NINTH-YEAR MATHEMATICS

BRESLICH

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NINTH-YEAR MATHEMATICS

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AUTHOR'S PREFACE

This volume is the third of a three-book series in mathematics. In the first two books the pupil has developed a clear understanding of most of the fundamental algebraic and geometric concepts. He has also become acquainted with some of the fundamental principles of both subjects. He is therefore well prepared for the study of ninth-year algebra.

Moreover, the study of algebra has been further simplified by correlating it with geometry. Literal numbers are presented in connection with lines, angles, perimeters, and circumferences. Algebraic expressions of the second degree are given real meanings through the study of areas of plane figures; expressions of the third degree are made concrete through a study of volumes; and signed numbers are taught in relation to the number scale.

The pupil's knowledge of geometry is particularly helpful to him in acquiring a clear conception of such fundamental laws of algebra as the laws of signs and of the fundamental operations of algebra.

Mathematical laws are developed inductively. First the experiences are provided which enable the pupil to attain an understanding of the laws. Then follows enough practice to develop the necessary proficiency in their use.

The practical values of mathematics are stressed to retain the learner's interest and to enrich the course.

To the pupils who plan to leave school at the end of the ninth year the course offers the opportunity to come in contact with a large variety of problems and to receive a broad mathematical preparation for the vocations which they expect to enter. At the same time those who continue in school work are thoroughly prepared for future courses.

There are numerous supplementary exercises in the book to allow for individual differences. Frequent tests measure the pupil's understanding, while the timed tests give him an idea as to how much he may reasonably be expected to accomplish in a specified time.

The author expresses his gratitude to Director Charles H. Judd, Professor H. C. Morrison, and Professor W. C. Reavis for their encouragement, assistance, and inspiration during the period of experimentation when the materials for this course were tried out with junior high school pupils.

Through a subsidy from the Commonwealth Fund the author was enabled to visit the leading junior high schools of the country. The study of these schools has been exceedingly helpful in selecting and organizing the materials used in the course.

E. R. BRESLICH

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NINTH-YEAR MATHEMATICS

CHAPTER I

A STUDY OF FORMULAS AND GRAPHS

RELATIONSHIPS IN MATHEMATICS

1. Relationships between numbers in everyday affairs. Relationships between numbers play an important rôle in the affairs of life, and problems involving relationships have to be solved by people in all occupations. For example, a man's taxes depend on the value of his property or on the amount of his income; the interest received on an investment is related to the amount invested and to the rate of interest; a life insurance

FIGURE 1
MAILING A PACKAGE

premium depends upon the amount of insurance carried and the age of the insured; the cost of a railroad ticket depends upon the distance to be traveled.

The boy sending a parcel (Fig. 1) verifies the charge by computing it from the weight of the parcel and the rate per ounce. The farmer building a fence finds the cost from the distance around the field and the price of fence material per yard. The cook preparing a meal determines the quantity of food from the number of people to be served. The horse power of an engine depends upon the number of cylinders and the diameter of the cylinders. The selling price of an article depends upon the purchase price and the cost of selling. A family's light bill depends upon the number of lamps in use. In the study of mathematics much attention is paid to the relationships involved in problems.

Give other examples that illustrate relationships, dependence, and correspondence.

2. Relationships in geometry are frequently expressed by means of equations and formulas. You are familiar with many relations of geometry. You know that, when two angles of a triangle are given, the size of the third is definitely fixed, because it depends upon the sum of the other two angles. You have previously learned that the size of an equilateral triangle depends upon the length of the side; that the area of a rectangle depends upon the lengths of base and altitude; and that the volume of a cube depends upon the length of the edge. Many geometric relationships are expressed by means of algebraic equations or formulas. Thus, in the case of the angles mentioned above, the formula expressing the relation is

$a + b + c = 180$. What does this mean in words? The perimeter of a triangle is given by the formula $p = a + b + c$ (Fig. 2). What does this formula mean? The circumference of a circle (Fig. 3) is given by the formula $c = 2\pi r$, where r is the radius of the circle and π equals 3.14 approximately. What does this formula mean?

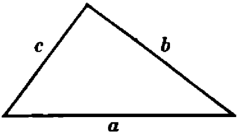

FIGURE 2
TRIANGLE

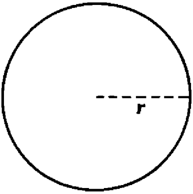

FIGURE 3
CIRCLE

EXERCISES

1. If the length of a rectangle is 8 inches and the width is 5 inches, what is the perimeter? State a simple rule for finding the perimeter of any rectangle when the length and the width are known. Write this rule as a formula, denoting the perimeter by p, the length by l, and the width by w (Fig. 4).

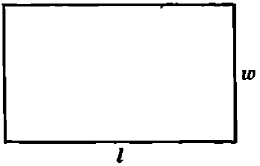

FIGURE 4
RECTANGLE

2. If one side of a square is 3 inches, what is the perimeter? State a simple rule for finding the perimeter of a square when the length of a side is known. Write this rule as a formula, denoting the perimeter by p and the length of a side by s (Fig. 5). How may $s + s + s + s$ be written in a simpler way?

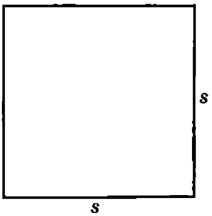

FIGURE 5
SQUARE

3. Measure the sides of the rectangle (Fig. 4) and thus find the numbers denoted by l and w. Place these numbers in the perimeter formula and find a numerical value for p.

4. As in Exercise 3 determine p for Figs. 2 and 5.

5. If $l = 4\frac{1}{2}$ and $w = 2\frac{1}{16}$, find p from the formula $p = 2l + 2w$.

6. If $r = 3.6$, find c from the formula $c = 2\pi r$.

7. The sides of a triangle are $3\frac{1}{2}$ inches, $4\frac{5}{8}$ inches, and $5\frac{1}{4}$ inches. Find the perimeter.

FORMULAS REPRESENTING RELATIONSHIPS

3. Formulas of the first degree. The formulas so far mentioned in this chapter are all said to be formulas of the **first degree,** since no term contains more than one dimension. Thus the right side of the formula $p = a + b + c$ has three terms; the first term contains one dimension, a; the second term contains one dimension, b; and the third term one dimension, c. Show that the formulas $p = 2l + 2w$, $p = 4s$, and $c = 2\pi r$ are of the first degree.

Using the formula for the circumference of a circle, $c = 2\pi r$, find the value of c when r is 1, 2, 3, 4, and 5, respectively. You see that the number c depends on r for its value. If you let r change in value, c also changes in value, the nature of the change being determined by the relation $c = 2\pi r$. This relation must always be satisfied by r and the corresponding value of c. The literal numbers r and c are said to change, or *vary*. The variable number c is said *to depend upon* r, which means that for every value of r the formula $c = 2\pi r$ determines a corresponding value of c.

EXERCISES

1. Show that the perimeter of a polygon having sixteen equal sides is given by the formula $p = 16s$. (What is the 16 called?) Find the values of p corresponding to the values of s given below. For example, if $s = 8$, then $p = 16s = 16 \times 8 = 128$.

s	12 ft.	3.8 ft.	1.46 mi.	$4\frac{1}{2}$ in.	$6\frac{2}{3}$ ft.	$2\frac{5}{6}$ ft.	.86 cm.
p							

2. The perimeter of a rectangle is given by the formula $p = 2l + 2w$. Find the values of p corresponding to the following values of l and w:

l	13.6′	$6\frac{1}{3}$′	8.97′	$7\frac{3}{4}$′	.83′	$4\frac{3}{8}$″
w	4.8′	$5\frac{1}{2}$′	6.42′	$5\frac{5}{6}$′	.42′	$2\frac{1}{16}$″
p						

3. Find the circumference of a circle corresponding to each of the following radii. Use $\pi = 3.14$.

r	24 yd.	$6\frac{3}{8}$ in.	4.82 m.	.065 mi.	$1\frac{2}{3}$ ft.	$6\frac{7}{8}$ in.
c						

4. The perimeter of a triangle is given by the formula $p = a + b + c$. Find p for the following sides:

a	$5\frac{1}{2}'$	$16.4'$	$.94'$	$2\frac{5}{6}'$	$11\frac{3}{4}'$
b	$6\frac{2}{3}'$	$12.3'$	$1.29'$	$1\frac{1}{3}'$	$12\frac{2}{3}'$
c	$2\frac{1}{6}'$	$8.7'$	$.73'$	$2\frac{1}{2}'$	$10\frac{5}{6}'$
p					

5. Complete the following statements, showing the dependence of one variable number upon one or more other variables:

(*a*) If a train travels at a uniform rate, the distance depends upon ————.

(*b*) The perimeter of an equilateral triangle depends upon ————.

(*c*) The amount of interest received from an investment depends upon ————.

(*d*) The premium of a life insurance policy depends upon ————.

(*e*) The amount of electricity used in a house depends upon ————.

(*f*) The temperature at a given place depends upon ————.

(*g*) The amount of work a man does depends upon the number of ————.

(*h*) The number of articles that can be bought for $3 depends upon ————.

(*i*) The weight of an iron rod depends upon ————.

6. Give examples of number relationships which occur in the following occupations: farming, cooking, selling, buying, building, banking, engineering.

7. Find out and be prepared to tell what the literal numbers stand for in each of the following equations. Name the variables, or variable numbers.

(*a*) $d = 20t$

(*b*) $C = \frac{5}{9}(F - 32)$

(*c*) $a = 180 - (b + c)$

(*d*) $p = \frac{5b}{100}$

(*e*) $c = 2\pi r$

(*f*) $v = gt$

4. Formulas of the second degree. You have learned that the area of a rectangle is equal to the product of the base and the altitude. More precisely, this means that the number of surface units is obtained by multiplying the number of units of length in the base by the number of units of length in the altitude. Draw a rectangle 4 inches long and 3 inches wide. Show, by dividing this rectangle into unit squares, how the rule is obtained. The formula $A = b \times h$ expresses the relation between the three numbers A, b, and h in a brief form. It is usually written, you know, $A = bh$, where bh means $b \times h$.

In the case of the square the foregoing formula changes to $A = s \times s$. Why? The product $s \times s$ is written briefly s^2. (What is the small 2 called?) Thus $A = s \times s$ is changed into $A = s^2$, where s^2 means that s is multiplied by itself.

The formulas $A = bh$ and $A = s^2$ are of the **second degree** because each contains the product of two dimensions. Other formulas of the second degree are those for the area of a triangle, $A = \frac{1}{2}bh$, the area of a trapezoid, $A = \frac{1}{2}h(a + b)$, and the area of a circle, $A = \pi r^2$. Tell what the literal numbers in each formula stand for.

EXERCISES

1. Find the area of a rectangle corresponding to each pair of values below:

b	9′	1.32″	$15\frac{1}{2}$″	$6\frac{1}{2}$′	.025 mi.	$4\frac{5}{6}$′	8.03 in.
h	18′	2.68″	$12\frac{3}{4}$″	$8\frac{2}{3}$′	.036 mi.	$6\frac{2}{3}$′	6.07 in.
A							

2. Find the areas of squares whose sides are $1\frac{1}{2}$ feet; $3\frac{2}{3}$ feet; 1.2 feet; 1.4 feet; 1.6 feet; 2.5 feet.

3. Find to three figures the areas of circles whose radii are 8 inches; 12 inches; 15 inches; 18 inches; $2\frac{2}{3}$ inches; $5\frac{5}{6}$ inches.

4. Draw a parallelogram $ABCD$ with base b and altitude h (Fig. 6). Cut it from the paper. Cut off the triangle from the right side and place it on the left side so as to form the rectangle $ABC'D'$. Show that the area of the parallelogram may be obtained by the formula $A = bh$.

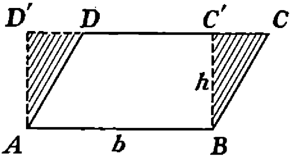

FIGURE 6.

5. Using the formula $A = bh$, find the areas of parallelograms corresponding to the following dimensions:

b	3′	2.35″	$6\frac{3}{4}$″	.324 mi.	$16\frac{5}{6}$″	7.2 m.
h	2′	1.68″	$8\frac{1}{2}$″	.026 mi.	$8\frac{2}{3}$″	3.6 m.

6. Draw a trapezoid $ABCD$ (Fig. 7), making $AB = 1\frac{1}{2}$ inches, $CD = 1$ inch, and $h = 1\frac{1}{4}$ inches. Extend AB to D', making $BD' = 1$ inch. Extend DC to A', making $CA' = 1\frac{1}{2}$ inches. Cut the resulting parallelogram from the paper. What is the area of the parallelogram? Show by cutting that the trapezoid $ABCD$ is equal to the trapezoid $BD'A'C$. What is the area of the trapezoid $ABCD$?

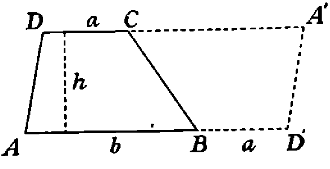

FIGURE 7

7. Repeat Exercise 6, letting $AB = a$ inches, $DC = b$ inches, and the altitude $= h$ inches.

8. Find the areas of the following trapezoids from the formula $A = \frac{1}{2}h(a + b)$:

h	15′	$4\frac{1}{2}$ yd.	$2\frac{2}{3}$′	12″	.84′	3.02′	$6\frac{5}{8}$″
a	3.1′	$5\frac{1}{3}$ yd.	$3\frac{1}{2}$′	2.75″	5.5′	.82′	$10\frac{1}{2}$″
b	2.3′	$4\frac{5}{6}$ yd.	$2\frac{5}{16}$′	1.8″	3.24′	.56′	$8\frac{3}{4}$″
A							

9. Divide the shaded surfaces (Fig. 8) into rectangles and find the area of each surface:

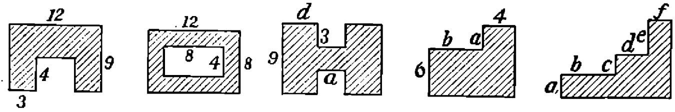

FIGURE 8

Make formulas expressing the areas of the last three surfaces (Fig. 8). Using the formulas, find the area A of these surfaces if $a = 2$, $b = 4$, $c = 3$, $d = 1.5$, $e = 3$, $f = 2\frac{1}{2}$; $a = 3$, $b = 7$, $c = 2.5$, $d = 4$, $e = 4.5$, $f = 2.8$.

ORDER OF OPERATIONS

5. Order of operations. Sometimes several operations occur in an expression. For example, in the expression $4 \times 5 - 6$ you find a multiplication and a subtraction; and in $8 \div 2 + 4$, you find a division and an addition. The question therefore arises as to which operation is to be performed first. The correct way is to multiply and divide before adding or subtracting. Thus $4 \times 5 - 6$ means $(4 \times 5) - 6$, and $8 + 4 \div 2$ means $8 + (4 \div 2)$.

When a multiplication and division follow each other, you should perform the operations in the order in which they occur. Thus $8 \div 2 \times 6$ means $(8 \div 2) \times 6$.

EXERCISES

Find the value of each of the following:

1. $8 + 12 \div 3 - 4$
2. $6 + 4 \times 8 - 2$
3. $8 \div 3 + 4 \times 9$
4. $15 \div 3 \times 4 + 6 \div 2$
5. $9 \times 8 \div 3 + 15 \times 3 \div 5$
6. $20 + 6 \times 4 + 8 \div 2 + 10 \div 5$

Find the value of each of the expressions in Exercises 7 to 15 if $a = 6, b = 2, c = 1, d = \frac{1}{3}$. In each case you should remember the rule that the multiplications are to be performed first and then the additions and subtractions. Use parentheses as shown in the solution of Exercise 7.

7. $ab + ac + bc + 4$

Solution: $ab + ac + bc + 4 = (6 \times 2) + (6 \times 1) + (2 \times 1) + 4$
$= 12 + 6 + 2 + 4$, by performing the multiplications,
$= 24$, by collecting the terms.

8. $ac + bd + bc$

9. $ab + bc + d + 5$

10. $\frac{ad + a + c}{ab}$

11. $\frac{b + d + bd}{bd}$

12. $\frac{ab + ac}{ab + dc}$

13. $\frac{bd + ba}{ad + c}$

14. $\frac{ac + d}{ab + bc}$

15. $\frac{2ab + c}{3ac + 2d}$

REVIEW EXERCISES

6. Supplementary exercises. The following exercises will give you further practice in the use of formulas.

EXERCISES

1. Draw a rectangle, using protractor and ruler. Measure the sides and find the area by means of the formula.

2. Find the cost of covering a rectangular floor 25 feet by 16 feet with linoleum at $2.25 a square yard.

3. A garden plot is 16 yards long and 18 yards wide. Find the cost of sodding it at 32 cents a square yard.

4. Find the cost of covering a floor 24 feet by 18 feet with linoleum selling at $1.50 a square yard.

5. How many acres are there in a field 95 rods long and 72 rods wide?

6. A rectangle is 6 inches long and 5 inches wide. Find the area. Double the length, but leave the width the same. Find the area. Compare the area of the second rectangle with the area of the first.

7. If the base of a rectangle is doubled and if the altitude remains the same, what change takes place in the area? Give numerical examples.

8. If the radius of a circle is doubled, by how much is the circumference changed? How is the area changed? Give numerical examples.

SUMMARY OF TERMS

7. Summary of terms used frequently in mathematics. You should be thoroughly familiar with the following terms that are used frequently in mathematics:

1. A letter denoting a number is a **literal number.**

2. Literal numbers and combinations of literal numbers are **algebraic expressions.** Expressions like $\frac{1}{2}bh$, $20t$, and a^2 are **terms** or **monomials.** Terms are either single arithmetical or literal numbers, products, or quotients. Give other examples of monomials.

3. An expression containing two or more terms is a **polynomial.** When a polynomial contains only two terms it is a **binomial,** and when it contains only three terms it is a **trinomial.** For example, $a + b$ and $\frac{5}{9}F - \frac{160}{9}$ are binomials, and $a + b + c$ and $x^2 - 3x + 4$ are trinomials. Give examples of binomials, trinomials, and other polynomials.

4. When two or more numbers are multiplied, they form a **product.** Thus $2x$, ab, $\frac{1}{2}h(a + b)$ are products. The numbers that are multiplied are **factors.** For example, the numbers $\frac{1}{2}$, h, and $(a + b)$ are factors of $\frac{1}{2}h(a + b)$. The numbers 2 and $\frac{1}{2}$ in the preceding terms are their **numerical coefficients.** Thus the coefficients of $3x$, πr^2, and $\frac{2}{3}b$ are, respectively, 3, π, and $\frac{2}{3}$.

5. An **exponent** indicates the number of *equal factors* in a product. Thus, in a^2 and r^2, the exponents are 2.

6. To indicate that a combination of numbers is to be considered as one number, the **parenthesis** ( ) or **brackets** [ ] are used. Thus $\frac{1}{2}h(a + b)$ means that $a + b$ is considered as one number to be multiplied by $\frac{1}{2}h$.

EXERCISES

In the following algebraic expressions name the binomials, trinomials, exponents, numerical coefficients, products, factors, and terms:

1. $2x^2 - 7$

2. $3x^2 + 5x - 2$

3. $a^2 + 6ab + 4b^2$

4. $1.6a - .3b$

5. $\frac{3}{4}ax^2$

6. $5a^2 + 3b^2 + 6c^2$

7. $2a(x + 6)$

8. $3a + 5(b - 4)$

OTHER FORMULAS REPRESENTING RELATIONSHIPS

8. Formulas of the third degree. The volume of a rectangular block (Fig. 9) is found by means of the formula $V = l \times w \times h$, or $V = lwh$. Show by a drawing how this formula is derived. Since the edges of a cube are equal to each other, this formula applied to the cube reduces to $V = e \times e \times e$, which is written briefly $V = e^3$. The formulas $V = lwh$ and $V = e^3$ are of the **third degree,** since they contain the products of three dimensions. Other formulas of the third degree are used in finding the volume of a cone, $V = \frac{1}{3}\pi r^2 h$; the volume of a sphere, $V = \frac{4}{3}\pi r^3$; and the volume of a cylinder, $V = \pi r^2 h$. Make drawings of all these figures and tell what the letters in each of these formulas stand for.

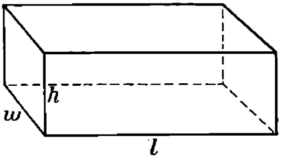

FIGURE 9
RECTANGULAR BLOCK

EXERCISES

1. Give the meaning of each of the following: $2a$, a^2, $3a$, a^3, ab, $a + b$, abc, $a + b + c$, a^2b, $\frac{h}{2}(a + b)$.

2. Find the volumes of cubes whose edges are 2 inches; 4.5 inches; $6\frac{1}{4}$ inches; 1.2 inches; $4\frac{3}{4}$ inches.

3. Find the volume of each of the following rectangular blocks:

l	3 in.	$4\frac{1}{2}$ in.	3.6 in.	$2\frac{3}{4}$ in.	.84 cm.
w	5 in.	$5\frac{3}{4}$ in.	4.8 in.	$3\frac{1}{8}$ in.	.73 cm.
h	4 in.	8 in.	2.7 in.	$1\frac{7}{8}$ in.	.65 cm.
V					

4. Write the following in the briefest form: $a + a + a$; $3 \times a$; $a \times b$; $l \times l \times l$; $\pi r \times r \times h$.

5. State the difference in meaning between a^2 and $2a$; a^3 and $3a$; $a + b$ and ab. Give numerical examples.

6. The horse power of an engine is found by means of the formula h.p. $= \frac{nd^2}{2.5}$, where n denotes the number of cylinders and d the diameter of the cylinders. Find the horse power of a 4-cylinder engine whose cylinders are $3\frac{3}{8}$ inches in diameter.

7. State the meaning of each of the following: x^2; x^3; $(2x)^2$; $2x^3$; $3x^3$; $(3x)^3$; $3x^2$; $6x^3$; $8x^2$. Find the value of each when $x = 2$; 2.5; $\frac{3}{2}$.

8. Find the value of $x^3 + 3x^2 + 3x + 9$ when $x = \frac{3}{2}$.

Solution: $x^3 + 3x^2 + 3x + 9$

$= (\frac{3}{2})^3 + 3(\frac{3}{2})^2 + 3(\frac{3}{2}) + 9$, by substituting $\frac{3}{2}$ for x,

$= \frac{27}{8} + 3(\frac{9}{4}) + 3(\frac{3}{2}) + 9$,

$= \frac{27}{8} + \frac{27}{4} + \frac{9}{2} + 9$, by simplifying each term,

$= \frac{189}{8} = 23\frac{5}{8}$.

9. Find the value of $a^3 + 6a^2 + 4a + 1$ when $a = 2$.

10. Find the value of $2m^3 + 5m^2 + 2m + 3$ when $m = 3.5$.

11. Find the value of $4t^3 + t^2 + 6t + 1$ when $t = 2\frac{2}{3}$.

12. Find the value of $8b^3 + 5b^2 + \frac{3}{4} + 3b$ when $b = \frac{1}{6}$.

13. Find the value of $\pi r^2h + rh + h$ when $r = 2$, $h = 2.5$.

Solution: $\pi r^2h + rh + h = (3.14)4(2.5) + 2(2.5) + 2.5$, by substitution,

$= 31.4 + 5 + 2.5$, by multiplying,

$= 38.9$, by combining terms.

14. Find the value of $x^3 + 3x^2y + 3xy^2 + y^3$ when $x = 2$, $y = 1.5$.

15. Find the value of $2a^3 + 4a^2b + 3ab^2 + b^3$ when $a = 3$, $b = 2$.

9. What is meant by a power. The products $a \times a$ and $a \times a \times a$ have been written in the brief forms a^2 and a^3, respectively. Similarly we may write the products $a \times a \times a \times a$ as a^4, read *a-fourth*, $a \times a \times a \times a \times a$ as a^5, read *a-fifth*, etc. The products a^2, a^3, a^4, etc., are **powers** of a. What is the value of a^4 when $a = 2$; when $a = 3$?

EXERCISES

If $a = 2$, $b = 1$, $c = 3$, find the value of each of the following:

1. $a + b^2 + c^3$

2. $\dfrac{a^3 + b^3 + c^3}{2a^2}$

3. $\dfrac{b^2 + a^2 + c}{a + b + c}$

4. $ab + \dfrac{a^2}{c} + \dfrac{2}{c^2}$

5. $2a^3 + 3b^2 + 5c$

6. $\dfrac{a^3 + 2b + c}{3c - 2a}$

7. $\dfrac{a^2 + 2ab + b^2}{ab}$

8. $\dfrac{a^4 + a^2 + 3}{b^2 + c^2}$

10. Making formulas. The following exercises will give you practice in making formulas. Think first what you would do if the literal numbers were replaced by arithmetical numbers; then do the same thing with the literal numbers.

EXERCISES

Complete the following sentences:

1. The sum of a and b is ————.

2. The product of a and b is ————.

3. The quotient of a by b is ————.

4. The difference between a and b is ————.

5. The cost of a yards of goods at b cents a yard is ———.

6. The cost of n tickets at c cents per ticket is ————.

7. A car traveling at the rate of r miles per hour for t hours goes ———— miles.

8. The area A of a rectangle m feet long and n feet wide is ———— square feet.

9. Mary bought a pounds of rice at b cents a pound, c pounds of butter at d cents a pound, e cans of tomatoes at f cents a can, and g bars of soap at h cents a bar. The total amount of her purchases was ————.

10. In long division the dividend D is equal to the divisor d multiplied by the quotient q plus the remainder r. Make this statement as a formula.

11. Finding values by means of formulas. In the following formulas the values of some of the variables are given; the others are to be found.

EXERCISES

1. The height of a rectangular block is found from the formula $h = \frac{V}{lw}$. Tell what the letters stand for. Find h when $V = 348$, $l = 16$, $w = 18$.

2. The rate of interest may be found by means of the formula $r = \frac{100i}{pt}$. Tell what the letters stand for. Find r when $i = 187.50$, $t = 3$, $p = 1250$.

3. One of the sides of the right angle of a right triangle may be found by the formula $a = \sqrt{c^2 - b^2}$. Find a if $b = 12$ and $c = 16$.

4. The altitude of a trapezoid may be found from the formula $h = \frac{2A}{a + b}$. Tell what the letters stand for. Find h when $A = 93$, $a = 14$, $b = 12$.

5. The altitude of a right circular cylinder may be computed by means of the formula $h = \frac{T - 2\pi r^2}{2\pi r}$. Tell what the letters stand for. Find h if $T = 124$ and $r = 3$.

THE GRAPHICAL WAY OF REPRESENTING NUMBERS

12. Bar graphs. In the preceding pages, numbers were represented arithmetically by means of figures and algebraically by means of literal numbers and formulas. The following exercise reviews the geometric, or graphical, method.

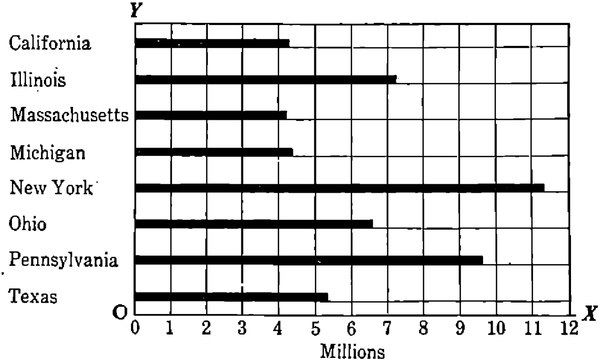

FIGURE 10

Fig. 10 represents the populations of some of our leading states. As you know, it is a bar graph. Two lines OX and OY are drawn at right angles. Along one of them, as OX, a unit is laid off repeatedly. Thus the unit on OX represents 1 million. The names of the states have been arranged alphabetically along the line OY.

The horizontal bars extending to the right represent the populations of these states. Thus the bar corresponding to California is 4.3 units long, which means that California's population is approximately 4.3 millions. What is the approximate population of each of the other states? Arrange the names of the states according to size of population. Compare the population of New York with that of Illinois, California, Michigan, and Texas.

EXERCISES

1. Make a bar graph like the one shown in Fig. 10.

2. Make a bar graph representing the following family budget: food $95, rent $80, clothing $34, running expenses $26, education and entertainment $46, savings $34. Tell what the graph shows.

3. In a certain high school 37% of the pupils are freshmen, 26% are sophomores, 20% are juniors, and 17% are seniors. Make a bar graph representing these facts and discuss the graph.

4. Look for bar graphs in newspapers and magazines. Study them carefully, bring them to class, and be prepared to tell what they mean.

5. Last year the grain crop on our farm was as follows: 262 bushels of corn, 180 bushels of oats, 72 bushels of rye, and 158 bushels of wheat. Represent these facts graphically and tell what the graph shows.

Suggestion: Round off the numbers, using 100 as a unit. This gives 2.6, 1.8, .7, and 1.6. Then use these numbers in making the graph.

On this graph what would a line 3 units long represent? Answer the same question for lines 2.4, 1.3, and .4 units long, respectively.

6. A merchant finds that the cost of operating his business is $22\frac{1}{2}\%$ of the receipts. Since the cost of goods amounts to $64\frac{1}{4}\%$ of the receipts, what is the profit actually made? Represent the three facts graphically.

7. Round steak is composed of 62.5% water, 18.5% protein, 9.2% fat, and 9.8% refuse. Make a graph representing these facts.

8. Five boys sold tickets for a game and reported the following number sold: John, 63; James, 74; Henry, 82; William, 6; and Paul, 51. Make a bar graph of these facts.

13. Supplementary exercises. The following exercises give further practice in making bar graphs.

EXERCISES

1. The weight of a cubic foot of water is 62.5 pounds, and the weight of a cubic foot of tin is 454.9 pounds. The ratio $\frac{454.9}{62.5} = 7.28$ means that tin weighs 7.28 times as much as an equal volume of water. This ratio is called the specific gravity of tin. In general, *the specific gravity is the ratio of the weight of a given substance to the weight of an equal volume of water.* Represent graphically the following specific gravities: alcohol, .79; aluminum, 2.58; benzine, .87; brass, 8.5; copper, 8.9; cork, .24; glass, 2.6; gold, 19.3; ice, .90; cast iron, 7.4; lead, 11.4; nickel, 8.9; nitric acid, 1.5; oak, .8; pine, .5; platinum, 21.5; silver, 10.53; tin, 7.29; zinc, 7.15.

2. Round off the following numbers to two figures and give the result as a number of millions: 4,316,459; 7,202,983; 11,303,296; 6,600,146; 9,613,579; 5,312,661.

Suggestion: 9,613,579 = 9.6 millions approximately.

By what number was 9,613,579 divided to change the unit to millions?

3. The estimated populations of the states listed in Fig. 10 for one year were as follows: California, 4,316,459; Illinois, 7,202,983; Massachusetts, 4,197,288; Michigan, 4,395,651; New York, 11,303,296; Ohio, 6,600,146; Pennsylvania, 9,613,570; Texas, 5,312,661. The densities (number of people per square mile) in these states were, respectively: 22.0, 115.6, 479.2, 63.8, 217.9, 141.4, 194.5, 17.8. Represent the populations and densities by graphs and discuss them by comparing them with each other.

Suggestion: To represent large numbers graphically, it is usually necessary to round them off.

4. The following table gives, for a period of years, the number of persons living in the United States who were from 5 to 18 years old, the number of pupils enrolled in the public schools, the average daily attendance, and the number of teachers:

YEAR	POPULATION 5 TO 18 YEARS	NUMBER ENROLLED	AVERAGE DAILY ATTENDANCE	NUMBER OF TEACHERS
1900	21,404,322	15,503,110	10,632,772	423,062
1905	23,410,800	16,256,038	11,481,531	460,269
1910	24,360,888	17,813,852	12,827,307	523,211
1915	26,425,100	19,693,007	14,964,886	604,001
1920	27,728,788	21,578,316	16,150,035	679,533
1925	30,150,672	24,988,707	19,856,311	797,618

Represent these facts by four graphs, using the same reference axes. Study the following ratios: the number enrolled to the number attending; the number enrolled to the number of teachers; the population to the number enrolled. How do the graphs picture the changes in these ratios? Tell any other facts shown by the graphs.

14. Making line graphs. To make a line graph, the end points of the bars are marked, but the bars are omitted. The points are then joined by a continuous line, which is the line graph (Fig. 11). (Note that, if the bars were drawn, they would be vertical.) Line graphs are used when the facts to be represented are related to each other. Thus, to represent the lengths of various rivers, you would use the bar graph because the lengths have no relation to each other. However, to represent temperature readings taken at regular intervals during the day, you would use the line graph, because it pictures not only the readings but also the *changes* of temperature for the day. The graph in Fig. 11 represents the growth of one of our cities within a period of five years from 1924 on: 30,820; 32,361; 39,543; 43,716; 46,926. Explain how the graph is made. Tell what it shows. What do you think will be the approximate population in 1929?

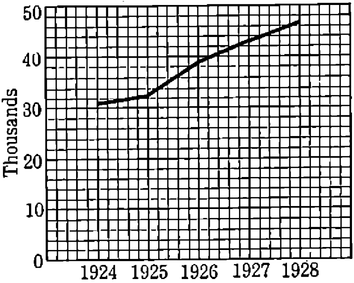

FIGURE 11

EXERCISES

1. The following scores were made by two ninth-grade classes on a test in mathematics:

Number of pupils	1	0	1	2	3	4	5	7	8	6	5	4	2
Test scores	14	15	16	17	18	19	20	21	22	23	24	25	26

Make a line graph representing these facts.

Suggestion: Represent the scores on the horizontal axis.

2. Make a line graph representing the table below, which gives the enrollment in our school for ten years.

Year	1919	1920	1921	1922	1923	1924	1925	1926	1927	1928
Enrollment	425	518	612	691	718	757	830	891	916	932

3. Make a line graph showing the changes in the population of the United States for a period of ten years.

4. From newspapers and magazines gather statistics on changes in pupil enrollment, wages, prices, sales of stock, population, and other subjects and represent some of them graphically.

5. Make a line graph representing the following temperature readings:

Time	6 A.M.	8	10	12 M.	2 P.M.	4	6
Temperature	46°	49°	54°	63°	64°	59°	55°

6. Make a line graph representing the average heights of girls given in feet for every two years from 2 to 18 years: 2.7, 3.2, 3.5, 3.8, 4.2, 4.6, 5.0, 5.1, 5.1. When do girls grow most rapidly? Most slowly?

GRAPHS OF FORMULAS AND EQUATIONS

15. Graph of the formula of uniform motion. If a train travels uniformly at a rate of 40 miles per hour for 3 hours, how far will it go? Give a rule for finding distance when you know the rate and the time. The formula for uniform motion is $d = rt$, where d is the number of units of distance (inches, feet, meters), r the rate, and t the number of units of time (seconds, hours, days). Suppose that a train is traveling at the rate of 30 miles an hour. Then the formula becomes $d = 30t$. What is the value of d when $t = 1$? What is the value of d when $t = 2$? You see that in this formula there are two variables, t and d. If you are given a value of t, you can always find a corresponding value of d, as you have just done. The value of d, therefore, depends upon the value of t. For every value of t, there is a corresponding value of d. If t is made to vary, d varies also.

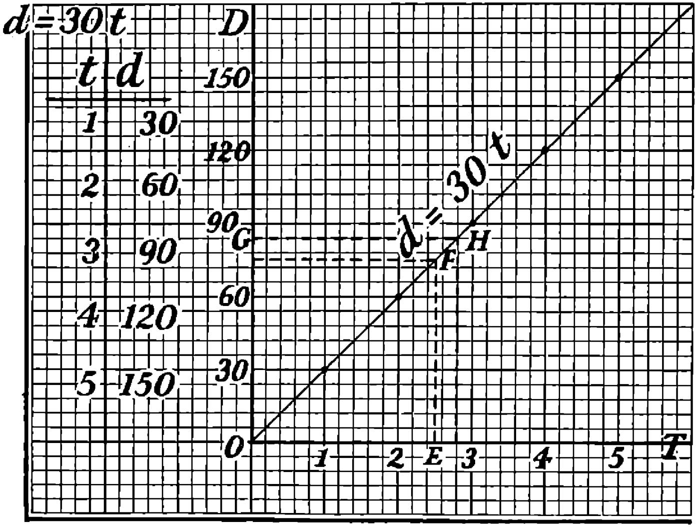

FIGURE 12

If $t = 3$, what is the value of d? If $t = 6$, what is the value of d? You see that, when t is doubled, d is also doubled. Test this statement with other values. If t is trebled, what is the effect upon d? This illustrates an important relationship in mathematics. If, when t is doubled, trebled, etc., the number d is also doubled, trebled, etc., the number d is said to **vary directly** with t or to be **directly proportional** to t.

The formula $d = 30t$ may be represented graphically as follows:

First make a table of several pairs of corresponding values of t and d (Fig. 12). Thus, if $t = 1$, then $d = 30$; if $t = 2$, then $d = 60$; etc.

The next step is to draw the two reference axes, OT and OD.

Convenient units are now chosen for representing graphically the number pairs in the table. The units are marked off on the axes, and the points corresponding to number pairs are marked, or plotted. Thus, to plot the pair (1, 30), pass from O one unit to the right and then up 30 units. To plot (2, 60), pass from O two units to the right and then up 60 units.

Finally, the line passing through the points is drawn. This is the required graph.

The graph may be used to find values of d for given values of t; or to find t when d is given. Thus, when $t = 2\frac{1}{2}$, pass from O two and one-half units to the right to E. From E pass upward to F. This vertical height may now be read off on the line OD. The result is $d = 75$.

To find t when $d = 84$, pass from O upward along the line OD a distance of 84 units to G. Then pass to the right to H and from H pass downward to the line OT. The result is $t = 2.8$.

16. Summary of the steps taken in making the graph of a formula. You have seen that the process of making graphs of formulas, such as $d = 30t$, involves the following steps:

1. Writing the formula

2. Tabulating pairs of corresponding values of the two variable literal numbers

3. Drawing the two reference axes

4. Selecting convenient units for laying off the numbers in the table

5. Plotting the pairs of corresponding numbers given in the table

6. Drawing the graph

EXERCISES

1. From the graph (Fig. 12) find the value of d for the following values of t: $1\frac{1}{2}$, $3\frac{1}{2}$, $5\frac{1}{2}$, 2.4, 4.6.

2. From the graph (Fig. 12) find the values of t which give the following values of d: 36, 72, 45, 105.

3. Make a graph of the formula $d = 20t$, following the directions given above.

Make a graph of each of the following equations, using the same axes for all.

4. $y = 4x$. **5.** $y = 2x$. **6.** $y = 6x$.

Suggestion: Lay off the values of x on the horizontal axis.

17. Graph of the percentage formula. The percentage formula is a relationship between the base and the percentage. If $r = 4$, the formula $p = \frac{rb}{100}$ takes the form $p = .04b$. This formula may be represented graphically as follows:

1. Let b take the values 0, 50, 100, 150, 200, etc. Tabulate the pairs of corresponding values of b and p.

2. Select convenient units and plot the number pairs in the table.

3. Through the points thus obtained draw a line. This is the graph of the formula $p = .04b$.

b	p
0	0
50	
100	
150	
200	
250	
300	
350	

EXERCISES

1. From the graph you have just made find 4% of the following numbers: 75, 125, 275.

2. Make a graph of the formula $p = .05b$.

3. Make a graph of the formula $p = .06b$.

18. Graph of the interest formula. The graph of the interest formula is similar to that of the percentage formula. Hence, in making the graph, the suggestions of §17 should be followed.

EXERCISES

Using the same axes and the arrangement shown in §17, make graphs of the following formulas:

1. $i = .02p$.

2. $i = .03p$.

3. $i = .08p$.

4. $i = .10p$.

5. $i = .04\frac{1}{2}p$.

6. $i = .05\frac{1}{2}p$.

19. Graph of the circumference formula. By means of the relation $c = 3.14d$, you are able to determine for any given value of d the corresponding value of c. In a table like the one shown below write the values of c corresponding to the values of d given in the first row:

d	1	2	3	4	5	6	7	8
c								

To represent graphically the facts stated in this table, lay off, to a convenient scale, the values of d horizontally and the values of c vertically (Fig. 13).

From the graph find c when $d = 1\frac{1}{2}$; $2\frac{1}{2}$; $6\frac{1}{2}$.

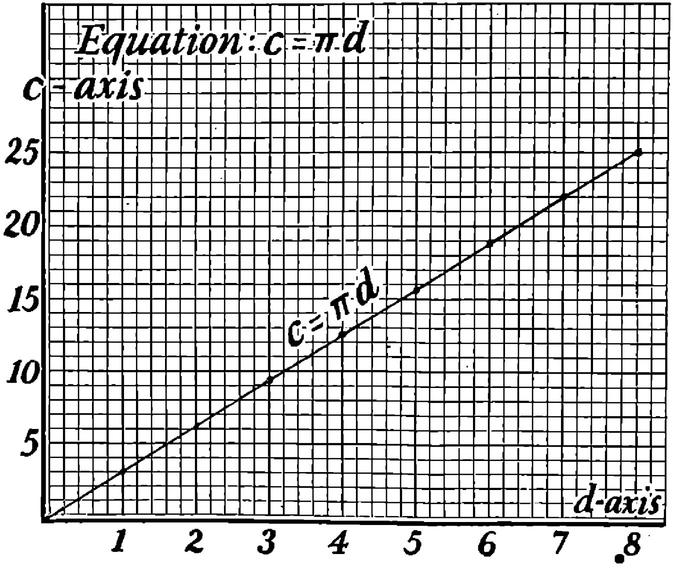

FIGURE 13

20. Graph of the area formula $A = bh$. Let the altitude of a rectangle vary, the base remaining the same, for example, equal to 3. Then $A = 3h$. Show that when $h = 1, 2, 3$, etc., the corresponding values of A are 3, 6, 9, etc.

Make a table of corresponding values of h and A (Fig. 14).

Make the graph of the formula $A = 3h$.

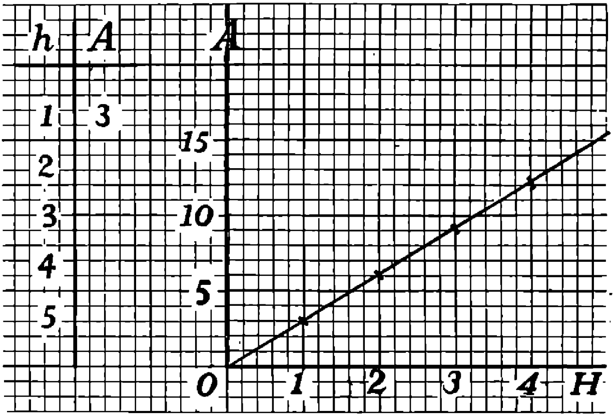

FIGURE 14

EXERCISES

1. From the graph (Fig. 14) find A when $h = 1\frac{1}{2}$; 2.4; 3.8.

2. Make graphs of the formulas $A = 5h$; $A = 8h$; $A = \frac{3}{2}h$.

3. Let the altitude of a parallelogram vary, the base remaining the same. Let the base be 4.6 and arrange in the form of a table the values of the area for the following values of the altitude: 1, 1.3, 1.6, 2, 2.8, 3.7, 3.9, 4.

Make the graph of the formula $A = 4.6h$.

4. Make the graph of the formula $A = 5.2h$.

From the graph determine, as nearly as you can, the area of the rectangle when $h = 3.5$; 4.5.

From the graph determine the value of h when $A = 15$.

21. Making graphs of equations. The first step in representing an algebraic expression graphically is, as you have seen from your work with formulas, to make a table of corresponding values of the variables. If $y = 2x - 1$, what is the value of y when $x = 1$? What are the values of y when $x = 2, 3, 4, 5$, and 6? These values may be shown in a table as follows:

x	1	2	3	4	5	6
y	1	3	5	7	9	11

They may also be given as number pairs placed in a parenthesis, the x value first, the y value second, as follows:

(1, 1); (2, 3); (3, 5); (4, 7); etc.

Show how you would write as a number pair the fact that, when $x = 5$, $y = 9$.

The next step is to plot these pairs with reference to two perpendicular lines (Fig. 15), OX and OY. Thus, to plot the pair (1, 1), start from O, pass 1 to the right and 1 upward; to plot the pair (2, 3), start again from O, pass 2 to the right and 3 upward. Similarly plot the pair (3, 5).

Finally, draw a straight line through the three points. This line is the graph of the equation $y = 2x - 1$.

By means of the graph find the values of y when $x = 1.8$; 1.6; .8; .6; .5.

If x is less than .5, the graph is *below* the line OX and y extends *downward*. To distinguish between downward and upward values of y, a minus sign ($-$) is prefixed to a downward value. Thus the graph shows that $y = -.8$ when $x = .1$, and that $y = -1$ when $x = 0$.

Similarly, when x is laid off to the *left* of O, a minus sign is prefixed to it. The graph shows that $y = -3$ when $x = -1$, and that $y = -5$ when $x = -2$.

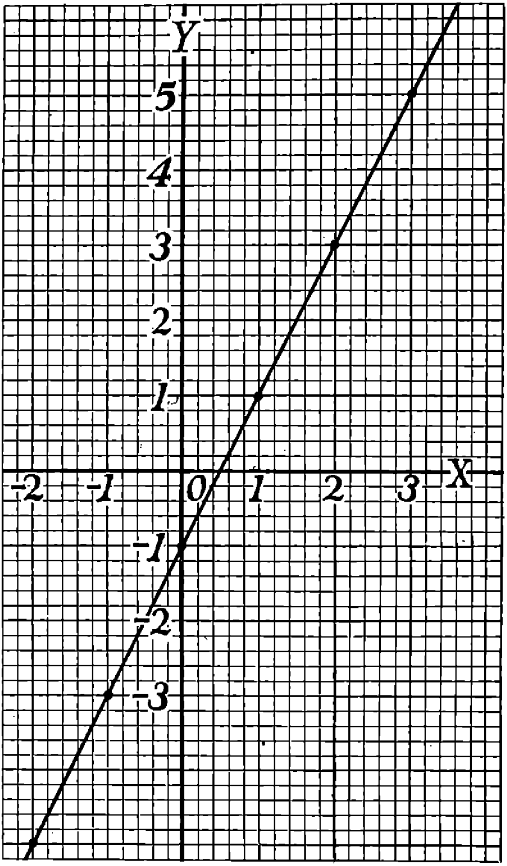

FIGURE 15

From the graph find the value of y when $x = -.8$; when $x = -.6$; when $x = -1.5$.

EXERCISES

1. Plot on graph paper the following number pairs: (3,5); (1,2); (2,1); (5,3); (0,2); (3,0); (6,7); (4,8); (0,4); (8,0).

2. Make a table of values for the equation $y = 2x + 3$, using the following numbers for x: 1, 2, 3, 4, 5. Write the corresponding values as number pairs.

Make a graph for each of the following equations and formulas:

3. $y = 2x + 1$.
4. $y = 32 + x$.
5. $s = 3t + 5$.
6. $y = 5x + 8$.
7. $F = \frac{9}{5}C + 32$.
8. $w = .3v + .5$.
9. $D = 20 + .01d$.
10. $y = 2x + 10$.
11. $y = 3(6x + 1)$.
12. $y = 2\pi r$.

EXERCISES TO TEST YOUR UNDERSTANDING OF CHAPTER I

22. What you should know and be able to do. You should be familiar with the meaning of each of the following terms:

perimeter
area
volume
square
rectangle
triangle
circle
trapezoid
variable
power
coefficient
exponent
product
factor
term
monomial
binomial
trinomial
polynomial
parenthesis

You should be able to find the values of algebraic expressions for given values of the literal numbers.

You should be able to interpret and to make bar graphs, line graphs, and graphs of equations and formulas.

EXERCISES

1. Find the values as indicated:

(*a*) $p = 4l$. Find p when $l = 1$; $\frac{1}{4}$; .6.
(*b*) $l = a + b + c$. Find l when $a = 1.2$, $b = .4$, $c = 3.07$.
(*c*) $c = 2\pi r$. Find c when $r = 42$, $\pi = 3\frac{1}{7}$; $r = 1.06$, $\pi = 3.14$.
(*d*) $s = \frac{1}{2}gt^2$. Find s when $g = 32$, $t = 5$.
(*e*) $A = \frac{1}{2}bh$. Find A when $b = 4.3$, $h = .25$.
(*f*) $V = \frac{4}{3}\pi r^3$. Find V when $\pi = 3\frac{1}{7}$, $r = 1\frac{3}{4}$.
(*g*) $y = x^2 + 3x + 1$. Find y when $x = 1.25$.

2. Select the words or numbers which complete the following sentences correctly:

(*a*) The sum of the sides of a polygon is called its (area, monomial, perimeter, square, degree).
(*b*) In $2y^3$ the 3 is called a (coefficient, power, degree, exponent, square).
(*c*) The number $3y^2$ means ($3y$ multiplied by itself, 2 times $3y$, $y^2 \times y^2 \times y^2$, $3 \times y \times y$, $3 + y^2$).
(*d*) The value of $4a^2$ for $a = 3$ is (24, 36, 144, 13, 48).

3. For each of the words below select the example which illustrates it best:

(*a*) Product
(*b*) Parenthesis
(*c*) Binomial
(*d*) Power
(*e*) Sum
(*f*) Trinomial
(*g*) Coefficient
(*h*) Exponent
(*i*) Term
(*j*) Factor
(*k*) Expression of second degree
(*l*) Expression of third degree

Examples:

1. $a + b$
2. $x + y + z$
3. y^3
4. (   )
5. 6 mn

4. Find the value of $3 + 6 \times 5 - 4 \div 2$.

5. Express as an equation this statement: The altitude of a cone is equal to 3 times the volume divided by π times the square of the radius of the base.

6. Round off the following numbers by expressing them as millions and tenths of millions: 3,825,356; 1,418,298; 15,627,419.

7. Make a graph of Henry's grades in his courses: Latin, 88; mathematics, 95; history, 82; English, 90.

8. Make a graph representing the average weights of girls from 2 to 18 years:

Age	2	4	6	8	10	12	14	16	18
Weight in pounds	25	32	41.5	56	69	82	111	121	129

9. Make a graph of the equation $y = 2x + 3$.

CHAPTER II

POSITIVE AND NEGATIVE NUMBERS. THE FUNDAMENTAL OPERATIONS OF ALGEBRA

MEANING OF SIGNED NUMBERS[1]

23. Signed numbers in graphs. In § 21 you learned that a minus sign $(-)$ is placed before a value of y if it is measured downward from the axis OX and also before a value of x if it is measured to the left from the axis OY. In this chapter you will study other uses of signed numbers.

24. Signed numbers are used in thermometer readings. On a winter day the weather report published in the newspapers gave the following table stating the temperature for 24 hours:

Maximum	2 P.M.	7
Minimum	2 A.M.	−5

3 A.M.	− 4	11 A.M.	5	7 P.M.	6
4 A.M.	− 3	Noon	6	8 P.M.	5
5 A.M.	− 2	1 P.M.	6	9 P.M.	4
6 A.M.	− 1	2 P.M.	7	10 P.M.	1
7 A.M.	2	3 P.M.	6	11 P.M.	0
8 A.M.	4	4 P.M.	6	Midnight	− 3
9 A.M.	4	5 P.M.	5	1 A.M.	− 4
10 A.M.	6	6 P.M.	6	2 A.M.	− 5

---

[1] Pupils who have studied signed numbers previously may omit §§ 23 to 30 or review them briefly.

The numbers in the right-hand columns in the preceding table refer to the thermometer scale (Fig. 16). A certain point on this scale is designated as the *zero point.* The numbers −5, −4, −3 in the table mean, respectively, 5° below zero, 4° below zero, and 3° below zero. Thus degrees *below* zero are denoted by numbers prefixed by a *minus sign* (−). The numbers 2, 4, 6, etc., in the table mean 2° above zero, 4° above zero, and 6° above zero. Sometimes readings *above* zero are denoted by numbers prefixed by a *plus sign* (+). Accordingly, readings of +6°, +4°, +2° mean that the temperature is 6° above zero, 4° above zero, 2° above zero.

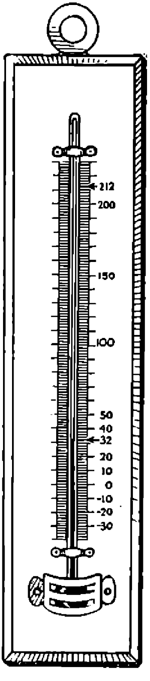

FIGURE 16

THERMOMETER SCALE

EXERCISES

1. State the meaning of each of the following temperature readings: +60°, −3°, −8°, 0°, +4°, −10°, +88°.

2. If the top of the mercury column in a thermometer is at 5° above zero and the temperature then rises 4°, express the final reading by means of a + or − sign. If the temperature then falls 11°, state the final reading.

3. If the top of the mercury column is at 3° below zero and the temperature rises 5°, what is the reading after the rise in temperature? Express your answer with a + or − sign. If the temperature continues to rise 2° and then falls 8°, what is the reading? If the temperature then rises 6°, what is the final reading?

4. State the final readings for the table below:

First reading .	8°	6°	2°	1°	−1°	3°	−8°	12°	−3°	0°	−2°
Change . . .	rise of 2°	fall of 10°	fall of 6°	fall of 1°	rise of 1°	rise of 4°	fall of 6°	fall of 12°	fall of 8°	rise of 10°	rise of 8°
Final reading .											

5. Represent the readings in the table on page 34 graphically as follows:

On the line *OX* (Fig. 17) lay off distances representing the hours. Then at each hour point lay off, at right angles to *OX*, the corresponding temperature reading. Degrees below 0 are laid off below the line *OX*. Connect the points thus located, forming the *temperature line.*

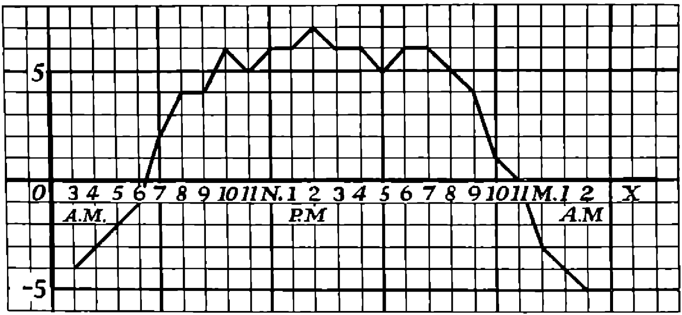

FIGURE 17

GRAPH OF THERMOMETER READINGS

Examine the graph and answer the following questions: When was the temperature highest? When was it lowest? When was the change in temperature greatest? What was the maximum temperature? The minimum?

6. The graph in Fig. 17 does not give all the changes in the temperature, since readings were taken only once an hour. A more complete record is obtained with the thermograph, which records the temperature automatically. If you have seen or read about a thermograph, describe it to the class.

Fig. 18 gives the complete thermograph record for the day which is shown partly in Fig. 17 and also for the days just preceding and following. Study Fig. 18 and tell what the graph shows.

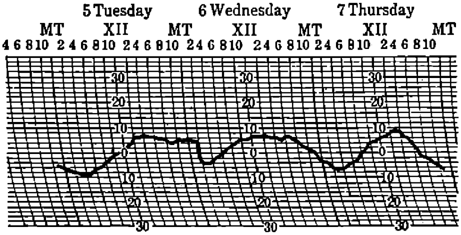

FIGURE 18

THERMOGRAPH CHART

7. For a certain day the hourly temperature readings beginning at 8:00 A.M. were as follows: 10°, 12°, 14°, 18°, 20°, 22°, 21°, 18°, 14°, 8°, −2°, −4°. Make the graph and tell what it shows.

8. Represent graphically the following daily average temperatures for one week: +10°, 0°, −8°, −4°, +6°, +14°, +15°.

9. Make a graph for the following table:

Time	Mid-night	1 A.M.	2	3	4	5	6	7	8	9
Temperature .	−10°	−12°	−15°	−13°	−10°	−9°	−8°	−6°	−3°	0°

Time	10	11	Noon	1 P.M.	2	3	4	5	6	7
Temperature .	2°	5°	7°	7°	9°	10°	8°	8°	7°	5°

25. Signed numbers are used to denote direction above or below water level. Heights of mountains and depths of lakes are measured in opposite directions from the surface of the water. The height of point *A* (Fig. 19) is +8′, that is, 8 feet above water level. The position of point *Q* is −5′, that is, 5 feet below.

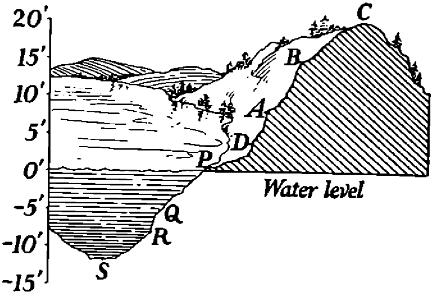

FIGURE 19

EXERCISES

1. By means of + and − signs state the position (height and direction) of points *B*, *C*, *D*, *P*, *R*, *S* (Fig. 19).

2. In a certain survey heights and depths were measured at intervals of 10 feet and recorded as in the table on page 39.

Make a diagram like that in Fig. 19 representing the facts contained in the table.

Horizontal distances.	0	10	20	30	40	50	60	70	80	90	100	110	120	130
Height or depth in feet . . .	−6	−12	−22	−20	−25	−20	−7	−2	+4	+5	+7	+6	+3	0

26. Review of the number scale. A number preceded by a + sign is called a **positive** number, and one preceded by a − sign is a **negative** number.

It must be remembered that positive and negative numbers are composed of two parts, the sign and the arithmetical value. The latter is at times called the *numerical* or *absolute* value. The + sign is frequently omitted. A number symbol without a sign is, therefore, positive.

To represent positive and negative numbers graphically, arrange them along a straight line (Fig. 20), the *positive* numbers to the *right* of the zero and the *negative* numbers to the *left*. This is the **number scale.** Any number in this scale is considered as *less* than all numbers to the right and as *greater* than all numbers to the left. At whatever point of the scale you may start, if you pass to the right the numbers are increasing and if you pass to the left they are decreasing. Thus −4 is *greater* than −6, although 4 is numerically *less* than 6.

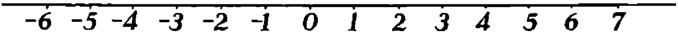

FIGURE 20

EXERCISES

1. Name the greater number in each of the following number pairs: 0, +5; −3, 0; −2, −5; +6, −4; −10, +10.

2. By how much is −5 less than −3? Verify your answer with the number scale (Fig. 20).

3. Locate the following numbers on the scale: +6, −5, $\frac{2}{3}$, $-\frac{3}{4}$, $2\frac{1}{2}$, $+3\frac{1}{3}$, 0, −6.3.

4. At a certain hour the temperature was + 3°. Two hours later it was −2°. What was the drop (difference) in temperature? Verify your answer with the number scale (Fig. 20).

27. Directed lines. The location of a place may be denoted by means of a + or − sign. Thus directions to the north are usually considered + and directions to the south −; directions to the east may be + and to the west −.

EXERCISES

In the following exercises directions are to be denoted by + and − signs, as shown in Exercise 1.

1. A boy riding a bicycle starts from home, rides 20 miles west, that is, in the negative direction, turns, and rides 14 miles in the opposite direction. How far is he from home?

Solve the problem in two ways as shown below:

Geometric statement:

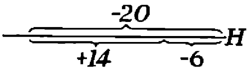

FIGURE 21

Algebraic statement: $-20 + 14 = -6$.

2. If a man travels 50 miles north one day and 76 miles south the next day, how far is he from the starting point?

Arrange the solution as shown in Exercise 1.

3. An elevator goes up 128 feet (eight floors) and then down 32 feet (two floors). How far is it from the first floor?

4. An elevator goes up 56 feet and down 70 feet. How far is it from the starting point?

28. Directed angles. The rotation of a radius about point B (Fig. 22) from the position of BA to that of BC forms angle ABC. Turning the radius the same amount, a, from BA to BC' forms angle ABC'. The two angles have the same numerical measure, but the directions of turning are opposite. We shall denote *clockwise* rotation by the $-$ sign and *counterclockwise* rotation by the $+$ sign. Hence $\angle ABC' = -a$, and $\angle ABC = +a$.

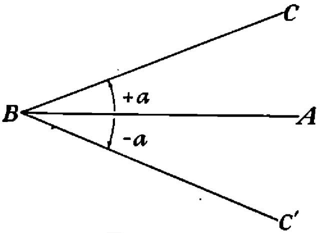

FIGURE 22

In geography, navigation, and astronomy directed angles are used to locate objects and places; for example, latitudes are + or − according as they are north or south of the equator.

EXERCISES

Use plus or minus signs in stating the results in the following exercises:

1. Draw the following angles, starting in each case with the initial line taken in the direction from left to right, as BA (Fig. 22): $+25°$, $-40°$, $-120°$, $180°$, $-220°$. Indicate in the drawing both size and direction.

2. What sign should be prefixed to the latitude of each of the following cities: New York, St. Louis, Cape Town, Boston, Rio de Janeiro, Galveston?

3. In what latitude is a traveler who starts in latitude – 3° and travels north 7°?

4. Find the latitude and longitude of your home city and express each with a signed number.

29. Directed forces. The opposing forces in a tug-of-war, the weight of a stone and the upward pull of a balloon, and the rate of a current and the rate of rowing upstream are examples of forces acting in opposite directions.

