GEOMETRY FOR SCHOOLS

PALMER AND PARR

GEOMETRY
FOR SCHOOLS

G. Bell and Sons Ltd
Portugal Street, London

Calcutta, Bombay & Madras
Longmans, Green & Co. Ltd

Toronto
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GEOMETRY FOR SCHOOLS

BY

A. H. G. PALMER, M.A.

Headmaster of Gt. Yarmouth Grammar School
Formerly Chief Mathematics Master, Whitgift School
Late Senior Scholar of Trinity College, Cambridge

AND

H. E. PARR, M.A.

Chief Mathematics Master, Whitgift School
Late Scholar of Jesus College, Cambridge

LONDON

G. BELL AND SONS LTD

1938

This book is published complete
and in two parts

Printed in Great Britain by T. and A. CONSTABLE LTD.
at the University Press Edinburgh

PREFACE

This book has been written with two main objectives. The first is to supply a very large number of very easy riders, especially theoretical riders. A pupil's interest in Geometry depends almost entirely on whether he or she can do riders or not. Here, we believe, are riders which 90 per cent. of pupils will be able to do, and the sense of power thus aroused should enable more difficult work to be faced with hope of success.

The second objective is reduction in the amount of bookwork. We assume that a preliminary experimental course has been done, and have therefore omitted explanations of some of the simpler terms such as 'point,' 'line,' 'parallels,' 'right angles.' Further, material which the slower pupils may well omit altogether has been placed in the Appendix, on the ground that it is of better moral effect to allow quick pupils to do 'extra' than to ask slow pupils to make omissions.

With the reduced amount of time which is devoted to Geometry in many schools, we feel that there is a serious danger that the time left over for riders when bookwork has been learnt will be too small to be effective, so that what is most vital in Geometry may fall into neglect. We suggest therefore that the bookwork should be explained by the teacher, but that the proofs should not be learnt until the pupil is able to tackle very easy riders with confidence. The older the pupil is, and the greater his or her capacity for inventing proofs, the more readily will the need for logical sequence be perceived and the easier the proofs will be to learn.[1] This is not a new doctrine, but it is one rarely practised. The argument usually raised against it is that pupils must be given a fair chance of passing the School Certificate, and therefore must at all costs be sure of their bookwork. Few examiners, if any, would support this view; knowledge of bookwork alone is insufficient to ensure a pass, and failure is nearly always

[1] This is especially true of the section on the parallelogram.

attributable to inability to tackle riders. Further, if a pupil does fail, the teacher who has followed the plan advocated above will at least have the satisfaction of having spent his time in trying to inculcate what is of greatest educational value in Geometry rather than something which, for the unmathematical pupil, has not even a potential value. Those teachers, however, who remain unconvinced by the arguments in this paragraph need not fear that we have neglected bookwork; great care has been taken with its presentation, and the needs of the less able pupils have constantly been borne in mind.

Examples marked *a* are very easy, and the theorem on which they depend is indicated clearly, so that some riders can be taken immediately after each theorem is explained. These examples are copiously supplied with figures, especially in the early stages when drawing the figure is a big difficulty; for few things are more disheartening to beginners than poring over a figure which turns out to be wrong. The *b* examples are of the standard usually to be found in textbooks. In *a* and *b* examples a dividing line indicates a change in the type of question. The *c* and Miscellaneous examples are mainly School Certificate questions, intended for revision. Where they are divided by lines into groups, the questions in any one group are as far as possible representative of all the bookwork with which the exercise deals, so that one group would make a useful revision test. In all the exercises the questions are carefully graded.

When 'Stage C' is reached the bookwork should be taken in the order in which it is printed, but during 'Stage B' many variations in the order may be made with advantage. The following are suggested:—

(*a*) After pp. 1 to 16, assume the isosceles triangle theorem and take the angle properties of the circle (pp. 170 to 178); then return to p. 18 and do congruence.

(*b*) After congruence do a first course of similar triangles, *i.e.* pp. 283 to 285 and Examples 24*a*, Nos. 1 to 30.

(*c*) After the section on the mid-point and intercept theorems (pp. 79 to 91), do a first course of ratio properties, *i.e.* pp. 263 to 269 and Examples 23*a*, Nos. 1 to 19. Special care has been taken to make the ratio section easy enough for early introduction. Pages 263 to 267 deal solely with the one idea, expressed in Fig. 322 on p. 263, of dividing lines up into equal parts. The proof of Theorem 38, which depends on this idea, should only be explained for special cases, *e.g.* $p:q=5:3$. Pages 268, 269 introduce a second idea, that of expressing equal ratios by equal fractions, and this can be applied to the riders which follow. It is assumed that the pupil has some previous acquaintance with ratio as taught in the Arithmetic course; algebraic notation, which usually causes difficulty, has been avoided.

(*d*) Defer the Extensions of Pythagoras' theorem, which are difficult, to a later part of the course.

Such changes make for increased variety and interest, and the early introduction of similarity is particularly recommended.

It may be added that the book conforms closely to most of the recommendations made in the Mathematical Association's *Second Report on the Teaching of Geometry.*

We acknowledge gratefully the help given by our colleagues, Messrs. R. F. Hobbs, M.A., M. Nelkon, B.Sc., A.K.C., and R. Porter, M.A., in reading the proofs and verifying the answers. Our thanks are due to the Cambridge Local Examinations Syndicate, the Delegates of the Oxford Local Examinations, the Senate of the University of London, the Joint Matriculation Board, the Oxford and Cambridge Schools Examination Board, and the Central Welsh Board for permission to use questions (distinguished in the text by the letters C, O, L, N, OC, W respectively) set in their examinations.

A. H. G. P.
H. E. P.

May 1938.

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LIST OF EXAMPLES DEALING WITH LOCI

In addition to the special exercises (**9, 21, 22**) on Loci, there are the following examples :—

Page	*Exercise*	*No.*	*Page*	*Exercise*	*No.*
124	**11 b**	23, 24	244	**20 c**	22
153	**Misc. II**	2, 18, 26	258	**Misc. III**	28
164	**14 a**	8, 16, 22	279	**23 b**	6, 14
169	**14 c**	10	282	**23 c**	17
202	**17 c**	6	293	**24 b**	18
210	**18 a**	5	316	**26 a**	22
212	**18 b**	5, 7, 8, 22	318	**26 b**	20-22
223	**19 a**	4-6	320	**26 c**	7

LIST OF EXAMPLES
DEALING WITH THREE DIMENSIONS

Page	*Exercise*	*No.*	*Page*	*Exercise*	*No.*
32	**3 b**	22-26	138	**12 c**	3, 6, 9, 14, 20 23, 24, 27, 30
67	**5 b**	13			
73	**6 a**	17-19	149	**13 b**	13
75	**6 b**	14-19	167	**14 b**	10-13
77	**Misc. I**	5, 10	169	**14 c**	6, 12, 17, 18
83	**7 b**	12, 13	214	**18 b**	21-23
84	**7 c**	11, 12	226	**19 b**	8
90	**8**	10	262	**Misc. III**	62
97	**9 a**	19, 21	287	**24 a**	4, 31
102	**9 b**	1-9, 26, 27	291	**24 b**	1, 2, 32-34
134	**12 a**	10	317	**26 b**	7
136	**12 b**	5, 9-12	324	**Misc. IV**	21

TO THE PUPIL

ADVICE ON TACKLING RIDERS

(i) As you draw *each bit* of your figure, mark on it what is given, using one or two coloured pencils if you like, and jot down any obvious conclusions.

(ii) Next think about the figure of the theorem on which your rider is based. If, for example, it is 'angles in the same segment,' look in your figure for angles in the same segment, and write down the equal angles.

(iii) Look at what you have to prove, and see if you can work backwards from it.

(iv) Try to connect the facts you have jotted down, and to make a clear proof out of them. If you cannot make a proof, make a neat copy of the facts and the reasons for them. If any of them form part of the proof, you ought to get some marks for them.

(v) Beware of making constructions. They are hardly ever wanted in easy riders, and they tend to make the figure look so complicated that you may not be able to see the wood for the trees.

(vi) Never show up a blank sheet of paper. Hints (i), (ii), and (iii) above are bound to enable you to write down something useful.

PART I

ELEMENTARY ASSUMPTIONS

In Fig. 1, O is the **vertex,** OA and OB are the **arms** of ∠AOB.

∠AOB and ∠BOC are **adjacent** angles (they lie next to one another, *ad* to, *jaceo* I lie).

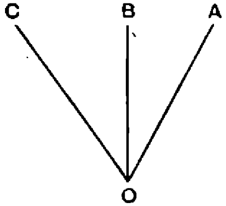


FIG. 1

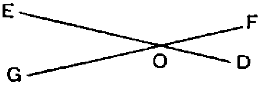


FIG. 2

In Fig. 2, ∠DOF and ∠EOG are **vertically opposite** angles; they have the same vertex O, and are on opposite sides of it.

An **acute** (sharp) angle is less than a right angle;

An **obtuse** (blunt) angle is between one and two right angles;

A **reflex** (bent back) angle is between two and four right angles. In Fig. 3, ∠PQR would refer to the obtuse angle; if

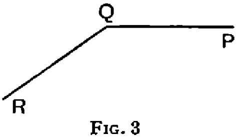


FIG. 3

we mean the reflex angle we must write *reflex* ∠PQR.

Two angles are **complementary** if their sum is one right angle; they are **supplementary** if their sum is two right angles.

A

ASSUMPTION 1 (Properties of Straight Lines)

If two straight lines intersect, any two adjacent angles are supplementary, any two vertically opposite angles are equal.

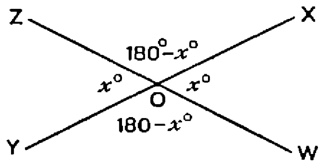


FIG. 4

References.

$\angle$XOW+$\angle$XOZ=180° (*adj.,* ZOW *a st. line*).

$\angle$XOZ=$\angle$YOW (*vert. opp.*).

Corollary.

The sum of the angles round a point is equal to four right angles. For the sum of the angles round O in Fig. 4

$$=(180°-x°)+x°+(180°-x°)+x°=360°.$$

ASSUMPTION 2 (Tests for a Straight Line)

(*a*) **If the adjacent angles** FOD **and** DOG (Fig. below) **are supplementary, then** FOG **is a straight line.**

(*b*) **If** DOE **is a straight line** (Fig. below) **and the angles** DOF **and** EOG **are equal, then** FOG **is a straight line.**

References.

(*a*) $\angle$FOD+$\angle$DOG=180°.

But these are adj.,

∴ FOG is a st. line.

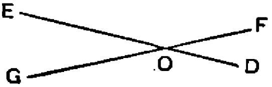


FIG. 2 (repeated)

(*b*) $\angle$DOF=$\angle$EOG.

But DOE *is a st. line,*

∴ FOG is a st. line.

In Fig. 5, ∠ABC and ∠BCD are **alternate** angles.
In Fig. 6, ∠EFG and ∠FHK are **corresponding** angles.

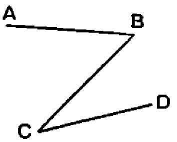


Fig. 5

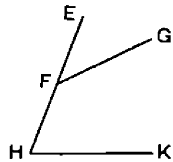


Fig. 6

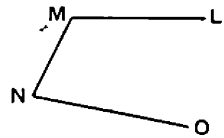


Fig. 7

In Fig. 7, ∠LMN and ∠MNO are **conjoined** angles.

ASSUMPTION 3 (Properties of Parallels[1])

If a straight line crosses two parallel straight lines,
alternate angles are equal,
corresponding angles are equal,
conjoined angles are supplementary.

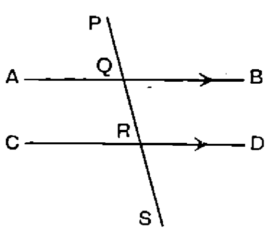


Fig. 8

References.

∠BQR = ∠QRC (*alt.*, AB ∥ CD).
∠PQA = ∠QRC (*corr.*, AB ∥ CD).
∠AQR + ∠QRC = 180° (*conj.*, AB ∥ CD).

The straight line PQRS which crosses the parallels (Fig. 8) is called a **transversal** (*trans* across, *verto* I turn).

[1] Par-allel, beside one another.

ASSUMPTION 4 (Tests for Parallels)

(*a*) **If a straight line crosses two other straight lines so as to make two alternate angles equal, the two straight lines are parallel.**

(*b*) **If a straight line crosses two other straight lines so as to make two corresponding angles equal, the two straight lines are parallel.**

(*c*) **If a straight line crosses two other straight lines so as to make two conjoined angles supplementary, the two straight lines are parallel.**

References.

(*a*) (See Fig. 5.) $\angle ABC = \angle BCD$.

But these are alt.,

$\therefore AB \parallel CD$.

Refer to (*b*) and (*c*) in a similar manner.

To **bisect** means to divide into two equal parts (bi-sect, from *bis* twice, *seco* I cut). Thus, if $\angle AOB = \angle BOC$ in Fig. 1, p. 1, OB 'bisects' $\angle AOC$; if O is the mid-point of XY in Fig. 4, p. 2, ZW 'bisects' XY at O.

N is the foot of the perpendicular from A to BC means that N is the point on BC such that $\angle ANB$ is a right angle. $\angle ANB + \angle ANC = 2$ right angles (adj., BNC a st. line), so $\angle ANC$ must also be a right angle.

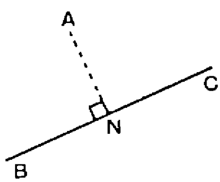


FIG. 9

The **perpendicular bisector** of BC is the straight line which is perpendicular to BC and also bisects BC. Thus, if $BN = NC$ in Fig. 9, AN is the 'perpendicular bisector' of BC.

To **produce** BC means to continue the straight line BC beyond C, whereas to **produce** CB means to continue CB beyond B.

AB is parallel to DC means that to go from A to B takes you in the same direction as to go from D to C. In Fig. 8, p. 3, exactly the opposite occurs, and we should say that **AB is parallel to CD.**

The **data** are the things which are given in a problem.

How to Mark Figures

The **same mark** is put on two straight lines or two angles if they are **equal.** See Fig. 10 below, where ∠COY = ∠YOB.

The special mark for a **right angle** is shown in Fig. 9, ∠ANB.

Arrows denote **parallels**; see Fig. 8, p. 3.

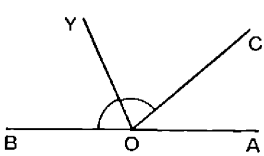
· When you start a question, draw a figure, and then mark the data on it by means of the code just explained.

EXAMPLES 1a

Draw a separate figure for each question, accurate enough to make the data clear. If in difficulty, draw an accurate figure.

1. In Fig. 10, if ∠AOC = 40°, calculate ∠BOY.

2. In Fig. 10, if ∠AOC = 40° and OX is drawn to bisect ∠AOC, prove that ∠XOY = 90°.



AOB is a straight line

FIG. 10

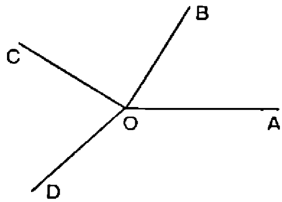


FIG. 11

3. In Fig. 11, if AOC is a straight line and ∠AOB = ∠COD, what can be said about OB and OD, and why ?

4. In Fig. 11, if ∠AOB = 60°, BOC = 110°, ∠COD = 60°, calculate ∠DOA. Are AOC and BOD straight lines ? Give reasons.

5. In Fig. 11, if ∠AOB = 60° and ∠AOC = ∠BOD = 120°, which three points lie in a straight line, and why ?

6. In Fig. 11, if ∠AOC = ∠BOD, prove that ∠AOB = ∠COD.

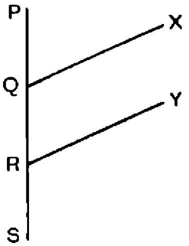
7. In Fig. 11, if ∠AOB = ∠BOC = ∠COD and ∠DOA = 111°, calculate ∠AOB.

8. In Fig. 12, if ∠RQX = 115° and RY is parallel to QX, calculate ∠QRY.

9. In Fig. 12, if ∠PQX = 70° and RY is parallel to QX, calculate ∠SRY.

10. In Fig. 12, if ∠PQX = 60°, ∠QRY = 59°, and if QX and RY are produced in both directions, on which side of PQRS will they meet ?

11. In Fig. 13, if ∠A = 63°, ∠B = 116°, ∠C = 64°, and ∠D = 117°, which lines are parallel, and why ?



PQRS is a straight line

FIG. 12

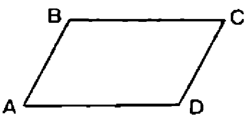


FIG. 13

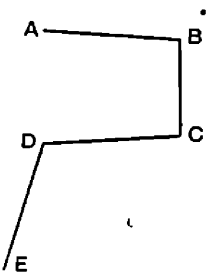


FIG. 14

12. In Fig. 14, if ∠ABC = 82°, ∠BCD = 98°, and ∠CDE = 108°, which lines are parallel, and why ?

13. In Fig. 14, if both pairs of lines are parallel and ∠ABC = 76°, calculate ∠CDE.

14. In Fig. 14, if ∠ABC = ∠BCD = ∠CDE, which lines are parallel, and why ?

EXAMPLES 1b

1. In Fig. 15, mark two pairs of equal angles. What fact follows about the sum of the angles of △ABC ?

2. In Fig. 15, if AC bisects ∠BAY, prove that two of the angles of △ABC are equal.

3. In Fig. 16, prove that ∠ACD = ∠A + ∠B.

4. In Fig. 16, if ∠BCA = ∠ACE = ∠ECD, what can be said about the angles of △ABC ?

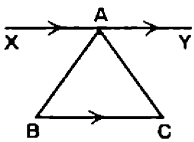
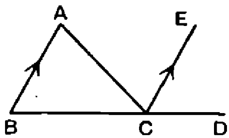


FIG. 15



BCD is a straight line

FIG. 16

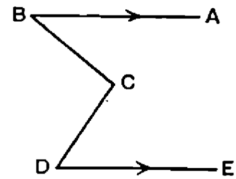


FIG. 17

5. In Fig. 17, by drawing a line CX through C parallel to AB,[1] prove that ∠BCD = ∠ABC + ∠CDE.

6. In Fig. 18, by joining BY and producing it, prove that ∠ABC = ∠XYZ.

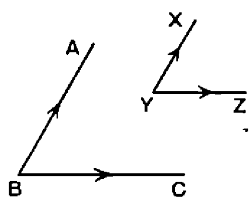
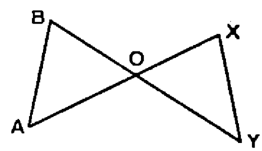


FIG. 18



AOX, BOY are straight lines

FIG. 19

7. Prove the same result as in No. 6 by producing XY to meet BC. What similar construction could be used for the same purpose ?

8. In Fig. 19, if ∠ABO = ∠OYX, prove that ∠OXY = ∠OAB.

9. In Fig. 11, if OB bisects ∠AOC and OD bisects the reflex angle AOC, prove that BOD is a straight line.

10. In Fig. 11, if ∠AOB : ∠BOC : ∠COD : ∠DOA = 1 : 2 : 2 : 4, calculate ∠AOB.

11. In Fig. 13, if both pairs of straight lines are parallel, prove that (i) opposite angles are equal, (ii) the sum of the four angles is 360°.

12. With the data of No. 11, prove that, if AC bisects ∠A, then AC also bisects ∠C.

[1] Remember to draw CX in the same direction as AB, *i.e.* to the left of C. See p. 5.

EXAMPLES 1c

1. In Fig. 13, if both pairs of straight lines are parallel and **AC** is joined, prove that the angles of $\triangle$s **ABC**, **CDA** are respectively equal.

2. Prove that the bisectors of two vertically opposite angles lie in the same straight line.

3. If one arm of an angle is produced backwards, prove that the bisectors of the two angles in the figure are perpendicular.

4. **ABCD** is a four-sided figure having **AB** parallel to **DC**. Prove that $\angle A - \angle C = \angle B - \angle D$.

ANGLE-SUM

An **acute-angled triangle** has all three angles acute. An **obtuse-angled triangle** has one obtuse angle; a **right-angled triangle** has one right angle.

A plane figure bounded by straight lines is called a **polygon** (many sides). A polygon with

3	sides	is called	a **triangle,**
4	,,	,,	a **quadrilateral,**
5	,,	,,	a **pentagon,**
6	,,	,,	a **hexagon,**
8	,,	,,	an **octagon,**
10	,,	,,	a **decagon.**

A polygon is **convex** if none of its angles are reflex.

A polygon is **regular** if all its sides are equal and all its angles are equal.

A corner of a polygon is called a **vertex** (plural **vertices**).

In naming a polygon, go round the vertices in order. In Fig. 20 the left-hand quadrilateral is ABCD, the right-hand quadrilateral is ABDC.

A **diagonal of a quadrilateral** is a straight line joining two opposite vertices. In Fig. 20 the diagonals are shown by dotted lines.

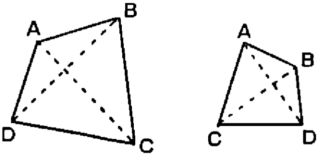


FIG. 20

If a side of a polygon is produced, the angle so formed is an **exterior angle.** In contrast, the ordinary angles are sometimes called **interior angles.**

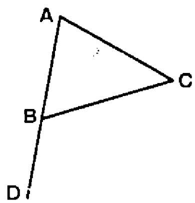


FIG. 21

If one side AB of a triangle ABC is produced to D, the exterior angle at B is ∠CBD ; ∠A and ∠C are then called the **interior opposite angles.**

By the **external bisector** of an angle of a polygon is meant the bisector of the exterior angle. In contrast, the bisector of the interior angle is sometimes called the **internal bisector.**

THEOREM 1

The sum of the three angles of a triangle is equal to two right angles.

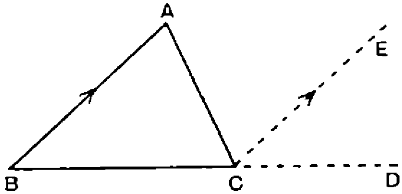


FIG. 22

Given a triangle ABC.

To prove that $\angle A + \angle B + \angle ACB = 2$ right angles.

Construction. Produce BC to D, and suppose CE $\parallel$ BA.

Proof. $\angle A = \angle ACE$ (alt., BA $\parallel$ CE),

and $\angle B = \angle ECD$ (corr., BA $\parallel$ CE);

$\therefore$ $\angle A + \angle B = \angle ACD$.

Adding $\angle ACB$ to both sides,

$\angle A + \angle B + \angle ACB = \angle ACD + \angle ACB$

$= 2$ right angles (adj., BCD a st. line).

Corollary.

If one side of a triangle is produced, the exterior angle is equal to the sum of the two interior opposite angles.

For it has been proved that $\angle ACD = \angle A + \angle B$.

References.

$\angle A + \angle B + \angle ACB = 180°$ ($\angle$*s of* $\triangle$).

$\angle ACD = \angle A + \angle B$ (*ext.=sum of int. opp.*).

Note. The corollary to the theorem is of greater use in solving riders than the theorem itself.

THEOREM 2

The sum of the interior angles of a polygon with n sides is equal to $2n-4$ right angles.

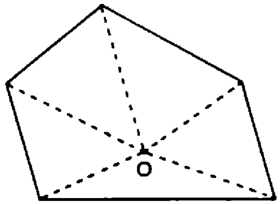


FIG. 23

Given a convex polygon with n sides.

Construction. Let O be any point inside the polygon, and join it to each of the vertices.

Proof. The polygon has n sides,

$\therefore$ there are n triangles in the figure.

The sum of the angles of one triangle is 2 right angles,

$\therefore$ the sum of the angles of all the triangles is $2n$ right angles.

These angles make up the interior angles of the polygon together with the angles round O.

But the sum of the angles round O is 4 right angles;

$\therefore$ the sum of the interior angles of the polygon is $2n-4$ right angles.

Corollary.

The sum of the interior angles of a quadrilateral is equal to four right angles.

THEOREM 3

If all the sides of a convex polygon are produced in order, the sum of the exterior angles is equal to four right angles.

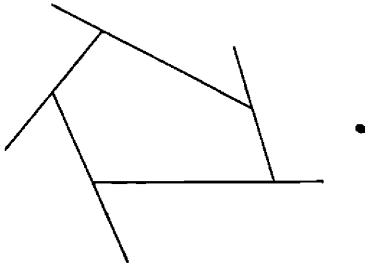


FIG. 24

Proof. The polygon has n vertices.

At each vertex, int. $\angle$+ext. $\angle$=2 right angles ;

$\therefore$ the sum of all the int. and ext. $\angle$s=$2n$ right angles.

But the sum of all the int. $\angle$s=$2n-4$ right angles ;

$\therefore$ the sum of all the ext. $\angle$s=4 right angles.

Theorem 3 is frequently more useful than Theorem 2 in numerical examples. Notice, too, that 4 is easier to remember than $2n-4$. The worked example which follows illustrates the value of this theorem.

Worked Example

If the interior angle of a regular polygon is 140°, *calculate the number of sides.*

Since each interior angle=140°,

$\therefore$ each exterior angle=40°.

But the sum of all the exterior angles=360° ;

$$\therefore \text{ the number of exterior angles} = \frac{360}{40} = 9.$$

$\therefore$ the polygon has 9 sides.

EXAMPLES 2 a

(*Theorem* 1)

1. If the angles of a triangle are equal, how large is each angle ?

2. In △ABC, ∠A=73°, ∠B=54° ; calculate ∠C.

3. In △PQR, ∠P=76° and ∠Q=∠R ; calculate ∠Q.

4. ABC, XYZ are two triangles in which ∠B=∠Y, ∠C=∠Z. Prove that ∠A=∠X.

5. ABC is a triangle and BC is produced to D. If ∠ACD=113° and ∠A=64°, calculate ∠B.

6. XYZ is a triangle ; YZ is produced to P, and ZY is produced to Q. If ∠XYQ=125° and ∠XZP=142°, calculate ∠X.

7. Prove that the largest angle of a right-angled triangle is equal to the sum of the other two angles.

8. In Fig. 25, if ∠OAB=∠OYX, prove that ∠OBA=∠OXY.

9. In Fig. 26, if ∠A=∠ABC=60°, prove that BD is perpendicular to AC.

10. In Fig. 26, if BD is perpendicular to AC, prove that ∠A=∠C.

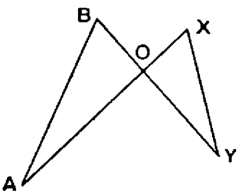


FIG. 25

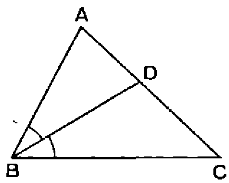


FIG. 26

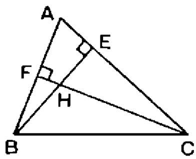


FIG. 27

11. In Fig. 26, if ∠ABC=80° and ∠A=62°, calculate ∠ADB.

12. In △LMN, ∠M=130°, ∠N=30°, and the bisectors of ∠M and ∠N meet at I. Calculate ∠MIN.

13. In Fig. 27, prove that ∠ECH=∠FBH.

(*Theorems* 2 *and* 3)

The polygons should be assumed convex.

14. Three angles of a quadrilateral are equal, and the fourth angle is 105° ; how large are the equal angles ?

15. How large is (i) an exterior, (ii) an interior angle of a regular pentagon ?

Answer the same questions for the regular hexagon and regular octagon.

16. What is the sum of all the angles, exterior and interior, of a polygon with n sides ?

17. If each exterior angle of a polygon is 40°, how many sides has the polygon ?

18. If each interior angle of a polygon is 144°, how many sides has the polygon ? (Use the method of p. 12.)

19. If the sum of four angles of an octagon is 5 right angles, what is the sum of the other four angles ?

20. If the sides of a triangle are produced in order, what is the sum of the exterior angles so formed ?

21. ABCDE is a regular pentagon, and AB produced meets DC produced at T ; calculate $\angle$BTC.

22. The sides of a quadrilateral are produced in order. If one pair of exterior angles are supplementary, prove that the remaining pair are also supplementary.

EXAMPLES 2 b

(*Theorem* 1)

1. By drawing a diagonal of a quadrilateral, prove that the sum of its angles is 4 right angles.

2. By joining one vertex of a hexagon to the remaining vertices, prove that the sum of the angles of a hexagon is 8 right angles.

3. Two equal angles of a triangle are each four times the third angle. Calculate the third angle.

4. If the angles of a triangle are in the ratio 1 : 2 : 3, prove that the triangle is right-angled.

5. In Fig. 27, prove that $\angle$BHF = $\angle$A.

6. In $\triangle$ABC, D is a point on BC such that $\angle$ADC = $\angle$BAC ; prove that $\angle$DAC = $\angle$B.

7. In Fig. 26, if the straight line through **D** parallel to **CB** meets **AB** at **E**, and if ∠**A**=80°, ∠**C**=40°, calculate ∠**BDE**.

8. In Fig. 28, if ∠**XYZ**=∠**XZY**, ∠**X**=36°, **YW** bisects ∠**XYZ**, and **ZW** is parallel to **YX**, prove that ∠**W**=∠**X**.

9. If a diagonal of a quadrilateral bisects two of the angles, prove that the other two angles are equal.

10. In Fig. 26, if ∠**A**=∠**ABC**, prove that ∠**BDC**=3∠**DBC**.

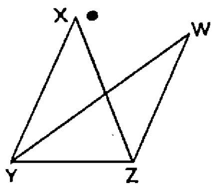


Fig. 28

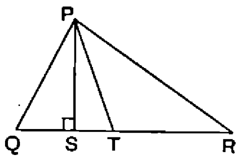


Fig. 29

11. In Fig. 29, if **PT** bisects ∠**QPR**, ∠**Q**=74°, and ∠**R**=42°, calculate ∠**SPT**.

12. In Fig. 26, if **AC** is produced to **E**, prove that ∠**BCE**−∠**BDC** =∠**BDC**−∠**BAD**.

13. In Fig. 30, **LMN** is a triangle, and **O** a point on **MN** such that ∠**OLM**=∠**M**, ∠**OLN**=∠**N**. Prove that ∠**MLN**=90°.

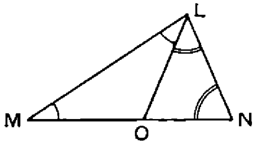


Fig. 30

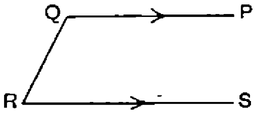


Fig. 31

14. In Fig. 31, if the bisectors of ∠**Q** and ∠**R** meet at **O**, prove that ∠**QOR**=90°.

15. If **O** is any point inside △**PQR**, prove, by joining **PO** and producing it, that ∠**QOR**=∠**OQP**+∠**QPR**+∠**PRO**. State two similar results.

16. **ABCD** is a quadrilateral having **AD** parallel to **BC**. If the diagonals intersect at **O**, prove that ∠**COD**=∠**ACB**+∠**ADB**.

(*Theorems* 2 *and* 3)

The polygons should be assumed convex.

17. If the numbers of degrees in the angles of an octagon are $x+65$, $2x+20$, $2x+25$, $3x-20$, $4x-60$, $3x-5$, $3x$, $3x+5$, find the value of x.

18. A floor is covered with equal tiles in the shape of a regular hexagon. How many tiles meet at each vertex ? Illustrate by a sketch.

19. Prove that the sum of three of the angles of a quadrilateral is equal to the reflex angle at the fourth vertex.

20. In what kind of polygon is each interior angle three times each exterior angle ?

21. In what kind of polygon is the sum of the interior angles half as large again as the sum of the exterior angles ?

22. If the sum of the interior angles of a polygon is 1440°, calculate the number of sides.

EXAMPLES 2 c

1. The interior angle of a regular polygon is 156°. How many sides has the polygon ? (L)

2. ABC is a triangle right-angled at A, and D is the foot of the perpendicular from A to BC. Prove that $\triangle$s ABD, CAD, and CBA are equiangular.

3. Which of the following angles can be the interior angle of a regular polygon : 120°, 130°, 140°, 150° ? State the number of sides in each possible case. (OC)

4. In $\triangle LMN$, $\angle L + \angle M = 130°$ and $\angle M + \angle N = 115°$; calculate the three angles of the triangle.

5. Two angles of a convex polygon are right angles, and each of the other angles is 120°. How many sides has the polygon ? (OC)

6. In Fig. 32, if BC and AD produced meet at Y and $\angle ABC = \angle ADC$, prove that $\angle X = \angle Y$.

7. In Fig. 32, if $\angle$**A**$=64°$, $\angle$**D**$=57°$, and $\angle$**XBC**$=81°$, calculate $\angle$**XCB**.

8. In Fig. 32, if $\angle$**XCB**$=\angle$**A**, prove that $\angle$**XBC**$=\angle$**D**.

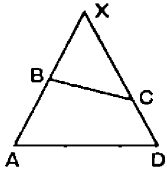


FIG. 32

9. In a pentagon **ABCDE** the angle **D** is double the angle **B**, and the angles **A**, **C**, and **E** are each half the sum of the angles **B** and **D**. Find all the angles of the polygon. (L)

10. If both pairs of opposite angles of a quadrilateral are equal, prove that both pairs of opposite sides are parallel.

11. A convex pentagon **ABCDE** has $\angle$**ABC**$=110°$ and the sides **AB** and **CD** perpendicular. Find the numbers of degrees in each of the other four interior angles of the pentagon, given that $\angle$**CDE** and $\angle$**EAB** are each twice $\angle$**AED**. (N)

12. In Fig. 29, if $\angle$**QPS**$=\angle$**SPT**$=\angle$**TPR**, prove that $\angle$**PTR** $-\angle$**PRT**$=\angle$**QPR**.

13. In Fig. 28, if **YW** bisects $\angle$**XYZ** and **ZW** bisects the angle between **YZ** produced and **ZX**, prove that $\angle$**YXZ**$=2\angle$**YWZ**.

14. In a pentagon **ABCDE**, $\angle$**A**$=120°$, $\angle$**E**$=140°$, **AB** is parallel to **DC**, and **BC** is parallel to **AE**; calculate $\angle$**B** and $\angle$**D**.

15. If the bisectors of the angles **B** and **C** of $\triangle$**ABC** meet at **I**, prove that $\angle$**BIC**$=90°+\frac{1}{2}\angle$**A**.

16. The interior angles of a convex pentagon taken in order are in the proportion 4 : 8 : 6 : 4 : 5. Prove that two pairs of sides are parallel.

If each interior angle of a regular polygon is p times as large as each exterior angle, prove that the number of sides is $2(p+1)$. (N)

17. Two regular polygons are such that the number of sides in one is double that in the other and an angle of the first is $1\frac{1}{2}$ times that of the second; find the number of sides in each. (OC)

18. The bisectors of the exterior angles at **B** and **C** of $\triangle$**ABC** meet at **D**; prove that $\angle$**BAC**$+2\angle$**BDC**$=180°$.

19. The side **BC** of a triangle **ABC** is produced to any point **D**, and the interior bisector of $\angle$**BAC** meets **BC** at **L**. Prove that $\angle$**ACD**$+\angle$**ABC**$=2\angle$**ALC**. (N)

20. From the accompanying figure, find a simple expression for x in terms of a, b, and c. (N)

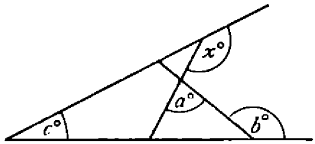


Fig. 33

21. ABCDE is a five-sided polygon all of whose angles are obtuse. The sides are produced so that AB and DC, BC and ED, CD and AE, DE and BA, EA and CB meet in P, Q, R, S, T respectively. Prove that the sum of the acute angles at P, Q, R, S, T is two right angles. (O)

22. If 4° is the difference between an exterior angle of a regular polygon of n sides and an exterior angle of a regular polygon of $n+1$ sides, find the value of n.

CONGRUENCE AND ISOSCELES TRIANGLES

The three sides and three angles of a triangle are called its six **parts.**

If the six parts of one triangle are respectively equal to the six corresponding parts of another triangle, the two triangles are said to be **congruent.**

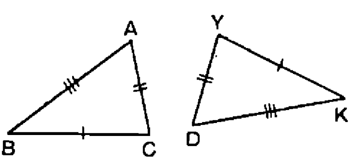


Fig. 34

If two congruent triangles ABC, DKY are such that

BC = KY, CA = YD, AB = DK,

then the part of $\triangle$DKY corresponding to $\angle$A in $\triangle$ABC is $\angle$D, because these angles are opposite the equal sides BC, KY; hence

$\angle$A = $\angle$D, and similarly $\angle$B = $\angle$K, $\angle$C = $\angle$Y.

The symbol ≡ is used for 'is congruent to'; thus

$\triangle$ABC ≡ $\triangle$DKY.

When we wish to make the correspondence of parts clear, we shall write

$$\triangle s \begin{matrix} \mathsf{ABC} \\ \mathsf{DKY} \end{matrix} \text{ are congruent,}$$

the letters denoting the vertices of the second triangle being underneath those denoting the corresponding vertices of the first triangle.

The word **equidistant** means 'at equal distances.' Thus 'P is equidistant from A and B' means that PA = PB.

ASSUMPTION 5 (First Test for Congruence)

If two sides and the included angle of one triangle are respectively equal to two sides and the included angle of another triangle, then the two triangles are congruent.

Reference. $\triangle s \begin{matrix} \mathsf{ABC} \\ \mathsf{DKY} \end{matrix}$ are congruent (*SAS*).

Here *S* means 'side,' *A* means 'angle,' and the order of the letters indicates that the angle referred to is the angle *included*, *i.e.* 'enclosed,' by the two sides.

THEOREM 4 (Second Test for Congruence)

(For proof, see Appendix, p. 328.)

If two angles and a side of one triangle are respectively equal to two angles and the corresponding side of another triangle, then the two triangles are congruent.

Reference. $\triangle s \begin{matrix} \mathsf{ABC} \\ \mathsf{DKY} \end{matrix}$ are congruent (*AA corr. S*).

Note on Theorem 4

It is important to realise that the triangles will only be congruent if the side of the second triangle corresponds to the equal side of the first triangle. In Fig. 35, where ∠A=∠D, ∠C=∠Y, and BC=DY, BC and DY are not corresponding sides, for BC is opposite ∠A, and DY is not opposite ∠D.

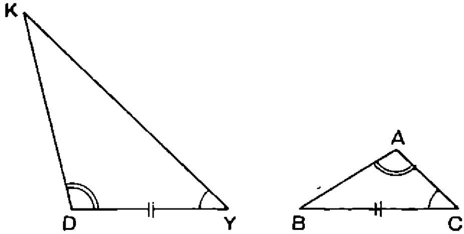


FIG. 35

Example showing how to write out the Proof that two Triangles are Congruent

If ABCDE *is a regular pentagon, prove that* AC=AD.

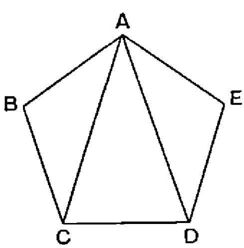


FIG. 36

In the △s ABC, AED,

AB=AE (sides of reg. pentagon),
BC=ED (,, ,, ,,),
∠ABC=∠AED (angles ,,);

∴ △s $\frac{\text{ABC}}{\text{AED}}$ are congruent (SAS).

∴ AC=AD.

Notes. The parts of △ABC are written on the left, those of △AED on the right. The letters need not be made to correspond throughout, but correspondence is essential in the statement that the triangles are congruent.

A triangle which has two sides equal is said to be **isosceles** (*iso-sceles,* equal legs). The third side is called **the base,** and the opposite vertex is called **the vertex.** The angle at the vertex is the **vertical angle,** and the angles opposite the equal sides are the **base angles.**

A triangle which has all three sides equal is said to be **equilateral** (*aequus* equal, *latus* side).

The rather similar word 'equiangular' is used in quite a different sense, and only with reference to a pair of triangles. Two triangles are said to be **equiangular** when the angles of the one triangle are respectively equal to the angles of the other triangle.

Converses

If Theorems 5a and 5b which follow are examined, it will be noticed that what is given in Theorem 5a is the same as what has to be proved in Theorem 5b, and vice versa. A pair of theorems in which this occurs are said to be **converses** of each other. Where a theorem and its converse are placed next to each other in this book, they are distinguished by the letters *a* and *b*, as here.

The converse of Theorem 5a has been proved to be true, but not all theorems have true converses. For instance, 'vertically opposite angles are equal,' but equal angles are not necessarily vertically opposite.

THEOREM 5 a (Isosceles Triangle Theorem)

If two sides of a triangle are equal, then the angles opposite those sides are equal.

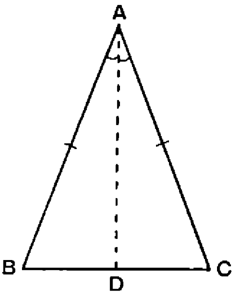


FIG. 37

Given a triangle ABC in which AB=AC.

To prove that ∠C=∠B.

Construction. Suppose that the bisector of ∠BAC meets BC at D.

Proof. In the △s ABD, ACD,

AB=AC (given),

∠BAD=∠CAD (constr.),

AD=AD ;

∴ △s ABD, ACD are congruent (SAS).

∴ ∠B=∠C.

Reference. *AB=AC,*

∴ ∠C=∠B.

THEOREM 5 b

(Converse of the Isosceles Triangle Theorem)

If two angles of a triangle are equal, then the sides opposite those angles are equal.

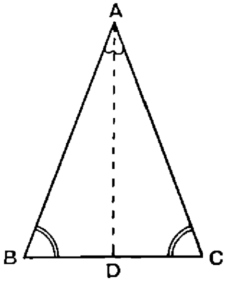


FIG. 38

Given a triangle ABC in which $\angle B = \angle C$.

To prove that $AC = AB$.

Construction. Suppose that the bisector of $\angle BAC$ meets BC at D.

Proof. In the Δs ABD, ACD,

$\angle B = \angle C$ (given),

$\angle BAD = \angle CAD$ (constr.),

$AD = AD$;

$\therefore$ Δs $\begin{matrix} ABD \\ ACD \end{matrix}$ are congruent (AA corr. S).

$\therefore$ $AB = AC$.

Reference. $\angle C = \angle B$,

$\therefore$ $AB = AC$.

THEOREM 6 (Third Test for Congruence)

(For proof, see Appendix, p. 330.)

If the three sides of one triangle are respectively equal to the three sides of another triangle, then the two triangles are congruent.

Reference. $\triangle$s $\begin{matrix}ABC\\DKY\end{matrix}$ are congruent (*SSS*).

In a right-angled triangle, the side opposite the right angle is known as the **hypotenuse.**

THEOREM 7 (Fourth Test for Congruence)

(For proof, see Appendix, p. 332.)

If two right-angled triangles have the hypotenuse and another side of the one triangle respectively equal to the hypotenuse and another side of the other triangle, then the two triangles are congruent.

Reference. $\triangle$s $\begin{matrix}ABC\\DKY\end{matrix}$ are congruent (*RHS*).

RHS stands for 'Right angle, Hypotenuse, and one other Side.'

Example showing how to use Theorem

If ABCDE *is a pentagon in which* ∠B=∠E=90°, BC=DE, *and* ∠ACD=∠ADC, *prove that* AB=AE.

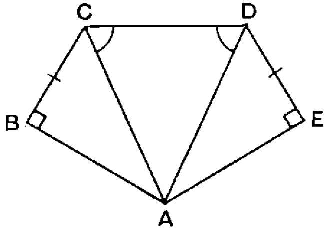


Fig. 39

In the right-angled △s ABC, AED,

∠ABC and ∠AED are the right angles (given),

BC=ED (given),

AC=AD (∵ ∠ADC=∠ACD);

∴ △s $\begin{matrix}\text{ABC}\\\text{AED}\end{matrix}$ are congruent (RHS).

∴ AB=AE.

Solid Geometry

Fig. 40 shows a **tetrahedron** or pyramid on a triangular base, a figure which may be compared with the triangle in plane geometry; it has four faces, each of which is a triangle. If BCD is regarded as the base, A is called the vertex. A tetrahedron is **regular** if all its edges are equal.

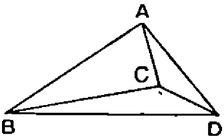


Fig. 40

A straight line AM, which meets a plane at M, is said to be **perpendicular to the plane** if it is perpendicular to each of *two* straight lines passing through M and lying in the plane.

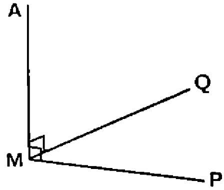


Fig. 41

From this definition it can be proved that a perpendicular to a plane is perpendicular to *all* straight lines passing through its foot and lying in the plane. This fact may be assumed in riders.

