The Teachers' Library

Edited by

F. W. Westaway

The Teaching of Arithmetic and Elementary Mathematics

BLACKIE & SON LIMITED
17 Stanhope Street, GLASGOW
50 Old Bailey, LONDON

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Warwick House, Fort Street, BOMBAY

BLACKIE & SON (CANADA) LIMITED
TORONTO

The Teaching of Arithmetic and Elementary Mathematics

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BLACKIE & SON LIMITED
LONDON AND GLASGOW
1932

Printed in Great Britain by Blackie & Son, Ltd., Glasgow

PREFACE

The volumes of *The Teachers' Library* have been planned for the guidance of teachers whose daily work is concerned with children of eight and upwards. The teacher's personal responsibility is now greater than it has ever been, and the general demand that the work of the school shall be linked up more and more closely with the future work of the child is becoming increasingly clamant. This necessitates a clearer understanding of the special needs of individual pupils; and in order to cope adequately with the many consequential problems that will thus confront him, the teacher will find it necessary not only to revise many of his working principles and much of his practice, but to strengthen his professional equipment. Confidence is felt that in all these ways the volumes will be of great service; they have not been written for the educational theorist, but for the teacher enmeshed in practical difficulties.

The writers of the different volumes bear names well known to the majority of teachers. They have long been recognized as experts in their respective departments, and their opinions and advocacy of particular principles and methods are the fruit of their successful experience in teaching. All the writers have purposely devoted themselves mainly to the practical side of their subjects, and have touched upon theoretical considerations only very lightly. For principles of a more general kind readers may supplement these volumes by reference to such a book as Ward and Roscoe's *The Approach to Teaching*, and to the works of Professor Sir Percy Nunn and Professor Sir John Adams.

Differences of opinion among the authors are not numerous, but there are some. This is of no consequence—indeed, it is something of an advantage for two opposing suggestions sometimes to be made.

Contributors have not been altogether consistent in their references to the various types of schools, but this will cause no difficulty to readers. The three principal grades of Elementary Schools in the future will be determined by fairly clear-cut age ranges. The names now in most common use are:

(*a*) Up to 8 years—Infants, Kindergarten.
(*b*) 8–11 years—Junior, Preparatory.
(*c*) 11–15 years—Senior, Central.

With the first of these *The Teachers' Library* is not concerned. Greater stress has not unnaturally been laid on the third group than on the second, though an adequate treatment of the various subjects has necessitated the giving of considerable attention to junior work as well as to senior.

F. W. W.

April, 1932.

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CHAPTER I

Some Preliminary Considerations

There is probably no subject in the whole school curriculum that tests so finely the teacher's capacity as does the teaching of arithmetic and elementary mathematics. There is certainly no subject in which it is easier for the teacher to make not merely mistakes but disastrous mistakes in teaching method, and yet be in ignorance of them. The teacher of mathematics who goes blindly on, unconscious of the error of his ways, may not only be failing to do anything of value so far as real mathematical education is concerned, but may even be doing a great deal of harm.

Let then the teacher who must teach arithmetic, or who from choice specializes therein, realize that he is undertaking the teaching of a subject which will test his powers to the utmost. Let him bear in mind that there is no master less respected than the bad mathematical master and no subject so disliked as badly taught mathematics. Let the teacher of arithmetic and elementary mathematics remember that from infancy—from the " this-pig-went-to-market " stage of counting units—his pupils have been interested in calculation and computation; that long before man invented for himself the simplest of methods for recording his thoughts he kept his tallies upon notched sticks, and by the use of his fingers he made his simple calculations; and that even the dullest child has an instinct for figures and delights in calculating and computing. Let the young teacher remember that badly prepared lessons in arithmetic can be as ruinous to the pupil's almost natural appetite for numbers as a badly cooked meal can be ruinous to his appetite for food. Above all let the teacher realize that any evidence of a dislike for the subject on the part of the pupil or of a lack of self-confidence is a sure indication that something is wrong, not necessarily with the pupil. The teacher would do well to examine his methods.

Particularly important is it that the teacher should avoid falling into the error, too frequently made, of assuming that the main qualification for teaching either arithmetic or elementary mathematics is a knowledge of the subject. The mathematical genius is not necessarily the best teacher of his subject, as not a few undergraduates know to their cost. Let, therefore, the non-mathematician who aspires to become an efficient teacher of arithmetic take courage. Provided that at the outset he recognizes that the subject is one which, as already stated, will test his teaching capacity to the utmost, and provided also that he is prepared to give to his task all the care and thoughtful preparation it demands, there is no reason at all why he should not become a successful teacher of his subject.

The Teacher's Qualification

First and foremost, the teacher must have a thorough understanding of child psychology, and ever keep in mind that he is concerned with the child first and arithmetic afterwards. He must guard against that most fatal of mistakes, viz. the dogmatic forcing of abstract rules which in the end leads to nothing more than the acquirement of knowledge by rote learning. In the teaching of geometry this danger is now fully realized. Euclid and learning by rote have been abandoned in favour of geometry and learning by experience.

The teaching of algebra is still largely divorced from experience, whilst the tendency to rush children, even in the kindergarten, on to symbolic arithmetic, and the still more prevalent tendency to devote too much time in the junior school to written arithmetic to the exclusion of what is termed mental arithmetic, give evidence of the fact that the subject is being considered more important than the pupil. The teacher armed with a sound knowledge of the underlying principles of child psychology will plan his work and adapt his methods to suit the mental development of the child. He will recognize that the development of confidence and self-reliance which follows the thorough grasping of arithmetical truths gained by the pupil's own powers of observation and reasoning, initiative, and inventiveness, is of far more importance than mere mechanical skill in figure manipulation which may be obtained by the pupil's blind acceptance of abstract rules.

In the second place, the teacher of arithmetic should be acquainted with the history of his subject. As a student of psychology he will be prepared to give due attention to the fundamental principle that the individual recapitulates in his own development the essential phases through which the race has developed. This principle is readily accepted as being biologically true. Its application to educational theory is of the utmost importance in the teaching of arithmetic, particularly in the early stages. The task of deciding the length of time to be devoted to the various phases of development is undoubtedly a difficult one, but the fact remains that not only will the teacher of arithmetic find a knowledge of the history of the subject intensely interesting but extraordinarily stimulating to his own teaching methods. He will find that acquaintance with the history of the development of the subject provides him with a sound guide in planning his syllabuses, and will give him an unexpected insight into the difficulties which confront his pupils from time to time.

The teacher's own personality is a very important factor in the equipment that goes to make the successful teacher of any subject in the curriculum. It is possible to argue that the degree to which a teacher's personality determines his success varies with different subjects. This is not the place to discuss such a question, but the writer is convinced that the personality of the teacher of arithmetic does play an enormous part in the success with which the subject is taught. It is necessarily so, for it is doubtful whether any boy will develop any real liking for the subject until he feels that he is becoming self-reliant. It is in this development of self-reliance and self-confidence and the encouragement of initiative that the personality of the teacher plays such an important part. The same set of boys who under one teacher will almost loathe the arithmetic lesson, will under another become keen and alert and will look forward with pleasurable anticipation to the daily lesson. To be a successful teacher of mathematics one must be not merely keen, alert, and enthusiastic, but must be, as it were, contagiously so. Added to this the teacher must possess ample patience and ever be ready to give sympathetic encouragement.

It follows from the foregoing that to be a successful teacher of arithmetic one must possess a genuine liking for the subject itself. It is not unusual to find successful mathematical masters who prefer teaching algebra and geometry rather than arithmetic: the result possibly of the vicious circle in which many otherwise successful teachers work. As a result of being taught the subject badly in their own school days they developed no enthusiasm for it, and, in their turn, having no enthusiasm for the subject, they teach it badly.

To such the writer would say—learn to appreciate that there is a vast difference between the dogmatic imparting of rote-knowledge and teaching for the purpose of developing, through experience and experiment, that sense of confidence and self-reliance which purposely has been strongly stressed above. It is the mechanical and abstract treatment of arithmetic which for many produces an utter distaste for it—a distaste which persists through life, and if the mathematical master finds he is cursed with this distaste he should make every effort to ensure that he is not responsible for the development of a like distaste in his pupil. Further, he should remember that, as a mathematical master, algebra is but generalized arithmetic, and to a very large extent, and particularly to the young pupil, geometry is applied arithmetic. It is therefore in his interest as a would-be successful teacher of mathematics that he develop a strong liking for his embryo subject.

CHAPTER II

Guiding Principles in the Teaching of Arithmetic

There are certain guiding principles in the teaching of arithmetic that every teacher should appreciate thoroughly. "What shall I teach?" and "How shall I teach it?" are questions which must be answered satisfactorily by every would-be successful teacher, whatever the subject. One way of settling such questions, so far as arithmetic is concerned, is of course by adopting a good textbook and "working through" it. Few teachers, however, who aspire to teach the subject satisfactorily will adopt blindly even some strongly recommended textbook without satisfying themselves as to the principles which should guide them in the work they are undertaking. "One of the distinguishing features of the work in the best European schools," says an eminent American professor of mathematics, "is the freedom with which the teacher omits matter from the textbook." Where this exercise of freedom is born of a desire to act, not blindly, but in accordance with accepted guiding principles, it is safe to assume that the teaching will be full of life and vigour.

What may be termed the three major guiding principles in the selection of material and method of presentation concern: (*a*) why we teach arithmetic; (*b*) what to include in the course of arithmetic we propose to teach; (*c*) when to teach what has been included in the course.

The Routine, Scientific, and Creative Stages

The first principle to be laid down is that the study of arithmetic should be regarded, so far as ordinary school life is concerned, mainly as the acquirement of a technique, and that the study of underlying scientific principles is pursued chiefly for the purpose of perfecting this technique. We recognize this very readily in what are termed purely practical subjects. Our handicraft master concentrates primarily on the mastery of tools and materials, recognizing first of all the routine stage through which the pupil must necessarily pass. The teacher does not, at this stage, weary the boy with the scientific principles of the saw, the advantages of the saw over a knife, and so on. " Use it thus, and not thus," says the master, and the boy makes his first attempt to become proficient in the use of the tool and is absorbed in the routine of his task. Later, and how much later will depend upon the mental development of the boy, the master will direct the boy's attention to the mechanical principles of the saw—the wedge-shaped teeth and their peculiar setting—and he will talk to him about the end grain and the cross grain of the wood, and so on.

Armed now with knowledge of the scientific principles governing the use of the tool, the boy gains confidence in himself and perfects his use of the instrument. This second stage in the development of his power over the tool, the logical, the scientific stage, arises out of the knowledge gained in the routine stage and perfects the technique. When he is entering the routine stage it will be demonstrated to him that pressing the saw into the wood, " jabbing " it or forcing it in any way, will make the implement impotent. The master demonstrates the firm but gentle stroke that is required. The dull, unintelligent, or even inattentive pupil may soon meet with disaster; at the best he fails to get the tool to work. He learns almost by experience and as he gains experience he improves his technique. The keen intelligent boy, however, meets no disaster. He carries out the instructions thoughtfully. He may even inquire " Why must it be done this way, why a firm gentle stroke?" and so on. In other words the boy enters the scientific stage almost simultaneously with his entry into the routine stage.

Carry the inquiry farther. We know that the first boy by virtue of superior manipulative skill and dexterity may outstrip the latter in the mastery of the technique, and there is every prospect that in time he will appreciate the advantage of knowing more and more of the scientific principles and the extent to which such knowledge aids him to a more complete mastery of his tools. In other words he is the type of boy who may develop into an intelligent, thoughtful, practical mechanic. The second boy, however, may become more and more absorbed in the scientific principles—he has exhausted his interest, it may be, in the routine stage —and he finally enters a third stage, wherein having at least satisfied himself that he can master the use of the tools, or having made the mastery thereof *automatic*, and having also grasped the scientific principles underlying the operation, he applies this scientific theory to creative work. He appreciates to the full what the tools can be *made* to do—he designs, he creates—he is now no mere mechanic but an engineer.

In the acquirement then of the mastery of any activity, we have these three stages: (1) the routine; (2) the scientific; and (3) the creative.

The above illustration has been set out at length because of the importance of the principle involved when applied to arithmetic. *It is most essential that the teacher should remember that in all mathematical work—and particularly in arithmetic—the pupil is acquiring mastery in a certain sphere of activity.* Each new arithmetical rule is to be regarded as a tool. There will be first of all the routine stage during which the boy is gaining, as it were, dexterity in the use of his new tool. Either at the time of his introduction to this new rule or soon after, or it may be even at a much later stage, he studies the scientific principles underlying the new rule, and this knowledge helps him to perfect the mastery of his new tool. And the deeper he penetrates into the third stage, the creative stage, the more he develops into the mathematician who may be termed the artistic or creative arithmetician.

It will be seen readily that the acceptance of this guiding principle will enable the teacher of arithmetic to determine what to teach and what to omit in what is usually called arithmetic—a point which will, of course, be considered when we come to discuss syllabuses.

Stages in the Development of the Individual

The second major principle to be recognized is one which in a sense follows from the first. It is that each individual passes through three well-marked phases in his development from childhood through adolescence to manhood, corresponding to the three stages mentioned above. *Childhood is essentially the period of full activity—the routine age; adolescence essentially the period of inquiry—the logical age; manhood the period of application—the creative age.*

The recognition of these three stages can best be emphasized by the recital of certain well-known facts. We all remember, for instance, how much more readily we were able to learn by heart during our childhood than later in life, and every experienced teacher of arithmetic will agree that the pupil who has not been thoroughly drilled in " tables " prior to the age of twelve is for ever after handicapped so far as accuracy of working is concerned. It is during childhood that mental energy is directed chiefly to routine work. Again, our own experience teaches us that during the age of adolescence there is a steady decline in accuracy; the mind seems to be losing its interest in the skill acquired during the routine period. What is really happening is that the pupil is passing out of the period of sense impression, with consequent lessening of his interest in the world without, into the period of feeling and emotion with consequent increase in his interest of the world from within.

It is the common experience of every secondary school mathematical master that whereas boys of thirteen upwards are ready to take a keen interest in algebra and geometry, their interest in arithmetic steadily declines. It is true that this may be due in part to bad teaching during childhood, for there can be no doubt "that the mechanical abstract treatment of arithmetic which has been so common in the past has produced for many a distaste for the subject which has persisted throughout life".[1] But this does not account for the fact that many pupils who up to the period of early adolescence have shown a keen interest in arithmetic almost suddenly develop a dislike for the subject—or at least take but a placid interest in it whilst at the same time developing a keen interest in well-taught algebra and geometry. The phenomenon can only be explained by the fact that the pupil has left the routine age and entered the rationalizing, systematizing age, and that more interest is being found in the logical and scientific development of the subject to the subordination of the interest in mere mechanical work.

Acceptance of the first of the major guiding principles here set forth will help the teacher to determine what to include and what to omit in his syllabus. The acceptance of this second guiding principle will further assist him in the same task by emphasizing that there is a period for maxi-

[1] Hadow Report, *The Education of the Adolescent.*

mum attention to routine work and a period when more generalized arithmetic and the underlying principles of the subject will make the greater appeal.

Why we teach Arithmetic

The third guiding principle is this: *The teacher of arithmetic as a teacher of mathematics should ever be mindful of the fact that his task is mathematical education—not the education of mathematicians.* In other words the task is, as has already been said, not so much to teach arithmetic as to teach boys. The recognition of this guiding principle will enable the teacher to appreciate the fact that arithmetic is included in the curriculum partly because of its usefulness in daily life.

"It is desirable that much of the traditionary arithmetic of the schools should be replaced by new material, especially such as is necessary for the intelligent comprehension of some of the problems of our everyday life."[1]

At the same time arithmetic is included for what is termed cultural reasons—because of the training it gives to the mind in reasoning and logical thinking; in habits of application and accuracy; in exactness and conciseness of statement. "The assertion that a statement is 'mathematically exact' is not without meaning, and the habit acquired under some helpful, sympathetic, inspiring teacher, of setting forth the work in arithmetic neatly, clearly, and with no superfluous labours is one of those mental acquisitions that may easily 'carry over' into the ordinary work of practical life."[2]

Without entering into a discussion of the claims of arithmetic and elementary mathematics to be considered as either cultural or utilitarian, the teacher of arithmetic should take it that the subject is included in the curriculum both for the mental discipline it imparts and because of its usefulness. There is hardly any subject of study which has not educational,

[1] The Hadow Report, *Suggestions on Teaching Elementary Mathematics.*
[2] Professor David Eugene Smith.

and therefore cultural, value, so long as it is treated educationally. *Let it therefore be recognized at the outset that arithmetic is taught because of its usefulness*, and that its value from the point of mental discipline and the training that it gives the mind in reasoning processes, the habits of application it inculcates and the training it gives in clearness and conciseness of expression, depend on how the subject is taught. Do not let the boy think he is being taught arithmetic because it is considered to be good for him. This is the surest way of getting the average boy to dislike the subject. Let the boy feel that he is working with a subject that is useful to him. It is quite true, of course, that " arithmetic has been too long dominated by the traditional utilitarian value of the subject ". But as the same authority points out almost immediately—" Our modern industrial system with its complex ramifications and the part played by science in the modern civilized community make greater demands upon the mathematical knowledge of the ordinary citizen." In other words to keep in mind that we teach the subject because of its usefulness in modern life is not to emphasize its utilitarian nature, but rather to give due regard to the dual purpose for including it in the school curriculum. Let us then quite unashamedly teach the subject, and let the boy feel he is learning the subject, because of its usefulness; but at the same time let us keep in mind that its value from a cultural point of view depends upon what we include as being useful and the manner in which we present it.

The Place of Mental Arithmetic

From the earliest the pupils should be encouraged to " think arithmetically ". For this reason written work should not be introduced too soon in the junior or infant school, and the bad old practice of regarding mental arithmetic as a subject distinct from arithmetic should never be adopted. So-called mental arithmetic should be regarded to a very large extent as bearing the same relationship to arithmetic as oral composition bears to language teaching, and it might very well be called oral arithmetic. See p. 15.

The Syllabus

What was said above regarding arithmetic as a useful subject should be the guiding principle in drawing up syllabuses at all stages. In the early stages the syllabus should be modelled on the principle of what will be wanted for later stages; and the syllabus for the later stages should be modelled on the two-fold principle of (*a*) making the subject useful from the point of view of citizenship, and (*b*) making the subject form a basis for other branches of mathematics and mathematical science.

Not only should such subjects as complicated fractions, recurring decimals, cube roots, and complicated work in practice, in H.C.F. and L.C.M., be entirely omitted from the syllabus, but throughout the course all purely artificial problems, such as are frequently found in compound proportion exercises, should be excluded. Even such problems as inverse simple interest, as finding the time taken for a sum of money to earn a given interest at a given rate per cent, though such may serve the purpose of emphasizing the fundamental work already done in the examples on simple interest itself, should not be regarded as seriously important.

This vigorous elimination of non-essentials from the arithmetic is not to be interpreted as an attempt to reduce the amount of time to be devoted to the subject. Rather it is a question of the change of emphasis, giving more time and practice to the really essential work. "When we consider the necessary work in arithmetic," says Professor David Eugene Smith, "we are struck by its simplicity and brevity. When we think that this is all that the world usually demands of the school and that we are allowed eight years in which to impart this knowledge, we are led to ask ourselves why the world is not satisfied with the results. Is the difficulty with ourselves in that we include a lot of matter of relatively little value, but which consumes the time without any just return? Or do we fail to insist on the fundamentals while we are teaching the more advanced topics that find place in our schools?"

Whether the world is or is not dissatisfied with the results attained after eight years' teaching, experienced teachers know that the inclusion of matter of relatively little value is not only unsatisfying to the pupils but destructive of their interest in the subject. At the same time the neglect of fundamentals tends to destroy pupils' self-confidence.

The guiding principle in drawing up the syllabus, therefore, should be: give the fundamentals proper emphasis and rigorously exclude the relatively unimportant.

Teaching Method

Inasmuch as the development of initiative and self-reliance is one of the most important aims in mathematical education, the teacher should endeavour always to play the part of a sympathetic guide. Dogmatic teaching of the " do-it-this-way " type will lead to unintelligent and unresponsive imitation. At the same time extreme " Daltonian " methods may tend to destroy self-confidence. A judicious blending of intelligent imitation and independence of effort, resulting from happy co-operation between the pupil and the teacher, will ensure that interest is maintained, but not at the expense of the development of self-reliance.

The application of this principle of co-operation between teacher and taught implies (*a*) that arithmetical facts should be gained by direct appeal to the senses, that is, through concrete experience, and (*b*) that before any new rule is introduced the pupil should feel the necessity for it rather than have it imposed upon him. Every experienced teacher knows that you cannot force a child to co-operate. The teacher must secure natural and willing response from the pupil, and he can only do so by appealing to sense experience, and by developing a consciousness of the necessity for, or advantage of, undertaking the work. To take an example: the first lesson on, say, long division should develop naturally out of an appreciation on the part of the pupils of the necessity for some method of dealing with computations that cannot be solved by the short division with which they are already familiar. If the necessity is properly appreciated the teacher may even find it advisable there and then to say: "Well look, this is how we do it, watch carefully!" He then proceeds to work the sum without a word of explanation, the pupils watching eagerly, if they are really alive to the *necessity* for something other than short division. "There's the answer —let's prove whether it's right," and the product of answer and divisor proves the correctness of the method. "Let us do another one," says the teacher. The machinery is started once more, the handle, as it were, is turned, and again the answer comes out at the other end and is proved correct. "Here's an easy one," at length says the teacher; "see if you can discover how I do it. Watch carefully as I go slowly." They are told that this is an invention thousands of years old and are asked to find out how it works. And having found out how it works, they proceed to use their new-found tool. In due course they will go into the question of why it works. This is what is meant by a judicious blending of intelligent imitation and independence of effort—of natural response awakened by a sense of necessity for something more than is already known.

Too often we confuse the minds of our pupils by forcing upon their attention arithmetical rules and explanations before they appreciate the need for the one or have any interest in the other. If co-operation is to be the keynote of the teacher's method in arithmetic, it must come from willing response and not from conscripted mental forces.

Marking and Correction

The interval between the execution of the work by the pupil and its subsequent marking and correction must be as short as possible. It goes without saying that no work must go unmarked and no work must go uncorrected. The correction by the pupil of work that is wrong must always be insisted on. The young teacher must never make the mistake that *his* work on the blackboard is more important than the pupil's in the exercise book. It must become a habit to mark all work promptly and thoroughly and to insist upon " corrections ".

This is not to say that the teacher himself must do all the marking. Tests must, of course, be marked by the teacher, at least in all but the more advanced forms. The pupils should be trusted to mark their own work. The few who cannot be trusted will soon be discovered if tests are carefully marked, and they will soon learn that dishonesty does not even pay. Further, unless the pupils are trusted to mark their own exercises, it will be impossible to allow pupils to make their own pace.

Pupils, of course, should be encouraged to check all their working, and wherever possible to " rough check " answers, proving their correctness. Here again if corrections are insisted upon pupils will soon form the habit of checking all their own work. Nothing contributes more to development of self-reliance than the feeling that what has been done is certainly correct.

CHAPTER III

Mental or Natural Arithmetic

Reference has already been made to the importance of what is usually termed mental arithmetic, a part of arithmetical education which is either almost wholly neglected or is put in its wrong place.

It is by no means an uncommon experience to find whole forms doing neat careful work in arithmetic and yet displaying nothing but timidity and hesitation when dealing with mental problems suitable to the age. The prevalence of the use of "scrap paper" and the large amount of figuring which fills the margins on so many arithmetic papers give ample evidence of many a pupil's lack of self-confidence in the matter of arithmetical calculations which should be done mentally and quite naturally.

Now, why is it that the majority of teachers either neglect this aspect of arithmetic altogether or assign to it a comparatively unimportant part in the normal lesson, using it either as a stepping-stone to more difficult examples which will require written work or utilizing it for a purpose of mental gymnastics? The reason is not far to seek. In this as in other phases of school organization we are working in a vicious circle. The teacher of to-day was "brought up" to regard written arithmetic as the more important and as such he presents it to his pupils. The skilful teacher, getting nearer the true function of mental arithmetic, devotes the first portion of a lesson to easy examples which will make clearer the new process he is about to teach. In both cases, however, mental arithmetic is regarded as useful merely so far as it is an aid to written arithmetic.

Now neither of these practices is to be despised. Each serves a useful purpose but neither recognizes the real function of either mental or written arithmetic. To devote a portion of each lesson to such so-called mental arithmetic is mentally stimulating, and serves the very admirable purpose of making the pupils keen and alert and of helping to perfect their knowledge of, and to facilitate their use of, the arithmetical facts known to them. It is a form of mental drill which undoubtedly serves a most useful purpose. But treated as an end in itself or as something more or less distinct from written arithmetic its true function is being misunderstood.

Again, when it is used as a means for preparing the way for the introduction of a new rule, this so-called mental arithmetic without doubt fulfils a very important function. It ensures that the pupil is concentrating all his attention on the real work in hand, suffering no distraction on account either of difficult computations, or of even the use of pen and paper. Problems are deliberately chosen which involve the minimum amount of " working " in order that the process itself shall not be confused. Paradoxical as it may seem, it is in this latter use of so-called mental arithmetic that we are getting nearer to its real function, even though the amount of purely arithmetical working involved is considerably less than in examples given as mental drill.

Real Function of Mental Arithmetic

What then is the real function of mental arithmetic, and what relationship does it bear to the subject as a whole? The only answer that can be given is that the function of mental arithmetic is the function of arithmetic itself. *The distinction should not be between mental arithmetic and arithmetic, but between arithmetic and written or mechanical arithmetic.* What we call mental arithmetic is not merely the handmaid of real arithmetic but *is* real arithmetic. Primarily written arithmetic is the handmaid, that is to say, it comes in to assist the mind to carry out more expeditiously the particular problem under consideration. This is so historically. In the course of evolution man eventually reached a stage in his ability to compute when a mechanical aid became a necessity, and so we find, throughout the ancient and modern world, machinery invented for this purpose—some form or other of the abacus that we still use very largely in our kindergarten to-day. This was the first form of written or mechanical arithmetic. After this somewhat complicated and clumsy machine had been in use probably for thousands of years, we find that a symbolic representation of numbers was evolved. Later came the invention of place values for the figures, giving what has been termed a " graphical abacus ".

Thus, fundamentally all written arithmetic is only some form of mechanical aid to the real arithmetic, which is usually termed mental arithmetic. This is a fact of the first importance to the teacher of arithmetic from the kindergarten stage upwards. Not until we have so-called mental arithmetic adequately recognized as real arithmetic shall we have the subject properly taught.

Mental Multiplication

There is here room for bold experiment, particularly in the junior school taking pupils up to the age of eleven years. Are we sure that we do not underestimate the ability of the average pupil in mental computation, supplying to him the mechanical aids which written arithmetic offers before such are really needed? For example, if a boy of eight or nine years of age, of average intelligence, who has been taught no written arithmetic, but *who has really been taught to think in numbers as distinct from figures*, is asked the product of say 17 × 8, he will, if 17 to his mind suggests not merely a one and a seven, but ten and seven, quite naturally say 8 tens are 80, 8 sevens are 56, total 136. We do not unfortunately give our pupils of eight or nine a chance to develop this natural method of mental multiplication. Instead we take it for granted that they can only multiply 17 × 8 with the aid of pen and paper and carefully ruled lines. We even penalize the pupil who, using his native intelligence and refusing such props, writes down the answer. We refuse to accept his answer because he has not *shown* how he obtained it.

Now if a pupil of this age is capable of multiplying 17 × 8 by registering mentally first the number 80 and then 56 and then 136, is it not evident that in showing him how to do it on paper we are introducing him to written arithmetic before it is absolutely necessary? And are we quite sure that with practice our pupils could not mentally multiply 17 × 18? Most boys of twelve upwards, and for that matter most adults, would find it difficult to do so. And in our attempt what should we be doing? We should visualize the figures 17 and 18 placed under one another, that is we proceed to visualize the whole of the paper work. In other words we are not working mentally but we are visualizing the whole mechanical process. Had we been taught from our infancy to think in the language of numbers and not in symbols, we should have obtained the product by mentally registering the numbers 170 (i.e. 10×17), then 250 (i.e. $170 + 8 \times 10$), and finally 306 (i.e. $250 + 8 \times 7$).

As already mentioned, there is room here for bold experiment in the junior school. We do not know how far this natural mental method of multiplication is capable of development. We introduce our pupils far too early to the written mechanical methods, and the writer is inclined to think that we tend thereby to arrest the development of the mental processes.

