IMPROVING NSTRUCTION IN ATHEMATICS

VOL I

DEPARTMENT OF FIELD SERVICES
ONAL INSTITUTE OF EDUCATION
NEW DELHI

IMPROVING INSTRUCTION IN MATHEMATICS

Experimental Edition

IMPROVING INSTRUCTION
IN
MATHEMATICS

Volume I

DEPARTMENT OF FIELD SERVICES
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FOREWORD

The Extension Services Project in India has put in substantive efforts in the area of school improvement. Its main focus has been on vitalising the classroom instruction through a programme of in-service education of teachers and other educational personnel. This programme has been considered a very effective programme in the direction of school improvement.

One neglected aspect of a programme of school improvement is enriching the content background of school teachers. The most significant activity in this direction is the involvement of teachers in developing instructional materials. It is only when teachers are involved in developing instructional materials that real improvement in classroom takes a shape. During the two years—1966-67 and 1967-68, all the Extension Services Departments in the country have been involved in a programme of developing instructional material in the form of teaching units. In many cases the teachers have been assisted in the programme by training college personnel and even lecturers from the colleges of Arts and Science. These materials will prove useful in improving the classroom instruction on the one hand and on the other in motivating the teachers to develop a wide variety of materials so useful in improving the instructional programme.

On repeated requests from the Extension Services Departments, the Department of Field Services has decided to select a few specimen teaching units developed by classroom teachers for a wider dissemination and try out by teachers. The suggestions of teachers in improving these materials and in removing the deficiencies will go a long way in improving this programme in the country.

The Department of Field Services is happy to bring out the experimental editions of the materials developed by teachers under its scheme of centralised publications. The Department hopes that the Extension Centres will collect reactions of the teachers and their suggestions for further improvement of these materials. The Department also hopes that the Extension Centres will intensify the programme of involvement of teachers into this programme which constitutes a challenging dimension in the programme of Educational Extension in this country.

M. B. BUCH
Head
Department of Field Services

NEW DELHI
5th February, 1969

INTRODUCTION

During recent years the National Council of Educational Research and Training had given considerable thought to the development of improved instructional materials to accelerate the process of improvement of the quality of school education. In the light of our experience in the field of educational extension work, we are fully convinced that the development of improved instructional materials in order to raise the effectiveness of classroom instruction must involve the classroom teacher. With this in view one of the major programmes undertaken by NIEaimed at improvement of quality in education has been the development of teaching units in school subjects for the benefit of classroom teachers. The Department of Field Services has been encouraging Extension Services Departments attached to more than 140 training institutions in the country to undertake similar work by involving experienced and competent classroom teachers in this process. It is gratifying to report that almost all the Extension Services Departments could set up strong working groups of teachers to develop instructional materials as a broad programme of educational extension. This has been found to be an excellent extension technique for promoting professional growth among practising teachers. The emphasis has been more on development of understanding and skills of teachers through their involvement rather than the production of materials. As such the materials produced should be treated as experimental.

During 1967-68 more than 350 teaching units were developed by the Extension Services Departments in various school subjects. Most of these units were screened and refined by experts locally available before they were circulated among teachers for trying out. By this process the classroom teachers are now in a position to remedy deficiencies in teaching by drawing freely new subject matter and teaching techniques form relevant teaching units. Further, at present, alongwith the explosion of school-going population there has been winessed a corresponding explosion of knowledge. Many textbooks and teaching techniques have become out of date. By developing teaching units the Extension Services Departments attempted to provide the teacher with new instructional materials which give him guidance and direction to improve his knowledge and methods of teaching.

The Department of Field Services collected some samples of teaching units in Mathematics from Extension Centres and got them reviewed by experts with a view to printing them for the use of teachers in the country. The present booklet which is one in the series entitled "Improving Instruction" contains the following four units :

1. 'Trigonometrical Ratios'—developed by the Extension Services Department, Kalyani University (W. Bengal).
2. 'Similarity'—developed by the Extension Services Department, R.N. Training College, Cuttack, Orissa.
3. 'H.C.F. and L.C.M.'—developed by Extension Services Department, Tilak College of Education, Poona.
4. 'Elimination'—developed by Extension Services Department, Teachers' Training College, Jammu.

In view of the nature and scope of a teaching unit, it is obvious that there is no finality about it. The materials will require continuous review and revision in the light of teachers' experiences and reactions. The materials are not to be taken as prescriptive in any sense. The teaching units are a kind of an enriched material which the teacher can use according to the exigencies of the classroom teaching and availability of resources. Extension Services Departments, we are confident, will not only continue this programme more vigorously but will also make a comprehensive plan of follow-up after training teachers in the technique of using these materials.

The four teaching units that we present in the following pages embody the collective thinking of a large number of classroom teachers, teacher educators and subject specialist. The

CONTENTS

1

TRIGONOMETRICAL RATIOS

I. Introduction

Trigonometry means the science of measurement of triangles. It establishes a relationship between the sides and angles of triangles and utilises the relationship between the ratios of the sides of a right angled triangle for its study. It also illustrates mathematical system built on a few assumptions. Trigonometry has possibilities for providing an enlightening cultural experience for pupils with non-science interests and at the same time furnishing needed skills and understandings for pupils preparing for further mathematics. It unveils a wide range of applied problems. Trigonometry is purposefully and effectively applied in navigation, engineering, surveying, military science, etc.

Trigonometrical ratios are ratios between the sides of a right angled triangle in terms of the angles of that triangle. An elevation or depression is expressed in the form of a ratio and is given a symbolic representation by utilising the angle of elevation or depression or the like. These ratios and their inter-relationship is of importance in the whole of the study of trigonometry. As a matter of fact trigonometrical ratios are as basic and important to the study of trigonometry as whole numbers and fractions are to the study of arithmetic.

A thorough study of trigonometrical ratios and their basic relationships is thus of prime importance.

II. Objectives

1. *To enable the pupils to understand the terms, concepts, principles, etc.*

SPECIFICATIONS

(*i*) The pupil recognises the terms, concepts, principles, etc.

(*ii*) He recalls the terms, concepts, principles, etc.

(*iii*) He illustrates with examples the terms, concepts, principles, etc.

(*iv*) He discriminates between closely related concepts.

2. *To enable the pupil to find the values of trigonometrical ratios.*

SPECIFICATIONS

(*i*) The pupil finds the values of trigonometrical ratios with the help of his knowledge of geometry.

(*ii*) He does computation with the help of these values.

3. *To enable the pupil to apply the knowledge of trigonometrical ratios in every day life situations.*

SPECIFICATIONS

(*i*) The pupil analyses the problem and finds what is given and what is to be found.

(*ii*) He finds the adequacy or otherwise of the data.

(*iii*) He selects relevant data.

(*iv*) He estimates results.

(*v*) He verifies the solution.

(*vi*) He can detect errors in a process or operation.

(*vii*) He can identify the properties involved in a given process or operation.

(*viii*) He develops ability to form a set of rules out of a number of given assumptions.

III. Content—Its analysis and organisation

(1) *Directed line segments*

The concepts that are associated with a line segment are distance and direction. The distance from one point to another or the length of the line segment between the two points, is the number of times that an accepted unit can be laid off along the given segment. We select either direction along the line as the +ive direction, and then the opposite direction as −ive. (+) and (−) signs are used to indicate opposite directions. Conventionally, in a line drawn horizontally the direction towards the right is taken as +ive and to the left as −ive. Similar conventions do exist for the types of lines.

(2) *Directed angle*

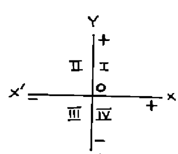
In trigonometry, we shall regard an angle as being generated by rotating one of the rays about its end point into the position of the other. Thus we define an angle to be the geometric angle together with the rotation used to bring one side (the initial side) into the position of the other side (the terminal side).

If the rotation is in the anti-clockwise direction, then the angle formed is defined as +ive angle and if the rotation is in the clockwise direction the angle formed is defined as −ive angle.

An angle that measures between 'greater than 0° but less than 90°' is an acute angle, one 'greater than 90° and less than 180°' is obtuse; and an angle 'greater than 180° and less than 360°' is called reflex angle. An angle is defined as the rotation, and not as just inclination of a line about a fixed point in another line. There can be angles greater than 360° also.

(3) *Quadrants*

Two directed line segments, one horizontal and another vertical intersecting at right angles form quadrants when extended in opposite directions from the point of intersection.



This point of intersection is named as the origin and the lines are called the axes of co-ordinates. It is the Cartesian system of co-ordinates and gets its name from the French mathematician

Descartes, the founder of co-ordinate geometry. Rene 'Descartes showed how algebraic methods could be applied to geometrical situations and thus made possible a great deal of the growth of modern mathematics. The quadrant bound by the +ive horizontal and + ive vertical rays is defined as first quardrant; +ive vertical and −ive horizontal as second quardrant; −ive horizontal and −ive vertical as third quadrant and +ive horizontal and −ive vertical as fourth quadrant. The directed angles between 0° and 90° lie in the I quadrant, between 90° and 180° in the II quadrant, between 180° and 270° in the III quadrant and those between 270° and 360° lie in the IV quadrant.

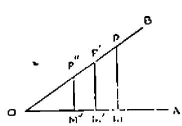
Any angle θ can be expressed as $\theta = 2n\pi + \alpha$ (where $2\pi = 360°$) where $n = 0$ or any positive integer. The determination of quadrant of θ will depend upon the quadrant of α according to the above said rules.

(4) *Ratio*

The ratio is defined as a comparison of two quantities in the sense that one is a certain fraction of the other or that one is a certain number of times the other and also as a single number which may be used as a multiplier.

(5) *Knowledge of constancy of ratios of similar sides of a right angled triangle for an acute angle*

Let P, P' and P'' be any three points on the line OB. PM, $P'M'$ and $P''M''$ are perpendiculars from these points on OA.



(*i*) $$\frac{PM}{OP} = \frac{P'M'}{OP'} = \frac{P''M''}{OP''}$$

$$\frac{OP}{PM} = \frac{OP'}{P'M'} = \frac{OP''}{P''M''}$$

(*ii*) $$\frac{OM}{OP} = \frac{OM'}{OP'} = \frac{OM''}{OP''}$$

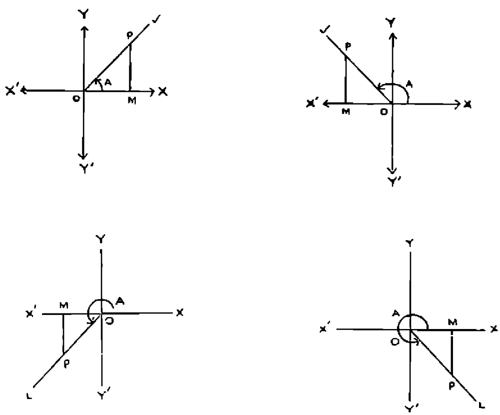
$$\frac{OP}{OM}=\frac{OP'}{OM'}=\frac{OP''}{OM''}$$

(*iii*) $$\frac{PM}{OM}=\frac{P'M'}{OM'}=\frac{P''M''}{OM''}$$

$\therefore$ $$\frac{OM}{PM}=\frac{OM'}{P'M'}=\frac{OM''}{P''M''}.$$

[This constancy is derived from the concept of similarity as applicable to triangles]

(6) *Knowledge of Trigonometrical ratios*



Let there be a fixed line OX and let the revolving line take up the position OL after revolving through an angle A in the anti-clokwise direction. From a point P on OL draw PM perpendicular to OX.

Trigonometrical ratios are defined as follows :

[sine A is read as sin A and similarly other ratios are read as cos A, tan A, cosec A, and cot A respectively]

(*i*) sine of $A=\dfrac{MP}{OP}$

i.e., $\sin A=\dfrac{\text{Perpendicular}}{\text{Hypotenuse}}$

(*ii*) cosine of $A=\dfrac{OM}{OP}$

$$i.e.\ \cos A=\frac{\text{Base}}{\text{Hypotenuse}}$$

(*iii*) tangent of $A=\dfrac{MP}{OM}$

$$i.e.\ \tan A=\frac{\text{Perpendicular}}{\text{Base}}$$

(*iv*) cosecant of $A=\dfrac{OP}{MP}$

$$i.e.\ \operatorname{cosec} A=\frac{\text{Hypotenuse}}{\text{Perpendicular}}$$

(*v*) secant of $A=\dfrac{OP}{OM}$

$$i.e.\ \sec A=\frac{\text{Hypotenuse}}{\text{Base}}$$

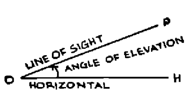
(*vi*) cotangent of $A=\dfrac{OM}{MP}$

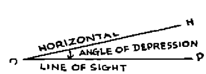
$$i.e.\ \cot A=\frac{\text{Base}}{\text{Perpendicular}}$$

Since sin A, cos A, tan A, etc. are simple ratios, they will be treated as numbers, and will follow all the operations of simple algebra. But sin A, cos A, tan A, etc. will not be treated as sin $\times A$, cos $\times A$ and tan $\times A$. Moreover, $(\sin A)^2$, $(\cos A)^2$, $(\tan A)^2$, etc. are written in the form $\sin^2 A$, $\cos^2 A$, $\tan^2 A$, etc.

Since hypotenuse of any right angled triangle is greater than its other two sides, therefore, the values of sin A and cos A cannot exceed unity, but the values of tan A may exceed unity.

(7) *Angle of elevation and depression*

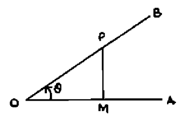




If an observer at O looks at a point P, then OP is called the line of Sight as indicated in the figure. If H is a point on a horizontal line through O then the angle between OH and OP is called the angle of elevation of P at O when P is above OH and is called the angle of depression of P at O when P is below OH.

(8) *Knowledge of conversion of one trigonometrical ratio into another*

P is any point on OB and PM is drawn perpendicular from P on OA.



(*i*) $\sin A = \dfrac{MP}{OP}$, $\operatorname{cosec} A = \dfrac{OP}{MP}$

$$\sin A = \frac{1}{\frac{OP}{MP}} = \frac{1}{\operatorname{cosec} A}$$

(*ii*) $\cos A = \dfrac{OM}{OP}$, $\sec A = \dfrac{OP}{OM}$

$$\cos A = \frac{1}{\frac{OP}{OM}} = \frac{1}{\sec A}$$

(*iii*) $\tan A = \dfrac{MP}{OM}$. $\cot A = \dfrac{OM}{MP}$

$$\tan A = \frac{1}{\frac{MP}{OM}} = \frac{1}{\cot A}$$

(*iv*) $OM^2 + MP^2 = OP^2$ (Theorem of Pythagorus)

$\dfrac{OM^2}{OP^2} + \dfrac{MP^2}{OP^2} = 1$ (Dividing both sides by OP^2)

i.e. $\left\{\dfrac{OM}{OP}\right\}^2 + \left\{\dfrac{MP}{OP}\right\}^2 = 1$

i.e. $(\cos A)^2 + (\sin A)^2 = 1$

i.e. $\cos^2 A + \sin^2 A = 1.$

(v) $$OM^2+MP^2=OP^2$$

or $$\frac{OM^2}{OM^2}+\frac{MP^2}{OM^2}=\frac{OP^2}{OM^2}$$

or $$1+\left(\frac{MP}{OM}\right)^2=\left(\frac{OP}{OM}\right)^2$$

or $$1+\left\{\frac{MP}{OM}\right\}^2=\left\{\frac{OP}{OM}\right\}^2$$

or $$1+\tan^2 A=\sec^2 A.$$

(vi) $$OM^2+MP^2=OP^2$$

or $$\left(\frac{OM}{MP}\right)^2+1=\left(\frac{OP}{MP}\right)^2$$

or $$\left\{\frac{OM}{MP}\right\}^2+1=\left\{\frac{OP}{MP}\right\}^2$$

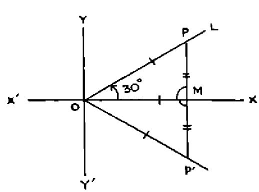
or $$\cot^2 A+1=\text{cosec}^2 A$$

(vii) $$\tan A=\frac{MP}{OM}=\frac{\frac{MP}{OP}}{\frac{OM}{OP}}=\frac{\sin A}{\cos A}$$

(viii) $$\cot A=\frac{1}{\tan A}=\frac{1}{\frac{\sin A}{\cos A}}=\frac{\cos A}{\sin A}$$

(9) *Knowledge of values of Trigonometrical ratios*

(a) Let the revolving line start from the initial position OX and after revolving through an angle of 30° in the anti-clockwise direction take up the terminal (final) position OL. Take a point P on OL and draw $PM \perp OX$. Produce PM to P' such that $MP=P'M$. In the two triangles OMP and OMP'



$OM=OM$

$MP=P'M$ and $\angle OMP=\angle OMP'$ (90° each)

$\therefore$ The triangles are congruent.