EXERCISES

Answer the following questions, using + and – signs to denote directions:

1. If the weight of a stone is denoted by a – sign, how should you denote the weight (upward pull) of a balloon?

2. If a stone weighing 5 ounces is attached to a balloon pulling upward with a force of 7 ounces, what is the magnitude and direction of the combined weight?

3. A boy can row in still water at the rate of 4.5 miles an hour. How fast can he go upstream against a current flowing at the rate of 2 miles an hour? How fast can he go downstream?

4. A carrier pigeon can fly at a rate of 55 miles an hour. How fast can it fly against a wind blowing at the rate of 35 miles an hour?

5. In a tug-of-war one side pulls to the north with a force of 320 pounds, the other to the south with a force of 312 pounds. Express these forces and the resulting force with + and – signs.

6. If the rate of a current is x miles per hour and the rate of rowing y miles, how fast can a man go upstream? Downstream?

30. Use of signed numbers in business. On page 43 is part of a report of the New York stock transactions, taken from a newspaper. The plus and minus signs in the last column are used to indicate the daily rise or fall of stocks.

New York Stock Transactions

Stock	Sales	High	Low	Close	Change
Adams Exp.	100	22½	22½	22½	+½
Ajax Rubber	900	25	24¾	24¾	−¼
Am. Sugar	2,600	90½	88¾	89	+1
Crucible Steel	15,500	74¼	71½	71⅞	−3⅝
Gen. Motors	7,500	15⅜	13⅜	13⅝	+⅛
People's Gas	500	33½	32⅛	32¾	−¾
Western Union	300	82¼	81½	81½	....

From the last column tell the rise or fall of each stock.

EXERCISES

1. State the opposites of the following terms: gain, increase, deposit, possession, export, asset, after Christ, north, forward, fall, above zero. If either of two opposites is denoted by a + sign, the other should have a − sign.

2. A man who was in debt $300 borrowed $450. He then received $1000. How much money had he then?

Solution: $-300 - 450 = -750.$
$-750 + 1000 = +250.$

In the following exercises arrange the solutions as in Exercise 2.

3. A man's property was worth $8500, and his debts amounted to $3600. Express his financial standing by means of a signed number.

4. A man's account book contained the following items: salary, $416; rent, $85; food, $75; insurance, $30; interest on bonds, $22. Denote these items with + or − signs and determine his financial standing.

5. The assets of a company are $38,328, and the liabilities are $35,220. What is the financial standing of the company?

6. A merchant gained $8115 one year and lost $1876 the next. Find his net gain or loss for the two years.

7. A man's monthly bank statement read as follows:

DATE	CHECKS		DATE	DEPOSITS	DATE	BALANCE
	Balance brought forward . .			$124.07		
Oct. 1	$35.00 –	$5.00 –	Oct. 1	$329.16	Oct. 1	$413.23
Oct. 2	10.00 –		Oct. 2	67.50	Oct. 2	470.73
Oct. 4	10.00 –	30.00 –	Oct. 4	80.25	Oct. 4	510.98
Oct. 5	18.50 –	19.25 –			Oct. 5	473.23
Oct. 6	112.66 –	50.00 –			Oct. 6	310.57
Oct. 8	40.00 –	5.00 –			Oct. 8	265.57
Oct. 9	45.00 –	28.50 –	Oct. 9	117.50	Oct. 9	309.57

The last amount in the last column to the right is the balance on October 9. For each date verify the correctness of the statement. Note that instead of prefixing the – sign the bank places it after the number; + signs in deposits are omitted but understood.

8. Hand work is being more and more replaced by machinery. The following table shows the changes in the volume of output and the changes in the number of workers employed during the years 1923–27.

INDUSTRY	PER CENT CHANGE IN OUTPUT	PER CENT CHANGE IN EMPLOYMENT
Oil . .	+ 84	– 5
Tobacco	+ 53	– 13
Meat .	+ 20	– 19
Railroads .	+ 30	– 1
Automobiles	+ 69	+ 48
Rubber tires .	+ 28	+ 7
Bituminous coal .	+ 4	– 15
Electricity	+ 70	+ 52
Steel . .	+ 8	– 9
Cotton mills	+ 3	– 13
Agriculture	+ 10	– 5
Lumber	– 6	– 21

Tell what each number means.

ADDING POSITIVE AND NEGATIVE NUMBERS

31. How to add signed numbers graphically. To add (+5) and (+3) graphically means to lay off on the number scale (Fig. 23) first (+5) in the positive direction OA and then (+3) in the positive direction. *The distance and direction* (+8) *from the starting point to the stopping point is the required sum.*

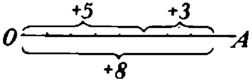

FIGURE 23

Thus you have the following equation:

$$(+5) + (+3) = +8.$$

This equation states briefly that 5 *to the right and then* 3 *more to the right gives* 8 *to the right*, or +8.

Similarly (+5) + (−3) = (+2), for (+5) is laid off first (Fig. 24) in the positive direction to B. Then (−3) is laid off from B in the negative direction BO. Finally, the direction and distance from the starting point O to the stopping point C is +2.

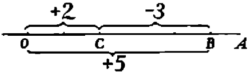

FIGURE 24

You may say that the statement (+5) + (−3) = (+2) means that 5 *to the right and then* 3 *to the left gives* 2 *to the right.*

The two preceding examples show that two signed numbers may be added graphically as follows:

Let a *and* b *be two numbers, positive or negative. To find the sum of* a *and* b, *lay off on the number scale first* a *in its own direction and then* b *in its own direction. The distance and direction from the starting point to the stopping point is the required sum.*

EXERCISES

Find the sums in Exercises 1, 2, and 3 graphically.

1. (−8) + (+3)

Solution: From O (Fig. 25) lay off −8 in its own direction to A. From A lay off + 3 in its own direction to B. Then OB, or − 5, is the required sum.

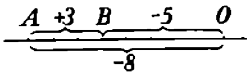

FIGURE 25

Hence (−8) + (+3) = (−5).

2. (+10) + (−4); (−6) + (−3); (−7) + (8); 12 + (+3); −10 + (6)

3. 6 + (−6); − 4 + 4; 5 + (−5)

4. Show by means of several examples, as in Exercise 3, that the sum of two numbers having the same arithmetical value and opposite signs, as $+a$ and $-a$, is always zero.

5. What must be added to each of the following to make the resulting sum zero: +4; −6; +3; −1?

6. Add the following mentally; that is, use the number scale without actually drawing the figure: +16 + (−3).

Solution: Pass 16 to the right and from there 3 to the left. The result is 13 to the right, or +13.

$$\therefore +16 + (-3) = +13.$$

Similarly add: +3 + (−6); +4 + (+8); −2 + (+1); −8 + (+12); −5 + (−7).

7. Add mentally: −8+(+4)+(−3); +3+(−2)+(+4); − 6 + 8 + (−7); 4 + 8 − 4; 6 + (−6) + 2.

8. Add mentally the lower numbers to the upper:

+16	+8	+6	−4	−3	−7	2
+4	−3	−9	−6	+5	3	−2

32. Combining terms. Show that 3·2 + 5·2 is the same as 8·2. (The dot, you remember, is used in place of the "times" sign.) Show that $2 \cdot 5 + 7 \cdot 5 - 3 \cdot 5$ is the same as $6 \cdot 5$. Show that $4 \cdot 3 + 5 \cdot 3$ may be written $(4 + 5) \cdot 3$. These numerical exercises illustrate a very important principle in algebra. Give other numerical examples.

Terms such as $2x$ and $3x$, $4a$ and $7a$, $5r^2h$ and $7r^2h$, whose literal parts are exactly the same, are called **similar terms,** as you know. Give other examples of similar terms. The exercises in the preceding paragraph illustrate the method of adding similar terms. Thus $2x + 3x = (2 + 3)x = 5x$; $4a + 7a = (4 + 7)a = 11a$; $5r^2h+7r^2h = 12r^2h$. Adding similar terms in this way is called **combining** or **collecting terms.**

Similar terms can be combined, but dissimilar terms cannot be combined. The sum of $3a$ and $2b$, for example, is $3a + 2b$. These two terms cannot be combined.

EXERCISES

In the following combine similar terms:

1. $16a + (4b) + (-3a) + (-b)$

Solution: $16a + (4b) + (-3a) + (-b)$

$= 16a - 3a + 4b - b$, by changing the order of the terms,

$= 13a + 3b$, by combining similar terms.

2. $10 + 3 + (-4) + 6 + (-3)$

3. $12 + (7) + (-3) + 2$

4. $5 + (-2) + 6 + (-3)$

5. $7x + 3x$

6. $8a + 2a + (-3a)$

7. $2b + 3b + 5b + (-4b)$

8. $6x + 2x + 3y + 4y$

9. $3d + 2c + 5d + 4c$

10. $7x + (-3x) + 4y + 2y$
11. $-2a + 5a$
12. $3b + (-9b)$
13. $4y + (-11y)$
14. $-2d + (-3d) + (-5d)$
15. $5x + (-2y) + 9y + (-7y)$
16. $7x + 2x + 5y + (-3y)$
17. $8a + (-b) + (-3a) + (-5b)$

33. A rule for adding numbers. You have learned to add numbers graphically by means of the number scale. You have also seen that you may add numbers more rapidly by using a mental picture of the graphical addition, without actually making the drawing. A third way is to use a rule in adding numbers. The following four examples in addition illustrate this rule:

Add graphically the lower numbers to the upper:

$$\begin{array}{r}+7\\+5\\\hline+12\end{array}\qquad\begin{array}{r}-7\\-5\\\hline-12\end{array}\qquad\begin{array}{r}+7\\-5\\\hline+2\end{array}\qquad\begin{array}{r}-7\\+5\\\hline-2\end{array}$$

Note (1) that in the *first two* examples the numbers to be added have *like* signs; (2) that the arithmetical value of the sum may be found by *adding* the arithmetical values of the two numbers; (3) that the sign of the sum is the *same* as the sign common to the given numbers. The three facts are stated in the form of a rule as follows:

To add two or more algebraic numbers having like signs, add the arithmetical values and prefix the common sign to the sum.

In the *third and fourth* examples above, note (1) that the two given numbers to be added have *unlike* signs;

(2) that the arithmetical value of the sum is equal to the *difference* between the arithmetical values of the numbers; and (3) that the sign prefixed is the sign of the number having the *greater* arithmetical value. This is expressed by the following rule:

To add a positive and a negative number, find the difference between the numerical values of the numbers and prefix to it the sign of the numerically greater number.

EXERCISES

1. Find the following sums by rule and then verify each by graphical addition: $+8 + (+3)$; $+9 + (-3)$; $7 + (+5)$; $-5 + 2$; $-6 + (-4)$; $+6 + (-3)$; $-4 + (+6)$; $-2 + 4$.

2. Show that $6 + (-2) = 6 - 2$; $8 + (-5) = 8 - 5$.

This exercise illustrates the fact that *adding a negative number gives the same result as subtracting the corresponding positive number.*

3. Collect the terms in each of the following:

$6 - 2 + (-8)$; $-8 + (-3) - 10$; $+3 + (+2) + (+8)$; $-6 + (-4) + (-1)$

4. Collect the terms in the following:

$3x - 2x + 6x + 4x - x$

Solution: $3x - 2x + 6x + 4x - x$

$= 3x + 6x + 4x - 2x - x$, by changing the order of the terms,

$= 13x - 3x$, by combining terms,

$= 10x$.

5. $-8x + 17x + 4x - 9x$

6. $14x - 10x - 3x + 7x$

7. $-3a - 12a - 5a + 20a$

8. $12.5b - 9.5b - 7.3b + b$

9. $.04m - .03m + .16m + .23m$

10. $\frac{2}{3}x + \frac{1}{4}x - \frac{3}{4}x + \frac{1}{3}x$

11. $7a^2 - 3b + 6b - 14a^2$

Solution: $7a^2 - 3b + 6b - 14a^2 = 7a^2 - 14a^2 + 6b - 3b$
$= -7a^2 + 3b.$

12. $2x^2 + 3x - 10x^2 - 12x + 4x^2$

13. $3x - 7z + 6y + 4y + 3z$

14. $16ab - 17.4ab - 1.6ab + 2ab$

34. Supplementary exercises. The following exercises will give you further practice in adding positive and negative numbers.

EXERCISES

Combine the similar terms:

1. $4x + 3x + 2x$

2. $2m + 5m + m$

3. $16a + 3a - 5a$

4. $10b - 6b + 3b$

5. $2t + \frac{1}{2}t + \frac{3}{4}t$

6. $\frac{1}{2}x - \frac{3}{4}x + \frac{5}{8}x$

7. $\frac{3x}{5} + \frac{x}{6} - \frac{2x}{3}$

8. $\frac{x}{3} - \frac{2x}{7} - \frac{6x}{5}$

9. $5w + 7 - 6w + 4 + 12w$

10. $7x - 3 - 12x + 7 - 12$

11. $3p - 8 - 5p + 4 - 4p$

12. $6p - 3p + 5 + 2 - 8p - 10$

13. $3x^2 + 5 + 7x^2 - 8 - 4x^2$

14. $6a^3 - 3b + 10a^3 + 5b - 16a^3$

15. $3m^4 - 8t - 7m^4 - 3t + 4m^4$

16. $5a - 3t^5 + 4a - 7a + 4t^5 - 7t^5$

17. $-4xy - 2z + 6xy + 8z + 3xy - z$

18. $8x + (+4x) + (-3x) + (10x)$

19. $14a + (2b) + (-3b) + (-7a) - 8$

20. $-3m - (4a) + (5m) + (-2m) + 6a$

Letting $a = 3$, $b = 2$, $c = 1.5$, $d = 1$, and $x = 2$, find the value of each of the following:

21. $2a - b$
22. $5c - 8d$
23. $a^2 + ax$
24. $2ad - 3bx$
25. $3ab + ab^2$
26. $b^2 - 4ac$
27. $2ac + ac^2$
28. $6d^2 + 3a^2b$
29 πd
30 $\frac{1}{2}ab$
31. $\frac{3}{4}\pi a^3$
32. πa^2b
33. $\frac{1}{2}a(b + c)$
34. πa^2x
35. $\dfrac{abc}{100}$
36. $\frac{1}{2}ax^2$
37. $2ab + 2ac + 2bc$
38. $\frac{1}{2}a + \frac{1}{3}b - \frac{1}{4}c$
39. $3.4b - 1.4a + 6.8d$
40. $2\frac{1}{2}x - 4\frac{1}{3}a - 5\frac{1}{4}d$
41. $3.1c + 6.5a - 10.1b$
42. $4\frac{1}{2}x - 3\frac{1}{3}b + 5\frac{1}{3}d$
43. $3.4a - 3.1b + 4.2c$
44. $5.3a + 5.6b - 3.2x$
45. $a^3 + 3a^2 - 6a - 4$
46. $3x^3 - 5x^2 + x - 6$
47. $2a^2 - 3b^2 + 8c - d$

ADDING POLYNOMIALS

35. A law for adding polynomials. If one boy has 5 apples and 7 oranges and a second boy has 9 apples and 3 oranges, how would you express what they both have together? The method of getting the answer to this exercise illustrates the method of adding polynomials. When two polynomials, as $6a + 3b + c$ and $a + 4b + 6c + d$, are to be added, the similar terms are combined first and then the sum of the resulting terms is expressed. What is the sum of these two polynomials? You will find it convenient to rearrange the terms before adding, as shown in Exercise 1 on page 52.

EXERCISES

Add the following polynomials:

1. $6x - 7y + 3z$ and $4x + 3y - z$

Solution:

(*a*) $(6x - 7y + 3z) + (4x + 3y - z)$
$= 6x + 4x - 7y + 3y + 3z - z$, by rearranging the terms,
$= 10x - 4y + 2z$, by combining similar terms.

(*b*) The solution may be arranged briefly as follows:

$$\begin{array}{r}
6x - 7y + 3z \\
4x + 3y - z \\
\hline
10x - 4y + 2z
\end{array}$$

2. $2a + 6b - 7c$ and $3a + 4b + 2c$

3. $6m - 3n + 10$ and $8m - 5n - 2$

4. $a + b$ and $a - b$

5. $a + b + c$ and $a - b - c$

6. $2a - 5b + 6c$ and $- 4a - 6b - 3c$

7. $x^2 + xy + y^2$ and $x^2 - xy + y^2$

8. $4m + 6n - 5 + (2m + 4n + 3) + (n - 6)$

9. $3x^2 + 3x - 7 + (2x^2 + 4x + 3) + (x - 3)$

10. $- x^2 - 2x + 4 + (- x^2 - 3x + 8) + (x^2 - x + 1)$

11. $a + 3(x - y) + b$ and $2b - 8(x - y) + 4a$ and $- 5(x - y) + 6b$

In each of the following exercises find the sum of the left members of the equations and the sum of the right members:

12.
$$\begin{array}{rcl}
x + 7y &=& 26. \\
2x + 3y &=& 16. \\
\hline
\end{array}$$

13.
$$\begin{array}{rcl}
x - 4y &=& 1. \\
4x + 3y &=& 40. \\
\hline
\end{array}$$

14.
$$\begin{array}{rcl}
3x \phantom{{}-4} &=& 8. \\
- x - 4 &=& 2. \\
\hline
\end{array}$$

15.
$$\begin{array}{rcl}
.8x + .2y &=& 10.2. \\
4x - 3.5y &=& 11.4. \\
\hline
\end{array}$$

16.
$$\begin{array}{rcl}
\frac{2x}{3} + 4y &=& \frac{26}{3}. \\
3x - \frac{7y}{2} &=& - 4. \\
\hline
\end{array}$$

36. Supplementary exercises. The following exercises will give you further practice in combining similar terms.

EXERCISES

Find the following sums:

1.
$$\begin{array}{r} 1.5mn \\ -3.4mn \\ .6mn \\ 2.1mn \\ \hline \end{array}$$

2.
$$\begin{array}{r} -2\frac{2}{3}x^2 \\ 1\frac{1}{5}x^2 \\ -4\frac{5}{6}x^2 \\ \frac{3}{2}x^2 \\ \hline \end{array}$$

3.
$$\begin{array}{r} 16abc \\ -12abc \\ -\ 3.1abc \\ -\ .8abc \\ \hline \end{array}$$

4.
$$\begin{array}{r} 3\frac{1}{2}x^2 - \frac{3}{2}xy + \frac{2}{3}y^2 \\ -1\frac{3}{4}x^2 - \frac{1}{4}xy - \frac{5}{6}y^2 \\ \hline \end{array}$$

5.
$$\begin{array}{r} \frac{1}{4}s^2 + \frac{2}{5}st - 6.5t^2 \\ -\frac{1}{5}s^2 - \frac{2}{3}st + 8.1t^2 \\ \hline \end{array}$$

6.
$$\begin{array}{r} 6(a - b) \\ -2(a - b) \\ 8(a - b) \\ (a - b) \\ \hline \end{array}$$

7.
$$\begin{array}{r} 3\frac{1}{2}(m + n) \\ 2\frac{3}{4}(m + n) \\ -1\frac{1}{5}(m + n) \\ -\ (m + n) \\ \hline \end{array}$$

8.
$$\begin{array}{r} 10(x^2 - y^2) \\ -2(x^2 - y^2) \\ -5(x^2 - y^2) \\ 13(x^2 - y^2) \\ \hline \end{array}$$

9. Add $3a^3 - 5a^2b + b^3$ to $a^3 - 6b^3$.

10. Add $6x^2 - 4x^3 + 1 - 3x$ to $2x^3 - 8$.

11. Add:

$$\begin{array}{rrr} -3.5x & & +\ 5.8z \\ 8.1x & +\ 3y & -\ z \\ & -\ 6y & +\ 2.3z \\ \hline \end{array}$$

Check by letting $x = 1$, $y = 2$, and $z = 3$.

12. Add:

$$\begin{array}{rrr} 2\frac{1}{2}A & +\ 6\frac{3}{4}B & -\ C \\ -4\frac{1}{3}A & & +\ \frac{2}{3}C \\ 5\ A & -\ \frac{2}{5}B & -\ \frac{3}{4}C \\ \hline \end{array}$$

Check by letting $A = 1$, $B = 2$, and $C = 3$.

SUBTRACTION

37. A clerk's way of subtracting. John was sent by his mother to make a purchase, for which she gave him a two-dollar bill. The price of the article was 38 cents. To make sure that he would receive the correct change, he wrote on a piece of paper the following:

FIGURE 26

$$\begin{array}{r} \$2.00 \\ .38 \\ \hline \$1.62 \end{array}$$

John's way of determining the correct change was to subtract $.38 from $2.

The clerk placed the two-dollar bill in the cash drawer and then laid on the counter the following amounts: 2 cents, saying, "38, 40," a dime, saying, "50," a half dollar, saying, "One dollar," a dollar bill, saying, "Two dollars." John counted the change and found it to be correct, $1.62. The clerk's method of making change apparently not only gave the same amount as John's method, but was very simple. John first subtracted $.38 from $2 and later counted the change to see if he had the correct amount. The clerk just added enough money to $.38 to make $2. His way of *subtracting* was to find how much he had to *add* to the $.38 to get a sum equal to $2. He really avoided subtraction by changing it into addition.

A five-dollar bill is handed a clerk to pay for an article costing $.63. Explain a simple method of making change.

38. Subtracting by means of the number scale. What is the result of subtracting 2 from 6? This question may be stated in another way, "What is the difference between 6 and 2?" A third way of stating the question, "How many units is it from 2 to 6?" suggests the clerk's method of subtracting. This method of subtraction by changing it to addition is very useful in subtracting signed numbers. Since thermometer readings are signed numbers, you may subtract signed numbers by finding the *difference between two thermometer readings*, that is, by determining the *change* in the position of the top of the mercury column of a thermometer. If the first thermometer reading (Fig. 27) is +16° and the second +2° (Fig. 28), you may find the difference by counting from the second reading, +2°, to the first reading, +16°. Since in passing from +2° to +16° you are counting *upward*, the difference is +14°. The result is the answer to the question, "How many degrees is it from 2° to 16°?" Similarly, if the first reading is −3° and

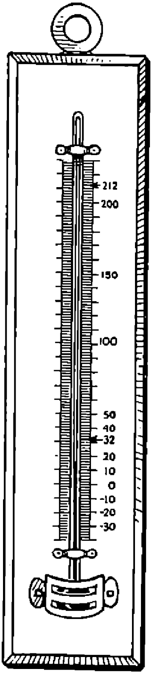

FIGURE 27

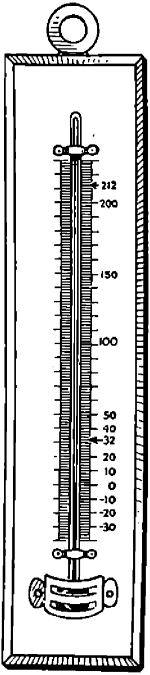

FIGURE 28

the second $+5^\circ$, the difference is -8°, because you count *downward* from the second reading to the first.

It is customary to arrange the written work of subtraction in either of the following two forms:

(*a*) $+16^\circ - (+2^\circ) = +14^\circ$.

(*b*)
$$\begin{array}{r} +16^\circ \\ \underline{+\ 2^\circ} \\ +14^\circ \end{array}$$

Similarly you write:

(*a*) $-3^\circ - (+5^\circ) = -8^\circ$.

(*b*)
$$\begin{array}{r} -3^\circ \\ \underline{+5^\circ} \\ -8^\circ \end{array}$$

The number to be *diminished*, as the $+16^\circ$ or -3° above, is the **minuend.** The number to be *subtracted*, as $+2^\circ$ or $+5^\circ$, is the **subtrahend.** The result, as $+14^\circ$ or -8°, is the **difference.**

The following examples illustrate further the process of subtracting numbers:

1. A man who died in 1916 was born in 1876. His age when he died is determined by *counting* from the time of birth to the time of death, which gives 40.

Thus 1916 − 1876 = 40.

2. It is said that Augustus, the Roman emperor, was born 63 B.C. (-63) and that he died A.D. 14 ($+14$). To find his age at the time of his death, you may lay off -63 (Fig. 29) on the number scale to the left of the zero point and $+14$ to the right. Counting from the point -63 to $+14$, you have the difference, 77; that is, $+14 - (-63) = +77$.

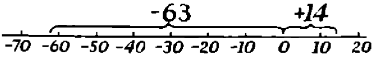

FIGURE 29

Thus *the difference between two numbers may be found on the number scale by counting from the subtrahend to the minuend.*

Note that here, as in making change, subtracting is really replaced by counting or adding.

EXERCISES

Solve the following exercises with the number scale.

1. On two successive days the average temperatures were $+8°$ and $-2°$. Find the difference.

2. The famous mathematician Archimedes was born 287 B.C. and died 212 B.C. How old was he when he died?

3. The longitude of a prominent building in Paris is $2°\ 20'$ East, and that of the City Hall, New York, is $74°$ West. Find the difference in longitude.

4. A ship sailed from latitude $8°\ 30'$ North to latitude $15°\ 20'$ South. Through how many degrees did it sail?

39. A simple rule for subtracting numbers. You have seen that subtraction is the process of finding the number which added to the subtrahend gives the minuend, and that you may subtract by counting along the number scale from the subtrahend to the minuend. We are now able to use a simpler method, which will be shown from a study of the following four examples:

1. From $+2$ subtract $+5$.

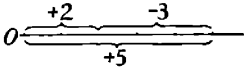

FIGURE 30

Solution: On the number scale mark $+5$ and $+2$ (Fig. 30). From $+5$ count to $+2$; that is, count 3 units to the *left*. Hence the difference is -3.

$$\text{Thus } +2-(+5)=-3.$$

Comparing this with $+2+(-5)=-3$,

you find that $+2-(+5)=+2+(-5) \quad ..(1)$

2. From -2 subtract -5.

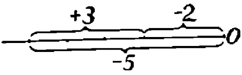

FIGURE 31

Solution: As before, mark off -2 and -5 (Fig. 31). Begin at point -5 and count to -2, that is, 3 units to the *right*.

$$\text{Hence } (-2) - (-5) = +3.$$
$$\text{Comparing this with } (-2) + (+5) = +3,$$
$$\text{you find that } (-2) - (-5) = (-2) + (+5) \quad . . (2)$$

3. From $+2$ subtract -5.

Solution: Beginning at the point -5 (Fig. 32), count 7 units to the *right* to the point $+2$.

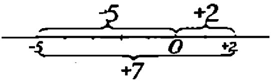

FIGURE 32

$$\text{Hence } +2 - (-5) = +7.$$
$$\text{But } +2 + (+5) = +7.$$
$$\text{Therefore } +2 - (-5) = +2 + (+5) \quad . . . . (3)$$

4. From -2 subtract $+5$.

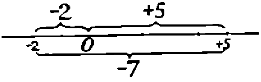

FIGURE 33

Solution: Beginning at $+5$ (Fig. 33), count 7 units to the *left* to -2.

This shows that the difference is -7

$$\text{Hence } -2 - (+5) = -7.$$
$$\text{Since } -2 + (-5) = -7,$$
$$\text{it follows that } -2 - (+5) = -2 + (-5) \quad . . . . (4)$$

Equations (1), (2), (3), and (4) illustrate the fact that in every case a subtraction problem may be replaced by an addition problem which gives the same result, by using the following rule: *To subtract a number, change the sign of the subtrahend and add the result to the minuend.*

The changing of the sign should be done mentally. Since subtraction is so much simpler by this rule than by the method shown in § 38, you should use the rule in all future subtraction exercises.

Explain why it is possible in algebra to subtract a number from a smaller one.

EXERCISES

In Exercises 1 to 13 subtract the lower numbers from the upper by using the rule on page 58, doing most of the work orally.

1. $\begin{array}{r} +6 \\ \underline{-3} \end{array}$

Solution: Change -3 mentally to $+3$ and then add $+3$ to $+6$. The result is $+9$.

2. $\begin{array}{r} +28 \\ \underline{-10} \end{array}$

3. $\begin{array}{r} -16 \\ \underline{-18} \end{array}$

4. $\begin{array}{r} +3 \\ \underline{+15} \end{array}$

5. $\begin{array}{r} -11 \\ \underline{+3} \end{array}$

6. $\begin{array}{r} -13 \\ \underline{-5} \end{array}$

7. $\begin{array}{r} -18 \\ \underline{+4} \end{array}$

8. $\begin{array}{r} +36 \\ \underline{+9} \end{array}$

9. $\begin{array}{r} +58 \\ \underline{-13} \end{array}$

10. $\begin{array}{r} +5 \\ \underline{+5} \end{array}$

11. $\begin{array}{r} -8 \\ \underline{-8} \end{array}$

12. $\begin{array}{r} -9 \\ \underline{+9} \end{array}$

13. $\begin{array}{r} +24 \\ \underline{-17} \end{array}$

In Exercises 14 to 20 combine similar terms, doing all you can orally.

14. $+8 - (+3) + (-6) - (-4) + (+6)$

Oral solution:

$$\begin{array}{r} +8 - (+3) = 5. \\ 5 + (-6) = -1. \\ -1 - (-4) = 3. \\ 3 + 6 = 9. \end{array}$$

15. $3 + (-6) + (+5) - (+8) - (-10)$

16. $15a - (-6a) + (-12a) + (+8a)$

17. $(-7m) + (+15m) - (+14m) - (-6m)$

18. $-32xy + 45xy - 13xy + 33xy$

19. $12a^2 - 8a^2 + 9a - 6a - 13$

20. $18x^2y + 16xyz - 20x^2y + 2xyz$

In Exercises 21 to 24 subtract the terms of the lower polynomials from the similar terms of the upper:

21.
$$\begin{array}{r} 17a^2 - 6a + 3 \\ \underline{8a^2 - 4a - 6} \end{array}$$

22.
$$\begin{array}{r} 2ab - 5bc - 3ac \\ \underline{-4ab + 6bc - 5ac} \end{array}$$

23.
$$\begin{array}{r} x^2 + 12x - 5 \\ \underline{-2x^2 + 7x - 8} \end{array}$$

24.
$$\begin{array}{r} x^2 + 2xy - y^2 \\ \underline{x^2 - 2xy + y^2} \end{array}$$

Subtract each side of the lower equations from the corresponding side of the upper:

25.
$$\begin{array}{rcl} 8x + 5y &=& 44. \\ \underline{2x - y} &\underline{=}& \underline{2.} \end{array}$$

26.
$$\begin{array}{rcl} 7x + 3y &=& -36. \\ \underline{-x + 3y} &\underline{=}& \underline{7.} \end{array}$$

27.
$$\begin{array}{rcl} 2x - 3y &=& 4. \\ \underline{2x + 5y} &\underline{=}& \underline{30.} \end{array}$$

28.
$$\begin{array}{rcl} 5x + 3y &=& 26. \\ \underline{-5x + 3y} &\underline{=}& \underline{-14.} \end{array}$$

29.
$$\begin{array}{rcl} 9a - 2b &=& 42. \\ \underline{6a - b} &\underline{=}& \underline{31.} \end{array}$$

30.
$$\begin{array}{rcl} 8m - 21n &=& 30. \\ \underline{6m + 35n} &\underline{=}& \underline{17.} \end{array}$$

40. Supplementary exercises. Most of the exercises on page 61 illustrate further that a polynomial is subtracted by changing the signs of all terms and then adding the results to the corresponding terms of the minuend. In the following exercises add or subtract as indicated and combine similar terms.

EXERCISES

Find:

1. $0 + (-3)$

2. $3 + 0$

3. $3 - 0$

4. $0 - (+3)$

5. $0 - (-3)$

6. $9a - (8 + 2a)$

Suggestion:

$9a - (8 + 2a)$

$= 9a - 8 - 2a.$ Why?

7. $-8n + (5 - 4n)$

8. $3a - (2a + 6)$

9. $(5 - 4x) - 8x$

10. $2m - (3n - 6m)$

11. $3r + (5r + 4t)$

12. $6g - (3g + 4h)$

13. $2a + (-4a + 6b)$

14. $2b - (b - 2a)$

15. $(5x - y) + (7x - 2y)$

16. $(8a - 9b) - (4a + 5b)$

17. $(3\frac{1}{2}a + 4\frac{1}{3}b) + (4\frac{1}{2}a - 3\frac{2}{3}b)$

18. $(5m - 10n) - (m + 2n)$

19. $(a + b + c) - (a - b - c)$

20. $(2x^2 + x - 4) + (3x^2 + 4x)$

21. $(5c - d) - (8 + 12c - 6d)$

22. $(a - 2b) - (1 + 3a - 8b)$

23. $(8b - 6f) + (7b - 3c - f)$

24. $4 - (2 + 3a) + (3a - 5)$

25. $3x - (4 + 2x) - (x + 3)$

26. From $10a - 8b + 6bc$ subtract $4a + 5bc - 10b$.

27. Subtract $3ay - 2bx - 4xy$ from $6xy + 4ay + 7bx$.

28. Subtract $a^2 + 2ab + b^2$ from $a^2 - 2ab + b^2$.

29. From $x^2 - 3xy + y^2$ subtract $x^2 + xy - 3y^2$.

30. From $1 + 3a + 6a^2 + 2a^3$ subtract $a - a^2 + a^3$.

31. Subtract $3m - n + 4q$ from $m + 6n + 3q$.

32. From $6 - \frac{2}{3}x^2 + 4x^3$ subtract $8 + \frac{3}{5}x^2 + x^3$.

Multiplying Monomials by Monomials

41. Multiplication of signed numbers by arithmetical numbers. If a boy deposits 4 dollars each month for 3 months, he changes his bank account by $3(+4)$ dollars, which is an *increase* of 12 dollars.

Thus $3(+4) = +12$.

If a boy withdraws 4 dollars a month, the account is changed by $3(-4)$ dollars, or a *decrease* of 12 dollars.

Thus $3(-4) = -12$.

If the mercury rises 2° each hour, in 4 hours it changes $4(+2)°$, which is a *rise* of 8°.

Thus $4(+2)=+8$.

If the mercury falls 2° each hour, in 4 hours it changes $4(-2)°$, which is a *fall* of 8°.

Thus $4(-2)=-8$.

These four examples show how to determine the sign when you multiply a positive or negative number by an *arithmetical* number. State the rule in your own words.

42. How to multiply signed numbers graphically. The following four examples illustrate the process of multiplying two signed numbers by means of the number scale.

1. Multiply $+2$ by $+3$.

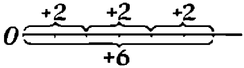

FIGURE 34

This is interpreted to mean that $+2$ is to be laid off three times in its own direction (Fig. 34). The result is $+6$.

Hence $(+3)\ (+2)=+6$ . . . . (1)

2. Multiply -2 by $+3$.

This means that -2 is to be laid off 3 times in its own direction (Fig. 35). The result is -6.

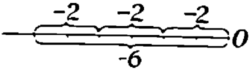

FIGURE 35

Hence $(+3)\ (-2)=-6$ . . . . (2)

3. Multiply $+2$ by -3.

Whatever may be the meaning to be assigned to $(-3)\ (+2)$, it must be in agreement with the laws of algebra. In particular, it must not violate the law of order in multiplication. Accordingly you must have

$$(-3)\ (+2)=(+2)\ (-3).$$

From Example 2 it follows that the product $(+2)(-3)$ is equal to -6. Hence the product $(-3)(+2)$ must also be -6.

The same result will be obtained graphically if you interpret $(-3)(+2)$ to mean that $+2$ is to be laid off 3 times in the direction *opposite* to that of the sign of $+2$.

Thus $(-3)(+2)=-6$ . . . . (3)

4. Multiply -2 by -3.

As in Example 3, the product $(-3)(-2)$ should mean that -2 is to be laid off 3 times in the direction *opposite* to that of the sign of -2.

Hence $(-3)(-2)=+6$ . . . . (4)

43. Laws for finding the sign of a product. By examining equations (1), (2), (3), and (4) in § 42 you obtain the following laws for multiplying signed numbers:

1. *The product of two numbers having like signs is positive.*

2. *The product of two numbers having unlike signs is negative.*

Since the sign of a product is much more easily found by means of these rules than by the graphical method shown in § 42, you should use these rules in all future multiplication exercises.

EXERCISES

In performing the multiplications in the following exercises, determine *first the sign,* using the laws stated above, and *then the arithmetical product.* Do most of the work orally.

1. $(-17)(-2)$

Solution: The sign is $+$; the arithmetical product is 34. Hence $(-17)(-2) = +34$.

2. $8(-2)$
3. $(-3)(6)$
4. $(-4)(-7)$
5. $(+3)(+6)$
6. $(-8)(+4)$
7. $-8(-7)$
8. $(+6)(4)$
9. $2(-19)$
10. $-5(+21)$
11. $(-3)(-15)$
12. $6(+5)$
13. $+8(-3)$
14. $(-15)(13)$
15. $(-6)(-4)$
16. $(+12)(+8)$

17. $(\frac{2}{9})(-\frac{3}{10})$

Solution: $\left(\dfrac{2}{9}\right)\left(-\dfrac{3}{10}\right) = -\dfrac{\cancel{2}\times\cancel{3}}{\underset{3}{\cancel{9}}\times\underset{5}{\cancel{10}}} = -\dfrac{1}{15}.$

18. $(-\frac{2}{15})(+\frac{5}{8})$
19. $(-\frac{7}{8})(-\frac{4}{15})$
20. $(+6\frac{3}{4})(-\frac{4}{9})$
21. $(-4.8)(2.8)$
22. $(6\frac{1}{3})(-\frac{9}{38})$
23. $(-\frac{7}{12})(-\frac{15}{28})$

24. $(-2)(3)(-4)$

Solution: $(-2)(3) = -6$; $(-6)(-4) = 24$.

25. $-6(-2)(-5)$
26. $(-4)(+8)(-3)$
27. $(+10)(-6)(3)$
28. $(8)(-4)(-2)$

44. Multiplying literal numbers with exponents. What is the meaning of x^3? What is the meaning of x^2? The product $x^3 \cdot x^2$ means $(x \cdot x \cdot x) \cdot (x \cdot x) = x^5$. What is the relation between the exponents 3, 2, and 5? In the same way obtain the product of a^3 and a^5. A rule for multiplying such numbers will be given in a later chapter. For the simple exercises in this chapter no rule is necessary.

In order to multiply $2a^2b^3$ by $-3a^3b^4$, you make use of the law of order in multiplication. The product $(2a^2b^3)(-3a^3b^4)$ is the same as $2(-3) \cdot a^2 \cdot a^3 \cdot b^3 \cdot b^4$, which is equal to $-6a^5b^7$. Find in the same way the product $(-3ab^2)(-4a^3bc^2)$.

EXERCISES

Find the products of the following:

1. $b^2 \cdot b^4$
2. $a^3 \cdot a^2$
3. $x^2 \cdot x^5$
4. $a \cdot a$
5. $b^3 \cdot b$
6. $y \cdot y^4$
7. $x^2 \cdot x^2$
8. $a^3 \cdot a^3$
9. $x^2 \cdot x^3 \cdot x^4$
10. $y \cdot y^2 \cdot y^3$
11. $2a^2 \cdot 3a^3$
12. $3x^3 \cdot 4x^3$
13. $5b \cdot 3b^2$
14. $-8n^3 \cdot \ -3n^3$
15. $a^2 \cdot a^2 \cdot a$
16. $(a^2)^3$

Suggestion: What is the meaning of $(a^2)^3$?

17. $(b^2)^4$

In Exercises 18 to 30 find the values when $a = -2, b = -3$.

18. a^2
19. b^2
20. b^3
21. a^3
22. a^4
23. b^4
24. ab^2
25. a^2b^2
26. $2a^3$
27. $3b^2$
28. $2a^3b^2$
29. $3a^2b^3$
30. $-2a^2b$

In Exercises 31 to 40 multiply as indicated:

31. $(-3x^2)(-2y^3)(+xy)$

Solution: The sign is $+$.
The arithmetical product is $3 \times 2 \times 1$.
The literal product is x^2y^3xy, or x^3y^4.
Hence $(-3x^2)(-2y^3)(xy) = 6x^3y^4$.

32. $a(-b)(-c)$
33. $-\ a(2b)(-3c)$
34. $mn(-2m^2)(3mn)$
35. $x(-\ y^2)(-2x^2y)$
36. $(4a)(-3n)(-bc)$
37. $(-5x)(-3x)(-7b)(-3b)$
38. $(-2)(-\frac{3}{2}a)(a^2b)(-\frac{5}{3})$
39. $(-2x)(-2y)(3xy)$
40. $\frac{3}{2}(a^2b)\left(-\frac{ab}{3}\right)(-4)$

45. Multiplying numbers by zero. Since 4×0 means $0 + 0 + 0 + 0$, it follows that $4 \times 0 = 0$. Since the value of a product remains the same when the factors are interchanged, it follows that 4×0 has the same value as 0×4. Hence $0 \times 4 = 0$. This means that *the value of a product is zero if one of the factors is zero.*

EXERCISES

1. Find the value of $\frac{1}{2} \times 0$; $\frac{2}{3} \times 0$; $0 \times \frac{3}{4}$.

2. Find the value of $3 \times 0 \times 8$; $\frac{1}{2} \times 6 \times 0$.

3. Show that $\frac{0}{3} = \frac{1}{3} \times 0 = 0$. Similarly find the value of $\frac{0}{2}$; $\frac{0}{5}$; $\frac{0 \times 6}{6}$.

MULTIPLYING A POLYNOMIAL BY A MONOMIAL

46. How to use a rectangle to multiply a polynomial by a monomial. Show that the area of the rectangle (Fig. 36) is $a(b + c)$. Show that the sum of the two parts is $ab + ac$.

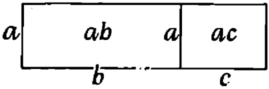

FIGURE 36

It follows that $a(b + c) = ab + ac$.

EXERCISES

By means of rectangles multiply as indicated:

1. $a(m + n)$

2. $m(c + d)$

3. $a(b + c + d)$

4. $x(y + z + t)$

47. A rule for multiplying a polynomial by a monomial. Exercises 1 to 4 (§ 46) illustrate the following important law for multiplying a polynomial by a monomial: *A polynomial is multiplied by a monomial by first multiplying each term of the polynomial by the monomial and then adding the resulting products.* This law is to be used in the following exercises.

EXERCISES

In Exercises 1 to 13 multiply as indicated and check as shown in Exercise 1.

1. $7(5a + 3)$

Solution: $7(5a + 3) = 35a + 21$.

Check: Substitute for a some value, first in the exercise, then in the result. Both should reduce to the same number. For example, let $a = 2$.

Substituting in $7(5a + 3)$, you have $7(5 \times 2 + 3)$, which means that 5 is multiplied by 2, the product then added to 3, and the sum multiplied by 7. Thus $7(5 \times 2 + 3) = 7(10 + 3) = 7 \times 13 = 91$.

Substituting 2 for a in the result, you have $35a + 21 = 35 \times 2 + 21 = 70 + 21 = 91$.

Note that the check of the solution of Exercise 1 makes use of the principle that, when various operations occur in an expression, multiplications and divisions are performed first and additions and subtractions later (§ 5).

2. $5(x + 2)$
3. $7(a + 10)$
4. $b(3a + 4)$
5. $2m(a + b)$
6. $t(6p + 9)$
7. $\frac{1}{2}b(r + 6)$
8. $5(2a + 3x + 2)$
9. $3x(4y + 2) + 5(x + 6)$
10. $\frac{1}{3}m(6a + 9) + 2(5x + 1)$
11. $p(3m + 4) + q(2n + 10)$
12. $2x(4a + 2b) + a(3x + 5b)$
13. $3m(2x + 8y) + 4x(2m + y)$

Multiplying a Polynomial by a Polynomial

48. Finding the product of two polynomials by means of a rectangle. Divide the rectangle (Fig. 37) into smaller rectangles, as shown in the diagram.

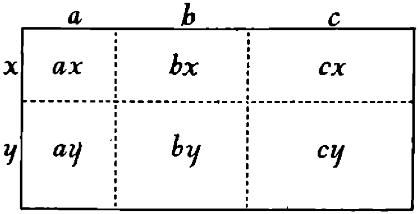

Figure 37

Then the area of the whole rectangle, which is $(x + y)(a + b + c)$, is equal to the sum of the parts

$$ax + ay + bx + by + cx + cy.$$

Hence $(x+y)(a+b+c) = ax+ay+bx+by+cx+cy.$

EXERCISES

1. Represent the product $(m + n)\ (x + y)$ geometrically as the area of a rectangle and express it in the form of a polynomial.

2. Express $(a + b)(m + n)$ as a polynomial without making the drawing.

49. A rule for finding the product of two polynomials. A study of the results in Exercises 1 and 2 above reveals the following law for multiplying polynomials without using a geometric figure:

Two polynomials are multiplied by multiplying each term of one by every term of the other and then adding the resulting products.

Apply this law to the following exercises.

EXERCISES

Multiply:

1. $(a + b)(c + d)$

Solution: $(a + b)(c + d) = ac + bc + ad + bd.$

2. $(x + 2)(y + 5)$

3. $(x + 2)(r + t)$

4. $(r + 8)(s + \frac{1}{2})$

5. $(x + y)(m + n + 4)$

6. $(f + g + 6)(a + 4)$

7. $(m + n + p)(x + y + z)$

8. Find the values of the products in Exercises 1 to 7, substituting the value 3 for a, b, c, and d; 4 for f, g, m, and n; $\frac{1}{2}$ for p, r, s, and t; $\frac{1}{3}$ for x, y, and z.

9. Find the value of the product in Exercise 2 when $x = .18$, $y = 3.12$.

10. Find the value of the product in Exercise 3 when $x = 3.52$, $r = 1.7$, $t = 1.34$.

Multiply as indicated:

11. $(3x + 2y + z)(a + b)$

12. $(m + 3n + 4)(2x + 3)$

13. $(2a + 8b + 3c)(14x + 7y)$

14. $(2x + y)(4a + 3b + c)$

15. $(x + 3y + \frac{1}{2}z)(a + 2b)$

16. $(a + b)(c + d + e)$

17. $(3m + p + 5r)(x + y)$

18. $(\frac{1}{2}f + \frac{1}{3}g)(6a + 12b + 18c)$

19. $(.5a + .25b)(10p + 60q + 8r)$

20. $(\frac{1}{3}x + \frac{1}{4}y + \frac{1}{2}z)(\frac{1}{2}m + n)$

50. Supplementary exercises. The following exercises will give you further practice in multiplying algebraic expressions.

EXERCISES

Multiply as indicated and change all results to the simplest form:

1. $3 \cdot 5x$
2. $6a^2 \cdot 3b$
3. $2a \cdot 7ab^2$
4. $3c(-4ac^2)$
5. $5a \cdot 2a^2bc$
6. $(6m^2)(-3mn)$
7. $(m^2)^3$
8. $3(-p^2)^2$
9. $(ab^2)^3$
10. $(-2mn^2)^3$
11. $2x \cdot 16xy^2$
12. $(-3a)\left(\frac{2b}{3a}\right)$
13. $(-\frac{1}{2}xy)(6x^2y^2)$
14. $4a(-5ab^2)$
15. $10a(3x)(-2ax)$
16. $ab^2(-2b^2)(3ab)$
17. $ab(-3b^2)(2ab)$
18. $a(b^2)(-2a^2b)$
19. $\left(\frac{7x}{12}\right)\left(-\frac{20}{21x^2}\right)$
20. $3x\left(-\frac{2}{5x}\right)\left(\frac{10}{3x^2}\right)$
21. $a(3x + 2)$
22. $- b(b + 4c)$
23. $\frac{1}{2}x(x + 4)$
24. $3p(4a - b)$
25. $- ab(3a^2 - ab)$
26. $\frac{c}{6}(12a + 6b)$
27. $3m\left(\frac{m}{6} + \frac{n}{9}\right)$
28. $2x(3y - 4) + 5(x + 3)$
29. $- y(3y^2 - 4y + 6)$
30. $(a + b)m + (x + y)n$
31. $x(x^2 + bx) - a(a^2 - x)$
32. $a(a^2 - 4a + 3) - 5a^2$
33. $9x(2x^2 - x - 1) + 16x$
34. $\frac{m}{3}(6x - 2) - \frac{m}{2}(4x - 1)$
35. $a^2(a + 6)$
36. $x(x^2 + y^2)$
37. $x^2(x + y)$
38. $a(2a^2 + 3a + 1)$
39. $3x(x^2 + 2x + 4)$
40. $3y(x + 5y + 2y^2)$
41. $2m(m + mn + n)$
42. $ab(1 + 2a + b)$
43. $y(3x^2 + 3xy + y^2)$
44. $2a(3a + 4b) + a^2(a + 3b)$
45. $x(x^2 + ax) + a(a^2 + x)$
46. $4x(x + 2y) + y(3x^2 + y)$
47. $(\frac{1}{2}a + \frac{1}{3}b)(6a + 2b)$
48. $(\frac{2}{3}x + 3z)(6x + 12z)$
49. $(2 + x + x^2)(x + 4)$
50. $(3x^2 + 2x + 1)(x + 4)$
51. $(4a^2 + 10a + 1)(2a + 6)$
52. $(a + b + c)(m + n + t)$
53. $(a^2 + ab + b^2)(a + b)$
54. $(a + b + c)^2$
55. $(2x + y + 5)^2$

Find the value of each of the following:

56. $3x^3 - 2x^2 - x + 3$ when $x = -2$

Solution:
$$\begin{aligned} 3x^3-2x^2- x + 3 &= 3(-2)^3- 2(-2)^2- (-2) + 3 \\ &= 3(-8)- 2(4) + 2 + 3 \\ &= -24 - 8 + 5 \\ &= -27. \end{aligned}$$

57. $x^3 + 5x^2 + 2x - 8$ when $x = -3$

58. $2x^3 - 6x^2 + 3x - 4$ when $x = -\frac{1}{3}$

59. $-3x^3 + 2x^2 - 8x - 7$ when $x = - 1$

60. $-x^3 - 8x^2 + 2x + 3$ when $x = - \frac{1}{2}$

DIVISION

51. Dividing literal numbers with exponents. In order to divide a^6 by a^4, you may write the exercise as follows: $\frac{a \cdot a \cdot a \cdot a \cdot a \cdot a}{a \cdot a \cdot a \cdot a}$. Then you may divide the numerator and denominator by a four times. The result is a^2. What is the relation between the exponents 6, 4, and 2? Divide x^7 by x^3 in the same way. The rule for dividing literal numbers with exponents will be given in a later chapter. For the simple exercises in this chapter you will not need to learn the rule.

EXERCISES

Perform the indicated divisions:

1. $x^7 \div x^5$

2. $a^3 \div a^2$

3. $b^4 \div b$

4. $\frac{y^4}{y^2}$

5. $\frac{m^6}{m^2}$

6. $\frac{a^2}{a^5}$

Solution:
$$\frac{\overset{1}{\cancel{a^2}}}{\underset{a^3}{\cancel{a^5}}} = \frac{1}{a^3}.$$

7. $\frac{y^2}{y^5}$

8. $\frac{n}{n^3}$

9. $\frac{a^2}{a^7}$

10. $\frac{b}{b^4}$

11. $\frac{2a^2b^3}{6a^4b}$

Solution:
$$\frac{\overset{b^2}{\cancel{2}\cancel{a^2}\cancel{b^3}}}{\underset{3a^2}{\cancel{6}\cancel{a^4}\cancel{b}}} = \frac{b^2}{3a^2}.$$

12. $\frac{3x^4y^6}{12x^2y^2}$

13. $\frac{6mn^3}{2mn}$

14. $\frac{3xy^2}{x^3}$

52. Laws of signs in division. The process of division is the opposite of multiplication; that is, to divide 8 by 2 is to determine the number which multiplied by 2 gives 8. Thus $\frac{6}{2} = 3$, because $3 \times 2 = 6$.

Applying this meaning of division to signed numbers, you have the following:

$$\frac{+6}{+2} = +3, \text{ because } (+3)(+2) = +6.$$

$$\frac{-6}{-2} = +3, \text{ because } (+3)(-2) = -6.$$

$$\frac{+6}{-2} = -3, \text{ because } (-3)(-2) = +6.$$

$$\frac{-6}{+2} = -3, \text{ because } (-3)(+2) = -6.$$

The four examples illustrate the following laws of signs in division:

1. *The quotient of two numbers having like signs is positive.*

2. *The quotient of two numbers having unlike signs is negative.*

EXERCISES

In performing the indicated divisions, determine (1) the sign, (2) the arithmetical quotient, and (3) the literal quotient:

1. $\dfrac{144m^4n^6}{-\ 4m^4n^2}$

Solution: $\dfrac{\overset{36}{\cancel{144}}\cancel{m^4}\overset{n^4}{\cancel{n^6}}}{-\cancel{4}\cancel{m^4}\cancel{n^2}} = -\ 36n^4.$

2. $\dfrac{21}{-3}$ 3. $\dfrac{+\ 10}{-5}$ 4. $\dfrac{625}{-25}$ 5. $\dfrac{-\ 34.3}{-.7}$

6. $(-90) \div (+45)$

7. $8(45) \div (-15)$

8. $196 \div (-14)$

9. $-275 \div (-25)$

10. $(-24) \div (8)$

11. $(90) \div (-15)$

12. $(-144) \div (-72)$

13. $(-a) \div (-a)$

14. $(1.21) \div (-11)$

15. $(-2.25) \div (4.5)$

16. $\dfrac{20y^2}{-5y}$

17. $\dfrac{-ar^2}{+a^2r}$

18. $-\dfrac{-a^2y}{+ay^3}$

19. $\dfrac{3^2 \cdot 5^2 \cdot 7^2}{-5^2 \cdot 7^3}$

20. $-\dfrac{-13p^3q}{-26p}$

21. $\dfrac{-8m^4n^2}{-2mn^2}$

22. $\dfrac{96a^4b^4}{-6ab}$

23. $\dfrac{-9x^6}{-27x^4}$

24. $\dfrac{(+2)^3y}{(+2)^2y^2}$

25. $(-\frac{4}{3}a^2) \div (-\frac{20}{21}a)$

Solution: $\left(-\dfrac{4}{3}a^2\right) \div \left(-\dfrac{20}{21}a\right) = \left(\dfrac{4a^2}{3}\right) \times \left(\dfrac{21}{20a}\right)$

$$= \frac{\overset{a}{\cancel{4}\cancel{a^2}} \times \overset{7}{\cancel{21}}}{\cancel{3} \times \underset{5}{\cancel{20}\cancel{a}}} = \frac{7a}{5}.$$

26. $(-\frac{5}{6}) \div (-\frac{2}{3})$

27. $(-\frac{3}{4}x^2y) \div (-\frac{1}{2}x^3y^2)$

28. $(-7\frac{2}{3}yz^2) \div (11\frac{1}{2}z)$

29. $(-3\frac{2}{5}ab) \div (\frac{34}{15}a^2b)$

53. Supplementary exercises. The following exercises will give you further practice in multiplication and division.

EXERCISES

Change each of the following to the simplest form:

1. $\dfrac{2}{7} \times \dfrac{7}{4}$

2. $\dfrac{a}{c} \times \dfrac{b}{a}$

3. $\dfrac{6a}{5b} \div \dfrac{9b}{5a}$

4. $\dfrac{16a^2b^2}{-27x^3y^2} \div \dfrac{8ab^2}{9x^2y}$

5. $\dfrac{20xy^3}{-21a^4c} \div \dfrac{4x^2y}{3ac^2}$

6. $\dfrac{81abc}{-9ab} \div -18a^2b$

7. $(x - 3y)^2$

8. $(x + 7)(x - 3)$

9. $(2a + 6)(a - 4)$

10. $(6x - 1)(3x + 2)$

11. $(2a - b + 3c)^2$

12. $\left(\frac{3m}{2} - 4\right)(6m + 2)$

13. $(a + 2x)(x - a)$

14. $(2r + 1)(r^3 - 7r)$

15. $(2a - 5)(a^2 + 3a - 2)$

16. $(\frac{1}{2}a + \frac{1}{3}b)(12a - 6b + 18c)$

17. $(3x^2 - 2x + 4)(5x - 3)$

18. $(m^2 - mn + n^2)(m - n)$

19. $(x^3 + x - 4 - 3x^2)(x^2 + x)$

20. $(a + b)(a^2 - ab + b^2)$

21. $(x - y)(x^2 - xy + y^2)$

22. $(a^3 - 3)(a^3 + a^2 - a + 1)$

23. $(m^2 - 2)(m^3 + 2m^2 - 6m + 4)$

24. $(2a^2 - b + c)(a - 2b - c)$

25. $(x^2 + xy + y^2)(x^2 + xy - y^2)$

26. $(\frac{2}{3}a^2 + \frac{1}{2}a + \frac{1}{4})(12a^2 + 6a + 24)$

54. Dividing a polynomial by a monomial. To divide a polynomial by a monomial, divide every term of the polynomial by the monomial. Thus

$$\frac{12a^2b + 6ab^2 + 3a^2b^3}{3ab} = 4a + 2b + ab^2.$$

EXERCISES

Perform the indicated divisions:

1. $\frac{8a^2 - 20ab}{4a}$

2. $\frac{5x^2y - 15xy^2}{-5xy}$

3. $\frac{9xy^2 + 6xy^2 + 3xy}{3xy}$

4. $\frac{12ma + 6mb}{-3m}$

5. $\frac{35a^2b^2 - 49ab^3}{7ab^2}$

6. $\frac{-a^4b^2 - a^2b^2 + 5a^2b^3}{-a^2}$

7. $\frac{-12m^3n + 4m^2n^2}{-4mn}$

55. Dividing one polynomial by another. The process of dividing polynomials is the same as that of dividing arithmetical numbers. In fact, division of arithmetical numbers may be considered a special case of division of algebraic polynomials.

When $x = 10$, the polynomial $6x^3 + 7x^2 + 5x + 2 = 6 \cdot 10^3 + 7 \cdot 10^2 + 5 \cdot 10 + 2 = 6000 + 700 + 50 + 2 = 6752$.

The division of 6752 by 32 is arranged in several ways in the following four examples:

(1)

$$\begin{array}{r|l}
6752 & 32 \\ \cline{2-2}
\underline{64} & 211 \\
35 & \\
\underline{32} & \\
32 & \\
\underline{32} &
\end{array}$$

(2)

$$\begin{array}{r|l}
6000 + 700 + 50 + 2 & 30 + 2 \\ \cline{2-2}
\underline{6000 + 400\phantom{{}+ 50 + 2}} & 200 + 10 + 1 \\
300 + 50\phantom{{}+ 2} & \\
\underline{300 + 20\phantom{{}+ 2}} & \\
30 + 2 & \\
\underline{30 + 2} &
\end{array}$$

(3)

$$\begin{array}{r|l}
6 \cdot 10^3 + 7 \cdot 10^2 + 5 \cdot 10 + 2 & 3 \cdot 10 + 2 \\ \cline{2-2}
\underline{6 \cdot 10^3 + 4 \cdot 10^2\phantom{{}+ 5 \cdot 10 + 2}} & 2 \cdot 10^2 + 1 \cdot 10 + 1 \\
3 \cdot 10^2 + 5 \cdot 10\phantom{{}+ 2} & \\
\underline{3 \cdot 10^2 + 2 \cdot 10\phantom{{}+ 2}} & \\
3 \cdot 10 + 2 & \\
\underline{3 \cdot 10 + 2} &
\end{array}$$

(4)

$$\begin{array}{r|l}
6x^3 + 7x^2 + 5x + 2 & 3x + 2 \\ \cline{2-2}
\underline{6x^3 + 4x^2\phantom{{}+ 5x + 2}} & 2x^2 + x + 1 \\
3x^2 + 5x\phantom{{}+ 2} & \\
\underline{3x^2 + 2x\phantom{{}+ 2}} & \\
3x + 2 & \\
\underline{3x + 2} &
\end{array}$$

The last example (4) shows that in dividing one polynomial by another the following steps are taken:

1. Divide the first term of the dividend by the first term of the divisor. This gives the first term of the quotient................ $6x^3+7x^2+5x+2 \;\big|\; \underline{3x+2}$ $\quad \boxed{2x^2+x+1}$

2. Multiply the divisor by the first term of the quotient............... $\underline{6x^3+4x^2}$

3. Subtract and bring down the next term.... $3x^2+5x$

4. Divide the first term of the remainder by the first term of the divisor. This gives the second term of the quotient.

5. Multiply the divisor by the second term of the quotient............... $\underline{3x^2+2x}$

6. Subtract and bring down the next term..... $3x+2$

7. Repeat steps 4, 5, and 6................. $\underline{3x+2}$

EXERCISES

Divide as indicated:

1. $(x^3 + 2x^2 - x - 2) \div (x + 1)$
2. $(x^3 + 6x^2 + 5x - 12) \div (x + 3)$
3. $(x^3 - 18x - 35) \div (x - 5)$
4. $(x^6 + 4x^4 + x^2 - 6) \div (x^2 + 2)$
5. $(x^6 - 6x^4 - 19x^2 + 84) \div (x^2 + 4)$
6. $(6x^4 + 19x^3 + 16x^2 + 3x - 12) \div (3x^2 + 5x - 4)$
7. $(3x^5 + 3x^4 - 11x^3 + x^2 + 4x) \div (3x^2 + 9x + 4)$

8. $(2x^5 + 8x^4 - x^3 - 26x^2 - 13x + 6) \div (x^2 + 4x + 3)$
9. $(x^4 - 1) \div (x - 1)$
10. $(x^5 - 1) \div (x - 1)$
11. $(a^5 - b^5) \div (a - b)$
12. $(a^3 - 3a^2b + 3ab^2 - b^3) \div (a - b)$
13. $(6a^3 - a^2b - 2ab^2 - 15b^3) \div (2a - 3b)$
14. $(12m^3a^3 - 17m^2a^2 + 10ma - 3) \div (4ma - 3)$
15. $(a^2 - b^2 - 2bc - c^2) \div (a - b - c)$
16. $(12m^4 - 25m^3n + 12m^2n^2) \div (3m^2 - 4mn)$

REVIEW EXERCISES

56. Supplementary exercises. The following exercises give additional practice in the four processes used in the preceding work.