We have, of course, the classic example of Bidder, the famous engineer, who, in his presidential address to the Institute of Civil Engineers in 1856, explained at length how it was that he was able amongst other feats of mental computation to give almost instantaneously the product of three-figure numbers. " You begin at the left-hand extremity and conclude at the unit, allowing one fact to be impressed on the mind at a time. Thus in multiplying 373 by 279, I know almost at once that the product is 104,067," and he went on to show that by mentally registering first 60,000 (200×300), then obliterating this in favour of 74,000 ($60{,}000 + 200 \times 70$) and so on, he rapidly arrived at the final product. *His arithmetic was nearly all self-taught.* " My first and only instructor was my elder brother, a working mason, who after teaching me to count to ten urged me to go to a hundred, and there he stopped. . . ." He became perfectly familiar, however, with numbers up to 100. " They became my friends and I knew all their relations and acquaintances," so much so that without knowing one figure or symbol or the word " multiply ", he taught himself all his tables up to 10 times 10 with the use of marbles. Bidder so developed his power of mental registration of numbers that he could even multiply twelve places of figures by twelve places, mentally.

He declares significantly: "*The reason for my obtaining the peculiar power of dealing with numbers may be attributed to the fact that I understood the value of numbers before I knew the symbolical figures. I learnt to calculate long before I could distinguish one figure from another.*" Bidder's gift is, of course, exceptional, but there are three important points worth noticing regarding his remarkable ability in mental computation, viz. (1) his arithmetic was self-taught and therefore the written mechanical processes were a closed book to him; (2) he learnt number values before symbolical figures; and (3) he developed to a remarkable degree the natural power of number registration and he maintained that the utility of mental processes only ceases when such mental registration becomes slower than registration on paper.

Do we then, in our junior schools, give sufficient attention to the development of this power of mental number registration? Does not the tendency to depend upon scrap paper, to fill margins with small sums in subtraction, addition, multiplication, and division, to work small "side" sums, to shirk addition and subtraction of figures arranged in horizontal lines as compared with the column arrangement and the frequency of mistakes in simple calculations, all tend to show that there is lacking in our work the self-confidence that mental proficiency alone can give?

The writer does not suggest that boys of twelve should be able to work multiplication sums of three figures by three figures mentally. He does not underestimate the value, at the right time, of written arithmetic and the importance of logically arranged written explanations. He does suggest, however, that the mechanical aids to mental processes should not be given until the pupil feels the need for them and can therefore fully appreciate their value.

Incidentally it is worth noticing that when the mathematical master comes to teach multiplication in algebra he insists on his pupils being able to *write down* the product of $(2x + 7)(3x + 5) = 6x^2 + 31x + 35$ and with practice the ability to do so is soon acquired. Who will say that the boy who can do this cannot mentally register multiplying 27 by 35, i.e. $(2 \times 10 + 7)(3 \times 10 + 5)$ with comparative ease? Bidder himself pointed out that his mental calculations were the same processes as used in algebra, but "fortunately for me," he says, "I began by dealing with natural instead of artificial algebra."

Mental Addition

The mental working of fairly simple multiplication has been dealt with at length because it illustrates more clearly the advantage of developing the habit of thinking in numbers and not in figures, but the same applies to other arithmetical rules. For instance, in adding numbers together every encouragement should be given to carrying out the process in the natural way of dealing in numbers. Thus, the stage having been reached when the pupil is able to add a series of numbers together, say $6 + 3 + 8 + 5$, the mental process should be 6, 9, 17, 22, *not* 6 and 3 are 9, 9 and 8 are 17, 17 and 5 are 22.

Then later in the addition of, say, 25, 47, and 62, the process should be 25 and 40 gives 65 and 7 gives 72 and 60 gives 132 and 2 gives 134. The actual numbers registered in the mind are 25, 65, 72, 132, 134, and with practice the intermediate numbers are omitted and the mind registers only 25, 72, and 134. *The important point is that the pupil all the time is thinking in numbers and not merely manipulating figures and symbols.*

Mental Subtraction

The same natural process should be encouraged in subtraction, and all the nonsense relating to " borrowing and paying back " abolished. Thus, the pupil who is called upon to subtract 37 from 185 will give the answer immediately as 148, arriving at it naturally by mentally registering first the number 155 (the difference between 185 and 30) and then 148 (subtracting the other 7). Or, to take a more difficult example, subtract 368 from 527. The numbers registered mentally are 227, 167, and 159. The pupil of course will be encouraged to check his answer by going through the reverse process of addition, registering the numbers 159, 459, 519, and 527.

In all such processes there is something mentally satisfying and invigorating. The pupil feels he is grappling with the problem as a whole and is depending upon his own natural ability. As the process continues so he gains confidence in himself, and this in turn stimulates him to proceed to the next step. This stimulation in its turn impels concentration and the work proceeds more accurately. There are no rules to remember, no "carrying" and such like. Compare all this with the totally different, largely unintelligible, and altogether unsatisfying piecemeal method of "8 from 7 I cannot, borrow ten—8 from 17 leaves 9".

Mental Division

Division, of course, is itself an entirely mechanical means of doing repeated subtraction, not a natural process at all. Thus in dividing 568 by 32, the pupil is really being taught a shorthand method of finding how many times 32 can be taken from 568, in other words, how many times the number 32 is contained in 568. Even so the pupil who has been encouraged to adopt natural and therefore mental methods in the other rules until such time as mental limitations require other and mechanical means, will secure a more intelligent grasp of this purely mechanical method of doing division. In the example given, for instance, he will have very little difficulty in appreciating that the first figure he obtains in the dividend is not 1 but the number 10, and that the complete answer will be between 10 and 20.

Conclusion

To prevent misunderstanding it should here be noted that, in all that has been said above, *abstract* numbers have been considered because it is the subject of computation and the relation of this subject to real mental arithmetic that is under discussion. In other words, only that stage in the solution of the concrete problem has been considered in which the computation of the actual numbers involved is for the moment the work in hand. The plea here put forward is for an appreciation of the right place of so-called mental arithmetic in the teaching of the subject as a whole, and in particular for more reliance on purely mental processes, acting through the subconscious mind, than on the more mechanical processes of written arithmetic. It must also be pointed out that the extent to which such natural mental processes should be encouraged to the exclusion of the artificial mechanical processes depends upon the mental ability of the pupils. A boy of average intelligence should not find it necessary to break off in the course of a problem to work a five-line sum to find the product of 28 and 37. It should not be necessary for such a boy to perform a long addition sum in order to find the total amount of such sums as £1, 3s. 6d.; 5s. 9d.; £2, 3s. 8d.; and £1, 6s. 5d. His mind is quite capable of registering in turns the amounts £1, 6s. 5d.; £3, 10s. 1d.; £3, 15s. 10d.; and finally £4, 19s. 4d. It is only a question of developing the habit of mentally registering numbers or quantities, relying upon self rather than upon the mechanical manipulation of figures. More should be done to encourage this habit of "thinking quantitatively" from the earliest years upwards. At present our capacity to do so is much underestimated, with the result that the majority of us go through life with an extraordinary want of confidence in ourselves in dealing with numbers and quantities.

To sum up then, so-called mental arithmetic is natural arithmetic, and therefore arithmetic proper. Written arithmetic is largely mechanical arithmetic—the arithmetic which gives to the mind the mechanical aid which it requires in order not to put undue strain upon mental energy. Mental arithmetic is here termed natural arithmetic, to distinguish it from symbolic or artificial arithmetic. It involves treatment of numbers rather than the manipulation of figures. On historical grounds it should come first. Written work should be delayed as long as possible, and when introduced should be largely for the purpose of recording the results of mental computation. On physiological and cultural grounds, too, this mental or natural arithmetic should precede written arithmetic, and should form the bulk of the work in the early stages of school life, for not only does it ensure a more effective storage of arithmetical facts to be used later, but it develops precision, accuracy, and above all self-reliance. Inaccuracy in arithmetic arises mainly from misapplied mechanical processes or inaccurate manipulation of figures. The more time is devoted to this natural arithmetic, the more familiar will the individual become with the fundamental facts, and the more accurate will computation become. It will be remembered that mention was made earlier regarding the decline in accuracy as the individual approaches and enters the adolescent period, and the surest way of combating this tendency is to make the mental arithmetical processes become more and more automatic.

Finally, on mathematical grounds, too, this natural arithmetic has the priority of claim. The more natural our arithmetical methods are, the more natural will be the transfer from arithmetic to algebra, i.e. to generalized arithmetic and to other branches of mathematics. The more we encourage our pupils to think numerically and quantitatively, the easier it will be for them to think mathematically. If we are to succeed in " developing in the pupils an appreciation of the meaning and teaching of a coherent system of mathematical ideas, and the realization of the subject as an instrument of scientific, industrial and social progress ",[1] we must, in the early years, regard the teaching of arithmetic as the first step in mathematical education, and get away from the mere manipulation of figures.

CHAPTER IV

Written or Symbolic Arithmetic

Consideration of written arithmetic is necessarily complementary to the preceding discussion on mental arithmetic. Historically the invention of symbols followed the purely mechanical aids such as the abacus. The assistance which such an invention rendered to man's intellect in the matter of computation ultimately led to the science of numbers which the Greeks called *arithmetic*, as distinct from their *logistic*, which was concerned entirely with numerical calculations: the fundamental four rules and their application to trade and commerce. The arithmetic of the Greeks was distinctly philosophic; it dealt with the science of numbers, which in time developed into the study of generalized arithmetic which we call algebra.

It should frankly be recognized that in our schools we are only to a very small extent concerned with the science of numbers, and that, for the most part, it is not until we enter upon the study of algebra that we are dealing with the subject from a purely scientific point of view. We are mainly concerned in teaching our pupils the use of the tools. Our written or symbolic arithmetic is a mechanical instrument invented for the purpose of assisting the mind to carry through mental computations more expeditiously and with an economy of effort.

[1] Hadow Report, *Suggestions on Teaching.*

Function of Written Arithmetic

It will thus be seen that what has been said previously regarding the importance of mental arithmetic is not intended to convey the idea that written arithmetic is relatively unimportant. On the contrary, the importance of mental arithmetic has been stressed in order that, the true relationship to written arithmetic being established, the important function of the latter will be the better appreciated. The more the true value of mental or natural arithmetic is appreciated, the easier will it be understood that the function of written and symbolic arithmetic is to enable computation to be carried out more expeditiously and without undue mental strain. It has been urged that pupils of a fair intelligence can add a series of two-figure numbers together by mentally registering first, the sum of the first two numbers, then the sum of that total with the third, and so on. It is further claimed that this is the natural process, that to do so causes no undue mental strain, and that with practice the method can be carried out quite as expeditiously and accurately as first adding all the unit figures and then the tens figures. The mental registration of three-figure numbers is, of course, more difficult, and hence we reach the stage, for some at least, when mechanical help is required. It may be contended, of course, that if ultimately the pupil has to adopt purely mechanical methods, he may as well get accustomed to them from the first. But apart from the question of the importance of encouraging young pupils in the habit of thinking in numbers and not in figures, do we not find, as the result of the over-emphasis placed upon written arithmetic, that there is a tendency to fill margins with unnecessary figuring or to depend largely on the use of scrap paper? Do we not also find that even with intelligent pupils the dependence on paper work is so habitual that not only is work done in the margins which might well be done mentally, but the longer mechanical processes are used to a ridiculous extent, even to dividing by 200, say, by long division? Do we not also find that pupils who have been encouraged to do all their arithmetic on paper, and discouraged even if they attempt to rely on mental process, find it extremely difficult to obtain an approximate answer to problems or "rough check" the answers obtained? Pupils to whom mental arithmetic is as familiar as written arithmetic will almost instinctively work out approximate answers before they attempt the written work, whilst their keen appetite for mental calculation tempts them to "rough check" not merely answers, but all their working. Intellectual sluggishness results from the deadly slow methods associated with excessive written work. It is a fair inference to make that pupils who produce such working have none of that self-confidence in their mental processes which it is the special function of the teaching of arithmetic to inculcate.

In the written arithmetic of the normal school life the pupil, then, uses those mechanical processes which have been devised to assist the mind in the solution of more difficult and more complicated problems. As the problems become more and more difficult, so the mechanical processes pass from the simple hand-tool variety, as it were, as in the fundamental rules, to those of a fairly complicated machine variety, as in the simplification of fractions and the use of logarithms. The teacher's problem is to assist his pupils to become proficient in the use of these tools and this machinery. Do not let it be thought that learning to use a machine necessarily produces a mere machine-like mind. Any novice can start, steer, and stop a car, but only the intelligent and thoughtful application of this knowledge will produce the expert driver. And the more intelligent and thoughtful the application, the more the scientific principles underlying the processes will be understood, and may even be extended.

Combination of Mental and Written Work.—In order to carry out his function of assisting his pupils to become proficient in the use of arithmetical tools and machinery the teacher must, with the eye of an expert, select the tools most suitable for the young craftsman and also the machinery which will prove the most efficient under the control of the young mechanic. Let us examine three types of machine performing the same work, that is to say, let us examine three arithmetical methods of solving the same problem.

Problem: Find the cost of excavating a V-shaped trench 120 yd. long at 2s. 6d. per cubic yard, given that the width of the top of the trench is 3 ft. 6 in. and the depth 2 ft. 4 in.

Method 1

Cross section of trench is a triangle:

$$\frac{3'\,6'' \times 2'\,4''}{2} = \frac{42 \times \overset{14}{\cancel{28}}}{\cancel{2}}$$

$$\begin{array}{r} 42 \\ 14 \\ \hline 168 \\ 42 \\ \hline 588 \\ \hline \end{array}$$

$$588 \text{ sq. in.} = \frac{\overset{\overset{49}{\cancel{196}}}{\cancel{588}}}{\underset{36}{\cancel{144}} \times \underset{3}{\cancel{9}}} \text{ sq. yd.}$$

$$= \frac{49}{108} \text{ sq. yd.}$$

$$\text{Volume of trench} = \frac{49}{\underset{9}{\cancel{108}}} \times \overset{10}{\cancel{120}} = \frac{490}{9} = 54\tfrac{4}{9} \text{ cu. yd.}$$

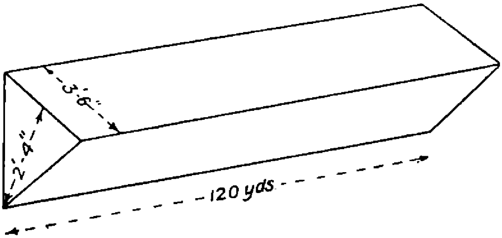
$$\text{Cost of excavation} = 54\tfrac{4}{9} \times 2\text{s. } 6\text{d.}$$

$$= \frac{\overset{245}{\cancel{490}}}{9} \times \frac{5}{\cancel{2}} \text{ shillings} = \frac{1225}{9} \text{ shillings}$$

$$= 136\tfrac{1}{9} \text{ shillings}$$

$$= £6,\ 16\text{s. } 1\tfrac{1}{3}\text{d.} \quad \text{Answer.}$$

Method 2



$$\text{Area of cross section} = \tfrac{1}{2}(3\tfrac{1}{2} \times 2\tfrac{1}{3}) \text{ sq. ft.}$$

$$= \frac{1 \times 7 \times 7}{2 \times 2 \times 3 \times 9} \text{ sq. yd.}$$

$$\text{Volume} = \left(\frac{49}{\cancel{108}_{9}} \times \cancel{120}^{10}\right) \text{ cu. yd.}$$

$$= \frac{490}{9} \text{ cu. yd.}$$

$$\text{Cost of excavation} = £\frac{\cancel{490}^{245}}{9} \times \frac{1}{\cancel{8}_{4}}$$

$$= £\frac{245}{36} = £6\tfrac{29}{36}$$

$$= £6,\ 16\text{s. } 0\text{d. approx.} \quad \text{Answer.}$$

Method 3

$$\text{Cost} = £\tfrac{1}{2}(3\tfrac{1}{2} \times 2\tfrac{1}{3}) \times \tfrac{1}{9} \times 120 \times \tfrac{1}{8}$$

$$= £\frac{1 \times 7 \times 7 \times 1 \times \cancel{\cancel{120}^{10}}^{5}}{\cancel{2} \times \cancel{2} \times \cancel{3} \times 9 \times \cancel{8}_{4}}$$

$$= £\frac{245}{36} = £6{\cdot}8 = £6,\ 16\text{s. } 0\text{d.}$$

The first method is typical of that adopted by the pupil who is too dependent upon paper work. In other words, it illustrates the results of written work being done to the exclusion of mental work, instead of aiding such work. It is an example of clumsy arithmetical method, showing a lack of precision, and involving a considerable waste of effort. It is neither business-like nor logical. Without attempting to criticize details which illustrate bad practices, such as unnecessary marginal figuring and the useless degree of accuracy in the final answer, it may be said that the chief fault of the method is that the written work is not fulfilling its purpose of merely assisting the mental process, but is acting as a substitute for practically all the mental work which the problem involves. To that extent it illustrates just what written work should not be.

The second method illustrates at almost every step that the written work is there to assist the natural or mental arithmetic. First the diagram serves the purpose of helping to concentrate on the problem as a whole. Such simple diagrammatic representation helps to make the problem more concrete, and encourages the pupil to consider the problem first as a whole, in order later to analyse it into its components. This analysis is clearly indicated. The cost will be the cost per cubic yard multiplied by the number of cubic yards in the volume excavated. The latter is then analysed into area of cross section multiplied by length. Mentally, then, the whole problem has been broken down into its several parts and now committed to paper to assist the mental arithmetic, (1) area of cross section, (2) volume, (3) cost. At each of these steps almost all the work is done mentally and there is no wastage of energy. All through, the written working is fulfilling its function of *assisting* the mental processes. The arithmetic is not figure manipulation, it is natural mental arithmetic aided by the machinery of written symbols.

The third method is what might be termed the practical business man's method, the work of the more expert craftsman. Here it will be noted that the mechanical paper work is reduced to a minimum. The problem as a whole is held in the mind without the aid of a diagram, and with the maximum of self-reliance the figures are, as it were, placed in at one end of the machine and the answer comes out at the other.

The second and third methods, then, both illustrate the real function of written arithmetic, and the question now is —which of these two methods is the teacher to encourage. The second method differs only from the third in so far as after the pupil has grasped the problem as a whole, the various steps in the solution of the problem are set out clearly and separately. The third method omits these statements and, proceeding direct to the necessary working, obtains the answer more expeditiously. Whilst, however, rapidity of working is, of course, much to be encouraged, particularly as it involves and therefore develops the habit of concentration of thought, it must never be gained at the expense of reasoning. The business man is concerned only to get his answer as quickly as possible, and he goes straight for it without any waste of effort. The teacher of mathematics, however, has to keep in mind that mental discipline is a very important part of his aim, and he must therefore take every opportunity to encourage clear logical reasoning expressed in equally clear and concise language. His choice of method, therefore, is determined by consideration of rapidity of working consistent with accuracy of working and mental discipline. To the question, which of these two latter methods is desirable? the answer is that the teacher should take the best of both methods. The written work should, for the most part, show the pupil's line of reasoning, and at the same time ensure accuracy and speed in operation. To require no analysis of the problem and no explanation at all of the mechanical work is to encourage looseness of reasoning. On the other hand, to exact too much detailed written explanation is to minimize the importance of rapidity and accuracy of working, if not to take the pupil's mind entirely from the main purpose in hand. In this connexion it should be noted that *mere labelling of numbers is not explanation*, and should be discouraged as being wasteful of time and serving no really useful purpose. The explanation should be a simple straightforward statement, clearly and concisely set out in logical sequence, of the steps involved in the solution of the problem in hand.

Again, the mind of the pupil should not be confused by insisting on forms of statements the exactness of which he does not comprehend. For instance, the statement "Area of cross section $= \frac{1}{2}(3\frac{1}{2} \times 2\frac{1}{3})$ sq. ft." is of course to be preferred to "Area of cross section $= \frac{1}{2} \times 3\frac{1}{2}$ ft. $\times\ 2\frac{1}{3}$ ft.", but it is unwise to bother young children about these finer points to the extent of expecting them to know *why* the former is more correct. If the concrete work on which the application of the formula for the area is based has been thoroughly grasped, the pupil will fall into the habit of expressing the product which gives the area in its exact form either by imitation or intelligent acceptance thereof. The aim should be never to allow the explanations to impede the progress of the real work in hand. On the contrary, they should assist the pupil to keep a more intelligent grasp of the problem as a whole. A very valuable form of exercise which assists pupils to appreciate a well arranged solution of a problem and sets for them a standard, is to require them to supply the question, given the solution.

Practice in what has been called the business man's method, which consists in most cases of giving the solution in one step, should be given as a special exercise for the purpose of emphasizing the importance of speed and accuracy.

Homework

A pupil's interest in the subject can easily be killed through want of care in setting mathematical homework. It is particularly necessary in arithmetic and mathematics that the teacher should remember that one of the main purposes in setting homework is the development of the pupil's sense of power and feeling of progress. Homework which merely supplements class-work without producing the legitimate feeling of self-satisfaction resulting from a knowledge of successful achievement, tends to become a drudgery. It is impossible to exaggerate the importance of this feeling of confidence. Even when teaching the dullest pupils, the key-note of the successful teacher's work is, encouragement. Self-confidence begets keenness, and keenness in mathematical work is invaluable.

All this should be kept in mind in setting homework or other forms of individual work in arithmetic. Neither in quantity nor quality should the homework tend to destroy the pupil's interest. Too much work will, of course, produce fatigue, and fatigue will tend to destroy interest. In this connexion it should be remembered that the conditions under which children do homework vary considerably. Some will have the advantage of a quiet room, others will be surrounded by distractions. It is a good plan to let pupils choose for themselves the number of examples to be done or to require as many as can be done in the allotted time. If the pupils have developed a real, keen interest in the subject, there need be no fear that individuals will misplace the trust thus reposed in them.

The degree of difficulty of the work set for homework demands the most careful consideration. As a rule the work so set should not be beyond the power of the average or less than average pupils of the form. There is no surer way of killing interest in the subject than in repeatedly setting homework which is beyond the capacity of the pupil. It may flatter and encourage the more brilliant pupil next morning to find that he has achieved what the majority have found beyond them, but the aim should be to encourage even the most backward. In every set of homework, therefore, there should as a rule be sufficient for every pupil to do and to do well.

The brighter pupils can be given the opportunity to do more examples or occasionally one or two more difficult examples may be added.

Finally the good old maxim "practice makes perfect" should not be overlooked when arithmetic homework has to be set. The new rule has to be practised in order to fix it firmly in the mind, and in its repeated application the pupil finds work well within his capacity and work which brings its own satisfaction.

CHAPTER V

The Arithmetic Course for the Infant and Junior School

"Schemes and syllabuses there must be, and by them all must be bound within reasonable limits or anarchy will result; but their interpretation is in the hands of the teachers themselves." [1]

The Infant School Course

One of the most gratifying features of the arithmetical work done in our schools to-day is the fact that the arithmetic taught in our infant schools goes by the name of "Number Work", and it would be well if infant teachers would recognize the full significance of this title. One would like to see the term still used in the more advanced stages, for the more our infants and juniors are taught to think and to speak in numbers, the better will the arithmetic of the seniors be.

Up to the age of seven, then, the children should be gaining experience of numbers of concrete objects. No working by

[1] G. St. L. Carson, *Essays on Mathematical Education.*

manipulation of figures should be tolerated. The introduction of the written symbol should be delayed as long as possible, and only introduced when necessity for it is really felt. The child in the kindergarten should play, think, count, and talk in numbers up to ten. The appeal should always be through the senses—hence the value of number games and number objects. Too much use cannot be made at this stage of the abacus. *We want our pupils to experience so frequently, for instance, that two and three make five, that the fact becomes a part of their subconscious being, welling up into consciousness whenever the appropriate stimulus appears.*

Prolonged experience of number facts is what is desired at this stage. The teacher should remember that the child who does not come to school until the age of seven often comes nevertheless with a store of these number-facts gained from experience in the home, and although not knowing, it may be, a single figure symbol, will progress quite as rapidly in the junior school as the child who has received two or more years of systematic teaching in number work. The work that such a child from an average home is able to do without previous schooling can be safely taken as a guide in determining the work of pupils up to the age of seven. Such a child can, as a rule, at least count up to a hundred and has some knowledge of addition up to ten. Further, he possesses a certain amount of knowledge, gained from experience, of our monetary system, and is usually familiar with some of the more simple fractions such as a half and a quarter.

Syllabus.—The following, then, is suggested as a syllabus covering the years up to the age of seven:

(*a*) A thorough knowledge of all numbers up to ten gained through concrete examples and games.

(*b*) The extension of the idea of ten to two tens, three tens, and so on, up to 100.

(*c*) Counting up to 100 and recording such numbers on the abacus, thus familiarizing the pupil with the idea of positional value.

(*d*) Counting by twos, fives, and tens, up to 100.

(*e*) Learning from concrete examples the combinations of two numbers, the sum of which does not exceed ten, later extended to twelve; thus two and four make six, five and two make seven, and so on.

(This should not be done by counting but preferably by analysis of a group of things into components, and then the combinations learnt and memorized. The child should be made so familiar with these combinations in the concrete, experiencing them so frequently, that for ever afterwards the combination of, say, the numbers two and four immediately conveys the idea, six. Counting two groups of objects to find the sum should never be allowed. We appreciate later the value of familiarity with multiplication tables. The pernicious habit of counting on fingers, which persists not infrequently throughout school life and even beyond, demonstrates how little the value of *addition* tables is appreciated.)

(*f*) The inverse process, the first step in subtraction: two—how many more to make six? (The teacher says " I have six marbles here in my hands, two in this hand, how many in this?"—and so on. This complementary process to the former addition is very important.)

(*g*) Simple shopping transactions involving the use of the penny, shilling, and pound; the halfpenny and farthing; the penny in relationship to the shilling, the shilling to the pound; the sixpence, florin, and half-crown; all through concrete examples.

(*h*) Simple measurements involving the use of yard, foot, and inch.

(*i*) Practical application of the idea of one-half and one-quarter.

(*j*) Recognition of common solids and polygons: cube, prism, cylinder, pyramid, sphere. Square, rectangle, triangle, circle. These should be familiar to the children; used maybe as objects in counting and for games.

The introduction of written symbols will come just when the necessity for them is felt. No hard and fast rule can be given regarding this point, beyond stating the general fact that there is more danger in using the symbols too soon than too late. The necessary drill work should be almost entirely oral, and should be frequent. Although the value of concrete examples has been again and again emphasized it must not be thought that the importance of this drill work—sometimes called abstract drill work—is underrated. On the contrary, it is the necessary complement to the normal concrete work. Speed and accuracy are closely related, and speed in the arithmetical processes can only be secured by such thorough familiarity with the results of concrete experience that there is no stopping even to think, much less obtaining by counting what, say, four and five make.

The Junior School Course

In the discussion on the infant school course, emphasis has been placed on the importance of gaining knowledge by direct appeal to the senses of the elementary arithmetical facts concretely, thus ensuring an intelligent understanding of the very foundation of the subject. Skill in written symbolic arithmetic has been regarded as relatively unimportant at this stage.

In considering the junior school course, however, the teacher has to keep in mind that the pupil has now reached the stage when he is expected to become proficient in the use of the mechanical tools designed for the purpose of performing computation less laboriously and more speedily and accurately than is possible by ordinary unaided mental processes. Psychologically this is the period during which the pupil can best acquire those habits of thought which will ensure accurate and speedy work. The teacher of the senior school knows only too well how extraordinarily difficult it is to give to boys above the age of eleven and twelve that training in arithmetical speed and accuracy which should have been acquired between the ages of seven and eleven.

In the junior course, then, whilst the extension of the knowledge of fundamental arithmetical facts must still be arrived at through concrete experience and appeal to the senses, thorough and systematic drill is of the utmost importance. Accuracy in computation must become a habit. The addition and multiplication " tables " and the inverse processes must be known so thoroughly as a result of continuous drill that the pupil shows no hesitation in supplying, quickly and accurately, whatever arithmetical fact is at the moment required. It is no use being misled by the cry that any intelligent pupil can find out what six sevens are whenever he requires it. Having discovered it once, he should commit it so well to memory that he will never have to waste time to rediscover the fact in order to use it when necessity arises. Life demands that we carry out automatically as many processes as possible, and in our desire to relieve the child of the drudgery and monotony of learning by rote, formerly called arithmetic, we must not overlook the very important fact that, without making it the beginning and the end of our teaching, constant drill work is absolutely essential. Experienced teachers will agree that pupils in the junior school thoroughly enjoy this type of work. They take a pride in it, and the more proficient they become the more they gain in self-confidence and even self-respect. The junior school stage is predominantly the period for acquiring skill in the use of the " tools " of the subject.

Finally, there should be no sudden break in the methods adopted in the first year of the junior school and those employed in the infant school. Work should still be largely oral, the written work being confined almost entirely to writing answers to examples which have been dictated.

The junior school course should include the following:

First Year (ages 7–8).—(*a*) A thorough revision of the combinations of two numbers which do not exceed 10, with subsequent extension to 20. The inverse process, viz. "Seven and how many to make fifteen?"

(*b*) Addition of series of three or four numbers, (i) in vertical column, and (ii) horizontal column, total not exceeding 20.

(*c*) Addition by series to 100 as follows:

$2 + 2$, $12 + 2$, $22 + 2$, &c.
$2 + 3$, $12 + 3$, $22 + 3$, &c.
$2 + 4$, $12 + 4$, and so on.

