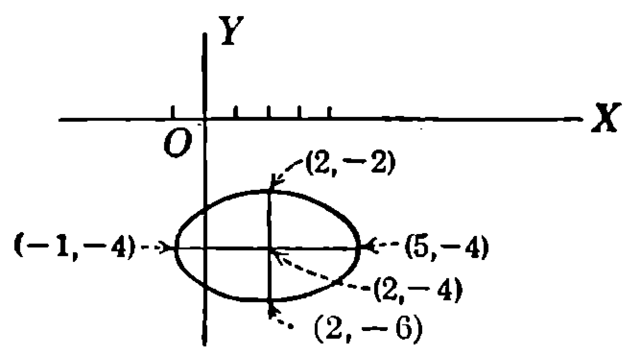

FIG. 61.

2. $\frac{(x+5)^2}{16} + \frac{(y-9)^2}{4} = 1.$ **3.** $\frac{x^2}{25} + \frac{y^2}{36} = 1.$

4. $\frac{(x-6)^2}{9} + \frac{(y+2)^2}{16} = 1.$ **5.** $\frac{(x+7)^2}{49} + \frac{(y+1)^2}{36} = 1.$

6. $\frac{x^2}{16} + \frac{(y-4)^2}{9} = 1.$

In Probs. 7 to 9, complete the squares as in the example in the text following (23), and find the values of h, k, a, and b. Do not draw the curves.

7. $25x^2 + 4y^2 - 50x + 16y - 59 = 0.$

8. $x^2 + 81y^2 - 2x - 80 = 0.$

9. $36x^2 + 9y^2 - 36x + 6y - 26 = 0.$

Use equation (10) in Probs. 10 to 15.

10. Given $a = 5$, $b = 4$, find c and e.

11. Given $a = 5$, $c = 4$, find b and e.

12. Given $b = 3$, $e = \frac{1}{3}$, find a and c.

Find the equations of the ellipses subject to the conditions indicated in Probs. 13 to 15.

13. Vertices are $(-4,2)$ and $(10,2)$; foci are $(-2,2)$ and $(8,2)$.

14. Vertices are $(2,-3)$ and $(2,5)$; $e = \frac{1}{2}$.

15. Foci are $(0,-2)$ and $(0,6)$; $e = \sqrt{3}/2$.

B. The Hyperbola

Sketch the graph of the hyperbolas in Probs. 16 to 21.

16. $\frac{(x-3)^2}{25} - \frac{(y+2)^2}{16} = -1.$

Solution: Since this is a Y hyperbola, we compare the constants in it with those of (21) and obtain $h = 3$, $k = -2$, $a = 4$, $b = 5$. Hence we may easily construct the graph (Fig. 62).

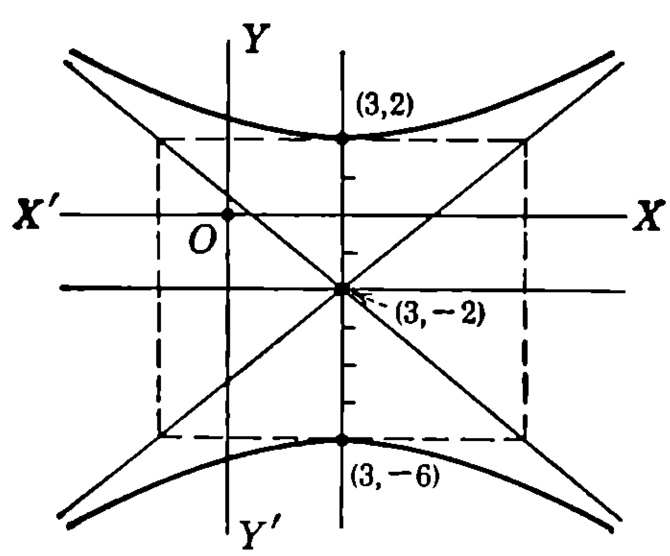

FIG. 62.

17. $\frac{(x-1)^2}{9} - \frac{(y-4)^2}{16} = 1.$

18. $\frac{x^2}{25} - \frac{y^2}{16} = 1.$

19. $\frac{x^2}{64} - \frac{(y-3)^2}{49} = -1.$

20. $\dfrac{(x-3)^2}{16} - \dfrac{(y-2)^2}{36} = 1.$

21. $\dfrac{(x+3)^2}{25} - \dfrac{(y-2)^2}{1} = -1.$

Complete the squares and find the values of h, k, a, and b. Do not draw.

22. $x^2 - 9y^2 + 2x + 36y - 71 = 0.$

23. $9x^2 - 4y^2 - 6x - 4y + 36 = 0.$

24. $x^2 - 16y^2 + 8y + 15 = 0.$

Use equation (14) in Probs. 25 to 28.

25. Given $a = 4$, $b = 5$, find c and e.

26. Given $a = 5$, $e = \sqrt{3}$, find b and c.

27. Given $b = 3$, $e = \frac{8}{5}$, find a and c.

28. Given $e = \sqrt{5}$, $c = 5$, find a and b.

Find the equations of the hyperbolas subject to the conditions indicated in Probs. 29 and 30.

29. Foci are $(-3,-2)$, $(7,-2)$; vertices are $(-2,-2)$, $(6,-2)$.

30. Vertices are $(0,4)$, $(0,-4)$; it passes through $(-2,6)$.

C. The Parabola

Sketch the graphs of the parabolas in Probs. 31 to 36.

31. $(x-2)^2 = -8(y+1).$

Solution: By comparing this with (23), we see that this equation represents a Y parabola whose vertex is $(2,-1)$, which opens downward, and in which $p = 4$. Thus, by (17), and the subsequent discussion, the coordinates of the focus are $(2,-3)$ and the equation of the directrix is $y = 1$. Furthermore, when $y = -3$, $x = 6$ and -2. Hence the curve passes through $(6,-3)$ and $(-2,-3)$. Therefore we get the graph drawn in Fig. 63.

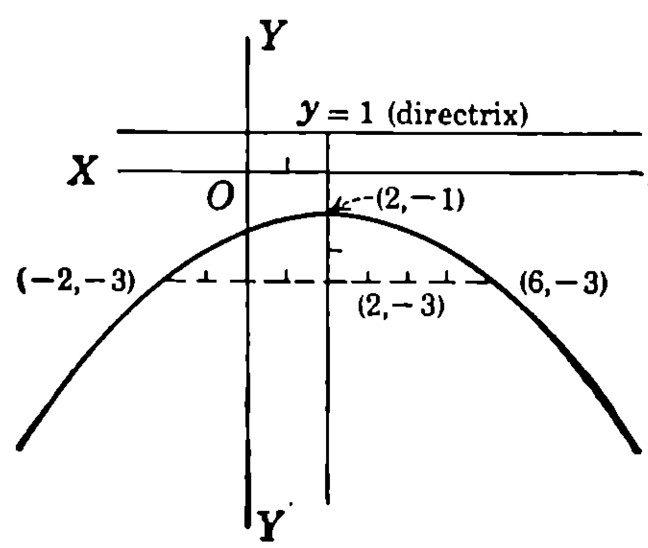

FIG. 63.

32. $(y-2)^2 = 4(x+2).$

33. $(y-1)^2 = -x.$

34. $(x+2)^2 = 6y.$

35. $2(x-3)^2 = -24(y-4).$

36. $3y^2 = 15x.$

In each of Probs. 37 to 39, complete the square, find the value of p, and tell whether the curve opens to the right, to the left, upward, or downward.

37. $x^2 - 4x - 8y - 4 = 0$.

38. $3y^2 - 2x + 6y + 15 = 0$.

39. $4x^2 - 4x + 6y - 1 = 0$.

Find the equations of the parabolas described in Probs. 40 to 44.

40. Its directrix is $x = 6$; its focus is (0,0).

41. Its directrix is $y = 2$; its vertex is (0,0).

42. Its vertex is (0,0); it passes through $(-2,-4)$, opening downward.

43. Its focus is $(3,-2)$ its vertex is $(0,-2)$.

44. Its focus is (0,8); it passes through (3,4), opening upward.

D. Miscellaneous Conics

Find the equations of the conics defined by the conditions given in Probs. 45 to 50.

45. Its vertices are $(-3,4)$ and (5,4); its foci are (0,4) and (2,4).

46. Its directrix is $y = 4$; its focus is $(2,-4)$; $e = 1$.

47. Its vertices are $(1,-2)$ and (1,8); its foci are $(1,-5)$ and (1,11).

48. Its covertices are $(5,-1)$ and (5,7); $c = 3$.

49. Its foci are $(2,-3)$ and (2,4); $e = \sqrt{3}$.

50. It has no center, its axis is $x = 1$, its vertex is (1,0), and it passes through (2,2).

74. Another aid and an application. Our chest of analytic tools, which now includes formulas for the distance from point to point and from point to vertical or horizontal lines, would be inexcusably inadequate if it omitted the one for the distance from a point to a slant line, since this practically completes the basic formulas of the subject. Incidentally, the tool will allow us to round out our brief discussion of the conic sections. But since its simplicity and ease of application may be lost sight of in the details of its derivation, we'll state it first and defer the proof until you can do the thing mechanically.

The ***distance,*** *and by that we mean the* ***perpendicular distance d*** *from the point* (x_1,y_1) *to the line*

$$(1) \qquad Ax + By + C = 0$$

is

$$(2) \qquad d = \frac{Ax_1 + By_1 + C}{\pm\sqrt{A^2 + B^2}}$$

Furthermore, if the sign of the radical is so chosen that $\dfrac{B}{\pm\sqrt{A^2 + B^2}}$ *is positive, then* ***d*** *will be positive or negative according as the point is above or below the line.*

Thus, the distance from $(4,-3)$ to the line $3x - 5y + 7 = 0$, which is above $(4,-3)$, is

$$\frac{3(4) - 5(-3) + 7}{-\sqrt{3^2 + (-5)^2}} = \frac{34}{-\sqrt{34}} = -\sqrt{34}$$

In Fig. 64, EF is a straight line whose slope is $-A/B$ and whose equation is (1), and OR is the distance from this line to the origin. It should be noted that nothing in the following argument prohibits the segment MN of EF from being in any one of the four quadrants. The length OM, or the *X-intercept* of the line, is $-C/A$, as we see when we set $y = 0$ in (1). Similarly, $ON = -C/B$. Then, in the similar right triangles ORM and OMN,

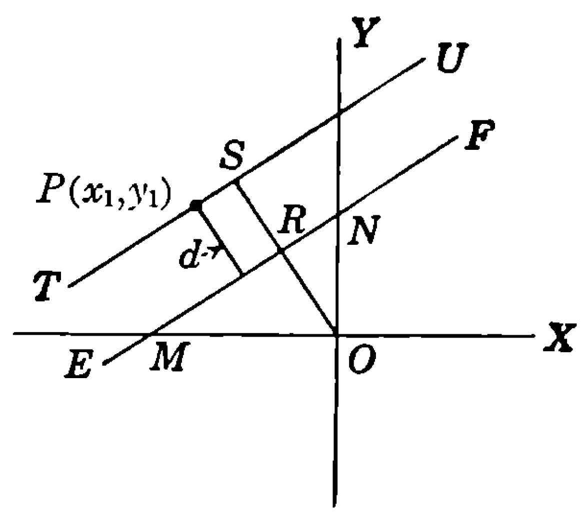

FIG. 64.

$$\frac{OR}{OM} = \frac{ON}{MN}$$

so that

$$OR = OM\frac{ON}{MN} = \frac{\left(\frac{-C}{A}\right)\left(\frac{-C}{B}\right)}{\pm\sqrt{\left(\frac{-C}{A}\right)^2 + \left(\frac{-C}{B}\right)^2}} = \frac{C}{\pm\sqrt{A^2 + B^2}}$$

Now let $P(x_1,y_1)$ be any point on the plane which is not on the line (1). Through P draw a line TU parallel to EF, so that its slope is also $-A/B$, its equation is

$$(3) \qquad Ax + By + C_1 = 0$$

and its distance OS from the origin is $\dfrac{C_1}{\pm\sqrt{A^2 + B^2}}$

First, we'll assume that neither OR nor OS equals zero. If we use the same sign in the expression for OS as that which makes OR positive, then OS will evidently be positive if TU and EF are both on the same side of the origin, and negative if the origin is between them. (Why?) From Fig. 64 we see that d, the required distance from EF to P, is either $OS - OR$, as in the figure, or $OR - OS$ if TU is between EF and the origin, or $OR + (-OS)$ if the origin is between TU and EF. In any case,

$$(4) \qquad d = \pm(OR - OS) = \frac{C - C_1}{\pm\sqrt{A^2 + B^2}}$$

But if $OR = 0$, EF goes through the origin and hence $C = 0$, and similarly $C_1 = 0$ if $OS = 0$. In these respective instances the distance d is OS or OR, and hence (4) still holds.

Now since $P(x_1,y_1)$ is on the line TU, whose equation is (3), it follows that $Ax_1 + By_1 + C_1 = 0$, so that $C_1 = -Ax_1 - By_1$, and hence (4) becomes

$$d = \frac{Ax_1 + By_1 + C}{\pm\sqrt{A^2 + B^2}}$$

There remains the matter of proving the statement immediately below (2) concerning the sign of the result. Assuming that $\dfrac{B}{\pm\sqrt{A^2 + B^2}}$ is positive, we note that if P_1 is moved upward on a vertical line, the distance d in (2) increases algebraically. Since $d = 0$ when P_1 is on the line (1), it must be negative or positive respectively when P_1 is below or above the line. This completes the proof.

We are now prepared to derive the equation of a conic section for which the directrix is oblique to the coordinate axes.

In Fig. 65, let the line MN, whose equation is $Ax + By + C = 0$, be the directrix, let $F(a,b)$ be the focus, and let $P(x,y)$ be our sample point on a conic whose eccentricity is e. Then, by definition,

$$FP = e(QP)$$

which, on application of the two important distance formulas (3) of Art. 68 and (2), becomes

$$\sqrt{(x-a)^2 + (y-b)^2} = e\left(\frac{Ax + By + C}{\pm\sqrt{A^2 + B^2}}\right)$$

Squaring both sides and simplifying, we get a formidable-looking second-degree equation which is not worth recording in itself, but in which the difference from the other conic equations we have met lies in the presence of an xy term. Thus it is shown that every conic section is described by a second-degree equation. The converse of this statement, which is that every second-degree equation represents a conic section, can be shown to be true if we liberalize our definition to include the so-called *degenerate* conics consisting of two straight lines, distinct or coinciding, a single point, or an *imaginary* curve whose graph does not appear on the coordinate plane. Examples of each of these cases are the following:

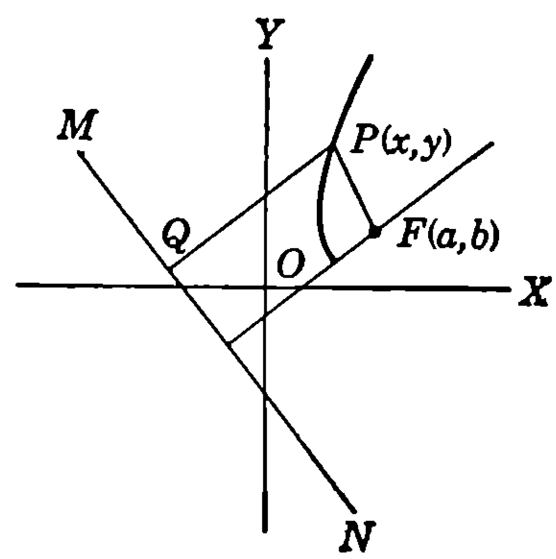

FIG. 65.

(1) $(2x - 3y + 1)(5x + y) = 0$	(two straight lines)
(2) $4(x-2)^2 + 9(y+1)^2 = 0$	[the point $(2,-1)$]
(3) $4(x-2)^2 + 9(y+1)^2 = -36$	(an imaginary ellipse)

EXERCISE 51

1. Find the distance from $(-3,4)$ to $2x - 3y = 6$.

2. Find the distance from $(0,0)$ to $5x + 12y - 3 = 0$.

3. Find the distance from $(0,6)$ to $5x - 2y = 0$.

4. Find the distance between the lines $4x + 3y = 6$ and $4x + 3y + 2 = 0$.

5. Find the three altitudes of the triangle with vertices $(-1,6)$, $(2,-1)$, and $(-4,-3)$.

6. Find the expression for the distance from the point (x,y) to the line $Ax + By + C = 0$.

7. Find the equation of the line with a positive slope which bisects the angle whose sides are $4x + 3y = 12$ and $12x - 5y = 6$.

8. Work Prob. 7 when *positive* is replaced by *negative.*

9. Find the equations of the lines which are two units from $2x - 3y = 4$.

10. Find the coordinates of the point in the first quadrant which is one unit from each of the lines $4x - 3y = 0$ and $3x - 4y = 0$.

11. Find the relation between A and B so that the lines $Ax + By - 1 = 0$ shall be five units from the origin.

12. Find the equation of the *locus* (or curve traced by) a point which moves so that it is twice as far from (3,2) as from $4x - 3y - 2 = 0$.

13. Work Prob. 12 when *twice* is replaced by *half.*

14. Find the locus of a point which moves so that it is always twice as far from the origin as from the point (5,2).

15. What conics are defined by Probs. 12 to 14?

75. What's the use? Near the beginning of our discussion of the conic sections we said that the Greeks, who studied these curves as mere toys of the mind, had no inkling that their results were to prove highly useful to the later scientists of the machine age, which includes our time. We have thus implied that these simple second-degree curves are met over and over again in the problems, machines, and instruments of everyday living. And now we shall bolster our flat assertions with a few specific examples.

At once the whole earth comes to our aid, for it travels around the sun in a curve which, measured inside a moving plane, is nearly an ellipse, and which would be exactly one if the earth and sun were all alone on the celestial stage. In general it is roughly (though not precisely) true that, with reference to the sun, the planets travel in ellipses, comets move in elongated ellipses or parabolas, and meteors come into the earth's atmosphere in great arcs which include, in the various cases, all three of the different types of conics.

Coming down to earth, we find that projectiles such as stones, arrows, rifle balls, and even the flowerpot and flatiron weapons of the comic page, all travel in the trusty parabola. And the engineer, not to be outdone by nature, swings into the curve of a railroad or highway along a parabolic arc, rounds the curve on a circle, and completes the turn on a parabola again.

It is indeed the parabola, or rather a surface associated with it, which perhaps takes the prize for versatile usefulness. This surface, called the *paraboloid*, is used in automobile headlights, searchlights, and in the mirrors of reflecting telescopes. In Fig. 66, the curve is an ordinary parabola, and the paraboloid is the three-dimensional surface made by revolving this curve about its axis OX. Now the particularly helpful fact is that any angle such as FBD, made at a point B on the surface by the lines BF through the focus and BD parallel to the axis of the parabola, is exactly bisected by the line BR perpendicular to the curve. The proof requires the methods of calculus, which will be discussed later. It follows from a well-known law of physics that if a source of light be placed at F, the angle of incidence FBR made by any simple ray FB will be equal to the angle of reflection RBD, and all the rays will emerge in parallel lines, giving the desired effect of a large light source. Or again, the direction of the rays may be reversed, as in a reflecting telescope, so that the image of a distant object appears at the focus. If in this case the rays are "lines of sound" coming from a distant airplane, its direction may be determined by a delicate listening device at the focus, and our paraboloid becomes a huge ear. Evidently the applications are numerous and important.

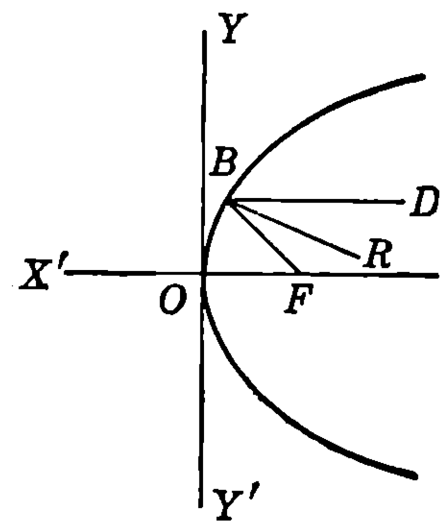

FIG. 66.

A similar property of an *ellipsoid*, the surface obtained by revolving an ellipse about its axis, is that rays from one focus F are reflected at the surface to the other focus F'. (See Fig. 67.) The property has a limited application in the field of building acoustics, as in the Mormon tabernacle at Salt Lake City, and in trick mirrors used to mystify crowds with such sights as bodyless and talking human heads floating apparently unsupported in the air; but these interesting accomplishments naturally do not give the ellipse the high commercial dignity of the parabola. Still, when

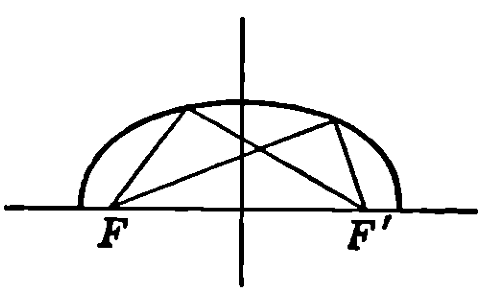

FIG. 67.

one considers a special ellipse—the circle—as it materializes in the form of a wheel, he can readily see that the argument is not one-sided. Only the hyperbola seems left by the workaday world as a mathematical abstraction, but even that is used by nature in some of her rock paths in the sky. For showing in graphical form the relation between real quantities such as pressure and volume in a gas (Boyle's law), the curve comes into its own, and since the point of contact of a plane and a line fixed in space will trace both branches of a hyperbola when the plane is revolved about a line on it, the chances for direct mechanical applications are not so remote as one might think.

76. Grand finale. But the conic sections, important as they are, do not represent exhibit A in the hall of the accomplishments of analytic geometry. In simple essence the subject is a device by which geometry falls heir to all the resources of algebra. Each geometric condition gives us an algebraic sentence, and we handle the results simultaneously by the technique of the immortal x and y.

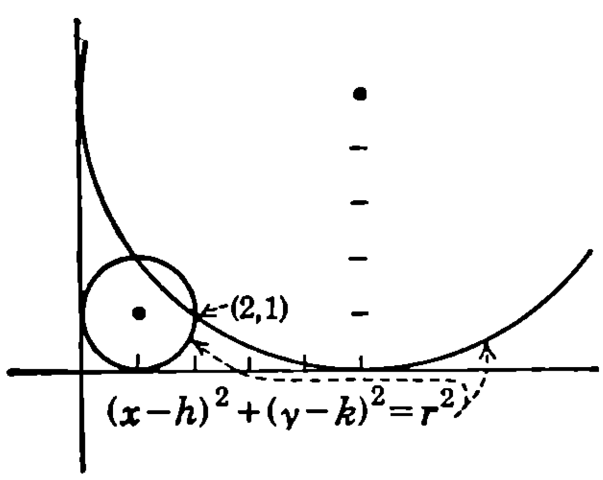

FIG. 68.

For instance, consider this geometric problem: Find the radius of a circle which is tangent to each of two mutually perpendicular lines and which passes through a point two units from one line and one unit from the other. In Fig. 68 we see that we can solve the problem if we can find the equation of the circle through (2,1) which touches both axes. We'll start with the general circle equation

$$(x - h)^2 + (y - k)^2 = r^2 \tag{1}$$

and our task is to find first the particular values of the constants h, k, and r.

Let the first condition be the requirement that the circle touch the Y axis on the right side. Here it is—check it for yourself:

$$h = r \tag{2}$$

Similarly, since the circle touches the X axis from above, we have

$$k = r \tag{3}$$

(You might note that the equation $k = -r$ would say that "the circle touches the X axis from the lower side.")

Finally, the circle must pass through (2,1) so that those coordinates must satisfy (1). Hence

$$(2 - h)^2 + (1 - k)^2 = r^2 \tag{4}$$

Now, solving (2), (3), and (4) simultaneously by the algebraic methods of Chap. III, we have

$$(2 - h)^2 + (1 - h)^2 = h^2$$
$$h^2 - 6h + 5 = 0$$
$$(h - 1)(h - 5) = 0$$

and

$$h = k = r = 1 \text{ or } 5$$

There are *two* circles—radii one and five, respectively—and *both* work, as you can easily see by trial. This illustrates our final point. Analytic geometry not only gives us *one* solution, but it turns them *all* up in our lap, sometimes unexpectedly. Could we possibly ask for anything more?

EXERCISE 52

Express in the form of an algebraic equation the condition stated in each of Probs. 1 to 8.

1. The point (h,k) is three units to the right of the line $x = 4$.

2. The point (a,b) is five units below the line $y + 3 = 0$.

3. The point (a,b) is above, and four units distant from, the line $3x + 4y = 0$.

4. The circle $(x - h)^2 + (y - k)^2 = r^2$ is tangent to, and above, the line $y + 6 = 0$.

5. Work Prob. 4 when $y + 6 = 0$ is changed to $5x - 3y - 4 = 0$.

6. The center of the circle $(x - h)^2 + (y - k)^2 = r^2$ is on the line $4x - 3y = 7$.

7. The axis of the parabola $(y - k)^2 = 2p(x - h)$ is the line $y = 6$.

8. The curve $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ passes through the point $(1,-2)$.

9. Find the equation of all circles in the first and third quadrants which are tangent to both axes. [*Answer:* $(x - h)^2 + (y - h)^2 = h^2$. Here h is called a *parameter*. For each constant value assigned to h we get a particular circle meeting the condition.]

10. Work Prob. 9 when *first* and *third* are replaced by *second* and *fourth*.

11. Find the equation of all circles with center $(3, -4)$.

12. Find the equation of all lines passing through $(4, -5)$.

13. Find the equation of all lines with slope $-\frac{4}{3}$.

14. Find the equation of all X ellipses with eccentricity $\frac{1}{2}$.

15. Find the equation of all Y hyperbolas with eccentricity 2.

16. Find the equation of all lines which are five units from the origin.

CHAPTER IX

WE CREEP UP ON SOLUTIONS

77. Into algebra with geometry. In analytic geometry, as we have just finished pointing out, geometry falls heir to all the resources of algebra. The ends of poetic justice would be served, then, if algebra in its turn should get substantial aid from geometry—and sure enough it does. For the happy union of the two subjects not only allows us to go from the algebraic sentence to the geometric picture (Chap. IV) and from conditions about the picture back to the sentence (Chap. VIII), but it also enables us to solve by the picture method the problem proposed by a single equation in one unknown. For instance, the sentence

$$x^3 + 3x^2 - 2x - 7 = 0 \tag{1}$$

asks us this question: What value (or values) of x will satisfy (1)? Algebra furnishes an answer, as we have seen in Chap. III, for the first- and second-degree equations and also rather complicated ones (which we shall omit) for equations of the third and fourth degree, but beyond this point its machinery stalls except in special cases. In this chapter we'll show how to use a picture to solve, to any required degree of accuracy, practically *any kind* of an equation in one variable, however hopeless may be the attack by purely algebraic methods.

78. Streamlined division for plotting purposes. The left side of (1), Art. 77, is an example of a *polynomial in one variable,* or an algebraic function of the type

$$f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n \tag{1}$$

where the a's are constants, $a_0 \neq 0$ if the degree is n, and n is a positive integer.

It will be convenient, in order to get the pictures we need, to divide $f(x)$ by $x - r$. For the case $r = 2$, $f(x) = x^3 + 3x^2$

$-\ 2x - 7$, the operation can be shown in long-division form thus:

$$\begin{array}{r|l}
x^3 + 3x^2 - \ \ 2x - 7 & x - 2 \\
x^3 - 2x^2 \qquad\qquad\quad & x^2 + 5x + 8 \\ \hline
5x^2 - \ \ 2x - 7 & \\
5x^2 - 10x \qquad & \\ \hline
8x - 7 & \\
8x - 16 & \\ \hline
9 & \text{(remainder)}
\end{array}$$

If we examine the above rather tedious process in detail, we'll note that certain letters and numbers are repeated in it more often than the efficiency expert would desire. Since the same letter, x, is involved throughout, we can start the elimination of deadwood with the x itself, so that the first line becomes

$$1 \quad 3 - 2 - 7 \,|\, \underline{1 - 2}$$

But now, since we are dealing only with divisors of the type $x - r$, we see also that the second digit 1 will always appear in the place shown, so that it can be understood and therefore omitted. The essential digits in the answer $x^2 + 5x + 8 + \dfrac{9}{x - 2}$ are 1, 5, 8, and 9, and these we can get by the skeletonized mechanical process thus:

$$\begin{array}{rrrr|l}
1 & 3 & -2 & -7 & \underline{-2} \\
 & -2 & -10 & -16 & \\ \hline
1 & 5 & 8 & 9 &
\end{array}$$

Here the initial 1 is carried down to the third line; then this 1 times -2 gives the -2 appearing in the second line; this subtracted from 3 gives 5 (third line); $(5)(-2) = -10$ (second line), etc. And finally, if we change the -2 of the abbreviated division to $+2$, and *add* instead of subtracting in each column, we arrive at the brief mechanical process known as *synthetic division.* The long division first shown is then replaced by the streamlined short cut:

(2)
$$\begin{array}{rrrr|l}
1 & 3 & -2 & -7 & \underline{2} \\
 & 2 & 10 & 16 & \\ \hline
1 & 5 & 8 & 9 &
\end{array}$$

which tells us that

$$\frac{x^3 + 3x^2 - 2x - 7}{x - 2} = x^2 + 5x + 8 + \frac{9}{x - 2}$$

To illustrate certain other points in this process, we'll divide

$$f(x) = 4x^5 - 35x^3 - 2x - 3 \text{ by } x + 3$$

thus:

$$\begin{array}{rrrrrr|r} 4 & 0 & -35 & 0 & -2 & -3 & \underline{-3} \\ & -12 & 36 & -3 & 9 & -21 & \\ \hline 4 & -12 & 1 & -3 & 7 & -24 & \end{array}$$

(The two zeros are the coefficients of the missing terms in x^4 and x^2.) From this we learn that

$$\frac{f(x)}{x + 3} = 4x^4 - 12x^3 + x^2 - 3x + 7 - \frac{24}{x + 3}$$

Now that we have this short division method, we'll put it to work in the rapid plotting of the curve

$$(3) \qquad y = f(x) = x^3 + 3x^2 - 2x - 7$$

We need, of course, a table of number-pairs applying to this curve. For small values of x we can find y fairly easily as follows:

When $x = 0$, $y = f(0) = 0^3 + 3(0^2) - 2(0) - 7 = -7$.
When $x = 1$, $y = f(1) = 1^3 + 3(1^2) - 2(1) - 7 = -5$.
When $x = 2$, $y = f(2) = 2^3 + 3(2^2) - 2(2) - 7 = 9$.

It will be noticed, however, that when x is a number containing several digits, such as 1.53, the computation of the corresponding y becomes long and tedious. At this point an interesting result known as *the remainder theorem* comes to our aid, since in conjunction with synthetic division it gives us a short way to compute y for any given x.

Remainder theorem. *If* $f(x)$ *is divided by* $x - r$ *until a constant remainder is obtained, that remainder is* $f(r)$.

Proof: Evidently

$$(4) \qquad \frac{f(x)}{x - r} = Q(x) + \frac{R}{x - r}$$

where $Q(x)$ is the quotient obtained in the division and R is the as yet undetermined constant remainder. Next we get

$$(5) \qquad f(x) = Q(x)(x - r) + R$$

when we multiply both sides of (4) by $x - r$. Now equation (4) holds for every value of x except $x = r$, the exception being due to the fact that in this case we have the nonpermissible zero in the two denominators. Hence (5) is also true for an infinite number of values of x. It seems reasonable, then, and it can be proved by use of an additional theorem,[1] that (5) is an identity, so that the left side equals the right side for *all* values of x, including $x = r$. It follows that, when $x = r$,

$$f(r) = Q(r)(r - r) + R = 0 + R = R$$

This means, you will note, that when x is assigned the particular value r, the corresponding value of $y = f(x)$ [which will be $f(r)$] can be found by getting the remainder by synthetic division instead of by direct substitution in $f(x)$. For instance, if $f(x) = x^3 + 3x^2 - 2x - 7$, the value $f(2) = 2^3 + 3(2^2) - 2(2) - 7 = 9$ is more easily found as the remainder in the synthetic division (2).

A brief table of number-pairs for the curve (3) constructed by this synthetic division method [except that $f(0) = -7$ can be obtained directly from $f(x)$ by substituting $x = 0$] follows:

x	-4	-3	-2	-1	0	1	2	3
y	-15	-1	1	-3	-7	-5	9	41

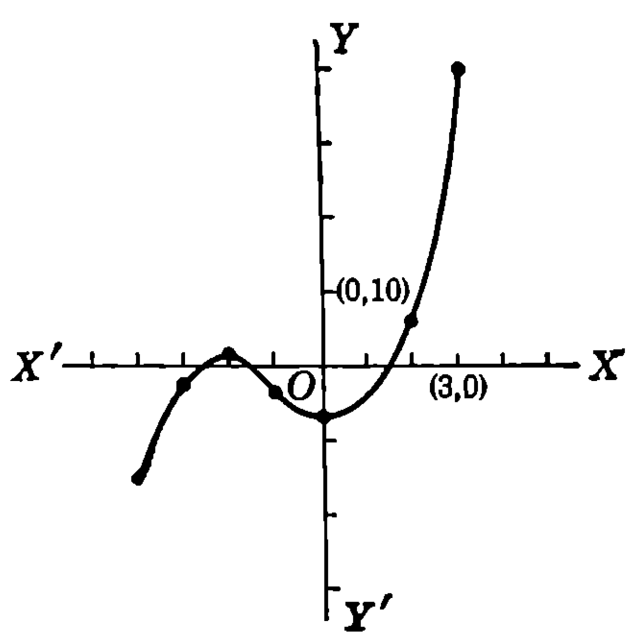

FIG. 69.

Connecting these points with a smooth curve, we get the graph of Fig. 69. It will be noted that here, as is often the case, it is convenient to make the horizontal and vertical scales different.

EXERCISE 53

1. Show that the remainder theorem holds true when $r = 3$ and $f(x) = 2x^3 - 3x^2 + 5$.

2. Show that the remainder theorem holds when $r = -2$ and $f(x) = 2 - x^3$.

[1] As for instance in *College Algebra* by Rees and Sparks, 2d ed., p. 198.

Find the quotient and remainder by synthetic division when $f(x)$ and r are as shown in Probs. 3 to 6.

3. $2x^3 - 3x^2 + x + 2$; 3.

4. $x^3 + 2x^2 - 4$; -2.

5. $3x^4 + x^2$; 1.

6. $x - x^4 + 2x^3 - 2$; -1.

In Probs. 7 to 14, using synthetic division, plot the curve $y = f(x)$, using integral values of x from -3 to 3, where $f(x)$ has the indicated values.

7. $x^3 + 2$.

8. $x^3 - 3x^2 + 2x - 1$.

9. $2x^4 + 3x^2 - 2$.

10. $x + 3 - 4x^3$.

11. $3x^5 - 2x^2 + x - 1$.

12. $1 - x^5$.

13. $4x^3 + 2x^2 + 3$.

14. $-x^4 - 4$.

79. The hemming-in process. Now the highly useful bit of information that we get from the graph of

$$y = f(x) \tag{1}$$

concerns, not the equation which is plotted, but the related one,

$$f(x) = 0 \tag{2}$$

For evidently the roots of (2) will be the values of x for which $y = 0$ in (1). In other words, they will be the abscissas of the points at which the curve (1) touches or crosses the X axis. Thus the graph of Fig. 69 shows that the equation,

$$x^3 + 3x^2 - 2x - 7 = 0 \tag{3}$$

has at least three roots and that they are, respectively, about -2.6, -1.5, and 1.5.

We'll now use a hemming-in process to get a better decimal approximation to the positive root of (3). For this purpose, we'll first enlarge the horizontal scale of the figure in the immediate neighborhood of the investigated crossing, so that the unit length from 1 to 2 is divided into tenths. The vertical scale is still held at any convenient size which will keep the significant part of the curve on the available space. Since small segments of the curve depart only slightly from the corresponding straight-line segments, we can assume for our first approximation that this magnified piece of the curve is actually straight. Noting from the foregoing table that y is -5 when x is 1, and 9 when x is 2, and connecting the points with a line, we get Fig. 70. Since the line crosses the axis between 1.3 and 1.4, we suspect that the particular root of (3) whose trail we are on will be near one of these numbers. Remembering, however, that the actual curve is not perfectly straight, but really slightly concave on the upper side, as appears in Fig. 69, we cannot be sure that the crossing is not at the right of 1.4 and even, possibly, of 1.5. To locate it more precisely, we need the values of $f(x)$ for several values of x in this neighborhood. To find $f(1.3)$ we use synthetic division and find it to be -2.333, thus:

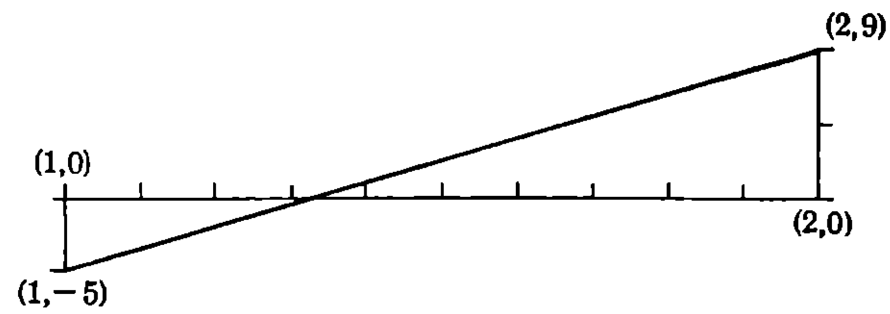

FIG. 70.

1	3	−2	−7	|1.3
	1.3	5.59	4.667	
1	4.3	3.59	−2.333	

Here the superiority of the synthetic-division method to direct substitution becomes more pronounced. Compare, for instance, the time required to compute $f(1.3) = (1.3)^3 + 3(1.3)^2 - 2(1.3) - 7$ with that consumed in the above synthetic division. Of course the difference is even greater when there are two or three decimal places in the trial value of x.

Again using synthetic division, we find $f(1.4)$ to be -1.176. Since this value is still negative, and since the curve, in this particular case, crosses the X axis from the lower side on the left to the upper side on the right, we see that 1.4 is still smaller than the hunted root, which we'll call r. Hence $f(1.5)$ is also needed. It turns out to be .125, so that $r < 1.5$. By now we have pinned down r to the interval between 1.4 and 1.5, and have learned the values of $f(1.4)$ and $f(1.5)$ to be used in the "second magnification" of Fig. 71.

This figure indicates r to be very nearly 1.49. Since the deviation from the straight line is even less in this case than in the preceding one, owing to the smallness of the interval considered, we are fairly safe in assuming $r = 1.49$ to the nearest hundredth. The argument is clinched when we find that $f(1.48) = -.147$, $f(1.49) = -0.012$, and $f(1.50) = 0.125$ (to three decimal places

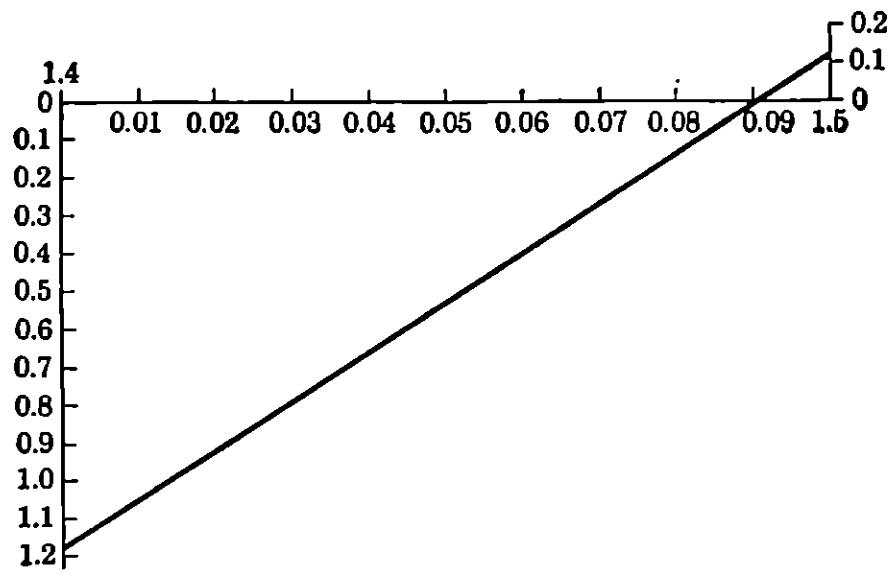

FIG. 71.

each) so that $r = 1.49(+)$. The plus sign indicates that the true value of r is nearer to this figure than to 1.50. If it were between 1.49 and 1.50 and closer to the latter, we would say $r = 1.50(-)$.

This hemming-in process, as illustrated above for one particular root of one particular polynomial equation, could evidently be used to find all real roots of all equations of this type, carried out to as many decimal places as might be required. Two decimal places would probably be enough for most practical problems, but the mathematician is never satisfied until he is prepared for all emergencies.

A great many improvements and refinements of the technique of solving rational integral equations such as (1) are dealt with in the field of algebra called the *theory of equations*. The "snaring" process of the above discussion suffices to find all real roots of such equations, and it applies with particular force when the root sought is irrational and therefore cannot be expressed exactly in decimal form. When the root is rational it would eventually be caught *exactly* at one or the other end of the tightening net. In practice a useful theorem about rational roots enables one to find all of these before the remaining irrational roots are sought, as illustrated in the following article; but one should not lose sight of the fact that the technique for finding the real roots as here developed is now complete. The rest is a matter of polish and efficiency.

80. A refinement or two. Since we lack space to prove the many efficiency-theorems of the theory of equations, we'll give a few of the most useful without proof.

Theorem 1. *The equation*

(1) $$f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n = 0$$

in which n *is a positive integer, the* a*'s are real constants, and* $a_0 \neq 0$*, has exactly* n *roots.*

It should be understood, however, that not all of these roots need be different. For instance, the equation

$$(x - 1)^{100}(x - 2) = 0$$

has the root *one* repeated 100 times and called a root of *multiplicity* 100.

In the following theorems it is to be understood that $f(x)$ is as defined in (1) and that the a's are integers..

Theorem 2. *If the graph of* $y = f(x)$ *does not touch the* X *axis, the roots of* (1) *are all imaginary.*

Theorem 3. *Any imaginary roots which* (1) *may have occur in conjugate pairs. That is, if* $a + bi$ *is a root, so will be* $a - bi$.