EXERCISES

1. If $y = ax^2$, find the value of y when $x = -2$, $a = 5$; $x = \frac{2}{3}$, $a = 3\frac{5}{6}$.

2. If $y = 3x^2 - x + 7$, find y when $x = 2$; -5; $1\frac{1}{2}$; $\frac{2}{3}$.

In the following add the lower polynomials to the upper:

3.
$$\begin{array}{r} 4x^2 - 6x + 1 \\ \underline{-5x^2 - 3x - 7} \end{array}$$

4.
$$\begin{array}{r} 6a^2b - 5ab^2 + b^3 \\ \underline{-3a^2b + 7ab^2 - 3b^3} \end{array}$$

In the following subtract the lower polynomials from the upper:

5.
$$\begin{array}{r} 7x^2 + 3x - 2 \\ \underline{-5x^2 - x + 6} \end{array}$$

6.
$$\begin{array}{r} 2x^2y^2 + 6xy - 8 \\ \underline{-3x^2y^2 - 2xy + 10} \end{array}$$

Add and subtract as indicated:

7. $3a + b + (4a - 2b)$
8. $6m^2 + 3n + (m^2 + n)$
9. $4r^2 + 6r - (3r^2 + 4r)$
10. $2x^2 - 5x - (7x^2 - 2x + 1)$
11. $4y^2 - 3y + 7 - (y - 6)$
12. $t^2 - 8t + 4 + (3t + 16)$

Multiply or divide as indicated:

13. $(-2\frac{1}{2})(3\frac{1}{3})$

14. $(-4x)(2x)$

15. $(+3xy)(-2x)$

16. $(ab)(-3ab)$

17. $(2x^2y)(-4xy^2)$

18. $(-18x) \div (-6)$

19. $(72xy) \div (-12y)$

20. $(-6x^3) \div (-2x)$

21. $(42a^2bc^2) \div (-3abc)$

22. $(-3\frac{1}{2}ax) \div (-2\frac{1}{3}x)$

Multiply:

23. $(2y^2 + y + 7)(y + 3)$

24. $(6a^2 - 3a + 4)(a - 2)$

25. $(5x^3 + 3x^2 - 2x + 4)(x^2 - 3x - 2)$

26. $(x^2 + 2xy + y^2)(x - y)$

27. $(2 + x)^2(1 + x - x^2)$

Divide:

28. $(x^2 - x - 30) \div (x + 5)$

29. $(a^2 - 4a - 21) \div (a - 7)$

30. $(6a^2 + 5a - 6) \div (3a - 2)$

31. $(4x^2 - 16x + 15) \div (2x - 5)$

Exercises to Test Your Understanding of Chapter II

57. What you should know and be able to do. Chapter II has shown the meaning and uses of positive and negative numbers. You should now understand these uses with the thermometer scale, angles, lines, forces, business terms, and the number scale.

You should be able to do the following:

1. To work exercises with signed numbers with accuracy and a fair degree of speed.
2. To add numbers graphically and by rule.
3. To subtract numbers by changing the sign of the subtrahend and adding the result to the minuend.

4. To use the laws of signs in multiplying and dividing.
5. To add and subtract polynomials.
6. To multiply polynomials.
7. To divide polynomials.

58. Supplementary exercises. Work the following exercises.

EXERCISES

1. Name uses for positive and negative numbers other than those mentioned in Exercise 1, §30.

2. Represent the following table graphically:

Time	4 P.M.	5	6	7	8	9	10	11	Mid-night	1 A.M.	2	3
Temperature	+8°	+8°	+6°	+4°	+3°	0°	−1°	−2°	−2°	−3°	−4°	−4°

For each of Exercises 3 to 6 write an equation stating the facts of the problem and the result:

3. An automobile travels 22 miles east and then 45 miles west. How far is it from the starting point?

4. A ship starts in latitude 5° north and travels 12° south. What is its latitude?

5. The rate of a current is 2.5 miles an hour. If a boy can row in still water at a rate of 3.5 miles an hour, how fast can he go upstream?

6. A man's bank account is $400, but he owes $250. What is his financial standing?

Perform the following operations and explain each:

7. (*a*) $+6+(-8)$
(*b*) $-5+(-2)$

8. (*a*) $-3-(-7)$
(*b*) $+5-(+3)$

9. (*a*) $(+3)(+2)$
(*b*) $(+5)(-3)$

10. (*a*) $(-3)\div(-4)$
(*b*) $(-6)\div(+5)$

Perform the operations indicated below:

11. $15a - (-3a) + (-5a) - (2a)$

12. $12x^2y + (-3z) - (6z) - (8x^2y)$

13. $(16a^2 - 3a + 5) + (10a^2 - 2a - 3)$

14. $(2xz + 3yz - 8xy) - (6xz - 4yz + 6xy)$

15. $(-8)(-5)(-2)$

16. $(-2)(-3)(+4)$

17. $\left(-\frac{2}{15x}\right) \div \left(+\frac{5}{8xy}\right)$

18. $(4a)\left(-\frac{3}{2m}\right)\left(-\frac{5}{an}\right)$

19. $(-625) \div (-25)$

20. $(2x^2 - 17x)4x$

21. $(x^2 - 3xy + y^2)(2xy)$

22. $(2a - b)(a + 2b)$

23. $(8a - 3y)(a - y)$

24. $(2a^2 + 7a - 9)(5a - 1)$

25. $\frac{-8\ m^4np^2}{-26m^2n^3p}$ **26.** $\left(\frac{4x^4}{5}\right) \div (2x^2)$ **27.** $\frac{7a}{15} \div \frac{14a^2}{20}$

28. $(3x^3 + x^2 - 8x + 4) \div (3x - 2)$

29. $(2x^3 - 11x^2 + 17x - 5) \div (2x - 5)$

59. Timed test in the four operations. Study the following exercises carefully, and when you feel that you understand them, work them as rapidly as you can. Time: 5 minutes.

1. Combine similar terms: $12a^2 - 7 + 3a^2 - 2a + 4 + a$.

2. Add $2x^2 + xy + 3y + 1$ and $4x^2 - 2xy + 4x - 3$.

3. From $5a^2 - 3ab + b^2 - 8$ subtract $a^2 + 5ab - 2b^2 + 4$.

4. Multiply as indicated:

(*a*) $(-4)(-3x)$

(*b*) $-\frac{2}{3}a(6a - 15)$

(*c*) $(4x + 2)(3x - 1)$

(*d*) $10 + 3a(9a - 5)$

5. Divide:

(*a*) $\frac{4a^2}{-5a}$

(*b*) $\frac{3x - 6y}{-3}$

(*c*) $\frac{5x}{7} \div \frac{15x^2}{14y}$

(*d*) $(x^2 - 8x + 12) \div (x - 6)$

CHAPTER III[1]

SOLVING SIMPLE EQUATIONS AND PROBLEMS

THE LAWS USED IN SOLVING EQUATIONS

60. A knowledge of equations is important in solving problems. The following problems illustrate the use of equations in solving problems:

1. Our boat has run 30 miles in $3\frac{1}{4}$ hours. At the same rate how far should we travel in 8 hours?

Solution: Since the distance is equal to the rate multiplied by the time, the equation $30 = 3\frac{1}{4}r$ enables you to determine the rate at which the boat travels. Having found the rate, you can easily find the distance traveled in any given time.

2. In taking care of a neighbor's lawn John worked twice as long each day as his brother James, but Henry worked three times as long as James. If together they were paid $48 for their work, how should they divide the money?

Solution: If the amounts are denoted by a, $2a$, and $3a$, the total is $a + 2a + 3a$. It follows, then, that $a + 2a + 3a = 48$. The solution of this equation determines the amount paid to James, from which you can obtain the amounts paid to John and Henry.

If you wish to be successful in solving problems, you must learn how to obtain the equations and how to solve them.

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[1] Chapter III is a review and extension of a similar chapter in *Eighth-Year Mathematics.*

61. Using the division axiom. To solve the equation in Problem 1 (§ 60), proceed as follows:

Solution: $3.25r = 30$.

Divide both members of the equation by 3.25. This gives

$$\frac{\cancel{3.25}r}{\cancel{3.25}} = \frac{30}{3.25}.$$

By changing the fraction to the simplest form, you have

$$r = \frac{3000}{325} = \frac{120}{13} = 9\tfrac{3}{13}.$$

In dividing both members of the equation by 3.25, you have used the principle that, *when equal numbers are divided by the same number or equal numbers, the quotients are equal.* This, you remember, is called the division axiom.

EXERCISES

Using the division axiom, determine the unknown numbers in the following exercises:

1. $1.23y = 532$.
2. $7.5m = 28.2$.
3. $3.14d = 4.71$.
4. $.25p = 938$.
5. $3.5x = 70$.
6. $1.06a = 530$.
7. $3.14d = 785$.
8. $.75a = 18$.
9. $.57x = 24.2$.
10. $.231a = 462$.

In solving the equation in Problem 2 (§ 60) proceed as follows:

11. $a + 2a + 3a = 48$.

Solution: By combining the similar terms you have

$$6a = 48.$$
$$a = 8. \quad \text{Why?}$$

12. $x + 3x = 24$.
13. $p + .45p = 43.5$.
14. $5x + 2x = 28$.
15. $8y - 2y = 12$.
16. $11x - 9x = 16$.
17. $.7b + .8b = 4.5$.

18. A farmer wishes to mark off a rectangular piece of ground containing 450 square feet. What must be the width if it is to be 24 feet long?

19. An object fell from a balloon and struck the ground with a velocity of 336 feet a second. In how many seconds did it reach the ground?

Suggestion: Use the formula $v = 32t$.

62. Using the subtraction axiom. To solve the equation $15 = a + 12$, it is necessary only to subtract 12 from 15 and from $a + 12$. This gives $3 = a$.

The complete solution may be written briefly as follows:

$$\begin{array}{rcl} 15 & = & a + 12. \\ \underline{12} & \underline{=} & \underline{\phantom{a +{}} 12.} \\ 3 & = & a. \end{array}$$

Check: Substitute 3 for a in the given equation. Then

$$15 = 3 + 12.$$

In the solution you have used the principle that, *if the same number or equal numbers are subtracted from equal numbers, the remainders are equal.* This, you know, is the subtraction axiom.

EXERCISES

Solve the following equations. Check your solutions.

1. $x + 3 = 12.$
2. $x + 6 = 9.$
3. $x + 1.2 = 3.7.$
4. $14 + x = 67.$
5. $10 = 7 + x.$
6. $4\frac{2}{3} = 1\frac{5}{6} + x.$
7. $x + .83 = 14.2.$
8. $x + 5\frac{1}{3} = 9\frac{1}{4}.$
9. $2x + 1 = 7.$
10. $3a + 6 = 9.$
11. $5 = 2 + 3b.$
12. $64 = 3 + 2x.$
13. $3x + 8 = 14.$
14. $5a + 4 = 24.$
15. $9m + 2 = 38.$
16. $6t + 7 = 19.$
17. $2t + 14 = 30.$
18. $7y + 3 = 38.$
19. $4n + 12 = 52.$
20. $8r + 9 = 25.$
21. $2x + 1.4 = 5.6.$
22. $3x + 2.7 = 3.9.$

63. Using the addition axiom. This axiom is used in solving equations like $x - 5 = 10$ or $3x - 6 = 15$.

Adding 5 to both sides of the equation

$$x - 5 = 10,$$

you have $x - 5 + 5 = 15$.

If 5 is first subtracted from a given number and if it is then added, the result must be the given number. For example,

$$8 - 5 + 5 = 8, 20 - 5 + 5 = 20, \text{ and } x - 5 + 5 = x.$$

It follows that $x = 15$.

The solution of the equation may now be summarized as follows:

$$\begin{array}{lrcl} & x - 5 & = & 10. \\ \text{Adding,} & 5 & = & 5 \\ \hline \text{you have} & x - 5 + 5 & = & 15, \\ \text{or} & x & = & 15. \end{array}$$

The axiom used here is: *If the same number or equal numbers are added to equal numbers, the sums are equal.* This is called the addition axiom, you know.

EXERCISES

Solve the following equations and check them:

1. $x - 4 = 12$.
2. $a - 1.5 = 6$.
3. $3m - 7 = 5$.
4. $6b - \frac{2}{3} = 17\frac{1}{3}$.
5. $5t - 18 = 3$.
6. $7y - 16 = 5$.
7. $2x + 3 = 14$.
8. $10x - 5 = 22$.
9. $2.5a + 6 = 27$.
10. $6b - 5.3 = 16$.
11. $5m + 8 = 12\frac{1}{2}$.
12. $4x - 11 = 15$.
13. $2a + 3.3 = 5.7$.
14. $3.4x - 4.6 = .92$.
15. $19m - 52 = 24$.
16. $3.2 = 2.1x - 3.1$.
17. $.75 = 4x - 1\frac{1}{4}$.
18. $11.62 = 2m + 5.04$.
19. $5x + 7 = 2x + 12$.
20. $10a - 5 = 7a + 13$.

64. Using the multiplication axiom. When an equation contains fractions, as $\frac{x}{3} + \frac{x}{4} = 21$, it is generally changed to one that does not contain fractions by multiplying every term by the least common multiple of the denominators of the fractions. Thus in the equation above, if you multiply each term by 12, you have

$$\frac{12x}{3} + \frac{12x}{4} = 12 \times 21,$$

or

$$4x + 3x = 252.$$

$$7x = 252,$$

$$\text{and } x = 36.$$

In multiplying every term by 12 you have used the principle that, *if equal numbers are multiplied by the same number or equal numbers, the products are equal.* This is the multiplication axiom, you remember. The four axioms of §§ 61 to 64 enable you to solve any given equation containing no terms of higher degree than the first.

EXERCISES

Solve the following equations. Check your solutions by substituting in the original equations.

1. $\frac{x}{5} = 8.$

2. $\frac{n}{3} = 15.$

3. $\frac{a}{7} = 12.$

4. $\frac{x}{11} = 11.$

5. $\frac{y}{2.5} = 1.3.$

6. $\frac{t}{1\frac{2}{3}} = 2\frac{3}{4}.$

7. $\frac{x}{.84} = .32.$

8. $\frac{2x}{3} = 7.$

9. $\frac{3n}{2} = 8.$

10. $x + \frac{x}{3} = 60.$

Solution: $x + \frac{x}{3} = 60.$

Multiplying each term by 3, $3x + x = 180.$
Combining terms, $4x = 180.$
Dividing by 4, $x = 45.$

Check: Substitute 45 for x in the original equation. Then the left member is $45 + \frac{45}{3}$, or 60. Since the right member is also 60, the equation is satisfied when $x = 45$.

11. $x - \frac{x}{50} = 21.$

12. $a - \frac{a}{2} = 27.$

13. $\frac{x}{2} + x = 96.$

14. $2x - 8 = \frac{2x}{3}.$

15. $\frac{x}{2} + \frac{x}{5} = 35.$

Solution: $\frac{x}{2} + \frac{x}{5} = 35.$

Multiplying by 10, $5x + 2x = 350.$
Combining terms, $7x = 350.$
Dividing by 7, $x = 50.$

16. $\frac{x}{5} + \frac{x}{12} = 1\frac{2}{3}.$

17. $\frac{x}{3} + \frac{x}{4} = 21.$

18. $\frac{x}{3} = 9 + \frac{x}{6}.$

19. $\frac{x}{4} + \frac{x}{8} = 3.$

20. $\frac{x}{3} - \frac{x}{10} = \frac{7}{2}.$

21. $\frac{4x}{5} - \frac{3x}{10} = 50.$

Solution: $\frac{4x}{5} - \frac{3x}{10} = 50.$

Multiplying by 10, $8x - 3x = 500.$
Combining terms, $5x = 500.$
Dividing by 5, $x = 100.$

22. $\frac{3x}{4} - \frac{x}{8} = x - 3.$

23. $\frac{26y}{3} + \frac{5y}{2} = 67.$

24. $\frac{2m}{3} + \frac{3m}{4} = 23.$

25. $\frac{3m}{4} + \frac{2m}{5} = 35 - \frac{3m}{5}.$

65. Supplementary exercises. The following exercises will give you additional practice in solving equations.

EXERCISES

Solve the following equations:

1. $3x = 7.$

2. $a + 3 = 9.$

3. $b - 5 = 7.$

4. $\frac{x}{2} = 5.$

5. $2n + 3 = 9.$

6. $3x - 4 = -1.$

7. $2 + 3m = 8.$

8. $10 = 2x - 6.$

9. $\frac{a}{5} + \frac{a}{6} = 2.$

10. $\frac{x}{4} + \frac{x}{7} = 9.$

11. $\frac{n}{1.3} + \frac{n}{4} = 2.$

12. $\frac{t}{6} + \frac{t}{14} = 3.$

13. $\frac{b}{2} + \frac{b}{8} = 6.$

14. $\frac{y}{2} + \frac{y}{8} = 35.$

The following equations are proportions. Solve each by using the multiplication axiom.

15. $\frac{x}{8} = \frac{3}{4}.$

16. $\frac{a}{9} = \frac{5}{3}.$

17. $\frac{c}{6} = \frac{1}{2}.$

18. $\frac{p}{12} = \frac{5}{6}.$

19. $\frac{t}{15} = \frac{3}{5}.$

20. $\frac{x}{20} = \frac{4}{5}.$

21. $\frac{n}{14} = \frac{5}{7}.$

22. $\frac{r}{6} = \frac{2}{3}.$

23. $\frac{a}{3.6} = \frac{1}{1.2}.$

24. $\frac{y}{.75} = \frac{6.8}{.15}.$

25. If a gallon of water weighs $8\frac{1}{3}$ pounds, what is the weight of 2 gallons of benzine whose specific gravity is .87?

Suggestion: Let B be the weight of 2 gallons of benzine.

$$\text{Show that } \frac{B}{2 \times 8\frac{1}{3}} = .87.$$

26. Find the weight of a cake of ice 2 feet by 3 feet by $1\frac{1}{2}$ feet if the specific gravity of ice is .91.

27. Find the weight of a piece of cast iron 3 inches by 5 inches by $2\frac{1}{2}$ inches if a cubic inch of water weighs .578 ounces and if the specific gravity of cast iron is 7.4.

28. The density (number of grams in one cubic centimeter) of a substance of given mass (number of grams) of given volume (number of cubic centimeters) is given by the formula $D = \frac{M}{V}$. Find the mass of an iron object whose volume is 100 cubic centimeters if the density is 7.4.

66. A summary of the steps involved in solving equations. You have learned that in solving an equation like

$$50 - 6n = n - 20$$

you must first *bring all terms containing the unknown number* n *to one side of the equation.* This is done by adding the same number to, or subtracting the same number from, both sides of the equation. In the equation above you may add $6n$ to both sides, which gives the equation

$$50 = 7n - 20.$$

The next step is to *bring all terms not containing the unknown number to the other side.* Adding 20 to both sides, you have

$$70 = 7n.$$

Finally, *divide both sides by the coefficient of* n. The result is

$$10 = n, \text{ or } n = 10.$$

EXERCISES

Solve the following equations:

1. $16 - 9n = 9n - 2.$

Solution: $16 - 9n = 9n - 2.$

$16 = 18n - 2$, by adding $9n$ to both sides.

$18 = 18n$, by adding 2 to both sides.

$1 = n$, by dividing both sides by 18.

Hence $n = 1.$

Check:

Left Member		Right Member
$16 - 9$		$9 - 2$, by substituting 1 for n in the original equation.
7	$=$	7, by combining terms.

2. $8x - 9 = -2x + 11.$

3. $-5a = -2a - 39.$

4. $6x + 40 - 11x = 0.$

5. $-12b + 18 = 6b.$

6. $4m - 7 = 53 - 6m.$

7. $17 - 8s = 2s - 47.$

8. $11x - 9 = 5x + 117.$

9. $42 - 3x = 48 - 9x.$

10. $5x - 25 = 3x - 5.$

11. $2x + 21 = 58 - 2x.$

12. $2m - 17 + m - 34 = -54.$

13. $2s + 3 = -s + 1 + 2s - 5.$

14. $-2y + 1 = -4y + 3.$

15. $2 - 64r = 144 + 7r.$

16. $4a - 15 - a = 35 - 2a.$

17. $9x + 10 = 88 + 2x - 8.$

18. The dimensions of a rectangular lot are 25 feet and 125 feet. By how much must the width of the lot be increased to form a rectangle whose area is 4000 square feet?

Suggestion: Let x be the number of feet in the increased part of the base (Fig. 38).

Then the new base is $x + 25.$

Hence $125(x + 25) = 4000.$

$125x + 3125 = 4000.$

$125x = 875.$

$x = 7.$

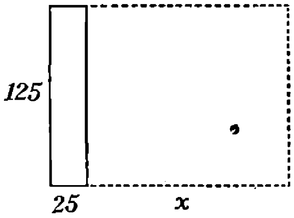

FIGURE 38

19. The base of a rectangle is 15 feet and the altitude 1 foot. By how much must the altitude be increased to form a rectangle whose area is 105 square feet?

20. The base of a rectangle is 8 and its altitude is $3x + 2$. If the area is 56, find x and the altitude.

21. The base of a triangle is 10 and its altitude is $2x + 3$. If its area is 35, what is the numerical value of the altitude?

67. Equations containing parentheses. In solving the following equations consider carefully the operations indicated. Then decide upon the correct order in which to perform these operations.

EXERCISES

Solve the following equations:

1. $x - 2(3 - 4x) = 12.$

Solution: The parenthesis in this equation indicates that $3 - 4x$ is to be multiplied by -2.

Hence $x - 6 + 8x = 12.$

$$9x - 6 = 12.$$

$$\therefore x = 2.$$

2. $6a - 3(3a - 1) - 1 = 0.$

3. $2(x - 1) = 3x - 6(2x + 3).$

4. $2y - 2(6y - 17) = 3(2y - 6).$

5. $3a - 2(a + 5) = 6a - 20.$

6. $3(a - 2) + 15 = 5a - 3.$

7. $8(3 - 2y) - 2(5 - y) = 28.$

8. A square is to be changed into a rectangle having the same area as the square by making one side 12 feet longer and the other 4 feet shorter. What will be the dimensions of the rectangle?

Solve the following equations and check as shown:

9: $\frac{2(x+2)}{3} = 8.$

Solution: $\frac{\cancel{3}\cdot 2(x+2)}{\cancel{3}} = 3\cdot 8$, by multiplying both members by 3.

$2x + 4 = 24$, by changing the left member to the simplest form.

$2x = 20$, by subtracting 4 from each member.

$x = 10$, by dividing both members by 2.

Check: Substitute 10 for x in the original equation and see if both sides of the equation reduce to the same number.

Thus $\frac{2(x+2)}{3} = \frac{2(10+2)}{3} = \frac{2 \times 12}{3} = 8.$

10. $\frac{3(x+2)}{5} = 6.$

11. $\frac{2(x+5)}{3} = 4.$

12. $\frac{4(x+3)}{3} = 5.$

13. $\frac{5(x-2)}{2} = 7.$

68. Supplementary exercises. The following exercises give further practice in solving equations.

EXERCISES

Solve:

1. $a - 9 = -3.$
2. $y - 4 = 9.$
3. $7x - 8 = 20.$
4. $3x - 10 = x - 6.$
5. $4x - 8 = 6 - 3x.$
6. $3a - 2 = -7a + 5.$
7. $5x - 7 = 2 - 3x.$
8. $2x - 3 = 18 - x.$
9. $6t - 10 = 8 - 5t.$
10. $10m - 4 = 2m + 7.$
11. $5r - 4 = 9r - 7.$
12. $10 - 5x = -6 + 3x.$
13. $.4x + .3 = .5 - .7x.$
14. $7a - 13a = 18 - 42.$
15. $12a - 10 - 15a = 7.$
16. $13m + 18 = -9m - 17.$
17. $9x - 7 = -13x + 6.$
18. $2a - 20 = 3(2a - 5).$
19. $6(a - 3) - 4(a + 2) = -a + 4.$

20. $4(2x - 5) + 15 = 3(x + 10)$.

21. $2(m - 3) + 3(m - 2) = 8$.

22. $8(11a - 4) = 7(13a - 3)$.

23. $\frac{x + 8}{5} = 12$.

24. $\frac{2 + x}{5} = \frac{2 - x}{2}$.

25. $\frac{5}{3}a = \frac{2}{3}a + 2$.

26. $\frac{11a - 4}{7} = \frac{13a - 3}{8}$.

27. $\frac{4a + 2}{3} - \frac{a - 3}{4} = 0$.

Solution: Multiply both members of the equation by 12.

$$\begin{aligned} 4(4a + 2) - 3(a - 3) &= 0. \\ 16a + 8 - 3a + 9 &= 0. \\ 13a + 17 &= 0. \\ 13a &= -17. \\ a &= -\tfrac{17}{13}. \end{aligned}$$

28. $\frac{7a - 8}{7} - \frac{a + 6}{2} = 0$.

29. $\frac{4x - 5}{2} - \frac{3x - 3}{4} - 3 = 0$.

30. $\frac{2x - 3}{2} + \frac{5x - 1}{5} = 3$.

Solve the following formulas for the required literal numbers:

31. $i = \frac{prt}{100}$. Find t.

Solution: Multiply both members of the equation by 100.

Then $100i = prt$.

Divide both members by pr. Then $\frac{100i}{pr} = t$.

32. $S = 2\pi rh$. Find h.

33. $V = abc$. Find c.

34. $V = \frac{1}{3}\pi r^2h$. Find h.

35. $S = \pi rs$. Find s.

36. $V = \frac{1}{3}\pi r^2h$. Find r.

37. $C = \frac{E}{R}$. Find R.

38. $s = \frac{w}{L}$. Find w.

39. $M = \frac{SI}{T}$. Find I.

40. $K = \frac{Mv^2}{2}$. Find M.

41. $C = \frac{E}{R + r}$. Find R.

42. $A = \frac{1}{2}h(a + b)$. Find a.

In the following exercises find the values of the unknown numbers:

43. $p = 20a$; $a = 4.23$.

44. $l = 2ab + 2ac$; $a = 5$, $b = \frac{3}{2}$, $c = 4$.

45. $c = 2\pi r$; $c = 98$.

46. $A = \pi r^2$; $r = 50$.

47. $A = \frac{1}{2}bh$; $A = 20$, $b = 3\frac{1}{2}$.

48. $A = \frac{1}{2}h(a + b)$; $A = 120$, $h = 6$, $a = 2$.

49. $V = \pi r^2 h$; $r = 140$, $h = 10$.

50. $S = 2\pi rh$; $S = 60$, $h = 8$.

51. $V = \frac{1}{3}bh$; $V = 27$, $h = 6$.

52. $F = 32 + \frac{9}{5}C$; $F = 34$.

53. $s = \frac{1}{2}gt^2$; $t = 11$, $g = 32$.

54. $A = P + PRT$; $A = 200$, $P = 180$, $T = 2$.

TRANSLATING VERBAL STATEMENTS INTO SYMBOLS

69. How to derive equations from verbal problems. Since many problems are solved by means of equations, you must know (1) how to *derive* the equations and (2) how to *solve* the equations. The second part offers no serious difficulty because there are definite directions for solving equations. For the first, no general directions can be given, but the following simple suggestions will be helpful as they apply to all verbal problems. Additional special directions, to be given later (§§ 71 to 78), will apply to certain types of problems designed to give you practice in deriving the equation from a problem.

1. *Read the problem carefully.* The purpose of this is to get general information as to the content of the problem.

2. *Read the problem again to determine what it asks for.* The number or numbers to be found are the *unknown* numbers.

3. *Denote one of the unknown numbers by a letter.* Make a definite, clear statement as to what the letter stands for. Thus, if the *rate* of motion is to be found, write, "Let r be the number of miles per hour," or "the number of yards per minute," or "the number of feet per second." Do not say briefly, "Let r be the rate," because this statement does not indicate the *unit.* Similarly, if the *price* is called for, write, "Let n be the number of cents," or "Let n be the number of dollars," not, "Let n be the price."

4. *Read the problem again, one sentence at a time, and express the various facts it contains in terms of the unknown literal number.* For example, if the first sentence reads, "A sum of $5330 is to be divided into two parts," the facts are stated as follows:

Let x be the number of dollars in the first part.

Then $5330 - x$ is the number of dollars in the second part.

5. *When the data of the problem have been translated into symbols, state the equation.* One of the following suggestions will usually help you to state the equation:

(a) Translate the verbal problem into symbols. For example, the statement, "A number diminished by 20 is equal to 50 decreased by 6 times the number," when translated into symbols, gives the equation

$$n - 20 = 50 - 6n.$$

The equation is simply an abbreviated form of the verbal statement.

(*b*) State a formula, or a principle, relating the facts given in the problem. To illustrate, let x, $2x$, and $3x$ denote the number of degrees in the angles of a triangle. The principle that the sum of the angles of any triangle is 180° gives the equation

$$x + 2x + 3x = 180.$$

(*c*) Equate two number expressions which denote the same fact. Thus, if two trains traveling at different rates are equally distant from the station from which they started and if they have traveled, respectively, $5x$ and $8(x - 6)$ miles, the equation is

$$8(x - 6) = 5x.$$

It states that both trains are the same number of miles from the station.

EXERCISES

Translate the following verbal statements into algebraic symbols:

1. The selling price of an article decreased by a profit of 85 cents is equal to the purchase price of 99 cents.

2. Five times a number diminished by the number is 10 greater than 2 times the number.

3. Four times a number exceeds 3 times the number by 2.

4. The ratio of the length of a rectangle to its width of 4 feet is 6.

5. A sum of money placed at 6% interest amounts to \$673.10 at the end of one year.

6. If a number is divided by 2 and the quotient increased by 3, the result is 16.

7. The number of boys in a class of 38 pupils is 2 greater than twice the number of girls.

8. The number of inches in the base of a triangle exceeds the number of inches in the altitude by 4.

In Exercises 9 to 12 find the equations from a general principle or a formula. Do not solve the equations.

9. The hypotenuse of a right triangle is 10 feet, and one of the sides is 3 feet longer than the other.

Suggestion: Use the theorem of Pythagoras.

10. A tree was broken over (Fig. 39) so that the top touched the ground 40 feet from the foot of the stump. The stump was 12 feet high. Find the height of the tree.

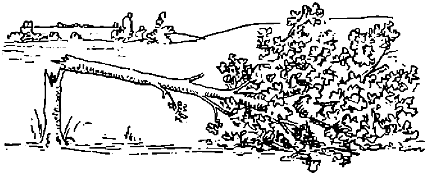

FIGURE 39

11. One angle of a triangle is 5° greater than 5 times another, and the third angle is 20°. Find the angles.

12. A bird flies a distance of 90 miles in $2\frac{1}{2}$ hours. Find the rate.

In Exercises 13 to 16 state the equation by equating two expressions denoting the same number:

13. A man rows downstream $\frac{x}{6}$ hours and returns in $\frac{x}{3}$ hours, using 9 hours for the complete trip.

14. A merchant made a profit of $350 in an investment. His gain was 9% of the amount invested.

15. The sides of a rectangular field are x and $2x$ rods. By making the rectangle 20 rods longer and 24 rods wider the area is doubled.

16. A freight train traveling x miles an hour was overtaken $1\frac{3}{5}$ hours after it started by an express train which left the station one hour later than the freight train and which traveled 40 miles an hour.

70. Supplementary exercises. The following exercises will give you additional practice in deriving equations from verbal statements.

EXERCISES

Translate the following statements into algebraic equations.

1. If a number is diminished by 4, the remainder is equal to 16.

2. If a number is increased by 2, the sum is equal to 9.

3. A number exceeds 26 by 5.

4. The width of a rectangle is 3 feet less than the length.

5. The interest on a sum of money for one year at 5% is $4.20.

Solve the following problems:

6. A merchant gained 9% on an investment, thereby making a profit of $350. How large was the investment?

7. Two trains leave a station at the same time and travel in opposite directions. If their rates are 33 miles per hour and 42 miles per hour, when will they be 175 miles apart?

Suggestion: Let t represent the number of hours traveled by each. What will express the distance covered by each? What will express the total distance?

8. How many pounds of tea selling at 58 cents a pound must be added to 22 pounds of tea selling at 75 cents a pound to make a mixture worth 70 cents a pound?

Suggestion: Let n represent the number of pounds of tea to be added. What will represent the selling price of this number of pounds? What will represent the number of pounds in the mixture? What is the total selling price of the mixture?

9. John did twice as much work as Tom. Together they were paid $3.60. How much of it did each earn?

10. Two angles of a triangle are equal, and the third angle is 40°. How large is each of the first two?

11. A fence 618 feet long incloses a rectangle whose length is 3 times its width. Find the dimensions.

12. For the second week's work John is to receive twice as much pay as for the first and twice as much for the third as for the second. How much will he be paid the first week if he receives $42 for the three weeks?

13. The width of a rectangle is one-half of the length. After adding 2 feet to each side of the rectangle the area is increased by 34 square feet. Find the original dimensions.

14. For each amount a boy saves his father puts twice that amount into his bank. If the boy has $13.50 in the bank, how much did he actually save out of his earnings?

15. The length of a rectangle is 5 times the width. The perimeter is 192 feet. Find the dimensions.

16. How long will it take the interest on $500 at 6% to amount to $40?

17. How should a sum of $459 be divided between two persons so that one may receive twice as much as the other?

18. One part of $1200 has been invested at 5% and the remainder at 4%. The yearly income is $56. How much was invested at each rate?

PROBLEMS GIVING PRACTICE IN DERIVING AND SOLVING EQUATIONS

71. Problems stating number relations. In each of the following problems express all unknown numbers in terms of one letter, derive the equation, and solve.

EXERCISES

1. A man left $18,500 to his wife and son. The mother was to receive three times as much money as the son. How should the money have been divided?

2. A man owns a lot and has saved \$6000 with which to build a house. He can borrow from a bank an amount of money equal to one-third of the cost of the house. What is the largest amount of money he can spend on the house?

3. Three men plan to buy a business costing \$8600. One has \$1700. The remainder is to be furnished by the others so that one pays twice as much as the other. How much money should each furnish?

4. Henry does twice as much work clearing the snow from the sidewalk as his smaller brother and therefore earns twice as much money. If a neighbor pays them \$4.50 for clearing the walks around his house, how much should each receive?

5. In mixing cement for a basement floor a man uses two shovels of gravel and four of crushed stone for every shovel of cement. How many cubic feet of each will he need to make a floor 10 feet by 8 feet and 3 inches deep?

6. A man contracts to pay \$320 for repairs on his house. The materials needed cost \$50. How much per day will a carpenter be able to earn if it takes him and his helper 15 days to do the work and if he is to earn 4 times as much as the helper?

7. In planning a hike for a holiday some boys decided to do less walking in the afternoon than in the morning. If the hike was $13\frac{1}{4}$ miles long and if they walked 4 miles more in the morning than in the afternoon, how far did they walk in the morning?

72. Perimeter problems. The perimeter of a polygon is the sum of the lengths of the sides, as you know. In each of the following problems express all unknown numbers in terms of one letter, derive the equation, and solve.

EXERCISES

1. To inclose a playground, 200 rods of wire fencing are available. If the length is to be $1\frac{1}{2}$ times as great as the width, find the dimensions.

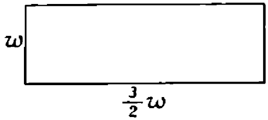

FIGURE 40

Solution: Let w be the number of rods in the width (Fig. 40).

Then $\frac{3w}{2}$ is the number of rods in the length, and $\frac{3w}{2} + w$ is the half-perimeter.

Hence $\frac{3w}{2} + w = 100$, by equating two numbers expressing the half-perimeter.

$2\left(\frac{3w}{2}\right) + 2w = 2 \times 100$, by multiplying each term by 2.

$$\therefore 5w = 200,$$
$$w = 40,$$
$$\text{and } \frac{3w}{2} = 60.$$

2. A triangular piece of ground is to be laid off with one side two and one-half times as long as the second and the third side 3 times as long as the second. What must be the length of the sides if the perimeter is to be 72 rods?

3. Find the lengths of the sides of an isosceles triangle (having two equal sides) whose base is to be 60 feet and whose perimeter is to be 240 feet.

4. The length of a rectangular field is 3 times the width. The perimeter is 264 feet. Find the dimensions.

5. The width of a rectangle is to the length as 3 is to 5. The difference between the length and width is 8 feet. Find the dimensions.

Suggestion: Let $3x$ be the number of feet in one side.
Then $5x$ is the number of feet in the other side.

6. The length of a rectangle exceeds the width by 14 feet. If the perimeter is 240 feet, determine the dimensions.

7. The perimeter of a rectangle is 184 feet, and the width is 8 feet less than the length. Find the dimensions.

8. A rectangle is 4 feet longer than twice the width. Find the dimensions if the perimeter is $13\frac{1}{6}$ feet.

9. A rectangle is 8 feet longer than twice the width. If the perimeter is 232 feet, what are the dimensions?

10. A rug is 2 feet longer than it is wide. The sum of the length and width is 14 feet. Find the dimensions.

73. Motion problems. Uniform motion depends upon distance, time, and rate. If an automobile travels 20 miles in one hour, it is said to travel, you remember, at the *rate* of 20 miles per hour. Rate is sometimes called speed, or velocity, and means a distance passed over in a unit of time.

EXERCISES

1. Using the formula $d = rt$, find the distances passed over in a given time by objects moving at uniform rates and tabulate the results as shown below:

Rate	20 mi. an hr.	18 ft. a sec.	3 yd. a min.	25 mi. an hr.	r mi. an hr.
Time	4 hr.	$2\frac{1}{2}$ sec.	$18\frac{1}{2}$ min.	24 hr.	t hr.
Distance					

2. Using the formula $d = rt$, determine the rates, having given the time and distance for each as shown:

Time	3 hr.	$1\frac{1}{2}$ hr.	4 hr.	2 sec.	t hr.
Distance	10 mi.	26 mi.	15 mi.	90 ft.	d mi.
Rate					

3. Using the formula $d = rt$, find the time for each distance and rate shown below:

Distance . .	258 mi.	20 yd.	58 mi.	16 ft.	d ft.
Rate	30 mi. an hr.	3 yd. a sec.	40 mi. an hr.	6 ft. a sec.	r ft. a sec.
Time					

4. A carrier pigeon flew 70 miles in $1\frac{1}{2}$ hours. How fast did it travel per hour?

5. The sound of a stroke of lightning was heard 8 seconds after the flash was seen. How far away was the stroke if sound travels at a rate of 1080 feet a second?

6. Two stations are 32 miles apart. Two trains leaving the stations at the same time travel toward each other at the rate of 30 and 50 miles an hour, respectively. How soon after starting will they meet?

Solution: Instead of writing out in full the given and required facts of the problem, arrange them briefly in the form of a table as shown below. The distances are derived from the given rates and the unknown time by means of the formula $d = rt$.

	FIRST TRAIN	SECOND TRAIN
Time in hours .	x	x
Rate in miles per hour .	30	50
Distance in miles	$30x$	$50x$

The equation is obtained by stating the fact that the stations are 32 miles apart (Fig. 41).

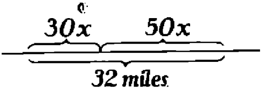

FIGURE 41

$$\text{Thus } 30x + 50x = 32.$$
$$80x = 32.$$
$$x = .4.$$

Summarize the steps taken in the solution on page 102 and use the same method in Exercises 7 to 12.

7. Two men starting from the same place travel in opposite directions, one going twice as fast as the other. In 5 hours they are 300 miles apart. Find the rate of travel of each.

Suggestion: Arrange in the form of a table the given time, the unknown rates, and the distances derived from them.

8. Two men living 96 miles apart travel toward each other at rates of 18 miles and 20 miles an hour. If A leaves home an hour earlier than B, when will they meet?

Solution: Verify the following table:

	A	B
Time in hours . .	x	$x - 1$
Rate in miles per hour	18	20
Distance in miles	$18x$	$20(x - 1)$

State the equation and solve.

9. A train traveling at the rate of 30 miles an hour is followed by a second train traveling 35 miles an hour. If the second train leaves a station 3 hours later than the first, in how many hours will it overtake the first?

10. A freight train leaves a station and travels at a rate of 32 miles an hour. An hour later it is followed by an express train traveling 60 miles an hour. When and how far from the station will the express train pass the freight train?

11. A train leaves Buffalo for New York and travels at a rate of 30 miles per hour. Three hours later another train follows, traveling 50 miles an hour. When and how far from Buffalo will the second train overtake the first?

12. A train leaves the station and travels at the rate of 35 miles an hour. Two hours later a second train follows and travels at the rate of 45 miles an hour. How long will it take the second train to overtake the first?

74. Mixture problems. By a "5% solution" of water and salt is meant a mixture 5% of which is salt. The following exercises show how to determine the amount of liquid needed to reduce a mixture to a desired solution.

EXERCISES

FIGURE 42

1. A druggist wishes to dilute a 25% mixture of water and listerine to a 15% mixture. How much water must be added to 8 ounces?

Solution: Tabulate the facts as follows:

	25% MIXTURE	15% MIXTURE
Number of ounces in mixture	8	$8 + x$
Per cent of listerine	25	15
Number of ounces of listerine	$.25 \times 8$	$.15(8 + x)$

Show that the equation is $.15(8 + x) = .25 \times 8$.

Multiply both members of the equation by 100.

Solve the resulting equation.

2. How much water must be added to 8 gallons of milk containing 5% butter fat to change it to a mixture testing 4% butter fat?

3. How much water must be added to a quart of a 20% solution of ammonia to reduce it to a 10% solution?

4. How much water must be added to 12 gallons of milk testing $5\frac{1}{4}\%$ butter fat to change it to a mixture testing 4% butter fat?

5. A druggist wishes to reduce 12 ounces of medicine containing 25% alcohol to one containing 20% alcohol. How much water must he add?

75. Interest problems. The interest formula is $i = \frac{prt}{100}$, where p is the sum invested, r the number of per cent, and t the number of years. The percentage formula is $p = \frac{rb}{100}$, where b denotes the base and p the percentage. Using these formulas, solve the following problems.

EXERCISES

1. At what rate will $8000 yield an interest of $910 in one year nine months?

Solution: The table below states the facts of Exercise 1. Note that the fourth fact is derived from the first three facts by means of the interest formula.

Principal	8000
Per cent	x
Time	$1\frac{3}{4}$
Interest .	$\frac{8000 \times x \times 1\frac{3}{4}}{100}$

$$\text{Show that } \frac{8000 \times x \times 1\frac{3}{4}}{100} = 910.$$

Solve the equation.

2. In how long a time will the interest on $600 at 6% amount to $48?

3. Find the rate at which $3645 gives $340.20 interest in one year four months.

4. An amount of money invested at 5% yields in one year an income $300 less than twice as large a sum invested at 4%. Find the amount invested at each rate.

Solution: As shown in the following table state the principal and number of per cent. Then derive the interest from them by means of the formula.

	FIRST SUM	SECOND SUM
Principal	x	$2x$
Per cent	5	4
Interest .	$.05x$	$.08x$

Show that $.05x = .08x - 300$.

Multiply each term by 100, reduce, and solve the resulting equation.

5. A man has a yearly income of $54 from an investment of $1000. If one part of the investment yields 6% and the remainder 5%, find the amount invested at each rate.

Suggestion: Let x be the sum invested at 6%. Then $1000 - x$ is the sum invested at 5%.

6. A man invested part of $10,000 at 6% interest and the remainder at 5%. The total yearly income is $570. Find the amount invested at each rate.

7. A man invested two sums at 5% and at 4%, respectively. From this investment he has a yearly income of $500. If the total sum invested amounts to $12,000, how much is invested at each rate?

8. A sum of $1400 is divided into two parts. The total annual income will be $75 if one part is invested at 5% and the other at 6%. Find the two parts.

9. A sum of $1200 is divided into two parts. The first part, invested at 5%, yields an income $39.10 greater than that of the second part, invested at 4.5%. How much money is invested at each rate?

76. Work problems. To determine the amount of work done within a given time, you must know the amount done in the *unit* of time, as a day, an hour, or a minute. Thus, if a machine does a complete piece of work in four hours, it does one-fourth of it in *one* hour. From this it is possible to find the amount done in *any* given number of hours. This principle may be used in the solution of the following exercises.

EXERCISES

1. If a printing press does a piece of work in 4 hours, how much of the whole amount will it do in 1 hour; in 2 hours; in 3 hours; in 4 hours; in x hours?

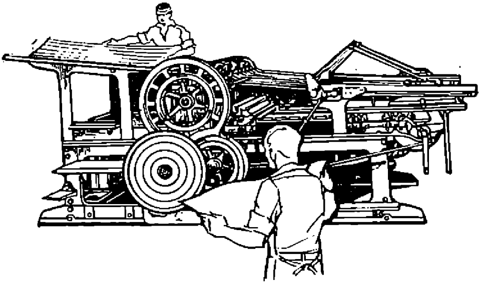

FIGURE 43

PRINTING PRESS

2. A machine can print the issue of a paper in 2 hours, and another machine can do it in 3 hours. In how many hours can an edition be printed with both machines?

Solution: The facts involved in this problem are tabulated as follows:

	FIRST MACHINE	SECOND MACHINE
Number of hours it takes to do all the work	2	3
Amount done in 1 hour	$\frac{1}{2}$	$\frac{1}{3}$
Amount done in x hours	$\frac{x}{2}$	$\frac{x}{3}$

Denoting the whole piece of work done by 1, show that

$$\frac{x}{2}+\frac{x}{3}=1.$$

State in words what this equation means. In solving the equation, first multiply every term by the least common denominator, 6.

3. If a farmer can plow a field in 6 days, how much can he plow in 2 days; in 3 days; in 6 days; in x days?

4. If a man can plow a field in 10 days and another man can do it in 8 days, how long will it take them if they work together?

5. A carpenter can build a fence in 6 days, and his apprentice can do it in 10 days. In how many days can they do it together?

6. A can do a piece of work in 8 days, and B can do it in 12 days. In how many days can they do it together?

7. One pipe can fill a tank in 11 hours. Another can fill it in 3 hours. How long will it take both to fill it?

77. Age problems. In the following exercises express all unknown numbers in terms of one letter.

EXERCISES

1. If a boy's age is 13 years, how old will he be in 3 years; in 5 years; in x years? How old was he 4 years ago; y years ago?

2. If a man's age is x years, how old was he 8 years ago? How old will he be in 5 years?

3. A is twice as old as B, but 7 years ago A was 3 times as old as B was. Find the present age of each.

4. A father is twice as old as his son. Ten years ago he was 3 times as old as the son. What is the present age of each?

78. Miscellaneous problems and equations. State the equations for the following problems and solve them.

EXERCISES

1. Draw two lines which meet, making one of the adjacent angles four times as large as the other.

2. On a line 50 feet long a rectangular garden is to be laid out to contain 1200 square feet of ground. Find the width of the garden.

3. If the corresponding sides of two rectangles form a proportion, the rectangles are similar. What must be the width of a rectangle $12\frac{1}{2}$ feet long which is to be similar to a rectangle 8 feet by 6 feet?

4. What sum of money invested at 4.5% simple interest will amount to $714.78 in two years?

5. The formula $C = \frac{E}{R}$ in electricity means that the current C is equal to the voltage E of the cell divided by the resistance R. To be more specific, it means that the amount of current in *amperes* is equal to the force of the current measured in *volts* divided by the resistance of the wire measured in *ohms*. What is the voltage if a wire with a resistance of 100 ohms carries a current of .3 ampere?

6. If n cells are connected in series, the amount of current T is given by the formula $T = \frac{nE}{R + nr}$, where E is the voltage of a cell, R the resistance of the external current, and r the resistance of a cell. Find E when $n = 8$, $R = 2$, $r = .5$ and $T = 2$.

7. If an object is thrown downward with a velocity v_0, it attains in t seconds a velocity V related to v_0 and t by the formula $V = v_0 + gt$. The number g is approximately equal to 32. Find t if $v_0 = 1.4$ and $V = 5.2$.

8. The angles $2(a + 10)$ and $\frac{3a + 68}{2}$ are supplementary. Find the number of degrees in each angle.

9. Two opposite angles formed by two intersecting lines are denoted by $\frac{b}{3} + \frac{b}{6}$ and $18 + \frac{b}{4}$. Find b and the number of degrees in the angles.

10. The acute angles of a right triangle are $x + \frac{x}{8}$ and $\frac{x}{12} + 17\frac{1}{2}$. Find x and the number of degrees in each angle.

11. One angle of a triangle is 15° larger than the second. The third angle is one-sixth as large as the second. Find the number of degrees in each angle.

Solve for x:

12. $\frac{2x + 3}{5} - \frac{x - 2}{3} = \frac{7}{5}$.

Solution: $$\frac{\overset{3}{\cancel{15}}(2x + 3)}{\cancel{5}} - \frac{\overset{5}{\cancel{15}}(x - 2)}{\cancel{3}} = \frac{\overset{3}{\cancel{15}}(7)}{\cancel{5}}.$$

$$\therefore\ 6x + 9 - 5x + 10 = 21.$$

Note especially that the sign of the fourth term is +. It is very easy to overlook the fact that both terms in the number $(x - 2)$ are multiplied by -5. The product is $-5x + 10$ and not $-5x - 10$.

13. $\dfrac{3x - 2}{7} - \dfrac{1 - 4x}{3} = 8\tfrac{4}{21}.$

14. $\tfrac{1}{2}(5y - 3) + \tfrac{1}{3}(5y - 2) = 5.$

15. $\dfrac{3x - 1}{5} - \tfrac{1}{2}(x + 6) = \dfrac{1}{2}.$

16. $\dfrac{a + 13}{13} - \dfrac{6 - 3a}{65} = 1.$

17. $\dfrac{2x}{31} - \dfrac{x + 11}{62} = \dfrac{1}{31} - x + \dfrac{3x - 6}{2}.$

18. $\dfrac{12 + y}{2y} = \dfrac{12 + 2y}{3y}.$

Suggestion: Multiply both members by $6y$.

19. $\dfrac{x}{x + 1} = \dfrac{3x}{x + 2} - 2.$

Solution: The least common multiple of the denominators is $(x + 1)(x + 2)$.

Hence $$\frac{\cancel{(x+1)}\,(x + 2)x}{\cancel{x+1}} = \frac{(x + 1)\,\cancel{(x+2)}3x}{\cancel{x+2}} - (x + 1)(x + 2)2.$$

$$\therefore x^2 + 2x = 3x^2 + 3x - 2x^2 - 6x - 4.$$

20. $\dfrac{4}{3 - x} = \dfrac{2}{1 + x}.$

21. $\dfrac{x + 1}{x - 4} = \dfrac{x}{x - 3}.$

22. $\dfrac{x + 3}{x - 2} = \dfrac{x + 5}{x - 4}.$

23. $\dfrac{1}{x + 6} + \dfrac{5}{2(x + 6)} = -1.$

24. $\dfrac{x - 1}{x - 2} + \dfrac{x}{x - 1} = 2.$

EXERCISES TO TEST YOUR UNDERSTANDING OF CHAPTER III

79. What you should know and be able to do. This chapter summarizes previous discussions relating to the solution of verbal problems. You should now be able to do the following:

1. To translate numerical facts stated in words into arithmetical and algebraic symbols.

2. To derive the equation which when solved gives the answers called for by the problems.

3. To solve equations in one unknown of the form $42 - 3x = 48 - 9x$.

4. To solve equations containing parentheses, as $3a - 2(a + 5) = 6a - 20$.

5. To solve fractional equations of a simple type which reduce to equations of the first degree.

EXERCISES

Solve the equations in Exercises 1 to 5:

1. $3.14b = 2355$.

2. $x + 22.5x = 705$.

3. $4a - 15 - a = 35 - 2a$.

4. $8(3 - 2y) - 2(5 - y) = 28$.

5. $\frac{x}{4} - \frac{x}{7} = 15$.

6. By traveling twice as fast in the afternoon as in the morning we were able to reach our destination 95 miles away on time. How far did we travel in the morning?

7. How many pounds of coffee selling at 58 cents a pound should be added to 10 pounds of 49-cent coffee to make a mixture selling at 55 cents a pound?

8. A family has an income of $3500. If the family budget allows twice as much for food as for rent and one-half as much for all other expenses as for rent, how much may be paid for rent?

80. Timed tests in deriving and solving equations. Solve as many of the following exercises as you can in the time stated. In the first test state the equations but do not solve them.

A. *Test in deriving equations.* Time: 4 minutes.

1. The length of a rectangular field is 3 times the width. The perimeter is 272 feet. Find the dimensions.

Equation: ______________

2. How should $15,648 be divided between two persons so that one receives 5 times as much as the other?

Equation: ______________

3. Two trains are leaving two stations 98 miles apart and traveling toward each other at the rates of 34 and 38 miles an hour. When will they meet?

Equation: ______________

4. How much water must be added to a gallon of 22% solution to reduce it to an 18% solution?

Equation: ______________

5. A man invested $14,000 in two parts, one part at 5% and the other at $4\frac{1}{2}$%. If the yearly income is $675, how much did he invest at each rate?

Equation: ______________

6. A carpenter can build a fence in $5\frac{1}{2}$ days, and another can do it in 5 days. In how many days can they do it together?

Equation: ______________

B. *Test in solving equations.* Time: 5 minutes.

1. $2.14x = 14.98.$

2. $13a - .4a = 13.86.$

3. $6y + 17 = 35.$

4. $2.1m - 4 = 59.$

5. $6 + 4x = 11x - 22.$

6. $\frac{x}{3} = 3.4.$

7. $\frac{t}{4} - \frac{t}{3} = 98.$

8. $3b + 20 = 6b + 2(b + 5).$

9. Solve $A = \frac{1}{2}h(a + b)$ for b.

CHAPTER IV

EXPONENTS. RADICALS

Laws of Exponents

81. The meaning of an exponent. You have learned that symbols like a^2, a^3, a^4, a^5 denote *products* in which the same factor is used several times. Thus $a^2 = a \cdot a$, $a^3 = a \cdot a \cdot a$. $a^n = a \cdot a \cdot a \ldots$ to n factors. The number of factors used in a^n depends upon the value of n. If n takes the values 1, 2, 3, etc., a^n has the corresponding meanings a, $a \cdot a$, $a \cdot a \cdot a$, etc.

a^n is called a **power,** a is the **base** of the power, and n is the exponent.

The meaning given above to the symbol a^n implies that n is a *positive integer.* It excludes such symbols as a^{-2}, $a^{\frac{1}{2}}$, a^0, $a^{\sqrt{2}}$.

You have already worked simple exercises with exponents. In this chapter you will review what you have already learned and then proceed further.

82. The law of exponents for finding the product of two powers having the same base. The product $a^3 \cdot a^2$ means $(a \cdot a \cdot a)(a \cdot a)$, or $a \cdot a \cdot a \cdot a \cdot a$, or a^5.

$\therefore a^3 \cdot a^2 = a^5$.

Similarly give the meaning and the final result for each of the following products:

1. $a^2 \cdot a^4$

Solution: $a^2 \cdot a^4 = (a \cdot a)(a \cdot a \cdot a \cdot a) = a^6$.

2. $m^2 \cdot m^3$
3. $t^2 \cdot t^7$
4. $5^3 \cdot 5$
5. $3^4 \cdot 3^2$
6. $p^4 \cdot p^6$
7. $r \cdot r^{10} \cdot r^3$

Careful examination of the results in these exercises leads to the following law: *When two or more powers having the same base are multiplied, the product is a power having the same base as the given powers and an exponent equal to the sum of the exponents of the given powers.*

In symbols, the law is expressed by means of the formula

$$a^m \cdot a^n = a^{m+n}.$$

EXERCISES

Find the products of the following as indicated:

1. $a^3 \cdot a^4$

Solution: $a^3 \cdot a^4 = a^{3+4} = a^7$.

2. $y^7 \cdot y^4$
3. $q^4 \cdot q^9$
4. $x^2 \cdot x^3 \cdot x$
5. $q \cdot q^2 \cdot q^9$
6. $s^5 \cdot s \cdot s^3$
7. $(\frac{1}{2})^2(\frac{1}{2})^6$
8. $(-2)^4(-2)$
9. $(\frac{2}{3})^3(\frac{2}{3})^2$
10. $(-a)^2(-a)^3$
11. $(-a)(-a)^3(-a)^2$
12. $x^{2a} \cdot x^{3a}$
13. $a^{3x} \cdot a^{5x}$
14. $x^{a+b} \cdot x^{a-b}$

15. $(6a^2c^2y)(-3a^3cy^2)$

Suggestion: First change the order. $(6a^2c^2y)(-3a^3cy^2) = 6 \cdot (-3) \cdot a^2 \cdot a^3 \cdot c^2 \cdot c \cdot y \cdot y^2$.

16. $(-5x^3yz^4)(8xy^4z^2)$

Multiply as indicated:

17. $8ab^2 - 3a^2bc^3$ by $5a^3b^3c$

Suggestion: Multiply each term of the binomial by the monomial.

18. $5x^2y + 3xy^2 + y^3$ by $x^2 - y$

Suggestion: Multiply every term of the trinomial by each term of the binomial.

19. $m^3 + 3m^2n + 6mn^2 + n^3$ by $m^2 + n^2$

83. A law for finding the power of a product. The power $(2 \cdot 3)^3$ means $(2 \cdot 3)\ (2 \cdot 3)\ (2 \cdot 3)$,

or $2 \cdot 3 \cdot 2 \cdot 3 \cdot 2 \cdot 3$,
or $2 \cdot 2 \cdot 2 \cdot 3 \cdot 3 \cdot 3$, by rearranging the factors,
or $2^3 \cdot 3^3$.

$\therefore (2 \cdot 3)^3 = 2^3 \cdot 3^3 = 8 \cdot 27 = 216.$

Similarly find the power of each of the following products as indicated:

1. $(3 \cdot 4)^2$
2. $(2 \cdot 5)^2$
3. $(3 \cdot 2)^4$
4. $(2 \cdot 3 \cdot 4)^2$
5. $(ab)^2$
6. $(mnt)^4$

These exercises lead to the following law: *The power of a given product is equal to the product of its factors, each factor having been raised to an exponent equal to that of the given product.*

In symbols this law is expressed briefly by the formula

$$(abc)^m = a^m b^m c^m.$$

EXERCISES

In the following raise the products to powers as indicated:

1. $(-3xy)^2$

Solution: $(-3xy)^2 = (-3)^2x^2y^2 = 9x^2y^2.$

2. $(2xy)^2$
3. $(3abc)^3$
4. $(-2x)^2$
5. $(-xyz)^3$
6. $(\frac{1}{3}mn)^3$
7. $(-\frac{1}{2}nx)^2$
8. $(2ab \cdot 3cd)^2$
9. $(3abc)^4$

10. $(ab)^2 + (ab)^3$

11. $(xy)^2 - (xy)^4$

12. $(2x)^2 + (-3y)^2 + (-z)^3$

13. $(3xy)^3 - (2xy)^2 + (-xy)^4$

84. A law for finding the power of a power. The expression $(3^2)^3$ means $3^2 \cdot 3^2 \cdot 3^2$, or $3^4 \cdot 3^2$, or 3^6.

Thus $(3^2)^3 = 3^6$.

Similarly $(a^2)^4$ means $a^2 \cdot a^2 \cdot a^2 \cdot a^2 = a^8$.

Thus $(a^2)^4 = a^8$.

Find the powers of the following as indicated:

1. $(x^4)^5$

2. $(e^4)^4$

3. $(x^6)^4$

4. $(a^5)^6$

5. $(m^3)^3$

6. $(s^6)^4$

7. $(m^4)^4$

8. $(b^2)^3$

9. $(a^5)^5$

The preceding exercises show that a power of a power may be found briefly by the following law: *The power of a given power is a power whose base is the base of the given power and whose exponent is the product of the given exponents.*

In symbols this is expressed by the formula

$$(a^m)^n = a^{mn}.$$

EXERCISES

Apply the law above to the following:

1. $(x^4)^2$

Solution: $(x^4)^2 = x^{4 \cdot 2} = x^8$.

2. $(a^5)^2$

3. $(a^4)^3$

4. $(m^4)^8$

5. $(a^3)^7$

6. $(x^2)^{2a}$

7. $(a^{n+1})^2$

8. $(y^{p+q})^{p-q}$

9. $(a^{x+1})^{x+4}$

10. $(x^{m+2})^{m-3}$

11. $(2a^2b^2)^3$

Solution: $(2a^2b^2)^3 = 2^3(a^2)^3(b^2)^3 = 8a^6b^6$.