(*d*) Counting by 2's, 3's, 4's, &c. For example, counting to 100 by 2's beginning with 0, with 1, with 2, and so on. The reverse process of counting down from 100. Much time should be devoted to drill in these various ways of counting, all making for speed and accuracy in addition and subtraction and laying a sound foundation for the multiplication tables.

(*e*) Addition of series of two-figure numbers, (i) in vertical columns, (ii) in horizontal columns, the sum not exceeding 100. Subtraction from any two-figure number.

(*f*) Construction of and thorough mastery through constant drill of multiplication tables up to 6×12. Equal attention to be given to these tables as preparation for division, e.g. How many 5's in 45? and so on. Short multiplication and division.

(*g*) Pence table up to 120 pence. Oral and written work in addition, subtraction, multiplication, and division of money, sums not to exceed £1. This work, however, to be mainly oral.

(*h*) Further exercises in measuring lengths. The pint and quart.

Simple problems involving use of arithmetical facts and processes mentioned above and of the familiar fractions $\frac{1}{2}$ and $\frac{1}{4}$, but requiring little written work.

Second Year (ages 8–9).—(*a*) Extension of the above work in numeration to numbers up to 1000. Counting should first be by 100's to 1000, then completely to 1000. Drill in counting forwards and backwards by 2's, by 3's, &c., from any starting-point should be continued as further exercise in the addition tables as well as in some cases forming the multiplication tables.

(*b*) Construction and thorough mastery of tables at least up to 10 × 10, extending to 12 × 12 if time permits. Again too much emphasis cannot be placed on pupils' thorough familiarity both with addition and multiplication tables.

(*c*) Addition in vertical and horizontal arrangements, sum not to exceed 1000. Subtraction from numbers not exceeding 1000. Multiplication and division similarly extended but multiplier and divisor being unit figures only.

(*d*) Extension of knowledge of monetary system up to sums of £10 and use of four rules therein.

(*e*) Extension of work previously done in weights and measures: inch, foot, yard; ounce, pound, cwt., and ton; pint, quart, gallon; seconds, minutes, hours, days. Whilst little can be done by way of problematic work in weights and measures at this stage, yet it is not too early to familiarize pupils with these various units.

(*f*) The meaning of $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{3}$, $\frac{1}{5}$, $\frac{1}{8}$, $\frac{1}{10}$, and $\frac{1}{12}$.

(*g*) Addition of two-column figures, multiplication of two-figure numbers.

In this second year the work again should be mainly oral, the problems dealt with requiring little written work. The pupils should become accustomed to the forms "the sum of", "the difference between", as variations of "add" and "subtract", and also to the signs +, −, ×, ÷, and of course =.

Third Year (ages 9–10).—In this year there should be a thorough revision of the fundamental processes applied to numbers and money. This revision should take the form:

1. Ample drill in the fundamental arithmetical facts included in addition and multiplication mainly through oral or mental work.

2. Speed and accuracy tests in purely mechanical exercises.
3. Problems requiring the use of the fundamental processes.

It is most important at this stage that, whilst much time has to be devoted to entirely new work, practice in the fundamental processes should be continued. Much of the inaccuracy which is evident in later years is due to the fact that, whilst the pupil is devoting time to the study of such new work as fractions, for instance, he is using multiplication tables to less extent than previously, and therefore loses to a certain degree the facility with which he previously employed them. Moreover, this repeated return to the more elementary work proves very stimulating to the pupil. He finds that he is more and more sure of himself, and in consequence tackles the newer work with more confidence and greater interest.

The new work will include:

(*a*) Further extension of numeration counting up to a million by thousands. Ample practice in writing large numbers in figures, from dictated words and vice versa.

(*b*) Vulgar fractions. Easy examples in addition and subtraction.

(*c*) Further examples in use of weights and measures introduced in previous year. Fundamental rules therein. Reduction.

(*d*) Mensuration. Perimeter and area of rectangle.

(*e*) The metre, centimetre, and millimetre.

(*f*) Decimal fractions—the first four rules applied to easy practical examples in measuring lengths and calculating areas.

For methods to be adopted in vulgar and decimal fractions see p. 61. With regard to the work in numeration and the extension thereof, as stated above, the counting should be done by thousands. The pupil should gain a very clear conception of the relationship between a thousand and a million. Ample practice should be given at this and subsequent stages in the translation from numbers dictated in words into figures, and from figures into words. Recent investigations by the Board of Education Inspectors revealed the fact that 34 per cent of children in Standard V were unable to write in figures the dictated number, ten thousand and ten; whilst 80 per cent of them failed to take ten thousand and one from one million. The weakness is due partly to want of practice in numerical dictation and translation, but mainly to the much more serious fault of paying more attention to figures and insufficient attention to numbers. The proper punctuation of numbers, too, by use of the comma, should be insisted upon.

Some idea of the size of a million should be given. Pupils should calculate how long it would take them to count a million, first timing themselves in counting to a hundred. Other methods of comparison might with advantage be employed. The pupils might be set to calculate, for instance, how many journeys round the earth, given its approximate circumference, would be required to cover a distance of a million miles. Interesting excursions into astronomical distances might also be made.

All such calculations will help to form a clearer conception of very large finite numbers. When, for instance, the pupil realizes that it would take up about one whole term of school life to count a million, he has a better conception of the number—it is now something more than " one, followed by six noughts ". And it is very important at this stage, when the pupil is becoming accustomed to using larger numbers, that they should be something more to him than mere figures. Whilst stressing the importance of speed and accuracy in mechanical working, at the same time it is essential that the pupil is " thinking in numbers " the whole time and not merely manipulating figures and symbols.

Fourth Year (ages 10–11).—The foregoing syllabuses aim at ensuring that, by the age of ten, a pupil of average intelligence will have thoroughly mastered the tools he is to employ, including addition and multiplication tables and the fundamental processes known as the first four rules. In addition he has covered a good deal of work in weights and measures, has a fair knowledge of vulgar and decimal fractions, and has done some elementary mensuration.

It is recognized that the ground thus covered is more than is usually attempted in the first three years. It is maintained, however, that this can be done if written arithmetic is reduced to a minimum, especially in the first two years. Even in the third year, the working of long " sums " and the setting out of lengthy explanations are not required. If the constant aim has been to get as much *arithmetic* as possible done in the arithmetic lesson rather than as much figure manipulation between neatly ruled lines, then speed and accuracy in the use of the arithmetical tools will have been ensured.

Nevertheless, it is unwise to assume that all can run the same race in the same time. It is at this stage that the teacher should carefully differentiate between the pupils who will need considerably more practice in the work already attempted and those who can, with profit, go on to more advanced work. Two syllabuses may therefore now be advisable; one for pupils of no more than just average intelligence, and the other for pupils of good average, or above average, intelligence.

SYLLABUS A.—For pupils (ages 10–11) of just average ability.

A thorough revision of the previous work applied as far as possible to concrete examples, either of a practical nature or drawn from everyday life. In particular attention will be given to:

(*a*) Multiplication and division (money, weights, and measures). Reduction.

(*b*) Factors, multiples, and all processes in fractions. Easy examples only.[1]

[1] Complex fractions should be rigorously excluded.

(*c*) Mensuration of the rectangle. Square measure.
(*d*) Further examples, of an easy type, in decimals.

SYLLABUS B.—For pupils (ages 10–11) above average intelligence preparing to enter secondary, central, or senior schools.

(*a*) Memorizing weights and measures tables. For such pupils this can be regarded almost entirely as memorizing work. The amount of time devoted to working examples thereon need not be very great.

(*b*) Prime numbers. Factors, multiples. Simple cases of L.C.M. and H.C.F. Fractions. Reduction and the four rules.

(*c*) Mensuration of rectangle, triangle, and circle. Square measure. Square root.

(*d*) The metric system and further and more difficult examples in decimals.

(*e*) Simple proportion and proportional parts. Method of unity. Percentages, easy examples.

(*f*) Averages. Practice.

Throughout both courses speed tests in the more elementary work should be given as frequently as possible, interest in these tests being secured by such means as recording results or forming teams to compete with one another.

Pupils who have worked through syllabus A may proceed to syllabus B during the following year. The subsequent work done by these pupils of just average ability will depend very largely on the type of school to which they are transferred, but as a rule their course will be to follow the more practical parts of the senior course. Considerable attention at the same time should be given to the practice of the more elementary work.

CHAPTER VI

The Arithmetic Course for the Post-Primary School

When the syllabus for the post-primary course (senior, central, or secondary school) for pupils of eleven to fifteen years of age and over comes to be considered, it has to be remembered that such pupils are approaching the age when interest is more and more being transferred from the purely mechanical use of the tools to the scientific principles underlying their application.

Nevertheless proficiency in mere numerical calculation has to be maintained, and pupils who have been promoted to central schools as being those who are at least a little above the average in intelligence, should never be allowed to consider accuracy as being of secondary importance. Frequent speed and accuracy tests, with interest and enthusiasm for such aroused it may be by purely artificial means, should be given. Such tests should be of two types, (*a*) those in which questions require answers only to be written, and (*b*) those designed to test speed and accuracy in the purely mechanical written work.

Both sets of tests should, in addition, serve the very useful purpose of constantly revising, and therefore keeping fresh, work done in previous years. (The mathematical master should take the necessary steps to ensure that uniform methods are employed in the mechanical work throughout the school. Agreement should be reached between the science and mathematics departments, for instance, on such points as the particular method to be employed in multiplication and division of decimals.)

It should also be remembered that whereas, during the junior course, explanations of the processes employed have never been required of the pupil, he has now reached the stage when such explanations will have a real interest for him. Previously the teacher was satisfied if the explanation *he* gave to the pupil of any particular process was understood at the time; now, however, fuller consideration of the underlying reasons for the particular method employed in the various arithmetical processes will help to arouse fresh interest therein.

The arithmetic of the post-primary course proper will be concerned too with underlying principles. At the same time the application of arithmetical processes to the world of everyday affairs is now of paramount importance. Call it what we will, business or commercial arithmetic, technical or industrial arithmetic, the pupil's interest is all the keener if he feels that the work in which he is engaged bears some relationship to the larger world outside. Not only should problems which bear no resemblance to real experience be rigorously excluded from the course, but every effort should be made to secure examples from the business world of the immediate neighbourhood as well as from the world at large, drawn from municipal and national affairs. It is, for instance, absurd to ask pupils to find the cost of 15 tons, 5 cwt., 14 lb., 3 oz. of coal at £2, 15*s*. 6*d*. per ton—for the very obvious reason that the purchase of such a quantity of coal is never made. To ask the pupils the cost of carriage on 3 tons, 16 cwt., 3 qr., 18 lb., at £1, 3*s*. 6*d*. per ton, is to ask a question which has some relationship to real experience. When, however, problems are drawn from actual business transactions, as they can be if a little trouble is taken, the interest is increased very considerably.

The relationship of the subject to other branches of mathematics is another important aspect of the work of the post-primary course. The written arithmetic must be considered from the point of view of the exactness and conciseness of the statements used and their logical sequence. Such work will take on, as it were, a new aspect. Hitherto the written arithmetic has been regarded mainly as a mechanical aid to the mind. Now it is to be considered more as a statement of mathematical reasoning, and more importance will be attached to the manner in which the solution of the problem in hand is set forth.

The following syllabuses are intended to be suggestive only. In the post-primary course the teacher must exercise a good deal of initiative in his selection of material and in the details of the work to be attempted. He must, for instance, be guided partly by the mental capacity of his pupils and also by the demands which their future careers are likely to make. Broadly speaking, he will have at least two types of pupils to consider, the more practically minded but not necessarily less intelligent pupil, and the pupil to whom the more academic side of the work appeals. The syllabuses set out below are intended to cover a general course suited more or less to both types of pupils, and it is left to the teacher to decide to which parts of the syllabus he must devote the major portion of the time at his disposal.

First Year (ages 11–12)

(*a*) A thorough revision and extension of previous work with special attention to factors, L.C.M., and H.C.F.[1] Fractions, decimals, and decimalization. The metric system.

(*b*) Measurement of angles, heights, and distances with practical examples, using simple or home-made instruments for measuring angles. Areas of rectangles and triangles, parallelogram, rhombus, trapezium, circle; field work. Use of symbols to express areas.

(*c*) Revision and extension of work of junior course in averages and percentages, ratio and proportion, profit and loss.

(*d*) Statistical graphs.

[1] Where pupils are drawn from several schools, considerable time may have to be devoted to such revision if unsuitable methods have to be eliminated.

Second Year (ages 12–13)

(*a*) Revision and extension of previous work, with special reference to ratio and simple proportion, averages and percentages; square root.

(*b*) Compound proportion.

(*c*) Simple interest.

(*d*) Further examples in areas. More difficult examples in field work. The right-angled triangle and the equilateral triangle. Volume of prism and cylinder, pyramid and cone.

(*e*) Factors and indices. Law of Indices treated from such statements as:

$$8 \times 16 = 2^3 \times 2^4 = 2^7$$

and thence $$a^3 \times a^4 = a^7.$$

(*f*) Logarithms. Easy examples in multiplication and division.

(*g*) Statistical graphs and simple deductions therefrom.

Third Year (ages 13–14)

(*a*) Revision and extension of previous work.

(*b*) More difficult computations using logarithms, including finding roots and powers of numbers. The slide rule. Use of logarithms in formulæ evaluations.

(*c*) Mensuration. More difficult examples in work previously done, including examples requiring use of logarithms or slide rule.

(*d*) Variation and graphs. $y \propto x$. $y \propto \frac{1}{x}$. The "gradient" of a curve.

(*e*) Simple interest revised. Discount and commission; banking; compound interest; simple and compound interest compared graphically.

Fourth Year (ages 14–15).—Revision and extension of previous work, with special reference to future careers. The work will therefore be designed to meet the requirements either of:

1. Those entering upon engineering or industrial careers, in which case logarithms, the slide rule, and graphical work will receive most attention.

2. Those entering upon commercial careers, in which case percentages, profit and loss, simple and compound interest, discount and commission, banking, stocks and shares, will form the chief part of the syllabus.

3. Those who during the following year may be taking a School Certificate Examination, in which case a more general revision of the whole with special reference to method would be the most satisfactory course to be followed.

It should, however, be noted that the syllabuses of the first three years cover a very wide field, and some teachers may prefer to spread the work over four years. As already mentioned, the syllabuses, which are fairly exhaustive, are only intended to be suggestive. Teachers should take into consideration both the mental capacity of the pupils and their possible future careers before determining the details of the work to be attempted.

CHAPTER VII

Notes on Arithmetical Methods

In the following notes on arithmetical methods only such branches of the subject are dealt with as those in which a choice of method is before the teacher, or those in which the method presents some special difficulty, or is of special importance. The notes are not intended to take the place of the textbook, but rather to supplement it.

The Fundamental Rules

The immense importance of the teaching of sound methods in the early stages of arithmetic cannot be over-estimated.

It is a very significant fact that the methods taught us in our arithmetical infancy, be they ever so bad, have a tendency to persist through life. So marked is this tendency that experienced teachers are inclined to leave the pupil to follow whatever methods have been taught him in his infancy, provided he can perform the operation expeditiously and accurately. It is claimed, and not without good reason, that attempts to change familiar methods for others tend to confuse the mind of the pupils and entail a good deal of wasted effort. If the change in methods can be effected not later than the age of ten, and pursued vigorously for a couple of years, there is considerably less risk of ultimate decline in accuracy. If, however, these unsound methods have been used until the age of twelve, change of method is apt to aggravate the tendency towards inaccuracy peculiar to the years of puberty. It is this difficulty which emphasizes the importance of teaching sound methods from the earliest stages.

Addition and Subtraction

The two are here purposely taken together, for they should be regarded as very closely allied processes. The facts that 6 requires 3 to make 9 and 3 requires 6 to make 9 should be taught almost at the same time as the facts that 6 and 3 make 9 and 3 and 6 make 9. Subtraction and addition should be taught almost simultaneously, and the former will then be a natural process instead of a very mechanical and difficult one. The special points concerning methods in teaching addition and subtraction which must be regarded as important are as follows:

(*a*) The addition (and subtraction) tables should be thoroughly well known. The facts should first be understood from experience, and then so well memorized that whenever the pupil is required to add or subtract any two single digit numbers, he should be able to give the answer quite automatically. The pupil not only learns, for instance, that 9 and 8 make 17, but the fact should become so thoroughly a part of his experience that the stimulus 9 + 8 calls forth immediately the response 17.

(*b*) Much practice should be given in the addition of single digit figures arranged in vertical columns and in horizontal lines. The mental work employed in this process should from the first be merely that of *registering the successive sums.* For example, 6 + 3 + 8 + 7 = 24. The pupil should say 9, 17, 24—not, 6 and 3 are 9 and 9 and 8 are 17. This, be it noted, is only an extension of what has just been advocated. The stimulus 6 + 3 should automatically call up the response 9, the stimulus of this number in the mind and the 8 should automatically give the response 17, and so on.

Such automatic response cannot be secured without constant practice. Not infrequently teachers who advocate thorough familiarity with multiplication tables do not recognize the importance of similar intimate knowledge of the addition and subtraction tables.

(*c*) The first step in adding two digit numbers together should be by way of examples such as the following:

$$4 + 3 = \ , \qquad 14 + 3 = \ , \qquad 24 + 3 = \ ,$$

and similarly in subtraction. The pupil should not only know that, say, 4 and 3 make 7 but that the addition of any two numbers ending in 4 and 3 will be a number ending in 7.

(*d*) Practice in the addition of series of two-figure numbers either in vertical columns or horizontal lines should now follow. In this connexion encouragement should be given to those who are able to add number to number instead of adding units first and then tens. There is room here for considerable experiment. The process is as follows:

$$24 + 35 + 17 + 49 = 125.$$

The pupil mentally registers in turn the numbers 54, 59, 69, 76, 116, 125, adding first the ten and then the unit. Ability to carry out this method depends, of course, upon the mental capacity of the pupil and is largely a question of training pupils to register figures in their minds, but more should be done to encourage individuals to adopt this method.

(*e*) In all cases of addition of series of numbers answers should be checked by adding the numbers in the reverse order. Accuracy should be encouraged right from the earliest stages.

(*f*) Subtraction. Three methods are available.

I. Decomposition Method. The extended working is as follows:

$$\begin{array}{rcrcr}
434 & = & 400 + 30 + 4 & = & 300 + 120 + 14 \\
\underline{186} & = & \underline{100 + 80 + 6} & = & \underline{100 + 80 + 6} \\
 & & & & \underline{200 + 40 + 8}
\end{array}$$

This is usually condensed thus:

$$\begin{array}{ccc}
3 & 12 & 14 \\
4 & \not{3} & \not{4} \\
1 & 8 & 6 \\
\hline
2 & 4 & 8 \\
\hline
\end{array}$$

The procedure is as follows: (i) 6 from 4 we cannot. Add 1 ten from the tens in the top line to make 14. 6 from 14 leaves 8. (ii) 8 tens from 2 tens we cannot. Add 10 tens, making 12 tens, from the hundreds in the top line. 8 tens from 12 tens leaves 4 tens. 100 from 300 leaves 200.

II. Method of Equal Additions.

$$\begin{array}{rcrcr}
434 & = & 400 + 30 + 4 & = & 400 + 130 + 14 \\
\underline{186} & = & \underline{100 + 80 + 6} & = & \underline{200 + 90 + 6} \\
 & & & & \underline{200 + 40 + 8}
\end{array}$$

Condensed this becomes:

$$\begin{array}{ccc}
 & 13 & 14 \\
4 & \not{3} & \not{4} \\
2 & 9 & \\
\not{1} & \not{8} & 6 \\
\hline
2 & 4 & 8 \\
\hline
\end{array}$$

The procedure is as follows: 6 from 4 we cannot. Add 10 units to the top and 1 ten to the botton. 6 from 14 leaves 8. 9 tens from 3 tens we cannot, add 10 tens to the top line making 13 tens and 100 to the bottom. 9 tens from 13 tens leaves 4 tens. 2 hundreds from 4 hundreds leaves 2 hundreds.

III. Complementary Addition Method.

$$\begin{array}{rl} 434 & = 400 + 30 + 4 \\ 186 & = 100 + 80 + 6 \\ \hline 248 & \ 200 + 40 + 8 \\ \hline 11 & \ 100 \quad\ 10 \end{array}$$

This method presupposes the familiarity with the subtraction table allied to the addition table already advocated. The pupil knows that the answer when added to the bottom line gives the top line. His task, therefore, is to find the missing number. Thus: 6 and (8) make 14; carry 1 ten; 9 and (4) make 13; carry 100; 2 and (2) make 4. He is adding the whole time, and therefore has no difficulty in appreciating the necessity for "carrying".

Dr. Ballard in his *Mental Tests* refers to an extensive investigation made into the relative merits of the first two methods and reaches the conclusion that the second method, that of equal addition, was from the point of view of speed and accuracy much more effective; in fact, for children of the lower standards it proved to be 50 per cent better. The decomposition method, from the teacher's point of view, is attractive, especially in the early stages. It can easily be explained in most cases, though it is very confusing and difficult to understand when subtracting from large round numbers, such as 1000 — 157. The method doubtless presents less difficulty to the pupil if called upon to explain what he has done. This does not necessarily prove that he understands it better at the time. Moreover explanations from the pupil at this stage are not required. We are placing in his hands a tool which he is required to use, once he has understood its use. The better tool is surely the one which he uses with the greater degree of accuracy and confidence. Teachers will therefore be well advised to leave the decomposition method alone.

The third method, that of complementary addition, is to be advocated on the ground that it is the method which should have been made familiar to the pupil from the very commencement of his work in arithmetic. It is not yet widely enough in use to test its effectiveness from the point of accuracy and speed, but inasmuch as it is so closely akin to the method of equal addition, this can hardly be questioned. It almost compels an immediate check being made and therefore has much to commend it. It is of course the shopkeeper's method, and there is certainly much to be said in favour of teaching the pupil a method which will enable him to test his change as rapidly as the shopkeeper presents it. *The one very great point in its favour, and on account of which it should certainly be taught from the earliest, is that it is the mathematician's method.* It can be applied immediately to algebra; thus:

$$\begin{array}{r} 3x - 5y + 2z \\ 5x - 3y + 8z \\ \hline -2x - 2y - 6z \\ \hline \end{array}$$

Here we say, $5x$ and $(-2x)$ make $3x$; $-3y$ and $(-2y)$ make $-5y$; $8z$ and $(-6z)$ make $2z$.

The writer has found that pupils of twelve who dislike change of method in multiplication and division not only are easily reconciled to the change in favour of the complementary addition method in subtraction, but become very keen and quickly adapt themselves to the change.

Addition and Subtraction of Money, &c.—A doubt exists in the minds of many teachers as to whether, in the addition of money, a pupil should be taught to add as in ordinary addition, converting the sum of pence into shillings and so on, or whether to make up shillings as the addition proceeds. There is much to be said in favour of encouraging the latter direct method—again the shopkeeper's method. Thus, the addition of 1s. 6½d. + 2s. 5d. + 4s. 9¾d. should proceed, 3s. 6½d., 3s. 11½d., 7s. 11½d., 8s. 9¼d. The process, however, may be difficult to some, and where the amounts are large it is better to add up each denomination as units and convert the sum, thus confining the work to the common notation.

In subtraction, however, the third method described above, that of complementary addition, i.e. the shopkeeper's method, is unquestionably the best.

The above remarks on addition and subtraction of money apply also to addition and subtraction in weights and measures.

Multiplication

Thorough familiarity with multiplication and division tables is essential. Recitation of tables is probably more useful than the writing of them. The teacher should devise ample variation of methods, drilling and testing orally. Pupils should be required to say the tables forwards, backwards, odd numbers, even numbers, and so on. In addition to rapid questioning, which helps to keep pupils alert, speed tests in writing out a series of items from the tables should be given frequently. The teacher must use his ingenuity in devising methods which will help to maintain interest and secure alertness in this very necessary drill.

Short Multiplication.—The first real step after tables in multiplication is that required to find the product of, say, 17 and 3. The problem may be presented thus. In each of three baskets there are 17 oranges. How many are there altogether? The pupils should be asked to find the answer. As pointed out on p. 18, if to the pupil's mind the number 17 conveys not the *figures* 1 and 7 but the *numbers* 10 and 7, many, if not all, will obtain the answer by mentally adding 30 and 21. Others may write down the 17 three times and add them together. But in so adding they will quickly see that their multiplication table shortens the process. A number of such problems should be given as rapidly as possible: 5 baskets with 16 in each, How many? 9 baskets with 15 in each; and so on. One of two things will happen: either some pupils will write down only once the number contained in each basket and use only their multiplication table to obtain the answer, or they will all be ready to appreciate the advantage of so doing. *In other words the necessity for and the advantage of the mechanical process will be realized.*

Long Multiplication.—The steps in teaching long multiplication are as follows:

1. $3 \times 3 = 9$; $30 \times 3 = 90$; 300×3 is 900. Repeat until the rule is obtained, to multiply by 200, 300, 400, &c., multiply by 2, 3, 4, &c., as the case may be and add two noughts, or, put 0 for the unit figure, 0 for the tens, and then multiply by 2, 3, 4, as the case may be.

2. Give practice in such examples as 127×30; 4569×100; 359×400; 270×500.

3. Take the first example in (2), i.e. 127×30, and ask how the following can be worked, 127×32. The answer should be readily forthcoming. Multiply first by 30 and then by 2. The working will be done on the blackboard as follows:

$$\begin{array}{r l}
127 & \\
32 & \\
\hline
3810 & = 30 \text{ times } 127 \\
254 & = \;\;2 \quad \text{,,} \quad 127 \\
\hline
4064 & = 32 \quad \text{,,} \quad 127 \\
\hline
\end{array}$$

The explanatory matter to the right will not, of course, be given in later examples. Multiplying first by the figure of highest denomination is here advocated because, (*a*) It comes naturally to the pupil to do so as explained on p. 18. (*b*) The first product thus obtained is nearest the complete answer.

(*c*) The method is much to be preferred in multiplication of decimals, when the reasons given above in (*a*) and (*b*) have far more significance. (*d*) The method is the only one that can be employed in contracted multiplication.

Division

It has already been mentioned that at the time pupils are learning their multiplication tables they should learn the reverse form of the table, that is to say that when learning $8 \times 6 = 48$, they should as soon after as possible learn that 8 is contained in 48, 6 times, or " 8's in 48 goes 6 ". The close association of these two facts helps to fix each in the mind, and the pupil is laying a sound foundation for quick and accurate work in division as in multiplication. The pupil should appreciate the fact that division is the reverse process to multiplication.

Short Division.—Treatment of the " remainder ". Little if any difficulty is experienced in teaching short division. The only point that calls for comment here is the treatment of the remainder. The statement $49 \div 4 = 12 + 1$ is, of course, ridiculous, and remainders should never be expressed in this manner.

If the written mechanical work is developing, as it should do, out of the mental work, then in such simple cases as the above the statement $49 \div 4 = 12\frac{1}{4}$ is perfectly intelligible to the pupil. Further, it is of course true that, at the stage at which the pupil is taught short division, only such simple fractions as $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, $\frac{3}{4}$ are understood, but the pupil can quite easily understand that when, for instance, 379 is divided by 5 the answer is 75, and that 4 more remain to be divided. He will quite readily accept the statement that $\frac{4}{5}$ means that 4 more remain to be divided by 5. We are, in fact, giving him another symbol. His work then becomes:

$$\begin{array}{r} 5\underline{)379} \\ 75\frac{4}{5} \end{array}$$

The same applies to "remainders" in long division. We are satisfied if the pupil accepts $\frac{4}{15}$ as the symbolic way of writing "4 more remain to be divided by 15", $\frac{29}{51}$ as the symbolic way of writing "29 more remain to be divided by 51".

Short division of money should proceed with ordinary short division.

Long Division.—The teacher will be wise in recognizing quite frankly that long division is a very difficult process for young pupils to understand, and that it is better to assume that the understanding will come later. The pupil is introduced to the process step by step and accepts it rather than learns it in the sense of understanding *why* the process "works". That it does work is convincingly proved by multiplication. The steps in the demonstrations are as follows:

1. Work on the blackboard by the long division method a sum, such as 2536 ÷ 4, which can be worked and is worked by short division.

2. Give the pupils several examples to work for themselves.

3. Call attention to the similarity between the two processes.

4. Set the pupils to attempt to work an example such as the following by the *short* and then the long division methods: 198 ÷ 15.

Note that the example should be an easy one involving a division by a number of which the table is not known, i.e. above 12.

5. Gradually increase the difficulty of the example, still using fairly easy divisors in order that both methods can still be employed. The pupils will begin to appreciate the utility of the long division method.

6. Finally set an example of the type 1487 ÷ 48, in which short division involves considerable difficulty and much margin work.

7. The pupils, convinced of the value of the new tool, are quite prepared to become proficient in its use *without*,

at this stage, entering into an exhaustive explanation of the underlying principles. Considerable practice should now be given first with two-figure divisors in which the units figure is not above 3 or 4, in order that the quotient figure can be readily found, and later with two-figure divisors with a units figure above 5.