$\therefore$ $\angle MOP = \angle MOP' = 30°$ each

$\therefore$ $\angle P'OP = 60°$, and $\angle OP'M = \angle OPM$. Thus each of these is also of 60°. If the side of the equilateral $\triangle OP'P$ is taken up as $2a$, then $MP = a, OP = 2a$ and $OM = \sqrt{3}a$.

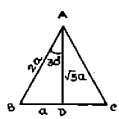
$$\sin 30° = \frac{MP}{OP} = \frac{a}{2a} = \frac{1}{2}$$

$$\cos 30° = \frac{OM}{OP} = \frac{\sqrt{3}a}{2a} = \frac{\sqrt{3}}{2}$$

$$\tan 30° = \frac{MP}{OM} = \frac{a}{\sqrt{3}a} = \frac{1}{\sqrt{3}}$$

Alternatively

ABC is an equilateral triangle of side $2a$. AD is perpendicular to BC, then



(*i*) $\sin 30° = \frac{BD}{AB} = \frac{a}{2a} = \frac{1}{2}$

(*ii*) $\cos 30° = \frac{AD}{AB} = \frac{\sqrt{3}a}{2a} = \frac{\sqrt{3}}{2}$

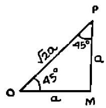
(*iii*) $\tan 30° = \frac{BD}{AD} = \frac{a}{\sqrt{3}a} = \frac{1}{\sqrt{3}}$

(*iv*) $\sin 60° = \frac{AD}{AB} = \frac{\sqrt{3}a}{2a} = \frac{\sqrt{3}}{2}$

(*v*) $\cos 60° = \frac{BD}{AB} = \frac{a}{2a} = \frac{1}{2}$

(*vi*) $\tan 60° = \frac{AD}{BD} = \frac{\sqrt{3}a}{a} = \sqrt{3}$

(*b*) *POM* is an isosceles right angled triangle. The length of whose equal sides is *a*. The length of the hypotenuse is $\sqrt{2}a$, then



(*i*) $\sin 45° = \frac{MP}{OP} = \frac{a}{\sqrt{2}a} = \frac{1}{\sqrt{2}}$

(*ii*) $\cos 45° = \frac{OM}{OP} = \frac{a}{\sqrt{2}a} = \frac{1}{\sqrt{2}}$

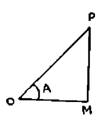
(*iii*) $\tan 45° = \frac{MP}{OM} = \frac{a}{a} = 1$

(*c*) *OPM* is a right angled triangle.

Let $\angle MOP = A$

$$\sin A = \frac{MP}{OP},\ \cos A = \frac{OM}{OP},\ \tan A = \frac{MP}{OM}$$

When, *OP* approaches towards *OM* and ultimately becomes coincident with *OM*. Then



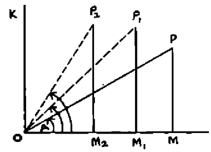
$$A = 0°, OP = OM, MP = 0.$$

(*i*) $\sin 0° = \frac{MP}{OP} = \frac{0}{OP} = 0$

(*ii*) $\cos 0° = \frac{OM}{OP} = \frac{OP}{OP} = 1$

(*iii*) $\tan 0° = \frac{MP}{OM} = \frac{0}{OM} = 0$

(*d*) Let *POM* be a right angled triangle and $\angle MOP = A$. *OK* is perpendicular to *OM* at *O*.



Now $\sin A = \frac{MP}{OP}$, $\cos A = \frac{OM}{OP}$, $\tan A = \frac{MP}{OM}$

When, *OP* approaches towards *OK* and becomes ultimately coincident with *OK*. Then

$$\angle A = 90°,\ MP = OP \text{ and } OM = 0$$

(*i*) $\sin 90° = \frac{MP}{OP} = \frac{OP}{OP} = 1$

(*ii*) $\cos 90° = \frac{OM}{OP} = \frac{0}{OP} = 0$

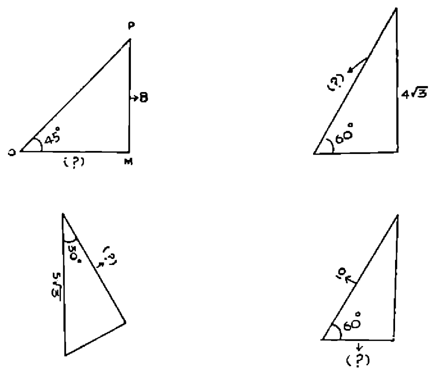
(*iii*) $\tan 90° = \frac{MP}{OM} = \frac{MP}{0} = \infty$

Angle / T. Ratio	0°	30°	45°	60°	90°
sin	0	1/2	$1/\sqrt{2}$	$\sqrt{3}/2$	1
cos	1	$\sqrt{3}/2$	$1/\sqrt{2}$	1/2	0
tan	0	$1/\sqrt{3}$	1	$\sqrt{3}$	∞

As cosec *A*, sec *A*, and cot *A* are reciprocals of sin *A*, cos *A*, and tan *A* respectively, their values for these angles can easily be calculated from the table given above. The values of the trigonometrical ratios are different for different angles. The values of sine and tangent of any angle increases as the angle increases from 0° to 90°, but the value of cosine decreases as the angle increases.

(10) *Application of Trigonometrical ratios*

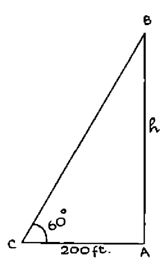
(*a*) If one side and an acute angle of a right angled triangle are given, pupils can find the other side by taking that trigono-



metrical ratio that involves both the unknown side and the known side.

(*b*) At a point 200 ft. from the foot of a tower the angle of elevation of its top is observed to be 60°. Find the height of the tower.

Here AB represents the tower. C is the point of observation. Then $CA=200$ ft. $\angle ACB=60°$, the angle of elevation of the



top of the tower. We have to find AB. Let $AB=h$. Now from the right angled $\triangle BCA$,

$$\tan 60^\circ = \frac{AB}{AC} = \frac{AB}{200}$$

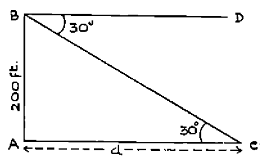
or $$\sqrt{3} = \frac{h}{200}$$

$\therefore$ $$h = 200 \times \sqrt{3} \text{ ft.}$$

or $$h = 200\sqrt{3} \text{ ft.}$$

(*c*) An observer on the top of a cliff 200 ft. above sea level notices that the angle of depression of a boat is 30°. How far is the boat from the foot of the cliff?

AB represents the height of the cliff, *B* represents an observer



and *C* represents a boat. $AB = 200$ ft. The angle of depression of the boat at *B* is 30°. We have to find *AC*.

Let $$AC = d \text{ ft.}$$

Now $$\angle DBC = \angle BCA = 30^\circ.$$

$$\cot 30^\circ = \frac{AC}{AB}$$

$$\frac{d}{AB} = \sqrt{3}$$

$$d = \sqrt{3}\, AB$$

$$= \sqrt{3} \times 200$$

$\therefore$ $$d = 200\sqrt{3} \text{ ft.}$$

Alternatively

$$\tan 30^\circ = \frac{AB}{AC}$$

or $$\frac{1}{\sqrt{3}} = \frac{200}{d}$$

or $$d = 200\sqrt{3} \text{ ft.}$$

IV. Teaching Hints and Learning Experiences

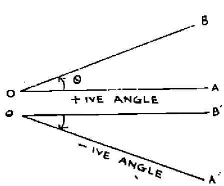
(*i*) *Directed line segment*

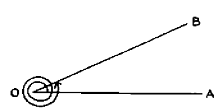
The teacher draws a straight line *AD*. He takes the length of the line segment *AB* in the figure as a unit. The length of *AC* is 4 units since the unit *AB* can be laid off four times along AC. Likewise the length of AD in the figure is 5 units.

In going from A to D, we travel in the direction opposite to that which we take in going from D to A. The teacher selects either direction along the line as the +ive direction and then the opposite direction is −ive. Consequently, if he takes the direction from A to D as +ive, then the direction from D to A is −ive. The conventions regarding directions of the lines may be explained for (*i*) horizontal line, (*ii*) vertical line, and (*iii*) revolving line. In the last case the outward direction is taken as the +ive direction.

(*ii*) *Directed angle*

In geometry the angle is taken to be the inclination between two lines. But in trigonometry an angle is defined by the rotation of a revolving line starting from the initial position and reaching a terminal position. Also the direction of rotation is given due consideration. An anti-clockwise rotation is taken to be positive and the clockwise one as negative. With these basic points the pupils may be given the understanding that in trigonometry we can have an angle of any size and positive and negative too.

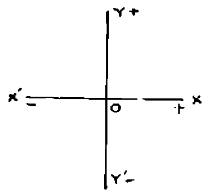




In geometry the maximum size of an angle could be 360° only. But in trigonometry that is not so.

(iii) Quadrant

Two directed line segments $X'OX$ and $Y'OY$ may be drawn intersecting at O having $X'OX$ in the horizontal direction and



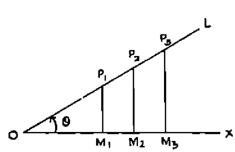
$Y'OY$ in the vertical direction. Take the direction of OX and OY as +ive. Then it is evident that OX' and OY' are —ive.

Then XOY is called the I quadrant, $X'OY$ as the II quadrant, $X'OY'$ as the III quadrant and XOY' as the IV quadrant. The students may be told that all angles between 0° and 90° lie in the I quadrant, angles between 90° and 180° lie in the II quadrant, angles betwen 180° and 270° lie in the III quadrant and angles between 270° and 360° lie in the IV quadrant. Angles which are greater than 360° can be placed in proper quadrants by deducting multiples of 360° from them and then deciding the quadrant on the basis of the remaining figure which is less than 360°.

(iv) *Knowledge of constancy of ratios between sides of a right angled triangle when one of the acute angles is given.*

The knowledge acquired by the students in the study of similarity and similar triangles is to be utilised here.

Right angled triangles OM_1P_1, OM_2P_2, OM_3P_3,......are similar triangles because of being equi-angular. Thus their sides are proportional. This makes for the constancy of the trigonometrical ratios of any angle whatever the size of the right angled triangle in which it exists.



Another point to be emphasized here is that as sin θ, cos θ, etc., are ratios, these do not carry any units with them. These are just written as numbers.

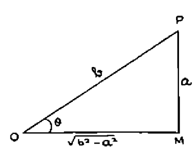
The teacher now defines the symbols sine, cosine, tangent, cosecant, secant and cotangent. He tells them that to save space, the symbols are written in brief as sin, cos, tan, cosec, sec and cot. It is important to emphasize here that when we are defining trigonometrical ratios we have to start arguing from the first principles and all cases in the different quadrants have to be considered as the angle may lie in any of the four quadrants.

Another point needing emphasis is that even though the size of the angle may be as large as possible and also that it may be +ive or —ive, yet it must lie in one of the four quadrants. And thus the signs of the trigonometrical ratios of an angle of any size will depend on the quadrant in which it lies.

Again, these signs in the four quadrants may be explained to pupils clearly by considering the directions of the line segments in each quadrant.

Knowledge of conversion of one trigonometrical ratio to another

It is important that everywhere an emphasis may be laid on starting and proving the ideas from first principles. But, at



the same time, it may be useful from computational point of view to tell the students as to how to utilise the idea of a right angled triangle in calculating all trigonometrical ratios of an angle when one of them is given.

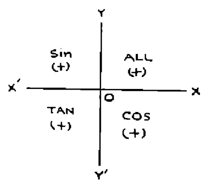
For example. Let $\sin\theta = \frac{a}{b}$

The students may be explained that they may just draw a right angled triangle and name one of the acute angles as θ.

The side opposite to that may be equated to the numerator '*a*' and the hypotenuse to the denominator '*b*'.

The third side now is $\sqrt{b^2-a^2}$ (By Pythagorus theorem)

All the trigonometrical ratios may now be written.



For the signs, the rule. 'All, sine, tangent, cosine' may be helpful. In the first quadrant all trigonometrical ratios are +ive, in the second sine only, in the third tangent and in the fourth cosine.

The signs of their reciprocals are, naturally, determined in the same way (Ask the students to verify the sign of each ratio in each quadrant).

While working out the trigonometrical ratios of 0°, 30°, 45°, 60° and 90° it is necessary to start *ab initio* and give details. In this unit one example has been given in the case of an angle of 30°. But hints regarding the use of an equilateral triangle or a right angled isosceles triangle are useful and may be given.

Again, even after explaining these it would be useful to discuss with the students that it is not necessary to have the right angled triangle put in a position where one of the sides is in horizontal position and the other in the vertical. Proceeding from the first principles is necessary when we are proving some relationships. At the application stage it is just necessary to have the right angled triangle and tell the trigonometrical ratios of an angle. The students may be given sufficient practice in this work.

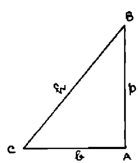
A common error committed by students is that of treating sin θ as sin×θ, cos θ as cos×θ and so on. This point has to be given proper emphasis beyond any scope for some wrong notions being formed. sin θ is an individual entity which cannot be divided into two factors like sin and θ nor it can be divided into parts sin+θ, etc. It is just like the symbol ∥ gm for parallelogram which cannot be written as ∥ ×gm or (∥) gm. sin θ is a symbolic representation or a ratio and is to be treated as an indivisible whole.

Application

Some of the instruments used in surveying and available in the institution or nearby may be explained to the students. They may be permitted to handle instruments like sextant, etc. so as to get the feel of the use of trigonometry in every day life.

Case I. Given one side and an angle.

Let the side BC be of length h.



Suppose the given angle is C.

We have to solve the triangle ABC.

In this case it is of no use beginning with the formula

$$h^2=p^2+b^2$$

where p and b are the lengths of the other two sides, because we know only one side. Suppose

$$\angle C=45° \quad \text{and} \quad h=\sqrt{2}$$

and we begin with the formula

$$h^2=p^2+b^2$$

We find

$$2=p^2+b^2$$

and we are unable to proceed further as we know neither p nor b. But we know the value of $\sqrt{2}$ and that

$$\frac{p}{h}=\sin C$$

In this formula we know two out of three things

$$h=\sqrt{2} \quad \text{and} \quad \angle C=45^\circ$$

$$\sin C=\sin 45^\circ=\frac{1}{\sqrt{2}}$$

But here $\qquad \sin C=\frac{p}{h}=\frac{p}{\sqrt{2}}$

$$\frac{1}{\sqrt{2}}=\frac{p}{\sqrt{2}}$$

$\therefore \qquad p=1$

Case II. Let the side b or p, and the angle C be given. Here also we cannot use the formula

$$h^2=p^2+b^2$$

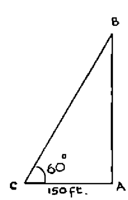
but because b is given and

$$\cos C=\frac{b}{h},$$

we can find h, etc., as before.

Illustrated Example. An observer standing 150 ft. from the foot of a tower notices that the angle of elevation of the top is 60°. Find the height of the tower.

Let AB be the tower whose height is h ft. $AC=150$ ft., $\angle BCA=60^\circ$. Here we have to find out AB, the base AC is given.



We have to use the ratio

$$\frac{\text{perpendicular}}{\text{base}}=\tan C$$

$$\tan C=\frac{AB}{CA}$$

$$\tan 60^\circ=\frac{h}{150}$$

or $$\frac{h}{150}=\sqrt{3}$$

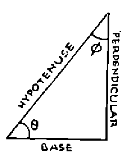
$$h=150\sqrt{3} \text{ ft.}$$

V. Evaluation

(Sample Test Items)

(A) *Knowledge aspect*

(*i*) Look at the figure given below and fill in the gaps in the following identities.