Theorem 4. *If* (1) *has a rational root* c/d *in which* c *and* d *are integers which have no common factor except* ± 1*, then* c *is a divisor of* a_n *and* d *is a divisor of* a_o.

For instance, any rational roots which $2x^3 - 3x^2 - 3x - 5 = 0$ may have are included in the list: ± 1, ± 5, $\pm \frac{1}{2}$, $\pm \frac{5}{2}$. The theorem does not guarantee, however, that any of these numbers is a root. The graph of $y = 2x^3 - 3x^2 - 3x - 5$ suggests which of these rational possibilities to try first. From the division

$$\begin{array}{rrrr|l} 2 & -3 & -3 & -5 & \frac{5}{2} \\ & 5 & 5 & 5 & \\ \hline 2 & 2 & 2 & 0 & \end{array}$$

we see that $2x^3 - 3x^2 - 3x - 5 = (x - \frac{5}{2})(2x^2 + 2x + 2) = 0$, so that one root is rational and equal to $\frac{5}{2}$, while the remaining two roots, which turn out to be imaginary, are found from the *depressed* equation, $2x^2 + 2x + 2 = 0$, or $x^2 + x + 1 = 0$. This suggests how to use the depressed equation to find the remaining roots after any particular one is found.

EXERCISE 54

1. Given the equation

$$f(x) = 4x^4 - 4x^3 - 11x^2 - 4x - 15 = 0$$

(1), list all possible rational roots, using Theorem 4; (2), decide which in this list are most likely to be roots by plotting the curve $y = f(x)$ from $x = -2$ to $x = 3$; and (3), find the two rational roots by trial of these prospects, using the depressed equation after one root is found.

How can we tell, from the bottom line of the synthetic division used in finding $f(3)$ and $f(-2)$ for the graph of $y = f(x)$, that all real roots of $f(x) = 0$ are in the interval from $x = -2$ to $x = 3$?

In Probs. 2 to 4, use the method suggested in Prob. 1 to find all the rational roots.

2. $2x^3 + x^2 + x - 1 = 0.$

3. $3x^4 + 2x^3 + 6x^2 + x - 2 = 0.$

4. $3x^4 + 2x^3 - 8x^2 - 5x = 0.$

In Probs. 5 to 10, find to two decimal places the irrational root which is algebraically the largest (or the farthest to the right in the corresponding graph).

5. $x^3 + x^2 + x - 1 = 0.$

6. $x^3 + 2x - 1 = 0.$

7. $x^3 - 3x^2 + 1 = 0.$

8. $2x^3 + 3x - 2 = 0.$

9. $x^3 = 2.$

10. $x^3 + 7x - 2 = 0.$

81. The all-conquering method. Most mathematical textbooks explain the hemming-in scheme as used in finding the real roots of rational integral equations, and then drop the method at the threshold of its usefulness. Other applications should certainly be given, for it cannot be too strongly emphasized that this is the one method which *always* locates for us the real roots of a mathematical equation in one unknown. Sometimes that unknown is so snarled in a tangle of exponents and functions of all conceivable kinds that a quest for it by known algebraic methods would be practically hopeless. But even that slim prospect for success is greater than the chance for the elusive unknown to conceal its general neighborhood when once the inexorable geometric net begins to tighten on it. It may defy to the end all efforts to route it out exactly for inspection, in the sense that one root of $x^3 - 8 = 0$ is exactly 2; but it cannot escape the probing forceps which locate it closely enough for practical purposes.

Among the more stubborn of the sentence types that, though relatively simple, still resist the algebraic attack, are the *exponential equations*, so called because they involve the unknown as an exponent.

Consider, for example, the equation

(1) $$2^x = 1 + 3x$$

We could transpose all terms to one side, getting $2^x - 1 - 3x = 0$, and then find the roots by locating the points at which the curve $y = 2^x - 1 - 3x$ crosses the X axis. A shorter way, however, is to plot the two separate curves $y = 2^x$ and $y = 1 + 3x$, since we can then make use of our knowledge of familiar curves, such as the straight line $y = 1 + 3x$. Substitution of integral values of x from -4 to $+4$ yields the following table of coordinate pairs for the curve $y = 2^x$:

x	-4	-3	-2	-1	0	1	2	3	4
y	$\frac{1}{16}$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8	16

It is interesting to note that this curve never crosses the X axis, since y is positive for any value of x, positive or negative. A portion of the curve and of the line, with the two points of crossing, is shown in Fig. 72.

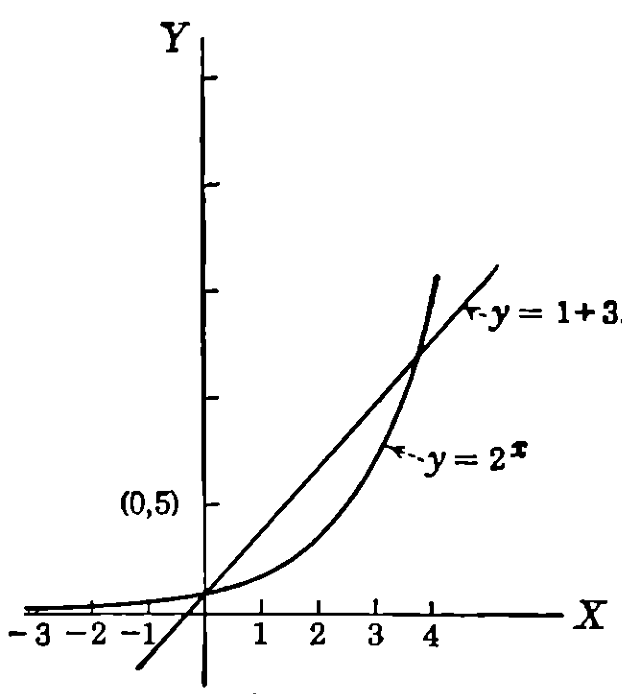

FIG. 72.

Now it should be fairly obvious that Eq. (1) will be satisfied by abscissas of the points of intersection of the curve and line, since these are the values of x which make the two ordinates equal, or which make the value of 2^x equal to that of $1 + 3x$. One of these points is at $x = 0$. Sure enough, we see by direct substitution that one root of (1) is 0. A second root is apparently between 3 and 4. Using logarithms, we find for

$y = 2^x$ some coordinate pairs in the neighborhood of the crossing as follows:

x	3	3.2	3.4	3.6	3.8	4
y	8	9.18	10.5	12.1	13.9	16

The usual magnification of the curves in the critical region yields Fig. 73. The crossing is evidently in the neighborhood of 3.5. But since $2^{3.5} = 11.31$, while $1 + 3(3.5) = 11.5$, the curve is still below the line for $x = 3.5$. Also $2^{3.6} = 12.1$, while $1 + 3(3.6) = 11.8$ so that the curve has crossed the line at $x = 3.6$. Comparing the two pairs of values in the light of the figure, we see that the root is 3.5(+). Evidently again, as in the example of Art. 79, the process may be continued to get roots to any desired degree of accuracy.

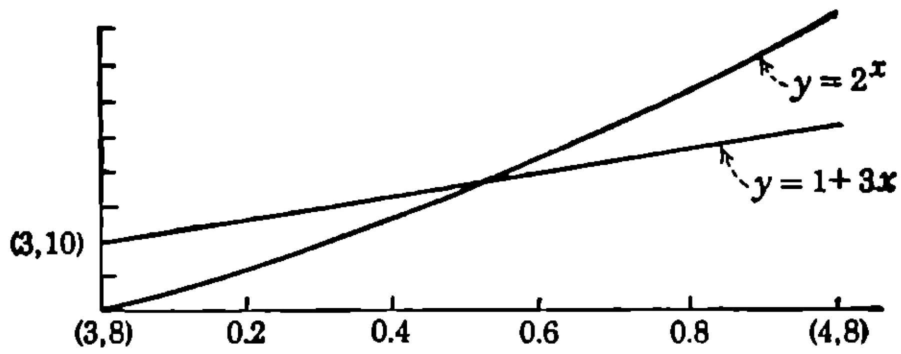

FIG. 73.

In the case of an equation involving trigonometric functions, such as $x + \sin x = 1$, the solution, of course, depends upon the angular unit in terms of which we express the value of x. It is customary in such cases to use the *radian*, which is an angle of such size that, if its vertex is at the center of a circle, it will subtend an arc equal to the radius of the circle. The important relation between radians and degrees is expressed in the equation

(2) π **radians = 180 degrees (or 180°)**

where $\pi = 3.1416$, to four decimal places. When x is expressed in fractions of π radians, the graph of $y = \sin x$, which is a very important, useful, and well-known curve, is found by plotting to be the gracefully bending line of Fig. 74. Evidently this curve crosses the line $y = 1 - x$ at

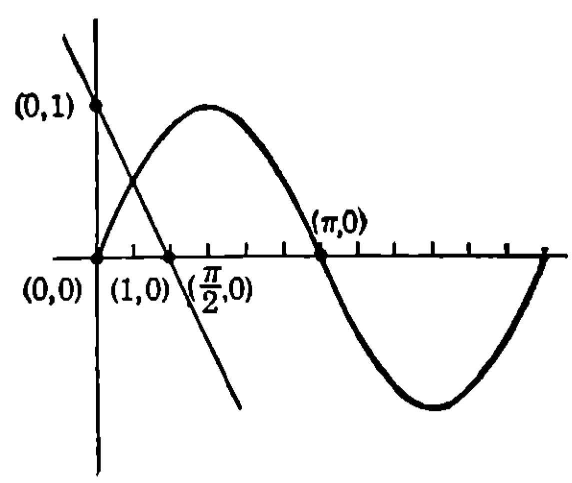

FIG. 74.

a point slightly to the left of $x = \pi/6 = .5236$, so that the single root of

$$x + \sin x = 1 (\text{or } \sin x = 1 - x)$$

is .5(+) radians to the nearest tenth.

Clearly this experimental method of solving an equation is long and tedious, particularly when the mathematical sentence is complicated. A simple and elegant device such as the quadratic formula for second-degree rational integral equations would be very much better, it is true; but unfortunately such formulas are not likely to be found for many cases. At that we are lucky to have a strong-arm method which will never fail us in the unpleasant emergencies when it is needed. Only the imaginary roots successfully defy its inelegant prying. There is, to be sure, a way to route out even these imaginaries; but we'll pass over that with this one official mention. After all, the imaginary solutions are seldom the ones needed in everyday problems.

EXERCISE 55

Solve the following equations to two decimal places, using the graphical aid explained in the text. In Probs. 5, 6, and 9, express x in radians, where 1 radian = 57° 17.7′ and .1 radian = 5° 43.8′.

1. $2^x = 2 - x$.

2. $3^x = 3 - x$.

3. $4^x = 4 - x$.

4. $x + \log x - 2 = 0$.

5. $x + 2 \sin x - 2 = 0$.

6. $x + \tan x - 3 = 0$.

7. $x^3 + \log x = 0$.

8. $x \log x = 1$. (Hint: Here $\log x = 1/x$.)

9. $x - \cos x = 0$.

10. $x^2 = 2^x$. (There are three solutions.)

CHAPTER X

VARIABLES CAUGHT IN ACTION

82. We deal with little things. In algebra, we dealt chiefly with *constants,* or quantities whose values remain fixed throughout the particular problems in which they appear. Even the apparently changeable x of an algebraic equation is usually nothing more than a convenient advance notice of one or more fixed quantities which are still in hiding. In the equation $2x^3 - x^2 - 2x + 1 = 0$, for instance, this letter is the temporary nom de plume of the numbers 1, $\frac{1}{2}$, and -1. In analytic geometry, on the other hand, x and y are frequently *variables,* in the sense that they represent the coordinates of unlimited numbers of points in a single graph. If, now, we consider a type of problem in which the quantities are constantly changing, like the coordinates of a point which is moving along a curve, the still hunt of algebra becomes the action drama of *calculus.* By a method suggesting the separate "shots" of a moving picture, this remarkable invention of the human mind breaks up the smooth motion of a graceful dive, for instance, into a succession of flash analyses, each of which is far more informative about the dive itself than its photographic counterpart. For not only does it show *status quo* like the picture, but it also tells "which way headed" and "how fast"; and this, it must be admitted, is a surprising bit of information to be contained in a single sample of frozen motion.

When the rate is constant, the relation between distance, rate, and time is neatly expressed in the equation

(1) $$d = rt$$

Useful algebraic variations of the same statement are

(2) $$r = \frac{d}{t}$$

(3) $$t = \frac{d}{r}$$

The chief practical objection to these admirable formulas is the fact that moving objects very seldom travel at a constant rate. The automobile which makes in 30 min. a 20-mile trip from one town to another has of course maintained an average speed of 40 miles per hour; but the actual rate varied from minute to minute according to traffic conditions. If we try to determine the rate of an automobile, then, by use of Eq. (2), we are merely determining its average velocity in the interval concerned. If the speed is constantly changing during that period, how can we possibly describe in terms of d and t the flash-by-flash indications of the moving speedometer arrow? During a zero time interval the car naturally goes zero feet ahead, so that Eq. (1) gives us $0 = r(0)$. This is all very true, but totally unhelpful in the matter of the value of r. But if, instead of taking a zero time interval, we merely take a small one, *then* Eq. (2) gives us an average rate which is necessarily within the two extremes indicated by the speedometer during the interval. And, obviously, the shorter the interval the smaller, in general, will be the range of speed. Now let us think of the "instant" as the sharp line of cleavage between the past and the future, without duration in time, just as the geometric point is a location without dimensions in space. A natural definition of the car velocity at a given instant is that velocity indicated by the speedometer as "frozen" by a photograph. From the above discussion it is evident that the average speed during an interval and the *defined* speed at the start of that interval are nearly the same when the time period considered is short, and that we can make the difference between these two quantities as close to zero as we please by shortening the interval sufficiently. To get at this elusive thing called *speed*, then, we are brought face to face with the necessity of dealing with increasingly tiny things, both in time and in space. Similarly, in studying how one variable changes with respect to another, we find it necessary to examine smaller and still smaller portions of the curve which translates to picture language the equation connecting the variables (just as in the last chapter we hemmed in the roots of an equation). Since, however, there is no more an end to possible smallness than there is to size in the direction of hugeness, the process seems discouragingly without point or hope of conclusion until we find that there is a solid and immovable goal toward which we can point our mental prow on our endless spiraling inward toward the inexpressibly tiny. And that goal lies in the comforting, stationary Rock of Gibraltar of calculus—that constant called the *limit,* which we define in the next article.

83. We reach the limit. Preliminary definition: *The quantity* $|x|$ *(read,* ***the absolute value of x****) is the positive number which equals* x *or* $-x$ *according as* x *is positive or negative.*

Thus $|3| = 3$, and, furthermore, $|-3| = 3$.

Important definition: *The* ***limit*** *of a variable* v *is a constant* L *such that* $|v - L|$ *becomes and remains less than any positive number named in advance, however small that number may be.*

So much for the technical definition. The concept of the limit, however, is so extremely important in calculus that something more than a brief (and probably misunderstood) definition is needed. Let's try again.

The *limit* of a variable v is a constant L. (Never forget that a limit is a *constant.*) This constant exists if, as v runs through an indicated sequence of values in a given order, there is some point in the process beyond which all subsequent values of v differ from L by quantities, positive or negative, which are each less numerically than a small positive number chosen in advance, however close to zero that number may be.

For example, let's consider what happens to the variable

$$v = \frac{x^2 + x - 2}{x^2 - 1}$$

as x approaches 1. When $x = 2$, $v = \frac{4}{3} = 1.33$; when $x = 1.1$, $v = 0.31/0.21 = 1.47$; when $x = 1.01$, $v = 0.0301/0.0201 = 1.498$. These few sample values of v are enough to suggest that both numerator and denominator in v get close to zero as x gets close to one (so that we actually are dealing with increasingly tiny things) and that v itself is perhaps approaching the constant 1.5. This we see to be actually the case when we write

$$v = \frac{x^2 + x - 2}{x^2 - 1} = \frac{(x + 2)(x - 1)}{(x + 1)(x - 1)} = \frac{x + 2}{x + 1}$$

since the latter expression equals 1.5 when $x = 1$. The *limit* of v

when x approaches 1 is therefore 1.5. This result is indicated briefly by the notation

$$\lim_{x \to 1} v = 1.5 \tag{1}$$

We now see that, in the foregoing discussion of average and instantaneous velocities, the latter is really a limit. For if we designate the small interval of time considered by Δt (read *delta t*) and the corresponding small distance covered in that interval by Δs, the average velocity during the interval is, by formula (2) of Art. 82, $\Delta s/\Delta t$. The *instantaneous* velocity v at the beginning of the interval is described exactly and mathematically in the equation

$$v = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} \tag{2}$$

When we find an instantaneous velocity, we say that we know for that instant how the distance s is changing *with respect to* the time t. If v comes out 60 when s is measured in miles and t in hours, it means that, if the velocity were maintained at the exact value of the flash sample, s would change by sixty miles when t changes by one hour. In general, the field of mathematics named *differential calculus* is concerned largely with how one quantity changes with respect to one or more others. Our discussion of velocity suggests that the relations between variables represented graphically by microscopic sections of the curve should probably be studied in detail; but it fails utterly to indicate the amazing power and usefulness of the results actually obtained. The great Newton (1642–1727) in England and Leibnitz (1646–1716) in Germany, working independently at practically the same time, were the first to make important contributions to the field. To them we are indebted for the beginning of *differential* and *integral* calculus—those mighty mathematical tools whose potent calculations have made possible a large share of the scientific and engineering developments of the last two centuries.

EXERCISE 56

Find the limits indicated in Probs. 1 to 8, when they exist. (In some cases below they do not exist.)

1. $\lim_{x \to 0} 3 + x - x^3 + 2x^5$. **2.** $\lim_{x \to 1} \frac{x^2 - x - 2}{x^2 - 3x + 1}$.

3. $\lim\limits_{x \to 1} \dfrac{2x^2 - x - 2}{x^2 - 3x + 1}$.   **4.** $\lim\limits_{x \to 0} \dfrac{x^3 + x}{x^2 + 3x}$.

5. $\lim\limits_{x \to \infty} \dfrac{x^2 - 2x + 1}{4x^3 + 3}$.   **6.** $\lim\limits_{x \to \infty} \dfrac{3x^5 + 2x^3}{x^3 + 2}$.

7. $\lim\limits_{x \to \infty} \dfrac{2x^4 + 5x}{5x^4 + 5}$.   **8.** $\lim\limits_{x \to \infty} \dfrac{3x^2 - \frac{4}{x}}{5x^2 + 3}$.

9. A point P moves 1, $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, etc., in. in successive seconds. Find the limit of the distance covered by P as time goes on indefinitely (see Art. 56).

10. A point P moves alternately forward and backward, covering 1, $\frac{1}{3}$, $\frac{1}{9}$, $\frac{1}{27}$, etc., in. on each move. Find the limit of the distance of P from its starting point as the number of moves becomes infinite.

Find the limits indicated in Probs. 11 to 13, when they exist.

11. $\lim\limits_{x \to \infty} \dfrac{\sin x}{x}$.   **12.** $\lim\limits_{x \to \infty} \dfrac{\cos x}{x}$.   **13.** $\lim\limits_{x \to \infty} \dfrac{\tan x}{x}$.

14. Given the polynomials $A = a_0x^n + a_1x^{n-1} + \quad + a_n$ and $B = b_0x^m + b_1x^{m-1} + \cdot \cdot \cdot + b_m$, find, if it exists, $\lim\limits_{x \to \infty} (A/B)$, (*a*), when $n > m$, (*b*) when $n = m$, and (*c*) when $n < m$.

84. Our method. Up to this point the illustrative problems calling for the new technique have dealt almost entirely with the changing velocity of a moving body. This was due to convenience rather than to necessity. For primarily calculus deals with *change*, and not solely with moving objects or even motion in the abstract. To be sure, the limit, that fundamental constant of calculus, can be, as we have seen, a velocity which is the limit of average velocities. But it also can be, and as a matter of fact often is, an area which is the limit of successively closer approximations. Again, it may be a length, volume, force, unit of work, or in fact any quantity whatever, real or imaginary, with which the exact thinker wishes to deal. Obviously, then, we need a scheme for representing variable quantities by means of letters which are not restricted by custom to specialized meanings. This rules out s, often used to mean *length* or *distance*, and also t, which frequently stands for *time*. To represent two variable quantities of any sort, what could possibly be a happier choice than the old reliable x and y? These tried, true, and noncommittal letters need only our offhand decision to do duty for anything under the sun. They have the tremendous added advantage that they are the letters used to represent general points on the coordinate plane, so that the results of calculus lend themselves automatically to interpretation by the picture machinery of analytic geometry. One should remember, however, that the quantities dealt with in calculus are definite measurable items of ordinary living such as are encountered daily by engineers, physicists, etc., and not merely the coordinates of a moving point in a plane. The variables of life go into the grist mill of calculus, and the fact that they come out for purposes of manipulation and illustration as the time-honored letters of analytic geometry should not blind one to their essential nature.

Trusting that the vicarious nature of x and y is now clear, we shall get our rules of manipulation in terms of these two letters. It will be convenient to think of x as the *independent* variable to which we may assign values at will, and of y as the *dependent* variable whose value is fixed when that of x is named. The relation between the two is expressed by the generalized equation

$$(1) \qquad y = f(x)$$

in which it will be understood that $f(x)$ is a mathematically expressible function of x (such as $x^2 + 3$ or $\sin x$ or $\log x$), allowing us actually to compute its value for a given x.

Furthermore, we shall assume in what follows that y is a *continuous* function of x. This means, roughly speaking, that the graph of $y = f(x)$ can be drawn in the region considered without lifting the pencil from the paper. Technically, $f(x)$ is continuous at $x = a$ if $\lim_{x \to a} f(x) = f(a)$; and it is continuous in the interval $c < x < a$ if it is continuous at every point in this interval.

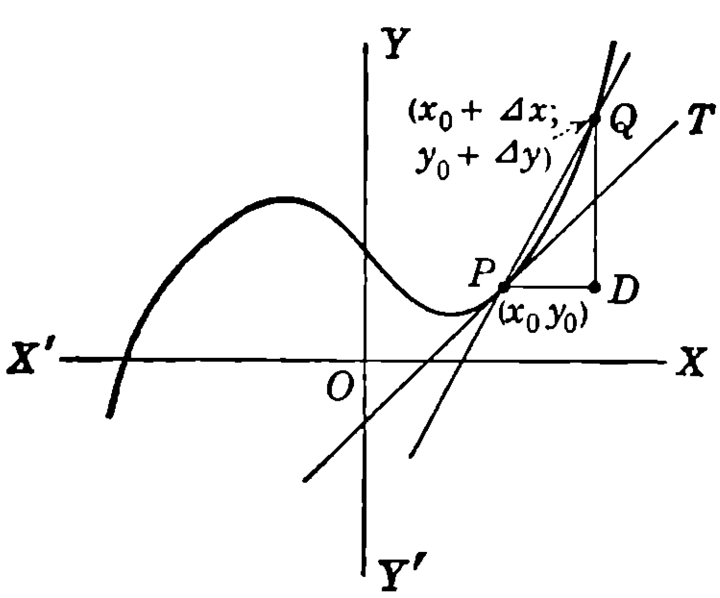

FIG. 75.

Suppose, then, that the graph of (1) looks, in the picture language of analytic geometry, about like Fig. 75. Let $P(x_0y_0)$ be a particular point on the curve, let Δx be an arbitrary increment, or change, in x (shown as PD on the figure), and let Δy (or DQ) be the corresponding change in y. It should be noted that the letters Δ and x are treated together as the one quantity Δx, so that, for instance, Δx^2 means $(\Delta x)^2$ and not $\Delta(x)^2$. Let Q be the point whose coordinates are $(x_0 + \Delta x, y_0 + \Delta y)$. The line PQ then becomes a secant of the curve. Usually, as in the figure, it will not coincide with the line tangent to the curve at P, here shown as PT. Now if we let PD approach zero, so does DQ in the figure. Evidently in that case the point Q moves down the curve toward P, and the secant PQ swings around toward the tangent PT in such a way that we can make the angle QPT as close to zero as we please by taking PD (or Δx) sufficiently small. In other words, the fixed angle TPD is the limit of the variable angle QPD as PD approaches zero, so that

(2) $$\text{limit}_{PD \to 0} \text{ tangent } QPD = \text{tangent } TPD$$

or

(3) $$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = m, \text{ the slope of the tangent line at } P$$

Referring to (2) of Art. 83 and remembering that s and t there used could very well be the quantities for which y and x are doing duty, we note that in such a case

(4) $$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = v, \text{ the velocity at time } t$$

Evidently the limit of the ratio of Δy to Δx, which is a slope from the geometric angle and a velocity in another aspect, must be a rather important quantity in this domain of the minute called calculus. And so it is, most decidedly. Rating a special name of its own, the *derivative*, it qualifies easily as the central concept of differential calculus, entitled to all the pomp and dignity that we can muster in the following

Definition. *If x and y are two variables related by the equation $y = f(x)$, and if $f(x)$ is continuous, then*

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

when this limit exists, is designated by the symbol dy/dx *and is called the derivative of* y *with respect to* x.

Geometrically, this derivative may be pictured as the slope of the line tangent to the curve $y = f(x)$ at the point (x,y). More briefly

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} \tag{5}$$

The process of finding the derivative is called *differentiation.*

The fractional form dy/dx is used partly as a matter of convenience, so that we can say

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx}, \lim_{\Delta w \to 0} \frac{\Delta v}{\Delta w} = \frac{dv}{dw}, \text{ etc.}$$

where the letters concerned are related in the same manner as y and x. A second justification for this form is that it may actually be used and interpreted as a fraction, according to more advanced theory that we shall not here discuss. The all-important pitfall for one to avoid is the idea that the dy and dx are the respective limits of Δy and Δx, when as a matter of fact those limits are each zero. The *limiting form* of $\Delta y/\Delta x$ is $0/0$, which gives us no information whatever, since division by zero is always barred; but the *actual limit* which $\Delta y/\Delta x$ is approaching is a constant indicated by the symbol dy/dx.

A second point worth noting in the definition is the fact there implied that

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

does not always exist as a unique constant at certain points in the graphs of some functions. This may be due to the fact that the value of $\Delta y/\Delta x$ increases without limit as Δx approaches zero, as at points B and C in Fig. 76. Or again, there may be several

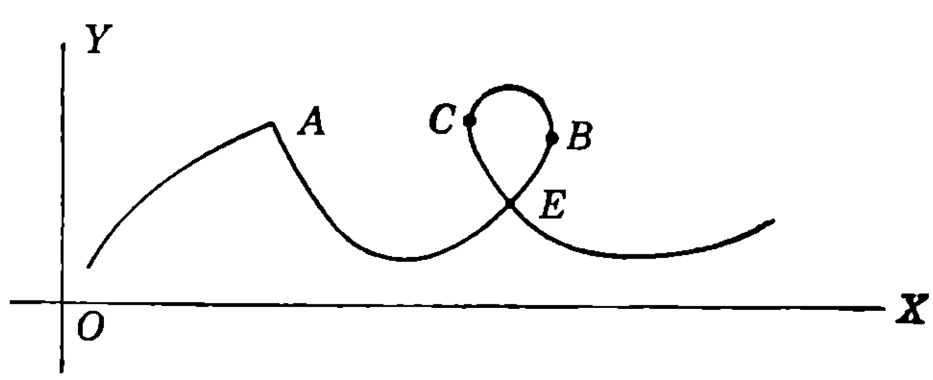

FIG. 76.

limits for $\Delta y/\Delta x$ at points such as A and E, through each of which there pass two distinct tangents, owing to the two distinct branches of the curve there converging. At other points unique derivatives may be missing for still different reasons which we lack the space to discuss. Our purpose here is merely to point out that the limit which is the derivative can sometimes fail to appear, and at the same time to assure you that it usually will come forth dutifully for well-behaved functions such as those one is likely to meet in practice.

Let's consider now a specific function of x, such as

$$y = 2x^2 \tag{6}$$

and proceed to find dy/dx at a sample point (x_0,y_0) on this parabola. If we remember that $dy/dx = \lim_{\Delta x \to 0} \Delta y/\Delta x$, this definition should give us the key to the necessary process, since we'll need to find the algebraic formula for Δy, divide this by Δx, and then seek the limit this ratio is approaching. Since (x_0,y_0) and $(x_0 + \Delta x, y_0 + \Delta y)$ are respectively the coordinates of points on curve (6) (corresponding with the points P and Q of Fig. 75), we have

$$y_0 = 2x_0^2$$

and also

$$\begin{aligned} y_0 + \Delta y &= 2(x_0 + \Delta x)^2 \\ &= 2x_0^2 + 4x_0\,\Delta x + 2\,\Delta x^2 \end{aligned}$$

Subtracting $y_0 = 2x_0^2$ from the respective sides, we get

$$\Delta y = 4x_0\,\Delta x + 2\,\Delta x^2$$

and

$$\frac{\Delta y}{\Delta x} = 4x_0 + 2\,\Delta x$$

No information is obtained from the left side of the last equation about

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

the quantity we are seeking, but from the alternate form of

$\Delta y/\Delta x$ on the right, we see that we can make it as close to $4x_0$ as we desire by taking Δx sufficiently small. In other words,

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = 4x_0$$

Graphically, this means that the slope of the line tangent to the parabola (6) at any point is equal to four times the abscissa of this point. The reader may check this result by drawing a part of the curve near the origin together with a tangent line or two on both sides of the Y axis.

In the above example we used the subscript 0 to emphasize the fact that we were considering a fixed point (x_0,y_0). However, $dy/dx = 4x$ for any point (x,y) on the curve. Hereafter we shall omit the subscripts.

The example illustrates the so-called *delta method* of differentiation. Even though it is replaced eventually in the conventional calculus course by the more or less mechanical short cuts of formulas, it is well worth practicing at first, since it recalls the meaning of the derivative with each application. Summarizing, to get dy/dx we proceed as follows:

1. Start with the function $y = f(x)$ and a sample point (x,y).
2. Substitute $x + \Delta x$ for x, getting $y + \Delta y = f(x + \Delta x)$.
3. Subtract $y = f(x)$ from both sides of the foregoing equation, obtaining $\Delta y = f(x + \Delta x) - f(x)$.
4. Divide both sides of the equation by Δx.
5. Find

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

indicating this important quantity by the notation dy/dx on the left, and getting the real information by setting $\Delta x = 0$ in the right member of the equation.

By way of a final example, suppose we apply these steps to the function $y = 1/x^2$.

1. $$y = \frac{1}{x^2}$$

2. $$y + \Delta y = \frac{1}{(x + \Delta x)^2}$$

3. $$\Delta y = \frac{1}{(x + \Delta x)^2} - \frac{1}{x^2} = \frac{-2x\,\Delta x - \Delta x^2}{x^2(x + \Delta x)^2}$$

4. $$\frac{\Delta y}{\Delta x} = \frac{-2x - \Delta x}{x^2(x + \Delta x)^2}$$

5. $$\frac{dy}{dx} = \frac{-2x}{x^2(x)^2} = \frac{-2}{x^3}$$

EXERCISE 57

When x is replaced by $x + \Delta x$ in each of Probs. 1 to 10 find (*a*) Δy, (*b*) $\Delta y/\Delta x$ and (*c*) dy/dx.

1. x. **2.** $3x$. **3.** $5x^2$. **4.** $\frac{1}{x}$.

5. $3x^2 - 2x$. **6.** $(x + 3)^2$. **7.** $(3x - 1)^2$.

8. $(3 - x)^2$. **9.** $\frac{2}{x^3}$. **10.** $\frac{1}{(x + 1)^2}$.

11. The distance s in feet covered by a body falling from rest is $16.1t^2$, where t is the elapsed time in seconds. Find the velocity of the body at the end of (*a*) 2 sec., (*b*) 5 sec.

12. Find the slope of the curve $y = x^3$ at (2,8).

13. Find the slope of the curve $y = 1/x^3$ at (1,1).

14. Find the equation of the line which is tangent to the curve $y = x^2 + x$ at (*a*) the point (1,2); (*b*) the point (−1,0).

15. Find the equation of the line which is perpendicular to the curve $y = 1 - x^2$ at (*a*) (1,0); (*b*) (2,−3).

85. Our results. Lest we create the wrong impression by the title of this article we'll state at once that "our results" consist of a list of formulas which are used for finding dy/dx when the function y belongs to certain special groups, and which are more efficient for this purpose than the longer delta method. They are special technical tools, then, rather than solutions of specific problems. Incidentally, some solutions of the latter type will be obtained in the next article.

Before launching into the derivation of these useful formulas, we'd better explain a little more fully the use of the symbol dy/dx. While in the final analysis it stands for $\lim_{\Delta x \to 0} \Delta y/\Delta x$, it is sometimes convenient to break it up into the two parts d/dx and y, and to think of the first part as a command to operate and of the second part as the subject of the operation. Thus $\frac{d}{dx}(y)$ may be read, "Take the derivative of y with respect to x," or more briefly, "Take the x derivative of y." Since, upon doing so, we get dy/dx, it follows that $\frac{d}{dx}(y) = \frac{dy}{dx}$. If y is a somewhat cumbersome expression, such as $(x\sqrt{x^2 + 1})/(x + 7)$, we may then write

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{x\sqrt{x^2 + 1}}{x + 7}\right)$$

instead of the awkward

$$\frac{d\left(\frac{x\sqrt{x^2 + 1}}{x + 7}\right)}{dx}$$

Alternate forms for $\frac{dy}{dx}$, assuming $y = f(x)$, are $\frac{df}{dx}$, $\frac{d}{dx}f(x)$ y', and $f'(x)$. One is likely to run across any of these when he rambles into mathematical territory.

Without further ado, then, we present herewith some formulas which are so useful to many brainworkers that, it must be admitted, they have already been widely advertised. For convenience we give them one in compact burst of information, with the explanation to follow in due time.

(1) $$\frac{d}{dx}(c) = 0, \text{ if } c \text{ is a constant}$$

(2) $$\frac{d}{dx}(x) = 1$$

(3) $$\frac{d}{dx}(cv) = c\frac{dv}{dx}$$

(4) $$\frac{d}{dx}(u + v) = \frac{du}{dx} + \frac{dv}{dx}$$

(5) $$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

(6) $$\frac{d}{dx}(v^n) = nv^{n-1}\frac{dv}{dx}$$

(6a) $$\frac{d}{dx}(x^n) = nx^{n-1}$$

(7) $$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

That completes our list. Experts will notice immediately, perhaps, that we have omitted many old favorites, such as the x derivatives of logarithmic, exponential, and trigonometric functions. The ones we have deigned to include, however, will illustrate well enough the principles involved, and certainly no one should demand all of the details in a brief survey which attempts to furnish merely an insight into the highlights of a subject. Besides, the incomplete nature of our previous sally into trigonometry prevents us from furnishing them at this point, anyway.

In regard to the letters u and v, they stand for functions of x of such a type that the derivative indicated in the formula exists. This state of affairs is the usual one when u and v are ordinary, run-of-the-mill functions with smooth-curve pictures, such as $x^2, x^3, \sqrt{x^2+1}$, etc. Under the specified conditions, as Δx approaches zero, the corresponding changes in u and v, designated as Δu and Δv, respectively, will naturally likewise approach zero, reaching it simultaneously with Δx. From the definition of the derivative,

$$\frac{du}{dx} = \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x}, \quad \frac{dv}{dx} = \lim_{\Delta x \to 0} \frac{\Delta v}{\Delta x}, \text{ etc.}$$

This information should enable the reader to derive the formulas all by himself, especially when he learns that the method is nothing more than the old delta one, modified just enough to take care of whole classes of functions instead of one at a time. Nevertheless, having had experience with mathematical tenderfeet before, we'll furnish the details and relieve the suspense:

Case 1.

$$\text{Let } y = c \qquad \text{(a constant)}$$

Since y is the same for all values of x (its graph being a straight line parallel to the X axis), it follows that as x changes by Δx, the corresponding change in y is zero. Thus $\Delta y = 0$;

$$\frac{\Delta y}{\Delta x} = \frac{0}{\Delta x} = 0$$

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} (0) = 0, \text{ proving (1)}$$

Case 2.

$$\begin{aligned} \text{Let } y &= x \\ y + \Delta y &= x + \Delta x \\ \Delta y &= \Delta x \\ \frac{\Delta y}{\Delta x} &= 1 \\ \frac{dy}{dx} &= \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = 1, \text{ proving (2)} \end{aligned}$$

Case 3.

$$\begin{aligned} \text{Let } y &= cv \\ y + \Delta y &= c(v + \Delta v) \\ \Delta y &= c(\Delta v) \\ \frac{\Delta y}{\Delta x} &= c\frac{\Delta v}{\Delta x} \\ \frac{dy}{dx} &= c\frac{dv}{dx} \end{aligned}$$

Case 4.

$$\begin{aligned} \text{Let } y &= u + v \\ y + \Delta y &= u + \Delta u + v + \Delta v \\ \Delta y &= \Delta u + \Delta v \\ \frac{\Delta y}{\Delta x} &= \frac{\Delta u}{\Delta x} + \frac{\Delta v}{\Delta x} \\ \frac{dy}{dx} &= \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} + \lim_{\Delta x \to 0} \frac{\Delta v}{\Delta x} \end{aligned}$$

since the limit of a sum is the sum of the respective limits of its parts (as can be proved, though we'll omit the details). Hence $\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$. An obvious extension of (4) is that *the* x *derivative of the sum of any number of functions of* x *is the sum of the derivatives of the separate functions.*

Case 5.

$$
\begin{aligned}
\text{Let } y &= uv \\
y + \Delta y &= (u + \Delta u)(v + \Delta v) \\
\Delta y &= u\,\Delta v + v\,\Delta u + \Delta u\,\Delta v \\
\frac{\Delta y}{\Delta x} &= u\frac{\Delta v}{dx} + v\frac{\Delta u}{\Delta x} + \Delta u\left(\frac{\Delta v}{\Delta x}\right) \\
\frac{dy}{dx} &= u\frac{dv}{dx} + v\frac{du}{dx} + 0\frac{dv}{dx} \\
&= u\frac{dv}{dx} + u\frac{du}{dx}, \text{ proving (5)}
\end{aligned}
$$

Here you must be sure to keep in mind the fact that the limit of each of the separate deltas (Δx, Δy, Δu, Δv) is zero, but that the limit of *the ratio of any two* is the as yet undetermined quantity which is designated as the derivative and determined as a separate problem in each separate case.

Case 6. To prove that (6) holds when n is a positive integer, we'll resort to the method called *mathematical induction*, which will be honored in due time with a whole chapter of discussion. This, however, is an excellent opportunity for an easy-to-follow preview of the method in operation. Since two steps are always involved, we'll break up the proof accordingly.

(a) Formula (6) is true when $n = 1$ (since $v^0 = 1$ by definition) and also when $n = 2$, since by (5),

$$\frac{d}{dx}(v^2) = \frac{d}{dx}(vv) = v\frac{dv}{dx} + v\frac{dv}{dx} = 2v\frac{dv}{dx}$$

It follows, furthermore, that

$$
\begin{aligned}
\frac{d}{dx}(v^3) = \frac{d}{dx}(vv^2) &= v\frac{d}{dx}(v^2) + v^2\frac{d}{dx}(v) \\
&= v\left(2v\frac{dv}{dx}\right) + v^2\frac{dv}{dx} = 3v^2\frac{dv}{dx}
\end{aligned}
$$

so that (6) still holds when $n = 3$. (These two latter results are not an essential part of the proof; but they suggest the proper step in the final clinching argument below.)

(b) Let's assume that (6) holds when n is a certain fixed integer k. That is, we'll assume that

$$\frac{d}{dx}(v^k) = kv^{k-1}\frac{dv}{dx}$$

and we'll show that, in this case, (6) will still hold when $n = k + 1$. For

$$\frac{d}{dx}(v^{k+1}) = \frac{d}{dx}(vv^k) = v\frac{d}{dx}(v^k) + v^k\frac{d}{dx}(v)$$

Hence, using our above assumption about $\frac{d}{dx}(v^k)$, we have

$$\begin{aligned}\frac{d}{dx}(v^{k+1}) &= v\left(kv^{k-1}\frac{dv}{dx}\right) + v^k\frac{dv}{dx}\\ &= kv^k\frac{dv}{dx} + v^k\frac{dv}{dx}\\ &= (k+1)v^k\frac{dv}{dx}\end{aligned}$$

and this is exactly what we get from (6) when we replace n by $k + 1$. In other words, if (6) holds when n is some positive integer, it will continue to hold for the next larger integral value of n. But in part (*a*) we demonstrated the truth of (6) for $n = 1$; and hence the argument of part (*b*) proves its truth for $n = 2$, then for $n = 3$, $n = 4$, and so on, until no positive integer is omitted, no matter how large it may be.

It happens that the reliability of this redoubtable formula (6) is not limited to the cases for which the above proof applies, since it works perfectly by virtue of certain logical definitions when n has any real value, whether positive or negative, and whether integral, fractional, or even irrational. But to those thorough ones who insist upon a treatment of the extended cases at this point, all we can say is that they will just have to look it up in another book. After all, we must stop somewhere.

Now, after that bout with (6), the proof of (7) would be too much of an anticlimax. Since it requires only the straightforward application of the delta method, we'll leave it to the amateurs with perfect confidence. It will be worth while to put the formula in words and to remember that we start with the *denominator*.