Conversely, to prove that **AM** is perpendicular to the plane, it is only necessary to prove that it is perpendicular to *two* straight lines through **M** and lying in the plane.

Coplanar means 'lying in the same plane.'

EXAMPLES 3 a

(*Congruence, SAS and AA corr. S*)

1. In Fig. 42, if **AB**=**DC** and ∠**ABC**=∠**DCB**, prove that **AC**=**DB** and state two other results.

2. In Fig. 42, if **OA**=**OD** and **OB**=**OC**, prove that ∠**A**=∠**D** and state two other results.

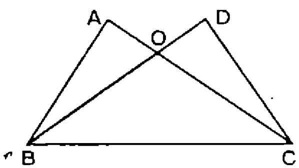


FIG. 42

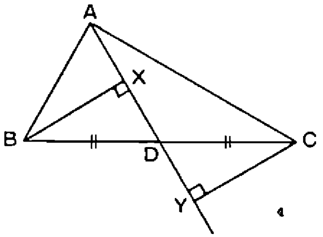


FIG. 43

3. In Fig. 43, prove that **BX**=**CY**.

4. In Fig. 44, if **AC** bisects the angles at **A** and **C**, prove that **AB**=**AD**, **CB**=**CD**.

5. From the result of No. 4 deduce that, if **BD** cuts **AC** at **O**, ∠**AOB**=∠**AOD**. What fact about **AC** and **BD** follows ?

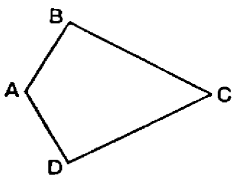


FIG. 44

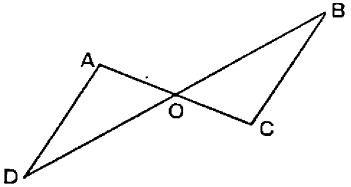


FIG. 45

6. In Fig. 45, if **OA**=**OC** and **OD**=**OB**, prove that **AD**=**BC**. Which angle is equal to ∠**ADO**, and what follows from this equality ?

7. In Fig. 46, if **AB** is parallel to **DC** and **AD** is parallel to **BC**, prove that **AB**=**DC** and **AD**=**BC**.

8. Deduce from No. 7 that **AC** and **BD** bisect each other.

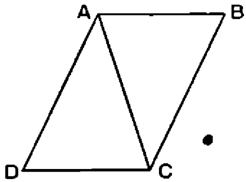


FIG. 46

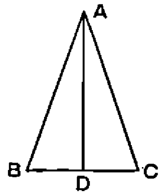


FIG. 47

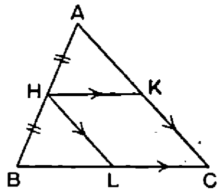


FIG. 48

9. In Fig. 47, if **AD** bisects ∠**BAC** and is perpendicular to **BC**, prove that **AB**=**AC**.

10. In Fig. 48, prove that **HK**=**BL**.

11. In Fig. 49, if **AB**=**AC**, prove that (i) **BE**=**CF**, (ii) **BF**=**CE**.

12. With the data of No. 6, prove that **AB** is parallel to **DC**.

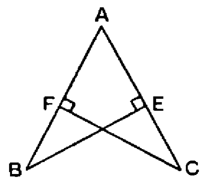


FIG. 49

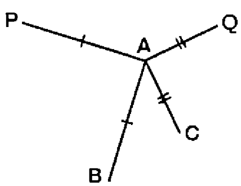


FIG. 50

13. In Fig. 50, if ∠**PAB**=∠**QAC**=90°, prove that **PC**=**BQ**.

14. If the diagonals of a quadrilateral bisect each other at right angles, prove that all the sides are equal.

(*Theorems* 5*a and b*)

15. The angle at the vertex of an isosceles triangle is 45° ; how large are the base angles ?

16. If one angle of an isosceles triangle is 70°, how large are the other angles ? (There are two possible sets of answers.)

17. In Fig. 51, if $\angle ACE = 43°$, calculate $\angle BAC$.

18. In Fig. 52, if $\angle ABD = 28°$, calculate $\angle ABC$.

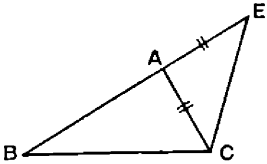


Fig. 51

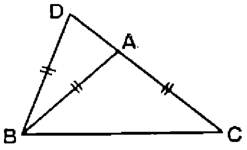


Fig. 52

19. If two isosceles triangles have their vertical angles equal, prove that all the four base angles are equal.

20. ABC is an acute-angled triangle and O a point inside it such that OA = OB = OC. If $\angle AOB = 140°$ and $\angle AOC = 130°$, calculate $\angle BOC$ and the angles of $\triangle ABC$.

21. In $\triangle ABC$, AB = AC and AB is produced to X. If $\angle CBX = 105°$, prove by calculation that $5\angle A = 2\angle C$.

22. In Fig. 53, prove that angles ABD and ADC are supplementary.

23. In $\triangle ABC$, AB = AC and the bisectors of $\angle B$ and $\angle C$ meet at I. Prove that IB = IC.

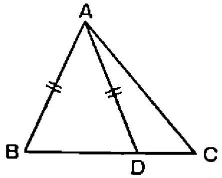


Fig. 53

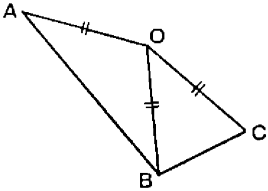


Fig. 54

24. In Fig. 54, prove that $\angle OAB + \angle OCB = \angle ABC$.

25. In Fig. 55, if CE is parallel to BA, prove that $\triangle ABC$ is equilateral.

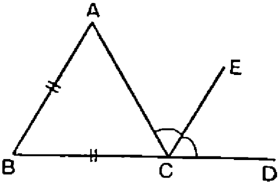


Fig. 55

26. In Fig. 44, if AB = AD and $\angle CBD = \angle CDB$, prove that $\angle ABC = \angle ADC$.

27. P is a point on the bisector of an angle ABC. If the straight line through P parallel to AB meets BC at Q, prove that $\triangle BPQ$ is isosceles.

28. In △ABC, AB=AC and AC is produced to D. Prove that ∠ABD−∠ACB=∠CBD.

29. ABCDEF is a regular hexagon. If AB and DC produced meet at X, prove that XB and XC are equal to the sides of the hexagon.

(*Congruence, SSS and RHS*)

30. In Fig. 44, if AB=AD and ∠B=∠D=90°, prove that AC bisects ∠BAD.

31. In Fig. 46, if the opposite sides of ABCD are equal, prove that they are also parallel.

32. In Fig. 56, if PQ=RS, prove that XQ is parallel to RY.

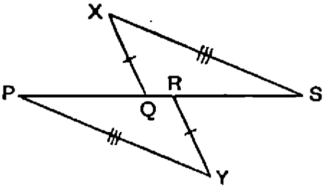


FIG. 56

33. In Fig. 47, if AD is perpendicular to BC and A is equidistant from B and C, prove that D is the mid-point of BC.

34. In Fig. 57, if AOX is a straight line, prove that BOY is a straight line.

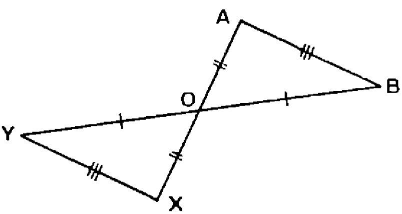


FIG. 57

35. Prove that the straight line joining the vertex of an isosceles triangle to the mid-point of the base is perpendicular to the base.

36. ABCD is a quadrilateral with all its angles right angles. M is the mid-point of AB, and P is any point on AD. If Q is the point on BC such that MQ=MP, prove that ∠AMP=∠BMQ.

37. In Fig. 49, join BC. If BE=CF, prove that △s BEC and CFB are congruent, and deduce that AB=AC.

38. In Fig. 58, prove two triangles congruent, and deduce that AB=AC.

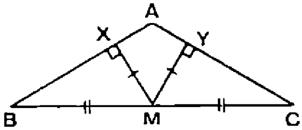


FIG. 58

39. ABCD and PQRS are two quadrilaterals in which AB=PQ, BC=QR, CD=RS, DA=SP, and the diagonals AC, PR are equal. Prove that the angles of ABCD are respectively equal to the angles of PQRS (*i.e.* prove the quadrilaterals congruent).

EXAMPLES 3 b

(*Theorems* 5*a and b*)

1. In Fig. 59, if AB=AC, prove that ∠CBD=∠A.

2. ABCDEFGH is a regular octagon; by calculating angles, prove that AC is perpendicular to AG.

3. In Fig. 60, if AB=AC and ∠A=36°, prove that AD=BC.

4. In Fig. 51, if the straight line through A parallel to EC meets BC at D, prove that AD bisects ∠BAC.

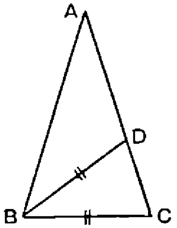


FIG. 59

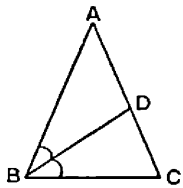


FIG. 60

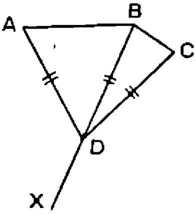


FIG. 61

5. In Fig. 61, prove that (i) ∠XDA=2∠DBA, (ii) reflex ∠ADC=2∠ABC.

6. In △ABC, AB=AC and the bisectors of ∠B and ∠C meet at I. If BC is produced to D, prove that ∠BIC=∠ACD.

7. In Fig. 54, if ∠ABC=90°, prove that AOC is a straight line.

8. In Fig. 62, prove that ∠BDC=2∠A. If AB is produced to E, prove also that ∠CBE=3∠A.

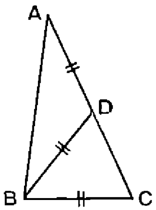


FIG. 62

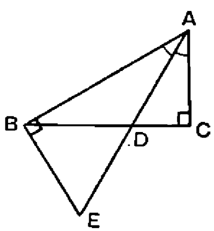


FIG. 63

9. In Fig. 63, prove that (i) △EBD is isosceles, (ii) ∠EBD =∠BAC.

(*Theorems 5a and b, and Congruence*)

10. ABC is an equilateral triangle, and X, Y, Z are points in BC, CA, AB respectively such that BX=CY=AZ. Prove that △XYZ is equilateral.

11. A straight line crosses two parallel straight lines at A and B, and M is the mid-point of AB. If another straight line through M crosses the parallels at P and Q, prove that M is the mid-point of PQ.

12. In Fig. 64, prove that (i) ∠CAX=∠YAB, (ii) CX=YB.

13. In Fig. 42, AC=BD and AB=CD. Prove that (i) ∠ABC =∠DCB, (ii) OB=OC.

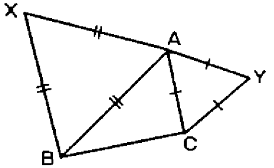


FIG. 64

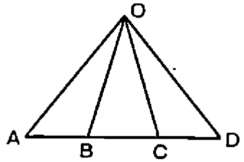


FIG. 65

14. In Fig. 65, if OB=OC and AC=BD, prove that OA=OD.

15. ABCD is a quadrilateral in which AB and DC are equal and parallel. Join AC, and prove that AD and BC are also equal and parallel.

16. ABC and XYZ are two congruent triangles. If D is the foot of the perpendicular from A to BC, and W the foot of the perpendicular from X to YZ, prove that AD=XW.

17. In Fig. 66, prove that (i) AE bisects ∠BDC, (ii) E is equidistant from B and C.

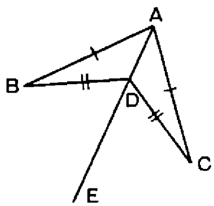


FIG. 66

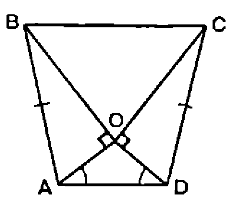


FIG. 67

18. In Fig. 67, prove two triangles congruent, and deduce that ∠OBC=∠OCB.

19. In $\triangle$ABC, AB=AC; the straight line through B perpendicular to BC meets CA produced at D, and the straight line through C perpendicular to BC meets BA produced at E. Prove that BD=CE and AD=AE.

20. In Fig. 68, prove that (i) BE=AD, (ii) $\triangle$CDE is isosceles.

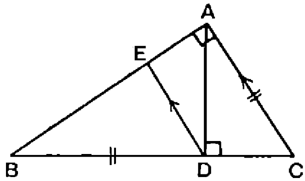


FIG. 68

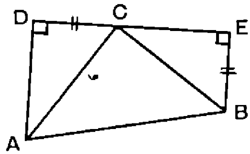


FIG. 69

21. In Fig. 69, if $\angle$CAB=$\angle$CBA=45°, prove two triangles congruent and deduce that DCE is a straight line.

(Three-dimensional)

22. In Fig. 70, A, B, C are three points, and O a point in their plane equidistant from them. If PO is perpendicular to the plane ABC, prove that P is equidistant from A, B, and C.

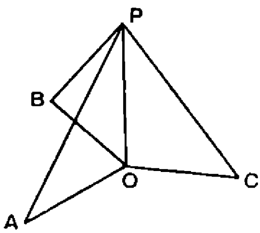


FIG. 70

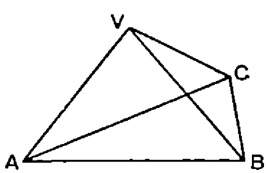


FIG. 71

23. In Fig. 71, V is the vertex and ABC the base of a tetrahedron. If VA=BC and VB=AC, prove that (i) $\angle$VAC=$\angle$VBC, (ii) $\angle$AVB=$\angle$ACB.

24. PAB, QAB are two congruent triangles on the same base AB, but in different planes. Prove that the perpendiculars to AB from P and Q meet AB at the same point. (Let PM, QN be the perpendiculars, and prove that AM=AN.)

25. The base BCD of a tetrahedron is horizontal, and the vertex A is vertically above B. If M is the mid-point of CD and BM is perpendicular to CD, prove that AM is perpendicular to CD. (Consider in succession the pairs of triangles BMC and BMD, ABC and ABD, AMC and AMD.)

26. PAB, QAB are two congruent triangles on the same base AB, but in different planes. R, S are points on BP, BQ respectively such that BR=BS. Prove that △AQR is congruent to △APS. (First consider △s ARB and ASB, BQR and BPS.)

EXAMPLES 3 c

1. The side BC of △ABC is produced to D, and BD is made equal to BA; the angle ACD is 80° and the angle BAC is 24°. Calculate, in degrees, the magnitudes of the angles ABC, ACB, CAD, and CDA. (OC)

2. ABC is a triangle in which ∠CAB=35°, ∠ABC=100°. BD is drawn perpendicular to AC. Prove that BD=DC. State the propositions on which your proof depends. (OC)

3. In Fig. 43, prove that AX+AY=2AD.

4. In △ABC, AB=AC and AB is produced to D. Prove that ∠ACD+∠D=2∠ABC.

5. The bisectors of the angles B and C of a triangle ABC meet at I; P, Q, R are the feet of the perpendiculars from I to BC, CA, AB respectively. By proving that each is equal to IP, prove that IQ=IR.

Hence prove that IA bisects ∠A.

6. If the perpendicular bisectors of the sides AB, AC of a triangle ABC meet at O, prove that OA=OB=OC, and deduce that the perpendicular from O to BC bisects BC.

7. ABCDEF is a regular hexagon; prove that AC is perpendicular to CD.

8. ABC is a triangle; the straight line through C parallel to the bisector of ∠A meets BA produced at D, and the straight line through C parallel to the bisector of ∠B meets AB produced at E. Prove that DE is equal to the perimeter of △ABC.

9. In Fig. 51, if CA bisects ∠BCE and ∠B=33°, calculate ∠BAC.

C

10. ABCDE is a regular pentagon. Prove that (i) AC=AD, (ii) AC and AD trisect ∠BAE, (iii) AC is parallel to ED.

11. ABC is an equilateral triangle, and D the foot of the perpendicular from A to BC. If E is a point, on the side of AC opposite to D, such that △ADE is equilateral, prove that AC and DE intersect at right angles.

12. In Fig. 72, if AB=AC, calculate ∠A.

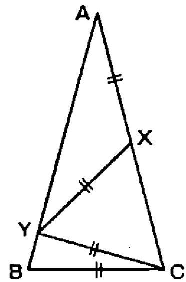


FIG. 72

13. The sides AB, AC of a triangle ABC are equal. Through any point X on AB a line is drawn perpendicular to BC to meet CA produced at Y. Prove that AX=AY. (N)

14. The angles A, B of a triangle ABC are 70° and 60° respectively. Lines AD, AE are drawn within the angle BAC to cut BC at D and E, such that ∠BAD=∠ACB and ∠CAE=∠ABC. Prove that AD=AE. (N)

15. O is a point inside an equilateral triangle ABC; on OC another equilateral triangle OCK is drawn, such that K and A are on opposite sides of BC; prove that OA=BK. (L)

16. In an isosceles triangle ABC the sides AB and AC are equal. If P is a point in AB and Q a point in AC such that AP=AQ, prove that the triangles BPC and CQB are congruent. Deduce that, if BQ and CP meet at R, the triangles PQR and BRC are each isosceles. (N)

17. The side BA of a triangle ABC is produced beyond A to D, so that AD=AC, and the side CA is produced beyond A to E, so that AE=AB. Prove that CD and BE are parallel. (W)

18. ABC is a triangle and F a point inside it; BF and CF, both produced, meet AC, AB at D, E respectively. If FD=FE and FB=FC, prove that AB=AC.

19. In a triangle ABC the angle ABC is greater than the angle ACB. D is a point in AC such that the angle ABD is equal to the angle ACB, and the bisector of the angle DBC meets AC at E. Prove that AE=AB. (W)

20. From the vertex A of a right-angled triangle a perpendicular is drawn to the hypotenuse BC. The bisector of the angle B meets this perpendicular at R and meets AC at S. Prove that the triangle ARS is isosceles. (L)

21. ABCDE is a five-sided figure having all its angles equal to one another and all its sides equal to one another. Prove that, if M is the middle point of CD, then AM is perpendicular to CD. (C)

22. ABC is a triangle, in which $\angle A$ is a right angle and BC=2AC. CA is produced to D so that CA=AD. Prove that $\angle BDC=60°$. (L)

23. The angle A of a triangle ABC is a right angle, and D is a point in BC such that BD=BA ; N is the foot of the perpendicular from A on BC. Prove that $\angle NAD=\angle DAC$.

24. Two isosceles triangles ABC, ADC with AB=AC=AD have a common vertex A and a common side AC. The angle BAC is 40°, the angle DAC is 70° and B and D are joined. Calculate the angles of the triangle BCD when (i) the triangles are on the same side of AC, (ii) the triangles are on opposite sides of AC. (C)

25. In a triangle ABC the sides AB and AC are equal, and P is any point in the side BC; Q is taken in AC such that CQ=BP; and R is taken in AB such that BR=CP. Prove that in the triangle PQR the sides PQ and PR are equal and the angle QPR equals the angle ABC. (C)

26. In Fig. 46, if AB=BC=CD=DA, prove that AC bisects BD at right angles.

27. ABCDEF is a regular hexagon; prove that $\angle BAC=\angle CAD=\angle DAE=\angle EAF$.

28. ABC is an acute-angled triangle, and on the sides AB, AC equilateral triangles APB, AQC are described outside the given triangle. Prove that the triangles PAC, BAQ are congruent. If PC and BQ cut at R, deduce that $\angle BRC$ is 120°. (N)

29. On the sides of an isosceles triangle ABC are described externally equilateral triangles DBC, EAC, FAB. Prove that the triangle formed by joining D, E, F is isosceles.

30. AB and XY are two equal straight lines, and the perpendicular bisectors of AX and BY meet at O. Prove that Δs AOB, XOY are congruent.

31. The quadrilateral ABCD is such that the straight line bisecting AB at right angles also bisects DC at right angles. Prove that AD=BC.

CONSTRUCTIONS: STAGE I

A **construction** is a drawing carried out with the help of two instruments only, namely a straight edge and a pair of compasses. Moreover, these two instruments may only be used for certain purposes. A straight edge may be used

(i) to join two given points,

(ii) to produce a given straight line.

A pair of compasses may be used to draw a circle, or an arc of a circle. The centre must be given, and the radius may be either a given length or some suitable length chosen at will.

Fig. 73 is an illustration of the incorrect use of a straight edge.

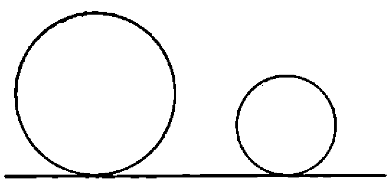


FIG. 73

If it is required to construct a straight line which touches the two given circles, why may this not be done by moving the straight edge about until it appears to be in the required position?

Fig. 74 illustrates the incorrect use of compasses. If we require a circle of radius 1 inch which touches the two straight lines OA, OB, why is it not correct to extend the compasses to a distance of 1 inch and then take various positions for the centre of the circle until at last, by trial and error, a position is found which gives a circle touching the straight lines with reasonable exactness? Later in this book a correct method for carrying out these constructions will be given.

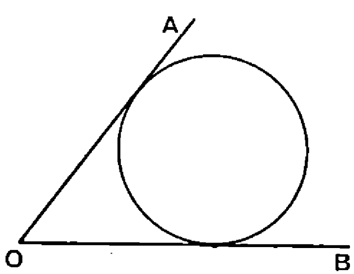


FIG. 74

CONSTRUCTION 1

To copy a given angle.

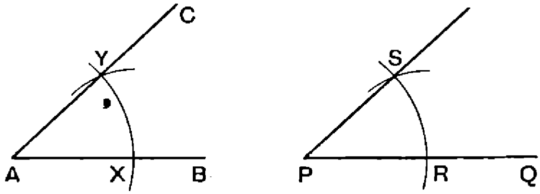


FIG. 75

Let BAC be the given angle, and suppose it is required to make at P, with the straight line PQ, an angle equal to ∠BAC.

With centre A and any radius, draw an arc to cut AB, AC at X, Y respectively.

With centre P and the same radius, draw an arc to cut PQ at R.

With centre R and radius XY, draw an arc to cut this arc at S.

Join PS.

Then ∠SPR is the required angle.

The proof depends on showing that Δs $\frac{\text{AXY}}{\text{PRS}}$ are congruent (SSS).

CONSTRUCTION 2

To bisect a given angle.

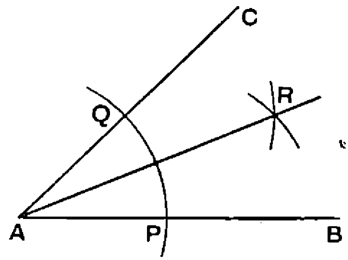


FIG. 76

Let BAC be the given angle.

With centre A and any radius, draw an arc to cut AB, AC at P, Q respectively.

With centres P and Q in turn and any suitable radius (the same in each case), draw arcs to cut at R.

Join AR.

Then AR bisects ∠BAC.

The proof depends on showing that △s $\begin{matrix}\text{APR}\\\text{AQR}\end{matrix}$ are congruent (SSS).

CONSTRUCTION 3

Through a given point to draw a straight line parallel to a given straight line.

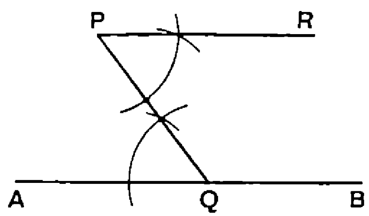


FIG. 77

Let P be the given point and AB the given straight line.

Join P to any point Q in AB.

Construct an angle QPR at P equal to ∠PQA and alternate to it.

Then PR is parallel to AB.

CONSTRUCTION 4

To find the mid-point of a given straight line.

To draw the perpendicular bisector of a given straight line.

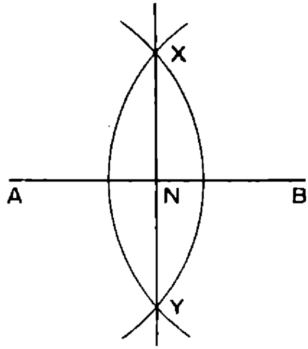


FIG. 78

Let AB be the given straight line.

With centres A and B in turn and any suitable radius (the same in each case), draw arcs to cut at X and Y.

Join XY, and let it cut AB at N.

Then N is the mid-point of AB, and XY is the perpendicular bisector of AB.

The proof depends on first showing that Δs $\begin{matrix}AXY\\BXY\end{matrix}$ are congruent (SSS), and then showing that Δs $\begin{matrix}AXN\\BXN\end{matrix}$ are congruent (SAS).

CONSTRUCTION 5

To erect a perpendicular to a given straight line at a given point in it.

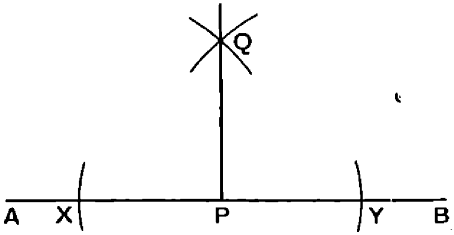


FIG. 79

Let AB be the given straight line and P the given point in it.

With centre P and any suitable radius, draw arcs to cut PA, PB at X, Y respectively.

With centres X and Y in turn and any suitable radius (the same for both arcs), draw arcs cutting at Q.

Join PQ.

Then PQ is perpendicular to AB.

The proof depends on showing that $\triangle$s $\begin{matrix} XPQ \\ YPQ \end{matrix}$ are congruent (SSS).

CONSTRUCTION 6

To draw a perpendicular to a given straight line from a given point outside it.

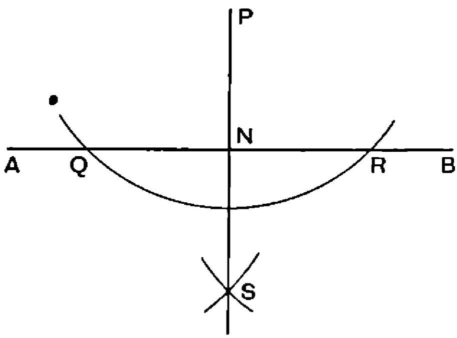


Fig. 80

Let P be the given point and AB the given straight line.

With centre P and any suitable radius, draw an arc to cut AB at Q and R.

With centres Q and R in turn and any suitable radius (the same in each case), draw arcs to cut at S.

Join PS, and let it cut AB at N.

Then PN is perpendicular to AB.

The proof depends on first showing that Δs $\frac{PQS}{PRS}$ are congruent (SSS), and then showing that Δs $\frac{PNQ}{PNR}$ are congruent (SAS).

CONSTRUCTION 7

To construct some special angles without the aid of a protractor.

An angle of 60°.

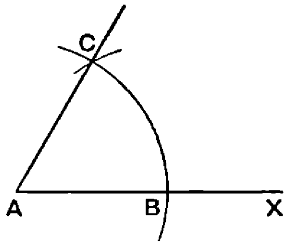


FIG. 81

Suppose it is required to construct, at the point A, an angle of 60° with the straight line AX.

With centre A and any radius, draw an arc to cut AX at B.

With centre B and the same radius as before, draw an arc to cut the first arc at C.

Join AC.

Then ∠BAC=60°.

Proof. △ABC is equilateral, by construction.

An angle of 30° is obtained by constructing an angle of 60° as above, and then bisecting it.

An angle of 45° is obtained by constructing an angle of 90°, and then bisecting it.

CONSTRUCTION 8

To construct a regular hexagon with a side of given length.

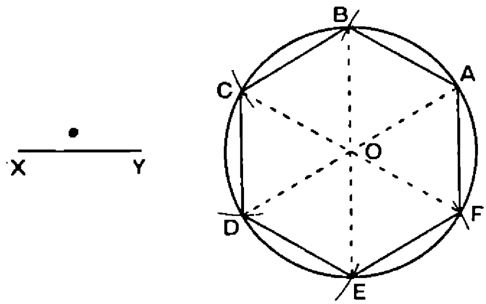


FIG. 82

Let XY be the given length.

With any point O as centre and radius equal to XY, draw a circle.

Mark any point A on the circle.

With centre A and radius XY, draw an arc to cut the circle at B.

With centre B and radius XY, draw an arc to cut the circle at C; and so on, thus obtaining the points D, E, F.

Join AB, BC, CD, DE, EF, and FA.

Then ABCDEF is the required hexagon.

Proof. Join the vertices of the hexagon to O.

$\triangle$s AOB, BOC, COD, DOE, EOF are all equilateral, since all the sides are equal to XY.

$\therefore$ $\angle$s AOB, BOC, COD, DOE, EOF are each 60°.

$\therefore$ their sum is 300°.

But the sum of the angles at O$=360^\circ$;

$\therefore$ $\angle$FOA$=60^\circ$.

$\therefore$ $\angle$OAF$+\angle$OFA$=120^\circ$ ($\angle$s of $\triangle$).

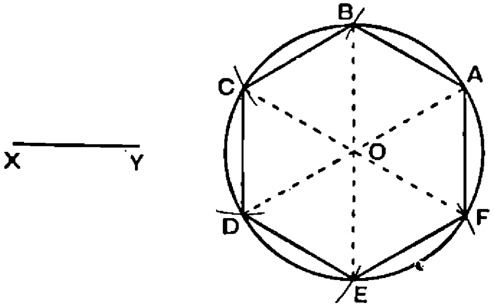


FIG. 82 (repeated)

But OF$=$OA,

$$\therefore \angle OAF = \angle OFA$$
$$= 60°.$$

$$\therefore \triangle OAF \text{ is equilateral, and } AF = OA.$$

Hence all the sides of the hexagon are equal to the given length XY, and all the angles of the hexagon are 120°.

$$\therefore \text{ABCDEF is the required regular hexagon.}$$

Use of the Graduated Ruler and Protractor

As it is hardly possible for the teacher to draw the given lengths and angles on each pupil's paper, the given lengths are usually stated in inches or centimetres and the given angles in degrees. The pupil must therefore begin by drawing the given lengths and angles, on some convenient part of the paper, with the help of a graduated ruler and a protractor. When that has been done, *the 'construction' begins and the rules on p.* 36 *apply.*

Once the 'construction' is finished, the graduated ruler and the protractor may be used again, this time to test the accuracy of the work by measuring some part of the completed figure.

CONSTRUCTION 9

To construct a triangle, given the lengths of the three sides.

EXAMPLE. *Construct* △ABC *in which* BC=2·2 *cm.*, CA=1·9 *cm.*, AB=2·8 *cm.*

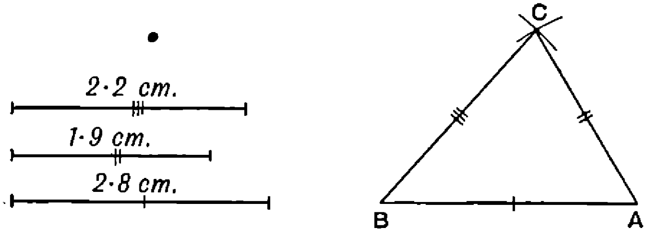


FIG. 83

First draw the given lengths on the left.

Then draw any straight line, and cut off from it a part AB equal to 2·8 cm.

With centre A and radius 1·9 cm., draw an arc.

With centre B and radius 2·2 cm., draw an arc.

Let these two arcs cut at C.

Then ABC is the required triangle.

NOTE. *The construction is only possible if the sum of any two sides of the triangle is greater than the third side. If this were not so, the two arcs above with centres* A *and* B *would not cut.*

CONSTRUCTION 10

To construct a triangle, given two sides and the included angle.

EXAMPLE. *Construct* △ABC *in which* AB=2·2 *in.*, AC= 1·5 *in., and* ∠A=48°.

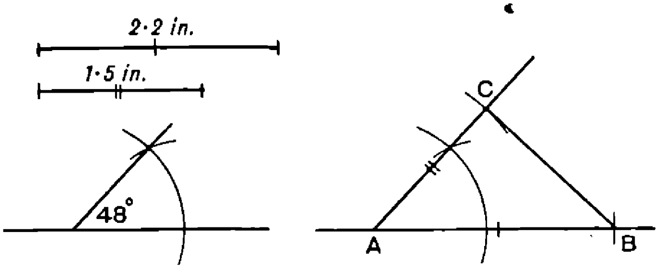


FIG. 84

First draw the data on the left.

Draw any straight line, and cut off from it a part AB equal to 2·2 in.

Copy at A an angle equal to the given angle of 48°.

With centre A and radius equal to 1·5 in., draw an arc to cut the arm of this angle at C.

Then ABC is the required triangle.

CONSTRUCTION 11

To construct a triangle, given a side and two angles.

EXAMPLE. *Construct* △ABC *in which* AB=1·4 *in.*, ∠A=27°, *and* ∠C=51°.

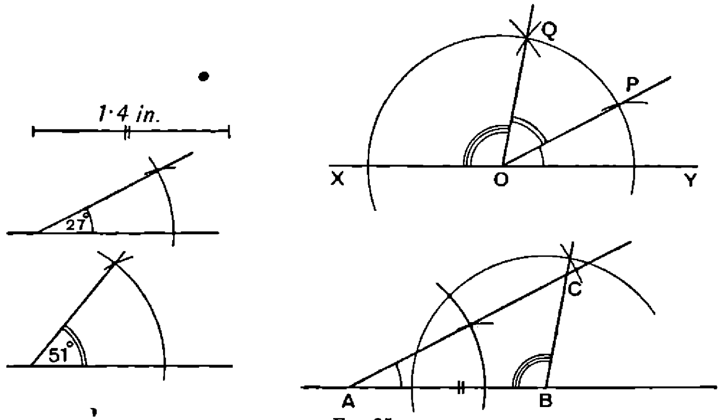


FIG. 85

First draw the data on the left.

Draw any straight line XY and mark a point O in it.

Copy the angle of 27° on the straight line OY in the position POY, and, adjacent to it, the angle of 51° in the position POQ.

Since ∠POY+∠QOP+∠XOQ=180°, then ∠XOQ is equal to the third angle of the triangle.

Now draw a straight line, and cut off from it a part AB equal to 1·4 in.

Copy at A an angle equal to the given angle of 27°, and copy at B an angle equal to ∠XOQ.

Let the arms of these angles intersect at C.

Then ABC is the required triangle.

If we are required to construct a triangle ABC given AB, ∠A and ∠B, the first part of the construction above is unnecessary.

CONSTRUCTION 12

To construct a triangle, given two sides and an angle opposite to one of them.

EXAMPLE 1. *Construct* $\triangle ABC$ *in which* $AB = 4$ *cm.*, $BC = 2{\cdot}5$ *cm.*, *and* $\angle A = 30°$.

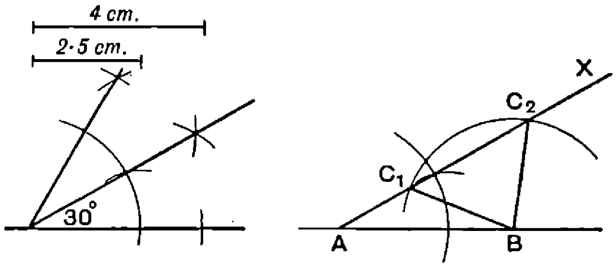


FIG. 86

First draw the data on the left. (The given angle of 30° may be drawn either by means of a protractor or by a geometrical construction.)

Then draw any straight line, and cut off from it a part AB equal to 4 cm.

Copy at A on the straight line AB an angle BAX equal to the given angle of 30°.

With centre B and radius 2·5 cm., draw an arc to cut AX at C_1 and C_2.

Then the *two* $\triangle$s ABC_1 and ABC_2 both satisfy the required conditions.

This is called the **Ambiguous Case.**

EXAMPLE 2. *Construct* $\triangle$**ABC** *in which* **AB**$=4$ *cm.*, **BC**$=$ 2 *cm., and* $\angle$**A**$=30°$.

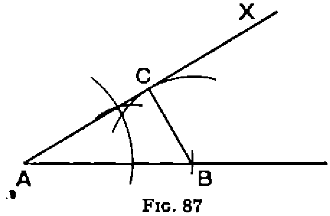


FIG. 87

Proceed as in Example 1.

The arc, drawn with centre **B** and radius 2 cm., will now be found to *touch* **AX**.

Only *one* $\triangle$**ABC** can be found which satisfies the required conditions, and it is right-angled.

EXAMPLE 3. *Construct* $\triangle$**ABC** *in which* **AB**$=4$ *cm.*, **BC**$=$ 5·9 *cm., and* $\angle$**A**$=30°$.

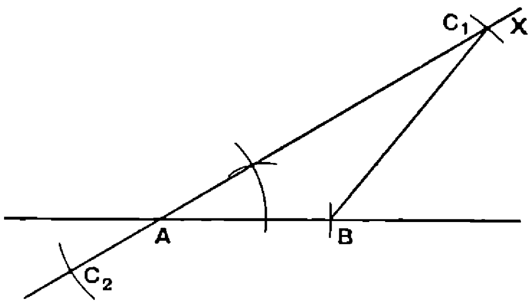


FIG. 88

Proceeding as before, the arc which has centre **B** and radius 5·9 cm. will be found to cut **AX** (at C_1, say) and **XA** produced (at C_2, say).

There is only *one* $\triangle$, namely $\triangle ABC_1$, which satisfies the required conditions, since $\triangle ABC_2$ has the angle **A** equal to 150° instead of 30°.

EXAMPLE 4. *Construct* △ABC *in which* AB=4 *cm.*, BC=1·5 *cm.*, *and* ∠A=30°.

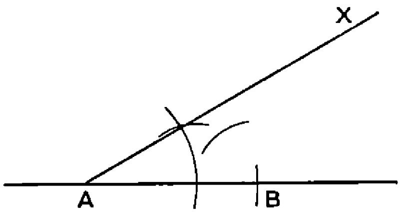


FIG. 89

Proceeding as before, the arc which has centre B and radius 1·5 cm. does not cut the straight line AX.

There is therefore *no* triangle which satisfies the required conditions.

Note on Examples 4

In Examples 4 the given lengths and angles should be drawn with ruler and protractor, and then copied where required on the figure by means of straight edge and compasses, as is done in the worked examples above. The pupil should draw a rough free-hand sketch, and mark on it the magnitudes given in the question. In the case of a quadrilateral it will then be seen that the pupil can begin by constructing a certain triangle (for example, one formed by two sides and a diagonal of the quadrilateral), after which the quadrilateral can easily be completed.

When asked to measure any straight line in the completed figure, the pupil should do so either correct to the nearest tenth of an inch or correct to the nearest millimetre, according as inches or centimetres have been used in the question.

Definitions

The **perimeter** of a figure is the sum of its sides.

A **median** of a triangle is a straight line drawn from one vertex of the triangle to the mid-point of the opposite side.

EXAMPLES 4 a

Construct, *when possible,* the triangle **ABC** with the following measurements. Where there are two different triangles fulfilling the requirements, *draw both* :—

1. **BC**=4 cm., **AC**=$3\frac{1}{2}$ cm., **AB**=5 cm. Measure ∠**A**.
2. **BC**=3 cm., **AC**=$3\frac{1}{2}$ cm., **AB**=8 cm. Measure ∠**A**.
3. **AC**=$4\frac{1}{2}$ cm., ∠**C**=32°, ∠**A**=43°. Measure **AB**.
4. **AB**=2 in., ∠**C**=46°, ∠**A**=30°. Measure **BC**.
5. **BC**=6 cm., ∠**A**=115°, ∠**B**=35°. Measure **AC**.
6. **BC**=2 in., ∠**B**=126°, ∠**C**=72°. Measure **AC**.
7. **BC**=6 cm., **AC**=4·7 cm., ∠**B**=42°. Measure **AB**.
8. **BC**=5 cm., **AC**=7 cm., ∠**B**=70°. Measure **AB**.
9. **AC**=7 cm., **AB**=3 cm., ∠**C**=63°. Measure **BC**.
10. **BC**=6·2 cm., **AC**=4·4 cm., ∠**C**=130°. Measure **AB**.
11. ∠**A**=30°, ∠**B**=70°, ∠**C**=80°. Measure **AC**.
12. **BC**=3 in., **AC**=2 in., ∠**A**=90°. Measure **AB**.

---

13. Given lengths of 5 cm. and 4 cm., draw without using a protractor a triangle **ABC** in which **BC**=5 cm., **AB**=4 cm., and ∠**B**=60°. Construct the bisector **CP** of ∠**C**, meeting **AB** at **P**, and draw through **B** a parallel to **PC** meeting **AC** produced at **Q**. Measure **CQ**.

14. Draw a triangle **ABC** in which **AB**=$4\frac{1}{2}$ cm., **BC**=5 cm., and **CA**=$3\frac{1}{2}$ cm. Construct the perpendicular from **A** to **BC**, meeting **BC** at **D**. Measure **AD**.

15. Draw a triangle **ABC** in which **AB**=$2\frac{1}{2}$ in., ∠**A**=40°, and ∠**B**=64°. Construct the perpendicular bisector of **AB**, and the bisector of ∠**C**, and let these straight lines cut at **P**. Measure **CP**.

16. Draw a triangle **ABC** in which **AB**=**AC**=$2\frac{1}{4}$ in., ∠**A**=36°. Find by construction a point **P** in **AC** such that ∠**PBC**=∠**BAC**, and measure **BP**.

17. Draw a triangle **ABC** in which **AB**=6·5 cm., ∠**A**=43°, ∠**B**=59°. Construct through **C** a parallel to **BA** and through **A** a parallel to **BC**, and let these straight lines meet at **D**. Measure **AD**.

18. Without using a protractor, construct angles of 120°, 30°, 45°, $22\frac{1}{2}$°.

19. Draw any obtuse angle **ABC** and construct a straight line **BD** through **B** such that $\angle$**DBC**$=\frac{1}{4}\angle$**ABC**.

20. Draw a triangle **ABC** in which **BC**=5·5 cm., $\angle$**B**=58°, $\angle$**C**=50°. Find by construction the mid-point **P** of **AB**, and construct through **P** a parallel to **BC**. Measure the lengths into which this parallel divides **AC**.

21. Given a straight line of length 2 in., construct an equilateral triangle with side 2 in.

22. Given a straight line of length 5 cm., construct a right-angled isosceles triangle with the sides containing the right angle each 5 cm.

23. Given a straight line of length 2 in., construct a square with side 2 in.

24. Construct a quadrilateral **ABCD**, given **AB**=4 cm., **BC**=5·6 cm., **CD**=7 cm., $\angle$**B**=96°, $\angle$**C**=63°. Measure **AD**.

25. Construct a quadrilateral **ABCD**, given **AB**=5·8 cm., **BC**=**CD**=3·5 cm., **DA**=6 cm., $\angle$**C**=142°. Measure **AC**.

26. Without using a protractor, construct a quadrilateral **ABCD** in which **AB** is parallel to **DC**, **AB**=1·4 in., **DC**=2·6 in., **AD**=1·2 in., $\angle$**D**=60°.

27. Draw a regular pentagon with sides $1\frac{1}{2}$ in. long. (First calculate the angle of a regular pentagon, and use the protractor to draw this angle.)

28. Construct a regular octagon with sides $1\frac{1}{2}$ in. long, using compasses and ruler only.

29. Construct a regular hexagon with sides 1 in. long.

EXAMPLES 4b

In Nos. 1 to 6, construct an isosceles triangle in which :—

1. The base is 5 cm., and the perpendicular from the vertex is 6 cm. Measure one of the equal sides.

2. The vertical angle is 64° and the perpendicular from the vertex is 2 in. Measure the base.

3. Each of the equal sides is 3 in. and the perpendicular from the vertex is 2 in. Measure the base.

4. Each of the base angles is 50° and each equal side $2\frac{1}{2}$ in. Measure the base.

5. The base is 2 in. and the vertical angle is 41°. Measure the equal sides.

6. The base is 4 cm. and the perimeter is 10 cm. Measure one of the base angles.

---

7. Construct a triangle ABC in which AB=3 in., AC=2·5 in., and the median from B=2·1 in. Measure BC.

8. Draw a quadrilateral ABCD in which AB=2 in., BC=1·6 in., DA=2·4 in., AC=2·6 in., BD=3·2 in. Measure CD.

9. Through a given point P, draw a straight line which shall make a given angle with a given straight line AB.

10. Given straight lines of lengths 1 in. and 2 in., use ruler and compasses only to construct an equiangular hexagon with four of its sides, taken in order, 1 in., 2 in., 2 in., and 1 in. respectively. Measure the remaining sides.

11. Construct angles of 75°, 15°, 105°, using ruler and compasses only.

12. Draw a circle of radius 2 in., and on the circumference mark three points A, B, C. Construct (i) the bisector of ∠BAC, (ii) the perpendicular bisector of BC. Produce these lines until they meet.