12. $(a^2b^3)^5$

13. $(a^3b^4c^2)^5$

14. $(9p^2q^4)^2$

15. $(aq^4)^3$

16. $(x^3yz^5)^2$

17. $(x^2y^{12}z)^4$

85. A law for finding the quotient of two powers having the same base. To divide a^5 by a^2, proceed as in arithmetic, that is,

$$a^5 \div a^2 = \frac{a^5}{a^2}$$

$$= \frac{\cancel{a}\cdot\cancel{a}\cdot a\cdot a\cdot a}{\cancel{a}\cdot\cancel{a}} = a^3.$$

$$\frac{a^5}{a^2} = a^3.$$

Similarly carry out the following divisions:

1. $\frac{x^4}{x^3}$
2. $\frac{a^7}{a^5}$
3. $\frac{b^5}{b}$
4. $\frac{m^7}{m^3}$
5. $\frac{b^3}{b}$
6. $\frac{y^4}{y^3}$

By examining the results in the preceding exercises show that in finding the quotient of two powers having the same base *the exponent in the quotient is equal to the exponent in the dividend minus the exponent in the divisor.*

In symbols this may be expressed by the formula

$$\frac{a^m}{a^n} = a^{m-n}.$$

EXERCISES

Divide as indicated:

1. $\frac{a^9}{a^4}$

Solution: $\frac{a^9}{a^4} = a^{9-4} = a^5.$

2. $\frac{m^6}{m^3}$
3. $\frac{m^4}{m^2}$
4. $\frac{y^{10}}{y^6}$
5. $\frac{12a^4}{6a^2}$
6. $\frac{20a^5}{4a^2}$
7. $\frac{a^{3n}}{a^n}$

8. $\frac{b^{m+1}}{b^2}$ **9.** $\frac{t^n}{t^{n-1}}$ **10.** $\frac{a^{2r+3}}{a^r}$

11. $\frac{2x^2y^3}{4xy^2}$

Solution: $\dfrac{\overset{x^{2-1}y^{3-2}}{\cancel{2x^2y^3}}}{\underset{2\cdot1\cdot1}{\cancel{4xy^2}}} = \dfrac{x^{2-1}y^{3-2}}{2} = \dfrac{xy}{2}.$

12. $\frac{x^2y^3}{x^2y}$ **14.** $\frac{-36x^6y^4z^2}{6x^4y^3z^2}$ **16.** $\frac{-96m^4n^3t}{-12m^2n}$

13. $\frac{a^3b^4c^2}{ab^3c}$ **15.** $\frac{65a^6b^5c^3}{-13ab^4}$ **17.** $\frac{-96x^8y^{12}}{-16x^3y^5}$

86. A law for finding the power of a quotient. The meaning of $\left(\frac{a}{b}\right)^3$ is $\frac{a}{b}\cdot\frac{a}{b}\cdot\frac{a}{b}$.

Multiplying, you have $\frac{a\cdot a\cdot a}{b\cdot b\cdot b}$, or $\frac{a^3}{b^3}$.

$$\text{Thus } \left(\frac{a}{b}\right)^3 = \frac{a^3}{b^3}.$$

Similarly find:

1. $\left(\frac{a}{b}\right)^4$ **2.** $\left(\frac{x}{y}\right)^2$ **3.** $\left(\frac{r}{x}\right)^5$

It is seen from the results of these exercises that, *to raise a quotient to a given power, you first raise the dividend and the divisor to the given power and then divide the first by the second.*

In symbols this may be expressed by the formula

$$\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}.$$

EXERCISES

Using the law on page 120, change the following powers:

1. $\left(\frac{m}{t}\right)^6$

2. $\left(\frac{x}{y}\right)^4$

3. $\left(\frac{a}{b}\right)^8$

4. $\left(\frac{r}{t}\right)^5$

5. $\left(\frac{t}{s}\right)^7$

6. $\left(\frac{q}{p}\right)^{10}$

7. $\left(\frac{x^2}{y^3}\right)^5$

Solution: $\left(\frac{x^2}{y^3}\right)^5 = \frac{(x^2)^5}{(y^3)^5} = \frac{x^{10}}{y^{15}}.$

8. $\left(\frac{a^5}{b^3}\right)^2$

9. $\left(\frac{m^3}{n^4}\right)^3$

10. $\left(\frac{x^2b^3}{xb^2}\right)^5$

11. $\left(\frac{2x^2y^3}{3a^2b}\right)^4$

12. $\left(\frac{-4b^3x^2}{2b^2y^3}\right)^3$

13. $\left(\frac{4p^4q}{9a^3b^2}\right)^3$

14. $\left(\frac{x^a}{y^b}\right)^2$

15. $\left(\frac{m^2r}{n^3s}\right)^3$

16. $\left(\frac{a^2b^2c}{x^3yz^2}\right)^4$

REVIEW EXERCISES

87. Supplementary exercises. The following exercises review the processes taught in the preceding chapters.

EXERCISES

Perform the following operations as indicated:

1. $(-2)(+7)$
2. $(-3) + (+2) + (-8)$
3. $(-22n) - (-30n)$
4. $(-3x) - (+7x)$
5. Add $2x - 3a + 5b$ to $- 8x - 3a - 12b$.
6. Add $\frac{1}{2}a - \frac{1}{3}b + \frac{1}{6}c$ to $1\frac{1}{2}a + 1\frac{4}{5}b - \frac{2}{3}c$.
7. Subtract $8x - 4y - 10z$ from $-2x + 8z$.
8. Subtract $3a^2 - 6a + 5$ from $a^2 - 5a - 7$.
9. $(-3a)(-2b)$

10. $(5a)(-3x)$
11. $(\frac{1}{3}x^2)(-\frac{2}{5}xy)$
12. $3x(x - 5)$
13. $-7x(2x^2 - 3x + 4)$
14. $-2a(4 - 3a) - 6a^2$
15. $(-36a^2 + 9a) \div (-3a)$
16. $(45x^2 - 27x) \div (-9x)$
17. $(6a^2 - 12a + 54) \div (-6)$
18. $(15b^2 + 45b - 90) \div (-15)$
19. $(a^3 + 2a^2 - 6a + 1)(a^2 - 4a + 3)$
20. $(x^2 - 2xy - 3y^2)(x^2 + 4xy - y^2)$
21. $(a^3 - 5a^2 + a + 10) \div (a - 2)$
22. Find the value of $4m^3 - 3m^2 + 7m + 2$ when $m = -2$.

Expand the powers in Exercises 23 to 26 by multiplying:

23. $(9a - 2b)^2$
24. $(6x^2 - 4y)^2$
25. $(2x + 3)^3$
26. $(a - 5b)^3$

27. If $P = \frac{A}{1 + rt}$, find the value of P if $A = 280$, $r = .06$, and $t = 4$.

28. If $s = \frac{n}{2}(a + l)$, find l when $a = 3$, $n = 5$, and $s = 75$.

29. If $C = \frac{E}{R + r}$, find r if $E = 110$, $R = 30$, and $C = 3\frac{1}{7}$.

30. If $a = 4$, $b = 5$, which is greater, $a^2 + b^2$ or $(a + b)^2$?

Solve the following equations:

31. $x - 13 = 4$.
32. $9a + 8 = 4a + 23$.
33. $6c - (4c - 3) = 8c$.
34. $\frac{3}{4}a - 11 = \frac{a}{2}$.

35. Solve for x: $\frac{18}{5x} = \frac{14}{9}$.

Solve the following problems:

36. How long will it take \$2000 to amount to \$2600 if invested at 6% simple interest?

37. If I have \$36 to spend for a desk and chair and if the cost of a desk is three times that of the chair, how much can I pay for each?

38. The length of a rectangular piece of ground is 2 yards greater than the width. If the perimeter is 36 yards, what is the length?

39. The sum of two consecutive numbers is 63. Find each number.

40. One of two complementary angles is 14° greater than the other. Find the number of degrees in each angle.

41. Two kinds of coffee sell at 40 cents and 35 cents a pound. How many pounds of each are needed to make a mixture of 10 pounds selling at 38 cents a pound?

42. If it takes a man $1\frac{1}{3}$ hours to row downstream a distance of 12 miles and 4 hours to return, find his rate of rowing in still water and the rate of the current.

43. If in a scale drawing 2 inches represents 25 feet, what length will $5\frac{1}{2}$ inches represent?

44. Make a graph of $y = 2x - 3$.

FRACTIONAL EXPONENTS

88. The meaning of fractional exponents. You have learned that exponents may indicate products of several equal factors. Thus a^3 means *aaa* and a^4 means *aaaa*. This gives meaning only to the case when exponents are *whole numbers*. The meaning of exponents will now be extended to include *fractions*.

To interpret the expression $a^{\frac{1}{2}}$, assume that the law $a^m \cdot a^n = a^{m+n}$ may be applied to $a^{\frac{1}{2}}$. Then $a^{\frac{1}{2}} \cdot a^{\frac{1}{2}} = a^{\frac{1}{2}+\frac{1}{2}} = a$.

This suggests that $a^{\frac{1}{2}}$ may be considered "one of the two equal factors whose product is a," that is, that $a^{\frac{1}{2}}$ is but another symbol for expressing the number $\sqrt{a}$. A root of a number indicated by a radical sign, as $\sqrt{a}$, is a radical, as you know.

If it is assumed that the law $a^m \cdot a^n = a^{m+n}$ may be applied to $a^{\frac{1}{3}}$, then $a^{\frac{1}{3}} \cdot a^{\frac{1}{3}} \cdot a^{\frac{1}{3}} = a^{\frac{1}{3}+\frac{1}{3}+\frac{1}{3}} = a$. This means that $a^{\frac{1}{3}}$ may be considered as one of the three equal factors whose product is a, or that $a^{\frac{1}{3}}$ and $\sqrt[3]{a}$ denote the same thing.

In general, $a^{\frac{1}{n}} = \sqrt[n]{a}$.

If you apply the multiplication law to $a^{\frac{2}{3}}$ and to $a^{\frac{3}{4}}$, you have $a^{\frac{2}{3}} \cdot a^{\frac{2}{3}} \cdot a^{\frac{2}{3}} = a^{\frac{6}{3}} = a^2$ and $a^{\frac{3}{4}} \cdot a^{\frac{3}{4}} \cdot a^{\frac{3}{4}} \cdot a^{\frac{3}{4}} = a^{\frac{12}{4}} = a^3$.

Thus $a^{\frac{2}{3}}$ may be defined as one of the three equal factors whose product is a^2; that is, $a^{\frac{2}{3}} = \sqrt[3]{a^2}$.

Figure 44. — John Wallis

John Wallis (1616–1703) was professor of geometry at Oxford. He deserves credit for being among the first to extend the meaning of exponents so as to include fractional and negative cases. He wrote on many subjects relating to mechanics, physics, astronomy, and other fields in which mathematics is used. His main interest was in mathematics, especially arithmetic and algebra.

Similarly $a^{\frac{3}{4}}$ may be regarded as one of the four equal factors whose product is a^3; that is, $a^{\frac{3}{4}} = \sqrt[4]{a^3}$.

From the preceding discussion you see that the following meaning may be assigned to a fractional exponent: *In the power* $a^{\frac{m}{n}}$ *the denominator is the index of a root and the numerator is an exponent.*

In symbols this may be expressed by the formula

$$a^{\frac{m}{n}} = \sqrt[n]{a^m}.$$

EXERCISES

Express the following with radical signs:

1. $a^{\frac{1}{4}}$ **3.** $m^{\frac{3}{2}}$ **5.** $y^{\frac{3}{2}}$ **7.** $x^{\frac{3}{2}}y^{\frac{1}{4}}$ **9.** $(3x)^{\frac{2}{3}}$
2. $a^{\frac{1}{6}}$ **4.** $m^{\frac{2}{3}}$ **6.** $y^{\frac{5}{3}}$ **8.** $a^{\frac{1}{2}}b^{\frac{1}{3}}$ **10.** $(2x)^{\frac{3}{4}}$

Find the value of each of the expressions in Exercises 11 to 24:

11. $49^{\frac{1}{2}}$

Solution: $49^{\frac{1}{2}} = \sqrt{49} = 7.$

12. $4^{\frac{3}{2}}$

Solution: $4^{\frac{3}{2}} = \sqrt{4^3} = 8.$

13. $27^{\frac{1}{3}}$ **17.** $(\frac{27}{8})^{\frac{1}{3}}$ **21.** $16^{\frac{3}{4}}$
14. $8^{\frac{2}{3}}$ **18.** $81^{\frac{3}{4}}$ **22.** $.008^{\frac{1}{3}}$
15. $32^{\frac{3}{5}}$ **19.** $(-125)^{\frac{1}{3}}$ **23.** $(\frac{9}{16})^{\frac{3}{2}}$
16. $(.25)^{\frac{1}{2}}$ **20.** $(-64)^{\frac{1}{3}}$ **24.** $.125^{\frac{2}{3}}$

CHANGING A RADICAL TO THE SIMPLEST FORM

89. An important law for simplifying radicals. You have seen that $\sqrt{a} = a^{\frac{1}{2}}$, by § 88,

and that $\sqrt{b} = b^{\frac{1}{2}}$.

$$\therefore \sqrt{a}\sqrt{b} = a^{\frac{1}{2}} \cdot b^{\frac{1}{2}} = (ab)^{\frac{1}{2}}, \text{ by § 83,}$$
$$= \sqrt{ab}, \text{ by § 88.}$$

Hence $\sqrt{a}\sqrt{b} = \sqrt{ab}$,

or $\sqrt{ab} = \sqrt{a}\,\sqrt{b}$.

Similarly $\sqrt[n]{ab} = \sqrt[n]{a}\sqrt[n]{b}$.

These laws may be used to change radicals to the simplest form. For example, $\sqrt{75}$ may be changed to $\sqrt{25 \times 3} = \sqrt{25}\sqrt{3} = 5\sqrt{3}$.

Suppose that you wish to change $\sqrt{50}$ to a simpler form. What factor of 50 is a perfect square? You may then write $\sqrt{50} = \sqrt{25 \cdot 2} = \sqrt{25}\,\sqrt{2}$. What is the value of $\sqrt{25}$? Therefore $\sqrt{50} = 5\sqrt{2}$. Similarly $\sqrt{18a^3} = \sqrt{9a^2 \cdot 2a} = \sqrt{9a^2}\sqrt{2a} = 3a\sqrt{2a}$.

To change $\sqrt[3]{81a^5b^6c^8}$ to the simplest form, proceed as follows:

1. Factor the term under the radical sign, finding all the factors that are cubes:

$$\sqrt[3]{81a^5b^6c^8} = \sqrt[3]{27 \cdot 3a^3a^2b^6c^6c^2} = \sqrt[3]{27a^3b^6c^6 \cdot 3a^2c^2}.$$

2. Extract the cube roots of the factors that are cubes:

$$\sqrt[3]{27a^3b^6c^6 \cdot 3a^2c^2} = 3ab^2c^2\sqrt[3]{3a^2c^2}.$$

EXERCISES

Change each of the following to the simplest form:

1. $\sqrt{52}$
2. $\sqrt{80}$
3. $\sqrt[3]{24}$
4. $\sqrt{63}$
5. $\sqrt[3]{270}$
6. $\sqrt{96}$
7. $\sqrt{512}$
8. $\sqrt{108}$
9. $\sqrt{a^6}$
10. $\sqrt{3r^2}$
11. $\sqrt{99a^2}$
12. $\sqrt{a^4b^3}$
13. $\sqrt{27a^4b^3}$
14. $\sqrt{128a^6x^3}$
15. $\sqrt{8a^2x^3y^5}$
16. $\sqrt{54x^5y^3}$
17. $\sqrt{100a^2x^3}$
18. $\sqrt{64x^6y^5}$
19. $\sqrt{250a^2b^6}$
20. $\sqrt{64x^{10}y^{18}}$
21. $\sqrt{8}\sqrt{16}$
22. $\sqrt{3a^2}\sqrt{12b^2}$
23. $\sqrt{5a^3b}\sqrt{20ab}$
24. $\sqrt{2ax^2y^3}\sqrt{50a^3xy}$

25. Find the value of $\sqrt{b^2 - 4ac}$ if $b = 6$, $a = 3$, $c = 2$.

26. The hypotenuse and one other side of a right triangle are, respectively, 8 inches and 4 inches. Find the length of the remaining side.

27. Find the altitude of an equilateral triangle whose sides are each 20 inches.

28. The approximate velocity v of an object falling from rest a distance of h feet is given by $v = \sqrt{32h}$. Find the velocity attained by an object that has fallen 62 feet.

29. The time required for one vibration of a simple pendulum l inches long is given by the formula $t = \frac{\pi}{8}\sqrt{6l}$. How long does it take a 36-inch pendulum to make one swing?

90. Simplifying radicals containing fractions. The square root of a fraction, such as $\sqrt{\frac{2}{3}}$, may be found by taking the square root of the numerator, $\sqrt{2}$, and dividing it by the square root of the denominator, $\sqrt{3}$. Thus $\sqrt{\dfrac{2}{3}} = \dfrac{\sqrt{2}}{\sqrt{3}}$. A simpler method is to change the fraction to an equivalent fraction whose denominator is a square. Since $\frac{2}{3} = \frac{6}{9}$, $\sqrt{\frac{2}{3}} = \sqrt{\frac{6}{9}} = \dfrac{\sqrt{6}}{\sqrt{9}} = \dfrac{\sqrt{6}}{3}$, or $\frac{1}{3}\sqrt{6}$.

The cube roots of fractions may be simplified in a similar manner. Multiply the numerator and the denominator of the fraction by a number so that the denominator of the resulting fraction will be a cube. Thus

$$\sqrt[3]{\frac{2a}{3b}} = \sqrt[3]{\frac{2a}{3b}\cdot\frac{9b^2}{9b^2}} = \sqrt[3]{\frac{18ab^2}{27b^3}} = \frac{\sqrt[3]{18ab^2}}{\sqrt[3]{27b^3}} = \frac{\sqrt[3]{18ab^2}}{3b} = \frac{1}{3b}\sqrt[3]{18ab^2}.$$

This method of simplification is called **rationalizing the denominator.**

EXERCISES

Change the following fractions to equivalent fractions whose denominators are squares:

1. $\frac{1}{2}$

2. $\frac{2}{3}$

3. $\frac{3}{5}$

4. $\frac{5}{8}$

5. $\frac{3}{10}$

6. $\frac{2b}{3}$

7. $\frac{a}{b}$

8. $\frac{3x}{4y}$

Change the following fractions to equivalent fractions whose denominators are cubes:

9. $\frac{1}{2}$

10. $\frac{1}{3}$

11. $\frac{3}{4}$

12. $\frac{2}{5}$

13. $\frac{2}{3}$

14. $\frac{a}{b}$

Simplify the following radicals:

15. $\sqrt{\frac{1}{2}}$

16. $\sqrt{\frac{2}{3}}$

17. $\sqrt{\frac{3}{5}}$

18. $\sqrt{\frac{3}{10}}$

19. $\sqrt{\frac{2b}{3}}$

20. $\sqrt{\frac{a}{b}}$

21. $\sqrt{\frac{3x}{4y}}$

22. $\sqrt[3]{\frac{1}{2}}$

23. $\sqrt[3]{\frac{1}{3}}$

24. $\sqrt[3]{\frac{3}{4}}$

25. $\sqrt[3]{\frac{2}{5}}$

26. $\sqrt[3]{\frac{a}{b}}$

91. Adding and subtracting radicals.[1] Two radicals, as $2\sqrt{5}$ and $3\sqrt{5}$, can be added in the same way that the similar terms $2x$ and $3x$ are added; that is, the coefficients of the common factor are added and the sum is then multiplied by the common factor. It is necessary to change all radicals to the simplest form

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[1] See Chapter XI for additional work with radicals.

before combining them. The following examples illustrate the method:

1. Subtract as indicated:

$$3\sqrt{98} - 7\sqrt{72} - 3\sqrt{18}$$

Solution:

$$\begin{aligned} 3\sqrt{98} - 7\sqrt{72} - 3\sqrt{18} &= 3\sqrt{49\cdot 2} - 7\sqrt{36\cdot 2} \\ &\qquad - 3\sqrt{9\cdot 2} \\ &= 21\sqrt{2} - 42\sqrt{2} - 9\sqrt{2} \\ &= -30\sqrt{2}. \end{aligned}$$

2. Add and subtract as indicated:

$$3\sqrt{5} + 2\sqrt{\tfrac{4}{5}} - 5\sqrt{\tfrac{1}{5}}$$

Solution: As in Example 1, first simplify each radical and then collect the similar terms.

$$\begin{aligned} 3\sqrt{5} + 2\sqrt{\tfrac{4}{5}} - 5\sqrt{\tfrac{1}{5}} &= 3\sqrt{5} + 2\sqrt{\frac{4\cdot 5}{25}} - 5\sqrt{\frac{5}{25}} \\ &= 3\sqrt{5} + \tfrac{4}{5}\sqrt{5} - \tfrac{5}{5}\sqrt{5} \\ &= 2\tfrac{4}{5}\sqrt{5}. \end{aligned}$$

EXERCISES

Add and subtract as indicated:

1. $2\sqrt{2} + 3\sqrt{2}$

2. $3\sqrt{a} + 4\sqrt{a}$

3. $\frac{1}{2}\sqrt{5} + 10\sqrt{5} - \frac{3}{4}\sqrt{5}$

4. $2\sqrt{5} + \sqrt{5}$

5. $4\sqrt{6} - 2\sqrt{6} + \sqrt{6}$

6. $3\sqrt{2} + 3\sqrt{3} + 4\sqrt{2} - 2\sqrt{3}$

7. $\sqrt{8} + \sqrt{18}$

8. $\sqrt{48} + \sqrt{12}$

9. $\sqrt{\frac{5}{2}} + \sqrt{\frac{5}{8}}$

10. $\sqrt{\frac{4}{3}} + \sqrt{\frac{16}{3}}$

11. $\sqrt{50} + \sqrt{98} - \sqrt{32}$

12. $8\sqrt{2} + 6\sqrt{8} - 5\sqrt{32}$

13. $5\sqrt{27} - 2\sqrt{192} + 3\sqrt{300}$

14. $\sqrt[3]{16} - 3\sqrt[3]{2} + \sqrt[3]{54}$

15. $8\sqrt{45} - \frac{3}{4}\sqrt{80} + 3\sqrt{63}$

16. $2\sqrt[3]{16x^3} - 7\sqrt[3]{54x^3} + \sqrt[3]{250x^3}$

17. $5\sqrt{\frac{6}{5}} + 2\sqrt{\frac{15}{2}} - 3\sqrt{\frac{3}{10}}$

18. $\sqrt{\frac{4}{6}} - 6\sqrt{\frac{4}{3}} + 3\sqrt{\frac{1}{3}}$

19. $6\sqrt{\frac{2}{7}} - 4\sqrt{\frac{1}{11}} + 3\sqrt{\frac{7}{2}}$

20. $2\sqrt{63} + \frac{3}{5}\sqrt{45} - 3\sqrt{\frac{1}{5}}$

21. $\sqrt[3]{128} - 3\sqrt[3]{81} + 2\sqrt[3]{\frac{1}{4}}$

22. $3\sqrt{4\frac{1}{6}} - 8\sqrt{\frac{4}{5}} + 2\sqrt{3\frac{1}{5}}$

NEGATIVE AND ZERO EXPONENTS[1]

92. The meaning of a zero exponent. The following shows that a meaning may be given to a *zero* exponent.

You may divide a^2 by a^2 as in arithmetic, and you will find the quotient to be 1; that is, $\frac{a^2}{a^2} = \frac{a \cdot a}{a \cdot a} = 1.$

If the law $\frac{a^m}{a^n} = a^{m-n}$ is assumed to hold when $m = n$, then $\frac{a^2}{a^2} = a^{2-2} = a^0.$

To make the two results for $\frac{a^2}{a^2}$ the same, it is agreed that a^0 should have the value 1. In other words, a^0 may be thought of as indicating that a power of a has been divided by itself.

Show that x^0, $(a + b)^0$, $(2a - 3b + c)^0$ all have the same value, that is, that they are all equal to 1.

In general,

$$a^0 = 1.$$

---

[1] §§ 92 and 93 are optional and may be used as supplementary material.

EXERCISES

Give the value of each of the following:

1. y^0
2. 5^0
3. $(-18)^0$
4. $(\frac{1}{2})^0$
5. $(x-y)^0$
6. $\left(\frac{x}{y}\right)^0$
7. $5(\frac{3}{4})^0$
8. $\frac{27}{3^0}$
9. $2^2 \times 4^2 \times 6^0$
10. $a^0 + b^0$
11. $6(x-y)^0 + x - y$
12. $\frac{x^0y^3z^4}{xyz^2}$

93. Finding the meaning of negative exponents. Let it be assumed that the law $\frac{a^m}{a^n} = a^{m-n}$ holds when n is greater than m.

Then $\frac{a^2}{a^5} = a^{2-5} = a^{-3}$.

But the value of $\frac{a^2}{a^5}$ may be found by changing the fraction to the simplest form. This gives

$$\frac{a^2}{a^5} = \frac{\cancel{a}\cdot\cancel{a}}{\cancel{a}\cdot\cancel{a}\cdot a\cdot a\cdot a} = \frac{1}{a^3}.$$

To make the two results agree, the symbol a^{-3} is given the meaning $\frac{1}{a^3}$.

Similarly $a^{-4} = \frac{1}{a^4}$, $a^{-5} = \frac{1}{a^5}$, . . .

In general, $a^{-n} = \frac{1}{a^n}$.

EXERCISES

Give the value of each of the following:

1. 5^{-2}
2. 4^{-3}
3. $2^{-1}\cdot 3^{-2}$
4. $6\cdot 4^{-2}$
5. $2^{-1}\cdot 8^0\cdot 3$
6. $6^2\cdot 4^{-3}\cdot 3^0$
7. $(-8)^{-2}$
8. a^2b^{-3}
9. $6a^{-2}xy^{-1}$

10. $3(-x)^{-2}y$
11. $a^{-1} + b^{-1}$
12. $(2a)^{-2}x^2$
13. $(a + a^{-1})^2$
14. $a^{-2} \cdot a^8 \cdot a^{-1}$
15. $(\frac{2}{3})^{-1}$
16. $(\frac{4}{5})^{-2}$
17. $\dfrac{1}{3^{-4}}$
18. $\dfrac{a}{b^{-2}}$

Exercises to Test Your Understanding of Chapter IV

94. What you should be able to do. Having studied Chapter IV, you should be thoroughly familiar with, and able to use correctly, the laws of exponents stated in §§ 82 to 86. They are as follows:

1. $a^m \cdot a^n = a^{m+n}$. (Product of two powers)
2. $(abc)^m = a^m b^m c^n$. (Power of a product)
3. $\left(\dfrac{a}{b}\right)^m = \dfrac{a^m}{b^m}$. (Power of a quotient)
4. $(a^m)^n = a^{mn}$. (Power of a power)
5. $\dfrac{a^m}{a^n} = a^{m-n}$. (Quotient of two powers)

You should be able to simplify radicals such as $\sqrt{20}$, $\sqrt[3]{24}$, $\sqrt{\frac{2}{3}}$.

You should be able to add and subtract radicals.

EXERCISES

Simplify each exercise and explain the laws that are used:

1. $x^2 \cdot x^3 \cdot x$
2. $(3abc)^3$
3. $(x^5y^2z)^6$
4. $(-c^2d^3)^6$
5. $\dfrac{12a^4}{-3a^2}$
6. $\left(\dfrac{x^2}{y^3}\right)^4$
7. $(5a^2b)(-7a^3b^2)$
8. $(-3x^2)(2x^2yz)$
9. $(\frac{3}{2}x^2yz^2) \div (1\frac{1}{3}xz)$

Simplify the following:

10. $\sqrt{5x^2}$

11. $\sqrt{\frac{4}{9}}$

12. $\sqrt{72}$

13. $\sqrt{54x^3y^2}$

14. $\sqrt{\dfrac{128a^4b^3}{8x^2y^2}}$

15. $\sqrt{x}\sqrt{xy}$

16. $\sqrt{x^3}\sqrt{xy^2}$

95. Timed tests in exponents and radicals. Examine the items of each of the following tests before you begin to work them. Then work rapidly and accurately.

A. *Multiplication.* Time: 3 minutes.

1. $y^2 \cdot y^3$

2. $a \cdot a^2 \cdot a^5$

3. $(-a)^2 \cdot a$

4. $(-3x^2yz^3)(5xy^2z)$

5. $(3a^2bc^4)^2$

6. $(2a^2 - 5ab^2)(-3ab^2c)$

7. $(3a^3 + 4a^2 + 2a + 1)(a + 3)$

8. $(x^2 - xy + y^2)(x + y)$

B. *Division.* Time: 5 minutes.

1. $\dfrac{a^8}{a^3}$

2. $\dfrac{22ax^2}{-2a}$

3. $\dfrac{-48a^4b^3c^2}{-6ab^3c}$

4. $\left(\dfrac{2x^2y^2}{3mn^3}\right)^2$

5. $\dfrac{96a^6b^4}{-16a^7b}$

6. $\left(\dfrac{-6x^2y^3}{2x^3y^2}\right)^3$

7. $(3x^5z^9) \div (-15x^2yz^3)$

8. $(-5a^2bc^3)^3 \div (-10abc^2)^4$

C. *Radicals.* Time: 6 minutes.

Simplify:

1. $\sqrt{96}$

2. $\sqrt{27x^4y^3}$

3. $\sqrt{2x}\sqrt{2xy^2}$

4. $\sqrt{8x^2b}\sqrt{2a^2b^3}$

Add and subtract as indicated:

5. $3\sqrt{45} - \sqrt{80} + 2\sqrt{125}$

6. $\sqrt{48} + 5\sqrt{75} - 2\sqrt{12}$

7. $30\sqrt{\frac{1}{5}} - 20\sqrt{\frac{4}{5}} + \sqrt{5}$

8. $15\sqrt{\frac{4}{3}} + 6\sqrt{\frac{1}{3}} - 12\sqrt{\frac{25}{3}}$

CHAPTER V

SIMPLE EQUATIONS IN TWO UNKNOWNS

Graphical Solution

96. Problems containing several unknown numbers. A problem frequently calls for several unknown numbers. For example, the problem, "If one angle of a triangle is twice as large as the second and the third is three times the second, find the number of degrees in each angle," calls for three unknown numbers. Expressed in terms of one letter, they are $2x$, x, and $3x$. They may be found by solving the equation

$$2x + x + 3x = 180.$$

In all previous work, when you solved problems containing several unknowns, you denoted one of them by a letter and expressed the others in terms of that letter. Sometimes the task of expressing all unknowns in terms of one letter is not simple, and it

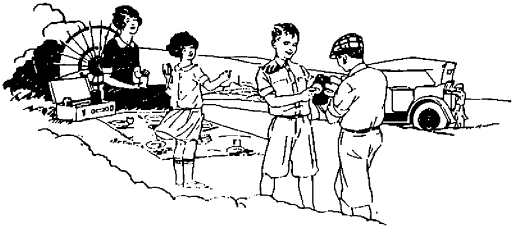

Figure 45

saves time and work to solve the problem by using several letters, each denoting one of the unknown numbers, as will be shown in the following problem.

Two boys were making purchases for a picnic. One bought 60 cents' worth of oranges and apples; the other bought 50 cents' worth. The first received 6 oranges and 10 apples, the other received 7 oranges and 5 apples of the same kind as the first. Later they found that they needed more fruit, and they therefore wished to know the price of one orange and of one apple. They worked the problem as follows:

Solution: Let x be the number of cents paid for an orange.
Let y be the number of cents paid for an apple.
Then $6x + 10y = 60$,
and $7x + 5y = 50$.

Thus the problem led to two equations containing two unknown numbers. To find the answer, the boys had to know how to solve this pair of equations.

In this chapter you will learn three methods of solving such pairs of equations. One is called a *graphical solution*, because graphs are used; the others are called *algebraic solutions*, because the values of the unknowns are found by means of algebraic processes.

97. How to use graphs in solving a pair of equations. If in the equation $7x + 5y = 50$ you assign to x a value, as $x = 2$, the equation takes the form

$$7 \cdot 2 + 5y = 50.$$

From this you have

$$\begin{aligned} 14 + 5y &= 50. \\ 5y &= 36. \\ y &= 7.2. \end{aligned}$$

The number pair (2, 7.2) *satisfies* the equation, for

$$(7)(2) + (5)(7.2) = 14 + 36 = 50.$$

For this reason the *pair* of numbers $(x, y) = (2, 7.2)$ is called a *solution* of the equation $7x + 5y = 50$.

Similarly other solutions may be found by assigning other values to x:

$$\begin{aligned} \text{If } x = 0, \text{ then } (7)(0) + 5y &= 50. \\ 0 + 5y &= 50. \\ \therefore y &= 10. \\ \text{If } x = 1, \text{ then } (7)(1) + 5y &= 50. \\ 5y &= 43. \\ \therefore y &= 8.6. \\ \text{If } x = 3, \text{ then } (7)(3) + 5y &= 50. \\ 5y &= 29. \\ \therefore y &= 5.8. \end{aligned}$$

In a similar way verify the correctness of the other pairs of values in Table 2 (Fig. 46).

Note that the equation has an *unlimited* number of solutions.

The number pairs in Table 2 may be represented graphically as follows:

Draw two reference axes, OX and OY (Fig. 46). Values of x are laid off on the x-axis, and values of y are laid off at right angles to the x-axis. Thus, to plot the pair (2, 7.2), pass from 0 two units to the right and then 7.2 units upward, locating point *A*. When all number pairs given in the table have been plotted as points, draw the line *AB* passing through these points.

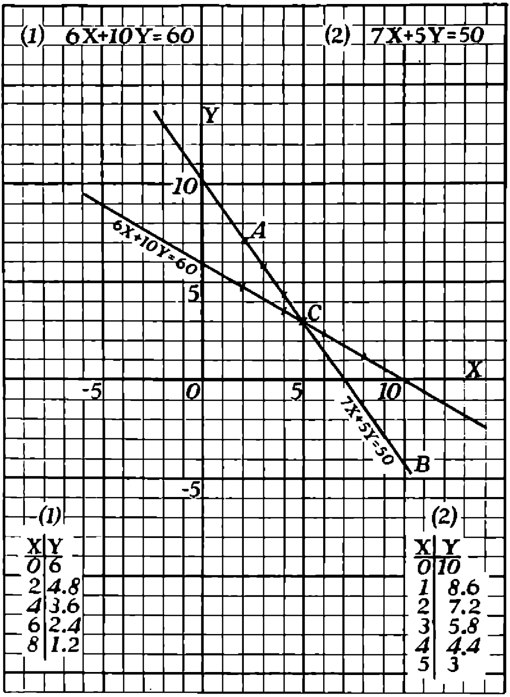

FIGURE 46

This line is said to be the graph of the equation $7x + 5y = 50$.

Remember that positive values of x are marked off to the right of the 0 point, and that positive values of y are marked off above the 0 point. Negative values of x

are marked off to the left of the 0 point, and negative values of y are marked off below the 0 point.

Verify the following two characteristics of line AB:

1. A solution of the equation, when plotted, locates a point on the line.

2. Any point on the line determines a pair of numbers satisfying the equation.

For example, for the point B you have $x = 10$ and $y = -4$.

Substituting in the equation, you have

$$(7)(10) + (5)(-4) = 70 - 20 = 50,$$

which shows that the equation is satisfied.

Because the graph is a straight line, equations like $7x + 5y = 50$ are called **linear equations.** A pair of such equations is called a **system of simultaneous equations.**

Verify Table 1 (Fig. 46), which gives solutions of the equation $6x + 10y = 60$.

Plot the points and make the graph of the equation.

The two straight lines intersect at C. Show that to this point corresponds the pair $(x, y) = (5, 3)$.

Since a pair of numbers corresponding to a point on a graph is a solution of the equation represented by the graph, it follows that the pair which corresponds to the point of intersection of *two* graphs is a solution of *both* equations.

Therefore the solution of the equations is the number pair $(x, y) = (5, 3)$.

This is also the solution of the problem on page 135. Hence the price of an orange was 5 cents, and the price of an apple was 3 cents.

98. Summary of the steps in the graphical solution of a system of equations. Let $6x + 10y = 60$ and $7x + 5y = 50$ be the system of equations to be solved. The following is a summary of the steps in the process of solving:

1. *Assume values of* x *and find the corresponding values of* y. Since the lines are straight lines, two or three points are sufficient to determine the line. For convenience:

First let $x = 0$. Then $(6)(0) + 10y = 60$, and $y = 6$.
Next let $y = 0$. Then $6x + (10)(0) = 60$, and $x = 10$.

Two points are sufficient to determine the line, but it is better to use a third point as a check. If the three points are found not to be on the same straight line, the values of x and y should be carefully rechecked. Choose as a third value for x one not too near the other two values, as $x = 4$. Then $24 + 10y = 60$, and $y = 3.6$.

Similarly for the second equation:

Let $x = 0$. Then $y = 10$.
Let $y = 0$. Then $x = 7.1$.
Let $x = 3$. Then $y = 5.8$.

2. *Tabulate the number pairs*, as in Tables 1 and 2 (Fig. 46).

3. *Draw the axes, select a convenient unit, and plot the number pairs given in the tables.*

4. *Draw the graphs and locate the point of intersection* C.

5. *Starting from* 0 *and passing along the* x-*axis, count the number of units to the foot of the perpendicular from* C *to the* x-*axis.* This is the required value of x.

Count the number of units contained in the perpendicular. This is the required value of y.

6. *The pair of numbers* (x,y) = (5,3) *thus determined is the required solution.*

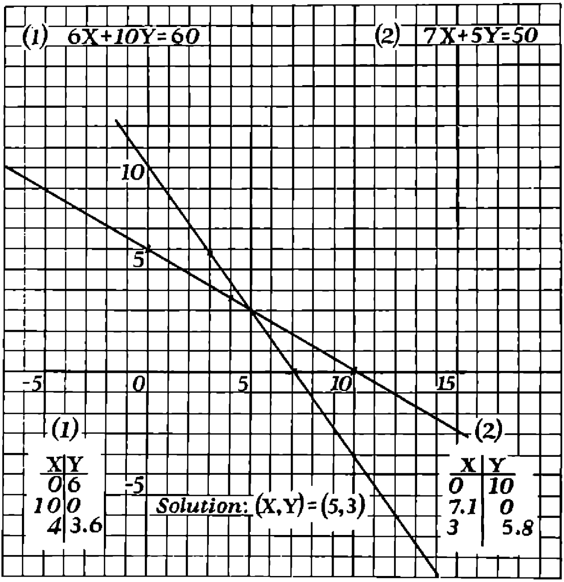

FIGURE 47

The written work may be conveniently arranged as in Fig. 47.

EXERCISES

Plot the following points on graph paper:

1. (3, 2)
2. (3, 4)
3. (1, 5)
4. (7, 0)
5. (9, 0)
6. (0, 3)
7. (0, 6)
8. (6.2, 4)
9. (3, 4.5)
10. (2, −3)
11. (4, −1)
12. (−3, 7)
13. (−5, 2)
14. (−2, −3)
15. (−8, −2)
16. (0, −4)
17. (0, −2.3)
18. (−4.5, 0)

19. Draw a pair of axes on a piece of graph paper. Place a point on the paper and then tell what number pair corresponds to it. Practice until you can do this quickly and accurately.

20. Draw a pair of axes on a piece of graph paper. Draw any two intersecting lines on the paper and tell what number pair corresponds to the point of intersection of the lines. Practice until you can do this quickly and accurately.

Solve the following equations graphically:

21. $x - y = 4$,
$x + y = 18$.

22. $x + y = 6$,
$2x - 3y = 2$.

23. $x + y = 5$,
$2x - 5y = -11$.

24. $6x - 5y = 15$,
$3x + 2y = 21$.

25. $x + y = 6$,
$3x - 2y = -2$.

26. $2x + 3y = 4$,
$5x - 2y = -9$.

27. $3x - 2y = -1$,
$2x + y = 11$.

28. $3x - 2y = 8$,
$4x - 3y = 10$.

99. Supplementary exercises. The following exercises give further practice in the graphical method.

EXERCISES

Solve the following systems graphically:

1. $2x - 3y = 4$,
$3x + 2y = 32$.

2. $5x - y = 1$,
$2x + 3y = 27$.

3. $x + 2y = 7$,
$3x - 2y = 5$.

4. $7x + 4y = 2$,
$5x - 3y = 19$.

5. $6x - 5y = 9$,
$5x + 2y = 26$.

6. $5x + 7y = 12$,
$2x + 21y = 23$.

Solve the following problems:

7. The weight of water in pounds corresponding to the volume in cubic inches is given by the equation $w = \frac{62.4}{1728} v$. Represent the equation graphically. From the graph find the weights of 100 cubic inches of water, 200 cubic inches, 400 cubic inches. Find the volume of 10 pounds, 15 pounds, 25 pounds.

8. The rate at which sound travels at a given temperature is given by the formula $v = \frac{12{,}648 + 13t}{12}$, where v is the number of feet a second and t is the number of degrees Fahrenheit. Make a graph of the equation and from it find the velocity at 32°; 72°.

9. The velocity of an object thrown downward is given by the formula $v = k + 32t$, where k is the initial velocity. Make a graph of the equation when the initial velocity is 64 feet a second.

100. Historical note on systems of equations. The study of systems of equations (equations containing two or more unknowns) dates back to the time of Diophantus of Alexandria, who lived about the middle of the third century after Christ. Practically nothing is known of his life. Among other works he wrote one on arithmetic which contains a considerable amount of algebra. The treatise is devoted to the solution of problems leading to equations of the first and second degree. Being several centuries ahead of his time in his knowledge of algebra, he stands out as one of the great mathematicians of Greek civilization, but his writings were not sufficiently known to influence the development of Greek and European mathematics. When his work finally became known in Europe in the fifteenth century, mathematics had passed beyond the stage to which it had been advanced by Diophantus.

René Descartes (1596–1650) had more influence on the development of mathematics during the seventeenth century than any other man of his time. He was a mathematical genius, a physicist, and a philosopher. He is the inventor of the graphic treatment of equations by which algebra, arithmetic, and geometry were closely related to each other.

ALGEBRAIC SOLUTION OF EQUATIONS IN TWO UNKNOWNS[1]

101. How to solve a pair of equations by algebraic methods. The graphical method explained in §§ 97 and 98 has several disadvantages. The process is long, and it is difficult to determine accurate solutions when the unknown numbers are not whole numbers. The algebraic method avoids both these difficulties.

Let it be required to solve the system

$$\begin{aligned} 5x + 2y &= 34, \\ 7x - 3y &= \ \ 7. \end{aligned}$$

Multiply all terms of the first equation by 7 and all terms of the second by 5. Then

$$\begin{array}{r|l} \text{you have } 7 & 5x + 2y = 34 \\ \text{and } 5 & 7x - 3y = \ \ 7 \\ \cline{2-2} \text{or} & 35x + 14\,y = 238 \\ \text{and} & 35x - 15\,y = \ \ 35. \\ \cline{2-2} \end{array}$$

By subtracting the last equation from the one above you *eliminate* (remove) x. This gives

$$29\,y = 203.$$
$$\therefore y = 7.$$

---

[1] See Chapter XI for other algebraic methods.

Instead of eliminating x, you might have eliminated y by multiplying the first equation by 3 and the second by 2 and *adding* the resulting equations.

Having determined the value of one unknown (x or y), you may substitute it in one of the given equations. Thus, when $y = 7$, the first equation changes to

$$\begin{aligned} 5x + 2\cdot 7 &= 34. \\ 5x + 14 &= 34. \\ \therefore x &= 4. \end{aligned}$$

Hence the solution is $(x, y) = (4, 7)$.

The check consists in verifying *both* the given equations.

Check: Left Member		Right Member
$5\times4 + 2\times7$		34
$20 + 14$		34
34	$=$	34
$7\times4 - 3\times7$		7
$28 - 21$		7
7	$=$	7

Consider as a second illustration the system of equations $4x + 3y = 6$ and $3x - 5y = 19$. If you wish to eliminate x, by what number must you multiply each equation to make the coefficients of x the same in both equations? Would you then add or subtract? If you wish instead to eliminate y, by what number must you multiply each equation? Would you then add or subtract? Complete the solution and the check as in the first illustration.

EXERCISES

Solve the following systems algebraically:

1. $3x - 7y = 40,$
$4x - 3y = 9.$

2. $9a - 4b = 3,$
$7a + 2b = 33.$

3. $3x - 2y = 8,$
$4x - 3y = 10.$

4. $2x - 3y = -11,$
$5x - 2y = 6.$

5. $3a - 13b = 41,$
$8a + 11b = 18.$

6. $13x + 3y = 14,$
$7x - 2y = 22.$

7. $5a + 2b = 36,$
$2a + 3b = 43.$

8. $2x + 3y = 27,$
$5x - 2y = 1.$

9. $3m + 2n = 23,$
$2m + 3n = 27.$

10. $x + y = 12,$
$x - y = 4.$

11. $x + 3y = 7,$
$x - 2y = 2.$

12. $x + 2y = 4,$
$3x - 2y = 2.$

13. $3x - 2y = -1,$
$2x + y = 11.$

14. $x + 4y = 4,$
$x - 2y = 16.$

15. $x = 2y,$
$3x - y = 15.$

Suggestion: Eliminate x by substituting $2y$ for x in the second equation.

16. $3x = y,$
$15x - 4y = 27.$

17. $5x = 36 - 2y,$
$2x + 3y - 43 = 0.$

18. $2x + 6y - 28 = 0,$
$2x - y - 7 = 0.$

102. Problems leading to equations in two unknowns. The following problems lead to two equations in two unknowns (see § 96). Solve them by the algebraic method.

EXERCISES

1. Two pencils and 3 tablets cost 24 cents. Three pencils and 5 tablets cost 39 cents. Find the cost of each.

2. A boy bought 8 apples and 6 oranges for 98 cents. His sister paid 82 cents for enough to make a dozen of each. What was the price of one orange and of one apple?

3. A grocer wishes to make a 5-pound mixture of coffee selling for $2.20 from two kinds of coffee, one selling at 40 cents a pound and the other at 50 cents a pound. How much of each kind must he use?

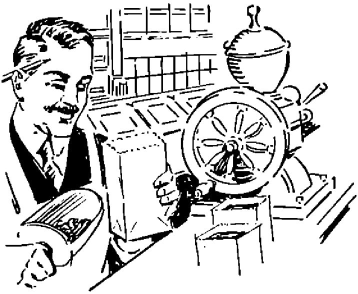

FIGURE 48

Solution: The unknown and the given facts of this problem are as follows:

Let x be the number of pounds in the 40-cent grade of coffee.

Let y be the number of pounds in the 50-cent grade of coffee.

5 is the number of pounds in the required mixture.
40 is the price per pound of the first grade,
and 50 is the price per pound of the second grade.
220 is the total price of the required mixture.
$40x$ is the total price of the first grade,
and $50y$ is the total price of the second grade.

To avoid repetition and to make the facts of the problem stand out clearly, it is convenient to arrange them in tabular form as follows:

	40¢ GRADE	50¢ GRADE	REQUIRED MIXTURE
Number of pounds	x	y	5
Price per pound	40	50	
Total price . .	$40x$	$50y$	220

Show that the equations are:

$x + y = 5$, expressing the number of pounds in the required mixture;

$40x + 50y = 220$, expressing the total price of the required mixture.

Solve this system of equations.

4. A grocer wishes to obtain 5 pounds of coffee, worth 39 cents per pound, by mixing two other grades, one worth 48 cents and the other 36 cents per pound. How much of each must he use?

5. Tea selling at 40 cents a pound is to be mixed with a grade selling at 60 cents a pound. How many pounds of each must be used to make 60 pounds selling at 54 cents a pound?

6. A boy can row 5 miles downstream in 1 hour and return in $1\frac{2}{3}$ hours. What is his rate of rowing in still water, and what is the rate of the current?

Suggestion: Let x be the rate of rowing in still water and y the rate of the current.

Then $x + y$ is the rate going downstream, and $x - y$ is the rate returning.

7. A man rows 10 miles downstream in 2 hours. Find the rate of the current and the rate of rowing in still water if it takes him 2 hours 30 minutes to return.

Solution:

	Going Down	Returning
Distance . . .	10	10
Time	2	$2\frac{1}{2}$
Rate	5	$\frac{10}{2\frac{1}{2}}$

$$\therefore x + y = 5,$$

$$\text{and } x - y = \frac{10}{2\frac{1}{2}}.$$

Solve this system of equations.

8. In 50 minutes a man rows 5 miles downstream and in 1 hour 30 minutes 6 miles upstream. Find the rate of the current and the rate of rowing in still water.

9. From 6 acres of potatoes and 9 acres of corn a farmer derived an income of $690. A year later his income from 8 acres of potatoes and 6 acres of corn was $620. How much per acre did he derive from each?

10. The sum of $6000 is to be divided among three partners so that the first receives twice as much as the second and the third as much as the first two together. How much should each receive?

11. Walnuts selling at 35 cents a pound are to be mixed with other nuts selling at 25 cents so that a mixture of 8 pounds can be sold for $2.20. How many pounds of each should be used?

12. A grocer wishes to make a mixture from two kinds of coffee, one selling for 35 cents a pound and the other for 25 cents a pound. The mixture is to contain twice as much of the first coffee as of the second and is to sell for $9.50. How much of each should be used?

13. A farmer has two qualities of milk, one 3% and the other 5%; that is, the qualities yield 3% and 5% butter fat, respectively. How much of each shall he use to make 10 gallons of 4.5% milk?

14. A farmer wants to send an order for 48 bags of cement to a wholesale house, but he is puzzled to know how much money he must inclose. He has no catalogue, but his son remembers that in making a cement walk costing $26 they used 8 bags of cement and 2 cubic yards of sand and in making a reservoir for water they used 24 bags of cement and 5 cubic yards of sand at a total cost of $73. From these statements determine how much money the farmer should send.

15. A man invested two sums at 5% and at 4%, respectively, and from this investment he has a yearly income of $500. If the total sum invested amounted to $12,000, how much did he invest at each rate?

16. A man invested a part of $10,000 at 6% interest and the remainder at 5%. The total yearly income is $570. Find the amount invested at each rate.

17. A man has a yearly income of $54 from an investment of $1000. If one part of the investment yields 6% and the remainder 5%, find the amount invested at each rate.

EXERCISES TO TEST YOUR UNDERSTANDING OF CHAPTER V

103. What you should know and be able to do. It is expected that at the end of Chapter V you should be able to do the following:

1. To translate the facts contained in verbal problems into algebraic symbols.

2. To derive the equation or equations from which the solution may be obtained.

3. To solve a pair of simple equations in two unknowns:

   (*a*) By the graphical method.

   (*b*) By eliminating one of the unknowns.

EXERCISES

1. Using the method of § 98, solve graphically the system:

$$\begin{aligned} 3x - 2y &= -1, \\ 2x + y &= 11. \end{aligned}$$

2. Solve by eliminating x:

$$\begin{aligned} 3x - 7y &= 40, \\ 4x - 3y &= 9. \end{aligned}$$

3. Solve by eliminating y:

$$\begin{aligned} 13x + 3y &= 14, \\ 7x - 2y &= 22. \end{aligned}$$

4. Eliminate y by substitution:

$$\begin{aligned} 3x \qquad &= y, \\ 15x - 4y &= 27. \end{aligned}$$

5. If two kinds of coffee sell at 48 and 55 cents a pound, respectively, how many pounds of each must be used to make a mixture of 20 pounds selling at 52 cents a pound?

6. Into what two parts should $2000 be divided so that by investing them at $4\frac{1}{2}\%$ and $5\frac{1}{2}\%$, respectively, an income of $107 may be derived?

7. Make the graph of the equation $5F - 9C = 160$. If F denotes degrees Fahrenheit and C degrees centigrade, tell from the graph the number of degrees Fahrenheit corresponding to 0° centigrade; the number of degrees Fahrenheit corresponding to 10° centigrade.

8. Two men working together can build a fence in 2 days. After working together the first day one of the men quit work and the other finished the work in 3 days. In how many days could each build the fence working alone?

9. The sum of three numbers is 2. The second is 6 more than the third, and the sum of the first and three times the second is 4. Find the numbers.

10. A man has an opportunity to invest a sum of money in two different ways, one yielding 4% and the other 5%. What part of $10,000 should he invest at 4% and what part at 5% to obtain an annual income of $450?

11. A druggist wishes to obtain one quart of an 80% solution of ammonia by mixing a 90% solution with a 60% solution. How much must he take from each?

12. One side of a field is 10 rods long. It is to be divided so that the parts are in the ratio 2:3. How long should each part be?

13. The perimeter of a rectangle is 283 feet, and the adjacent sides are to each other as 3:8. How long is each side?

14. Two partners in business made a profit of $10,000 last year. They have an agreement that one of them is to receive $1000 more than two-thirds of the amount received by the other. How much did each receive last year?

15. Going downstream, a steamer travels 60 miles in 3 hours. The return trip is made in 5 hours. Find the rate of the current and the rate at which the steamer can travel in still water.

16. A farmer wishes to get 10 gallons of 3.6% milk by mixing 3.2% milk and 4% milk. How much of each kind of milk should he use?

17. How can I invest $3200 in two parts, one at 5% and one at 6%, yielding together a total income of $180 a year?

104. Timed test in equations with two unknowns. Time: 8 minutes. Exercises 1 to 4 test your understanding of the graphical method.

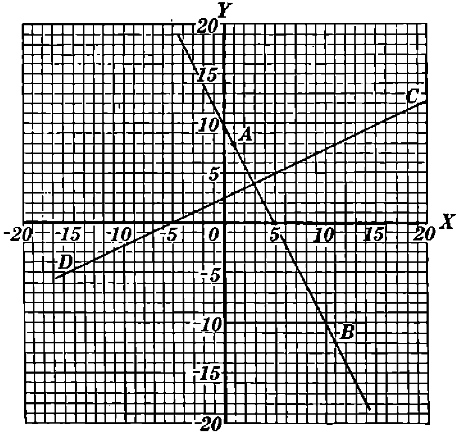

FIGURE 49

1. Find three number pairs satisfying the equation $6x - 5y = 9$ that could be used conveniently to draw the graph of the equation. Show all your work.

2. On graph paper draw a pair of perpendicular lines. Using them as reference lines, plot the following points: $(2, 4)$, $(-3, 12)$, $(-8, -16)$, $(10, -18)$.

3. State the pair of numbers that locates point A (Fig. 49).

4. If the two lines AB and CD (Fig. 49) represent two equations in two unknowns, what is the solution of this system of equations?

Solve the following systems of equations algebraically:

5. $x + y = 30,$
$x - y = 6.$

6. $3a + b = 21,$
$8a - 3b = 22.$

7. $2x + 3y = 17,$
$5x - 2y = -5.$

8. $4m + 7t - 40 = 0,$
$-3m + 2t + 1 = 0.$

CHAPTER VI

FRACTIONS. FACTORING

WHAT YOU ARE GOING TO STUDY IN THIS CHAPTER

105. Why you study fractions. In your previous work problems were usually solved by adding, subtracting, multiplying, or dividing whole numbers (integers). However, there are problems which lead to polynomials or equations containing fractions. Knowledge of fractions is necessary to solve them. The following example illustrates a problem of this type.

An automobile concern has a contract for 126 automobiles. One of the factories of the company can make that many automobiles in 15 days and another in 10 days. The question arises as to how many days it would take to fill the contract if both factories worked together.

To answer the question, denote the number of days in which both factories together can make the 126 automobiles by n. Then $\frac{126}{n}$ is the number they make in one day.

However, the first factory makes $\frac{1}{15}$ of 126 in one day, and the second factory makes $\frac{1}{10}$ of 126.

Hence together they make $\frac{126}{15} + \frac{126}{10}$ in one day. It follows that

$$\frac{126}{n} = \frac{126}{15} + \frac{126}{10}.$$

This is called a **fractional equation** because it contains an unknown in the denominator.

In this chapter you will learn enough about fractions to enable you to solve this and other fractional equations. The chapter is a review of what you have learned about fractions in arithmetic and extends the study of fractions further.

You know from arithmetic that one has to be able to perform the following operations with fractions:

1. To change fractions to the simplest form.
2. To multiply fractions.
3. To divide fractions.
4. To add and subtract fractions.

All four processes are as essential in algebraic work as in arithmetical work. The first is especially important because in all operations the final result is generally changed to the simplest form.

106. Why you study factoring. In changing an *arithmetical* fraction, as $\frac{126}{15}$, into the simplest form and in adding fractions, as in finding such sums as $\frac{126}{15} + \frac{126}{10}$, it is necessary to find the divisors (factors) of numbers. Certain principles have been found helpful in discovering them. They enable you to tell by inspection whether such numbers as 2, 3, and 5 are factors of a number. You must know them thoroughly if you are to find quickly the factors common to the numerator and denominator of a fraction or the least common multiple of the denominators of several fractions which are to be added. Similarly in *algebraic* work you must be familiar with the principles which help you to find the factors of polynomials which occur in algebraic fractions. Hence factoring of polynomials is another important part of the work of this chapter.

CHANGING FRACTIONS TO LOWEST TERMS

107. How fractions are changed to lowest terms. If in the fraction $\frac{6}{15}$ you divide numerator and denominator by the factor 3, which is contained in both, the fraction is said to be *changed* (reduced) *to lower terms. To change a fraction to lowest terms* means to divide numerator and denominator by the *largest* factor contained in both. It involves two steps:

1. *You must factor the numerator and the denominator.*
2. *You must divide the numerator and denominator by the largest factor common to both.*

For example, to change the fraction $\frac{2ac - 3bc}{2ab - 3b^2}$ to lowest terms,

(1) change numerator and denominator to the factored forms $\frac{c(2a - 3b)}{b(2a - 3b)}$ and

(2) divide numerator and denominator by the factor $2a - 3b$.

This changes the fraction to $\frac{c}{b}$.

Briefly the work may be arranged as follows:

$$\frac{2ac-3bc}{2ab-3b^2}=\frac{c\overset{1}{\cancel{(2a-3b)}}}{b\underset{1}{\cancel{(2a-3b)}}}=\frac{c}{b}.$$

In dividing the numerator $c(2a - 3b)$ by $2a - 3b$ the following principle has been applied: *Dividing one factor of a product by a number divides the product by that number.*

You must keep in mind that the quotient found by dividing $2a - 3b$ by itself is 1. To save time, the 1 is usually not written, but it should be understood. The important thing to remember is that you can divide only by a *factor*, never by a term or a factor of a term. It would be wrong in the fraction on page 155 to attempt to divide by 2 or a or b, because they are not *factors* of the numerator and denominator.

EXERCISES

Change each of the following fractions to the simplest form:

1. $\frac{4(x + y)}{12(x + y)}$

Suggestion: Divide the numerator and denominator by 4. Then divide the numerator and denominator by $x + y$.

2. $\frac{7(a - b)}{14(a - b)}$

3. $\frac{4(m + n)^2}{2(m + n)}$

4. $\frac{3x^2yz(a + b)}{2xyz(a + b)}$

5. $\frac{12a^2b}{18a^2b^2}$

6. $\frac{45a^4bc^3}{63a^3b^2c^5}$

7. $\frac{45xy^4z^2}{20y^5z^3}$

8. $\frac{36m^2n^2}{54m^3n^4k}$

9. $\frac{5(a + 2)(a + 7)}{15(a - 2)(a + 7)}$

10. $\frac{2(x^2 - y^2)}{5(x^2 - y^2)}$

POLYNOMIALS HAVING A COMMON FACTOR

108. Factoring a polynomial whose terms have a common factor. If the area of a rectangle (Fig. 50) is $2x + 2y + 2z$, one side may be 2 and the other may be $x + y + z$. This illustrates the fact that $2x + 2y + 2z = 2(x + y + z)$.

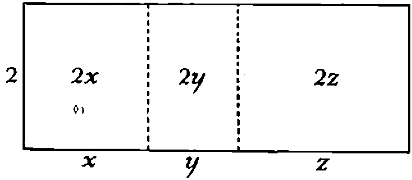

FIGURE 50

If all the terms of a polynomial, as $2x + 2y + 2z$, contain the same (common) factor, then you may say:

(1) that this factor is a factor of the polynomial, and

(2) that the other factor is obtained by dividing each term of the polynomial by the common factor.

Thus 2 is one factor of $2x + 2y + 2z$, and the other factor is $x + y + z$. Why?

Similarly in the polynomial $x^3(x + 1) - 1(x + 1)$ you see that the common factor is $x + 1$. By dividing each of the two terms by $x + 1$ you find the other factor to be $x^3 - 1$.

Hence $x^3(x + 1) - 1(x + 1) = (x + 1)(x^3 - 1)$.

EXERCISES

Factor the following and in each case check the results by multiplying mentally one factor by the other:

1. $15x^2 + 25x + 40$

Solution: You observe that the greatest factor contained in all terms is 5. Dividing by 5, you find the other factor to be

$$3x^2 + 5x + 8.$$

$$\therefore 15x^2 + 25x + 40 = 5(3x^2 + 5x + 8).$$

2. $y^2 + 2y$

Suggestion: You observe that the greatest factor common to both terms is y. By dividing you find the other factor to be $y + 2$.

3. $8m^2 + 24m$

4. $a^2b + ab^2$

5. $6xy^2 - 3x$

6. $4ab^2 - 3a^2b$

7. $16x^2y^2 + 48x^2y - 16xy + 16x$

8. $4x^2y^2 - 6x^2y^3 + 12x^4y^2z^2$

9. $x^5 + x^4 + x^3 + x^2 + x$

10. $14a^7 - 49a^5 + 21a^3 - 7a$

Change each of the following fractions to simplest form:

11. $\dfrac{3ab + 3b^2}{5a^2 + 5ab}$

Solution: First factor the numerator and the denominator. This gives

$$\frac{3ab + 3b^2}{5a^2 + 5ab} = \frac{3b(a + b)}{5a(a + b)}.$$

Then divide the numerator and the denominator by $a + b$, which gives the result $\dfrac{3b}{5a}$.