Having disposed of the formulas, we could leave the tyro in a daze before the examples in the next exercise—but we won't. Skill in differentiation, as in the use of any other highly specialized tool, is usually attained only after a considerable amount of practice, and the way is long enough even with a starting push in the form of illustrative examples. A very special warning about one thing will probably help the beginner even more than the problems. It is this. Always, at first, make a final checkup to see that the dv/dx part of the formula is represented in the result, since its omission is by all odds the most popular of the many interesting and original blunders which are associated with these problems. More briefly: *Watch that dv/dx.*

Illustrative Examples

1. Let $y = 4x^3 + 2x^2 + 7$.

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}(4x^3 + 2x^2 + 7)\\ &= \frac{d}{dx}(4x^3) + \frac{d}{dx}(2x^2) + \frac{d}{dx}(7) && \text{[by (4)]}\\ &= 4\frac{d}{dx}(x^3) + 2\frac{d}{dx}(x^2) + \frac{d}{dx}(7) && \text{[by (3)]}\\ &= 4(3x^2)\frac{dx}{dx} + 2(2x)\frac{dx}{dx} + 0 && \text{[by (6) and (1)]}\\ &= 12x^2 + 4x. && \text{[by (2)]}\end{aligned}$$

2. Let $y = \sqrt{2x^3 - x + 2}$.

Since (6) holds when n is a fraction, we have

$$\frac{dy}{dx} = \frac{d}{dx}(2x^3 - x + 2)^{1/2} = \frac{1}{2}(2x^3 - x + 2)^{-1/2}\frac{d}{dx}(2x^3 - x + 2)$$

Notice how slavishly we follow the formula, indicating with the operational sign d/dx the uncompleted part of the differentiation. The beginner should have the formula (6) before him, either on paper or in his mind. Here

$$v = 2x^3 - x + 2,\ n = \tfrac{1}{2},\ n - 1 = -\tfrac{1}{2}$$

and

$$\frac{dv}{dx} = \frac{d}{dx}(v) = \frac{d}{dx}(2x^3 - x + 2)$$

Continuing the process, and abbreviating the steps in finding

$$\frac{dv}{dx} = 6x^2 - 1$$

which are like those in *Example* 1, we have

$$\frac{dy}{dx} = \frac{1}{2}(2x^3 - x + 2)^{-1/2}(6x^2 - 1) = \frac{6x^2 - 1}{2\sqrt{2x^3 - x + 2}}$$

3. Let $y = (x^2 - 2)\sqrt{x^2 + 1}.$

Here both $x^2 - 2$ and $\sqrt{x^2 + 1}$ involve x and hence are functions of x, so that (5) applies. [If $x^2 - 2$ were replaced by a constant, (5) could still be used, but (3) would be shorter.] Then

$$u = x^2 - 2,\ v = \sqrt{x^2 + 1}$$

and

$$\frac{dy}{dx} = (x^2 - 2)\frac{d}{dx}\sqrt{x^2 + 1} + \sqrt{x^2 + 1}\frac{d}{dx}(x^2 - 2)$$

$$= (x^2 - 2)\frac{1}{2}(x^2 + 1)^{-1/2}\frac{d}{dx}(x^2 + 1) + \sqrt{x^2 + 1}\left[\frac{d}{dx}(x^2) - \frac{d}{dx}(2)\right]$$

$$= (x^2 - 2)\frac{1}{2}(x^2 + 1)^{-1/2}(2x + 0) + \sqrt{x^2 + 1}(2x - 0)$$

$$= \frac{x(x^2 - 2)}{\sqrt{x^2 + 1}} + 2x\sqrt{x^2 + 1} = \frac{3x^3}{\sqrt{x^2 + 1}}$$

4. Let $y = \dfrac{3x^2 - x}{x^3 + 1}$

Here (7) applies, with $u = 3x^2 - x$ and $v = x^3 + 1$

$$\frac{dy}{dx} = \frac{(x^3 + 1)\frac{d}{dx}(3x^2 - x) - (3x^2 - x)\frac{d}{dx}(x^3 + 1)}{(x^3 + 1)^2}$$

$$= \frac{(x^3 + 1)(6x - 1) - (3x^2 - x)(3x^2)}{(x^3 + 1)^2}$$

(We're beginning to telescope operations. With practice you'll do this more and more.) Finally,

$$\frac{dy}{dx} = \frac{-3x^4 + 2x^3 + 6x - 1}{(x^3 + 1)^2}$$

EXERCISE 58

Find dy/dx by use of the proper formulas when y has the values indicated in Probs. 1 to 16.

1. π. **2.** $3x$. **3.** $x + \dfrac{1}{x}$. **4.** x^5. **5.** $\dfrac{1}{x^3}$ (or x^{-3}).

6. $\sqrt{5x}$.

7. $\dfrac{1}{\sqrt{x^5}} + \dfrac{3}{x^{10}}$ (HINT: Write this $x^{-5/2} + 3x^{-10}$).

8. $14x^3 + 3x^2 - 2$.

9. $3x^{10} - (\frac{2}{3})x^{3/2} + \frac{3}{8}$. **10.** $\sqrt[5]{6x - 9}$.

11. $\sqrt{2x} - 5x^{-2/5}$. **12.** $(2x + 3)/(3x - 2)$.

13. $\sqrt{3x^2 - 5x}$. **14.** $1/\sqrt{7x}$.

15. $x\sqrt{2x + 1}$. **16.** $(3x - 2)\sqrt[3]{2x + 1}$.

17. Find the points at which the slope of the curve $y = \dfrac{2x^3}{3} + \dfrac{x^2}{2} - 3x$ is zero.

18. Find the equation of the line with slope 8 which is tangent to the curve $y = 2x^4$.

19. At what points does the ellipse $\dfrac{x^2}{4} + \dfrac{y^2}{25} = 1$ have the slope $\frac{5}{2}$?

20. Find the coordinates of the lowest point on the curve $y = x^3 - x^2$ which is at the right of the Y axis.

86. What we can do with them. Now that we have disposed of some of the differentiation formulas which are among the fundamental tools of calculus, we certainly should also include some of the practical results without which the technical machinery would lose most of its significance. These results are twofold in nature. In the first place, they solve important problems in a thousand fields of industry and research. Secondly, they throw incidental light on the lighting system itself. By this we mean that they help in the quick analysis and plotting of the analytic geometry curves used to put the results in picture form.

Since the second (and probable lesser) of these achievements helps in the accomplishment of the primary one, we'll take them up in order of convenience rather than importance. And this brings us to the subject of curves and their relation to the derivative.

If the reader recalls that dy/dx, when interpreted geometrically, is the slope of the line tangent to the curve $y = f(x)$ at (x,y), and if furthermore he gets clearly in mind what is meant by the slope of a line, he will be ready for the first announcement. From the definition we see that the slope of a line is positive if a point can be moved on it upward and to the right at the same time. It follows that at any point on a curve at which the derivative is positive the curve is rising to the right, since it practically coincides with the tangent line if we consider a short enough segment of it containing the point of tangency. Similarly, where the slope is negative the curve is "falling," or descending to the right.

And, of course, when the slope is zero, the curve is neither falling nor rising at that particular spot. In this case the point thus determined is called a *critical point* and may mark a local high place on the curve (*maximum*), a low place (*minimum*), or merely a point at which the curve momentarily pauses in its ascent or descent. In Fig. 77, A marks a maximum of the function, B a minimum, and C and D are zero-slope points on a generally falling or rising curve.

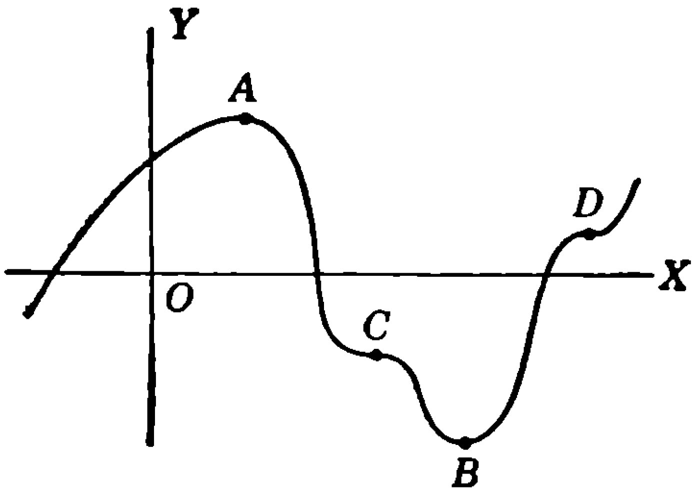

FIG. 77.

Now in many problems concerning two variables the maxima and minima of the corresponding curve, as distinguished from zero-slope points like C and D, are of special importance. Evidently we can find all four points A, B, C, and D by finding dy/dx and setting it equal to zero. If the graph is like most curves encountered in practice,[1] a maximum point such as A is determined from the fact that the slope of the curve is positive at the left of A and negative at the right when we test a part of the curve which does not lie beyond the nearest critical point on each side. Similarly, we could learn before plotting that B is a minimum because the slope is negative at the left and positive at the right if again we stay within the prescribed bounds.

To illustrate, let's examine the curve

$$(1) \qquad y = 3x^3 - 9x + 2$$

which is a normal and well-behaved specimen. The derivative is $dy/dx = 9x^2 - 9 = 9(x-1)(x+1)$. Obviously $dy/dx = 0$ when $x = 1$ or -1. Computing the corresponding y, we get the points A $(1,-4)$ and B $(-1,8)$ at which the slope is zero. Now when $x < -1$, $dy/dx > 0$, since the factors $x - 1$ and $x + 1$ are each negative. Thus we see that the curve is rising from the left to B. Next we examine $x > -1$, being careful, however, to make $x < 1$,

[1] We implicitly assume here that the curve is continuous and has a continuous derivative (barring corners like the bottom of a "V"). When writers forget or ignore these possibilities they are guilty of the mathematical sin called "lack of rigor." Such oversights sometimes lead to erroneous conclusions, so that the careful thinker should be on his guard against them.

since 1 is the next critical point at the right. In this allowable range ($-1 < x < 1$), $x - 1$ is negative and $x + 1$ is positive, so that dy/dx is negative. Now we know that B is a maximum. When $x > 1$, $dy/dx > 0$, since both factors of dy/dx are positive, so that the point A is evidently a minimum. The graph of a part of curve (1) appears in Fig. 78. Notice that while we could have sketched the curve without the help of calculus, we needed the latter to be sure of the exact points at which the curve reaches a "crest" (maximum) or "trough" (minimum).

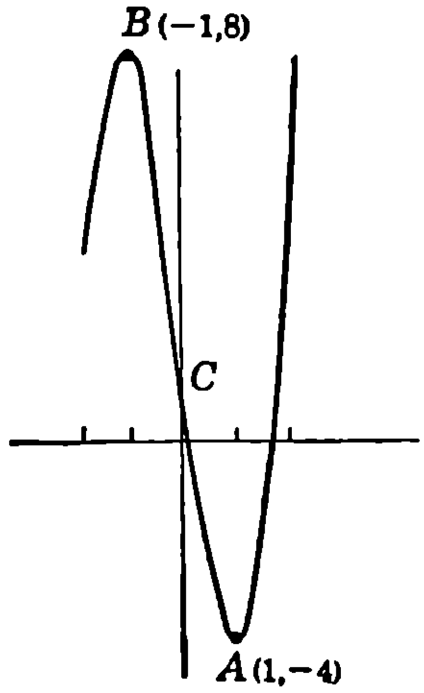

FIG. 78.

A second bit of definite information about the curve yielded by the methods of calculus concerns the points at which the curve changes from *concave downward* to *concave upward* or vice versa. If we consider a point moving from left to right along a curve which bends upward like the bottom half of a circle, we notice that the slope is algebraically increasing (in the case cited from $-\infty$ to 0 to ∞). Just as velocity, or ds/dt, is positive when s increases with t, so dm/dx is positive when the slope m ($= dy/dx$) increases with x. This quantity dm/dx is called *the second derivative of* y *with respect to* x and is designated usually by the symbol d^2y/dx^2, though its meaning is made clearer at first by the form

$$\frac{d}{dx}(m) \text{ or } \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

Now, as we have seen, d^2y/dx^2 is positive when the curve bends upward and (similarly) negative when it bends downward. We might carelessly assume, then, that it has to be zero when the curve changes from *concave upward* to *concave downward* at a so-called *point of inflection*. Though this is not necessarily true,[1] it often is the case. To put it in another way, any point on the curve at which $d^2y/dx^2 = 0$ marks a place of peculiar importance

[1] For the benefit of the critical reader who would like an example of a point of inflection at which $d^2y/dx^2 \neq 0$, we oblige with the point of contact of two semicircles placed left-end to right-end, with the two slopes equal at the point of contact. Here d^2y/dx^2 is discontinuous and double valued, the two values differing by an amount ranging from $(1/r + 1/R)$ to infinity.

one way or another. If it is a point of inflection the value of d^2y/dx^2 will have opposite signs within certain bounds on the two sides of the point. This is illustrated by C in Fig. 78. From (1) we have

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}(9x^2 - 9) = 18x$$

This is zero when $x = 0$, and negative and positive, respectively, when $x < 0$ and $x > 0$. Hence we see that the curve (1) bends downward as far as we care to follow it to the left of C and bends upward without ceasing at the right of C.

All of this information about the illustrative machinery for calculus (analytic geometry curves) comes from the formulas of calculus itself by a sort of backhanded poetic justice. It all goes to show how these various branches of mathematics work hand in glove together.

EXERCISE 59

With reference to the curves defined in Probs. 1 to 10, find the points at which $dy/dx = 0$ (critical points), $d^2y/dx^2 = 0$, and $dy/dx = \infty$. Then sketch the curves, showing the geometric significance of each point thus algebraically determined.

1. $y = 2x^3 - 9x^2 + 12x - 6.$ **2.** $y = 1 + 12x - 3x^2 - 2x^3.$

3. $y = (x + 3)^4.$ **4.** $y = \dfrac{x}{1 + x^2}.$

5. $y = \dfrac{x + 1}{x - 1}.$ **6.** $y = \pm\sqrt{x}.$

7. $y = (x + 2)^4 (x - 1)^3$. HINT: Use (5), Art. 85, in finding dy/dx and then factor without expanding the binomials.

8. $y = (2x + 1)^2 (1 - 3x)^3.$ **9.** $y = \dfrac{2x + 1}{1 - 2x}.$

10. $y = \dfrac{x}{1 - x^2}.$

11. Sketch the curve $y = (1 - x^{2/3})^{3/2}$.

12. Sketch the curve $y = (1 \pm x^{1/2})^2$.

87. We come down to cases. In the previous article we promised to suggest what may be done with the elementary formulas of calculus by way of meeting directly many challenging problems of life. It seemed advisable at first, however, to point out some sidelights on curves which cropped up on the way. And now, by a happy coincidence, these very results, which may have seemed interesting but unimportant, suddenly assume a practical bread-and-butter meaning in a large group of problems. The poorer students falter before them; but perhaps they would try harder if they could realize the importance of the so-called *maxima and minima* problems. For these brain teasers are in fact the main concern of business, which is that of getting the maximum return on an investment, and of engineering, which seeks to do a thing in the best and most efficient way. The builder who wants to know how many stories high he should make the skyscraper, transport companies which seek the most economical speeds for their ships, trains, and trucks, even manufacturers of the lowly can, who want to use as little tin as possible for a product of given capacity—all of these, and many others, are faced with problems essentially like those which harass the schoolboy. Judging by the fact of business failures, some of these puzzlers are a bit troublesome to a few of the elders as well.

To avoid complicating details and yet bring out the principle of the method, we'll begin with a simple situation. A boy has 50 ft. of wire netting with which he wishes to build three sides of a rectangular rabbit pen, the fourth side of which will be the yard fence. What should be the ratio of length to width so that the area enclosed shall be as large as possible?

Solution: Let x be the width of the pen. Then $50 - 2x$ is the length, and $x(50 - 2x)$ is the area, or the function of x which is to be stretched to the limit. Then, using our new-found technique,

$$A = x(50 - 2x) = 50x - 2x^2$$

$$\frac{dA}{dx} = 50 - 4x$$

Now $50 - 4x = 0$ when $x = \frac{50}{4} = \frac{25}{2}$, in which case $50 - 2x = 25$, so that the graph of the function representing the area has a zero slope when the length is exactly twice the width. A little consideration (and also some other trial dimensions, if necessary) shows us that this is a *maximum* area and that the smallest possible area would be enclosed either by the impractical method of putting the rabbit fence slapdash against the yard fence or else by placing two sides together perpendicular to the fence. Can you see why these minimum solutions do not also come to light when we set dA/dx equal to zero?

Before leaving the struggling learner to his own resources in the looming shadow of our next stack of problems, we'll let him take one more practice swing:

A depot is located at the intersection of an east-west railroad and a north-south highway. Two budding mathematicians leave the depot for a boy-scout camp located in the woods 5 miles south of the railroad and 3 miles from the highway. If they estimate that they can walk 5 and 4 miles per hour respectively on the highway and in the forest, where should they leave the road in order to reach the camp as soon as possible? (Hitchhiking possibilities are barred, of course.)

Solution: First we must decide, *as always* in such problems, what quantity it is that is to be as large or as small as possible. In the foregoing problem the area was to be a maximum; in this one the *time* is to be a minimum. Evidently we'll need our old standby: Distance = (rate)(time). Abbreviated, it is $d = rt$, or, solving for the quantity in which we are most interested, $t = d/r$.

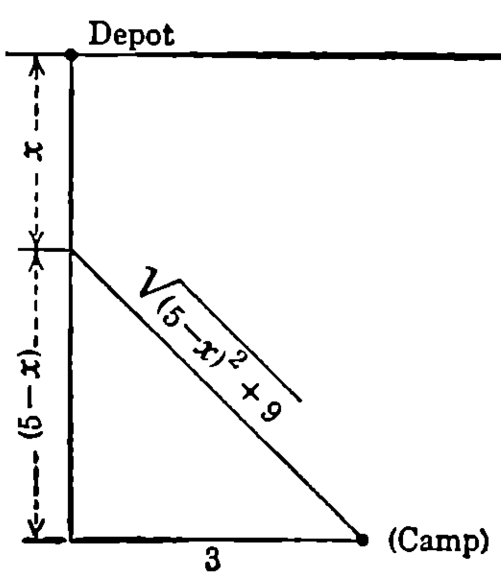

FIG. 79.

Clearly we'll need two mathematical expressions, one for the time spent on the road and the other for the time in the woods. About here a figure will help. (Fig. 79.)

If we let x be the road-walking distance, the other distances come out as shown on the figure, and as found by the Pythagorean theorem. Our formula for time then yields:

$$T \text{ (total time)} = \text{time on highway} + \text{time in woods}$$

$$T \text{ (total time)} = \frac{\text{distance on highway}}{\text{rate on highway}} + \frac{\text{distance in woods}}{\text{rate in woods}}$$

$$T = \frac{x}{5} + \frac{\sqrt{(5-x)^2+9}}{4}$$

$$= \frac{4x + 5\sqrt{x^2 - 10x + 34}}{20}$$

$$\frac{dT}{dx} = \frac{1}{20}\left[4 + \frac{5(2x - 10)}{2\sqrt{x^2 - 10x + 34}}\right]$$

$$= \frac{1}{20}\left[\frac{4\sqrt{x^2 - 10x + 34} + 5(x - 5)}{\sqrt{x^2 - 10x + 34}}\right]$$

Now when dT/dx equals zero, the numerator of the expression on the right is zero. Hence the equation

$$4\sqrt{x^2 - 10x + 34} = 5(5 - x)$$

appears when we follow our mechanical routine method. Squaring, simplifying, and solving, we get $x = 9$ or 1. The former value of x is obviously not what we want, whatever it is. Since our common sense tells us that there must be a value of x between zero and five at which the time required is a minimum, and since only one such answer appears, the decision is easy: The time is least when they leave the road 1 mile from the station. Of course the doubter can check this by our mathematical rule about slopes if he insists, but it shouldn't be necessary. We find frequently in this type of problem that common sense will save us from the entanglements of algebra in its more complicated and annoying forms.

EXERCISE 60

1. Find the dimensions of the largest rectangular field which can be enclosed with 100 yd. of fencing.

2. What must be the dimensions of a rectangular field containing 100 sq. rods in order that as little fence as possible shall be required to enclose it?

3. Find the dimensions of the largest rectangular field which will require 300 rods of fencing if one-half of one side is left unfenced.

4. Find the volume of the largest box which can be made from a square piece of cardboard with a 12-in. side by cutting a square from each corner and folding up the sides.

5. Work Prob. 4 if the square piece of cardboard has a side of h in.

6. A piece of tin 12 in. wide is bent into a V-shaped gutter. At what depth will it allow a maximum amount of water to run through it?

7. A man in a lighthouse 3 miles from a point A directly west of him on the north-south shore wishes to reach a point B on the shore

5 miles below A. How many miles below A is the spot to which he should point his prow if he can row 4 and walk 5 miles per hour?

8. Find the length of the shortest line segment whose ends lie on the coordinate axes and which passes through the point (3,4). HINT: Express the equation of the line through (3,4) in the form $y - 4 = m(x - 3)$.

9. Find the length of the shortest ladder which, when it is standing on the ground, will pass over a board fence 5 ft. high and touch the side of a house which is 4 ft. from the fence.

10. The strength of a beam is proportional to the width and the square of the height of its cross section. Find the dimensions of the strongest rectangular beam which can be hewn from a tree trunk 10 in. in radius.

11. A window is in the form of a rectangle surmounted by a semicircle. If the perimeter of the window is fixed in advance, what is the ratio of the width to the height of the rectangle in order that as much light as possible may be admitted? (The area and circumference of a circle of radius r are πr^2 and $2\pi r$, respectively.)

12. What is the ratio of the diameter to the height of a cylindrical can with a circular base in order that a minimum amount of tin may be used in a can with a given capacity?

13. Work Prob. 12 if one end of the can is open.

14. How many stories high should a hotel be made in order to yield the largest possible percentage of return on the investment if the lot and first story cost \$20,000 and \$10,000 respectively and if, due to foundation strengthening and elevator requirements, the cost of the nth story, when $n > 1$, is $n/2$ times the cost of the first? Assume that the income is proportional to the number of stories.

15. Two towns A and B are 6 and 12 miles, respectively, east of a river running north and south. Points C and D on the river are due west of A and B respectively, and C is 20 miles north of D. How far south of C should a filter plant F be placed in order that the sum of the lengths of the two water mains FA and FB may be a minimum?

16. The cost per mile for fuel to run a certain coast steamer varies within certain limits as the square of the velocity, and amounts to \$1 per mile at 20 miles per hour. Other expenses amount to \$20 per hour. Find the most economical rate (least cost per mile).

88. Backing into new territory. Earlier in this chapter we said that the symbol dy/dx, not only stands for the single number representing the slope of the line tangent to $y = f(x)$ at (x,y), but may actually be taken apart and given a meaning in its fractional or two-story aspect. To get a geometric interpretation of the respective parts dy and dx, we need again the figure which shows a portion of the curve in the neighborhood of the general point (x,y). In Fig. 80, let the coordinates of P and Q be (x,y) and $(x + \Delta x, y + \Delta y)$ respectively. It will be recalled, we trust, that $dy/dx = \tan \phi =$ slope of line PT. Now if we define dx by the equation

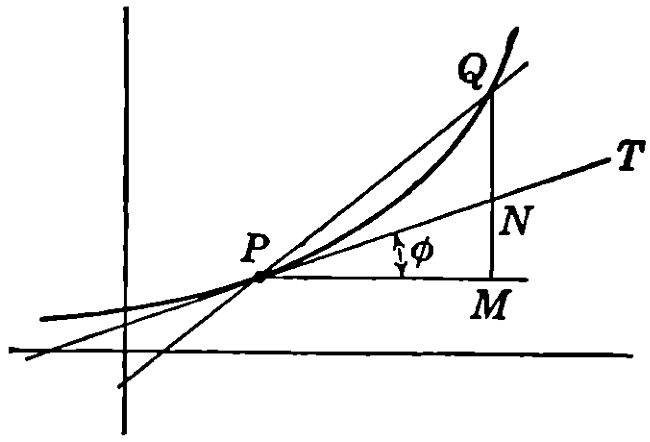

FIG. 80.

$$dx = \Delta x = PM$$

then, in order to treat dy/dx like an ordinary fraction, we must have

$$dy = (\tan \phi)dx = MN \quad \text{(geometrically)}$$

This quantity dy, called the *differential* of y, evidently differs from Δy by an amount (NQ in the figure) which becomes relatively unimportant for small values of dx, so that dy serves as an approximation for Δy when both are small. We should note that for a given point (x,y) the quantities Δx, Δy, dx, and dy are all *infinitesimals* (which is a mathematical way of saying that we can vary them and make them as near to zero as we please), while the value of dy/dx is unalterably fixed for that point and that curve. And furthermore, since dy/dx is actually a ratio, we can find dy in a given case by simply multiplying the x derivative of y by dx. For example, if $y = 3x^3 - 2x$, then $dy/dx = 9x^2 - 2$, and $dy = (9x^2 - 2)dx$.

At this point we're ready to switch into reverse and investigate the appearance of the territory we've come over when looked at from the opposite direction. In this case, the simple process of backing up leads to some very important discoveries. Instead of starting with a function and getting its derivative as before, we'll start with the derivative and look for the original function. The result is surprising.

To get under way, let's say that a given function $f(x)$ is the derivative of some unknown function $F(x)$, so that the differential of $F(x)$ is $f(x)dx$. Then, if we designate by the symbol $\int f(x)dx$ the function whose differential is $f(x)dx$, we find that

$$\int f(x)dx = F(x) + C \quad (C = \text{any constant})$$

since the differential of a constant is zero, and hence

$$d[F(x) + C] = dF(x) + d(C) = f(x)dx$$

The quantity $\int f(x)dx$ is called the *integral* of $f(x)dx$, the symbol resembling an elongated *s* is known as the *integral sign*, and the process of finding the specific $F(x)$ which is indicated by that symbol is called *integration.*

With these definitions out of the way we are ready to assert that, when n is a constant,

$$\textbf{(1)} \qquad \int v^n \, dv = \frac{v^{n+1}}{n+1} + C$$

We may verify the assertion by noting that, according to (4), (1), (3), and (6) of Art. 85, and with the assumption that v is a function of the independent variable x

$$\frac{d}{dx}\left(\frac{v^{n+1}}{n+1} + C\right) = \frac{d}{dx}\left(\frac{v^{n+1}}{n+1}\right) + \frac{d}{dx}(C)$$

$$= \frac{1}{n+1}\frac{d}{dx}(v^{n+1}) + 0$$

$$= \frac{1}{n+1}(n+1)v^n \frac{dv}{dx} = v^n \frac{dv}{dx}$$

Therefore, since the differential of a function of x is merely its x derivative multiplied by dx,

$$d\left(\frac{v^{n+1}}{n+1} + C\right) = v^n \, dv$$

which proves (1). This important integration formula holds for any value of n except -1, which evidently puts zero into the denominator on the right. Thus, when $v = x$,

$$\int x^5 \, dx = \frac{x^6}{6} + C$$

and

$$\int \sqrt{x^3} \, dx = \int x^{3/2} \, dx = \frac{x^{5/2}}{\frac{5}{2}} + C = \frac{2x^2\sqrt{x}}{5} + C$$

Two more useful formulas are:

$$(2)\qquad \int c\,dv = c\int dv$$

$$(3)\qquad \int (du + dv) = \int du + \int dv$$

In words, we may say that *a constant factor may be taken to the left of the integration sign*, and that *the integral of a sum is the sum of the integrals*. These rules of integration follow rather obviously from the fact that $d(cv) = c\,dv$ and $d(u + v) = du + dv$, as we learn when we multiply both sides of Eqs. (3) and (4), Art. 85, by dx. By use of (3), (2), and (1) in that order we find, for example, that

$$\begin{aligned}\int(3x^3 - 2x^2 - 4x + 3)dx &= \int(3x^3\,dx - 2x^2\,dx - 4x\,dx + 3\,dx)\\ &= \int 3x^3\,dx + \int(-2x^2)dx + \int(-4x)dx + \int 3\,dx\\ &= 3\int x^3\,dx + (-2)\int x^2\,dx + (-4)\int x\,dx + 3\int dx\\ &= 3\left(\frac{x^4}{4}\right) - 2\left(\frac{x^3}{3}\right) - 4\left(\frac{x^2}{2}\right) + 3x + C\\ &= \tfrac{3}{4}x^4 - \tfrac{2}{3}x^3 - 2x^2 + 3x + C\end{aligned}$$

To illustrate another point we'll integrate $\int 5x\sqrt{3x^2 - 1}\,dx$ thus: Let $v = 3x^2 - 1$. If (1) is to apply here, we must have for the dv of the formula: $dv = d(3x^2 - 1) = 6x\,dx$. Therefore we take out what we don't want (5 in this case), put in what we do want (6), and put the reciprocal of the latter outside the integration sign, so that we are not changing the value of the integral, thus:

$$\begin{aligned}\int 5x\sqrt{6x^2 - 1}\,dx &= \int(6x^2 - 1)^{1/2}5x\,dx\\ &= 5\int(6x^2 - 1)^{1/2}x\,dx \qquad \text{(since 5 is not wanted)}\\ &= \tfrac{5}{6}\int(6x^2 - 1)^{1/2}6x\,dx \qquad \text{(since 6 is needed)}\\ &= \frac{5}{6}\,\frac{(6x^2 - 1)^{3/2}}{\frac{3}{2}} + C \qquad \text{[by (1)]}\\ &= \tfrac{5}{9}(6x^2 - 1)^{3/2} + C\end{aligned}$$

EXERCISE 61

Perform the indicated integrations in Probs. 1 to 20.

1. $\int(2x^3 - 3x^2 + 5x - 4)dx$. **2.** $\int\sqrt{5x}\,dx$. HINT: $\sqrt{5x} = \sqrt{5}x^{1/2}$.

3. $\displaystyle\int\sqrt{\frac{3}{x}}\,dx$. HINT: $\sqrt{\dfrac{3}{x}} = \sqrt{3}x^{-1/2}$. **4.** $\int(3 - 2x)^{10}\,dx$. HINT: Since $d(3 - 2x) = -2dx$, we insert -2 and use formula (1).

5. $\int(5 + 3x)^2 x\, dx$. Hint: Since $d(5 + 3x) = 3\, dx$, which does not involve x, we cannot use (1) at once, but must square the binomial and collect terms.

6. $\int\sqrt[3]{2x}\, dx$.

7. $\int\sqrt[4]{\frac{3}{2x}}\, dx$.

8. $\int 2(5 + 2x)^3\, dx$.

9. $\int(2x)^5\, dx$.

10. $\int x(1 - x)^3\, dx$.

11. $\int\frac{dx}{\sqrt{1 - x}}$.

12. $\int 5x^3\sqrt{3x^4 - 2}\, dx$.

13. $\int 2x^2\sqrt[4]{5x^3 + 1}\, dx$.

14. $\int(2x^2 - 3)3x\, dx$.

15. $\int(x^3 - 2)^2 4x\, dx$.

16. $\int\frac{3x\, dx}{\sqrt{5 - 2x^2}}$.

17. $\int\frac{4x^2\, dx}{\sqrt[3]{x^3 + 7}}$.

18. $\int\frac{(4x + 10)dx}{\sqrt{x^2 + 5x - 2}}$.

19. $\int(x^3 - 7x + 1)^{2/3}(9x^2 - 21)dx$.

20. $\int\left(\sqrt{3}x - \frac{2}{\sqrt{5x}} + \frac{3}{4}\right)dx$.

Thus far our purely mechanical integration formulas have involved nothing which was not to be expected. The surprise we promised lies in the truly remarkable geometric interpretation of the integral—an interpretation which opens the gates of research to a vast array of particular results in the field of geometry which it would be difficult to obtain otherwise. These results include compact formulas for an unlimited number of areas enclosed within curves (like the area in a circle) or contained in the surfaces of solid figures (such as spheres, pyramids, etc.). While, as we shall show, the solutions of such geometric problems by no means exhaust the uses of integration, they do lead us into one wide field of usefulness to which we can direct a passing glance or two.

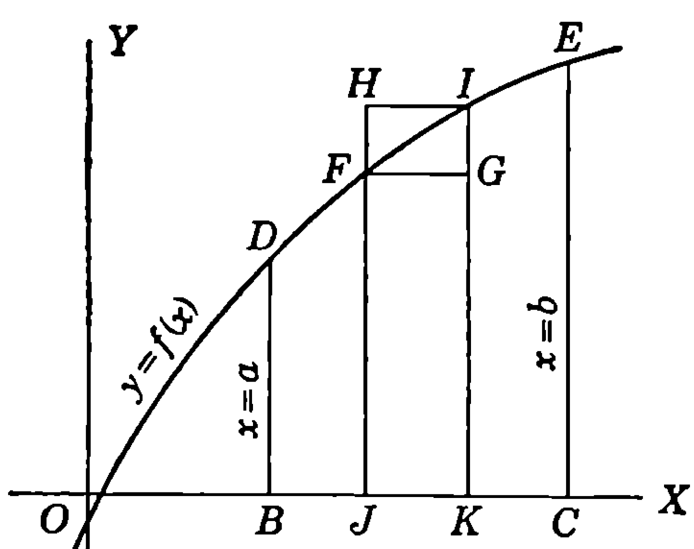

Fig. 81.

Consider, for instance, the area A' bounded by the X axis, the lines $x = a$, $x = b$, and the graph of $y = f(x)$. In Fig. 81 $A' =$

$BCED$. Let the coordinates of F and I be (x,y) and $(x + \Delta x, y + \Delta y)$ respectively, and let A and ΔA be the respective parts of A' directly below the respective arcs DF and FI. Then

$$FJKG < \Delta A < HJKI$$

or

$$y\,\Delta x < \Delta A < (y + \Delta y)\Delta x$$

But since Δx is positive, we may divide through by it without reversing the inequality sign. Hence

$$y < \frac{\Delta A}{\Delta x} < (y + \Delta y)$$

Now, as we have pointed out before, the limit of each separate delta is zero, but the limit of the ratio of two, such as $\Delta A/\Delta x$ in this instance, has to be determined according to the conditions. It follows that, since $\Delta A/\Delta x$ is between y and a quantity whose limit is y, the limit of $\Delta A/\Delta x$ (or dA/dx by our own convenient derivative notation) is also y. Thus

(4) $$\frac{dA}{dx} = y$$

emerges as the first result, hinting of the important geometric meaning of integration. The differential notation immediately translates this result into

(5) $$dA = y\,dx = f(x)dx$$

and integration of both sides yields

(6) $$A = \int f(x)dx = F(x) + C$$

where evidently $F(x)$ is the function whose x derivative is $f(x)$ and whose differential is $f(x)dx$.

The fortunate fact that C can be any constant we'll use very cunningly (with apologies and thanks to our predecessors). Since by previously stated conditions we are concerned with the area $A' = BCED$, we shall begin measuring A at the line BD, so that $A = 0$ when $x = a$. Substituting these values in (6), we have $0 = F(a) + C$, so that $C = -F(a)$ and (6) goes into

(7) $$A = F(x) - F(a)$$

In other words, the area enclosed between the X axis, the curve $y = f(x)$, and the ordinates $x = a$ and $x = x$ is expressed by the right side of (7). Evidently when the variable x reaches b the varying area A has increased to A', and we have

(8) $$A' = F(b) - F(a)$$

Another notation for this result is

(9) $$A' = \int_a^b f(x)dx = F(x)\Big]_a^b = F(b) - F(a)$$

in which the expression $\int_a^b f(x)dx$ is called a *definite integral* and is evaluated as indicated in the subsequent parts of (9). Thus if $f(x) = x^2$, $F(x) = \frac{x^3}{3}\left(\text{since} \int x^2\,dx = \frac{x^3}{3} + C\right)$ and hence

$$\int_2^3 x^2\,dx = \frac{x^3}{3}\Big]_2^3 = \frac{3^3}{3} - \frac{2^3}{3} = \frac{27 - 8}{3} = \frac{19}{3}$$

As an illustration, we'll find by integration the area of the trapezoid $ABCD$ in Fig. 82, in which the equations of the sides and top are as shown. By (9) the area is

$$\int_2^6 \left(\frac{x}{2} + 3\right)dx = \left[\frac{x^2}{4} + 3x\right]_2^6$$
$$= \left(\frac{36}{4} + 18\right) - \left(\frac{4}{4} + 6\right)$$
$$= 20$$

In this case, to be sure, you may get the same result by using the trapezoid rule of plane geometry. (As a matter of fact, we gave this easy example just so that the doubting Thomas could check it in that way.) You should note, however, that the method required only that the top boundary of the area be a line, straight or curved, whose equation we know or can find. Then compare, if you will, the power and generality of this method with anything you ever ran across in plane geometry!

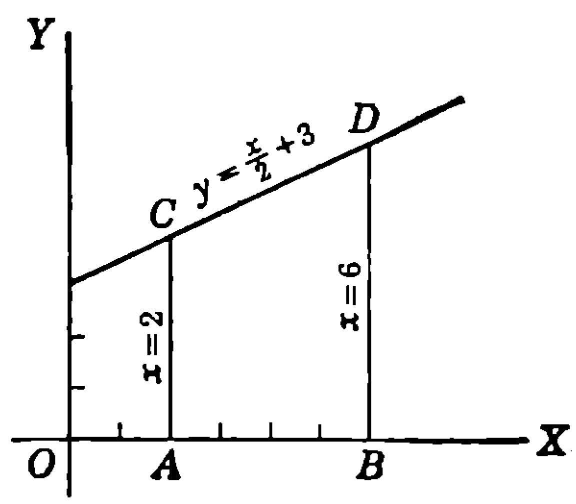

FIG. 82.

Even so, the best of the news is to come. While it is easy enough to see offhand how to extend this method so as to get the area between two vertical lines and two separate curves (merely by subtracting one area from another), it is less obvious that this same integration process could give us a volume, or a force, or a city population, or any one of the multitudinous other things which the mathematical magicians are able to conjure out of it. To get an inkling of the simple principle which is the key to the mystery, we must edge up with all due caution to a somewhat pretentious proposition whose true significance is usually lost somewhere in the shadow of its august title. It is called *the fundamental theorem of integral calculus* and appears to the properly awed student to have something or other to do with geometry. This is because it comes up first in a geometric setting and is illustrated in terms of an area. Unless and until, however, he eventually sees that in essence it has no more to do with area than with volume or length or force or any other mathematical quantity whatever, he will fail utterly to realize why it is of such great importance that it truly deserves its glamorous and imposing title.

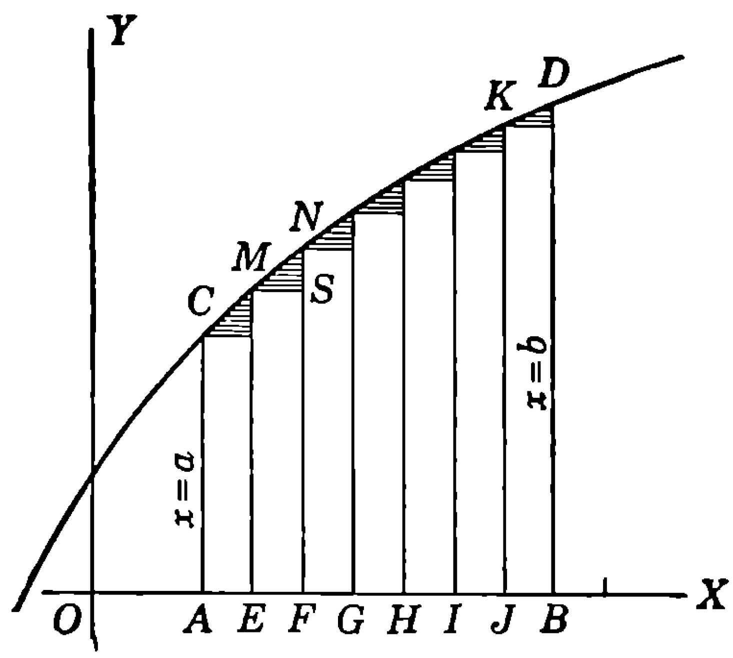

FIG. 83.

Now that we have sounded our warning about a deeper meaning to come, we'll return to our geometric trappings. In Fig. 83 we see again an area $ABDC$ enclosed within the X axis, the curve $y = f(x)$, and the ordinates $x = a$ and $x = b$. But now this area is subdivided into n vertical strips of equal width. To be specific, we have let n be 7 in this particular figure. Let's designate this width by Δx, and the lengths OA, OE, $\cdots$, OJ by $x_1, x_2, \cdots, x_n$. Then, of course, $AC = f(x_1)$, $EM = f(x_2)$, $\cdots$, $JK = f(x_n)$. Now the area of the particular rectangle $EFSM$ is $f(x_2)\Delta x$, and that of the sum of the n rectangles—which sum we'll designate by A_n—is

$$A_n = [f(x_1) + f(x_2) + \cdots + f(x_n)]\Delta x = \sum_{i=1}^{n} f(x_i)\Delta x$$

in which the meaning of the briefer notation is explained by the preceding expression. By inspection we can see that the area A_n differs from the area A under the curve by the sum of the areas of the shaded near-triangles above. By inspection also, we see that the sum of all the shaded areas (in this case of a curve rising to the right throughout the interval considered) is less than the area of the single strip at the right. It follows that A_n differs from A by a quantity which can be made to approach zero by increasing n, and therefore decreasing Δx. In the limit notation,

$$\lim_{n \to \infty} A_n = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i)\Delta x = A$$

But we have already seen in Eq. (9) that the value of A [which replaces the A' of (9)] is given by the definite integral $\int_a^b f(x)dx$. It follows that

$$\int_a^b f(x)dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i)\Delta x \tag{10}$$

where the strips of width Δx are summed up in the interval from $x = a$ to $x = b$.

But wherein, you ask, has that increased our information or insight? We *already* were able to get the area by means of the definite integral. Ah, but here lies the somewhat subtle transition into realms which may be nongeometric. The newly discovered and highly important fact is that the definite integral represents the limit of a summation. This idea is the essence of the fundamental theorem, which we put in our own words below. Remember that you probably can't read it and understand it in a flash. Read it over and over and over again, and *then*, perhaps, will come the dawn.

Fundamental theorem of integral calculus. *If we can express a quantity of any kind whatever (not just an area) as the limit of the sum of* n *separate parts; and if, further, when* x *is used to designate a convenient independent variable, we can find a particular* $f(x)$ *such that each of these separate parts is expressible as* $f(x)$ *times an increment* Δx *which approaches zero as* n *increases, then a definite integral can be written down whose evaluation gives the desired quantity.*

Incidentally that integral is $\int_a^b f(x)dx$, where the limits a and

b depend upon the particular problem; but there's no need of introducing that more or less foreboding statement into the comforting blanket promise of the fundamental theorem. We're merely saying what can be done, and to fill in the details of how to do it would require an extensive course in the applications of integral calculus.

Maybe now the reader begins to catch a faint previewing glimpse of the possibilities. The desired quantity of the theorem might be a volume, such as that of a sphere which is cut into thin slices or into concentric peelings. It might be the total force exerted by the water on the slanting face of a dam, where the pressure on each infinitesimal horizontal strip increases with the depth below the surface. It might be a force on a piston or a test-tube bacteria population in a culture which increases according to a certain law. It might be many other things, and often it is. In fact, the accomplishments in this field are second only to the possibilities.