13. Draw a triangle ABC in which AB=2·7 in., AC=1·9 in., and ∠A=76°. Construct the bisectors of the three angles of the triangle.

14. Repeat the triangle of No. 13, and construct the perpendiculars from the three vertices to the opposite sides.

15. Repeat the triangle of No. 13, and construct (i) the perpendicular bisectors of the three sides, (ii) the medians.

16. ABC is a triangle, E is the mid-point of BC, and D is the point where the bisector of the angle ABC meets AC. Using straight edge and compasses only, construct the triangle ABC given that BD=0·9 in., DE=1·2 in., and ABC=110°. Measure AB.

17. Draw a circle of radius $1\frac{1}{2}$ in. and draw any triangle **ABC** with its vertices on the circumference of the circle. Mark any point **P** on the circumference of the circle (not too near to a vertex of the triangle) and construct perpendiculars from **P** to **AB**, **BC**, **CA** (produced where necessary). Verify that the feet of these perpendiculars lie in a straight line.

EXAMPLES 4 c

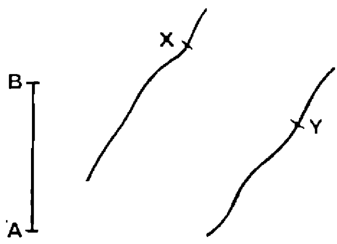
1. Draw a triangle **ABC** in which **AB**=6 cm., **BC**=7 cm., **CA**=8 cm. Using ruler and compasses only, construct (i) the bisector **AD** of ∠**BAC** to meet **BC** at **D**, (ii) the perpendicular **AL** from **A** to **BC** to meet **BC** at **L**. (N)

2. Using ruler and compasses only, construct *in one figure* two quadrilaterals in which **AB**=**BC**=4 cm., **B**=60°, **C**=90°, and **AD**=3 cm. Measure the remaining side in each case. (N)

3. Construct a quadrilateral **ABCD** in which **AB**=2·2 in., ∠**A**=30°, ∠**C**=80°, and ∠**D** is divided by the diagonal **DB** so that ∠**ADB**=65° and ∠**BDC**=45°. Measure **AC**.

4. **X**, **Y** are points on opposite sides of a river, **AB** is a measured line of 100 yards running due N. from **A**. The bearings of **X** are N. 30° E. from **A**, and N. 60° E. from **B**, and those of **Y** are N. 65° E. from **A**, and S. 85° E. from **B**. Draw a figure on the scale of 1 inch to 100 yards and find by measurement the width of the river at **XY**.

[N. 30° E. means 30° to the E. measured from N.] (OC)



[The figure is not drawn to scale]

FIG. 90

5. The sides **AB** and **AC** of a triangle are 4 and 3 in. respectively, and the angle **B** is 30°; construct two triangles which satisfy these conditions. Measure the third side of each. (C)

6. (i) It is required to construct a quadrilateral **ABCD** from the following data: **AB**=7·0 cm., **BC**=9·4 cm., **AC**=11·0 cm., ∠**ACD**=47°, **AD**=9·1 cm. Show that there are two solutions, draw the quadrilaterals, and measure and write down the length of the diagonal **BD** for each.

(ii) Suppose the first four data are as before, but **AD** has now to be made as short as possible. Find by measurement this shortest length, and also the length of **BD** for this case. (W)

7. **ABC** is a triangle in which the angle **B** is greater than the angle **C**. **CA** is produced to **D** so that **AD**=**AB**. Prove that the angle **CBD**$=90°+\frac{1}{2}($**B**$-$**C**$)$.

Making use of this formula, construct a triangle **ABC** in which **BC** is 5 cm., **AB**+**AC**=8 cm., and **B**$-$**C**$=30°$. Measure its sides and angles. Give a brief description of your construction. (O)

8. Prove the validity of the following method for constructing a triangle **ABC** in which $\angle$**B**$=64°$, $\angle$**C**$=72°$, and the perimeter $=12$ cm. :—

On the same side of a straight line **XY**, 12 cm. long, construct straight lines **XP**, **YQ** such that $\angle$**PXY**$=64°$, $\angle$**QYX**$=72°$. Bisect angles **PXY**, **QYX**, and let these bisectors intersect at **A**. Construct through **A** straight lines **AB**, **AC** parallel respectively to **PX**, **QY**, cutting **XY** at **B** and **C**. Then **ABC** is the required triangle.

Draw an accurate figure.

9. Prove the validity of the following method for constructing a triangle **ABC** in which $\angle$**B**$=63°$, **AC**$=5$ cm., and **AB**+**BC**$=9$ cm. :

Construct an angle **YXZ** equal to one half of $\angle$**B**. From the arm **XZ** of this angle cut off **XC**$=9$ cm. With centre **C** and radius 5 cm., draw an arc of a circle cutting **XY** at **A**. Construct the perpendicular bisector of **XA**, and let it cut **XC** at **B**. Join **AB**, **AC**. Then **ABC** is the required triangle.

Draw an accurate figure. Two triangles can be obtained; verify by measurement that they are congruent.

PARALLELOGRAMS

Definition of a Parallelogram

A **parallelogram** is a quadrilateral with both pairs of opposite sides parallel.

THEOREM 8 a

A quadrilateral with both pairs of opposite angles equal is a parallelogram.

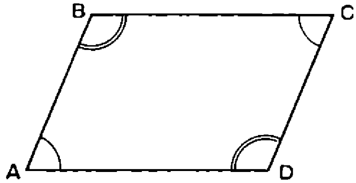


FIG. 91

Given a quadrilateral ABCD in which $\angle A = \angle C$, $\angle B = \angle D$.

To prove that AB ∥ DC, BC ∥ AD.

Proof. $\angle A + \angle B + \angle C + \angle D = 360°$ (∠s of quad.) ;

but $\angle A = \angle C$ and $\angle D = \angle B$ (given) ;

$\therefore\ 2\angle B + 2\angle C = 360°$,

$\therefore\ \angle B + \angle C = 180°$.

But these are conjoined angles ;

∴ AB ∥ DC.

Similarly, BC ∥ AD.

∴ ABCD is a parallelogram (both pairs of opp. sides parallel).

Reference.

ABCD is a parallelogram (*both pairs of opp. ∠s equal*).

THEOREM 8b

The opposite angles of a parallelogram are equal.

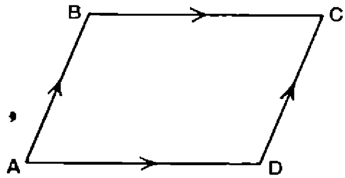


FIG. 92

Given a parallelogram ABCD, *i.e.* AB ∥ DC, BC ∥ AD.

To prove that $\angle A = \angle C$, $\angle B = \angle D$.

Proof. $\angle A + \angle B = 180°$ (conj., BC ∥ AD),

and $\angle B + \angle C = 180°$ (conj., AB ∥ DC);

$\therefore\ \angle A + \angle B = \angle B + \angle C$;

$\therefore\ \angle A = \angle C$.

Similarly, $\angle B = \angle D$.

Reference. $\angle A = \angle C$ (*opp.* ∠*s of* ∥*gram*).

THEOREM 9 a

A quadrilateral with both pairs of opposite sides equal is a parallelogram.

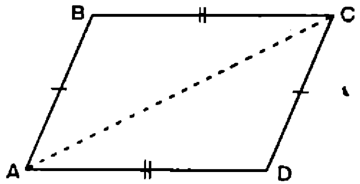


FIG. 93

Given a quadrilateral ABCD in which AB = CD, BC = AD.

To prove that AB ∥ DC, BC ∥ AD.

Construction. Join AC.

Proof. In the △s ABC, CDA,

AB = CD (given),

BC = DA (given),

AC = CA ;

∴ △s $\begin{matrix}\text{ABC}\\ \text{CDA}\end{matrix}$ are congruent (SSS).

∴ ∠BAC = ∠DCA.

But these are alternate angles ;

∴ AB ∥ DC.

Similarly, BC ∥ AD.

∴ ABCD is a parallelogram (both pairs of opp. sides parallel).

Reference.

ABCD is a parallelogram (*both pairs of opp. sides equal*).

THEOREM 9 b

The opposite sides of a parallelogram are equal.

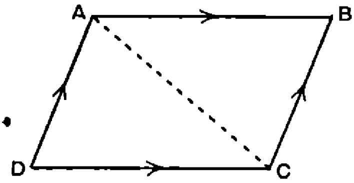


FIG. 94

Given a parallelogram ABCD, *i.e.* AB ∥ DC, BC ∥ AD.

To prove that AB = CD, BC = DA.

Construction. Join AC.

Proof. In the △s ABC, CDA,

∠BAC = ∠DCA (alt., AB ∥ DC),

∠BCA = ∠DAC (alt., BC ∥ AD),

AC = CA ;

∴ △s ABC, CDA are congruent (AA corr. S).

∴ AB = CD, BC = DA.

Reference. AB = CD (*opp. sides of* ∥*gram*).

THEOREM 10 a

A quadrilateral whose diagonals bisect each other is a parallelogram.

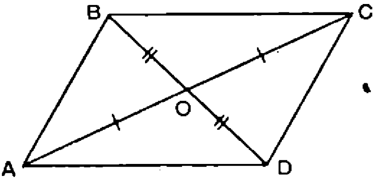


FIG. 95

Given a quadrilateral ABCD with diagonals cutting at O so that AO = OC, BO = OD.

To prove that AB ∥ DC, BC ∥ AD.

Proof. In the △s AOB, COD,

AO = CO (given),

BO = DO (given),

∠AOB = ∠COD (vert. opp.);

∴ △s AOB, COD are congruent (SAS).

∴ ∠OAB = ∠OCD.

But these are alternate angles;

∴ AB ∥ DC.

Similarly, BC ∥ AD.

∴ ABCD is a parallelogram (both pairs of opp. sides parallel).

Reference. ABCD is a parallelogram (*diags. bisect each other*).

THEOREM 10 b

The diagonals of a parallelogram bisect each other.

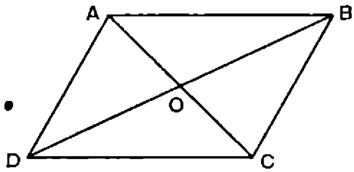


FIG. 96

Given a parallelogram ABCD with diagonals AC, BD cutting at O.

To prove that OA=OC, OB=OD.

Proof. In the $\triangle$s AOB, COD,

$\angle$BAO=$\angle$DCO (alt., AB $\parallel$ DC),

$\angle$ABO=$\angle$CDO (alt., AB $\parallel$ DC),

AB=CD (opp. sides of $\parallel$gram);

$\therefore$ $\triangle$s $\begin{matrix}\text{AOB}\\ \text{COD}\end{matrix}$ are congruent (AA corr. S).

$\therefore$ OA=OC, OB=OD.

Reference. OA=OC (*diags. of* $\parallel$*gram*).

THEOREM 11

If one pair of opposite sides of a quadrilateral are equal and parallel, then the other pair of opposite sides are equal and parallel.

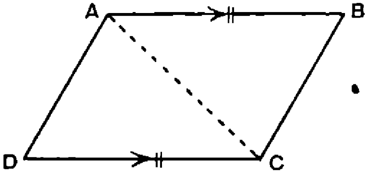


FIG. 97

Given a quadrilateral ABCD in which AB=DC, AB ∥ DC.

To prove that BC=AD, BC ∥ AD.

Construction. Join AC.

Proof. In the △s ABC, CDA,

AB=CD,

AC=CA,

∠BAC=∠DCA (alt., AB ∥ DC);

∴ △s $\frac{\text{ABC}}{\text{CDA}}$ are congruent (SAS).

∴ BC=DA,

and ∠BCA=∠DAC.

But these are alternate angles;

∴ BC ∥ AD.

∴ BC=AD and BC ∥ AD.

Corollary.

A quadrilateral with one pair of opposite sides equal and parallel is a parallelogram.

References.

{ *AB=DC and AB ∥ DC,*
∴ AD=BC and AD ∥ BC.

{ *AB=DC and AB ∥ DC,*
∴ ABCD is a parallelogram.

SUMMARY

A quadrilateral is a parallelogram if—

(*a*) both pairs of opposite sides are parallel,
or (*b*) both pairs of opposite sides are equal,
or (*c*) one pair of opposite sides are equal and parallel,
or (*d*) both pairs of opposite angles are equal,
or (*e*) the diagonals bisect each other.

A parallelogram has the following properties :—

(*a*) opposite sides are parallel,
(*b*) opposite sides are equal,
(*c*) opposite angles are equal,
(*d*) the diagonals bisect each other.

Solid Geometry

Fig. 98 shows a **parallelepiped,** which may be compared with the parallelogram in plane geometry. Each of its six faces is a parallelogram, and it has four diagonals, shown in the figure by dotted lines.

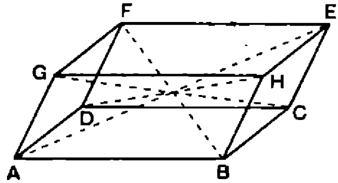


FIG. 98

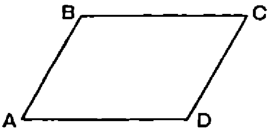
Notice the way in which the figure is lettered. One face is lettered ABCD ; then E, F, G, H are the vertices *diagonally opposite* to A, B, C, D respectively. The parallelepiped is then referred to as ABCDEFGH.

EXAMPLES 5a

In Nos. 1 to 16 it is preferable that congruence should not be used unless it is mentioned in the question.

1. In Fig. 99, if $\angle A = 58°$, calculate the other angles.

2. In Fig. 99, if AC bisects $\angle A$, prove that it also bisects $\angle C$.



ABCD is a parallelogram

Fig. 99

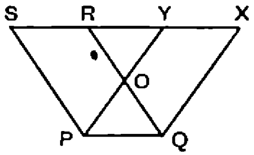


Fig. 100

3. In Fig. 100, if PQRS and PQXY are parallelograms, and if RY = PQ, prove that SX = 3PQ.

4. In Fig. 99, if the straight line through B parallel to CA meets DA produced at E, prove that EA = AD.

5. In Fig. 99, if E is any point on BC, and the straight line through D parallel to AE meets BC produced at F, prove that BE = CF.

6. In Fig. 99, X is any point on AD, and Y a point on AD produced such that $\angle DYC = \angle AXB$. Prove that $\triangle ABX \equiv \triangle DCY$.

7. In Fig. 99, if AB is produced to E so that BE = AB, prove that DE bisects BC.

8. Copy Fig. 99, mark a point X inside ABCD, and complete the parallelogram ABXY. Prove that DCXY is a parallelogram.

9. In Fig. 99, if X is the mid-point of AB and Y the mid-point of DC, prove that BYDX is a parallelogram.

10. If the diagonals of a hexagon bisect one another, prove that the opposite sides are equal and parallel.

11. A straight line parallel to the side LN of a triangle LMN cuts LM at K and MN at H. If NL is produced to G so that LG = HK, prove that GH bisects LK.

12. In Fig. 99, let X be any point on AB, and Y a point on CD such that CY = XA. Prove that AC and XY bisect each other.

13. In Fig. 101, ABCD is a parallelogram, and BXC, DYA are congruent triangles lettered in corresponding order. Prove that (i) ∠XBD = ∠BDY, (ii) XY and BD bisect each other.

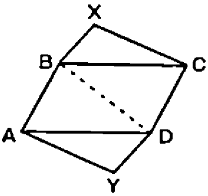


Fig. 101

14. In Fig. 99 the diagonals cut at O, X is the mid-point of OA, and Y the mid-point of OC. Prove that BXDY is a parallelogram.

15. In Fig. 99 the diagonals cut at O, and XOY is a straight line through O such that XO = OY. Prove that BX is parallel to YD, and write down three similar results.

16. PQRS is a quadrilateral having PQ parallel to SR. If PQ is produced to T so that QT = SR, and RS is produced to U so that SU = QP, prove that ∠PUR = ∠PTR.

Nos. 17 to 20 require the use of congruence.

17. ABCD is a quadrilateral whose diagonals cut at O. If OA = OC and AB is parallel to DC, prove that ABCD is a parallelogram.

18. In Fig. 99, if the diagonals cut at O and a straight line through O meets BC at X and AD at Y, prove that OX = OY.

19. In Fig. 99, if the bisector of ∠A meets BD at X, and the bisector of ∠C meets BD at Y, prove that BY = DX.

20. In Fig. 99, if M and N are points on the diagonal AC such that ∠AMB and ∠CND are right angles, prove that (i) BM = DN, (ii) BMDN is a parallelogram.

Construct a parallelogram ABCD with the following data :—

21. AB = 4·1 cm., AD = 5·2 cm., and ∠A = 62°. Measure AC.

22. AB = 1·7 in., the diagonal AC = 3·4 in., and AD = 2·2 in. Measure BD.

23. The diagonal AC = 8·9 cm., the diagonal BD = 3·1 cm., and BC = 3·7 cm. Measure AB. (Start by constructing ΔBOC, where O is the point where the diagonals cut.)

24. The diagonal AC $=3\cdot3$ in., the diagonal BD $=2\cdot7$ in., and the acute angle between the diagonals $=72°$. Measure one of the longer sides.

25. Construct an accurate figure to show that a quadrilateral, which has one pair of opposite sides equal and the other pair of opposite sides parallel, is not necessarily a parallelogram.

EXAMPLES 5 b

1. Through each vertex of a triangle is drawn a straight line parallel to the opposite side. Prove that the perimeter of the triangle formed by these lines is double that of the original triangle.

2. ABCD is a parallelogram. If AC bisects $\angle A$, prove that AB $=$ AD.

3. ABCD is a parallelogram. If M is the mid-point of BC and DM is produced to N so that MN $=$ DM, prove that (i) AB and BN lie in the same straight line, (ii) AN $=$ 2DC.

4. ABCD is a parallelogram. BC is produced to E so that CE $=$ BC, and CEFG is a parallelogram on the side of CE opposite to AD. Prove that AG is equal and parallel to DF.

5. ABCD is a parallelogram. If AC is produced to E so that AC $=$ CE, and DC to F so that CF $=$ DC, prove that BE and CF bisect each other.

6. ABCD is a parallelogram. If AB is produced to X so that BX $=$ BC, and XC and AD produced meet at Y, prove that AX $=$ AY.

7. ABCD is a parallelogram; F is a point on DA such that FB, FC bisect $\angle B$ and $\angle C$ respectively. If the straight line through F parallel to DC meets BC at E, prove that (i) EB $=$ EF, (ii) BC $=$ 2AB.

8. The bisectors of the angles A and B of the parallelogram ABCD meet at X. If the straight line through X parallel to CB meets AB at Y, prove that (i) Y is the mid-point of AB, (ii) $\angle AXB = 90°$.

9. ABCD is a parallelogram, and P, Q, R, S are points in AB, BC, CD, DA respectively such that AP $=$ CR, BQ $=$ DS. Prove that (i) $\triangle APS \equiv \triangle CRQ$, (ii) PQRS is a parallelogram.

10. ABCD is a parallelogram, and equilateral triangles APB, BQC, CRD, and DSA are constructed outside the parallelogram on the sides AB, BC, CD, DA respectively. Prove that (i) $\triangle$PBQ $\equiv\triangle$RDS, (ii) PQRS is a parallelogram.

11. H, K are the mid-points of the sides AB, AC respectively of a triangle ABC. If HK is produced to L so that KL=HK, (i) prove by congruence that CL is equal and parallel to HA, (ii) prove that HK is parallel to and a half of BC.

12. Prove that the straight lines, which bisect a pair of opposite angles of a parallelogram, are parallel.

13. In Fig. 98, prove that the diagonals BF and DH bisect each other. Deduce that all four diagonals bisect one another and have a point in common.

---

14. Construct a parallelogram ABCD in which AB=3·9 cm., AD=5·4 cm., and $\angle$D=125°. Measure AC.

15. Construct a parallelogram with diagonals 9·2 and 5·9 cm., and one side 6·1 cm. Measure the other side.

16. Without using a protractor, construct a parallelogram ABCD in which AC=3·5 in., BD=1·2 in., and $\angle$ABD=45°. Measure AD.

17. Without using a protractor, construct a parallelogram ABCD in which AB=2·5 in., $\angle$A=60°, and BD=2·2 in. Measure AC. (Two solutions are possible.)

EXAMPLES 5 c

1. The diagonals AC, BD of a parallelogram ABCD cut at O. $\angle$BAC=24°; $\angle$DAC=32°; $\angle$AOB=70°. Calculate the numbers of degrees in the angles ABD, CBD. (N)

2. The diagonal DB of a parallelogram ABCD is bisected at O. Through O is drawn a line cutting BC and AD at P and Q respectively. Prove that BPDQ is a parallelogram. (N)

3. ABCD is a parallelogram, and points P, in AB, and Q, in CD, are such that AP=CQ. Two parallel straight lines AX and CY meet PQ in X and Y. Prove that AX=CY. (C)

4. Construct a parallelogram **ABCD** in which the diagonals **AC** and **BD** are 5·6 in. and 5·0 in. respectively, and **AB**=3·7 in. Measure **AD**.

5. On the diagonal **BD** (not produced) of a parallelogram **ABCD** are taken two points **E**, **F** such that **BE**=**DF**. **AF**, **AE**, **CF**, **CE** are drawn. Prove that **AECF** is a parallelogram. (L)

6. **ABCD** is a parallelogram ; **P**, **Q**, **R**, **S** are points on **AB**, **BC**, **CD**, **DA** respectively, such that **AP**=**CR**, **BQ**=**DS**. Prove that the lines **AR**, **BS**, **CP**, and **DQ** intersect in four points which are the vertices of a parallelogram.

7. **ABCD** is a parallelogram whose diagonals meet in **O**. Any equal lengths **AE**, **CF** are marked off along **AB**, **CD** respectively. Show that **EOF** is a straight line. (W)

8. If **X**, **Y** are the mid-points of the respective sides **AB**, **CD** of a parallelogram **ABCD**, prove that **AC**, **XY**, and **BD** intersect at the same point.

9. Three straight lines **OA**, **OB**, **OC** meet at a point **O**, with **OB** between **OA** and **OC** ; **Q** is any point on **OB**. Draw accurately, explaining your construction without proof, a line through **Q** meeting **OA** in **P** and **OC** in **R** in such a way that **PQ**=**QR**. (L)

10. **D** and **E** are the mid-points of the sides **BC**, **CA** respectively of the triangle **ABC** ; **AD** is produced to **G** so that **DG**=**AD**, and **BE** is produced to **H** so that **EH**=**BE**. Prove that **G**, **C**, **H** are in the same straight line.

11. **OA**, **OB**, **OC** are three coplanar straight lines, **OBXC**, **OCYA**, **OAZB** are three parallelograms. Prove that **AX**, **BY**, **CZ** bisect one another. (OC)

12. **ABCD** is a parallelogram. From **A** a line **AE** is drawn perpendicular to **AB** and equal to **AD**, **E** being on the opposite side of **AB** from **D**. From **C** a line **CF** is drawn perpendicular to **BC** and equal to **CD**, **F** being on the opposite side of **BC** from **D**. Prove that the angles **ADE**, **CDF** are equal and that $\angle$**EDF** is a right angle. (L)

13. **ABCD** is a parallelogram. If **AB** is produced to **X** so that **BX**=**AB**, and **AD** is produced to **Y** so that **DY**=**AD**, prove that **XCY** is a straight line.

14. ABCDE is a five-sided figure with no two sides equal, and the interior angles at B, C, D, and E are each 120°. Calculate the size of the interior angle at A, and prove AB=ED+DC. (W)

15. AB and CD are two equal and parallel straight lines. Any point X is joined to B and D, and the lines through A parallel to BX and through C parallel to DX meet at Y. Prove that YX is equal and parallel to AB. (Let the line through X parallel to BA meet AY at H and CY at K, and prove that H and K coincide.)

16. ABCD is a parallelogram having AB greater than AD. From AB cut off AE equal to AD. Show that DE bisects the angle ADC.

Find a point P in AB produced such that PD bisects the angle APC. (OC)

17. X, Y are the mid-points of the respective sides AB, CD of a parallelogram ABCD. If BY and DX meet AC at Z and W respectively, prove that XZYW is a parallelogram.

18. Draw a triangle ABC with AB=4 cm., BC=7 cm., AC=10 cm. Show how this triangle can be used to construct two points X and Y such that BXY is a straight line, AX=5 cm., AY=7 cm., and BX=XY. (W)

19. Construct a parallelogram whose area is 8 sq. in. and the lengths of whose diagonals are 5 in. and 4 in., and measure its angles. Give a brief description of your construction. (W)

SPECIAL FORMS OF THE QUADRILATERAL

Definitions

A **parallelogram** is a quadrilateral with both pairs of opposite sides parallel.

A **rhombus** is a quadrilateral with all its sides equal.

A **rectangle** is a quadrilateral with all its angles right angles.

A **square** is a quadrilateral which has all its sides equal and all its angles right angles.

A **kite** is a quadrilateral which has two pairs of adjacent sides equal.

A **trapezium** is a quadrilateral which has one pair of opposite sides parallel. A trapezium which has its non-parallel sides equal is said to be **isosceles.**

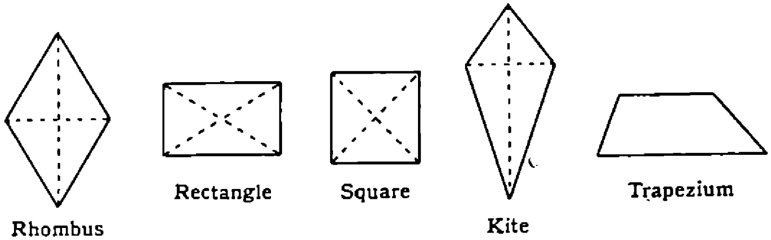


FIG. 102

These figures can be distinguished by the properties of their diagonals. A quadrilateral in which the diagonals

(*a*) bisect each other must be a parallelogram ;

(*b*) bisect each other at right angles must be a rhombus ;

(*c*) bisect each other and are equal must be a rectangle ;

(*d*) bisect each other at right angles and are equal must be a square ;

(*e*) cut at right angles so that only one diagonal is bisected must be a kite.

Starting with the definition of a parallelogram, we have in Theorems 8 to 11 proved all the important properties of parallelograms. It would be proper to pursue the same course with the other special forms of quadrilateral, but this procedure would be lengthy, and is usually omitted. The reader should note, however, that **all properties of the rhombus, rectangle, square, and trapezium may be taken for granted in riders, short of assuming what has to be proved.** The properties of the kite also should be assumed in this book, but might not be accepted in a public examination. Proofs of some, at least, of these properties should be discussed in class.

Solid Geometry

A **rectangular parallelepiped** or **cuboid** is a parallelepiped whose faces are rectangles.

A **cube** is a parallelepiped whose faces are squares.

A **pyramid** is obtained by joining the vertices of a polygon (the 'base') to a point (the 'vertex') not in the plane of the base. If the base is a regular polygon, and if the straight line joining the vertex to the centre of the base is perpendicular to the base, the pyramid is a **right pyramid.**

EXAMPLES 6 (Oral)

Apply the following questions in turn to the parallelogram, rhombus, rectangle, square, kite, and trapezium.

1. Do the diagonals bisect each other ?

2. Do the diagonals cut at right angles ?

3. Are the diagonals equal ?

4. Do the diagonals bisect the angles ?

5. Are the opposite angles equal ?

6. Are the opposite sides equal ?

Of what type must a quadrilateral be if it has the properties mentioned below ?

7. The diagonals are equal and bisect each other.

8. One diagonal only is bisected at right angles by the other.

9. The diagonals bisect each other at right angles.

10. Both pairs of opposite angles are equal.

11. All the angles are equal and are bisected by the diagonals.

12. One pair of opposite sides are equal and parallel.

EXAMPLES 6 a

1. ABCD is a rectangle whose diagonals cut at O. If ∠CBD = 32°, calculate ∠AOB.

2. ABCD is a rhombus. If ∠BAD = 112°, calculate ∠ADB.

3. ABCD is a square whose diagonals cut at O. If E is a point on AB such that AE = AO, calculate ∠BOE.

---

4. AOB and COD are straight lines which intersect at right angles at O; OA = OC and OB = OD. Complete the squares AOCE, BODF, and prove that AC is parallel to DB.

5. With the data of No. 4, prove that OE and OF lie in the same straight line.

6. ABC, ADC are triangles, on opposite sides of a common base AC, such that $\begin{matrix}\text{ABC}\\\text{ADC}\end{matrix}$ are congruent. Prove that AC bisects BD at right angles.

7. If APB, AQB are two unequal isosceles triangles on opposite sides of a common base AB, prove that PQ bisects the angles at P and Q.

8. If ABCD is a rhombus and X a point on AC produced, prove that XB = XD.

9. In Fig. 103, ABCD and AXYZ are squares. Prove that BX = DZ. (Use congruence.)

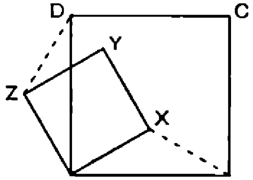


FIG. 103

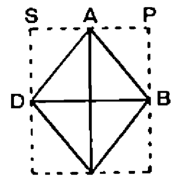


FIG. 104

10. In Fig. 104, ABCD is a rhombus. If PQ and SR are both parallel to AC, and PS and QR both parallel to BD, prove that PQRS is a rectangle.

11. A, B, C, D are four points in order on a straight line, and AB = BC = CD. A rhombus BCEF is drawn on BC as base. (i) Prove that DE is parallel to CF; (ii) write down a similar result, and deduce that DE and AF, if produced, would meet at right angles.

12. ABCD is a square whose diagonals cut at O. If X is a point in AB and Y a point in AD such that AX = AY = AO, prove by congruence that XY = AB.

---

13. Construct a rhombus with diagonals 2·7 and 2·2 in. long. Measure a side.

14. Without using a protractor, construct a rectangle with diagonals 3·2 in. long, and the angle included by the diagonals equal to 60°. Measure the sides.

15. Construct a trapezium ABCD in which AB = 4·1 cm., BC = 3·6 cm., AC = 6·0 cm., BD = 7·2 cm., and DC is parallel to AB.

16. With ruler and compasses only, construct a square ABCD of side 1·5 in., and construct a second square with AC as one side. Measure the diagonal of the second square.

---

17. ABCD is a square horizontal courtyard, and AE a vertical post. Prove that EB = ED.

18. In Fig. 105, V is the vertex of a right pyramid with a square base ABCD. Point out some pairs of congruent triangles in different planes, and prove the congruences.

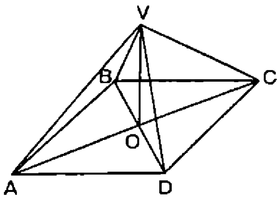


FIG. 105

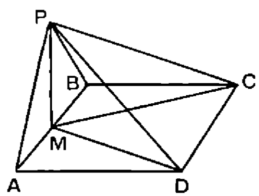


FIG. 106

19. Fig. 106 shows a pyramid with a rectangular base ABCD; M is the mid-point of AB, and the vertex P lies on the perpendicular at M to ABCD. Mention two pairs of congruent triangles in different planes, and prove the congruences.

EXAMPLES 6 b

1. ABCD is a kite in which AB=AD and CB=CD. If the diagonals cut at O and BC=2BO=2AO, calculate the angles of the kite.

2. ABCD is an isosceles trapezium, AB, DC being the parallel sides, and AD, BC the equal sides; the angles at C and D are acute. (i) Prove that ∠C=∠D. (Let the parallel to AD through B cut DC at E.) (ii) Prove by congruence that the diagonals are equal.

3. ABCD is a square. P, Q, R, S are points on AB, BC, CD, DA respectively such that AP=BQ=CR=DS. Prove that (i) △APS≡△BQP, (ii) PQRS is a square.

4. If, through each vertex of a rectangle, a line is drawn parallel to a diagonal, prove that a rhombus is formed.

5. ABCDEF is a regular hexagon with D joined to B and F. Prove a pair of triangles congruent, and deduce that BF is perpendicular to AD.

6. If ABCD and AECF are two rectangles with a common diagonal AC, prove that DEBF is also a rectangle.

7. ABCD is a parallelogram in which BC=2AB, and E, F are the mid-points of AD, BC respectively. If AF and BE cut at G, and CE and DF cut at H, prove that (i) the angles at G and H are right angles, (ii) EGFH is a rectangle.

8. Prove that the points of intersection of the internal bisectors of the angles of a parallelogram are the vertices of a rectangle. (Use the property of conjoined angles.)

---

9. Draw any straight line. Using straight edge and compasses only, construct a square with this line as one diagonal.

10. Construct a rhombus ABCD, given that AB=5 cm., AC=8 cm. Measure BD.

11. With ruler and compasses only, construct a rectangle ABCD with AB = 5·4 cm. and AC = 6·8 cm. Measure AD.

12. Construct a trapezium ABCD, given that AB is parallel to DC, AB = 1·2 in., BC = 1·7 in., CD = 3·0 in., and DA = 1·1 in.

(Suppose the straight line through B parallel to AD meets CD at E; start by constructing △BCE.)

13. Construct a trapezium ABCD, given that AB = 2·1 in., ∠A = 108°, ∠B = 38°, and DC is parallel to AB and equal to 1·2 in. Measure BC. (If E is a point on AB such that AE = DC, what kind of figure is AECD?)

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14. Fig. 107 shows a cube ABCDEFGH. Prove that (i) the edges of the tetrahedron BEDG are equal, (ii) the opposite edges of BEDG are perpendicular.

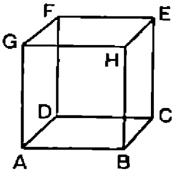


FIG. 107

15. Draw a rectangular parallelepiped lettered as in Fig. 107 (but with AB, AD, AG unequal). Prove that ∠FCH = ∠CHA.

16. With the data of No. 14, state the shortest distance between two opposite edges of the regular tetrahedron BEDG, given that AB = a cm.

17. A flagstaff is erected in the centre of a square grass-plot, and stayed by ropes running from the top of the flagstaff to the corners of the grass-plot. If the height of the flagstaff is equal to half the diagonal of the grass-plot, prove that each rope is as long as a side of the grass-plot.

18. ABCD is the square base of a right pyramid with vertex V; M, N are the feet of the perpendiculars from A, C respectively to VB. Prove that (i) △VBA ≡ △VBC, (ii) △ABM ≡ △CBN, and deduce that the points M and N coincide.

19. Draw a parallelepiped, and inscribe a tetrahedron in it (compare BEDG, in Fig. 107). Hence prove that, if the opposite edges of a tetrahedron are equal, (i) the parallelepiped in which it is inscribed must be rectangular, (ii) the straight lines joining the mid-points of opposite edges of the tetrahedron must be perpendicular to one another.

EXAMPLES 6 c

1. AB and AC are two straight lines enclosing an angle of 70°. Squares ABDE, ACFG are drawn outside the angle BAC. The diagonal FA is produced to meet the diagonal EB in H. Find the angles EAH, HAB. (W)

2. ABCD is a square; P is a point in DC and Q is a point in AD such that AP=BQ. Prove that AP and BQ cut at right angles.

3. ABCD is a square, and H is any point in CD. If E is a point in DA such that DE=CH, and a square DEFG is drawn outside ABCD, prove that HB and HF are equal and perpendicular.

4. AC is a diagonal of a square ABCD. P is any point within the triangle ACD, and APRS is a square such that R and D are on the same side of AC. Prove that the triangles APB, ASD are congruent. Prove also that the perpendicular from A on SD is parallel to BP. (O)

5. ABCD is a square and the point E on the diagonal AC is such that AE=AB; the perpendicular to AC through E meets BC at F and DC at G. Prove that the angle GAF=45°. (C)

6. ABCD is a square; a line AX meets the diagonal BD at X. Prove that the triangles ABX and CBX are congruent.

If a line BY, at right angles to AX, meets the diagonal AC at Y, prove that the triangles ABX and BCY are congruent. (C)

7. PABQ is a straight line and PA=AB=BQ. A parallelogram ABCD is drawn in which BC=2AB. Prove that PC and QD, when drawn, intersect at right angles. (W)

8. Given that the four sides of a quadrilateral are equal, show that its angles are bisected by its diagonals.

A line drawn parallel to the base BC of an isosceles triangle ABC meets the equal sides AB, AC in E, F respectively. Lines through E, F parallel to AC, AB meet in G. Show that AG, produced if necessary, will bisect BC. (W)

9. If APB and CQD are two parallel straight lines, prove that the bisectors of the angles APQ, BPQ, CQP, and PQD form a rectangle. (C)

10. If the diagonals of a parallelogram are equal, prove that the parallelogram is a rectangle.

11. XAYB and XCYD are two kites, XY being in each case the diagonal which bisects a pair of opposite angles. Prove that ABDC is an isosceles trapezium.

12. A triangle ABC is right-angled at A and BE is drawn from B at right angles to BC, equal to BC and on the side of BC opposite to A; the perpendicular from E to AC (produced if necessary) meets it at F. Prove that EF is equal to the sum of AB and AC. (C)

13. E is any point in the side BC of a rectangle ABCD. EF is drawn parallel to the diagonal CA to meet AB in F, and EG parallel to BD to meet CD in G. Show that EF and EG together make up the length of a diagonal. (W)

14. P is any point on the base BC of an isosceles triangle ABC; CN is an altitude. If L, M are the feet of the perpendiculars from P to AB, AC respectively, prove that PL+PM=CN.

MISCELLANEOUS EXAMPLES I

1. PQRS is a square; PS is produced to T, and Q, S are joined.

(i) Bisect, by accurate geometrical construction, the angle QST.

(ii) If, in the above figure, the bisector of the angle QST meets PR produced at X, *calculate* the number of degrees in the angle RXS, and prove that RS=RX. (L)

2. ABCDE is a regular pentagon. On AB the square ABPQ is described lying entirely within the pentagon. Calculate the number of degrees in each of the angles CBP, PBD. (N)

3. Any point O is taken inside a triangle ABC and the parallelograms OBDC, OCEA, OAFB are completed. Prove that the triangles ABC, DEF are equal in all respects. (C)

4. ABC is a triangle right-angled at A, BCDE is the square on BC described outside the triangle, EG is drawn parallel to BA to meet AC (produced if necessary) in G, and BF is drawn parallel to AC to meet EG in F; prove that ABFG is a square. (OC)

5. O is the vertex of a symmetrical pyramid of height 5·7 cm. with a square base ABCD of side 5 cm. Make an accurate drawing of the face OAB.

6. Draw a triangle **ABC** such that **B**$=40°$, **C**$=120°$, **BC**$=2$ in. Showing all construction lines and using ruler and compasses only, bisect the angle **ABC** by a line **BX** meeting **AC** in **X**, and draw the perpendicular from **C** to **AB**, meeting **BX** in **O**.

Prove in this figure that **XO**$=$**XC**. (C)

7. If **ABCD** is a quadrilateral in which $\angle$**A**$=\angle$**B**$=90°$ and $\angle$**D**$=135°$, prove that **BC**$=$**AB**$+$**AD**.

8. **ABCD** is a quadrilateral such that the bisectors of the four interior angles meet at a point **O**. If $\angle$**AOB**$=57°$, *calculate* $\angle$**COD**. (OC)

9. The sides **AB**, **AC** of the triangle **ABC** are produced to **F**, **E** respectively; the bisectors of the angles **ABC**, **ACB** meet in **X**, and those of the angles **CBF**, **BCE** meet in **Y**; show that the angle **BXC** is obtuse and the angle **BYC** acute. (OC)

10. **OABCD** is a right pyramid on a square base **ABCD** of side 2 in., and the slant faces **OAB**, **OBC**, **OCD**, **ODA** are all equilateral triangles. Determine by drawings the height of the pyramid and the angle **AOC**. (OC)

---

11. The sides **AB**, **AC** of a triangle **ABC** are equal; **BC** is produced through **C** to **D** so that **CD**$=$**CA**. The points **D** and **A** are joined. If the angle **BAD**$=132°$, calculate the angle **BAC**. (C)

12. In Fig. 108, **AB**$=$**AC**; also **ABD**, **ACE** are equilateral triangles. Prove that (i) the angle **AFG**$=$the angle **AGF**, (ii) **DFGE** is parallel to **BC**. (OC)

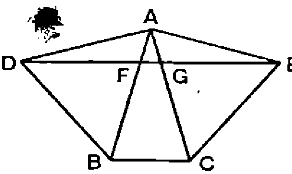


FIG. 108

13. **ABC** is a triangle right-angled at **A**, and **D** is the foot of the perpendicular from **A** to **BC**. If the bisector of $\angle$**BAD** meets **BC** at **E**, prove that **CE**$=$**CA**.

If the bisector of angle **CAD** meets **BC** at **F**, state a similar result.

14. If **ABCD** is a quadrilateral such that $\angle$**A**$=60°$, $\angle$**B**$=\angle$**C**$=120°$, and **AB**$=$**BC**, prove that **AD**$=$**AB**$+$**CD**.

15. In Fig. 109, **SA**$=$**SB**, **SD**$=$**SC**, **AD**$=$**BC**. Prove that **AC**$=$**BD**. (OC)

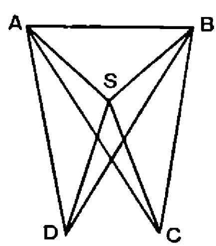


FIG. 109

THEOREM 12 a (Mid-point Theorem)

The straight line joining the mid-points of two sides of a triangle is parallel to and one-half of the third side.

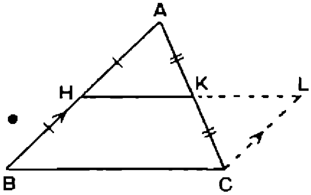


FIG. 110

Given that H, K are the mid-points of the sides AB, AC of △ABC.

To prove that HK ∥ BC, HK$=\frac{1}{2}$BC.

Construction. Draw CL ∥ BA to meet HK produced at L.

Proof. In the △s AHK, CLK,

AK=CK (given),

∠AKH=∠CKL (vert. opp.),

∠AHK=∠CLK (alt., HA ∥ CL);

∴ △s AHK, CLK are congruent (AA corr. S).

∴ HK=LK,

and AH=CL.

But AH=BH;

∴ BH=CL, and BH ∥ CL (constr.);

∴ BHLC is a parallelogram.

∴ HL ∥ BC and HL=BC (opp. sides of ∥gram).

But HK=LK (proved);

∴ HK ∥ BC and HK$=\frac{1}{2}$BC.

Reference. *AH=HB and AK=KC,*

∴ HK ∥ BC and HK$=\frac{1}{2}$BC (*mid-point theorem*).

THEOREM 12 b (Converse of Mid-point Theorem)

A straight line drawn through the mid-point of one side of a triangle, and parallel to another side, bisects the third side.

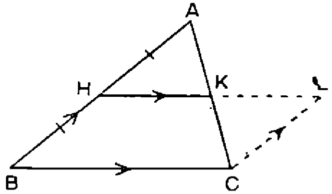


FIG. 111

Given a triangle ABC, with H the mid-point of AB, and HK ∥ BC.

To prove that AK = CK.

Construction. Draw CL ∥ BA to meet HK produced at L.

Proof.

BHLC is a parallelogram (both pairs of opp. sides parallel);

∴ CL = BH (opp. sides of ∥gram).

But BH = AH (given);

∴ AH = CL.

In the △s AHK, CLK,

AH = CL (proved),

∠AKH = ∠CKL (vert. opp.),

∠AHK = ∠CLK (alt., HA ∥ CL);

∴ △s AHK, CLK are congruent (AA corr. S).

∴ AK = CK.

Reference.

AH = HB and HK ∥ BC,

∴ AK = CK (*converse of mid-point theorem*).

EXAMPLES 7 a

1. In Fig. 112, if BD=DC and DE is parallel to BA, what follows? If EF is then drawn parallel to CB, what follows? If FD is joined, what follows?

2. In Fig. 112, if D, E, F are the mid-points of the sides of the triangle ABC, prove that △s DEF and ABC are equiangular.

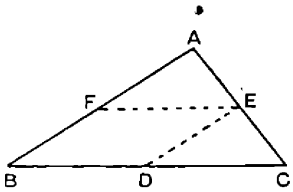


FIG. 112

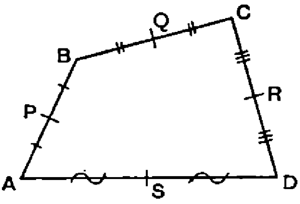


FIG. 113

3. In Fig. 113, what follows by applying the mid-point theorem to △ABC? Prove that PQRS is a parallelogram.

4. In Fig. 113, prove that PR and QS bisect each other.

5. From a point O, unequal straight lines OX, OY, OZ are drawn. If L, M, N are their respective mid-points, prove that triangles LMN and XYZ are equiangular.