Hence $\dfrac{3ab + 3b^2}{5a^2 + 5ab} = \dfrac{3b\cancel{(a + b)}}{5a\cancel{(a + b)}} = \dfrac{3b}{5a}$.

12. $\dfrac{2a^2b - 2a^2}{3a^3b - 3a^3}$

13. $\dfrac{x^2y^2 + xy}{(xy + 1)^2}$

14. $\dfrac{5a^2 - 5ab}{5ax + 5ay}$

15. $\dfrac{5x^2y - 5xy - 60y}{3ax^2 - 3ax - 36a}$

MULTIPLYING FRACTIONS

109. Multiplication of arithmetical fractions. Before learning to multiply algebraic fractions you should have clearly in mind the steps taken in multiplying fractions in arithmetic. To multiply two fractions, as $\frac{2}{5}$ and $\frac{3}{7}$, the product of the numerators is divided by the product of the denominators.

Thus $\dfrac{2}{5} \times \dfrac{3}{7} = \dfrac{2 \times 3}{5 \times 7} = \dfrac{6}{35}$.

If in multiplying fractions the resulting product is not in the simplest form, the change should be made before the multiplication is carried out.

Thus $\dfrac{2}{5} \times \dfrac{10}{6} = \dfrac{\overset{1}{\cancel{2}} \times \overset{2}{\cancel{10}}}{\underset{1}{\cancel{5}} \times \underset{3}{\cancel{6}}} = \dfrac{2}{3}$.

Observe carefully that in the multiplication of arithmetical fractions the following two steps are involved:

1. *Multiply the numerators, multiply the denominators, and write the first product over the second.*

In the last example on page 158 this gives $\frac{2 \times 10}{5 \times 6}$.

2. *Change the resulting fraction to simplest form and multiply the remaining factors.* This gives $\frac{2}{3}$.

110. Multiplication of algebraic fractions. The algebraic operation of multiplying fractions is the same as that of multiplying arithmetical fractions. The following example illustrates it:

Multiply $\frac{3a - 3b}{5c - 5d} \times \frac{10ac - 10ad}{6a - 6b}$.

Solution: 1. Multiply the numerator by the numerator and the denominator by the denominator, leaving the operation indicated:

$$\frac{(3a - 3b)(10ac - 10ad)}{(5c - 5d)(6a - 6b)}$$

2. Change the product to the simplest form:

(*a*) Factor the numerator and the denominator:

$$\frac{3(a - b)10a(c - d)}{5(c - d)6(a - b)}$$

(*b*) Divide the numerator and the denominator by factors common to both:

$$\frac{\cancel{3}\cancel{(a-b)}\overset{2}{\cancel{10}}a\cancel{(c-d)}}{\cancel{5}\cancel{(c-d)}\underset{2}{\cancel{6}}\cancel{(a-b)}} = a.$$

Usually the work should be arranged as follows:

$$\frac{3a-3b}{5c-5d}\times\frac{10ac-10ad}{6a-6b}=\frac{(3a-3b)(10ac-10ad)}{(5c-5d)(6a-6b)}$$

$$=\frac{\cancel{3}\cancel{(a-b)}\overset{2}{\cancel{10}}a\cancel{(c-d)}}{\cancel{5}\cancel{(c-d)}\underset{2}{\cancel{6}}\cancel{(a-b)}}=a.$$

EXERCISES

Multiply the following:

1. $\frac{2a}{3}\times\frac{6}{5}$
2. $\frac{3a^2}{2}\times\frac{7}{6a^3}$
3. $\frac{a+b}{2}\times\frac{10}{a-b}$
4. $\frac{3x+3y}{5}\times\frac{3}{x+y}$
5. $\frac{6x+6y}{2ax-2ay}\times\frac{x-y}{x+y}$
6. $\frac{(a+2)(a+3)}{2a+8}\times\frac{a+4}{a+2}$
7. $\frac{2\pi r^2+2\pi rh}{x-y}\times\frac{3hx-3hy}{\pi r^2}$
8. $\frac{(x+2)(x-3)}{(x+3)(x-1)}\times\frac{x+3}{x-3}$
9. $\frac{3a-6b}{4a+2b}\times\frac{6(2a+b)^2}{2(a-2b)}$
10. $\frac{x^4-x^2}{x+2}\times\frac{9(x+2)^2}{3x^2-3}$

THE DIFFERENCE OF TWO SQUARES

111. How to multiply by inspection the sum of two numbers by the difference. The following set of exercises will help you to understand better a second type of factoring. If a and b are any two numbers, the product of $a+b$ and $a-b$ is $a^2+ab-ab-b^2$, or simply a^2-b^2. The binomial $a+b$ is the sum of the two numbers and $a-b$ is the difference of the same two numbers. The square of the first number is a^2, the square of the second is b^2. What is the difference of these squares? Thus, you see, *if the sum of any two* *numbers is multiplied by the difference of the same two numbers, the product is the difference of the squares of the two numbers.*

Multiply by inspection $(2x + 3y)(2x - 3y)$. The factor $(2x + 3y)$ is the sum of two numbers. The factor $(2x - 3y)$ is the difference of the same two numbers. What is the first number? What is the second number? What is the square of the first number? What is the square of the second number? Then $(2x + 3y)(2x - 3y) = 4x^2 - 9y^2$ according to the rule. Verify this result by actual multiplication.

EXERCISES

Multiply by inspection:

1. $(x + y)(x - y)$
2. $(a + 3)(a - 3)$
3. $(2x - 5)(2x + 5)$
4. $(x + \frac{1}{2})(x - \frac{1}{2})$
5. $(2x + y)(2x - y)$
6. $(2m + 3n)(2m - 3n)$
7. $(2a^2 - b)(2a^2 + b)$
8. $(5xy^2z + 2)(5xy^2z - 2)$
9. $(6a^3bc - 7x)(6a^3bc + 7x)$
10. $(4 + 8xy^2z)(4 - 8xy^2z)$

112. How to factor the difference of two squares. You have seen that $a^2 - b^2$ is the result of multiplying $(a + b)$ by $(a - b)$. Hence the *factors of the difference of two squares* may be found by inspection as follows:

1. Find the square root of each square and add the two roots. This gives the first factor $a + b$.

2. To find the second factor, subtract b from a. This gives $a - b$.

Similarly, to find the factors of $x^4 - 9$, find first the square roots of x^4 and 9. They are x^2 and 3. Hence the factors are the sum of x^2 and 3 and the difference of x^2 and 3, that is, $x^2 + 3$ and $x^2 - 3$.

Show that $x^2 - 25$ is the difference of two squares. Show that $64a^2 - 25b^2$ is the difference of two squares.

If each term of the binomial contains a common factor, remove this common factor before factoring as the difference of two squares. Thus $3a^2 - 3b^2 = 3(a^2 - b^2) = 3(a - b)(a + b)$.

EXERCISES

Find the square roots of the following numbers:

1. x^2
2. m^2
3. 25
4. 36
5. y^4
6. $121x^2$
7. $\frac{1}{a^2}$
8. $\frac{1}{b^2}$
9. v^4t^2
10. a^6
11. x^4
12. .25
13. $.25x^2$
14. .01
15. $.01b^2$
16. $(a + b)^2$
17. $(b - c)^2$
18. $(x + 3y)^2$
19. $(3a - 1)^2$
20. $(3b + 2)^2$

Factor each of the following binomials and test the correctness of your factors by multiplying them mentally:

21. $16a^6 - 9b^2$

Solution: The square roots of $16a^6$ and $9b^2$ are $4a^3$ and $3b$. The sum is $4a^3 + 3b$, and the difference is $4a^3 - 3b$.

$$\therefore\ 16a^6 - 9b^2 = (4a^3 + 3b)(4a^3 - 3b).$$

Check: $(4a^3 + 3b)(4a^3 - 3b) = 16a^6 + 12a^3b - 12a^3b - 9b^2$
$= 16a^6 - 9b^2.$

22. $x^2 - 25$
23. $m^2 - 36$
24. $16 - a^2$
25. $v^2 - 1$
26. $v^4 - 1$
27. $v^2 - \frac{1}{4}$
28. $1 - 121x^2$
29. $\frac{1}{a^2} - \frac{1}{b^2}$
30. $64x^2 - 25$
31. $v^4t^2 - s^2$
32. $a^6 - \frac{1}{4}m^2$
33. $x^4 - \frac{4y^2}{9}$
34. $.25x^2 - 4$
35. $a^2 - .01b^2$
36. $\pi r^2 - \pi x^2$
37. $a^4 - b^4$
38. $16aw^2 - at^2$
39. $(a + b)^2 - c^2$
40. $(m - n)^2 - 16$
41. $a^2 - (b - c)^2$
42. $(x + 3y)^2 - (m + 2)^2$
43. $(3b - 2c)^2 - (a + t)^2$
44. $(2x - 1)^2 - (3b + 2)^2$
45. $(3a - c)^2 - (b - d)^2$

Change each of the following fractions to lowest terms by dividing numerator and denominator by the same factor:

46. $\dfrac{a^2 - 1}{(a - 1)^2}$

47. $\dfrac{a^2 - b^2}{(a - b)^3}$

48. $\dfrac{x^2 - 25}{3x - 15}$

49. $\dfrac{a^2 - 64}{am + 8m}$

50. $\dfrac{x^2 - 4y^2}{(x - 2y)^2}$

51. $\dfrac{9 - x^2}{(3 - x)^2}$

52. $\dfrac{(m - n)a - (m - n)b}{a^2 - b^2}$

53. $\dfrac{(a + b)\,(a + c)}{(a^2 - b^2)\,(a^2 - c^2)}$

54. $\dfrac{(a + b)^2 - c^2}{2a + 2b + 2c}$

55. $\dfrac{(4x - t)^2 - (2y - 3z)^2}{(4x - 2y)^2 - (3z - t)^2}$

Multiply as indicated and change each result to the simplest form:

56. $\dfrac{(x + 8)^2}{(x - 4)^2} \cdot \dfrac{x^2 - 16}{x^2 - 64}$

57. $\dfrac{(x - 2)^2}{x^2 + x} \cdot \dfrac{x^2 - 1}{x^2 - 4}$

58. $\dfrac{ab}{a + b} \cdot \dfrac{6a^2b - 4ab^2}{45a^2 - 20b^2} \cdot \dfrac{3a + 2b}{4a^2b^2}$

59. $\dfrac{(3 - 2a)^2}{2a} \cdot \dfrac{3 + 2a}{3 - 2a} \cdot \dfrac{8a^2b}{9 - 4a^2}$

To save time and work in computation, the formulas in Exercises 60 to 63 are stated in factored form. Show that the formulas are true.

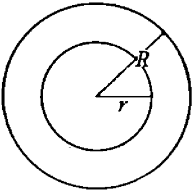

FIGURE 51

60. The area of a circular ring (Fig. 51) is $\pi(R + r)(R - r)$. Find the area of a circular ring whose outer and inner radii are 48 inches and 18 inches.

61. The difference between the areas of two squares (Fig. 52) is $(x + y)(x - y)$. The sides of two squares are 36 feet and 16 feet. Find the difference of the areas.

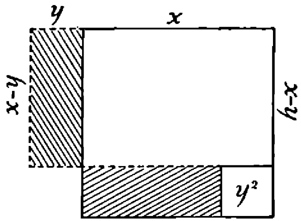

FIGURE 52

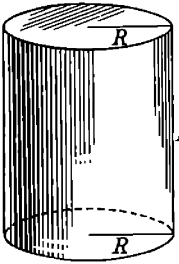

FIGURE 53

62. The total area of a right circular cylinder (Fig. 53) is $2\pi R(R + H)$. Find the total area of a right circular cylinder whose altitude is 15 inches and the radius of whose base is 5 inches.

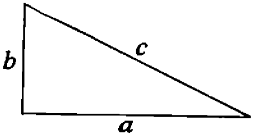

FIGURE 54

63. The side a of a right triangle (Fig. 54) is $\sqrt{(c + b)(c - b)}$. Find the third side of a right triangle if the hypotenuse is 21 inches and one side is 9 inches.

DIVIDING FRACTIONS

113. Dividing by an arithmetical fraction. You know from arithmetic that, when a number, as 25, is to be divided by a fraction, as $\frac{2}{3}$, the divisor is inverted and the number is then multiplied by the inverted divisor.

For example, $25 \div \frac{2}{3} = 25 \times \frac{3}{2} = \frac{25 \times 3}{2} = \frac{75}{2}$.

The process involves two steps:

1. *The divisor is inverted.*
2. *The dividend is multiplied by the inverted divisor. The result is then changed to simplest form.*

$$\text{Thus } \frac{4}{9} \div \frac{2}{3} = \frac{4}{9} \times \frac{3}{2} = \frac{\overset{2}{\cancel{4}} \times \overset{1}{\cancel{3}}}{\underset{3}{\cancel{9}} \times \underset{1}{\cancel{2}}} = \frac{2}{3},$$

$$\text{and } \frac{6}{10} \div \frac{3}{15} = \frac{6}{10} \times \frac{15}{3} = \frac{\overset{\cancel{2}}{\cancel{6}} \times \overset{3}{\cancel{15}}}{\underset{\cancel{2}}{\cancel{10}} \times \cancel{3}} = 3.$$

114. Dividing by an algebraic fraction. The process of dividing by an algebraic fraction is the same as that of dividing by an arithmetical fraction. Each of the following three examples illustrates the steps:

1. $$\frac{6x}{3y} \div \frac{8x}{15y^2} = \frac{6x}{3y} \times \frac{15y^2}{8x} = \frac{\overset{\cancel{2}}{\cancel{6x}} \cdot 15\overset{y}{\cancel{y^2}}}{\underset{1}{\cancel{3y}} \cdot \underset{4}{\cancel{8x}}} = \frac{15y}{4}.$$

2. $$\frac{x^2 + 2x}{x^2 - 2x} \div \frac{(x + 2)^2}{(x - 2)^2} = \frac{x^2 + 2x}{x^2 - 2x} \cdot \frac{(x - 2)^2}{(x + 2)^2}$$

$$= \frac{\cancel{x(x + 2)}(x - 2)^{\cancel{2}}}{\cancel{x(x - 2)}(x + 2)^{\cancel{2}}} = \frac{x - 2}{x + 2}.$$

3. $$\frac{m^2a + ma^2}{m^2b + b^3} \div \frac{3m + 3a}{m^4 - b^4} = \frac{(m^2a + ma^2)(m^4 - b^4)}{(m^2b + b^3)(3m + 3a)}$$

$$= \frac{ma\cancel{(m + a)}\cancel{(m^2 + b^2)}(m^2 - b^2)}{b\cancel{(m^2 + b^2)}3\cancel{(m + a)}} = \frac{ma(m^2 - b^2)}{3b}.$$

Explain the steps in the three examples.

EXERCISES

Divide as indicated:

1. $\frac{3x}{2y} \div \frac{4x}{9y}$

2. $\frac{6a^2}{5b^4} \div \frac{9a^3}{2b}$

3. $\frac{x+y}{2} \div \frac{x-y}{4}$

4. $\frac{12a}{a+b} \div \frac{15a^2}{a+b}$

5. $\frac{(a+b)(a-b)}{4} \div \frac{3(a+b)}{8}$

6. $\frac{(x-y)^2}{2} \div \frac{(x-y)}{6}$

7. $\frac{(x+3)^2}{x^2-9} \div \frac{x+3}{x-3}$

8. $\frac{(x+4y)^2}{x^2+2xy} \div \frac{x^2+4xy}{(x+5y)(x+2y)}$

9. $\frac{a^2-b^2}{4a} \div \frac{a+b}{6a^2b}$

10. $\frac{4a+b}{3x} \div \frac{16a^2-b^2}{12x^2y^3}$

11. $\frac{a^4-b^4}{a^3-ab^2} \div \frac{a^2-b^2}{a^2+b^2}$

12. $\frac{a^2-121}{a^2-4} \div \frac{a+11}{a+2}$

13. $\frac{4x^2-9y^2}{x^2-4} \div \frac{2x-3y}{x-2}$

14. $\frac{4x^2-25y^2}{16x^2-9y^2} \div \frac{2xy-5y^2}{4x^2+3xy}$

15. $\frac{a^2b+ab^2}{a^2b+b^3} \div \frac{5ab(a+b)^2}{a^4-b^4}$

16. $\frac{xy^2+x^2y}{x^3+xy^2} \div \frac{2xy(x+y)}{x^4-y^4}$

FACTORING QUADRATIC TRINOMIALS

115. Multiplying two binomials of the form $ax + b$. The following set of exercises will help you to understand better a third type of factoring. If you multiply two binomials, as $2x + 3$ and $5x + 7$, the work may be arranged as follows:

$$\begin{array}{r}
2x + 3 \phantom{{}+ 21} \\
5x + 7 \phantom{{}+ 21} \\
\hline
10x^2 + 15x \phantom{{}+ 21} \\
+ 14x + 21 \\
\hline
10x^2 + 29x + 21
\end{array}$$

Thus the product $10x^2 + 29x + 21$ is a **quadratic** trinomial (containing terms of the second degree) whose factors are $2x + 3$ and $5x + 7$.

Notice that the coefficient of x^2 is the product of 2 and 5, that the number 21 is the product of 3 and 7, and that the coefficient of x is the *sum* of the two cross products, 2×7 and 3×5. This will enable you to find the product by inspection.

EXERCISES

Multiply the following by inspection:

1. $(a + 2)(a + 3)$
2. $(x - 5)(x - 2)$
3. $(m + 8)(m - 3)$
4. $(b - 11)(b + 1)$
5. $(m + 2)(m + 3)$
6. $(2a - 5)(3a - 2)$
7. $(6x + 2)(2x - 5)$
8. $(x^2 + 1)(x^2 + 4)$
9. $(a + 7b)(a - b)$
10. $(3ab - x)(ab + 2x)$
11. $(\frac{1}{2}x + y)(\frac{1}{3}x - 2y)$
12. $(2a^2 + 3b)(a^2 - 4b)$
13. $(5xy - 3z)(2xy + z)$
14. $(4m^2 + 3n^2)(2m^2 - 5n^2)$

116. Factoring trinomials by trial. Suppose that the trinomial $10x^2 + 29x + 21$ is given and that the factors are to be found. This means that you must know how to determine the 2 and 3 of the factor $2x + 3$ and the 5 and 7 of $5x + 7$. This may be done as follows:

Write two factors of the coefficient of x^2, as 2 and 5, and two factors of the third term, as 3 and 7:

$$\boxed{\begin{array}{ccc} 2 & & 3 \\ & \times & \\ 5 & & 7 \end{array}}$$

If the sum of the cross products is 29, the required numbers have been found. If not, other factor pairs of

10 and 21 have to be tried. The following examples illustrate further how to find the factors of a quadratic trinomial:

1. Factor $6x^2 + 7x + 2$.

Solution: Two factors of 6 are either 6 and 1 or 2 and 3. The factors of 2 are 2 and 1 or 1 and 2. By trial select

$$\boxed{\begin{array}{ccc} 2 & & 1 \\ & \times & \\ 3 & & 2 \end{array}}$$

because the sum of the cross products is 7. It follows that $6x^2 + 7x + 2 = (2x + 1)(3x + 2)$.

2. Factor $9r^2 - 6r - 8$.

Solution: Two factors of 9 are 3 and 3 or 9 and 1. Two factors of -8 are 8 and -1 or 1 and -8; 2 and -4 or 4 and -2. By trial select

$$\boxed{\begin{array}{ccc} 3 & & 2 \\ & \times & \\ 3 & & -4 \end{array}}$$

because the sum of the cross products is -6.

Hence $9r^2 - 6r - 8 = (3r + 2)(3r - 4)$.

3. Factor $112m^2 + 49m^4 + 64$.

Solution: In this trinomial it is necessary to rearrange the terms, which gives $49m^4 + 112m^2 + 64$.

By trial select the numbers:

$$\boxed{\begin{array}{ccc} 7 & & 8 \\ & \times & \\ 7 & & 8 \end{array}}$$

$$\begin{aligned} \text{Hence } 49m^4 + 112m^2 + 64 &= (7m^2 + 8)\ (7m^2 + 8) \\ &= (7m^2 + 8)^2. \end{aligned}$$

EXERCISES

Factor each of the following:

1. $2x^2 - 3x - 5$
2. $6a^2 - 7a - 5$
3. $a^2 + 6a + 8$
4. $x^2 + 7x - 18$
5. $y^2 - 8y + 12$
6. $b^2 - 3b - 10$
7. $10x^2 - 13x - 30$
8. $6x^2 + 17x + 7$
9. $2x^2 - 3x - 2$
10. $4x^2 + 20x + 25$
11. $35x^2 - 39x - 36$
12. $4x^4 - 8x^2 + 4$
13. $5x^2 + 18xy + 16y^2$
14. $64a^2 + 16ab + b^2$
15. $x^4y^2 + 2x^2yz + z^2$
16. $9x^6 - 6x^3 - 8$
17. $a^2b^2 - 9ab - 22$
18. $10x^2 - 7xy - 12y^2$
19. $10r^2 + 3rs - 18s^2$
20. $27x^2 - 42xy - 24y^2$
21. $16y^2 + 5y^4 + 3$
22. $1 - 6xy + 5x^2y^2$

Multiply or divide as indicated:

23. $\dfrac{x^2 - 7x + 12}{x - 1} \times \dfrac{x^2 - 1}{x^2 - 16}$

24. $\dfrac{a^2 - b^2}{ab + 4b^2} \div \dfrac{a^2 + 4ab + 5b^2}{2b}$

25. $\dfrac{3b^2 + 10bx + 3x^2}{3b^2 - 10bx + 3x^2} \div \dfrac{(3b + x)x}{(3b - x)^2}$

26. $\dfrac{x^2 - 4}{x^3 + 3x^2} \times \dfrac{x^3 - x^2 - 12x}{x^2 - 4x + 4}$

27. $\dfrac{x^2 - 5xy - 14y^2}{x^2 + 5xy - 24y^2} \div \dfrac{x^2 - 3xy - 28y^2}{x^2 - 8xy + 15y^2}$

28. $\dfrac{a^2b + 10ab + 21b}{a^3 - 4a^2 + 3a} \div \dfrac{a^2b^3 - 9b^3}{a^3 - a^2}$

ADDING AND SUBTRACTING FRACTIONS

117. How fractions are added or subtracted in arithmetic. Since the operations of adding and subtracting algebraic fractions are the same as those used with arithmetical fractions, you should recall first what is done when fractions are added in arithmetic.

The simplest case is that of adding fractions having the same denominators, as $\frac{5}{8} + \frac{1}{8}$. The problem is not different from that of adding 5 marbles and 1 marble, or 5 oranges and 1 orange, or 5 dollars and 1 dollar.

Thus, as 5 marbles + 1 marble = 6 marbles,
and as 5 dollars + 1 dollar = 6 dollars,
so you have 5 eighths + 1 eighth = 6 eighths,
or $\frac{5}{8} + \frac{1}{8} = \frac{6}{8}$.

Briefly, *to add (or subtract) fractions having the same denominator, first add (or subtract) the numerators and then divide the result by the common denominator.*

118. Addition and subtraction of algebraic fractions having the same denominator. By the method of § 117 the fractions $\frac{5}{a}$ and $\frac{1}{a}$ may be added by adding the numerators 5 and 1 and dividing the sum by the denominator a, that is,

$$\frac{5}{a} + \frac{1}{a} = \frac{6}{a}.$$

Similarly $\frac{5}{a} - \frac{1}{a} = \frac{4}{a}.$

EXERCISES

Perform the following additions and subtractions orally:

1. $\frac{2}{x} + \frac{8}{x}$
2. $\frac{6}{y} - \frac{2}{y}$
3. $\frac{a}{b} + \frac{c}{b}$
4. $\frac{a}{c} + \frac{b}{c}$
5. $\frac{3}{a+1} + \frac{4}{a+1}$
6. $\frac{a}{m+n} - \frac{b}{m+n}$
7. $\frac{x}{a+b} + \frac{x-y}{a+b}$
8. $\frac{m}{x+y} - \frac{m-n}{x+y}$
9. $\frac{a}{a-1} - \frac{ab}{a-1}$
10. $\frac{x}{x+y} + \frac{y}{x+y}$
11. $\frac{2a-b}{x} - \frac{a+2b}{x}$
12. $\frac{5+3c}{a+b} - \frac{1-c}{a+b}$

119. Addition and subtraction of algebraic fractions having different denominators. As in arithmetic, fractions with different denominators must be changed to fractions having the same denominator before they can be combined.

To illustrate: to add $\frac{3}{7}$ and $\frac{5}{14}$, you change $\frac{3}{7}$ to $\frac{6}{14}$ and then combine $\frac{6}{14}$ and $\frac{5}{14}$.

Thus $\frac{3}{7} + \frac{5}{14} = \frac{6}{14} + \frac{5}{14} = \frac{11}{14}$.

Similarly $\dfrac{a}{x} + \dfrac{b}{2x} = \dfrac{2a}{2x} + \dfrac{b}{2x} = \dfrac{2a + b}{2x}$.

The following examples further illustrate the process:

1. $\dfrac{1}{a^3} + \dfrac{1}{a^2} + \dfrac{1}{a}$

Solution: Note that a^3 is the least common multiple of the denominators. Hence multiply numerator and denominator of $\dfrac{1}{a^2}$ by a and of $\dfrac{1}{a}$ by a^2.

Thus $\dfrac{1}{a^3} + \dfrac{1}{a^2} + \dfrac{1}{a} = \dfrac{1}{a^3} + \dfrac{a}{a^3} + \dfrac{a^2}{a^3}$.

Adding the numerators, you have the result $\dfrac{1 + a + a^2}{a^3}$.

2. $\dfrac{1}{4} + \dfrac{2x}{3y} - \dfrac{5}{xy}$

Solution: The least common multiple of the denominators is $12xy$.

$$\text{Hence } \frac{1}{4} + \frac{2x}{3y} - \frac{5}{xy} = \frac{1 \cdot 3xy}{12xy} + \frac{2x \cdot 4x}{12xy} - \frac{5 \cdot 12}{12xy}$$

$$= \frac{3xy + 8x^2 - 60}{12xy}.$$

3. $x + \frac{3}{y} + \frac{1}{xy}$.

Solution: Consider the term x as a fraction with a denominator equal to 1. Change the fractions to the same denominator.

$$\text{Then } x + \frac{3}{y} + \frac{1}{xy} = \frac{x}{1} + \frac{3}{y} + \frac{1}{xy}$$

$$= \frac{x \cdot xy}{xy} + \frac{3 \cdot x}{xy} + \frac{1}{xy} = \frac{x^2y + 3x + 1}{xy}.$$

120. Finding the least common multiple of numbers. Before you can add or subtract fractions easily, you must have practice finding the least common multiple of numbers. A multiple of a number, you remember, is a number which can be divided by the given number without a remainder. Thus 6, 9, 12, etc., are multiples of 3; and $a^2 - b^2$, $a^2 - 2ab + b^2$, etc., are multiples of $a - b$. A common multiple of two numbers is a number which can be divided by both numbers without a remainder. For example, 6, 12, 18, etc., are common multiples of 3 and 2. The least common multiple of two or more numbers is the *smallest* number which can be divided by all the numbers without a remainder. What is the least common multiple of 2, 3, and 5?

In many cases the least common multiple of numbers can be found by inspection. You know at once that the least common multiple of 3 and 4 is 12, that the least common multiple of 6 and 8 is 24, and that the least common multiple of $x + 1$ and $x + 2$ is $(x + 1)(x + 2)$.

Sometimes it is necessary to factor expressions before you can find the least common multiple. Suppose that you wish to find the least common multiple of

$ab - b^2$ and $a^2 - b^2$. First factor these two expressions.

$$ab - b^2 = b(a - b).$$
$$a^2 - b^2 = (a - b)(a + b).$$

You can now see that the least common multiple is $b(a - b)(a + b)$.

EXERCISES

Find the least common multiples of these numbers:

1. $5b, b$
2. $a, 5a$
3. $4, x + 1$
4. $a - 1, a(a - 1)$
5. $a + 2, a - 2$
6. $m + 2, m - 2$
7. $x^2 - 9, x + 3$
8. $2a - 3b, 4a^2 - 6ab$
9. $a^2 + 5ab + 6, a + 2$
10. $x - y, x + y, x^2 - y^2$

Change the following fractions to equivalent fractions having common denominators:

11. $\frac{a}{5b}, \frac{2}{b}$
12. $\frac{3}{4}, \frac{x}{x + 1}$
13. $\frac{a}{a - 1}, \frac{b}{a(a - 1)}$
14. $\frac{4}{a + 2}, \frac{3}{a - 2}$
15. $\frac{3}{a^2 + 5a + 6}, \frac{a}{a + 2}$
16. $\frac{3}{5x}, \frac{2x}{x - y}, \frac{2y}{x + y}$

Add and subtract as indicated:

17. $\frac{a}{5b} - \frac{2}{b}$
18. $\frac{1}{a} - \frac{1}{5a}$
19. $\frac{3}{4} + \frac{x}{x + 1}$
20. $8 + \frac{1}{a}$
21. $\frac{a^2}{b^2} + 1$
22. $\frac{(x - y)^2}{2xy} - 1$
23. $\frac{a}{a - 1} - \frac{b}{a(a - 1)}$
24. $\frac{4}{a + 2} + \frac{3}{a - 2}$
25. $\frac{3m}{m + 2} + \frac{4}{m - 2}$
26. $\frac{6}{x^2 - 9} + \frac{2}{x + 3}$

27. $\dfrac{1}{2a-3b}+\dfrac{a+b}{4a^2-6ab}$ **29.** $\dfrac{1}{x-y}+\dfrac{1}{x+y}+\dfrac{1}{x^2-y^2}$

28. $\dfrac{3}{a^2+5a+6}-\dfrac{a}{a+2}$ **30.** $\dfrac{3}{5x}+\dfrac{2x}{x-y}-\dfrac{2y}{x+y}$

31. $\dfrac{x}{x^2-1}+\dfrac{x+3}{x-1}-\dfrac{x-3}{x+1}$

32. $\dfrac{2}{5a+10b}-\dfrac{7}{3a+6b}+\dfrac{9}{2a+4b}$

33. $\dfrac{2}{x^2-10x+21}+\dfrac{1}{x-7}+\dfrac{2}{x-3}$

34. $a-3+\dfrac{a^3-27}{a^2+3a+9}$ **35.** $\dfrac{2x^2-3y^2}{9x+12y}-\dfrac{2x+y}{9x^2-16y^2}$

36. $\dfrac{1}{m^2-4m-5}-\dfrac{1}{m^2-6m+5}$

37. $\dfrac{x^2-y^2}{(x+y)^2}+\dfrac{x-y}{x+y}-\dfrac{x^2+y^2}{x^2-y^2}$

38. $\dfrac{2x+3}{x-6}-\dfrac{x^2-11x+18}{x^2-36}-\dfrac{x-6}{x+6}$

121. Multiplying the numerator and denominator by −1. Consider the exercise $\dfrac{2}{a-b}+\dfrac{3}{b-a}$. You will note that the denominators are nearly the same and that the second denominator could be made the same as the first by multiplying it by −1. If you multiply the denominator of a fraction by −1 and wish to keep the value of the fraction the same, by what must you multiply the numerator? Thus the fraction $\dfrac{3}{b-a}$ can be changed to the equivalent fraction $\dfrac{-3}{a-b}$. Therefore

$$\frac{2}{a-b}+\frac{3}{b-a}=\frac{2}{a-b}+\frac{-3}{a-b}=\frac{-1}{a-b}.$$

EXERCISES

Multiply the following numbers by -1:

1. -3	**3.** $-a$	**5.** $a - b + c$	**7.** $b^2 - a^2$
2. -4	**4.** $-xy^2$	**6.** $y - x$	**8.** $4b^2 - 9a^2$

Change the following fractions to equivalent fractions with positive denominators:

9. $\dfrac{3}{-2}$

10. $\dfrac{-5}{-6}$

11. $\dfrac{ab}{-5}$

12. $\dfrac{x - y}{-7}$

13. Change to an equivalent fraction whose denominator is $x - y$: $\dfrac{2x}{y - x}$.

14. Change to an equivalent fraction whose denominator is $a - b$: $\dfrac{a + b}{b - a}$.

15. Change to an equivalent fraction whose denominator is $4m^2 - n^2$: $\dfrac{2m}{n^2 - 4m^2}$.

Add or subtract as indicated:

16. $\dfrac{3}{a - b} + \dfrac{2}{b - a}$

17. $\dfrac{2x^2}{x - y} - \dfrac{-3}{y^2 - x^2}$

18. $\dfrac{2x}{3x - 2y} + \dfrac{x}{2x - 3y}$

19. $\dfrac{5x}{x + y} + \dfrac{xy}{x^2 - y^2} - \dfrac{4y}{y - x}$

THE USE OF FACTORING IN SOLVING EQUATIONS

122. Solving equations by factoring. Let it be required to solve the equation $x^2 - 16 = 0$.

Factoring the left member, you have

$$(x + 4)(x - 4) = 0.$$

This equation is satisfied if $x + 4 = 0$, or if $x - 4 = 0$, or if both factors are zero. Thus the value of x which makes a factor equal to zero also satisfies the original equation $x^2 - 16 = 0$.

To find this value, put a factor equal to zero, as

$$x + 4 = 0.$$

It follows that $x = -4$.

Hence $x_1 = -4$ satisfies the original equation.

Similarly from $x - 4 = 0$, you have the solution

$$x_2 = 4.$$

The preceding solution of the equation $x^2 - 16 = 0$ may now be briefly arranged as follows:

Given: $x^2 - 16 = 0$.

Solution: By factoring, $(x + 4)(x - 4) = 0$.

Let $x + 4 = 0$,

and $x - 4 = 0$.

Then $x_1 = -4$,

and $x_2 = 4$.

EXERCISES

Solve the following equations by factoring:

1. $4x^2 - 9 = 0$.

Solution: $4x^2 - 9 = 0$.

By factoring, you have $(2x + 3)(2x - 3) = 0$.

Let $2x + 3 = 0$.

Then $x_1 = -\frac{3}{2}$.

Let $2x - 3 = 0$.

Then $x_2 = \frac{3}{2}$.

Check for $x_1 = -\frac{3}{2}$:

$$4(-\tfrac{3}{2})^2 - 9 = 4\cdot\tfrac{9}{4} = 9.$$

Check for $x_2 = +\frac{3}{2}$:

$$4(\tfrac{3}{2})^2 - 9 = 4\cdot\tfrac{9}{4} = 9.$$

2. $16y^2 - 25 = 0.$
3. $225a^2 - 9 = 0.$
4. $49x^2 - 16 = 0.$
5. $9x^2 - 4 = 0.$
6. $y^2 - 25 = 0.$
7. $2a^2 - 50 = 0.$

8. $x^2 + 6x = 0.$

Solution:
$$\begin{aligned} x^2 + 6x &= 0. \\ x(x + 6) &= 0. \\ x_1 &= 0. \\ x + 6 &= 0. \\ x_2 &= -6. \end{aligned}$$

9. $2y^2 - 10y = 0.$
10. $3x^2 = 5x.$
11. $d^2 - 7d = 0.$
12. $24r^2 - 2r = 0.$
13. $x^2 + \frac{5x}{2} = 0.$
14. $3x^2 + 7x = 0.$

15. $x^2 - 13x + 12 = 0.$

Solution:
$$\begin{aligned} x^2 - 13x + 12 &= 0. \\ (x - 12)(x - 1) &= 0. \\ x - 12 &= 0. \\ x - 1 &= 0. \\ \therefore x_1 &= 12, \\ \text{and } x_2 &= 1. \end{aligned}$$

16. $x^2 - 11x + 30 = 0.$
17. $x^2 + 45 = 14x.$
18. $x^2 - 8x + 15 = 0.$
19. $x^2 - 4x - 21 = 0.$
20. $a^2 = 6 + a.$
21. $9y^2 - 12y + 4 = 0.$
22. $x^2 + 8x - 48 = 0.$
23. $3 - x^2 = - 2x.$
24. $81x^2 + 18x + 1 = 0.$
25. $50x + 24 = 25x^2.$
26. $9x^2 + 3x - 2 = 0.$
27. $12x - 28 = - x^2.$
28. $x^2 + 20 = 9x.$
29. $y^2 + 48 = 16y.$
30. $9x^2 + 34x - 8 = 0.$
31. $4a^2 + 5a - 9 = 0.$
32. $7x^2 + 5x - 2 = 0.$
33. $8x^2 - 22x + 15 = 0.$
34. $6a^2 - 7a - 20 = 0.$
35. $3x^2 - 11x - 20 = 0.$
36. $2x^2 + 5x + 2 = 0.$
37. $3y^2 + 7 = 22y.$
38. $3x^2 + 4x + 1 = 0.$
39. $6x^2 + 47x + 35 = 0$
40. $6y^2 + y = 35.$
41. $2x^2 = 9x + 35.$

REVIEW EXERCISES

123. Practice exercises for review. The following exercises review the processes taught in the preceding chapters.

EXERCISES

Solve the following equations:

1. $.3x + .4 = .9.$

2. $3.7y - 7.4 = 11.1.$

3. $\frac{3x}{.04} = 6.3.$

4. $2.5a - 5 = .5a - 2.$

5. $.5x + .2 = .6 + .1x.$

6. $\frac{1}{3}a + \frac{2}{3} = \frac{1}{3} + \frac{1}{2}a.$

7. Find the value of $x^2 + y^2 + z^2 + a^2$ when $x = \frac{1}{3}$, $y = \frac{2}{5}$, $z = 2\frac{2}{3}$, $a = 3.75$.

Perform the indicated operations:

8. $10 + (-6)$

9. $-8 - (-3)$

10. $-3x - (5x)$

11. $2\frac{1}{2}a + (-3\frac{2}{3}a)$

12. $3mn(-\frac{4}{3}m^2)$

13. $x(-3x^2y)(-2xyz)$

14. $(-\frac{3}{4}a^2bc^2) \div (-\frac{1}{2}abc^2)$

15. $(-144xt) \div (12x)$

16. Combine these terms: $2ax - 4ax + 13ax - 5ax.$

17. Add $4x^3 - 3x^2 - 6x + 7$ to $3x^3 - 4x^2 - 8x + 5.$

18. Subtract $a - b + c$ from $4a - 6c + 4b.$

19. Simplify $3x + 4(2x - 1) + \frac{2}{3}.$

20. Multiply $(3y^2 + 2y - 4)$ by $(5 - y).$

21. Expand $(2a - 3b)^2$; $(3x - 4)^3.$

22. Divide $-4x^2 + 3x$ by $-x.$

23. Divide $(b^3 - 3b^2 + 2b - 6)$ by $(b^2 + 2).$

24. Solve for a the equation $\frac{1}{a} = \frac{1}{10} + \frac{1}{15}.$

25. Solve for x: $\frac{8}{x} = \frac{25}{32}.$

26. Make a graph of the equation $3x - 2y + 2 = 0$.

27. Solve by graphs:

$$\begin{aligned} x + 2y &= 3, \\ 2x + y - 3 &= 0. \end{aligned}$$

28. Solve algebraically:

$$\begin{aligned} 2a - 3b &= 12, \\ 3a + 5b &= 8. \end{aligned}$$

29. Eliminate a from the equations

$$\begin{aligned} l &= a + 16, \\ 256 &= \tfrac{3}{2}(a + l). \end{aligned}$$

SUPPLEMENTARY WORK WITH FRACTIONS AND FACTORING[1]

124. Factoring the sum or difference of two cubes. Multiplying $a + b$ by $a^2 - ab + b^2$, you have

$$\begin{aligned} (a + b)(a^2 - ab + b^2) &= a^3 - a^2b + ab^2 + a^2b - ab^2 + b^3 \\ &= a^3 + b^3. \end{aligned}$$

Similarly

$$\begin{aligned} (a - b)(a^2 + ab + b^2) &= a^3 + a^2b + ab^2 - a^2b - ab^2 - b^3 \\ &= a^3 - b^3. \end{aligned}$$

The binomials $a^3 + b^3$ and $a^3 - b^3$ are called the *sum of two cubes* and the *difference of two cubes.*

It is interesting to find the factors of binomials in the form of the sum or of the difference of two cubes. The multiplication above shows that the factors of $a^3 + b^3$ are $a + b$ and $a^2 - ab + b^2$, and that the factors of $a^3 - b^3$ are $a - b$ and $a^2 + ab + b^2$.

The following statements explain the method of finding the factors of binomials of these two forms:

1. To find the first factor, $a + b$, of $a^3 + b^3$, take the cube root of each of the terms a^3 and b^3 and then add them.

---

[1] §§ 124 and 125 are intended for those who wish to do more than the required work. Topics on fractions are discussed still further in Chapter XI.

2. Then derive the second factor from the first factor by squaring the first term, subtracting the product of the two terms, and adding the square of the second term.

For example, to find the factors of $27a^3 + 64$, first note that this is the sum of two cubes.

To find the first factor, take the cube root of $27a^3$ and the cube root of 64 and add the results. This gives $3a + 4$.

To find the second factor, square the $3a$, subtract the product $(3a)(4)$, and add the square of 4. This gives $9a^2 - 12a + 16$.

Hence $27a^3 + 64 = (3a + 4)(9a^2 - 12a + 16)$.

To find the factors of the difference of two cubes, as $27x^3 - 64$, proceed similarly as follows:

1. Find the cube roots of $27x^3$ and 64 and take the difference. This gives the factor $3x - 4$.

2. Square the first term of this factor, add the product of the two terms, and add the square of the second term.

Hence $27x^3 - 64 = (3x - 4)(9x^2 + 12x + 16)$.

Note the following: When a binomial is the *sum* of two cubes, the *sum* of the cube roots is taken as the first factor; in the second factor their product is *subtracted.*

For the *difference* of two cubes, the *difference* is taken as the first factor, and the product in the second factor is *added.*

EXERCISES

Find the cube roots of the following numbers:

1. 8
2. 27
3. 64
4. $\frac{1}{8}$
5. a^3
6. x^3
7. $8x^3$
8. $64y^3$
9. a^6
10. b^6
11. x^9
12. y^9
13. $a^3b^6c^9$
14. $8d^6$
15. $(x + y)^3$
16. .008

Factor each of the following and test the results by multiplying the factors mentally:

17. $a^3 + 27$

18. $x^3 - 64$

19. $x^3 - 64y^3$

20. $8x^3 - 1$

21. $y^3 + \frac{1}{8}$

22. $3x^3 + 24$

23. $1 - 64a^6$

24. $x^6 - y^6$

25. $x^3 + 8y^3z^3$

26. $512a^3 - b^6$

27. $a^6 + b^6$

28. $.008a^3 - b^3$

29. $40x^3 - 5$

30. $x^4 - 8a^3x$

31. $a^9 - b^9$

32. $\frac{x^3}{27} - y^3$

33. $a^3b^6c^9 + 8d^6$

34. $(x + y)^3 - 27$

Change the following fractions to lowest terms:

35. $\frac{x^3 - 1}{x^2 + x + 1}$

36. $\frac{4a^2b^2 + 1}{64a^6b^6 + 1}$

37. $\frac{m^2 + mn}{m^3 + n^3}$

38. $\frac{x^6 - y^6}{x^4 - y^4}$

Multiply and divide as indicated:

39. $\frac{y^2 - 9}{y^3 - 27} \cdot \frac{y^2 + 3y + 9}{y + 3}$

40. $\frac{a^3 - 3ab}{a^3 - b^3} \div \frac{a^2 - 10ab + 21b^2}{a^2 + ab + b^2}$

41. $\frac{a^3 + 8b^3}{a^3 - 8b^3} \cdot \frac{a - 2b}{a + 2b} \cdot \frac{a^2 + 2ab + 4b^2}{a^2 - 2ab + 4b^2}$

42. $\frac{a^6 - b^6}{(a - b)^2} \div \frac{a^2 + ab + b^2}{a - b}$

125. Miscellaneous exercises on fractions. The following exercises are more difficult and complicated than those given in the previous sections. Before attempting to carry out the operations, study each carefully and select the best method of work. When there are brackets containing parentheses, it is usually best to perform the operations in the parentheses first.

EXERCISES

Perform the indicated operations:

1. $$\left[\frac{1}{mx}\left(\frac{m}{x}+\frac{x}{m}\right)\div\left(\frac{m^6+x^6}{m^3x^3}\right)\right]\left(m^2-x^2+\frac{x^4}{m^2}\right)\left(\frac{am+mx}{cx-ax}\right)$$

Solution:

$$\left[\frac{1}{mx}\left(\frac{m^2+x^2}{mx}\right)\div\frac{m^6+x^6}{m^3x^3}\right]\frac{m^4-m^2x^2+x^4}{m^2}\cdot\frac{am+mx}{cx-ax}$$

$$=\frac{(m^2+x^2)\cdot m^3x^3\cdot(m^4-m^2x^2+x^4)(am+mx)}{mx\cdot mx\cdot(m^6+x^6)\cdot m^2\cdot(cx-ax)}$$

$$=\frac{\cancel{(m^2+x^2)}\quad\cancel{m^3}\cancel{x^3}\quad\cancel{(m^4-m^2x^2+x^4)}\,\cancel{m}(a+x)}{\cancel{m}\cancel{x}\cdot\cancel{m}\cancel{x}\cancel{(m^2+x^2)}\cancel{(m^4-m^2x^2+x^4)}\cancel{m^2}\cancel{x}(c-a)}$$

$$=\frac{a+x}{c-a}.$$

2. $\left(\frac{1}{x^2}-\frac{2}{xy}+\frac{1}{y^2}\right)\left(1+\frac{2x}{y-x}\right)\div\left(\frac{y}{x}-\frac{x}{y}\right)$

3. $\left(2a-1+\frac{6a-11}{a+4}\right)\div\left(a+3-\frac{3a+17}{a+4}\right)$

4. $\left(x-1+\frac{5}{x+1}\right)\left(\frac{x^2}{6}+\frac{x}{4}+\frac{1}{12}\right)\div\left(\frac{2x^3+8x}{3x+3}\right)$

5. $\left[\left(\frac{x^3}{y^3}+\frac{y^3}{x^3}\right)\div\left(\frac{x}{y}+\frac{y}{x}\right)\right]\left[y^2-\frac{y^4}{y^2-x^2}\right]$

6. $\left(a^2+\frac{b^4}{a^2-b^2}\right)\left(a^2+b^2\right)\div\left(\frac{a}{a+b}+\frac{b}{a-b}\right)$

7. $\frac{8c^3-1}{9c^2-12c+4}\left(1-\frac{4}{3c+2}\right)\div\left(\frac{2c-1}{9c^2-4}\right)$

8. $\left[\frac{a+b}{a-b}-\frac{a-b}{a+b}+\frac{4b^2}{b^2-a^2}\right]\div\left[\frac{a-b}{a+b}-1\right]$

9. $\frac{r}{(r-s)(r-t)}+\frac{s}{(s-r)(s-t)}+\frac{t}{(t-r)(t-s)}$

Suggestion:

$$\frac{r}{(r-s)(r-t)}+\frac{-s}{\underset{(r-s)}{\cancel{(s-r)}}(s-t)}+\frac{t}{\underset{(r-t)(s-t)}{\cancel{(t-r)(t-s)}}}$$

$$=\frac{r(s-t)-s(r-t)+t(r-s)}{(r-s)(r-t)(s-t)}, \text{ etc.}$$

10. $\frac{x+y}{(x-z)(y-z)}-\frac{y+z}{(x-y)(z-x)}-\frac{x+z}{(z-y)(y-x)}$

11. $\dfrac{\frac{a-1}{a}}{a-\frac{1}{a}}$

Suggestion: 1. Perform the division as indicated; that is, subtract $\frac{1}{a}$ from a, invert the resulting fraction, and multiply.

2. Obtain the result more easily by multiplying numerator and denominator by the least common multiple of the denominators. Thus

$$\frac{\frac{a-1}{a}}{a-\frac{1}{a}}=\frac{\frac{a(a-1)}{a}}{a^2-\frac{a}{a}}=\frac{a-1}{a^2-1}=\frac{1}{a+1}.$$

12. $\dfrac{\frac{1}{a}+\frac{1}{b}}{\frac{1}{b}-\frac{1}{a}}$ **13.** $\dfrac{p+\frac{a}{b}}{q-\frac{c}{d}}$ **14.** $\dfrac{\frac{x+3y}{m-5n}}{\frac{x-3y}{m+5n}}$ **15.** $\dfrac{\frac{a}{\frac{a}{b}+1}}{\frac{b}{\frac{a}{b}-1}}$

Exercises to Test Your Understanding of Chapter VI

126. What you should be able to do. In Chapter VI you have been taught to do the following:

1. To change fractions to lowest terms.
2. To add, subtract, multiply, and divide fractions.
3. To factor polynomials of the following forms:
   - (*a*) Polynomials having a factor common to all terms.
   - (*b*) The difference of two squares.
   - (*c*) Trinomials of the form $ax^2 + bx + c$.

EXERCISES

Factor each of the following:

1. $15xy^2 - 3x$

2. $a^2 - \frac{1}{16}$

3. $(a + b)^2 - 9t^2$

4. $9a^2 - 2a - 7$

5. $3a^2 + 18ab + 24b^2$

6. $4x^2 - 8x + 4$

7. $81a^2 + 36a + 4$

8. $b^2 + 9b - 52$

9. $p^2 - 4p - 192$

10. $9a^2 - 6a^3 + a^4$

11. $a^4 - 8a^2 + 7$

12. $2x^2 - 8xy + 8y^2$

Change each of the following fractions to the simplest form:

13. $\dfrac{3x^2 - 3xy}{3xm + 3xn}$

14. $\dfrac{a^2 - 36}{am + 6m}$

15. $\dfrac{x^2 + x - 6}{x^2 - 2x - 15}$

16. $\dfrac{5a^2b - 5ab - 60b}{8ma^2 - 8ma - 96m}$

Multiply:

17. $\dfrac{a^4 - a^2}{a + 2} \times \dfrac{9(a + 2)^2}{3a^2 - 3}$

18. $\dfrac{(a - 2)^2}{a^2 + a} \cdot \dfrac{a^2 - 1}{a^2 - 4}$

Divide:

19. $\dfrac{x^2 - 121}{x^2 - 4} \div \dfrac{x + 11}{x + 2}$

20. $\dfrac{a^2 - 7a + 12}{a - 1} \div \dfrac{a^2 - 16}{a^2 - 1}$

Add and subtract:

21. $\frac{6}{a^2 - 9} + \frac{2}{a + 3}$

22. $\frac{3}{a^2 + 5a + 6} - \frac{a}{a + 2}$

Solve the following equations:

23. $3a^2 - 75 = 0.$

24. $x^2 - 4x - 21 = 0.$

25. $9a^2 - 12a + 4 = 0.$

127. Timed tests on fractions and factoring. The following test items are typical of the processes taught in Chapter VI. The time limits indicate how long it takes some ninth-grade pupils to do them.

A. *Factoring.* Time: 7 minutes.

1. $2xy - 3y^2$

2. $4x^2 + 20x + 25$

3. $x^2 - 25$

4. $m^2 - 5mn - 14n^2$

5. $8x^2 + 24x$

6. $2\pi r^2 + 2\pi rh$

7. $x^2 - 7x + 12$

8. $v^2 - \frac{1}{4}$

9. $a^3 + a^2 + a$

10. $9x^2 - 16y^2$

11. $\pi r^2 - \pi x^2$

12. $am + mx$

13. $a^2 - 10ab + 21b^2$

14. $x^4 - 25y^4$

15. $5a^2b - 5ab + 60b$

16. $a^2 - 3ab - 28b^2$

17. $a^4 - b^4$

18. $2x^2 - 10x + 8$

19. $2a^2 + 6ab + 5a$

20. $(x + y)^2 - z^2$

B. *Solving equations by factoring.* Time: 5 minutes.

1. $x^2 - x - 6 = 0.$

2. $a^2 + 48 = 16a.$

3. $3a^2 + 4a + 1 = 0.$

4. $2m^2 = 9m + 35.$

C. *Reducing fractions.* Time: 6 minutes.

1. $\frac{12x^2y}{27x^2y^2}$

2. $\frac{6x + 6y}{x + y}$

3. $\frac{m^2a + ma^2}{3m + 3a}$

4. $\frac{-5xy + 5y^2}{-3x^2 + 3xy}$

5. $\frac{x^2 - 7x + 12}{x^2 - 16}$

6. $\frac{9 - a^2}{(3 - a)^2}$

7. $\frac{a^2 - a - 12}{a^3 + 3a^2}$

8. $\frac{x^2 - 5xy - 14y^2}{x^2 - 3xy - 28y^2}$

9. $\frac{a^2b^2 + ab}{(ab + 1)^2}$

D. *Multiplying and dividing.* Time: 12 minutes.

1. $\frac{3x^2}{2y} \cdot \frac{4ay}{x}$
2. $\frac{10r^2s}{9x} \div \frac{5rs}{3x^2}$
3. $\frac{m^2 - 7m + 12}{m - 1} \div \frac{m^2 - 16}{m^2 - 1}$
4. $\frac{3x - 6y}{2(a + b)} \cdot \frac{8(a + b)^2}{5(x - 2y)}$
5. $\frac{x^2 - 3x}{b - 5} \div \frac{x - 3}{b^2 - 7b + 10}$
6. $\frac{(3 - 2b)^2}{2b} \div \frac{9 - 4b^2}{8a^2b}$
7. $\frac{6a}{3b} \cdot \frac{15b^2}{8a}$
8. $\frac{b^2 - 9}{b - 3} \div \frac{b + 3}{b^2 - 6b + 9}$
9. $\frac{a^2 - 4}{a - 4} \cdot \frac{a + 4}{a^2 + 4a + 4}$
10. $\frac{a^2 - 16}{a^2 - 64} \cdot \frac{(a + 8)^2}{4a - 16}$
11. $\dfrac{1 + \frac{a}{b}}{1 + \frac{m}{b}}$
12. $\frac{4a^2 - 9b^2}{a^2 - 4} \cdot \frac{a - 2}{2a - 3b}$

E. *Adding and subtracting.* Time: 10 minutes.

1. $\frac{3}{a} + \frac{m}{a}$
2. $6 + \frac{1}{x}$
3. $\frac{1}{a} + \frac{2a}{a - b} + \frac{b}{a + b}$
4. $\frac{x}{x - 1} - \frac{xy}{x^2 - x}$
5. $\frac{1}{2m - 3x} + \frac{m + x}{4m^2 - 9x^2}$
6. $a + b - \frac{3}{a - b}$
7. $\frac{x}{x + y} + \frac{y}{x - y}$
8. $\frac{5}{m^2 - 9} + \frac{3}{m + 3}$
9. $\frac{m}{m^2 - 1} - \frac{m - 3}{m + 1} + \frac{m + 3}{m - 1}$
10. $\frac{2a + 2b}{a - b} - \frac{a^2}{2a^2 - 2ab}$

CHAPTER VII

QUADRATIC EQUATIONS

WHAT YOU HAVE PREVIOUSLY LEARNED ABOUT QUADRATIC EQUATIONS

128. Quadratic equations were found in studying areas and volumes. The first quadratic equations appeared in such problems as the following: "A field of the form of a square has an area of 100 square rods. Find the length of the side." The equation for solving the problem is $a^2 = 100$. The method of solving is to extract the square root of both members, which gives $a = 10$.

A similar equation occurred in studying the area of a circle. It was shown that the area of a circle is found from the formula $A = \pi r^2$. To find the radius when the area is 164, it is necessary to solve the quadratic equation $\pi r^2 = 164$.

Solution: $\pi r^2 = 164$.

Dividing both members by the coefficient of r^2, you have

$$r^2 = \frac{164}{\pi}.$$

Taking the square root of both members, you have

$$r = \sqrt{\frac{164}{\pi}}.$$

To complete the solution, use $\pi = 3.14$, find the quotient $\frac{164}{\pi}$ to three figures, and extract the square root.

The following exercises give practice in solving problems which lead to equations similar to those just explained.

EXERCISES

1. Find the radius of a circle whose area is 37.5 square inches.

2. We learn in science that the number of feet s through which an object falls in t seconds is given by the formula $s = \frac{1}{2}gt^2$, where $g = 32.16$ approximately. How long will it take a stone to fall 850 feet?

3. The volume of a cylinder is 24 cubic inches. The altitude is 6 inches. Find the radius of the base.

4. The area of the surface of a sphere is 28 square inches. Find the radius.

Solve the following equations:

5. $x^2 - 25 = 0.$

6. $x^2 - 121 = 0.$

7. $2x^2 - 30 = 0.$

8. $6a^2 - 72 = 0.$

9. Solve the equation $s - \frac{1}{2}gt^2 = 0$ for t.

Solution: $s = \frac{1}{2}gt^2$. Why?

Dividing both members of the equation by the coefficient of t^2, you have

$$\frac{s}{\frac{1}{2}g} = t^2,$$

$$\text{or } \frac{2s}{g} = t^2.$$

Taking the square root, you have

$$\sqrt{\frac{2s}{g}} = t.$$

10. Using the result of Exercise 9 as a formula, find t when $s = 8$; 24; 16; 48. In each case use $g = 32.16$.

11. The area of an equilateral triangle is given by the formula $A = \frac{a^2}{4}\sqrt{3}$, where a is the length of a side. Find the length of each side of an equilateral triangle whose area is 620 square feet.

12. Solve the equation $A = \frac{a^2}{4}\sqrt{3}$ for a, using the method explained in Exercise 9.

13. The volume of a right circular cylinder is given by the formula $V = \pi r^2 h$. Solve the equation for r.

14. The area of the surface of a sphere is found from the formula $S = 4\pi r^2$. Solve the equation for r.

15. Solve the equation $18 = \frac{6v^2}{r}$ for v.

16. The sides of a right triangle are a and b, and the hypotenuse is $2b$. Find a in terms of b.

129. The relation between the sides of a right triangle. The following problem shows how to use the quadratic equation when it is required to find a side of a right triangle, having given the other two sides.

The dimensions of a room are 12 feet and 9 feet. How far apart are two diagonally opposite corners?

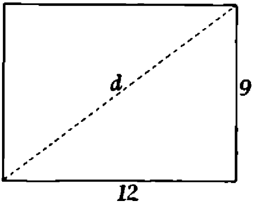

FIGURE 55

Solution: Let d be the number of feet in the diagonal (Fig. 55).

Then by the theorem of Pythagoras you have

$$d^2 = 9^2 + 12^2 = 81 + 144 = 225.$$

$$\therefore d^2 = 225.$$

$$\therefore d = 15.$$

EXERCISES

1. One side of a right triangle is 18 inches long, and the hypotenuse is 30 inches long. Find the length of the remaining side.

2. If the equal sides of an isosceles triangle are 60 inches long and the altitude 36 inches long, find the length of the base.

Suggestion: Draw the altitude, dividing the given triangle into two right triangles. Use the theorem of Pythagoras with the sides of one of the right triangles.

Solve the following equations:

3. $2x^2 - 81 = 0.$

4. $5x^2 - 36 = 0.$

5. $4a^2 - 242 = 0.$

6. $6x^2 - 200 = 0.$

7. $d = 16t^2$ for t.

8. $E = \frac{1}{2}mv^2$ for v.

9. $F = \dfrac{mM}{d^2}$ for d.

10. $f = \dfrac{mv^2}{R}$ for v.

130. Solving quadratic equations by factoring. When all the terms of a quadratic equation have been brought to the same side and when the similar terms have been combined, they may form two or three terms. Thus you may have equations of the form $ax^2 - b = 0$, or of the form $ax^2 + bx = 0$, or of the form $ax^2 + bx + c = 0$. In all cases the factoring method is worth trying because it is easily carried out. The following examples show how it is used for each of the three forms:

1. Solve the equation $3x^2 - 75 = 0$.

Solution: Factor the left side: $3(x - 5)(x + 5) = 0.$

The last equation is satisfied if $x - 5 = 0$

or if $x + 5 = 0,$

for the value of a product is zero if one of the factors is zero.

$$\therefore x_1 = 5,$$

and $$x_2 = -5.$$

2. Solve the equation $5x^2 + 15x = 0$.

Solution: Factor the left side: $5x(x + 3) = 0$.

This equation is satisfied if $x = 0$

or if $x + 3 = 0$.

$\therefore x_1 = 0,$

and $x_2 = -3.$

3. Solve the equation $2x^2 - 5x - 12 = 0$.

Solution: Factor the left side: $(2x + 3)(x - 4) = 0$.

Put each factor equal to zero; that is, let $2x + 3 = 0,$

and $x - 4 = 0.$

$\therefore x_1 = -\frac{3}{2},$

and $x_2 = 4.$

EXERCISES

Solve the following equations and check the results:

1. $2x^2 + 5x = 0$.

2. $3x^2 + 3x = 0$.

3. $6x^2 - 8x = 0$.

4. $x^2 - 3x + 2 = 0$.

5. $x^2 - 26x + 25 = 0$.

6. $x^2 - 5x + 6 = 0$.

7. $2x^2 - 3x - 5 = 0$.