EXERCISE 62

Make a picture for each problem of this exercise.

In each of Probs. 1 to 5, find by integration the area between the X axis, the designated curve, and the lines $x = a$ and $x = b$, where a and b have the indicated values.

1. x^2; $a = 0$, $b = 2$.

2. x^3; $a = 1$, $b = 2$.

3. $x^2 + 3x + 3$; $a = -2$, $b = 3$.

4. $x\sqrt{2x^2 + 1}$; $a = 0$, $b = 2$.

5. $2 - x - x^2$; $a = 1$, $b = 3$. (Explain the sign of your result.)

6. Find the area in the first quadrant between the axes and the curve $y = 4 - \dfrac{4x^2}{9}$.

7. Find by subtracting one integral from another one the area inside the quadrilateral whose sides are $x - y - 1 = 0$, $x + 2y - 10 = 0$, $x = 0$, and $y = 0$. Check by geometric methods.

8. Find the area between the curve $y = x^3$ and the lines $x = 0$ and $y = 8$ by subtracting the area under the curve from the proper rectangle. Then express this area as the limit of the sum of horizontal strips and get the value of this limit by a single integration, using the fundamental theorem of integral calculus.

Use the second method suggested in Prob. 8 to find, in each of Probs. 9 to 12, the area between the Y axis, the designated curve, and the lines $y = a$ and $y = b$, where a and b have the indicated values.

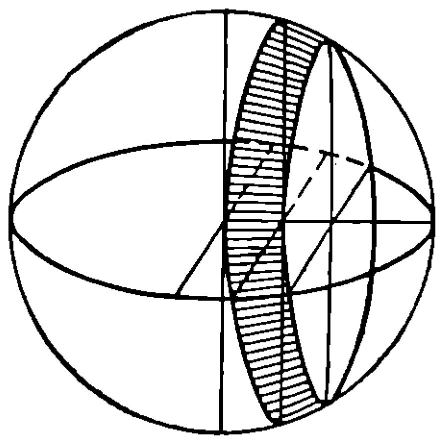

FIG. 84.

9. x^2; $a = 2$, $b = 4$.

10. $\sqrt{x}$; $a = 1$, $b = 2$.

11. $4 - \frac{4x^2}{9}$; $a = 2$, $b = 3$.

12. $1 - x$; $a = 0$, $b = 1$.

13. Find the volume of a sphere of radius r by expressing it as the limit of the sum of sample slices as shown in Fig. 84. (The volume of one slice is $\pi y^2\, dx$.)

14. Find the volume of a cone of base radius r and height h, using the method of Prob. 13.

CHAPTER XI

OUR RESULTS GROW PROPHETIC

89. We lose our smug certainty. Up to this point we have been a trifle supercilious, perhaps, about the absolutely dependable nature of mathematics as compared with many upstart fields of surmising called "sciences." When once we get a *premise* or *working hypothesis* imbedded in an equation, only the answer can come forth. The certainty of taxes is absurdly overrated as compared with that of mathematics. And when the independent variable in a guaranteed equation is time, as in some calculations involving the earth, sun, and moon, we can even soar to the heights of prophecy and outline the future in the matter of eclipses. This has been demonstrated so often that it is one completely conceded triumph of mathematics. And so, indeed, are many formulas, such as those in geometry which give areas and volumes. You tell us the radius (r) of a circle and we'll give you its area (πr^2) easily, haughtily, and infallibly. Our mathematics will not play us false.

It is with a bit of chagrin, therefore, that we approach a subject whose very name—probability—involves a tacit admission that mathematics does not always deal in certainty. Its conclusions, to be sure, are eternal truths if and when its premises are correct. But in that last clause lies the catch, and also the one bar to perfect faith in the soundness of a mathematical conclusion. With the proper premises, for instance, we might be able to work out an equation connecting the temperature at a given spot with the time, so that priceless information about future weather could be hung on a mathematical peg. The functional correspondence between time and temperature exists, but our regular brand of deductive mathematics is powerless to cope with it. Even when there is a suspected causal relation between two quantities, such as rainfall and crop production, or earthquakes and epidemics, it often happens that, because of our ignorance of the physical principles involved, we have no certain premises to put into the mathematical hopper, and hence no worth-while product can come out of the elaborate machine.

One might be tempted to conclude, then, that the usefulness of mathematics is limited to such fields as engineering, in which the conditions (premises) can be controlled, or to business, where they can be agreed upon in advance, or to geometry, where they can be invented, or to astronomy and other sciences, where they can be discovered in the form of laws which describe how certain events have been observed to follow each other. These are, indeed, important fields of service for the tools of mathematics; but even they do not contain all of the ground in which those tools can dig with profit. For mathematics has successfully invaded those phases of life in which the premises themselves are so hazy that the conclusions are necessarily uncertain; and it has helped us to *guess with intelligence.*

A little consideration makes it obvious that this particular service of mathematics can be of great practical importance. For in most problems of life we do not deal with a situation in which the entering factors are so clear cut that a little pencil-jiggling brings out one solution representing the best thing to do. The manufacturer, who can compute exactly by means of calculus the most economical dimensions for a cylindrical quart can, is blessed in this respect far above the insurance company. For the latter, desiring to fix a premium rate low enough to get business from its competitors, and yet high enough to give a reasonable chance for profit, is faced with the fact that nobody knows or can find out in advance the exact number of deaths which will occur among the members of a given group in the ensuing year. There we see one example of the stark business necessity for the development of the art of intelligent guessing. And there, furthermore, we see ample justification for the mathematical theory of probability.

90. From insecurity to insurance. But, as often happens, this highly useful phase of mathematics did not develop directly out of the later obvious need. It came up first in connection with entertaining, pleasant, and apparently useless mental calisthenics involving games of chance. A dabbler named Pacioli, back in 1494, proposed the problem, "Two gamblers are playing for a stake that is to go to the first one who wins n points. If the game is interrupted when one player has p points and the other q points, how is the stake to be divided?" Pascal and Fermat, two redoubtable mathematicians of the seventeenth century, got interested in Pacioli's poser, and their discussions of it "aroused wide interest" (among the intellectually elite, we suppose) and laid the foundation for the development of the mathematical theory.[1]

During this period in European history business was developing rapidly. The merchant class was becoming wealthy and had money to lend. The shipping industry was not only growing in importance, but was beset with two major perils—shipwreck and piracy. From this union of wealth and insecurity came the beginning of the insurance business, a risky and experimental gambling venture which expanded rapidly into a legitimate community service destined for great things. Whipped into scientific soundness by the backwash of speculations about penny flipping, dice throwing, and similar acts of a more or less scandalous nature, it survived the miscalculations and bankruptcies of youth and grew into a human institution facing an almost incredibly brilliant financial future. In the United States alone this alumnus of the gaming table sells security to millions and reckons its accounts in the billions of dollars.

91. Two kinds of probability. Having seen the importance of the simple principles of probability, we can proceed now to the precise statement required for a mathematical treatment. Immediately we are confronted with the fact that two definitions are needed to describe two entirely different concepts. One of these, called *mathematical probability*, is easily illustrated by the paraphernalia of the more naive forms of gambling (as we might expect, in view of its early career). The other, called experimental or *empirical probability*, is the concept which is most impressively and certainly useful in the business world, since it underlies all forms of insurance. We'll begin with the first-mentioned type.

If an event can happen in h *ways and fail in* f *ways, and if any one of the* $h + f$ *ways is as likely to occur as any other, the* (***mathe-***

[1] Quotations and paraphrasing from D. E. Smith, *History of Mathematics*, Vol. II, p. 529.

***matical) probability** that the event will happen is **$h/(h+f)$**; and that it will fail, **$f/(h+f)$**.*

For example, suppose that a bag is known to contain ten balls altogether, of which two are white, three are black, and five are yellow; and suppose further that a person closes his eyes and draws out a ball at random. Then the probability that he will get a white ball is $\frac{2}{10} = \frac{1}{5}$, since he can get a white ball in two ways and can fail to get one in eight ways. Similarly, the probability that a flipped penny will fall with heads or tails uppermost is $\frac{1}{2}$ in each case, and that a single die will fall with the four-spot above is $\frac{1}{6}$. (A *die*, in case you didn't know, is the singular of *dice*, and consists usually of an ivory cube with one to six dots on each of its six faces, no two of which are alike.)

The first important (and often overlooked) point is that probability of this kind can be determined once for all in advance of a trial, the figure thus obtained being in no wise modified as a result of experience. For example, in the case of our bag problem, the probability of drawing a white ball remains exactly $\frac{1}{5}$ even though an experimenter has drawn a white ball ten times in succession (assuming, of course, that he replaces the one drawn and mixes the balls thoroughly after each trial). The guide for intelligent betting in this case is an immutable figure, though the weak thinker usually looks at the run of luck and bets accordingly.

A second point worth noting is that there is no law to assure us that results will accord even approximately with the precalculated ratios. Occasionally a head will fail to appear a single time in ten flips of a penny; more rarely still it will not show up in one hundred successive trials, and so on. Hence the probability that an event will happen guarantees nothing whatever about the next trial *or even about the next thousand or million trials.* It merely points out the ratio of successes to trials which is *usually* approximated rather closely when the number of trials is large. Though it may be interesting for you to see what you get when you flip a penny a hundred times, your experiment is essentially worthless, because here we have something better than mere trial to guide us.

But frequently this happy situation does not exist, for the conditions allowing the computation of mathematical probability are so drastic that they sharply limit its usefulness. As the next best thing to probability based on reason we are often forced, as insurance companies are, to deal with another kind based on experience. This brings up the next definition:

If in the past an event has happened ***h*** *times in* ***t*** *trials, and if no change in pertinent conditions has occurred in the meantime, then the (****experimental****)* ***probability*** *that it will happen in the next trial is* ***h/t.***

For example, suppose a bag contains a large number of balls known to be individually either black or white, though the proportion of balls of each color is unknown. An experimenter makes ten successive draws, replacing the one drawn and mixing the balls thoroughly after each trial. If the colors come out in order as follows: *wwbbbwbwbb*, then the (experimental) probability of drawing a white ball on the next trial becomes successively 1, 1, $\frac{2}{3}$, $\frac{1}{2}$, $\frac{2}{5}$, $\frac{1}{2}$, $\frac{3}{7}$, $\frac{1}{2}$, $\frac{4}{9}$, and $\frac{2}{5}$. And if the last figure arrived at after a thousand trials is $\frac{1}{2}$, then this figure is *likely* to approximate the ratio of white to black balls in the bag more closely than is the tentative result at the end of a hundred trials. Thus we are here dealing with a constantly changing quantity, whose dependability is roughly proportional to the amount of data used in getting it, and whose usefulness in at least one field will appear in the next article.

The following two theorems, applying to either type of probability, will help in the solution of many problems, both in our text and elsewhere.

Theorem I. *If the probability of an event is* p, *and if, after the first event has occurred, the probability of a second one is* q, *then the probability that the two will take place in the given order is* pq.

For instance, the probability that the one-spot will appear twice in succession when a die is cast in $(\frac{1}{6})(\frac{1}{6}) = \frac{1}{36}$. This result accords with the first definition of probability, since there are 36 possible sequences of two faces.

Theorem II. *If the probabilities of two events, only one of which can happen, are* p *and* q *respectively, then the probability that one or the other will happen is* $p + q$.

For example, the probability that a penny will turn up one face or the other when flipped and not allowed to stand on edge, is $\frac{1}{2} + \frac{1}{2} = 1$, representing certainty. Events of the type described in Theorem II above are called *mutually exclusive*.

EXERCISE 63

Besides giving the numerical answer for each of Probs. 1 to 20, state the type of probability which is involved.

1. Circular cardboard disks are numbered from 1 to 100, placed in a box and thoroughly mixed. If one disk is drawn from it, what is the probability that its number will end in zero? Will be divisible by 5?

2. What is the probability that a three will appear in one toss of a pair of dice?

3. What is the probability that you will be killed in an automobile wreck next year on the basis of 40,000 such deaths per year among 140,000,000 people?

4. A bag contains three white, four black, and five yellow balls. If one is drawn out at random, what is the probability that it will be (*a*) black, (*b*) black or white?

5. Statistics in the registrar's office revealed that over a period of years, an average of 250 boys graduated out of each 500 that entered, and that ten of the 250 that graduated became doctors. What is the probability that a boy who enters this college will become a doctor?

6. The probability that a certain man will be nominated for governor is one-third, and if nominated, the probability that he will be elected is three-fifths. What is the probability that he will become governor? That he will win the nomination but lose the election?

7. Three pennies are flipped on a table. What are the probabilities that exactly (*a*) one head, (*b*) two heads, (*c*) three heads will appear?

8. A traffic count revealed that in 3 hr. 100 passenger cars, 36 light trucks and 12 heavy trucks passed through an intersection. What are the probabilties that the next vehicle passing through the intersection will be (*a*) a passenger car, (*b*) a light truck, (*c*) not a passenger car?

9. One domino is drawn from a well shuffled set. What is the probability that there will be 10 spots on it?

10. Assuming that 10 out of 30 people who have been bitten by rattlesnakes in a community isolated from medical resources have died, what are the probabilities that (*a*) the next one, (*b*) the next two, (*c*) the next three, (*d*) at least one of the next three, will die?

11. A man throws a pair of dice twice in succession. What is the probability that (*a*) a seven will appear on the first toss, (*b*) on both tosses, (*c*) on the second toss only?

12. If three balls are drawn out at once from the bag in Prob. 4, what is the probability that two of them will be yellow and one white?

13. If three balls are drawn at once from a bag containing five red, six white and seven blue balls, what is the probability that there will appear (*a*) three red balls, (*b*) two red balls and one white one, (*c*) a ball of each color?

14. Assuming that a meteor crosses a given plane of 10 sq. miles once a day, what is the probability that a space ship with a surface area (as projected on that plane) of one ten-thousandth of a square mile would be hit on a trip lasting 30 days?

15. If only one disk is drawn from the box in Prob. 1, what is the probability that it will contain the digit 6?

16. If two disks are drawn from the box described in Prob. 1, what is the probability that the sum of the numbers drawn will be 6?

17. A bag contains one black ball and two white ones. Assuming that the drawing of the black ball is fatal and that a man has to make three draws, restoring the ball after each attempt, what is the probability that he will be unlucky?

18. Assuming that 1,000 people in a nation of 140,000,000 are killed each year by lightning, what would be the chance of an individual of that nation to reach the age of 70 if lightning were the only hazard to life?

19. Some virgin territory is blocked off into seven areas. A careful search reveals gold in one of the five areas prospected to date. In the light of this evidence alone, what is the probability that gold will be found (*a*) in a given one of the two untouched areas, (*b*) in the untouched region?

20. If two dominoes are drawn from a well shuffled set, what is the probability that there will be (*a*) exactly 15 spots, (*b*) exactly 20 spots, (*c*) at least 20 spots?

92. The billion-dollar test. We have already stated that insurance companies must depend upon the theory of experimental probability to make the rules of the game for a business which deals in the aggregate with billions of dollars. Evidently we have here a means of putting the mathematician's armchair theory to the test of experience on a supermagnificent scale. And the virtually uniform success of well regulated insurance companies, as compared with the "ups and downs" of business in general, is striking evidence of the practical dependability of this theory when the numbers involved are sufficiently large.

But what is this theory back of the gigantic investment in insurance? Briefly, it is this: *If a certain fraction of a very large amount of goods, or a certain proportion of a very large number of people, is lost to society through accidental damage or natural death in one year, it will be good business to risk money on the assumption that the ratio of loss will be nearly the same in the next year.*

Let's see it in operation in the case of property insurance. We'll suppose that in a community the value of the dwelling property is $5,000,000, and that for 10 years the average annual loss by fire has been $25,000. The fire insurance company will apply the above theory and figure that about 25,000/5,000,000 or 0.5 per cent of the property will *probably* be destroyed each year. On this basis, the company will charge 50 cents per $100 of insurance, plus a slight additional charge for operating expenses, profit, and "safety margin." If this additional margin is not a comparatively small fraction of the 50 cents, the company will face loss of business through competition.

Of course in practice the procedure in determining premium rates is much more complicated than in the above simple illustration of principle. The insurance companies gather their statistics from all parts of the country over a long period of years, and in adjusting local rates they take into account many factors which influence fire losses. Because of the danger of a single disastrous conflagration, they seek a wide territorial distribution of risks, and avoid too heavy concentration of business in one area. In these precautions they recognize the fact that experimental probability is of little use unless it is based on many trials, and they also take into account the importance of this qualifying clause, "and if no change in pertinent conditions has occurred in the meantime," which appears in our definition of experimental probability.

Turning to the second important phase of the big business based on a mathematical idea, we find that life insurance companies also use extensive statistical data in making their rates. The American Experience Mortality Table, of which a condensed portion is shown in Table VIII, has been revised in recent years, to be sure, but the figures as given do not vary greatly from present-day experience, and it doesn't matter anyway, since we're concerned only with the principle. This table starts with a typical group of 100,000 American youths, ten years of age, including both boys and girls and both the healthy and the "ailing," and it follows them through the years, setting forth the number who (according to a certain recorded part of past experience) will still be living at various ages.

As an example of the use of the table, let's solve the following problem: "Find the probability that a forty-eight-year-old man will live to be sixty-five years of age." In the table we find that of 71,627 persons alive at the age of forty-eight, there were 49,341 alive at 65. Thus the man's probability of living to that age is 49,341/71,627 = 0.689.

Now just what is the value of this figure? Evidently, since it leaves the individual's private appendix and other parts out of account, it is worthless for any particular forty-eight-year-old specimen of humanity. The fellow in the problem is purely a "statistical man"—a fictional character whom we could slay with ease by means of a more careful statement of the problem, but nevertheless a convenient and harmless creature of the statistician's imagination. But though our answer is worthless to you and to me, its high value to the American insurance companies is attested by their healthy growth.

EXERCISE 64

1. In a certain city with property valued at $25,000,000, the fire damage in one year was $50,000. What would be the theoretical annual fire-insurance premium on a $10,000 house, neglecting the operating expenses and profit motive of the insurance company?

By use of Table VIII, find the probability that the prophecies of Probs. 2 to 10 will be fulfilled.

2. A 10-year-old boy will live to be 73.

3. A 34-year-old man will live at least 10 years.

4. A 19-year-old youth will die in his fortieth year.

5. Two 50-year-old men will both reach 60.

6. At least two out of three given 20-year-old youths will live to be 30.

7. Work Prob. 6 with "at least" replaced by "exactly."

8. Work Prob. 6 with "at least" replaced by "at most."

9. A 50-year-old man will not reach 70.

10. Three boys of 10 will all die in that year.

11. A sum of money is to be paid 10 years hence to the oldest living one of three brothers who are now respectively 30, 35 and 40 years of age. What is the probability that the youngest brother will receive the money?

12. Three air corps veterans agreed to meet, if all three are alive, 10 years after their discharge, to celebrate the event. If each one was 22 years of age when he was discharged, what is the probability that they can have the celebration?

13. An ambitious parent expects his son, age 10, to enter college at the age of 18 and then to complete his medical training in 8 years. If the probability that the boy will comply successfully with his father's wishes, if he lives long enough, is 0.8, what is the probability that he will become a doctor?

14. The ownership of a ranch is to pass from a father to his son upon the death of the former. If their ages are 45 and 20 years respectively, what is the probability that the son will own the ranch 20 years hence?

CHAPTER XII

A LADDER WITHOUT A TOP

93. Back to certainty. In our last chapter we temporarily left the field of certainty in order to deal with practical problems in which action is based upon the best guess possible under given circumstances. Even there we had, to be sure, that pride of mathematics, the inevitable consequence; but in that case it was merely a figure which followed from a definition, and not an essential certainty which could be verified experimentally from the premises. Now we are returning once more to the field of the absolutely dependable result which mathematics alone of all the sciences can furnish. But in this chapter we shall point out the creative aspect of our subject.

Let's consider for a moment the nature of the mathematical results we have obtained thus far. For the most part, they have been *deductive*. They tell what follows if something else is true to begin with; but they make no guarantee about the starting proposition. For example, you might say that "If the money in John Jones' pocket is twice that in my own, his cash resources will amount to not less than 60 cents." Even though this should be a true mathematical statement, it would still fail to throw any light whatever on the real financial status of the Jones boy. This illustrates the somewhat unsatisfactory nature of a deductive conclusion, which is ironclad and dependable only as a statement of the situation which would hold under given circumstances, regardless of whether the assumed conditions are true or false. This type of conclusion, though often highly useful, would never by itself lift our subject above the role of a clerk among the sciences, precise and dependable but definitely noncreative.

Fortunately, however, this is not all that mathematics has to offer. Like chemistry, physics, and other experimental sciences, it also can make use of *induction*, or the building up of generalizations from observed special cases, thus leading investigators to new and authentic truth.

In the case of all sciences but one, the establishment of this inductive conclusion is to some extent an act of faith. By noting experimentally that certain results occur with regularity, the investigator is led to believe in the existence of a *general principle* in nature whose operation would explain his separate observations. His description of the principle is nothing more than a statement of the way things will always (he trusts) work. If continued observations do not contradict his hopeful surmise, his conjecture rises to the status of a *theory;* and finally, if all the evidence at hand is impressive in quantity and points without exception to the dependability of the theory, it graduates automatically into a *law*. Thus was born the *law of gravity*. Galileo measured the falling time of objects and noted that his results were consistent with the theory that acceleration is constant. Kepler studied the motions of planets and formulated his three laws pertaining to their orbits. The genius Newton, picking up the loose ends, put them together in his law of gravitation, which describes the quantitative rules governing the attraction of bodies as measured up to date in all parts of observed creation. Einstein threw further light on the subject with his *relativity* refinements. These were intellectual achievements of tremendous value, pointing creatively to many conclusions which have been verified by experiment. We need not disparage them when we point out that the only real "proofs" involved concern the deductive conclusions which came mathematically out of the superb generalizations.

As was perhaps noted, we said above that an inductive result is an act of faith "in all sciences but one." You have guessed it; the hero is mathematics. The scientist shrewdly suspects a great truth and checks it as best he can; but only the mathematician can go him one better and actually prove a generalization! Herein we stand alone and supreme, if for a moment you'll pardon us while we bask in impersonal glory.

The particular process of marching to universal truth along the steppingstones of a few special cases, and then capping the climax by proving it for the untried infinity of such cases, is called *mathematical induction*. It is generally resorted to when the usual or deductive mathematical procedure is inadequate for the occasion, owing to the lack of premises from which, or technique by which, to make a deduction. The induction, it should be understood, lies in the method and not in the nature of the result. The mere fact that the latter covers an infinity of cases (as do many deduced theorems of geometry, for example) has no bearing on the matter.

94. An ironclad argument. Comes the time for an example of this highly interesting process. We have given one already, back in Chap. X, in our proof of the formula $\frac{d}{dx}(v^n) = nv^{n-1}\frac{dv}{dx}$, but now we'll go into the matter more fully.

One can easily verify for himself that $a - b$ is a factor of $a - b$, $a^2 - b^2$, $a^3 - b^3$, and $a^4 - b^4$. After doing so, he may suspect the truth of the following generalization:

The binomial $\boldsymbol{a^n - b^n}$ *is divisible by* $\boldsymbol{a - b}$ *when* $\boldsymbol{n}$ *is any positive integer.*

The job of proving the above statement is an ideal one for our mathematical induction method. We'll apply it in three typical steps.

I. We verify that it is true when $n = 1$.

II. We prove that if the theorem is true when n is equal to any special integer t, it will be true when $n = t + 1$.

III. We argue that, because of II, the result in I holds good when $n = 2$, and hence when $n = 3$, and so on until n equals any positive integer whatever, no matter how large.

Carrying through the suggested steps, we have no trouble with I, since $a - b = (1)(a - b)$. To prove II, we get the necessary assumption clearly before us thus:

Assume

$$a^t - b^t = (a - b)Q \tag{1}$$

where Q is a second undetermined factor.

Then, to prove that $a^{t+1} - b^{t+1}$ is divisible by $a - b$, we must make use somehow of the above assumption. A little experimenting and scratch work shows us that the thing can be done if we juggle $a^{t+1} - b^{t+1}$ algebraically, and express it as follows:

$$a^{t+1} - b^{t+1} = a(a^t - b^t) + b^t(a - b) \tag{2}$$

Now then, by virtue of our assumption (1) and Eq. (2), we have

$$(3)\quad a^{t+1} - b^{t+1} = a(a - b)Q + b^t(a - b) = (a - b)(Qa + b^t)$$

and II is proved.

As for III, well, the reader can outline the argument for himself. And while doing it, he should see that he has demonstrated that $a^n - b^n$ is divisible by $a - b$ when $n = 1{,}000{,}000$ (for example) just as certainly as if he had actually checked it by long division.

One more illustrative example will probably help. By way of getting started, let's see what happens when we add up the terms of a series of which the nth term has the form $3n^2 - 3n + 1$. With $n = 1$ the first term comes out to be $3(1)^2 - 3(1) + 1 = 1$; with $n = 2$ we get $3(2)^2 - 3(2) + 1 = 7$ as the second term; and so on. The first n terms of the series are then:

$$(4)\qquad 1 + 7 + 19 + 37 + 61 + \cdots + (3n^2 - 3n + 1)$$

Now the sum of the first one, two, three, and four terms of (4) are respectively 1, 8, 27, and 64; and most any alert person will soon notice something special about these numbers. Each, in fact, is a perfect cube, and they come in succession thus: 1^3, 2^3, 3^3, 4^3. With our curiosity aroused, we try the next one. Sure enough, it is $125 = 5^3$. But if, like Grant, we try to "fight it out along this line," we can verify and verify until all our teeth come out, and yet we'll be as far as ever from having proved that the thing will always work. We'll have taken care of Step I of the induction proof in grand and exuberant style, overdoing it most thoroughly, while all we really needed to settle the matter completely was a simple application of Step II in our machined process. Remember that we suspected, and are trying to prove, that

$$(5)\qquad 1 + 7 + 19 + \cdots + (3n^2 - 3n + 1) = n^3$$

Taking our stance for Step II, then, we'll assume that

$$(6)\qquad 1 + 7 + 19 + \cdots + (3t^2 - 3t + 1) = t^3$$

where $3t^2 - 3t + 1$ is the tth term. The next, or $(t + 1)$th term, will be $3(t + 1)^2 - 3(t + 1) + 1 = 3t^2 + 3t + 1$. The question now is: If we add this quantity to both sides of (6), so that the left side will have $t + 1$ terms, will the right side be $(t + 1)^3$?

Trying it, we find that $t^3 + (3t^2 + 3t + 1) = (t + 1)^3$. There, that proves it. Now we know that (5) will be always true, since we have checked it for $n = 1$ (completing Step I) and can go without effort, and as rapidly as we please, up the ladder rounds of Step II to the promised land, $n =$ any positive integer we care to name.

EXERCISE 65

Prove the statements of Probs. 1 to 10 by mathematical induction.

1. $1 + 2 + 3 + \cdots + n = \dfrac{n(n+1)}{2}$.

2. $2 + 4 + 6 + \cdots + 2n = n(n+1)$.

3. $3 + 6 + 9 + \cdots + 3n = \dfrac{3n(n+1)}{2}$.

4. $3 + 5 + 7 + \cdots + (2n+1) = n(n+2)$.

5. $4 + 7 + 10 + \cdots + (3n+1) = \dfrac{n(3n+5)}{2}$.

6. $2 + 5 + 8 + \cdots + (3n-1) = \dfrac{n(3n+1)}{2}$.

7. $a^n + b^n$ is divisible by $a + b$ if n is odd.

8. $1^2 + 2^2 + 3^2 + \cdots + n^2 = \dfrac{n(n+1)(2n+1)}{6}$.

9. $1^3 + 2^3 + 3^3 + \cdots + n^3 = \dfrac{n^2(n+1)^2}{4}$.

10. $1^4 + 2^4 + 3^4 + \cdots + n^4 = \dfrac{n(n+1)(2n+1)(3n^2+3n-1)}{30}$.

11. $1^5 + 2^5 + 3^5 + \cdots + n^5 = \dfrac{n^2(n+1)^2(2n^2+2n-1)}{12}$.

12. $1^6 + 2^6 + 3^6 + \cdots + n^6$
$$= \frac{n(n+1)(2n+1)(3n^4+6n^3-3n+1)}{42}.$$

13. $1 \times 2 + 2 \times 3 + 3 \times 4 + \cdots + n(n+1) = \dfrac{n(n+1)(n+2)}{3}$.

14. $1^2 \times 2 + 2^2 \times 3 + 3^2 \times 4 + \cdots + n^2(n+1)$
$$= \frac{n(n+1)(n+2)(3n+1)}{12}.$$

15. $\dfrac{1}{1 \times 2} + \dfrac{1}{2 \times 3} + \cdots + \dfrac{1}{n(n+1)} = \dfrac{n}{n+1}$.

16. $\frac{1}{1^2 \times 2^2} + \frac{7}{2^2 \times 3^2} + \cdots + \frac{2n^2 - 1}{n^2(n+1)^2} = \frac{n^2}{(n+1)^2}.$

17. $\frac{1}{1 \times 2} + \frac{5}{2 \times 3} + \cdots + \frac{n^2 + n - 1}{n(n+1)} = \frac{n^2}{n+1}.$

18. $\frac{1}{1 \times 2} + \frac{13}{2 \times 3} + \cdots + \frac{2n^3 - 2n + 1}{n(n+1)} = \frac{n^3}{n+1}.$

95. Pointing the ladder. Here, then, we have a powerful, original method of thought, and a logical ladder by which we can climb to new truth. When we suspect a generalization, this tool lies at our hand, and often it is the only one which will cut through the fog to certainty. But in that "when" clause lies a weakness, also. It does not furnish us the "suspector." Stodgy fiddling with special cases, numerical coincidences, experience, experiment, and flashes of intuition—all these yield guessed-at generalizations which may or may not be true. To these surmises we point our induction ladder; and then, climbing the first round by Step I, we see whether the mathematical details of Step II are within our powers (confidentially, they can be too much on occasion for the best mathematician extant). If so, we have arrived at an unquestionable new truth, or else have proved the falsity of a bad guess. In any case, we have added one more conquest in the endless attack on the Things That are Yet to Be Found Out.

CHAPTER XIII

FUN WITH FIGURES

96. Time to celebrate. We have surveyed somewhat briefly and jauntily, but nonetheless conscientiously, the sober and useful fields of mathematics which are assigned to routine duty in the work of the world, and which, though closely and logically interrelated, are herded for convenience into pedagogical corrals labeled *algebra*, *trigonometry*, *analytic geometry*, *business mathematics*, and *calculus*. With these formidable hurdles out of the way, the time has come to relax. We now feel privileged to dabble with that which is surprising and interesting in itself, regardless of possible applications. Not that the applications do not often turn up of themselves in the mere sideswipings of the sport of figure juggling. They do. But we shall not go out of our way to look for them. Primarily, this chapter is for recreation, and therefore those who do not care for an occasional bout with numbers and ideas for the sheer fun of it should leave this part of our text strictly alone. Reluctant and protesting brainwork here would be like the efforts of a confirmed shade lover who rounds the golf course on the doctor's orders. The purpose of the game would be lost in the misery of the execution.

97. The intriguing integer. Take, for instance, such an apparently simple concept as that of the whole number, whose unfolding properties occupied the center of the stage in the drama of the birth of arithmetic. Is there meat for investigation here? Apparently so, since the world's literature, published and scribbled, on the *theory of numbers*, the subject dealing exclusively with integers, would probably sink an undetermined but highly imposing number of battleships if it were stacked book by book and page by page on the sturdy but limited deck space. Where so many brains have found satisfying calisthenics, there must be ample room for exercise.

Let's begin with the prime number, that special kind of integer which has no divisor except itself and one. By unanimous agreement the number *one*, itself, is excluded from the fold as being so extra special that it rates separate consideration, so the first primes in order are 2, 3, 5, 7, 11, 13, 17, 19, etc. How many such integers are there? How are they distributed? Is there a formula which will produce them? These three questions are samples of those which came up in the study, and we may as well say right here that they are so hard that only the first of the three has been answered up to date, at least satisfactorily. The second one has a partial answer in the method of getting successive primes called the *sieve of Eratosthenes*. The first was disposed of by the industrious and versatile Euclid of long-enduring fame, who proved that the number of primes is infinite.

Eratosthenes' sieve is applied as follows. First, the positive integers, excluding 1, are written down thus:

$$2, 3, 4, 5, 6, 7, 8, 9, 10, 11, \cdots$$

Next all multiples of 2 are stricken out, thus

$$\not{2}, 3, \not{4}, 5, \not{6}, 7, \not{8}, 9, \not{10}, 11, \cdots$$

The first one of this group, namely 2, is the first prime. Again, the multiples of the 3 are deleted, leaving the first number thus scratched out, or 3, as the second prime. The number 4 was eliminated from the ranks of the possibilities when the even numbers were marked out, thus leaving 5 as the smallest untouched one left. After the 5-multiples are marked, 7 looms immaculate on the left, 6 having already gone down twice in the slaughter. Thus the primes appear in succession as the "leftmost survivors," and we have a systematic but laborious method which would bring us eventually to any number, however large, and enable us to say whether it is prime or *composite* (having two or more integral factors, each different from $+1$ or -1).

Euclid's ingenious and simple proof of the infinitude of primes hinges on the demonstration that there is always at least one more prime larger than any given prime. Let p be the given prime, and let π be one more than the product of all positive primes in order, up to and including p. Thus if $p = 3$, $\pi = 1$

$+ (2)(3) = 7$; if $p = 5$, $\pi = 1 + 2(3)(5) = 31$; if $p = 7$, $\pi = 1 + (2)(3)(5)(7) = 211$. In general,

$$\pi = 1 + (2)(3)(5)(7)(11) \cdots (p)$$

Now, obviously, π is not divisible by any of the primes which appear explicitly in the expression for it, since there will always be a remainder of *one* when the division is attempted. It follows that either (a), π itself is a prime, or (b), the prime divisors of π, if it is composite, are larger than p. In either case the theorem is proved. While a great many interesting theorems have been obtained regarding prime numbers, there are wide gaps in our knowledge of them, for the simple reason that the whole subject is surprisingly difficult. It was thought once, for instance, that numbers of the form $2^{2^n} + 1$ might be prime for every n, until someone showed that $2^{2^5} + 1 = 2^{32} + 1$ is divisible by 741. Similarly $n^2 - n + 41$ turns out to be prime for every positive integer n up to 41, but $(41)^2 - 41 + 41 = (41)^2$, which is of course composite.

The so-called *perfect* numbers were of considerable interest to the ancients. A number of this type equals the sum of all of its divisors except itself. Thus $6(= 1 + 2 + 3)$ and $28(= 1 + 2 + 4 + 7 + 14)$ are perfect numbers. The ubiquitous Euclid, popping up here as usual, proved that $2^{n-1}(2^n - 1)$ is perfect if $2^n - 1$ is prime, and he suspected that all even perfect numbers are included in this class. It was not until the eighteenth century that the suspicion was finally confirmed by Euler. The numbers called *perfect* and *amicable* (we'll mention the latter again) are examples of what might be called dead-end research topics. They represent end-products of questionable interest in themselves and lead nowhere in particular. As such they are in direct contrast to the basic idea in the modern theory of numbers, which opens vistas for endless exploration, pleasurable and even useful in spots. With apologies for that last backsliding phrase, we'll explain the aforesaid basic idea. This is the *congruence*, a modification of the equation. Credit for the notation is due chiefly to the great German mathematician Karl Friedrich Gauss.

The statement

(1) $$a \equiv b \ (mod\ m)$$

read ***a*** *is congruent to* ***b****, modulo* ***m*****,** means that m is a divisor of $a - b$. If b is less than $|m|$ and either zero or positive, it is called *the residue of* ***a****, modulo* ***m***. (We may as well say here, once for all, that every letter used in number theory to represent a number stands invariably for an integer unless the contrary is specifically stated.)

We'll now give a few simple theorems about congruences which will be useful in proving the particular arithmetic results we have in mind.

Theorem I. *If* $a \equiv b \pmod{m}$ *and* n *is any integer, then*

$$na \equiv nb \pmod{m} \tag{2}$$

Proof: Since by (1) $a - b$ is divisible by m, so is $na - nb = n(a - b)$. The congruence (2) follows.

Theorem II. *If* $a \equiv b \pmod{m}$, *then*

$$a^n \equiv b^n \pmod{m} \tag{3}$$

Proof: We showed in the previous chapter that $a^n - b^n$ has the factor $(a - b)$. Since this factor is divisible by m, so is $a^n - b^n$.

Theorem III. *If* $a \equiv b \pmod{m}$ *and* $c \equiv d \pmod{m}$ *then* $ac \equiv bd \pmod{m}$.

Proof: The difference $ac - bd = (a - b)c + b(c - d)$. Since the right side of the latter equation is divisible by m, so is the left side.

Theorem IV. *If* $a \equiv b \pmod{m}$ *and* $c \equiv d \pmod{m}$, *then* $(a \pm c) \equiv (b \pm d) \pmod{m}$.

Proof: $(a + c) - (b + d) = (a - b) + (c - d)$

Also

$$(a - c) - (b - d) = (a - b) - (c - d)$$

The right, and hence the left, sides of these two equations are divisible by m.

Theorem V. *If* $x \equiv y \pmod{m}$, *then*

$$a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n \equiv a_0y^n + a_1y^{n-1} + \cdots + a_{n-1}y + a_n \pmod{m}$$

This follows directly by application of theorems II, I, and IV, in the order named.

Theorem V has many interesting applications in arithmetic, of which we'll cite just a few.

Owing to the fact that 10 is the base of our number system, any number N may be written in the form

$$(4) \qquad N = a_0(10)^n + a_1(10)^{n-1} + \cdots + a_{n-1}(10) + a_n$$

where the a's are the ordered digits of N. For instance, in the number 205, $a_0 = 2$, $a_1 = 0$, $a_2 = 5$, and n, or one less than the number of digits, is 2. Now since $10 \equiv 1$ (mod 3 or mod 9), $10^P \equiv 1$ (mod 3 or mod 9) by Theorem II. Hence

$$(5) \qquad N \equiv a_0 + a_1 + \cdots + a_{n-1} + a_n \pmod{3 \text{ or mod } 9}$$

Thus if the sum of the digits of N is divisible by 3 or 9, so is N itself.

Also, since $10^2 \equiv 0 \pmod 4$, it follows by Theorem I that $10^e \equiv 0 \pmod 4$ if $e > 2$. Hence

$$\begin{aligned} N &\equiv a_0 10^n + a_1 10^{n-1} + \cdots + a_{n-1}10 + a_n \\ &\equiv a_{n-1}10 + a_n \pmod 4 \end{aligned}$$

Hence if the number formed by the last two digits of N is divisible by 4, so is N itself. (Yes, we admit that this one would be easy anyway. A little elementary practice on congruence theory, however, will do no harm.)

Again, since by (5) a number is congruent, modulo 9, to the sum of its digits, and that sum is also congruent, modulo 9, to the sum of *its* digits, and so on, we eventually reach, by continuing this process long enough, a single digit associated with any number N and congruent to N, modulo 9, which we may call the *sum digit* of N. It will be agreed that the sum digit of 9, 18, 27, etc., is 0 rather than 9. (Note "Casting out nines" and also Prob. 1, Ex. 66.) Thus the sum digit of 78,758 is 8, since $7 + 8 + 7 + 5 + 8 = 35$ and $3 + 5 = 8$. Now it follows from Theorems III and IV that:

Theorem VI. *The sum or product of two or more numbers is congruent, modulo* 9, *to the sum or product, respectively, of their sum digits.*

This fact leads to the time-honored method of checking addition and multiplication by *casting out nines*—an ingenious device which may still be useful occasionally when computing machines are not available. To illustrate, if we multiply 7,926 by 3,487 and get, let us say, 27,627,962, we can see readily that the result is incorrect, since the sum digit of (7,926)(3,487) = sum digit of (6)(4) = sum digit of 24 = 6; whereas the sum digit of 27,627,962 = 5 (actually the fourth digit in the product should have been 3). We can find the sum digits involved by striking out of a given number all digits 9 and all combinations of digits whose sums are 9. Thus: the sum digit of 27,627,962 = sum digit of 662 = 5. In a similar manner, we may cast out nines in adding long columns of figures, and may turn up an error in the process. It should be needless to say that this type of check helps chiefly in the quick rejection of most incorrect answers. Its satisfaction does not mean that the result obtained is correct, though it does bolster the confidence.

At this point, in line with our policy of merely opening new trails for you to ignore or explore in the future, we are temporarily done with the congruence.

EXERCISE 66

1. Prove that the sum digit of $n!$ $[= 1 \times 2 \times 3 \cdots (n-1)n]$ is 0 when $n > 5$.

2. Write out all primes between 50 and 100. Are there more or less of them than the number of primes < 50?

3. Without carrying through the multiplication, find the sum digit of $(8{,}639{,}428{,}614{,}317{,}912)^{10}$.

4. Without making the divisions, test the number 486,934,287,164 for divisibility by 2,3,4,5, and 9.

5. Find the residues, modulo 5, of the numbers 2, 2^2, 2^3, and 2^4.

6. Find the residues, modulo 7, of the numbers 2, 2^2, 2^3, 2^4, 2^5, and 2^6.

7. Find the residues, modulo 5, of the numbers 1^3, 2^3, 3^3, and 4^3.

8. Find the residues, modulo 7, of the numbers 1^3, 2^3, 3^3, 4^3, 5^3, and 6^3.

9. Find the value of π as used in the proof of the infinitude of primes for $p = 11$, 13, and 17.

Solve for x the congruences in Probs. 10 to 13, getting all positive solutions less than the modulus.

10. $x^2 \equiv 4 \pmod{11}$.

11. $x^3 \equiv 5 \pmod{11}$.

12. $x^3 \equiv 671 \pmod{7}$.

13. $x^2 \equiv 89{,}642{,}893 \pmod{9}$.

14. Give five numerical illustrations of Fermat's theorem: *If p is a prime and x is not divisible by p, then* $x^{p-1} \equiv 1 \pmod{p}$.

15. A boy counted the number of eggs in his basket by twos, threes, fours, fives, and sixes and each time one remained. When he counted by sevens there were none left over. Find the least number of eggs he could have.

16. Using congruences prove that the difference between any number and that number with the digits reversed is divisible by 9. Hence what is the sum digit of the difference? Will the sum digit be changed if the difference is multiplied by any integer whatsoever?