(*a*) $\frac{\text{perp.}}{\text{hyp.}}$ =? (for angle θ) (*e*) $\frac{\text{hyp.}}{(?)}$ =sec φ

(*b*) $\frac{\text{perp.}}{(?)}$ =tan θ (*f*) $\frac{\text{base}}{\text{perp.}}$ =(?)(for angle φ)

(*c*) $\frac{\text{perp.}}{\text{base}}$ =(?) (for angle θ) (*g*) $\frac{(?)}{\text{hyp.}}$ = sin φ

(*d*) $\frac{\text{base}}{(?)}$ =cot θ (*h*) $\frac{\text{perp.}}{\text{base}}$ =(?)(for angle φ) (*ii*) Fill in the gaps of the following

(*a*) $\sin 30° = (?)$

(*b*) $\tan 45° = (?)$

(*c*) $\cot 90° = (?)$

(*d*) $\cos 45° = (?)$

(*e*) $\text{cosec}\,(?) = 2$

(*f*) $\frac{1}{\sqrt{3/2}} = \text{cosec}\,(?)$

(*iii*) Fill in the gaps of the following

(*a*) $\cot\theta = \frac{1}{(?)}$

(*b*) $\sec\theta = \frac{1}{(?)}$

(*c*) $(?) + \cos^2\theta = 1$

(*d*) $\sec^2\theta = (?) + \tan^2\theta$

(*e*) $\text{cosec}^2\theta - (?) = 1$

(*iv*) (*a*) $\cos^2\theta + \sin^2\theta$ is equal to

(*A*) $\tan^2\theta$

(*B*) 1

(*C*) 2

(*D*) $\sec^2\theta + \text{cosec}^2\theta$

(*b*) $\sec^2\theta - 1$ equals

(*A*) $\text{cosec}^2\theta$

(*B*) $\frac{1}{\cot^2\theta}$

(*C*) $\cot^2\theta$

(*c*) $\tan 60°$ equals

(*A*) $\frac{1}{2}$

(*B*) $\sqrt{3/2}$

(*C*) $\sqrt{3}$

(*v*) Tick mark (√) the best possible word from among the given words.

(*a*) Trigonometry deals with

(*A*) rectangle

(*B*) sides and angles of a triangle

(*C*) computation only

(*b*) Give a tick (√) mark on the right answer :

sin 30° is equal to

(*A*) $\frac{1}{2}$

(*B*) $\frac{1}{2}$ feet

(*C*) $\frac{1}{2}$ cm.

(*vi*) Is the equation sin 30°=3/2 true or false ? If false, state the correct answer.

(*vii*) (*a*) What is the difference between an angle of elevation and an angle of depression ?

(*b*) What is the difference between $\sin^2 30°$ and $\sin (30^2)°$?

(*viii* (*a*) If $\sin\theta=\frac{1}{2}$, $\text{cosec}\,\theta=(?)$

(*b*) If $\sec^2\phi=4/3$, $\tan\phi=(?)$

(*c*) If $\sin 30°=\frac{1}{2}$, find cos 30°, tan 30° and sec 30°.

(*ix*) (*a*) Find the values of sin 60°, tan 45° and sec 30° with the help of geometrical figures.

(*b*) Is $\cos\theta=\cos\times\theta$ correct? If not, give reasons for your answer.

(*B*) *Skill aspect*

(*i*) Express the following expression in terms of sec θ and tan θ

$$\frac{1+\sin^2\theta}{\cos^2\theta}$$

(*ii*) Rewrite the following expression in terms of sin θ and cos θ

$$\frac{\tan\theta}{\sec\theta}+\frac{\cot\theta}{\text{cosec}\,\theta}$$

(*iii*) Express $(\sin x-\cos x)\,\frac{\sec x}{\sin x}$ as the difference of sec *x* and cosec *x*.

(*iv*) Is the expression $\frac{\sec^3 x}{1+\tan^2 x}=\sec^2 x$ true or false?

If false, why?

(*v*) Prove that $\frac{\sin^2 A}{1-\cos A}-1=\cos A$

(*vi*) Evaluate sin 30° cos 60°+cos 30° sin 60°

(*vii*) Verify the following statement

(*i*) $\cos 90° - \cos 30° = -2 \sin 60° \sin 30°$

(*ii*) $\sin 90° + \sin 30° = 2 \sin 60° \cos 30°$

(*viii*) In which quadrant the following angles lie? Indicate the direction of rotation.

(*a*) 125° (*b*) −130° (*c*) 220° (*d*) 320° (*e*) −75°
(*f*) 375° (*g*) 750° (*h*) 835°.

(*C*) *Application*

(*a*) A boy flying a kite has let out 435 feet of string. The angle of elevation of the kite is 45°. If the string is straight what is the height of the kite?

(*b*) Find the shadow cast by a stick 12 ft. high, the rays of the sun being inclined to the ground at an angle of 60°.

(*c*) The upper part of a tree broken at a certain height, makes an angle of 60° with the ground at a distance of 10 ft. from its root. Find the original height of the tree.

(*d*) A tree on the bank of a river is 45 ft. high and the angle of elevation of the top of the tree from a point just on the opposite bank is found to be 60°. Find the breadth of the river.

(*e*) From an aeroplane vertically above a straight road the angles of depression of two consecutive mile-stones on the road are observed to be 45° and 30°. Find the height of the aeroplane above the road.

2

H.C.F. and L.C.M.

I. Introduction

The idea of the Highest Common Factor can be very well illustrated by our *common* needs and wants in everyday life. People in the different strata of the society have different types of needs and wants. But which of them are *common* to all the strata and which of them are of *highest* importance to all? These wants of *highest* urgency are also *common* to all of us. Food, shelter, and clothing, thus, prove to be, in a non-mathematical way, the H.C.F. of all the necessities and wants of all the human beings in the society. This feature is very unique.

On the other hand, in this modern civilization, taking for granted that our wants are increasing in multiples, the *lowest multiple* of the multiples of our *common* wants may be regarded as "Education,"

Like "Average" and "Percentage," H.C.F. and L.C.M. are also basic to the understanding of several situations in life.

II. Background and importance

The knowledge of H.C.F. and L.C.M. is very useful in solving several examples in mathematics.

$3+2=5$ but $1/3+1/2 \neq 1/5$. This has a bearing on the knowledge of the L.C.M. In fact, L.C.M. plays an important role in addition and subtraction of fractions.

The procedure of finding out H.C.F. by factorisation would become very cumbersome when the natural numbers are very large.

However, there is a simpler procedure available for finding the H.C.F. of any two given numbers. This process of ***repeated division*** of finding the H.C.F. of two numbers is called "Euclid's Algorithm." It appeared in Euclid's Elements about 2,300 years ago.

Algorithm for the determination of H.C.F.

Let the two numbers be 15844, 13281

As a result of successive division :

$$15844 = 13281 \times 1 + 2563$$
$$13281 = 2563 \times 5 + 466$$
$$2563 = 466 \times 5 + 233$$
$$466 = 233 \times 2$$

∴ 233 is the H.C.F. of the given numbers. The process may be arranged as follows :

	2	5	5	1
233	466	2563	13281	15844
	466	2330	12815	13281
		233	466	2563

III. Scope and Limitations

1. *H.C.F. and L.C.M. of numbers* and their applications in every day life of the students.

Stds : V to VIII
Age-group : 10 + to 13 +

Previous knowledge expected : Idea of numbers,—four fundamental operations and factors.

2. *H.C.F. and L.C.M. of algebraic expressions or algebraic phrases* and their applications in life-situations and higher mathematics.

Stds : IX to XI
Age-group : 14+ to 16+

Previous knowledge expected : H.C.F. and L.C.M. of natural numbers, idea of directed numbers, four fundamental algebraic operations and factorisation.

IV. Objectives

I. Knowledge

To enable the pupils to understand the terms, concepts, principles and theorems.

SPECIFICATIONS

(*a*) The pupil recognises the terms.

(*b*) The pupil recalls the terms.

(*c*) The pupil illustrates with examples the meaning of concepts.

(*d*) The pupil discriminates between terms and concepts and identifies H.C.F. and L.C.M. of numbers.

II. Application

To enable the puplis to apply the concepts of H.C.F. and L.C.M. in solving different problems.

SPECIFICATIONS

(*a*) The pupil analyses the problem, factorises the given expressions and locates correctly the similarity with reference to H.C.F. and L.C.M.

(*b*) The pupil tackles the reverse process, *i.e.* given the H.C.F. and L.C.M. he finds the expressions.

(*c*) The pupil finds the adequacy of data and selects relevant data.

(*d*) The pupil solves problems and verifies the solution (*i.e.*. detects the error in the process).

(*e*) The pupil generalizes the rules.

III. Skill

(*a*) The pupil solves sums with speed and accuracy.

(*b*) The pupil either prepares the table or fills in the gaps in a given table.

V. Content matter

Concepts	*Terms*
1. Set	Element
	Symbol "$\cup$"
(*a*) Union of sets	Symbol "$\cap$"
(*b*) Intersection of sets	Symbol '$\in$'
2. Factorisation	Prime numbers
	Co-prime numbers
	Composite numbers
	Index (exponent)
(*a*) Power $a \times a \times a \ldots\ldots n$ times$=a^n$	Power
(*b*) Common factors.	
3. Highest or Greatest common factor	Unique
(*a*) (*i*) by way of factors	
(*ii*) by way of division- "Euclid's Algorithm"	Euclid's algorithm.
(*b*) Every common factor of two numbers is a factor of their H.C.F.	Subset $\subset$
(*c*) The highest common factor of three numbers is the H.C.F. of any one of them and the H.C.F. of the other two.	
(*d*) The number h is the H.C.F. of two numbers a and b *if and only if* h is a common factor of a and b ; and the two numbers $a \div h$ and $b \div h$ are relatively prime.	Relatively prime
(*e*) H.C.F. of a and b is also the H.C.F. of $a \pm b$ and it is also the H.C.F. of $ax \pm by$ provided x and y are natural numbers.	

4. Lowest Common Multiple	Multiple
(*a*) The smallest of the common multiples of two numbers is called their lowest common multiple (l).	Common multiple Lowest Common Multiple
(*b*) H.C.F. of two or more numbers is a factor of their L.C.M.	
5. H.C.F. and L.C.M.	
(*a*) The product of two numbers is equal to the product of their H.C.F. and L.C.M. $hl = ab$	
(*b*) The product of the smaller powers of the common prime numbers is the H.C.F.; and the product of the numbers in either or both of these expressions taken with greater powers is the required L.C.M.	Common prime numbers Algebraic phrase (*i.e.* expression)

VI. Content Development

Introduction of the Set-theory leading to the basic understanding of H.C.F. and L.C.M.

Sets

A fundamental concept in all branches of mathematics is that of a set. A set is any well defined list, collection or class of objects. These objects are called the *elements*, or *members* of the set.

These elements may be any thing : numbers, letters, rivers, books, etc.

We generally use capital letters A, B, C, ... to designate sets, while the lower case letters $a, b, x, y, \ldots$ are used to denote elements of a set. If a set A consists of the elements $a, b, c,$

d, e, f, we write

$$A = \{a, b, c, d, e, f\}$$

Other examples of the sets are :

$B = \{1, 2, 3, 4, \ldots\ldots\}$, the set of natural numbers
$C = \{1, 3, 5, 7, \ldots\ldots\}$, the set of odd numbers
$D = \{2, 4, 6, \ldots\ldots\ldots\}$, the set of even numbers
$E = \{2, 3, 5, 7, 11, 13, 17\ldots\}$, the set of prime numbers.

If an object x is a member of a set A, then we write :

$$x \in A$$

Which reads "x belongs to A" or "x is a member of A".

Example : Let A = {Ganga, Yamuna, Krishna, Kaveri}, then Ganga $\in A$ but Mahanadi $\notin A$, *i.e.*, Mahanadi is not a member of the set A or Mahanadi does not belong to A.

Let $B = \{a, e, i, o, u\}$, then $a \in B$, $e \in B$ but $p \notin B$.

Let $C = \{2, 4, 6, 8, \ldots\ldots\}$, the set of even numbers, then $2 \in C$, $8 \in C$, $12 \in C$ but $5 \notin C$.

It may be noted that the change of order of the elements in a set does not alter the set, *i.e.*

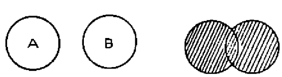
$$B = \{a, e, i, o, u\} = \{e, i, a, u, o\}$$

Set Operations

In arithmetic we learn how to add, subtract and multiply the numbers like x and y and we denote the sum of the numbers x and y by a number $x+y$, their difference by a number $x-y$ (or $y-x$) and their product by a number xy. These operations are called the operations of addition, subtraction and multiplication of numbers. In the study of set theory, we shall learn the operations of "Union denoted by $\cup$" and "Intersection denoted by $\cap$" on a pair of sets A and B.

Union

The union of two sets A and B (written $A \cup B$ and read as A



union B) is the set of all elements which belong to A or to B or to both.

Example : Let $A = \{1, 2, 3, 4\}$ and $B = \{1, 3, 5, 6, 7\}$
then $A \cup B = \{1, 2, 3, 4, 5, 6, 7\}$

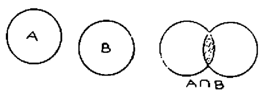
Example : Let $A = \{a, e, i, o, u\}$, $B = \{a, b, c, d\}$
then $A \cup B = \{a, b, c, d, e, i, o, u\}$

Example : Let $A = \{1, 3, 5, 7, ...\}$, set of odd numbers
$B = \{2, 4, 6, 8, ...\}$, set of even numbers
then $A \cup B = \{1, 2, 3, 4, ...\}$, set of natural numbers

Example : Let $A = \{1, 3, 5, 7, 8\}$
then $A \cup A = \{1, 3, 5, 7, 8\} = A$

Intersection

The intersection of two sets A and B (written $A \cap B$ and read as A intersection B) is the set of the elements which belong to both the sets A and B.



Example : Let $S = \{1, 2, 4, 5\}$, $T = \{2, 4, 6, 8\}$
then $S \cap T = \{2, 4\}$

Example : Let $A = \{a, b, c, d\}$, $B = \{a, b, c, f\}$
then $A \cap B = \{a, b, c\}$

Example : Let $A = \{x, y, z\}$, $B = \{1, 2, x\}$
then $A \cap B = \{x\}$

Example : Let $A = \{o, p, q, r\}$
then $A \cap A = \{o, p, q, r\}, \quad = A$

We have observed here that $A \cup A = A$, $A \cap A = A$

It is easy to verify that

$$A \cup B = B \cup A$$
$$A \cap B = B \cap A$$

Thus the operations of 'union' and 'intersection' follow the commutative law.

Subsets

Let us examine the following two sets:

$$A = \{1, 2, 3, 4, 5, 6, 8, 9, 10\}$$
$$B = \{1, 3, 4, 5\}$$

We observe that *each* element which belongs to B also belongs to A. We call B a subset of A, and write $B \subset A$ (read as B is a subset of A). Thus if B is a subset of A, then each element of B is also an element of A.

Example : Let $S = \{1, 5, 7, 9\}$

$T = \{1\}$

$V = \{1, 7\}$

$W = \{1, 5, 7\}$

$T \subset S,\ V \subset S,\ W \subset S,\ T \subset V,\ T \subset W,\ V \subset W.$

Example : Let $A = \{a, e, i, o, u\}$

$B = \{a, i\}$

$C = \{1, 3\}$

$D = \{1, 3, i, o\}$

$B \subset A$, $B \not\subset C$ (B is not a subset of C since $a \in B$ but $a \notin C$). $D \not\subset A$ and $C \subset D$.

It is interesting to note that each set is a subset of itself, *i.e.* $A \subset A$.

Factors

$5 \times 2 = 10$. Instead of calling one of the numbers the multiplicand and the other one the multiplier, we give both of them the name : *factor*.

When we say "the factors" we mean "all the factors" of a number.

The set of factors of 20 is $\{1, 2, 4, 5, 10, 20\}$

Definition

If a, b, and c are whole numbers and if $ac = b$, then the number a is called a factor of b. c is another factor of b.

Factorisation

Definition : If a counting number is written as a product of prime numbers, this product is called a complete factorisation of the given number.

$20 = 4 \times 5 = 2 \times 2 \times 5 = 2^2 \times 5.$

Similarly, $72 = 2^3 \times 3^2$

The unique factorisation property of the counting numbers

Every counting number greater than 1 can be factorised into primes in only one way, except the order in which they occur in the product.

Unique : It means that there *is one and only one* factorisation except for order.

Greatest Common Factor (G.C.F. OR H.C.F.)

All whole numbers are multiples of 1. Thus 1 is a *common factor* of the members of any set of whole numbers. Therefore, when we are looking for common factors, we generally look for numbers other than 1.

Definition

The greatest common factor of two or more whole numbers is the largest whole number which is a factor of each of them.

Set of factors of 24 is {1, 2, 3, 4, 6, 8, 12, 24}

Set of factors of 60 is {1, 2, 3, 4, 5, 6, 10, 12, 15, 20, 30, 60}

The intersection of the two sets or the set of common factors is {1, 2, 3, 4, 6, 12}

The greatest of these factors is 12. Therefore, 12 is the G.C.F. of 24 and 60.