6. In Fig. 114, P, Q, R, S, T, U are the mid-points of BC, OC, CA, OA, AB, OB. Prove that the opposite sides of the hexagon PQRSTU are equal and parallel.

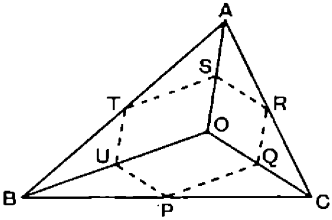


FIG. 114

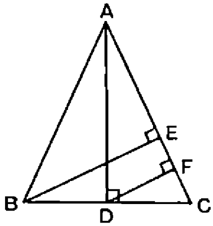


FIG. 115

7. In Fig. 115, if AB=AC, prove that BE=2DF.

8. H, K are the mid-points of the sides AB, AC of a triangle ABC. If P is any point on BC, prove that HK bisects AP.

9. H, K are the mid-points of the sides AB, AC of a triangle ABC, and O, any point on BC, is joined to A. If L, M are the mid-points of OB, OC respectively, prove that HKML is a parallelogram.

10. ABC is an equilateral triangle, and O is any point not in the plane ABC. If D, E, F are the mid-points of OA, OB, OC respectively, prove that $\triangle$DEF is equilateral.

EXAMPLES 7b

1. H, K are the mid-points of the sides AB, AC of a triangle ABC. If BC is produced to D so that CD=$\frac{1}{2}$BC, prove that CK and DH bisect each other.

2. X is the mid-point of the side AB of a parallelogram ABCD, and the straight line through B parallel to XD meets AD produced at Y. Prove that AY=2BC.

3. In Fig. 116, prove that DP is parallel to CA.

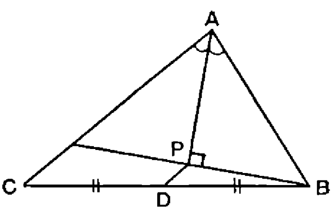


Fig. 116

4. D, E, F are the mid-points of the sides BC, CA, AB respectively of a triangle ABC, and the triangle DEF is drawn. Prove that four triangles in the figure are congruent.

5. ABCD is a quadrilateral having DC parallel to AB. Prove that the straight line joining the mid-points of CA and CB bisects AD.

6. In Fig. 113, if AC=BD, prove that PQRS is a rhombus.

7. In Fig. 113, if AC is perpendicular to BD, prove that PQRS is a rectangle.

8. In Fig. 113, if PQRS is to be a square, what properties *must* ABCD possess ?

9. In Fig. 117, prove that (i) EX=$\frac{1}{2}$DA, (ii) EB=$\frac{1}{2}$(DA+FC).

10. In Fig. 117, if CD cuts BE at Y, prove that BY=EX.

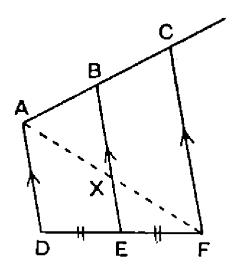


Fig. 117

11. E, F are the mid-points of the respective sides AC, AB of a triangle ABC; BE and CF intersect at G, and AG is produced to X so that GX = AG. By applying the mid-point theorem to Δs ABX and ACX, prove that (i) BGCX is a parallelogram, (ii) EG = $\frac{1}{2}$BG.

12. ABC is the base and V the vertex of a tetrahedron. If D, E, F, G are the mid-points of VB, VC, AB, AC respectively, prove that DEGF is a parallelogram.

13. From No. 12 deduce that the straight lines joining the mid-points of the opposite edges of a tetrahedron bisect one another and have a point in common.

EXAMPLES 7 c

1. ABCD is a parallelogram, and E, F are the mid-points of CB, CD respectively. If EF cuts AC at G, prove that AC = 4GC.

2. D is the middle point of the side BC of a triangle ABC. The line AD is bisected at E, and BE is produced to cut AC at S. The line DT drawn parallel to BS cuts AC at T. Prove that AS = ST = TC. (N)

3. ABCD is a quadrilateral, and P, Q, R, S are the mid-points of AD, AC, BD, BC respectively. Prove that (i) PR and QS are equal and parallel, (ii) if AB is parallel to DC, then P, Q, R, S lie in a straight line.

4. Given the positions of the mid-points of the sides of a triangle, state how to construct the triangle.

5. ABC is a triangle, D is the mid-point of BC, and G is the mid-point of AD. If the straight lines through D, G, parallel to BA, meet AC at E, H respectively, prove that (i) AH = $\frac{1}{4}$AC, (ii) GH = $\frac{1}{4}$ BA.

6. ABC is a triangle and D, E are the feet of the perpendiculars from A, B to BC, CA respectively. If AD cuts BE at H, and P, Q, R are the mid-points of AH, AB, BC respectively, prove that ∠PQR = 90°.

7. In Fig. 113, if PR = QS, prove that AC is perpendicular to BD.

8. D, E, F are the mid-points of the sides BC, CA, AB respectively of a triangle ABC. If DF cuts BE at X and DE cuts CF at Y, prove that XY is parallel to BC and XY $=\frac{1}{4}$BC.

9. ABCD is a parallelogram, and E, F are the mid-points of AB, DC respectively. If AC cuts DE at H and FB at K, prove that DEBF is a parallelogram, and deduce that (i) AH=HK and HK=KC, (ii) HE $=\frac{1}{2}$KB and HE $=\frac{1}{3}$DE.

10. Draw an angle ABC and mark a point O inside it. Find by construction a point X in AB and a point Y in BC such that XOY is a straight line bisected at O.

11. ABCD is a parallelogram, and V a point not in the plane ABCD. If P, Q, R, S are the mid-points of VA, VB, VC, VD respectively, prove that PR bisects QS.

12. The legs of a table with a rectangular top are 36 in. long. If 6 in., 3 in., and 9 in. respectively are cut off three consecutive legs, what length must be cut off the fourth leg in order that the table, though with its top no longer horizontal, may rest evenly on its four legs ?

INTERCEPT THEOREM

An **intercept** is a piece 'cut off.' In Fig. 118 the four parallels cut off from ABCD the pieces AB, BC, CD ; hence AB, BC, CD are the 'intercepts' made by the parallels on ABCD.

THEOREM 13 (The Intercept Theorem)

If three or more parallel straight lines make equal intercepts on one transversal, they will make equal intercepts on any other transversal.

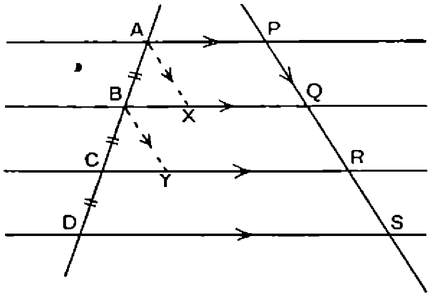


FIG. 118

Given AP ∥ BQ ∥ CR ∥ DS, AB=BC=CD.

To prove that PQ=QR=RS.

Construction. Through A, B draw AX, BY parallel to PQRS to cut BQ, CR at X, Y respectively.

Proof. In the △s ABX, BCY,

AB=BC (given),

∠ABX=∠BCY (corr., BQ ∥ CR),

∠BAX=∠CBY (corr., AX ∥ BY);

∴ △s ABX, BCY are congruent (AA corr. S).

∴ AX=BY.

Now APQX, BQRY are ∥grams. (both pairs of opp. sides parallel);

∴ AX=PQ, BY=QR (opp. sides of ∥gram.).

∴ PQ=QR.

Similarly, QR=RS.

Reference.

AB=BC=CD and AP ∥ BQ ∥ CR ∥ DS,

∴ PQ=QR=RS (*Intercept theorem*).

CONSTRUCTION 13

To divide a given straight line into n equal parts.

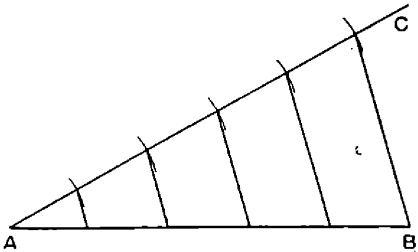


FIG. 119

Let AB be the given straight line.

Through A draw another straight line AC at a suitable angle to AB.

Using any suitable radius, and starting with A as centre, draw arcs so as to mark off n equal lengths along AC.

Draw the straight line joining the last point of intersection to B.

Through the other points of intersection draw parallels to this straight line to cut AB.

Then these parallels divide AB into n equal parts.

Proof.

The intercepts made by the parallels on AC are equal;

∴ the intercepts made by them on AB are equal.

CONSTRUCTIONS: STAGE II

If several parallels have to be constructed, as occurs in Construction 13, the process is apt to be laborious, and the use of so many arcs (not shown in the printed figure) tends to obscure the figure. **In future set-squares may be used, unless the question says otherwise, to construct parallels, perpendicu-**

lars, and angles of 60°, 45°, and 30°. As you know how to perform these constructions without set-squares, it is clear that the new rule does not contradict, in spirit, the definition of a construction given on p. 36.

There is another way in which trouble may be saved and accuracy increased without infringing the spirit of the definition. **At any stage of a construction, the protractor may be used to mark off a *given* angle, and the ruler to mark off a *given* length.** This makes it unnecessary to draw the data in the margin and then transfer them by copying. Whenever a ruler or protractor is used in this way, the measurement should be marked on the line or angle.

CONSTRUCTION 14

To construct a straight line parallel to a given straight line and at a given distance from it.

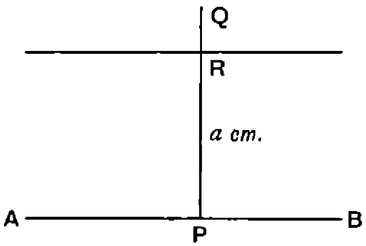


FIG. 120

Let AB be the given straight line, and *a* cm. the given distance.

Mark any point P in AB, and erect the perpendicular PQ to the straight line AB.

Along PQ, mark off PR equal to *a* cm.

Construct the straight line through R parallel to AB.

This is the required parallel.

CONSTRUCTION 15

To slide a straight line of given length and direction into a given angle.

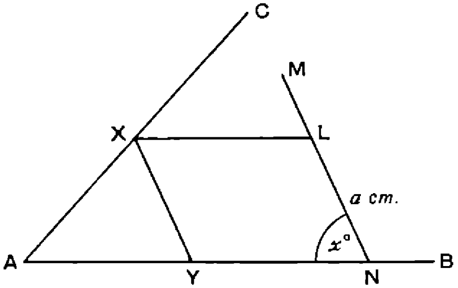


FIG. 121

Suppose it is required to slide a straight line of length a cm. into the angle BAC in such a way that the straight line has its ends on AB and AC and is inclined to AB at an angle $x°$.

Mark any point N in AB.

Make at N an angle ANM equal to $x°$.

Along NM mark off NL equal to a cm.

Draw through L a straight line parallel to BA, to cut AC at X.

Draw through X a straight line parallel to LN, to meet AB at Y.

Then XY is the required straight line.

Proof. By construction, NL is equal and parallel to the required straight line.

But NLXY is a ||gram (both pairs of opp. sides parallel);

$\therefore$ YX is equal and parallel to NL.

But X, Y lie on AC, AB respectively;

$\therefore$ YX is the required straight line.

EXAMPLES 8

1. A, B, C are three points on a straight line, and AB = BC. The perpendiculars from A, B, C to a second straight line meet it at P, Q, R respectively. Prove that BP = BR.

2. The side AC of a triangle ABC is produced to E so that CE = $\frac{1}{2}$AC; D is the mid-point of BC, and ED produced meets AB at F. By drawing straight lines through C and D parallel to BA, prove that DE = 2DF.

3. In Fig. 122, prove that AC = CD = DB.

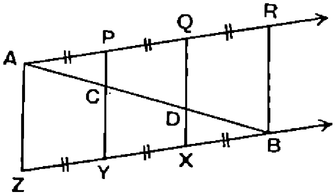


FIG. 122

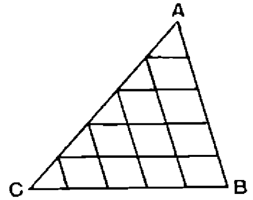


FIG. 123

4. In Fig. 123, four straight lines parallel to BC divide AB into five equal parts. Prove that (i) the four straight lines parallel to AB divide BC into five equal parts, (ii) the shortest line parallel to BC equals $\frac{1}{5}$BC.

5. Fig. 124 shows four parallels at *any* equal distances apart. If AB = 2 in. and GH = 3 in., prove that CD = 2$\frac{1}{3}$ in. and EF = 2$\frac{2}{3}$ in.

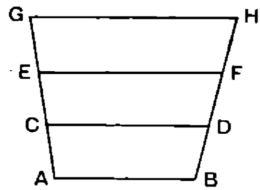


FIG. 124

Given a ruler marked in inches only, the figure shows a way of constructing a scale (called a 'diagonal scale') from which measurements in inches and thirds of an inch can be made.

Construct a diagonal scale for measuring lengths up to 4 in., in inches and sevenths of an inch.

6. A straight line parallel to the side BC of a triangle ABC cuts AB at H and AC at K. If AH/HB = 5/7, prove, by drawing parallels to BC, that AK/KC = 5/7.

7. In the intercept theorem, prove that the intercepts on the one transversal are *not* in general equal to the intercepts on the other transversal. What are the two exceptional cases?

8. In Fig. 125 there are three rectangles. If AB = BC, prove that BQ − AP = CR − BQ.

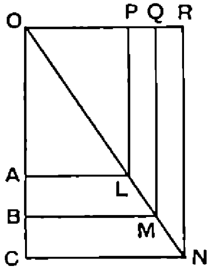


Fig. 125

9. B, C, D are three points in order on a straight line; BC = 3 in., CD = 2 in. Construct on the same side of the line two congruent triangles ABC, CDE right-angled at B and D. Construct also a straight line which is divided into three equal parts by the sides of the triangles other than BC and CD. (G)

10. Any straight line through O, the point of intersection of the diagonals of a parallelepiped ABCDEFGH (Fig. 98, p. 63, shows how the letters should be placed) meets the plane ABCD at X and the plane EFGH at Y. By drawing a plane through O parallel to ABCD, prove, with the help of the three-dimensional form of the intercept theorem in which the two transversals cut three parallel *planes*, that AX = EY.

---

11. Draw a straight line 3 in. long, and divide it into 7 equal parts.

12. Draw a straight line AB, 5·3 cm. long, and construct a point X on it such that AX = 2XB.

13. Draw a straight line AB, 3·7 in. long, and construct a point X, on AB produced, such that

$$\frac{AX}{BX} = \frac{7}{3}.$$

14. Draw a large square, and divide it into 9 equal squares.

15. Draw a non-convex pentagon ABCDE (see Fig. 126), and construct the pentagon APQRS which has AP equal to $\frac{2}{5}$AB, and sides parallel to the respective sides of ABCDE.

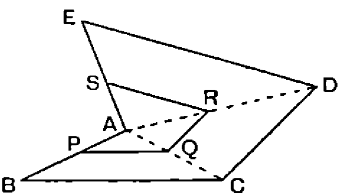


Fig. 126

16. Draw an angle ABC and mark a point O within it. Find by construction a point X in AB and a point Y in BC such that XOY is a straight line and 2OX = 3OY.

17. Draw any straight line, and construct a parallel straight line 4 cm. distant from it.

18. Draw an angle **ABC** of 37°. On the same side of **BA** as **C**, construct a straight line parallel to **BA** and 3 cm. from it; on the same side of **BC** as **A**, construct a straight line parallel to **BC** and 2 cm. from it, to cut the previous parallel at **D**. Measure **BD**.

19. Construct a trapezium **ABCD** which has **AB** and **DC**, the two parallel sides, equal respectively to 1 in. and 2·1 in., ∠**C**=72°, and ∠**D**=50°. (Hint: Begin by drawing **DC**, ∠**C** and ∠**D**, and use Construction 15.)

20. Construct a quadrilateral **ABCD** in which **AB**=3·8 cm., **CD**=6 cm., ∠**A**=140°, ∠**C**=80°, and ∠**D**=75°.

LOCI

A dozen points, marked at random on a sheet of paper, are not likely to form a recognisable pattern ; and, even if they do, the addition of more points, still chosen at random, will tend to spoil whatever pattern there is. Suppose, however, we mark some points which all have a particular property, for instance the property of being 0·6 cm. from the nearest point of a straight line AB 1·4 cm. long. As one point after another is marked, it will become increasingly clear that a pattern is being formed ; and, *the more points we add, the more complete the pattern will become.* The pattern formed by *all* points which have a common property is called the **locus** of the points ; the process of marking enough points to see what a locus will be is described as **plotting the locus.** In the example given, the locus consists of two parallel straight lines joined by two semicircles, as shown in Fig. 127.

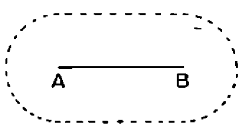


FIG. 127

If the given straight line AB, instead of being 1·4 cm. long, were to be regarded as continuing in both directions to an unlimited extent, the semicircular ends of the locus would disappear, and the locus would take the simpler form of two parallel straight lines. Where, in connection with loci, we speak of a 'given straight line' without mentioning its length, it is to be understood that the straight line is unlimited in both directions.

EXAMPLES 9 a

1. O is a point, and AB a straight line not passing through O. Plot the locus of the mid-points of straight lines joining O to points on AB.

2. ABC is a triangle; any straight line parallel to BC meets AB at X and AC at Y. Plot the locus of the mid-point of XY.

3. P is any point on the arm OA of an acute angle AOB. The straight line through P perpendicular to OA meets OB at Q, and is produced to R so that QR = PQ. Plot the locus of R.

4. Draw a circle of radius 2 in. and mark a point A $2\frac{1}{2}$ in. from the centre. Plot the locus of the mid-points of straight lines joining A to points on the circumference of the circle.

5. Mark a point O near the centre of a sheet of squared paper, and through it draw two perpendicular straight lines XOX′, YOY′ (see Fig. 128). P is any point on the paper, and M, N are the feet of the perpendiculars from P to XOX′, YOY′ respectively. Plot the locus of P if (i) PM = 2PN, (ii) PM + PN = 3 in., (iii) PM − PN = $\frac{1}{2}$ in.

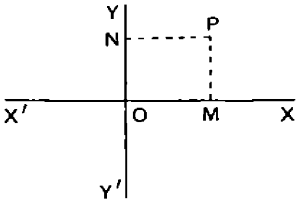


FIG. 128

6. Draw XOX′, YOY′ as in No. 5. Place a ruler so that the zero mark is on YOY′ and the 3 in. mark is on XOX′. Plot the locus of the $1\frac{1}{2}$ in. mark.

When we speak of **the distance of a point from a straight line,** we mean the length of the perpendicular from the point to the straight line (produced if necessary). Thus, in Fig. 129, the distance of X from PQ is XM. If, in the same figure, XM=XN, we say that the point X is **equidistant** from the straight lines PQ and RS.

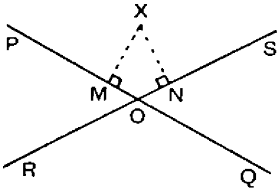


FIG. 129

There are four loci of particular importance :—

(i) **The locus of points at a given distance from a given point A is a circle.**

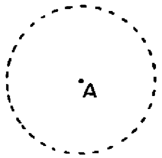


FIG. 130

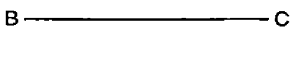


FIG. 131

(ii) **The locus of points at a given distance from a given straight line BC is two straight lines parallel to BC.**

(iii) **The locus of points equidistant from two given points D and E is the perpendicular bisector of DE.**

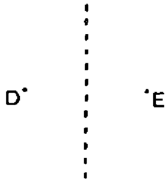


FIG. 132

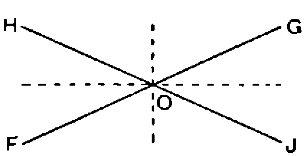


FIG. 133

(iv) **The locus of points equidistant from two given intersecting straight lines FOG and HOJ is the pair of perpendicular straight lines bisecting the angles between the given straight lines.**

Statement (i) above is the definition of a circle. The given point A is called the ‘ centre ’ of the circle, and the given distance is called the 'radius.' Statement (ii) may be justified by the properties of the rectangle; (iii) and (iv) are proved in Theorems 14 and 15.

Worked example to illustrate the method of the Intersection of Loci

Draw a triangle ABC *in which* BC=2·4 *in.*, CA=1·8 *in.*, *and* AB=1·3 *in. Find as many positions as possible of a point* P *which is equidistant from* BA *and* BC *and is such that* AP=AC.

P has to satisfy two conditions: (i) it is to be equidistant from BA and BC, (ii) it is to be such that AP=AC.

All points which satisfy (i) lie on the two perpendicular straight lines which bisect ∠ABC and the other three angles formed by producing AB and CB.

All points which satisfy (ii) lie on the circle which has centre A and radius AC.

The point P, which has to satisfy both (i) and (ii), must therefore lie on both loci.

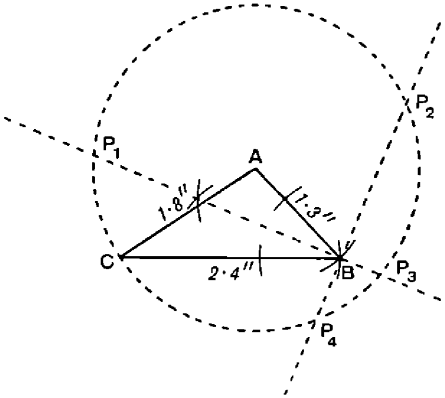


FIG. 134

In the figure the two loci are shown by dotted lines. It will be seen that there are four possible positions for P, namely P_1, P_2, P_3, P_4.

EXAMPLES 9 a (*continued*)

In the following questions produce the given straight lines in both directions as far as necessary.

Where more than one solution is possible, find all the possible solutions.

7. Draw a straight line AB, and mark a point C 3 cm. from it. Construct the locus of (i) points 1 cm. from AB, (ii) points 5 cm. from C. Indicate the points which satisfy both conditions.

8. Draw a triangle ABC having BC=4 cm., ∠B=110°, ∠C=40°, and produce the sides in both directions. Construct the locus of (i) points 1 cm. from AC, (ii) points equidistant from AB and AC. How many points satisfy both conditions? Measure their distances from C.

9. A treasure is hidden in a field in which there are three trees, an ash (A), a beech (B), and a chestnut (C); BC=300 yd., CA=210 yd., and AB=165 yd. Draw a plan on a scale of 1 in. to 100 yd. If the treasure is at the same distance from the chestnut as from the beech, and is 60 yd. from the ash, determine its position on the plan by constructing two loci, and find its distance from the beech.

10. Draw two straight lines, AOB and COD, cutting at 60° at the point O. By constructing two loci, show that there are four points which are 1 cm. from AB and 2 cm. from CD. Measure their distances from O.

11. A and B are two points, $1\frac{1}{2}$ miles apart, on a straight stretch of shore, B being due east of A. A ship is 2 miles from the shore and bears N. 60° E. from A. Find by construction its distance from B. (Scale: 1 in. to 1 mile.)

12. Draw a quadrilateral ABCD. (i) Find by construction a point on BD equidistant from AB and AD. (ii) Also find on BD (produced either way if necessary) a point equidistant from B and C.

13. Construct a triangle ABC in which ∠A=90°, BC=2·8 in., and AB=2·1 in. Measure AC.

14. Construct a triangle ABC, given that BC=2·2 in., AC=2 in., and the perpendicular distance of A from BC is 1·45 in. Measure AB. (First draw BC; then construct two loci for A.)

There is another aspect of a locus which is often helpful. When locus (i) on p. 94 is drawn with compasses, we see it described by a point (the point of the pencil) which moves along the paper so as always to be at the given distance from A. Thus **a locus may be regarded as the path traced out by a point which moves according to a given law.** As an exercise, the pupil should try to re-state loci (ii), (iii), (iv) of p. 94 in language of this kind.

EXAMPLES 9 a (*continued*)

(*Nos.* 15 *to* 21 *should be taken orally*)

State as precisely as possible the locus of the following :—

15. A door-knob.

16. The head of a boy sliding down the banisters.

17. The mid-point of the portion of a straight line intercepted by two given parallel straight lines.

18. A boy who walks so as to keep 3 ft. from the nearest point of the edge of a lawn in the shape of (i) a circle, (ii) a rectangle.

19. The vertex of a set-square whose longest side is hinged at both ends to a fixed horizontal rod.

20. The centre of a circular disc rotating in its own plane about a point not at its centre.

21. The centre of the base of a right circular cone rolling on a table.

Suppose we have guessed, by plotting, that a locus is a certain curve. There is a further step to take, namely, to make sure that *all* points on the curve form part of the locus. For the locus consists of all points satisfying the given conditions, *and no other points.* To see the need for this further step, mark two points B and C, 4 cm. apart, and consider the locus of points which are 3 cm. from one of the two given points and never less than 3 cm. from the other. Points 3 cm. from B lie on the circle with centre B and radius 3 cm.; points 3 cm. from C lie on the circle with centre C and radius 3 cm. Draw these two circles, and then ask yourself if *all* points on the two circles belong to the locus.

G

THEOREM 14

(i) A point on the perpendicular bisector of the straight line joining two given points is equidistant from them.

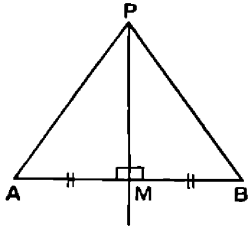


FIG. 135

Given two points A and B, M the mid-point of AB, and P any point on the perpendicular bisector PM of AB.

To prove that PA = PB.

Proof. In the $\triangle$s PAM, PBM,

AM = BM (given),

$\angle$AMP = $\angle$BMP (90°),

PM = PM ;

$\therefore$ $\triangle$s $\begin{matrix}\text{PAM}\\\text{PBM}\end{matrix}$ are congruent (SAS).

$\therefore$ PA = PB.

(ii) **A point equidistant from two given points lies on the perpendicular bisector of the straight line joining them.**

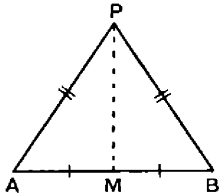


FIG. 136

Given that PA = PB.

Construction. Let M be the mid-point of AB. Join PM.

To prove that PM is perpendicular to AB.

Proof. In the $\triangle$s PAM, PBM,

AM = BM (constr.),

PA = PB (given),

PM = PM;

$\therefore$ $\triangle$s $\begin{matrix} \text{PAM} \\ \text{PBM} \end{matrix}$ are congruent (SSS).

$\therefore \angle PMA = \angle PMB$.

But $\angle PMA + \angle PMB = 180°$ (adj., AMB a st. line);

$\therefore \angle PMA = \angle PMB = 90°$.

Summary. The two parts of this theorem together prove that **the locus of points equidistant from two given points is the perpendicular bisector of the straight line joining them.** For, by (ii), *all* the points whose locus is required lie on the perpendicular bisector, and by (i) *no other* points lie on the perpendicular bisector.

THEOREM 15

(i) **A point on the bisector of one of the angles formed by two intersecting straight lines is equidistant from the lines.**

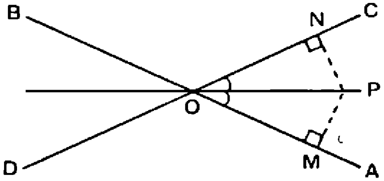


FIG. 137

Given two straight lines AOB, COD, and a point P on the bisector of ∠AOC.

Construction. Let M, N be the feet of the perpendiculars from P to OA, OC.

To prove that PM = PN.

Proof. In the △s POM, PON,

∠POM = ∠PON (given),

∠PMO = ∠PNO (90°),

PO = PO ;

∴ △s POM, PON are congruent (AA corr. S).

∴ PM = PN.

(ii) **A point equidistant from two given intersecting straight lines lies on one of the bisectors of the angles formed by them.**

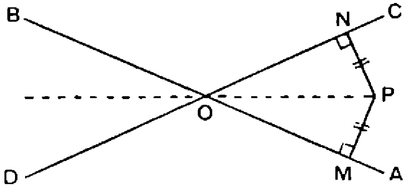


FIG. 138

Given two straight lines AOB, COD, and a point P, within the angle AOC, such that its perpendicular distances PM, PN from OA and OC are equal.

Construction. Join PO.

To prove that $\angle POM = \angle PON$.

Proof. In the right-angled Δs POM, PON,

PMO and PNO are the right angles (given),

PO = PO,

PM = PN (given);

$\therefore$ Δs $\frac{POM}{PON}$ are congruent (RHS).

$\therefore$ $\angle POM = \angle PON$.

$\therefore$ P lies on the bisector of $\angle AOC$.

Similarly, if P is within one of the angles COB, BOD, or DOA, it will lie on the bisector of that angle.

Note.—The fact that the four bisectors form two perpendicular straight lines may be proved as follows :—

Let the bisectors be OP, OQ, OR, OS.

$\angle AOC + \angle COB = 180°$ (adj., AOB a st. line);

$\therefore$ $\frac{1}{2}\angle AOC + \frac{1}{2}\angle COB = 90°$.

$\therefore$ $\angle POQ = 90°$.

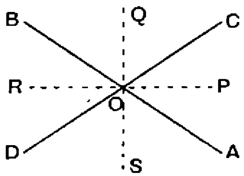


FIG. 139

Similarly, angles QOR, ROS, SOP are right angles ;

$\therefore$ the four bisectors form two perpendicular straight lines.

Summary. The two parts of this theorem together prove that **the locus of points equidistant from two given intersecting straight lines is the pair of perpendicular straight lines which bisect the angles between the given straight lines.** For, by (ii), *all* the points whose locus is required lie on the bisectors, and by (i) *no other* points lie on the bisectors.

EXAMPLES 9 b

(*Nos.* 1 *to* 9 *should be taken orally*)

State the locus in space of points which are :—

1. At the same distance from a given point.

2. At the same distance from a given plane.

3. Equidistant from two given points.

4. Equidistant from two given parallel planes.

5. Equidistant from two given intersecting planes.

6. At the same distance from a given straight line of unlimited length.

7. Equidistant from two given parallel straight lines of unlimited length.

8. Equidistant from two given intersecting straight lines of unlimited length.

9. Equidistant from three given points.

---

10. **SOH** is a straight line across the middle of a sheet of squared paper; **O** is near the centre of the paper, and **SO** = **OH** = 2 in. With centre **S**, draw circles of radii 1, $1\frac{1}{2}$, 2, $2\frac{1}{2}$ . . . 5 in., and do the same with centre **H**. Hence plot the locus of points **P** such that **PS** + **PH** = 6 in. (The locus is an ellipse with **S** and **H** as foci.)

11. Draw a straight line near one edge of a sheet of squared paper, and mark a point **S** 1 in. from it (near the mid-point of the line). Plot the locus of points whose distances from the line are equal to their distances from **S**. (The locus is a parabola with **S** as focus.)

12. Mark two points, **A** and **B**, 3 in. apart. Through each point draw straight lines, produced both ways, making with **AB** angles of 15°, 30°, 45° . . . 165°. Hence plot the locus of points **P** such that $\angle$**APB** = 60°.

13. Draw an angle **BAC**, and mark off centimetres from **A** along both arms, up to 10 cm.

If **P**, **Q** are points on **AB**, **AC** respectively such that **AP** + **AQ** = 10 cm., plot, by means of a ruler, the locus of the mid-point of **PQ**.

14. Mark two points, **A** and **B**, 6 cm. apart. Plot the locus of points **P** such that **PA**$=\frac{1}{2}$**PB**, taking 4, 5, 7, 9, 11, 12 cm. as the lengths of **PB**. (This locus is associated with Apollonius.)

In the following questions produce the given straight lines in both directions as far as necessary.

Where more than one solution is possible, find all the possible solutions.

15. **OA** is a straight breakwater, 750 yd. long, with a lighthouse at **A**. Find by construction the distance from **O** of a ship which is 190 yd. from the breakwater and 280 yd. from the lighthouse.

16. Copy the triangle **ABC** of Ex. 9*a*, No. 8, p. 96, and find by construction the points which are equidistant from **A** and **B** and also equidistant from **CA** and **CB**. Measure their distances from **A**.

17. Construct a triangle **ABC** such that **BC**=7·4 cm., ∠**B**=25°, and the perpendicular distance from **A** to **BC** is 2·3 cm. Measure **AC**.

18. Construct a convex quadrilateral **ABCD** in which **AB**=1·2 in., **BC**=2·6 in., **AC**=3·25 in., **AD**=1·5 in., and **D** is equidistant from **BA** and **BC**. Measure **CD**.

19. State how to construct a parallelogram, given the lengths of the diagonals and the distance between one pair of opposite sides.

20. Construct a triangle **ABC**, given that **BC**=4·5 cm., the median from **A**=5·1 cm., and the perpendicular from **A** to **BC** produced=3·7 cm. Measure **AB**.

21. Copy the triangle **ABC** of Ex. 9*a*, No. 8, and find by construction a point **O** which is equidistant from **A**, **B**, and **C**. Measure its distance from **A**, and draw the circle with centre **O** and radius **OA**.

22. Draw a triangle with sides 2·3, 2·3, and 1·8 in., and find by construction a point **I** inside the triangle such that **I** is equidistant from the three sides. Measure the distance of **I** from one of the sides, and draw a circle with this as radius and **I** as centre.

---

23. If X and Y are equidistant from A and B, prove that XY meets AB at right angles.

24. In a triangle ABC, the perpendicular bisectors of BC and CA meet at O. Prove that OA = OB.

25. If the internal bisectors of the angles B and C of a triangle ABC meet at O, prove that OA bisects ∠A.

26. ABP, ABQ are two isosceles triangles with a common base AB, but in different planes. Prove that P and Q lie in the plane which bisects AB at right angles.

27. M is the mid-point of a vertical straight line AB, and MC, MD are straight lines drawn from M in two different horizontal directions. Prove that △CAD ≡ △CBD.

EXAMPLES 9 c

1. Construct the triangle ABC for which AB = 4 cm., BC = 5 cm., and CA = 4·5 cm.

Using *only* ruler and compasses, construct (*a*) the locus of points which are equidistant from CA and CB, (*b*) the point E on this locus and *outside* the triangle, which is 3 cm. from B, (*c*) the perpendicular EF from E to CA produced. Measure the length of EF. (N)

2. Using ruler and compasses only, construct a quadrilateral ABCD in which AB = 3 in., BC = 2 in., the angle ABC = the angle DAB = 60°, and AD = DC. Draw and measure the diagonal DB. (C)

3. Draw a triangle ABC in which BC = 4·3 in., CA = 3·5 in., and AB = 3·0 in. Find by construction the point O, inside the triangle, which is 1·2 in. from AB and 0·6 in. from BC. Measure the distance of O from CA.

4. Construct a triangle ABC in which the angle A is 70°, the angle C is 35°, and the length of the perpendicular from A to BC is 2 in. Measure the sides of the triangle. (W)

5. Draw an angle ABC of 50° and make the arm AB 2 in. long. By a *geometrical construction*, find a point P which is equidistant from the lines AB and BC and which is also the same distance from A as it is from B. (W)

6. OX, OY are two straight lines at right angles. S is any point in their plane. Show how to find a point P in OY such that its distance from S and its perpendicular distance from OX are equal. (W)

7. AD and BE are altitudes of a triangle ABC. If AD is produced to P so that DP=AD and BE is produced to Q so that EQ=BE, prove that AQ=BP.

8. Construct a quadrilateral ABCD such that DA=2·1 in., DB=3·7 in., ∠DBC=∠ACB=30°, and A is 1·8 in. from BC. Measure AC.

9. Construct a triangle ABC in which BC=8 cm., the angle BAC=40°, and the length of the perpendicular from B to AC=7·3 cm. (C)

10. (i) Prove that the point O, where the bisectors of the exterior angles B and C of a triangle ABC meet, is equidistant from all three sides of the triangle, and prove also that OA bisects the angle BAC.

(ii) In the same figure, BC is produced to E where CE=AC, and CB is produced to F where BF=AB. Prove that O is equidistant from A, E, and F. (W)

11. Construct a convex quadrilateral PQRS from the following data : the *diagonal* PR=8 cm., RS=6 cm., ∠QPR=65°, S is 3·5 cm. from PQ and the diagonals intersect at right angles. Describe your method briefly. Measure QS. (N)

AREAS

Figures which are equal in area are sometimes said to be **equivalent.**

A straight line is said to **bisect** a figure when it divides the figure into two parts of equal area.

The symbol = is used for 'is equal in area to.' Thus '||gram ABCD=△XYZ' means that the parallelogram ABCD is equal in area to the triangle XYZ.

When, as in Fig. 140, two parallelograms are such that a side of the one coincides exactly with a side of the other, the parallelograms are said to be **on the same base.** Here the 'same base' is AB.

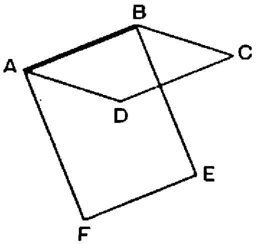


Fig. 140

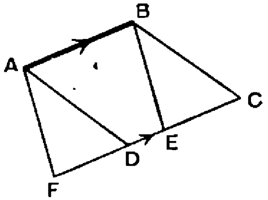


Fig. 141

If, in addition, the sides opposite the common base lie in the same straight line, as in Fig. 141, the parallelograms are said to be **on the same base** (AB) **and between the same parallels** (AB and FC).

The same phrases are also used in connection with triangles. Thus in Fig. 142 $\triangle$s ABC, ABD are on the same base AB ; in Fig. 143 they are on the same base AB and between the same parallels, AB and CD.

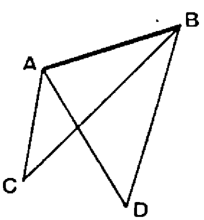


Fig. 142

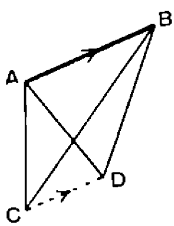


Fig. 143

Notice that the base must be one of the parallels.

ASSUMPTION 6

Congruent triangles are equal in area.

THEOREM 16

A diagonal bisects a parallelogram.

For it has been proved in the course of Theorem 9*b*, p. 59, that a diagonal divides a parallelogram into two congruent triangles.

THEOREM 17

Parallelograms on the same base and between the same parallels are equal in area.

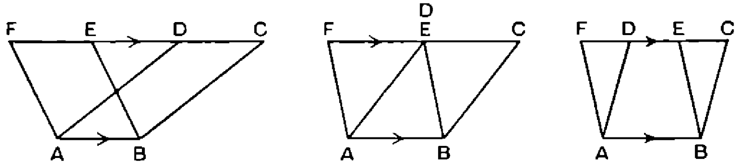


FIG. 144

Given two ||grams ABCD, ABEF, on the same base AB and between the same parallels AB, FC.

To prove that ABCD=ABEF.

Proof. In the $\triangle$s ADF, BCE,

$\angle$AFD=$\angle$BEC (corr., AF || BE),

$\angle$ADF=$\angle$BCE (corr., AD || BC),

AD=BC (opp. sides of ||gram);

$\therefore$ $\triangle$s $\begin{matrix}ADF\\BCE\end{matrix}$ are congruent (AA corr. S);

$\therefore$ they are equal in area.

Subtracting each in turn from the whole figure ABCF,

ABCD=ABEF.

Reference.

||gram ABCD=||gram ABEF (*same base AB, same* ||s *AB, FC*).

THEOREM 18

If a triangle and a parallelogram are on the same base and between the same parallels, the area of the triangle is half that of the parallelogram.

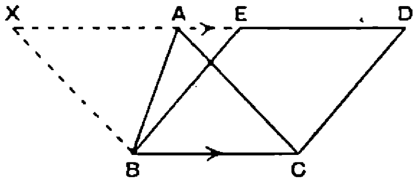


FIG. 145

Given a △ABC and a ||gram BCDE, on the same base BC and between the same parallels BC, AD.

To prove that △ABC$=\frac{1}{2}$BCDE.

Construction. Complete the ||gram BCAX.

Proof. △ABC$=\frac{1}{2}$BCAX (diag. bisects ||gram),

and ||gram BCAX=||gram BCDE (same base BC, same ||s BC, XD);

∴ △ABC$=\frac{1}{2}$BCDE.

Reference.

△ABC$=\frac{1}{2}$ ||gram BCDE (*same base BC, same ||s BC, AD*).

THEOREM 19 a

Triangles on the same base and between the same parallels are equal in area.

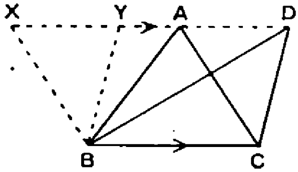


FIG. 146

Given two △s ABC, DBC, on the same base BC and between the same parallels BC, AD.

To prove that △ABC = △DBC.

Construction. Complete the ||grams BCAX, BCDY.

Proof.

||gram BCAX = ||gram BCDY (same base BC, same ||s BC, XD).

But △ABC = $\frac{1}{2}$ ||gram BCAX (diag. bisects ||gram),

and △DBC = $\frac{1}{2}$ ||gram BCDY (diag. bisects ||gram),

∴ △ABC = △DBC.

Reference.

△ABC = △DBC (*same base BC, same ||s BC, AD*).

THEOREM 19 b (Area Test for Parallels)

If two triangles are on the same base and on the same side of it, and are equal in area, then they are between the same parallels. (For proof, see Appendix, p. 334.)

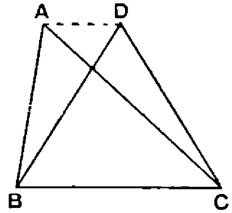


FIG. 147

Reference. △BAC = △BDC.

But these are on the same base BC and on the same side of it;

∴ AD ∥ BC.

Any of the three sides of a triangle may be regarded as the **base.** Having chosen one side as base, the perpendicular to that side (produced if necessary) from the opposite vertex is called the **altitude** or **height** of the triangle.[1] Thus, in Fig. 148, if BC is the base, AD is the altitude; if AB is the base, CF is the altitude.

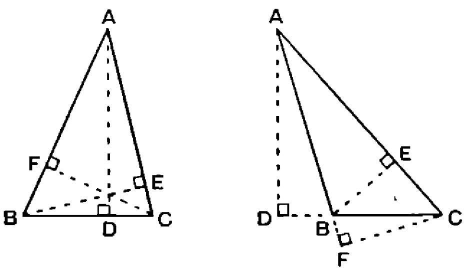


FIG. 148

The phrase 'XH is an altitude of △XYZ' means that H is the foot of the perpendicular from X to YZ.

[1] The common sense of this definition will become clear if the paper on which the triangle is drawn is turned round until the base appears to be at the bottom of the triangle.

If one side of a parallelogram has been chosen as base, the **altitude** or **height** is the perpendicular distance between the base and the opposite side. Thus, in Fig. 149, if AB is the base, the height is equal to HK.

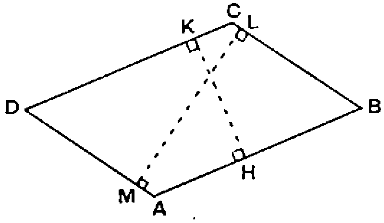


FIG. 149

Notice that a triangle has three distinct altitudes (AD, BE, CF in Fig. 148), but a parallelogram only two (HK, LM in Fig. 149).

Corollaries to Theorems 17 to 19.

(i) **Parallelograms on equal bases and between the same parallels are equal in area.**

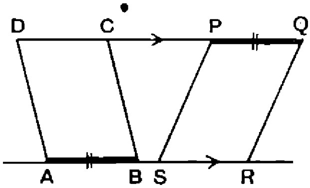


FIG. 150

Thus, in Fig. 150, if ABCD and PQRS are parallelograms on equal bases AB, PQ and between the same parallels AR, DQ, ||gram ABCD= ||gram PQRS.

(ii) **If a triangle and a parallelogram are on equal bases and between the same parallels, the area of the triangle is half that of the parallelogram.**

Thus, in Fig. 151, if PQR and ABCD are a triangle and a parallelogram on equal bases PQ, AB and between the same parallels AQ, DR, $\triangle$PQR$=\frac{1}{2}$ ||gram ABCD.

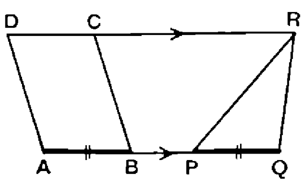


FIG. 151

(iii) Triangles on equal bases and between the same parallels are equal in area.

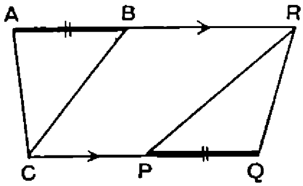


FIG. 152

Thus, in Fig. 152, if ABC and PQR are two triangles on equal bases AB, PQ and between the same parallels AR, CQ, $\triangle ABC = \triangle PQR$.

AREA FORMULAE

We shall assume that

area of rectangle = base × height.