Example 1.

```
      74
62)4593
   434
    253
    248
      5
```

Answer. $74\frac{5}{62}$.

The quotient is obtained thus: 62 is approximately 60. 60 into 4 thousands will not "go" any thousands. 60 into 45 hundreds will not "go" any hundreds. 60 into 459 tens will go 7. In practice this is shortened to 6 into 45, &c.

Example 2.

```
      67
68)4593
   408
    513
    476
     37
```

Answer. $67\frac{37}{68}$.

Here 68 is nearer 70 than 60, and hence we say 7 into 45 goes 6, and therefore 68 into 459 goes 6, and so on.

Division by factors as a method of bridging the difficulty experienced by young pupils in going from short to long division is not advocated. Division by factors only postpones the difficulty for a short while, since it is only applicable in certain cases.

Long Multiplication and Division of Money, &c.

In what is termed compound long multiplication the single-line method in column form is much to be preferred.

Example 1. £17, 14s. 8½d. × 63.

£17,	14s.	8½d.
		63
£1117,	6s.	7½d.
46	44	31
630	630	504
441	252	535
1117	926	

Answer. £1117, 6s. 7½d.

It will be noticed that no working is shown for the conversion of the 535 pence into 44 shillings and 7 pence, nor for the 925 shillings into 46 pounds and 6 shillings. Pupils should be encouraged to do work mentally as much as possible. In these cases it is not difficult to do so inasmuch as the figures when obtained are placed in their immediate columns.

Example 2.—13 tons 5 cwt. 3 qr. 7 lb. × 23.

13 tons	5 cwt.	3 qr.	7 lb.
			23
305	13	2	21
6	18	5	28)161
299	115	69	140
305	133	74	21

Answer. 305 tons 13 cwt. 2 qr. 21 lb.

The column method of arranging long division has much in its favour, and is here recommended partly because of its neat and compact form and also because it avoids " putting down multiplication sums " which should be done mentally.

Example 3.—£538, 14s. 1½d. ÷ 52.

```
       £ 10,    7s.   2¼d.
   52)£538    14s.   1½d.
        52   360   120   68
        18   374   121   70
             364   104   52
              10    17   18
```

Answer. £10, 7s. 2¼d. Remainder 18 farthings.

The same method is applicable to division in weights and measures.

Fractions

The youngest child comes to school equipped with some very useful knowledge concerning fractions. He knows, for instance, what is meant by " half an apple ", and he may be acquainted with the fact that a scone easily divides into four parts. In other words his idea of fractions is connected with partitions and divisions. The teacher will do well to make full use of the elementary notion of fractions which every child possesses, and, building on this, develop the mathematical conception of a fraction as " one or more of aliquot parts into which the unit is divided ". A knife and a few apples or potatoes will prove far more effective instruments for teaching fractions than lines, circles, and rectangles drawn on the blackboard, and will save much time. The shilling and its twelve parts, the foot rule and its twelve parts, the pound with its two half-sovereigns, its ten florins, its eight half-crowns, its twenty shillings, all natural and well-known ready-to-hand instruments, are infinitely better than the less familiar and quite artificial diagrammatic units. Briefly, the steps[1] in teaching " fractions " are as follows:

I. (*a*) The recognition of $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, $\frac{1}{10}$, $\frac{1}{12}$, by actual partition and reference to known fractions of familiar units.

[1] N.B.—Each " step " constitutes at least one lesson to be followed by ample exercise and " drill ".

(*b*) $\frac{1}{2} + \frac{1}{2}$, $\frac{1}{2} + \frac{1}{4}$, $\frac{1}{4} + \frac{1}{8}$, &c. $1\frac{1}{2} + 1\frac{1}{2}$, $2\frac{1}{2} + 1\frac{1}{4}$, and so on. At first all such exercises should be worked as concrete examples. Blackboard and paper work will be employed only after ample time has been devoted to the concrete examples.

II. $\frac{1}{2} = \frac{2}{4}$, $\frac{1}{4} = \frac{2}{8}$, $\frac{1}{2} = \frac{4}{8}$, $\frac{1}{3} = \frac{2}{6} = \frac{4}{12}$, and the reverse $\frac{2}{4} + \frac{1}{2}$, &c.

Examples such as:

$$£\tfrac{1}{2} + £\tfrac{1}{8} = 10\text{s.} + 2\text{s. } 6\text{d.} = 12\text{s. } 6\text{d.}$$

$$\text{also} \qquad £\tfrac{1}{2} + £\tfrac{1}{8} = £\tfrac{5}{8} = 12\text{s. } 6\text{d.};$$

that is, examples in adding fractions first in the concrete form and then in the fractional abstract form.

III. Ample exercises in adding and subtracting easy fractions, such as $\frac{1}{2} + \frac{3}{8} + \frac{3}{4}$; $\frac{3}{4} - \frac{1}{8}$.

Later such examples as $1\frac{1}{2} + 2\frac{3}{8} + 5\frac{3}{4}$; $5\frac{3}{4} - 1\frac{1}{8}$; and later still $4\frac{1}{4} - 2\frac{1}{2}$; $3\frac{1}{8} - 2\frac{3}{4}$.

IV. Multiplication of fractions. $\frac{1}{2}$ of $\frac{1}{4}$, &c.

Division of fractions, i.e. how many times is $\frac{1}{8}$ contained in $\frac{1}{4}$, in $\frac{1}{2}$, &c.

V. Harder examples in all four rules without reference to L.C.M. and H.C.F., e.g. $\frac{1}{4} + \frac{1}{3}$.[1]

VI. Factors. Multiples. H.C.F. and L.C.M.

VII. All four rules applied to any fractions.

Such a course of lessons covering the whole of the necessary work in fractions need not be continuous. Easy examples, plentiful in number, are to be preferred to a few difficult

[1] If ample mental work has been given up to this stage there should be no difficulty in obtaining the suggestion that each fraction here should be converted into twelfths. Even at this stage revision of earlier and easier examples and reference to the foot rule will be of more use than diagrams.

and complicated ones. Complex fractions should be omitted entirely. Further, whilst the examples should certainly supply the necessary drill in working fractions, the application of fractions to a variety of problems should form a fair proportion of the work.

Decimal Fractions

Perhaps the soundest advice that can be given in connexion with the teaching of decimal fractions is: make sure that the full significance of the decimal point is appreciated before the work in multiplication and division of "decimals", as they are usually termed, is undertaken.

The work should begin with a thorough revision of facts concerning notation. Ample time should therefore be devoted to exercises such as the following:

(*a*) Powers of 10:

$$10 = 10^1; \quad 100 = 10 \times 10 = 10^2;$$
$$1000 = 10 \times 10 \times 10 = 10^3;$$

and so on.

(*b*) Writing numbers in extended form:

$$365 = 3 \times 10^2 + 6 \times 10 + 5$$
$$51{,}279 = 5 \times 10^4 + 1 \times 10^3 + 2 \times 10^2 + 7 \times 10 + 9$$
$$1{,}000{,}000 = 10^6$$
$$6{,}532{,}429 = 6 \times 10^6 + 5 \times 10^5 + 3 \times 10^4 + 2 \times 10^3 + 4 \times 10^2 + 2 \times 10 + 9.$$

N.B.—There should be no difficulty in getting boys to use indices. There is nothing to be explained. They are simply told that 10^2 is the "shorthand" or symbolic way of writing 100, 10^3 the "shorthand" or symbolic way of writing 1000. They will be better pleased if they are told they are using the mathematician's way of writing such numbers.

(*c*) Analysis of the working of a multiplication sum, e.g. 1063 × 248.

```
   1063
    248
 ------
 2126
  4252
   8504
 ------
263,624
```

Before the analysis commences some preliminary work on examples such as the following will be necessary:

$$100 \times 10 \quad = 1000 \text{ or } 10^2 \times 10^1 = 10^3$$

$$1000 \times 100 = 100{,}000 \text{ or } 10^3 \times 10^2 = 10^5$$

and so on.

The analysis will proceed as follows: (i) Before working. Approximately what is the answer?

$$1000 \times 2 \times 100 = 10^3 \times 2 \times 10^2 = 2 \times 10^5.$$

The answer is two hundred thousand and " something ". (ii) What is the 6 in the first line of the product?

$$3 \times 200 = 600.$$

What is the value of the 4 in the second line? and so on.

It will be recognized that work of the character indicated above will lay the foundation for a clear understanding of " decimals ", inasmuch as the pupil will see that the whole notation is a decimal notation. The figures to the left of the decimal point are as much " decimals " as the figures to the right. No mention of decimal point, however, is yet necessary, though the use of the other punctuation mark, the comma, should be emphasized; it assists in making the reading of numbers an easier process. The decimal point when it is introduced may be referred to as another number-punctuation mark—" the full-stop " to the whole numbers.

The next step should include preliminary work in the metric system. The metre, decimetre, centimetre, and millimetre should be studied and easy examples in addition, subtraction, multiplication, and division dealt with. Careful contrast with our own system of measuring length should be made with a view to emphasizing the decimal notation of the one and in consequence the simplicity of "carrying", and the varying notation of the other with its difficulties of conversion from inches to feet, feet to yards, &c., before the "carrying" can be done.

The Decimal Point.—Having secured that the pupil has thoroughly grasped the decimal nature of our ordinary system of notation, the decimal fraction and the decimal point can now be introduced as follows. Study the number 6666.

$$6666 = 6 \times 10^3 + 6 \times 10^2 + 6 \times 10 + 6 \times 1.$$

The attention is directed to the fact that each figure is $\frac{1}{10}$ that of the figure to its left. Other examples of a similar type are studied in the same way, the attention in each case being directed to the descending order of the figures.

There should now be no difficulty in introducing the pupil to the extension to decimal fractions. In the above example, for instance, the pupil is asked: What will be the value of another 6 placed to the right of the existing number, that number remaining unchanged? He is then told that a dot, or full stop, is used to indicate that the figure to its left is the unit figure, and the commas can then be inserted. Exercises now follow in reading examples such as the following:

$$27{\cdot}23 = 2 \times 10 + 7 \times 1 + 2 \times \tfrac{1}{10} + 3 \times \tfrac{1}{100}.$$

Ample drill should be given at this stage in examples of this kind and the converse, viz. writing down numbers as punctuated figures. As already stated, the teacher should be in no hurry to introduce multiplication and division of decimals. Thorough familiarity with decimal notation and decimal fractions should be secured before any work in these two rules is attempted. Most of the difficulty which arises in the mind of the pupil regarding the decimal point is due to the fact that its place in the notation system is not sufficiently understood.

Further work should now follow on the metric system of measuring lengths. The decimetre is $\frac{1}{10}$ and the centimetre $\frac{1}{100}$ of the metre. Hence 2·35 metres = 2 metres, 3 decimetres, 5 centimetres; ·05 metres = 5 centimetres; and so on. Examples in addition and subtraction in the metric system can now be given as examples in addition and subtraction of decimals.

Work on the special decimal fractions should now follow.

$$\cdot 5 = \tfrac{1}{2},\ \cdot 05 = \tfrac{1}{20},\ \cdot 25 = \tfrac{1}{4},\ \cdot 025 = \tfrac{1}{40},\ \cdot 125 = \tfrac{1}{8}.$$

With the more intelligent pupils, exercises on decimalization of money might follow.

Multiplication and Division.—The pupil's greatest difficulties will arise from the variety of the methods which may be taught in the junior and senior schools, and in the mathematics and science classes. By some the old-fashioned method is abandoned in favour of " standard form " because the latter is the best for " contracted " methods. But who wants to do work by contracted methods when the work can be done more accurately and more expeditiously by logarithms and the slide rule? Others, convinced that the pupil must always be ready to explain the reasons for the working he employs, and must never use methods which tend to become mechanical and automatic, insist on the pupil arriving at the decimal point by first principles in all cases.

Apart from the fact that it is largely a ready-made and memorized explanation which the pupil gives, why should he be required to give any explanation at all? If he understood what he was doing when he first used the method, surely in the interests of accuracy and speed the sooner the work becomes automatic the better. After teaching all three methods for varying periods during the past twenty-five years, the writer is convinced that the old-fashioned method of finding the decimal point is to be preferred. The method is a sure, straightforward one which develops naturally out of the previous work on decimal and vulgar fractions, is easily understood by young pupils, who quickly become proficient in its use and thereby gain confidence in themselves. If explanation of the rules adopted is required it can quickly be recalled. The rule for multiplication is arrived at as follows:

$$\cdot 3 \times \cdot 2 = \frac{3}{10} \times \frac{2}{10} = \frac{6}{100} = \cdot 06$$

$$\cdot 03 \times \cdot 2 = \frac{3}{100} \times \frac{2}{10} = \frac{6}{1000} = \cdot 0006$$

$$\cdot 03 \times \cdot 02 = \frac{3}{100} \times \frac{2}{100} = \frac{6}{10{,}000} = \cdot 006,$$

and so on.

The pupil observes that there are as many decimal places in the answer as there are noughts in the denominators of the corresponding vulgar fractions, a fact known previously, of course, and that there will always be as many noughts in the denominator of this vulgar fraction as there are decimal places in the two fractions together.

Hence the rule: *Multiply as in ordinary multiplication. The total of the number of decimal places in the factors of the product is the number of decimal places in the product.* The work should be set out as follows:

Example: $75 \cdot 12 \times \cdot 024$.

Approximate answer (the following work is done mentally):

$$75 \times 2 = 150, \ 75 \times 3 = 225.$$

Therefore $75 \times \cdot 02 = 1 \cdot 5$ and $75 \times \cdot 03 = 2 \cdot 25$.

The answer therefore is between $1 \cdot 5$ and $2 \cdot 25$.[1] In this

[1] It is a good plan to insist on getting an approximate answer, not only as a check on the work, but as forming a habit likely to be useful when work is done by logs or the slide rule.

particular case the approximate answer might be obtained as follows: ·24 is nearly $\frac{1}{4}$, therefore ·024 is nearly $\frac{1}{40}$. Hence 75 × ·024 is approximately 1·8.

The actual working is as follows:

```
  7512
    24
 -----
 15024
  30048
 ------
 180288
```

There being a total of five decimal places in the two factors of the product there must be five decimal places in the answer.

Answer. 1·80288

The rule for division is arrived at as follows:

$$4{\cdot}72 \div 2 = 2{\cdot}36$$
$${\cdot}472 \div 2 = {\cdot}236$$
$${\cdot}0472 \div 2 = {\cdot}0236.$$

Further examples of a similar kind where the divisor is a whole number convince the pupil that there is no difficulty in the division of decimal fractions.

Now take an example such as 4·72 ÷ ·02. The answer can of course be deduced from the first of the above.

If 4·72 ÷ 2 = 2·36, then when divided by a quantity which is $\frac{1}{100}$ of the former divisor the answer must be 100 times the former answer, i.e.

$$4{\cdot}72 \div {\cdot}02 = 236.$$

But normally we shall not have the answer to multiply thus by 100. We can instead multiply the dividend. In other words we multiply both divisor and dividend by such a multiple of ten as will convert the latter into a whole number.

Another method or a supplementary method of arriving at the same rule is as follows:

$$(4{\cdot}72 \div 0{\cdot}02) = \frac{4{\cdot}72}{0{\cdot}02} = \frac{4{\cdot}72}{{\cdot}02} \times \frac{100}{100}$$
$$= \frac{472}{2}$$
$$= 236.$$

RULE: *If the divisor is not a whole number, move decimal point a sufficient number of places to the right to convert it into a whole number and move the decimal point in the dividend the same number of places*, i.e. multiply both by such a multiple of ten as will convert the divisor into a whole number. Proceed as in ordinary division.

Some teachers prefer the " standard form " method, i.e. multiplying divisor and dividend by such a multiple of ten as will convert the former into a quantity consisting of a whole number of one digit followed by decimal places. For example: $42{\cdot}635 \div 0{\cdot}0258$ becomes $4263{\cdot}5 \div 2{\cdot}58$. By this method the pupil has to accustom himself to division by a divisor which is partly a whole number and partly a decimal fraction. For pupils who experience no difficulty in dealing with such a divisor, this method has obvious advantages. Other pupils, however, accustomed to division by 258, find division by 2·58 confusing. The presence of the decimal point " worries " them, even though in practice they concentrate on the 2, or rather for the time being ignore the 58, each time they have to find a quotient figure.

Proportion

In proportion the unitary method should not be adhered to too long. Examples such as the following can be worked mentally and do not need the unitary method.

(i) If I walk 15 miles in 5 hours, how long will it take me at the same rate to walk (*a*) 48 miles, (*b*) 8 miles?

(ii) If the wages of 3 men amount to £12, 18s. per week, what will be the total wages of (*a*) 6 men for 2 weeks, (*b*) 9 men for half a week.

Such work easily leads up to the solution of examples in proportion as follows.

Example.—If the cartage costs on 30 tons of goods carried a distance of 115 miles is £85, what will be the cost of cartage on 69 tons carried a distance of 80 miles.

The cost on 30 tons carried 115 miles is £85.

The cost on 69 tons carried 80 miles is £x.

$$x = \pounds\overset{17}{\cancel{85}} \times \frac{\overset{3}{\cancel{69}}}{\underset{10}{\cancel{30}}} \times \frac{\overset{8}{\cancel{80}}}{\underset{23}{\cancel{115}}}$$

$$= \pounds 136. \quad \text{Answer.}$$

The multiplying ratios are obtained as follows. In the second case 69 tons have to be carried as gainst 30 tons in the first instance. The cost will be more being in the ratio of $\frac{69}{30}$. Again the distance is less and the cost will be less being in the ratio of $\frac{80}{115}$.

Percentages

As much use as possible should be made of the fact that a *percentage is a fraction whose denominator is* 100, $5\% = \frac{5}{100}$. "Interest at the rate of 5% per annum" means that the borrower has to pay to the lender each year $\frac{5}{100}$ of the money as rent for the use of that money.

Example.—The national capital before the war was estimated at £15,019 millions. Transport, industries, &c., absorbed £3753 millions and agriculture £876 millions. What percentage did each form of the whole?

1. Fraction of national capital represented by transport, &c.

$$= \frac{£3753}{£15019}$$

$$= \frac{3753}{15019} \times 100\%$$

$$= 25\% \text{ approx.}$$

2. Fraction of national capital represented by agriculture

$$= \frac{876}{15019} \times 100\%$$

$$= 5{\cdot}8\%.$$

Business Arithmetic

The various applications of percentages constitute to a very large degree what is termed the arithmetic of the business world. Discount, profit and loss, commission, simple and compound interest, discount and bills of exchange, stocks and shares, usually form the bulk of this section of the arithmetic syllabus. The work at this stage will lose most of its value if the pupil does not begin to feel that his arithmetic is bringing him into contact with the world outside school, the world of business and home affairs as viewed by the ordinary citizen. The problems should be genuine applications to the affairs of the real world. For this reason compound interest should certainly be extended to house purchase and mortgage interest, bank loans, insurance and annuities, saving and deposit accounts, and an explanation of stocks and shares given which will make them more than mere arithmetical examples.

Little need be said as to " method " of this applied arithmetic. As already stated, the actual arithmetic consists very largely of the application of percentages, and the " fraction " definition simplifies most of the work. In simple interest, for example, formulæ are unnecessary.

Example.—Find the simple interest on £535 at 6% for 7 years.

$$\text{The simple interest per annum} = \frac{6}{100} \times £535.$$

$$\text{The simple interest for 7 years} = 7 \times \frac{6}{100} \times £535$$

$$= £224,\ 14s.\ 0d.$$

Little time should be spent on examples usually included in textbooks on finding time and rate per cent, since these have little practical value.

The usual textbook examples of finding compound interest for a few years only should not be allowed to exhaust the subject. The practical application of compound interest to insurance, mortgage loans, and so on, necessitates the use of: (*a*) compound interest tables; (*b*) compound interest formulæ and the use of logarithms. The necessity for the use of logarithms of more than four figures in order to give a satisfactory degree of accuracy in certain examples will of course be appreciated, but the compilation of compound interest tables will be the better understood if a few examples at least are worked by logarithms.

ELEMENTARY MATHEMATICS

CHAPTER I

Arithmetic and Elementary Mathematics

In the preceding section on Arithmetic, it has been made abundantly clear that although in this book one section is devoted to arithmetic and another to elementary mathematics, no such complete isolation of arithmetic from elementary mathematics in the school course is for one moment intended. Of all the reforms that have been brought about in mathematical teaching during the past twenty-five years, one of the most important and far-reaching has been the unification of all branches of mathematical science and art. The reform has aptly and justly been described as " the arithmetization of geometry and the geometrization of arithmetic by the intermediacy of generalized arithmetic and of algebra ".[1] Although therefore this present section is devoted to the teaching of elementary mathematics it is to be regarded as entirely complementary to the preceding section.

The term " elementary mathematics " here used is to be understood to refer to the mathematics which can normally

[1] Benchara Branford in *A Study of Mathematical Education.*

be undertaken by pupils in a four years' course beginning at the age of eleven. This of course does not mean that the mathematical syllabus for pupils up to the age of eleven is to be confined entirely to arithmetic. On the contrary, the process of generalization will begin as soon as the pupil feels the necessity for or the advantage of passing from the figure symbol to the general symbol. Mensuration will begin the process of "geometrizing" the arithmetic, and such generalized arithmetic as the formulæ developed from mensuration will form the introduction to algebra. In the main, however, this section covers a four years' course in geometry, algebra, and trigonometry. Mathematical education and not examination requirements has been kept in mind. The free development of the post-primary school will not, it is hoped, be hampered by the imposition of a compulsory school leaving examination. The mathematical masters and mistresses in such schools will therefore be in the very enviable position of being quite free to give due attention to the cultural value of the subject.

Development of Numerical Symbolism

Historically arithmetic has developed gradually and continuously, first from mental arithmetic to written or symbolic arithmetic, then into generalized arithmetic, passing into symbolic algebra or algebra proper. Whilst this process has been developing slowly through the ages, generalized arithmetic first, and at a later stage algebra have been closely associated with the measurement of space, which study in due course developed into geometry and trigonometry. The development of the race from the period when the individual's only record of the number of sheep he possessed was the very possession of the sheep, to the period when the number was recorded by a number of things, such as a pile of stones representing the sheep, marks the first step—an enormous one—in symbolic representation. As man developed in experience and in intelligence so he developed his symbolization. He symbolized a group of objects by a single object in order to simplify his process of counting. That is, from the stage of representing objects by other objects man passed to the stage of representing a group of objects, say 10, by one object, i.e. by a compound unit. From this stage he passed in the course of years to the stage of written numerical symbols, pictures, and other signs. Thus a picture of a man holding up his hands in astonishment was the Egyptian symbol for 1,000,000. Then came the wonderful discovery that instead of having innumerable symbols to represent numbers the relative position of the symbols could be utilized to indicate different values, thus making it possible to represent any number by the use of only nine such symbols. So the abacus was invented. The graphical abacus or place value of written symbols followed, and the process thus started developed finally into the present highly complex abstract system of symbols.

Such a brief sketch of the development of numerical symbolism conveys but a poor idea of the enormous strides in man's intellectual development which each stage really represents, but it is sufficient to show how important it is that the teacher of mathematics should appreciate the fact that the step from arithmetic to highly symbolical algebra and other branches of elementary mathematics is not one that can be taken suddenly. Not only will an acquaintance with the history of the development of mathematics through the ages from the mathematical empiricism of the Egyptians to the scholarly and scientific investigations of the Greeks give an insight into the development of the mind of the pupil, but it will guide the teacher in solving problems of method and the syllabus. He will not, for instance, be making the blunder of attempting to teach highly symbolic algebra before ample preparation for such has been made by way of a preliminary training in generalized arithmetic, neither will he attempt to teach "textbook mechanics" but will see to it that such highly abstract mathematics is approached through practical experience and experiment. Neither will he in the early stages of so-called practical geometry keep the pupils at rule-of-thumb use of mathematical instruments and at work which demands very little thought or reasoning. On the contrary, he will be keenly alive to the necessity of ensuring that the development of intellect is not arrested through want of a judicious blending of thought process and sense experience.

Teaching methods then must be based on a wise understanding of the spirit and order of the historical development of mathematics. On the one hand, young pupils must not be expected to possess the reasoning powers necessary to understand the principles underlying a highly developed science. On the other hand, their intellectual growth must not be arrested by confining the activities of the pupils to so-called practical work requiring little or nothing in the way of thought processes. Mathematical education should ensure the development of the individual both as a " doer and a thinker ".

CHAPTER II

The Non-selective Post-primary School Course

In the Hadow Report the following is suggested as a suitable four years' course of mathematics in a non-selective central or senior school.

Numbers. Growth of the number system.
Elementary operations with the usual applications.
The meaning of a fraction. Simple operations with fractions.
Decimals.
The measurement of length, area, volume, weight, capacity, and time with appropriate tables.

The metric system.
Areas of rectangles, squares, triangles, surface of prisms, &c.
Appropriate geometrical work.
Volumes of prisms.
Generalization of results in above work on areas, &c. Introduction of symbols, construction of elementary formulæ. Use and manipulation of formulæ. Easy equations. Transformations of formulæ for purposes of computations.
Easy factors.
Use of squared paper. Construction, meaning, and use of graphs. Drawing to scale.
Meaning and use of averages.
Factors: common factors; H.C.F. and L.C.M.
Simple algebraic examples.
Further work on fractions.
Decimalization of money. Calculation of cost.
Ratio: constant ratios. Ratios connected with angles. Sine, cosine, and tangent of an angle.
The right-angled triangle.
Surveying problems and their practical application.
Proportional division. Similar triangles.
Mensuration of the circle, cylinder, pyramid, cone, and sphere with appropriate geometry.
Percentages with applications to interest, insurances, &c.
Compound interest.
Indices, logarithms.
Investments. Foreign currencies and methods of exchange.
True discount and present worth.

Basic Principles.—The principles on which this suggested course of work is based are as follows:

1. Arithmetic should not be regarded, as it has been and frequently still is regarded, solely as a bread-and-butter subject.

2. The amount of arithmetical knowledge indispensable for providing the necessary facility and accuracy in arithmetical work required by the pupil in his after-school life is comparatively small.

3. Instead of justifying the time usually devoted to the subject by adding to this comparatively small amount of necessary arithmetical knowledge matter which is "often without meaning to the child and is seldom of value to him

in after life, attention should be devoted to giving pupils a wider mathematical training ".

4. The modern industrial system and the part played by science in the modern civilized community make greater demands upon the mathematical knowledge of the ordinary citizen.

5. Civic, national, and even international finance are now closely associated with our daily existence, and demand for their intelligent comprehension an increasing amount of mathematical knowledge.

6. This wider mathematical training which modern life demands for intelligent citizenship necessitates the replacement of much of the traditionary arithmetic of the schools by suitable parts of mensuration, algebra, geometry, and trigonometry.

7. The introduction of this new material necessitates a modification of the methods of treatment: the " mechanical, lifeless, and abstract treatment of arithmetic giving place to more vivid, more logical, and more practical methods in teaching the subject—methods which will cause the pupil to appreciate both the beauty of mathematical truths and their practical applications ".

8. " *On the one hand are the abstract relations which these mathematical truths have between themselves, and on the other are relations to realities outside themselves. The history of mathematical progress is a record of the development of these two aspects of mathematical truths in close association with each other, and the view that they exist as distinct forms of intellectual activity has exerted a harmful influence upon mathematical teaching.*"

9. Every course should aim at developing in the pupil an appreciation (i) of the meaning and teaching of a coherent system of mathematical ideas, and (ii) of the importance of the subject as an instrument of scientific, industrial, and social progress.

This Hadow syllabus is, as already stated, only a suggested syllabus and somewhat limited in its range, as it is only intended to provide a course suitable for a non-selective type of school in an urban area. It is instructive, however, to note that a syllabus of such modest pretensions, designed for a restricted purpose, is nevertheless based on the principles mentioned above. Thus the first part of the syllabus covers almost all the work in fundamental arithmetic indispensable for the wider application of the subject to the world of affairs.

This wider mathematical training is directed to the study of problems which in their application bear a close relationship to problems of everyday life. This, of course, is but a syllabus and as such deals with the material of the mathematical course. The method of treatment will determine to what extent the pupil will realize that mathematics is not just a bread-and-butter subject but an instrument which, if properly appreciated, will enable him to take a more intelligent interest in and to understand more clearly problems of "scientific, industrial, and social progress". At the same time the syllabus ensures that mathematics is treated as a coherent whole. There are no watertight compartments. It is very important to note that the syllabus, however, contains more geometry than algebra. Here again it is following very closely the historical development of the subject, and the teacher will be well advised to make a special note of the fact that in this syllabus algebra is definitely recognized as being a highly abstract branch of mathematics. There is indeed a complete absence of academic algebraic work. Algebra is introduced as a natural development of the work in elementary mensuration leading to the establishment of formulæ. The growing complexity of the work creates the necessity for a knowledge of what may be termed formula-manipulation, and thus the pupil passes on to the solution of equations, factorization, and other simple operations.