17. A teacher surprised five boys playing "for keeps" and confiscated their marbles. When he sought to return them, the boys had forgotten how many each had and decided to divide them equally, but found that there were three left. Then one boy was persuaded to accept a fine agate for his share, but when the remainder was divided by 4 there were two left. Finally a second boy agreed to accept three smaller agates, and then the remaining marbles could be divided equally among the other three boys. How many marbles were there if the total number was between 100 and 150?

98. Surprising mixups. In spite of our previous assertion that this chapter is for mental relaxation and not for applications in the crasser features of living such as working for money, it must be admitted that the substance of this article has a great many practical applications. As a matter of fact, it appears in staid textbooks on algebra under the title "Permutations and Combinations," so that it is bound to have uses. (By way of a single example, the President of the Board of Trade, or whoever it is who does the figuring when a quarter million automobile license plates, all different, are to be issued, simply refers to this book and finds that four digits preceded by a letter will be sufficient, thus saving a considerable amount of hard-to-get tin.) But in spite of its uses the subject has the essential features of a pastime, such as surprise and challenge, to such marked degree that even in school interest usually perks up at this point; and sometimes, it is suspected, the official dullard of the class leans forward with the hint of a gleam in his eye.

In Chap. XI we pointed out that the mathematical probability of an event is the ratio $h/(h + f)$, where h and f are the number of ways in which the event can happen and fail respectively. But in how many ways *can* an event happen? Often, in simple-sounding problems, the answer is a surprisingly large figure.

For example, suppose six cards, numbered 1 to 6 respectively, are mixed in a hat. In how many different orders can they be drawn out? Evidently any one of the six numbers may come first, and after that any one of five. Hence there are $6 \times 5 = 30$ different orders of appearance for the first two numbers. Continuing in this manner, we find that the six cards can be drawn from the hat in $6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720$ orders, no two of which are alike. It follows that the probability of a random drawing yielding the order 1, 2, 3, 4, 5, 6, is $\frac{1}{720}$. The reasoning in the problem illustrates a useful

General principle. *If an event can happen in* ***a*** *ways, and if, after that, a second event can happen in* ***b*** *ways, the two events can happen in succession in* ***ab*** *ways.*

By means of this principle (as easily extended to cover n events) we can solve many problems for which no "turn-the-crank" formula is available; and we can also fashion a formula or two for use in job-lot situations.

A problem of the first kind is this: "How many odd numbers of 3 different digits each can be formed with the digits 1, 2, 3, 4, 5?"

Solution: For the right digit we have three choices (1, 3, or 5); after that selection we have four choices for the middle digit, and then three choices for the left one. Applying the general principle, there are $3 \times 4 \times 3 = 36$ odd numbers of the type specified.

A problem solvable by formula is this: How many different arrangements are there for the letters a, b, and c? Each separate arrangement is called a *permutation.* Our particular problem calls for *the number of permutations of three elements taken three at a time.* This is designated by the symbol ${}_3P_3$, and equals $3 \times 2 \times 1$, or 6, as we may see by applying our principle and considering the number of choices available in succession for the first, second, and third letter. The six arrangements are easily found to be *abc, acb, bac, bca, cab,* and *cba.*

The extension to n instead of three elements is obvious. The symbol of ${}_nP_n$ is read *the number of permutations of* ***n*** *elements taken* ***n*** *at a time,* and the formula is:

$$(1) \qquad {}_nP_n = n(n-1)(n-2)\cdots(2)(1) = n!$$

The latter symbol is read *factorial n* and means, as the formula indicates, the product of all positive integers up to and including n. Thus $2! = 1 \times 2 = 2$; $3! = 1 \times 2 \times 3 = 6$, etc.

But not all the elements involved need be used in one permutation. For instance, to get the number of three-letter arrangements possible for the 26 letters of the alphabet, we get, by the general principle, $26 \times 25 \times 24 = 15{,}600$. This is called *the permutations of twenty-six elements three at a time,* and is designated thus: ${}_{26}P_3$. In general

$$(2) \qquad {}_nP_r = n(n-1)\cdots(n-r+1)$$

represents *the number of permutations of* **n** *things taken* **r** *at a time.* It may also be written in the more elegant form

$$(3) \qquad {}_nP_r = \frac{n!}{(n-r)!}$$

which is obtained from (2) by multiplying the numerator and denominator of the right side by $(n-r)!$. We'll use this form in deriving (4), but for actual computation (2) is preferable, and ${}_nP_r$ may be thought of as *the product of* **r** *factors, beginning with* **n** *and decreasing successively by* 1.

In many situations, as when we find the number of committees of three which can be formed from a group of five persons, the order of the members of a set is not involved. Thus "Smith, Jones, and Brown" make one committee, and if we permute the order of names to "Jones, Smith, and Brown" we still get the same committee. Such a group of elements, irrespective of order, is called a *combination.* Thus there are four combinations of letters, taken three at a time, which can be selected from the letters *abcd,* namely *abc, abd, acd,* and *bcd.* The symbol ${}_nC_r$ is read *the number of combinations of* **n** *elements taken* **r** *at a time,* and its formula is

$$(4) \qquad {}_nC_r = \frac{n!}{(n-r)!\,r!}$$

The proof is easily obtained by a flank attack. Since there are $r!$ permutations in each combination of r elements, it follows that ${}_nP_r = r!\,{}_nC_r$. Hence

$${}_nC_r = \frac{{}_nP_r}{r!} = \frac{n!}{(n-r)!\,r!}$$

by use of (3). Again this is a neat, easily remembered form, but not the best one for actual computation. For that we may use

$$(5) \qquad {}_nC_r = \frac{{}_nP_r}{r!} = \frac{n(n-1)\cdots(n-r+1)}{1 \times 2 \times 3 \cdots r}$$

(Note that the numerator starts with n and continues for r factors.)

We are now ready to apply this formula to the solution of our committee problem. All that is needed is the value of ${}_5C_3$, which by (5) is $(5 \times 4 \times 3)/(1 \times 2 \times 3) = 10$. In fact, this crank-turning is so easy that, in applying it, one may be lulled into forgetting what it means. It may be worth your while, in order to fix in your mind the meaning of a combination, to write out specifically the ten committees of three which can be formed from the five individuals designated, say, as A, B, C, D, and E.

EXERCISE 67

1. Find the number of rearrangements possible for the letters of the word *banter*.

2. Find the number of four-letter arrangements of the 26 letters of the alphabet.

3. How many numbers of four different digits each can be made from the digits 1 to 9 inclusive?

4. Work Prob. 3 if the adjective "different" is omitted.

5. How many odd five-digit numbers can be made from the digits 1 to 9 inclusive?

6. In how many ways may six boys be seated in a row of 10 chairs?

7. In how many ways may 10 books be arranged on a shelf if two of them must be together?

8. A person is allowed to draw out three books at a time from a circulating library owning 1,000 volumes. How many different selections could he make?

9. In a group of 25 people, each one shakes the hand of each of the others. How many handshakings are there?

10. Three cars can carry seven, five, and three people, respectively. In how many ways may 15 people be divided into three groups for the respective cars?

11. A man has 2 hats, 20 ties, 3 suits, 10 shirts, and 4 pairs of shoes. In how many ways can he dress?

12. Four homes are available for 50 children in an orphanage. In how many ways may the distribution be made?

13. Work Prob. 12 if three children are to be taken into each home.

14. In how many ways may 10 persons be seated at a round table if in deciding whether two arrangements are different we take into account (*a*) the order of seating only, (*b*) both order and background?

15. In how many essentially different ways may 10 different beads be strung on a necklace?

16. How many numbers having four significant digits can be made from the digits 0 to 8 inclusive if repetitions of digits are (*a*) allowed, (*b*) not allowed?

17. A game requires 4 boys and 3 girls. How many times may the game be played by 6 boys and 5 girls if the same group is not used in any two games?

18. At a reception given during an international congress of students the receiving line consisted of one representative from England, one from the United States, one from Australia, one from China and an interpreter. In how many ways can the line be formed with an English-speaking person first and with the Chinese and the interpreter standing together?

19. In how many ways can the students in the above problem seat themselves around a circular table if the Chinese and the interpreter are together?

20. A committee is composed of 6 Rotarians and 6 Kiwanians and it elects a chairman from each group to serve jointly. In how many ways can they be seated around three sides of a table with the co-chairmen at one end and no two members of the same club together?

21. A primary Sunday-school class is composed of three boys and three girls. In how many ways can they be seated on a bench if one boy is timid and refuses to sit next to a girl?

99. Pascal's peculiar pyramid. Before we can get to the meat of this article we must dispose of the notorious binomial theorem. Most students encounter this theorem rather early in their algebraic careers, learn to use it mechanically if at all, and forget it forthwith. After all, it conveys no information in itself which one could not get by strong-arm multiplication, so that it probably is not worth remembering unless one has occasion to use it often. It does, however, bring into the limelight a set of integers which have most interesting relations with each other. To set the stage, we'll exhibit the first five powers of the binomial $(a + b)$, which may be verified by actual multiplication.

$$(a + b)^0 = 1, \text{ by definition}$$
$$(a + b)^1 = a + b$$
$$(a + b)^2 = a^2 + 2ab + b^2$$
$$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$
$$(a + b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$$
$$(a + b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$$

Now if we write down only the coefficients on the right, we get the pyramidal form known as *Pascal's triangle* because the mathematician Blaize Pascal called attention to many of its interesting properties. Actually it was known long before his time. Here it is:

(1)

1					
1	1				
1	2	1			
1	3	3	1		
1	4	6	4	1	
1	5	10	10	5	1

As many lines may be added to the bottom as we wish, according to the rule: The first and last number of each line is 1, and every other number may be obtained by adding two adjacent numbers in the line above and writing their sum immediately below the right one of the two.

The numbers in this triangle will be found to coincide with the coefficients of the successive powers of a binomial as shown above. For this reason they are called *binomial coefficients*, and also, incidentally and surprisingly, they coincide with the values of the expression ${}_nC_r$ for all values of n and r. This we'll prove next.

In the above expansion of, say, $(a + b)^4$, we find by trial that the result is the same whether we build up to it by multiplying each preceding expansion by $(a + b)$ or simply define the expansion of $(a + b)^4 = (a + b)(a + b)(a + b)(a + b)$ as the algebraic expression obtained by adding together all the fourth-degree terms having as component factors one letter from each of the four binomials. For instance, the product $abab = a^2b^2$, written in the first form, shows that the selection of the four factors from left to right was "a, then b, then a, then b." Each such product has the coefficient 1; but some of them, such as $aaab = a^3b$ and $abaa = a^3b$, give terms which may be combined in the end. Now how many terms are there which will yield a^3b? Evidently the answer is ${}_4C_1 = 4$, since we select one b at a time in each of the four ways possible.

To generalize, consider the expansion $(a + b)^n = (a + b)(a + b) \cdots$ (with n factors). Evidently each term, having one factor from each binomial, will be of the nth degree; and among the terms will be some of the form $a^{n-r}b^r$. The number of such terms will be ${}_nC_r$, since each combination of r b's will yield one of them, and there are n b's to select from. It follows that the coefficient of $a^{n-r}b^r$ is ${}_nC_r$ when $r > 0$, and we have

$$(2)\quad (a + b)^n = a^n + {}_nC_1a^{n-1}b + {}_nC_2a^{n-2}b^2 + \cdots + {}_nC_ra^{n-r}b^r + \cdots$$

From the definition of ${}_nC_r$ we may also write (2) thus:

$$(3)\quad (a + b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{2}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{3!} + \cdots$$

which is the form perhaps more often seen. A third form used often in connection with binomial coefficients is $\binom{n}{r}$ for ${}_nC_r$.

A second way of seeing that the coefficient of $a^{n-r}b^r$ in the expansion of $(a + b)^n$ is ${}_nC_r$ leads easily to the fearsome-sounding thing known as the *multinomial theorem*. If we interchange two like letters in the product $aabb$, for instance, we do not change that particular arrangement which indicates the order of choice "a, then a, then b, then b;" but if we interchange any a with any b we get an essentially different order such as "a, then b, then a, then b." Now let us say that there are K essentially different orders of choice of the letters yielding a term $a^{n-r}b^r$. If the $(n - r)$ like a's were all different we could replace any one order of choice by $(n - r)!$ permutations; and a similar statement applies to the r b's. In that case there would be $n!$ permutations of the n different letters, and we would have

$$K(n - r)!r! = n! \tag{4}$$

Thus

$$K = \frac{n!}{(n - r)!r!} = {}_nC_r \tag{5}$$

Similarly, in the expansion of $(a + b + c + d)^{14}$, for instance, the coefficient of $a^4b^3c^2d^5$ is $14!/4!3!2!5!$. The generalization to the expansion of $(a_1 + a_2 + \cdots + a_m)^n$ is obvious, though somewhat cumbersome to write, so we shall omit it.

Referring again to Pascal's triangle (1), we find that there are other interesting relations of these numbers besides that mentioned in the rule for the formation of successive lines. For instance:

1. The sum of the numbers in each row is a power of 2.

2. The sum of first, third, fifth, etc., numbers in each row equals the sum of the remaining numbers.

3. If we begin numbering the rows with the second one in (1), then the sum of the squares of the numbers in the kth row equals the $(k + 1)$th number in the $(2k)$th row.

To prove 1 and 2, consider $(1 + 1)^n$ and $(1 - 1)^n$ respectively. To prove 3, let $(a + b)^{m+n}$ be considered as the product of $(a + b)^m$ and $(a + b)^n$. Get the coefficient of $a^{m+n-r}b^r$ in two ways, and then let $m = n$ in your general result. With these hints we leave the rest to you. Evidently we have merely skimmed the surface of the possibilities. The field is now yours.

EXERCISE 68

1. Using either (2) or (3), expand $(a + b)^{10}$.

2. Add five more rows to the triangle (1) and compare the bottom row with the coefficients obtained in Prob. 1. (They should be the same.)

Get the first four terms of the expansions indicated in Probs. 3 to 7.

Simplify each term, using the laws of exponents, but do not simplify in the first step.

3. $\left(2 - \frac{x}{2}\right)^9$.

Solution: $\left(2 - \frac{x}{2}\right)^9 = \left[2 + \left(\frac{-x}{2}\right)\right]^9$

$$= 2^9 + 9 \cdot 2^8\left(-\frac{x}{2}\right) + \frac{9 \cdot 8}{2} \times 2^7\left(\frac{-x}{2}\right)^2 + \frac{9 \cdot 8 \cdot 7}{2 \cdot 3} \times 2^6\left(\frac{-x}{2}\right)^3 + \cdots$$
$$= 2^9 - 9 \times 2^7x + 36 \times 2^5x^2 - 84 \times 2^3x^3 + \cdots$$
$$= 512 - 1{,}152x + 1{,}152x^2 - 672x^3 + \cdots$$

4. $\left(2 - \frac{x}{2}\right)^{10}$. **5.** $\left(2 - \frac{x}{2}\right)^{11}$ **6.** $\left(\frac{x}{2} + 2\right)^8$. **7.** $\left(\frac{2}{x} - \frac{x}{4}\right)^6$.

8. Find the coefficient of $a^3b^2c^7$ in the expansion of $(a + b + c)^{12}$, using the method suggested in the text below (5).

9. Show that the result 3 in the text holds when $k = 4$, $k = 5$, and $k = 6$.

10. Complete the proof of 3 in the text by the method there suggested.

100. Fermat's poser. The floods of the Nile River Valley, which have occurred throughout the period of recorded history, brought into existence the rudiments of surveying. Those who helped to locate the obliterated land lines were known as "rope stretchers" because of the fact that they used ropes divided by knots into twelve equal intervals. In making right angles they used the well-known triad 3, 4, and 5 and the relation

(1) $$3^2 + 4^2 = 5^2$$

between the sides and hypotenuse of a particular right triangle.

The early dabblers in number theory were very much interested in triads of this sort, especially after Pythagoras proved his famous theorem dealing with sides of a right triangle. Diophantus of Alexandria discovered the formula

(2) $$x = 2mn,\ y = m^2 - n^2,\ z = m^2 + n^2$$

for grinding out such sets of integers. For instance, with $m = 2$, $n = 1$, we get $x = 4$, $y = 3$, $z = 5$; while $m = 3$, $n = 2$ yields

$x = 12$, $y = 5$, $z = 13$. In both cases $x^2 + y^2 = z^2$, as may be verified by trial.

The interest in such integral solutions of $x^2 + y^2 = z^2$ soon broadened, naturally enough, to include equations like $x^3 + y^3 = z^3$, $x^4 + y^4 = z^4$, and in general,

$$x^n + y^n = z^n \tag{3}$$

But here the mathematicians struck an unexpected snag. Not only could no one find a general solution of (3), but not even one set of positive integral solutions came to light for any n larger than two. A French mathematician named Pierre Fermat (1601–1665)—a man particularly adept in number theory—looked into this rather surprising situation. Unfortunately, we do not know a great deal about his results in this connection, and what we do know is just enough to irritate rather than enlighten us. On the margin of his copy of *Arithmetica*, by Diophantus, this notation was found: "It is impossible to partition a cube into two cubes, or a fourth power into two fourth powers, or generally, any power of higher degree into two powers of like degree. I have discovered a truly wonderful proof of this, which, however, this margin is too narrow to hold."

And "truly wonderful" that proof must have been! Probably most of the best mathematical minds in the last three centuries have dallied more or less seriously with the problem, and none has solved it. The statement on the margin, now secure in fame as "Fermat's last theorem," has probably caused more headaches than all the low-hanging beams in the world. It is so easy to understand, so challenging, so insidiously tantalizing, that it is like a sleep-robbing puzzle. "There exist no positive integral solutions of the equation $x^n + y^n = z^n$ for $n > 2$." But we have warned you!

Not that nothing has been done with it. That is far from the case. So far, in fact, that we now know the theorem to be true when $n = 3, 4, 5$, and on and on and on even beyond $n = 7{,}000$. But the clinching and final argument is always just beyond the point we have reached. Maybe Fermat saw it, but more likely not (in our opinion). For the fact that he never took time to write up and publish a "truly wonderful" argument is fairly convincing evidence that he must have detected a flaw in the logic while putting the argument into formal dress. But whether he did or not, the challenge of the casual marginal statement worked wonders, even if it did not get the problem solved. Far more than any other one problem, it was responsible for much of the development and popularity of the modern theory of numbers.

EXERCISE 69

1. Using (2), write 10 sets of solutions of the equation $x^2 + y^2 = z^2$ in which x, y, and z are relatively prime (*i.e.*, no pair has a factor in common).

2. Prove that if x, y, and z are a set of relatively prime integral solutions of the equation $x^2 + y^2 = z^2$, then x or y is even, z is odd, x or y is divisible by 3, and one of x, y, and z is divisible by 5. HINT: Note that $(5m + k)^2 \equiv 1$ or 4 (mod 5) when $k = 1, 2, 3$, or 4.

3. Show that if x, y, and z are a set of relatively prime integral solutions of the equations $x^3 + y^3 = z^3$, then one of x, y, z must be divisible by 7.

4. Work Prob. 3 with "7" replaced by "13."

5. Show that if x, y, and z are a set of relatively prime integral solutions of the equation $x^{p-1} + y^{p-1} = z^{p-1}$ where p is prime, then x or y must be divisible by p. (See Prob. 14, Exercise 66.)

6. Show that $x^2 + y^2 = z^2$ is satisfied when $x = 2n^2 + 2n$, $y = 2n + 1$, and $z = 2n^2 + 2n + 1$. Does this include all cases for which z is one more than x? Show that in this case x is a multiple of 4.

7. Show that the equation $x^2 + y^2 = (x + a)^2$ cannot be satisfied unless y and a are both even or both odd.

101. Numerology, or what does not follow. Song writers assure us that there is "magic in moonlight;" and more than one dazed youth who has been in no condition to weigh his words with scientific caution has noted the "glamour and mystery" in a maiden's eyes. These things we expect. But surely the most optimistic dealer in the occult can find nothing mystical in the staid integers 1, 2, 3, 4, 5, 6, etc. That's what *you* think, maybe. But listen to the eminent fourth-century divine and philosopher, St. Augustine: "Six is a number perfect in itself, and not because God created all things in six days; because this number is perfect and would remain perfect even if the work of six days did not exist."[1] Or to Alcuin of York, who explained the imperfection of the second creation by the observation that "eight souls, and not six," were rescued in Noah's ark—"eight" being doubtless too earthly and imperfect, in its inner significance, to be safely entrusted with the shaky destiny of mankind. Or to Uncle and Aunt Swisher of Dog Hollow, who, like a fair scattering of their sophisticated New York cousins, and like their parents' folks and *their* parents' folks way back to some unknown short story of history, will not put too much trust in the number 13, though they'll admit judiciously that the rumor about it could be faulty.

Here, if anywhere, we more or less rational bystanders catch a broadside view of the incurable mysticism of the human race. Any group of beings who can rhapsodize over the integer 6 because it is *perfect*, or because, in other words, it is equal to the sum of its divisors ($1 + 2 + 3 = 6$), can be expected to draw other conclusions about numbers which will seem a bit shaky, here and there, to the unpleasantly critical few. And draw conclusions they do and have done, all through recorded history—conclusions weird and fanciful, but conclusions, nevertheless, which may tangle their inconsequential logic along with some interesting surprises in the numbers themselves.

Take those highly touted ancient Greeks. They (or at least some of them) solemnly concluded that the even integers were "feminine" and the odd ones "masculine." The number *one* represented reason; *two*, opinion; *four*, justice; and *five*, marriage. The reason, as will perhaps be obvious, is that *five* is the union of the first even (2) and the first odd (3)—the number *one*, of course, being an outlander. This seems fair enough, since *one* stood for reason.[2]

Or take, again, the numbers 220 and 284. They are a sample of the many *amicable* pairs sought assiduously by the ancients. What are the mystic properties of this potent pair? Well, the sum of the divisors of 220 is 284 and the sum of the divisors of 284 is nothing more or less than (lean close while we whisper) 220! Isn't that just too, too significant! No wonder that a legendary

[1] Dantzig, *Number, the Language of Science*, p. 45.
[2] *Ibid*, p. 40.

gentleman of an earlier but no less uncritical era than ours is said to have hunted high and low for a life-mate whose number was 220, his own being 284. Oh yes, we forgot to tell you that, according to ancient and modern numerologists, every person has a number, even though he may never have been within eleven miles of a penitentiary. You see, we can let $a = 1$. Then, of course, $b = 2$, $c = 3$, and so forth. It follows that "Mary Jones" adds to 120, and $1 + 2 + 0 = 3$, so that Mary is 3, or rather, Mary vibrates 3—that is to say, Mary and 3 are harmonious, so that they vibrate together, if you follow us. Or perhaps you don't. It is all very deep. But this is only what you might expect, considering how expensive are the experts who can explain it more simply. If one has sufficient financial stamina to ferret out of the cosmos his complete numerological status, he can change his name so that his new synthetic number will fit his personality like a wet suit, and the consequent harmonious combination will naturally rocket him infallibly to the peak of success. Or with even moderate assets he can study the voluminous numerological literature of the bookstands and get at least to a different status with his employer.

But perhaps we have let our enthusiasm carry us too far. What we wanted to point out is that the science of numbers owes part of its development to interest in magic as well as to the requirements of humdrum living. Like astronomy, which was fathered by astrology, the study of numbers grew out of an ancestor who is lusty and thriving even at this late day, though his standing with the scientists is atrocious, and he really ought to be dead.

We haven't the heart to give you an exercise in numerology. Work out your own key number, if you will, but we'll have nothing to do with it.

CHAPTER XIV

EXTENDED ANALYTIC GEOMETRY—A NEW FIELD OF MATHEMATICS

102. Why we are disgruntled. Analytic geometry, which we dealt with briefly in Chap. VIII, is a delightful find for the beginner in mathematics who likes both algebra and geometry, and who enjoys translating algebraic results into picture language. But as he delves more deeply into the subject he eventually comes upon a disturbing limitation of his flair for drawing-board interpretation. He finds that he can draw curves and construct three-dimensional models to give geometric meaning to equations in one, two, or three variables; but there he must stop. Even so simple an equation as

$$x + y + z + u = 1$$

baffles him as an artist, if not as an algebraist, so long as he is content to follow the main line of mathematics through the first half of the twentieth century. Unless, after all, he is at heart a pure algebraist, he may well be repelled from the branch of mathematics which bars him from resorting to his favorite language of the line when just one more letter comes into the picture along with x and y and z.

All of the foregoing, as the reader may readily surmise, is fanfare designed to call attention to a system of analytic geometry[1] which retains many features of Descartes' simple and immortal plan, but does not suffer from the three-variable limitation. But before we set forth the elements of this plan we shall describe very briefly the historical and widely accepted line of development in this general field.

103. What we like. The things that we like in plane analytic geometry have already been outlined rather briefly in Chap. VIII.

[1] See "An Analytic Geometry for N Variables," by R. S. Underwood; *American Mathematical Monthly*, May, 1945.

Here, if anywhere, we have the ultimate in simplicity and patness; one has the instinctive impression that to seek a better scheme than that of Descartes to illustrate the subtle interplay of two varying quantities is to attempt the palpably absurd task of improving upon perfection. About things such as poems and works of art we can say: "This is good; that is great; and that one is greater still." But with reference to the rectangular coordinate system such comparisons seem pompous and rather absurd; one merely says flatly: "This is *IT*," and the search is finished in the field concerned. Why this is so, or seems to be so, it is hard to say. Perhaps the explanation will appear when we review the aspects of analytic geometry which contribute so forcefully to the impression that here, in one sadly circumscribed realm, lies a portion of the absolute and the unimprovable.

To begin with, a good geometric counterpart of a real number is almost "made to order." It is a point on a straight line drawn (or rather approximated in a sample segment) on a sheet of paper and marked with a starting point and a scale. The various real values of a single variable such as x can be represented conveniently by points on the line. Or, to put it in another way, as a point moves along a segment, say from one to two, it sweeps grandly over an endless array of numbers. Certainly without this analogy our grasp of the abstract number concept would be impaired, to say the least. Though the sand grains of the Sahara would not suffice to match the numbers between one and two, geometry assigns to each individual his separate home on the range.

Now if a second point representing the number y moves on a second line at right angles to the first, the positions of both points at a given time may evidently be shown by the point (x,y), located according to the rules of plane analytic geometry. Thus the relation between two varying quantities can be indicated on a plane by the path of a single moving point. Again, by the use of a third axis at right angles to each of the others, like one of the three adjacent edges of a box, we can still use one point to designate a trio of numbers; and on this simple framework we may hang the weightless fabrics of solid analytic geometry.

104. What we are complaining about. But what then? Certainly it is not natural for a mathematician to stop with the number 3, or with any given number, for that matter. Professional instinct tells him that he must generalize at all costs until he gets to n, which can stand for any positive integer that a student or rival professor cares to mention. He noticed long ago, of course, that he was running out of slots for mutually perpendicular axes. But his prestige was at stake; the idea had proved fruitful thus far; and besides, he never was one to care much about the mere plausibility of his premises. The point was to have some on hand, so that he could go about his main business of drawing conclusions. Clearly, if this good thing was to continue, he would require a lot of perpendicular axes; and clearly he had the right, as a working mathematician, to postulate as many of them as he needed. And so he did; and so "n-space" was born; and it grew lustily forthwith.

More seriously, in the foregoing summary treatment of a major historical development, it is of course not to be assumed that we think we have disposed of, or even roughly outlined, the merits and accomplishments of a point of view which has permeated much of modern mathematics. It is sufficient for our purpose merely to note that in this abstract approach the working tie-up of algebra and geometry, which functions so beautifully for the cases of two and three variables, has been abandoned, reluctantly or otherwise. Explorers in n-space enjoy some of geometry's benefits, to be sure, in the form of carried-over suggestion and of analogy enhanced by the use of pseudo-geometric terms; but if we mean by geometry a mathematical field which permits rough verification of its rules by actual measurements, then surely, in the current scheme, geometry has fizzled out when n exceeds 3.[1]

105. What can be done about it. But was this necessary? Apparently so, if the only kind of generalization of the step from

[1] Two passages from Dr. E. T. Bell's stimulating book, *Handmaiden of the Sciences* (Williams & Wilkins) are apropos here:

(Page 102) "What has just been outlined is the almost universally accepted method of attacking geometry today. Although it has been a mathematical commonplace for at least forty years, the bold and, some might say, barren abstraction behind it has only recently come into fashion in science."

(Page 117) "With one eye on what Descartes did for his 2-ples and 3-ples, we can talk about the entuples in a given manifold in the language of geometry. This may seem like cheating after having turned our backs on geometry, and perhaps it is. Possibly it is merely lack of imagination."

one to two axes is that which retains the feature of perpendicularity. But—and note this well—on the plane that feature appears in the orthodox system precisely and exactly *once!* Actually it is possible to make another generalization of the step from one to two axes which seems equally logical, and which does not promptly run the generalizer into a blind alley from which he can escape only "by analogy," with some loss of face and much loss of geometry. For the positive and negative ends of the two axes make a four-spoked wheel whose hub is the origin, and there is nothing in nature or mathematics to bar one from adding more spokes ad infinitum. There pops up at once, to be sure, the objection that it seems a pity for solid analytic geometry to be left a respected but outcast orphan with no place at all in the sequence. But even this aesthetic carping may be silenced rather easily. For with the axle added to each of the $2n$-spoked wheels, a new solid sequence is started in which the erstwhile orphan takes the primary, or leading, position. Thus in the exposition to follow we have deviated not one iota from the classical plane and solid systems, as far as they go; but we *have* tampered, and drastically so, with the generalizations to which they give rise.

106. Why we concentrate on the 3-axes plane. The case of three axes on a plane should be considered first of all for several reasons, such as: (1) Next to the familiar one of two axes, it is the simplest of the systems on a plane. (2) It provides a plane analog for solid analytic geometry, inviting comparison and suggesting parallel studies at many points. (3) Its results can be carried over bodily to the corresponding solid case, in which a function of three variables can be interpreted in terms of surfaces and lines in space. And finally, (4), the simple pattern of the points designated by integral coordinates, which are at the vertices of equilateral triangles filling the plane, combined with the new principle of one degree of freedom in the coordinates of a point, provides a means for attack upon problems in number theory, as we shall see.

107. The basic system. It is convenient to place the three axes so that their positive ends radiate from the origin at standard position angles of 0°, 60°, and 120°, as shown in Fig. 85. Here we see also how to locate a point with three given coordinates. The point $P(-3,1,-2)$, for example, is reached by finding first the point A which is 3 units from the origin O in the negative-x (or left) direction; then measuring 1 unit from A in the positive-y direction (parallel with the y axis), reaching B; and then measuring 2 units in the negative-z direction, to P. It will be seen, upon reflection, that in this system a set of three coordinates designates

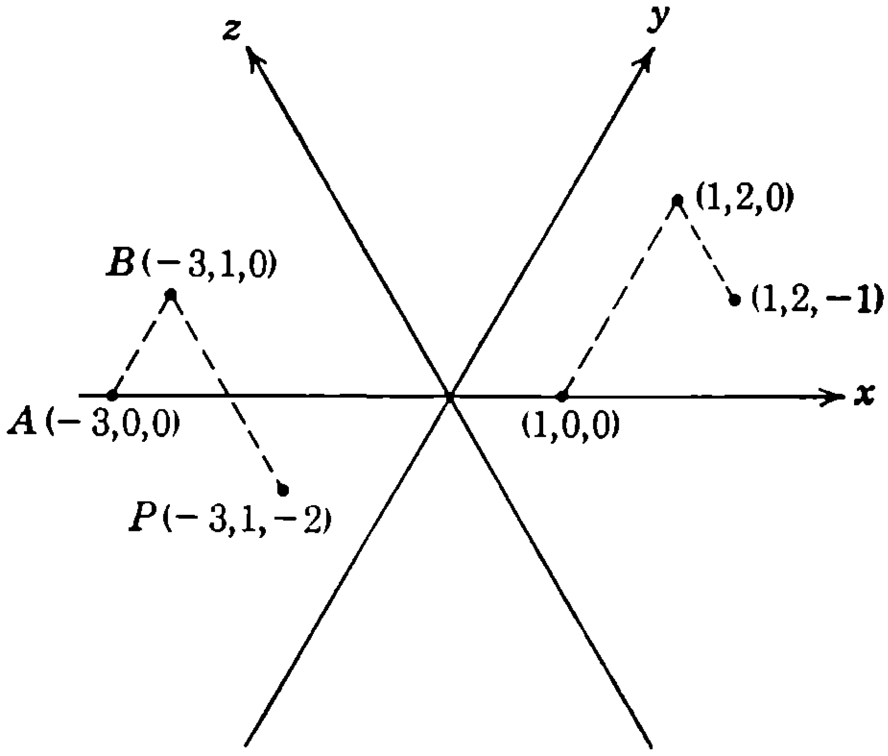

Fig. 85.

one and only one point, regardless of the order of application. Somewhat fortunately, as we shall see, the converse statement is not true.

At this point, in advance of any findings about the nature of loci in the system, the following definition is in order.

Definition: *The locus of an equation on the* 3-*axes plane is the totality of points having coordinates which satisfy the equation.*

This means, of course, that each point on a given locus has at least one set of coordinates satisfying the equation. It appears upon further study that a typical locus consists of a family of curves which completely fill a given area, and which, with allowance for the slanting of axes, exhibit the normal properties of the curves with which we are familiar on the 2-axes plane. It may be mentioned in passing that, for the n-axes plane with $n > 3$, they fill out the hitherto incomplete geometric interpretation of the meaning of partial derivatives in calculus.

108. The basic equations. Fig. 86 shows the relation of any set (x,y,z) of the coordinates of a given point P to the unique coordinates (X,Y) designating P in a 2-axes system superimposed upon the other as shown. Evidently, $X = x\cos 0° + y\cos 60°$

$+ z\cos 120° = x + \frac{1}{2}(y-z)$, and $Y = x\sin 0° + y\sin 60° + z\sin 120° = \frac{1}{2}\sqrt{3}\,(y+z)$. Thus we get the basic equations:

$$\textbf{(1)} \qquad 2x + y - z = 2X;\ \ y + z = \frac{2}{\sqrt{3}}Y$$

These equations are passports to and from the familiar territory of just two axes. One needs to use them for occasional return trips to known country while learning to walk alone in the strange new land.

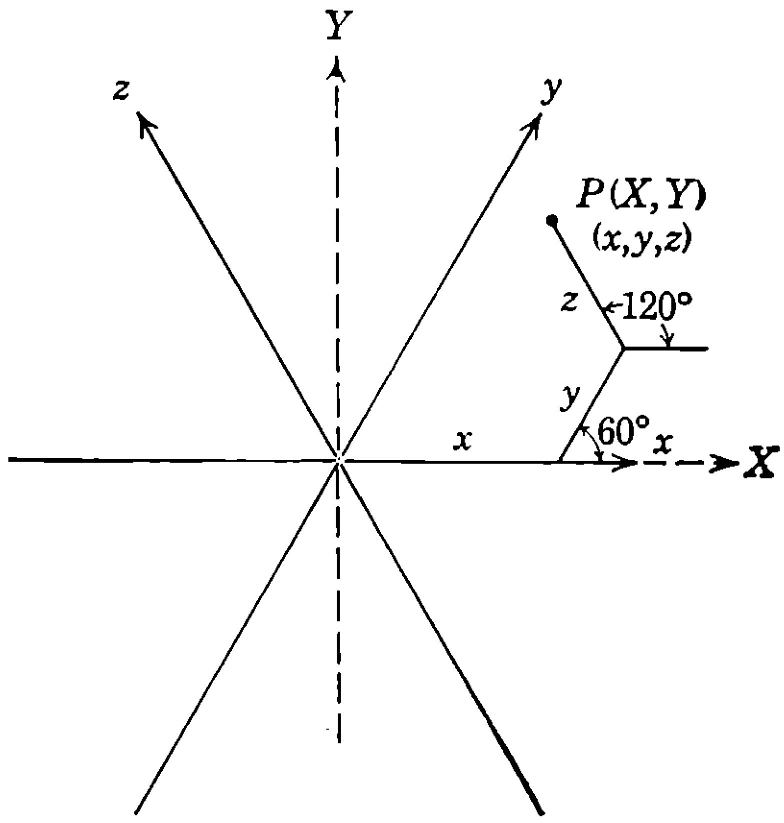

FIG. 86.

109. Generalized coordinates and a new freedom. The reader should now have clearly in mind the fact that a point on the 3-axes plane has an unlimited number of sets of coordinates. For example, he may verify by trial the fact that the point (1,2,−1) shown in Fig. 85 is also designated by the trios (0,3,−2) and (−3,6,−5), among others. We'll now show how these additional coordinates may be found.

From (1), Art. 108, it follows that if (a,b,c) is a particular set of coordinates of a fixed point (X,Y), and if (x,y,z) is another set of coordinates of the same point, then

$$(1) \qquad 2a + b - c = 2x + y - z$$

and

$$(2) \qquad b + c = y + z$$

Eliminating z and y successively from (1) and (2), we find that $y = a + b - x$ and $z = c - a + x$. From this we see that the numbers

$$\textbf{(3)} \qquad (x,\ a + b - x,\ c - a + x)$$

may fittingly be called the *generalized coordinates* (abbreviated: G.C.) of the point (a,b,c). For example, all coordinates of the point (2,−1,4) are included in the set $(x,1-x,2+x)$, with x

arbitrary. To illustrate its use, we'll find the coordinates of a given point, say $(1,-2,3)$ which satisfy the equation

$$(4) \qquad 2x + 3y + 4z = 5$$

The G.C. of $(1,-2,3)$ are $(x,-1-x,2+x)$. Replacing y and z in (4) by $-1-x$ and $2+x$, we get $x = 0$, so that the answer is $(0,-1,2)$.

Evidently every point on the plane has one and only one set of coordinates which satisfies (4), so that its locus covers the 3-axes plane and may be said to be "one layer deep." Similarly, the loci of second degree equations are normally two layers deep, like deflated balloons spread flat; the loci of third degree equations have three layers or one, according as the roots of an associated cubic[1] are all real or not; and so on. Any of these loci may be infinitely many layers deep, however—in which case they are curves, like the lines of the next article, rather than broadside areas.

110. The loci of first degree equations.

Theorem 1. *The locus of the equation*

$$(1) \qquad Ax + By + Cz = D$$

where A, B, and C are not all zero, is a straight line if $B = A + C$; otherwise it is the whole 3-axes plane.

Corollary. The general equation of a straight line is

$$(2) \qquad Ax + By + (B - A)z = C$$

The proof, which we shall omit in detail, is obtained by solving (1) simultaneously with the two equations in (1), Art. 108. If $B \neq A + C$ it turns out that there is a trio of real numbers, x, y, and z, satisfying (1) for any point (X,Y) on the plane; while if $B = A + C$ the equations are inconsistent unless (X,Y) is on a line determined by the numbers A, B, C, and D.

Several lines, with their equations, are shown in Fig. 87.

111. Strictly on the side. These lines remind one of planes seen on edge. Moreover, other curves, such as the locus of

$$(1) \qquad x^2 + y^2 + z^2 + xy - xz + yz = 4$$

[1] The equation referred to is the cubic in x obtained when y and z are replaced respectively by $\frac{Y}{\sqrt{3}} + X - x$ and $\frac{Y}{\sqrt{3}} - X + x$, as obtained from (1), Art. 108.

which is a circle of radius 2, suggest cylindrical surfaces seen on edge, while in general the loci of second degree equations resemble the projections on a plane of the corresponding surfaces in solid analytic geometry. And as a matter of fact, they are. The observer's eye is "at infinity" in the same direction from the origin as is the point $(-1,1,-1)$. However, the notion that loci

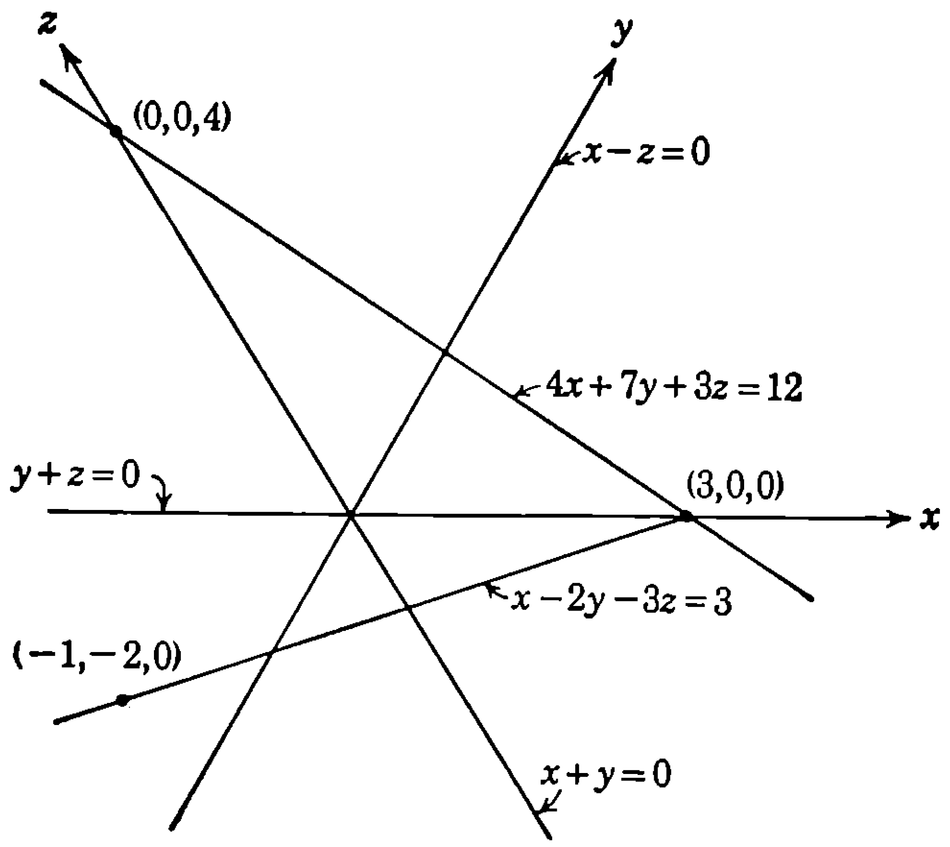

FIG. 87.

on the n-axes plane are "projections of configurations in n-space" becomes far-fetched and unprofitable when n exceeds 3. It is intriguing, of course, to think of a simple curve, drawn according to simple rules, as the splashed and distorted silhouette of some deeper and more fuzzy reality, like those rare and dubious photographs of ghosts; but it seems wise also to recall the fact that this is a point of view and a reflex of conditioned thinking rather than an intellectual toe-hold on necessary truth.