The greatest common factor is also called the highest common factor and in short form is written as G.C.F. or H.C.F.

Division method

Another way to find the G.C.F. is to make use of a relationship among the parts of a division problem.

dividend =(diviser $\times$ quotient)+ remainder

(Proof of Euclid's Algorithm has been given in Appendix)

THEOREM I

Every common factor of the two numbers is a factor of their H.C.F.

Example : Let $a=45$, $b=63$

If A represents the factor set of 45, and B the factor set of 63, we have

$$A=\{1, 3, 5, 9, 15, 45\}$$
$$B=\{1, 3, 7, 9, 21, 63\}$$
$$A \cap B=\{1, 3, 9\}$$

The highest element in the $A \cap B$ is 9. The H.C.F. is, therefore, 9.

We observe here that each element 1, 3, 9 is a factor of H.C.F. 9.

Corollary

Let d be a common factor of a and b. Then H.C.F. of $a \div d$, $b \div d$ is $h \div d$, h being the H.C.F. of a and b.

For example : the H.C.F. of 36 and 60 is 12, and 2 being a common factor of 36 and 60, the H.C.F. of $36 \div 2$ and $60 \div 2$ is $12 \div 2$, *i.e.* 6.

The H.C.F. of 36, 60 is 12 and the H.C.F. of $36 \div 12$, $60 \div 12$ *i.e.* of 3, 5 is $12 \div 12 = 1$.

This means that 1 is the only common factor of 3 and 5.

THEOREM II

The highest common factor of three numbers is the highest common factor of any one of them and the highest common factor of the other two.

Example : Let there be three numbers : 12, 16 and 18.

2 is the H.C.F. of 16 and 18. The H.C.F. of 12 and 2 is again 2.

2 is the H.C.F. of all the three numbers.

THEOREM III

The number h is the H.C.F. of two numbers a and b *if and only if* h is a common factor of a and b ; and the two numbers $a \div h$ and $b \div h$ are *relatively prime.*

Definition

The two numbers which are *co-prime* are also said to be *relatively prime.*

The pairs of numbers 12, 35 ; 63, 6 are co-prime or relatlvely prime, but the pairs of numbers 6, 8 and 45, 65 are not co-prime.

Example : 5 is the H.C.F. of 130 and 255.

The two numbers $\frac{130}{5}$ and $\frac{255}{5}$.

i.e. 26 and 51 are *relatively prime.*

THEOREM IV (GAUSS'S THEOREM)

Let a, b, c be three numbers such that :

(*i*) c is a factor of product ab

(*ii*) c is co-prime to a.

Then, c is a factor of b.

Example: Let $a=12$, $b=30$, $c=5$.

we have

$$\frac{ab}{c}=\frac{360}{5}=72,$$ *i.e.* c is a factor of the product ab.

Also 5 is co-prime to 12. It follows by Gauss's theorem that 5 is a factor of 30.

Lowest Common Multiple

6 is a multiple of both 2 and 3. There are many such numbers divisible by both 2 and 3. The set of these numbers is written as follows :

$\{6, 12, 18, 24, 30....\}$

Definition

Numbers which are multiples of more than one number are called *common multiples* of those numbers. "Common" means belonging to more than one.

Thus 6 and 12 are common multiples of 2 and 3.

Example :

Set of multiples of 3 : {3, 6, 9, 12, 15, 18, 21,...}

Set of multiples of 4 : {4, 8, 12, 16, 20, 24, 28,...}

The numbers that these sets have in common are the *common multiples* of 3 and 4. This set is written as {12, 24, 36,...}.

This set is the intesection of the two previous sets. This clearly shows that 12 is the L.C.M. of 3 and 4.

Definition

The least common multiple of a set of counting numbers is the smallest counting number which is a multiple of each member of the set of given numbers.

The L.C.M. of any two given numbers *exists* and is *unique*.

The least common multiple of a set of numbers cannot be less than the largest member of the set of given numbers.

H.C.F. and L.C.M.

THEOREM V

The product of the smaller powers of common prime numbers is the H.C.F. ; and the product of the prime numbers in either or both of these numbers taken with greatest powers is the required L.C.M.

Example :

$$a=2^3\times 5\times 11\times 13^2$$

$$b=2^2\times 5^2\times 11^2\times 13\times 17$$

we have H.C.F. $=2^2\times 5\times 11\times 13$

L.C.M. $=2^3\times 5^2\times 11^2\times 13^2\times 17$

THEOREM VI

The product of two numbers is equal to the product of their H.C.F. and L.C.M.

i.e. $hl=ab$;

where h and l stand for the highest common factor and the lowest common multiple of the two numbers a and b respectively.

This theorem gives us a method for finding the lowest common multiple of two given numbers.

Let a and b be two given numbers. Then their lowest common multiple is $(ab) \div h$; h being the highest common factor of a and b.

It also follows that every multiple of the lowest common multiple of the two numbers is a multiple of each of the two numbers.

Application to Algebra

So far, numericals were being dealt with. The same rules are now to be applied to algebraic terms and phrases.

The words "phrase" denotes the algebraic expression.

Highest Common Factor

The term "H.C.F." is to be used in algebra in place of "G.C.F."

In the case of natural numbers, we try to find out the Greatest Common Factor; but in algebra it is difficult to know which of the phrases is greater than the other, that is why, we try to substitute the word Highest Common Factor in place of Greatest Common Factor.

Method of factorisation

1. Let $A=6a^2b^3$, $B=12a^3b^3c$ and $C=18a^4b^3c$;

Now $B=6a^2b^3\times 2ac$
and $C=6a^2b^3\times 3a^2c$

In all the expressions A, B and C the only common factor is $6a^2b^3$ which is the required H.C.F.

2. H.C.F. of two or more phrases.

Consider the following phrases:
a^2-4b^2, $a^2+3ab+2b^2$
1st phrase $=(a+2b)(a-2b)$
2nd phrase $=a^2+2ab+ab+2b^2$
$=a(a+2b)+b(a+2b)$
$=(a+2b)(a+b)$

In both of these phrases, $(a+2b)$ is the only common factor.

$\therefore \quad$ H.C.F.$=(a+2b)$

Method of Division

3. H.C.F. of two phrases, when they cannot be factorised easily. Sometimes, it become difficult for us to factorise the given phrases.

Then we proceed to compute the H.C.F. by 'Euclidean Algorithm', *i.e.* division method.

Procedure adopted in solving such phrases :

(*i*) Arrange the phrases in ascending powers of the unknown.

(*ii*) Phrase having the higher power may be divided by the other and if the phrases are of the same power, then divide one by the other.

(*iii*) Divide the divisor by the remainder.

(*iv*) Continue this process, *i.e.* dividing the last divisor by the last remainder until the remainder is zero.

(*v*) The last divisor used is the Highest Common Factor.

HINT : During the process of division this fact should be borne in mind that the H.C.F. of the phrases is the same as that of the remainder and the divisor.

Example : x^3+3x^2-9x+5; $x^3-19x+30$.

Here both the phrases are of the same order, therefore divide one by the other.

$$
\begin{array}{l}
x^3-19x+30)\ x^3+3x^2-9x+5\ (1 \\
\qquad\qquad\quad \underline{\pm x^3 \qquad \mp 19x \pm 30} \\
\qquad\qquad\quad +3x^2+10x-25\ )\ x^3-19x+30\ (\ x-10 \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \underline{\times 3} \\
\qquad\qquad\qquad\qquad\qquad\qquad 3x^3-57x+90 \\
\qquad\qquad\qquad\qquad\qquad\qquad \underline{\pm 3x^3 \mp 25x \pm 10x^2} \\
\qquad\qquad\qquad\qquad\qquad\qquad -10x^2-32x+90 \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \underline{\times 3} \\
\qquad\qquad\qquad\qquad\qquad\qquad -30x^2-96x+270 \\
\qquad\qquad\qquad\qquad\qquad\qquad \underline{\mp 30x^2 \mp 100x \pm 250} \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 4x+20
\end{array}
$$

Taking out 4 common, we get $x+5$, then

$$
\begin{array}{r|l}
x+5) & 3x^2+10x-25 \;(\; 3x-5 \\
 & \underline{+3x^2 \mp 15x} \\
 & \quad -5x-25 \\
 & \quad \underline{\mp 5x \mp 25} \\
 & \qquad \times
\end{array}
$$

$\therefore$ H.C.F. is $x+5$.

4. H.C.F. of three or more than three phrases when it is difficult to factorise the phrases. For example :

Suppose A, B, C, D are the phrases. Find out the H.C.F. of the two phrases, *i.e.* say A and B. Let it be denoted by X. Now proceed to find out the H.C.F. of X and C. Let it be denoted by Y. Now proceed to find out the H.C.F. of Y and D. Extend this process till you get the desired H.C.F.

Lowest Common Multiple

Factorisation method

1. Consider the following phrases $16l^2mn$, $24\ lm^2n$ and $48\ l^2m^4n^4$.

 (*a*) L.C.M. of the numbers 16, 24, 48 is 48.

 (*b*) Letters which occur in these phrases are l, m, and n and their highest powers in order are 2, 4 and 4 respectively.

$\therefore$ L.C.M. of these phrases is $48\ l^2\ m^4n^4$.

2. L.C.M. of the phrases which can be easily factorised.

$1+a,\ 1-a^2,\ 1-2a+a^2$

1st phrase$=(1+a)$

2nd phrase$=(1+a)\ (1-a)$

3rd phrase$=(1-a)^2=(1-a)(1-a)$

$\therefore$ L.C.M.$=(1-a)^2\,(1+a)$

Division method

3. L.C.M. of those phrases which cannot be factorised easily.

Suppose 'A' and 'B' are two such phrases and 'H' is their highest common factor. This can be illustrated like this:

$$A=aH.$$

$$B=bH.$$

Here 'a' and 'b' will not have any common factor.

$\therefore$ L.C.M. of A and $B=a\ b\ H.$

$$=aH\,.\,bH\,.\,\frac{1}{H}$$

$$=A.B.\,\frac{1}{H}$$

$$i.e.\ L=\frac{A}{H}\ .\ B \text{ or } A\ .\ \frac{B}{H}$$

where L represents L.C.M.

Thus to compute L.C.M. of two phrases, we divide one phrase by their highest common factor and then the result is multiplied by the second phrase.

Corollary: The product of L.C.M. and H.C.F. of two phrases is always equal to their product.

Example: Find the L.C.M. of $1+a+a^5$, and $1+a^4+a^5$. Arranging them in order we have a^5+a+1 ; a^5+a^4+1.

1. H.C.F. of two phrases:

a^2	a^5+a+1 $+a^5 \mp a^2 \times$	a^5+a^4+1 $\pm a^5 \pm a \pm 1$	1
	$a^2+a\ +1$	a^4-a	Taking out a as common
		a^3-1 $\pm a^3 \pm a^2 \pm a$	a
		$-a^2-a-1$ $\mp a^2 \mp a \mp 1$	-1
		$\times$	

$\therefore$ H.C.F. $=a^2+a\ +1$

$$\text{L.C.M.}=\frac{a^5+a^4+1}{a^2+a\ +1}\times(a^5+a+1)$$

$$=(a^3-a^2+1)(a^5+a+1)$$

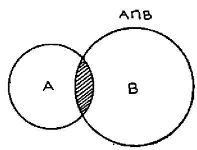
4. To find out the L.C.M. of three or more than three phrases which cannot be easily factorised.

Suppose *A*, *B*, *C* and *D* are four phrases. Compute the L.C.M. of *A*, and *B*. Let it be *X*. Now proceed to find out the L.C.M. of *X* and *C*. Let it be denoted by *Y*. Now find out the L.C.M. of *Y* and *D*. Extend this process. The last result will be the L.C.M. of the given phrases.

VII. Teaching Hints

1. In starting teaching H.C.F. and L.C.M., it is advisable to make use of natural numbers.

It is easier to introduce the idea of common factors through intersection of sets of factors. The diagram (Wenn-Euler's diagram) indicates clearly the commonality.



Intersection, *i.e.* Common factors

2. A clear distinction between prime and co-prime should be drawn. While so doing, the following should be made the focal points.

(*a*) The prime numbers are always co-prime.

(*b*) A pair of co-primes may or may not be constituted of both primes or even one prime and one composite number.

3. The 'existence' and 'uniqueness' of H.C.F. and L.C.M. should be stressed thoroughly.

4. It may be made very clear that there is no restriction imposed on the members of a set to be of the same kind. This is to be emphasised through typical sets of the type :

(*i*) Teacher, book, chalk.

(*ii*) Pupil, desk, bag.

6. The term *well defined* in the definition of a set needs proper emphasis. Once a set is named it should be possible to state in the unambiguous terms if an object is a member of the

given set or not. At the same time it must be made clear that a part of an element of a set is not a member of the set. For example :

Set A = {Teacher, Table, Chair}

Even though the teacher is an element of the set, his hand, if asked about, is not an element of the set. Teacher is treated as a unit and he alone is the member of the set and not any part of his body.

7. An infinite set may be mentioned completely so that no ambiguity is involved.

8. It should be made clear to the pupils that the change of order of the elements of a set does not alter the set. For example a set A {1, 2, 3, 4} may be written in any of the forms {1, 2, 4, 3}, {2, 1, 3, 4} and so on as long as the elements remain the same.

9. The difference between addition and subtraction in case of numbers, and union and intersection of sets should be brought out clearly. This is particularly true for intersection of sets because the whole unit involving common factors or common multiples is to be built up on the basis of intersection of sets.

10. It may be stated very clearly that the whole discussion is limited to natural numbers (thereby avoiding '0'). After a thorough study of H.C.F. and L.C.M. in the domain of natural numbers, the discussion may be extended to algebraic expressions or phrases.

VIII. Assignments

The following specimen assignments would serve as guidelines :

Some of the examples are meant for bright students.

1. Write the set of factors for each of the following :
   9, 7, 18, 21, 33.
2. Write the set of common factors in the following cases :
   (9, 21), (28, 32), (66, 77).
3. Find out the greatest common factor in each of the following cases :

   (*i*) 8, 12, 16, 36.

(*ii*) 9, 21, 33, 39.

(*iii*) 42, 48,54, 60.

4. (*i*) What is the greatest common factor of 6 and 6 ?

(*ii*) What is the greatest common factor of 29 and 29 ?

(*iii*) What is the greatest common factor of 1 and 7 ?

5. Let A be the set of factors of 18, and B be the set of all factors of 42. Then

(*i*) write the sets A and B.

(*ii*) find out the common factors of 18 and 42.

6. Find out the greatest common factor of each group of numbers.

(2040, 2184) ; (46, 69, 92) ; (21, 42, 63)

7. Write the set of all the primes up to 50 and name the number which is even and prime.

8. Use Euclid's Algorithm to determine which of the following pairs of numbers are co-prime :

(*i*) 385, 931 ; (*ii*) 3753, 3380 ;

(*iii*) 564, 7963 ; (*iv*) 17463, 27325.

9. Find the H.C.F. of the following sets of numbers by expressing the numbers as product of primes.

(*i*) 429, 528, 1094.

(*ii*) 183, 488, 793, 915, 1220.

10. Find the L.C.M. of the following sets of numbers by expressing the numbers as product of primes :

(*i*) 32, 48, 176, 36, 24.

(*ii*) 72, 117, 236, 351.

11. Obtain two numbers such that their H.C.F. is 20 and L.C.M. is 420.

12. Obtain two numbers such that their product is 12600 and L.C.M. is 6300.

13. Obtain two natural numbers a and b such that $a^2+b^2=10530$ and their L.C.M. is 297.

14. Determine two numbers, knowing their H.C.F. and their sum or their product as given in the following tables:

I	Sum	72	360	552	420	180	86	168
	H.C.F.	9	18	24	12	15	12	24

II	Product	64800	1512	360	2700	840
	H.C.F.	18	6	5	6	2

15. Ram and Gopal are playing with iron-rings of diameters 140 cms. and 210 cms. respectively.

What is the least distance they will cover with complete revolutions of their rings ?

16. There are 42, 48 and 54 students respectively in each division of VIII standard. Equal number of students from each division is to be sent every time for the medical check-up. What would be the number of students from each division going for medical check-up every time, if no student is left out in any class after finishing all turns ?

17. Find H.C.F. of the following :

(*i*) $40\, ax^2$ and $25\, a^2bx$.

(*ii*) $24\, a^3x^4y^3z$, $30a^2x^5y^6z^2$, $6\, a^7x^3y^4z^5$ and $42a^4x^6y^5z^8$.

(*iii*) $8a^4+6a^3-20a^2$ and $4a^5-9\, a^4+5a^3$.