In the same way the geometry is closely associated with the work of arithmetic and more particularly in mensuration.

It is largely experimental geometry, the only formal deductive work being that which arises from this experimental geometry. Such deductive work is of course important inasmuch as it prevents the subject becoming purely mechanical work in geometrical exercises. The work is practical and provides the concrete material for abstract reasoning.

Opinions may differ regarding details such as the somewhat delayed treatment of indices and logarithms, the inclusion of true discount and present worth, and the somewhat meagre amount of solid geometry—yet it is essentially a syllabus which follows the historical development of the subject, is suited to the age and mentality of the pupils, and ensures that their mathematical work, on the one hand, bears some relation to the world outside, and at the same time secures for them the opportunity of appreciating the meaning of a " coherent system of mathematical ideas ". One feels that one would rarely, if ever, hear such questions as " What is the good of it?" " What is it for?" put by pupils following such a course.

CHAPTER III

The Selective Post-primary School Course

The authors of the Hadow course discussed above state that central or senior schools taking this course as a basis should make such additions, especially in algebra and geometry, as " would be required by the character of the school, its objects, and the length of the course ". We come now to consider these additions which will convert the syllabus suitable for a four years' course in elementary mathematics for pupils in the non-selective type of school into one for a four years' course for pupils, presumably of higher intellectual capacity, in the selective type of school—the central or modern school.

Syllabus

For the first two years of the course the work must necessarily be fundamental, and therefore will differ but very little indeed from that of the syllabus discussed above. The rate of progress should, of course, be greater in these schools, and for this reason it will be possible to extend the syllabus in geometry and algebra during the third and fourth years.

If reference is made to the section on Arithmetic, it will be found that the syllabus there suggested for the four years' course in arithmetic covers in the first two years nearly all that is included in the Hadow four years' course, the third and fourth years being devoted to a revision and extension of the work of the previous two years, with special reference in the fourth year to the needs of the pupil's future career. The first-year course is very largely a course in mensuration, the other part of the syllabus being devoted to a revision of the work of the junior school. The second-year course completes the mensuration course and includes logarithms.

Mensuration.—Now this concentration on Mensuration in the first two years of the arithmetical course provides both an interesting and most stimulating course for boys and even girls at the age of eleven to twelve. The writer's experience convinces him that, provided that such work is based largely on practical experience and does not degenerate into applying and manipulating a number of formulæ, there is something intensely appealing and satisfying in this work to pupils at this age. There is historical authority, of course, for saying that this is as it should be.

We are, moreover, on safe psychological grounds in making mensuration the prominent part of our arithmetical work once the pupils have the usual arithmetical tools at their command. In so doing we are but developing the pupils' mathematical knowledge along the same lines as such knowledge has developed according to mankind's own experience. Mathematical masters will find that the more they can take their pupils out into the fields, with home-made surveying instruments for preference, and the more the work is done in the mathematical laboratory, the keener will be the interest shown and more vivid and real to the pupils will the subject become.

The work of measurement need not of course be confined to field work and the mensuration of solids. Much will depend upon the ability of the pupils and the type of school, but there is no reason why, preferably in the second year, work on density and specific gravity and measurement of mass, force, and velocity should not be recognized as a part of the mathematical course, transferring it if need be from the elementary science course. Most science masters who now attempt such work in the first-year science course know that it is not only uninteresting but forms a very difficult introductory science course. Its transfer to the second-year mathematics course as a part of the more advanced work in measurement would add variety to the mathematical work and would assist in further developing algebra and the use of formulæ along natural lines. It would also release time in the science course for more important work, preferably in the direction of extending the syllabus to include a course in elementary biology.

There is another reason for devoting the greater part of the arithmetic syllabus to mensuration. From the point of view of mathematical education, this is the starting-point of other branches of the subject. Algebra and geometry both develop naturally from such work. This of course is but saying again what was said above. In making mensuration the starting-point in the teaching of algebra and geometry we are but taking the pupil once again along the paths traversed by mankind in the development of mathematical knowledge. The practical inductive geometry or mensuration, largely empirical, gradually developed into the abstract geometry of the Greeks, and mathematical masters will do well to appreciate the importance to the young pupil of recognizing that the step from practical mensuration to *abstract* geometry and *abstract* algebra must be a gradual one. Only those who, having attempted to teach to young pupils abstract geometry and highly symbolic algebra along the old traditional lines, have abandoned such work in favour of practical mensuration and experimental work can appreciate how much more interesting and stimulating, how much more effective and convincing the latter can be. It is so because it is based on the pupil's own experience and is suited to his stage of intellectual development.

The Age Factor.—There is still one other important factor which must be taken into account when the work—and particularly the mathematical work—to be undertaken in the senior school is being determined, and that is what may be termed briefly the age factor, in some respects the most important factor of all. Pupils in such schools normally will be transferred at the age of eleven and remain at least until the age of fifteen, the most critical of the school years. That is to say, so far as elementary mathematics is concerned these pupils will begin their work just about the age, for the majority of them, of the onset of puberty and continue the study at least until the early years of adolescence. Every experienced teacher knows that during these years there is not only evidence of physiological changes but of psychical changes. Interest in " doing " diminishes and there is, as was mentioned in the section on Arithmetic, a steady decline in accuracy of achievement which reaches its maximum at about the age of fifteen for boys and from six to twelve months earlier for girls. The observant teacher notices that in the case of boys particularly, the falling off is most marked where physical growth is most rapid, and that the smaller boys seem to suffer less mental disturbance during these critical years. The observant teacher of wide experience also knows that the violence of this mental disturbance varies very considerably for different pupils. Some recover rapidly whilst others are in a state of mental unrest for quite a considerable time. Again, it is often the boys of rapid physical growth who are well into their sixteenth year before they have recovered, whilst the smaller boys have reached the period of stability much earlier.

All this is of the utmost importance to those engaged in the work of the senior schools, whilst for the teachers of mathematics it is of considerable importance. Roughly speaking, the senior school period falls into two well-marked divisions—one covering approximately the first two years from eleven to thirteen and the other from thirteen to fifteen and later. During the earlier period the pupil is for the most part still in his childhood, though the onset of puberty begins to show itself. There is, however, little evidence either of marked physiological or psychical changes. Interest in technique, in " doing ", is still dominant and there is as yet little, if any, evidence of a decline in accuracy. At about the age of thirteen, however, when the pupil is entering upon the first stages of adolescence, whilst interest in technique declines and with it the " curve of accuracy " begins to fall, other qualities appear. The pupil has for one thing reached the logical stage of his development—he begins to reason for himself, to find interest in underlying principles, to have opinions and to hold views. It is during these years that personality begins to show itself and the pupil's particular bent or talent begins to emerge.

For mathematics in particular these facts have special significance. It is evident that the mathematical curriculum for the first two years of the senior school period should be quite different from that of the later period. In the earlier period the pupil is for the most part finishing his childhood—he is still, as it were, living in the world of sense impression. At the same time, however, it must be remembered that the pupil is at the end of the period of childhood and that the change is not so sudden as it appears. He is beginning to ask " Why ", and he does find an interest even in a fairly long chain of reasoning so long as the facts upon which such reasoning is based are the outcome of experience. All such considerations point again to the advisibility of making mensuration an important part of the arithmetical course, and developing from this the beginnings of elementary mathematics, geometry, and algebra, and of postponing both formal geometry and purely abstract symbolic algebra.

In the later period of the senior school course, when the pupil's interest in sense impressions is declining and he enters the world of feeling and emotion, interest is now found in formal mathematical reasoning. There must be no slackening of effort—no mere taking things easily. The period is altogether too critical and is essentially the period of character formation. The decline in accuracy which now becomes evident must not be treated unsympathetically—neither must it be ignored altogether. There must be nothing in the nature of a deliberate attempt to overcome inaccuracy by forceful methods—neither must there be anything in the nature of a benevolent ignoring of the facts and condoning slackness. The highest must be exacted during this period as before, but whereas in the previous period the aim was to lay sound foundations and secure automatic mastery over fundamental facts, now we have to provide the maximum opportunities for the development of other powers. As was said above—a *difference* in curriculum must now be made and the more marked this difference is the more effective it will be. Hence once again the importance of " hastening slowly " so far as formal geometry and purely abstract algebra are concerned. The cultural value of mathematics, even in these days of improved methods, is too often destroyed for many a pupil, his interest in and his taste for the subject being killed by premature and dogmatic forcing of abstract rules. It is true that much can be done during the period of approximately eleven to thirteen to ensure an intelligent knowledge of the elementary facts of geometry and a clear understanding of the main working tools of the subject. It is also true that during the same period the necessity for and the advantage of the establishment and manipulation of formulæ can be appreciated. Some teachers would even go so far as to say that it is better to postpone elementary mathematics entirely until about the age of thirteen rather than spend the previous two years in the more formal type of work. When the pupil has reached the logical stage of his mental life, then, and only then, should he be plunged into this purely formal geometry and abstract algebra. This is not to say that during the earlier period simple types of reasoning within the capacity of the pupils should not be attempted. At no stage should there be anything in the nature of an abrupt change. Such a change at the age of thirteen, however, is less dangerous than at the age of eleven.

The most difficult problem of all is that which concerns the pupil who, as he passes from puberty to adolescence, suffers the greater amount of mental disturbance. For such, an alternative mathematical curriculum is certainly necessary. The outstanding characteristics of such pupils are their abundance of initiative and their marked social or group instincts. Not merely a change in syllabus but a change in method is necessary. Further, the work undertaken must provide the maximum outlet for individual initiative combined with ample scope for what may be termed team work. As yet such pupils have no great interest in purely formal mathematical work, though what may be termed collective reasoning as is shown in class work, as distinct from individual reasoning, will create the keenest possible interest. It is this somewhat curious combination of initiative and team spirit that tends to single these boys out as being those of marked personality, and work amongst them calls forth all the skill and sympathy which the teacher can command. For the teacher of mathematics, work with such boys is distinctly exacting. Above all, he himself must be a team leader, and during this somewhat difficult period he must devise methods which will provide them with ample opportunities both for team work and the exercise of initiative.

CHAPTER IV

First and Second Years—Early Stages

We come now to a consideration of the elementary mathematics course of the first and second years, not only of the senior selective type of school, i.e. the modern school of the Hadow Report, but of any school in which a mathematical course of at least four years' duration is a part of the normal school curriculum. Mathematical teachers will be in general agreement with the statement, " It is very desirable that the course of elementary mathematics in all types of school should be approximately the same. For the first two years of the course the work will be mainly fundamental and will not vary materially whether in a grammar school or a modern school."[1] Not only is such general agreement desirable in the interests of those pupils who at the end of the second year may be transferred from the senior to the secondary school or from the modern to the grammar school, but sufficient has been said in the preceding chapter to make it quite evident that there is both historical and psychological authority for asserting that the elementary mathematics course should be approximately the same during the first two years for all types of schools. Divergencies in the latter years there must be, for rate of progress, the type of school, length of the subsequent school course, and the future careers of the pupils are all contributory factors which help to determine the nature and extent of the later mathematical course.

[1] Hadow Report.

Whatever the subsequent structure, however, the foundation must be the best, and, as the best, it will be approximately the same for each type of school.

As previously stated, this fundamental course of the first and second years should develop almost entirely out of the work of measurement, i.e. the measurement of length, area, and volume, extended if possible to measurement in elementary hydrostatics, of density and specific gravity, and in elementary mechanics, force, and velocity.

Measurement of Length or Distance

For this course the pupil will require as measuring instruments a surveyor's chain and measuring tape, some simple form of "angle meter", as well as the usual mathematical instruments—compass, dividers, set squares, and ruler.

The work begins with problems in finding position. If playing-field accommodation permits, treasure-hunting expeditions will help to arouse enormous interest. The pupils are told, for instance, that "treasure" C has been hid 40 ft. from the corner A of the building and 50 ft. from the tree B. Find the "treasure". The pupils devise their own methods of finding the treasure, using measured lengths of ropes for the purpose. Three or four such exercises are given and the pupils return to their room to produce drawings to scale to illustrate their work.

Exercises of the following type now follow:

(*a*) A and B are two towns 25 miles apart. Another town C is 15 miles from A and 20 miles from B. Draw a plan showing the positions of the three towns.

(*b*) Find a point Z which is 4 in. from X and 5 in. from Y, X and Y being 6 in. apart.

In like manner a treasure hunt is organized for the discovery of hidden treasure, given its distances from a given line (such as the base of a wall) and a given point, or given its distances from two given lines. Variations of this problem in the form of geometrical exercises similar to those above will then follow. The preliminary " treasure hunt " ensures that the geometrical work which follows is developed out of real experience. Any such practical exercise will of course serve the purpose so long as the subsequent work in practical geometry is thereby made real, ensuring that such practical geometry is something more than merely " doing geometrical exercises ".

Incidental work on elementary facts concerning the parts

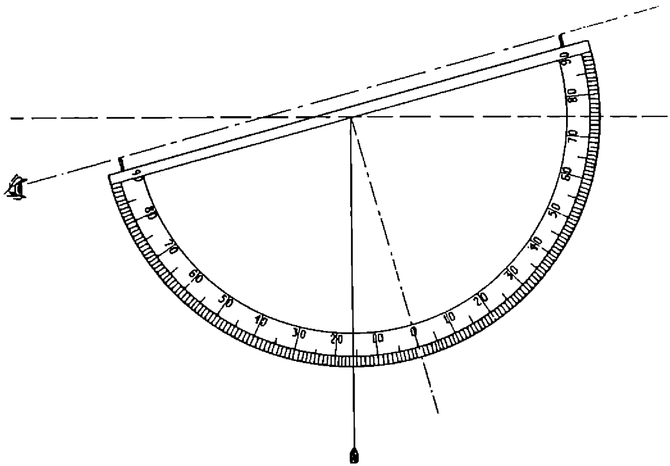


Fig. 1.—For Measuring Angles of Elevation and Depression in Vertical Plane

of the circle used in these exercises, the arc, circumference, radius, &c., will be treated as they arise. Simple ideas of loci can also be discussed.

Angles.—Some of the foregoing exercises can now be used for introductory work on angles. The first problem, for instance, can be solved by using AB as a base line and by knowing the angles ABC and BAC. The usual elementary work on angles, points of compass and compass bearing, and the more fascinating examples in measurement of inaccessible distances and heights of buildings follow.

For this purpose a simple and easily constructed home-made instrument for measuring angles—an angle meter—is required and can be made as follows.

A semicircular piece of wood about 9 in. or 12 in. in diameter and about $\frac{1}{8}$ in. thick is fitted with a simple plumb line suspended from the centre and two sighting points—screw eyelets or thin nails having their heads removed will do. The semicircle is graduated as shown, the plumb line falling over the mark 0° when the instrument is in a horizontal position. The reading indicated by the plumb line when the instrument is inclined will then give the angle of inclination (see fig. 1).

For the measurement of angles in a horizontal plane, the other side of the instrument can be graduated as an ordinary protractor and a simple form of clock-hand made to pivot on the centre of the diameter. A more elaborate but still "home-made" instrument is illustrated in fig. 2. With such an instrument, easily made in the handicraft room, quite accurate results can be obtained.

Too much time, however, must not be devoted to outside practical work. Care must be taken that the development of reasoning and imagination is not arrested by mere repetition of exercises predominating in sense perceptions. It is a good plan to give pupils a few days in which to collect data in their own time. The heights and distances thus measured can then be utilized in class. In any case, however, all such practical work should be followed by exercises demanding careful thought, and by exercises which will tend to extend the body of knowledge already acquired.

The practical work mentioned above will easily lead on to the consideration of vertical, horizontal, and perpendicular lines and also to parallel lines.

Consideration of vertically opposite angles and their equality and the nature of complementary, supplementary, and adjacent angles will likewise be possible. The terms acute, obtuse, and right angles will also be dealt with, and the construction and classification of triangles, the bisection of lines and angles can be undertaken. Consideration of parallel lines

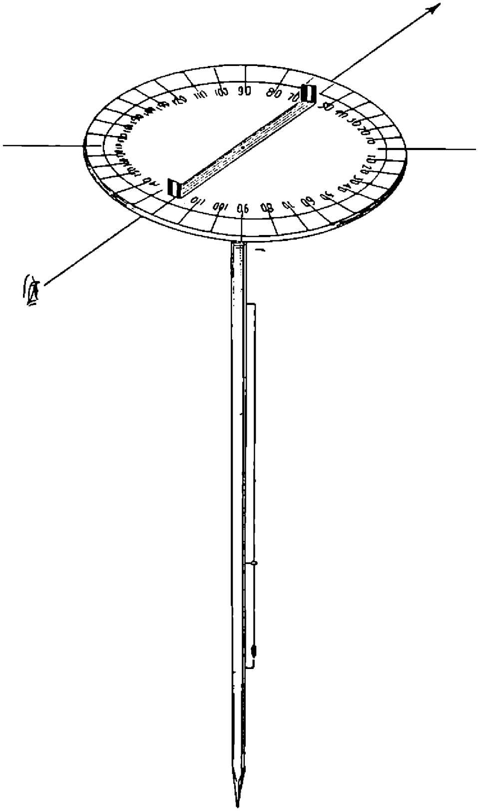


Fig. 2.—For Measuring Angles in Horizontal Plane

will include discussions on alternate angles and the division of a line into any number of equal parts.

Nothing in the nature of formal or rigorous deductive proofs will be attempted. The aim is to give the pupils an intelligent knowledge of such elementary facts of geometry as arise naturally from their intuitions and their practical work in mensuration. The boys do not *prove* that vertically opposite angles are equal by measuring the angles with a protractor. They *know* by intuition that such angles are equal, and although they may demonstrate the fact by the rotation method, the truth for them at this stage is in the nature of a postulate.

Definitions.—Definitions at this stage, if used at all, and indeed at subsequent stages, should be " working definitions ". Rigour of definition beyond the capacity of the pupil to assimilate must be avoided as much as rigour of deductive proof. The exactness of the definition must be appropriate to the stage of development of the pupil, and all that is demanded is that the pupil is guided to the discovery of the definition himself, and is able to apply it. There must be just that degree of refinement and precision, no more and no less, that is within the capacity of the pupil to assimilate.

The work done thus far covers all that is necessary at this elementary stage in what is termed the geometry of position. The geometrical problems will have been stated, not in terms of points, lines, and angles, or geometrical exercises thereon, so much as in terms of familiar objects—trees, roads, rivers, &c. What may be termed the pupil's geometrical imagination has been stimulated by simple research suggested by the presentation of problems arising from real experience. The skilful teacher will have no difficulty in introducing the pupil quite naturally to symbolic expression. Thus in the work on the construction of triangles the introduction of the conventional capital letters A, B, and C for angles, and of *a*, *b*, and *c* for the length of the opposite sides will be made when the pupil is ready to appreciate the advantages of such shorthand symbols. The pupil is not called upon to juggle with meaningless symbols, but is using symbols just when he feels the real need of them, or at least can appreciate their value. His research under the guidance of the teacher, for instance, may have led to the discovery that in any triangle $A + B + C = 180°$, and that $a + b > c$, a and b being the lengths of any two sides of a triangle, and c the length of the third. Thus even at this early stage the arithmetic, geometry, and algebra constitute one whole—" a coherent system of mathematical ideas ".

Measurement of Area

From the measurement of length the pupil passes naturally to measurement of area, and in so doing will be studying the geometry of space and extending his use of symbolic representation.

Rectangle.—The first lesson at this stage will deal with the area of a rectangle. Plans are drawn to scale, say, of the school hall and the formroom, and a rough estimate of their comparative sizes is made. (Details such as recesses and fireplaces should be omitted from the plan.) Carefully chosen exercises of the following type will follow. Find by drawing to scale how many yards of linoleum 2 yd. wide will be required to cover a room measuring 12 ft. wide and 15 ft. long (only examples in which either length or breadth is a multiple of width of the covering will at this stage be given). These exercises will lead easily to the solution of the more important problem—compare the floor-space of a room measuring 12 ft. by 15 ft. with that of one measuring 20 ft. by 8 ft. From a comparison of sizes of rectangles the pupil passes easily to the problem of finding the area of a room of given length and breadth. Provided that the foregoing work has been thoroughly understood, the pupil feels the need for the rule which, under the guidance of the teacher, he now sets out to discover. In other words the pupil has not been plunged into the problem of finding a rule in which as yet he has no interest. He has been led step by step to feel the necessity for the rule. Similarly, having found the rule, he is ready to appreciate the utility of the symbolic way,

$A = l \times b$, or $A = l \,.\, b$, of expressing the rule. Its application follows, and at the same time scope is found for further work in fractions, decimals, and the metric system. Some useful work in algebra in the form of manipulation of formulæ can also be undertaken. For example, $A = l \,.\, b$ is changed into $\frac{A}{b} = l$ and $\frac{A}{l} = b$.

By the use of rectangles the following algebraic expressions are illustrated and learnt:

1. $a(b + c) = ab + ac$. Thus:

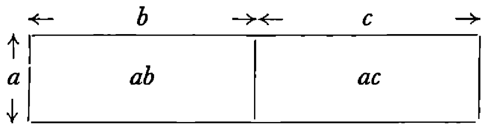


Fig. 3

Similarly $a(b + c + d) = ab + ac + ad$.

2. $(a + b)(a + b)$ or $(a + b)^2 = a^2 + 2ab + b^2$.

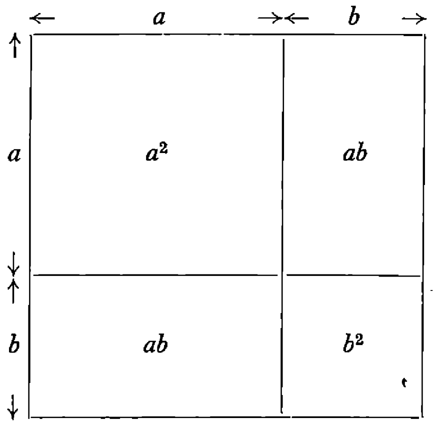


Fig. 4

3. $(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ac.$

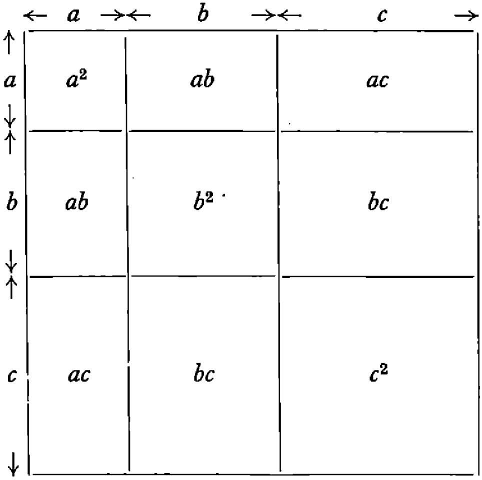


Fig. 5

4. $a^2 - b^2 = (a + b)(a - b).$

Here the $a^2 - b^2$ is read as the difference between the squares on the lengths of a and b, and $a - b$ as the length equal to the difference of the lengths a and b.

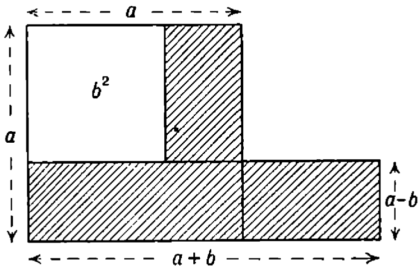


Fig. 6

It is perhaps advisable at this stage to defer consideration of the formula $(a - b)^2 = a^2 - 2ab + b^2$. The pupil is scarcely ready to appreciate the meaning of or necessity for the square of such a quantity as $(a - b)$. The formula is too abstract. In No. 4 above, the negative sign is used in the sense of " a difference between " and not as indicating a negative quantity.

5. $(x + 3)(x + 5) = x^2 + 8x + 15$.

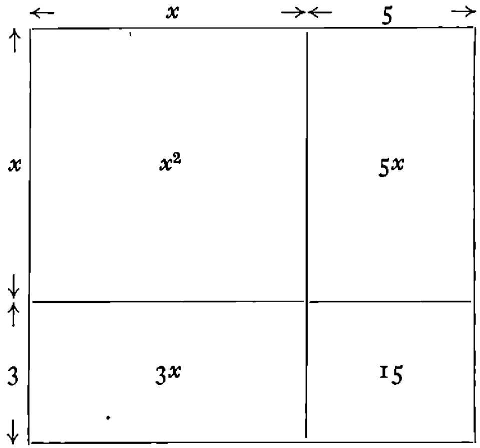


Fig. 7

These exercises will arouse keen interest at this stage and ample practice should be given in their reproduction. It will be observed that the pupil is dealing here quite early in his elementary mathematics course with the algebraical identities which appear much later in most elementary algebra textbooks. Instead, however, of being abstract algebraic expressions, they are here treated as examples in the manipulation of formulæ in mensuration. [In the section on Arithmetic it was urged that the pupil should be so familiar with his tables that whenever he received the stimulus, say 6 times 9, he should *automatically* respond 54. So with these formulæ. The pupil should be so familiar with them that he is able without the least hesitation to repeat or write down any one of them. Such familiarity means much at a later stage. Above all it ensures accuracy and confidence in working.]

Quadrilateral and Triangle.—Passing from the area of the rectangle, the pupil is given the larger problem of finding the area say of a quadrilateral shaped field. If the playing-field is thus shaped so much the better. If not, then the pupil is taken into the familiar world of "make-believe", and for the time being the playing-field becomes an "estate", and a portion, of convenient shape, is roped or pegged off to represent tenant Farmer Jones's field. Most textbooks suggest passing from the rectangle to the triangle, but as yet the pupil has not met the necessity for finding the area

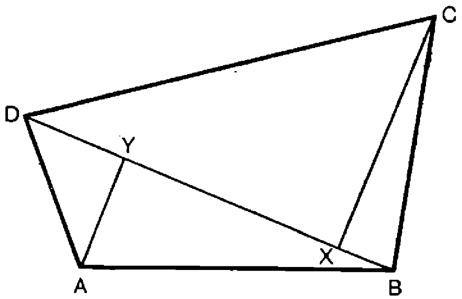


Fig. 8

of a triangle. It is more probable that, after having dealt with the rectangle, the problem of finding the area of an irregular four-sided figure will present itself to his mind, and it is in his attempt to solve this problem that he meets with the necessity for finding the area of a triangle. Let us suppose the field is shaped as in fig. 8.

A rough sketch plan is made and the problem is presented —what measurements must be made in order to draw a correct plan of the field? The probability is that the lengths of the sides only will be first suggested, but experiment will prove that the data thus obtained are insufficient. Further discussion will suggest that the following measurements will prove sufficient: (i) Length of AB; (ii) angle BAD; (iii) angle ABC; (iv) length of AD; (v) length of BC. With the use of the home-made instruments and the tape measure, the necessary measurements for making a correct plan of the field are now obtained. Under the guidance of the teacher the pupils attempt the problem of finding other measurements which will enable them to produce a plan, this time without measuring angles. Reference is made to the sketch plan already drawn and the measurements first of AB, AD, and then the diagonal DB, and then the two remaining sides BC and DC are suggested. The pupils will readily see that this method has certain obvious advantages, as measurement of lengths can be undertaken with greater accuracy than measurement of angles. The obvious disadvantage, viz. a considerable amount of journeying, is also appreciated, and attention is now drawn to the problem of economizing effort. The pupils are told that the surveyor uses a method which overcomes both the disadvantage of measurement of angles and the waste of effort incurred in going all the way round the field. Skilful questioning will arouse interest in this new aspect of the problem. It may be necessary to give the hint that a diagonal, say DB, is used as a base line, but there should be no great difficulty in leading the pupils to discover that the measurements BX, XY, and YD, and the two offsets XC and YA, are all that are necessary.

The work of taking these measurements is simplified by the use of a cross staff, which can be made easily in the handicraft room. It consists of a cubical block with two vertical slits perpendicular to each other (see fig. 9*a*), and supported on a broomstick pointed at one end. [An alternative instrument is also illustrated (see fig. 9*b*), the crosspieces being either two metal strips or wooden laths.] The instrument is used thus. It is placed as near as possible to the point X, the exact position of which is yet to be found, with one slit pointing in the direction DB. It is then moved along this base line until, looking through the slit, the point C can be seen. The point X is thus located exactly and the necessary measurements are taken. Y is similarly located and the corresponding measurements taken.