112. And now we apply it. Consider the problem of finding all integral solutions of two linear equations, such as, for example,

(1) $$7x + 9y + 4z - 5 = 0$$

and

(2) $$10x + 6y - 7z + 7 = 0$$

The following solution of this problem will serve to illustrate a general method which applies our analytic results, and which, for many such problems, including this one, is shorter than the "natural" algebraic attack based upon the initial elimination of one letter.

Upon multiplying the members of (2) by k, and adding corresponding members of (1) and (2), we get

$$(3)\quad (7 + 10k)\, x + (9 + 6k)\, y + (4 - 7k)\, z + (-5 + 7k) = 0$$

which in solid analytic geometry would represent the family of planes passing through the line of intersection of the planes (1) and (2). In the new system (1) and (2) are still planes (lying flat on the 3-axes plane), and they still intersect in a line, as indicated below, but in this case we can find the equation of the line directly simply by fixing k so that

$$(4)\qquad (9 + 6k) = (7 + 10k) + (4 - 7k)$$

This yields $k = \frac{2}{3}$, and from this the equation of the line is

$$(5)\qquad 41x + 39y - 2z = 1$$

Note that, from the manner of its derivation, (5) will be satisfied by all common solutions of (1) and (2). Hence the line (5) contains all points with coordinates which satisfy (1) and (2) simultaneously. Each point on the line has exactly one such set of coordinates, as may be proved formally for the general case by restating the following specific steps in more general form.

First, we find all integral solutions of (5). Since 41 is the largest of the coefficients it is convenient to let $x = 0$, thus using the one degree of freedom. The remaining two coordinates, y and z in this case, are sufficient to "scan" the line, reaching every point on it. When $x = 0$, $z = (39y - 1)/2$, so that if $y = 2n + 1$ (n an arbitrary integer) z will be the integer $39n + 19$.

The remaining details are routine. The coordinates

$$(6)\qquad (0,\ 2n + 1,\ 39n + 19)$$

designate all points on (5) having coordinates which are all integers. To find the special coordinates of these points which satisfy (1) and (2) as well as (5), we get the G.C. of (6):

$$(7)\qquad (x,\ 2n + 1 - x,\ 39n + 19 + x)$$

Substituted in (1) [and also in (2) as a check], these yield $x = -40 - 87n$, whence by (7) the desired answer is

$$(8)\qquad x = -40 - 87n;\ y = 41 + 89n,\ z = -21 - 48n$$

Comment. The typical, central, and time-consuming item in problems of this sort is reduced, in the foregoing case, to the solution of the congruence

$$(9) \qquad 39y \equiv 1 \pmod 2$$

We could have solved the problem by eliminating x or y or z; but in each case we would have been faced by a congruence much longer and more tedious to solve.

Is it true, then, that all that can be claimed for our picture-guided method in the way of accomplishment in number theory is a gain in efficiency in the solution of such problems as the one above, in about half of the cases selected at random? Fortunately we are able to answer "No—see Probs. 16 to 20 and 24, Ex. 70, for example." But clearly, since the geometric illumination merely suggests algebraic technique, it would be unfair to demand of this new approach that it point to results which could not be reached by other algebraic methods. Clearly, moreover, the statement that such results have been obtained would be an unprovable assertion. Yet it is possible, as we shall see, to prove rather easily, in the light of the new geometry, some theorems which appear to be resistant, to say the least, to frontal and unguided algebraic attacks.

113. A generalization, and what comes of it. We have seen that, as interpreted by our system, the loci of two linear equations in three variables normally intersect in a line containing all points designated by common solutions of the two equations. When the equation of the line is written with three variables, as in the case of (5), Art. 112, it is certain to be satisfied by the common solutions, since it is "infinitely many layers deep" in the sense that all coordinates of every point on the line satisfy the equation. On the other hand, when one of the three variables is given a fixed value, the line may still be drawn from the simplified equation, though the locus of the latter is only one layer deep.

For the case of two equations in general, linear or otherwise, we may define *the curve of intersection* of the two loci as the curve containing all the points, and only the points, designated by common solutions of the equations. The *complete curve of intersection*, whose equation is obtained by the method of the following theorem, usually coincides with the curve of intersection, though it may contain some additional points.[1]

Theorem. *To get the equation of the complete curve of intersection of the loci of two equations in* x, y, *and* z, *we replace these letters by* x', $x + y - x'$, *and* $z - x + x'$, respectively, and then eliminate x'.

Proof: Replacing x, y, and z by x, $Y/\sqrt{3} + X - x$, and $Y/\sqrt{3} - X + x$ respectively [from (1), Art. 108], and eliminating x, we get the equation of the complete curve of intersection in terms of X and Y. Since, from (1), $Y/\sqrt{3} + X = x + y$ and $Y/\sqrt{3} - X = z - x$, the simple rule of the theorem follows.

The significance of the phrase "what comes of it" must be left to the reader. He can get a rough idea of the possibilities from some of the sample problems in the exercise below, in which other bypaths of the subject may also be glimpsed. The trails are open, and the paths are mostly untrod.

EXERCISE 70

1. Find the special coordinates of the point (0,0,0) which satisfy each of the following equations of planes:

(*a*) $x + y + z = 1$; (*b*) $x + 3z = 4$; (*c*) $x = 5$; (*d*) $2y + z = 0$.

2. Work problem 1 with (0,0,0) replaced by (1,2,−3).

3. Draw the following lines:

(*a*) $x - y - 2z = 2$; (*b*) $x - z = 1$;
(*c*) $y + z = -2$; (*d*) $3x + 5y + 2z = 12$.

4. Verify that the equation of the straight line through the points (a,b,c) and (d,e,f) is

$$(b + c - e - f)x - (a + f - d - c)y + (d + e - a - b)z = bd + dc + ec - ae - af - fb.$$

5. If we define the slope m of the line $Ax + By + (B - A)z = C$ as $-\frac{A}{B}$, or $-\frac{\text{the } y\text{-intercept}}{\text{the } x\text{-intercept}}$ when the intercepts are not both zero, show that, for the line through the points (a,b,c) and (d,e,f),

$$m = \frac{(b + c) - (e + f)}{(a + f) - (d + c)}$$ HINT: Refer to Prob. 4.

6. Writing two sets of coordinates thus: abc/def, note the pattern formed by the definition of m in Prob. 5. Then find, by use of this

[1] For example, the complete curve of intersection of the loci of $x^2 + y + z + 1 = 0$ and $x^2 + 1 = 0$ is the x axis ($y + z = 0$); though $x^2 + 1 = 0$ has no locus.

pattern, the slopes of the lines through the following pairs of points: (*a*) (0,0,0), (1,2,3); (*b*) (−1,4,0), (0,−2,1); (*c*) (−1,−1,−1), (2,3,4); (*d*) (0,1,1), (2,2,0); (*e*) (1,0,0), (1,1,0); (*f*) (1,2,3), (−1,−2,−3).

7. Note that the left side of the equation of the line through the points (1,−2,3) and (3,2,0) may be written immediately, when the slope ($m = \frac{1}{5}$) is known, as $x - 5y - 6z$; and that the right side is −7, as found by substituting the coordinates of the first point in the left member of the equation, and then checking by use of the second point. Now write the equations of the lines through the pairs of points in Prob. 6.

8. Prove that the perpendicular distance d from the point (a,b,c) to the line $Ax + By + (B - A)z + C = 0$ is given by the formula

$$d = \frac{Aa + Bb + (B - A)c + C}{\pm \sqrt{\frac{2}{3}[A^2 + B^2 + (B - A)^2]}}$$

HINT: Using (1), change the coordinates and the equation to their equivalents in the 2-axes system, apply the line-to-point formula, and then change back to the 3-axes system.

9. Using the formula in Prob. 8, find the distance from the point (1,−2,3) to each of the lines whose equations were found in Prob. 7.

10. Find all integral solutions of the following pairs of equations:

(*a*) $2x + 3y + 4z = 6$, $3x + 4y + 3z = 1$;
(*b*) $2x + 3y + 4z = 6$, $x + y + z = 2$;
(*c*) $3x + 6y + 4z = 10$, $4x + 5y - 3z = 3$.

11. Noting that the locus of one of the two equations in each of the following pairs is a straight line, begin with that equation and find all integral solutions, when they exist.

(*a*) $x + 2y + z = 1$, $3x + 4y + 5z = 9$;
(*b*) $x + 2y + z = 1$, $3x + 4y + 5z = 2$.

12. Given the pair $4x + 3y + z = -10$ and $5x + 2y + 2z = -2$, find all integral solutions by three methods, as indicated. (*a*) First interchange x and y, noting that the first locus is then a straight line. (*b*) Solve directly. (*c*) Solve by eliminating z first. (*d*) Show that the answers obtained are equivalent.

13. Noting that the loci of each pair below, are straight lines, find all integral solutions. HINT: Find the coordinates of the point of intersection when $y = 0$, and then find the G.C. of this point.

(*a*) $3x + 5y + 2z = 5$, $2x - 3y - 5z = -3$;
(*b*) $3x + 5y + 2z = 5$, $2x - 3y - 5z = 0$.

14. Given the equations $A_1x + B_1y + C_1z = D_1$ and $A_2x + B_2y + C_2z = D_2$, show that if $\dfrac{B_1 - A_1 - C_1}{A_2 + C_2 - B_2}$ is an irreducible fraction, then when we substitute the G.C. in the original equations by the method of Art. 112, x will come out in integral form at once, without further adjustment of n. (In other words, only one congruence need be solved in the problem.)

15. Show that the equation of the circle $X^2 + Y^2 = a^2$ becomes $x^2 + y^2 + z^2 + xy - xz + yz = a^2$ on the 3-axes plane.

16. In view of Prob. 15, prove that the equation $x^2 + y^2 + z^2 + xy - xz + yz = n$ has infinitely many integral solutions for a given n when there exist integers a and b such that $n = a^2 + ab + b^2$, and that it has no integral solutions otherwise.

17. Given $s = 2x + y - z$ and $t = y + z$, show that if $f(s,t) = 0$ has an integral solution when $x = 0$, then it also has an integral solution when $y = 0$ or $z = 0$, and has, moreover, infinitely many integral solutions; but that if the first premise fails there are no integral solutions.

18. Find simultaneous integral solutions of the equations $3(y + z)^2 = 2x + y - z$ and $x + y + z = k$ (k an integer). Hint: Since the locus of the first equation is a curve (the parabola $2Y^2 = X$), let $z = 0$ in that equation and solve for x, noting that x is integral when $y = n$. Use the G.C. of these points in the second equation.

19. Using the theorem of Art. 113, show that the curve of intersection of the loci of the equations $x + y + z = 1$ and $x^2 - y^2 + z = 2p + 1$ (p a prime $\geqq 3$) is the hyperbola $(1 - y - z)^2 - (x + 2y + z - 1)^2 - x - y - 2p = 0$. From this, letting $y = 0$, we get $z = \dfrac{1 - x}{2} - \dfrac{p}{x}$, whence x and z are both integral only if $x = \pm 1, \pm 2, \pm p$, or $\pm 2p$. Find the eight integral solutions of the first two equations in terms of p.

20. Eliminating z in the two equations of Prob. 19, we get $x = \frac{1}{2}(1 \pm \sqrt{4y^2 + 4y + 8p + 1})$. Hence show, in view of the solution of Prob. 19, that the expression $4y^2 + 4y + 8p + 1$ (p an odd prime) is a perfect square only when $y = -p$, $p - 1$, $\frac{1}{2}(1 - p)$, and $\frac{1}{2}(p - 3)$.

21. Find formulas yielding integral values for the sides of a triangle with an interior angle of 60° (see Fig. 88).

Solution: Let x, y, and z be the lengths of the sides of the triangle, with side z opposite the 60° angle. Then, by the law of cosines,

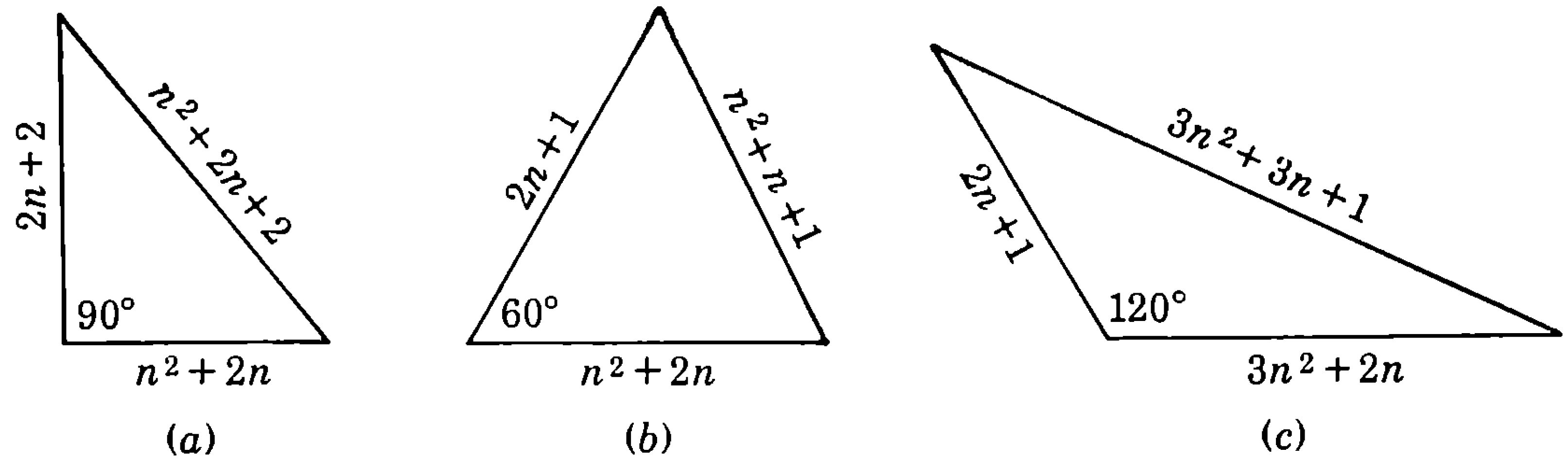

FIG. 88.

$x^2 - xy + y^2 = z^2$. If this equation is solved simultaneously with $x + y = z + m$, the equation of the curve of intersection of their loci is found by the theorem of Art. 113 to be

$$(2x+y-z-m)^2-(2x+y-z-m)\ (z-x+m)+(z-x+m)^2=(x+y-m)^2$$

Setting $x = 0$, we get $y = \dfrac{3z^2 + 6mz + 2m^2}{3z + m}$, which becomes $\dfrac{z^2 + 6nz + 6n^2}{z + n}$ when $m = 3n$. Hence if $z = 1 - n$, $y = n^2 + 4n + 1$. Substituting the G.C. of the point $(0, n^2 + 4n + 1, 1 - n)$ in $x + y = z + 3n$, we get $x = n^2 + 2n$, whence, from the G.C., $x = n^2 + 2n$, $y = 2n + 1$, $z = n^2 + n + 1$ (answer).

22. Work problem 21 by solving $x^2 - xy + y^2 = (x - n)^2$ for x, thereby getting a different set of solutions. Which set yields the triangle: $x = 8$, $y = 3$, $z = 7$? Which yields: $x = 8$, $y = 5$, $z = 7$?

23. Find formulas yielding integral values for the lengths of the sides of (*a*) a right triangle, (*b*) a triangle having an interior angle of 120°.

24. Show that the equations $x^2 + y^2 - xz = 0$ and $z^2 + xy + yz = n$ have no common integral solutions if n is any integer (such as 2,5,6,8, $\cdots$) which cannot be expressed in terms of integers a and b in the form $a^2 + ab + b^2$. (HINT: See Prob. 16. In this case the equation of the curve of intersection may be obtained simply by adding corresponding members of the two given equations.)

25. Prove the conclusion of the preceding problem if the simultaneous equations are $f(x,y,z) = 0$ and $f(x,y,z) + x^2 + y^2 + z^2 + xy - xz + yz = n$.

26. Prove that the locus of the equation $x^2 + y^2 + z^2 - a^2$ is a family of ellipses [for example: $x = \frac{1}{2}a$, $y^2 + z^2 = \frac{5}{4}a^2$] which traverse every point on and inside the circle $X^2 + Y^2 = \dfrac{3a^2}{2}$, and no point outside it. HINT: Replace y and z by $\dfrac{Y}{\sqrt{3}} + X - x$ and $\dfrac{Y}{\sqrt{3}} - X + x$

respectively, and then equate to zero the discriminant of the resulting equation in x.

27. Show that the locus of the equation $x = y^2 - z^2$ is the whole 3-axes plane.

28. Show that the locus of the equation $x^2 + y^2 - z^2 = 0$ consists of curves (hyperbolas and ellipses) passing through all points on the plane except those in the first and third quadrants which are respectively below and above the line $Y = \sqrt{3}X$.

APPENDIX

TABLE I.—COMMON LOGARITHMS

N	0	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396
N	0	1	2	3	4	5	6	7	8	9

TABLE I.—COMMON LOGARITHMS.—(*Continued*)

N	0	1	2	3	4	5	6	7	8	9
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996
N	0	1	2	3	4	5	6	7	8	9

TABLE II.—COMPOUND INTEREST: $(1 + r)^n$
(Amount of One Dollar Principal at Compound Interest after n Years)

n	2%	2½%	3%	3½%	4%	4½%	5%	6%	7%
1	1.0200	1.0250	1.0300	1.0350	1.0400	1.0450	1.0500	1.0600	1.0700
2	1.0404	1.0506	1.0609	1.0712	1.0816	1.0920	1.1025	1.1236	1.1449
3	1.0612	1.0769	1.0927	1.1087	1.1249	1.1412	1.1576	1.1910	1.2250
4	1.0824	1.1038	1.1255	1.1475	1.1699	1.1925	1.2155	1.2625	1.3108
5	1.1041	1.1314	1.1593	1.1877	1.2167	1.2462	1.2763	1.3382	1.4026
6	1.1262	1.1597	1.1941	1.2293	1.2653	1.3023	1.3401	1.4185	1.5007
7	1.1487	1.1887	1.2299	1.2723	1.3159	1.3609	1.4071	1.5036	1.6058
8	1.1717	1.2184	1.2668	1.3168	1.3686	1.4221	1.4775	1.5938	1.7182
9	1.1951	1.2489	1.3048	1.3629	1.4233	1.4861	1.5513	1.6895	1.8385
10	1.2190	1.2801	1.3439	1.4106	1.4802	1.5530	1.6289	1.7908	1.9672
11	1.2434	1.3121	1.3842	1.4600	1.5395	1.6229	1.7103	1.8983	2.1049
12	1.2682	1.3449	1.4258	1.5111	1.6010	1.6959	1.7959	2.0122	2.2522
13	1.2936	1.3785	1.4685	1.5640	1.6651	1.7722	1.8856	2.1329	2.4098
14	1.3195	1.4130	1.5126	1.6187	1.7317	1.8519	1.9799	2.2609	2.5785
15	1.3459	1.4483	1.5580	1.6753	1.8009	1.9353	2.0789	2.3966	2.7590
16	1.3728	1.4845	1.6047	1.7340	1.8730	2.0224	2.1829	2.5404	2.9522
17	1.4002	1.5216	1.6528	1.7947	1.9479	2.1134	2.2920	2.6928	3.1588
18	1.4282	1.5597	1.7024	1.8575	2.0258	2.2085	2.4066	2.8543	3.3799
19	1.4568	1.5987	1.7535	1.9225	2.1068	2.3079	2.5270	3.0256	3.6165
20	1.4859	1.6386	1.8061	1.9898	2.1911	2.4117	2.6533	3.2071	3.8697
21	1.5157	1.6796	1.8603	2.0594	2.2788	2.5202	2.7860	3.3996	4.1406
22	1.5460	1.7216	1.9161	2.1315	2.3699	2.6337	2.9253	3.6035	4.4304
23	1.5769	1.7646	1.9736	2.2061	2.4647	2.7522	3.0715	3.8197	4.7405
24	1.6084	1.8087	2.0328	2.2833	2.5633	2.8760	3.2251	4.0489	5.0724
25	1.6406	1.8539	2.0938	2.3632	2.6658	3.0054	3.3864	4.2919	5.4274
26	1.6734	1.9003	2.1566	2.4460	2.7725	3.1407	3.5557	4.5494	5.8074
27	1.7069	1.9478	2.2213	2.5316	2.8834	3.2820	3.7335	4.8223	6.2139
28	1.7410	1.9965	2.2879	2.6202	2.9987	3.4297	3.9201	5.1117	6.6488
29	1.7758	2.0464	2.3566	2.7119	3.1187	3.5840	4.1161	5.4184	7.1143
30	1.8114	2.0976	2.4273	2.8068	3.2434	3.7453	4.3219	5.7435	7.6123
31	1.8476	2.1500	2.5001	2.9050	3.3731	3.9139	4.5380	6.0881	8.1451
32	1.8845	2.2038	2.5751	3.0067	3.5081	4.0900	4.7649	6.4534	8.7153
33	1.9222	2.2589	2.6523	3.1119	3.6484	4.2740	5.0032	6.8406	9.3253
34	1.9607	2.3153	2.7319	3.2209	3.7943	4.4664	5.2533	7.2510	9.9781
35	1.9999	2.3732	2.8139	3.3336	3.9461	4.6673	5.5160	7.6861	10.6766
36	2.0399	2.4325	2.8983	3.4503	4.1039	4.8774	5.7918	8.1473	11.4239
37	2.0807	2.4933	2.9852	3.5710	4.2681	5.0969	6.0814	8.6361	12.2236
38	2.1223	2.5557	3.0748	3.6960	4.4388	5.3262	6.3855	9.1543	13.0793
39	2.1647	2.6196	3.1670	3.8254	4.6164	5.5659	6.7048	9.7035	13.9948
40	2.2080	2.6851	3.2620	3.9593	4.8010	5.8164	7.0400	10.2857	14.9745
41	2.2522	2.7522	3.3599	4.0978	4.9931	6.0781	7.3920	10.9029	16.0227
42	2.2972	2.8210	3.4607	4.2413	5.1928	6.3516	7.7616	11.5570	17.1443
43	2.3432	2.8915	3.5645	4.3897	5.4005	6.6374	8.1497	12.2505	18.3444
44	2.3901	2.9638	3.6715	4.5433	5.6165	6.9361	8.5572	12.9855	19.6285
45	2.4379	3.0379	3.7816	4.7024	5.8412	7.2482	8.9850	13.7646	21.0025
46	2.4866	3.1139	3.8950	4.8669	6.0748	7.5744	9.4343	14.5905	22.4726
47	2.5363	3.1917	4.0119	5.0373	6.3178	7.9153	9.9060	15.4659	24.0457
48	2.5871	3.2715	4.1323	5.2136	6.5705	8.2715	10.4013	16.3939	25.7289
49	2.6388	3.3533	4.2562	5.3961	6.8333	8.6437	10.9213	17.3775	27.5299
50	2.6916	3.4371	4.3839	5.5849	7.1067	9.0326	11.4674	18.4202	29.4570

TABLE III.—COMPOUND DISCOUNT: $1/(1 + r)^n$
(Present Value of One Dollar Due at the End of n Years)

n	2%	2½%	3%	3½%	4%	4½%	5%	6%	7%
1	.98039	.97561	.97087	.96618	.96154	.95694	.95238	.94340	.93458
2	.96117	.95181	.94260	.93351	.92456	.91573	.90703	.89000	.87344
3	.94232	.92860	.91514	.90194	.88900	.87630	.86384	.83962	.81630
4	.92385	.90595	.88849	.87144	.85480	.83856	.82270	.79209	.76290
5	.90573	.88385	.86261	.84197	.82193	.80245	.78353	.74726	.71299
6	.88797	.86230	.83748	.81350	.79031	.76790	.74622	.70496	.66634
7	.87056	.84127	.81309	.78599	.75992	.73483	.71068	.66506	.62275
8	.85349	.82075	.78941	.75941	.73069	.70319	.67684	.62741	.58201
9	.83676	.80073	.76642	.73373	.70259	.67290	.64461	.59190	.54393
10	.82035	.78120	.74409	.70892	.67556	.64393	.61391	.55839	.50835
11	.80426	.76214	.72242	.68495	.64958	.61620	.58468	.52679	.47509
12	.78849	.74356	.70138	.66178	.62460	.58966	.55684	.49697	.44401
13	.77303	.72542	.68095	.63940	.60057	.56427	.53032	.46884	.41496
14	.75788	.70773	.66112	.61778	.57748	.53997	.50507	.44230	.38782
15	.74301	.69047	.64186	.59689	.55526	.51672	.48102	.41727	.36245
16	.72845	.67362	.62317	.57671	.53391	.49447	.45811	.39365	.33873
17	.71416	.65720	.60502	.55720	.51337	.47318	.43630	.37136	.31657
18	.70016	.64117	.58739	.53836	.49363	.45280	.41552	.35034	.29586
19	.68643	.62553	.57029	.52016	.47464	.43330	.39573	.33051	.27651
20	.67297	.61027	.55368	.50257	.45639	.41464	.37689	.31180	.25842
21	.65978	.59539	.53755	.48557	.43883	.39679	.35894	.29416	.24151
22	.64684	.58086	.52189	.46915	.42196	.37970	.34185	.27751	.22571
23	.63416	.56670	.50669	.45329	.40573	.36335	.32557	.26180	.21095
24	.62172	.55288	.49193	.43796	.39012	.34770	.31007	.24698	.19715
25	.60953	.53939	.47761	.42315	.37512	.33273	.29530	.23300	.18425
26	.59758	.52623	.46369	.40884	.36069	.31840	.28124	.21981	.17220
27	.58586	.51340	.45019	.39501	.34682	.30469	.26785	.20737	.16093
28	.57437	.50088	.43708	.38165	.33348	.29157	.25509	.19563	.15040
29	.56311	.48866	.42435	.36875	.32065	.27902	.24295	.18456	.14056
30	.55207	.47674	.41199	.35628	.30832	.26700	.23138	.17411	.13137
31	.54125	.46511	.39999	.34423	.29646	.25550	.22036	.16425	.12277
32	.53063	.45377	.38834	.33259	.28506	.24450	.20987	.15496	.11474
33	.52023	.44270	.37703	.32134	.27409	.23397	.19987	.14619	.10723
34	.51003	.43191	.36604	.31048	.26355	.22390	.19035	.13791	.10022
35	.50003	.42137	.35538	.29998	.25342	.21425	.18129	.13011	.09366
36	.49022	.41109	.34503	.28983	.24367	.20503	.17266	.12274	.08754
37	.48061	.40107	.33498	.28003	.23430	.19620	.16444	.11579	.08181
38	.47119	.39128	.32523	.27056	.22529	.18775	.15661	.10924	.07646
39	.46195	.38174	.31575	.26141	.21662	.17967	.14915	.10306	.07146
40	.45289	.37243	.30656	.25257	.20829	.17193	.14205	.09722	.06678
41	.44401	.36335	.29763	.24403	.20028	.16453	.13528	.09172	.06241
42	.43530	.35448	.28896	.23578	.19257	.15744	.12884	.08653	.05833
43	.42677	.34584	.28054	.22781	.18517	.15066	.12270	.08163	.05451
44	.41840	.33740	.27237	.22010	.17805	.14417	.11686	.07701	.05095
45	.41020	.32917	.26444	.21266	.17120	.13796	.11130	.07265	.04761
46	.40215	.32115	.25674	.20547	.16461	.13202	.10600	.06854	.04450
47	.39427	.31331	.24926	.19852	.15828	.12634	.10095	.06466	.04159
48	.38654	.30567	.24200	.19181	.15219	.12090	.09614	.06100	.03887
49	.37896	.29822	.23495	.18532	.14634	.11569	.09156	.05755	.03632
50	.37153	.29094	.22811	.17905	.14071	.11071	.08720	.05429	.03394

TABLE IV.—AMOUNT OF AN ANNUITY: $s_{\overline{n}|}$

(Amount of an Annuity of One Dollar per Year after n Years)

n	2%	2½%	3%	3½%	4%	4½%	5%	6%	7%
1	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
2	2.0200	2.0250	2.0300	2.0350	2.0400	2.0450	2.0500	2.0600	2.0700
3	3.0604	3.0756	3.0909	3.1062	3.1216	3.1370	3.1525	3.1836	3.2149
4	4.1216	4.1525	4.1836	4.2149	4.2465	4.2782	4.3101	4.3746	4.4399
5	5.2040	5.2563	5.3091	5.3625	5.4163	5.4707	5.5256	5.6371	5.7507
6	6.3081	6.3877	6.4684	6.5502	6.6330	6.7169	6.8019	6.9753	7.1533
7	7.4343	7.5474	7.6625	7.7794	7.8983	8.0192	8.1420	8.3938	8.6540
8	8.5830	8.7361	8.8923	9.0517	9.2142	9.3800	9.5491	9.8975	10.2598
9	9.7546	9.9545	10.1591	10.3685	10.5828	10.8021	11.0266	11.4913	11.9780
10	10.9497	11.2034	11.4639	11.7314	12.0061	12.2882	12.5779	13.1808	13.8164
11	12.1687	12.4835	12.8078	13.1420	13.4864	13.8412	14.2068	14.9716	15.7836
12	13.4121	13.7956	14.1920	14.6020	15.0258	15.4640	15.9171	16.8699	17.8885
13	14.6803	15.1404	15.6178	16.1130	16.6268	17.1599	17.7130	18.8821	20.1406
14	15.9739	16.5190	17.0863	17.6770	18.2919	18.9321	19.5986	21.0151	22.5505
15	17.2934	17.9319	18.5989	19.2957	20.0236	20.7841	21.5786	23.2760	25.1290
16	18.6393	19.3802	20.1569	20.9710	21.8245	22.7193	23.6575	25.6725	27.8881
17	20.0121	20.8647	21.7616	22.7050	23.6975	24.7417	25.8404	28.2129	30.8402
18	21.4123	22.3863	23.4144	24.4997	25.6454	26.8551	28.1324	30.9057	33.9990
19	22.8406	23.9460	25.1169	26.3572	27.6712	29.0636	30.5390	33.7600	37.3790
20	24.2974	25.5447	26.8704	28.2797	29.7781	31.3714	33.0660	36.7856	40.9955
21	25.7833	27.1833	28.6765	30.2695	31.9692	33.7831	35.7193	39.9927	44.8652
22	27.2990	28.8629	30.5368	32.3289	34.2480	36.3034	38.5052	43.3923	49.0057
23	28.8450	30.5844	32.4529	34.4604	36.6179	38.9370	41.4305	46.9958	53.4361
24	30.4219	32.3490	34.4265	36.6665	39.0826	41.6892	44.5020	50.8156	58.1767
25	32.0303	34.1578	36.4593	38.9499	41.6459	44.5652	47.7271	54.8645	63.2490
26	33.6709	36.0117	38.5530	41.3131	44.3117	47.5706	51.1135	59.1564	68.6765
27	35.3443	37.9120	40.7096	43.7591	47.0842	50.7113	54.6691	63.7058	74.4838
28	37.0512	39.8598	42.9309	46.2906	49.9676	53.9933	58.4026	68.5281	80.6977
29	38.7922	41.8563	45.2189	48.9108	52.9663	57.4230	62.3227	73.6398	87.3465
30	40.5681	43.9027	47.5754	51.6227	56.0849	61.0071	66.4388	79.0582	94.4608
31	42.3794	46.0003	50.0027	54.4295	59.3283	64.7524	70.7608	84.8017	102.0730
32	44.2270	48.1503	52.5028	57.3345	62.7015	68.6662	75.2988	90.8898	110.2182
33	46.1116	50.3540	55.0778	60.3412	66.2095	72.7562	80.0638	97.3432	118.9334
34	48.0338	52.6129	57.7302	63.4532	69.8579	77.0303	85.0670	104.1838	128.2588
35	49.9945	54.9282	60.4621	66.6740	73.6522	81.4966	90.3203	111.4348	138.2369
36	51.9944	57.3014	63.2759	70.0076	77.5983	86.1640	95.8363	119.1209	148.9135
37	54.0343	59.7339	66.1742	73.4579	81.7022	91.0413	101.6281	127.2681	160.3374
38	56.1149	62.2273	69.1594	77.0289	85.9703	96.1382	107.7095	135.9042	172.5610
39	58.2372	64.7830	72.2342	80.7249	90.4091	101.4644	114.0950	145.0585	185.6403
40	60.4020	67.4026	75.4013	84.5503	95.0255	107.0303	120.7998	154.7620	199.6351
41	62.6100	70.0876	78.6633	88.5095	99.8265	112.8467	127.8398	165.0477	214.6096
42	64.8622	72.8398	82.0232	92.6074	104.8196	118.9248	135.2318	175.9505	230.6322
43	67.1595	75.6608	85.4839	96.8486	110.0124	125.2764	142.9933	187.5076	247.7765
44	69.5027	78.5523	89.0484	101.2383	115.4129	131.9138	151.1430	199.7580	266.1209
45	71.8927	81.5161	92.7199	105.7817	121.0294	138.8500	159.7002	212.7435	285.7493
46	74.3306	84.5540	96.5015	110.4840	126.8706	146.0982	168.6852	226.5081	306.7518
47	76.8172	87.6679	100.3965	115.3510	132.9454	153.6726	178.1194	241.0986	329.2244
48	79.3535	90.8596	104.4084	120.3883	139.2632	161.5879	188.0254	256.5645	353.2701
49	81.9406	94.1311	108.5406	125.6018	145.8337	169.8594	198.4267	272.9584	378.9990
50	84.5794	97.4843	112.7969	130.9979	152.6671	178.5030	209.3480	290.3359	406.5289

TABLE V.—PRESENT VALUE OF AN ANNUITY: $a_{\overline{n}|}$
(Present Value of One Dollar per Year for n Years)

n	2%	2½%	3%	3½%	4%	4½%	5%	6%	7%
1	0.9804	0.9756	0.9709	0.9662	0.9615	0.9569	0.9524	0.9434	0.9346
2	1.9416	1.9274	1.9135	1.8997	1.8861	1.8727	1.8594	1.8334	1.8080
3	2.8839	2.8560	2.8286	2.8016	2.7751	2.7490	2.7232	2.6730	2.6243
4	3.8077	3.7620	3.7171	3.6731	3.6299	3.5875	3.5460	3.4651	3.3872
5	4.7135	4.6458	4.5797	4.5151	4.4518	4.3900	4.3295	4.2124	4.1002
6	5.6014	5.5081	5.4172	5.3286	5.2421	5.1579	5.0757	4.9173	4.7665
7	6.4720	6.3494	6.2303	6.1145	6.0021	5.8927	5.7864	5.5824	5.3893
8	7.3255	7.1701	7.0197	6.8740	6.7327	6.5959	6.4632	6.2098	5.9713
9	8.1622	7.9709	7.7861	7.6077	7.4353	7.2688	7.1078	6.8017	6.5152
10	8.9826	8.7521	8.5302	8.3166	8.1109	7.9127	7.7217	7.3601	7.0236
11	9.7868	9.5142	9.2526	9.0016	8.7605	8.5289	8.3064	7.8869	7.4987
12	10.5753	10.2578	9.9540	9.6633	9.3851	9.1186	8.8633	8.3838	7.9427
13	11.3484	10.9832	10.6350	10.3027	9.9856	9.6829	9.3936	8.8527	8.3577
14	12.1062	11.6909	11.2961	10.9205	10.5631	10.2228	9.8986	9.2950	8.7455
15	12.8493	12.3814	11.9379	11.5174	11.1184	10.7395	10.3797	9.7122	9.1079
16	13.5777	13.0550	12.5611	12.0941	11.6523	11.2340	10.8378	10.1059	9.4466
17	14.2919	13.7122	13.1661	12.6513	12.1657	11.7072	11.2741	10.4773	9.7632
18	14.9920	14.3534	13.7535	13.1897	12.6593	12.1600	11.6896	10.8276	10.0591
19	15.6785	14.9789	14.3238	13.7098	13.1339	12.5933	12.0853	11.1581	10.3356
20	16.3514	15.5892	14.8775	14.2124	13.5903	13.0079	12.4622	11.4699	10.5940
21	17.0112	16.1845	15.4150	14.6980	14.0292	13.4047	12.8212	11.7641	10.8355
22	17.6580	16.7654	15.9369	15.1671	14.4511	13.7844	13.1630	12.0416	11.0612
23	18.2922	17.3321	16.4436	15.6204	14.8568	14.1478	13.4886	12.3034	11.2722
24	18.9139	17.8850	16.9355	16.0584	15.2470	14.4955	13.7986	12.5504	11.4693
25	19.5235	18.4244	17.4131	16.4815	15.6221	14.8282	14.0939	12.7834	11.6536
26	20.1210	18.9506	17.8768	16.8904	15.9828	15.1466	14.3752	13.0032	11.8258
27	20.7069	19.4640	18.3270	17.2854	16.3296	15.4513	14.6430	13.2105	11.9867
28	21.2813	19.9649	18.7641	17.6670	16.6631	15.7429	14.8981	13.4062	12.1371
29	21.8444	20.4535	19.1885	18.0358	16.9837	16.0219	15.1411	13.5907	12.2777
30	22.3965	20.9303	19.6004	18.3920	17.2920	16.2889	15.3725	13.7648	12.4090
31	22.9377	21.3954	20.0004	18.7363	17.5885	16.5444	15.5928	13.9291	12.5318
32	23.4683	21.8492	20.3888	19.0689	17.8736	16.7889	15.8027	14.0840	12.6466
33	23.9886	22.2919	20.7658	19.3902	18.1476	17.0229	16.0025	14.2302	12.7538
34	24.4986	22.7238	21.1318	19.7007	18.4112	17.2468	16.1929	14.3681	12.8540
35	24.9986	23.1452	21.4872	20.0007	18.6646	17.4610	16.3742	14.4982	12.9477
36	25.4888	23.5563	21.8323	20.2905	18.9083	17.6660	16.5469	14.6210	13.0352
37	25.9695	23.9573	22.1672	20.5705	19.1426	17.8622	16.7113	14.7368	13.1170
38	26.4406	24.3486	22.4925	20.8411	19.3679	18.0500	16.8679	14.8460	13.1935
39	26.9026	24.7303	22.8082	21.1025	19.5845	18.2297	17.0170	14.9491	13.2649
40	27.3555	25.1028	23.1148	21.3551	19.7928	18.4016	17.1591	15.0463	13.3317
41	27.7995	25.4661	23.4124	21.5991	19.9931	18.5661	17.2944	15.1380	13.3941
42	28.2348	25.8206	23.7014	21.8349	20.1856	18.7235	17.4232	15.2245	13.4524
43	28.6616	26.1664	23.9819	22.0627	20.3708	18.8742	17.5459	15.3062	13.5070
44	29.0800	26.5038	24.2543	22.2828	20.5488	19.0184	17.6628	15.3832	13.5579
45	29.4902	26.8330	24.5187	22.4955	20.7200	19.1563	17.7741	15.4558	13.6055
46	29.8923	27.1542	24.7754	22.7009	20.8847	19.2884	17.8801	15.5244	13.6500
47	30.2866	27.4675	25.0247	22.8994	21.0429	19.4147	17.9810	15.5890	13.6916
48	30.6731	27.7732	25.2667	23.0912	21.1951	19.5356	18.0772	15.6500	13.7305
49	31.0521	28.0714	25.5017	23.2766	21.3415	19.6513	18.1687	15.7076	13.7668
50	31.4236	28.3623	25.7298	23.4556	21.4822	19.7620	18.2559	15.7619	13.8007