(*iv*) x^3-3x^2+3x-1, x^3-x^2-x+1, x^4-2x^3+2x+1 and $x^4-2x^3+2x^2-2x+1$.

(*v*) $2x^4-13x^2+45$ and $2x^3-3x^2-18x+27$.

(*vi*) $4x^5-4a^2x^3+4a^3x^2-a^2$ and $6x^5+4x^4a-9x^3\, a^2-3x^3\, a^3+2x$.

(*vii*) a^4-b^4 ; $a^4+a^3b-ab^3-b^4$ and $a^4+2a^3b-2ab^3-b^4$.

18. Find L.C.M. of the following :

(*i*) $9xy\,p^4q^3\,r^2$, $81x^2p^3r$, $11x^2\,q^4r^5$ and $99\,pqr$.

(*ii*) $6x^2+11x-21$, $2x^2+x-15$ and $4x^2+14x+6$.

(*iii*) x^4+4x^2-5, $2x^3+x^2-8x+5$ and x^2-4x+3.

(*iv*) $15x^3-31ax^2+5a^2x+2a^3$ and $6x^4-25ax^3-26a^2x^2-a^4$.

(*v*) $1+4x+8x^2+8x^3$, $1+4x+4x^2-16x^4$ and $1+2x-8x^3-16x^4$.

(*vi*) $x^2+\ (a+b)x+ab$ and $x^2+(c+b)x+cb$.

(*vii*) $x-1$, $x+1$, x^2-1, $x^2\ +1$, x^3+1, x^3-1.

19. The product of two phrases of second degree is $(x+4)^2(x+7)\,(x-1)$. The H.C.F. is $(x+4)$. Find the L.C.M. and the phrases.

20. The L.C.M. and H.C.F. of two phrases respectively are $5x^5-9x^4-17x^3+7x^2+12x+2$ and 1. If one of the phrases is $5x^4-4x^3-21x^2-14x$. Find the other.

IX Evaluation

An attempt is made to give some written tests which will assess the fulfilment of the objectives, with which the teaching of the topic in question was started.

Tests given below should serve as specimens. They are not exhaustive.

UNIT TEST

Instructions

A. All question are compulsory.

B. Read the questions carefully before trying to answer them.

C. Answers have to be written on the question paper itself.

D. Be quick to answer the questions.

E. Do not waste time on a question about the answer of which you are not sure. Pass on to the next.

F. The unit has been divided into two sections *A* and *B*. Attempt *A* first and then pass on to the next section *B*. Section *A* consists of objective type questions and

Section *B*-short answer type and long answer type questions.

Time :

M. Marks :

SECTION A

Note : In each of the questions, there are four answers marked *i*, *ii*, *iii* and *iv*. Correct answer should be written in the space provided.

---

1. The set of factors of 60 is ( )

   (*i*) {1, 2, 3, 4, 5, 6, 15, 20, 30, 60}
   (*ii*) {0, 1, 2, 3, 4, 5, 6, 10, 15, 30, 60}
   (*iii*) {1, 2, 3, 4, 5, 6, 10, 12, 15, 20, 30, 60}
   (*iv*) {2, 3, 4, 5, 6, 10, 15, 20, 30, 60}

2. 12 is the greatest common factor o the pair of numbers

   (*i*) 12, 60 ( )
   (*ii*) 18, 60
   (*iii*) 24, 240
   (*iv*) 36, 72

3. Let *A* be the set of all factors of 18. Let *B* be the set of all factors of 42. Then the intersection set of set *A* and set *B* is

   (*i*) {1, 2, 3} ( )
   (*ii*) {1, 2, 3, 6}
   (*iii*) {1, 2, 3, 6, 7}
   (*iv*) {1, 2, 3, 6, 9}

4. The greatest common factor for 306 and 1173 is

   (*i*) 3 ( )
   (*ii*) 9
   (*iii*) 17
   (*iv*) 51

5. For which of the following pairs, 20 is the least common multiple ?

(*i*) (2, 10) ( )
(*ii*) (4, 10)
(*iii*) (6, 10)
(*iv*) (8, 10)

6. The H.C.F. of the phrases, x^2-1, x^3+1, x^2-4x-5 is

(*i*) $(x-1)$ ( )
(*ii*) (x^2-1)
(*iii*) $(x+1)$
(*iv*) $(x-5)$

7. The L.C.M. of the phrases (x^2+5x+6), (x^2-9), $(x+3)$ is

(*i*) $(x+3)(x+3)(x+2)$ ( )
(*ii*) $(x+3)(x-3)(x+2)$
(*iii*) $(x+3)(x-3)$
(*iv*) $(x-3)(x+3)(x-2)$

8. The L.C.M. and H.C.F. of two phrases are :

$x^3-3x^2-16x+48$ and $x+4$ respectively. If one of the phrases is x^2-16, then the other phrase is

(*i*) $x^2+8x+16$ ( )
(*ii*) $x^2-8x+16$
(*iii*) x^2+x-12
(*iv*) x^2-x+12

9. Let a and b be two numbers and L and H, be their L.C.M. and H.C.F. respectively.
Then L is equal to

(*i*) $\frac{a}{H} \cdot \frac{b}{H}$ ( )
(*ii*) $a \cdot b \cdot H.$
(*iii*) $\frac{a \cdot H}{b}$
(*iv*) $\frac{a}{H} \cdot b$

SECTION B

I. Essay Type Questions :

1. Find the L.C.M. of $9x^4-28x^2+3$, $27x^4-12x^2+1$ and $27x^4+6x^2-1$.
2. If x is a factor of A and B, prove that x is also a factor of

   $PA+QB$ and $PA-QB$, provided P and Q are natural numbers.
3. Prove that the product of two numbers is equal to the product of their H.C.F. and L.C.M.
4. Find the H.C.F. of

   $x^5+x^4-4x^3+2x^2+6x-9$, x^4-x^2+6x-9, and $x^4+2x^3-5x^2-6x+9$.

II. Short Answer Type Questions :

(*a*) Find the L.C.M. of

1. $(a-b)^2\,(b-c)^2(c-a)^2$ and $(a-b)\,(b-c)\,(c-a)$.
2. $(x-1)^3$ and x^3-1.
3. $(1+x+x^2)$, $(1-x+x^2)$, $(1+x^2+x^4)$.

(*b*) Find the H.C.F. of

1. $8x^3-8a^3$ and $2x^2-2a^2$.
2. x^2-4 and x^3+8.
3. $(x^3+y^3+z^3-3xyz)$ and $(x^2-y^2-z^2-2yz)$.

(*c*) Find the H.C.F. and L.C.M. of

1. $(x^2-y^2)\,(a^2-b^2)-4abxy$ and $(x^2-y^2)\,(a^2+b^2)+2(x^2+y^2)\,ab$.
2. $px^2-(p+1)\,x+1$ and $px^2-\,(1+2p)x+2$.

III. Matching Type Questions :

Match the phrases with their appropriate H.C.F.

Pairs of Phrases	*H.C.F.*
1. x^3-1, x^6-1.	(*i*) $2x^2-2x+2$
2. x^4+x^2+1 and x^2+x+1	(*ii*) $x+(y+z)$
3. $x^3+y^3+z^3-3\,xyz$ and	(*iii*) $x(x+1)+1$
$x^2+y^2+z^2+2(xy+yz+zx)$	(*iv*) $(x-1)(x^2+x+1)$
4. $6x^5+6x^3+6x$, $4x^3+4$	(*v*) $x^2+y^2+z^2+xy+yz+zx$
	(*vi*) x^2-x-1

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APPENDIX

Proof of "Euclid's Algorithm" or Division Method for finding H.C.F.

Let a and b two given numbers, such that $a>b$

Supose b is not a factor of a.

By the division method, there exist numbers q and r such that

$$a=bq+r \ldots r<b \qquad \ldots(i)$$

We now show that the H.C.F. of a and b is the same as the H.C.F. of b and r. Let x be the common factor of a and b and consider

$$a=xu$$
$$b=xv$$

Substituting in equation (i), we have

$$xu=xvq+r$$
$$\text{or} \quad r=x\,(u-vq) \qquad \ldots(ii)$$

This shows that x is a factor of r.

Thus, we see that x is a common factor of a and b, and b and r.

Therefore, the H.C.F. of a and b is also the H.C.F. of b and r; where r is the remainder obtained on dividing a by b.

This important clue gives us the necessary clue for computing the H.C.F. of two given numbers by division method.

Since $b>r$, we divide b by r.

Let the remainder be r_1. Therefore, H.C.F of r and r_1 is the same as that of b and r, and as such that of a and b.

We notice that $r_1<r$

If r_1 is a factor of r, the H.C.F. of r_1 and r will be r_1 and the process stops here. If, however, r_1 is not a factor of r, we again divide r by r_1 and obtain the remainder, say r_2, where $r_2<r_1$.

As the remainders go on decreasing the process must camplete after a finite number of steps, *i.e.* we shall obtain a remainder h which will be a factor of the previous remainder k.

The H.C.F. of h and k will be h, and will be the H.C.F. of a and b.

3

SIMILARITY

I. Introduction

The concept of similarity and the properties of similar figures constitute an important and useful branch of mathematical knowledge which has enormous academic and practical value. It is essential for persuing higher courses in Plane and Solid Geometry and in Surveying and Engineering. The student finds its immediate application in the determination of areas, in indirect measurement of heights and distances, and in drawing maps and sketches according to scale. The idea of similarity also imparts an aesthetic sensibility in the domain of decorative arts. In the study of similarity the student is called upon to utilise his algebraic knowledge of ratio and proportion. Thus the study of similarity affords an excellent opportunity for the application of algebraic principles in the solution of geometrical problems.

So far in the traditional treatment of similarity emphasis is laid on the mechanical memorisation of certain definitions and the proofs of certain theorems. The emphasis should shift to clear understanding of the basic concepts and the wide application of the properties of similarity in problem situations.

This unit deals with the fundamental concepts of similarity, the conditions of similarity and the properties of similar geometrical figures mainly triangles and polygons.

There may be many approaches to the development of concept of similarity and allied concepts, namely Euclidean approach, Set theoretic approach, etc. The present treatise of the topic has been restricted to only the Euclidean approach.

II. Objectives

1. The pupil acquires knowledge and understanding of terms, concepts, principles, etc.

SPECIFICATIONS

The pupil

(*i*) recognises the terms, concepts, principles, etc.

(*ii*) recalls the terms, concepts, principles, etc.

(*iii*) illustrates the terms, concepts, principles, etc.

(*iv*) compares and contrasts terms, concepts, principles, etc.

(*v*) discriminates between terms, concepts, principles, etc.

2. The pupil develops the ablity to apply the principles of similarity to new situations.

SPECIFICATIONS

The pupil

(*i*) locates problems where principles of similarity can be applied.

(*ii*) analyses the problems.

(*iii*) checks adequacy or otherwise of the data.

(*iv*) selects relevant data.

(*v*) selects an appropriate method.

(*vi*) estimates the result.

(*vii*) verifies the solution.

3. The pupil acquires constructional and computational skills.

SPECIFICATIONS

The pupil

(*i*) uses the geometrical instruments correctly, neatly and with desirable speed.

(*ii*) works out numerical problems accurately and with re-asonable soeed.

(*iii*) detects mistakes in a procedure.

III. Content analysis

Terms and concepts

(*i*) Parallel straight lines
(*ii*) Concentric circles
(*iii*) Congruency of triangles
(*iv*) Ratio and Proportion
(*v*) Division of straight lines in a given ratio
(*vi*) Similarity
(*vii*) Ratio of Similitude
(*viii*) One to one correspondence
(*ix*) Equiangularity
(*x*) Centre of similarity or homothetical centre
(*xi*) Line of symmetry
(*xii*) Idea of a scale
(*xiii*) Projection

Understandings

1. A straight line can be divided into any given ratio both internally and externally.
2. If a straight line is drawn parallel to one side of the triangle, it divides the other sides proportionally (Theorem 1).
3. If a straight line divides two sides of a triangle proportionately, it is parallel to the third side (Converse of Theorem 1–Theorem 2).
4. The internal and external bisectors of the vertical angle of a triangle divide the opposite side in the ratio of the sides containing the vertical angle (Theorem 3).
5. If the base of a triangle is divided internally or externally into two segments proportional to the other sides of the triangle, the line joining the point of section to the vertex bisects the vertical angle internally or externally (Theorem 4).
6. Two figures are similar if they have exactly the same shape but not necessarily the same size.

7. In similar figures, angles ol the one are equal to the corresponding angles of the other.
8. In similar figures, the sides of the one are proportional to the corresponding sides of the other.
9. All equiangular triangles are similar but all equiangular pentagons are not necessarily similar.
10. All congruent figures are similar but the converse is not necessarily true.
11. In two figures, the sides opposite to the equal angles are corresponding sides.
12. In a hollow cone, the parallel sections made by parallel planes are similar figures.
13. If two triangles are similar to a third triangle, the triangles are similar to each other.
14. If two triangles are equiangular, their corresponding sides are proportional (Theorem 5).
15. If two triangles have three sides of the one proportional to the three sides of the other, the triangles are equiangular, those angles being equal which are opposite to corresponding sides (Converse of Theorems 5—6).
16. When two triangles have one angle of the one equal to one angle of the other, and the sides about equal angles proportional, the triangles are similar (Theorem 7).
17. If a perpendicular is drawn from the right angle of a right angled triangle to the hypotenuse, the triangles on each side of the perpendicular are similar to the whole triangle and to each other (Theorem 8).
18. The areas of similar triangles are proportional to the squares on the corresponding sides (Theorem 9).
19. Areas of similar polygons are to one another as the squares on the corresponding sides (Theorem 10).
20. Similar polygons may be divided into the same number of similar triangles (Theorem 11).

21. If two triangles have their sides parallel or perpendicular each to each, they are similar.

Skills

1. To draw a polygon similar to a given polygon.
2. To divide a given straight line into parts proportional to any number of given line segments.
3. To divide a line externally in a given ratio.
4. To find the fourth proportional to three given line segments.
5. To find the third proportional to two given line segments.
6. To find the mean proportional to two given segments.
7. To divide a segment in extreme and mean ratio—internally and externally.

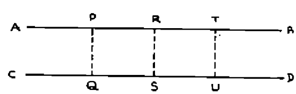
IV. Content Development and Organization

Terms and concepts (given under content analysis) can be grouped into two categories, namely terms and concepts studied earlier and new concepts. For concepts falling in the first category, only a brief treatment has been attempted.

Revision of concepts studied earlier

(I) Parallelism

Two lines or surfaces are parallel if the perpendicular distance between them is the same at all places. *AB* and

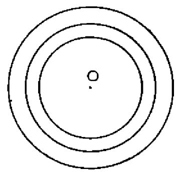


CD will be parallel if the perpendicular distances *PQ*, *RS*, *TU*—between them are equal at all places.

EXAMPLES : The two rails of the railway line, opposite walls of a rectangular room, the opposite edges of a rectangular table, opposite edges of a scale, etc. (Students should be asked to quote similar examples).

(II) Concentric Circles

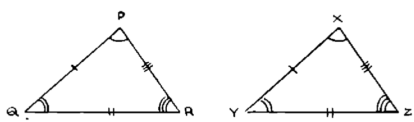
Circles having a common centre but different radii are called concentric circles. This idea shculd be conveyed through



figures. Students should be asked to quote examples from life situations (*i.e.* circular waves in a pond emanating from a centre of disturbance).

(III) Congruency of Triangles

Two triangles are said to be congruent when they are identically equal in all respects. The corresponding sides and angles as well as their areas are equal.



If the two $\triangle$s PQR and XYZ are congruent, then

$PQ = XY$ $\qquad \angle P = \angle X$

$QR = YZ$ $\qquad \angle Q = \angle Y$ $\qquad$ and area $\triangle$ PQR = area $\triangle$ XYZ

$PR = XZ$ $\qquad \angle R = \angle Z$

Conditions of congruency of two triangles

Two triangles are congruent when

(*a*) Two sides and the included angle of one are respectively equal to two sides and included angle of the other (S.A.S.).

(*b*) Three sides of one are equal to three sides of the other, each to each (S.S.S.).

(*c*) Two angles and one side of the one are respectively equal to the two angles and the corresponding side of the other (A.A.S.).

(*d*) Two right angled triangles are congruent when their hypotenuses are equal and one side of the one is equal to one side of the other.

(IV) Ratio and Proportion

A ratio is the quotient of the measures of two quantities provided the quantities are measured in the same unit. It is written as $\frac{a}{b}$ or $a : b$. In the ratio $\frac{a}{b}$, a is called the antecedent and b the consequent.