A parallelogram and a rectangle on the same base and between the same parallels are equal in area, and have the same height. Hence,

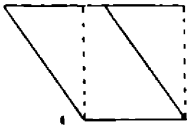


FIG. 153

area of parallelogram = base × height.

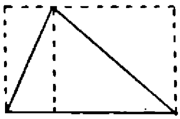


FIG. 154

If a triangle and a rectangle are on the same base and between the same parallels, the triangle is half the rectangle, and they have the same height. Hence,

area of triangle = $\frac{1}{2}$ base × height.

For a trapezium, let a in. and b in. be the lengths of the parallel sides, and h in. the distance between them. The trapezium can be divided into two triangles of areas $\frac{1}{2}ah$ sq. in. and $\frac{1}{2}bh$ sq. in., giving a total area of $\frac{1}{2}(a+b)h$ sq. in. Hence,

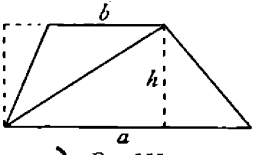


FIG. 155

area of trapezium = half the sum of the parallel sides multiplied by the distance between them.

CONSTRUCTION 16

To construct a triangle equal in area to a given quadrilateral.

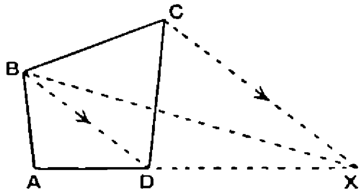


Fig. 156

Let ABCD be the given quadrilateral.

Draw the diagonal BD.

Through C draw a straight line parallel to BD to meet AD produced at X.

Join BX.

Then ABX is the required triangle.

Proof. $\triangle BXD = \triangle BCD$ (same base BD, same ||s BD, CX).

$\therefore \triangle BXD + \triangle ABD = \triangle BCD + \triangle ABD$;

$\therefore \triangle ABX =$ quad. ABCD.

Note. A polygon equal in area to a given polygon, but having one side fewer, may be drawn by a similar method. This is indicated in Fig. 157, where ABCX is constructed equal in area to ABCDE.

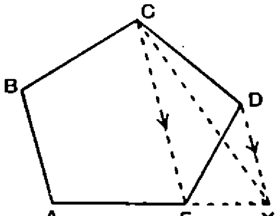


Fig. 157

By successive application of this method, **a triangle can be constructed equal in area to any given polygon.**

EXAMPLES 10 (Oral)

Given a parallelogram, discuss how to draw :—

1. An equivalent rectangle.

2. An equivalent rhombus.

3. An equivalent parallelogram with given acute angles.

4. An equivalent isosceles triangle.

5. An equivalent right-angled triangle.

Repeat Nos. 1 to 5 given a triangle, or requiring figures of half, double, treble . . . the area.

What conclusions can be drawn if you are given :—

6. Two triangles of equal area between the same parallels ?

7. Two triangles of equal area on equal bases ?

8. Two parallelograms on equal bases and of equal heights ?

9. A triangle and a parallelogram with equal bases and between the same parallels ?

10. A triangle and a parallelogram between the same parallels, the base of the triangle being three times that of the parallelogram?

EXAMPLES 10 a

(*Calculations and constructions based on formulae*)

1. The base of a parallelogram is 6 in. and the height is $2\frac{1}{2}$ in. ; calculate the area.

2. The area of a parallelogram is 60 sq. ft. and the base is 12 ft. ; calculate the height.

3. Repeat Nos. 1 and 2 with a triangle instead of a parallelogram.

4. A triangle and a parallelogram each have a base of 6 cm. and an area of 24 sq. cm. ; calculate their heights.

5. Construct a parallelogram ABCD in which AB=1·7 in., BC=1·1 in., and the diagonal AC=1·5 in.

Draw and measure the perpendiculars from C to AB and AD, and hence calculate the area of ABCD in two independent ways.

6. Construct a parallelogram ABCD in which ∠A=72°, AB=3 cm., and AD=5 cm.

Draw and measure the perpendicular from A to CB produced, and hence calculate the area of ABCD.

7. Construct a triangle ABC in which ∠A=25°, AB=1·6 in., and AC=3·2 in.

Draw and measure the altitudes from B and C, and hence calculate the area of △ABC in two independent ways. How could you calculate the area in a third independent way ?

8. Calculate the area of a right-angled triangle whose shortest sides are of lengths 1·6 and 2·4 yd.

9. In Fig. 158, calculate the area of ABCD.

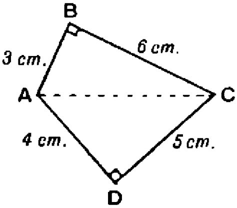


FIG. 158

10. Calculate the area of a rhombus whose diagonals are 16 and 10 ft. long. (Use a property of the diagonals of a rhombus.)

11. Construct a convex quadrilateral ABCD in which AB=8 cm., BC=7 cm., CD=4 cm., DA=6 cm., and the diagonal AC=8 cm. Draw and measure the perpendiculars from B and D to AC, and hence calculate the area of ABCD.

(*Constructions based on theorems and corollaries*)

12. Draw a parallelogram ABCD in which ∠A=64°, AB=2·4 in., and AD=1·7 in. With base AB construct an equivalent rectangle, and measure its height.

13. With the data of No. 12 (transfer ABCD by pricking), construct on base AB an equivalent parallelogram ABXY with the side AY 2·1 in. long. Measure the longer diagonal of ABXY.

14. With the data of No. 12, construct an equivalent parallelogram ABXY with the diagonal AX equal to AB, and measure BY.

15. Draw an equilateral triangle of side 6 cm., and construct an equivalent parallelogram having sides 6 and 4 cm. long. Measure the shorter diagonal.

16. Draw a triangle with sides 10, 7, and 6 cm. long. Construct an equivalent isosceles triangle with base 7 cm., and measure its height.

17. Construct any triangle **ABC**, and divide it into three triangles of equal area by drawing two straight lines through **A**.

(*Construction* 16)

18. Draw any quadrilateral, and construct a triangle equivalent to it. State which is the triangle required.

19. Draw a quadrilateral **ABCD** in which **AB**=1·3 in., **BC**=2·1 in., **CD**=1·9 in., **DA**=1·6 in., and the diagonal **BD**=2·4 in. Construct an equivalent triangle, draw one of its altitudes, and so calculate the area.

20. Draw any quadrilateral **ABCD**, and find by construction a point **E** on **CD** produced such that △**BCE**=quad. **ABCD**.

EXAMPLES 10 b

(*Calculations*)

1. In Fig. 159, if **AB**=5·2 cm. and **CF**=2·4 cm., calculate the area of △**ABC**.

Hence calculate the length of **AC** if **BE**=2·6 cm.

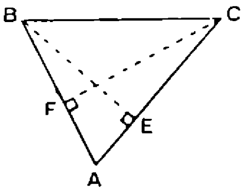


Fig. 159

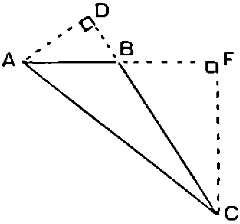


Fig. 160

2. In Fig. 160, if **AB**=3·2 in., **BC**=5·5 in., and **CF**=4 in., calculate the area of △**ABC** and the length of **AD** (to the nearest tenth of an inch).

3. ABCD is a parallelogram in which AB $=12$ cm. and AD $=10$ cm. If the perpendicular from A to BC is of length 6 cm., calculate the length of the perpendicular from A to CD.

4. In $\triangle$ABC, $\angle$A $=90°$, BC $=10$ cm., CA $=8$ cm., and AB $=6$ cm. Calculate (i) the area of $\triangle$ABC, (ii) the length of the perpendicular from A to BC.

5. $\triangle$ABC is acute-angled, and AD is an altitude. If AD $=4$ cm., CD $=3\cdot5$ cm., and the area of $\triangle$ABC $=12\cdot4$ sq. cm., calculate the length of BD.

6. ABCD is a trapezium in which AB $=8$ in., DC $=12$ in., and the distance between the parallel sides AB and DC is equal to 7 in. Calculate the area of ABCD.

7. ABCD is a trapezium having AB parallel to DC. If AB $=7\cdot2$ cm., DC $=8\cdot4$ cm., and the area of $\triangle$BCD $=12\cdot6$ sq. cm., calculate the area of ABCD.

8. ABCD is a quadrilateral in which $\angle$B $=\angle$C $=90°$, AB $=3$ cm., BC $=10$ cm., and CD $=5$ cm. If P is a point on BC such that BP $=4$ cm., calculate the area of $\triangle$APD.

9. If the area of a rhombus is 24 sq. m. and the length of one diagonal is 8 m., calculate the length of the other diagonal.

10. ABCD is a square of side a in. If P, Q are points on AB, CD respectively such that AP $=$ CQ, calculate the area of PBCQ.

11. Calculate the area of a kite, given that the diagonals are of lengths $2a$ and $2b$ in.

12. In Fig. 161, ABCD and PQRS are squares. If AB $=21$ in. and AP $=$ BQ $=$ CR $=$ DS $=12$ in., calculate the area of PQRS and the length of PQ.

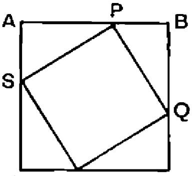


FIG. 161

13. ABCDE is a pentagon in which AB $=a$ cm., BC $=b$ cm., CD $=c$ cm., DE $=d$ cm., EA $=e$ cm., BE $=f$ cm., and $\angle$A $=\angle$EBC $=\angle$D $=90°$. Prove that its area is $\frac{1}{2}(ae+bf+cd)$ sq. cm.

14. Fig. 162 represents a pentagonal field. A surveyor measures distances along and perpendicular to **AD**, and makes the following record :—

	Yards	
	to **D**	
	360	
	240	150 to **C**
to **E** 100	180	
	80	120 to **B**
	from **A**	

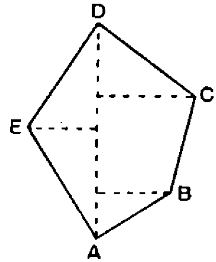


FIG. 162

Calculate the area of the field in acres.

(Constructions based on theorems and corollaries)

15. With ruler and compasses only, construct a triangle **ABC** in which **AB**=1·6 in., **AC**=2·9 in., and ∠**A**=60°. Also construct an equivalent isosceles triangle **ACX** and measure **AX**.

16. With ruler and compasses only, construct a square of side 1·5 in., and construct a parallelogram of half the area having an angle of 45°.

17. Draw any trapezium, and construct an equivalent rectangle.

18. Draw a rectangle of length 2 in. and breadth 1·2 in. Construct an equivalent rectangle having one side 1·5 in. long. (First draw an equivalent parallelogram having one side 1·5 in. long.)

19. Draw a triangle with sides 4, 4, and 7 cm., and construct an equivalent isosceles triangle with the equal sides 5 cm. long. Measure its other side. (First draw an equivalent triangle with two sides 4 and 5 cm. long.)

20. Draw any parallelogram with sides 4·2 and 2·6 cm., and use the result of Ex. 11*b*, No. 6 to construct an equivalent equiangular parallelogram with one side 3 cm.

21. Draw a triangle **ABC** in which **BC**=3 in., **CA**=2·5 in., and **AB**=2 in., and mark a point **K** on **AC** at a distance 0·9 in. from **C**. Use the result of Ex. 11*b*, No. 13 to construct a straight line through **K** which will bisect △**ABC**.

(*Construction* 16)

22. Draw a quadrilateral ABCD in which AB = 1·6 in., ∠B = 110°, BC = 1·1 in., ∠C = 120°, and CD = 1·35 in. Construct an equivalent triangle with base CD and vertex on CB produced, and measure its altitude. Hence calculate the area of ABCD.

23. Copy (*e.g.* by pricking) the quadrilateral ABCD of No. 22, and construct an equivalent parallelogram having one side of length 1·6 in.

24. Draw a pentagon, and construct an equivalent triangle.

25. Draw a non-convex quadrilateral, and construct an equivalent triangle. Prove that the construction is correct.

26. Draw a triangle ABC in which AB = 1·8 in., BC = 1·5 in., and CA = 1·2 in., and mark a point X on AB such that BX = 1·4 in. Find by construction a point Y, on BC produced, such that △BXY = △BAC, and measure BY.

EXAMPLES 11a

RIDERS

(*Theorem* 17)

1. In Fig. 163, ABCD, ABEF, and AFGH are parallelograms. Prove that they are equivalent.

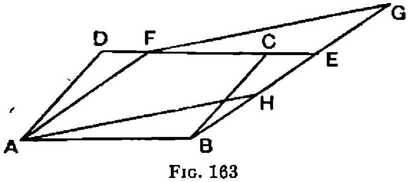


FIG. 163

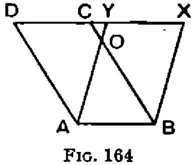


FIG. 164

2. In Fig. 164, ABCD and ABXY are parallelograms. Prove that AOCD = BOYX.

3. Two parallelograms ABPQ, ABXY lie on opposite sides of their common base AB. Prove that ABPQ + ABXY = PQYX. (Hint: let AB, produced if necessary, cut PX, QY at L, M.)

(*Theorem* 18)

4. **P** is a point on the side **AB** of a parallelogram **ABCD**. State the relation between the areas of △**PCD** and the parallelogram, and deduce that △**PCD**=△**ADP**+△**BCP**.

5. With the data of No. 4, if **DP** and **CB** produced meet at **Q**, prove that (i) △**QAD**=△**PCD**, (ii) △**BCP**=△**PQA**.

6. **ABCD** is a parallelogram. If **AB** is produced to **P** and **DA** to **Q**, prove that △**BQC**=△**CPD**.

(*Theorem* 19*a*)

7. In Fig. 165, if **XY** is parallel to **BC**, prove that (i) △**XBY**=△**XCY**, (ii) △**ABY** =△**ACX**, (iii) △**BOX**=△**COY**.

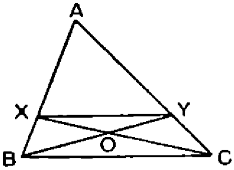


FIG. 165

8. In Fig. 166, if **AB** is parallel to **DC**, prove that there are three pairs of equivalent triangles.

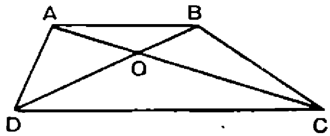


FIG. 166

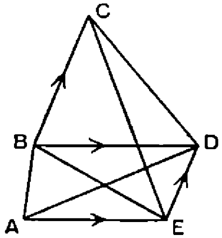


FIG. 167

9. In Fig. 167, prove that △**ABD**=△**CDE**.

(*Theorem* 19*b*)

10. In Fig. 165, if △**ABY**=△**ACX**, prove that **XY** is parallel to **BC**.

11. In Fig. 166, if △**AOD**=△**BOC**, prove that **AB** is parallel to **DC**.

12. If the diagonals of a quadrilateral divide it into four equivalent triangles, prove that the quadrilateral is a parallelogram.

(Theorem 16)

13. Through the vertices of a quadrilateral, straight lines are drawn parallel to the diagonals. Prove that the parallelogram which they form has twice the area of the quadrilateral.

14. Use the method of No. 13 to prove that the area of a quadrilateral whose diagonals cut at right angles is given by half the product of its diagonals.

15. X, Y are the mid-points of the sides AB, CD of a parallelogram ABCD. Prove, by joining XY, that $AXCY = \frac{1}{2}ABCD$.

(Corollaries)

16. Prove that a median bisects a triangle.

17. If P is any point on the median BE of a triangle ABC, apply the result of No. 16 to two triangles, and deduce that $\triangle BPA = \triangle BPC$.

18. BE, CF are medians of a triangle ABC; by comparing each with $\triangle ABC$, prove that $\triangle AEB = \triangle AFC$.

If BE and CF cut at G, prove that (i) $\triangle BGF = \triangle CGE$, (ii) $\triangle BGC =$ quad. AEGF.

(Formulae)

19. In Fig. 168, if $BE = \frac{7}{8}AD$, prove that $BC = \frac{7}{8}AC$.

20. APB, AQB are two equivalent triangles on opposite sides of a common base AB. If M, N are the feet of the perpendiculars to AB from P, Q respectively, prove that PQ and MN bisect each other.

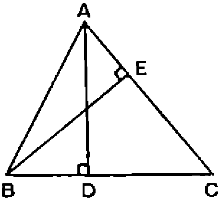


Fig. 168

21. AD is an altitude of a triangle ABC. If E is any point on AD, prove that $BACE = \frac{1}{2}AE \cdot BC$.

EXAMPLES 11 b

RIDERS

(*Theorems*)

1. In Fig. 169, prove that BCYX=BAYZ.

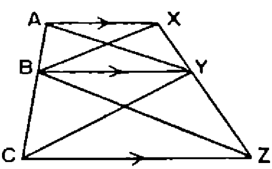


FIG. 169

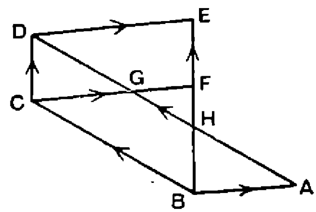


FIG. 170

2. In Fig. 170, prove that (i) ABCG=CDEF, (ii) △ABF=△CEF.

3. If P is any point inside a parallelogram ABCD, prove, by drawing a straight line through P parallel to AD, that △PAD+△PBC=$\frac{1}{2}$ ||gram ABCD.

4. A straight line, through any point Q on the side BC of a parallelogram ABCD and parallel to CD, meets AD at S. If P, R are any points on AB, CD respectively, prove that PQRS=$\frac{1}{2}$ABCD.

5. ABCD is a parallelogram. If a straight line parallel to AB cuts AD at F and BC at E, prove that △ABC=△AEF+△CDE.

6. ABCD is a parallelogram, and O is any point on the diagonal AC. The straight line through O parallel to AB cuts AD, BC at H, K, and the straight line through O parallel to AD cuts AB, DC at L, M. Prove that DMOH=BKOL.

7. With the data of No. 6, prove that AHMC=ALKC.

8. In Fig. 171, if AF is parallel to CD, prove that (i) △XAC=△XDF, (ii) XABC=XDEF.

9. In Fig. 171, if XABC=XDEF, (i) prove that △ABF=△AEF and deduce that AF is parallel to BE, (ii) prove that CD is parallel to BE.

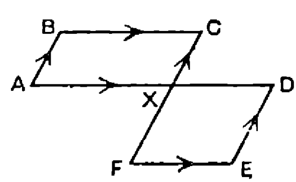


FIG. 171

(*Corollaries*)

10. In Fig. 166, if $\triangle$AOB = $\triangle$BOC, prove that $\triangle$AOD = $\triangle$COD.

11. Prove that the diagonals of a parallelogram divide it into four equivalent triangles.

12. M, N are the mid-points of the sides AB, CD of a quadrilateral ABCD. Join BD, and prove that $\triangle$AMD + $\triangle$BNC = quad. BNDM.

13. H is the mid-point of the side AB of a triangle ABC, and K is a point on AC nearer to C than to A. If the straight line through C parallel to KH meets AB at L, prove that (i) $\triangle$AKL = $\triangle$AHC, (ii) KL bisects $\triangle$ABC.

14. M is the mid-point of the diagonal AC of a quadrilateral ABCD. Prove that DABM = DCBM.

15. O is the mid-point of the median AD of a triangle ABC. Prove that $\triangle$AOB = $\triangle$COD.

16. Q is the mid-point of the side AB of a triangle ABC. If BC is produced to P so that CP = BC, and AP is joined, prove that $\triangle$PQB = $\triangle$ABC.

17. The diagonals of a parallelogram ABCD intersect at O. If P, Q are the mid-points of OB, OD respectively, prove that (i) APCQ is a parallelogram, (ii) APCQ = $\frac{1}{2}$ABCD.

18. H, K, L are the mid-points of the sides BC, CA, AB respectively of a triangle ABC. Prove that (i) BHKL is a parallelogram, (ii) BHKL = $\frac{1}{2}\triangle$ABC.

(*Formulae*)

19. ABC, XYZ are two triangles in which AB = XY, AC = XZ, and $\angle$A = 180° − $\angle$X. By drawing the altitudes from B and Y, prove that $\triangle$ABC = $\triangle$XYZ.

20. ABC is a straight line. If P is any point not on ABC, prove that $\triangle$ABP : $\triangle$ACP = AB : AC.

21. AC is a diagonal of a quadrilateral ABCD. If $\triangle$ABC = $\triangle$ADC, prove that AC bisects BD.

22. ABC is a triangle ; P is a point on AC such that AP = $\frac{2}{3}$AC, and Q is a point on AB such that AQ = $\frac{1}{3}$AB. By comparing each with $\triangle$ABP, prove that $\triangle$APQ = $\frac{2}{9}\triangle$ABC.

23. Given the base and area of a triangle, state the locus of its vertex (i) in the plane of your paper, (ii) in space.

24. ABCD is a quadrilateral, and P is any point, on the side of AC opposite to B, such that ABCP=ABCD. State the locus of P.

25. P is any point inside an equilateral triangle ABC of side a, and X, Y, Z are the feet of the perpendiculars from P to BC, CA, AB respectively. Write down and add together expressions for the areas of triangles PBC, PCA, PAB, and deduce that PX+PY+PZ has the same value for all positions of P.

EXAMPLES 11c

(*Miscellaneous*)

1. M, A, N are points in a straight line, the length of MA being 7 in. and that of AN 5 in.; MB, NC are drawn perpendicular to MN and on the same side of it, the length of MB being 5 in. and that of NC 9 in.; find the area of the triangle ABC. (OC)

2. Construct a quadrilateral ABCD with A and C on opposite sides of BD, and having AB=2 cm., BC=3 cm., CD=5 cm., DA=4 cm., and the diagonal BD=5 cm.

Find by construction a point E on AB produced such that the triangle ADE is equal in area to the quadrilateral ABCD. Measure the perpendicular distance from E to AD, and hence calculate the area of the quadrilateral. (C)

3. Draw, full size, a parallelogram ABCD having AB=3·6 in., BC=2·4 in., and ∠ABC=42°.

(*a*) On AB as base construct a triangle OAB of area equal to ABCD, and having one of its base angles equal to 65°. Find, by measurement, the other base angle of this triangle.

(*b*) Construct a parallelogram ABRS of area equal to ABCD, having its diagonal AR equal to AB. (N)

4. ABCD is a parallelogram, E any point in CD; AE, BC produced meet in F; show that the triangles ADF, ABE are equal in area, and also that the triangles DEF, BCE are equal in area. (OC)

5. ABCD is a rhombus, and E is the foot of the perpendicular from B to AD. Prove that AC . BD = 2AD . BE.

6. O is the point of intersection of the diagonals AOC, BOD of a parallelogram ABCD ; EOF is a line drawn through O meeting AB in E and CD in F.

Show that the triangles BOE, DOF are congruent ; and hence that EOF divides the area of the parallelogram into two equal parts. (W)

---

7. Points D, E are taken on the sides AC, AB of a triangle ABC such that AD = $\frac{1}{2}$AC, AE = $\frac{2}{3}$AB. Express the areas of each of the triangles BEC, CED, DEA as fractions of the area of the triangle ABC. (N)

8. Draw a plan of a field ABCD in which AD = 250 yd., AB = 220 yd., BC = 130 yd., BD = 200 yd., CD = 150 yd. Construct a triangle equal in area to your figure and find the area of the field in acres correct to one place of decimals. (OC)

9. Construct a parallelogram ABCD having AB = 3 in., ∠ABC = 79°, and an area of 8·4 sq. in. On the side AB of this parallelogram construct a parallelogram ABEF having the same area and each of its sides 3 in. long. (State your construction but do *not* prove it.) (L)

10. ABCD is a quadrilateral. If the parallel to AC through B meets DC produced at X, and the parallel to BD through A meets CD produced at Y, prove that △ADX = △BCY.

11. A point O inside a rectangle PQRS is joined to the angular points. Prove that the sum of the areas of two of the triangles so formed is equal to the sum of the areas of the other two triangles.

If O is outside the rectangle, but between one pair of parallel sides, show that the area of one of the triangles OPQ, OQR, ORS, OSP is equal to the sum of the areas of the other three. (O)

12. AFE is a triangle with B the mid-point of AF and C the mid-point of BE ; AC produced meets EF at D. Prove that the triangles FBD, ADB, ADE are equal in area. (W)

---

13. The perpendicular distances from the vertices of a triangle to the opposite sides are in the ratios 10 : 15 : 24, and its longest side is 60 in. Find its shortest. (W)

14. Construct the convex quadrilateral **PQRS** for which **PQ**=3 in., **QR**=1 in., **RS**=1·5 in., **SP**=2 in., and ∠**QPS**=50°.

By a geometrical construction and using any instruments you please, obtain a triangle equal in area to the quadrilateral and having **S** as one vertex. By a further geometrical construction obtain a triangle, also with **S** as one vertex, whose area is one-third that of the quadrilateral. (N)

15. The quadrilateral **ABCD** is divided by the diagonal **BD** into two parts of equal area ; **AB**=3·5 cm., **BC**=4 cm., **CD**=8 cm., and ∠**C**=110°. Construct the quadrilateral and measure the remaining side, stating briefly your method of construction. How many solutions are possible ? (N)

16. A point **E** is taken on the side **CD** of a parallelogram **ABCD**, and **CD** is produced to **F**, so that **DF**=**CE**. Prove that, if **BE** produced meets the side **AD** produced at **G** and the line through **F** parallel to **AD** at **H**, then **AFHG** is a parallelogram, whose area is equal to that of the parallelogram **ABCD**. (C)

17. **ABC** is a triangle and **EF** is any straight line parallel to **BC**. If **BR**, drawn parallel to **AC**, and **CS**, drawn parallel to **AB**, meet **EF** in **R** and **S**, prove that the triangles **ABR**, **ACS** are equal in area. (C)

18. **ABCD** is a parallelogram ; **X** is the foot of the perpendicular from **D** to **AB**, **Y** is any point on **BC**, and **Z** is the foot of the perpendicular from **D** to **AY**. Prove that **AY . DZ**=**AB . DX**.

---

19. **ABCD** is a rectangle, **E**, **F**, **G**, **H** are points in **AB**, **BC**, **CD**, **DA** respectively, such that **AE**=$\frac{1}{4}$**AB**, **BF**=$\frac{1}{3}$**BC**, **CG**=$\frac{1}{4}$**CD**, and **DH**=$\frac{1}{3}$**DA** ; show that the area of the quadrilateral **EFGH** is $\frac{7}{12}$ that of the rectangle. (OC)

20. Construct a quadrilateral **ABCD** in which **AB**=$1\frac{3}{4}$ in., **BC**=$2\frac{1}{2}$ in., **CD**=3 in., **DA**=$1\frac{1}{2}$ in., ∠**BCD**=62°, and mark a point **E** on **CD** 1 in. from **C**. Construct a triangle **XEA** (**A**, **B**, **X** to be in one straight line) equal in area to the quadrilateral

ABCE. Construct also the triangle **XEY** (**X**, **B**, **A**, **Y** to be in one straight line) which will be equal in area to the quadrilateral **ABCD**. (W)

21. **ABCD** is a parallelogram; **P** is any point in **AB**, and **Q** is the mid-point of **BC**. If **PQ** and **DC** produced meet at **R**, prove that **APRD**$=2\triangle$**AQD**.

22. **ABCD** is a parallelogram; **H** is any point on **CB**, and the straight line through **H** parallel to **BD** cuts **CD** at **K**. Prove that $\triangle$**AHB**$=\triangle$**AKD**.

23. If **P** is any point on the diagonal **AC** of a parallelogram **ABCD**, prove that $\triangle$**PAB**$=\triangle$**PAD**.

If **X**, **Y** are the feet of the perpendiculars from **P** to **CB**, **CD** respectively, prove that **PX** . **CB**$=$**PY** . **CD**.

24. In the triangles **ABC**, **DEF**, **AB**$=$**DE**, **BC**$=$**EF**, and the angles **ABC**, **DEF** are together equal to two right angles. Prove that the triangles are equal in area.

ABC is any triangle, and squares **ABDE**, **BCFG**, **ACHK** are described outwards on its sides. Prove that each of the triangles **BDG**, **FCH**, **AKE** is equal in area to the original triangle. (OC)

---

25. The diagonals **AC**, **BD** of a parallelogram meet in **O**. From **M**, the middle point of **AD**, a parallel is drawn to **DB** meeting **AO** in **H**, and a parallel to **AC** meeting **DO** in **K**. Express the area of the parallelogram **MHOK** as a fraction of that of **ABCD**. (W)

26. **ABCD** is a quadrilateral whose diagonals cut at **O**. If **P** is a point on **BD** produced such that **DP**$=$**BO**, prove that $\triangle$**PAC**$=$quad. **ABCD**.

27. Draw a quadrilateral **ABCD** in which **AB**$=2{\cdot}0$ in., **BC**$=2{\cdot}6$ in., **CD**$=4{\cdot}2$ in., **DA**$=2{\cdot}2$ in., and the diagonal **BD**$=3{\cdot}6$ in.

Then, using ruler and compasses only, find by construction a point **P** on **AC** such that $\triangle$**PBD**$=\frac{1}{2}$ quad. **ABCD**. Measure **PA**.

28. **ABCD** is a quadrilateral whose opposite sides **AB**, **DC** are parallel. Any line through the point of intersection of the diagonals **AC**, **BD** intersects the parallels to **AC** through **D** and **B** in **X** and **Y** respectively. Prove (i) that the areas of **XAY** and **DAB** are equal, (ii) that **AY**, **XC** are parallel. (O)

29. If two triangles are equal in area, and have two sides of the one triangle respectively equal to two sides of the other triangle, prove that the triangles are not necessarily congruent, but that the included angles are either equal or supplementary.

THE THEOREM OF PYTHAGORAS

What relation between the area of the square on the hypotenuse of a right-angled triangle and the areas of the squares on the other two sides is suggested by Figs. 172 and 173 ?

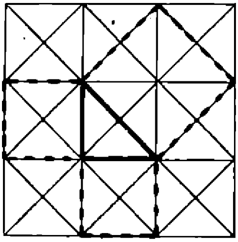


FIG. 172

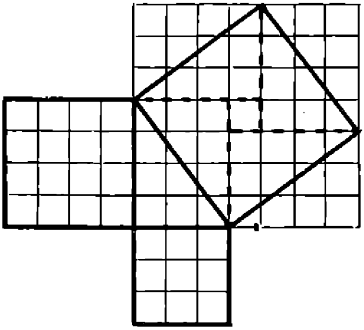


FIG. 173

In Fig. 172 the right-angled triangle is isosceles, and in Fig. 173 its sides are in the ratio 3 : 4 : 5. To see if the result suggested by these figures is true for any right-angled triangle, consider Fig. 174 :—

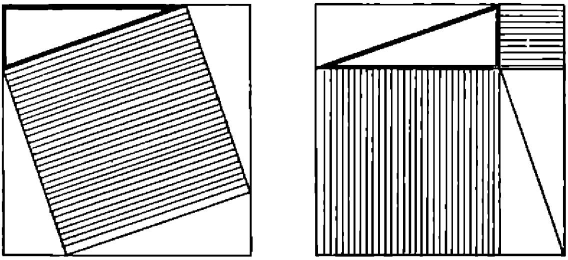


FIG. 174

Another verification is given by Perigal's dissection, shown in Fig. 175. This figure should be constructed on cardboard. First find the centre of the square on AB by drawing the two diagonals. Through the centre draw lines parallel and perpendicular to the hypotenuse BC. Cut out the squares on AB and AC, dividing the former into four pieces as shown. Then verify that the five pieces can be fitted together so as to cover the square on BC exactly, noting that the five pieces can be moved into their new positions without rotation.

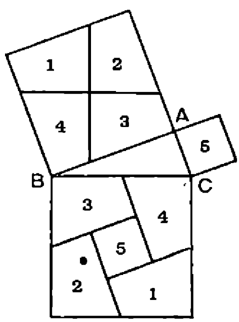


FIG. 175

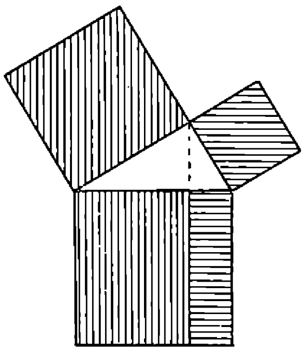


FIG. 176

While a formal proof by the method suggested in Fig. 174 is possible, the following method, due to Euclid, is less liable to be misunderstood by pupils. The essence of the method is indicated in Fig. 176.

THEOREM 20 a (Pythagoras' Theorem)

The square on the hypotenuse of a right-angled triangle is equal to the sum of the squares on the other two sides.

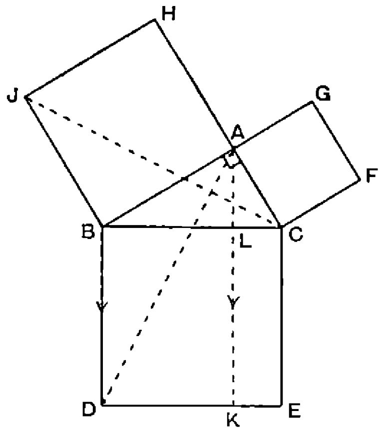


FIG. 177

Given a $\triangle$ABC with a right angle at A, and squares CBDE, ACFG, BAHJ drawn on the sides.

To prove that CBDE=ACFG+BAHJ.

Construction. Join AD, CJ, and draw a line through A parallel to BD to cut BC at L and DE at K.

Proof. $\because$ $\angle$BAC and $\angle$BAH are right angles,

$\angle$BAC+$\angle$BAH=180°.

But these are adjacent,

$\therefore$ CAH is a straight line.

$\therefore$ $\triangle$JBC=$\frac{1}{2}$ sq. BAHJ (same base JB, same ||s JB, HC).

Also $\triangle$ABD=$\frac{1}{2}$ rect. BLKD (same base BD, same ||s BD, AK).

In the $\triangle$s JBC, ABD,

JB=AB (sides of square),

BC=BD (sides of square),

$\angle$JBC=$\angle$ABD (each 90°+$\angle$ABC);

$\therefore$ $\triangle$s $\begin{matrix}\text{JBC}\\ \text{ABD}\end{matrix}$ are congruent (SAS);

$\therefore$ they are equal in area.

But $\triangle \text{JBC}=\frac{1}{2}\text{BAHJ}$, $\triangle \text{ABD}=\frac{1}{2}\text{BLKD}$ (proved);

$\therefore$ BAHJ=BLKD.

Similarly, ACFG=CLKE;

$\therefore$ ACFG+BAHJ=CLKE+BLKD

=CBDE.

Reference $\angle BAC=90°$,

$\therefore$ $BC^2=AB^2+AC^2$ (*Pythagoras'*).

Exercise. Prove in full that ACFG=CLKE.

Note on the use of Pythagoras' Theorem

Pythagoras' theorem has so far been regarded as a result connecting the areas of certain squares. It can also be regarded as giving an equation connecting the lengths of the sides of a right-angled triangle. For, if h in. is the length of the hypotenuse and x, y in. are the lengths of the other two sides,

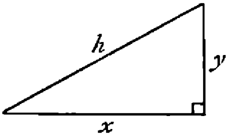


FIG. 178

$$h^2=x^2+y^2.$$

It is in this form that the theorem is of use in examples.

Notation. The area of the square on a line AB is AB$\times$AB, and this is usually written AB^2. Similarly, the area of the rectangle ABCD is AB$\times$BC, which is usually written AB.BC.

Examples.

1. Multiply out

(i) $(AB+BC)^2$, (ii) $(AB-CD)^2$, (iii) $(AB+BC)(AB-BC)$.

2. Factorise

(i) $AB^2+AB.AC$, (ii) $AB.BC-2BC^2$, (iii) AB^2-4AC^2.

3. What is the square of (i) 2AB, (ii) 3AB?

THEOREM 20 b (Converse of Pythagoras' Theorem)

If the square on one side of a triangle is equal to the sum of the squares on the other two sides, then these sides contain a right angle.

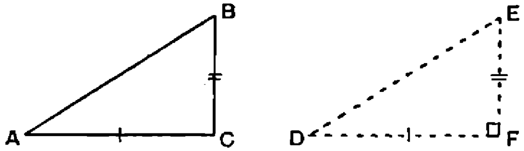


FIG. 179

Given a $\triangle ABC$ such that $AB^2 = CA^2 + CB^2$.

To prove that $\angle C = 90°$.

Construction. Draw a $\triangle DEF$ such that $FD = CA$, $\angle F = 90°$, $FE = CB$.

Proof. $\angle DFE = 90°$;

$\therefore DE^2 = FD^2 + FE^2$ (Pythagoras').

But $FD = CA$, $FE = CB$ (constr.);

$\therefore DE^2 = CA^2 + CB^2$.

But $AB^2 = CA^2 + CB^2$ (given);

$\therefore AB^2 = DE^2$;

$\therefore AB = DE$.

In the $\triangle$s ABC, DEF,

$AB = DE$ (proved),

$AC = DF$ (constr.),

$BC = EF$ (constr.);

$\therefore \triangle$s $\begin{matrix} ABC \\ DEF \end{matrix}$ are congruent (SSS).

$\therefore \angle ACB = \angle DFE$

$= 90°$ (constr.).

Reference. $AB^2 = CA^2 + CB^2$;

$\therefore \angle ACB = 90°$.

The following construction, for drawing straight lines $\sqrt{2}$, $\sqrt{3}$, $\sqrt{4}$, $\sqrt{5}$ . . . times as long as a given straight line, is interesting, but of little practical importance. More direct methods can usually be found for cases which the pupil is likely to meet.

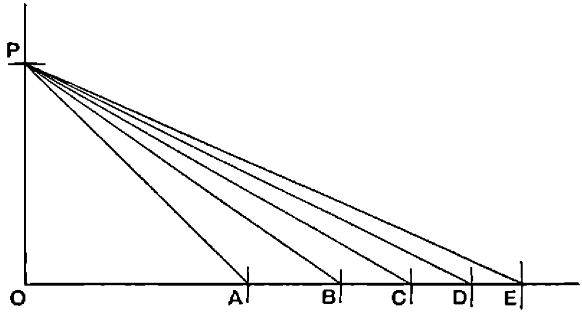


FIG. 180

Taking the length of the given straight line as unit, construct an isosceles right-angled triangle OAP in which OA = OP = 1 unit. Then PA = $\sqrt{2}$ units.

Along OA produced mark off OB equal to PA; then PB = $\sqrt{3}$ units.

Along OA produced mark off OC equal to PB; then PC = $\sqrt{4}$ units; and so on.

The proof is left to the reader.

EXAMPLES 12 a

(*Theorem* 20*a*)

1. If ABC is a triangle right-angled at A, and AB = 3 cm., AC = 4 cm., calculate the length of BC.

2. A rectangle is 12 cm. long and its diagonals are 13 cm. long; calculate its width.

3. Calculate to the nearest mm. the length of a diagonal of a square of side 6 cm.

4. If a man first travels 12 miles west, and then 14 miles north, calculate to the nearest tenth of a mile his distance, as the crow flies, from his starting-point.

5. Calculate the area of a triangle ABC in which $\angle A = 90°$, BC=17 yd., and AB=8 yd.

6. A right circular cone has a base of diameter 60 cm. and a vertical height of 100 cm. Calculate the slant height to the nearest cm.

7. AD is an altitude of an acute-angled triangle ABC. If AB=5 ft., AD=4 ft., and AC=6 ft., calculate the length of BC to the nearest tenth of a foot.

8. A rhombus has sides 2·4 cm. long. If one diagonal is 3·5 cm. long, calculate the length of the other diagonal to the nearest mm.

9. A and B are the feet of two vertical poles, AX, BY, of heights 34 and 50 ft. If AB=40 ft. and is horizontal, calculate to the nearest ft. the distance between X and Y.

10. If a rectangular room is 16 ft. long and 12 ft. wide (see Fig. 181), calculate the length of the diagonal AB of the floor.

If the room is 15 ft. high, use $\triangle$ABC to calculate the length of the diagonal AC of the room.

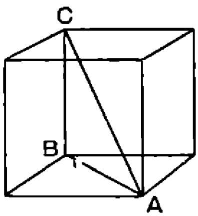


FIG. 181

(*Theorem 20b*)

11. Prove that a triangle, with sides 9, 12, and 15 in. long, is right-angled.

12. Find out by calculation if a triangle with sides 5, 13, and 14 in. long is right-angled.

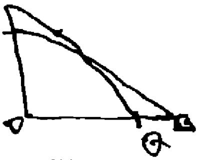
(*Theorem 20a*)

13. ABCD is a quadrilateral in which $\angle A = \angle C = 90°$. Prove that $AB^2 + AD^2 = CB^2 + CD^2$.

14. If AD is an altitude of a triangle ABC, prove that (i) $AB^2 - DB^2 = AC^2 - DC^2$, (ii) $AB^2 - AC^2 = DB^2 - DC^2$.

15. If the diagonals of a quadrilateral ABCD cut at right angles, prove that $AB^2 + CD^2 = AD^2 + BC^2$.

16. If ABCD is a rectangle, prove that $AB^2 + AD^2 = AC^2$.



17. $\triangle ABC$ has a right angle at A, and $AC=2AB$. Prove that $BC^2=5AB^2$.

18. $\angle AOB=90°$; P is a point on OA, and Q a point on OB. Prove that $PB^2+AQ^2=PQ^2+AB^2$.

19. ABC is a triangle right-angled at A; AXB, AYC are triangles outside $\triangle ABC$, right-angled at X and Y. Prove that $BC^2=BX^2+XA^2+AY^2+YC^2$.

20. If BE and CF are altitudes of a triangle ABC, prove that (i) $BE^2+EC^2=BF^2+CF^2$, (ii) $BE^2-BF^2=CF^2-CE^2$.

21. ABCD is a rectangle and P a point inside it. Draw the perpendiculars from P to AB, BC, CD, DA, and denote their lengths by a, b, c, d units respectively. (i) Prove that $PA^2=d^2+a^2$, (ii) write down three similar results, and deduce that $PA^2+PC^2=PB^2+PD^2$.

22. ABC is a triangle right-angled at A. If D is the foot of the perpendicular from A to BC, prove that

$$2DA^2+DB^2+DC^2=BC^2.$$

23. In Fig. 182, prove that $AB^2+BC^2=AX^2+XY^2+YC^2$.

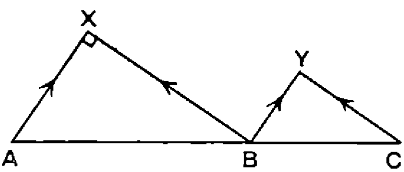


FIG. 182

(*Theorem* 20*b*)

24. ABCD is a quadrilateral in which $\angle A=90°$. If $AB^2+AD^2=CB^2+CD^2$, prove that $\angle C=90°$.

25. AC is a diagonal of a parallelogram ABCD. If $AB^2+AD^2=AC^2$, prove that ABCD is a rectangle.

26. ABC is a triangle in which $AB=2BC$. If $AC^2=3BC^2$, prove that $\angle ACB=90°$.

---

27. Draw any straight line. With straight edge and compasses only, construct a straight line (i) $\sqrt{2}$, (ii) $\sqrt{5}$ times as long as the original line.

28. (i) If $\triangle ABC$ is such that $\angle A=90°$, $BC=2$ units, and $AB=1$ unit, how long is AC?

(ii) Draw a square, and use the result of (i) to construct a square of three times the area.

29. Draw a square, and construct another square of twice the area. (Use an isosceles right-angled triangle.)

30. Draw a square, and construct another square of five times the area.

EXAMPLES 12b

1. **ABCD** is a quadrilateral and **AC** a diagonal. If $\angle$**B**$=\angle$**CAD** $=90°$, and **AB**$=3$ in., **AC**$=5$ in., **CD**$=13$ in., calculate the area of **ABCD**.

2. Calculate the area of a triangle whose sides are of lengths 17, 17, and 16 ft. (Draw the altitude which bisects the triangle.)

3. Find out by calculation if a triangle with sides 1, $\sqrt{2}$, and $\sqrt{3}$ in. long is right-angled.

4. In $\triangle$**ABC**, $\angle$**A**$=90°$, **AB**$=0{\cdot}9$ m., and **BC**$=1{\cdot}5$ m. Prove that the length of the median through **B** is $\sqrt{1{\cdot}17}$ m., and evaluate this correct to 2 significant figures.

5. Fig. 183 shows an end-elevation of a lean-to shed. If the shed is 20 ft. from end to end, calculate the area of the roof.

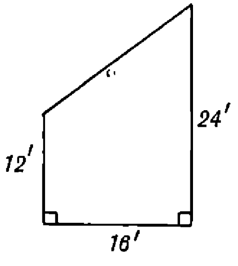


FIG. 183

6. A ladder 30 ft. long rests with one end on the ground and the other against a wall at a point 24 ft. above the ground. If the foot of the ladder is moved 2 ft. nearer the wall, how much higher up the wall, to the nearest ft., will the ladder reach?