8. $2a^2 - 3a = 35$.

9. $6m^2 + 7m + 2 = 0$.

10. $x^2 + 8x = 0$.

GRAPHICAL SOLUTION OF QUADRATIC EQUATIONS

131. Equation of motion of an object thrown vertically upward. When a ball is thrown upward, its velocity decreases. It becomes zero when the greatest height is reached. Then, as the ball drops back, the velocity increases. The greatest height reached by a ball thrown upward depends upon the velocity with which the ball is thrown. Experiments in science have shown that the ball rises and falls according to a law which can be expressed mathematically. If we denote by v the velocity with which the ball is thrown and by t

the number of seconds in which it reaches a height d, then d can be found from the formula $d = vt - \frac{gt^2}{2}$.

The number g is approximately 32.16. It denotes the velocity acquired by an object falling unresisted for one second under the action of gravity.

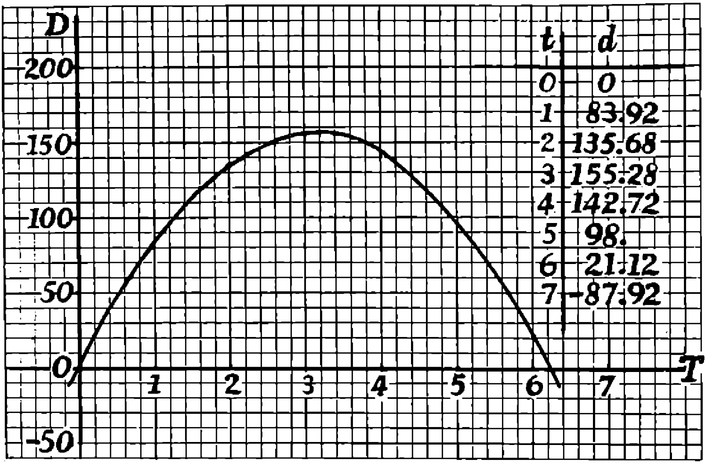

FIGURE 56

The equation shows that the height d depends on the time t and the initial velocity v. If you substitute for g the value 32.16, the equation changes to the form $d = vt - 16.08t^2$. The equation may be used (1) to determine the height d for a given value of t and (2) to find the time t in which the ball rises to any given height d.

Suppose that a ball is thrown vertically upward with a velocity of 100 feet a second. Then the relation between d and t is given by the equation

$$d = 100t - 16.08t^2.$$

Let $t = 1, 2, 3, 4$, etc., and find corresponding values of d.

Tabulate the corresponding values of d and t (Fig. 56).

Plot the number pairs in the table and draw the graph. Note that the graph is not a straight line.

From the graph determine approximate values of t for given values of d.

From the graph tell when the ball will be 50 feet from the ground; 75 feet from the ground.

132. How to use the graph to solve a quadratic equation. In § 131 a way is shown of solving a quadratic equation graphically. The method may be summarized in the following steps:

1. *Bring all the terms to one side of the equation, making the other side equal to zero.* For example, the equations $4x^2 - 4x = 15$ and $4x^2 = 4x + 15$ are changed to $4x^2 - 4x - 15 = 0$.

2. *Tabulate corresponding pairs of values of* x *and* $4x^2 - 4x - 15$.

Thus, if $x = 0$, $4x^2 - 4x - 15 = -15$;
if $x = 1$, $4x^2 - 4x - 15 = 4 - 4 - 15 = -15$;
if $x = 2$, $4x^2 - 4x - 15 = 16 - 8 - 15 = -7$;
if $x = 3$, $4x^2 - 4x - 15 = 36 - 12 - 15 = 9$,
etc.

3. *Plot these pairs and make the graph by drawing a smooth curved line through the points* (Fig. 57).

4. *Measure the distances from the origin* (*the* 0 *point*) *to the points of intersection of the curve with the* x*-axis.* These are the values of x for which the expression $4x^2 - 4x - 15$ is zero. If you denote these values by

x_1 and x_2, it follows that $x_1 = -1.5$ and $x_2 = 2.5$ are the required solutions.

The numbers -1.5 and 2.5 are called the **roots** of the quadratic equation $4x^2 - 4x - 15 = 0$. Note that this quadratic equation has *two* roots, corresponding to the two points of intersection of the curve with the x-axis.

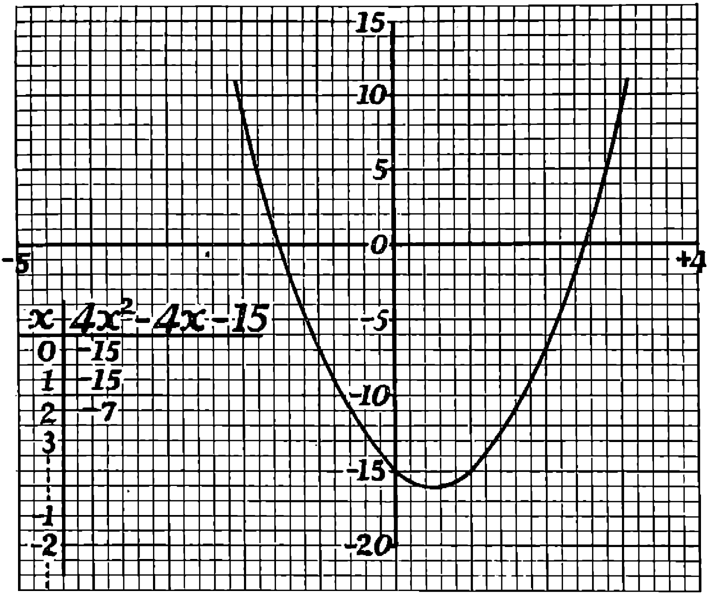

FIGURE 57

EXERCISES

Solve the following quadratic equations graphically:

1. $m^2 - 4m - 12 = 0$.
2. $y^2 + y = 56$.
3. $y^2 = 4y - 3$.
4. $x^2 - 8x + 12 = 0$.
5. $4x^2 - 12x + 5 = 0$.
6. $x^2 - 34x + 145 = 0$.

ALGEBRAIC SOLUTION OF QUADRATIC EQUATIONS

133. Every quadratic equation has two roots. The graphical solution of the quadratic equation (Fig. 57) shows that the equation $4x^2 - 4x - 15 = 0$ has *two* roots. This is true of every quadratic equation. For example, the equation $x^2 - 25 = 0$ is satisfied by *two* values of x, that is, $+5$ and -5. This is easily verified by substituting in the equation the numbers $+5$ and -5 in place of x.

The complete solution of the equation $x^2 - 25 = 0$ may now be arranged as follows:

$$\begin{aligned} x^2 - 25 &= 0. \\ x^2 &= 25. \\ \sqrt{x^2} &= \pm\sqrt{25} \text{ (read } + \textit{ or } - \textit{ the square root of 25}\text{).} \\ x_1 &= +5. \\ x_2 &= -5. \end{aligned}$$

By arithmetic you were able to find only one square root of a number. You have just seen that there is always a second square root. Thus the square roots of 36 are $+6$ and -6, because $(+6)^2 = 36$ and $(-6)^2 = 36$.

EXERCISES

Solve the following equations as shown in Exercise 1:

1. $(x + 1)^2 = 16$.

Solution: Extracting the square root, you have

$$\begin{aligned} x + 1 &= \pm 4. \\ \therefore x + 1 &= +4, \\ \text{and } x + 1 &= -4. \\ \therefore x_1 &= 3, \\ \text{and } x_2 &= -5. \end{aligned}$$

2. $(x + 3)^2 = 4$.	**6.** $(x - 4)^2 = 18$.
3. $(x - 8)^2 = 25$.	**7.** $(y + 6)^2 = 10$.
4. $(x + 2)^2 = 16$.	**8.** $(a + 8)^2 = 12$.
5. $(x - 5)^2 = 26$.	**9.** $(m - 11)^2 = 20$.

134. Solving quadratic equations in which one member is a quadratic trinomial square. The quadratic equations in § 128 were all of a simple form, the first degree term being missing. Before taking up the solution of the complete equation, let us consider further equations which reduce to the form shown in Exercises 1 to 9 in § 133.

Squaring both members of the equation $x + 1 = 4$, you have $(x + 1)^2 = 16, \dots \quad (1)$
or $x^2 + 2x + 1 = 16 \dots \quad (2)$

Equations like (2), in which the left member is a square, can always be solved by *first changing them to form* (1) and then using the method shown in Exercise 1 (§ 133).

EXERCISES

Exercises 1 and 2 below show how to change a trinomial square into the square of a binomial.

1. Change $x^2 + 2x + 1$ into the square of a binomial.

Solution: Show by multiplying $x + 1$ by $x + 1$ that $x^2 + 2x + 1 = (x + 1)^2$. The binomial $x + 1$ may be obtained from $x^2 + 2x + 1$ by inspection as follows:

Extract the square root of the first term, x^2.
Extract the square root of the third term, 1.
Add the results, which gives $x + 1$.

2. Change $x^2 - 14x + 49$ into the square of a binomial.

Solution: Extract the square root of the first term, x^2.
Extract the square root of the third term, 49.
State the *difference* of the results, which gives $x - 7$.

Test the answer by multiplying $x - 7$ by $x - 7$.

Change each of the following trinomials into the square of a binomial. Then verify the result by multiplying the binomial by itself.

3. $x^2 + 6x + 9$

4. $x^2 + 4x + 4$

5. $y^2 + 8y + 16$

6. $m^2 - 12m + 36$

7. $a^2 - 10a + 25$

8. $r^2 - 4r + 4$

The left members of the quadratic equations below are trinomial squares. Solve each as shown in Exercise 9.

9. $x^2 + 8x + 16 = 9.$

Solution: $x^2 + 8x + 16 = 9.$

Change the left member into the square of a binomial by taking the square root of the first term and the square root of the third term and adding the results. Thus you have

$$(x + 4)^2 = 9.$$

Extracting the square root, you find that

$$\begin{aligned} x + 4 &= \pm 3. \\ \therefore x_1 &= -1, \\ \text{and } x_2 &= -7. \end{aligned}$$

Check:

LEFT MEMBER		RIGHT MEMBER
$x^2 + 8x + 16$		9
$(-1)^2 + 8(-1) + 16$		9
$1 - 8 + 16$		9
9	=	9
$(-7)^2 + 8(-7) + 16$		9
$49 - 56 + 16$		9
9	=	9

10. $x^2 + 6x + 9 = 16.$

11. $x^2 - 4x + 4 = 25.$

12. $x^2 + 8x + 16 = 49.$

13. $x^2 - x + \frac{1}{4} = 9.$

14. $x^2 + 8x + 16 = 7.$

15. $x^2 - 10x + 25 = 11.$

16. $x^2 + \frac{2x}{3} + \frac{1}{9} = 16.$

17. $x^2 - \frac{x}{2} + \frac{1}{16} = 25.$

135. Solving a quadratic equation by completing the square. In § 134 you learned how to change a trinomial square into the square of a binomial. Thus $x^2 + 6x + 9$ may be changed to $(x + 3)^2$. Geometrically this means that the whole square $(x + 3)^2$ (Fig. 58) is equal to the sum of the parts x^2, $3x$, $3x$, 9. The third term, 9, of the trinomial $x^2 + 6x + 9$ is related to the first-degree term $6x$ and may be found from it by *taking one-half of the coefficient of* x *and then squaring it.*

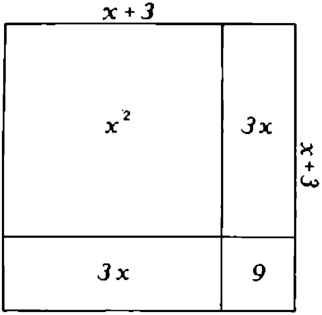

FIGURE 58

Similarly, to form the square whose first two terms are $x^2 + 16x$, take $\frac{1}{2}$ of 16 and square the result, which gives 64. Add 64 to $x^2 + 16x$. The result $x^2 + 16x + 64$ is a square.

You are now able to solve *any* quadratic equation containing one unknown. The solution is illustrated in the following examples:

1. Solve $x^2 + 6x - 55 = 0$.

Solution: Add 55 to both members. This gives

$$x^2 + 6x = 55.$$

Find one-half of the coefficient of x in $6x$ and square it. The result is 9.

Complete the square on the left side by adding 9 to both sides:

$$x^2 + 6x + 9 = 64.$$

Change the left side to the square of a binomial:

$$(x + 3)^2 = 64.$$

Extract the square root:

$$x + 3 = \pm 8.$$

Solving for x, you have $x_1 = 5$

$$\text{and } x_2 = -11.$$

State the five steps in the solution of the preceding equation.

2. Solve $2x^2 - 3x - 35 = 0$.

Solution: Note the difference between this equation and that in Example 1. Before solving Example 2, you must first make the coefficient of x^2 equal to 1. Divide each term by the coefficient of x^2:

$$x^2 - \tfrac{3}{2}x - \tfrac{35}{2} = 0.$$

Add $\frac{35}{2}$ to both sides of the equation:

$$x^2 - \tfrac{3}{2}x = \tfrac{35}{2}.$$

Take one-half of the coefficient of x, square it, and add it to both sides of the equation:

$$x^2 - \tfrac{3}{2}x + \tfrac{9}{16} = \tfrac{35}{2} + \tfrac{9}{16}.$$

Change the left side into the form of the square of a binomial: $(x - \tfrac{3}{4})^2 = \tfrac{289}{16}$.

Extract the square root: $x - \tfrac{3}{4} = \pm \tfrac{17}{4}$.

Solve for x: $x_1 = 5$,

$$\text{and } x_2 = -\tfrac{7}{2}.$$

Check by substituting $-\frac{7}{2}$ and 5 in the original equation.

EXERCISES

To each of the binomials in Exercises 1 to 9 add the term needed to complete the trinomial square:

1. $x^2 + 6x$

2. $x^2 - 8x$

3. $x^2 + 4x$

4. $x^2 - 3x$

5. $x^2 + x$

6. $x^2 + 7x$

7. $x^2 - \frac{1}{2}x$

8. $x^2 + \frac{1}{3}x$

9. $x^2 + \frac{2}{3}x$

Solve the equations in Exercises 10 to 28 by completing the square:

10. $x^2 + 3x + 2 = 0.$

11. $x^2 - 8x - 48 = 0.$

12. $a^2 + 8a - 20 = 0.$

13. $y^2 + 4y + 3 = 0.$

14. $x^2 + 6x - 16 = 0.$

15. $x^2 + 4x = 21.$

16. $x^2 + 6x + 5 = 0.$

17. $x^2 + 4x - 32 = 0.$

18. $3 + 2x = x^2.$

19. $x^2 = 3x + 4.$

20. $3x^2 - 2x - 3 = 0.$

Solution: Divide every term by 3.

$$\text{Then } x^2 - \frac{2x}{3} - 1 = 0.$$

Add 1 to both members: $x^2 - \dfrac{2x}{3} = 1.$

Add the square of $\frac{1}{2}$ of $-\frac{2}{3}$:

$$x^2 - \frac{2x}{3} + \frac{1}{9} = 1 + \frac{1}{9}.$$

$$\therefore (x - \tfrac{1}{3})^2 = \tfrac{10}{9}.$$

$$\therefore x - \frac{1}{3} = \frac{\pm\sqrt{10}}{3}.$$

Solve for x: $$x = \frac{1}{3} \pm \frac{\sqrt{10}}{3} = \frac{1 \pm \sqrt{10}}{3}.$$

$$x_1 = \frac{1 + \sqrt{10}}{3}, \text{ and } x_2 = \frac{1 - \sqrt{10}}{3}.$$

State the steps in the solution of the equation above.

21. $3x^2 - 7x - 20 = 0.$

22. $4x^2 - 4x - 79 = 0.$

23. $3a^2 - 7a = 6.$

24. $4y^2 = 1 - 4y.$

25. $x^2 + 18x - 15 = 0.$

26. $5a^2 + 25a = -9.$

27. $6x^2 = 3x + 45.$

28. $2x(x + 4) = 42.$

136. Supplementary exercises. Solve the following equations by whatever method seems best to you.

EXERCISES

1. $a^2 + 4a - 5 = 0.$

2. $x^2 - 2x = 11.$

3. $8a = a^2 - 180.$

4. $2x + 4 = 2 + 3x^2.$

5. $x^2 - 12x - 13 = 0.$

6. $x^2 - 5x = 12.$

7. $6x^2 - 51x + 99 = 0.$

8. $a^2 + 3a = 10.$

9. $4x^2 + 4x - 35 = 0.$

10. $2a(a + 4) = 42.$

11. $2x^2 - 7x + 3 = 0.$

12. $3a^2 - a = 2.$

13. $5x^2 + 2x = 3.$

14. $4a^2 - 3a - 10 = 0.$

15. $3x^2 + 5x - 7 = 0.$

16. $6x^2 - 7x - 5 = 0.$

137. Problems solved by means of quadratic equations. The following problems lead to quadratic equations which may be solved by the methods explained in this chapter.

EXERCISES

1. The sum of two numbers is 42, and the product is 416. Find the numbers.

2. The sum of the squares of two consecutive numbers is 421. Find the numbers.

Suggestion: Let x be one number. Then $x + 1$ is the consecutive number.

3. The sum of the squares of two consecutive numbers is 1013. What are the numbers?

4. Three times the square of a number, if increased by the number, is equal to 16. Find the number.

5. The perimeter of a rectangle is 84 rods, and the area is 432 square rods. Find the dimensions.

6. The length of a rectangle exceeds the width by 4 inches. The area is 140 square inches. Find the dimensions.

7. The base of a triangle exceeds twice the altitude by 4 inches, and the area is 63 square inches. Find the lengths of the base and altitude.

8. The base of a triangle exceeds the altitude by 4 inches. The area is 30 square inches. Find the lengths of the base and altitude.

SOLVING QUADRATIC EQUATIONS BY MEANS OF FORMULAS[1]

138. Deriving formulas for solving quadratic equations. By solving the equation $ax^2 + bx + c = 0$, using the method explained in § 135, it is possible to work out formulas which may be used to solve quadratic equations. The solution may be arranged as follows:

Divide each term by a: $x^2 + \frac{b}{a}x + \frac{c}{a} = 0.$

Subtract $\frac{c}{a}$ from both sides: $x^2 + \frac{b}{a}x = -\frac{c}{a}.$

Add $\left(\frac{1b}{2a}\right)^2$ to both sides:

$$x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = \frac{b^2}{4a^2} - \frac{c}{a}.$$

Change the left side into the square of a binomial:

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a}.$$

Carry out the subtraction on the right side:

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}.$$

Extract the square root of both sides:

$$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}}$$

$$= \pm\frac{\sqrt{b^2 - 4ac}}{2a}.$$

---

[1] §§ 138 and 139 are for those who wish to do more than the required work. Further discussion of the formula is found in Chapter XI.

Subtract $\frac{b}{2a}$ from both sides: $x = \frac{-b}{2a} \pm \frac{\sqrt{b^2-4ac}}{2a}$

$$\text{or} \qquad x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$$

$$\text{and} \qquad x_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

Because these formulas are very important, they should be memorized. They will enable you to solve any given quadratic equation.

139. How to use the quadratic formulas. To solve an equation by means of the quadratic formulas, proceed as follows:

1. Bring all terms to one side of the equation.

2. Collect similar terms.

3. Arrange the terms according to descending powers of x. The equation will then be of the form $ax^2 + bx + c = 0$.

4. By comparing the equation with $ax^2 + bx + c = 0$, determine the values of a, b, and c.

For example, in solving the equation $2x^2 + 5x - 3 = 0$, comparison with $ax^2 + bx + c = 0$ determines the values $a = 2$, $b = 5$, $c = -3$.

5. Substitute these values in the formulas. Then

$$x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} = \frac{-5 + \sqrt{25 - 4\cdot 2\,(-3)}}{2\cdot 2}$$

$$= \frac{-5 + \sqrt{25 + 24}}{4} = \frac{-5 + 7}{4} = \frac{1}{2},$$

$$\text{and } x_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a} = \frac{-5 - \sqrt{49}}{4} = -\frac{12}{4}$$

$$-3.$$

Thus $x_1 = \frac{1}{2}$, and $x_2 = -3$.

EXERCISES

Solve the equations in Exercises 1 to 12 by using the formulas and check both results:

1. $x^2 + 8x + 12 = 0.$	**7.** $x^2 + 6x = 16.$
2. $a^2 + 5a + 4 = 0.$	**8.** $2x^2 - 3x - 2 = 0.$
3. $x^2 - 10x + 21 = 0.$	**9.** $18x - x^2 = 77.$
4. $3a^2 - 15a = -18.$	**10.** $2x^2 - 3x - 9 = 0.$
5. $3x^2 - 17x + 10 = 0.$	**11.** $5x^2 + 3x = 2.$
6. $12x^2 - x - 1 = 0.$	**12.** $3x^2 - 5x = 8.$

Solve the following equations, using the formulas:

13. $3x^2 - 2x - 3 = 0.$

Solution: $$x = \frac{2 \pm \sqrt{4 + 36}}{6} = \frac{2 \pm \sqrt{40}}{6} = \frac{2 \pm 2\sqrt{10}}{6}$$

$$= \frac{1 \pm \sqrt{10}}{3}.$$

Therefore $$x_1 = \frac{1 + \sqrt{10}}{3}$$

and $$x_2 = \frac{1 - \sqrt{10}}{3}$$

are the exact values of the roots.

To find the approximate values, extract the square root of 10. This gives 3.16.

$$\therefore x_1 = \frac{1 + 3.16}{3} = \frac{4.16}{3} = 1.39,$$

$$\text{and } x_2 = \frac{1 - 3.16}{3} = \frac{-2.16}{3} = -.72.$$

14. $x^2 - 8x + 14 = 0.$	**19.** $2a^2x^2 + ax - 3 = 0.$
15. $a^2 - 2a - 4 = 0.$	**20.** $5x^2 - 12ax + 4a^2 = 0.$
16. $3x^2 + 8x = 15.$	**21.** $3x^2 - 6ax + 2a^2 = 0.$
17. $3x^2 - 2x - 3 = 0.$	**22.** $3c^2x^2 - 4cdx - 4d^2 = 0.$
18. $x^2 - 3x - 5 = 0.$	**23.** $10a^2x^2 - 4ax - 3 = 0.$

Find the values of x to the nearest tenth:

24. $x^2 + 5x = 7.$

25. $3x^2 - 5x - 8 = 0.$

26. $2x^2 + 5x - 14 = 0.$

27. $x^2 - 3x - 2 = 0.$

28. $\dfrac{x^2}{x-2} + \dfrac{4}{x-2} + 5 = 0.$

29. $\dfrac{x-2}{x+2} = \dfrac{x+3}{x-3} - 6\frac{2}{3}.$

VERBAL PROBLEMS

140. Problems leading to quadratic equations. In the following problems choose your own method of solving the equations.

EXERCISES

1. A rectangular piece of tin is to be cut 4 inches longer than it is wide (Fig. 59). Four 6-inch squares are to be cut out, one from each corner, so that when the sides are turned up an open box is formed which will contain 840 cubic inches. What should be the dimensions of the piece of tin?

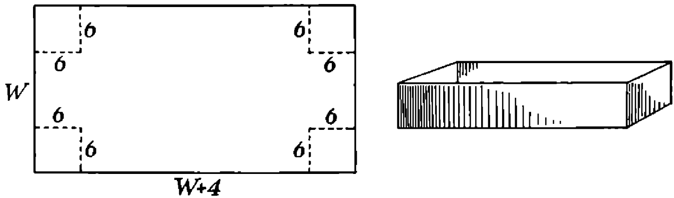

FIGURE 59

Suggestion: The facts of the problem can be expressed by means of a quadratic equation. This equation will be found to have two roots. Both roots are not necessarily answers to the problem. A root of the equation which does not satisfy the conditions of the problem is to be discarded.

2. The total surface of a cone (Fig. 60) is given by the formula $T = \pi r(s + r)$, where r is the length of the radius of the base and s the slant height. Of what length must the radius be to make the total surface 374 square inches if the slant height is 10 inches?

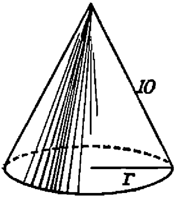

FIGURE 60

Suggestion: Use $\pi = 3\frac{1}{7}$.

3. By lengthening the radius of a sphere by 2 feet, we doubled its surface. Find the original radius to two figures.

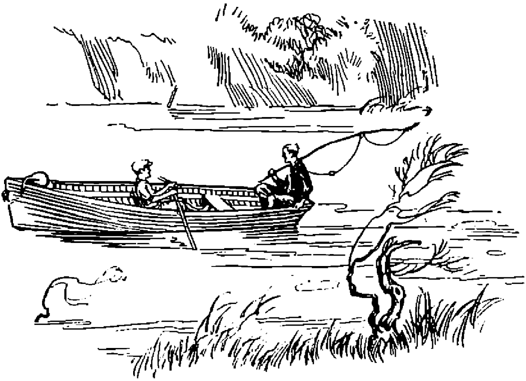

FIGURE 61

4. The rate of the current of a stream is 1 mile an hour. It took us $7\frac{1}{2}$ hours to row 18 miles downstream and to return. What would be our rate of rowing in still water?

5. An open box is to be made from a square piece of tin whose sides are 16 inches. How large a square must be cut from each corner so that the box formed by turning up the sides contains 256 cubic inches?

Suggestion: Form the equation and use the method of factoring for finding one root. Find the other roots by solving the quadratic equation by means of the formulas.

6. A coal bin is to be 8 feet deep and 10 feet longer than it is wide. If the bin is to hold 15 tons of coal and if one ton (2000 pounds) occupies 35 cubic feet, what must be the length and width of the bin?

7. A piece of land is to be laid off in the shape of a right triangle (Fig. 62) with the vertex D of the right angle on AB. If the hypotenuse BE is to be 90 yards and if the perimeter is to be 216 yards, what must be the length of BD?

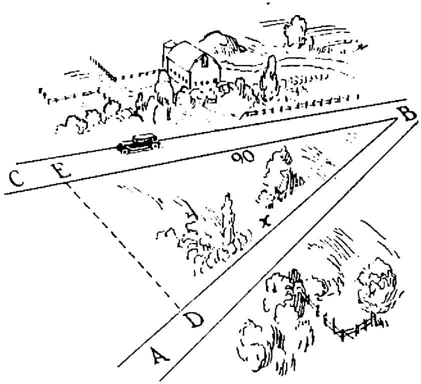

FIGURE 62

8. The radius of a circle is 21 inches long. By how much must it be decreased to diminish the area of the circle by 770 square inches?

Suggestion: Use $3\frac{1}{7}$ for π.

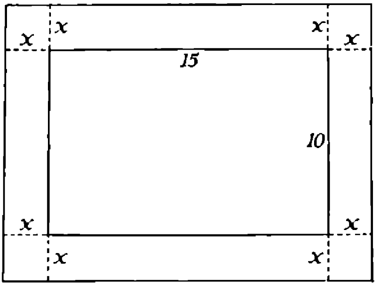

FIGURE 63

9. A rectangular garden bed is to be doubled in size by extending all sides, as shown in Fig. 63. If the dimensions are 15 feet by 10 feet, by how much must the sides be increased?

10. Two men together can do a piece of work in $6\frac{2}{3}$ hours. If one alone can do it in 3 hours less than the other, how long will it take him to do it alone?

141. Supplementary exercises. Solve the following problems:

1. The hypotenuse of a right triangle is 9 feet longer than one of the other sides and 2 feet longer than the third side. Find the lengths of the three sides of the triangle.

2. The hypotenuse of a right triangle is 10 inches, and the sum of the other two sides is 14 inches. Find the lengths of the sides.

3. The sum of the areas of two squares is 61 square rods. A side of one is 1 rod longer than a side of the other. Find the lengths of the sides of the squares.

4. Find the lengths of the sides of a square whose area is doubled if the dimensions are increased by 9 feet and 6 feet, respectively.

5. Find the dimensions of a coal bin, with a capacity of 6 tons, whose depth is 6 feet. The length is equal to the sum of the width and depth, and one ton of coal takes up 35 cubic feet of space.

6. A man drove his automobile to a city 160 miles from his home. Returning, he increased its speed by 4 miles an hour. Find the rates if the round trip took 9 hours.

7. Two men starting from the same place and at the same time walk at rates of 3 and 4 miles an hour, respectively. If the first walks east and the other north, how soon will they be 15 miles apart?

8. The radius of one circle is 7 inches longer than that of another, and the area of the first is 770 square inches greater than that of the second. Find the length of the radius of each circle.

9. A ball is thrown upward with a rate of 30 feet a second. How long will it take it to reach a height of 14 feet?

Suggestion: Use the formula $s = v_0 \cdot t - \frac{1}{2}gt^2$. Use 32 for the value of g.

10. An open box is to be made from a square piece of tin (Fig. 64) by cutting out a square from each corner and turning up the sides. If the box is to contain 180 cubic inches and is to be 5 inches high, how large a square of tin is to be used? What is the total area of the box?

11. From each corner of a square piece of tin a square is cut whose sides are 8 inches. By turning up the sides a box is formed containing 1152 cubic inches. Find the length of each side of the square piece of tin.

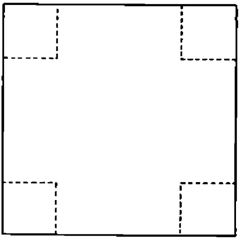
FIGURE 64

12. In using an old plan for a new and smaller house an architect must reduce it in size. The original plan was made for a piece of ground 30 feet wide and 40 feet long. If a new and similar house is to cover a piece of ground containing 925 square feet, by what amount must the dimensions of the plan be reduced?

EXERCISES TO TEST YOUR UNDERSTANDING OF CHAPTER VII

142. What you should be able to do. Chapter VII is a study of quadratic equations. You should now be able to do the following:

1. To solve quadratic equations:
   (*a*) By factoring.
   (*b*) By completing the square.
   (*c*) By formula[1].
2. To solve problems leading to quadratic equations.
3. To interpret the solution of a quadratic equation by means of a graph.

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[1] Optional.

143. Timed tests on quadratic equations. The following tests involve the processes used in Chapter VII in solving quadratic equations.

A. *Completing squares.* Time: 2 minutes.

1. $x^2 - 6x$
2. $a^2 - 2a$
3. $m^2 + m$
4. $y^2 + 3y$
5. $b^2 - 5b$
6. $x^2 + \frac{1}{2}x$
7. $m^2 - \frac{1}{3}m$
8. $p^2 + \frac{3}{4}p$

B. *Evaluation of quadratic trinomials.* Time: 2 minutes.

1. Find the value of $x^2 + 2x - 5$ when $x = 2$.
2. Find the value of $a^2 - 7a + 3$ when $a = -3$.
3. Find the value of $3y^2 - y + 2$ when $y = 5$.
4. Find the value of $2b^2 - \frac{1}{3}b + \frac{1}{4}$ when $b = -6$.

C. *Solving equations by algebra.* Time: $2\frac{1}{2}$ minutes.

1. $x^2 - 121 = 0$.
2. $3a^2 - 108 = 0$.
3. $(x + 5)^2 = 64$.
4. $4\pi r^2 = 100$.
5. $s = \frac{1}{2}gt^2$ for t.
6. $fR = mv^2$ for v.

D. *Solving equations by graphs.* Time: 2 minutes.

1. Make the graph representing the table below. Choose scales as indicated in Fig. 65.

x	$x^2 + x - 12$
0	− 12
1	− 10
2	− 6
3	0
4	8
− 1	− 12
− 2	− 10
− 3	− 6
− 4	0
− 5	8

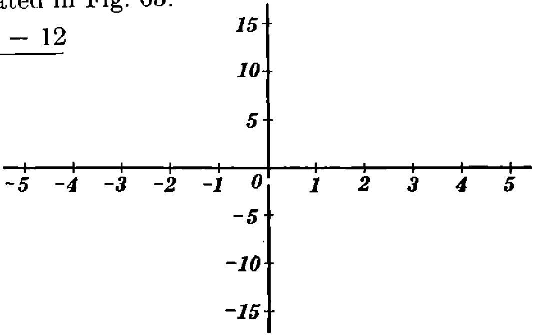

FIGURE 65

2. Using the graph, solve the equation $x^2 + x - 12 = 0$.

E. *Solving quadratic equations by completing the square.* Time: 6 minutes.

1. $x^2 - 6x = 27$.

2. $x^2 - 3x - 28 = 0$.

3. $2x^2 - 13x + 6 = 0$.

4. $3x^2 + 6x - 2 = 0$.

CHAPTER VIII

TRIGONOMETRIC RATIOS

RELATIONS BETWEEN THE SIDES AND ANGLES OF A RIGHT TRIANGLE

144. What is meant by the sine ratio. You will recall that in determining unknown distances by means of the right triangle you made use of tangent ratios (*Eighth-Year Mathematics*, pages 209–215). The values of tangent ratios corresponding to given angles were found in special tables. Because the relationships between the sides and acute angles of a right triangle are useful in other school subjects and in practical work, a further study is to be made of these relations.

EXERCISES

1. On squared paper (Fig. 66) draw angle *ABC* equal to 23°.

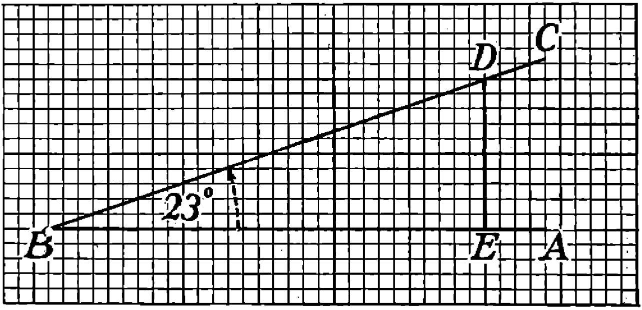

FIGURE 66

From a convenient point *D* on *BC* draw *DE* perpendicular to *BA*.

By measuring find the length of *DE*, the side opposite ∠ *B*. Measure the hypotenuse *BD*.

By dividing find the ratio $\frac{DE}{BD}$ to the nearest tenth.

Compare your result with the results of other members of the class.

The ratio obtained by dividing the side opposite *B* in the triangle *BDE* by the hypotenuse is called the **sine ratio.** The word *sine* is usually abbreviated "sin," and "sine of 23°" is written briefly "sin 23°."

2. Explain by means of a principle of similar triangles why all pupils of the class should find the same value for sin 23°.

3. On squared paper draw an angle equal to 30°. As in Exercise 1 draw a right triangle containing the 30° angle. Measure the side opposite the 30° angle, measure the hypotenuse, and find the ratio of the first to the second. What is the sine of 30°? Compare your result with the results of other pupils.

4. As in Exercises 1 and 3, find sin 67°.

145. A table of sine ratios. If you draw many angles of various sizes and find the sine of each by the method explained in the foregoing exercises, you may tabulate the corresponding values of angles and sine ratios. Such a table would be called a *table of sines*. The table on page 214 contains measures of angles in the first column and their sines in the second. From the table find the sines of 23°; 30°; 67°.

The values in the table have been computed to four figures. Your results should agree with them to two figures. You cannot draw and measure accurately enough to determine the correct third or fourth figure.

TABLE OF SINES, COSINES, AND TANGENTS OF ANGLES FROM 1° TO 90°

Angle	Sine	Cosine	Tangent	Angle	Sine	Cosine	Tangent
1°	.0175	.9998	.0175	46°	.7193	.6947	1.0355
2	.0349	.9994	.0349	47	.7314	.6820	1.0724
3	.0523	.9986	.0524	48	.7431	.6691	1.1106
4	.0698	.9976	.0699	49	.7547	.6561	1.1504
5	**.0872**	**.9962**	**.0875**	**50**	**.7660**	**.6428**	**1.1918**
6	.1045	.9945	.1051	51	.7771	.6293	1.2349
7	.1219	.9925	.1228	52	.7880	.6157	1.2799
8	.1392	.9903	.1405	53	.7986	.6018	1.3270
9	.1564	.9877	.1584	54	.8090	.5878	1.3764
10	**.1736**	**.9848**	**.1763**	**55**	**.8192**	**.5736**	**1.4281**
11	.1908	.9816	.1944	56	.8290	.5592	1.4826
12	.2079	.9781	.2126	57	.8387	.5446	1.5399
13	.2250	.9744	.2309	58	.8480	.5299	1.6003
14	.2419	.9703	.2493	59	.8572	.5150	1.6643
15	**.2588**	**.9659**	**.2679**	**60**	**.8660**	**.5000**	**1.7321**
16	.2756	.9613	.2867	61	.8746	.4848	1.8040
17	.2924	.9563	.3057	62	.8829	.4695	1.8807
18	.3090	.9511	.3249	63	.8910	.4540	1.9626
19	.3256	.9455	.3443	64	.8988	.4384	2.0503
20	**.3420**	**.9397**	**.3640**	**65**	**.9063**	**.4226**	**2.1445**
21	.3584	.9336	.3839	66	.9135	.4067	2.2460
22	.3746	.9272	.4040	67	.9205	.3907	2.3559
23	.3907	.9205	.4245	68	.9272	.3746	2.4751
24	.4067	.9135	.4452	69	.9336	.3584	2.6051
25	**.4226**	**.9063**	**.4663**	**70**	**.9397**	**.3420**	**2.7475**
26	.4384	.8988	.4877	71	.9455	.3256	2.9042
27	.4540	.8910	.5095	72	.9511	.3090	3.0777
28	.4695	.8829	.5317	73	.9563	.2924	3.2709
29	.4848	.8746	.5543	74	.9613	.2756	3.4874
30	**.5000**	**.8660**	**.5774**	**75**	**.9659**	**.2588**	**3.7321**
31	.5150	.8572	.6009	76	.9703	.2419	4.0108
32	.5299	.8480	.6249	77	.9744	.2250	4.3315
33	.5446	.8387	.6494	78	.9781	.2079	4.7046
34	.5592	.8290	.6745	79	.9816	.1908	5.1446
35	**.5736**	**.8192**	**.7002**	**80**	**.9848**	**.1736**	**5.6713**
36	.5878	.8090	.7265	81	.9877	.1564	6.3138
37	.6018	.7986	.7536	82	.9903	.1392	7.1154
38	.6157	.7880	.7813	83	.9925	.1219	8.1443
39	.6293	.7771	.8098	84	.9945	.1045	9.5144
40	**.6428**	**.7660**	**.8391**	**85**	**.9962**	**.0872**	**11.4301**
41	.6561	.7547	.8693	86	.9976	.0698	14.3006
42	.6691	.7431	.9004	87	.9986	.0523	19.0811
43	.6820	.7314	.9325	88	.9994	.0349	28.6363
44	.6947	.7193	.9657	89	.9998	.0175	57.2900
45	**.7071**	**.7071**	**1.0000**	**90**	**1.0000**	**.0000**	**∞**

EXERCISES

1. Using the table, find the sines of 10°; 25°; 45°; 80°.

2. Using the table, find the angles whose sines are .8290; .7314; .4848; .2250.

146. Three important trignometric ratios. If BAC (Fig. 67) represents any angle and if from any point B, on either side of the angle, a perpendicular BC is drawn to the other side, a right triangle ABC is formed.

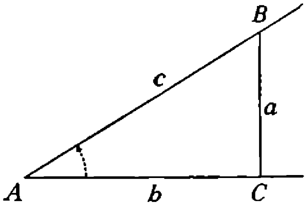

FIGURE 67

The following three ratios are important:

1. The ratio of the side opposite point A to the hypotenuse. It is called the **sine of $\angle$A.**

2. The ratio of the side passing through A to the hypotenuse. It is called the **cosine of $\angle$A.**

3. The ratio of the side opposite A to the side passing through A. It is called the **tangent of $\angle$A.**

Using the symbol A to denote angle A, you may write the three statements briefly:

$$1.\ \sin A = \frac{\text{side opposite } A}{\text{hypotenuse}} = \frac{a}{c}.$$

$$2.\ \cos A = \frac{\text{side passing through } A}{\text{hypotenuse}} = \frac{b}{c}.$$

$$3.\ \tan A = \frac{\text{side opposite } A}{\text{side passing through } A} = \frac{a}{b}.$$

147. How to find by measurement the value of the cosine and tangent of an angle. The method of finding the tangent or cosine of an angle is the same as that used in finding the sine in Exercises 1 and 3 (§ 144). Thus, to find sin 60°, cos 60°, and tan 60°, proceed as follows:

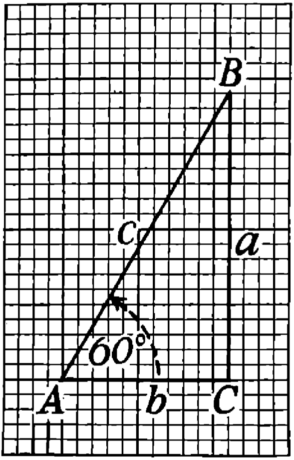

FIGURE 68

On squared paper (Fig. 68) draw $\angle A = 60°$.

Draw BC perpendicular to AC.

Determine a, b, c by measurement.

$$\sin 60° = \frac{a}{c} = \frac{19}{22} = .86.$$

$$\cos 60° = \frac{b}{c} = \frac{11}{22} = .50.$$

$$\tan 60° = \frac{a}{b} = \frac{19}{11} = 1.73.$$

Check your results by means of the table (§ 145).

EXERCISES

Find by measurement the values of the following and check the results by means of the table:

1. sin 40°; cos 40°; tan 40°
2. sin 70°; cos 70°; tan 70°
3. sin 64°; cos 64°; tan 64°
4. sin 57°; cos 57°; tan 57°

USE OF TRIGONOMETRIC RATIOS IN PROBLEMS

148. Problems in finding distances and angles. The trigonometric table can be used to simplify the solution of geometric problems. The following exercises illustrate methods used by surveyors.

EXERCISES

1. Find the height of the school flagstaff.

Solution: To find the height of the flagstaff AB (Fig. 69), some of the ninth-grade pupils marked off a distance $CD = 20.5$ feet.

They placed a transit directly over D and measured the angle FEA.

They found

$$\angle FEA = 74°, \quad CB = 2.5 \text{ feet}, \quad \text{and } FB = 3 \text{ feet}.$$

How long is EF?

The class used these measurements to determine AB as follows:

Denoting AF by h, the pupils wrote

$$\tan 74° = \frac{h}{EF} = \frac{h}{23}.$$

From the table they found

$$\tan 74° = 3.4874, \text{ or } 3.49.$$

$$\therefore \frac{h}{23} = 3.49.$$

$$\therefore h = (3.49)\,(23)$$
$$= 80.3 \text{ approximately.}$$
$$\therefore AB = 3 + 80.3 = 83.3.$$

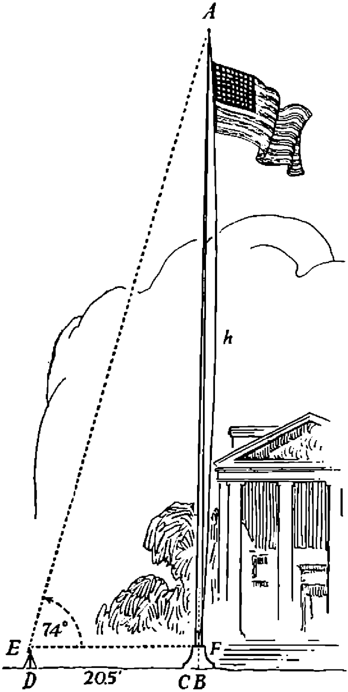

FIGURE 69

FINDING THE HEIGHT OF A FLAGSTAFF

2. Using the table, find the cosines of 12°, 32°, 48°, and 81°.

3. Using the table, find to the nearest degree the angles whose cosines are .3245, .7546, and .8473.

Suggestion: To find to the nearest degree the angle whose cosine is .3245, look in the cosine column for the number which is nearest to .3245. It is .3256, which corresponds to 71°. Hence to the nearest degree the angle whose cosine is .3245 is 71°.

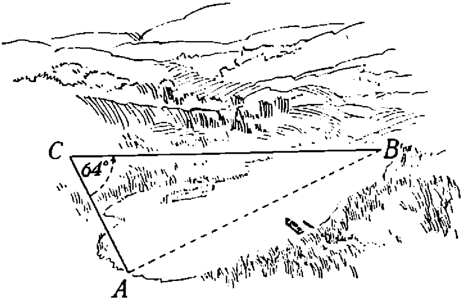

FIGURE 70
FINDING THE DISTANCE ACROSS A POND

4. A telephone pole is braced by a wire fastened to the ground at a point 16 feet from the foot of the pole. The wire makes an angle of 62° with the ground. How high is the point where the wire is attached to the pole?

5. A surveyor wishes to measure the distance *AB* across a pond (Fig. 70). With the help of his transit he lays off *AC* perpendicular to *AB*, making it 300 feet long. He then measures angle *ACB* and finds it to be 64°. How long is *AB*?

6. Make a drawing to show what is meant by *angle of elevation* and another to show what is meant by *angle of depression*.

7. The angle of elevation of the top of a mountain (Fig. 71) is 38°. The distance from the point of observation to the top is 4865 feet. Determine the height of the mountain. Which one of the three ratios must you use here?

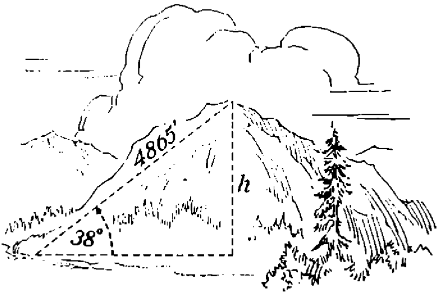

FIGURE 71
FINDING THE HEIGHT OF A MOUNTAIN

8. A balloon B (Fig. 72) is anchored to the ground at a point P. The point D is directly below B. If $\angle DPB = 54°$ and $PD = 310$ feet, what is the length of the wire PB? How high is the balloon?

9. A concrete road is inclined to the horizontal at an angle of 6°. How much does the road rise for a distance of 100 feet, measured along the road?

10. A monument 380 feet high casts a shadow 215 feet long. Find the angle of elevation of the sun to the nearest degree.

11. A balloon is fastened by a cable 640 feet long. The angle of elevation of the balloon is 68°. How high is the balloon?

12. A road rises 3.5 feet for every 100 feet, measured along the road. What is the angle of elevation of the road? Give your result to the nearest degree.

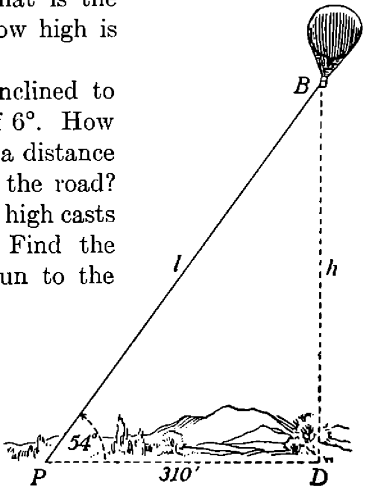

FIGURE 72

FINDING THE HEIGHT OF A BALLOON

13. From the top of a cliff 165 feet high the angle of depression of a point marked by a rock is 30°. How far is the rock from the foot of the cliff?

14. The angle C of an isosceles triangle (Fig. 73) is 40°; AC and $BC = 20.2$ inches. How long is AB?

Suggestion: Draw the line CD, making two congruent right triangles.

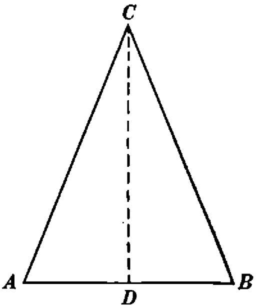

FIGURE 73

15. The angle through which a pendulum swings is 9 degrees. If the pendulum is 39.1 inches long, what is the distance between the two extreme positions?

16. The shadow of a vertical pole 32 feet high is 50 feet long. Find the angle of elevation of the sun.

17. Each side of an inscribed regular pentagon is 3.4 feet. Find the radius of the circumscribed circle.

Suggestion: Make a diagram. Draw the perpendicular from the center to one side of the polygon and a radius to one end of that side.

18. The angle of elevation of the highest point of a building is 30° when observed from a point 900 feet from the base of the building. How high is the building?

SUPPLEMENTARY WORK WITH TRIGONOMETRIC RATIOS [1]

149. How to find the values of the ratios when the angles are expressed in degrees and minutes. The table (§ 145) gives the values of the trigonometric ratios only for angles expressed in degrees. For example, it does not give the sine of 30° 20′. What is the sine of 30°? What is the sine of 31°? This shows that sin 30° 20′ lies between .5000 and .5150. It will be assumed that a small change in the sine is nearly proportional to the change of the angle. The change from 30° to 31° is 60′. Then it may be said that sin 30° 20′ is equal to sin 30° plus $\frac{20}{60}$ of the difference between sin 30° and sin 31°. What is the difference between sin 30° and sin 31°?

Denote by x the number which added to the fractional part of sin 30° gives sin 30° 20′.

---

[1] §§ 149 to 151 are intended for those pupils who wish to do more than the required work.

$$\text{Then } \frac{20}{60} = \frac{x}{.0150}, \text{ or } \frac{1}{3} = \frac{x}{.0150}.$$

$$\therefore x = \tfrac{1}{3}(.0150) = .0050.$$

$$\therefore \sin 30^\circ\, 20' = .5000 + .0050,$$

$$\text{or } \sin 30^\circ\, 20' = .5050.$$

The process which was used to find sin 30° 20′ from sin 30° and sin 31° is called **interpolation.**

EXERCISES

1. Find tan 68° 40′.

Solution:

$$\tan 68^\circ = 2.4751.$$

$$\tan 69^\circ = 2.6051.$$

$$\text{Difference} = .1300.$$

$$\therefore \frac{40}{60} = \frac{x}{.1300}.$$

$$\therefore x = \tfrac{2}{3}(.1300) = .0866.$$

$$\therefore \tan 68^\circ\, 40' = 2.4751 + .0866$$

$$= 2.5617.$$

2. Find cos 42° 16′.

Solution:

$$\cos 42^\circ = .7431.$$

$$\cos 43^\circ = .7314.$$

$$\text{Difference} = .0117.$$

$$\therefore \frac{16}{60} = \frac{x}{.0117}.$$

$$\therefore x = \frac{16(.0117)}{60} = .0031.$$

Since cos 43° is *less* than cos 42°, we must *subtract* x.

$$\text{Hence } \cos 42^\circ\, 16' = .7431 - .0031$$

$$= .7400.$$

Find the following values:

3. sin 18° 10′

4. cos 21° 40′

5. tan 34° 50′

6. cos 10° 32′

7. sin 74° 54′

8. tan 28° 18′

9. tan 15° 21′

10. cos 71° 45′

11. sin 89° 12′

12. From a point 210 feet from the foot of a wireless telegraph mast the angle of elevation of the top was 48° 14′. Find the height of the mast.

13. The top of a mountain 12,340 feet high is observed to be at an angle of elevation of 30° 20′ from a station located at an altitude of 5364 feet. What is the direct distance from the station to the mountain top?

14. From the top of a tower 220 feet high the angle of depression of one object in the plane below is found to be 50° 12′. Find the distance from the object to the foot of the tower.

15. To find the width of a stream, a surveyor laid off along the bank a line AB 400 feet long. At B he determined a point C on the opposite bank so that CB was perpendicular to AB. Angle BAC was then measured and found to be 53° 18′. How wide was the stream?

16. A steamer is driven northward by the wind with a force sufficiently great to carry it 10 miles in one hour. The engine is driving it eastward with a force of 14 miles per hour. What distance will it actually travel in one hour and in what direction?

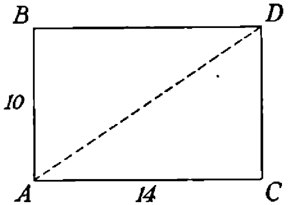

FIGURE 74

Solution: Let AB and AC (Fig. 74) represent the magnitude and direction of the two forces. Then it is shown by experiment that the diagonal AD (resultant) of the rectangle $ACDB$ represents the actual rate and direction of the boat. Hence you must find AD and $\angle DAB$.

150. How to find an angle to the nearest minute when the value of a trigonometric ratio is given. You already know how to find an angle to the nearest degree when a ratio is given. Sometimes it is desired to find the angle to the nearest minute. The method is similar to that described in § 149. Suppose that you wish to find A when you know that $\sin A = .6324$. If you look at the table, you will find that the two numbers nearest to .6324 in the sine column are .6293 and .6428. Sin 39° is .6293, and sin 40° is .6428. The difference between 39° and 40° is 60′. The difference between .6293 and .6428 is .0135. The difference between .6293 and the given value .6324 is .0031. Just as in § 149, it is assumed that a small change in the angle is nearly proportional to the change in the sine. You then have the following:

$$\frac{x}{60} = \frac{.0031}{.0135}.$$

$$\therefore x = \frac{60(.0031)}{.0135} = 14 \text{ approximately.}$$

Hence, if $\sin A = .6324$, $A = 39° \ 14'$ to the nearest minute.

EXERCISES

Find to the nearest minute the angles corresponding to the following:

1. $\tan A = .6372$.

Solution:

$$\begin{aligned} \tan 32° &= .6249. \\ \tan 33° &= .6494. \\ \text{Difference} &= .0245. \\ \tan 32° &= .6249. \\ \tan A &= .6372. \\ \text{Difference} &= .0123. \end{aligned}$$

Therefore $$\frac{x}{60} = \frac{.0123}{.0245} = \frac{123}{245}.$$

$$\therefore x = \frac{60(123)}{245}.$$

$$x = 30 \text{ approximately.}$$

$$\text{Hence } A = 32° \ 30'.$$

2. $\sin A = .4336$.

3. $\cos A = .9190$.

4. $\tan A = .5434$.

5. $\cos A = .4295$.

6. $\sin A = .8845$.

7. $\tan A = 1.1640$.

8. The hypotenuse of a right triangle is 50 inches, and one of the arms is 40 inches. Find the value of each of the acute angles to the nearest minute.

9. Two forces, AB and AC, act at right angles to each other. Find the resultant force AD if AB represents a force of 12 pounds and AC one of 38 pounds. Find the angle which the resultant makes with the second force.

10. A ship sails from latitude 48° 20′ N. along a course AB (Fig. 75), running N.W. a distance of 653 nautical miles. Find the latitude arrived at.

Suggestion: One nautical mile is approximately 1 minute. Find NA in miles, change it to degrees and minutes, and add that to 48° 20′.

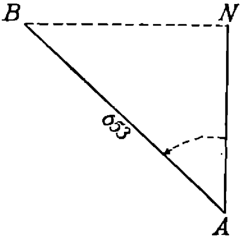

FIGURE 75

151. Miscellaneous exercises. The following exercises give further practice in the use of the trigonometric ratios in solving problems.

EXERCISES

1. If $\sin x = \frac{3}{4}$, make a drawing and find angle x by measurement.

2. The equal sides of an isosceles triangle are 20 feet long, and the base is 28 feet long. Find the length of the altitude and the number of degrees in each base angle.

3. What is the pitch of a roof if it makes an angle of 32° with the horizontal?

Suggestion: By the *pitch* of a roof is meant the ratio of the height to the horizontal.

4. The rope of a flagpole makes an angle of 42° with the ground at a point 29.5 feet from the foot of the pole. Find the height of the pole.

5. From the top of a cliff 95 feet high the angle of depression of a boat is 31°. How far is the boat from the foot of the cliff?

EXERCISES TO TEST YOUR UNDERSTANDING OF CHAPTER VIII

152. What you should be able to do. Having studied Chapter VIII, you should be able to do the following:

1. To state the meaning of the sine, cosine, and tangent of an angle.

2. To find the value of the sine, cosine, and tangent of a given acute angle:

   (*a*) by measuring.
   (*b*) by using the table.

3. To solve problems involving right triangles by means of the trigonometric ratios.

4. To state the meaning of angle of elevation and of angle of depression.

EXERCISES

1. Draw an angle equal to 58° and find by measurement sin 58°, cos 58°, tan 58°.

2. Find the angle whose sine is .9781.

3. Find the height of a pole whose shadow is 23 feet long when the angle of elevation of the sun is 58°.

4. Find the height of a tower if the angle of depression of an object on the ground 125 feet from the foot of the tower is 40°.

5. Find the diagonal of a square whose sides are 2.5 feet long.

6. By examining the table of sines tell how the sine ratio varies as the angle changes from 0° to 90°.

7. Tell how the cosine varies as the angle changes from 0° to 90°.

8. Find the number of degrees in the acute angles of a right triangle if the hypotenuse is 18 inches long and if one side of the right angle is $12\frac{1}{2}$ inches.

CHAPTER IX

THE USE OF LOGARITHMS

THE MEANING OF LOGARITHMS

153. How to use a table to multiply numbers. Tables are used to save time and work in numerical computation. The tables in this chapter enable you to change the processes of multiplication and division to the simpler processes of addition and subtraction and to find powers and roots of numbers by performing very simple exercises in multiplication and division.

The following shows how a table of exponents may be used to multiply two numbers.

Arrange in the form of a table a series of numbers obtained by raising 2 to various powers (Fig. 76), as shown in the middle column. By multiplication you have $2^1 = 2$, $2^2 = 4$, $2^3 = 8$, etc. Write the numbers equal to the powers of 2 in the first column and the exponents in the third.

Use the table to multiply 16 by 64 as follows:

In the column headed N locate 16 and to the right find 2^4.

(1) N	(2) POWERS	(3) EXPONENTS
2	2^1	1
4	2^2	2
8	2^3	3
16	2^4	4
32	2^5	5
64	2^6	6
128	2^7	7
256	2^8	8
512	2^9	9
1024	2^{10}	10
2048	2^{11}	11
4096	2^{12}	12

FIGURE 76

Likewise, in column N locate 64 and to the right find 2^6.

Note that $16 \times 64 = 2^4 \times 2^6 = 2^{10}$. Why?

Locate 2^{10} in column (2) and find, to the left in column (1), the number 1024. This is the required product. To simplify the work, repeat what has been said but use column (3) instead of column (2) as shown below:

In column (1) locate 16 and, passing to column (3), find 4.

In column (1) locate 64 and, passing to column (3), find 6.

Add: $4 + 6 = 10$.

In column (3) locate 10 and, in column (1), find 1024. This is the required product.

Only one arithmetical operation has been used in finding the product, the addition of 4 and 6. The table has done the rest. The table will have to be made much more complete before it can be used to multiply any two given numbers, but it illustrates the method.

EXERCISES

Using the table (Fig. 76), find the products:

1. 32×128

2. 16×256

3. 1024×4

4. 128×8

5. 512×4

6. 16×32

7. Represent graphically the number pairs of columns (1) and (3) (Fig. 76).

Suggertion: Select $\frac{1}{10}$ of a centimeter as a unit for marking off the numbers and 1 centimeter for the exponents.

Discuss the changes in the exponents as the numbers change from 0 to 130.

8. From the graph (Fig. 77) find the exponents corresponding to the numbers 3, 12, 26, 58, 96.

9. Show how the graph (Fig. 77) may be used to multiply 5 by 24.

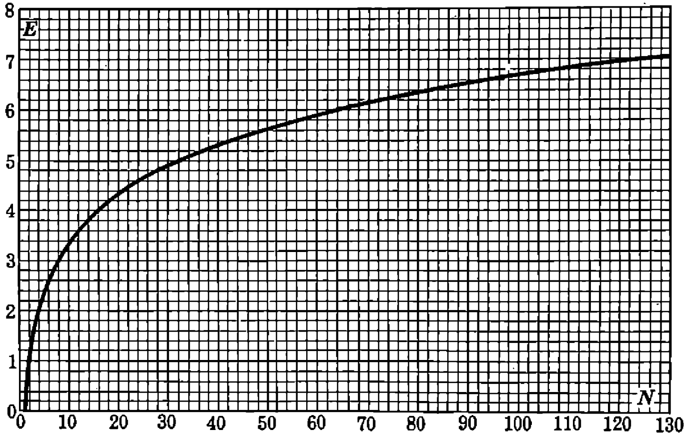

FIGURE 77

LOGARITHMIC CURVE

154. How to use a table to divide numbers. Fig. 76 may be used to divide one number by another. For example, to divide 4096 by 256, locate 4096 in column (1) and, to the right, find 2^{12} in column (2).

Then locate 256 in column (1) and, to the right, find 2^{8} in column (2).

$$\text{Since } \frac{4096}{256} = \frac{2^{12}}{2^{8}} = 2^{12-8} = 2^{4},$$

locate 2^{4} in column (2) and, to the left, find 16.

The required quotient is 16.

The only arithmetical operation used has been the subtraction of 8 from 12.

As in multiplying, the quotient may be found in a simple way from columns (1) and (3). The steps are as follows:

Locate 4096 in column (1) and, to the right, find 12.
Locate 256 in column (1) and, to the right, find 8.
Subtracting 8 from 12, you have 4.
Locate 4 in column (3) and, to the left, find 16.
This is the required quotient.

EXERCISES

By means of the table (Fig. 76) find the quotients of the following:

1. 4096 ÷ 64

2. 128 ÷ 32

3. 512 ÷ 128

4. 256 ÷ 64

5. 2048 ÷ 32

6. 1024 ÷ 256

155. What is meant by a logarithm. The *exponents* in column (3) (Fig. 76) are called **logarithms.** The table is a **logarithmic table.** Thus 8 is the logarithm which corresponds to the number 256. Briefly we say that "the logarithm of 256 is 8 if 2 is used as base." Tables could be constructed with other numbers as bases just as the table of Fig. 76 was formed with the base equal to 2.