When the two ratios $\frac{a}{b}$ and $\frac{c}{d}$ are equal we write

$$\frac{a}{b}=\frac{c}{d} \text{ or } a : b=c : d \text{ or } a : b : : c : d$$

If $\frac{a}{b}=\frac{c}{d}$ or $a : b : : c : d$,

then the four quantities a, b, c and d are said to be in proportion.

Two sequences of numbers $a, b, c,\ldots\ldots$ and $p, q, r,\ldots\ldots$ (where none of these is zero) are proportional if

$$\frac{a}{p}=\frac{b}{q}=\frac{c}{r}=\ldots\ldots\ldots\ldots$$

Hence, a proportion is an equation in which the left side and right side of the equation are single ratios.

The simplest proportionalities are those involving only four numbers, and these have special properties that are worth noting. They are given here for later reference.

Algebraic properties of a simple proportion

If $\frac{a}{b}=\frac{c}{d}$ with a, b, c, d all different from zero, then

(*i*) $ad=bc$

(*ii*) $\frac{a}{c}=\frac{b}{d}$

(*iii*) $\frac{a+b}{b}=\frac{c+d}{d}$

(*iv*) $\frac{a-b}{b}=\frac{c-d}{d}$

(*v*) $\frac{a+b}{a-b}=\frac{c+d}{c-d}$

(The proofs of these relationships must have been studied earlier by the pupils).

If a, b, c and x are positive numbers such that

$$a : b :: c : x \left(\text{or} \quad \frac{a}{b}=\frac{c}{x}\right)$$

then x is called the fourth proportional of a, b and c. In the special case, when $b=c$, then x is called the third proportion of a and b.

If a, b and c are positive numbers and

$$\frac{a}{b}=\frac{b}{c},$$

then b is the *geometric mean* between a and c. It is also called the *mean proportion* between a and c.

From the above relation it follows that

$$b^2=ac \quad \text{or} \quad b=\sqrt{ac}$$

If AB is any line segment, there exists a point P on AB (or on AB produced) such that

$$AP^2=AB.BP$$

then P is said to divide the line AB in *extreme* and *mean ratio.*

(*V*) (*a*) *Division of a straight line in a given ratio*

If AB is any straight line and P is any point in AB, as in Fig. 1 or in AB produced, as in Fig. 2 or BA produced as in Fig. 3, the line is divided in all cases into two segments AP and BP.

In Fig. 1, *P* divides *AB* internally in the ratio *AP*/*PB*.

Fig. 1

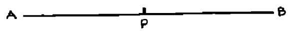
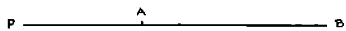


Fig. 2

Fig. 3



In Figs. 2 and 3, *P* divides *AB* externally in the ratio *AP*/*PB*.

(*V*) (*b*) There are always a pair of points (say *M* and *N*) which divide the straight line (say *AB*) internally and externally such that the ratio *AM*/*BM* is equal to the ratio *AN*/*BN*. Ask the students to verify the result by actual measurement.

Development of New Concepts

(*VI*) *Similarity*

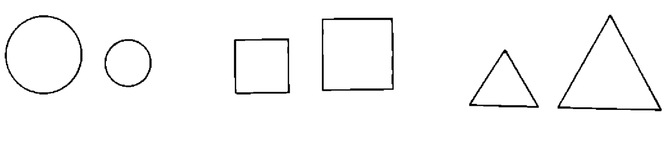
Two figures are similar if they have exactly the same shape, but not necessarily the same size, *e.g.*,

—a map of a district and the district itself ;

—a picture on a film and its projection on the screen ;

—a model of a ship and the ship itself.

Any two circles are similar, any two squares are similar, any two equilateral triangles are similar and any two segments of a straight line are similar.

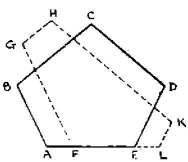


In order that two polygons may have the same shape and thus be called similar, all angles in one must be exactly equal to the corresponding angles in the other and all lengths in the one must be in the same ratio to the corresponding lengths in the other. This ratio between the lengths of the similar polygons is called the *ratio of similitude.*

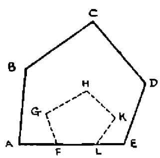
Similar figures must, therefore, satisfy two conditions :

(*i*) The angles of the one must be equal to the corresponding angles of the other.

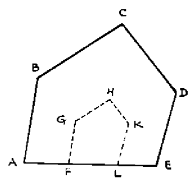
(*ii*) The sides of the one must be proportional to the corresponding sides of the other.



(1)



(2)



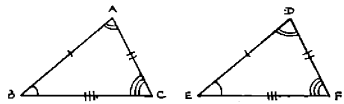
(3)

Both the conditions are necessary for the figures to be similar. For example, in Fig. 1, the two pentagons are equiangular, but the sides are obviously not in proportion and the pentagons are clearly of different shapes and are not similar figures. In Fig. 2, each side of the pentagon *FGHKL* is half the corresponding side of the pentagon *ABCDE,* but the pentagons are obviously not equiangular and are not of the same shape. They are, therefore, not similar figures. In Fig. 3, the two pentagons are equiangular, and each side of *FGHKL* is three quarters of the corresponding side of pentagon *ABCDE*, *i e.* both the conditions are satisfied. The pentagons are clearly of the same shape and, therefore, are similar figures. However, it must be borne in mind that all circles and line segments are similar irrespective of any conditions.

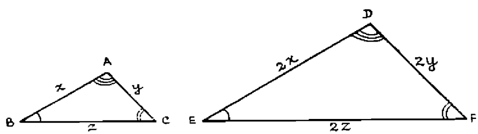
A distinction should be made between congruency and similarity. Examine the following pairs of triangles :

In pair (1) the angles of each of the two triangles *ABC* and *DEF* are equal. The sides are also equal. Thus the two triangles have the same shape and size. They are called congruent triangles.

In pair (2) the angles of each of the triangles *ABC* and *DEF* are equal but not their sides. Each side of the triangle *DEF* is exactly twice the corresponding side of triangle *ABC*. Here the two triangles are similar but not congruent.



pair (1)

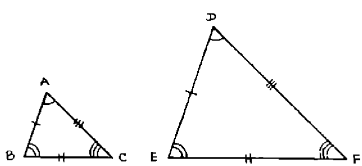


pair (2)

It should be noted that all congruent figures are similar but all similar figures are not congruent.

(VII) One to one correspondence

The idea of one to one correspondence is implicit in the above discussion on similarity. To make it more specific, in a pair of similar figures, for every part (angle or side) in one there is a corresponding part (angle or side) in the other.



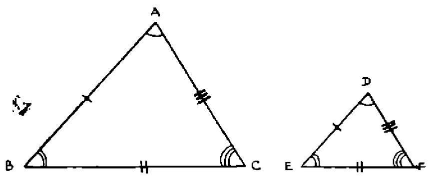
Thus in the two similar triangles *ABC* and *DEF*, $\angle A = \angle D$ $\angle B = \angle E$; $\angle C = \angle F$. These pairs of equal angles are called corresponding angles.

Similarly $AB : DE = BC : EF = AC : DF$. These pairs of sides are called corresponding sides.

The corresponding sides are usually identified with the help of equal angles as the sides opposite to the equal angles are correponding sides.

(VIII) Equiangularity

The concept of equiangularity has also been implicitly explained in the foregoing definitions. Two figures cannot be similar if they are not equiangular.



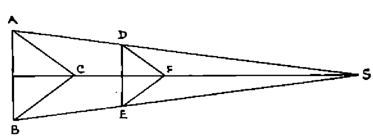
The two triangles *ABC* and *DEF* are said to be equiangular when their angles are equal each to each, *i.e.*

$$\angle A = \angle D,\ \angle B = \angle E \text{ and } \angle C = \angle F.$$

It will be seen later that in the case of similarity of triangles, if either of the two conditions of similarity is satisfied, the other is also satisfied, that is, if the angles are equal the sides must be proportional and if the sides are proportional, the angles must be equal.

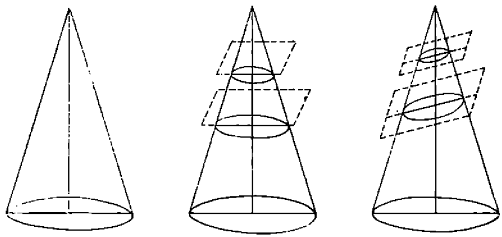
But this does not hold good for all figures. This has already been illustrated in the case of pentagons while discussing the conditions of similarity.

(IX) Centre of Similarity or Homothetical Centre



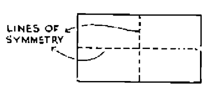
If triangles *ABC* and *DEF* are similar and if the corresponding sides are parallel, then the straight line joining their corresponding vertices meet in a point *S*. The point *S* is called the *centre of similarity* or the *homothetical centre*.

If we take a hollow cone, the parallel sections made by parallel planes are similar figures, may be circles, ellipses, parabolas or hyperbolas, and the vertex of the cone is the centre of similarity or the homothetical centre.

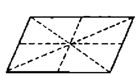


(X) Line of Symmetry

Some geometrical figures are symmetrical about a given line and if the figure is folded along that line, one part of the figure coincides with the other part (both the parts being similar and identical in all respects). The line under reference is known as the *line of symmetry* of the figure, *e.g.*, in a rectangle the line joining the mid-points of the opposite sides is line of symmetry of the rectangle.



If a point can be found such that all lines drawn through the point and terminating on the figure are bisected at the point, the fiigure has a point symmetry.



Some figures have both line and point symmetry, *e.g.*, a circle. A solid figure may have point, line or plane symmetry, *e.g.*, a human being when standing erect is symmetrical with respect to a plane perpendicular to floor through the center of the body.

(*XI*) *Idea of scale*

In a map or a sketch the ratio of the distance on the map to the actual distance is called the scale.

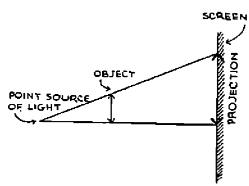
A distance of 20 metres can be represented by a length of 4 cms. There the scale is

$$\frac{4 \text{ cms.}}{20 \text{ m.}} = \frac{4}{20 \times 100} = \frac{1}{500}$$

Here $\frac{1}{500}$ is called the *Representative Fraction* (R.F.). The numerator of representative fraction is always taken to be 1. This principle is followed in drawing maps.

(*XII*) *Projection*

When an object is held in front of a point source of light it casts a shadow on a screen held behind. This shadow which has exactly the same shape as the object is called



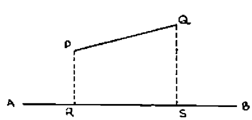


Fig. (*i*)

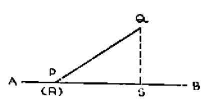


Fig. (*ii*)

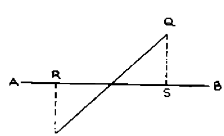


Fig. (*iii*)

Projection of Straight lines

the projection of the object. When the screen is moved away from the object the size of the projection goes on increasing and when it is moved nearer, the size of projection goes on decreasing. However, the projections at all positions of the screen are similar.

In each of the Figs. (*i*), (*ii*) and (*iii*), there are two straight lines *PQ* and *AB*. Perpendiculars *PR* and *QS* are drawn from the ends of *PQ* on the straight line *AB*. (In Fig. (*ii*), *R* coincides with *P*). Then *RS* is called the projection of *PQ* on *AB*.

Theorems

There are some basic theorems (enunciated under the 'understandings') for the study of similarity. These should be thoroughly mastered. The teacher should prove the theorems with the help of pupils by the method of analysis and synthesis (the deductive proofs are found in any text-book on Geometry dealing with the topic).

V. Learning Experiences

(*i*) Present before the students a number of objects and pictures and ask them to identify those that are similar and those that are not.

(*ii*) Ask them to give reasons for their inferences.

(*iii*) Give them practice in drawing similar geometrical figures.

(*iv*) Let them identify the corresponding parts of similar triangles, quadrilaterals and polygons.

(*v*) Ask them such questions as the following :

Given two $\triangle$s *ABC* and *DEF* are similar. Write down the proportionality between corresponding sides using the notation *AB*, *AC*, and so on. Then

(*a*) Express *AB* in terms of *AC*, *DE* and *EF*
(*b*) ,, *BC* in terms of *AB*, *DE* and *EF*
(*c*) ,, *AC* in terms of *BC*, *EF* and *DF*
(*d*) ,, *AB* in terms of *BC*, *BE* and *EF*
(*e*) ,, *BC* in terms of *AC*, *EF* and *DF*
(*f*) ,, *AC* in terms of *AB*, *DE* and *DF*
...........and so on.

(*vi*) Ask them to state under what conditions can two similar figures be congruent.

(*vii*) Let them determine the corresponding points of a figure similar to another figure whose points and the relation between the sides are given.

(*viii*) Let them study and determine the centres of similarity of different systems of similar figures—triangles, circles, polygons, etc.

(*ix*) Let them examine plane and solid figures such as cylinders, cubes, cones, frustums of spheres, etc. to determine their lines of symmetry.

(*x*) Let them study the relationship between the objects and their projections on the screen by actual experimentation.

(*xi*) Give them practice in drawing projections geometrically.

The Basic Proportionality Theorems

THEOREMS

(*i*) If a straight line is drawn parallel to one side of a triangle, it divides the other sides proportionally.

(*ii*) Conversely, if a straight line divides two sides of a triangle proportionally, it is parallel to the third side.

(*iii*) The internal and external bisectors of the vertical angle of a triangle divide the opposite side in the ratio of the sides containing the vertical angle.

(*iv*) Conversely, if the base of a triangle is divided internally or externally into two segments proportional to the other sides of the triangle, the line joining the point of section to the vertex bisects the vertical angle internally or externally.

Theorems on Similarity of Triangles

THEOREMS

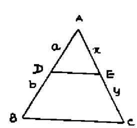
(*v*) If two triangles are equiangular, their corresponding sides are proportional and hence the two riangles are similar (*vi*) If two triangles have three sides of the one proportional to the three sides of the other, the triangles are equiangular, those angles being equal which are opposite to the corresponding sides. Then the two triangles are similar.

Note : These basic theorems on proportionality are fundamental to the study of the properties of similar figures. Hence they should be thoroughly mastered. The teacher is to prove the theorems with the help of the pupils by the methods of analysis and synthesis.

Learning Experiences on Theorems

Theorems 1 and 2

1. Ask the students to take different types of triangles and draw parallels to one side. Let them measure the segments in which the remaining two sides are divided ; compare the ratios and draw inferences.
2. Ask them to divide two sides of a triangle in the same ratio and study about the parallelism of the line connecting the points of division and the third side.
3. Ask them to prepare work models to illustrate the truth of the theorems.
4. In the figure given below, the lengths of segments are a, b, x and y as shown. Fill the places with (?) with appropriate insertions.

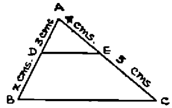


$$\frac{a+b}{a}=\frac{?}{x}, \qquad \frac{a}{b}=\frac{?}{?},$$

$$\frac{a+b}{b}=\frac{x+?}{?}, \qquad \frac{a}{x}=\frac{b}{?},$$

$$\frac{a+b}{x+y}=\frac{?}{x}, \qquad \frac{x+y}{a+b}=\frac{y}{?}.$$

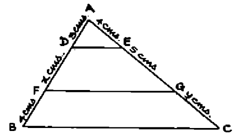
5(i)



$DE \parallel BC$

Find x.

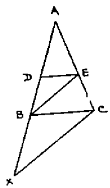
5(ii)



$DE \parallel FG \parallel BC$

Find x and y.

5(iii)



Let $DE \parallel BC$

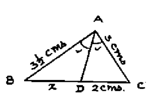
and $\frac{AD}{DB} = \frac{2}{1}$.

Prove that $\frac{AX}{XB} = \frac{3}{1}$.

Theorems 3 and 4

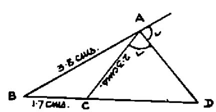
6. Ask the students to verify the theorems experimentally.
7. Ask them to bisect an angle of a triangle both internally and externally and study the relationships between the four parts into which the opposite side is divided by the two points of internal and external division (the two ratios will be equal).
8. Ask them such questions as follows :

(i)



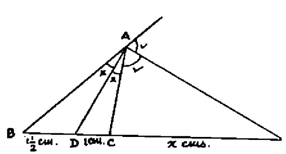
Find x.

(ii)



Find CD.

(iii)



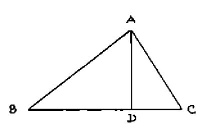
Find x.