7. **AD** is an altitude of $\triangle$**ABC**. If **AD**$=12$ in., **BD**$=9$ in., and **BC**$=25$ in., prove that $\angle$**BAC**$=90°$.

8. Prove that a triangle, whose sides are of lengths m^2+n^2, m^2-n^2, and $2mn$ units, is right-angled.

9. A rectangular room is l ft. long, b ft. broad, and h ft. high. Prove that the length of a diagonal is $\sqrt{(l^2+b^2+h^2)}$ ft.

10. A pyramid has a square horizontal base of side 2 ft., and the vertex is 1 ft. vertically above the mid-point of the base. Calculate, to the nearest tenth of a ft., (i) the length of a slanting edge, (ii) the area of a slanting face.

11. ABCD is a horizontal rectangle, M the mid-point of AD, and O a point vertically above M. If $AB=20$ ft., $AD=18$ ft., and $OM=12$ ft., calculate the areas of Δs DOA, AOB, and BOC to the nearest sq. ft.

12. Fig. 184 shows the development of a pyramid. If the square is of side 12 cm., and the other sides of the isosceles triangles are 10 cm., calculate to the nearest mm. the perpendicular height of the pyramid.

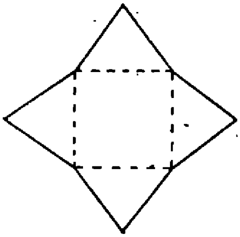


FIG. 184

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13. If D is the mid-point of the side BC of an equilateral triangle ABC, prove that $4AD^2=3BC^2$.

14. If ABCD is a rhombus, prove that $AC^2+BD^2=4AB^2$.

15. ABC is a triangle right-angled at B. If D is the mid-point of BC, prove that $AB^2+AC^2=2(AD^2+BD^2)$. (Express each side of the required result in terms of AB and BD.)

16. ABC is a triangle in which $\angle B=90°$, $BC=2BA$. If D is the mid-point of BC, and E the foot of the perpendicular from D to AC, prove that $EA^2-EC^2=AB^2$.

17. ABC is a triangle in which $BC=2BA$; E is the foot of the perpendicular to AC from the mid-point D of BC. If $EA^2-EC^2=AB^2$, prove that $\angle B=90°$.

18. In Fig. 185, prove that $b^2-(a+x)^2=AD^2=c^2-x^2$. Simplify this equation, and deduce that $AC^2=BA^2+BC^2+2BC.BD$.

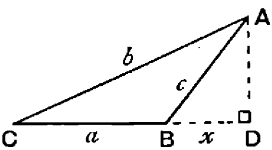

FIG. 185

19. ABC is a triangle, and any straight line is drawn through A. The perpendiculars from B and C to this line meet it at X and Y respectively. If $BC^2=BX^2+XA^2+AY^2+YC^2$, prove that $\angle BAC=90°$.

20. AD is an altitude of a triangle ABC. If $2DA^2+DB^2+DC^2=BC^2$, prove that $\angle BAC=90°$.

21. If BE and CF are altitudes of a triangle ABC, prove that $AB^2-AC^2=AE^2-AF^2+BF^2-CE^2$.

22. ABCD is a quadrilateral, and X, Y are the feet of the perpendiculars to AD from B, C respectively. Prove that $AB^2-CD^2=AX^2-CX^2+BY^2-DY^2$.

23. If ABCD is a parallelogram, and $AC^2+BD^2=4AB^2$, prove that ABCD must be a rhombus.

24. By writing $(DB+DC)^2$ instead of BC^2 in the result of Ex. 12*a*, No. 22, prove that $DA^2=DB.DC$.

25. O is any point inside a triangle ABC, and P, Q, R are the feet of the perpendiculars from O to BC, CA, AB respectively. Prove that (i) $PB^2-PC^2=OB^2-OC^2$, (ii) $PB^2+QC^2+RA^2=PC^2+QA^2+RB^2$.

26. O is any point inside a quadrilateral ABCD, and P, Q, R, S are the feet of the perpendiculars from O to AB, BC, CD, DA respectively. Prove that $PA^2+QB^2+RC^2+SD^2=PB^2+QC^2+RD^2+SA^2$.

---

27. Given two squares, state how to construct a square whose area is the sum of the areas of the given squares.

28. Given two squares, state how to construct a square whose area is the difference of the areas of the given squares.

29. Draw a straight line 1 in. long, and construct a straight line of length $\sqrt{6}$ in.

30. (i) AD is an altitude of an equilateral triangle ABC. Calculate the ratio of the sides of $\triangle ADB$.

(ii) Draw a square, and use the result of (i) to construct a square of one-third the area.

EXAMPLES 12 c

1. The parallel sides of an isosceles trapezium are 5 and 8 cm. long, and the equal sides are $2\frac{1}{2}$ cm. long. Calculate the area.

2. The vertices of a triangle ABC are the points (0, 0), (12, 5), (7, 17) respectively, referred to rectangular axes. Find by calculation the sides of the triangle, and show that it is right-angled. (N)

3. A cube has an edge of 2 in. A triangle is formed by joining the middle point of one face of the cube to the ends of one of the edges of the opposite face. Calculate the area of this triangle and the lengths of its sides. (OC)

4. AD is an altitude of a triangle ABC. If $AB=3CD$ and $AD=2CD$, prove that $AC=BD$.

5. ABC is a triangle right-angled at B, and D is a point outside the triangle such that the square on AD is equal to the sum of the squares on AB, BC, CD. Prove that the angle ACD is a right angle. (W)

6. Prove that the square on a diagonal of a cube is three times the square on an edge.

---

7. A is the foot of a vertical pole AX, 18 ft. high, and B the top of a vertical shaft BY, 30 ft. deep. If A and B are at the same level and 14 ft. apart, calculate the distance from X to Y.

8. The shortest distance AP from a point A to a line QR is 12 cm., and Q, R are respectively 15 cm. and 20 cm. distant from A on opposite sides of AP. Prove that QAR is a right angle. (N)

9. On each side of a square ABCD, side 4 in., an equilateral triangle is drawn outside the square. These triangles are folded about the sides of the square so as to form a pyramid OABCD, vertex O. Calculate the height of the pyramid, and show that the triangle AOC is right-angled. (O)

10. ABCD is a square. If BC is produced to E so that $CE = \frac{1}{2}BC$, prove that $AE^2 = ED^2 + 2DA^2$.

11. In the equilateral triangle ABC a point D is taken on BC such that $BC = 5BD$. Prove that $25AD^2 = 21AB^2$. (N)

12. Draw a square of side 1 in., and construct a square of 13 times the area.

---

13. Two straight roads intersect at right angles. Two cars, travelling at 25 and 30 m.p.h. respectively, leave the cross-roads at the same instant by different roads. After how many hours, correct to 2 significant figures, are the cars 60 miles apart ?

14. The 8 edges of a pyramid which has a square base are each 2 in. in length ; show that the height of the pyramid is $\sqrt{2}$ in.

Show that the perpendicular distance of each slant edge from the centre of the base is 1 in. (OC)

15. ABC is a triangle and D is the foot of the perpendicular from A on BC. Prove that $AC^2 - AB^2 = CD^2 - BD^2$.

If $AC = 10$ cm., $AB = 7$ cm., $BC = 9$ cm., calculate the length of the perpendicular AD. (OC)

16. D is the mid-point of the base BC of an isosceles triangle ABC, and E is the mid-point of BD. Prove that $AB^2=AE^2+3BE^2$.

17. ABC is a triangle right-angled at A. A right-angled triangle is constructed, in which the lengths of the sides containing the right angle are the sum and difference of the lengths AB and AC. Prove that the hypotenuse of this triangle is equal to the diagonal of the square on BC. (W)

18. Given three squares, state how to construct a square whose area is the sum of the areas of the given squares.

---

19. A wooden protractor is in the shape of a rectangle 6 in. by 2 in. Show that the 45° graduation should be 1 in. from the top corner and find by calculation the distance from that corner of the 60° graduation. (Fig. 186.) (OC)

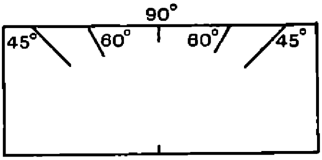

FIG. 186

20. A tetrahedron OABC has its edges OA, OB, OC at right angles to one another. OA=OB=2 in., OC=3 in. Find the area of each of the four faces of the tetrahedron. (W)

21. $\triangle$ABC has a right angle at A; X, Y are the mid-points of AB, AC respectively. Prove that $4(BY^2+CX^2)=5BC^2$.

22. D is a point on the side BC of an acute-angled triangle ABC. If $AB^2-AC^2=DB^2-DC^2$, prove that AD is an altitude. (Hint: let AM be an altitude. Prove that $MB^2-MC^2=DB^2-DC^2$, and use the fact that MB+MC=DB+DC.)

23. Prove that the sum of the squares on the diagonals of a rectangular parallelepiped is equal to the sum of the squares on its edges.

24. A cube of side 3 in. stands on a base ABCD with its edges AA′, BB′, CC′, DD′ vertical. The cube is cut in half across the plane DD′B′B so as to form two wedges, which are then placed with DD′B′B in a horizontal plane. Draw a diagram to represent the wedge DBB′D′CC′, and on it show the lengths of the edges of the wedge.

The part of the wedge DBB′D′AA′ which lies between A′ and the plane AB′D′ is cut away so that the remaining part is a pyramid having its vertex at A and its base DBB′D′. Draw a diagram of the pyramid so formed and find the lengths of the edges AB′, AD′. (OC)

---

25. ABC is a triangle in which $\angle A = 60°$ and $\angle C = 90°$. If D is the point on CA such that $4CD = CA$, prove that $BD = 7CD$.

26. A square, PQRS, of area 58 sq. cm., is placed in a square ABCD, of area 100 sq. cm., with its vertices P, Q, R, S on AB, BC, CD, DA respectively. Calculate the lengths of AP and AS. (L)

27. If the diagonals of the floor, a long wall, and a short wall of a rectangular room are of lengths 22, 20, and 15 ft., calculate to the nearest foot the length of a diagonal of the room.

28. ABCD is a rectangle; X, Y are points on BC, CD respectively such that $\angle AXY = 90°$. Prove that $BX.XC = CY.CD$. (Hint: denote by small letters the lengths in the required result.)

29. AEB, ADB are two right-angled triangles on the same side of a common hypotenuse AB, DE not being parallel to AB. P, Q are the feet of the perpendiculars from A, B respectively to DE (produced both ways). Prove that $DP^2 + DQ^2 = EP^2 + EQ^2$. (L)

30. AO, AB, AC are concurrent edges of a rectangular block, and P, Q, R are the middle points of OB, AB, AC respectively. Prove that PQR is a right-angled triangle and that $4PR^2 = OA^2 + BC^2$. (O)

EXTENSIONS OF PYTHAGORAS' THEOREM

If AB and CD are any two straight lines, and if H, K are the feet of the perpendiculars from A, B respectively to CD (produced either way if necessary), then HK is called the **projection** of AB on CD.

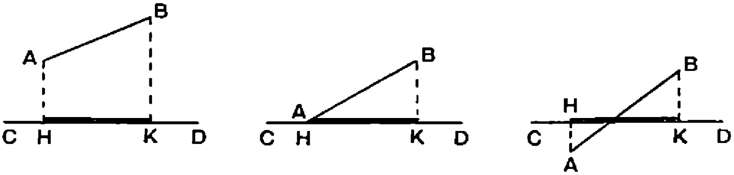

FIG. 187

Figs. 148 (p. 110), 177 (p. 130), 192-195 (p. 147) may be used for oral practice in the use of the word 'projection.'

THEOREM 21 (Extension of Pythagoras' Theorem)

In an obtuse-angled triangle, the square on the side opposite the *obtuse* angle is equal to the sum of the squares on the other two sides *plus* twice the rectangle contained by one of those sides and the projection of the other upon it.

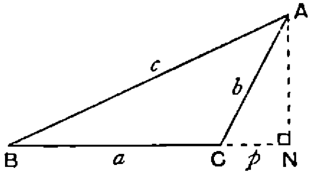

FIG. 188

Given a $\triangle$ABC with $\angle$C obtuse. CN is the projection of CA on CB.

To prove that $AB^2=CA^2+CB^2+2CB.CN$.

Proof. Let BC, CA, AB, CN be a, b, c, p units respectively; then we have to prove that $c^2=b^2+a^2+2ap$.

Applying Pythagoras' to the two right-angled $\triangle$s ABN, ACN,

$$\left.\begin{aligned} c^2&=AN^2+(a+p)^2,\\ b^2&=AN^2+p^2.\end{aligned}\right\}$$

$$\therefore \left.\begin{aligned} c^2&=AN^2+a^2+2ap+p^2,\\ b^2&=AN^2+p^2.\end{aligned}\right\}$$

Subtracting, $c^2-b^2=a^2+2ap$;

$$\therefore c^2=b^2+a^2+2ap.$$

Reference. $\because \angle ACB$ *is obtuse,*

$AB^2=CA^2+CB^2+2CB.CN$ (*ext. of Pythagoras'*).

Note.—To remember the method of proof, notice that small letters are used to denote only the lengths which are required in the result; the other length which occurs in the working, AN, must be eliminated.

To remember the result, notice that (i) each length on the right-hand side begins with C, the letter denoting the angle opposite the side we are considering, (ii) CB and CN, which occur in the rectangle-term, both lie along the same straight line.

This note also applies to the next theorem.

THEOREM 22 (**Extension of Pythagoras' Theorem**)

In any triangle, the square on a side opposite an *acute* angle is equal to the sum of the squares on the other two sides *minus* twice the rectangle contained by one of those sides and the projection of the other upon it.

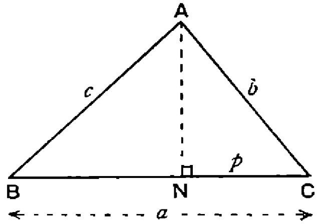

FIG. 189 (i)

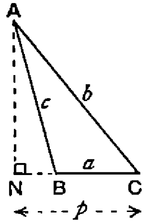

FIG. 189 (ii)

Given a $\triangle ABC$ with $\angle C$ acute. CN is the projection of CA on CB.

To prove that $AB^2 = CA^2 + CB^2 - 2CB.CN$.

Proof. Let BC, CA, AB, CN be a, b, c, p units respectively; then we have to prove that $c^2 = b^2 + a^2 - 2ap$.

Applying Pythagoras' to the two right-angled $\triangle$s ABN, ACN,

$$\left.\begin{array}{l} c^2 = AN^2 + (a-p)^2 \\ b^2 = AN^2 + p^2 \end{array}\right\} \text{in Fig. 189(i)}, \qquad \left.\begin{array}{l} c^2 = AN^2 + (p-a)^2 \\ b^2 = AN^2 + p^2 \end{array}\right\} \text{in Fig. 189(ii)}.$$

$$\therefore \left.\begin{array}{l} c^2 = AN^2 + a^2 - 2ap + p^2 \\ b^2 = AN^2 + p^2 \end{array}\right\} \text{in both Figs.}$$

Subtracting, $c^2 - b^2 = a^2 - 2ap$;

$$\therefore c^2 = b^2 + a^2 - 2ap.$$

Reference. $\because$ *$\angle ACB$ is acute,*

$AB^2 = CA^2 + CB^2 - 2CB.CN$ (*ext. of Pythagoras'*).

Theorems 20*a*, 21, and 22 may be summed up in the single statement :—

Pythagoras' Theorem and its Extensions.

The square on one side of a triangle is greater than, equal to, or less than the sum of the squares on the other two sides according as the angle opposite the side considered is greater than, equal to, or less than 90°. The difference, if any, is twice the rectangle contained by one of the two sides and the projection of the other upon it.

THEOREM 23 (Apollonius' Theorem)

The sum of the squares on two sides of a triangle is equal to twice the square on half the third side plus twice the square on the median which bisects the third side.

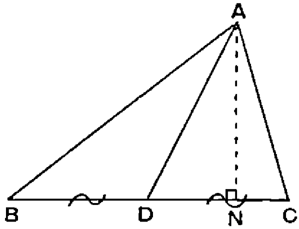

FIG. 190

Given a $\triangle ABC$ with a median AD.

To prove that $AB^2+AC^2=2BD^2+2AD^2$.

Construction. Draw AN perpendicular to BC to meet BC at N.

Proof. If the angles ADB, ADC are not equal,[1] one (ADB say) is obtuse, and the other acute.

$\because \angle ADB$ is obtuse,

$AB^2 = DB^2 + DA^2 + 2DB.DN$

(ext. of Pythagoras').

$\because \angle ADC$ is acute,

$AC^2 = DC^2 + DA^2 - 2DC.DN$

(ext. of Pythagoras').

But $DC = DB$ (given);

$\therefore AC^2 = DB^2 + DA^2 - 2DB.DN.$

$\therefore AB^2 + AC^2 = 2DB^2 + 2DA^2.$

Reference. *AD is a median,*

$\therefore AB^2 + AC^2 = 2BD^2 + 2AD^2$ (*Apollonius'*).

Note.—The square on a line is equal to *four* times the square on half the line.

For, if $BC = 2BD$,

$BC^2 = 4BD^2.$

Hence $2BD^2$ is *not* equal to BC^2, but only to $\frac{1}{2}BC^2$.

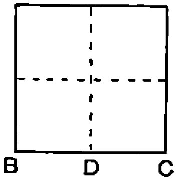

FIG. 191

[1] If $\angle$s ADB, ADC are equal, they are right angles, and the proof is the same except that the rectangle terms are zero because N coincides with D.

EXAMPLES 13 a

(*Theorems* 21 *and* 22)

1. Find out by calculation if triangles with the following sides are acute-, obtuse-, or right-angled :—

(i) 3, 4, 6 in. ; (ii) 4, 5, 6 in. ;
(iii) 5, 12, 13 in. ; (iv) 5, 11, 12 in.

2. Given four straight lines of lengths 2, 3, 4, 5 in., find by calculation how many triangles can be formed with three of them for sides, and the nature of these triangles.

3. Prove that a triangle with sides 7, 9, and 11 cm. is acute-angled. Draw a freehand figure, and calculate the projection of the 7 cm. side on the 11 cm. side. ($9^2 = \ . \ . \ .$)

4. Prove that a triangle with sides 14, 12, and 7 in. is obtuse-angled. Draw a freehand figure, and calculate the projection of the 7 in. side on the 12 in. side. ($14^2 = \ . \ . \ .$)

5. A triangle has sides 8, 10, and 11 cm. long. Calculate the projection of the 8 cm. side on the 10 cm. side.

6. A triangle has sides of 3, 5, and 7 in. Calculate the projection of the 7 in. side on the 3 in. side.

7. In $\triangle$ABC, BC=6 in., CA=7 in., and AB=9 in. ; AN is an altitude. Prove that the triangle is acute-angled, and calculate the lengths of BN and CN.

8. State a problem, on calculating a projection, which leads to the equation $81=49+16+8p$.

(*Theorem* 23)

9. If the sides of a triangle are 4, 7, and 9 in. long, calculate the length of the median which bisects the 9 in. side.

10. Calculate correct to the nearest mm. the lengths of the three medians of a triangle with sides 2, 3, and 4 cm. long.

11. AD is a median of a triangle ABC. If the lengths of AB, AC, and AD are 13, 29, and 12 in. respectively, calculate the length of BC.

12. A parallelogram has sides 13 and 9 in. long. If one diagonal is 20 in. long, calculate the length of the other.

(*Theorems* 21 *and* 22)

13. Apply the Extension of Pythagoras' in two different ways to each of Figs. 192 and 193.

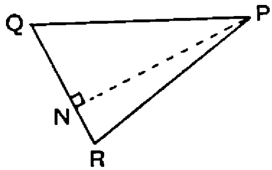

FIG. 192

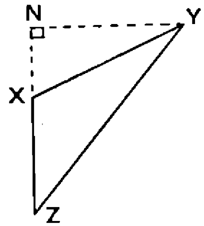

FIG. 193

14. In **ABC**, **AB**=**AC**. If **BN** is an altitude, prove that $BC^2=2CA.CN$. (Write down an expression for AB^2.)

15. If **BM**, **CN** are altitudes of a triangle **ABC**, prove that $AC.AM=AB.AN$. (Write down two different expressions for BC^2.)

16. **ABCD** is a rectangle; **P** is any point on **AB**, and **Q** is any point on **CD**. Prove that $PA^2+PC^2+2PA.PB=QA^2+QC^2+2QC.QD$.

17. In Fig. 194, if the lengths of **BC**, **CA**, **AB** were given, what equations would you write down in order to calculate the lengths of **BD**, **CD**, **AF**, **AE**, **BF**, **CE**?

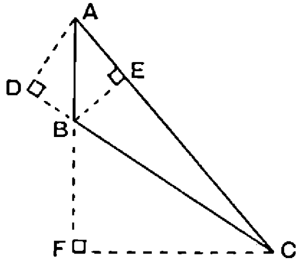

FIG. 194

18. **ABC** is a triangle with $\angle C$ obtuse and **CB**=**2CA**. If **BD** is an altitude, prove that $AB^2=5CA^2+2CA.CD$.

19. For Fig. 195, complete the statements $AC^2=DA^2+DC^2$ . . . , $BD^2=CB^2+CD^2$ . . . , and hence prove that

$$AC^2+BD^2=AD^2+BC^2+2AB.CD.$$

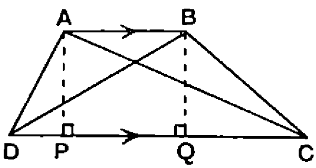

FIG. 195

20. $\triangle ABC$ has an obtuse angle at **C**; **AN** is an altitude, and **M** is the mid-point of **BC**. Write down an expression for AB^2, and deduce that $AB^2=AC^2+2BC.MN$.

21. If BN is an altitude of an acute-angled triangle ABC, prove that $BC^2 = AB^2 + AC(NC - NA)$.

22. ABC is a triangle with BC, CA, AB of lengths $2a$, $4a$, $3a$ units respectively. Prove that the triangle is obtuse-angled.

If AD is an altitude, prove that $BD = \frac{1}{4}AB$.

23. ABC is an equilateral triangle with BC produced to D. Prove that $AD^2 = BD^2 - CB.CD$. (Draw the altitude AN; denote CD by x units, and each side of $\triangle ABC$ by a units.)

(*Theorem* 23)

24. ABC is a triangle in which AB=AC. If BA is produced to D so that AD=BA, prove that $CB^2 + CD^2 = 4CA^2$.

25. ABC is an equilateral triangle. If CB is produced to D so that BD=CB, prove that $AD^2 = 3AC^2$. (Let the equal lengths be x units.)

26. ABCD is a rectangle. If P is any point in the plane ABCD, prove that $PB^2 + PD^2 = PA^2 + PC^2$. (Let the diagonals cut at O, and join PO.)

Is this result also true when P is not in the plane ABCD ?

27. ABC is a triangle in which AB=AC. If AC is produced to D so that CD=AC, prove that $BD^2 = AB^2 + 2BC^2$.

28. If ABCD is a parallelogram, prove that $2(AB^2 + BC^2) = AC^2 + BD^2$.

29. ABCD is a rectangle whose diagonals cut at O. If P is any point, prove that $PA^2 + PB^2 + PC^2 + PD^2 = AC^2 + 4PO^2$.

EXAMPLES 13 b

1. Two sides of a triangle are 10 and 7 in. long. What is the greatest possible length of the third side if it is to contain a whole number of inches and if the triangle is to be acute-angled ?

2. ABC is a straight line; AB=4 in., BC=2 in. If a perpendicular BD, 3 in. long, is erected at B, find out if $\angle ADC$ is acute or obtuse.

3. Calculate to the nearest mm. the length of the longest median of a triangle with sides 4, 5, and 8 cm. long.

4. A triangle has sides 9, 14, and 15 cm. long. (i) Calculate the length of the projection of the 9 cm. side on the 15 cm. side; (ii) prove that the area of the triangle is $\frac{5}{2}\sqrt{608}$ sq. cm.

5. Use the method of No. 4 to calculate to the nearest sq. cm. the area of a triangle whose sides are 5, 6, and 8 cm. long.

6. The diagonals of a parallelogram **ABCD** intersect at **O**. If **AB**=12 cm., **AC**=19 cm., and **BD**=13 cm., calculate the length of **BC**.

7. **D** is the mid-point of the side **AC** of a triangle **ABC**. If **AC**=4 in. and **BC**=**BD**=2 in., calculate the length of **AB** to the nearest tenth of an inch.

8. In △**ABC**, **AC**=40 ft., **AB**=20 ft., and the projection of **AB** on **BC** is 5 ft. Calculate the length of **BC**, and state whether △**ABC** is acute- or obtuse-angled.

9. Calculate the length of the median which bisects the longest side of a triangle with sides $\sqrt{11}$, $\sqrt{15}$, and 4 cm. long.

10. **ABC** is an acute-angled triangle such that **BC**=8 cm., **CA**=10 cm., and the projection of **BC** on **CA** is 1 cm. Calculate the projection of **BC** on **AB**.

11. **ABCD** is a rectangle, and **E** a point on **CD**. If **AD**=2 in., **DE**=3 in., and **EC**=1 in., prove that △**ABE** is acute-angled.

12. The sides of a triangle are of lengths $2a$, $2b$, $2c$ units. Express in terms of a, b, and c the length of the median bisecting the third side.

13. If **OA, OB, OC** are three edges of a rectangular parallelepiped, prove that △**ABC** is acute-angled.

---

14. **ABC** is a triangle in which **AB**=**AC**; **O** is the mid-point of **BC**, and **D** is any point between **B** and **O**. Prove, by considering △**ABD**, that $AB^2=DA^2+DB.DC$. (Let **BD**=x, **DO**=y units; then **OC**=$x+y$ units.)

15. E is a point on the median AD of a triangle ABC. By applying Apollonius' to triangles ABC and EBC, prove that $AB^2+AC^2-2AD^2=EB^2+EC^2-2ED^2$.

16. ABCD is a straight line, and AB=BC=CD. If O is any point not on ABCD, prove that $OA^2+OD^2=OB^2+OC^2+4BC^2$. (Use OB and OC as medians.)

17. If two medians of a triangle are equal, prove that the triangle is isosceles. (Apply Apollonius' twice, and subtract.)

18. AXB, AYB are two acute-angled triangles on opposite sides of the same base AB. If $XA^2-XB^2=YA^2-YB^2$, prove that XY is perpendicular to AB. (Let XM, YN be perpendiculars to AB; first prove that $XA^2-XB^2=AB(AM-MB)$, and a similar result.)

19. In any triangle, prove that three times the sum of the squares on the sides is equal to four times the sum of the squares on the medians.

20. If CH and BK are medians of a triangle ABC, prove that $BK^2+CH^2=AH^2+AK^2+BC^2$. (Let $AB=2b$, $AC=2c$ units.)

21. AD is a median of a triangle right-angled at A. Apply Apollonius', and deduce that $AD=\frac{1}{2}BC$.

22. ABC is a triangle, and D a point on BC such that DC=2DB. Join AD, draw an altitude AN, and prove that $2AB^2+AC^2=2DB^2+DC^2+3DA^2$.

23. If ABC is a triangle, and D a point on BC such that $m.BD=n.DC$, prove that $m.AB^2+n.AC^2=m.DB^2+n.DC^2+(m+n)AD^2$. (Use the same method as for No. 22.)

EXAMPLES 13 c

1. The sides of a triangle are 5, 7, and 9 in. long. Prove, without drawing it to scale, that it is obtuse-angled, and calculate the lengths of the segments into which the longest side is divided by the perpendicular from the opposite vertex. (N)

2. Fig. 196 shows a crane; the jib **AC** is 16 ft. long, the tie-rod **AB** is 7 ft. long, and **BC** is a vertical post 12 ft. long. Calculate the vertical height of **A** above **C**. (C)

3. The base **BC** of a triangle **ABC** is trisected at **P** and **Q**, **Q** being nearer to **C**. Prove that $2AB^2+AC^2=3AP^2+6PQ^2$. (W)

4. If **BE** and **CF** are altitudes of an acute-angled triangle **ABC**, prove that $BC^2=BA.BF+CA.CE$.

5. If **CH** and **BK** are medians of a triangle **ABC**, prove that $4(BK^2-CH^2)=3(AB^2-AC^2)$.

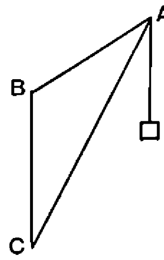

FIG. 196

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6. If the lengths of the sides of a triangle are $2\frac{2}{3}$, $4\frac{1}{3}$, and 5 in., calculate the length of the projection of the shortest side on the longest side.

7. If a triangle **ABC** is such that **AC**=8 in., **BC**=3 in., and **AB**=7 in., calculate to the nearest hundredth of an inch the length of the perpendicular from **A** to **BC**.

8. **ABC** is a triangle in which **AB**=**AC**, and **D** is a point on **AC** such that **BD**=**BC**. Prove that $CB^2=CA.CD$. (Draw **BX** perpendicular to **CD**.)

9. **ABCD** is a straight line, and **AB**=**BC**=**CD**. If **O** is any point not on **ABCD**, prove that $OA^2-OD^2=3(OB^2-OC^2)$.

10. In Fig. 197, **SR**=2**QS**. By considering the triangles **PQS** and **PSR** or otherwise, prove that $PR^2+2PQ^2=6QS^2+3PS^2$. (N)

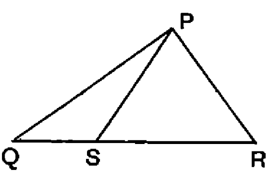


FIG. 197

11. In a triangle **ABC**, $AB^2-AC^2=$ 66 sq. in., and **BC**=6 in. State whether the triangle has an obtuse angle or not, and calculate the length of the projection of **AC** upon **BC**. (N)

12. If the angle **C** of a triangle **ABC** is 60° and **BC**=4**AC**, prove that $AB=\sqrt{13}\,AC$.

13. The sides **AB**, **AC** of a triangle **ABC** are equal. From any point **D** on the side **AB**, a line **DE** is drawn parallel to **BC** to cut **AC** at **E**. Prove that $BE^2=CE^2+BC.DE$. (N)

14. ABC is an isosceles triangle in which AB=AC. E is a point taken anywhere on CB produced beyond B. Prove that $AE^2-AB^2=EB.EC$. (W)

15. ABCD is a parallelogram. If P is any point, prove that the expression $(PA^2+PC^2)-(PB^2+PD^2)$ has the same value wherever P may be. Is this true if P is not in the same plane as ABCD ?

---

16. In a triangle ABC, AB, BC, and CA are respectively equal to 5, 4, and 7 in. Calculate the projection of BC on AB, and the area of the triangle to the nearest tenth of a sq. in. (L)

17. ABCD is a square of side $2a$ units ; E and F are the middle points of CD, CB respectively. The perpendicular from E upon AF meets AF at K. Prove that $AK=4a\div\sqrt{5}$ units. (L)

18. If X, Y are the mid-points of the diagonals of a quadrilateral ABCD, prove that $AB^2+BC^2+CD^2+DA^2=AC^2+BD^2+4XY^2$. (Begin by using $\triangle$s ABC and ADC.)

19. In a triangle ABC the angle C is 135°, BC is 14 in. long and the perpendicular from A to BC produced is 10 in. long. Show that the square on AB exceeds the sum of the squares on the other two sides by four times the area of the given triangle. Further, if N is the mid-point of AB, show that the squares on BN and CN differ by twice the area of the given triangle. (N)

20. BCML is a square, and P is any point outside the square. Prove that $PL^2-PB^2=PM^2-PC^2$.

Deduce that if P is any point within a triangle ABC and squares BCML, CNRA, ASTB are described externally on its sides, then $PL^2+PN^2+PS^2=PM^2+PR^2+PT^2$. (N)

MISCELLANEOUS EXAMPLES II

1. In Fig. 198, a triangle ABC is inscribed in a rectangle having the dimensions shown. Calculate the area of the triangle and the length of the line joining the middle points of AB and BC. (N)

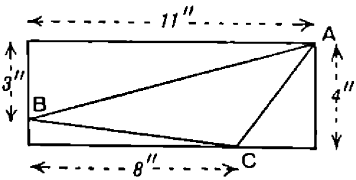


FIG. 198

2. Draw a parallelogram of area 7 sq. in., having a line $3\frac{1}{2}$ in. long as base and its longer diagonal 5 in. long.

On this diagonal as base, construct, without any calculation, one triangle of area equal to that of the parallelogram, and also the locus of the vertices of all such triangles. (In each case, state your construction without proof.)

3. In Fig. 199, AD and CE are at right angles to AC; also AD=BC, CE=AB, and DF is parallel to AC. Prove that DBE is an isosceles right-angled triangle.

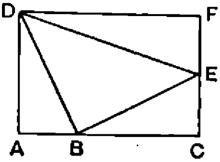


FIG. 199

Taking AB=a units, BC=b units, express in algebraical form the result of applying the Theorem of Pythagoras to the triangles DBE, DEF, *i.e.* $DB^2+BE^2=DF^2+FE^2$. (OC)

4. ABCD is a trapezium in which AB is parallel to DC. DF drawn parallel to CB meets AB at F. E is the middle point of AF. Prove that the area of the triangle EBC is half that of the trapezium. (OC)

5. On opposite sides of a straight line BC, acute-angled triangles ABC, DBC are drawn such that AB=CD, and BD is greater than AC. If AH and DK are drawn perpendicular to BC, prove that $BD^2-AC^2=2BC(BH-CK)$. (L)

---

6. Draw a line 5 in. long. Trisect it by a geometrical construction and on one of the parts construct an equilateral triangle. Calculate the altitude of the triangle correct to two places of decimals. (N)

7. Construct the triangle ABC in which $\angle ABC=100°$, AC=8 cm., and the altitude AD is 4 cm. Measure BC and calculate the area of the triangle. State briefly the steps of your construction. (N)

8. H and K are the mid-points of the sides AB, AC of a triangle ABC. CE is drawn equal and parallel to AH, EH cuts BC at L, and EK cuts BC at M. Prove that (i) L is the mid-point of BC, (ii) M is the mid-point of LC.

9. ABC is a triangle in which AB=AC, and CD is an altitude. Prove that $BC^2=2AC.BD$.

10. The sides AB, BC, CD, DA of a parallelogram ABCD are produced in that order to E, F, G, and H respectively, so that BE=AB, CF=BC, DG=CD, and AH=DA. Prove that :—

(i) the triangles DGH and BEF are congruent ;
(ii) EFGH is a parallelogram ;
(iii) the area of triangle DGH is equal to the area of the parallelogram ABCD. (N)

---

11. Draw a triangle ABC, having AB=5 cm., BC=3 cm., ∠ABC=52°. Find one position of a point P which satisfies the following conditions : PA=PB and the area of the triangle PBC equals 4·5 sq. cm. Describe (without proof) your method. Measure PA. (L)

12. A fixed straight line BC is 11 in. long and A is a variable point such that $AB^2-AC^2=231$ sq. in. Prove that ∠ACB is always obtuse.

If D is the foot of the perpendicular from A to BC produced, calculate the length of CD and state the locus of A.

13. Construct, full size, a regular hexagon of side 4 cm. and *calculate*, correct to the nearest mm., the distance between any pair of opposite sides. (N)

14. The side BC of the triangle ABC is produced to D, CD being less than BC ; CE is drawn parallel to DA to meet AB in E. Show that triangles ABC, BDE are equal in area.

Hence show how to draw a line from E so as to bisect the triangle ABC. (OC)

15. M, N are the mid-points of the equal sides AB, AC respectively of an isosceles triangle ABC. The straight line through B parallel to MC meets AC produced at P, and the straight line through C parallel to NB meets AB produced at Q. Prove that ∠APQ=∠AQP.

---

16. Prove that the square on the hypotenuse of a right-angled triangle is equal to the square on the median which bisects one of the other two sides together with three times the square on half this side.

17. If P, Q, R, S are the mid-points of the respective sides AB, BC, CD, DA of a quadrilateral ABCD, prove that $AC^2+BD^2=2(QS^2+PR^2)$.

18. O is a fixed point and AB a fixed straight line. If a variable point P moves along AB, state the locus of the mid-point of OP.

Use this result to help you to construct a triangle DEF, given that the base EF is of length 1·7 in., the altitude is 1·2 in., and the length of the straight line joining E to the mid-point of DF is 0·9 in.

19. P is the middle point of the side BC of a triangle ABC and X is any point on AP. Prove that the triangles AXB, AXC are equal in area.

State, without proof, a construction for determining a point O within the triangle ABC such that the triangles BOC, COA, AOB are equal in area. (OC)

20. ABC is a triangle in which AB=AC. P is any point on AB, and Q is the point, on AC produced, such that CQ=BP. Prove that BC bisects PQ.

---

21. In a convex quadrilateral ABCD the mid-points of the sides AB, BC, CD, DA are E, F, G, H respectively. Prove that EG and FH bisect one another. If EG = FH, find the angle between the diagonals of the quadrilateral. (N)

22. If A, B, C, D are successive vertices of any regular polygon, prove that AC bisects ∠BAD.

23. PQR is an isosceles triangle having PQ=PR=3 cm., QR=2 cm. A point S is taken in QR produced such that $PS=6\sqrt{2}$ cm. *Calculate* the length of the projection of SP upon SQ and prove, *by calculation*, that the angle QPS is a right angle. (L)

24. Construct a triangle ABC in which AB=AC=5·5 cm. and BC=4·3 cm. With ruler and compasses only construct (i) the point D on BC such that AD is perpendicular to BC, and (ii) a triangle ADE equal in area to ABC and having AE equal to AC. (C)

25. ABCD is a square, E, F, G, H are points in the sides AB, BC, CD, DA respectively, such that AE=BF=CG=DH; show that $FH^2=2(AE^2+BE^2)$. (OC)

---

26. ABCD is a quadrilateral, and P any point; M, N are the mid-points of AB, CD respectively, and O the mid-point of MN. If $PA^2+PB^2+PC^2+PD^2$ is constant, prove that the locus of O is a circle. (Hint: apply Apollonius' three times.)

27. ABCD is a parallelogram. If any straight line parallel to BA cuts BC, AC, AD at X, Y, Z respectively, prove that $\triangle AXY=\triangle DYZ$.

28. OA, OB, OC are lines mutually at right angles and respectively 5, 4, and 3 in. long; find (to the nearest tenth of an inch) the length of the perpendicular from A on BC and (to the nearest tenth of a square inch) the area of the triangle ABC. (OC)

29. Two straight lines, AB, CD, of equal length are neither parallel nor along the same straight line. The perpendicular bisectors of AC, BD meet at O. Show that the perpendicular distances of O from AB, CD are equal. (OC)

30. ABC is a triangle in which AB=10 cm., AC=4 cm., and BC=8 cm. A point X is taken on BC such that CX=2 cm. By two successive applications of Apollonius' theorem, calculate the length of AX.

31. Construct a triangle ABC in which BC=9 cm., CA=8 cm., AB=5 cm. Along BC mark off BD=7 cm. On BD as base, construct a triangle equal in area to the triangle ABC. Measure its height.

Give a description of the construction of the second triangle and prove the correctness of your method. (OC)

THE CIRCLE: CHORD PROPERTIES

Definitions

A **circle** is the locus of points at a given distance from a certain fixed point. The fixed point is called the **centre** of the circle; the given distance is called the **radius** of the circle.

Sometimes the word 'circle' is used to denote the part of the plane inside the curved line. The curved line is then called the **circumference** of the circle.

A straight line passing through the centre of a circle and terminated both ways by the circle is called a **diameter**.

An **arc** of a circle is any portion of the circumference of the circle.

A **chord** of a circle is a straight line joining any two points on the circle.

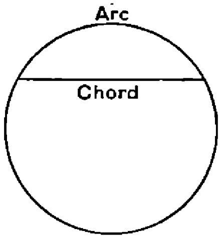


FIG. 200

Two circles are said to be **concentric** if they have the same centre.

If ABC is a triangle, the circle which passes through A, B, C is called the **circumcircle** of △ABC; its centre is the **circumcentre** of the triangle, and its radius is the **circumradius** of the triangle.

THEOREM 24 a

The perpendicular from the centre of a circle to a chord bisects the chord.

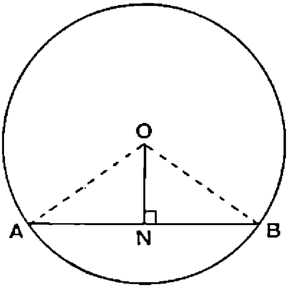


FIG. 201

Given a circle, centre O, and a chord AB ; ON is the perpendicular from O to AB.

To prove that AN=BN.

Construction. Join OA, OB.

Proof. In the right-angled △s OAN, OBN,

ONA and ONB are the right angles (given),

OA=OB (radii),

ON=ON ;

∴ △s $\begin{matrix}OAN\\OBN\end{matrix}$ are congruent (RHS).

∴ AN=BN.

Reference. AN=BN (*perp. from centre to chord*).

THEOREM 24 b

The straight line joining the centre of a circle to the mid-point of a chord is perpendicular to the chord.

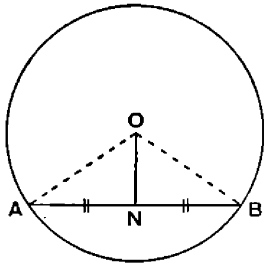


Fig. 202

Given a circle, centre O, and a chord AB ; N is the mid-point of AB.

To prove that ∠s ONA, ONB are right angles.

Construction. Join OA, OB, ON.

Proof. In the △s ONA, ONB,

OA = OB (radii),

AN = BN (given),

ON = ON ;

∴ △s ONA, ONB are congruent (SSS).

∴ ∠ONA = ∠ONB.

But ∠ONA + ∠ONB = 180° (adj., ANB a str. line) ;

∴ ∠s ONA, ONB are right angles.

Reference.

ON is perpendicular to AB (*line joining centre to mid-point of chord*).

Corollary.

The perpendicular bisector of a chord of a circle passes through the centre of the circle.

THEOREM 25 a

Equal chords of a circle are equidistant from the centre.

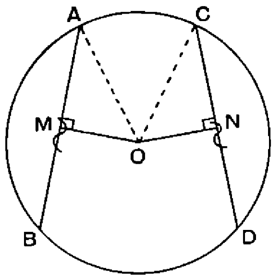


FIG. 203

Given a circle, centre O, and two equal chords AB, CD.

Construction. Draw OM, ON, the perpendiculars from O to AB, CD respectively. Join OA, OC.

To prove that OM=ON.

Proof. AM=$\frac{1}{2}$AB (perp. from centre to chord),

and CN=$\frac{1}{2}$CD (,,);

but AB=CD (given),

∴ AM=CN.

In the right-angled △s AMO, CNO,

AMO and CNO are the right angles (constr.),

AO=CO (radii),

AM=CN (proved);

∴ △s $\begin{matrix}\text{AMO}\\ \text{CNO}\end{matrix}$ are congruent (RHS).

∴ OM=ON.

Reference. OM=ON (*equal chords equidistant from centre*).

THEOREM 25 b

Chords of a circle which are equidistant from the centre of the circle are equal.

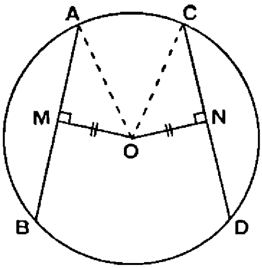


FIG. 204

Given a circle, centre O, and two chords AB, CD of the circle; the perpendiculars OM, ON to these chords are equal.

To prove that AB=CD.

Construction. Join OA, OC.

Proof. In the right-angled △s AMO, CNO,

AMO and CNO are the right angles (given),

AO=CO (radii),

OM=ON (given);

∴ △s $\begin{matrix}\text{AMO}\\\text{CNO}\end{matrix}$ are congruent (RHS).

∴ AM=CN.

But AB=2AM (perp. from centre to chord),

and CD=2CN ( ).

∴ AB=CD.

Reference. AB=CD (*chords equidistant from centre*).

L

Corollary 1.

Equal chords of two equal circles are equidistant from the centres of those circles.

Corollary 2.

Chords of two equal circles which are equidistant from the centres of those circles are equal.

Corollary 3.

The locus of the mid-points of equal chords of a circle is a concentric circle.

CONSTRUCTION 17

To construct the circumcircle of a triangle.

(In other words, to construct a circle to pass through three given points which do not lie in a straight line.)