In general, it is said that *the logarithm of a number* N *is the exponent to which a base must be raised to give a power equal to* N.

Thus, since $2^8 = 256$, $\log_2 256 = 8$. This is read "the logarithm of 256 to the base 2 is 8."

The statements $2^8 = 256$ and $\log_2 256 = 8$ give the same information in different forms.

EXERCISES

Using the table of Fig. 76, find the logarithms of the numbers below. State your results as shown in Exercise 1.

1. 128

Solution: $\log_2 128 = 7$.

Check: $2^7 = 128$.

2. 256

3. 1024

4. 8

5. 64

6. 2

7. 16

8. 4096

9. 4

10. 512

COMMON LOGARITHMS

156. A table of logarithms to the base 10. The table of logarithms (Fig. 76) was made by using 2 as a base. In the tables which are generally used for computation the base is 10.

Proceeding as in § 153, verify the table of Fig. 78. In this table the first column contains the numbers, the second the powers of 10 equal to the numbers, and the third the exponents, or logarithms.

N	POWERS	L
100,000	10^5	5
10,000	10^4	4
1,000	10^3	3
100	10^2	2
10	10^1	1
1	10^0	0
.1	10^{-1}	-1
.01	10^{-2}	-2
.001	10^{-3}	-3
.0001	10^{-4}	-4

FIGURE 78

What is the value of 10^0? What is the value of 10^{-1} and of 10^{-2}?

A logarithmic table whose base is 10 is called a table of **common logarithms.**

The base 10 is usually not written. It is understood that any base other than 10 must always be indicated. Thus $\log_{10} 100 = 2$ is written more simply: $\log 100 = 2$.

EXERCISES

Find the logarithms of the following numbers:

1. 10,000

2. 1

3. $\frac{1}{10}$

4. $\frac{1}{100}$

5. 100,000

6. $\frac{1}{1000}$

7. Find the logarithm of $\frac{1}{3}$ to the base 3.

Solution: Since $3^{-1} = \frac{1}{3}$, it follows that $\log_3 \frac{1}{3} = -1$.

Find the following logarithms and give the reason for each:

8. $\log_4 16$

9. $\log_3 9$

10. $\log_6 216$

11. $\log_3 243$

12. $\log_5 25$

13. $\log_5 625$

14. $\log_7 49$

15. $\log_8 512$

16. $\log_3 81$

17. $\log_3 729$

18. $\log_4 64$

19. $\log_6 36$

20. Simplify $\log_2 4 + \log_3 81 + \log_4 64$.

21. Simplify $\log_2 64 - \log_3 9 + \log_2 1$.

157. Logarithms of numbers not contained in Fig. 78. The table of Fig. 78 contains only a few logarithms. However, it gives some information about the logarithms of numbers which do not appear in the table. You can see that the logarithms of all numbers between 10 and 100 are greater than 1 and less than 2. It follows that the logarithm of 58 must be 1 plus a fraction. Likewise, the logarithm of 258 must be 2 plus a fraction, because 258 lies between 100 and 1000.

All logarithms, including those in the table, consist of two parts, the integral (whole) part and the fractional part. The fractional part is always expressed as a decimal fraction. For the numbers in the table (Fig. 78) the fractional part is zero.

The logarithm of 246.5 lies between 2 and 3, since 246.5 lies between 100 and 1000. Therefore the integral part of the logarithm is 2. The fractional part is given in special tables (pages 236 and 237).

The fractional part of log 246.5, expressed to four figures, is .3918. Hence log 246.5 = 2.3918.

The integral part of a logarithm is determined by a simple rule which you will learn in § 159.

158. Special names to denote the parts of a logarithm. The integral part of a logarithm is called the **characteristic,** the fractional part the **mantissa.**

EXERCISES

The following are specimens of logarithms. For each state the characteristic and the mantissa. Verify the correctness of the characteristics by the table of Fig. 78.

1. log 6315 = 3.8004.

2. log 65.24 = 1.8145.

3. log 6.608 = 0.8201.

4. log 2.36 = 0.3729.

5. log 48.731 = 1.6878.

6. log 1340.3 = 3.1272.

HOW TO FIND THE LOGARITHM OF A NUMBER

159. A rule for determining the characteristic. You have seen that the logarithm of 100 is 2, and that the logarithm of 1000 is 3, also that the logarithm of any number from 100 to 1000 is 2 plus a fraction. In other words, any number from 100 to 1000 has a logarithm whose characteristic is 2. Explain why it is that any number from 1000 to 10,000 has a logarithm whose characteristic is 3 and that any number from 10 to 100 has a logarithm whose characteristic is 1. What is the characteristic of the logarithm of any number from 1 to 10?

Find the characteristics of the logarithms of the following numbers: 428; 3485; 429.7; 63.42; 1.93; 83,760.

You have seen also that the logarithm of .01 is −2, and that the logarithm of .1 is −1. State a number which lies between .01 and .1. What is the characteristic of the logarithm of the number you have stated? State a number which lies between .1 and 1 and another which lies between .001 and .01. What are the characteristics of the logarithms of the numbers you have stated?

Find the characteristics of the logarithms of the following numbers: .0723; .1257; .0923; .0047; .324; .0125.

You can now find the characteristic of the logarithm of any number. The method you have used is one which helps you to understand the meaning of characteristic. It is easier, however, to find the characteristic by means of a simple rule. Consider the table (Fig. 79) of numbers and characteristics.

NUMBER	CHARACTERISTIC
63,462	4
4,371.6	3
561.3	2
72.47	1
9.82	0
.472	− 1
.0723	− 2
.0054	− 3

FIGURE 79

First show that the characteristics are correct by the method you have already learned. Then notice that in each case you can determine the characteristic by counting. *Start from the units' place of the number and count* 0, 1, 2, 3, *etc., until you come to the left-most figure which is not zero. The characteristic is + if you count to the left; it is − if you count to the right.*

For example, to find the characteristic of the logarithm of 326.45, point to the 6, which is in the units' place, and say, "0"; point to the 2 and say, "1"; point to the 3 and say, "2." You have counted to the left; hence the characteristic is +2.

You will see from the preceding discussion that *the characteristic depends entirely upon the position of the decimal point in the number.*

EXERCISES

Find the characteristics of the logarithms of the following numbers:

1. 6531.8
2. 3485
3. 3072
4. 429.634
5. .859
6. .0642
7. 1.648
8. 42.241
9. .00952
10. .000463
11. .0406
12. .003005

160. The mantissa does not depend upon the position of the decimal point. You have seen that the characteristic of a logarithm indicates the position of the decimal point in the number. The following shows that the mantissa is independent of the decimal point.

It is known that

$$\log 4326 = 3.6361, \text{ that is, } 10^{3.6361} = 4326.$$

Dividing the last equation by 10, you have

$$\frac{10^{3.6361}}{10^1} = \frac{4326}{10},$$

or $$10^{2.6361} = 432.6, \text{ that is, } \log 432.6 = 2.6361.$$

Dividing again by 10, you have

$$10^{1.6361} = 43.26, \text{ that is, } \log 43.26 = 1.6361.$$

Likewise, $10^{.6361} = 4.326$, that is, $\log 4.326 = 0.6361$.

Thus, whenever you divide by 10 or by a power of 10, you change the position of the decimal point in the number and the characteristic of the logarithm, but leave the mantissa the same. The numbers 432.6, 43.26, 4.326, etc., all have the *same* mantissa. In general, it is said that *numbers which differ only in the position of the decimal point have the same mantissa.*

LOGARITHMS OF NUMBERS

N	0	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474

LOGARITHMS OF NUMBERS

N	0	1	2	3	4	5	6	7	8	9
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633
92	9638	9643	9647	9652	9657	9661	9666	9671	9375	9680
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996
100	0000	0004	0008	0013	0017	0021	0026	0030	0035	0039

Mantissas are computed by methods studied in advanced courses in mathematics. We are interested only in the use of the mantissas in computations. For this purpose they have been arranged in tabular form on pages 236 and 237.

161. The table of logarithms. The four-figure numbers in the tables (pages 236 and 237) are mantissas approximated to four figures. For convenience the decimal points have been left off. Thus the mantissa corresponding to 35 is .5441. For ordinary purposes it is not necessary to have the mantissas carried beyond the fourth figure. When greater accuracy is required, larger tables must be used in which the mantissas are worked out to five or more figures.

162. Finding the logarithm of a two-figure number. To find the logarithm of 42, determine first the characteristic by rule. This gives 1. Then locate 42 in the column headed N and to the right, in the column headed 0, find the mantissa .6232.

$$\therefore \log 42 = 1.6232.$$

EXERCISES

Verify the following:

1. $\log 4.2 = 0.6232$.

2. $\log 89 = 1.9494$.

3. $\log 6.3 = 0.7993$.

4. $\log 63 = 1.7993$.

5. $\log 56 = 1.7482$.

6. $\log 7.2 = 0.8573$.

7. $\log 10 = 1.0000$.

8. $\log 200 = 2.3010$.

9. $\log 630 = 2.7993$.

10. $\log 100 = 2.0000$.

163. Finding the logarithm of a three-figure number. The first two figures of the numbers whose logarithms are to be found are in column N. The third figures are in the top row of the table. The column headed 0 contains the mantissas for the numbers 0 to 100. The mantissas for the numbers 100 to 109 are in the first row, for 110 to 119 in the second row, for 120 to 129 in the third, etc. To find the mantissa of 438, locate 43 in column N and pass to the right to the column headed 8, where you find 6415.

$$\therefore \log 438 = 2.6415.$$

EXERCISES

Verify the following:

1. log 34.6 = 1.5391.

2. log 98.3 = 1.9926.

3. log 532 = 2.7259.

4. log 5.87 = 0.7686.

5. log 3.65 = 0.5623.

6. log 284 = 2.4533.

7. log 340 = 2.5315.

8. log 2.50 = 0.3979.

164. Finding the logarithm of a four-figure number. The examples which are worked out in full in this section show how to find the logarithm of a number containing four figures. In a five-place table such logarithms can be found directly, but the method explained here would then apply to logarithms of five-figure numbers.

1. Find log 338.2.

Solution: Since 3382 lies between 3380 and 3390, the mantissa of 3382 lies between the mantissa of 3380 and the mantissa of 3390. Since the difference between 3380 and 3390 is 10, 3382 is $\frac{2}{10}$ of the way between 3380 and 3390. The mantissa of 3382, therefore, lies approximately $\frac{2}{10}$ of the way between the mantissas of 3380 and 3390. In the table you find the mantissa of 3380 = 5289 and the mantissa of 3390 = 5302.

$\therefore$ the mantissa of 3382 $= 5289 + \frac{2}{10}(5302 - 5289)$
$= 5289 + \frac{2}{10}(13) = 5289 + 2.6.$

Since 2.6 is nearer to 3 than to 2, add 5289 and 3.

$$\therefore \log 338.2 = 2.5292.$$

Nearly all the work of finding a logarithm should be done mentally. In practice you should write only the most necessary parts, as shown below.

Solution: Write log 338.2 =

The characteristic is 2. *Write* 2 to the right of the equality sign.

In row 33, column headed 8, find the mantissa 5289.

The next mantissa in the table is 5302.

The difference between these two mantissas (the tabular difference) is 13.

$$\frac{2}{10} \text{ of } 13 = 2.6.$$

Drop the 6 and increase the 2 to 3. Add 3 to 5289.

Write 5292 to the right of the characteristic.

2. Find log 46.18.

Solution: Write log 46.18 =

The characteristic is 1. Write the 1.

In row 46, column headed 1, find 6637.

Next to it find 6646.

The tabular difference is 9.

$$\frac{8}{10} \text{ of } 9 = 7.2.$$

Drop the 2, add 7 to 6637, and write 6644.

$$\therefore \log 46.18 = 1.6644.$$

EXERCISES

Find the following logarithms:

1. log 5.463
2. log 3628
3. log 80.31
4. log 933.0
5. log 18.42
6. log 289.4
7. log 3.521
8. log 40.54
9. log 236.5

10. log 2995
11. log 5.613
12. log 629.1
13. log 3.062
14. log 8774
15. log 20.49
16. log 4.206
17. log 135.2
18. log 98.15
19. log 6487
20. log 2.165
21. log 32.93
22. log 7.805
23. log 764.2
24. log 457.0
25. log 80.28
26. log 8.758
27. log 431.9

165. Logarithms with negative characteristics. When the characteristic of a logarithm is negative, it simplifies the work of computation to let $-1 = 9 - 10$, $-2 = 8 - 10$, $-3 = 7 - 10$, $-4 = 6 - 10$, etc.

Thus log .432 has the characteristic -1 and the mantissa .6355.

$$\therefore \log .432 = -1 + .6355 = 9.6355 - 10.$$

Similarly $\log .0432 = 8.6355 - 10,$

and $\log .00432 = 7.6355 - 10.$

EXERCISES

Find the following logarithms:

1. log .0541

Solution: The characteristic is -2, or $8 - 10$.
The mantissa is .7332.
$\therefore \log .0541 = 8.7332 - 10.$

2. log .362
3. log .0587
4. log .00364
5. log .04582
6. log .6329
7. log .2583
8. log .0006958
9. log 9.582
10. log .006524
11. log 8.317
12. log .9423
13. log 42.28
14. log .03416
15. log 75.34
16. log .8002
17. log .3200
18. log 28.00
19. log .07901

166. Finding the number corresponding to a given logarithm. This is the reverse of the process of finding a logarithm. The following examples show the method:

1. Find the number whose logarithm is 2.6599.

Solution: Look among the mantissas until you find 6599.

Pass to the left and, in column N, find 45. Hence the first two figures of the required number are 4, 5.

Pass upward from 6599 and find 7 at the top of the column. This is the third figure of the required number.

Hence 4, 5, 7 are the figures of the required number.

The characteristic, 2, is now used to place the decimal point. Starting from the left-most figure, 4, count to the right 0, 1, 2 and then place the decimal point to the right.

Therefore the required number is 457.

The solution is now repeated by omitting all explanations:

Among the mantissas find 6599.

In column N to the left find 45. *Write* 45.

At the top find 7. *Write* 7.

Beginning at 4, count to the right 0, 1, 2 and place the decimal point to the right of the 7.

2. Find the number whose logarithm is 4.5092.

Solution: Among the mantissas find 5092.

To the left, in column N, find 32. *Write* 32.

At the top find 3. *Write* 3.

Since the characteristic is 4, beginning at the left-most figure count to the right 0, 1, 2, 3, 4, annexing two zeros, and place the decimal point to the right.

The required number is 32,300.

Similarly find the numbers whose logarithms are 1.6107; 5.5465; 3.7143.

3. Find the number whose logarithm is 8.6637 −10.

Solution: Find the mantissa 6637 in the table.

To the left find 46; at the top find 1.

Hence the first three figures of the required number are 4, 6, 1.

Since the characteristic is −2, start, as before, from the leftmost figure and count to the *left* 0, 1, 2. In doing this place two zeros to the left of 461. This gives the figures 0, 0, 4, 6, 1. Then place the decimal point to the right, obtaining the required number: 0.0461.

Note that the essential difference between this problem and the preceding ones is the *direction* of counting.

Verify the following:

8.5658 − 10 = log 0.0368.

9.7372 − 10 = log 0.546.

7.3483 − 10 = log 0.00223.

FIGURE 80. — JOHN NAPIER

John Napier (1550–1617), a wealthy Scotch baron, was the inventor of logarithms. He had tried for years to find simple ways of performing multiplications and divisions. He derived his logarithms from the study of arithmetical and geometric progressions and the binomial theorem, not from exponents, for the theory of exponents was not sufficiently developed and understood in his time. The table of common logarithms with the base 10 was constructed by Henry Briggs (1560–1630), an English professor, who had read Napier's work and had gone to Scotland for a conference with Napier about his discovery. But the whole world is indebted to Napier for his great invention.

4. Find the number whose logarithm is 0.3201.

Solution: Locate the mantissa 3201 in the table. *Write* 209.

Beginning with the left-most figure, 2, count 0 and place the decimal point to the right.

Hence 0.3201 = log 2.09.

Find the numbers corresponding to the following logarithms: 0.4900; 0.5502; 0.4014.

5. Find the number whose logarithm is 2.4238.

Solution: Locate the mantissa which is less than 4238, but nearest to it. This is 4232.

To the left find 26; on top find 5. *Write* 265.

Hence the required number lies between 2650 and 2660.

The difference between the mantissas is

$$4238 - 4232 = 6.$$

The tabular difference is 4249 − 4238 = 11.

$$\therefore \frac{6}{11} = \frac{x}{10},$$

where x is the fourth figure of the required number.

$$\therefore x = \tfrac{60}{11} = 5.5, \text{ or } 5 \text{ approximately.}$$

The required number is made up of the figures 2, 6, 5, 5.

Placing the decimal point, you find it to be 265.5.

EXERCISES

Find the numbers corresponding to the following logarithms:

1. 0.6341
2. 1.8096
3. 2.8865
4. 8.1761 − 10
5. 0.6970
6. 2.3729
7. 1.9557
8. 0.5240
9. 9.7973 − 10
10. 2.8192
11. 8.8077 − 10
12. 1.4900
13. 9.5717 − 10
14. 8.3090 − 10
15. 2.4527

COMPUTATION BY MEANS OF COMMON LOGARITHMS

167. How to multiply numbers by means of logarithms. You have seen (§ 153) that numbers can be multiplied by means of a table of exponents, or logarithms. The following steps are involved:

1. The logarithms are found and added.
2. The number corresponding to the sum is found. This is the required product.

Thus the work of finding the product

42.3 × 6.89 × .341 may be arranged as follows:

Outline:

$$\begin{array}{lr} & \log 42.3 = \\ & \log 6.89 = \\ & \log .341 = \\ \hline \text{The sum} & = \\ \text{The required product} & = \end{array}$$

In all computation the outline should be made before the table is used.

Verify the following solution:

Solution:

$$\begin{array}{rcl} \log 42.3 & = & 1.6263. \\ \log 6.89 & = & 0.8382. \\ \log .341 & = & 9.5328 - 10. \\ \hline \log \text{N} & = & 11.9973 - 10 \\ & = & 1.9973. \\ \therefore \text{N} & = & 99.38. \end{array}$$

EXERCISES

Find the products of the following:

1. 637 × 2.43

2. 7.82 × 92.4

3. 9820 × .0234

4. 57.6 × 987

5. 13.16 × 8.742 × .0153

6. 1.38 × 84.32 × .651

7. 25.1 × 16.38 × 2.416

8. .6462 × .9865 × 12.22

9. 1.89 × 3.724 × 4.987

10. 8.675 × 53.67 × .1832

11. 15.8 × 2.53 × 12.84

12. 64.91 × 3.82 × 1.74

13. 1.582 × 23.84 × 5.46

14. .00832 × .0469 × 128.2

168. How to divide numbers by means of logarithms. You have seen (§ 154) that exponents, or logarithms, enable you to change the process of dividing numbers to that of subtracting. The following steps are involved:

1. Find the logarithms of dividend and divisor.
2. Subtract the second from the first.
3. Find the number corresponding to the difference.

Thus the work of dividing 3.18 by 1.63 may be arranged as below:

$$\begin{array}{rl} \textit{Outline:}\quad \log 3.18 & = \\ \log 1.63 & = \\ \hline \text{Difference} & = \\ \text{Quotient} & = \end{array}$$

EXERCISES

Find the quotients of the following:

1. 389.6 ÷ 4.265

$$\begin{array}{rl} \textit{Solution:}\quad \log 389.6 & = 2.5906. \\ \log 4.265 & = 0.6299. \\ \hline \text{Difference} = \log Q & = 1.9607. \\ Q & = 91.35. \end{array}$$

2. 398.6 ÷ 4.533

3. 54 82 ÷ 26.01

4. 3.852 ÷ 3.735

5. 16.51 ÷ 2.186

6. 22.22 ÷ 8.301

7. 256.2 ÷ 144.8

8. 2.958 ÷ 1.736

9. 5864 ÷ 3421

10. 900.6 ÷ 31.24

11. 8.431 ÷ .6324

In the solution below the characteristic of log 8.431 is changed from 0 to 10 − 10 to simplify the subtraction.

Solution:

$$\begin{array}{rl} \log 8.431 = & 10.9259 - 10. \\ \log \ .6324 = & 9.8010 - 10. \\ \hline \log Q = & 1.1249. \\ Q = & 13.33. \end{array}$$

12. 42.64 ÷ 85.63

Solution:

$$\begin{array}{rl} \log 42.64 = & 11.6298 - 10. \\ \log 85.63 = & 1.9326. \\ \hline \log Q = & 9.6972 - 10. \\ Q = & .4980. \end{array}$$

13. 40.31 ÷ .3172

14. .0649 ÷ .485

15. 368.8 ÷ 462.8

16. .08321 ÷ .0942

17. .00358 ÷ .000878

18. .6427 ÷ .7831

Perform the following operations by means of logarithms:

19. $F = \dfrac{3.48 \times 1.97 \times 6.314}{.8591 \times .324 \times 41.65}$.

Suggestion: Make an outline as below. Then use the table.

Outline:

$$\begin{array}{rl} \log 3.48 = & \\ \log 1.97 = & \\ \log 6.314 = & \\ \hline \text{log numerator} = & \\ \text{log denominator} = & \\ \hline \log F = & \\ F = & \end{array} \qquad \begin{array}{rl} \log \ .8591 = & \\ \log \ .324 = & \\ \log 41.65 = & \\ \hline \text{log denominator} = & \end{array}$$

20. $\dfrac{36.42 \times 28.4}{32.5 \times 31.4}$

21. $\dfrac{364.1 \times 4.87}{68.1 \times 32.58}$

22. $\dfrac{16.5 \times .832 \times .0531}{281 \times .0466}$

23. $\dfrac{1.306 \times 208 \times .00581}{3014 \times .0592 \times 3.61}$

24. $\dfrac{3.145 \times 8 \times 32.4}{1.042 \times 23 \times 16.1}$

25. $\dfrac{.00643 \times .0124 \times .00048}{3.612 \times .000312 \times .000894}$

169. How to find a power of a number. To illustrate the method, let us find the value of 3^4.

Since $3^4 = 3 \times 3 \times 3 \times 3$, the logarithm of 3^4 is $\log 3 + \log 3 + \log 3 + \log 3$, or $4 \log 3$; that is, *the logarithm of a power is found by multiplying the exponent by the logarithm of the base.*

The following is another way of seeing the truth of this principle:

Let $m = \log 3$.

This means that

$$10^m = 3.$$

$\therefore\ 10^{4m} = 3^4$ (by raising both members to the fourth power).

The last equation means that $\log 3^4 = 4m = 4 \log 3$, which is the result obtained above.

To find the value of 3^4, proceed as follows:

Solution: $\log 3 = 0.4771.$

$\log 3^4$, or $4 \log 3 = 1.9084.$

The number corresponding to this logarithm is 81.0.

$$\therefore\ 3^4 = 81.$$

EXERCISES

Find the following powers by means of logarithms:

1. 6^3
2. 18^2
3. 23^2
4. 29^2
5. 7^3
6. 4^4
7. 9^3
8. 3^6
9. 6^4

170. Changing the root of a number into a power. A knowledge of the meaning of fractional exponents, which you studied in § 88, helps you to understand the method of finding roots by logarithms. This section reviews the meaning of fractional exponents.

If we assume that the law $a^m \cdot a^n = a^{m+n}$ holds for fractional exponents, then

$$\begin{aligned} a^{\frac{1}{2}} \cdot a^{\frac{1}{2}} &= a, \\ a^{\frac{1}{3}} \cdot a^{\frac{1}{3}} \cdot a^{\frac{1}{3}} &= a, \\ a^{\frac{1}{4}} \cdot a^{\frac{1}{4}} \cdot a^{\frac{1}{4}} \cdot a^{\frac{1}{4}} &= a. \end{aligned}$$

Thus $a^{\frac{1}{2}}$ is one of the 2 equal factors whose product is a,

$a^{\frac{1}{3}}$ is one of the 3 equal factors whose product is a,

$a^{\frac{1}{4}}$ is one of the 4 equal factors whose product is a.

This means that $a^{\frac{1}{2}}$ is only another symbol for the square root of a, that is, $a^{\frac{1}{2}} = \sqrt{a}\cdot$

Similarly $a^{\frac{1}{3}}$ is the same as $\sqrt[3]{a}$; $a^{\frac{1}{4}}$ is the same as $\sqrt[4]{a}$.

In general, $a^{\frac{1}{n}} = \sqrt[n]{a}$.

171. How to use logarithms to find a root of a number. Since a root can be changed to a power with a fractional exponent, you can use the method of § 169.

The following example illustrates the process:

Find $\sqrt{8281}$.

Outline: Change $\sqrt{8281}$ to the form $8281^{\frac{1}{2}}$.

$$\begin{aligned} \log 8281 &= \\ \tfrac{1}{2}(\log 8281) &= \end{aligned}$$

Finally find the number corresponding to this logarithm.

In computation the first two steps are to be taken mentally. This leaves the solution in the form shown below:

Solution:

$$\begin{aligned} \log 8281 &= 3.9181. \\ \tfrac{1}{2}(\log 8281) &= 1.9590. \\ \therefore \sqrt{8281} &= 91.00. \end{aligned}$$

EXERCISES

Find the value of each of the following:

1. $\sqrt[5]{328.4}$

Suggestion: log. 328.4 =
$\frac{1}{5}$(log 328.4) =
$\sqrt[5]{328.4}$ =

2. $\sqrt[3]{4.325}$ **3.** $\sqrt[4]{9.863}$ **4.** $\sqrt{9584}$ **5.** $\sqrt[4]{8.762}$

6. $\sqrt{.0786}$

Suggestion: log .0786 = 8.8949 − 10.
Change to 18.8949 − 20. Why?
Dividing by 2, you have 9.4474 − 10.

7. $\sqrt{.00429}$ **8.** $\sqrt{.643}$ **9.** $\sqrt{.00582}$

10. $\sqrt{3 \times 4.12 \times 7.14}$

Suggestion: log 3 =
log 4.12 =
log 7.14 =
Sum =
$\frac{1}{2}$(Sum) =
$\sqrt{3 \times 4.12 \times 7.14}$ =

11. $\sqrt{841 \times 36 \times 16}$

12. $\sqrt[3]{6.83 \times 294.3}$

13. $\sqrt[4]{64.8 \times 7.34}$

14. $\sqrt{\dfrac{283 \times 4.631}{7.61^3}}$

Suggestion: log 283 =
log 4.631 =
Sum =
3(log 7.61) =
Difference =
$\frac{1}{2}$(Difference) =
$\sqrt{\dfrac{283 \times 4.631}{7.61^3}}$ =

log 7.61 =
3(log 7.61) =

15. $\sqrt{\dfrac{576 \times 8.14^2}{2.64 \times 13.7^2}}$ **17.** $\sqrt{\dfrac{3.14 \times 27.31}{2.738 \times 1.042 \times 12.4}}$

16. $\sqrt[3]{\dfrac{36.75 \times 25.6}{27.2 \times 18.67}}$ **18.** $\sqrt[3]{\dfrac{.0581 \times .00356}{.987 \times .00421}}$

172. The use of logarithms in problems. In the following problems logarithms should be used in all operations except in additions and subtractions.

EXERCISES

1. The area of a triangle is given by the formula $A = \dfrac{bh}{2}$. Find A if $b = 442.3$ feet and $h = 361.8$ feet.

2. The perimeter of a triangle is 438.1 units, and the radius of the inscribed circle is 39.05 units. Using the formula $A = rs$, where s denotes the semiperimeter, find A.

3. The total area of a cylinder is $T = 2\pi r(r + h)$. Find the total area of a cylinder whose height is 981 centimeters and whose base radius is 490 centimeters. Use $\pi = 3.14$.

4. Find the number of cubic feet of air in a classroom whose dimensions are 48 feet by 25.5 feet by 16.5 feet.

5. Find the simple interest on $865, at 5%, for $6\frac{1}{4}$ years.

6. The sides of a triangle are 82.4 feet, 63.1 feet, and 78.1 feet. Find the area by means of the formula $\sqrt{s(s - a)(s - b)(s - c)}$, where a, b, and c represent the sides and s half the perimeter.

7. The volume of a sphere is $V = \frac{4}{3}\pi r^3$. Find the volume of a metal sphere whose radius is 3.26 inches. Use $\pi = 3.14$.

8. One of the sides of a right triangle is found by the formula $a = \sqrt{c^2 - b^2} = \sqrt{(c + b)(c - b)}$. Find the side a if the hypotenuse c is 892.3 yards and if the other side b is 627.4 yards.

9. The amount A of an investment p, after n years, with an annual interest rate r, compounded k times a year, is given by the formula $A = p\left(1 + \frac{r}{k}\right)^{kn}$. Find the amount, after 18 years, on an investment of \$3760, with interest at 5%, compounded quarterly.

Suggestion: $p = 3760$, $r = \frac{5}{100}$, $k = 4$, $n = 18$.

10. Find the amount, after 10 years, on an investment of \$600, with interest at 5%, compounded annually.

11. What sum must be invested to yield \$5500, in 25 years, at an interest rate of 4%, compounded annually?

Suggestion: Solve the equation in Exercise 9 for p. Then use logarithms.

12. What sum would yield \$20,000, in 30 years, at 5% interest, compounded semiannually?

13. Find the sum which must be set aside for the education of a boy at the age of 1 year to yield \$2500, in 16 years, at 4% interest, compounded quarterly.

14. At what rate of interest, compounded semiannually, would an investment of \$2250 yield \$5500 after 20 years?

Solution: Using the formula of Exercise 9, you have

$$5500 = 2250\left(1 + \frac{r}{2}\right)^{40}.$$

$$\therefore \frac{\overset{\overset{22}{\cancel{110}}}{\cancel{5500}}}{\underset{\underset{9}{\cancel{45}}}{\cancel{2250}}} = \left(1 + \frac{r}{2}\right)^{40}.$$

$$\therefore \left(1 + \frac{r}{2}\right)^{40} = \frac{22}{9}.$$

$$1 + \frac{r}{2} = \sqrt[40]{\frac{22}{9}}.$$

Use logarithms to find $\sqrt[40]{\frac{22}{9}}$. Then find r.

15. At what rate will $95, compounded annually, amount to $135 in 8 years?

16. The formula $d = \sqrt[3]{\frac{65h}{n}}$ gives the number of inches in the diameter of a shaft required to transmit h horse power at a speed of n revolutions per minute. What diameter is required to transmit 120 horse power at a rate of 115 revolutions per minute?

17. The approximate velocity v of an object which has fallen s feet is found from the formula $v = \sqrt{2gs}$. Find the velocity of an object which has fallen 350 feet. Let $g = 32.16$.

EXERCISES TO TEST YOUR UNDERSTANDING OF CHAPTER IX

173. What you should be able to do. After completing the study of Chapter IX you should be able to do the following:

1. To find the logarithm of a given number.
2. To find the number corresponding to a given logarithm.
3. To use logarithms to perform the processes of multiplying, dividing, raising to a power, and extracting roots.
4. To solve problems involving formulas which are more easily evaluated by logarithms than by actually performing the indicated operations.

EXERCISES

1. Find the logarithms of 73.52; .7352; .007352.

2. Find the numbers whose logarithms are 2.4849; 9.4849 − 10; 7.4849 − 10.

3. Find the product 12.24 × .0583 × 28.12.

4. Find the quotient $9.643 \div .8416$.

Find the value of the following:

5. $\dfrac{26.31 \times 1.289}{78.42 \times .3651}$

6. $68^2 \times \sqrt[3]{249}$

7. The volume of a cylinder is given by the formula $V = \pi r^2 h$. Find the volume if $r = 1.36$ inches and $h = 16$ inches.

8. Simple interest, as you know, is computed by means of the formula $i = prt$. Find i when $p = 5283$, $r = .06$, $t = 8$.

9. The lateral area of a frustum of a cone (the part of a cone between the base and a plane parallel to the base) is given by the formula $S = \pi s\left(\dfrac{D + d}{2}\right)$. Find the lateral area if the diameters of the bases are $D = 6.21$ inches, $d = 5.38$ inches and the slant height $s = 5.38$ inches. Use $\pi = 3.14$.

CHAPTER X

THE SLIDE RULE

THE LOGARITHMIC SCALE

174. Where the slide rule is used. In Chapter IX you used logarithms to perform the operations of multiplying, dividing, extracting roots, and finding powers. People whose work calls for a large amount of computation use freely any short cuts and devices that save time and labor. Various mechanical computing machines have been invented to do much of the arithmetical work for them. One machine familiar to all of us is the adding machine used in stores and offices. Other devices for making computations are tables of powers, roots, and logarithms.

FIGURE 81
USING AN ADDING MACHINE

One of the best instruments for performing difficult arithmetical processes with ease and speed is the **slide rule** (Fig. 87).[1] It is being used more and more by statisticians and others engaged in scientific work, as well as by men doing practical work in shops and offices.

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[1] William Oughtred (1574–1660) is the inventor of the slide rule. He was one of the greatest writers of mathematics in the early part of the seventeenth century. He wrote books on arithmetic, algebra, and logarithms. Although he studied much and slept little, he reached the age of eighty-six years.

In school work you will find the slide rule helpful wherever arithmetical computations arise.

175. How to construct a logarithmic scale. The **logarithmic scale** is really a table of logarithms marked off to scale along the edge of a ruler. The following table gives the logarithms of the whole numbers 1, 2, 3, . . . 10.

Number. . .	1	2	3	4	5	6	7	8	9	10
Logarithm	0	.3010	.4771	.6021	.6990	.7782	.8451	.9031	.9542	1

Let the length of AB (Fig. 82) represent the unit 1. Then $AB = 1 = \log 10$. It is immaterial what length is selected for AB.

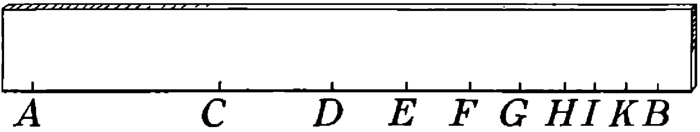

FIGURE 82

If AC is $\frac{30}{100}$ of AB and AD is $\frac{48}{100}$ of AB, these two lengths will represent, respectively, the logarithms of 2 and 3.

Similarly:

$AE = .60 = \log 4.$

$AF = .70 = \log 5.$

$AG = .78 = \log 6.$

$AH = .85 = \log 7.$

$AI = .90 = \log 8.$

$AK = .95 = \log 9.$

In Fig. 83 the numbers whose logarithms are represented by AC, AD, etc., are written above the points C, D, etc. Since log 1 is 0, the number 1 is written over point A.

Thus the distance from the index 1 to any number is the logarithm of that number. *AB* is a logarithmic scale.

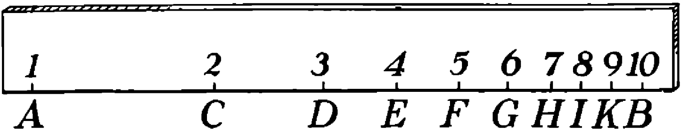

FIGURE 83

176. A close examination of a logarithmic scale. With the scale as in Fig. 83 you have the logarithms of one-figure numbers only. To make the scale more useful, the divisions already marked off must be subdivided. As you have seen, *AC* (.30 *AB*) represents the logarithm of 2. Since the logarithm of 1.5 is .1761, *AM* (.18 *AB*) (Fig. 84) represents the logarithm of 1.5. (In Fig. 84, *AC* is made longer than in Fig. 83 in order to show the subdivisions better.) What lengths represent 1.1, 1.2, 1.3, 1.4, 1.6, 1.7, 1.8, 1.9 (Fig. 84)?

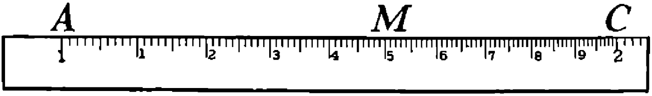

FIGURE 84

Similarly subdivide the distances *CD*, *DE*, . . . *KB* and let the subdivisions represent the logarithms of 2.1, 2.2, 2.3 . . .; 3.1, 3.2, 3.3, . . . You will then have on the logarithmic scale the logarithms of two-figure numbers.

To represent the logarithms of three-figure numbers, you must subdivide again. If you subdivide to more than three figures, the divisions will be too small to be of practical use. Furthermore, for most computations which have to be made with numbers found by measurement, the results are sufficiently accurate if given in three-figure numbers. On some slide rules a magnifying glass is provided to take the readings of the small divisions.

By making *AB* (Fig. 82) sufficiently large the graduations may be increased and a greater degree of accuracy attained in reading off the numbers corresponding to given logarithms. A scale 10 inches long gives the numbers to three figures. When greater accuracy is required a more elaborate instrument has to be used.

HOW THE SLIDE RULE IS CONSTRUCTED

177. A description of the slide rule. The slide rule (Fig. 87) has four logarithmic scales, *A*, *B*, *C*, and *D*. Scales *C* and *D* are single scales (Fig. 85) extending from 1 to 10. Scales *A* and *B* are double scales (Fig. 86) extending from 1 to 100.

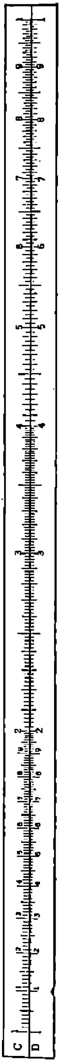

FIGURE 85

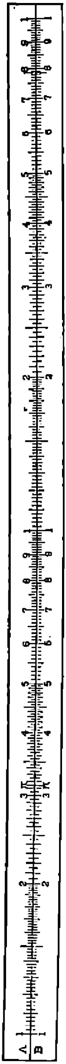

FIGURE 86

That the two parts of scales *A* or *B* must be identically the same follows from the fact that the number pairs 1, 10; 2, 20; 3, 30; etc., have the same mantissa but different characteristics.

Thus from the table you find that

$$\begin{aligned}\log\ 2 &= 0.3031,\\ \log 10 &= 1.0000,\\ \log 20 &= 1.3031.\end{aligned}$$

Comparing the sum of the first two logarithms with the third logarithm, you have

$$\log 20 = \log 10 + \log 2.$$

Hence, to locate log 20 on the scale, lay off log 10 first and then log 2 in the same direction.

Similarly, to locate log 30 on the scale, lay off log 10 first and then log 3 in the same direction. Thus scale *A* should read 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 20, 30, 40, 50, 60, 70, 80, 90, 100.

Note that a line representing a logarithm on scale *C* is twice as long as the line representing the same logarithm on scale *A*. Furthermore, log 6 on scale *C* is equal to 2 log 6, or log 6^2, or log 36, on scale *A*.

Scales *A* and *D* are fixed, but scales *B* and *C* can be made to slide (Fig. 87).

The taking of the readings is facilitated by using the runner, which can be moved to any desired point on the scales.

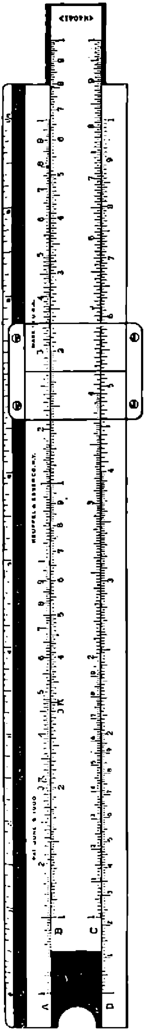

FIGURE 87
SLIDE RULE

EXERCISES

1. Locate the following numbers on scale *C*:

2	4	7	3.4	3.7	3.9
1.2	1.5	1.9	4.9	5.2	7.1
2.3	2.5	2.8	8.4	9.8	9.5

2. Locate on scale *A* the numbers given in Exercise 1.

3. Locate on scale *A* the following numbers:

12	15	19	49	52	61
23	25	28	78	82	87
34	37	39	91	95	99

4. Locate on scale *C*:

1.24	2.96	2.65	6.45
1.83	2.72	2.53	8.35
1.76	2.84	2.47	9.15

Performing Operations with the Slide Rule

178. How to multiply with the slide rule. You must remember that the scales on the slide rule are logarithmic tables represented geometrically. The numbers whose logarithms are required are printed on the scales. The mantissas are the lengths on the scale counted from the index 1 on the left end. The characteristics are determined by inspection.

Scales *C* and *D* are constructed to a larger scale than *A* and *B*. Therefore more accurate results can be obtained with them in multiplying numbers than with *A* and *B*.

The method of multiplying is illustrated in the following examples:

1. Find the product 2 × 3.

Solution: Move the sliding scale C until the index 1 of scale C is exactly over 2 on scale D (Fig. 88).

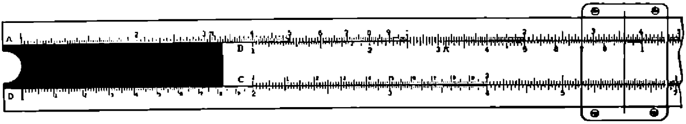

FIGURE 88

Pass along scale C to 3. Directly under 3 is the number 6. This is the required product.

Explanation: The distance from 1 to 2 on scale D is log 2.

The distance from 1 to 3 on scale C is equal to log 3.

Since the sum of the two distances is equal to the distance from 1 to 6 on scale D, it follows that this is equal to log 2 + log 3, or log 2 × 3, or log 6.

2. Find the product 1.65 × 2.34.

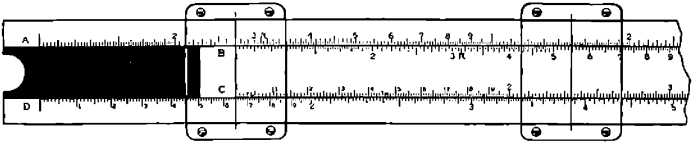

FIGURE 89

Solution: Place the runner on 1.65, scale D (Fig. 89).

Move scale C until the index 1 is exactly above 1.65 on scale D.

Move the runner to 2.34, scale C.

Then take the reading shown by the runner on scale D.

The result is 3.86 approximately. Verify the result by actual multiplication.

3. Find the product $2.4 \times 3 \times 1.61$.

Solution: Place the runner on 2.4 on scale D.

Move scale C until the index 1 is exactly over 2.4.

Move the runner to 3 on scale C.

Move scale C until the index 1 is under the line on the runner.

Move the runner to 1.61 on scale C.

The runner now extends beyond scale D, and the product $2.4 \times 3 \times 1.61$ cannot be read off. Whenever this happens, move the runner back to the preceding position, in this case to 2.4×3 on scale D. Then move the slide back until the index 1 on the *right* is under the line on the runner. Then move the runner to 1.61 on scale C.

On scale D the reading indicated by the runner shows the first three figures of the product to be 116.

Place the decimal point by inspection and obtain the final result, 11.6.

EXERCISES

Find the products of the following with the slide rule:

1. 3×4
2. 6×8
3. 2.5×1.6
4. 1.75×4
5. $4 \times 2 \times 3$
6. 1.62×2.48
7. $.45 \times 1.37$
8. 4.16×2.13
9. 6.15×1.32
10. 5.81×1.48
11. 4×84
12. 2.6×9.2
13. 5.4×3.12
14. 1.75×8.6
15. 2.34×8.12
16. 6.41×2.15
17. 11.2×1.46
18. 8.36×4.14
19. 15.1×12.6
20. $5.1 \times 4 \times 1.2$
21. $3.14 \times 6 \times 2.16$
22. $4.8 \times 3.6 \times 2.5$
23. $7.2 \times 4.1 \times 1.8$
24. $3.6 \times 1.2 \times .84$

179. How to divide with the slide rule. Division is the inverse of multiplication. You have found previously (§ 168) that in performing a division by logarithms you proceed as in multiplication, but you *subtract* the logarithms instead of adding them. The use of the slide rule for performing a division will be shown in the following examples:

1. Find the quotient 8 ÷ 4.

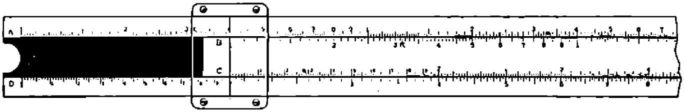

FIGURE 90

Solution: Locate 8 on scale *D* (Fig. 90).

Place the runner over 8.

Move the slide until the 4 is exactly above the 8.

Pass along scale *C* to the index 1.

Read on scale *D* the number below the index 1 on scale *C*.

This is the required quotient.

2. Find the quotient 268 ÷ 24.

Solution: As before locate 268 on scale *D*.

Move the slide until 24 falls above 268.

On scale *D* read the number exactly below the index 1 on scale *C*. This gives 112.

The decimal point is placed by inspection. The quotient $\frac{268}{24}$ must be approximately 10. Hence the correct answer is 11.2.

EXERCISES

Find the quotients for the following with the slide rule:

1. 6 ÷ 3
2. 3 ÷ 6
3. 14 ÷ 7
4. 36 ÷ 9
5. 10 ÷ 2.5
6. 16 ÷ 3.6
7. 14 ÷ 8.4
8. 18 ÷ 6.7
9. 12.4 ÷ 8
10. 18.3 ÷ 5
11. 14.8 ÷ 3
12. 7.12 ÷ 6
13. 14.1 ÷ 2.16
14. 7.18 ÷ 12.2
15. 16.5 ÷ 8.8
16. 9.42 ÷ 3.18

180. How to solve proportions by means of the slide rule. Solving the proportion $\frac{a}{b} = \frac{c}{d}$ for one of the literal numbers, as a, you have $a = \frac{bc}{d}$. To find the value of $\frac{bc}{d}$, divide b by d and multiply the quotient by c. The following example illustrates the method when the slide rule is used:

Find the value of x satisfying the equation

$$\frac{x}{3.42} = \frac{19.1}{61.6}.$$

Solution: Solving for x, you have $x = \frac{19.1 \times 3.42}{61.6}$.

Paying no attention to the decimal points in the numbers, move the runner to 191 on scale D (Fig. 91).

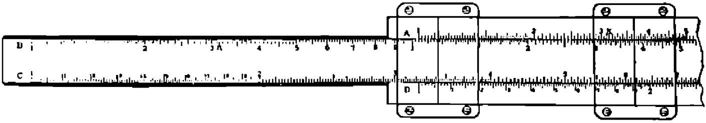

FIGURE 91

Move 616 on scale C to the runner.

Since you are not interested in the quotient, do not stop to read it but note the fact that it is on scale D below the index 1 of scale C.

Move the runner to 342 on scale C and take the reading on scale D. This is 106.

Locate the decimal point by inspection as follows: $19 \div 61$ is about $\frac{1}{3}$, and $\frac{1}{3} \times 3$ is 1.

Hence the result is 1.06.

EXERCISES

Using the slide rule, find the value of x in each of the following:

1. $x = \dfrac{8 \times 6}{7}.$

2. $x = \dfrac{9.5 \times 3.2}{8}.$

3. $\dfrac{x}{3} = \dfrac{2}{5}.$

4. $\dfrac{x}{2.5} = \dfrac{3}{6.7}.$

5. $x = .04 \times 2000 \times 4.5.$

6. $x = \dfrac{26.3 \times 4.16}{98.1}.$

7. $x = \dfrac{42.6 \times 3.41}{162}.$

8. $56.1x = 18.6 \times 3.42.$

9. $\dfrac{x}{18.2} = \dfrac{2.41}{36.5}.$

10. $\dfrac{x}{1.28} = \dfrac{21.5}{84.2}.$

181. How to carry on a continuous series of computations. The following example shows how a series of multiplications and divisions may be performed with the slide rule:

Find the value of

$$\frac{3.64 \times 14.9 \times 58 \times .314}{61.8 \times .038 \times 4.16}.$$

Paying no attention to the decimal points, move the runner to 364 on scale *D*.

Move scale *C* until 618 falls on the runner.

Place the runner at 149, scale *C*.

Move scale *C* until 38 falls on the runner.

Move the runner to 58, scale *C*.

Move 416 on scale *C* to the runner.

Move the runner to 314 on scale C.

On scale *D* take the reading indicated by the runner. This is 101.

Place the decimal point by inspection as follows:

$$
\begin{aligned}
3.64 \times 14.9 &= 45 \text{ approximately,}\\
58 \times .314 &= 20 \text{ approximately,}\\
45 \times 20 &= 900.\\
61.8 \times .03 &= 1.8 \text{ approximately,}\\
1.8 \times 4.16 &= 8 \text{ approximately,}\\
900 \div 8 &= 112 \text{ approximately.}
\end{aligned}
$$

Hence the answer is 101.

EXERCISES

Find the value of each of the following:

1. $\dfrac{3 \times 48}{7 \times 29}$

2. $\dfrac{4 \times 26}{15 \times 3}$

3. $\dfrac{6.1 \times 2.7}{1.25 \times 8.74}$

4. $\dfrac{10.2 \times 8.61}{21.4 \times 7.38}$

5. $\dfrac{27.1 \times 15.3}{4.85 \times 81.2}$

6. $\dfrac{16.1 \times .032}{4.16 \times 3.84}$

7. $\dfrac{72.4 \times 16.3 \times 4.12}{81.5 \times 50.7}$

8. $\dfrac{16.5 \times 12.8 \times .34}{89.1 \times 2.11}$

9. $\dfrac{.46 \times 18.1 \times 4.15}{12.6 \times 3.18}$

182. How to find squares, cubes, and square roots with the slide rule. Comparing scales A and B (Fig. 92)

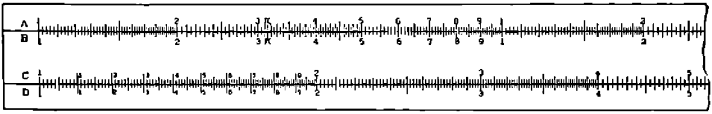

FIGURE 92

with C and D, you notice that the numbers on the first two scales are the squares of those directly below on the last two. This is because the logarithms for scales *A* and *B* were laid off to a scale half as large as that used for *C* and *D*. Hence, to find the square of a number, locate the number on scale *D* and move the runner to that point. Then read off the square on scale *A*.

Conversely, to find the square root of a number, locate the number on scale *A* and with the help of the runner read the number directly below on scale *D*. This is the required square root. The following examples illustrate the process:

1. Find the square of 35.4.

Solution: Move the runner to 354 on scale *D*.
Take the reading directly above on scale *A*.
This is 125.
Hence the answer is 1250 approximately.

2. Find the square root of 35.4.

Solution: Move the runner to 354 on scale *A*.
Directly below on scale *D* find 595.
The answer is 5.95.

3. Find the cube of 2.14.

Solution: Change $(2.14)^3$ to $(2.14)^2 \times 2.14$.
Move the runner to 214 on scale *D*.
Move the slide until the index 1 on scale *B* is on the runner.
Move the runner to 214 on scale *B*.
Take the reading directly above on scale *A*.

EXERCISES

Find the squares of the following numbers:

1. 3
2. 5
3. 4.1
4. 4.12
5. 8.63
6. 9.45
7. .34
8. .21
9. .56
10. 61.8
11. 43.9
12. 84.2

Find the square root of each of the following numbers:

13. 25 **15.** 3.16 **17.** .28 **19.** 35.2

14. 14 **16.** 4.28 **18.** .58 **20.** 61.8

183. Computations in problems performed with the slide rule. The computations in the following problems are readily carried out with the slide rule.

EXERCISES

Solve the following problems:

1. Find the diameter of a circle whose circumference is 34.3 inches.

Solution: $c = \pi d.$

$$\therefore d = \frac{c}{\pi} = \frac{34.3}{3.14}.$$

Use the slide rule to perform the division.

2. Find the area of a circle whose radius is 6.18 inches.

3. A flagpole casts a shadow 28 feet long. At the same time a rod 5.6 feet high casts a shadow 4.3 feet long. Find the height of the pole.

4. Find the simple interest on $328 at $4\frac{1}{2}\%$ for $3\frac{1}{2}$ years.

5. Find the radius of a circle whose area is 36.5 square feet.

6. What sum of money placed at simple interest at $4\frac{1}{2}\%$ yields an income of $250 in 3 years?

7. A farm valued at $12,300 is taxed for $74.80. At the same rate what will be the tax on a farm valued at $15,200?

8. If 225 pounds of milk produce 8.1 pounds of butter fat, how many pounds of milk will be required to produce 32 pounds of butter fat?

9. The pole of a tent is 32 feet high. To hold the pole in position, stakes are placed 46 feet from the foot of the pole and wires of equal length attached to the top of the pole are fastened to these stakes. Find the length of the wires, not counting the part used in fastening.

10. Find the base of a triangle whose area is 30.5 square feet and whose altitude is 10.4 feet.

11. How many cubic feet of ground must be excavated to make a ditch 2.4 feet wide, 3.5 feet deep, and 124 feet long?

12. Find the cost of painting the lateral surface of a silo 14.5 feet in diameter and 35.8 feet high, at the rate of $2.75 per 100 square feet.

13. Find the horse power of an eight-cylinder engine the diameter of whose pistons is 4.5 inches.

Suggestion: Use the formula $\text{h.p.} = \frac{d^2n}{2.5}$.

14. Find the volume of a sphere whose diameter is 1.26 inches.

EXERCISES TO TEST YOUR UNDERSTANDING OF CHAPTER X

184. What you should be able to do. The purpose of Chapter X is to teach you to use the slide rule for such operations as are usually found in practical problems. It is expected that you should now be able to perform the following operations with the slide rule:

1. To find the product of two numbers.
2. To find the quotient of two numbers.
3. To find the square and cube of a number.
4. To find the square root of a number.
5. To solve proportions.
6. To perform in succession the operations of a continuous series of multiplications and divisions.

EXERCISES

Find the products of the following numbers with the slide rule:

1. 4.82×3.91

2. 62.3×2.85

3. $4.2 \times 31 \times 6.4$

4. $3.8 \times 5.2 \times 1.7$

Divide as indicated:

5. $13.7 \div 2.6$

6. $28.3 \div 4.2$

7. $8.57 \div 2.34$

8. $165 \div 3.19$

Find the value of x:

9. $x = \dfrac{3.01 \times 27 \times 5.3}{1.8 \times 6.2}$.

10. $x = \dfrac{3.25 \times 8.14}{289}$.

11. $\dfrac{x}{2.7} = \dfrac{1.25}{.61}$.

12. $\dfrac{16.5}{x} = \dfrac{2.12}{4.50}$.

Find the square of each of the following numbers:

13. 2.9

14. .52

15. 5.3

Find the square root of each of the following numbers:

16. 43.1

17. 16.8

18. 2.84

Find the value of:

19. $\dfrac{.32 \times 1.8 \times 5.4}{12 \times 3.1}$

20. $\dfrac{7.2 \times 2.4 \times 1.6}{4.2 \times 3.4}$

21. Find the volume of a circular cone whose altitude is 3.57 inches and the radius of whose base is 1.65 inches. ($V = \frac{1}{3}\pi r^2 h$.)

CHAPTER XI

SELECTED SUPPLEMENTARY TOPICS

VARIATION

185. An important algebraic law. Relationships between variables can frequently be stated in the form of algebraic equations or formulas. For example, the relation between the circumference and radius of a circle is given, as you know, by the equation $c = 2\pi r$, the distance traveled by a train moving at a uniform rate of 40 miles an hour is given by $d = 40t$, and the interest on $2000 invested at a rate of 6% is given by the formula $i = 120t$.

Each of these equations contains two variables; one variable in each case is equal to the product of a **constant** (a number which does not vary) and the other variable. In the formula $c = 2\pi r$, the variable c equals the product of the constant 2π and the variable r. In the formula $d = 40t$, the variable d equals the product of the constant 40 and the variable t. Show that the third formula is in this same form. If you denote one variable by y and the other by x and the constant by c, all three of these formulas can be written as $y = cx$.

In the formula $d = 40t$, substitute for t the values 1, 2, 3, 4, 5, 6 and note that as t increases d also increases. Note also that, when t is doubled or trebled, d is also doubled or trebled. Give numerical examples to illustrate this fact. Show that this is also true of the other formulas and of the equation $y = cx$.

When two variables are so related that, when one is doubled, trebled, etc., the other is also doubled, trebled, etc., one is said to vary directly as the other. (The word *directly* is often omitted.) This means that the ratio of one to the other is always the same. If you divide both sides of the formula $i = 120t$ by t, you have $\frac{i}{t} = 120$. Note that the ratio $\frac{i}{t}$ is always 120; that is, it is constant. Similarly, if you divide the equation $y = cx$ by x, you have $\frac{y}{x} = c$; that is, the ratio $\frac{y}{x}$ is constant.

FIGURE 93

François Viète (Vieta) (1540–1603), a French mathematician, studied mathematics as a leisure occupation because it gave him great pleasure and satisfaction. Although he did not devote all his time to the study, he achieved the fame of being the greatest of French mathematicians of the sixteenth century.

His interest was chiefly in algebra and trigonometry. He was one of the first writers to use letters to denote numbers. He used vowels to denote unknowns and consonants to represent known numbers. Exponents, as we now know them, were not known at his time, but he introduced a better notation than the one in use. He denoted the square of an unknown number by Aq, the cube by Ac, and the fourth power by Aqq.

The previous discussion shows that the statement, "A variable y varies directly as a variable x," may be expressed algebraically by the equation $y = cx$. Conversely, if $y = cx$, y is said to vary directly as x. The literal number c is called the **constant of variation,** and the equation $y = cx$ the **law of variation.**

EXERCISES

Express the following relations algebraically:

1. The cost of sugar varies directly as the weight.

Solution: Using the first letter of the word to denote the variable, you have $C = cw$.

2. The weight of a liquid varies directly as the volume.

3. The weight of a beam varies as the length.

4. The diagonal of a cube varies directly as the edge.

5. The distance sound travels varies directly as the time.

6. The distance a spring is stretched by a weight varies as the weight.

7. The distance (in feet) an object falls from rest varies directly as the square of the time (in seconds).

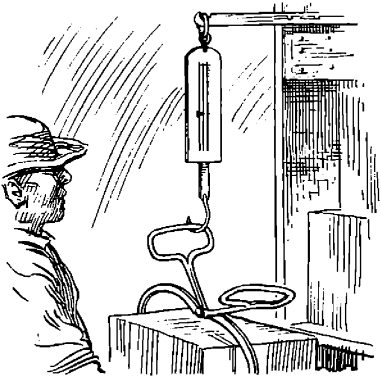

FIGURE 94

Suggestion: If t denotes the time, what will represent the square of the time?

8. The pressure in pounds per square inch of a column of water varies directly as the height of the column.

9. The weight of a sphere varies as the cube of the radius.

10. The time required by a pendulum to make one vibration varies directly as the square root of its length.

11. The area of an equilateral triangle varies as the square of a side.

12. The volume of a cylinder varies as the square of the radius of the base if the altitude remains constant.

186. How to determine the constant of variation. You have seen that the statement, "The distance traveled by a train varies directly as the time," may be expressed briefly by means of the equation $d = ct$. To determine the constant c, you must know one pair of corresponding values of t and d. Thus, if it is known that the train traveled 70 miles in 2 hours, you have $d = 70$ if $t = 2$. A substitution of these values of d and t in the equation $d = ct$ gives $70 = c \times 2$. It follows that $c = 35$.

In each of the following exercises determine the constant of variation.

EXERCISES

1. The amount of gasoline used in driving an automobile varies directly as the number of miles traveled by it. If an automobile uses 6 gallons to travel 72 miles, what is the constant of variation?

Solution: Since the amount of gasoline varies directly as the distance, it follows that $a = cd$.

Hence $6 = c \times 72$,

or $c = \frac{1}{12}$.

$\therefore a = \frac{1}{12}d$.

Here the constant of variation denotes the amount of gasoline used per mile.

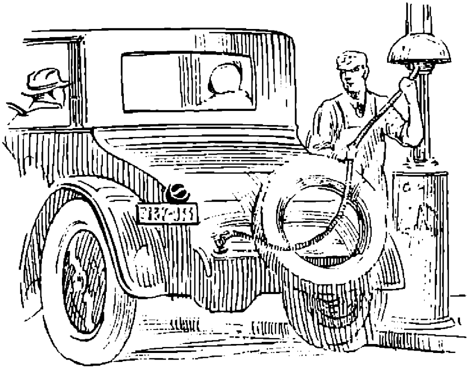

FIGURE 95

2. Find the value of c in the equation $y = cx$ for each of the following cases:

(*a*) $x = 3, y = 6.$ (*c*) $x = 10, y = 5.$

(*b*) $x = 5, y = 20.$ (*d*) $x = 5, y = 7.5.$

3. When a spring is stretched by a weight w, the amount of the stretch s varies as the weight. When a weight of 18 pounds stretches the spring 3 inches, find the constant of variation and state the general law of variation in symbols. What is the meaning of the constant of variation?

4. The cost C of a certain grade of butter varies as the number of pounds n. If 6 pounds of butter cost $2.64, find the constant and state the law of variation in symbols. State the meaning of the constant of variation.

5. If the altitude of a rectangle is constant, the area varies as the base. If the area is 20 when the base is 4, write the law of variation. What is the meaning of the constant?

187. Solving problems in variation by proportions. A convenient method of solving problems in variation is found in the use of proportions. The example below explains the method.