The plan is now drawn to these new measurements and the problem of finding the area is tackled. The pupils see that the problem is one of finding the area either of four

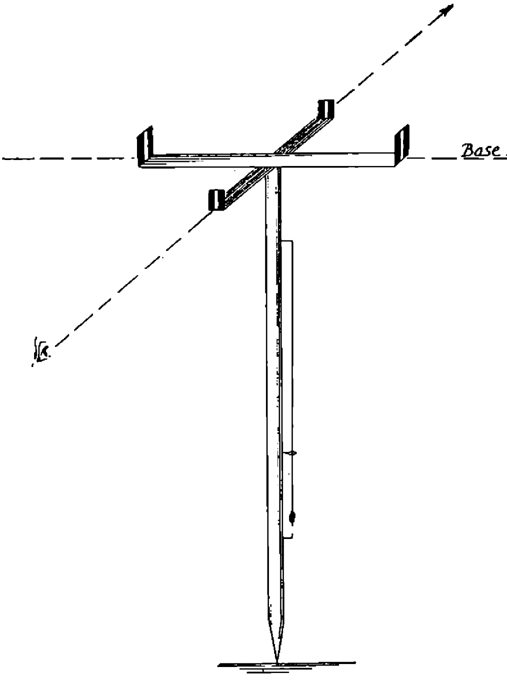


Fig. 9b

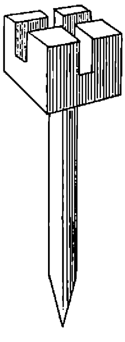


Fig. 9a

right-angled triangles or of the two triangles into which the quadrilateral is divided by the diagonal. It is at this point that the pupil meets the necessity for knowing how to find the area of a triangle. The lesson could have started—" We are going to learn how to find the area of a triangle "; but there is a vast difference in teaching method between thus forcing the attention of the pupils to such a problem and arousing their interest therein by leading them *to feel the need for such knowledge,* especially when such need has arisen out of a natural development of work which they have already understood and carried through.

At this point two methods of procedure are before the teacher. He may break off entirely from the problem in hand and guide the pupils to the solution of the general problem of finding the area of any triangle, returning to the quadrilateral and its two triangles as particular cases of the

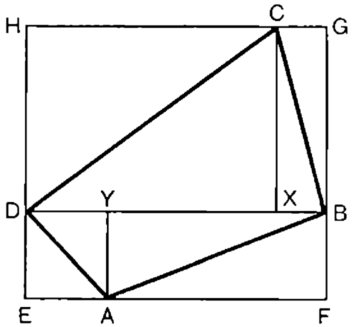


Fig. 10

general rule thus obtained. Or he may continue with the problem in hand and proceed from the particular triangles to the solution of the general problem of finding the area of any triangle. The latter is obviously the better course. The pupils have no difficulty in seeing that the triangle ADY is half a rectangle and the rectangle is then sketched in. The same applies to the triangle ABY, and the pupil quickly sees that the area of the triangle ABD is one-half the rectangle EFBD or half DB . AY (see fig. 10). The other triangle, BCD, is seen to be half the rectangle DBGH or half DB . CX.

In this particular case the pupils see that the area of the quadrilateral is half the area of a rectangle whose sides are respectively equal to the length of one diagonal and the sum of the offsets. In other words they have discovered the formula,

$$\text{area of quad.} = \tfrac{1}{2}\,(\text{diagonal} \times \text{sum of offsets})$$
$$= \tfrac{1}{2}\, d(x + y).$$

They have also seen that the area of each triangle was found to be half the rectangle erected on the base of the triangle and of the same height. The question presents itself: is this always true? and any triangle ABC is drawn (fig. 11), and its area investigated.

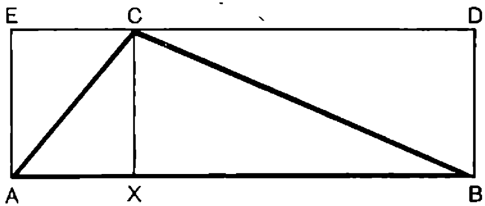


Fig. 11

Area of triangle AXC = ½ rectangle AXCE.
,, ,, BXC = ½ ,, XBDC.
∴ Area of triangle ABC = ½ ,, ABDE
= ½AB . BD
= ½AB . XC,

i.e. the area of a triangle is half the product of the length of the base and the altitude.

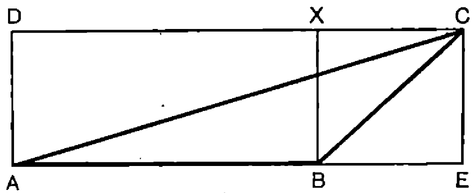


Fig. 12

The case of the obtuse-angled triangle, in which a side not opposite the obtuse angle is taken as a base, should be investigated (fig. 12).

$$\text{Area of triangle AEC} = \tfrac{1}{2}\text{ rectangle AECD.}$$
$$\text{,, ,, BEC} = \tfrac{1}{2}\text{ ,, BECX.}$$

By subtraction:

$$\begin{aligned}\text{Area of triangle ABC} &= \tfrac{1}{2}\text{ rectangle ABXD}\\ &= \tfrac{1}{2}\text{ AB . BX}\\ &= \tfrac{1}{2}\text{ base} \times \text{altitude.}\end{aligned}$$

Ample exercises on the area of quadrilaterals and triangles will now follow, with, of course, further practice in the manipulation of formulæ. Some mathematical masters may complain that the pupil is getting "no algebra", and that by this time work in simple and simultaneous equations should have been reached. The answer is that at this stage the pupil feels no need for purely abstract algebra. That to force him to it is to run the risk of creating a distaste for the subject by compelling him to undertake work which as yet is unintelligible to him, and would therefore become purely mechanical manipulation. He does understand the use of formulæ and his algebra for the present is confined to such work.

Application of Formulæ.—Application of formulæ may be extended by such exercises as the following:

1. Find a formula for the area of fig. 13. Give the formula in the form which can be applied most readily.

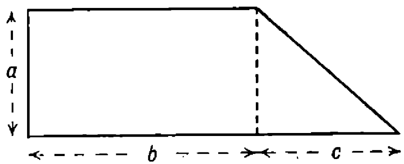


Fig. 13

2. Find the area when $a = 5$ units, $b = 7$ units, and $c = 4$ units.
3. Find a formula which will express the ratio of the areas of the two parts.
4. Find a formula for the area of figs. 14, 15, and 16.

There should be no difficulty in devising problems of this type. They can be made more complex by requiring, say,

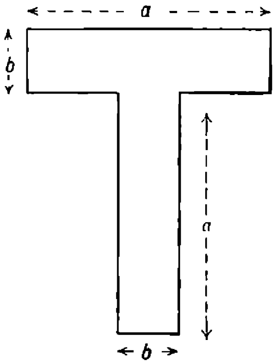


Fig. 14

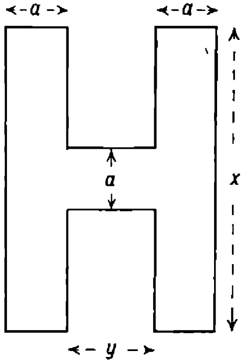


Fig. 15

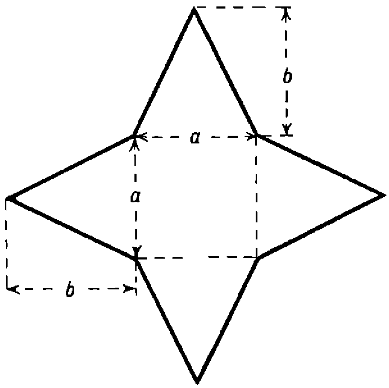


Fig. 16

a formula for the weight of material in each, given the weight as W oz. per square unit.

The building up of a few complicated formulæ of this type will prepare the way for such exercises as the following:

1. Find the value of $\frac{Wa}{b(W + P)}$ when $W = 4$, $a = 8$, $b = 2$, and $P = 12$.[1]

2. The breaking weight in a beam can be calculated from the formula $W = \frac{KBD^2}{L}$ where W is the weight in hundredweights, B is the breadth, D is the depth, and L is the length, all in inches. Calculate the breaking weight on a cast-iron beam 12 ft. in length, 3 in. broad, $4\frac{1}{2}$ in. deep, taking $K = 46$.[1]

Further practice should also be given in the identities already learnt. As already stated the pupil cannot be too familiar with such identities as $(a + b)^2 = a^2 + 2ab + b^2$, $a^2 - b^2 = (a + b)(a - b)$. He should also be able to write down without hesitation the answers to such products as $(a + 3)^2$; $(2a + 3b)^2$; $(x + 3)(x + 5)$; $(2x + 3)(3x + 5)$.

Easy factorization of the following types of expressions can also be attempted as further examples in the manipulation of formulæ: (i) $ax + bx$; (ii) $a^2 + 2ab + b^2$; (iii) $a^2 - b^2$; (iv) $ax^2 + bx + c$.

The identity $a^2 + 2ab + b^2 = (a + b)^2$ can now be used as the basis of teaching square root, if this has not already been dealt with in the arithmetic course.

Pythagoras' Theorem.—The introduction of square root and the previous work on the right-angled triangle makes a convenient point at which to introduce Pythagoras' Theorem. The teacher has a choice of one or two methods of approach. He can set the pupils to construct carefully a right-angled triangle, given the lengths of two of the sides. The pupils are divided into half a dozen groups, each group being given different dimensions for these two sides. Careful measurement of the hypotenuse is then made by each boy, and the average length of the measurements thus taken by each individual is accepted as the length of the hypotenuse of each group's triangle. The pupils are then instructed to square the length of each of the given sides and compare the sum of these with the square of the length of the hypotenuse.

[1] Gibbs, *Engineering Mathematics*.

The other method of approach is to tell the pupils that Pythagoras, a Greek philosopher who lived 500 years before the Romans first came to Britain, made the wonderful discovery that the square on the hypotenuse of a right-angled triangle is equal in area to the sum of the areas of the squares on the other two sides, the pupils then being shown the demonstration of the truth of the theorem as explained below.

It should be clearly appreciated that the first method of approach is neither a proof of the theorem, nor can it be dignified by the name of geometrical research, undertaken by young pupils to discover in the course of half an hour what only a man of genius, working with a limited knowledge of mathematics and very crude instruments, discovered. The experiment in measuring the sides is a useful one, but it will fail in its object if it is regarded as anything more than a method of approach to the theorem. This point is purposely stressed. Under an enthusiastic mathematical master the pupils can catch something of the spirit of mathematical research, and an appreciation of the genius of the mathematician, if they are impressed as they should be by the importance of the discovery. The theorem should be referred to as a "wonderful discovery", as indeed it was, and the pupils should not accept it as something quite ordinary, or as an interesting fact taken for granted. In some respects the statement of the historical fact of the discovery is a more impressive method of approach or introduction than the work in measurement, as this latter method is apt to deceive the pupils that the discovery was a fairly simple matter.

The theorem having been introduced, practical demonstration of its truth will help to impress the fact on the minds of the pupils. Two modes of such demonstration are given.

1. Through the point where the diagonals of the square on the larger of the two sides about the right angle meet, one line is drawn parallel to the hypotenuse of the triangle, and another at right angles to this line, thus dividing the square into four equal parts. These are then cut out, and if

placed in the corners of the square on the hypotenuse as shown in fig. 17, it will be found that a space in the centre of the latter square can just be filled by the square on the other side about the right angle.

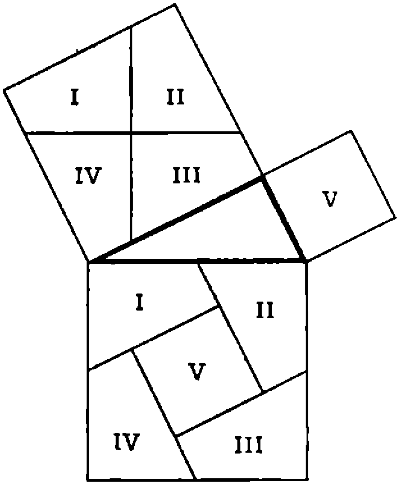


Fig. 17

2. A more ingenuous method is as follows. A figure ABCDEFG, made up of two squares ABFG and CDEF, is drawn as shown (fig. 18). This figure represents the squares on the two sides about the right angle of a right-angled triangle, and can be cut into three parts which when fitted together make a square equal in area to the square on the hypotenuse. The division is made by taking a point X in GE, such that GX = FE, and joining AX and DX. The triangles AGX, DEX are then removed, and placed in position as indicated in fig. 19, so as to complete the square.

Although at this stage the formal proof is not given, the nature of the above, as demonstrations only, should be emphasized. It is a good plan to indicate to the pupils that a proof can and will be given later. The basis of this method of proof can be demonstrated by measurement as follows. A line CX, drawn from the right angle C of the triangle at right angles to the hypotenuse if produced to meet the opposite side of the square in Y, divides the largest square into two rectangles which are respectively equal in area to the other two squares. The fact can be verified by careful measurement, and the pupils are informed that later they will prove that these rectangles are always respectively equal in area to the squares on the other two sides, and hence the truth

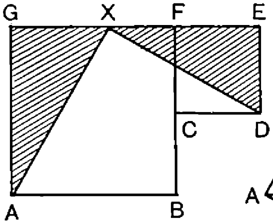


Fig. 18

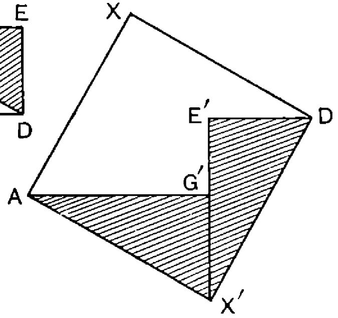


Fig. 19

of the theorem is established. It is not impossible at this stage, even though as yet no formal geometry has been undertaken, for intelligent pupils to want to know the nature of this proof, and the teacher who has thus successfully aroused their curiosity should most certainly satisfy it. In other words, if the pupils feel the need for the proof it should be given to them.

The truth of this important theorem having been demonstrated, its application to finding square root by graphical methods follows, and thus the geometry is linked up again with arithmetic and algebra. The theorem, of course, can also be applied to other problems, such as finding the altitude of triangles, the height of a roof, given the slant height and span, and vice versa. There is ample scope here, limited only by the ingenuity of the teacher, for much varied and useful application of a very important theorem.

Polygon and Trapezium.—The practice work in algebraic identities referred to on p. 104, should be undertaken at short intervals so as to prevent anything in the nature of a prolonged interruption in the further work in mensuration. One of the most fatal mistakes in the teaching of elementary mathematics is "intellectual dawdling". "Push ahead" should be the watchword, and much of the work that has been mentioned immediately above can well be set as homework exercises whilst the main work is being further developed and extended.

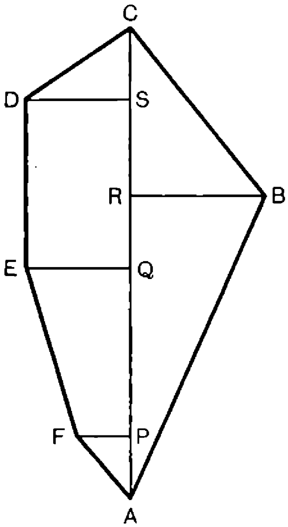


Fig. 20

The first step in such extension will be to find the area of a more irregular shaped field, such as that illustrated in fig. 20.

The pupils, who should have no difficulty in suggesting the necessary measurements, are introduced to the surveyor's "field book" method of recording the measurements in convenient form. Let us suppose that a field of some such shape as the above is near at hand or can be pegged out on the playing-field. The figures are recorded thus:

	Yards.	
	To C	
	250	
to D 58	210	
	160	75 to B
to E 58	120	
to F 35	30	
	From A	

The pupil here meets a new kind of quadrilateral, the trapezium FPQE (fig. 21), and the need for a formula for the area of such a figure is realized. The application of knowledge already obtained simplifies the solution of the problem.

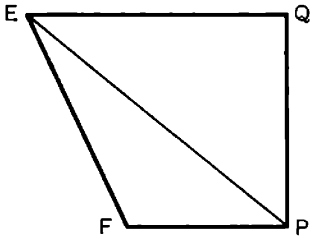


Fig. 21

Thus, by drawing the diagonal EP, two triangles are obtained, and the formula for the area of a trapezium, viz. $\frac{1}{2}$ sum of parallel sides × perpendicular distance between them, is readily obtained. The working necessary to find the area of the field is set out thus:

		Sq. Yd.		Sq. Yd.
Area of triangle	ABC	$= \frac{1}{2}(250 \times 75)$	=	9375
,, ,,	APF	$= \frac{1}{2}(30 \times 35)$	=	525
,,	trapezium FPQE	$= \frac{1}{2}(93 \times 90)$	=	4185
,,	rectangle EQSD	$= (58 \times 90)$	=	5220
,,	triangle DSC	$= \frac{1}{2}(58 \times 40)$	=	1160
		Total area	=	20,465
			=	4·23 ac.

Pupils thoroughly enjoy field work of this kind, and further textbook examples should be given. In all cases a well-drawn though not a scale-drawn diagram should be produced from the field book, and the attention directed to the importance of carefully arranging the necessary working, not the least important aspect of this particular piece of work.

Parallelogram.—The pupil has now dealt with the areas of the rectangle and the square, the triangle, the irregular quadrilateral, and the trapezium. He is quite ready to complete the list by including the parallelogram. The ample practice he has now had in finding the area of the triangle will enable him to find without difficulty the formula for the area of the parallelogram. He is guided to deal with it thus:

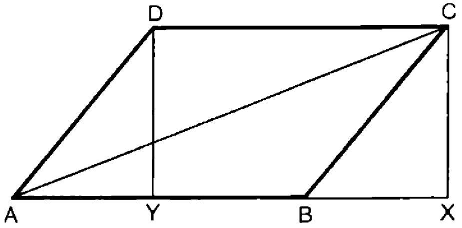


Fig. 22

In fig. 22

area of parallelogram = area of triangle ABC + area of triangle CDA

$$= \frac{AB \times CX}{2} + \frac{CD \times DY}{2}$$

$$= \text{twice } \frac{AB \times CX}{2}$$

$$= AB \times CX$$

$$= \text{base} \times \text{height}.$$

(The equality of CX and DY is obvious. By cutting out the triangle DCA, the equality of DC and AB can be demonstrated by superposition.)

The truth of this formula can of course be demonstrated by cutting out the triangle AYD and placing it in the position BXC, thus converting the parallelogram into a rectangle.

Exercises on the construction of parallelograms and their areas will now follow.

Circle.—The circle is such a familiar figure to the pupils that, having dealt with other familiar figures including the triangle, rectangle, square, and the parallelogram, they should, if their interest has really been aroused in the previous work, desire now to find the area of a circle.

The ratio of the diameter to the circumference must first be investigated, and in this connexion the teacher must not expect the pupils to measure to any degree of accuracy the circumference of a circle drawn on paper, even if the circle is a large one. The fetish of making every boy do his own experiment leads to a good deal of wasted time and effort, and produces very often results which are valueless and which at times unfortunately have to be "explained away". In the science laboratory it is fortunately being recognized that whilst experimental work is excellent, such work need not always be in the nature of individual experiment. Demonstration by the science master is a most valuable part of experimental work.

So here, instead of setting pupils to measure with cotton the length of the circumference of a circle of three- or four-inch diameter, the length of the circumference of the large circular end of the cylinder from the art room is found by wrapping a strip of paper around it and measuring the distance between the two pin-points made by puncturing the paper where it just overlaps. The circumference of a circle of ten- or twelve-inch diameter can certainly be measured sufficiently accurately by this method to give the value of π as $3{\cdot}14$. A bicycle wheel and other large circles are measured and the value of π as $3{\cdot}14$ or $3\frac{1}{7}$ is again established.

The pupils can now be set to verify approximately this value for themselves, by means of the following experiment. Draw a circle of say one-inch or two-inch radius. Cut the circle into sixteen equal sectors after drawing the necessary diameters. Paste the sectors carefully together as in fig. 23. The bases of the sectors make approximately a continuous straight line which when measured will be found to be approximately $3\frac{1}{7}$ times the diameter of the circle. No great degree of accuracy can be expected from this experiment, but it assists the pupil to fix the formula in his mind—he

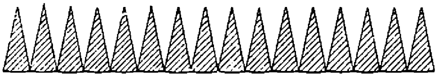


Fig. 23

readily appreciates the fact that the degree of accuracy in the result depends partly on his skill and partly on the number of small sectors taken. The experiment also serves the very useful purpose of suggesting to the pupil a method for finding the area of the circle.

It must of course be explained that the *exact* value of this ratio cannot be obtained, and for that reason the ratio is denoted by a symbol—the agreed symbol being the Greek letter π. Hence the circumference of a circle is $\pi \times$ diameter.

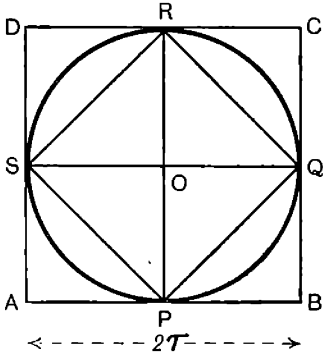


Fig. 24

The area of a circle is obtained in the following manner. Compare the area of a circle of radius r with the area of the circumscribed and inscribed square (fig. 24).

The area of the circumscribed square ABCD is $2r \times 2r = 4r^2$.

The area of the inscribed square PQRS $= 2\left(\frac{SQ \times OR}{2}\right)$

$$= SQ \times OR$$
$$= 2r \times r$$
$$= 2r^2.$$

The area of the circle is obviously between $2r^2$ and $4r^2$ and is approximately $3r^2$. The previous work on the value of the ratio indicated by π may tempt the pupils to hazard a guess as to the exact area of the circle. In any case this preliminary exercise will help to make the following work more convincing.

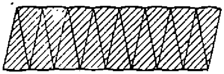


Fig. 25

A circle is divided into sixteen sectors as previously (see fig. 23). The sectors are now rearranged as in fig. 25, forming approximately a rhomboid whose height is r, and whose base is half the circumference of the circle or πr. The area of the circle is therefore $\pi r \times r$ or πr^2.

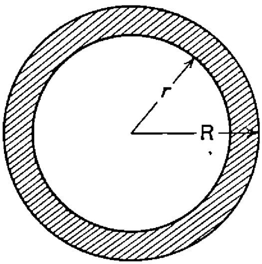


Fig. 26

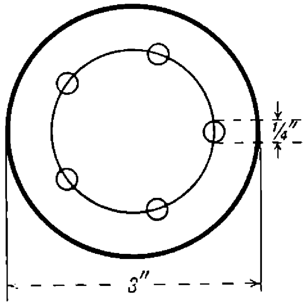


Fig. 27

The formula for the length of the circumference and the area of a circle having been established, much additional work in formulæ construction and manipulation can now be undertaken. The following are suggestive exercises:

1. Establish a formula for the area of the ring in fig. 26.
2. Change the formula $\pi R^2 - \pi r^2$ into a form which can be used more conveniently.
3. Find the surface area of the circular metal plate which has been drilled as in fig. 27.
4. Find a formula for the surface area of this plate of diameter 2R, when each of the small circular holes is of radius r.

Geometrical exercises on the construction of circles and a study of some of the more important properties of the circle can be undertaken whilst the more practical work on the mensuration of the circle is forming a part of the mathematical course. Here again no formal proofs are attempted. The terms diameter, arc, chord, secant, and sector are made familiar to the pupil, and he can easily discover under the guidance of the teacher how to find the centre of a circle and how to circumscribe a circle about a triangle. The angle properties of a circle, the meaning of the term tangent, and the construction of tangents can also form a part of the practical geometry course.

Work on the construction and the area of some of the more important polygons follows, and will complete the mensuration of the plane figures and the appropriate geometry related thereto.

CHAPTER V

First and Second Years—Later Stages

From the measurement of area the natural step is to the volume of solids, and allied to this section of the course in mensuration should be a course in solid geometry. In his eagerness to hurry on to formal geometry the mathematics master is sometimes content to hand that portion of the course in mensuration which deals with the measurement of volume over to his science colleagues. The inevitable result is that whilst the practical work in the measurement of volume is done with reasonable thoroughness, the mathematics course is devoid of any solid geometry. Its omission from the preliminary elementary mathematics course not only leaves this section of the work incomplete, but leads either to its entire omission from the usual school course or, if postponed until later in the course, tends to give the impression that it is unrelated to mathematical geometry. "Its persistent neglect by teachers, examining bodies, and writers of textbooks is one of the most marked and regrettable features in the developments of recent years."[1]

It is equally regrettable that the mathematics master not only neglects solid geometry but is content to relegate it to the handicraft room. However well the handicraft master there teaches it—and most of them do—the pupil is inclined to regard the subject not as a part of his mathematical course but as something quite apart from it. The application in the handicraft course of work treated by the mathematics master would vitalize the mathematical work and stimulate further interest in it. Inasmuch as the preliminary course in mathematics here suggested is intended to cover a period of two years, there is ample time for both the mensuration of solids and some solid geometry to be included.

Mensuration of Solids

The work in this section should begin with the construction of the following solids: the prisms, including the triangular, rectangular, pentagonal, hexagonal, and the cylinder; the corresponding pyramids including the cone; the following polyhedra—the octahedron, the dodecahedron, and the icosahedron in addition to the tetrahedron and the cube mentioned above; and finally the sphere. These can

[1] G. St. L. Carson, *Essays on Mathematical Education.*

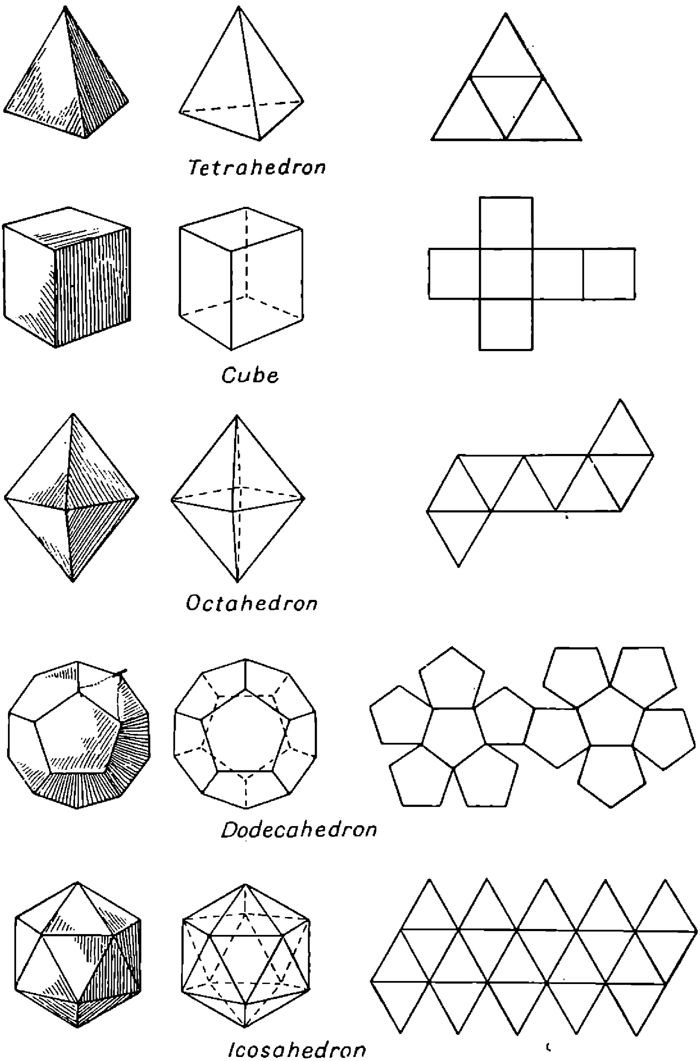


Fig. 28

easily be constructed out of stiff cartridge paper. The exercise is one which proves not only thoroughly interesting but one which is of immense practical value.

The theory of regular polyhedra—the fact that there must be at least three plane angles in any of the solid angles, that each of these angles must be less than 120°, and that the faces must therefore be either equilateral triangles, squares, or pentagons—is all within the grasp of the more intelligent pupils at this stage, and the work forms an interesting link between the plane geometry already studied and the three-dimensional work now undertaken. It may, however, be advisable to postpone such work until the third or fourth year.

The "nets" required for the construction of these solids are as follows:

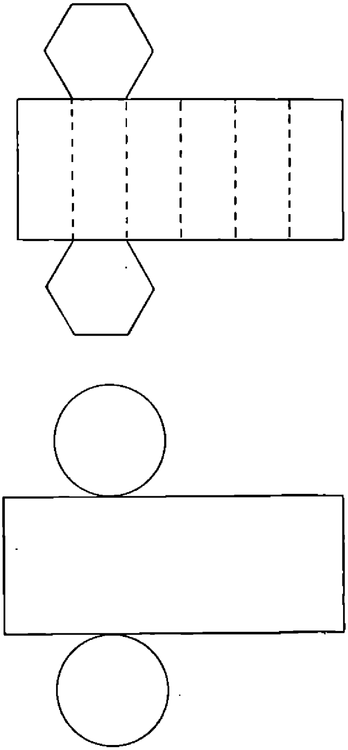


Fig. 29

I. Polyhedra.—The tetrahedron, octahedron, and icosahedron are made up of equilateral triangles as follows (fig. 28):

1. Tetrahedron.—A large equilateral triangle whose mid points are joined, thus dividing it into four such triangles.

2. Octahedron.—A series of equilateral triangles arranged as shown.

3. Icosahedron.—Twenty equilateral triangles arranged as shown.

4. The cube.—Six squares as shown.

5. The dodecahedron.—Twelve pentagons as shown.

The simplest method is to construct the centre pentagon. With a tracing of this on tracing-paper prick off the other pentagons arranged as shown.