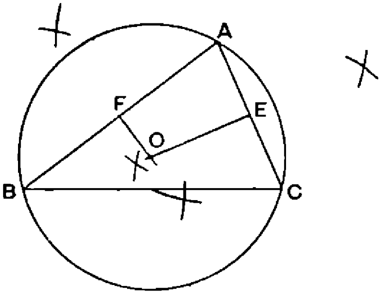


FIG. 205

Given a triangle ABC.

To construct a circle to pass through A, B, C.

Construction. Draw the perpendicular bisectors FO, EO of AB, AC to meet at O.

With centre O, radius OA, draw a circle.

This is the required circle.

Proof. ∵ O lies on the perpendicular bisector of AB,

∴ OB = OA.

Again, ∵ O lies on the perpendicular bisector of AC,

∴ OC = OA.

Hence, OA = OB = OC.

∴ the circle, with centre O and radius OA, passes through A, B, and C.

Note.—It should be noticed that a circle cannot be drawn through three points A, B, C which lie in a straight line, for the perpendicular bisectors of AB and AC would not meet. It follows that a straight line cannot cut a circle at more than two points, a fact which is of some importance in connection with the definitions of a tangent and a secant (p. 203).

Corollaries to the Isosceles Triangle Theorem.

The corollaries to Theorem 5 are useful in riders, but they should not be used to prove Theorems 14 or 24. Three are given below; the reader might usefully formulate some of the others.

The bisector of the vertical angle of an isosceles triangle bisects the base at right angles.

The perpendicular from the vertex of an isosceles triangle to the base bisects the base.

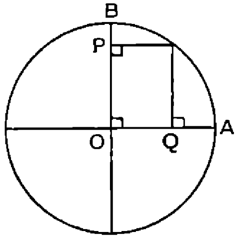
If the perpendicular bisector of the base of a triangle passes through the vertex, the triangle is isosceles.

EXAMPLES 14a

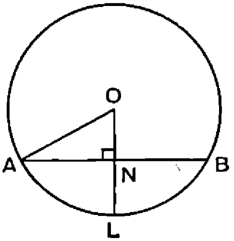
(*Definition of a circle ; Theorem* 24)

1. **AB** is a diameter of a circle, **C** is any point on the circle, and **CA**, **CB** are joined. Prove that $\angle CAB + \angle CBA = \angle ACB$, and deduce that $\angle ACB = 90°$. (Join **C** to the centre of the circle.)

2. In Fig. 206, calculate **PQ**, if $BP = \frac{1}{2}$ in. and the radius of the circle is 3 in. (This is a catch question.)



O is the centre
FIG. 206



O is the centre
FIG. 207

3. In Fig. 207, if $AB = 8$ cm., and $OA = 5$ cm., calculate **ON**.

4. In Fig. 207, if $OA = 13$ cm., and $ON = 5$ cm., calculate **AB**.

5. In Fig. 207, if $AB = 14$ in., and $ON = 24$ in., calculate **OA**.

6. Two parallel chords of a circle of radius 6 cm. are 10 cm. and 8 cm. long. Calculate the distance between them to the nearest mm. (There are two cases.)

7. **AB**, **BC** are two equal chords of a circle, centre **O**, and **M**, **N** are the feet of the perpendiculars from **O** to **AB**, **BC** respectively. Prove that $\angle MOB = \angle NOB$. (Prove that $\triangle$s **OMB**, **ONB** are congruent.)

8. State the locus of the mid-points of parallel chords of a circle, and illustrate by a sketch.

9. In Fig. 208, **PQRS** is a straight line perpendicular to the line of centres **AB** ; prove that $PR = QS$.

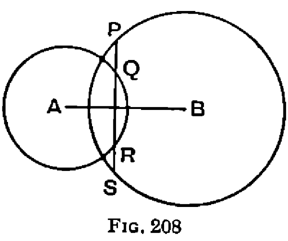


FIG. 208

10. If two circles intersect, prove that the line of centres bisects the common chord at right angles. (Hint: let **X**, **Y** be the centres of the circles, **AB** the common chord, and **M** the mid-point of **AB**. Join **MX**, **MY**, and prove that **XMY** is a straight line.)

(*Theorem* 25)

11. In Fig. 209, two chords **AB**, **CD** of a circle, centre **O**, intersect at **X**, and **OX** bisects ∠**AXC**. Prove that **AB**=**CD**. (Hint: draw perpendiculars **OP**, **OQ** from **O** to **AB**, **CD**, and show that **OP**=**OQ**.)

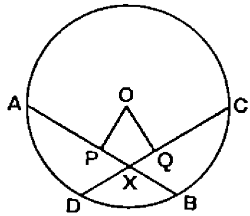


FIG. 209

12. **AB**, **CD** are two equal chords of a circle which cut at right angles at **X**. **O** is the centre of the circle, and **P**, **Q** are the mid-points of **AB**, **CD**. Prove that **OPXQ** is a square.

(*Constructions*)

13. Draw a circle of radius 2 in. and place end to end in it six chords each equal to the radius.

14. Draw any straight line **AB**, and, using straight edge and compasses only, construct the circle which has **AB** as diameter.

15. Draw a rhombus with each side 6 cm., and, using straight edge and compasses only, construct the four circles which have the sides of the rhombus as diameters. (They should all pass through the intersection of the diagonals of the rhombus.)

---

16. Draw any straight line **AB**, and construct the locus of the centres of all circles which have **AB** as a chord.

17. Draw a straight line **AB** 5 cm. long, and construct a circle of radius 4 cm. to pass through **A** and **B**.

18. Construct two circles of radii 3 cm. and $4\frac{1}{2}$ cm. such that their common chord is of length 4 cm. (Begin by drawing the common chord.)

19. Draw a chord **AB** of length 2 cm. in a circle of radius $2\frac{1}{2}$ cm. Construct a circle to pass through **A**, **B** and have its centre on the first circle. (Two solutions.)

20. Draw any quadrilateral **PQRS**, and construct two concentric circles, one passing through **P** and **Q**, and the other through **R** and **S**.

21. Given a circle and a point inside it, construct the chord which is bisected by the given point.

---

22. Construct the locus of the mid-points of chords 6 cm. long in a circle of radius 5 cm.

23. Draw a circle by means of a penny, and construct its centre.

24. Draw a triangle with sides 6, 7, 8 cm. Construct its circumcircle, and measure the radius.

25. Draw a triangle **ABC** in which **BC**=7 cm., ∠**ABC**=62°, ∠**ACB**=47°. Construct its circumcircle, and measure the radius.

26. Draw a triangle with sides $1\frac{1}{2}$, 2, and $2\frac{1}{2}$ in. Construct its circumcircle, and measure the radius.

27. Draw a triangle **ABC** in which **AB**=2 in., **BC**=3 in., and ∠**BAC**=120°. Construct its circumcircle, and measure the radius.

28. Draw a triangle with sides 5, 6, and 7 cm., and on its three sides construct equilateral triangles all pointing outwards. Draw the circumcircles of these equilateral triangles. (They should all intersect at a point inside the triangle.)

EXAMPLES 14b

1. In Fig. 207, if **AB**=l, **OA**=r, **ON**=p units, prove that $l^2+4p^2=4r^2$.

2. An arch is in the form of a circular arc, the centre of the circle being underground. If the height and span of the arch are 29 ft., 100 ft. respectively, calculate to the nearest tenth of a ft. the depth of the centre of the circle below the ground.

3. **AB** is a chord of a circle, centre **O**, and **AC** is the diameter through **A**. **N** is the foot of the perpendicular from **O** to **AB**. Prove that **ON**=$\frac{1}{2}$**CB**.

4. Two circles, centres **A** and **B**, intersect at **X** and **Y**. A straight line drawn through **X**, parallel to **AB**, cuts the circles at **P** and **Q**. **R**, **S** are the feet of the perpendiculars from **A**, **B** respectively, to the line **PQ**. Prove that (i) **RS**=**AB**, (ii) **PQ**=2**AB**.

5. Two circles are drawn with the same centre O, and a straight line cuts them at the points P, Q, R, S in order. Prove that PQ=RS. (What construction is necessary?)

6. Given a circle and a point P, construct a circle with centre P to cut the given circle at the ends of a diameter of that circle.

7. AB is a diameter and O the centre of a circle; P, Q are points on one of the semicircular arcs AB. The perpendiculars to PQ at P and Q cut AB at M, N respectively. Prove that OM=ON.

8. AB, CD are chords of a circle; M, N are the respective mid-points of these chords, and O is the centre of the circle. Prove that $OM^2-ON^2=CN^2-AM^2$, and deduce that the longer chord is the one nearer the centre of the circle.

9. Given a circle and a point inside it, use the result of No. 8 to draw the longest and shortest chords through the given point.

10. Calculate the radius of the circumsphere of a cuboid which is 6 in. by 3 in. by 2 in.

11. A sphere, whose centre is O, is cut by a plane, and ON is drawn perpendicular to the plane, meeting it at N (see Fig. 210). If P is any point on the curve of intersection of the plane and the sphere, prove that $PN^2=OP^2-ON^2$, and deduce that the curve of intersection is a circle.

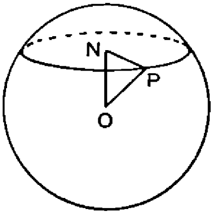


FIG. 210

12. A lens, shaped like the part of a sphere cut off by a plane, has its flat circular face 4 in. in diameter, and the lens is 1 in. thick. Find the radius of the sphere of which it forms part. (Hint: suppose Fig. 207 represents a section through the centres of the sphere and lens; then, if $OA=r$ in., $ON=(r-1)$ in.)

13. A rectangular plate 6 in. long and 4 in. wide rests in a horizontal position inside a hemispherical bowl of radius 5 in. Calculate to the nearest tenth of an inch the height of the centre of the plate above the bottom of the bowl.

---

14. A triangle ABC is drawn with its vertices lying on the circumference of a circle whose centre is O, and OA bisects ∠BAC. Prove that AB=AC. (Draw perpendiculars from O to AB, AC.)

15. A, B are the centres of two equal circles. A straight line is drawn parallel to AB, cutting one circle at P, Q and the other circle at R, S. Prove that PQ=RS. (Draw perpendiculars from A, B to PS.)

16. A, B are the centres of two equal non-intersecting circles, and O is the mid-point of AB. A straight line is drawn through O cutting one circle at P, Q and the other circle at R, S. Prove that PQ=RS.

17. In Fig. 209, two equal chords AB, CD of a circle, centre O, intersect at X. P and Q are the feet of the perpendiculars from O to AB, CD. Prove that (i) XP=XQ (use congruence), (ii) AX=CX.

---

18. The diagonals of a quadrilateral ABCD meet at O. Prove that the circumcentres of triangles AOB, BOC, COD, DOA are the vertices of a parallelogram. (Note that the circumcentre of △AOB lies at the intersection of the perpendicular bisectors of AO and BO. Similarly for the other triangles.)

19. △ABC has ∠A a right angle. D, E, F are the mid-points of BC, CA, AB respectively. Join DE, DF, and prove that D is the circumcentre of the triangle ABC.

EXAMPLES 14c

1. In a circle of radius 6 in. a chord is placed equal in length to the radius. Calculate its perpendicular distance from the centre. (W)

2. Calculate the length of a chord of a circle which is 1·2 in. from the centre, if the radius is 1·5 in.

Two circles intersect at A and B. Through A a line PAQ is drawn to cut the first circle at P and the second at Q. Through B, a line RBS is drawn to cut the first circle at R and the second at S. Prove that, if PQ is parallel to RS, then PQ=RS. (W)

3. O is the centre of a circle, AB is a diameter, and CD a chord. If M, N are the feet of the perpendiculars from A, B respectively to CD (produced if necessary), prove that MC=DN.

4. **AB**, **CD** are two equal chords of a circle which can be produced beyond **B** and **D** respectively to meet at **S** outside the circle. Prove that **SA**=**SC**. (N)

5. Draw a triangle **ABC** in which **AB**=3 in., **BC**=4 in., **CA**=4·5 in. Construct a circle to pass through the points **A, B, C** and measure its radius. (No proof is required, but all construction lines should be shown.) (OC)

6. A horizontal circle, diameter 3 in., is drawn. Through its centre **O** and in its plane a straight line **OA**, $2\frac{1}{2}$ in. long, is drawn. A point **P** is 2 in. vertically above **A**. Find by accurate drawing (i) the height above the plane of the circle of the centre of a sphere whose surface passes through the circle and through **P**, (ii) the radius of the sphere. (O)

---

7. A chord distant 2 cm. from the centre of a circle is 18 cm. long. Calculate the length of a chord of the same circle which is 6 cm. distant from the centre. (N)

8. Prove that the common chord of two intersecting circles is bisected at right angles by the line joining the centres of the circles. (C)

9. **PQRS** is a quadrilateral. The circles on **PQ** and **QR** as diameters cut again at **X**, and the circles on **SP** and **SR** as diameters cut again at **Y**. Use the result of No. 8 to prove that **QX** is parallel to **YS**.

10. Draw any triangle **PQR**, and produce the sides **PQ, PR** in both directions. Construct (i) the locus of the centres of circles which cut off equal chords from **PQ, PR**; (ii) the two circles which cut off equal chords from **PQ, PR** and pass through **Q** and **R**.

11. Prove that the circumcentres of the four triangles into which a quadrilateral is divided by its diagonals are the vertices of a parallelogram.

12. Two circles which are sections of a sphere of radius 7 in. have their planes at right angles. The radii of the two circles are 3 in. and 5 in. Find the distance between the centres of the circles. (O)

---

13. In Fig. 207, if **AB**=l, **OA**=r, **NL**=d units, prove that $l^2+4d^2=8rd$.

14. A, B are the centres of two circles which cut at X, Y. C is the mid-point of AB, and the line through X perpendicular to CX cuts the circles at P, Q. Prove that PX=XQ.

Draw two circles which intersect. Through one of the points of intersection, X, draw a line to cut the circles at P, Q so that X is the mid-point of PQ.

15. X and Y are two points on the circumference of a circle whose centre is O. XY is produced to a point Z, beyond Y, such that YZ is equal to the radius of the circle, and ZO is produced beyond O to T. Show that $\angle XOT = 3\angle Z$. (OC)

16. If the circumcentre of a triangle lies on the bisector of one of the angles, prove that the triangle is isosceles.

17. Two spheres of radii 6 in. and 4 in. have their centres at a distance 5 in. apart. Find (i) by drawing and measurement and (ii) by calculation, the radius of the circle which is their curve of intersection, and the distances of their centres from the plane of this circle. (O)

18. A rectangular block of wood rests in a hemispherical bowl with one face level with the rim (see Fig. 211). Given that the block is 4 cm. long, 3 cm. wide, and 6 cm. high, calculate the radius of the bowl.

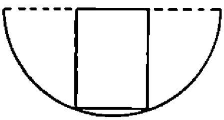


FIG. 211

THE CIRCLE: ANGLE PROPERTIES

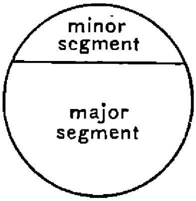


FIG. 212

Definitions

A **segment** of a circle is the part of the plane between an arc of a circle and the chord joining the ends of the arc; it is a **major,** or a **minor,** segment according as the arc is greater than, or less than, a semicircle.

The angle **subtended** at a point P by a line AB is the angle APB.

Angles APB, AQB in Fig. 213 are said to be **in the same segment** (*i.e.* the major segment cut off by AB). They are also said to **stand on the same arc** (*i.e.* the smaller arc AB), or to be 'subtended by' this same arc.

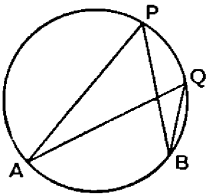


Fig. 213

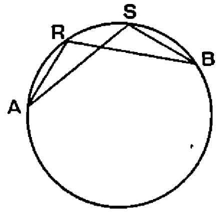


Fig. 214

Similarly, in Fig. 214, ∠s ARB, ASB are *in* the minor segment cut off by AB, and stand *on* the larger arc AB.

Notice carefully that an angle in a *major* segment is subtended by (*i.e.* stands on) an arc which is *less* than a semicircle; while an angle in a *minor* segment is subtended by an arc which is *greater* than a semicircle.

A polygon through whose vertices a circle can be described is said to be **cyclic.**

Four or more points through all of which a circle can be described are said to be **concyclic.**

Exercises on Fig. 215

1. Name the arcs on which the following angles stand: ∠ADB, ∠CDB, ∠ACD, ∠ABD, ∠DBC.

In which segments do these angles lie?

2. If O is the centre of the circle, which arc of the circle subtends the *obtuse* angle AOC, and which arc subtends the *reflex* angle AOC?

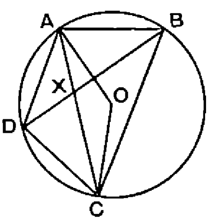


Fig. 215

3. Is AXB an angle in a segment of the circle?

4. What sort of quadrilateral is ABCD?

THEOREM 26

The angle which an arc of a circle subtends at the centre is double that which it subtends at any point on the remaining part of the circumference.

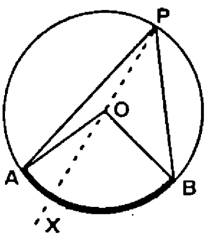


FIG. 216 (i)

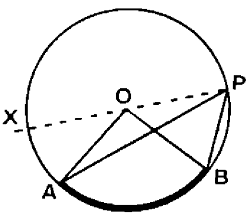


FIG. 216 (ii)

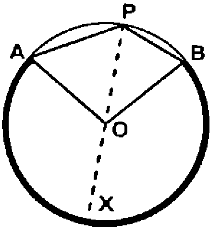


FIG. 216 (iii)

Given a circle, centre O, an arc AB, and a point P on the remaining part of the circumference.

To prove that $\angle AOB$ (reflex in (iii)) $=2\angle APB$.

Construction. Join PO and produce it to X.

Proof. $\because OA=OP$ (radii),

$\therefore \angle OAP=\angle OPA.$

But $\angle AOX=\angle OAP+\angle OPA$ (ext.=sum of int. opp.);

$\therefore \angle AOX=2\angle OPA.$

Similarly, $\angle BOX=2\angle OPB.$

Hence, in Fig. 216 (i),

$$\angle AOX+\angle BOX=2\angle OPA+2\angle OPB$$
$$=2(\angle OPA+\angle OPB).$$
$$\therefore \angle AOB=2\angle APB.$$

In Fig. 216 (ii),

$$\angle BOX-\angle AOX=2\angle OPB-2\angle OPA$$
$$=2(\angle OPB-\angle OPA).$$
$$\therefore \angle AOB=2\angle APB.$$

Fig. 216 (iii) shows the case where the given arc is a major arc. The proof is the same as for Fig. 216 (i), except that the word 'reflex' must be inserted before ∠AOB.

Reference.

[1] ∠AOB=2∠APB (∠ *at centre*).

THEOREM 27

Angles in the same segment of a circle are equal.

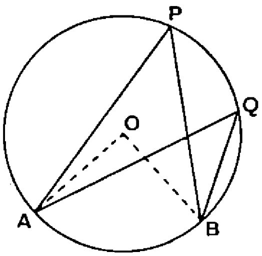


Fig. 217 (i)

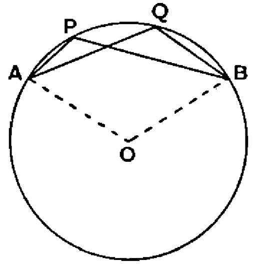


Fig. 217 (ii)

Given a circle, centre O, and two ∠s APB, AQB in the same segment.

To prove that ∠APB=∠AQB.

Construction. Join OA, OB.

Proof. ∠AOB=2∠APB (∠ at centre),

and ∠AOB=2∠AQB (,, ,,);

∴ ∠APB=∠AQB.

Note.—For (ii), the word 'reflex' should be inserted before ∠AOB.

Reference. ∠APB=∠AQB (*same seg.*).

[1] Insert the word 'reflex' if necessary.

THEOREM 28

The angle in a semicircle is a right angle.

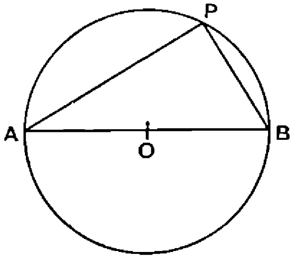


FIG. 218

Given a circle, centre O, a diameter AB, and a point P on the circle.

To prove that $\angle APB = 90°$.

Proof. $\angle AOB = 2\angle APB$ ($\angle$ at centre).

But $\angle AOB = 180°$ (AOB a str. line);

$\therefore$ $\angle APB = 90°$.

Reference. $\angle APB = 90°$ ($\angle$ *in semicircle*).

THEOREM 29

The opposite angles of a cyclic quadrilateral are supplementary.

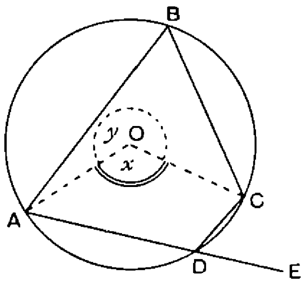


FIG. 219

Given a quadrilateral ABCD whose vertices lie on a circle of which O is the centre.

To prove that $\angle ABC + \angle ADC = 180°$,
and $\angle BAD + \angle BCD = 180°$.

Construction. Join OA, OC.

Proof. Denote by x the angle subtended by arc ADC at the centre, and denote by y the angle subtended by arc ABC at the centre.

Then $\angle ABC = \frac{1}{2}x$ ($\angle$ at centre),
and $\angle ADC = \frac{1}{2}y$ (,, ,,).

$\therefore \angle ABC + \angle ADC = \frac{1}{2}x + \frac{1}{2}y$
$= \frac{1}{2}(x+y)$.

But $x + y = 360°$;

$\therefore \angle ABC + \angle ADC = 180°$.

Similarly, by joining OB and OD, it may be proved that $\angle BAD + \angle BCD = 180°$.

Corollary.

If a side of a cyclic quadrilateral is produced, the exterior angle is equal to the interior opposite angle.

For, in the above figure, if the side AD is produced to E,

$\angle CDE + \angle ADC = 180°$ (adj., ADE a str. line).

But $\angle ABC + \angle ADC = 180°$ (proved);

$\therefore \angle CDE = \angle ABC$.

References.

$\angle ABC + \angle ADC = 180°$ (*opp. $\angle$s of cyclic quad.*).

$\angle CDE = \angle ABC$ (*ext. $\angle$ of cyclic quad.*).

Note 1.—In writing out the theorem above, it is most important that a figure should be drawn in which one of the angles AOC is obviously reflex, so that AOC does not look like a diameter; also it should be made quite clear which of the angles at the centre is twice $\angle ABC$ and which is twice $\angle ADC$.

2. The corollary to the theorem is of greater use in solving riders than the theorem itself.

EXAMPLES 15a

(*Theorem* 26)

1. In Fig. 220, if ∠AOB = 124°, calculate ∠APB and ∠AQB.

2. In Fig. 220, if ∠AQB = 107°, calculate the reflex ∠AOB.

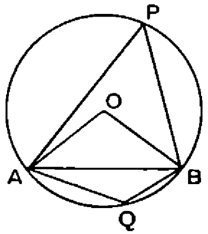
3. In Fig. 220, if ∠APB = 41°, calculate ∠AOB and ∠OAB.

4. In Fig. 220, if ∠AQB = 146°, calculate ∠AOB and ∠OAB.

5. In Fig. 220, if ∠OAP = 21°, and ∠APB = 48°, calculate ∠OAB and ∠OBP.

6. In Fig. 220, if ∠APB = 60°, prove that ∠AOB = ∠AQB.

7. In Fig. 220, prove that ∠AOB + 2∠AQB = 360°.



O is the centre of the circle

FIG. 220

8. In Fig. 220, if D is the mid-point of AB, prove that (i) ΔAOD ≡ ΔBOD, (ii) ∠AOD = ∠APB.

(*Theorem* 27)

9. In Fig. 221, prove that Δs CXA, BXD are equiangular.

10. In Fig. 221, if AX = AC, prove that DX = DB.

11. In Fig. 221, if CA is parallel to BD, prove that XA = XC and XB = XD.

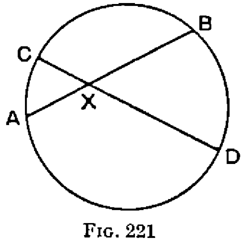


FIG. 221

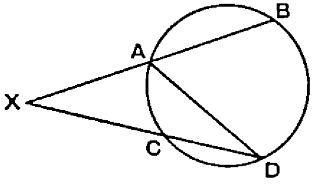


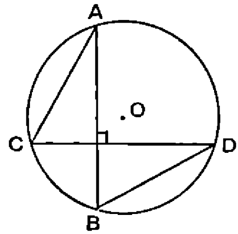
FIG. 222

12. In Fig. 222, prove that ∠XAD = ∠XCB.

13. Two circles intersect at **A**, **B**. Straight lines **PAQ, RAS** are drawn through **A** to cut the circles, **P** and **R** being on one circle, and **Q, S** being on the other. Prove that ∠**PBR** = ∠**QBS**.

14. In Fig. 223, if ∠**CAB** = 28°, calculate ∠**ABD**.

15. In Fig. 223, prove that ∠**ACD** + ∠**BDC** = 90°.



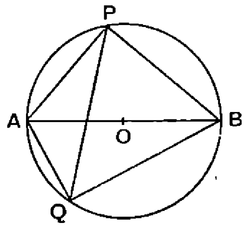
O is the centre of the circle
FIG. 223

(*Theorem* 28)

16. In Fig. 224, if ∠**PAB** = 38°, calculate ∠**PBA** and ∠**PQB**.

17. In Fig. 224, if ∠**QPB** = 62°, calculate ∠**ABQ**.

18. Two circles cut at **A, B**. Through **A** diameters **AP, AQ** of the two circles are drawn. Show that **PB, BQ** are in the same straight line. (Join **AB**.)



O is the centre of the circle
FIG. 224

19. **AB** is a chord of a circle whose centre is **O**. Another circle is drawn on **OA** as diameter, and it cuts **AB** at **X**. Prove that **AX** = **XB**. (Join **OX**.)

20. Prove that the circle drawn on one of the equal sides of an isosceles triangle as diameter passes through the mid-point of the base.

21. **A, B, C, D** are four points in order on a circle, and **AD** is a diameter of the circle. If ∠**ABC** = 109°, calculate ∠**DAC**.

(*Theorem* 29)

22. **ABCD** is a cyclic quadrilateral; ∠**BAD** = 72°, ∠**BCA** = 25°. Calculate ∠**ACD**.

23. In Fig. 225, if $\angle$AXP$=76°$, calculate $\angle$ABQ and $\angle$AYQ.

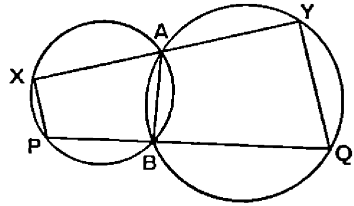


Fig. 225

24. The diagonals of a cyclic quadrilateral ABCD cut at O; $\angle$BOC$=86°$, $\angle$BDC$=54°$, $\angle$BCA$=40°$. Calculate $\angle$BCD and $\angle$BAD.

25. In Fig. 222, if XA$=$AC, prove that $\angle$CXA$=\angle$ABD. (Hint: use the property of the exterior angle of a cyclic quadrilateral.) Deduce that DX$=$DB.

26. In Fig. 222, if AC is parallel to BD, prove that $\angle$XBD$=\angle$XDB.

27. ABCD is a quadrilateral inscribed in a circle, and AC bisects $\angle$BAD and $\angle$BCD. Prove that $\angle$BAC$+\angle$BCA$=90°$, and deduce that AC must be a diameter of the circle.

28. In Fig. 225, prove that $\angle$PXA$+\angle$AYQ$=180°$, and hence that XP is parallel to YQ. (Hint: use the property of the exterior angle of a cyclic quadrilateral.)

EXAMPLES 15b

1. In Fig. 220, if C is the mid-point of AB and $\angle$APB$=x°$, find in terms of x the numbers of degrees in $\angle$AOC and $\angle$OAC.

2. In Fig. 220, if AO is parallel to QB and $\angle$OAB$=x°$, express in terms of x the numbers of degrees in $\angle$AOB, $\angle$ABQ, $\angle$AOQ, $\angle$QOB, $\angle$QAB.

3. P is any point on a chord AB of a circle whose centre is O. H and K are the circumcentres of Δs OAP and OBP respectively. Prove that $\angle$OHP$=\angle$OKP.

4. In Fig. 223, if $\angle$AOC$=2x°$, express in terms of x the numbers of degrees in $\angle$ADC, $\angle$DAB, and $\angle$OAC, and thus show that $\angle$OAC$=\angle$DAB.

5. Each of two equal circles, whose centres are X and Y, passes through the centre of the other. AB is the common chord. How many degrees are there in the angles AXB, AYB ?

If a straight line CAD is drawn through A cutting the circles again at C, D, prove that $\triangle$BCD is equilateral.

6. In Fig. 222, if O is the centre of the circle and $\angle$BAD$=x°$, $\angle$ADC$=y°$, express in terms of x and y the numbers of degrees in $\angle$BOD, $\angle$AOC, and $\angle$AXC, and thus show that $\angle$BOD$-\angle$AOC$=2\angle$AXC.

---

7. In Fig. 222, if $\angle$ABC$=35°$ and $\angle$AXC$=43°$, calculate $\angle$DAX.

8. The bisector of $\angle$A of $\triangle$ABC meets BC at D and the circumcircle of the triangle at E. Prove that $\triangle$s ACD, AEB are equiangular.

9. AD, BE are two altitudes of $\triangle$ABC, and H is the point where they intersect. AD is produced to meet the circumcircle of the triangle again at K. Prove that $\angle$DBK$=\angle$DBH.

10. In Fig. 222, the perpendicular bisectors of AX and AD are drawn to meet at O. Prove that $\angle$ABC$=\frac{1}{2}\angle$AOX.

---

11. AE is a diameter of the circumcircle of an acute-angled $\triangle$ABC, and D is the foot of the perpendicular from A to BC. Prove that $\triangle$s ABD, AEC are equiangular.

12. AOB is any angle, and circles are described on OA, OB as diameters. These circles meet again at X. Prove that AX, XB are in the same straight line.

13. In Fig. 224, PA and BQ are produced to meet at X. Write down the area of $\triangle$XAB in two different ways, and thus show that AQ.BX$=$AX.BP.

14. AB is a diameter of a circle, centre O, and C, D are points on the semicircle ACDB (C being the nearer to A) such that $\angle$COD$=90°$. BC meets AD at P. Calculate $\angle$CAD, and prove that CA$=$CP.

15. A, X, Y, B are four points in order on a circle of which AB is a diameter. AX, BY are produced to meet at T. Prove that ∠XBA + ∠YAB = ∠ATB.

16. Prove that congruent triangles have equal circumcircles. (Hint: let ABC, DEF be the congruent triangles, and BP, EQ diameters of the circumcircles; prove that △ABP≡△DEQ.)

17. PQRS is a parallelogram. O is the circumcentre of △PQS, L is that of △RSQ. Prove that the figure OQLS is a rhombus. (Use the result of No. 16.)

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18. In Fig. 222, if ∠AXC = $x°$ and ∠XAC = $y°$, express ∠ABD in terms of x and y.

19. Prove that, if a parallelogram is inscribed in a circle, it must be a rectangle.

20. In Fig. 226, if the circumcircles of △s BCX, DCY have the point Z as their other point of intersection, prove that ∠CZX + ∠CZY = 180°.

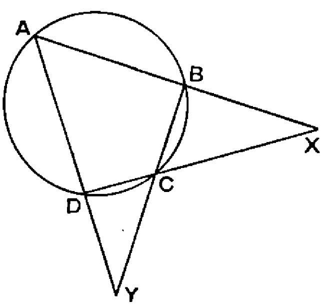


FIG. 226

21. ABCDEF is a hexagon inscribed in a circle. Prove that ∠ABC + ∠CDE + ∠EFA = 360°. (Join AD.)

22. An isosceles triangle ABC with AB equal to AC is inscribed in a circle. BC is produced to D, and AD meets the circle again at E; EC is joined. Prove that ∠CED = ∠ACB.

23. D is any point on the side BC of △ABC, and H, K are the circumcentres of △s ADB, ADC. Prove that

(i) ∠DHA + ∠DKA = 360° − 2∠BAC;

(ii) if AHDK is cyclic, ∠BAC is a right angle.

24. The bisector of the angle B of △ABC meets AC at P, and the bisector of the angle C meets AB at Q. BP and CQ cut at I. Prove that

(i) ∠IBC + ∠ICB = 90° − ½∠BAC;

(ii) ∠BIC = 90° + ½∠BAC;

(iii) if APIQ is cyclic, ∠BAC = 60°.

25. A hexagon ABCDEF has its vertices on the circumference of a circle. AB is parallel to ED, and BC is parallel to FE. Prove that

(i) ∠ABC = ∠FED (join BE);
(ii) ∠ADC = ∠DAF;
(iii) CD is parallel to AF.

26. ABCD is a quadrilateral inscribed in a circle. CD is produced to F, and the bisector of the angle ABC meets the circle at E. Prove that DE bisects ∠ADF.

EXAMPLES 15c

1. ABCD is a cyclic quadrilateral. If AC bisects ∠BAD, prove that △BCD is isosceles.

2. (Th. 26.) APB, CPD are two perpendicular chords of a circle which cut at P, within the circle, and O is the centre of the circle. Prove that ∠AOD and ∠BOC are supplementary. (N)

3. XYZ is an equilateral triangle inscribed in a circle. X is joined to a point P on the smaller arc XZ. YP is drawn meeting XZ at Q. Prove that ∠PXY = ∠XQY. (L)

4. (Th. 29.) AB, CD are parallel chords of a circle such that CA and DB when produced meet outside the circle at P. Prove that PA = PB. (N)

5. With centre A, a point on the circumference of a circle, a second circle is drawn to intersect the first circle in P and Q. Show that P and Q are equidistant from B, the point on the first circle diametrically opposite to A. (OC)

6. A, B, C are three points on a circle such that ABC is an acute-angled triangle; BQ and CR are diameters of the circle, and AQ, AR are joined. Show that ∠BAR = ∠CAQ. (OC)

7. An isosceles triangle in which the vertical angle is 32° is inscribed in a circle. What are the angles in the three minor segments of the circle cut off by the sides of the triangle? Show how your answers are obtained. (OC)

8. M is the mid-point of the side AB of $\triangle$ABC. If the bisector of $\angle$B meets the straight line through M parallel to BC at O, prove that $\angle$AOB is a right angle.

9. PQR is a triangle inscribed in a circle; the bisector of $\angle$P meets QR at S and the circle at T. Prove that $\angle RST = \angle PRT$.

10. (Th. 27.) ABCD is a quadrilateral inscribed in a circle in which AB=AC; the side CD is produced to E. Show that AD bisects $\angle$BDE. (OC)

11. AB, AC are equal chords of a circle. The bisector of $\angle$ABC meets the circle at D and AD produced meets BC produced at E. Prove that CE=CA. (L)

12. (Th. 26.) A, B, C, D are four points in order on the circumference of a circle, centre K. The chords AC, BD intersect at M. Prove that $\angle AKB + \angle CKD = 2\angle AMB$. (W)

13. A, B, C are three points lying on a circle whose centre is O. If D is the foot of the perpendicular from A to BC, prove that $\angle BAO = \angle DAC$.

14. ABC is a triangle inscribed in a circle, and AB is greater than AC. The bisector of the angle between CA produced and AB meets the circle again at T. Prove that TB=TC. (L)

15. (Th. 26.) OP is a chord of a circle PRS. With centre O and radius less than OP describe a circle cutting the given circle in R and S and OP in X. Prove that $\angle RSX = \frac{1}{2}\angle RSP$. (L)

16. (Th. 29.) Two equal chords PQ, PR are drawn in a circle. A point S is chosen on the minor arc PR and the lines PS and QR produced intersect at T. Prove that the triangles QPS, TRS are equiangular. (W)

17. In Fig. 227, ABCD is a cyclic quadrilateral; AD, BC produced meet at E; the circle through A, C, E cuts AB at P and CD produced at Q. Prove that EP=EQ.

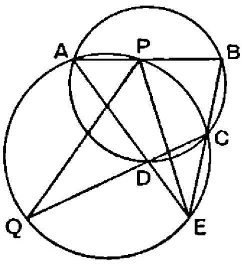


FIG. 227

18. ABCD is a cyclic quadrilateral and AB, DC when produced meet at E. The internal bisector of $\angle$BEC meets BC in F and AD in G. Prove that $\angle AGF = \angle BFG$. (L)

19. In a triangle **ABC**, if **AD**, **CF** are drawn perpendicular to the opposite sides, cutting each other in **H**, and if **BK** is a diameter of the circumcircle of **ABC**, and **CK**, **KA** are joined, prove that the quadrilateral **AHCK** is a parallelogram. (L)

20. **AB**, **AC** are equal chords of a circle. **BC** is produced to **D** so that **CD**=**CA**. **DA** cuts the circle at **E**. Prove that **BE** bisects ∠**ABC**. (L)

21. (Th. 29.) **ABCD** is a quadrilateral inscribed in a circle; the sides **AB**, **DC** produced meet in **E**, and the sides **AD**, **BC** produced meet in **F**; if ∠**E**=30° and ∠**F**=40°, find the angles of the quadrilateral **ABCD**. (OC)

PART II

THE CIRCLE: ARC PROPERTIES

ASSUMPTION 7 (Properties of Equal Arcs)

In equal circles (or in the same circle), if two arcs subtend equal angles at the centres or at the circumferences of the circles, they are equal.

Conversely, in equal circles (or in the same circle), if two arcs are equal, they subtend equal angles at the centres or at the circumferences of the circles.

References. (See Fig. 228.)

Arc AB = arc CD ($\because \angle AHB = \angle CKD$).

$\angle$APB = $\angle$CQD ($\because$ *arc AB = arc CD*).

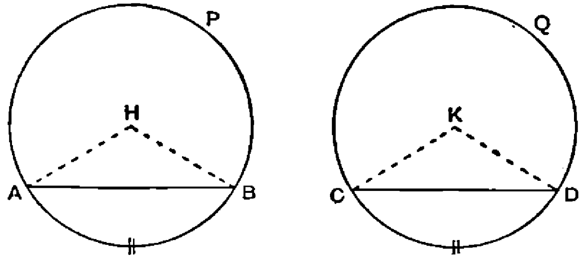


FIG. 228

THEOREM 30a

In equal circles (or in the same circle), if two chords are equal, the arcs which they cut off are equal.

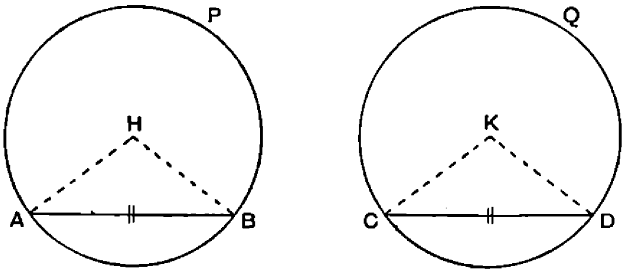


FIG. 229

Given two equal circles ABP, CDQ, with centres H and K, and two equal chords, AB, CD.

To prove that arc AB = arc CD.

Construction. Join HA, HB, KC, KD.

Proof. In the $\triangle$s HAB, KCD,

HA = KC (radii of equal circles),

HB = KD („ „),

and AB = CD (given);

$\therefore$ $\triangle$s $\begin{matrix}\text{HAB}\\\text{KCD}\end{matrix}$ are congruent (SSS).

$\therefore$ $\angle$AHB = $\angle$CKD.

$\therefore$ arc AB = arc CD.

THEOREM 30b

In equal circles (or in the same circle), if two arcs are equal, the chords cutting off those arcs are equal.

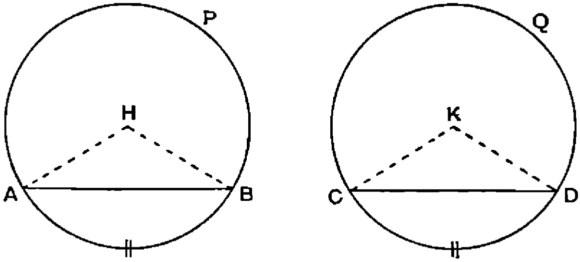


FIG. 228 (repeated)

Given two equal circles ABP, CDQ, with centres H and K, and two equal arcs AB, CD.

To prove that chord AB=chord CD.

Construction. Join HA, HB, KC, KD.

Proof. In the $\triangle$s HAB, KCD,

HA=KC (radii of equal circles),

HB=KD (,, ,, ,,),

and $\angle$AHB=$\angle$CKD ($\because$ arc AB=arc CD);

$\therefore$ $\triangle$s $\begin{matrix}HAB\\KCD\end{matrix}$ are congruent (SAS).

$\therefore$ AB=CD.

References.

Arc AB=arc CD ($\because$ *chord AB=chord CD*).

Chord AB=chord CD ($\because$ *arc AB=arc CD*).

EXAMPLES 16a

1. AB is a chord of a circle, and any point P on the minor arc AB is joined to the mid-point Q of the major arc AB. Why are ∠s APQ, BPQ equal ?

2. AC is a diameter of a circle, and B, D are the mid-points of the two arcs AC. Prove that ABCD is a square.

3. In Fig. 230, if ∠ABC=62°, ∠ACB=95°, and arc YZ=arc BC, calculate ∠X.

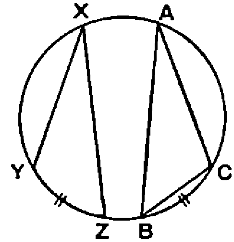


FIG. 230

4. AB, AC are two equal chords of a circle. P is the mid-point of the minor arc AB, Q that of the minor arc AC. Prove that the chords PQ, AB are equal.

5. In Fig. 231, prove that the chords AC, BD are equal.

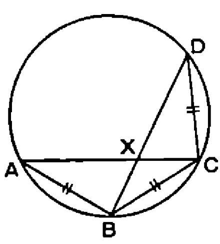
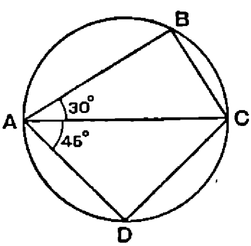


FIG. 231



AC is a diameter
FIG. 232

6. In Fig. 232, calculate the remaining angles. How large are the angles subtended at the centre by arcs AB, BC, CD, DA ? What fraction of the whole circumference is each of these arcs ?

7. What angle does the arc which represents a five-minute space on a clock-face subtend (i) at the centre of the clock-face, (ii) at one of the hour-divisions on the circumference ?

Calculate the angle subtended at I by the arc joining VI and VIII on the clock-face. Calculate also the angle subtended at VIII by the arc joining I and V. Deduce that the straight line joining I and VI is perpendicular to the straight line joining V and VIII.

8. AB, AC are equal chords of a circle drawn on opposite sides of a diameter AD. Prove that the arcs BD, CD are equal.

9. AB is a diameter of a circle, and C is a point on the circle such that the arc AC is twice the arc BC. Calculate ∠CAB and ∠CBA.

10. AB is a side of a regular hexagon inscribed in a circle, and AC is a side of a regular octagon inscribed in the same circle. C lies on the minor arc AB. What fraction of the circumference is arc BC ?

11. AB, CD are two parallel chords of a circle. Prove that arc AC=arc BD. (Join AD.)

12. In Fig. 231, prove that XB=XC.

13. Two equal circles cut at A and B. A straight line through A cuts one circle at P and the other circle at Q. Prove that BP=BQ. (Hint: why are ∠s BPA, BQA equal ?)

14. ABCD is a quadrilateral inscribed in a circle, and AB=CD. Prove that AD and BC are parallel. (Join AC.)

15. ABC is a triangle inscribed in a circle. The minor arcs AC, BC are respectively one-sixth and one-quarter of the circumference, and A lies on the major arc BC. Calculate the angles of △ABC.

16. An isosceles triangle, in which AB=AC and ∠A=36°, is drawn with its vertices lying on a circle. Calculate angles B and C.

The bisectors of angles B and C meet the circle at P and Q respectively. Prove that the pentagon APCBQ has all its sides equal.

17. O is the centre of a circle, N is the foot of the perpendicular from O to any chord AB. ON produced meets the circle at P. Prove that arc AP=arc BP.

18. A and B are two points on the circumference of a circle, and P is any point on the major arc AB. Prove that, whatever the position of P, the bisector of ∠APB always meets the minor arc AB at the same point.

19. AC is a diameter of a circle; P and Q are points on opposite sides of AC such that ∠PAC=∠QAC=30°. Prove that (i) ∠PCA=∠QCA, (ii) AP=AQ, (iii) APQ is an equilateral triangle.