TABLE VI.—PERIODICAL PAYMENT OF ANNUITY WHOSE PRESENT VALUE IS 1

$$\frac{1}{a_{\overline{n}|}} = \frac{i}{1-(1+i)^{-n}} = \frac{i}{1-v^n}. \qquad \left[\frac{1}{s_{\overline{n}|}} = \frac{1}{a_{\overline{n}|}} - i.\right]$$

n	½ %	1 %	1¼ %	1½ %	2 %	n
1	1.005 0000	1.010 0000	1.012 5000	1.015 0000	1.020 0000	1
2	0.503 7531	0.507 5124	0.509 3944	0.511 2779	0.515 0495	2
3	0.336 6722	0.340 0221	0.341 7012	0.343 3830	0.346 7547	3
4	0.253 1328	0.256 2811	0.257 8610	0.259 4448	0.262 6238	4
5	0.203 0100	0.206 0398	0.207 5621	0.209 0893	0.212 1584	5
6	0.169 5955	0.172 5484	0.174 0338	0.175 5252	0.178 5258	6
7	0.145 7285	0.148 6283	0.150 0887	0.151 5562	0.154 5120	7
8	0.127 8289	0.130 6903	0.132 1331	0.133 5840	0.136 5098	8
9	0.113 9074	0.116 7404	0.118 1706	0.119 6098	0.122 5154	9
10	0.102 7706	0.105 5821	0.107 0031	0.108 4342	0.111 3265	10
11	0.093 6590	0.096 4541	0.097 8684	0.099 2938	0.102 1779	11
12	0.086 0664	0.088 8488	0.090 2583	0.091 6800	0.094 5596	12
13	0.079 6422	0.082 4148	0.083 8210	0.085 2404	0.088 1184	13
14	0.074 1361	0.076 9012	0.078 3052	0.079 7233	0.082 6020	14
15	0.069 3644	0.072 1238	0.073 5265	0.074 9444	0.077 8255	15
16	0.065 1894	0.067 9446	0.069 3467	0.070 7651	0.073 6501	16
17	0.061 5058	0.064 2581	0.065 6602	0.067 0796	0.069 9698	17
18	0.058 2317	0.060 9820	0.062 3848	0.063 8058	0.066 7021	18
19	0.055 3025	0.058 0518	0.059 4555	0.060 8785	0.063 7818	19
20	0.052 6664	0.055 4153	0.056 8204	0.058 2457	0.061 1567	20
21	0.050 2816	0.053 0308	0.054 4375	0.055 8655	0.058 7848	21
22	0.048 1138	0.050 8637	0.052 2724	0.053 7033	0.056 6314	22
23	0.046 1346	0.048 8858	0.050 2967	0.051 7308	0.054 6681	23
24	0.044 3206	0.047 0735	0.048 4866	0.049 9241	0.052 8711	24
25	0.042 6519	0.045 4068	0.046 8225	0.048 2634	0.051 2204	25
26	0.041 1116	0.043 8689	0.045 2873	0.046 7320	0.049 6992	26
27	0.039 6856	0.042 4455	0.043 8668	0.045 3153	0.048 2931	27
28	0.038 3617	0.041 1244	0.042 5486	0.044 0011	0.046 9897	28
29	0.037 1291	0.039 8950	0.041 3223	0.042 7788	0.045 7784	29
30	0.035 9789	0.038 7481	0.040 1785	0.041 6392	0.044 6499	30
31	0.034 9030	0.037 6757	0.039 1094	0.040 5743	0.043 5964	31
32	0.033 8945	0.036 6709	0.038 1079	0.039 5771	0.042 6106	32
33	0.032 9473	0.035 7274	0.037 1679	0.038 6414	0.041 6865	33
34	0.032 0559	0.034 8400	0.036 2839	0.037 7619	0.040 8187	34
35	0.031 2155	0.034 0037	0.035 4511	0.036 9336	0.040 0022	35
36	0.030 4219	0.033 2143	0.034 6653	0.036 1524	0.039 2328	36
37	0.029 6714	0.032 4680	0.033 9227	0.035 4144	0.038 5068	37
38	0.028 9604	0.031 7615	0.033 2198	0.034 7161	0.037 8206	38
39	0.028 2861	0.031 0916	0.032 5536	0.034 0546	0.037 1711	39
40	0.027 6455	0.030 4556	0.031 9214	0.033 4271	0.036 5558	40
41	0.027 0363	0.029 8510	0.031 3206	0.032 8311	0.035 9719	41
42	0.026 4562	0.029 2756	0.030 7491	0.032 2643	0.035 4173	42
43	0.025 9032	0.028 7274	0.030 2047	0.031 7246	0.034 8899	43
44	0.025 3754	0.028 2044	0.029 6856	0.031 2104	0.034 3879	44
45	0.024 8712	0.027 7050	0.029 1901	0.030 7198	0.033 9096	45
46	0.024 3889	0.027 2278	0.028 7168	0.030 2512	0.033 4534	46
47	0.023 9273	0.026 7711	0.028 2641	0.029 8034	0.033 0179	47
48	0.023 4850	0.026 3338	0.027 8307	0.029 3750	0.032 6018	48
49	0.023 0609	0.025 9147	0.027 4156	0.028 9648	0.032 2040	49
50	0.022 6538	0.025 5127	0.027 0176	0.028 5717	0.031 8232	50
60	0.019 3328	0.022 2444	0.023 7899	0.025 3934	0.028 7680	60
70	0.016 9666	0.019 9328	0.021 5194	0.023 1724	0.026 6676	70
80	0.015 1970	0.018 2188	0.019 8465	0.021 5483	0.025 1607	80
90	0.013 8253	0.016 9031	0.018 5715	0.020 3211	0.024 0460	90
100	0.012 7319	0.015 8657	0.017 5743	0.019 3706	0.023 2027	100

Table VII.—Trigonometric Functions

Angles	Sines		Cosines		Tangents		Cotangents		Angles
	Nat.	Log.	Nat.	Log.	Nat.	Log.	Nat.	Log.	
0° 00′	.0000	∞	1.0000	0.0000	.0000	∞	∞	∞	90° 00′
10	.0029	7.4637	1.0000	0000	.0029	7.4637	343.77	2.5363	50
20	.0058	7648	1.0000	0000	.0058	7648	171.89	2352	40
30	.0087	9408	1.0000	0000	.0087	9409	114.59	0591	30
40	.0116	8.0658	.9999	0000	.0116	8.0658	85.940	1.9342	20
50	.0145	1627	.9999	0000	.0145	1627	68.750	8373	10
1° 00′	.0175	8.2419	.9998	9.9999	.0175	8.2419	57.290	1.7581	89° 00′
10	.0204	3088	.9998	9999	.0204	3089	49.104	6911	50
20	.0233	3668	.9997	9999	.0233	3669	42.964	6331	40
30	.0262	4179	.9997	9999	.0262	4181	38.188	5819	30
40	.0291	4637	.9996	9998	.0291	4638	34.368	5362	20
50	.0320	5050	.9995	9998	.0320	5053	31.242	4947	10
2° 00′	.0349	8.5428	.9994	9.9997	.0349	8.5431	28.636	1.4569	88° 00′
10	.0378	5776	.9993	9997	.0378	5779	26.432	4221	50
20	.0407	6097	.9992	9996	.0407	6101	24.542	3899	40
30	.0436	6397	.9990	9996	.0437	6401	22.904	3599	30
40	.0465	6677	.9989	9995	.0466	6682	21.470	3318	20
50	.0494	6940	.9988	9995	.0495	6945	20.206	3055	10
3° 00′	.0523	8.7188	.9986	9.9994	.0524	8.7194	19.081	1.2806	87° 00′
10	.0552	7423	.9985	9993	.0553	7429	18.075	2571	50
20	.0581	7645	.9983	9993	.6582	7652	17.169	2348	40
30	.0610	7857	.9981	9992	.0612	7865	16.350	2135	30
40	.0640	8059	.9980	9991	.0641	8067	15.605	1933	20
50	.0669	8251	.9978	9990	.0670	8261	14.924	1739	10
4° 00′	.0698	8.8436	.9976	9.9989	.0669	8.8446	14.301	1.1554	86° 00′
10	.0727	8613	.9974	9989	.0729	8624	13.727	1376	50
20	.0756	8783	.9971	9988	.0758	8795	13.197	1205	40
30	.0785	8946	.9969	9987	.0787	8960	12.706	1040	30
40	.0814	9104	.9967	9986	.0816	9118	12.251	0882	20
50	.0843	9256	.9964	9985	.0846	9272	11.826	0728	10
5° 00′	.0872	8.9403	.9962	9.9983	.0875	8.9420	11.430	1.0580	85° 00′
10	.0901	9545	.9959	9982	.0904	9563	11.059	0437	50
20	.0929	9682	.9957	9981	.0934	9701	10.712	0299	40
30	.0958	9816	.9954	9980	.0963	9836	10.385	0164	30
40	.0987	9945	.9951	9979	.0992	9966	10.078	0034	20
50	.1016	9.0070	.9948	9977	.1022	9.0093	9.7882	0.9907	10
6° 00′	.1045	9.0192	.9945	9.9976	.1051	9.0216	9.5144	0.9784	84° 00′
10	.1074	0311	.9942	9975	.1080	0336	9.2553	9664	50
20	.1103	0426	.9939	9973	.1110	0453	9.0098	9547	40
30	.1132	0539	.9936	9972	.1139	0567	8.7769	9433	30
40	.1161	0648	.9932	9971	.1169	0678	8.5555	9322	20
50	.1190	0755	.9929	9969	.1198	0786	8.3450	9214	10
7° 00′	.1219	9.0859	.9925	9.9968	.1228	9.0891	8.1443	0.9109	83° 00′
10	.1248	0961	.9922	9966	.1257	0995	7.9530	9005	50
20	.1276	1060	.9918	9964	.1287	1096	7.7704	8904	40
30	.1305	1157	.9914	9963	.1317	1194	7.5958	8806	30
40	.1334	1252	.9911	9961	.1346	1291	7.4287	8709	20
50	.1363	1345	.9907	9959	.1376	1385	7.2687	8615	10
8° 00′	.1392	9.1436	.9903	9.9958	.1405	9.1478	7.1154	0.8522	82° 00′
10	.1421	1525	.9899	9956	.1435	1569	6.9682	8431	50
20	.1449	1612	.9894	9954	.1465	1658	6.8269	8342	40
30	.1478	1697	.9890	9952	.1495	1745	6.6912	8255	30
40	.1507	1781	.9886	9950	.1524	1831	6.5606	8169	20
50	.1536	1863	.9881	9948	.1554	1915	6.4348	8085	10
9° 00′	.1564	9.1943	.9877	9.9946	.1584	9.1997	6.3138	0.8003	81° 00′
	Nat.	Log.	Nat.	Log.	Nat.	Log.	Nat.	Log.	
Angles	Cosines		Sines		Cotangents		Tangents		Angles

TABLE VII.—TRIGONOMETRIC FUNCTIONS.—(*Continued*)

Angles	Sines		Cosines		Tangents		Cotangents		Angles
	Nat.	Log.	Nat.	Log.	Nat.	Log.	Nat.	Log.	
9° 00′	.1564	9.1943	.9877	9.9946	.1584	9.1997	6.3138	0.8003	81° 00′
10	.1593	2022	.9872	9944	.1614	2078	6.1970	7922	50
20	.1622	2100	.9868	9942	.1644	2158	6.0844	7842	40
30	.1650	2176	.9863	9940	.1673	2236	5.9758	7764	30
40	.1679	2251	.9858	9938	.1703	2313	5.8708	7687	20
50	.1708	2324	.9853	9936	.1733	2389	5.7694	7611	10
10° 00′	.1736	9.2397	.9848	9.9934	.1763	9.2463	5.6713	0.7537	80° 00′
10	.1765	2468	.9843	9931	.1793	2536	5.5764	7464	50
20	.1794	2538	.9838	9929	.1823	2609	5.4845	7391	40
30	.1822	2606	.9833	9927	.1853	2680	5.3955	7320	30
40	.1851	2674	.9827	9924	.1883	2750	5.3093	7250	20
50	.1880	2740	.9822	9922	.1914	2819	5.2257	7181	10
11° 00′	.1908	9.2806	.9816	9.9919	.1944	9.2887	5.1446	0.7113	79° 00′
10	.1937	2870	.9811	9917	.1974	2953	5.0658	7047	50
20	.1965	2934	.9805	9914	.2004	3020	4.9894	6980	40
30	.1994	2997	.9799	9912	.2035	3085	4.9152	6915	30
40	.2022	3058	.9793	9909	.2065	3149	4.8430	6851	20
50	.2051	3119	.9787	9907	.2095	3212	4.7729	6788	10
12° 00′	.2079	9.3179	.9781	9.9904	.2126	9.3275	4.7046	0.6725	78° 00′
10	.2108	3238	.9775	9901	.2156	3336	4.6382	6664	50
20	.2136	3296	.9769	9899	.2186	3397	4.5736	6603	40
30	.2164	3353	.9763	9896	.2217	3458	4.5107	6542	30
40	.2193	3410	.9757	9893	.2247	3517	4.4494	6483	20
50	.2221	3466	.9750	9890	.2278	3576	4.3897	6424	10
13° 00′	.2250	9.3521	.9744	9.9887	.2309	9.3634	4.3315	0.6366	77° 00′
10	.2278	3575	.9737	9884	.2339	3691	4.2747	6309	50
20	.2306	3629	.9730	9881	.2370	3748	4.2193	6252	40
30	.2334	3682	.9724	9878	.2401	3804	4.1653	6196	30
40	.2363	3734	.9717	9875	.2432	3859	4.1126	6141	20
50	.2391	3786	.9710	9872	.2462	3914	4.0611	6086	10
14° 00′	.2419	9.3837	.9703	9.9869	.2493	9.3968	4.0108	0.6032	76° 00′
10	.2447	3887	.9696	9866	.2524	4021	3.9617	5979	50
20	.2476	3937	.9689	9863	.2555	4074	3.9136	5926	40
30	.2504	3986	.9681	9859	.2586	4127	3.8667	5873	30
40	.2532	4035	.9674	9856	.2617	4178	3.8208	5822	20
50	.2560	4083	.9667	9853	.2648	4230	3.7760	5770	10
15° 00′	.2588	9.4130	.9659	9.9849	.2679	9.4281	3.7321	0.5719	75° 00′
10	.2616	4177	.9652	9846	.2711	4331	3.6891	5669	50
20	.2644	4223	.9644	9843	.2742	4381	3.6470	5619	40
30	.2672	4269	.9636	9839	.2773	4430	3.6059	5570	30
40	.2700	4314	.9628	9836	.2805	4479	3.5656	5521	20
50	.2728	4359	.9621	9832	.2836	4527	3.5261	5473	10
16° 00′	.2756	9.4403	.9613	9.9828	.2867	9.4575	3.4874	0.5425	74° 00′
10	.2784	4447	.9605	9825	.2899	4622	3.4495	5378	50
20	.2812	4491	.9596	9821	.2931	4669	3.4124	5331	40
30	.2840	4533	.9588	9817	.2962	4716	3.3759	5284	30
40	.2868	4576	.9580	9814	.2994	4762	3.3402	5238	20
50	.2896	4618	.9572	9810	.3026	4808	3.3052	5192	10
17° 00′	.2924	9.4659	.9563	9.9806	.3057	9.4853	3.2709	0.5147	73° 00′
10	.2952	4700	.9555	9802	.3089	4898	3.2371	5102	50
20	.2979	4741	.9546	9798	.3121	4943	3.2041	5057	40
30	.3007	4781	.9537	9794	.3153	4987	3.1716	5013	30
40	.3035	4821	.9528	9790	.3185	5031	3.1397	4969	20
50	.3062	4861	.9520	9786	.3217	5075	3.1084	4925	10
18° 00′	.3090	9.4900	.9511	9.9782	.3249	9.5118	3.0777	0.4882	72° 00′
	Nat.	Log.	Nat.	Log.	Nat.	Log.	Nat.	Log.	
Angles	Cosines		Sines		Cotangents		Tangents		Angles

TABLE VII.—TRIGONOMETRIC FUNCTIONS.—(*Continued*)

Angles	Sines		Cosines		Tangents		Cotangents		Angles
	Nat.	Log.	Nat.	Log.	Nat.	Log.	Nat.	Log.	
18° 00′	.3090	9.4900	.9511	9.9782	.3249	9.5118	3.0777	0.4882	72° 00′
10	.3118	4939	.9502	9778	.3281	5161	3.0475	4839	50
20	.3145	4977	.9492	9774	.3314	5203	3.0178	4797	40
30	.3173	5015	.9483	9770	.3346	5245	2.9887	4755	30
40	.3201	5052	.9474	9765	.3378	5287	2.9600	4713	20
50	.3228	5090	.9465	9761	.3411	5329	2.9319	4671	10
19° 00′	.3256	9.5126	.9455	9.9757	.3443	9.5370	2.9042	0.4630	71° 00′
10	.3283	5163	.9446	9752	.3476	5411	2.8770	4589	50
20	.3311	5199	.9436	9748	.3508	5451	2.8502	4549	40
30	.3338	5235	.9426	9743	.3541	5491	2.8239	4509	30
40	.3365	5270	.9417	9739	.3574	5531	2.7980	4469	20
50	.3393	5306	.9407	9734	.3607	5571	2.7725	4429	10
20° 00′	.3420	9.5341	.9397	9.9730	.3640	9.5611	2.7475	0.4389	70° 00′
10	.3448	5375	.9387	9725	.3673	5650	2.7228	4350	50
20	.3475	5409	.9377	9721	.3706	5689	2.6985	4311	40
30	.3502	5443	.9367	9716	.3739	5727	2.6746	4273	30
40	.3529	5477	.9356	9711	.3772	5766	2.6511	4234	20
50	.3557	5510	.9346	9706	.3805	5804	2.6279	4196	10
21° 00′	.3584	9.5543	.9336	9.9702	.3839	9.5842	2.6051	0.4158	69° 00′
10	.3611	5576	.9325	9697	.3872	5879	2.5826	4121	50
20	.3638	5609	.9315	9692	.3906	5917	2.5605	4083	40
30	.3665	5641	.9304	9687	.3939	5954	2.5386	4046	30
40	.3692	5673	.9293	9682	.3973	5991	2.5172	4009	20
50	.3719	5704	.9283	9677	.4006	6028	2.4960	3972	10
22° 00′	.3746	9.5736	.9272	9.9672	.4040	9.6064	2.4751	0.3936	68° 00′
10	.3773	5767	.9261	9667	.4074	6100	2.4545	3900	50
20	.3800	5798	.9250	9661	.4108	6136	2.4342	3864	40
30	.3827	5828	.9239	9656	.4142	6172	2.4142	3828	30
40	.3854	5859	.9228	9651	.4176	6208	2.3945	3792	20
50	.3881	5889	.9216	9646	.4210	6243	2.3750	3757	10
23° 00′	.3907	9.5919	.9205	9.9640	.4245	9.6279	2.3559	0.3721	67° 00′
10	.3934	5948	.9194	9635	.4279	6314	2.3369	3686	50
20	.3961	5978	.9182	9629	.4314	6348	2.3183	3652	40
30	.3987	6007	.9171	9624	.4348	6383	2.2998	3617	30
40	.4014	6036	.9159	9618	.4383	6417	2.2817	3583	20
50	.4041	6065	.9147	9613	.4417	6452	2.2637	3548	10
24° 00′	.4067	9.6093	.9135	9.9607	.4452	9.6486	2.2460	0.3514	66° 00′
10	.4094	6121	.9124	9602	.4487	6520	2.2286	3480	50
20	.4120	6149	.9112	9596	.4522	6553	2.2113	3447	40
30	.4147	6177	.9100	9590	.4557	6587	2.1943	3413	30
40	.4173	6205	.9088	9584	.4592	6620	2.1775	3380	20
50	.4200	6232	.9075	9579	.4628	6654	2.1609	3346	10
25° 00′	.4226	9.6259	.9063	9.9573	.4663	9.6687	2.1445	0.3313	65° 00′
10	.4253	6286	.9051	9567	.4699	6720	2.1283	3280	50
20	.4279	6313	.9038	9561	.4734	6752	2.1123	3248	40
30	.4305	6340	.9026	9555	.4770	6785	2.0965	3215	30
40	.4331	6366	.9013	9549	.4806	6817	2.0809	3183	20
50	.4358	6392	.9001	9543	.4841	6850	2.0655	3150	10
26° 00′	.4384	9.6418	.8988	9.9537	.4877	9.6882	2.0503	0.3118	64° 00′
10	.4410	6444	.8975	9530	.4913	6914	2.0353	3086	50
20	.4436	6470	.8962	9524	.4950	6946	2.0204	3054	40
30	.4462	6495	.8949	9518	.4986	6977	2.0057	3023	30
40	.4488	6521	.8936	9512	.5022	7009	1.9912	2991	20
50	.4514	6546	.8923	9505	.5059	7040	1.9768	2960	10
27° 00′	.4540	9.6570	.8910	9.9499	.5095	9.7072	1.9626	0.2928	63° 00′
	Nat.	Log.	Nat.	Log.	Nat.	Log.	Nat.	Log.	
Angles	Cosines		Sines		Cotangents		Tangents		Angles

TABLE VII.—TRIGONOMETRIC FUNCTIONS.—(*Continued*)

Angles	Sines		Cosines		Tangents		Cotangents		Angles
	Nat.	Log.	Nat.	Log.	Nat.	Log.	Nat.	Log.	
27° 00′	.4540	9.6570	.8910	9.9499	.5095	9.7072	1.9626	0.2928	63° 00′
10	.4566	6595	.8897	9492	.5132	7103	1.9486	2897	50
20	.4592	6620	.8884	9486	.5169	7134	1.9347	2866	40
30	.4617	6644	.8870	9479	.5206	7165	1.9210	2835	30
40	.4643	6668	.8857	9473	.5243	7196	1.9074	2804	20
50	.4669	6692	.8843	9466	.5280	7226	1.8940	2774	10
28° 00′	.4695	9.6716	.8829	9.9459	.5317	9.7257	1.8807	0.2743	62° 00′
10	.4720	6740	.8816	9453	.5354	7287	1.8676	2713	50
20	.4746	6763	.8802	9446	.5392	7317	1.8546	2683	40
30	.4772	6787	.8788	9439	.5430	7348	1.8418	2652	30
40	.4797	6810	.8774	9432	.5467	7378	1.8291	2622	20
50	.4823	6833	.8760	9425	.5505	7408	1.8165	2592	10
29° 00′	.4848	9.6856	.8746	9.9418	.5543	9.7438	1.8040	0.2562	61° 00′
10	.4874	6878	.8732	9411	.5581	7467	1.7917	2533	50
20	.4899	6901	.8718	9404	.5619	7497	1.7796	2503	40
30	.4924	6923	.8704	9397	.5658	7526	1.7675	2474	30
40	.4950	6946	.8689	9390	.5696	7556	1.7556	2444	20
50	.4975	6968	.8675	9383	.5735	7585	1.7437	2415	10
30° 00′	.5000	9.6990	.8660	9.9375	.5774	9.7614	1.7321	0.2386	60° 00′
10	.5025	7012	.8646	9368	.5812	7644	1.7205	2356	50
20	.5050	7033	.8631	9361	.5851	7673	1.7090	2327	40
30	.5075	7055	.8616	9353	.5890	7701	1.6977	2299	30
40	.5100	7076	.8601	9346	.5930	7730	1.6864	2270	20
50	.5125	7097	.8587	9338	.5969	7759	1.6753	2241	10
31° 00′	.5150	9.7118	.8572	9.9331	.6009	9.7788	1.6643	0.2212	59° 00′
10	.5175	7139	.8557	9323	.6048	7816	1.6534	2184	50
20	.5200	7160	.8542	9315	.6088	7845	1.6426	2155	40
30	.5225	7181	.8526	9308	.6128	7873	1.6319	2127	30
40	.5250	7201	.8511	9300	.6168	7902	1.6212	2098	20
50	.5275	7222	.8496	9292	.6208	7930	1.6107	2070	10
32° 00′	.5299	9.7242	.8480	9.9284	.6249	9.7958	1.6003	0.2042	58° 00′
10	.5324	7262	.8465	9276	.6289	7986	1.5900	2014	50
20	.5348	7282	.8450	9268	.6330	8014	1.5798	1986	40
30	.5373	7302	.8434	9260	.6371	8042	1.5697	1958	30
40	.5398	7322	.8418	9252	.6412	8070	1.5597	1930	20
50	.5422	7342	.8403	9244	.6453	8097	1.5497	1903	10
33° 00′	.5446	9.7361	.8387	9.9236	.6494	9.8125	1.5399	0.1875	57° 00′
10	.5471	7380	.8371	9228	.6536	8153	1.5301	1847	50
20	.5495	7400	.8355	9219	.6577	8180	1.5204	1820	40
30	.5519	7419	.8339	9211	.6619	8208	1.5108	1792	30
40	.5544	7438	.8323	9203	.6661	8235	1.5013	1765	20
50	.5568	7457	.8307	9194	.6703	8263	1.4919	1737	10
34° 00′	.5592	9.7476	.8290	9.9186	.6745	9.8290	1.4826	0.1710	56° 00′
10	.5616	7494	.8274	9177	.6787	8317	1.4733	1683	50
20	.5640	7513	.8258	9169	.6830	8344	1.4641	1656	40
30	.5664	7531	.8241	9160	.6873	8371	1.4550	1629	30
40	.5688	7550	.8225	9151	.6916	8398	1.4460	1602	20
50	.5712	7568	.8208	9142	.6959	8425	1.4370	1575	10
35° 00′	.5736	9.7586	.8192	9.9134	.7002	9.8452	1.4281	0.1548	55° 00′
10	.5760	7604	.8175	9125	.7046	8479	1.4193	1521	50
20	.5783	7622	.8158	9116	.7089	8506	1.4106	1494	40
30	.5807	7640	.8141	9107	.7133	8533	1.4019	1467	30
40	.5831	7657	.8124	9098	.7177	8559	1.3934	1441	20
50	.5854	7675	.8107	9089	.7221	8586	1.3848	1414	10
36° 00′	.5878	9.7692	.8090	9.9080	.7265	9.8613	1.3764	0.1387	54° 00′
	Nat.	Log.	Nat.	Log.	Nat.	Log.	Nat.	Log.	
Angles	Cosines		Sines		Cotangents		Tangents		Angles

TABLE VII.—TRIGONOMETRIC FUNCTIONS.—(*Continued*)

Angles	Sines		Cosines		Tangents		Cotangents		Angles
	Nat.	Log.	Nat.	Log.	Nat.	Log.	Nat.	Log.	
36° 00′	.5878	9.7692	.8090	9.9080	.7265	9.8613	1.3764	0.1387	54° 00′
10	.5901	7710	.8073	9070	.7310	8639	1.3680	1361	50
20	.5925	7727	.8056	9061	.7355	8666	1.3597	1334	40
30	.5948	7744	.8039	9052	.7400	8692	1.3514	1308	30
40	.5972	7761	.8021	9042	.7445	8718	1.3432	1282	20
50	.5995	7778	.8004	9033	.7490	8745	1.3351	1255	10
37° 00′	.6018	9.7795	.7986	9.9023	.7536	9.8771	1.3270	0.1229	53° 00′
10	.6041	7811	.7969	9014	.7581	8797	1.3190	1203	50
20	.6065	7828	.7951	9004	.7627	8824	1.3111	1176	40
30	.6088	7844	.7934	8995	.7673	8850	1.3032	1150	30
40	.6111	7861	.7916	8985	.7720	8876	1.2954	1124	20
50	.6134	7877	.7898	8975	.7766	8902	1.2876	1098	10
38° 00′	.6157	9.7893	.7880	9.8965	.7813	9.8928	1.2790	0.1072	52° 00′
10	.6180	7910	.7862	8955	.7860	8954	1.2723	1046	50
20	.6202	7926	.7844	8945	.7907	8980	1.2647	1020	40
30	.6225	7941	.7826	8935	.7954	9006	1.2572	0994	30
40	.6248	7957	.7808	8925	.8002	9032	1.2497	0968	20
50	.6271	7973	.7790	8915	.8050	9058	1.2423	0942	10
39° 00′	.6293	9.7989	.7771	9.8905	.8098	9.9084	1.2349	0.0916	51° 00′
10	.6316	8004	.7753	8895	.8146	9110	1.2276	0890	50
20	.6338	8020	.7735	8884	.8195	9135	1.2203	0865	40
30	.6361	8035	.7716	8874	.8243	9161	1.2131	0839	30
40	.6383	8050	.7698	8864	.8292	9187	1.2059	0813	20
50	.6406	8066	.7679	8853	.8342	9212	1.1988	0788	10
40° 00′	.6428	9.8081	.7660	9.8843	.8391	9.9238	1.1918	0.0762	50° 00′
10	.6450	8096	.7642	8832	.8441	9264	1.1847	0736	50
20	.6472	8111	.7623	8821	.8491	9289	1.1778	0711	40
30	.6494	8125	.7604	8810	.8541	9315	1.1708	0685	30
40	.6517	8140	.7585	8800	.8591	9341	1.1640	0659	20
50	.6539	8155	.7566	8789	.8642	9366	1.1571	0634	10
41° 00′	.6561	9.8169	.7547	9.8778	.8693	9.9392	1.1504	0.0608	49° 00′
10	.6583	8184	.7528	8767	.8744	9417	1.1436	0583	50
20	.6604	8198	.7509	8756	.8796	9443	1.1369	0557	40
30	.6626	8213	.7490	8745	.8847	9468	1.1303	0532	30
40	.6648	8227	.7470	8733	.8899	9494	1.1237	0506	20
50	.6670	8241	.7451	8722	.8952	9519	1.1171	0481	10
42° 00′	.6691	9.8255	.7431	9.8711	.9004	9.9544	1.1106	0.0456	48° 00′
10	.6713	8269	.7412	8699	.9057	9570	1.1041	0430	50
20	.6734	8283	.7392	8688	.9110	9595	1.0977	0405	40
30	.6756	8297	.7373	8676	.9163	9621	1.0913	0379	30
40	.6777	8311	.7353	8665	.9217	9646	1.0850	0354	20
50	.6799	8324	.7333	8653	.9271	9671	1.0786	0329	10
43° 00′	.6820	9.8338	.7314	9.8641	.9325	9.9697	1.0724	0.0303	47° 00′
10	.6841	8351	.7294	8629	.9380	9722	1.0661	0278	50
20	.6862	8365	.7274	8618	.9435	9747	1.0599	0253	40
30	.6884	8378	.7254	8606	.9490	9772	1.0538	0228	30
40	.6905	8391	.7234	8594	.9545	9798	1.0477	0202	20
50	.6926	8405	.7214	8582	.9601	9823	1.0416	0177	10
44° 00′	.6947	9.8418	.7193	9.8569	.9657	9.9848	1.0355	0.0152	46° 00′
10	.6967	8431	.7173	8557	.9713	9874	1.0295	0126	50
20	.6988	8444	.7153	8545	.9770	9899	1.0235	0101	40
30	.7009	8457	.7133	8532	.9827	9924	1.0176	0076	30
40	.7030	8469	.7112	8520	.9884	9949	1.0117	0051	20
50	.7050	8482	.7092	8507	.9942	9975	1.0058	0025	10
45° 00′	.7071	9.8495	.7071	9.8495	1.0000	0.0000	1.0000	0.0000	45° 00′
	Nat.	Log.	Nat.	Log.	Nat.	Log.	Nat.	Log.	
Angles	Cosines		Sines		Cotangents		Tangents		Angles

TABLE VIII.—AMERICAN EXPERIENCE TABLE OF MORTALITY

Age	Number living	Number dying	Yearly probability of dying	Yearly probability of living	Age	Number living	Number dying	Yearly probability of dying	Yearly probability of living
10	100 000	749	0.007 490	0.992 510	53	66 797	1 091	0.016 333	0.983 667
11	99 251	746	0.007 516	0.992 484	54	65.706	1 143	0.017 396	0.982 604
12	98 505	743	0.007 543	0.992 457	55	64 563	1 199	0.018 571	0.981 429
13	97 762	740	0.007 569	0.992 431	56	63 364	1 260	0.019 885	0.980 115
14	97 022	737	0.007 596	0.992 404	57	62 104	1 325	0.021 335	0.978 665
15	96 285	735	0.007 634	0.992 366	58	60.779	1 394	0.022 936	0.977 064
16	95 550	732	0.007 661	0.992 339	59	59 385	1 468	0.024 720	0.975 280
17	94 818	729	0.007 688	0.992 312	60	57 917	1 546	0.026 693	0.973 307
18	94 089	727	0.007 727	0.992 273	61	56 371	1 628	0.028 880	0.971 120
19	93 362	725	0.007 765	0.992 235	62	54 743	1 713	0.031 292	0.968 708
20	92 637	723	0.007 805	0.992 195	63	53 030	1 800	0.033 943	0.966 057
21	91 914	722	0.007 855	0.992 145	64	51 230	1 889	0.036 873	0.963 127
22	91 192	721	0.007 906	0.992 094	65	49 341	1 980	0.040 129	0.959 871
23	90 471	720	0.007 958	0.992 042	66	47 361	2 070	0.043 707	0.956 293
24	89 751	719	0.008 011	0.991 989	67	45 291	2 158	0.047 647	0.952 353
25	89 032	718	0.008 065	0.991 935	68	43 133	2 243	0.052 002	0.947 998
26	88 314	718	0.008 130	0.991 870	69	40 890	2 321	0.056 762	0.943 238
27	87 596	718	0.008 197	0.991 803	70	38 569	2 391	0.061 993	0.938 007
28	86 878	718	0.008 264	0.991 736	71	36 178	2 448	0.067 665	0.932 335
29	86 160	719	0.008 345	0.991 655	72	33 730	2 487	0.073 733	0.926 267
30	85 441	720	0.008 427	0.991 573	73	31 243	2 505	0.080 178	0.919 822
31	84 721	721	0.008 510	0.991 490	74	28 738	2 501	0.087 028	0.912 972
32	84 000	723	0.008 607	0.991 393	75	26 237	2 476	0.094 371	0.905 629
33	83 277	726	0.008 718	0.991 282	76	23 761	2 431	0.102 311	0.897 689
34	82 551	729	0.008 831	0.991 169	77	21 330	2 369	0.111 064	0.888 936
35	81 822	732	0.008 946	0.991 054	78	18 961	2 291	0.120 827	0.879 173
36	81 090	737	0.009 089	0.990 911	79	16 670	2 196	0.131 734	0.868 266
37	80 353	742	0.009 234	0.990 766	80	14 474	2 091	0.144 466	0.855 534
38	79 611	749	0.009 408	0.990 592	81	12 383	1 964	0.158 605	0.841 395
39	78 862	756	0.009 586	0.990 414	82	10 419	1 816	0.174 297	0.825 703
40	78 106	765	0.009 794	0.990 206	83	8 603	1 648	0.191 561	0.808 439
41	77 341	774	0.010 008	0.989 992	84	6 955	1 470	0.211 359	0.788 641
42	76 567	785	0.010 252	0.989 748	85	5 485	1 292	0.235 552	0.764 448
43	75 782	797	0.010 517	0.989 483	86	4 193	1 114	0.265 681	0.734 319
44	74 985	812	0.010 829	0.989 171	87	3 079	933	0.303 020	0.696 980
45	74 173	828	0.011 163	0.988 837	88	2 146	744	0.346 692	0.653 308
46	73 345	848	0.011 562	0.988 438	89	1 402	555	0.395 863	0.604 137
47	72 497	870	0.012 000	0.988 000	90	847	385	0.454 545	0.545 455
48	71 627	896	0.012 509	0.987 491	91	462	246	0.532 468	0.467 532
49	70 731	927	0.013 106	0.986 894	92	216	137	0.634 259	0.365 741
50	69 804	962	0.013 781	0.986 219	93	79	58	0.734 177	0.265 823
51	68 842	1 001	0.014 541	0.985 459	94	21	18	0.857 143	0.142 857
52	67 841	1 044	0.015 389	0.984 611	95	3	3	1.000 000	0.000 000

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ANSWERS

Exercise 1

1. 3,210,865. **2.** 491.7976. **3.** 1,630.3102. **5.** 3,625.341. **6.** 476.5. **7.** 1,088.272. **13.** .02270367. **14.** .04119576. **15.** 4.80065625. **17.** Less by 3,000; less by 2,900; more by 4,500. **18.** Less by 4,995; less by 7,002. **19.** More by 35; more by 105; less by 105. **21.** .5, .33 · · ·, .25, .2, .166 · · ·, .142857142857 · · ·, .125, .11 · · ·, .1, .0909 · · ·, .0833 · · ·, $\frac{1}{3}$, $\frac{1}{6}$, $\frac{1}{7}$, $\frac{1}{9}$, $\frac{1}{11}$, $\frac{1}{12}$. **22.** Multiplied by 100. **25.** 214. **26.** 14.309. **27.** 14.0027. **29.** .002375375 · · · .

Exercise 2

1. $x + 2x = 93$. **2.** $x - x/3 = 2{,}000$. **3.** $x/3 + x/4 = 21$.
5. $(x + 10)(x + 6) = 95$. **6.** $x + 5 = 2(x - 2)$. **7.** $x^2 - 1{,}200 = 800$.
9. $2x - 10 = x + 10$. **10.** $x + 2x/3 = 100$. **11.** $2x = 3(x - 10)$.
13. $2x - 10 = 3(x - 10)$. **14.** $10/(x + 4) + 10/(x - 4) = 2$.
15. $20x + 600 = 25(x + 20)$. **17.** $1{,}000 + x + 2x + 3x = 10{,}000$.
18. $40x + 8 + 60(x/2) = 400$. **19.** $(\frac{1}{30}) + (\frac{1}{25}) + (1/x) = (\frac{1}{10})$.

Exercise 3

1. -2. **2.** 11. **3.** 9. **6.** $11x + 9y + 7$. **7.** $9a + 9b + 14c$. **9.** $10a + 17b + 10c$. **10.** $4x$. **11.** $r + s - 5t$. **13.** $6a + 8b - 7c - 5d$. **14.** $3a$. **15.** $-a - 2b$. **17.** $2x - 8y + 4z$. **18.** $-11r + 6s + 5t$. **19.** $-3a + 5b - 9c$. **21.** $5x + 4u + 4v + 8w$. **23.** 4. **25.** 0. **26.** 0. **29.** $3a - 4b$. **30.** $2x$. **31.** $a + 7b$. **33.** $4a - b$. **34.** $3x + 2y$. **35.** $2a - 4b$. **38.** $2m + n$.

Exercise 4

1. 20. **2.** -51. **3.** $14xy$. **5.** $28xy + 2ab$. **6.** $-17rs + 49tu$. **7.** $-4a - 7b$. **9.** a^6. **10.** $6b^6$. **11.** $3a^5b^3$. **13.** a^8b^{12}. **14.** $8x^6y^3$. **15.** $16a^4b^3$. **17.** $324a^8b^6c^2$. **18.** $-24a^b$. **19.** $8x^{a+b}$. **21.** $a^{2m}b^5$. **22.** a^5. **23.** $x^3 + y^3$. **25.** $x^2 - y^2 + 2yz - z^2$. **26.** $2x^4 - 5x^3 + 5x^2 - 3x + 1$. **27.** $3a^3b + a^2b^2 - 3ab^3 - b^4$. **29.** $x^5 - y^5$. **30.** $a^3 - ab^2 + a^2b + 2abc + 2b^3 + 3bc^2 - 3b^2c - c^3$.

Exercise 5

1. $a^2 + ab + b^2$. **2.** $a^2 - ab + b^2$. **3.** $x^2 + 2x + 4$. **5.** $x^2 - 2xy + y^2$. **6.** $x - y$. **7.** $2a^2 - 3ab + 2b^2$. **9.** $x^4 + x^3y + x^2y^2 + xy^3 + y^4$. **10.** $x^2 - 2xy + 2y^2$. **11.** $3x + 2y$; $5y^2$. **13.** $3x^2 + 6xy + 10y^2$; $21y^3$. **14.** $2a^3 - b^3$; $-7b^4$. **15.** $x^3 - x^2y + xy^2 - y^3$; $2y^4$. **17.** $2(a - 2b + 3c)$. **18.** $3(x - 3y + 4z)$. **19.** $4x(x^2 - 2x + 3y)$. **21.** $(a + b)(2c + 3d)$. **22.** $(x + y + z)(2u - 3v)$. **23.** $(a - 2)(a + 2)$. **25.** $(3x^2 - 2y^2)(3x^2 + 2y^2)$. **26.** $(5a^3 - 4b^2)(5a^3 + 4b^2)$. **27.** $(4x^4 - 5y^3)(4x^4 + 5y^3)$. **29.** $(x + y - 2)(x + y + 2)$. **30.** $(a - b - c)(a - b + c)$. **31.** $(2x - 3y - 3z)(2x - 3y + 3z)$. **33.** $(2 - x - y)(2 + x + y)$. **34.** $(x - y - z)(x + y + z)$.

35. $(2a - 2b + c)(2a + 2b - c)$. **37.** $(a + 2b)^2$. **38.** $(2x + y)^2$.
39. $(a + 3b)^2$. **41.** $(2a - 3b)^2$. **42.** $(3x - 4y)^2$. **43.** $(a - 2b)(a + b)$.
45. $(2a - b)(a + 2b)$. **46.** $(3x - y)(x + 3y)$. **47.** $(3a - 2b)(2a - 3b)$.
49. $(12a - b)(a + 8b)$. **50.** $(24x - y)(x - 6y)$. **51.** $(a - 2)(b - 3)$.
53. $(a - 2b)(2b + c)$. **54.** $(x - 2y)(3z - w)$.

Exercise 6

1. $\frac{2}{3}$. **2.** $\frac{3}{2}$. **3.** $\frac{4}{5}$. **5.** $2y$. **6.** $7/3a$. **7.** $3a^3/(8b^3c)$.
9. $y(x - y)/x$. **10.** ab. **11.** $1/(2y)$. **13.** x. **14.** $ab + b^2$. **15.** $\frac{7}{8}$.
17. $\frac{3}{4}$. **18.** $\frac{3}{5}$. **19.** $2/a$. **21.** $3/(2y)$. **22.** $1/(3x)$. **23.** $\frac{1}{2}$. **25.** $1/b$.
26. $\frac{3}{4}$. **27.** $\frac{2}{3}$. **29.** $y/(x + 2y)$. **30.** $1/(a + b)$. **31.** $2/(x - y)$.
33. $a/(a + b)$. **34.** $z/(x - y)$. **35.** $2/(a + 2b)$. **37.** $3/(2a - 3b)$.
38. $2/(3x + 4y)$.

Exercise 7

1. 3. **2.** $\frac{6}{5}$. **3.** $\frac{15}{16}$. **5.** $a/(a - 1)$. **6.** $x - 1$. **7.** $3/a$. **9.** a.
10. x/y. **11.** $(x + 1)/(x - 1)$. **13.** $(x + y)/(x - y)$.
14. $a/(a + 2)$. **15.** $(1 - x^2)/x$.

Exercise 8

1. 2^3. **2.** 2^5. **3.** x^7. **5.** m^{13a}. **6.** $(4b^4)/(9a^6)$. **7.** $x^3y^6/27$.
9. $324x^2y^6$. **10.** $512a^9$. **11.** $(54)^a x^a y^a$. **13.** $(12)^{x+y}a^{6x+6y}b^{2x+2y}$.
14. $36x^8$. **18.** $\frac{1}{4}$. **19.** $\frac{8}{9}$. **21.** $\frac{1}{16}$. **22.** $\frac{2}{3}$. **23.** 1. **25.** 0. **26.** 1.
27. 8. **29.** 5. **30.** 2. **31.** $1/a^2$. **33.** x^{2b^2}. **34.** $y^5/(x^6z^{10})$. **35.** 5.
37. $6a^3b^2\sqrt{a}$. **38.** $5x^4y^3\sqrt{3y}$. **39.** $4a^2b\sqrt[3]{b^2}$. **41.** $2x^2y\sqrt[4]{6y}$. **42.** $2ab\sqrt{b}$.
43. $3xy\sqrt[3]{y^2}$. **45.** $12xy^3$. **46.** $3ab^2$. **47.** $3xy\sqrt[3]{y^2}$. **50.** $4 + 3\sqrt{2}$.
51. $\sqrt{3} + 4\sqrt{2}$. **53.** $12\sqrt{3} - 3\sqrt[3]{3}$.

Exercise 9

3. (*a*) 3. **5.** 2. **6.** 3. **7.** 5. **9.** $\frac{6}{7}$. **10.** -3. **11.** 1. **13.** $\frac{3}{4}$.
14. $\frac{2}{3}$. **15.** $\frac{1}{2}$. **17.** 3. **18.** 0. **19.** -1. **21.** 1. **22.** $(a - b)/(a + b)$.
23. a. **25.** 20, 30, 50. **26.** 60. **27.** 9 and $10\frac{1}{2}$ days. **29.** $\frac{6}{5}$. **30.** (*a*) $(ab)/(a + b)$; (*b*) $\frac{40}{9}$. **31.** $3\frac{1}{7}$.