Theorems 5, 6 and 7

9. Ask the students to verify the truth of the propositions experimentally taking two triangles :
   (*a*) which are equiangular and measuring the corresponding sides,
   (*b*) whose sides are proportional and measuring the corresponding angles, and
   (*c*) in which one angle of one is equal to one angle of the other and the sides about them proportional and measuring the remaining sides and angle.
10. Give them numerical problems where they will be required to supply the missing points of similar triangles.
11. Give them opportunity to draw inferences as to whether two triangles satisfy the conditions of similarity by examining given data.

Theorem 8

12. Ask the students to draw a perpendicular from the vertex of right angle of a right angled triangle. Let them measure



the sides, the perpendicular and the segments of the hypotenuse and examine the relationship between the three triangles *ABC*, *ABD* and *ACD*.
13. Let them study the relationship between *AD* and the product of *BD* and *CD*.

Miscellaneous

14. In this way let them learn how to draw the mean proportion between two segments.
15. Ask the students to draw a square equivalent to a rectangle.

16. Ask the students to find geometrically the value of

$$\frac{3{\cdot}5\times4{\cdot}2}{5{\cdot}1};\ \frac{6{\cdot}5}{2{\cdot}3};\ \frac{5{\cdot}1}{3{\cdot}1\times4{\cdot}8};\ \sqrt{3{\cdot}5\times4{\cdot}1};\ \sqrt{8}$$

17. Ask the students to find a point P on $AB=3{\cdot}5$ cms. or on AB produced such that

$$AP^2=AB.BP$$

18. Ask the students to prove the theorems experimentally.

19. Ask the students to prove the following using the principle of similarity :

   (*a*) Pythagorus Theorem : The sum of the squares on the sides of a right triangle equals the square on the hypotenuse.

   (*b*) The perpendicular to the diameter from any point on the circle is the mean proportional between the segments into which it divides the diameter.

   (*c*) If two chords intersect in a circle, the product of the segments of one chord equals the product of the segments of the other.

   (*d*) If from a point outside a circle a secant and a tangent are drawn, the tangent is the mean proportional between the secant and its external segment.

20. Ask the students to solve the puzzle :

   In the middle of a square pond grew a reed. The pond was 10 ft. on each side. The reed protruded 1 ft. above the surface of water and yet when blown aside by the wind, it just touched one side of the pond at its middle point. How deep was the pond ?

I. Assigment Constructions

1. Draw a straight line AB of length 4 cms. and divide it externally at C in the ratio of 5 : 2.

2. A, B, C and D are four points lying in order on a

straight line. Find a point X on it such that

$$XA : XB = XC : XD$$

3. Find geometrically the fourth proportional of 3·5, 4·2, 5·1 and verify the result by calculations.

4. Find geometrically the third proportional of 3·5 and 4·2 and verify the result by calculations.

5. Find geometrically the mean proportional of 5 and 3.

6. Divide AB=3·5 cms. long internally at P such that

$$AP^2 = AB \times PB$$

7. Draw an equilateral triangle equal to (*a*) the sum of two equilateral triangles and (*b*) difference of two.

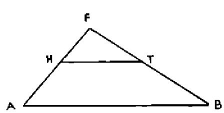
8. Bisect a triangle by a straight line drawn parallel to the base.

9. Draw any triangle and enlarge its sides in the ratio 3 : 5 geometrically.

10. Construct a triangle similar to a given triangle on a given line segment as base.

VII. Assignments

1. In this figure HT is parallel to AB. Make the appropriate



insertions in the (?) spaces.

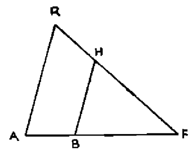
$$\frac{FA}{FH} = \frac{FB}{?} \qquad \frac{TB}{FT} = \frac{HA}{?}$$

$$\frac{FA}{HA} = \frac{?}{TB} \qquad \frac{FT}{FH} = \frac{FB}{?}$$

$$\frac{FH}{HA} = \frac{FT}{?} \qquad \frac{BT}{AH} = \frac{?}{HF}$$

2. In the figure,

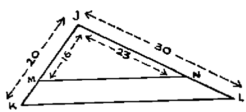
(*a*) If $RH=4$, $HF=7$
$BF=10$, then $AB=?$



(*b*) If $RH=6$, $HF=10$
$AB=3$, then $BF=?$

(*c*) If $RH=5$, $RF=20$
$AF=18$, then $BF=?$

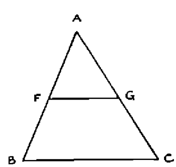
3. In the figure let the segments have measures as indicated.



Can MN be parallel to KL ?

Justify your answer.

4. Which of the following sets of data make FG parallel to BC ?



(*a*) $AB=14$, $AF=6$, $AC=7$, $AG=3$

(*b*) $AB=12$, $FB=3$, $AC=8$, $AG=6$

(*c*) $AF=6$, $FB=5$, $AG=9$, $GC=8$

(*d*) $AC=21, GC=9, AB=14$, $AF=5$

(*e*) $AB=24$, $AC=6$, $AF=8$, $GC=4$.

5. Which of the following cases are sufficient to show that the two $\triangle$s ABC and DEF are similar ?

(*a*) $\angle A=\angle D$, $\angle B=\angle E$

(*b*) $\frac{AB}{AC}=\frac{DE}{DF}$

(*c*) Corresponding sides are proportional

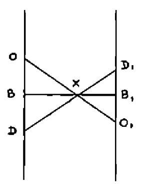
(*d*) Both triangles are equilateral

(*e*) Both triangles are isosceles

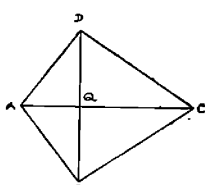
(*f*) $\angle C=\angle F=90°$ and $AB=DE$.

6. In the diagram, OD is parallel to $O_1 D_1$. Prove that

$$\frac{OB}{O_1B_1}=\frac{OD}{O_1D_1}$$



7. In this figure if DB is perpendicular to AC and $DQ=BQ=2\ AQ=\frac{1}{2}QC$



Prove that

(*a*) $\triangle AQD$ is similar to $\triangle DQC$

(*b*) $\triangle BQC$ is similar to $\triangle AQD$

(*c*) $AD \perp DC$

8. The bisectors of the base angles of a triangle meet the opposite sides at X and Y respectively. If XY is parallel to the base, prove that the triangle is isosceles.

9. AD is a median of $\triangle ABC$, and the angles ADB and ADC are bisected by lines which meet AB, AC at E and F respectively. Show that EF is parallel to BC.

10. The bisectors of the angles of the $\triangle ABC$ intersect the sides BC, CA, AB at L, M, N, respectively. Prove that $BL. CM. AN = LC. MA. NB$.

11. Construct a triangle having given the base, the vertical angle and the ratio of the other two sides.

12. Show that the line joining the middle points of any two sides of a triangle is parallel to the third side and is half of it.

13. In the trapezium $ABCD$, AB is parallel to DC and the diagonals intersect at O, show that $OA : OC = OB : OD = AB : CD$.

14. Triangle ABC is right angled at A and AD is an altitude. If $BD = 12$ cms. and $DC = 3$ cms.; calculate the length of AD.

15. Triangle XYZ is right angled at X, and XN is an altitude. If M is the mid-point of YZ, prove that

$$XN^2 = MZ^2 - MN^2.$$

16. Triangle ABC is right angled at C, and $AC = 60$ cms., $BC = 80$ cms. If CD is an altitude of $\triangle ABC$, and DE is an altitude of $\triangle BCD$; calculate BE and EC.

17. In $\triangle XYZ$, $\angle X$ is a right angle and XM is an altitude. If $YZ = 5\ MZ$; prove that $YZ^2 = 5\ XZ^2$.

18. Two parallel straight lines meet the straight lines OP, OQ, OR at P, Q, R and P', Q', R' respectively. Prove that $PQ : QR = P'Q' : Q'R'$.

19. Any straight line drawn parallel to the base of a triangle is bisected by the line drawn from the vertex to the middle point of the base.

20. $ABCD$ is a parallelogram. From D a straight line is

drawn to cut AB at E and CB produced at F. Show that $DA : AE = BF : BE = FC : CD$.

21. A, B, C and D are four points lying in order on a straight line. Find a point X on it such that $XA : XB = XC : XD$.

22. If one of the parallel sides of a trapezium is double the other, show that its diagonals cut each other at the point of trisection.

23. Any two medians of a triangle cut one another at a point of trisection.

24. The bases BC and EF of two similar triangles ABC and DEF are divided in the same ratio at X and Y. Prove that $AX : DY = BC : EF$.

25. Prove that in two similar triangles the corresponding medians and altitudes are proportional.

26. In $\triangle ABC$, AD is perpendicular to BC and $AB^2 = BC.\ BD$. Show that $\angle A$ is a right angle.

27. In $\triangle ABC$, AD is perpendicular from A to BC. If BD, AD and CD be in continued proportion, then angle BAC is a right angle.

28. Any triangle described on a diagonal of a square is double the similar triangle described on the side of the square.

29. Equilateral $\triangle$s are described on the sides of a right-angled triangle. Prove that the area of the triangle on the hypotenuse is equal to the sum of the areas of the other two triangles.

30. Prove that the areas of similar triangles have the same ratio as the squares on corresponding medians.

31. If two $\triangle$s are similar, their areas are proportional to the squares on their corresponding altitudes.

32. The area of the $\triangle ABC$ is 19.6 sq. cms. and XY drawn parallel to BC cuts AB in the ratio 4 : 3. Find the area of the $\triangle AXY$.

33. Two equiangular triangles have areas in the ratio of 3:2 and an altitude of the greater is 5.2 cms. What is the corresponding altitude of the other ?

34. If PQ is drawn parallel to BC, the base of the $\triangle ABC$, and if $\triangle APQ$: figure $PBCQ=4:5$, show that $AP:BP=2:1$.

35. ABC is a triangle whose area is 16 sq. cms. and XY is drawn parallel to BC, dividing AB in the ratio 3 : 5; if BY is joined, find the area of the triangle BXY.

36. ABC and XYZ are two similar triangles whose areas are respectively 32 sq. cms. and 60.5 sq. cms. If $XY=7.7$ cms. find the length of the corresponding side AB.

VIII. Teaching hints

1. The concept of similarity may be introduced through the word resemblance. When two persons or figures resemble each other (of course with minor differences) then there are two possibilities. Either the two are identical in all respects. This is discussed in geometry as congruency. As an alternative the figures may be of the same form though different in size. A non-mathematical analogy may be a man growing in size with the passage of time. Starting with this, an example from photography may be taken up. Enlargement of photographs almost every student may be familiar with.

A number of examples may be available in the building architecture and these may be referred to.

2. A clear concept of ratio and proportion is necessary and this may be considered on the basis of examples like measuring the height of a tower with the help of its shadow. Simultaneous measurement of the shadow of stick put parallel to the tower and the ratio between the size of the stick and its shadow may be utilised. This concept may be further strengthened through its utilisation in measuring instruments. Once this is developed properly similarity may be easier to deal with.

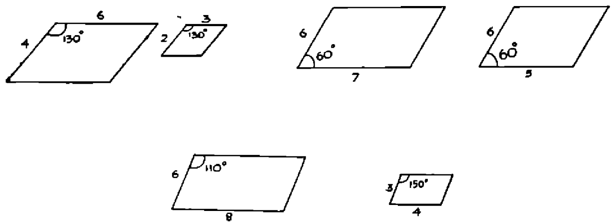
3. A clear distinction between congruency and similarity may be drawn. The elements of the two discussed in detail will help at the stage of solution of problems.

4. Symmetry in relation to similarity and congruency may be explained.

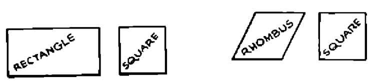
IX. Evaluation

(Sample test items)

1. Which pair of parallelograms is similar?



2. Are these pairs of figures similar? Which part of the definition of similar polygons is true in each case?



3. Say which of the following pairs of figures must be similar:

(*a*) a pair of right angled triangles

(*b*) a pair of isosceles triangles

(*c*) a pair of equilateral triangles.

4. Say which of the following definitions is the definition of similarity:

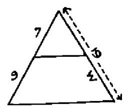
(*a*) Two geometrical figures are similar when they are equi-angular.

(*b*) Two geometrical figures are similar when their sides are proportional.

(*c*) Two geometrical figures are similar when they are equi-angular and their corresponding sides are proportional.

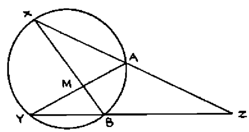
5. Given the figure, a person handled the problem of finding *W* in this way:

$$\frac{7}{9}=\frac{19-W}{W}$$

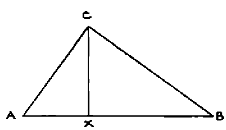


Propose a more convenient equation. Do you get the same result?

6. Find two sets of triangles that are similar on this figure.



7. Given the figure shown with $AC \perp BC$ and $CX \perp AB$,

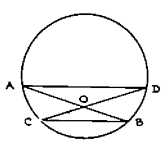


(*a*) Name an angle which is equal to $\angle ACB$.

(*b*) Name an angle equal to $\angle B$.

(*c*) Name a triangle which is similar to $\triangle ABC$.

8. Two chords *AB* and *CD* of a circle intersect at *O*. *AD* and *BC* are joined. Are the two $\triangle$s *AOD* and *BOC* congruent?



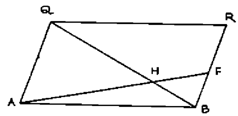
Which of the following conditions of similarity proves that they are similar ?

(*a*) sides are proportional

(*b*) angles are equal

(*c*) one angle is equal to one angle and the sides about them proportional.

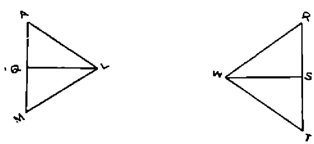
9. Given parallelogram *ABRQ* with diagonal *QB* and segment *AF* intersecting *QB* in *H* as shown.



Prove : $QH.\ HF = HB.\ AH$

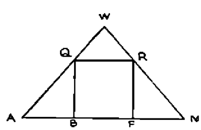
10. In these figures *WS* and *LQ* are medians and

$$\frac{RW}{AL} = \frac{RT}{AM} = \frac{WS}{LQ}$$



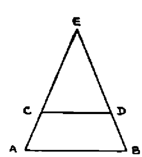
Prove that $\triangle RWT$ is similar to $\triangle ALM$

11. Given $AW \perp MW$. *BFRQ* is a square with *Q* on *AW* and *R* on *WM* as shown in the figure.

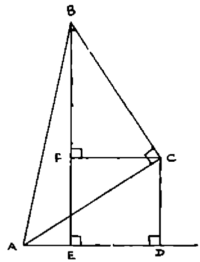


Prove : $AB.\ WR = QW.\ BQ$ and $AB.\ FM = FR.BQ$

12. Given $\angle A = \angle B$ and $AC = BD$; show that *CD* is parallel to *AB*.



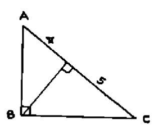
13. The angles in the figure marked with small squares are right angles.



Show that $\frac{BF}{BC} = \frac{AD}{AC}$

PART II

14. Given a right $\triangle ABC$ with altitude drawn to the hypotenuse and lengths as shown, find the unknown lengths.



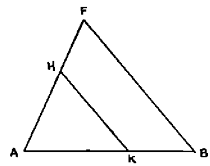
15. In a right triangle if the altitude to the hypotenuse is 12 and the hypotenuse is 25, find the length of each side, and of the segments of the hypotenuse.

16. What is the ratio of areas of two similar triangles whose bases are 3 cms. and 4 cms., X cms. and Y cms. ?

17. A side of one of the two similar triangles is 5 times the corresponding side of the other. If the area of the first is 6 square units, what is the area of the second ?

18. In the figure if H is the mid-point of AF and K is the mid-point of AB, the area of $\triangle ABF$ is how many

times as great as the area of $\triangle$ *AKH* ? If the area of



$\triangle$ *ABF* is 15 square units, find the area of $\triangle$ *AKH*.

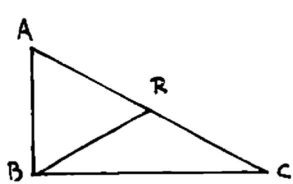
19. Two jugs, similar in shape, are 7 cms. and 10 cms. high respectively. If the smaller jug holds 2 pints, find the capacity of the larger.

20. Two dolls weigh 2 lbs. and $13\frac{1}{2}$ ozs. respectively, and are identical in design. If the larger wears an apron of area 20 sq. cms., find the area of the apron worn by the smaller.