In geometry it is shown that the diagonal of a cube varies as the edge. If the diagonal is about 8.5 inches when the edge is 5 inches, find the diagonal of a cube whose edge is 2 inches. You may solve the problem as follows:

1. State the law of variation. This is $d = ce$.

2. You know that this equation is true for any pair of corresponding values of d and e. Denote two such pairs by (d_1, e_1) and (d_2, e_2).

$$\begin{aligned} \text{Hence } d_1 &= ce_1, \\ \text{and } \quad d_2 &= ce_2. \end{aligned}$$

3. Dividing the first equation by the second, you have

$$\frac{d_1}{d_2} = \frac{ce_1}{ce_2} = \frac{e_1}{e_2}.$$

4. Since in the problem on page 275 $d_1 = 8.5$, $e_1 = 5$, and $e_2 = 2$, you may substitute these values in the proportion above. This gives

$$\frac{8.5}{d_2} = \frac{5}{2}.$$

5. Multiply each side of this equation by $2d_2$. Then you have

$$\frac{2d_2 \times (8.5)}{d_2} = 2d_2 \times \frac{5}{2}.$$

Reducing the fractions, you have

$$2 \times (8.5) = 5d_2.$$
$$\therefore d_2 = 3.4.$$

EXERCISES

Solve the following problems, using proportions:

1. The amount of gasoline used in driving an automobile varies directly as the number of miles traveled by it. If an automobile running at a uniform rate in the country uses 5 gallons of gasoline to travel a distance of 52 miles, how much gasoline will be needed for a trip of 90 miles?

Solution: $a = cd.$

$$\therefore \frac{a_1}{a_2} = \frac{d_1}{d_2}.$$

$a_1 = 5$, $d_1 = 52$, and $d_2 = 90$.

Then by substitution you have

$$\frac{5}{a_2} = \frac{52}{90}.$$

Hence $26a_2 = 225$,

and $a_2 = 8.7$ approximately.

2. The mass of an object (amount of matter in it) of uniform material varies as the volume. If the mass of an aluminum object is 17 when the volume is 6, find the mass of an aluminum object made of the same material whose volume is 9.35.

3. A column of water 2.5 feet high is known to exert a pressure of 1.08 pounds per square inch. What is the pressure of a column 3.33 feet high if the pressure varies as the height?

4. The diagonal of a cube varies as the edge. If the edge is about 10 when the diagonal is 22.3, find the diagonal when the edge is 8.

5. The voltage of a dynamo varies directly as the speed. If the speed is 225 revolutions a minute, the voltage is 183.8. Find the voltage if the speed is 800 revolutions a minute.

6. The elongation of a spring varies directly as the weight. The elongation is 8 when the weight is 100. Find the elongation when the weight is 70.

7. The weight of a liquid varies directly as the volume. If 10 cubic feet of water weigh 625 pounds, find the weight of 37 cubic feet.

8. The work done by a machine varies as the number of hours. Working 6 hours, a machine makes 1800 screws. How many screws will it make in 20 minutes?

9. The distance which an object falls varies as the square of the time. If an object falls 144.7 feet in 3 seconds, how far will it fall in 5 seconds?

Suggestion: $\frac{d_1}{d_2} = \frac{t_1^2}{t_2^2}$.

10. The area of a circle varies as the square of the radius. The area of a circle is approximately 12.56 when the radius is 2. Find the area when the radius is 5.

11. The time required by a pendulum to make a vibration varies as the square root of the length. If a pendulum 100 centimeters long makes one vibration in 1 second, what is the time of one vibration of a pendulum 64 centimeters long?

12. The number of tons of coal used per hour in a locomotive varies as the square of the number of miles per hour. If a locomotive consumes 4 tons of coal at a rate of 40 miles, what is the speed obtained when 5 tons are used per hour?

13. The load a beam can carry varies as the square of the thickness of the beam. If a 6-inch beam can carry 1600 pounds, how much weight can an 8-inch beam carry?

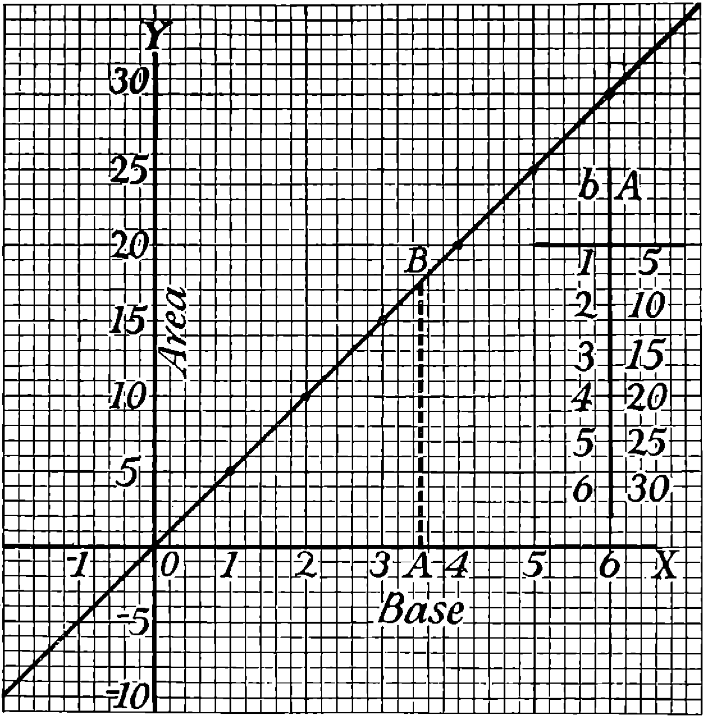

Figure 96

188. Solving problems in variation graphically. You have seen that the law of variation for Exercise 5 (§ 186) is $A = 5b$. Let it be required to find the area when $b = 3.5$.

Solution: 1. By substituting values for b, find the corresponding values of A as given in the table of Fig. 96. Thus, if $b_1 = 1$, then $A_1 = 5$; if $b_2 = 2$, $A_2 = 10$; etc.

2. Draw the reference axes OX and OY and plot the pairs of numbers given in the table.

3. Draw the line passing through the marked points. This is the required graph of the equation $A = 5b$.

4. To find the area when $b = 3.5$, pass from the origin O to the right a distance equal to 3.5 to point A. Then pass upward to the graph at B and read off the length of the vertical line segment AB.

EXERCISES

1. From the graph (Fig. 96) find values of A corresponding to the following values of b: 2.5; 5.2; 1.8; 4.4.

2. The diagonal of a cube varies as the edge. If the diagonal is approximately 8.5 inches when the edge is 5 inches, find the constant of variation. State the law of variation and represent it graphically. From the graph find the approximate lengths of diagonals of cubes having the following edges: 2; 4; 4.6; 3.2.

3. The weight of steel wire varies as the length. If 10 feet of wire weigh 2 pounds, find graphically the weights of the same kind of wire of the following lengths: 2; 8; 4.5; 5.

4. From the graph of Exercise 3 find the lengths of wire weighing 1 pound; 3 pounds; 4.5 pounds.

5. The speed (velocity) of a falling object varies as the time. If an object has fallen 3 seconds, it has a speed of approximately 96.6 feet a second. Find graphically the speed attained by an object when it has fallen the following numbers of seconds: 2; 6; 4; 3.5.

6. From the graph of Exercise 5 find the time it takes a falling object to attain a speed of 161 feet a second; 225.4 feet a second.

7. Represent graphically the equation $y = cx$ when $c = 3$.

189. What is meant by inverse variation. If each side of the rectangle *PQRS* (Fig. 97) is 4 inches, what is its area? If the side *PQ* is increased so that it is 6 inches, what must be the length of the other side in order that the area of the resulting rectangle shall be the same as the area of the original rectangle? Answer the same question when *PQ* is increased to 8 and 12 inches.

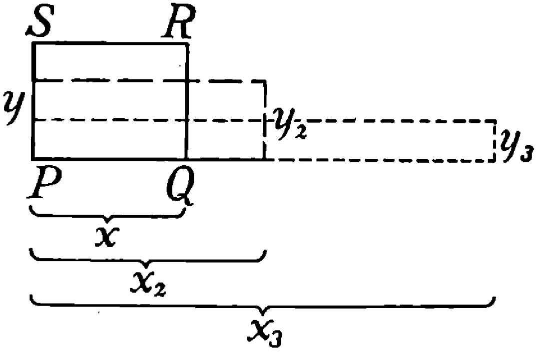

FIGURE 97

In general, if the sides x and y of the rectangle *PQRS* (Fig. 97) vary so that the area remains constant, y must decrease when x increases and y must increase when x decreases, but the product xy remains the same, always being equal to A. If two variables x and y are so related that their product xy remains constant, either is said to *vary inversely* as the other. Thus the statements, "x varies inversely as y," and, "$xy = c$," are equivalent.

The term *inversely* is used because the equation $xy = c$ may be changed to $x = c\frac{1}{y}$, which in the language of variation means that x varies as the *inverse* of y. The equation $xy = c$ is a quadratic equation in x and y.

EXERCISES

Express the following statements in symbols in the form $xy = c$:

1. The time required to do a piece of work varies inversely as the number of men employed.

2. The pressure of a gas on the walls of a retaining vessel varies inversely as the volume.

3. The illumination, on the page of a book, from an incandescent lamp varies inversely as the square of the distance from it.

4. The force of gravitation due to the earth varies inversely as the square of the distance from the earth's center.

190. How to represent the equation $xy = c$ graphically. You have seen (Fig. 97) that as x increases y must decrease in order that the product xy remain constant.

Let $c = 12$. Then the equation is $xy = 12$.

Let x take the values 1, 2, 3, 4, . . . 12.

Then y takes the corresponding values 12, 6, 4, 3, . . . 1.

Plot the pairs of corresponding values as shown in Fig. 98.

Draw a smooth curve through the marked points.

The curve is the graph of the equation $xy = 12$.

Note that the plotted points in Fig. 98 correspond to the upper right-hand corners in the rectangles in Fig. 97.

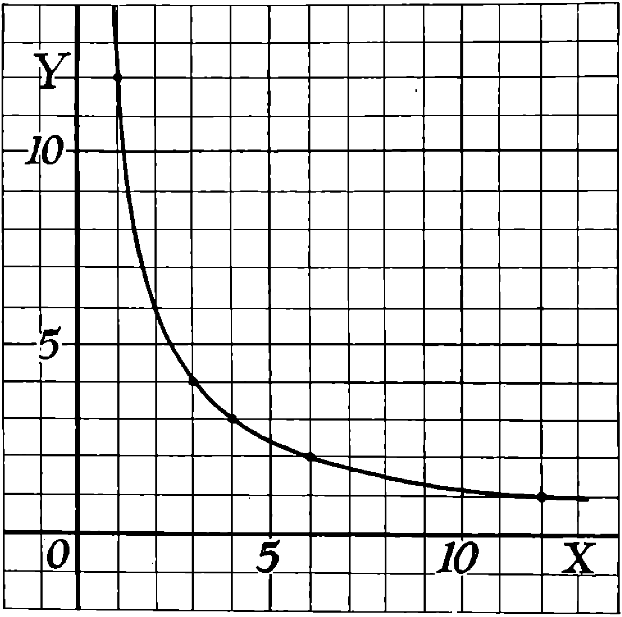

FIGURE 98

EXERCISES

Make graphs for the following equations:

1. $xy = 6$.

2. $xy = 8$.

3. $xy = 24$.

4. $xy = 4$.

191. **Solving problems in inverse variation by means of proportions.** The method is illustrated by the following example:

If 12 men build a fence in 8 days, how long will it take 32 men to do it?

Solution: 1. Denoting the number of men by m and the number of days by d, you have $md = c$ (Exercise 1, § 189).

2. Let (m_1, d_1) and (m_2, d_2) be two pairs of values satisfying the equation $md = c$; that is, let $m_1 d_1 = c$, and $m_2 d_2 = c$.

$$\text{Then } \frac{m_1 d_1}{m_2 d_2} = \frac{c}{c} = 1. \quad \text{Why?}$$

Multiply both members of the equation by $m_2 d_2$.

Then $$m_1 d_1 = m_2 d_2.$$

By dividing both members of this equation by $m_2 d_1$ you have

$$\frac{m_1 d_1}{m_2 d_1} = \frac{m_2 d_2}{m_2 d_1}.$$

This reduces to $\dfrac{m_1}{m_2} = \dfrac{d_2}{d_1}$.

3. Using the last equation, you can now solve the problem above by the method shown in Exercise 1 below.

EXERCISES

Solve the following problems by means of proportions:

1. If 12 men build a fence in 8 days, how long will it take 32 men to do it?

Solution: It is given that $m_1 = 12$, $d_1 = 8$, $m_2 = 32$.

Substituting these values in the equation $\dfrac{m_1}{m_2} = \dfrac{d_2}{d_1}$, you have

$$\frac{\overset{3}{\cancel{12}}}{\underset{8}{\cancel{32}}} = \frac{d_2}{8}.$$

$$\therefore d_2 = 3.$$

2. The rate of traveling a given distance varies inversely as the time. If a train traveling 30 miles an hour passes over the distance between two cities in 4 hours, how long will it take a train traveling 35 miles an hour?

3. Two boys of unequal weights are making a 12-foot teeter-board balance. They have learned in science that the weight varies inversely as the distance from the turning point to make the board balance. The two boys weigh 125 pounds and 102 pounds, respectively. The second boy sits at one end of the board. How far from the other end should the heavy boy sit in order that the board may balance?

4. A lamp shines on a book 5 feet from it. How far from it must the book be held to receive twice as much light? (See Exercise 3, § 189.)

5. The attraction of gravitation at points outside the earth's surface varies inversely as the square of the distance from the earth's center. If the attraction on a body at the surface of the earth is 9 pounds, at what height above the surface would the attraction be 4 pounds? (Use 4000 miles as the radius of the earth.)

6. If the pressure of a gas on the walls of a retaining vessel is 40 pounds per square foot when the volume is 10 cubic feet, what is the pressure when the volume is 50 cubic feet? (See Exercise 2, § 189.)

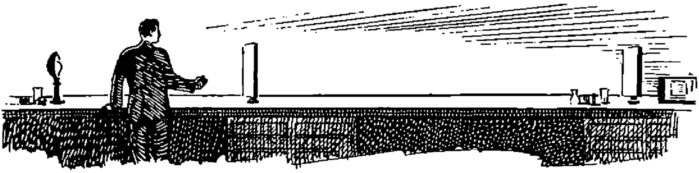

FIGURE 99

TESTING THE INTENSITY OF LIGHT

7. A screen 18 feet from a lamp is moved to a distance of 6 feet from it. How do the intensities compare if the intensity varies inversely as the square of the distance from the screen?

8. The volume of a gas at constant temperature varies inversely as the pressure. If the volume is 150 when the pressure is 30, find the volume when the pressure is 24.

9. The intensity of light on an object is inversely proportional to the square of the distance of the object from the source of light. If a screen 15 feet from a lamp is moved up a distance of 10 feet, compare the intensities by means of a ratio.

10. If 20 men can do a piece of work in 10 days, how long will it take 25 men to do it?

11. The horse power required to drive a ship varies directly as the cube of the speed. If a horse power of 2000 propels a ship at the speed of 10 knots, what horse power is required for a speed of 15 knots?

FACTORING POLYNOMIALS

192. Factoring polynomials by grouping terms. You have previously learned how to factor polynomials whose terms contain a common factor. For example, $3x^3 + 2x^2 + 5x = x(3x^2 + 2x + 5)$. In this case the common factor x is a monomial. Consider the expression $3a(x + y) + 2b(x + y)$. How many terms has it? How many factors has each term? Is there a common factor? You see that $3a(x + y) + 2b(x + y) = (x + y)\ (3a + 2b)$. In this case the common factor is a binomial. Factor also $a(m + n) - b(m + n)$ and $a(a + 1)\ (a - 1) + b(a - 1)$.

The following examples show how this method helps you to factor another type of polynomial.

1. Consider the polynomial $ax + ay + bx + by$.

Notice that the first two terms can be factored.

Hence consider them as a group. Similarly consider the last two terms as one group.

Thus $ax + ay + bx + by = \overgroup{ax + ay} + \overgroup{bx + by}$.

Factoring each group gives $a(x + y) + b(x + y)$.

Since $x + y$ is now a factor common to both terms, you have $a(x + y) + b(x + y)$ equal to $(x + y)(a + b)$.

The work may now be arranged briefly:

$$\begin{aligned}\overgroup{ax + ay} + \overgroup{bx + by} &= a(x + y) + b(x + y)\\ &= (x + y)(a + b).\end{aligned}$$

2. The terms of a polynomial must often be rearranged before this method can be applied.

$$\begin{aligned}&6a^3 - 8a - 15a^2 + 20 = 6a^3 - 15a^2 - 8a + 20\\ &= (6a^3 - 15a^2) - (8a - 20) = 3a^2(2a - 5) - 4(2a - 5)\\ &= (2a - 5)(3a^2 - 4).\end{aligned}$$

Note that, when terms are placed in a parenthesis preceded by a minus sign, the signs of the terms within the parenthesis are changed. Explain the reason why.

3. $a^2 - b^2 + 2a - 2b = (a^2 - b^2) + (2a - 2b)$
$= (a - b)(a + b) + 2(a - b) = (a - b)(a + b + 2)$.

If you have forgotten how to factor binomials like $a^2 - b^2$, refer to § 112.

EXERCISES

Factor each of the following polynomials:

1. $ax + bx + ay + by$
2. $my^2 + m + n^2y^2 + n^2$
3. $3a^2 + 3a + 4ab + 4b$
4. $ax + bx - ay - by$
5. $6y^3 - 15y^2 + 8y - 20$
6. $6y^3 - 15y^2 - 8y + 20$
7. $x^3 + x^2 + x + 1$
8. $3ax - ay + 9bx - 3by$
9. $x^3 - 3x^2 + 4x - 12$
10. $x^2 - 9y^2 + x + 3y$
11. $x^4 + x^3y - xy^3 - y^4$
12. $ax^2 - 2a^2x - x + 2a$
13. $ax + ay + bx + by - cx - cy$
14. $x^2y + y^2z + xz^2 - x^2z - xy^2 - yz^2$

Reduce the following to lowest terms:

15. $\dfrac{y^2 + ay + by + ab}{(y^2 - a^2)(y^2 - b^2)}$

16. $\dfrac{x^2 - px - qx + pq}{x^2 - px - rx + pr}$

17. $\dfrac{2mx - 6x - 2my + 6y}{3mnx - 9nx - 3mny + 9ny}$

18. $\dfrac{m^3 - mn^2 - m^2 + n^2}{(m^4 - n^4)(m^2 - 2m + 1)}$

Each of the polynomials in Exercises 19 to 31 can be changed into the difference of two squares. Exercises 19, 20, and 21 illustrate the method.

19. $x^2 - 6x + 9 - y^2$

Solution: Grouping the first three terms, you have

$$\begin{aligned}(x^2 - 6x + 9) - y^2 &= (x - 3)^2 - y^2\\ &= (x - 3 + y)(x - 3 - y).\end{aligned}$$

20. $a^2 - x^2 - y^2 + 2xy$

Solution: Grouping the last three terms, you have

$$\begin{aligned}a^2 - x^2 - y^2 + 2xy &= a^2 - (x^2 - 2xy + y^2)\\ &= a^2 - (x - y)^2\\ &= (a + x - y)(a - x + y).\end{aligned}$$

21. $m^2 + 6m - x^2 + 9 - 4xy - 4y^2$

Solution: Grouping the first, second, and fourth terms, you have

$$\begin{aligned}&\overparen{m^2 + 6m + 9} - (x^2 + 4xy + 4y^2)\\ &= (m + 3)^2 - (x + 2y)^2\\ &= (m + 3 + x + 2y)(m + 3 - x - 2y).\end{aligned}$$

Factor the following:

22. $x^2 + 2xy + y^2 - z^2$

23. $a^2 + 2ab + b^2 - 1$

24. $m^2 - 6m + 9 - a^2$

25. $a^2 - x^2 - 2xy - y^2$

26. $4a^2 - 9b^2 + 6b - 1$

27. $z^2 - a^2 - b^2 - 2ab$

28. $9x^2 - 4a^2 - 4ab - b^2$

29. $12ab + 25 - 4a^2 - 9b^2$

30. $a^4 - 12a^2 + 36 - x^2$

31. $x^4 - x^2 + 12xy - 36y^2$

The following polynomials can be changed to trinomials. Exercise 32 illustrates the method.

32. $a^2 + b^2 + 2ab + 8a + 8b - 9$

Solution: Grouping the first three terms and the fourth and fifth, you have the trinomial

$$\begin{aligned}&(a^2 + 2ab + b^2) + (8a + 8b) - 9\\ &= (a + b)^2 + 8(a + b) - 9\\ &= (a + b + 9)(a + b - 1).\end{aligned}$$

Factor the following:

33. $x^2 + 2xy + y^2 + 4mx + 4my + 4m^2$

34. $x^2 + 6xy + 9y^2 + 5x + 15y + 6$

35. $4x^2 + 4xy + y^2 - 8x - 4y - 21$

36. $a^2 - 6ab + 9b^2 - 4a + 12b - 12$

RADICALS

193. Multiplying radicals. The method of multiplying two radicals of the same order may be seen from the law $\sqrt{a}\sqrt{b} = \sqrt{ab}$.

Before multiplying change the radicals to the simplest form.

EXERCISES

Multiply as indicated:

1. $\sqrt{3}\sqrt{18}$

Solution: Simplifying the second radical, you have

$$\sqrt{3} \times 3\sqrt{2}.$$

$$\therefore \sqrt{3}\sqrt{18} = \sqrt{3} \times 3\sqrt{2} = 3 \times \sqrt{3} \times \sqrt{2} = 3\sqrt{6}.$$

2. $\sqrt{2}\sqrt{3}$

3. $\sqrt{5}\sqrt{2}$

4. $2\sqrt{3}\sqrt{2}$

5. $\sqrt{3}\sqrt{12}$

6. $\sqrt{m}\sqrt{n}$

7. $\sqrt{a^2b}\sqrt{b}$

8. $\sqrt{12}\sqrt{18}$

9. $2\sqrt{5}\sqrt{15}$

10. $4\sqrt{5}\sqrt{5}$

11. $\sqrt{a}\sqrt{ab^2}$

12. $\sqrt[3]{20}\sqrt[3]{12}$

13. $\sqrt[3]{9}\sqrt[3]{9}$

14. $\sqrt{5}\sqrt{35}$

15. $2\sqrt[4]{4}\cdot 6\sqrt[4]{8}$

16. $\sqrt{\frac{15}{16}}\cdot\sqrt{\frac{5}{8}}$

17. $2x\sqrt{x^3y}\cdot x^3\sqrt{xy^3}$

18. $4\sqrt[5]{a^6y^4}\cdot\sqrt[5]{ay^5}$

19. $3\sqrt{3a^2}\cdot 2\sqrt{15a}$

20. $(2\sqrt{2}-\sqrt{5}+3\sqrt{3})\sqrt{3}$

Solution: Multiply each term by $\sqrt{3}$. This gives

$$2\sqrt{6}-\sqrt{15}+9.$$

21. $(\sqrt{2}+\sqrt{3})\sqrt{5}$

22. $(\sqrt{3}+6)\sqrt{3}$

23. $\sqrt{2}(\sqrt{2}+7)$

24. $4\sqrt{3}(3\sqrt{2}-\sqrt{5})$

25. $\sqrt{5}(2+\sqrt{5})$

26. $(3\sqrt{30}+2\sqrt{5})2\sqrt{5}$

27. $(\sqrt{2}-\sqrt{5})(3\sqrt{5}+\sqrt{3})$

Solution: Multiply each term of the first binomial by each term of the second:

$$\begin{aligned}(\sqrt{2}-\sqrt{5})(3\sqrt{5}+\sqrt{3}) &= 3\sqrt{10}-3\times 5+\sqrt{6}-\sqrt{15}\\ &= 3\sqrt{10}-15+\sqrt{6}-\sqrt{15}.\end{aligned}$$

28. $(\sqrt{3}+6)(\sqrt{3}-4)$

29. $(\sqrt{2}+7)(\sqrt{2}-7)$

30. $(2\sqrt{3}+8)(2\sqrt{3}-8)$

31. $(\sqrt{2}-5)(3\sqrt{2}-4)$

32. $(2+\sqrt{5})^2$

33. $(2-5\sqrt{7})(2+5\sqrt{7})$

34. $(3\sqrt{2}-\sqrt{5})(4\sqrt{3}+\sqrt{7})$

35. $\frac{3+\sqrt{5}}{2}\cdot\frac{3-\sqrt{5}}{2}$

36. $(2\sqrt{a}+4\sqrt{c})(\sqrt{a}+3\sqrt{c})$

37. $(3\sqrt{30} + 2\sqrt{5})(2\sqrt{5} + 3\sqrt{6})$

38. $(\sqrt{x} + \sqrt{y})(\sqrt{x} - \sqrt{y})$

39. $(\sqrt{x - 3} + 2)^2$

ALGEBRAIC METHODS OF SOLVING EQUATIONS WITH TWO OR MORE UNKNOWNS

194. Review of what is meant by elimination. In solving algebraically equations in two unknowns you may, by combining the equations, derive a single equation containing but one unknown. Thus through the processes of combining the equations we eliminate (remove) one of the unknowns. The method to be used to bring about the elimination depends upon the form of the equations to be solved. You have already solved (§ 101) very simple equations by the method explained more fully in the next section.

195. Eliminating by substitution. As the name suggests, the method of eliminating one of the unknowns involves a substitution. For example, in the pair of equations $\begin{cases} 2x + 3y = 12 \\ y = 2x - 1 \end{cases}$ you may substitute $2x - 1$ in place of y in the first equation and obtain $2x + 3(2x - 1) = 12$, which contains only one unknown.

Although the method can be used for any pair of equations, it is especially useful when one of the unknowns is easily expressed in terms of the other. The following examples illustrate the process:

1. Solve by eliminating by substitution the system

$$8a - 3b = 30,$$
$$b = a - 5.$$

Solution: Substitute $a - 5$ for b in the first equation.

$$\begin{aligned} \text{Then } 8a - 3(a - 5) &= 30. \\ 8a - 3a + 15 &= 30. \\ 5a &= 15. \\ a &= 3. \\ b = a - 5 &= -2. \\ \therefore (a, b) &= (3, -2). \end{aligned}$$

Check the result by substituting 3 and -2 for a and b, respectively, in the original equations.

2. Solve the system of equations

$$\begin{aligned} 3x - 5y &= 1, \\ 2x &= y + 3. \end{aligned}$$

Solution: Solving the second equation for x, you have

$$x = \frac{y + 3}{2}.$$

Substitute $\frac{y + 3}{2}$ in place of x in the first equation.

$$\text{Then } 3\left(\frac{y + 3}{2}\right) - 5y = 1.$$

Multiply both members by 2. This gives

$$\begin{aligned} 3y + 9 - 10y &= 2. \\ 7 &= 7y. \\ y &= 1, \\ \text{and } x = \frac{y + 3}{2} &= 2. \\ (x, y) &= (2, 1). \end{aligned}$$

Check the result.

EXERCISES

Solve the following systems of equations:

1. $x + y = 6,$
$y = 2x.$

2. $2x + y = 7,$
$y = x + 1.$

3. $3x - 2y = 14,$
$2x = -9y - 1.$

4. $3x + 2y = 12,$
$3x = 2y.$

5. $x + y = 7,$
$5x - 2y = 10.$

6. $8x - y = 43,$
$10x + 3y = 75.$

7. $10x - y = 3,$
$2x + 2y = 17.$

8. $x + 4y = 4,$
$2x + 2y = 5.$

9. $2x + y = 13,$
$x + 2y = 14.$

10. $3a + b = 11,$
$5a - b = 13.$

11. $4x = 3y,$
$2y = 3x - 1.$

12. $3a + 2b = 22,$
$3a - b = 25.$

13. $m - 7n = 0,$
$3m - n = 60.$

14. $a - 3b = 9,$
$2a + 7b = -6.$

15. $m - 3n = 0,$
$4m + 5n = 38.$

16. $3x + 5y = 29,$
$x + 2y = 11.$

In the following formulas eliminate the literal numbers as indicated:

17. $A = bh$, $h = 4c$. Eliminate h.
18. $V = abc$, $4a = 3h$. Eliminate a.
19. $s = \frac{1}{2}gt^2$, $v = gt$. Eliminate t.
20. $T = \pi r(h + r)$, $3r = 2h$. Eliminate h.
21. $C = \frac{5}{9}(F - 32)$, $F = 3C$. Eliminate C.
22. $x^2 + y^2 = r^2$, $x = 6y$. Eliminate y.

196. Miscellaneous exercises. The following exercises will give you further practice in solving systems of equations.

EXERCISES

Solve by whatever method seems best to you:

1. $2a + 3b = 27,$
$5a - 2b = 1.$

2. $2x + 7y = 52,$
$3x - 5y = 16.$

3. $4a + 7b = 121,$
$8a - 3b = 55.$

4. $2x + 7y = -1,$
$5x + 8y = 7.$

5. $5a + 10b = 14,$
$2a + 5b = 4.$

6. $2m - 6n = -9,$
$3m - 18n = 10.$

7. $7a - 2b = 3,$
$6a - 19b = -89.$

8. $7x - 3y = 2,$
$3x + 7y = 5.$

9. $3m + 4n = 12,$
$2m - 5n = 54.$

10. $2x - 7y = -34,$
$7x - 2y = 16.$

11. $3a + 2b = 7,$
$2a + 3b = 8.$

12. $3a - 4b = 12,$
$4a + 3b = -6.$

13.
$$\frac{2}{x} + \frac{3}{y} = 27,$$
$$\frac{5}{x} - \frac{2}{y} = 1.$$

Solution: Multiply the first equation by 2 and the second by 3. This gives

$$\frac{4}{x} + \frac{6}{y} = 54,$$
$$\frac{15}{x} - \frac{6}{y} = 3.$$

Add the two equations and show that

$$\frac{19}{x} = 57$$
$$\text{or}\quad 19 = 57x.$$
$$x = \tfrac{1}{3}.$$

Substitute $\frac{1}{3}$ for x in the first equation. Then

$$\frac{4}{\frac{1}{3}} + \frac{6}{y} = 54.$$
$$12 + \frac{6}{y} = 54.$$
$$\frac{6}{y} = 42.$$
$$y = \tfrac{1}{7}.$$

Therefore $(x, y) = (\frac{1}{3}, \frac{1}{7}).$

14. $\frac{1}{a} + \frac{1}{b} = 25,$

$\frac{1}{a} - \frac{1}{b} = 15.$

15. $\frac{6}{x} + \frac{12}{y} = -1,$

$\frac{8}{x} - \frac{9}{y} = 7.$

16. $\frac{2}{a} + \frac{3}{b} = 13,$

$\frac{5}{a} + \frac{1}{b} = 13.$

17. $\frac{5}{x} + \frac{2}{y} = 7,$

$\frac{3}{x} + \frac{2}{y} = 5.$

197. Fractional equations. In solving fractional equations it is usually best to remove the denominators by multiplying every term of the equation by a number. Thus, in the equation $\frac{2x+6}{5} + \frac{y}{3} = 8$, remove the denominators by first multiplying every term by 15. This gives

$$\frac{\overset{3}{\cancel{15}}(2x+6)}{\cancel{5}} + \frac{\overset{5}{\cancel{15}}y}{\cancel{3}} = 15 \times 8, \text{ or } 6x + 18 + 5y = 120.$$

If you have forgotten how to find the lowest common multiple of the denominators, review § 120.

In some cases, as you saw in § 196, it is better to leave the fractions.

EXERCISES

Solve the following systems of equations:

1. $\frac{x+2}{y+2} = 2,$

$\frac{x+7}{y+7} = \frac{3}{2}.$

2. $\frac{x+4}{4} - \frac{2y-4}{7} = 1,$

$\frac{x-2}{3} + \frac{y-4}{5} = 3.$

3. $\frac{3x-8}{9} = \frac{5y-3}{2} + 31,$

$\frac{7x-1}{5} + \frac{3y+6}{10} + 28 = 0.$

4. $$\frac{3a-b}{2}+\frac{a+b}{3}=4,$$
$$\frac{3a+b}{11}+2=\frac{3a-b}{3}.$$

5. $$\frac{x}{4}-\frac{y+1}{3}=1,$$
$$\frac{x}{3}-\frac{3y+1}{4}=0.$$

6. $$\frac{1}{x-y}+\frac{1}{x+y}=15,$$
$$\frac{4}{x-y}-\frac{5}{x+y}=17.$$

7. $$\frac{3}{4x-y}-\frac{5}{2x-y}=2,$$
$$\frac{4}{4x-y}+\frac{3}{2x-y}=-\tfrac{23}{5}.$$

8. $$\frac{8a-3b}{2}+6b+9=0,$$
$$8a=3b+1.$$

198. How to solve equations in three unknowns. The algebraic methods for solving equations in two unknowns are also used to solve equations in three or more unknowns. The following examples show how to solve such systems:

1. Solve: $a+c=-2,$
$$b+2c=3,$$
$$a+2b+3c=4.$$

Suggestion: The solution is simplified in this case by the fact that the first two equations contain only two unknowns. You may eliminate a from the first and third equations and obtain an equation in b and c, which together with the second equation will give the values of b and c. Or you may eliminate b from the second and third equations and use the resulting equation with the first.

Solution: Following the last suggestion, write the third equation:

$$a + 2b + 3c = 4.$$

Multiply the second equation by 2. Then

$$2b + 4c = 6.$$

Subtracting, you have the equation

$$a - c = -2.$$

The first equation is

$$a + c = -2.$$

$$\text{Adding, you have } 2a = -4.$$
$$a = -2,$$
$$\text{and } c = 0.$$

Substituting the values of a and c in the equation $a + 2b + 3c = 4$, you find that

$$-2 + 2b = 4.$$
$$\therefore 2b = 6,$$
$$\text{and } b = 3.$$
$$\therefore (a, b, c) = (-2, 3, 0).$$

2. Solve the system:

$$x - y + z = 2,$$
$$x + y + z = 4,$$
$$2x + y + z = 1.$$

Solution: Subtracting the second equation from the third, you have

$$x = -3.$$

Subtracting the second equation from the first, you have

$$-2y = -2.$$
$$y = 1.$$

Substituting the values of x and y in the third equation, you have

$$\begin{aligned} -6 + 1 + z &= 1. \\ z &= 6. \\ (x, y, z) &= (-3, 1, 6). \end{aligned}$$

3. Solve the system:

$$\begin{aligned} 2x + 3y + 4z &= 16, && (1) \\ 5x - 8y + 2z &= 1, && (2) \\ 3x - y - 2z &= 5. && (3) \end{aligned}$$

Solution: Add equations (2) and (3):

$$8x - 9y = 6. \qquad (4)$$

Write equation (1) and multiply equation (2) by 2:

$$\begin{array}{rrcll} & 2x + 3y + 4z & = & 16. & (1) \\ & \underline{10x - 16y + 4z} & \underline{=} & \underline{2.} & (5) \\ \text{Subtract (5) from (1):} & -8x + 19y & = & 14. & (6) \\ \text{Write equation (4):} & \underline{8x - 9y} & \underline{=} & \underline{6.} & (4) \\ \text{Add:} & 10y & = & 20. & \\ & y & = & 2. & \end{array}$$

Substitute 2 for y in equation (4):

$$\begin{aligned} 8x - 18 &= 6. \\ 8x &= 24. \\ x &= 3. \end{aligned}$$

Substitute 3 for x and 2 for y in equation (1):

$$\begin{aligned} 6 + 6 + 4z &= 16. \\ 4z &= 4. \\ z &= 1. \\ \therefore (x, y, z) &= (3, 2, 1). \end{aligned}$$

Check by substituting 3, 2, 1 in the original equations.

EXERCISES

Solve the following systems:

1. $$\begin{aligned} a + b &= 4, \\ a + c &= 6, \\ b + c &= 8. \end{aligned}$$

2. $$\begin{aligned} 2x - y &= 11, \\ x - 10z &= -7, \\ x + y + z &= -1. \end{aligned}$$

3. $$\begin{aligned} 2a + b &= -1, \\ 6a + 4b + 2c &= 1, \\ 10a - 15b + 15c &= 57. \end{aligned}$$

4. $$\begin{aligned} 3a - b - c &= 1, \\ a + 2b - 2c &= -3, \\ a - 3b + c &= 2. \end{aligned}$$

5. $$\begin{aligned} 3a - 3b + 5c &= 12, \\ 5a + 2b - 4c &= -3, \\ 4a + 5b + 2c &= 20. \end{aligned}$$

6. $$\begin{aligned} 2x - 3y + 3z &= 14, \\ 2x + 3y + z &= 16, \\ 5x + 4y - 2z &= 15. \end{aligned}$$

7. $$\begin{aligned} x - y + 3z &= 0, \\ 5x + 2y + z &= 14, \\ 2x + 3y + 4z &= -14. \end{aligned}$$

8. $$\begin{aligned} a - 2b + c &= 1, \\ 3a + b + c &= 14, \\ a + b + 2c &= 15. \end{aligned}$$

199. Equations with literal coefficients. The methods used in solving equations containing literal coefficients are illustrated in the following examples:

1. Solve the system of equations

$$\begin{aligned} x - y + 1 &= 0, \\ ax + by - c &= 0. \end{aligned}$$

Solution: Solve the first equation for x. This gives

$$x = y - 1.$$

Substitute $y - 1$ for x in the second equation.

Then $a(y - 1) + by - c = 0,$

or $ay - a + by - c = 0.$

Collect the terms containing y:

$$(a + b)y = a + c.$$

$$\therefore y = \frac{a + c}{a + b},$$

and $x = \dfrac{a + c}{a + b} - 1 = \dfrac{a + c - (a + b)}{a + b} = \dfrac{c - b}{a + b}.$

2. Solve the system of equations

$$\begin{aligned} ax + by &= c, \\ dx + ey &= f. \end{aligned}$$

Solution: Multiply the first equation by d and the second by a.

$$\begin{array}{ll} \text{Then} & adx + bdy = cd, \\ \text{and} & \underline{adx + aey = af.} \\ \text{Subtract:} & bdy - aey = cd - af. \end{array}$$

Collect the terms:

$$(bd - ae)y = cd - af.$$

$$y = \frac{cd - af}{bd - ae}.$$

To find x, eliminate y; that is, multiply the first equation by e and the second by b.

$$\begin{aligned} \text{Then } aex + bey &= ce, \\ \underline{bdx + bey} &\underline{= bf.} \\ \therefore (ae - bd)x &= ce - bf. \\ \therefore x &= \frac{ce - bf}{ae - bd}. \end{aligned}$$

EXERCISES

Solve the following systems of equations:

1. $ax + ay = b^2,$
$x - y = a.$

2. $ax + by = 2ab,$
$y = x - a.$

3. $bx + ay = a + b,$
$abx - aby = a^2 - b^2.$

4. $\dfrac{x - a}{y + b} = \dfrac{a}{b},$
$\dfrac{x + a}{y - b} = \dfrac{a + 1}{b}.$

5. $\dfrac{a}{x} + \dfrac{b}{y} = c,$
$\dfrac{b}{x} + \dfrac{a}{y} = d.$

6. $bx - ay = b^2,$
$(a - b)x + by = a^2.$

DISCUSSION OF THE ROOTS OF QUADRATIC EQUATIONS

200. The nature of the roots of a quadratic equation. You have learned that the roots of the general quadratic equation $ax^2 + bx + c = 0$ are

$$x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \text{ and } x_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

The number $b^2 - 4ac$ which is found under the radical sign may be positive, zero, or negative, depending on the values of a, b, and c. These possibilities lead to the following classification of the roots:

1. If $b^2 - 4ac$ is positive, the number $\sqrt{b^2 - 4ac}$ is said to be a **real** number.

Thus, for the equation $3x^2 - 7x + 2 = 0$, the number $b^2 - 4ac = 49 - 4 \cdot 3 \cdot 2 = 49 - 24 = 25$; that is, $b^2 - 4ac$ is positive. The roots $\frac{7 \pm \sqrt{25}}{6}$ are the two *real numbers* 2 and $\frac{1}{3}$.

For the equation $3x^2 - 7x + 3 = 0$, the number $b^2 - 4ac = 49 - 36 = 13$; that is, $b^2 - 4ac$ is positive as before. But the exact values of the roots $\frac{7 \pm \sqrt{13}}{6}$ cannot be found because 13 is not a square. They are said to be **irrational** numbers.

Hence, if $b^2 - 4ac$ is a positive number and

(a) if $b^2 - 4ac$ is a *square*, the roots of the equation are *real* and *rational*;

(b) if $b^2 - 4ac$ is not a square, the roots of the equation are *real* and *irrational*.

2. If $b^2 - 4ac$ is equal to zero, then the number $\sqrt{b^2 - 4ac}$ is zero.

Thus, for the equation $x^2 - 6x + 9 = 0$, the number $b^2 - 4ac = 36 - 36 = 0$. The roots are $\frac{6+0}{2}$ and $\frac{6-0}{2}$; that is, the roots are both equal to 3.

Hence, if $b^2 - 4ac$ is zero, the roots of the equation are *real, rational,* and *equal.*

3. If $b^2 - 4ac$ is less than zero, then $\sqrt{b^2 - 4ac}$ denotes the square root of a negative number and is called an **imaginary** number. Thus, for the equation $x^2 - 6x + 10 = 0$, the number $b^2 - 4ac = 36 - 40 = -4$, and the roots are $\frac{6 \pm \sqrt{-4}}{2}$. They are called **complex** numbers.

Briefly you may now summarize what you have learned about the roots of a quadratic equation:

FIGURE 100.—CARL FRIEDRICH GAUSS

Carl Friedrich Gauss (1777–1855) was the son of a bricklayer. He early showed great mathematical ability. His favorite study was higher arithmetic, but his contributions were made in almost every field of mathematics. In time he became the greatest mathematician in Germany and the founder of modern mathematics of Germany. He was the first to use the term "complex number" in the sense it has today. Formerly, imaginary numbers appearing as roots of equations had been regarded as meaningless. Through his work the imaginary number was fully admitted into the number system.

(1) If $b^2 - 4ac < 0$, the roots are complex. (The symbol < means "is less than.")

(2) If $b^2 - 4ac$ is not < 0 and if

- $b^2 - 4ac$ is a square, the roots of the equation are real, rational, and unequal.
- $b^2 - 4ac$ is zero, the roots are real, rational, and equal.
- $b^2 - 4ac$ is not a square, the roots are real, irrational, and unequal.

$b^2 - 4ac$ is called the **discriminant** of the quadratic equation $ax^2 + bx + c = 0$.

201. The geometric interpretation of the nature of the roots of a quadratic equation. Let equation (1), $x^2 - 6x + 5 = 0$, be represented graphically as shown in Fig. 101. From the graph the roots of the equation

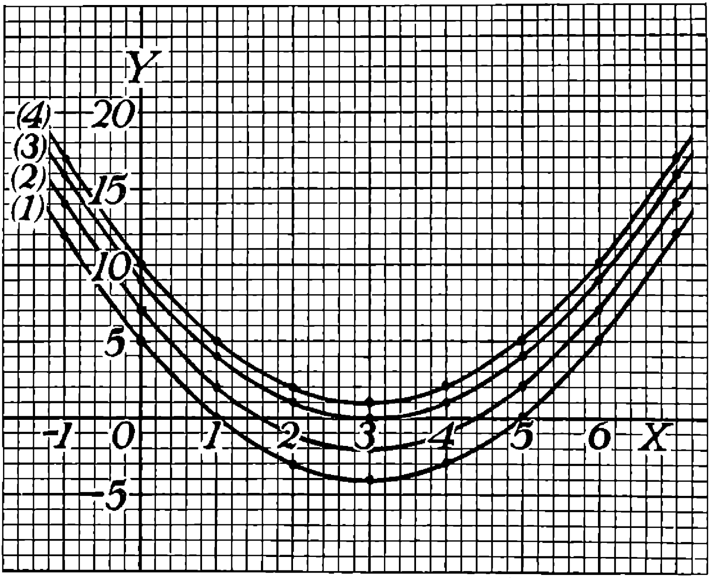

FIGURE 101

are shown to be $x_1 = 1$, $x_2 = 5$; that is, the roots are real, rational, and unequal.

Note that the discriminant $b^2 - 4ac = 36 - 20 = 16$ is a square. This verifies what was said in the summary above.

For equation (2), $x^2 - 6x + 7 = 0$, $b^2 - 4ac = 36 - 28 = 8$. Hence the roots are real, irrational, and unequal. In the diagram, the roots are the points of intersection of the x-axis with curve (2).

For equation (3), $x^2 - 6x + 9 = 0$, $b^2 - 4ac = 36 - 36 = 0$. The roots are real, rational, and equal. Geometrically this means that curve (3) just touches the x-axis.

For equation (4), $x^2 - 6x + 10 = 0$, $b^2 - 4ac = 36 - 40 = -4$. The roots are therefore complex. Geometrically this means that the curve lies entirely above the x-axis; that is, it does not touch the axis and it does not intersect it, as shown in curve (4).

EXERCISES

Determine the nature of the roots of the following equations:

1. $3x^2 + 8x + 5 = 0$.

Solution: $b^2 - 4ac = 64 - 60 = 4$.

Since 4 is a square, the roots are real, rational, and unequal.

2. $x^2 - 8x + 16 = 0$.

Solution: $b^2 - 4ac = 64 - 64 = 0$.

The roots are real, rational, and equal.

3. $x^2 - 5x + 8 = 0$.

Solution: $b^2 - 4ac = 25 - 32 = -7$. Hence the roots are complex.

4. $5x^2 - 3x - 2 = 0$.
5. $2x^2 - 4x + 1 = 0$.
6. $x^2 - 2x + 5 = 0$.
7. $7x^2 + 9x - 10 = 0$.
8. $3x^2 - x + 10 = 0$.
9. $3x^2 - 7x + 2 = 0$
10. $2x^2 - 13x + 15 = 0$.
11. $x^2 - 3x + 5 = 0$.
12. $9x^2 + 12x + 4 = 0$.
13. $5x^2 + 8x - 2 = 0$.
14. $5x^2 - 4x + 2 = 0$.
15. $x^2 - 5x + 8 = 0$.

202. Relations between the roots of a quadratic equation and the coefficients. You know that the roots of the equation $ax^2 + bx + c = 0$ are

$$x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a},$$

$$\text{and } x_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

Find the *sum* of the two roots as follows:

$$x_1 + x_2 = \frac{-b + \sqrt{b^2 - 4ac} - b - \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-\cancel{2}b}{\cancel{2}a} = -\frac{b}{a}.$$

Therefore $x_1 + x_2 = -\frac{b}{a}$.

Find the *product* of the two roots as follows:

$$x_1 \cdot x_2 = \frac{\left(-b + \sqrt{b^2 - 4ac}\right)\left(-b - \sqrt{b^2 - 4ac}\right)}{(2a)(2a)}$$

$$= \frac{(-b)^2 - \left(\sqrt{b^2 - 4ac}\right)^2}{4a^2}$$

$$= \frac{b^2 - b^2 + 4ac}{4a^2}$$

$$= \frac{\cancel{4a}c}{\underset{a}{\cancel{4a^2}}} = \frac{c}{a}.$$

Therefore $x_1 \cdot x_2 = \frac{c}{a}$.

This means that, if every term of $ax^2 + bx + c = 0$ is divided by a and the equation is thereby changed to $x^2 + \frac{b}{a}x + \frac{c}{a} = 0$, then the coefficient of x with the sign changed is the sum of the roots and the constant term is the product of the roots.

EXERCISES

State the sum of the roots and the product of the roots for each of the following equations:

1. $x^2 - 3x + 7 = 0.$

Solution: $x_1 + x_2 = -\frac{b}{a} = -\frac{-3}{1} = 3.$

$$x_1x_2 = \frac{c}{a} = \frac{7}{1} = 7.$$

2. $3x^2 - 2x + 8 = 0.$

Solution: $x_1 + x_2 = -\frac{-2}{3} = \frac{2}{3}.$

$$x_1x_2 = \frac{8}{3}.$$

3. $2x^2 - 9x + 8 = 0.$

4. $3x^2 - 2x + 6 = 0.$

5. $5x^2 - 2x - 16 = 0.$

6. $7x^2 + 9x - 10 = 0.$

7. $5x^2 - 3x - 2 = 0.$

8. $x^2 - x + 10 = 0.$

9. $x^2 + 12x + 35 = 0.$

10. $x^2 - 6x + 9 = 0.$

Form the equations whose roots are:

11. 2, 5

Solution: $x_1 = 2, x_2 = 5.$

$$\therefore -(x_1 + x_2) = -7 = -\frac{b}{a}.$$

$$x_1x_2 = 10 = \frac{c}{a}.$$

Substituting in the equation $x^2 + \frac{b}{a}x + \frac{c}{a} = 0$, you have

$$x^2 - 7x + 10 = 0.$$

12. $-2, 3$
13. $-5, -2$
14. $\frac{1}{2}, 3$
15. $\frac{3}{4}, \frac{1}{2}$
16. $-\frac{3}{4}, -\frac{1}{3}$
17. $\frac{1}{2}, -\frac{1}{4}$
18. a, b
19. $a + 2b, a$
20. $-m + n, -m - n$
21. $a - 2, a + 2$
22. $\sqrt{3}, -\sqrt{3}$
23. $1 + \sqrt{2}, 1 - \sqrt{2}$
24. $2 + \sqrt{3}, 2 - \sqrt{3}$
25. $-1 + \sqrt{7}, -1 - \sqrt{7}$
26. $a + \sqrt{b}, a - \sqrt{b}$

EQUATIONS INVOLVING RADICALS

203. What is meant by an irrational equation. If an equation is written in its simplest form and if the unknown appears under the radical sign, it is called an **irrational** equation. This includes the case where the unknown number has a fractional exponent. The equations $x - x^{\frac{1}{2}} + 12 = 0$ and $\sqrt{3x - 5} - 4 = 0$ are illustrations of irrational equations.

204. How to solve irrational equations. The following examples explain a method of solving irrational equations:

1. Solve the equation $\sqrt{x + 2} - 4 = 0$.

Solution: Add 4 to each side of the equation. This leaves the radical alone on one side:

$$\sqrt{x + 2} = 4.$$

Square both sides of the equation. This gives

$$x + 2 = 16.$$

Therefore $x = 14$.

Check:

LEFT MEMBER	RIGHT MEMBER
$\sqrt{14 + 2}$	4
$\sqrt{16}$	4
4	4

2. Solve the equation $x - 1 - \sqrt{x^2 - 5} = 0$.

Solution: Add $\sqrt{x^2 - 5}$ to both members. This leaves the radical alone on one side of the equation:

$$x - 1 = \sqrt{x^2 - 5}.$$

Squaring both members, you have

$$\begin{aligned} x^2 - 2x + 1 &= x^2 - 5. \\ -2x + 1 &= -5. \\ 2x &= 6. \\ x &= 3. \end{aligned}$$

Check by substituting 3 for x in the original equation.

EXERCISES

Solve the following equations:

1. $\sqrt{x^2 + 7} = x + 1$.
2. $\sqrt{2x + 10} = \sqrt{3x + 7}$.
3. $\sqrt{x - 5} = 5 - \sqrt{x}$.
4. $\sqrt{9x^2 - 5} - 3x - 1 = 0$.
5. $\sqrt{x} - 4 + \sqrt{x + 8} = 0$.
6. $\sqrt{4x + 3} = 2\sqrt{x - 1} + 1$.
7. $\sqrt{x + 16} + \sqrt{x} = 8$.
8. $\sqrt{x - 1} - \sqrt{x - 4} - 1 = 0$.
9. $2\sqrt{3a - 5} = 3\sqrt{a + 1}$.
10. $\sqrt{y} - 2 = \sqrt{y - 12}$.
11. $\sqrt{x + 4} = 3 - \sqrt{x}$.
12. $\sqrt{a + 6} - 5 = \sqrt{a + 11}$.

13. $\sqrt{2x} + 1 = \sqrt{2x + 9}.$

14. $\sqrt{x + 3} - 5 = -\sqrt{x - 2}.$

15. $5 - \sqrt{2x} = \sqrt{2x + 5}.$

16. $\sqrt[3]{x + 2} = 3.$

Suggestion: Raise both sides to the third power.

205. Irrational equations leading to quadratics. Irrational equations are sometimes of quadratic form. For example, the equation

$x^2 - 3x + 15 + \sqrt{x^2 - 3x + 15} - 30 = 0$ may be written

$$(\sqrt{x^2 - 3x + 15})^2 + \sqrt{x^2 - 3x + 15} - 30 = 0.$$

Factoring, you have

$$(\sqrt{x^2 - 3x + 15} + 6)(\sqrt{x^2 - 3x + 15} - 5) = 0.$$

$$\text{Therefore } \sqrt{x^2 - 3x + 15} + 6 = 0,$$

$$\text{and } \sqrt{x^2 - 3x + 15} - 5 = 0.$$

$$\sqrt{x^2 - 3x + 15} = -6,$$

$$\text{and } \sqrt{x^2 - 3x + 15} = 5.$$

Squaring both members in the last two equations, you have

$$x^2 - 3x + 15 = 36,$$

$$\text{and } x^2 - 3x + 15 = 25.$$

These equations are quadratic equations.

Solve them and substitute the roots in the original equation to see whether they check. The check is necessary for irrational equations because in squaring a radical equation a new root is sometimes introduced which is not a root of the original equation.

EXERCISES

Solve the following:

1. $x^2 - 5x - 2 + 2\sqrt{x^2 - 5x - 2} = 8$.

2. $y + 2\sqrt{y - 1} - 4 = 0$.

Solution: Subtract $y - 4$. This leaves the radical alone on one side, and $2\sqrt{y - 1} = 4 - y$.

Square both sides:

$$\begin{aligned} 4(y - 1) &= 16 - 8y + y^2. \\ y^2 - 12y + 20 &= 0. \\ y_1 &= 10, \\ \text{and } y_2 &= 2. \end{aligned}$$

Check by substituting in the original equation.

3. $a + \sqrt{a - 7} = 19$.

4. $\sqrt{x + 9} - \sqrt{x - 7} = 2$.

5. $\sqrt{a + 5} - \sqrt{a} = 1$.

6. $\sqrt{x^2 - 16} - \sqrt{x^2 - 9} = 7$.

7. $2\sqrt{a - 1} = \sqrt{4a + 3} - 1$.

8. $\sqrt{4a - 22} + \sqrt{2} - 2\sqrt{a} = 0$.

9. $\sqrt{x} + \sqrt{4x + 9} - \sqrt{x + 5} = 0$.

10. $\sqrt{2 + a} = \sqrt{3} - \sqrt{2 - a}$.

11. $\sqrt{2a + 8} = \sqrt{7a + 21} - \sqrt{a + 5}$.

12. $\sqrt{x + 1} + \sqrt{x + 10} = 9$.

13. $\sqrt{2x + 5} + \sqrt{2x - 3} = 4$.

14. $2\sqrt{a - 1} - \sqrt{a - 4} = \sqrt{a + 4}$.

TABLES FOR REFERENCE

MEASURES OF LENGTH

English System

12 inches (in.) = 1 foot (ft.)
3 feet = 1 yard (yd.)
$5\frac{1}{2}$ yards or $16\frac{1}{2}$ feet = 1 rod (rd.)
320 rods = 1 mile (mi.)
1 mile = 320 rods = 1760 yards = 5280 feet = 63,360 inches

Metric System

10 millimeters (mm.) = 1 centimeter (cm.)
10 centimeters = 1 decimeter (dm.)
10 decimeters = 1 meter (m.)
1 meter = 39.37 inches
1 yard = .9144 meter

Surveyor's Units of Length

7.92 inches = 1 link
25 links = 1 rod
100 links = 4 rods = 1 chain
80 chains = 5280 feet = 1 mile

Other Units of Length

4 inches = 1 hand (used in measuring height of horses)
6 feet = 1 fathom (used in measuring depth of water)
1.15 miles = 1 knot (used in measuring distances at sea)

Measures of Surface

144 square inches (sq. in.) = 1 square foot (sq. ft.)
9 square feet = 1 square yard (sq. yd.)
$30\frac{1}{4}$ square yards = 1 square rod (sq. rd.)
160 square rods = 1 acre (A.)
640 acres = 1 square mile (sq. mi.)
36 square miles = a township (tp.)
1 acre = a square approximately 209 feet on each side, or 4840 square yards, or 43,560 square feet

Measures of Volume

1728 cubic inches (cu. in.) = 1 cubic foot (cu. ft.)
27 cubic feet = 1 cubic yard (cu. yd.)
16 cubic feet = 1 cord foot (cd. ft.)
128 cubic feet = 1 cord (cd.)
1 board foot = 1 foot long, 1 foot wide, and 1 inch thick

Equivalents

1 bushel = $1\frac{1}{4}$ cubic feet, or 2150.42 cubic inches
1 gallon = 231 cubic inches
1 cubic foot of water = 62.5 pounds (approx.)
1 ton of hay = 500 cubic feet (approx.)
1 ton of hard coal = 35 cubic feet (approx.)
1 ton of soft coal = 38 cubic feet (approx.)

Liquid Measure

4 gills (gi.) = 1 pint (pt.)
2 pints = 1 quart (qt.)
4 quarts = 1 gallon (gal.)
231 cubic inches = 1 gallon
1 cubic foot = 7.48 gallons
$31\frac{1}{2}$ gallons = 1 barrel (bbl.)
4.27 cubic feet = 1 barrel

Dry Measure

2 pints (pt.) = 1 quart (qt.)
8 quarts = 1 peck (pk.)
4 pecks = 1 bushel (bu.)
$1\frac{1}{4}$ cubic feet = 1 bushel (approx.)

Weights

16 ounces (oz.) = 1 pound (lb.)
100 pounds = 1 hundredweight (cwt.)
2000 pounds = 1 ton (T.)

Weights per Bushel

Corn on ear = 70 pounds
Potatoes, wheat = 60 pounds
Onions = 57 pounds
Rye, shelled corn = 56 pounds
Turnips, sweet potatoes = 55 pounds
Apples, peaches = 48 pounds
Barley, buckwheat = 48 pounds
Oats = 32 pounds

Counting

12 units = 1 dozen (doz.)
12 dozen = 1 gross (gr.)
24 sheets of paper = 1 quire
480 (now often 500) sheets = 1 ream

United States Money

10 mills (m.) = 1 cent (ct. or ¢)
10 cents = 1 dime (di.)
10 dimes = 1 dollar ($)
100 cents = 1 dollar

Measures of Angles and Arcs

60 seconds (″) = 1 minute (′)
60 minutes = 1 degree (°)
$69\frac{1}{6}$ miles = 1 degree of latitude

Measures of Time

60 seconds (sec.) = 1 minute (min.)
60 minutes = 1 hour (hr.)
24 hours = 1 day (da.)
7 days = 1 week (wk.)
365 days = 1 common year (yr.)
366 days = 1 leap year
12 months = 1 year
360 days = 1 commercial year
10 years = 1 decade
100 years = 1 century

Formulas

Areas

Area of a rectangle	$A = bh$
Area of a square	$A = s^2$
Area of a parallelogram	$A = bh$
Area of a triangle	$A = \frac{1}{2}bh$
Area of a trapezoid	$A = \frac{1}{2}h(a + b)$
Area of a circle	$A = \pi r^2$
Theorem of Pythagoras	$a^2 + b^2 = c^2$

Volumes

Volume of a rectangular block	$V = lwh$
Volume of a cube	$V = e^3$

Angles

Sum of the angles of a triangle	$a + b + c = 180$
Complementary angles	$a + b = 90$
Supplementary angles	$a + b = 180$

Circle

Circumference $= 2\pi r = \pi d$ $\quad \pi = 3.14159$

Area $= \pi r^2$

Percentage

Interest $i = \dfrac{prt}{100}$

Percentage $p = \dfrac{rb}{100}$

Quadratic Equations

$$ax^2 + bx + c = 0.$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

$$x_1 + x_2 = -\frac{b}{a}.$$

$$x_1 \cdot x_2 = \frac{c}{a}.$$

Variation

Direct variation: $y = cx.$

$$\frac{y_1}{y_2} = \frac{x_1}{x_2}.$$

Inverse variation: $y = \dfrac{c}{x}.$

$$xy = c.$$

$$\frac{y_1}{y_2} = \frac{x_2}{x_1}.$$

Logarithms

$$\log ab = \log a + \log b.$$

$$\log \frac{a}{b} = \log a - \log b.$$

$$\log a^m = m \log a.$$

$$\log \sqrt[n]{a} = \frac{\log a}{n}.$$

Trigonometric Ratios

Let c denote the hypotenuse of a right triangle, a the side opposite angle A, and b the side adjacent to angle A.

$$\text{Then } \sin A = \frac{a}{c}.$$

$$\cos A = \frac{b}{c}.$$

$$\tan A = \frac{a}{b}.$$

Factoring

$$x^2 - y^2 = (x + y)(x - y).$$

$$x^3 - y^3 = (x - y)(x^2 + xy + y^2).$$

$$x^3 + y^3 = (x + y)(x^2 - xy + y^2).$$

$$x^2 \pm 2xy + y^2 = (x \pm y)^2.$$

$$ax + ay + az = a(x + y + z).$$

Exponents

$$a^m \cdot a^n = a^{m+n}.$$

$$\frac{a^m}{a^n} = a^{m-n}.$$

$$(abc)^m = a^m b^m c^m.$$

$$\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}.$$

$$(a^m)^n = a^{mn}.$$

$$a^0 = 1.$$

$$a^{-n} = \frac{1}{a^n}.$$

$$a^{\frac{m}{n}} = \sqrt[n]{a^m} = \left(\sqrt[n]{a}\right)^m.$$

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