Exercise 10

1. 9. **2.** $4a^2$. **3.** $\frac{4}{25}$. **5.** $\frac{9}{64}$. **6.** $25m^2/144$. **7.** $a^2/(4b^2)$. **9.** $(a - b)^2/(4c^2)$.
10. $(r + s)^2/(u - v)^2$. **11.** 0, -6. **13.** 0, $\frac{4}{5}$. **14.** 0, $-6b/7$. **15.** 0, $\frac{3}{4}$. **17.** 0, $-a/b$.
18. 0, $2(m + n)$. **19.** 0, $(a - b)/c$. **21.** $\pm\sqrt{2}$. **22.** $\pm\frac{5}{3}$. **23.** 2, -1.
25. -1, $\frac{3}{2}$. **26.** $\frac{1}{2}$, $-\frac{3}{2}$. **27.** $\frac{2}{3}$, $\frac{3}{2}$. **29.** $\frac{1}{2}$, $\frac{3}{4}$. **30.** $b/(2a)$, $-b/a$. **31.** $\pm 2\sqrt{2}$.
33. $\pm\frac{1}{2}$. **34.** ± 7. **35.** ± 1. **37.** $\pm\sqrt{3}$. **38.** 0. **39.** $\pm 2a$. **41.** 3, -7.
42. 2, -3. **43.** $(2 \pm \sqrt{7})/2$. **45.** 5, $-\frac{5}{21}$. **46.** $\frac{15}{4}$, -6. **47.** $\frac{2}{3}$, $-\frac{1}{2}$. **49.** 2, $\frac{8}{9}$.
50. $\frac{1}{2}$, $\frac{1}{6}$. **51.** $2a$, $a/2$. **53.** b, $-a$. **54.** $2/a$, $-26/(9a)$. **55.** 1, -3. **57.** -2, $-\frac{1}{2}$.
58. 1, $-\frac{1}{2}$. **59.** $(1 \pm \sqrt{13})/2$. **61.** 2, $-\frac{1}{3}$. **62.** $(7 \pm \sqrt{17})/4$. **63.** $\frac{1}{2}$, $-\frac{2}{3}$.
65. $(b \pm \sqrt{b^2 + 4ac})/(2a)$. **66.** $(-n \pm \sqrt{n^2 - 4mr})/(2m)$.
67. $[(a + b) \pm \sqrt{(a + b)^2 - 4(c + 2)}]/2$. **69.** $-y \pm\sqrt{y^2 + 3y^3 + 1}$.
70. $[1 - b \pm \sqrt{(b - 1)^2 - 4(a - 1)(c - 1)}]/(2a - 2)$. **77.** $1 + \sqrt{2}$.
78. 4. **79.** 10 ft. **81.** 20 hr., 30 hr. **82.** 30 miles per hour. **83.** $10\sqrt{2}$ miles per hour.

Exercise 11

3. 1, 2, 3, 4, 5. **5.** $(-1 \pm i\sqrt{3})/2$. **6.** $(1 \pm i\sqrt{23})/4$. **7.** $\pm i\sqrt{5}$.
9. $1 \pm i\sqrt{k-1}$. **10.** $-2 \pm i\sqrt{-k-4}$. **11.** (c) $-12, -6, -\sqrt{10}, -\sqrt{10}$.
13. $10 + 10i$. **14.** $4 + \sqrt{6} + i(4\sqrt{3} - \sqrt{2})$. **15.** $2 + 23i$.
17. $-2\sqrt{15} + 2i\sqrt{3}$. **18.** $-2i$. **19.** $(1 + i)/2$. **21.** $(1 - 3i\sqrt{3})/7$.
22. $(1 - i\sqrt{3})/2$. **23.** $i, -1, -i, 1, i, -1, -i, 1$.

Exercise 12

1. Irrational and unequal. **2.** Rational and unequal. **3.** Rational and equal. **5.** Imaginary. **6.** Rational and unequal. **7.** Imaginary. **9.** Rational and unequal. **10.** Imaginary. **11.** $-3, 1$. **13.** $-2, 1$. **14.** $0, \frac{5}{3}$. **15.** $-1, 1$. **17.** $\frac{3}{2}, 2$. **18.** $0, -4$. **19.** $-\frac{3}{2}, \frac{1}{2}$. **21.** $x^2 + 5x + 6 = 0$. **22.** $x^2 - 5 = 0$. **23.** $x^2 + 3 = 0$. **25.** $x^2 - 8x + 25 = 0$. **26.** $x^2 - 10x + 32 = 0$. **27.** $x^2 - 2ax + a^2 - b = 0$. **29.** -8. **30.** -2. **31.** 2. **33.** 3, 6. **34.** 6, 2. **35.** $x^4 - 10x^3 + 35x^2 - 50x + 24 = 0$. **37.** $x^4 - 4x^3 + 5x^2 - 2x - 2 = 0$. **38.** $x^4 - (a + b + c + d)x^3 + (ab + ac + ad + bc + bd + cd)x^2 - (abc + abd + acd + bcd)x + abcd = 0$

Exercise 13

1. 1, 2. **2.** $\frac{3}{4}, -1$. **3.** $2, \frac{3}{2}$. **5.** $a + b, -b$. **6.** $a + b, a - b$. **7.** Real roots are $1, -\sqrt[3]{4}$. **9.** $0, -1, i, -i$. **10.** $1, i, -i$. **11.** $i, -i, 2i, -2i$. **13.** $0, 2, -1$. **14.** $0, 2, i, -i$. **15.** $a, -a, -b$. **17.** Real roots are 2, -1. **18.** $(-1 \pm i\sqrt{11})/2$, $(-1 \pm i)/2$. **19.** $\pm 1, \pm i\sqrt{5}/5$. **21.** $-1 \pm \sqrt{3}$, $(-3 \pm \sqrt{6})/3$.
22. $(1 \pm i\sqrt{3})/4$. **23.** $(-3 \pm \sqrt{33})/4$. **25.** $(1 \pm i\sqrt{3})/2$, $(3 \pm i\sqrt{39})/6$.
26. $0, -\frac{1}{2}, (-1 \pm i\sqrt{47})/4$. **27.** $1, 2, (9 \pm \sqrt{137})/6$.
29. $(2 \pm \sqrt{2})/2$, $(3 \pm \sqrt{19})/2$. **30.** $(-3 \pm \sqrt{3})/2$, $(-3 \pm \sqrt{17})/2$.
31. $1, -1, i, -i$. **33.** $0, 1 \pm \sqrt{2}$. **34.** $-\frac{3}{2}$, $(-3a \pm \sqrt{3a - ab})/2a$.
35. $1, -2, 1 \pm i\sqrt{3}, (-1 \pm i\sqrt{3})/2$.

Exercise 14

1. $(2, -1)$. **2.** $(3, -2)$. **3.** $(-1, -2)$. **5.** $(2, -3)$. **6.** (5, 2). **7.** (2, 1). **9.** $(\frac{1}{2}, \frac{2}{3})$. **10.** $(\frac{3}{4}, -\frac{5}{3})$. **11.** Inconsistent. **13.** $(\frac{7}{6}, \frac{3}{8})$. **14.** (1, 2). **15.** (2, 4). **17.** $(b, 0)$. **18.** Inconsistent if $a \neq b$, dependent if $a = b$. **19.** $(a + b, a)$. **21.** (1, 1, 1). **22.** (2, 0, 1). **23.** $(\frac{2}{5}, -\frac{7}{5}, \frac{9}{5})$. **25.** $(a + b, a + b, -a - b)$. **26.** $(1/a, 1/b, 1/c)$. **35.** (12, 32). **37.** 60 miles per hour, 72 miles per hour. **38.** 3 Disabled, 19 disqualified. **39.** 18,000. **41.** 1 hr. **42.** 60 lb. @35 cents, 40 lb. @20 cents.

Exercise 15

1. (1, 2); (2, 1). **2.** (3, 1). **3.** $(2, -1)$. **5.** $(-1, -\frac{1}{3})$; $(-2, \frac{1}{3})$.
6. $(\frac{1}{2}, 1)$. **7.** $(\frac{1}{2}, -\frac{1}{2})$; $(\frac{15}{26}, -\frac{7}{26})$. **9.** $(a + b, a - b)$. **10.** $(1, a/b)$. **11.** $(a, 0)$; $(0, b)$. **13.** $(1/a, a/b)$; $[(2a^2 + a - b)/(a^2 + ab), (ab + 2b - a^2)/(b^2 + ab)]$. **14.** $(b, 2b - a)$. **15.** Imaginary. **17.** Real, equal. **18.** Real, unequal. **19.** Imaginary. **21.** $(-4, -3)$; $(\frac{4}{5}, \frac{1}{5})$. **22.** $(3, -2)$; $(-3, 2)$; $(2, -3)$; $(-2, 3)$. **23.** $(\frac{1}{3}, -\frac{2}{3})$; $(\frac{4}{3}, \frac{2}{3})$; (2, 1); $(-2, 4)$. **25.** $(\frac{3}{2}, \frac{1}{2})$; $(-\frac{3}{2}, -\frac{1}{2})$; $(\frac{1}{2}, \frac{3}{2})$; $(-\frac{1}{2}, -\frac{3}{2})$. **26.** (0, 0); (a, a); $(b, -b)$; $(a + b, a - b)$. **27.** Impossible. **29.** 24 by 16 in. **30.** 60 by 40 yd. **31.** 2 by 3 yd., $\frac{254}{145}$ by $\frac{483}{145}$ yd. **33.** 150 miles per hour, 60 miles per hour, 15 miles per hour. **34.** Plane, $\frac{400}{3}$; wind, $\frac{100}{3}$ and $\frac{200}{3}$. **35.** 40 miles per hour.

Exercise 16

1. $y = k/x$. **2.** $a = kbc$. **3.** $w = kxy/z$. **5.** $g = k/(pq)$. **6.** $P = kMn^2/d^2$. **7.** 16. **9.** 6. **10.** $\frac{4}{9}$. **11.** $\frac{1}{12}$. **13.** $3338\frac{7}{16}$ lb. **14.** 4 to 1; 64 to 1. **15.** 100 to 81. **17.** $1{,}548\frac{12}{25}$ lb. **18.** .25, .6, 1, 2, 11, 30, 90, 160, and 250 years, respectively. **19.** $\frac{1}{(280{,}000)^2}$. **21.** 64,000,000 to 1.

Exercise 17

1. $\frac{10}{3}$. **2.** $\frac{1}{8}$. **3.** 5. **5.** $\frac{3}{40}$. **6.** $\frac{1}{52}$, (approximately). **7.** $\frac{5}{4}$. **9.** $\frac{1}{64}$. **10.** $\frac{1}{72}$. **11.** 54. **13.** ± 6. **14.** $\frac{28}{3}$. **15.** $\frac{25}{4}$. **17.** $\frac{40}{5}$. **18.** 72; ± 6. **19.** $a^2/(1 + a)$. **21.** 40 miles per inch. **22.** $13\frac{1}{2}$ acres. **23.** $\frac{80}{3}$ min. **25.** $\frac{200}{3}$ cu. in. **26.** 6.4 ft. from pivot. **27.** (*a*) -15; (*b*) -6; (*c*) $-\frac{20}{7}$; (*d*) $\frac{1}{3}$.

Exercise 18

1. $x > \frac{9}{2}$. **2.** $x < -1$. **3.** $x < 5$. **5.** $x > 2$. **6.** $x < (3a - 3b)/2$. **7.** $x > \frac{24}{65}$. **9.** $x > -b^2/(a + b)$. **10.** $x > (2ab^2 + b)/(3ab - 2a)$. **13.** $-2 < x < 3$. **14.** $x < -3$; $0 < x < 1$. **15.** $-3 < x < 0$; $0 < x < 1$. **17.** $x < 1$; $3 < x < 4$; $x > 4$. **18.** $d > 20$ miles. **19.** $1{,}200 < A < 7{,}500$, where A is area enclosed by fence, in square feet.

Exercise 19

2. (*a*) I; (*b*) III; (*c*) IV; (*d*) II. **3.** Positive half of X-axis; negative half of Y-axis. **5.** (*a*) $y = 0$; (*b*) $x = 0$; (*c*) $y = -2$; (*d*) $x > 0$ and $y < 0$. **6.** (4, 6); (−4, 6); (4, −6); (−4, −6).

Exercise 20

13. They go through the origin; the slope is increased. **14.** (*b*) $x = 0$; (*c*) $y = x$; (*d*) $y = -2x$. **15.** $x - y = 1$. **17.** (*a*) 45 miles per hour; (*b*) When more than 73 ft. ahead of rear driver; (*c*) 23 miles per hour. **18.** (*a*) $(-160/9)°$; (*b*) 32°; (*c*) −40°; (*d*) 50°.

Exercise 21

11. (*a*) (3, 4); (*b*) $\frac{1}{3}$.

Exercise 23

6. (*a*) 13; (*b*) 1; (*c*) 3. **7.** (*a*) 11; (*b*) $\frac{5}{4}$; (*c*) $3z^2 + 5z + 3$; (*d*) 6; (*e*) $\frac{25}{3}$; (*f*) $4z^2 - 3z + 1$. **9.** $x(-1 \pm i\sqrt{3})/2$. **10.** $\pm 5x^5\sqrt{3}/3$. **11.** $2 - x$. **13.** $x^2 + 6x + 14$. **14.** $\frac{x^3 + 1}{x^3}$. **15.** $x + 3$.

Exercise 25

1. (*a*) 21,040; (*b*) 9; (*c*) 100,000. **2.** (*a*) 2; (*b*) 9; (*c*) 1. **3.** (*a*) 2; (*b*) 6; (*c*) −1; (*d*) −4; (*e*) 1; (*f*) 0; (*g*) 1; (*h*) 0; (*i*) −2; (*j*) −3. **5.** .001; 1; 100; 10,000,000. **7.** (*a*) $2.35(10)^2$; (*b*) $3.67(10)^6$; (*c*) $8.31(10)^{-1}$; (*d*) $4.62(10)^{-4}$; (*e*) $4.1(10)$; (*f*) $8.12(10)^0$; (*g*) $3.28(10)$; (*h*) $2.7(10)$; (*i*) $3(10)^{-2}$; (*j*) $5(10)^{-3}$.

Exercise 26

1. (*a*) 2.3711; (*b*) 6.5647; (*c*) 9.9196 − 10; (*d*) 6.6646 − 10; (*e*) 1.6128; (*f*) 0.9096; (*g*) 1.5159; (*h*) 0.4314; (*i*) 8.4771 − 10; (*j*) 7.6990 − 10. **2.** (*a*) 6.94; (*b*) .0238; (*c*) 8.610; (*d*) 37,400; (*e*) .00281; (*f*) 6,770,000; (*g*) .000000917; (*h*) 382; (*i*) .159; (*j*) 55.8. **3.** (*a*) 1.3702; (*b*) 4.5089; (*c*) 7.4744 − 10; (*d*) 9.4620 − 10; (*e*) 2.5008; (*f*) 6.4466; (*g*) 8.5868 − 10; (*h*) 1.6210; (*i*) 3.5178; (*j*) 9.8022 − 10. **4.** (*a*) 249.2; (*b*) .1651.

Exercise 27

1. 806.0. **2.** .06647. **3.** 1923. **5.** .03994. **6.** 42.88. **7.** .08136. **9.** 23,250. **10.** .1930. **11.** 3.026. **13.** 3.560. **14.** 5.024. **15.** .01004. **17.** .2702. **18.** 37,250. **19.** .5773. **21.** .000002609. **22.** 5.666. **23.** 805.9662. **25.** −.1152. **26.** −.6616. **27.** $1.259(10)^{383}$.

Exercise 28

1. 7.000. **2.** 3.000. **3.** .7498. **5.** .6667. **6.** ±3.999. **7.** ±2.000. **9.** 9.128. **10.** .2399. **11.** 1.378. **13.** −1.156. **14.** 2.364; −.364. **15.** 2.102.

Exercise 30

1. $\frac{1}{2}(n^2 + n)$; n^2. **2.** $3n^2 - 2n$.

Exercise 31

1. 4, 5; 20. **2.** 15, 19; 79. **3.** −10, −18; −138.
5. $3\sqrt{2} - 4$, $4\sqrt{2} - 6$; $19\sqrt{2} - 36$. **6.** $3n - 2m$, $4n - 3m$; $19n - 18m$.
7. 3.42, 2.56; −10.34. **9.** $a = 2$, $l = -6$. **10.** $l = 4$, $n = 8$.
11. $l = 21$, $d = 2$. **13.** $a = 6$, $n = 5$; $a = -4$, $n = 10$.
14. $a = 2$, $d = -5$. **15.** $a = 47$, $s = 63$. **17.** $n = 7$, $s = 45\frac{1}{2}$.
18. $d = 20$, $s = 1{,}144$. **19.** (*a*) $\frac{3}{2}$; (*b*) x; (*c*) $\frac{1}{2}(\sqrt{11} + \sqrt{5})$; (*d*) 0.
21. (*a*) $s = \frac{n}{2}\left[2a + (n - 1)d\right]$; (*b*) $s = \frac{n}{2}\left[2l - (n - 1)d\right]$;
(*c*) $s = (l + a)(l + d - a)/(2d)$. **22.** 3,999,000. **23.** \$895.50.
25. 1,610 ft. **26.** 21 sec.

Exercise 32

1. 81, 243; 3^{20}. **2.** 1, $\frac{1}{2}$; $(\frac{1}{2})^{16}$. **3.** 16, 32; 2^{20}.
5. $4\sqrt{2}$, 8; $1{,}024\sqrt{2}$. **6.** $-a$, a; $-a$. **7.** .001, .0001; $(10)^{-19}$.
9. Arithmetic. **10.** Neither. **11.** Geometric. **13.** Geometric.
14. Neither. **15.** Geometric. **17.** $n = 6$, $s = 2\frac{20}{81}$.
18. $a = .12$, $s = .12121212$.
19. $l = 256$, $s = 341\frac{5}{16}$. **21.** $r = \frac{1}{3}$, $s = \frac{1093}{243}$; $r = -\frac{1}{3}$, $s = \frac{547}{243}$.
22. $n = 6$, $s = \frac{665}{81}$. **23.** $a = 64$, $s = 44$. **25.** $l = .0000512$, $s = 520.83(+)$.
26. $n = 3$, $r = -2$. **27.** (*a*) $4\sqrt[5]{-\frac{1}{4}}$, $4\sqrt[5]{\frac{1}{16}}$, $4\sqrt[5]{-\frac{1}{64}}$, $4\sqrt[5]{\frac{1}{256}}$; (*b*) 2, $\frac{2}{5}$, $\frac{2}{25}$, $\frac{2}{125}$;
(*c*) $m^{4/5}n^{1/5}$, $m^{3/5}n^{2/5}$, $m^{2/5}n^{3/5}$, $m^{1/5}n^{4/5}$; (*d*) b, b^2/a, b^3/a^2, b^4/a^3.
29. (*a*) $s = (a - rl)/(1 - r)$; (*b*) $s = [l(1 - r^n)]/[r^{n-1}(1 - r)]$;
(*c*) $l(s - l)^{n-1} = a(s - a)^{n-1}$. **30.** $a = \frac{2}{3}$, $s = \frac{7}{16}$. **31.** $n = 5$, $l = -\frac{1}{2}$.
33. $r = -\frac{1}{2}$, $l = -\frac{1}{2}$. **34.** 740,300. **35.** 2^{29} cents, or \$5,368,709.12.
37. 4, 3, 2, 1; 4, 2, 1, $\frac{1}{2}$.

Exercise 33

1. $\frac{12}{5}$. **2.** $\frac{2}{3}$. **3.** $\frac{70}{33}$. **5.** $11\frac{11}{999}$. **6.** $\frac{3}{2}$. **7.** $a^3/(a^2 - 1)$. **9.** $\frac{1}{3}$. **10.** $\frac{5}{9}$. **11.** $\frac{37}{30}$. **13.** $2\frac{140}{999}$. **14.** $\frac{1189}{990}$. **15.** 1. **17.** No. It forever approaches a spot 5 miles from the shore. **18.** 40 ft. **19.** 4 tons.

Exercise 34

1. $37.50. **2.** $525. **3.** $123.90. **5.** $554.62.

Exercise 35

1. (*a*) $1,790.80; (*b*) $432.19; (*c*) 17.7 years; (*d*) 18.6 years; (*e*) 4.7 per cent; (*f*) 5.6 per cent. **2.** (*a*) $672.97; (*b*) $4,119.90; (*c*) $174.56. **3.** (*a*) $2,931; (*b*) $3,281; (*c*) 17.3 years; (*d*) 2.8 per cent.

Exercise 36

1. 21. **2.** (*a*) $4,390.27, $2,093.03; (*b*) $36,785.60, $11,469.90. **3.** (*a*) $77.83; (*b*) $416.39; (*c*) $66.48; (*d*) $157.70. **5.** $5,376.05. **6.** $320.30. **7.** $2,464.99. **9.** 89 per cent. **10.** 26. **11.** $611.57; $20,000.

Exercise 38

1. .3502, .9367, 3739, 2.6746. **2.** .6611, .7503, .8811, 1.1349. **3.** .9216, .3881, 2.3750, .4210. **5.** .7369, .6760, 1.0900, .9174. **6.** .6189, .7855, .7879, 1,2693. **7.** .3365, .9417, .3574, 2.7980. **9.** 1.0000, .0052, 206.27, .0052. **10.** .0049, 1.0000, .0049, 223.45. **11.** 13°50′. **13.** 65°. **14.** 89°48′. **15.** 86°30′. **17.** 1′. **18.** 83°37′. **19.** 17°37′. **21.** 15. **22.** 15. **23.** 5. **25.** $4\sqrt{2}$. **26.** $\frac{1}{4}\sqrt{2}$. **27.** 7. **29.** $\sqrt{21}$. **30.** $2\sqrt{10}/3$. **31.** 3, $3\sqrt{3}$. **33.** .4331, .9013, .4806, 2.0809; .9013, .4331, 2.089, .4806. **34.** 1, 1, 1. **35.** .999997.

Exercise 39

1. $c = 43$, $A = 29°35'$, $B = 60°25'$. **2.** $b = 168$, $A = 38°26'$, $B = 51°34'$. **3.** $b = 284.6$, $c = 357.5$, $B = 52°46'$. **5.** $a = 32{,}910$, $b = 28{,}780$, $A = 48°50'$. **6.** $b = 1.661$, $c = 1.678$, $B = 8°11'$. **7.** $b = .0138$, $c = .0142$, $A = 12°49'$. **9.** $c = 1.086$, $A = 37°17'$, $B = 52°43'$. **10.** $b = 365{,}000$, $A = 59°36'$, $B = 30°24'$. **11.** 41°58′, 15.09. **13.** 50.2 ft. **14.** 123 ft. **15.** 1,240 miles **17.** 40°36′, 28 ft. **18.** Dirt road better by about 13 min. **19.** 4 min. after 5 P.M. **21.** 67,000,000 miles approximately. **22.** 26.02 ft. **23.** 42.5 miles.

Exercise 40

1. $C = 118°50'$, $b = 260$, $c = 397$. **2.** $B = 120°20'$, $a = 9.78$, $c = 4.28$. **3.** $A = 80°$, $a = 280$, $b = 176$. **5.** $A = 51°47'$, $a = 105.9$, $b = 98.58$. **6.** $B = 80°29'$, $a = 11.45$, $c = 19.84$. **7.** $A = 102°9'$, $b = 2.325$, $c = 3.752$. **9.** $C = 104°5'$, $b = 68.40$, $c = 74.97$. **10.** $C = 65°32'$, $a = 21.42$, $c = 24.35$.

Exercise 41

1. $B = 55°4'$, $C = 82°16'$, $c = 432.7$; $B' = 124°56'$, $C' = 12°24'$, $c' = 93.78$. **2.** No solution. **3.** No solution. **5.** No solution. **6.** $A = 90°$, $B = 46°30'$, $b = 725.4$. **7.** $A = 61°22'$, $C = 50°43'$, $a = 798.6$. **9.** $B = 47°8'$, $C = 87°52'$, $c = 116$; $B' = 132°52'$, $C' = 2°8'$, $c' = 4.3$. **10.** No solution.

Exercise 42

1. $c = 15$. **2.** 231. **3.** $c = 352.3$. **5.** $a = 2{,}166$. **6.** 693. **7.** $a = .0047$. **9.** 1.00. **10.** $c = 7{,}000$.

Exercise 43

1. $A = 36°52'$, $B = 53°8'$, $C = 90°0'$. **2.** $A = 41°25'$, $B = 55°46'$, $C = 82°49'$. **3.** $A = 44°25'$, $B = 57°7'$, $C = 78°28'$. **5.** $A = 48°11'$, $B = 58°25'$, $C = 73°24'$. **6.** $A = 22°37'$, $B = 67°23'$, $C = 90°0'$. **7.** $A = 28°4'$, $B = 61°56'$, $C = 90°0'$. **9.** $A = 27°40'$, $B = 40°32'$, $C = 111°48'$. **10.** $A = 22°20'$, $B = 49°27'$, $C = 108°13'$.

Exercise 44

1. 123 ft. **2.** 2.728 miles. **3.** 27°15′. **5.** 78 miles. **6.** 32.3 miles. **7.** 5,355 miles. **9.** 101 miles. **10.** 88 miles. **11.** 47.9 miles per hour. **13.** About 40 miles per second. **14.** $a = 70$ ft., $b = 16$ ft. **17.** 239,000 sq. yd.

Exercise 46

1. 13. **2.** 6. **3.** 0. **5.** 3. **6.** $y + 5$. **7.** $6 - x$. **9.** 10. **10.** 14. **11.** 6. **13.** 10. **14.** 20. **15.** 13. **17.** 9. **18.** 5. **19.** 12. **21.** 12. **22.** 6. **23.** 13. **25.** 42. **26.** 18. **27.** 36. **29.** Right and isosceles. **30.** Isosceles. **31.** Scalene. **33.** Right. **34.** Right and isosceles. **35.** (3, 7). **37.** $(-5, -1)$. **38.** $(-3, -\frac{11}{2})$. **39.** (5, 5). **41.** $(-9, -\frac{3}{2})$. **42.** (0, 2). **43.** $(\frac{23}{5}, -\frac{14}{5})$.

Exercise 47

1. ∞ (or no slope). 0, 0, ∞, $\frac{4}{3}$, $\frac{4}{3}$, $-\frac{5}{12}$, $-\frac{3}{2}$.

Exercise 48

1. $x - 3y = -7$. **2.** $11x - 5y = -13$. **3.** $2x - 7y = -13$. **5.** $3x + 4y = 12$. **6.** $x - y = 0$; $x + y = 12$; $x - 5y = 0$. **7.** $4x + 7y = 19$; $11x + 8y = -4$; $7x + y = 22$. **9.** $7x - 4y = 52$; $8x - 11y = 13$; $x - 7y = -39$. **10.** $14x - 8y = -31$; $16x - 22y = 11$; $x - 7y = 21$. **11.** $x + y = 1$; $x - 2y = 1$; $2x + y = 2$. **17.** $2x + 3y = 12$. **18.** $3x + y = 11$. **19.** $x + 2y = 7$.

Exercise 49

1. $(x + 3)^2 + (y - 4)^2 = 25$. **2.** $x^2 + y^2 = 9$. **3.** $(x - 4)^2 + y^2 = 0$. **5.** $(x - 3)^2 + (y + 4)^2 = 25$. **6.** $(x + 6)^2 + (y - 5)^2 = 36$. **7.** $x^2 + y^2 = 25$. **9.** $5x^2 + 5y^2 + 11y - 36 = 0$. **10.** $x^2 + y^2 + 10x - 2y = 0$. **11.** $\sqrt{13}$; $(2, -3)$. **13.** No locus. **14.** 0; $(\frac{1}{2}, -2)$ (point-circle). **15.** $\frac{2}{3}$; $(\frac{2}{3}, -\frac{1}{3})$. **17.** $h = -D/2$, $k = -E/2$, $r = \frac{1}{2}\sqrt{D^2 + E^2 - 4F}$. **21.** $r = h - 3$. **22.** $5h - 3k = 4$. **23.** $(x + 1)^2 + (y - 1)^2 = 1$; $(x + 5)^2 + (y - 5)^2 = 25$.

Exercise 50

7. Y - ellipse; center $(1, -2)$; $a = 5$, $b = 2$.
9. Y - ellipse; center $(\frac{1}{2}, -\frac{1}{3})$; $a = 2$, $b = 1$. **10.** $c = 3$, $e = \frac{3}{5}$.
11. $b = 3$, $e = \frac{4}{5}$. **13.** $\frac{(x - 3)^2}{49} + \frac{(y - 2)^2}{24} = 1$.

14. $\frac{(x-2)^2}{12} + \frac{(y-1)^2}{16} = 1.$ **15.** $12x^2 + 3(y-2)^2 = 64.$
22. X - hyperbola; center $(-1, 2)$; $a = 6$, $b = 2$.
23. Y - hyperbola; center $(\frac{1}{3}, -\frac{1}{2})$; $a = 3$, $b = 2$.
25. $c = \sqrt{41}$; $e = \sqrt{41}/4$. **26.** $b = 5\sqrt{2}$; $c = 5\sqrt{3}$.
27. $a = 5\sqrt{39}/13$; $c = 8\sqrt{39}/13$. **29.** $\frac{(x-2)^2}{16} - \frac{(y+2)^2}{9} = 1.$
30. $5x^2 - y^2 + 16 = 0$. **37.** Vertex $(2, -1)$; $p = 4$; upward.
38. Vertex $(6, -1)$; $p = \frac{1}{3}$; to right.
39. Vertex $(\frac{1}{2}, \frac{1}{3})$; $p = \frac{3}{4}$; downward. **41.** $x^2 = -8y$.
42. $x^2 = -y$. **43.** $(y+2)^2 = 12x$. **45.** $\frac{(x-1)^2}{16} + \frac{(y-4)^2}{15} = 1.$
46. $(x-2)^2 = -16y$. **47.** $\frac{(x-1)^2}{39} - \frac{(y-3)^2}{25} = -1.$
49. $6(x-2)^2 - 12(y-\frac{1}{2})^2 = -49$. **50.** $(x-1)^2 = y/2$.

Exercise 51

1. $24\sqrt{13}/13$. **2.** $3/13$. **3.** $12\sqrt{29}/29$.
5. $8\sqrt{10}/5$, $12\sqrt{10}/5$, $24\sqrt{58}/29$. **6.** $\pm(Ax + By + C)/\sqrt{A^2 + B^2}$.
7. $4x - 32y + 63 = 0$. **9.** $2x - 3y - 4 \pm 2\sqrt{13} = 0$. **10.** $(5, 5)$.
11. $25(A^2 + B^2) = 1$. **13.** $100[(x-3)^2 + (y-2)^2] = (4x - 3y + 2)^2$.
14. $3x^2 + 3y^2 - 40x - 16y + 116 = 0$. **15.** Hyperbola; ellipse; circle.

Exercise 52

1. $h - 4 = 3$. **2.** $-3 - b = 5$. **3.** $3a + 4b = 20$.
5. $5h - 3k + r\sqrt{34} = 4$. **6.** $4h - 3k = 7$. **7.** $k = 6$.
10. $(x-h)^2 + (y+h)^2 = h^2$. **11.** $(x-3)^2 + (y+4)^2 = r^2$. **13.** $4x + 3y = k$.
14. $3(x-h)^2 + 4(y-k)^2 = 3a^2$. **15.** $(x-h)^2 - 3(y-k)^2 = -3a^2$.

Exercise 53

3. $2x^2 + 3x + 10$; 32. **5.** $3x^3 + 3x^2 + 4x + 4$; 4. **6.** $-x^3 + 3x^2 - 3x + 4$; -6.

Exercise 54

1. Part (3); $\frac{5}{2}$, $-\frac{3}{2}$. **2.** $\frac{1}{2}$. **3.** $-\frac{2}{3}$. **5.** .54. **6.** .45. **7.** 2.88. **9.** 1.26. **10.** .26.

Exercise 55

1. .54. **2.** .75. **3.** .83. **5.** .70. **6.** 1.09. **7.** .60. **9.** .74. **10.** 2 and 4 (exactly); $-.77$.

Exercise 56

1. 3. **2.** 2. **3.** 1. **5.** 0. **6.** No limit. **7.** $\frac{2}{5}$. **9.** 2 in. **10.** $\frac{3}{4}$ in. **11.** 0.
13. No limit. **14.** (*a*) No limit; (*b*) a_0/b_0; (*c*) 0.

Exercise 57

1. (*a*) Δx; (*b*) 1; (*c*) 1. **2.** (*a*) $3\,\Delta x$; (*b*) 3; (*c*) 3. **3.** (*a*) $10x\,\Delta x + 5\,\Delta x^2$; (*b*) $10x + 5\,\Delta x$; (*c*) $10x$. **5.** (*a*) $6x\,\Delta x + 3\,\Delta x^2 - 2\,\Delta x$; (*b*) $6x - 2 + 3\,\Delta x$; (*c*) $6x - 2$. **6.** (*a*) $2(x+3)\,\Delta x + \Delta x^2$; (*b*) $2(x+3) + \Delta x$; (*c*) $2(x+3)$.

7. (a) $6(3x - 1)\Delta x + 9\,\Delta x^2$; (b) $6(3x - 1) + 9\,\Delta x$; (c) $18x - 6$.
9. (a) $(-6x^2\Delta x - 6x\,\Delta x^2 - 2\,\Delta x^3)/[x^3(x + \Delta x)^3]$;
(b) $(-6x^2 - 6x\,\Delta x - 2\,\Delta x^2)/[x^3(x + \Delta x)^3]$; (c) $-6/x^4$.
10. (a) $[-2(x + 1)\Delta x - \Delta x^2]/[(x + 1)^2\,(x + 1 + \Delta x)^2]$;
(b) $[-2(x + 1) - \Delta x]/[(x + 1)^2\,(x + 1 + \Delta x)^2]$; (c) $-2/(x + 1)^3$.
11. (a) 64.4 ft. per sec.; (b) 161 ft. per sec. **13.** -3.
14. (a) $3x - y = 1$; (b) $x + y = -1$. **15.** (a) $x - 2y = 1$; (b) $x - 4y = 14$.

Exercise 58

1. 0. **2.** 3. **3.** $1 - 1/x^2$. **5.** $-3/x^4$. **6.** $(-5/2)x^{-7/2} - 30x^{-11}$.
7. $\sqrt{5x}/2x$. **9.** $30x^9 - \sqrt{x}$. **10.** $\frac{6}{5}(6x - 9)^{-4/5}$. **11.** $1/\sqrt{2x} + 2x^{-7/5}$.
13. $(6x - 5)/(2\sqrt{3x^2 - 5x})$. **14.** $-\frac{7}{2}(7x)^{-3/2}$. **15.** $(3x + 1)/\sqrt{2x + 1}$.
17. $(1, -\frac{11}{6})$, $(-\frac{3}{2}, \frac{27}{8})$. **18.** $8x - y = 6$. **19.** $(-\sqrt{2}, 5\sqrt{2}/2)$, $(\sqrt{2}, -5\sqrt{2}/2)$.

Exercise 60

1. 25 by 25 yd. **2.** 10 by 10 rods. **3.** 100 by 75 rods. **5.** $2h^3/27$ cu. in.
6. $3\sqrt{2}$ in. **7.** 4. **9.** $(5^{2/3} + 4^{2/3})^{3/2}$ ft. **10.** $10\sqrt{3}/3$ by $10\sqrt{6}/3$ in. **11.** $w = 2h$.
13. $d = 2h$. **14.** 3. **15.** $\frac{20}{3}$ miles.

Exercise 61

1. $\dfrac{x^4}{2} - x^3 + \dfrac{5x^2}{2} - 4x + C$. **2.** $2x\sqrt{5x}/3 + C$. **3.** $2\sqrt{3x} + C$.

5. $\dfrac{25x^2}{2} + 10x^3 + \dfrac{9x^4}{4} + C$. **6.** $3x\sqrt[3]{2x}/4 + C$. **7.** $2\sqrt[4]{24x^3}/3 + C$.

9. $(2x)^6/12 + C$. **10.** $\dfrac{x^2}{2} - x^3 + \dfrac{3x^4}{4} - \dfrac{x^5}{5} + C$. **11.** $-2\sqrt{1 - x} + C$.

13. $8(5x^3 + 1)^{5/4}/75 + C$. **14.** $3(2x^2 - 3)^2/8 + C$.

15. $\dfrac{x^8}{2} - \dfrac{16x^5}{5} + 8x^2 + C$. **17.** $2(x^3 + 7)^{2/3} + C$. **18.** $4\sqrt{x^2 + 5x - 2} + C$.

19. $9(x^3 - 7x + 1)^{5/3}/5 + C$.

Exercise 62

1. $\frac{8}{3}$. **2.** $\frac{15}{4}$. **3.** $34\frac{1}{6}$. **5.** $-\frac{26}{3}$. **6.** 8. **7.** $\frac{23}{2}$. **9.** $4(4 - \sqrt{2})/3$. **10.** $\frac{7}{3}$.
11. $2\sqrt{2} - 1$. **13.** $4\pi r^3/3$. **14.** $\pi r^2h/3$.

Exercise 63

1. $\frac{1}{10}$; $\frac{1}{5}$. **2.** $\frac{1}{18}$. **3.** $\dfrac{2}{7{,}000}$. **5.** $\frac{1}{50}$. **6.** $\frac{1}{5}$; $\frac{2}{15}$. **7.** (a) $\frac{3}{8}$; (b) $\frac{3}{8}$; (c) $\frac{1}{8}$.

9. $\frac{1}{14}$. **10.** (a) $\frac{1}{3}$; (b) $\frac{1}{9}$; (c) $\frac{1}{27}$; (d) $\frac{19}{27}$. **11.** (a) $\frac{1}{6}$; (b) $\frac{1}{36}$; (c) $\frac{5}{36}$.

13. (a) $\frac{5}{408}$; (b) $\frac{5}{68}$; (c) $\frac{35}{136}$. **14.** $\dfrac{3}{10{,}000}$. **15.** $\frac{19}{100}$. **17.** $\frac{19}{27}$.

18. $1 - \left(\dfrac{139{,}999}{140{,}000}\right)^{70}$. **19.** (a) $\frac{1}{5}$; (b) $\frac{9}{25}$.

Exercise 64

1. \$20. **2.** $\dfrac{31{,}243}{100{,}000}$. **3.** $\dfrac{74{,}985}{82{,}551}$.

5. $\left(\frac{57{,}917}{69{,}804}\right)^2$. **6.** $\frac{(85{,}441)^2(107{,}029)}{(92{,}637)^3}$.

7. $\frac{(21{,}588)(85{,}441)^2}{(92{,}637)^3}$. **9.** $\frac{31{,}235}{69{,}804}$.

10. $(749)^3(10)^{-15}$. **11.** $\frac{(7{,}649)(8{,}302)}{(81{,}822)(85{,}441)}$. **13.** .706512.

14. $\left(\frac{24{,}832}{74{,}173}\right)\left(\frac{78{,}106}{92{,}637}\right)$.

Exercise 66

3. 7. **5.** 2; 4; 3; 1. **6.** 2; 4; 1; 2; 4; 1. **7.** 1; 3; 2; 4. **9.** 2,311; 30,031; 510,511. **10.** 2; 9. **11.** 3. **13.** 2; 7. **15.** 301. **17.** 103.

Exercise 67

1. 720. **2.** 358,800. **3.** 3,024. **5.** 32,805. **6.** 151,200. **7.** 725,760. **9.** 300. **10.** 360,360. **11.** 4,800. **13.** ${}_{50}C_3 \cdot {}_{47}C_3 \cdot {}_{44}C_3 \cdot {}_{41}C_3$. **14.** (*a*) 9! 10! **15.** 181,440. **17.** 150. **18.** 36. **19.** 12. **21.** 144.

Exercise 68

5. $2{,}048 - 5{,}632x + 7{,}040x^2 - 5{,}280x^3$. **6.** $\frac{x^8}{256} + \frac{x^7}{8} + \frac{7x^6}{4} + 14x^5$.

7. $\frac{64}{x^6} - \frac{48}{x^4} + \frac{15}{x^2} - \frac{5}{2}$.

Exercise 70

1. (*a*) (1, −1, 1); (*b*) (1, −1, 1); (*c*) (5, −5, 5); (*d*) (0, 0, 0).
2. (*a*) (2, 1, −2); (*b*) (4, −1, 0); (*c*) (5, −2, 1); (*d*) (2, 1, −2)
6. (*a*) $-\frac{5}{2}$; (*b*) Infinity; (*c*) $-\frac{9}{2}$; (*d*) 0; (*e*) $\frac{1}{2}$; (*f*) $-\frac{5}{2}$.
7. (*a*) $5x + 2y - 3z = 0$; (*b*) $x - z = -1$; (*c*) $9x + 2y - 7z = -4$; (*d*) $y + z = 2$; (*e*) $x - 2y - 3z = -1$; (*f*) $5x + 2y - 3z = 0$. **9.** (*a*) $4\sqrt{57}/19$; (*b*) $\sqrt{3}/2$; (*c*) $6\sqrt{201}/67$; (*d*) $\sqrt{3}/2$; (*e*) $3\sqrt{21}/14$; (*f*) $4\sqrt{57}/19$.
10. (*a*) $(14 + 7n, -14 - 6n, 5 + n)$; (*b*) $(2 - n, -2 + 2n, 2 - n)$; (*c*) $(-6 + 2n, 12 - 7n, -11 + 9n)$. **11.** (*a*) $(1 + 3n, -1 - n, 2 - n)$; (*b*) no integral solutions.
13. (*a*) $(n, 1 - n, n)$; (*b*) no integral solutions.
18. $[k - n, -k + (3n^2 + 3n)/2, k - (3n^2 + n)/2]$.
19. $(p + 1, -p, 0)$; $(-p, p - 1, 2)$; $[\frac{1}{2}(p + 3), \frac{1}{2}(1 - p), -1]$; $[-\frac{1}{2}(p + 1), \frac{1}{2}(p - 3), 3]$; $[\frac{1}{2}(p + 3), \frac{1}{2}(p - 3), -p + 1]$; $[-\frac{1}{2}(p + 1), \frac{1}{2}(-p + 1), p + 1]$; $(p + 1, p - 1, -2p + 1)$; $(-p, -p, 2p + 1)$.
22. $3n^2 + 4n + 1, 2n + 1, 3n^2 + 3n + 1$.