II. Prisms.—As examples, the hexagonal prism and the cylinder are illustrated (fig. 29).

III. Pyramids.—As examples, the hexagonal pyramid and cone are illustrated.

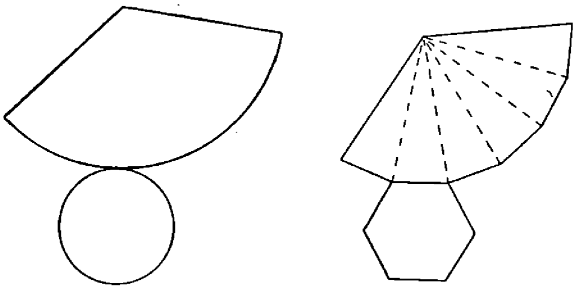


Fig. 30

To convert these nets into actual models, narrow lappets must be left on some of the edges—one to each pair of edges which meet.

Having constructed these solids and thus handled them, the pupils should tabulate details such as the number of edges, faces, and vertices. The transformation of one polyhedron from another can also be discussed and can actually be carried out. Plasticine or good yellow soap is the most suitable material, and a cube made out of these can easily be transformed into say a tetrahedron by cutting away four of the eight vertices. All this work familiarizes the pupils with the solids.

Similar work in the construction of prisms and pyramids, including the cylinder and the cone, should form a part of the same course. The construction of these being somewhat easier, some teachers may prefer to deal with them first.

Following the construction of these solids, exercises on surface areas will present no great difficulty, and will form an excellent connecting link between the previous mensuration on areas of plane figures and the work on volumes which is to follow.

Practical work in the measurement of the volumes of these solids necessitates a certain amount of laboratory work in weighing and measuring. The volume of the rectangular prism and cube is of course easily obtained, particularly if a number of pupils, having made a unit cube, build up from these a rectangular prism. The formula obtained should be memorized in the form

$$\text{Volume of rectangular prism} = \text{area of base} \times \text{height},$$

rather than

$$\text{Volume of rectangular prism} = l \times b \times h.$$

The question then naturally arises—is the volume of any prism given by the same formula? Is the volume of a cylinder which is twice the height of another twice the volume? A pile of coins makes a cylinder. Is the volume of twelve pennies twice the volume of six? Displacement is suggested as the means of testing. Then arises the question—Is a cylinder whose diameter is twice that of another, twice the volume if the heights are the same? Wooden or metal cylinders of such dimensions can easily be obtained or may possibly be made in the school workshops, and the volumes of the cylinders ascertained. One is found to be four times the volume of the other, and the connexion between this fact and that concerning the area of the bases is readily appreciated.

Finally experimental work is carried out on a series of prisms—triangular, hexagonal, and so on—the volumes found by applying the supposed formula and by displacement. So the formula for the volume of any prism is established as a general statement—area of base multiplied by height.

With the aid of pyramids of the same dimensions as those of the corresponding prisms, the volume of each pyramid is found by displacement to be one-third of its corresponding prism. Hence the formula for the volume of any pyramid is established as being $\frac{1}{3}$(area of base) $\times$ height.

All such " laboratory mathematics ", as already mentioned, is essentially the concern of the mathematical department of the school and should not form part of the preliminary science course. The practical work involved gives reality to the mathematics, whilst the formulæ so deduced are not only more clearly understood but arouse keen interest when used later as the basis of more abstract mathematical work.

The formula for the volume of the sphere can be obtained in much the same way as that in which the area of a circle was obtained, only the formula for the surface area of the sphere—$4\pi r^2$—will have to be accepted without proof.

The sphere is considered as being made up of a large number of small square pyramids (see fig. 31).

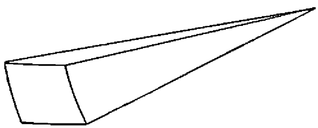


Fig. 31

The volume of these pyramids will each be $\frac{1}{3}$ area of base $\times$ height, or $\frac{1}{3}$ area of base $\times$ radius of sphere. Supposing the number of such pyramids is n, then the volume of the sphere will be $n \times (\frac{1}{3}$ area of base $\times$ radius), which is easily seen to be $\frac{1}{3}$ surface area of sphere $\times$ radius, i.e. $\frac{4}{3}\pi r^3$.

Application of Formulæ.—The establishment of additional formulæ gives further practice in generalized arithmetic, i.e. concrete, as distinct from abstract, algebra. If work on density and specific gravity is also added to the course at this stage, the problems will have a more varied character and the mathematical study a wider significance. By this time the pupil will have become familiar with the use of indices. The quantity r^3 has a real meaning to him, it is $r \times r \times r$, derived from real experience. Likewise such signs $\sqrt{}$ and $\sqrt[3]{}$ have also become familiar to him, and in both cases the way has been prepared for work in indices and in surds; and, what is equally important and of considerable value at this stage, the pupil's way has been prepared for an intelligent use of logarithms. The mathematical work now divides itself into three sections corresponding to the usual arithmetic, geometry, and algebra. Lessons in the mensuration of solids as indicated above will proceed along with lessons in practical solid geometry, projections, plans, and elevations, and in algebra including simple theory of indices and logarithms—the latter being in turn applied to more difficult examples in the mensuration of solids.

Solid Geometry.—The *practical solid geometry* course will consist mainly of exercises on orthographic and isometric projection. In the former will be included examples in drawing plans and elevations of lines in various positions relative to the H.P. and V.P. and the similar examples in drawing plans, elevations, and sections of some of the more familiar solids. In schools with a technical bias, exercises in drawing plans and elevations of simple machine parts can form part of the elementary mathematical course in order to prepare the way for the more serious study later of engineering drawing. Examples in isometric projection should be included in this section of the work.

Logarithms

We come finally to the treatment of logarithms. The work in formulæ manipulation and transformation has familiarized the pupils with the use of indices. They know that r^3 means

$r \times r \times r$, and they will have little difficulty in arriving at the meaning of a^5, a^7, and in general a^n.

It is not unusual for work in logarithms to be postponed until quite late in the course in algebra, it being the view that considerable knowledge of indices, integral and fractional, is necessary for an understanding of logarithms. Beyond what is mentioned above regarding indices and the simple facts that, say, $a^7 \times a^3 = a^{10}$, and $a^7 \div a^3 = a^4$, which present no difficulty at this stage, very little more algebra is required before logarithms can be well understood and used with ease. The early work proceeds as follows:

The meaning of 2^2, 2^3, 2^4, 2^5; of 3^2, 3^3, 3^4; 10^2, 10^3, 10^4, 10^5 is revised. Work in statistical graphs having formed part of the arithmetic course, the following exercises will present no great difficulty.

(i) Exhibit graphically 2, 2^2, 2^3, 2^4, 2^5. From the graph express 10, 18, 25 as powers of 2.

(ii) Using a larger scale, exhibit graphically 2, 2^2, and 2^3, and tabulate the following as powers of 2: 1·5, 2·5, 3, 3·5, 4·5, 5. [The interpolation of 1·5 will entail some discussion which should prove profitable.]

(iii) Similar exercises to (i) and (ii) with powers of 3.

These exercises will enable pupils to appreciate that it is possible to express any number as a power of 2 or as a power of 3. They will have no difficulty in arriving at the conclusion that any number can be expressed as a power of any other number.

The next step will be the consideration of numbers expressed as powers of 10. The work proceeds thus:

$$10 = 10^1$$
$$100 = 10^2$$
$$1000 = 10^3$$
$$10{,}000 = 10^4$$
$$100{,}000 = 10^5$$
$$1{,}000{,}000 = 10^6$$

The next exercises take the form of asking the pupil: "What must be the first figure in the index of 42 expressed as a power of 10?" Similarly of 54 and 78 and so on; and of any number of two digits. Again, what must be the first figure in the index of, say, 226, 358, 989, 999, expressed as powers of 10? The work is continued until the rule is finally obtained that any number of two digits can be expressed as a power of ten, the index of the power commencing with 1; any number of three digits can be expressed as a power of ten, the index of the power commencing with 2; and so on. After the term logarithm has been explained (*logos*, ratio; *arithmos*, number), these statements are put into the form:

1 is *characteristic*	of the	logarithms	of all	two	digit	numbers.
2	,,	,,	,,	three	,,	,,
3	,,	,,	,,	four	,,	,,

A few exercises in finding the logarithm of numbers then follow. *There should, however, be as little delay as possible in getting the pupils to use logarithms.* The teacher must not forget that the main purpose in hand is the introduction to a new mathematical tool, and it is better to hurry on to the use of the logarithms as quickly as possible, returning if necessary to the underlying principle. Boys will find much more interest in the explanation of why a thing works after they have satisfied themselves that it really does work and how it works. The preliminary work detailed above should therefore not be too prolonged. As quickly as possible the pupils should be using their logarithms. At first the exercises should be quite simple, the work in the first instance being set out as follows:

Example: Use logarithms to find the product of 87 and 52.

$$87 = 10^{1\cdot9395}$$
$$52 = 10^{1\cdot7160}$$
$$\therefore 87 \times 52 = 10^{3\cdot6555}$$
$$= 4524.$$

For some time the working should be set out in the above form, the numbers being expressed as powers of ten. Only when the pupils have thoroughly grasped the fundamental ideas should the base be omitted and the working arranged in the usual form. The important point at this stage is to give ample practice in the use of logarithms to enable the pupil to gain confidence in himself and his new weapon whilst understanding its nature. Exercises therefore should be simple in character in order that the working can be carried through with conviction. In the foregoing example, for instance, if the pupil has any doubt as to whether his new tool is really working he can—and it will not be surprising to find that he does—test his answer by ordinary multiplication. Once he has gained confidence in the use of logarithms in simple calculations he will use them with confidence in more complicated examples. Such examples as the following should therefore be worked:

Find the value of:

(*a*) $27{\cdot}5 \times 1{\cdot}36$; (*b*) $498 \times 2{\cdot}51$; (*c*) $(42{\cdot}5)^2$; (*d*) find the area of a circle of 2·51 cm. radius; (*e*) find the volume of a cylinder of radius 1·98 in. and 3·65 in. height; (*f*) $\frac{2{\cdot}368 \times 8{\cdot}951}{3{\cdot}682}$. It will be noticed that only numbers having positive characteristics are being dealt with at this stage.

The investigation of the use of logarithms to find square root and cube root can be based either on a knowledge of the laws of indices or, if this is thought too difficult, the work can be based on a study of the multiples of ten.

In the latter case the pupil knows that:

$$\begin{aligned} \text{The square of } 10^1 &= 10^2 \\ 10^2 &= 10^4 \\ 10^3 &= 10^6 \end{aligned}$$

$$\begin{aligned} \text{Conversely the square root of } 10^2 &= 10^1 \\ 10^4 &= 10^2 \\ 10^6 &= 10^3 \end{aligned}$$

Similar work on cube root will establish the rule that to find square root or cube root of a number by means of logarithms the index of the power of ten, i.e. the log, is divided by 2 or 3 as the case may be.

The more intelligent boys can approach the question from general consideration of the index laws, i.e. $a^2 \times a^2 = a^{2+2}$, $\therefore \sqrt{a^4} = a^2$, and so on. Again simple examples are dealt with in the first instance, e.g. Find by logs: $\sqrt{64}$, $\sqrt[3]{125}$. The work should be set out as follows:

$$64 = 10^{1\cdot8062}$$

$$\therefore \sqrt{64} = 10^{\frac{1\cdot8062}{2}}$$

$$= 10^{0\cdot9031} = 8\cdot0.$$

Further examples similar to those in multiplication and division can now be given, including such examples as: Find the radius of a sphere whose volume is 360·5 cu. in.

So far the work has been confined to numbers above unity, the characteristics of the logarithm being therefore positive. Work in negative characteristics is dealt with as follows:

$$3658 = 10^{3\cdot5633}.$$

$$365\cdot8 = \frac{3658}{10} = \frac{10^{3\cdot5633}}{10^1} = 10^{2\cdot5633}.$$

$$36\cdot58 = \frac{3658}{100} = \frac{10^{3\cdot5633}}{10^2} = 10^{1\cdot5633}.$$

$$3\cdot658 = \frac{3658}{1000} = \frac{10^{3\cdot5633}}{10^3} = 10^{0\cdot5633}.$$

At this point attention is again called to the fact that the characteristic only is changing.

$$0\cdot3658 = \frac{3658}{10{,}000} = \frac{10^{3\cdot5633}}{10^4} = 10^{\bar{1}\cdot5633}.$$

Here it will have to be explained that in order to preserve the same decimal part the characteristic again is the only part that is changed, this being less by 1 than the previous characteristic. So with

$$0 \cdot 03658 = \frac{3658}{100,000} = \frac{10^{3 \cdot 5633}}{10^5} = 10^{\bar{2} \cdot 5633}.$$

$$0 \cdot 003658 = \frac{3658}{1,000,000} = \frac{10^{3 \cdot 5633}}{10^6} = 10^{\bar{3} \cdot 5633}.$$

The deduction is now made that if the number is less than unity, the characteristic is negative and numerically one more than the number of ciphers immediately to the right of the decimal point.

Exercises in multiplication and division by logs of such decimal fractions involves the addition and subtraction of positive and negative numbers. Pupils so far have dealt with no such problems, but this need not present any great difficulty, nor need it be made difficult. "Tell a boy about ghosts," the late Professor Perry was fond of saying, "and the simplest things become complex and mysterious. Tell a boy he is sure to find difficulty in simple algebra, and of course he finds great difficulty with a problem that would be quite easy if you told him it was easy." Is not this the case here? Need we take it for granted that the average boy is going to find difficulty in adding and subtracting positive and negative characteristics because he has not had the usual lengthy application of algebraic rules for addition and subtraction? There is no difficulty but those which the teacher is inclined to make. Put less faith in the manufacture of rules and more in the exercise of the pupil's common sense.

At the most, the pupil has to add a series of positive and negative numbers and then "subtract" a positive number from a negative number or vice versa, or "subtract" two positive or two negative numbers. To take the former. Suppose the characteristics are 2, $\bar{1}$, $\bar{2}$, 3. Surely there is nothing but common sense needed. The sum is made in the same way as in ordinary addition, thus: 2 *and* 1 less gives 1 *and* 2 less gives minus 1, *and* 3 more gives 2.

Simple illustrations can of course be given, but the process is not so difficult as we are inclined to make it. A negative quantity is the opposite to a positive quantity. Here then a simple tug of war is going on; the centre point of the rope is two paces to the right of the mark on the ground, it is pulled back one pace, now two more, and then again to the right three paces. No rules are wanted—it's only common sense. To quote Professor Perry again: "It is only a teacher who remembers hundreds of rules."

The so-called subtraction—a word which with advantage could be abolished from mathematical language—again is straightforward common sense.

Take for example:

$$\begin{array}{r} \bar{1}\cdot 1486 \\ \underline{\bar{3}\cdot 8659} \\ 1\cdot 2827 \end{array}$$

Following the method advocated in the section on Arithmetic, the subtraction is done by the addition, or the "cashier's", method. 9 and *seven* make 16, 6 and *two* make 8, 6 and *eight* make 14, 9 and *two* make 11. The "carried one" makes $\bar{3}$ into $\bar{2}$. The question then is, How much must be added to $\bar{2}$ to make $\bar{1}$? Again no rule is wanted, just common sense.

In the use of logarithms accuracy in working is of course of the utmost importance. The work must be arranged neatly and compactly, and each step in the process must be completed before the next is begun. It is fatal to mix the work of determining characteristics with the reading of logarithms. *All* the numbers should be written down first, then *all* characteristics determined, then *all* the decimal parts dealt with, and so on. Unless the work proceeds thus, characteristics will be forgotten or anti-logarithms will be used instead of logarithms — errors that are all too frequently made.

An example is here worked out in full to illustrate what is meant by neat and compact arrangement:

Example:

$$\text{Evaluate } \sqrt{\frac{256{\cdot}5 \times 1{\cdot}326}{(0{\cdot}483)^2 \times 87{\cdot}62}}.$$

$$\text{Estimated value} = \sqrt{\frac{250 \times 1\frac{1}{3}}{(\frac{1}{2})^2 \times 88}}$$

$$= \sqrt{\frac{330}{22}}$$

$$= \sqrt{15}$$

$$= 4 \text{ approx.}$$

$$\sqrt{\frac{256{\cdot}5 \times 1{\cdot}326}{(0{\cdot}483)^2 \times 87{\cdot}62}} = \sqrt{\frac{M}{N}}.$$

$\log 256{\cdot}5 = 2{\cdot}4091$	$2 \log 0{\cdot}483 = \bar{1}{\cdot}3678 \quad \bar{1}{\cdot}6839$
$\log 1{\cdot}326 = 0{\cdot}1226$	$\log 87{\cdot}62 = 1{\cdot}9426$
$\log M = 2{\cdot}5317$	$\log N = 1{\cdot}3104$
$\log N = 1{\cdot}3104$	
$\log \frac{M}{N} = 1{\cdot}2213$	
$\frac{1}{2} \log \frac{M}{N} = 0{\cdot}06107$	

$$\therefore \sqrt{\frac{M}{N}} = 4{\cdot}081. \quad \text{Ans.}$$

The various steps in the working of the above are as follows:

1. Rough estimate. The rough preliminary estimate is of far more value than a rough check of subsequent working.

Such an estimate necessitates covering rapidly the whole of the working to be done, estimates the various portions of the calculation, and gives the pupil confidence before he begins the actual working.

2. All the numbers are then written down as shown, in two columns,,and the remainder of the " scaffolding " erected. That is to say every part of the working except the entry of the actual logarithms is stated. Thus the pupil concentrates on one task at a time, and incidentally has planned the working as a whole.

3. All four characteristics are then entered. (Note when a number has to be squared the characteristic is written in the margin and the necessary multiplication is done after the next step. The " blank " thus left calls attention to the necessity for such multiplication.)

4. All four logs are then completed.

5. The two additions are made, the log N is transferred, the subtraction and the division follows.

6. The anti-log is obtained and the answer compared with the rough estimate.

The work in logarithms rounds off, as it were, the preliminary two years' course in elementary mathematics. Examples such as the above can be given as practice in the use of logarithms, and of course such practice is necessary. Revision of all previous work, however, is now essential, and the use of logarithms in dealing with fairly complicated computation arising out of the application of formulæ in area, volume, density, and the manipulation and transformation of other formulæ will give new interest to the previous work in mensuration, whilst fostering appreciation of this new and valuable mathematical tool.

In the revision of the measurement of heights and distances some teachers may find it possible to introduce some elementary trigonometry and the use of sine, cosine, and tangent tables. Much will depend on the ability of the pupils, the type of school, and the nature of the subsequent mathematical work to be attempted. On the whole, work in numerical trigonometry can in most cases be postponed until the next year of the course (see Chapter VII).

Aims of the Course

The work of this two years' preliminary course in elementary mathematics has been set out at some length. The aim of the course throughout has been to keep an even balance between the intensive practical course in mathematics and the traditional abstract mathematical course. Most modern textbooks in algebra attempt to make the early work take the form of generalized arithmetic. There is, however, something artificial about the type of exercise which takes the form of $3 \times 3 = 3^2$, $5 \times 5 = 5^2$, $a \times a = a^2$. There is likewise something artificial about the usual early treatment of equations which takes the form of solution of problems dealing with perhaps the fact that twice A's age will be 7 years more than B's age was two years ago—or the fact that if 12 is added to twice a certain number, the result is 42.

The course in mensuration has developed naturally into the building of important formulæ, and in the transformation of these the pupil has become thoroughly accustomed to the use of literal symbols and has had considerable practice in what may be termed manipulative algebra. If this aspect of the work has been thoroughly appreciated by the mathematics teacher, and carried out naturally and with common sense by the pupil, the more academic and abstract algebra which is now to follow will have no mysteries for the pupil. His progress will be both rapid and sure, inasmuch as he has grasped the real meaning of generalized arithmetic and become familiar with and confident in the use of literal symbols.

In geometry his experience has been widened, ideas of position, shape, and size have been developed, and concepts which already existed have been systematized and extended.

He has now, or should have, a sound working knowledge of the subject and a grip of the significance of its underlying truths which will enable him with confidence to use these truths in practical applications.

This confidence in the use of its underlying principles, based on the sound working knowledge secured, will give interest in and reality to the building up of the logical systematic body of knowledge termed formal geometry.

Finally, in this preliminary course of elementary mathematics such close bonds have been forged between the various branches, arithmetic, mensuration, geometry, algebra, and possibly trigonometry, in the help they have rendered one another, that to whatever extent they may now be treated as distinct branches in the subsequent course of later study, these bonds will not be weakened. A sound foundation to a "coherent system of mathematical ideas" has been laid, so much so that the pupil is in no danger of losing his sense of the unity of the subject as he proceeds to turn his attention more and more to the full development of the various branches of the subject.

CHAPTER VI

Third and Fourth Years—Geometry

In the foregoing chapters the point has been emphasized that historically mathematics gradually passed from concrete experience to the study of the abstract. The point has also been emphasized that on psychological grounds mathematical education should proceed in like manner from concrete experience gradually to the study of the abstract.

The work outlined for the first two years of the post-primary school course has been based largely on concrete experience, and mathematics has been largely analytical. We come now to the period of transition—the gradual change from this concrete and analytical mathematical work to the abstract and synthetic. At the outset it is important that the mathematical teacher should appreciate the importance of the fact that the transfer must be a gradual one. There must be no sudden break. Thus in the geometry of the third and fourth years the aim must still be to provide a broad basis of geometrical facts. The development of the subject, on rigorously Euclidean deductive lines, from first principles is not intended. Precise reasons for statements should now be insisted upon, but " young boys are never happy and are often suspicious if they feel they are being asked to prove the obvious, but they can follow a fairly long chain of reasoning ". They are suspicious because they fail to appreciate the abstract reasoning which is necessary in order that the subject may be developed from the minimum number of postulates. These obvious truths or intuitions must at this stage play a very prominent part in the development of the subject and must for the time being be accepted as postulates. To attempt to prove them is to engage in subtleties in which the pupil at this point has no interest.

The nature of these intuitions must be appreciated by the mathematical master. They are for the time being the fundamental hypotheses of the subject. They bear the same relation to mathematics as fundamental physical laws bear to physics and fundamental chemical laws bear to chemistry. Such laws or hypotheses are the starting-points from which the sciences develop. These mathematical intuitions, however, differ from scientific laws inasmuch as whilst the latter are arrived at as the result of experiment, mathematical intuitions are arrived at subconsciously. We do not know that vertically opposite angles are equal as the outcome of measurement. We are aware of the truth of the fact intuitively as the outcome of experience, because mentally we are what we are.

The fundamental facts concerning angles at a point, parallels, and congruent triangles can for the time being then be accepted as postulates, and treated as the working hypotheses of the subject. They are not accepted without proof merely to make the subject easier or more interesting. We fail to do justice to the educational value of mathematics if in our teaching methods we are for ever endeavouring to make the subject easier for the pupil. Our aim should be to adapt our methods to suit the stage of mental development of our pupils. And at the stage now under consideration the pupil is gradually passing from the period of concrete experience to the period when his interest will be more and more concerned with the abstract treatment of the subject.

Making all possible use, however, of these intuitions, straightforward proofs of theorems as well as easy riders are well within the grasp of the pupils taking this course. The degree of precision aimed at will necessitate a proper appreciation of classification and definition. A clear understanding of the nature of such classification and of definitions constitutes in fact a most important part of the work in geometry.

Definitions.—Ready-made definitions, learnt parrot-fashion, must not be allowed. On the contrary, the pupils should be taught the nature of a good definition, the importance of placing the thing defined into its appropriate class, and of stating just what distinguishes it from all other members of that class. If classification and definition go hand in hand, pupils will have no difficulty in formulating their own definitions as accurately as their knowledge permits. The definitions must be learnt. Full appreciation of theorems depends to a large extent on a clear understanding and knowledge of definitions, whilst the solution of riders depends as much on knowledge of definitions as of theorems. Accuracy and precision of statement is an essential part of mathematical training. The pendulum must not swing too far in the direction opposite to that of the days of rigid Euclidean geometry. As previously stated, it is a grave mistake to assume that the change from Euclid to modern geometry has been brought about merely for the purpose of making geometry easier, and that therefore practical demonstrations can take the place of rigid proof, or vague ideas take the place of clear definitions. Modern geometry simplifies the early study of the subject. The effect of such simplification will not be necessarily easier geometry, but a more extensive course including a more systematic study of solid geometry.

Setting out Work.—Accuracy and precision are likewise necessary in the case of theorems and riders, and must therefore be insisted upon. Not only must the theorems themselves be stated clearly and concisely, but the pupil must be conscious of the unity of the structure of the whole body of geometrical knowledge with which he is dealing. Theorems should be presented in sequences, each group of which can be appreciated as a unit. This will have already been partly realized in the preliminary course if, as suggested, attention has been concentrated in turn on questions of position, of shape, and size. No theorem should stand isolated. Its place in the group should be recognized; its relation to previous theorems and its significance so far as subsequent theorems is concerned must be fully understood. In this manner the conception of a chain of proofs as well as a sense of logical proof will be developed, and not only will the real purpose of the study be better appreciated, but the applications of the theorems to the solution of riders will be made more intelligently and with more confidence. *Too much emphasis cannot be placed on the importance of this question of classification, definition, and appreciation of the unity of the whole body of theorems.* Only by due attention to these aspects of the study can the desired degree of precision and accuracy be secured and the subject be made to play its full part in mathematical training as well as general culture.

As regards the theorems themselves, a good standard of setting out the whole argument must be insisted upon. In the first place good, clear, carefully drawn figures, preferably executed in pencil, are essential. Freehand drawing of circles should most certainly be discouraged. The habit should be encouraged of recalling to mind all relevant facts as the figure is being constructed. For instance, if a rider begins with the statement " P and Q are points on an arc AB of a circle whose centre is O ", the pupil almost " from force of habit " should be recalling to his mind theorems relating to angles at the circumference of a circle and so on. This can best be done if figures are well drawn. Neat conventional marking of figures, showing plainly the known or given facts, should be encouraged. The whole statement of proof must likewise be set out carefully. The particular enunciation should be stated under the headings of " *given* ", and " *to prove* ". Any necessary " *construction* " is briefly indicated, and the " *proof* " is then stated clearly and concisely, reasons in support of each statement being given in summarized form. The use of recognized abbreviations should, of course, be allowed, but care must be taken to see that the use of such neither impairs the precision of statements nor encourages careless, slipshod, or untidy methods of arranging written work. An example of a well-arranged proof is given below and attention is drawn to the clear indication of each part of the proof, the arrangement of the various statements made, the abbreviations used, and the manner of giving references in support of such statements. Attention is also directed to that part of the proof in which facts concerning congruent triangles are employed. The phrase " In the triangles . . ." should always be followed by the numbering of the next three lines, and the facts then entered, the equality of sides being first stated. If only one side of each triangle is known to be equal the pupil then knows that statements 2 and 3 must concern angles. If two sides are known to be equal, each to each, then the third fact must be either the third side or the included angle, except in the case of a right-angled triangle.

Only by insisting on the facts concerning congruent triangles being stated in some such systematic manner will the common errors concerning congruency be avoided.

The formation of good habits in the purely mechanical work of writing out theorems should be regarded as a part of mathematical training, and attention to details must therefore be given, particularly in the early stages.

Example.—*If the square on the side of a triangle is equal to the sum of the squares on the other two sides, then the angle contained by these sides is a right angle.*

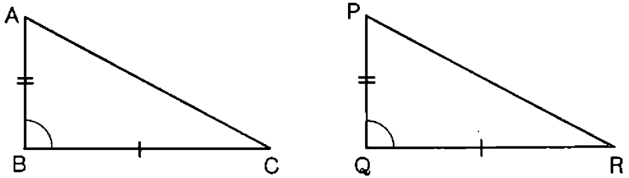


Fig. 32

Given: ΔABC such that $AC^2 = AB^2 + BC^2$.
To prove: $\angle ABC$ is a rt. $\angle$.
Construction: Construct a ΔPQR, so that

$PQ = AB$,
$QR = BC$,
and $\angle PQR$ is a rt. $\angle$.

Proof: $AC^2 = AB^2 + BC^2$ (*given*)
$= PQ^2 + QR^2$ (*constr.*)
$= PR^2$. (*Pythagoras' theorem*)
$\therefore AC = PR$.

In the Δs ABC and PQR

1. $AB = PQ$, (*constr.*)
2. $BC = QR$, (*constr.*)
3. $AC = PR$. (*proved*)

$\therefore \Delta ABC = \Delta PQR$ (*three sides*)
so that $\angle ABC = \angle PQR$
$=$ a rt. $\angle$. (*constr.*) Q.E.D.