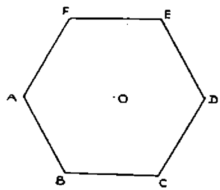
21. A doll's house is a scale model of a real house. The volumes of air in the drawing-rooms are respectively 1125 cu. cms. and 6·61 cu. cms. The front door of the real house has an area of 54 sq. cms. Find the area of the front door of the doll's house.

22. Prove the theorem : The mid-point of the hypotenuse of a right triangle is equi distant from the vertices.

23. In this triangle $AR = RC = RB$. Prove that $\triangle$ *ABC* is a right triangle.



24. Construct a figure similar to a given geometrical figure.



Given any figure *ABCDEF*.

To construct a similar figure.

4

ELIMINATION

I. Overview

Elimination of the middleman is one of the practices coming up in everyday life. In trade, co-operatives are trying to establish a direct link between the producer and the consumer. Even though it may not be a very good analogy yet the same term is used in Mathematics for quite a similar process. There we often meet with instances where, given several statements concerning several distinct quantities, we wish to discover precisely what is affirmed of a smaller group of these quantities. The process of elimination is widely made use of in Higher Mathematics and in science subjects. The topic is regarded important for its functional utility. It is a process by means of which we can find out the conditions under which a certain set of equations may be simultaneously true or consistent.

II. Objectives

1. To enable the pupil to understand the basic concepts, terms, principles, etc. involved in learning the process of elimination.

2. To enable the child to find the value of a variable and substitute for it.

3. To enable the child to eliminate one or more than one variables.

4. To enable the child to apply the knowledge of elimination in solving practical problems in Science and Higher Mathematics.

III. Content analysis

Terms : Equation

Variables (unknowns) and Co-efficients
Cross multiplication
Elimination and Eliminant

1. Equation

When two algebraic expressions are connected by the sign of equality (=), they form an equation, *e.g.*,

$$3x+8=17.$$

Thus, there are two members or sides of an equation, namely first member or the left hand side and the second member or the right hand side.

Equations may be sub-divided into two classes : (*i*) Simple Equations, and (*ii*) Simultaneous Equations.

(*i*) *Simple equations*

An equation in which the expressions of the unknown are stated in the first power only is called a simple equation.

(*ii*) *Simultaneous equations*

Two or more equations which are satisfied by one and only one set of values of the variables involved in them are called simultaneous equations.

For example :

$$2x-5y=1$$
$$4x-3y=23$$

II. (i) Variables

The equation $2x-5y=1$ has got two unknown quantities x and y. These are called the variables. The value of 'y' depends upon the value of 'x' and hence 'y' is called the dependent variable and 'x' the independent variable. We may interchange x and y as regards their dependence.

(*ii*) *Co-efficients*

In the equation, say, $4x-3y=23$, the numerical factor of $4x$ is $+4$ and that of $-3y$ is -3. This $+4$ is the *co-efficient* of x and -3 that of y. 23 is the 'constant or independent term.'

III. Cross multiplication

Let there be two simultaneous equations, say

$$lx+my+nz=0$$

and $$px+qy+rz=0.$$

Here the variables are x, y and z which are to be found out in terms of the co-efficients. For this purpose, we use what is known as 'Cross Multiplication', explained as under :

(1) For writing the denominator of 'x' cover with your hand the terms containing x and then multiply the co-efficients of (y, z) and subtract the product of the co-efficients of (z, y).

(2) For the denominator of 'y' cover with your hand the terms containing y, then multiply the co-efficients of (z, x) and subtract the product of the co-efficients of (x, z).

(3) For denominator of 'z' cover with your hand the terms containing z, then multiply the co-efficients of (x, y) and subtract the product of the co-efficients of (y, x). As we get these denominators by multiplying the co-efficients crosswise, this process is known as 'Cross Multiplication' method. Thus it is as under :

$$\frac{x}{\begin{array}{ccc} m & \searrow\nearrow & n \\ q & \nearrow\searrow & r \end{array}}=\frac{y}{\begin{array}{ccc} n & \searrow\nearrow & l \\ r & \nearrow\searrow & p \end{array}}=\frac{z}{\begin{array}{ccc} l & \searrow\nearrow & m \\ p & \nearrow\searrow & q \end{array}}$$

$$\frac{x}{(mr-qn)}=\frac{y}{(np-rl)}=\frac{z}{(lq-pm)}$$

If it is put$=k$, then values of x, y and z can be found out in terms of the co-efficients and k.

IV. Elimination

To eliminate means to remove or to throw off. Elimination is the process by which one or more variables or unknown quantities are removed and relation is established between or among the constants or co-efficients of the given equations or among a smaller number of variables. In the equation $ax+b=0$, the variable is x, which is $=-b/a$. If this value of x is substituted in another equation $cx+d=0$, we get

$$c(-b/a)+d=0,$$

or $$bc-ad=0,$$

or $$bc=ad.$$

$bc=ad$ gives us a condition for which the set of equations

$$ax+b=0$$

and $$cx+d=0$$

are simultaneously true or consistent.

This relation is independent of the variable x, and is called the 'Eliminant'. The process by which the relation has been established is called 'Elimination'. It may be the elimination of one, two or more variables.

(*i*) *Elimination of one variable*

(1) The equations are

$$qx+p=0 \qquad (i)$$
$$px-q=0 \qquad (ii)$$

Eliminate 'x'.

From (*i*), we get the value of 'x' in terms of p and q as under :

$$x=\frac{-p}{q}$$

Now substituting for x in (*ii*), we get

$$p\left(\frac{-p}{q}\right)-q=0,$$

$$\frac{-p^2-q^2}{q}=0,$$

or $$p^2+q^2=0.$$

This contains no 'x' and hence is the eliminant.

(2) Eliminate t from

$v=u+ft \quad ......(i)$

$s=ut+\frac{1}{2}ft^2......(ii)$

From (*i*), we get $ft=v-u$

i.e. $t=\dfrac{v-u}{f}$. Now putting for t in (*ii*) we get

$$s = \frac{u(v-u)}{f} + \tfrac{1}{2}f\left(\frac{v-u}{f}\right)^2$$

$$= \frac{uv-u^2}{f} + \frac{1}{2f}\,(v^2+u^2-2uv)$$

$$= \frac{2uv-2u^2+v^2+u^2-2uv}{2f}$$

$$= \frac{v^2-u^2}{2f}$$

or

$$2\,fs = v^2-u^2,$$

which is independent of 't' and is the eliminant.

(*ii*) *Elimination of two variables* (*reducible to one variable*)

The equations are

$$lx+my=0\ldots\ldots(i)$$

$$ax^2+2xy+by^2=0\ldots\ldots(ii)$$

Eliminate x and y.

Here we divide (*i*) by y and (*ii*) by y^2 on both the sides and get

$$l(x/y)+m=0\ldots\ldots(iii)$$

$$a\,(x/y)^2+2\,(x/y)+b=0\ldots\ldots(iv)$$

Now (*iii*) is a simple equation in (x/y). Substituting for x/y in (*iv*), we have

$$a\left(\frac{-m}{l}\right)^2+2\,(-m/l)+b=0,$$

i.e. $$a\left(\frac{m^2}{l^2}\right)-2\,(m/l)+b=0,$$

i.e. $$\frac{am^2}{l^2}-\frac{2m}{l}+b=0,$$

i.e. $$\frac{am^2-2ml+bl^2}{l^2}=0,$$

i.e. $$am^2-2ml+bl^2=0,$$

i.e. $$am^2+bl^2=2ml$$

which is independent of x and y and is the eliminant.

(*iii*) *Elimination of three variables (reducible to two variables)*

Eliminate x, y and z from the equations

$$ax+by+cz=0 \qquad \text{... } (i)$$

$$bx+cy+az=0 \qquad \text{... } (ii)$$

$$cx+ay+bz=0 \qquad \text{...} (iii)$$

From (*i*) and (*ii*), by cross multiplication, we get :

$$\frac{x}{ab-c^2}=\frac{y}{bc-a^2}=\frac{z}{ac-b^2}$$

Let each of these ratios be equal to k. Thus $x=k\,(ab-c^2)$, $y=k\,(bc-a^2)$ and $z=k\,(ac-b^2)$. Substituting for x, y and z in (*iii*) we get

$$kc\,(ab-c^2)+ka\,(bc-a^2)+kb\,(ac-b^2)=0.$$

Dividing by k and opening the brackets, we get :

$$abc-c^3+abc-a^3+abc-b^3=0,$$

i.e.

$$3abc-a^3-b^3-c^3=0,$$

which is independent of x, y and z and is the required eliminant.

IV. Suggested activities

1. Discuss with the pupils the need for eliminating variables.

2. Discuss with the pupils the various kinds of equations, variables and co-efficients.

3. Ask the students to collect examples and situations in the study of other branches of Mathematics where elimination is used.

4. Ask the students to collect examples and situations in the study of science subjects where process of elimination is used.

5. Ask the pupils to define the following :

   (*i*) Variable

   (*ii*) Elimination

   (*iii*) Eliminant.

6. Discuss with the pupils the various methods of elimination of one, two and three variables.

7. Ask the students to apply the knowledge gained from the elimination in solving practical problems of Physics as under :

   "A train starts from rest and gains a velocity of 45 miles per hour after travelling a distance of 30 miles ; find the acceleration".

V. Hints for the teacher

1. It must be emphasised that the number of equations should be one more than the number of variables or unknowns or elements to be eliminated unless the number of variables can be reduced by one variable as discussed in the examples.

2. It may be pointed out clearly that whereas for the solution of equations we need as many equations as the variables to be found out, for elimination we need one equation extra.

3. As elimination is a process which the pupils ought to know before proceeding to the solution of simultaneous equations algebraically its importance in that connection may also be made very clear.

4. Functional utility of the process of elimination should be stressed to the students. Process of elimination is employed in various science subjects and in other branches of Mathematics. For example, in Physics by the process of elimination a direct relation between v (final velocity), u (initial velocity) and s (distance covered) can be established ($v^2-u^2=2fs$) from the equations $v=u+ft$ and $s=ut+\frac{1}{2}ft^2$. In Co-ordinate Geometry, from parametric equations of Parabola, Ellipse and Hyperbola, etc. we can find the equations of these curves in Cartesian coordinates.

5. It should be emphasised that by taking specific examples in a particular problem the process of elimination can be completed in different ways, but the eliminants thus obtained

would be the same.

VI. Assignments

1. Eliminate x from $ax+b=0$ and $cx+d=0$
2. Eliminate x from $ax+b=0$ and $px^2+q=0$
3. If $v=u+ft$ and $s=ut+\frac{1}{2}ft^2$, show that $v^2=u^2+2fs$
4. Eliminate x from the equations

   $ax^4-b=0$ and $px^6-q=0$
5. Eliminate x from the equations

   (a) $x+\dfrac{1}{x}=m, \qquad x-\dfrac{1}{x}=n$

   (b) $x+\dfrac{1}{x}=a, \qquad x^2+\dfrac{1}{x^2}=b^2$
6. Eliminate t from the equations

   (a) $x=\dfrac{a(1+t^2)}{1-t^2}, \quad y=\dfrac{2bt}{1-t^2}$ (Hyperbola)

   (b) $x=\dfrac{a(1-t^2)}{1+t^2}, \quad y=\dfrac{2bt}{1+t^2}$ (Ellipse)

   (c) $x=at^2, \quad y=2at$ (Parabola)
7. Eliminate z from

   $$z^2+fz+x=0$$
   $$z^2+gz+y=0$$
8. Eliminate x and y from

   $$a_1x+b_1y+c_1=0$$
   $$a_2x+b_2y+c_2=0$$
   $$a_3x+b_3y+c_3=0$$
9. Eliminate x from the equations

   $$x^3+\frac{3}{x}=4(a^3+b^3), \qquad 3x+\frac{1}{x^3}=4(a^3-b^3)$$

VII. Evaluation

UNIT TEST

SUBJECT : Algebra

UNIT : Elimination — Time : 30 minutes

CLASS : Matric — Max. Marks : 25

BLUEPRINT

Objective	Knowledge			Understanding			Application			Total
Form of Item / Content Sub-Unit	E	S	O	E	S	O	E	S	O	
(1) Definitions of Concepts	1(4)	1(2)	1(1)							7
(2) Elimination of one and more than one variables					4(2)	4(1)				12
(3) Solution of Problems in Science and Higher Mathematics using Elimination								3(2)		6
Total	1 (4)	1(2)	1(1)		4(2)	4(1)		3(2)		25

Note : Indicate skill part of the essay question in the same row. Figures within brackets indicate the number of questions and figures outside the brackets indicate marks.

Summary :

Essay (E) No.....................1 Marks.........4

Short Answer (S) No..........8 Marks.......16

Objective type (O) No.........5 Marks.........5

Scheme of Sections : A and B.

SECTION A

Q. 1. Given the equation $ax+b=0$ *in which*

(*A*) a is the variable

(*B*) b is the variable

(*C*) x is the variable — *O* (1)

(*D*) x is the constant. — ½ Min.

C

Q. 2. The eliminant is defined as :

(*A*) the terms which contain no variable

(*B*) the co-efficients of the variable

(*C*) the product of the co-efficients

(*D*) the relation between the co-efficients and the independent term which does not contain the term to be eliminated. — *O* (1) ½ Min. *D*

Q. 3. In the equations

$lx-a=0$ and

$mx-b=0,$

the eliminant after eliminating x is

(*A*) $l=a$

(*B*) $m=b$

(*C*) $am=bl$ — *O* (1)

(*D*) $lm=ab$ — 1 Min.

C

Q. 4. The condition that the equations

$x^2-yz=a$

$y^2-zx=b$

$z^2-xy=c$ and

$ax+$ by $+cz=0$ may have the same values of a, b and c is :

(*A*) $x^2+y^2+z^2=xy+yz+zx$

(*B*) $3xyz=x^3+y^3+z^3$

(*C*) $ax^2+by^2+cz^2=a+b+c$

(*D*) $a^2+b^2+c^2=abc$. *O* (1)

2 Min.

B

Q. 5. *Given*

$x=cy$

$y=az$

$z=bx$, then abc is equal to

(*A*) 2 *O* (1)

(*B*) 1 2 Min.

(*C*) 3 *B*

(*D*) xyz.

SECTION B

Q. 6. *Define the terms* :

(*i*) Variable

(*ii*) Co-efficient

(*iii*) Elimination

(*iv*) Eliminant. *S* (2)

2 Min.

Q. 7. *Solve the given equations for x and compare the results*.

$ax+b=0$..............(*i*)

$lx+m=0$..............(*ii*)

What is your observation? *S* (2)

2 Min.

Q. 8. *Eliminate t from*

$x=t+1/t$

$y^2=t^2+1/t^2$. *S* (2)

2 Min.

Q. 9. *Eliminate x and y from the equations* :

$x=ay$, $px^2+qy^2=rxy$.

S (2)

2 Min.

Q. 10. Eliminate x and y from :

$$ax + by + c = 0$$
$$px + qy + r = 0$$
$$x + y + 1 = 0$$

S (2)
3 Min.

Application

Q. 11. In physics, the formulas for velocity and distance are :

$$v = u + at \text{ and}$$
$$s = ut + \tfrac{1}{2} at^2.$$

Find out a formula for distance when time is not given.

S (2)
4 Min.

Q. 12. Show that the expression :

$$\frac{x}{(x-2)(2x-2)} = \frac{1}{x-2} - \frac{}{2x-2}$$

S (2)
2 Min.

Q. 13. In mechanics :

$v = s/t$ and $a = v/t$, then show that $a = s/t^2$, when a, s, t and v stand for acceleration, distance, time and velocity respectively.

S(2)
2 Min.

Knowledge

Q. 14. Define a simultaneous equation. How will you eliminate two variables from these simultaneous equations? Write in detail.

E(4)
5 min.

QUESTION-WISE ANALYSIS

Sl. No.	*Objective*	*Specification*	*Topic*	*Form*	*Marks*	*Est. Time*	*Est. Diff.*
1	Knowledge	Recalls	Elimination	O	1	½	C
2	Understanding	Discrimination		O	1	½	B
3	-do-	Identifies		O	1	1	B
4	-do-	Compares		O	1	2	A
5	-do-	-do-		O	1	2	B
6	Knowledge	Recalls		S	2	2	B
7	Understanding	Campares		S	2	2	C
8	-do-	-do-		S	2	2	B
9	-do-	Guesses the correct answer		S	2	2	B
10	-do-	-do-		S	2	3	B
11	Application	Establishes relationship		S	2	4	A
12	-do-	-do-		S	2	2	C
13	-do-	-do-		S	2	2	B
14	Knowledge	Recalls	,,	E	4	5	B

A-Difficult, B-Average and C-Easy

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