Proofs of riders are set out in the same form—there must not be one standard for theorems and another for riders. That very valuable part of the work in connexion with the solution of riders, viz. the necessary analysis, does not of course appear on paper as a rule. The teacher therefore should give considerable attention to this in oral work. The boys who have failed to find the solution of a rider should not merely be given the solution, but should arrive at the solution by being taken through the necessary steps in the analysis. In the early stages of such work it is a good plan to insist on such analysis being given and full credit allowed for it even in the absence of a correct solution.

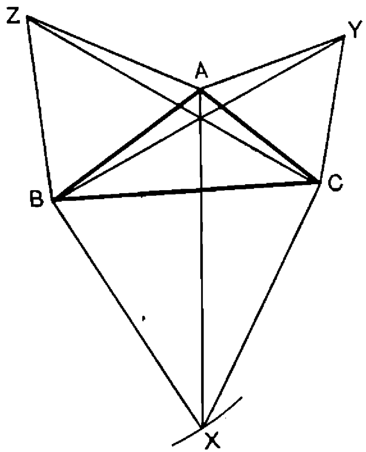


Fig. 33

Example.—*ABC is any triangle on whose sides equilateral triangles BXC, CYA, and AZB are drawn. Prove that AX = BY = CZ.*

To prove: (i) AX = BY.
(ii) AX = CZ.

Analysis.—The usual method by which two lines are proved equal is the application of the congruent triangle theorems.

AX and BY are respectively sides of the triangles ACX and BCY, which *look* congruent.

(Note here the importance of well-drawn figures.)

Are they congruent?

Now in these triangles we know AC = CY and CX = BC (sides of equilateral triangles). If the triangles are congruent the included ∠ACX must be equal to the included ∠BCY. Are they? The ∠ACB is common to both ∠s. The remaining parts, ∠ACY and ∠BCX are each ∠s of equilateral triangles and are therefore equal. Hence the ∠s ACX and BCY are equal. The triangles *are* therefore congruent and it follows that AX = BY. The proof for the second part will be similar. The analysis being completed, the pupil is in a position to give proof synthetically.

A synthetic proof without previous analysis will often be found quite readily if the habit has been formed of drawing good figures, and at the same time fixing clearly in mind exactly what is given, and all that this implies, and what has to be proved. There are no sudden inspirations in solving riders. What are sometimes regarded as such may possibly be the working of the subconscious mind on the facts which the intelligent drawing of the figure suggests, and the recalling to mind all that the figure implies, all that the hypothesis grants, and all that is relevant to the conclusion to be reached.

Syllabus

The geometry syllabus of the third- and fourth-year course here suggested is as follows:

(*a*) Fundamental ideas. Planes and perpendiculars, horizontal, vertical, and oblique lines and planes. Parallel planes and lines, solids, surfaces, lines and points. Loci.

(*b*) Direction. Angles. Parallel lines and transversals. Angles of rectilineal Parallelograms. Parallel lines and equal intercepts. Ratios. Proportional division of lines. Proportion and similarity. Construction of similar figures. Construction of quadrilaterals.

(*c*) Revision of areas and geometrical illustrations of algebraic identities. Construction of equivalent triangles. Theorem of Pythagoras and its extension. Application of Pythagoras' theorem to various constructions, square root, &c.

(*d*) Loci.

(*e*) The circle. Chord, angle, and tangent properties. Construction of circumscribed, inscribed, escribed circles and other circle problems. The common tangents to two circles.

(*f*) Solid geometry. Regular polyhedra, their construction and transformation. Lines, planes, and solids, and the projection of these in horizontal, vertical, and inclined planes. Sections of solids.

The above syllabus may be regarded by some as being more than can be attempted in a two years' course. It has been planned, however, on the sound principle that one of the surest ways of maintaining interest in geometry is to push ahead.

If the conception of a chain of proofs is to be formed in addition to the development of a sense of logical proof, we can scarcely hope to do so if too much time is devoted to any one sequence of theorems or if time is wasted in wearisome revision of a limited examination syllabus. And if further evidence is required to prove the wisdom of the push-ahead policy, it can readily be supplied by any experienced teacher who knows the meagre success which so often attends attempts to answer simple questions on the early part of the syllabus, as compared with the large measure of success with which questions on later parts of the syllabus are attempted.

For the less academically-minded type of pupil, and particularly for the boys whose period of mental unrest following the onset of puberty referred to in Chapter III is somewhat lengthy, there is grave risk that lack of interest in the subject will arrest mental development. For such, the exploration of the unknown, even to the neglect of logic of the strictly formal kind, is the only wise course. They can return later to the more formal geometry if necessary, when for them the period of disturbance has passed and the "logical stage" in their mental life has arrived.

The syllabus can of course be varied to suit various types of school. The senior selective school with a technical bias will naturally give more time to perspective geometry and to exercises of the numerical type.

CHAPTER VII

Third and Fourth Years—Algebra and Numerical Trigonometry

Algebra

If the work of the preliminary two years has been well done, the pupils, by the time they are ready to begin what perhaps may be termed algebra proper, are thoroughly at home in the use of formulæ, their construction, manipulation, and transformation. For such pupils algebra is not likely to be a "tyranny of x and y". Even so, the rule during these two years should be the same as during the previous two years. That is to say, new processes should not be introduced to the pupil until he feels the necessity for such, or at least is able to appreciate them. Frequent reference to the work of the previous two years should therefore be made, and wherever possible the new work should develop out of the more practical work in mensuration. The abstract algebra, in other words, is not merely the generalized arithmetic of the mensuration course but is the extension of the concrete work. Thus, for example, the preliminary course ended with work in logarithms, and this *necessitated* some reference to negative quantities. The starting-point in algebra therefore should be a general consideration of negative quantities. This makes possible a considerable extension of the work on formulæ. The subject of simple, simultaneous, and quadratic equations is developed, naturally, as a more general treatment of formulæ. In like manner the work on algebraic identities, treated previously as general examples in the area of rectangles, is now extended to a general consideration of such expressions and their factorization. This leads to a consideration of common factors and multiples and the application to fractions.

Too often algebra is made unnecessarily puzzling because it is not sufficiently recognized that in its highly abstract form it is a really difficult subject for young pupils. With such pupils, condemned to study algebra in its abstract form, the most that can be attained is a certain dexterity of manipulation of symbols acquired through constant application of rote knowledge. Negative numbers, for instance, present quite considerable difficulty to pupils of ten and eleven. Mathematical teachers are apt to forget that until the beginning of the seventeenth century mathematicians dealt exclusively with positive quantities. Everything is in favour of delaying the more highly abstract algebra until the pupil is sufficiently developed intellectually either to feel the need for the generalizations involved or to appreciate them. Thus, if the pupil is not required to deal with the negative quantities until, in the course of the development of some branch of his mathematics, he finds a real necessity to investigate them, he has an immediate interest in such an investigation, inasmuch as such quantities are real to him. Instead of being compelled to study these negative quantities as they are thrust at him he meets them as it were across his path. Moreover, he meets them in such a manner that he immediately recognizes them as being what they really are—the opposite to positive quantities. The fundamental processes of addition, "subtraction", multiplication, and division of such negative quantities present no difficulty. As the negative quantity is the opposite to a positive quantity, it is a matter of common sense, for instance, that if $a \times b = ab$ then $a \times (-b)$ must be the opposite to the previous product, i.e. $-ab$. Further $(-a) \times (-b)$, being the opposite to $a \times (-b)$, must be ab, again the opposite of the previous product. And so to the pupil the manipulative rule has no mysteries. What is equally important, he will not be found blindly applying some such nonsensical rule as " like signs give plus and unlike signs give minus ".

The best advice one can give to the teacher of mathematics is " Do not make the subject difficult for the pupil by introducing highly abstract work too soon ". Progress in the end will be more rapid and more certain. Mathematical masters who have taught algebra in evening classes to ex-elementary schoolboys of fourteen to fifteen, who have previously been taught no algebra, know how rapid such progress can be.

A word is necessary regarding so-called " mental " algebra. The term mental as applied to algebra must not be confused with the term as applied to arithmetic. What is termed mental algebra is more in the nature of short sums which need little or no paper work. In the section on Arithmetic, mental arithmetic is termed " natural arithmetic ", being distinct from written or mechanical arithmetic. It is so termed because it is the arithmetic which the mind does naturally without the assistance of the artificial aid of mechanical processes. In the teaching of algebra, the modern tendency is to abandon the longer type of algebraic sum in favour of the more numerous shorter sums, and these shorter sums are still further shortened, constituting the important oral work. It is obvious that such oral work is very valuable, but it does not perform the same function as mental arithmetic. The aim in the teaching of mental arithmetic is to develop the individual's natural ability to carry through mentally, and therefore without unnecessary artificial aid, as much computation as possible. The aim of so-called mental algebra is not to develop mental processes, but rather to give as much practice as possible in a given time in the application of some new algebraical rule. It is oral work rather than mental work.

Syllabus.—The main function of algebra, so far as the normal school curriculum is concerned, is to furnish the mathematical equipment which will enable the pupil to become acquainted with at least the fundamental ideas of other branches of mathematics, especially trigonometry and the elements of calculus and mechanics. With this in view, the following is suggested as the syllabus in algebra covering the third and fourth years of the elementary mathematics course.

Negative numbers. The four rules applied to such numbers. Statistical and functional graphs involving negative quantities. Simple equations. Problems, including graphical problems.

Simultaneous equations.

Identities revised. Factors and multipliers.

H.C.F. and L.C.M. and fractions.

(Note.—As each type of factor is dealt with, so it should be applied to H.C.F. and L.C.M., and fractions. In other words, work on fractions should not be postponed until all types of factors have been learnt, nor should separate work on fractions be undertaken at this stage.)

Harder examples in factors and fractions may be included if time permits.

Literal equations.

Quadratic equations. Problems. Graphical treatment of quadratic functions and problems.

Variation and proportion.

Further functional representations and notation.

Limits and gradients and, if time permits, the elements of calculus.

It is recognized that this is a very full syllabus for a two years' course, but it is felt that more rapid progress will be possible following the two years' preliminary course than is usually the case. Interest will certainly be keener because the work has been postponed until the pupil is sufficiently developed mentally to appreciate such abstract work.

The nature of the latter part of the syllabus will depend largely on the type of school. In schools with a technical bias, for instance, some of the less important parts of the syllabus can be omitted in favour of calculus and its application.

Negative Numbers.—(i) *Addition.* It has already been pointed out that if a negative quantity is regarded as the opposite of a positive quantity the rules for dealing with the combination of such quantities are largely common-sense rules. The boy who thoroughly appreciates that $6 + 4$ means 6 increased by 4, and who knows that -4 is the opposite to $+4$, will have no difficulty in understanding that 6 and -4 must mean 6 decreased by 4.

(ii) In *subtraction*, if the addition method has been taught in arithmetic, there is again no great difficulty. For example, to find the value of $4 - 7$, we are required to find a number which "added" to 7 makes 4, i.e. -3. Again $4 - (-7)$ means that we are required to find a number which added to -7 will give 4, i.e. 11. Here the pupil by the application of common sense finds that $-(-7)$ is the same as $+7$, and it will be pointed out again that this is but the application of the idea of a negative quantity being the opposite to a positive quantity. We know that $-(-7)$ must be the opposite of $+(-7)$, and the latter being -7, the former must be $+7$.

(iii) *Multiplication and division.*—Reference has already been made to multiplication. Division can be approached either in the same way as multiplication or can be treated as being dependent upon multiplication. Thus:

$$20 \div 5 = 4,$$
$$\therefore\ -20 \div 5 = -4,$$

i.e. -20 being the opposite to 20, the result must be an "opposite" result.

Similarly, $-20 \div -5 = +4$ for the same reason, and so on. Or the processes can be treated thus:

$$-20 \div 5.$$

The answer must be the number which, when multiplied by 5, will give -20, i.e. -4, and so with other combinations.

It should be pointed out that this treatment of negative numbers as the opposite of positive numbers must not prevent the idea of an extended notation scale being overlooked. In other words, -5 must not only be regarded as the opposite of $+5$ because of the sign before it. In the sense of being as much below zero as the other $+5$ is above zero, it is the opposite of the latter. Graphical exercises with examples involving negative numbers should immediately follow this work in the four fundamental rules. Such graphical work should include statistical graphs such as temperature charts, and also the drawing of graphs of given algebraical expressions such as $2x - 5$, $2x - x^2$, &c. (See note below on *Graphs*.)

Brackets.—Complicated exercises in the removal of brackets are not recommended. Examples of the type found in the older textbooks provide useful exercise in rules, the application of which is mainly mechanical. It is more important to ensure that boys can distinguish between $3(x + y)$ and $3x + y$, between $a + b - (c + d)$ and $a + b - c + d$, than to be skilled in the mechanical application of rules for removal of brackets. Understanding the use of brackets is more important than skill in manipulating them, and ability in the intelligent insertion of brackets is more important than mechanical skill in removing them.

Oral questions, designed to ensure that the pupils know exactly what the meaning of such expressions as $3(x + y)$, $(x + 5)y + c$, $a + b - a + c$, are most important. In like manner the pupils should be taught the use of brackets in such a question as "Express by the use of brackets the difference between half of the sum of two numbers and one-third their difference."

Graphs.—Graphical work, mainly of a statistical nature, has been included in the first- and second-year elementary mathematical course, and the pupil therefore has already become familiar with the use of squared paper. He already appreciates the importance of neatness and accuracy, a wise choice of scale, and the clear indication of the axes and the scale chosen.

The graphical work to be included in the third and fourth years of the course will deal in the first place with the plotting of simple algebraical functions of a variable, with the object of illustrating the change in such functions as the variable changes. The questions should be of the type:

(*a*) Show how $\frac{x}{5} - 2$ changes as x changes; or

(*b*) Show how the value of x^2 changes as the value of x changes from -5 to $+5$; or

(*c*) Show how the function of $x^2 - 3x - 10$ changes as x changes from -5 to $+10$.

At this stage the pupil should not get the idea that graphical work serves the purpose of solving simultaneous and quadratic equations. Such work should be regarded as incidental, the main function of graphical work being the illustration of functional variation. The graphical treatment of quadratic equations arising as a development from exhibition of such functional illustration can of course be made to serve a very useful purpose, and may even be used as a method of introducing quadratic equations. Thus, in the above example, after the pupil has exhibited graphically the values of the function $x^2 - 3x - 10$, he may be asked to state the values which will make $x^2 - 3x - 10 = 0$. Such a question, however, is an extension of the main problem and, treated as such, there is no risk of the pupil regarding his square paper work as merely supplying another means of solving equations.

After the pupils have had considerable experience of various types of graphs, they should be ready to recognize the type of function which will give straight-line graphs, parabolas, &c., and other similar problems. At a later stage more difficult examples can be introduced, and simple problems relating to limits and gradients can be attempted.

Equations.—Here again the pupil, being already familiar with formulæ and their transformation, should be able to make very rapid progress with all types of equations. Teachers who may be apprehensive concerning the small amount of algebra attempted in the first and second years of the course will find that much less time will now be required to enable the pupil to deal satisfactorily with simple, simultaneous, and quadratic equations. Little need be said regarding the treatment of simple and simultaneous equations beyond the importance of insisting on good arrangement of the various steps in the solution, and the importance of insisting upon all answers being tested. Burdening the pupil with rules which have to be applied to the solution of equations tends to make the work unnecessarily difficult. The application of common sense and not rules is all that is necessary. Thus, in such an example as, " Find the value of x which will make $3x + 5$ equal to $2x - 7$ ", the pupil exercises common sense to the extent of treating both sides of the equation alike in order that the " balance " may be preserved, whilst at the same time so treating each side of the equation that it may be written finally in the form which will give what is wanted, viz. x stated in terms of a number. Obviously the transformation made in the statements must be for the purpose of getting the x terms to the left of the equality sign and the numbers to the right.

Hence $3x + 5 = 2x - 7$.

This first becomes $3x + 5 - 2x = -7$, because as $2x$ must be taken off the right-hand side of the equation a like quantity must be deducted from the other side. In like manner the next step is:

$$3x - 2x = -7 - 5.$$

Finally $x = -12$.

In the early stages all such steps should be shown either as above, or in the following form:

$$3x + 5 = 2x - 7$$
$$3x(+5 - 5) - 2x = (2x - 2x) - 7 - 5,$$
$$\text{i.e. } 3x - 2x = -7 - 5$$
$$x = -12.$$

The important point is that intelligent application of common sense is to be preferred to mechanical application of rules. Such rules as " take over to the other side and change the sign " should develop out of this application of " common sense ", and should never be forced upon pupils to be applied unintelligently by them.

A wise choice of carefully graded exercises helps very considerably towards making the solution of simultaneous equations again a matter of applying common sense. The first exercise should be:

$$\text{Solve } x + y = 7$$
$$x - y = 3;$$

or, " The sum of two numbers is 7 and their difference 3. Find the numbers." The pupils will readily see that y can be eliminated by adding and that subtraction completes the process. The next example should be of the type:

$$2x + y = 6$$
$$x + y = 4,$$

and the next

$$2x + y = 15$$
$$x + 3y = 4.$$

Such a series of exercises will introduce, step by step, addition and subtraction and multiplication as the process to be employed in order to bring about the elimination of one of the unknown.

It is the usual practice in modern textbooks for equations to be introduced in the form of easy problems, then for exercises giving practice in the solution of problems to follow. More difficult problems are then attempted. It is doubtful whether in the course here under consideration any useful purpose is served by introducing equations by way of problems such as: "Find a number such that when 7 is added to it the result is equal to 21." It should be remembered that we are dealing with pupils who are more advanced and who are already acquainted with equations as formulæ. Preliminary problems as an introduction to the work on equations are therefore unnecessary.

Quadratic equations will not be introduced until later, and certainly not until factorization has been mastered. The factorization method should first be taught and ample practice be given in this type of quadratic, not only because it is the easier method but because in the solution of problems it is the method which the pupil should attempt first, applying either the "completion of the square" method or the formula should the factorization method fail. As indicated below, the "completion of the square" method is taught as a special form of "factorization method". The master should insist on all steps in the argument being inserted.

Thus, in the example:

Solve the equation $x^2 - 3x - 10 = 0$.

$$x^2 - 3x - 10 = 0$$
$$\therefore (x - 5)(x + 2) = 0$$
$$\therefore \text{ either } \left.\begin{array}{r} x - 5 = 0 \\ \text{or } x + 2 = 0 \end{array}\right\}$$
$$\therefore x = 5 \text{ or } -2.$$

The last step but one must not be omitted in the statement. Its omission tends to foster unintelligent mechanical work, and also leads to the error of giving the answer as -5 or 2.

The method of "completing the square" should be recognized as a variation of the factorization method applicable when ordinary factorization is not possible.

Example: Solve the equation $2x^2 - 5x - 18 = 0$.

$$2x^2 - 5x - 18 = 0$$

Dividing by 2
$$x^2 - \tfrac{5}{2}x - 9 = 0$$

Completing the square and subtracting
$$x^2 - \tfrac{5}{2}x + (\tfrac{5}{4})^2 - \tfrac{25}{16} - 9 = 0$$

$$\therefore\ (x^2 - \tfrac{5}{4})^2 - \tfrac{169}{16} = 0$$

$$\therefore\ (x^2 - \tfrac{5}{4})^2 - (\tfrac{13}{4})^2 = 0$$

$$\therefore \text{ either } \left.\begin{aligned} (x - \tfrac{5}{4} - \tfrac{13}{4}) &= 0 \\ \text{or } (x - \tfrac{5}{4} + \tfrac{13}{4}) &= 0 \end{aligned}\right\}$$

$$\text{i.e. either } \left.\begin{aligned} x - \tfrac{18}{4} &= 0 \\ \text{or } x + \tfrac{8}{4} &= 0 \end{aligned}\right\}$$

$$\text{i.e. } x = \tfrac{9}{2} \text{ or } -2.$$

Solution by formula should appeal to boys who have had ample opportunity of becoming familiar with the use of formulæ.

Literal equations may be introduced as a means of testing knowledge of the various methods previously employed in solving equations, but apart from such use they serve no great purpose at this stage.

Factors. H.C.F. and L.C.M. Fractions.—The only point here that calls for comment is that work in H.C.F., L.C.M., and fractions should follow each type of factor. As soon as one type of factor has been mastered it should be applied immediately to H.C.F., L.C.M., fractions, and other processes. In other words the question "Factorize $ax + bx$" should also be put in the forms, divide $ax + bx$ by x, divide $ax + bx$ by $a + b$. Again, instead of giving merely a number of expressions to factorize, questions in the following form are to be preferred: "Factorize the following, $ax + bx$; $cx - dx$; $3x^2 - x$. What is (1) the H.C.F., (2) the L.C.M. of these expressions." So with each type of factor, application of such factorization to H.C.F., L.C.M., and fractions should immediately follow. Only in this way will the pupil appreciate the purpose of such factorization, and he will find the work far more interesting. Each set of questions in H.C.F., L.C.M., and fractions should of course involve expressions of the type the factorization of which has previously been learnt. In this way constant revision is ensured.

The work in fractions as well as in H.C.F. and L.C.M. helps to illustrate the nature of algebra as generalized arithmetic, since the processes applied to algebra are practically identical with those used in arithmetic. One of the most frequent mistakes made in reduction of algebraic fractions, viz. the cancellation of terms instead of factors, will be made far less frequently if pupils have been taught reduction in arithmetical fractions intelligently. Attention drawn to the analogy between the arithmetical and algebraical processes in all work involving fractions will help to avoid many such mistakes.

Numerical Trigonometry

Numerical trigonometry will be introduced as early as possible in the elementary mathematics course of the third and fourth years, the aim being to give the pupils a good practical knowledge of the subject, based on the use of logarithms, for the solution of triangles and practical problems such as measuring heights and distances.

The mathematical course of the first and second years leads to the use of logarithms. In the early part of the geometry course the fundamental work in angles is dealt with and problems connected with construction of triangles follow. It is at this point that elementary numerical trigonometry is most conveniently introduced. Exercises aiming at the acquirement of skill in the solution of artificial problems should be rigorously excluded. Manipulation of identities should find no place in the course. The aim will be, as already stated, to introduce the pupil to the more practical parts of the subject. The problems dealt with will be very similar to those already attempted in the elementary geometry and mensuration of the first and second years' course. The pupil will feel that he is merely bringing into use a more efficient mathematical tool.

Syllabus.—The course should include the following:

A. Acute angles. Trigonometrical ratios, sine, cosine, and tangent. Exercises in finding by geometrical construction the sine, cosine, and tangent of given angles. The trigonometrical ratios of angles 0°, 90°, 60°, 30°, 45°.

The use of sine, cosine, and tangent tables and checking of former geometrical constructions by means of such tables.

The right-angled triangle.

Simple examples in heights and distances.

The reciprocal ratios, cosecant, secant, and cotangent.

B. The extension of the above work to obtuse angles.

The solution of triangles and harder examples in heights and distances, including simple examples in three dimensions.

Areas.

The above syllabus is divided into two sections A and B, to emphasize the importance of recognizing that whereas trigonometrical ratios of acute angles present little difficulty to beginners, the extension of these ratios to obtuse angles is not grasped so quickly. The teacher should aim first of all at thoroughly familiarizing the pupils with the ratios, sine, cosine, and tangent and with the use of trigonometrical tables. The work at this early stage can well be confined to acute angles and the right-angled triangle. There should be ample graphical illustrative work in order to ensure that the use of trigonometrical tables does not become merely mechanical. Examples in heights and distances should, from time to time, be checked by scale drawings.

Whilst this part of the syllabus is being covered the work in geometry will be proceeding, and in algebra the pupil will have become thoroughly familiar with negative quantities. The more difficult work involved in the application of trigonometrical ratios to obtuse angles will therefore present less difficulty, whilst the study of the theorem of Pythagoras, and more particularly its extensions, will assist in the work to be undertaken in the solution of triangles.

Mention has been made of problems in heights and distances. The application of trigonometry to the solution of other problems will of course depend very largely on the type of school. In senior schools where elementary mechanics forms a part of the curriculum, the mathematical master will have no difficulty in extending trigonometry to this branch of the work. The above syllabus is extended to meet the requirements of a general elementary mathematical course, so far as trigonometry is concerned.

In schools where more advanced work is likely it may be advisable to introduce questions on compound angles and trigonometrical identities, but even in such schools the more practical work in trigonometry should not be sacrificed during the third and fourth years in favour of the more theoretical considerations of the subject.

CHAPTER VIII

Conclusion

This volume of *The Teachers' Library* necessarily deals very largely with the subject-matter of the elementary mathematics course, but throughout, the aim has been to impress upon those to whom mathematical teaching is entrusted that a knowledge of the pupil and his mental development is as important as a knowledge of mathematics. If learning is to be through experience rather than by rote, the teacher's task is to know what constitutes a real experience for the pupil. This necessarily implies that understanding the pupil is as important as understanding the subject. As was pointed out in the previous section on Arithmetic, the teacher armed with a sound knowledge of the underlying principles of child psychology will plan his work and adapt his methods to suit the mental development of the child. Above all he will appreciate the important fact that interest in mathematics is maintained not merely by making the subject-matter palatable to the pupil, but by developing in the pupil a keen sense of power of achievement and a desire for further achievement. It is the pupil's own powers of observation and reasoning which have to be developed, the pupil's own initiative and inventiveness which have to be encouraged to the utmost, and all this requires of the teacher a sound knowledge and understanding of the pupil himself.

The value to the teacher of a knowledge of the history of mathematical development has also been emphasized. What is known in psychology as the " culture epoch " theory is particularly applicable in the teaching of mathematics.

The elementary mathematics course here suggested follows very largely the historical development of the subject. Geometry, for instance, has been treated in the early stages as " earth measurement ", and the work has been based on a fairly complete course of mensuration. Again, recognition has been given to the fact that abstract algebra is, comparatively, a late development, and for that reason the teaching of such algebra has been deferred to quite late in the course. For the same reason, work in connexion with negative quantities has been postponed to the third year of the course. In stressing the importance of a knowledge of the historical development of the subject and the significance of the culture epoch theory, one is of course only emphasizing once again the importance to the teacher of mathematics of a knowledge of child psychology.

Lack of sympathetic understanding of the pupil and not the difficulty of mathematics tends to create distaste for the subject. In this connexion the importance of the teacher's own personality must not be overlooked. As previously mentioned, to be a successful teacher of the subject one must not be merely keen, alert, and enthusiastic, but must be contagiously so. At the same time the teacher's own enthusiasm and keenness must never tempt him to be impatient with the slower pupil or to indulge in sarcastic disparagement of the pupil who finds the subject difficult. On the contrary, the teacher must exercise the utmost patience and ever be ready with sympathetic encouragement.

This is not to say that all difficulties are to be removed from the pupil's path or that the teacher must do the thinking for him. " Spoon feeding " cannot be too strongly condemned. Under the sympathetic guidance of the teacher the pupil must fight his own mental battles, and as he gains the mastery so he gains in confidence, self-reliance, and even self-respect. The test of successful mathematical teaching is the progress which the dullards and those of just average ability make rather than the progress made by those who are likely to develop into mathematicians. The mathematical teacher must never forget that his task is " mathematical education " rather than " the education of mathematicians ".

Finally, the importance of careful and systematic correction of the pupil's work must be fully appreciated. No work should go unmarked, and no work, having been marked, should go uncorrected. As soon as possible after the exercise has been done the work should be marked, and whatever is wrong should be immediately corrected by the pupil. Closely linked with the questions of " corrections " and " marking " is the question of choice of homework. Unless the utmost care is exercised in selecting mathematical homework, utter distaste and even a loathing for the subject can be created. Both as regards quantity and degree of difficulty, the exercises set should be of such a nature that the pupil feels the task has been worth while. Homework should increase confidence and self-reliance; it should never cause discouragement or a feeling of hopelessness, particularly in the early years.

If throughout this volume the importance of the development of the child has been emphasized as being the principal concern of the mathematics teacher, it must not be thought that the cultural value of mathematics is underestimated; far from it. The ideal mathematics master is perhaps he who has faith in the child and a like faith in mathematics as an educational instrument. "We want mathematics," says Giovanni Gentile, "but we want it *in* the man. And the same for religion, economics, poetry, and all the rest. Culture," he says, "is not in books, nor in the brains of others. It is in our own souls as it is gradually being formed there."

And what are we to understand by mathematics being *in* our pupils, being in their souls? "A man," says G. St. L. Carson, speaking of the purpose of teaching mathematics, "who has in his mind this chain of processes, observations, speculation, proof of consistence in speculation, rejection of redundant speculation, and finally the erection of deductions on this foundation, is in possession of an intellectual creation, which, in beauty alone, is worthy to rank with the creations of poetry, music, or art; and beyond this, it is a possession which in so far as it guides his life will make him a more efficient labourer and a better citizen."

The writer desires to acknowledge the instruction he derived some years ago from "A Study of Mathematical Education" by Benchara Branford, and from the series of essays contained in "Mathematical Education" by G. St. L. Carson. The re-reading of both these works has been particularly helpful in the preparation of this volume.

To Mr. J. F. Chalk, B.Sc., Senior Mathematical Master at Taunton's School, Southampton, the writer is also indebted for valuable suggestions made whilst the book was in manuscript form.