EXAMPLES 16b

1. ABCD is a quadrilateral inscribed in a circle, and AB = CD. Prove that ∠ABC = ∠BCD. (Consider arcs ADC, BAD.)

2. In Fig. 231, prove that, if O is the centre of the circle, ∠AXB = ∠AOB. (Regard ∠AXB as an exterior angle of △XCB.)

3. ABCDE is a regular pentagon inscribed in a circle. BD cuts AC at X. Calculate ∠CXD.

4. In Fig. 233, by using the property of an exterior angle of a triangle, prove that ∠CXA is equal to the angle subtended at the circumference of the circle by an arc equal to the sum of the arcs AC and BD.

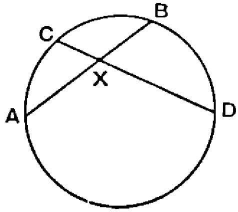


FIG. 233

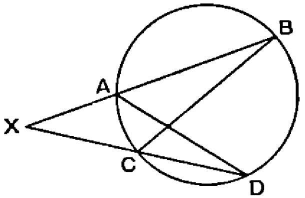


FIG. 234

5. In Fig. 234, by considering ∠BAD as an exterior angle of △AXD, prove that ∠CXA is equal to the angle subtended at the circumference of the circle by an arc equal to the difference of the arcs BD and AC.

6. A circle whose centre is O cuts another circle at A, B, and the second circle passes through O. P is any point on the major arc AB of the second circle. Join OA, OB, and prove that OP bisects ∠APB.

7. Prove that the angles which equal chords of a circle subtend at the circumference are either equal or supplementary.

8. Disprove the statement that, if one chord AB of a circle is double another chord CD, then the smaller angle subtended by AB at the circumference is double the smaller angle subtended by CD at the circumference. (Take AB to be a diameter, and CD a chord equal to the radius, and prove that, in this particular case, the one angle is *three* times the other angle.)

9. ABCDEF is a hexagon inscribed in a circle, and AB = ED, BC = FE. Prove that CD is parallel to AF. (Join AD, and consider arcs AC, DF.)

10. ABCD is a rectangle inscribed in a circle. AE is a chord of the circle such that ∠EAC = ∠BAC, with E on the other side of AC from B. Prove that EC = AD.

11. In Fig. 235, two equal circles cut at A and B. Prove that BE = BC. (First prove that ∠BCA and ∠BEC are each equal to ∠ADB.)

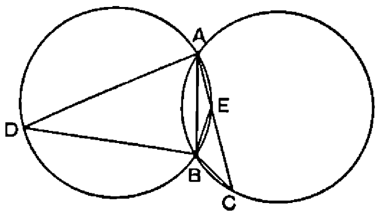


FIG. 235

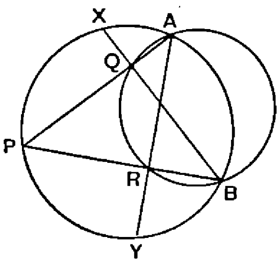


FIG. 236

12. In Fig. 236, two circles cut at A and B. P is a point on one circle, and AP, BP meet the second circle at Q, R. BQ and AR are produced to cut the first circle at X and Y respectively. Prove that PX = PY.

13. In Fig. 237, P and Q are points on the two arcs AB and AC. Prove that ∠AXY is equal to the angle subtended at the circumference by an arc equal to arc PB + arc AQ.

State a similar result for ∠AYX, and prove that, if P, Q are the *middle* points of arcs AB, AC, then AX = AY.

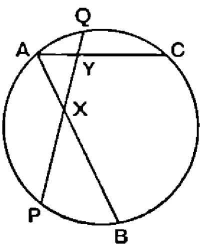


FIG. 237

14. AB, CD are two perpendicular chords of a circle, cutting at O. Prove that arc AC + arc BD = arc BC + arc AD. (First show that the sum of the angles standing on the arcs AC and BD is 90°.)

15. ABC is a triangle inscribed in a circle. The bisector of ∠A meets the circle again at D, and DE is drawn parallel to BA to meet the circle at E. Prove that DC = AE.

16. ABC is a triangle inscribed in a circle. The perpendicular from A to BC is produced to meet the circle at P, and the perpendicular from B to AC is produced to meet the circle at Q. Prove that arc PC=arc CQ.

17. ABC is an equilateral triangle inscribed in a circle. P, Q are the mid-points of the minor arcs AB, AC, and PQ meets AB at X and AC at Y. (See Fig. 237.) Prove that

(i) PQ is parallel to BC (join PC);
(ii) $\triangle$AXY is equilateral;
(iii) $\angle$PAX=$\angle$APX and $\angle$AQY=$\angle$QAY;
(iv) PX=XY=YQ.

18. In Fig. 238, P, Q, R, S are the mid-points of the arcs AB, BC, CD, DA. Prove that

(i) $\angle$QPR=$\frac{1}{2}\angle$BAD (compare arcs);
(ii) $\angle$PQS=$\frac{1}{2}\angle$BCD;
(iii) PR is perpendicular to QS (show that $\angle$QPR+$\angle$PQS=90°).

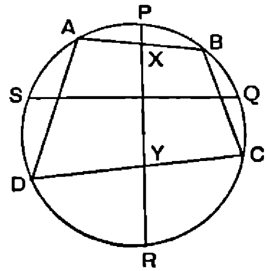


FIG. 238

19. In Fig. 238, P, R are the mid-points of the arcs AB, CD; and PR meets AB, CD at X and Y. By considering $\angle$PXA as an exterior angle of $\triangle$PXB, prove that $\angle$PXA is equal to the angle subtended at the circumference of the circle by an arc equal to the sum of the arcs BC, AP, and CR.

Write down a similar result for $\angle$DYR, and deduce that $\angle$PXA=$\angle$DYR.

EXAMPLES 16c

1. In Fig. 239, the bisectors of angles ABC and ACB of $\triangle$ABC meet at I. The circle which passes through B, I, C meets AB again at P and meets AC produced at Q. Prove that

(i) arc PI=arc IC;
(ii) arc QI=arc IB (show that $\angle$IBQ and $\angle$IQB each equal $\frac{1}{2}\angle$ACB);
(iii) $\angle$BQC=$\angle$IBC+$\angle$ICB;
(iv) $\angle$ABQ=$\angle$IBA+$\angle$ICA;
(v) AB=AQ and AP=AC.

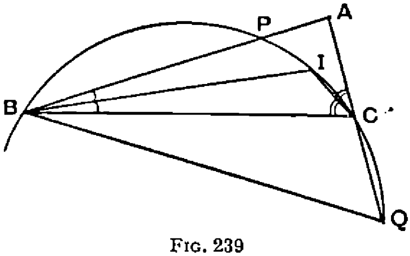


FIG. 239

2. **ABCDEFGH** is a regular octagon inscribed in a circle. Calculate in degrees the size of each of the angles in triangles **BCE**, **BHE**.

3. If a cyclic quadrilateral **ABCD** has its diagonals at right angles, show that the sum of the two opposite arcs **AB**, **CD** is equal to half the circumference of the circle. (W)

4. **A**, **B**, **C** are three points on a circle, and $\angle$**ABC**$=38°$ and $\angle$**ACB**$=68°$. **P**, **Q** are, respectively, the middle points of the minor arcs **AC**, **AB**. Calculate $\angle$**BCP** and $\angle$**CPQ**. (N)

5. **A**, **B**, **C**, **D**, **E**, **F**, **G**, **H**, **K** are points in order on the circumference of a circle dividing it into nine equal arcs. Find the angles of $\triangle$**ACK**, without using measurement. (OC)

6. A quadrilateral **ABCD** is inscribed in a circle, and **AC** bisects $\angle$**BAD**. **AD** is produced to **E** so that **DE**$=$**AB**. Prove that **CE**$=$**CA**. (N)

7. **ABCD** is a rectangle inscribed in a circle. If a circle is drawn with centre **A** and radius **AB** to cut the first circle at **E**, prove that (i) $\angle$**ACE**$=\angle$**DAC**, (ii) **CE**$=$**AD**.

8. **ABCD** is a quadrilateral inscribed in a circle such that the centre lies within the quadrilateral and such that **AB** and **CD** are not parallel. **P** and **Q** are the mid-points of the minor arcs **AB** and **CD**. Prove that the chord **PQ** makes equal angles with the chords **AB**, **CD**. (L)

THEOREM 31 (Converse of Theorem 27)

(For proof, see Appendix, p. 335)

If the straight line joining two points subtends equal angles at two other points on the same side of it, then the four points are concyclic.

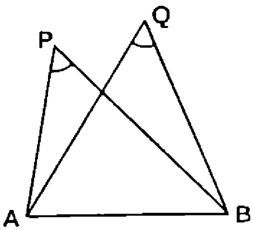


FIG. 240

Reference. (See Fig. 240.)

$\angle APB = \angle AQB$;

$\therefore$ A, P, Q, B are concyclic.

Corollary. (The constant-angle locus.)

The locus of points at which a fixed straight line subtends a constant angle consists of a segment of a circle on the fixed straight line as chord and containing the given angle, together with an equal segment with the same chord but on the opposite side of it.

The continuous arcs in Fig. 241 are the locus of points at which AB subtends a constant acute angle; the dotted arcs are the locus of points at which AB subtends a constant obtuse angle.

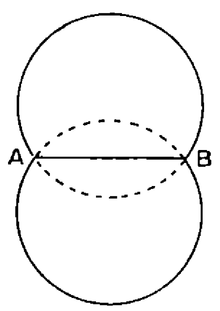


FIG. 241

THEOREM 32 (Converse of Theorem 29)

(For proof, see Appendix, p. 336)

If a pair of opposite angles of a quadrilateral are supplementary, the quadrilateral is cyclic.

Corollary.

If, when a side of a quadrilateral is produced, the exterior angle is equal to the interior opposite angle, the quadrilateral is cyclic.

References. (See Fig. 242.)

(i) $\angle ABC + \angle ADC = 180°$;

$\therefore$ ABCD is cyclic.

(ii) $\angle CBX = \angle ADC$;

$\therefore$ ABCD is cyclic (*ext.* $\angle$ = *int. opp.* $\angle$).

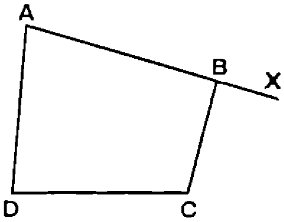


FIG. 242

THEOREM 33 (Converse of Theorem 28)

(For proof, see Appendix, p. 337)

The circle described on the hypotenuse of a right-angled triangle as diameter passes through the opposite vertex.

Corollary.

The locus of points at which a fixed straight line subtends a right angle is the circle on the fixed straight line as diameter.

EXAMPLES 17a

(*Theorem* 31)

1. In Fig. 243, if ∠**ACO**=57° and ∠**BDO**=33°, prove that **ACBD** is cyclic.

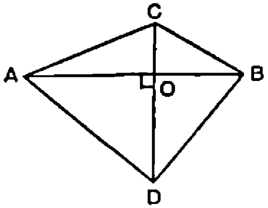
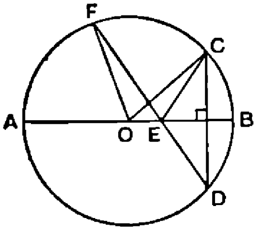


Fig. 243



AB is a diameter, **O** the centre

Fig. 244

2. In Fig. 244, if ∠**EDC**=∠**ECD**=32°, calculate ∠**CEF** and ∠**COF**, and prove that **C**, **E**, **O**, **F** are concyclic.

3. In Fig. 245, **ABC** is an equilateral triangle; the bisector of ∠**ABC** meets at **D** the line through **C** perpendicular to **BC**. Calculate ∠**ACD**, and prove that **A**, **B**, **C**, **D** are concyclic.

4. **ABC** is a triangle in which **AB**=**AC** and ∠**BAC**=36°; the bisector of ∠**ABC** meets at **D** the straight line through **C** parallel to **BA**. Prove that the points **A**, **B**, **C**, **D** are concyclic.

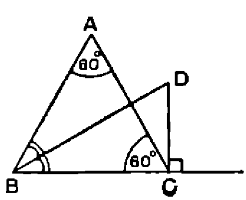


Fig. 245

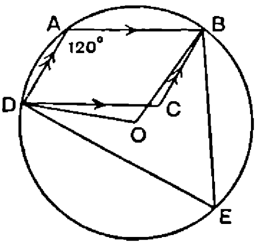


Fig. 240

5. In Fig. 246, **ABCD** is a parallelogram in which ∠**BAD**=120°. **O** is the centre of the circle through **D**, **A**, and **B**, and **E** is any point on the major arc **BD** of the circle. Calculate ∠**BED**, ∠**BOD**, and ∠**BCD**, and prove that **B**, **C**, **O**, **D** are concyclic.

6. In Fig. 247, if $\angle$**ACB**$=26°$, what are $\angle$**DBC**, $\angle$**FAE**, and $\angle$**AFB**? Calculate $\angle$**BPA** (an exterior angle of $\triangle$**PBC**) and $\angle$**BQA** (an exterior angle of $\triangle$**FAQ**), and show that **B**, **P**, **Q**, **A** are concyclic.

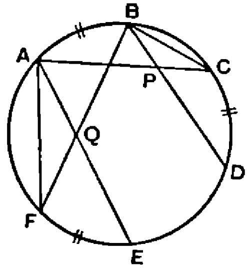


FIG. 247

(*Theorem* 32)

7. In Fig. 248, prove that **AEHF** is a cyclic quadrilateral.

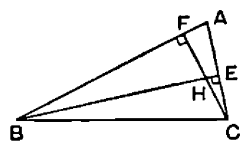


FIG. 248

8. Two straight lines meet at **O**. From any point **P** perpendiculars **PL**, **PM** are drawn to the lines, meeting them at **L** and **M**. Prove that **OLPM** is cyclic.

9. In Fig. 249, **ABCD** is a square whose diagonals meet at **O**. A straight line parallel to **AD** meets **AB** at **X** and **OB** at **Y**. Prove that **AXYO** is cyclic.

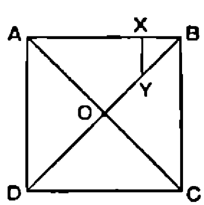


FIG. 249

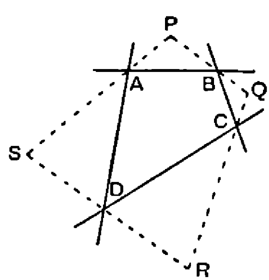


FIG. 250

The exterior angles of the quadrilateral **ABCD** are bisected to form the quadrilateral **PQRS**

10. In Fig. 250, if $\angle$**DAB**$=94°$, $\angle$**ABC**$=130°$, $\angle$**BCD**$=100°$, calculate $\angle$**ADC**, $\angle$**APB** and $\angle$**CRD**, and prove that the quadrilateral **PQRS** is cyclic.

11. In Fig. 251, **AB** is a diameter of a circle, and it is produced to any point **P**. **C** is any point on the circle, and the straight line through **P** perpendicular to **AP** meets **AC** produced at **Q**. Prove that the quadrilateral **BCQP** is cyclic.

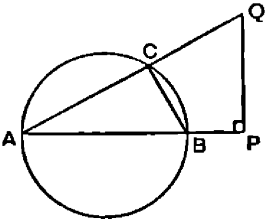


Fig. 251

(*Theorem* 33)

12. In Fig. 248, prove that **B**, **F**, **E**, **C** lie on a circle. Where is its centre ?

13. Prove that the circle drawn with one of the equal sides of an isosceles triangle as diameter passes through the mid-point of the base of the triangle.

14. **ABCD** is a rhombus whose diagonals cut at **O**. Prove that the circles on **AB**, **BC**, **CD**, and **DA** as diameters all pass through **O**.

EXAMPLES 17b

1. Given two points **A** and **B**, state the complete locus of points **P** such that $\angle APB = 90°$.

2. In Fig. 248, prove that (i) $\angle AEF = \angle ABC$, (ii) $\angle EFC = \angle EBC$.

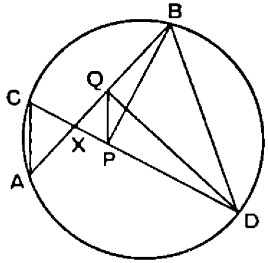


Fig. 252

3. In Fig. 252, **AB**, **CD** are two chords of a circle which meet at **X**, and $\angle BXD$ is an acute angle. **BP** is drawn perpendicular to **XD** meeting it at **P**, and **DQ** is drawn perpendicular to **XB** meeting it at **Q**. By means of concyclic points prove that $\angle QPX = \angle QBD$, and then prove that **PQ** is parallel to **AC**.

4. In Fig. 247, prove that ∠**BPA** is equal to the sum of the angles standing on the arcs **AB**, **CD**. Write down a similar result, and deduce that **B**, **P**, **Q**, **A** are concyclic.

5. In Fig. 253, **ABCD** is a square, and the side **CD** is produced to **E**. **P** is any point on **CD**. The straight line through **P** perpendicular to **PB** meets at **Q** the bisector of ∠**ADE**. Prove that

(i) **BPDQ** is cyclic,

(ii) ∠**QBP**=45°.

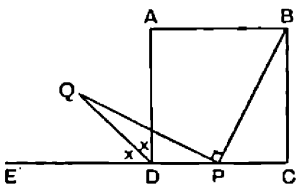


FIG. 253

6. In Fig. 248, if **X** is the foot of the perpendicular from **B** to **EF** produced, prove that ∠**FBX**=∠**EBC**.

---

7. In Fig. 248, prove that (i) ∠**CFE**=∠**HAE**, (ii) ∠**EHC**=∠**BAC**.

8. In Fig. 254, **ABCD** is a parallelogram, and any circle through **A** and **D** cuts **AB** at **P** and **DC** at **Q**. Prove that **PBCQ** is cyclic.

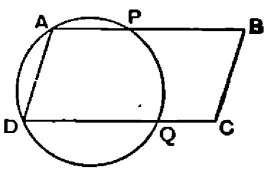


FIG. 254

9. In Fig. 255, the bisectors of ∠**ABC** and ∠**ACB** meet at **I**, and the bisectors of the exterior angles **CBX** and **BCY** meet at I_1. Prove that **BIC**I_1 is a cyclic quadrilateral.

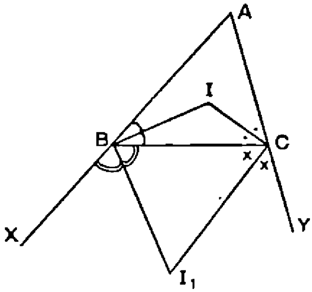


FIG. 255

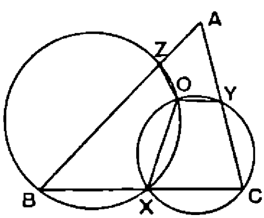


FIG. 256

10. In Fig. 256, **ABC** is a triangle with any points **X**, **Y**, **Z** marked on the sides **BC**, **CA**, **AB** respectively. The circumcircle of △**BXZ** meets the circumcircle of △**CXY** at **O**. Prove that the quadrilateral **AZOY** is cyclic. (Hint: use the property of the exterior angle of a cyclic quadrilateral.)

11. In Fig. 257, prove that EB is parallel to FO.

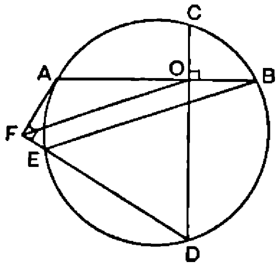


FIG. 257

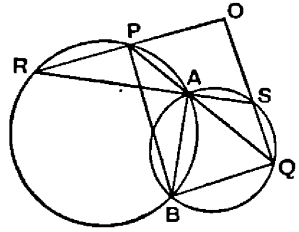


FIG. 258

12. In Fig. 258, two circles intersect at A and B. Straight lines PAQ, RAS meet one circle at P, R and the other at Q, S. RP, QS are produced to meet at O. Prove that OPBQ is cyclic. (Hint: prove that $\angle RPB = \angle BQS$.)

---

13. In Fig. 248, if X is the mid-point of BC, prove that (i) $XE = XF$, (ii) $\angle EXF = 2\angle ABE = 180° - 2\angle BAC$.

14. Two straight lines AOB, COD intersect at O, and OE is the bisector of $\angle AOC$. P is any point (not on OE) within the angle AOC. X, Y, Z are the feet of the perpendiculars from P to the lines OA, OC, OE respectively. Prove that O, X, P, Y, Z all lie on a circle, and deduce that $XZ = YZ$.

15. AB, CD are two perpendicular diameters of a circle whose centre is O. P is any point on AO, Q is any point on OB. CP, CQ are produced to meet the circle at X, Y respectively. Prove that (i) $\angle CPO = \angle ODX$, (ii) PQYX is cyclic.

16. In Fig. 259, AB is a diameter of a circle; AP and AQ are two chords which are produced to cut at X and Y the straight line through B perpendicular to AB. Prove that PXYQ is cyclic.

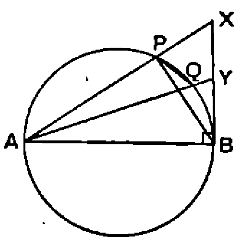


FIG. 259

17. AB, BC, CD are three equal chords of a circle whose centre is O. BD meets AC at X. Prove that A, O, X, B are concyclic.

18. In Fig. 250, if $\angle DAB = a°$, $\angle ABC = b°$, $\angle BCD = c°$, and $\angle CDA = d°$, express the angles P and R in terms of these letters, and prove that the quadrilateral PQRS is cyclic.

19. In Fig. 260, AB and AC are equal chords of a circle. AP, AQ are two chords cutting BC at X, Y. Prove that PQYX is cyclic.

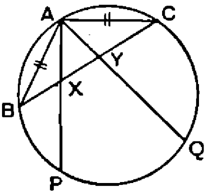


FIG. 260

20. In Fig. 261, if ABQP is cyclic, prove that CDSR is cyclic.

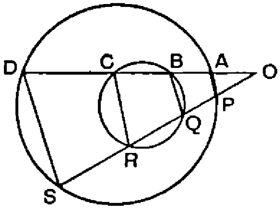


FIG. 261

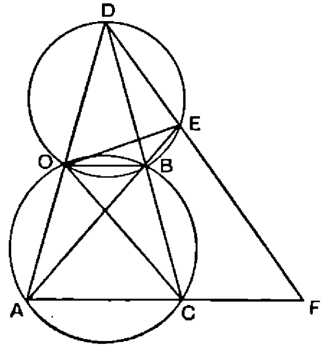


FIG. 262]

21. In Fig. 262, the side CB of $\triangle ABC$ is produced to any point D, and a straight line is drawn through D cutting AB produced at E and AC produced at F. The circumcircles of $\triangle$s ABC, BDE meet again at O. Prove that (i) ODFC is cyclic, (ii) OAFE is cyclic.

EXAMPLES 17c

1. (Th. 29.) Equilateral triangles are described outwardly on the sides of a given acute-angled triangle. Prove that the circles which circumscribe these equilateral triangles meet in a common point. (OC)

2. Two circles cut at **X** and **Y**, and a straight line **ABCD** cuts one circle at **A**, **C** and the other at **B**, **D**. The parallel to **YA** through **D** meets **XY** at **Z**. Prove that **C**, **D**, **X**, **Z** are concyclic.

3. **A**, **B** are the ends of a diameter of a circle, centre **O**, and **P** is any point on the circumference of the circle. **AP**, **BP**, produced where necessary, meet the perpendicular to **AB** through **O** at **L** and **M** respectively and the radius **OP** is drawn. Prove that (i) the points **A**, **O**, **M**, **P** are concyclic, (ii) $\triangle$s **OPM**, **OLP** are equiangular. (C)

4. In Fig. 244, p. 196, prove that **F**, **O**, **E**, **C** are concyclic.

5. (Th. 31.) **ABC** is any triangle. **D** is the point of intersection of the interior bisector of angle **B** and the exterior bisector of angle **C**. Prove that the circumcircle of **ABC** passes through the centre of the circumcircle of **BCD**. (O)

6. State and prove the converse of the theorem 'Angles in the same segment are equal.'

The circumcircle and base **BC** of an acute-angled triangle **ABC** are fixed. Perpendiculars are drawn from **B** and **C** to the opposite sides and cut each other at **P**. Prove that, as **A** moves on the circumcircle, **P** lies on another circle. (N)

7. (Th. 29.) **ABC** is an isosceles triangle having **AB**$=$**AC**. A straight line through **B** is drawn cutting **AC** at **D**. Through **C** a straight line is drawn meeting **BD** produced at **E**, and making $\angle$**ACE** equal to $\angle$**ABE**. Prove that $\angle$**AEC**$=90^\circ+\frac{1}{2}\angle$**BAC**. (L)

8. (Th. 32.) **OC** is the internal bisector of an angle **AOB** which equals 120°. **X** and **Y** are any points on **OA**, **OB** respectively. The perpendiculars to **OA**, **OB** at **X** and **Y** intersect in **P**. **Z** is the foot of the perpendicular on **OC** from **P**. Prove that $\triangle$**XYZ** is equilateral. (O)

9. **ABCD** is a rectangle. The line through **C** perpendicular to **AC** meets **AB**, **AD** produced at **P**, **Q**. Prove that **PBDQ** is cyclic. (L)

10. **P** is a point on the circumference of a circle whose centre is **O**. **OA**, **OB** are two fixed radii, and **R**, **S** are the feet of the perpendiculars from **P** to **OA**, **OB** (produced if necessary). Prove that **PROS** is cyclic.

If **P** is now made to move round the circle, prove that the circle **PROS** has a constant size, and deduce that **RS** is of constant length.

11. ABC is an acute-angled triangle; AD is drawn perpendicular to BC, and DK, DL are drawn perpendicular respectively to AB and AC. Prove that BKLC is a cyclic quadrilateral. (L)

12. (Th. 31.) A, B, C are three points in the order named on a straight line. P and Q are points on the same side of the line ABC such that PAB, QBC are equilateral triangles. Prove that PBC and ABQ are congruent triangles, and that if AQ and PC intersect in R, $\angle ARB = \angle BRC = 60°$. (O)

THE CIRCLE: TANGENT PROPERTIES

Definitions

A **tangent** (Latin *tango*, I touch) to a circle is a straight line which, however far it may be produced in either direction, has one and only one point in common with the circle.

The point which is common to the straight line and the circle is called the **point of contact.** The tangent is said to **touch** the circle at this point. Notice carefully that the phrases ' AB *touches* the circle at A ' and ' AB *is a tangent to* the circle at A ' have the same meaning.

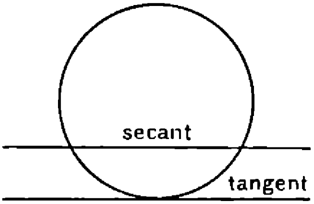


FIG. 263

A **secant** (Latin *seco*, I cut) is a straight line which has two points in common with a circle and extends beyond the circle. The secant is said to **cut** the circle at these two points. The distinction between a secant and a chord (see p. 157) should be noticed.

THEOREM 34a (Test for a tangent)

(For proof, see Appendix, p. 338)

The straight line drawn perpendicular to a radius of a circle at its extremity is a tangent to the circle.

THEOREM 34b

(For proof, see Appendix, p. 339)

A tangent to a circle is perpendicular to the radius drawn through the point of contact.

Reference. (See Fig. 264.)

$\angle OPT = 90°$ (*tan. perp. radius*).

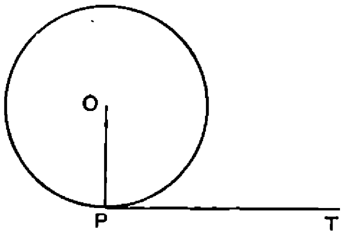


FIG. 264

Corollary.

The perpendicular to a tangent at its point of contact passes through the centre of the circle.

CONSTRUCTION 18

To construct a tangent to a circle from an external point.

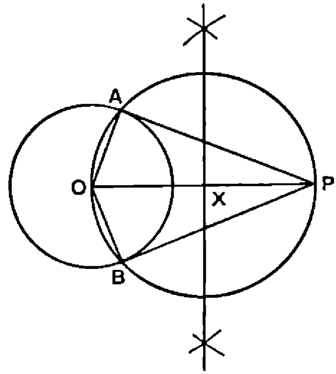


FIG. 265

Given a circle, centre O, and a point P outside it.

To construct a tangent from P to the circle.

Construction. Join OP.

Bisect OP at X.

With centre X, radius XO, draw a circle, cutting the given circle at A and B.

Join PA.

Then PA is a tangent to the given circle.

Proof. Join OA.

$$XO = XP \text{ (constr.)},$$

∴ the circle, centre X, radius XO, passes through P, and OP is a diameter.

$$\therefore \angle OAP = 90^\circ \text{ (}\angle\text{ in semicircle).}$$

But OA is a radius of the given circle;

∴ PA is a tangent (tan. perp. radius).

Note.—It may similarly be shown that PB is also a tangent.

Definitions

If all the sides of a polygon touch a circle, the polygon is said to be **circumscribed about** the circle, and the circle is said to be **inscribed in** the polygon.

The circle which touches the three sides of a triangle is called the **inscribed circle** or **incircle**; its centre is called the **incentre.**

A circle which touches one side of a triangle and touches the other two sides produced is called an **escribed circle**; its centre is called an **excentre.**

Fig. 266 shows the four circles which can be drawn to touch the sides of $\triangle ABC$; they are the inscribed circle, centre I, and the three escribed circles, centres I_1, I_2, and I_3.

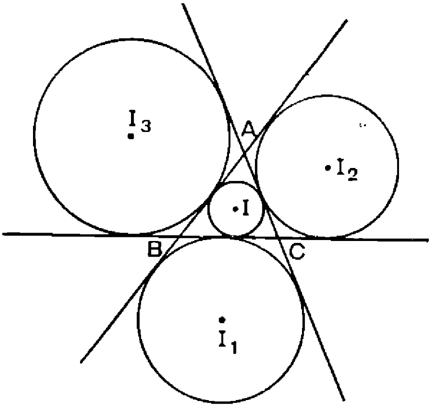


FIG. 266

CONSTRUCTION 19

To construct the inscribed circle of a triangle.

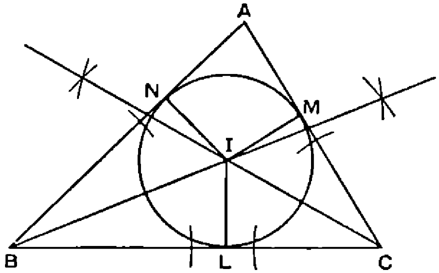


FIG. 267

Given a △ABC.

To construct the inscribed circle of △ABC.

Construction. Draw the bisectors of ∠s ABC, ACB to meet at I.

Draw IL perpendicular to BC, meeting it at L.

With centre I, radius IL, draw a circle.

This is the inscribed circle of △ABC.

Proof. Let M, N be the feet of the perpendiculars from I to AC, AB.

I lies on the bisector of ∠ABC,

$$\therefore \text{IN} = \text{IL}.$$

Again, I lies on the bisector of ∠ACB,

$$\therefore \text{IM} = \text{IL}.$$

Hence $\text{IL} = \text{IM} = \text{IN}$.

∴ the circle, centre I and radius IL, passes through M and N.

Also IL is a radius and $\angle \text{ILB} = 90°$ (constr.);

∴ BC is a tangent to the circle (tan. perp. radius).

Similarly, CA and AB are tangents to the circle.

∴ the circle touches BC, CA, and AB.

CONSTRUCTION 20

To construct an escribed circle of a triangle.

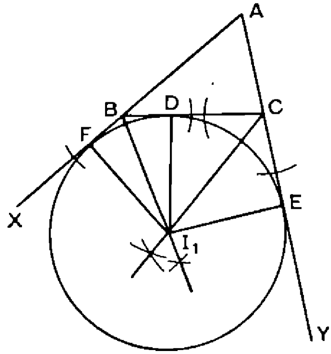


Fig. 208

Given a $\triangle$ABC, with the sides AB, AC produced to X, Y.

To construct a circle touching BC, AB produced, AC produced.

Construction. Draw the bisectors of $\angle$s CBX, BCY to meet at I_1.

Draw I_1E perpendicular to CY, meeting it at E.

With centre I_1, radius I_1E, draw a circle.

This is the escribed circle required.

Proof. Let D, F be the feet of the perpendiculars from I_1 to BC, BX.

I_1 lies on the bisector of $\angle$CBX,

$$\therefore \; I_1F = I_1D.$$

Again, I_1 lies on the bisector of $\angle$BCY,

$$\therefore \; I_1E = I_1D.$$

Hence $\qquad I_1D = I_1E = I_1F.$

$\therefore$ the circle, centre I_1 and radius I_1E, passes through D and F.

Also I_1E is a radius and $\angle I_1EA = 90°$ (constr.);

$\therefore$ AY is a tangent to the circle (tan. perp. radius).

Similarly, AX and BC are tangents to the circle.

$\therefore$ the circle touches BC, AB produced, and AC produced.

THEOREM 35

The two tangents from an external point to a circle are equal.

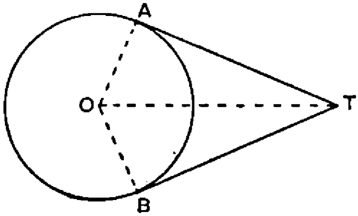


FIG. 269

Given two tangents TA, TB to a circle, centre O, from an external point T.

To prove that TA = TB.

Construction. Join OA, OB, OT.

Proof. In the right-angled △s OAT, OBT,

OAT and OBT are the right angles (tan. perp. radius),

OA = OB (radii),

OT = OT;

∴ △s $\begin{matrix}\text{OAT}\\\text{OBT}\end{matrix}$ are congruent (RHS).

∴ TA = TB.

Reference. TA = TB (*equal tangents*).

EXAMPLES 18a

(*Theorem* 34)

1. A point T is 5 in. from the centre of a circle of radius 3 in. Calculate the length of the tangent from T to the circle.

2. From a point T, a tangent 8 cm. long is drawn to a circle of radius 15 cm. Calculate the distance of T from the centre of the circle.

3. In Fig. 269, if OA=4 in. and OT=7 in., calculate TA to the nearest tenth of an inch.

4. Two circles of radii 4 in. and 3 in. have the same centre. Calculate to the nearest tenth of an inch the length of a chord of the outer circle which touches the inner one.

5. State the locus of the centres of circles which touch a given straight line at a given point.

6. Prove that tangents to a circle at the ends of a diameter are parallel.

7. In Fig. 269, prove that OA touches the circle on AT as diameter.

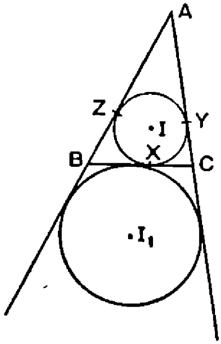
8. AB is a diameter of a circle, AP is any chord. The tangent at B to the circle cuts AP produced at T. Prove that ∠ATB=∠PBA.

9. AB is a chord of a circle whose centre is O. AN is drawn perpendicular to the tangent at B to the circle. Prove that ∠OBA=∠BAN.

(*Theorem* 35)

10. In Fig. 270, if AY=5 cm., BZ=6 cm., and CX=3 cm., calculate the lengths of BC, CA, and AB.

11. In Fig. 270, if the circle centre I_1 touches BC at P, AC produced at Q, and AB produced at R, and if AR=15 in., BP=4 in., CQ=6 in., calculate the lengths of BC, CA, AB.

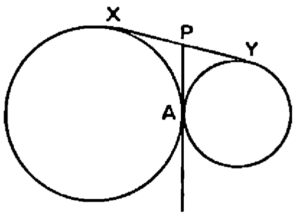


I, I_1 are centres of circles; X, Y, Z are points of contact

Fig. 270

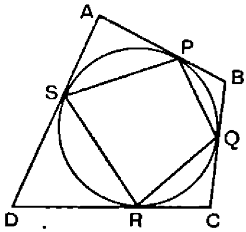
12. In Fig. 271, prove that PX = PY.

13. In Fig. 272, prove that AB + CD = AD + BC. (Hint: let AP, BQ, CR, DS be a, b, c, d in. respectively.)



PA touches both circles at A; XPY touches the circles at X and Y

FIG. 271



AB, BC, CD, DA are tangents to the circle; P, Q, R, S are the points of contact

FIG. 272

14. In Fig. 272, if ABCD is a parallelogram, prove that the parallelogram must be a rhombus. (Use the result of No. **13.**)

15. Two circles both pass through a point A and have the same tangent at that point. Any line through A cuts the circles at X and Y. Prove that the tangents to the circles at X and Y are parallel. (Hint: draw the common tangent to the two circles at A, and let it cut the tangent at X at P, and cut the tangent at Y at Q.)

16. In Fig. 270, if Y is the mid-point of AC and Z the mid-point of AB, prove that X is the mid-point of BC. What kind of triangle must ABC be?

(*Constructions* 19 *and* 20)

17. In Fig. 270, if ∠ABC = 40° and ∠ACB = 62°, calculate ∠BIC and $\angle BI_1C$.

18. In Fig. 270, prove that AII_1 is a straight line.

(*Constructions*)

19. Draw a circle with radius 4 cm. and centre O. Mark two radii OA and OB such that ∠AOB = 130°. Construct the tangents at A and B to the circle, and verify by measurement that these tangents contain an angle of 50°.

20. Draw a circle with radius 4·5 cm., and mark a point **P** 7·5 cm. from the centre of the circle. Construct the two tangents from **P** to the circle, and measure their lengths.

21. Draw a triangle **ABC** in which **AB**=2·2 in., **BC**=2·5 in., **CA**=1·8 in., and construct the inscribed circle. Measure its radius.

22. Draw a triangle with sides 8 cm., 8 cm., and 6 cm., and construct the inscribed circle. Measure its radius.

23. Draw a triangle **ABC** in which **AB**=5·5 cm., **BC**=6 cm., **CA**=4·5 cm., and construct the escribed circle which touches **BA** produced, **BC** produced, and **CA**. Measure its radius.

EXAMPLES 18b

1. If the tangents from a point to a circle include an angle of 40°, calculate the angle between the radii through the points of contact.

2. If, in Fig. 269, **AO** is produced to **X**, prove that ∠**BOX** = ∠**ATB**.

3. Two parallel tangents to a circle whose centre is **O** are met by a third tangent at **P** and **Q**. Prove that ∠**POQ**=90°.

4. **O** is the centre of a circle, and **PR** a chord. If the tangents at **P** and **R** meet at **T**, prove that ∠**PTR**=2∠**OPR**.

5. In Fig. 269, if **OA**=a in., **AT**=d in., express in terms of a and d the length of **OT**. Hence prove that, if **P** is a point from which the tangents to a given circle are of given length, then the locus of **P** is a circle. State its centre and radius.

6. Two circles have the same centre. Prove that the length of the tangent from any point on the outer circle to the inner circle is constant.

7. Prove that the points of contact of tangents from a given point to a series of concentric circles lie on a circle, and state its centre and radius.

8. State the locus of points the tangents from which to a given circle include a given angle, and illustrate by a sketch.

9. In Fig. 269, prove that **OT** bisects **AB** at right angles.

10. In Fig. 270, if **BC**$=a$ in., **CA**$=b$ in., **AB**$=c$ in., and the radius of the inscribed circle is r in., write down expressions for the areas of △s **BIC**, **CIA**, and **AIB**, and show that the area of △**ABC** is $\frac{1}{2}r\,(a+b+c)$ sq. in.

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11. In Fig. 270, if ∠**ABC**$=54°$ and ∠**ACB**$=72°$, calculate ∠**YXC**, ∠**ZXB**, ∠**ZXY**.

12. In Fig. 272, if ∠**A**$=80°$, ∠**B**$=104°$, ∠**C**$=118°$, calculate the angles of the quadrilateral **PQRS**.

13. In Fig. 273, **ABCD** is a quadrilateral and the inscribed circles of △s **ABD**, **BCD** touch **BD** at the same point **P**. Prove that **AB**+**CD**=**AD**+**BC**. (Use the method suggested for Ex. 18a, No. 13.)

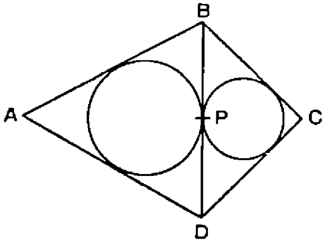

FIG. 273

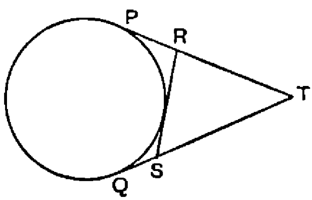

FIG. 274

14. In Fig. 274, **TP**, **TQ**, and **RS** are tangents, and **P**, **Q** are points of contact. Prove that the perimeter of △**RST** is equal to 2**TP**.

15. **ABCDEF** is a hexagon circumscribing a circle, *i.e.* its sides are tangents to the circle. Prove that **AB**+**CD**+**EF**=**BC**+**DE**+**FA**.

16. In Fig. 270, if **BC**=12 cm., **CA**=14 cm., and **AB**=18 cm., calculate the lengths of **AY**, **BZ**, and **CX**.

17. In Fig. 270, let **BC**$=a$ in., **CA**$=b$ in., **AB**$=c$ in., **AY**$=x$ in., **BZ**$=y$ in., **CX**$=z$ in. Prove that (i) $2x+2y+2z=a+b+c$, (ii) $y+z=a$, and from these two equations show that $x=\frac{1}{2}(b+c-a)$.

18. In Fig. 271, use the fact that P is the centre of a certain circle in order to prove that $\angle XAY=90°$. Hence prove that $AX^2+AY^2=4AP^2$.

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19. In Fig. 270, prove that the circle on II_1 as diameter passes through B and C.

20. If, in Fig. 270, $\angle ACB=90°$, prove that $\angle AIB=135°$.

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21. A sphere of radius 2 in. rests inside a hollow inverted cone of vertical angle 90°. Calculate to the nearest tenth of an inch the distance of the centre of the sphere from the vertex of the cone.

22. P is the point of contact of any tangent from a fixed point A to a sphere. What is the locus of P, and what kind of surface does AP trace out as P describes its locus ?

23. A hollow right circular cone with a plane base is of height 12 in. and base-radius 5 in. By means of an accurate scale drawing, find the radius of the largest sphere which can be placed inside it.

---

24. Draw a circle with radius 4 cm. and construct a pair of tangents to contain an angle of 110°. (Hint: first draw two radii inclined at an appropriate angle.)

25. Draw a circle of radius 1 in., and about it circumscribe a triangle ABC such that $\angle A=72°$ and $\angle B=44°$. (Hint: first draw radii inclined at appropriate angles.)

26. Draw a straight line AB, 6 cm. long. Construct through B a straight line such that the perpendicular to it from A is 4 cm. long.

27. Draw a circle of radius 2 in. and construct any chord of the circle $3\frac{1}{2}$ in. long. Next mark a point P $1\frac{1}{2}$ in. from the centre of the circle, and construct a chord $3\frac{1}{2}$ in. long to pass through P. (Start by drawing the circle of which all $3\frac{1}{2}$ in. chords are tangents.)

28. Draw a circle and a straight line. Construct tangents to the circle, (i) parallel, (ii) perpendicular to the straight line you have drawn.

29. Draw a circle of radius 3 cm., and construct the locus of points from which the lengths of the tangents drawn to the circle are 5 cm. (Begin by drawing *any* tangent of length 5 cm.)

30. Draw a straight line AB 5·5 cm. long and construct the circle on AB as diameter. Now find by construction the point P on AB produced which is such that the tangent from P to the circle is of length 6 cm. (Begin by drawing *any* tangent of length 6 cm., and use the fact that all points, from which the tangents to the circle are of length 6 cm., are at the same distance from the centre of the circle.)

31. Draw a rhombus with each side $2\frac{1}{2}$ in. and an angle of 50°, and inscribe a circle in it.

EXAMPLES 18c

1. The tangent to a circle of radius 1·5 in. from an external point T is 2 in. long. Calculate the distance of T from the nearest point of the circumference. (N)

2. The circle inscribed in the triangle ABC touches BC at X, CA at Y, and AB at Z. If the angle XZY=53° and the angle AZY=77°, calculate the angles of the triangle ABC. (C)

3. If I_1, I_2 are the excentres opposite the vertices A, B respectively of a triangle ABC, and if I is the incentre, prove that the common chord of the circles on II_1, II_2 as diameters is the bisector of the angle ACB.

4. (Th. 35.) If the four sides of a parallelogram touch a circle, prove that the four sides are all equal. (L)

5. (Th. 25a.) The radius of a circle is 3 in. and P is a point distant 4 in. from the centre. Construct a chord of length 2·5 in. to pass through P. (C)

6. In Fig. 272, if O is the centre of the circle, prove that $\angle AOB + \angle COD = 180°$.

7. If the inscribed circle of the triangle **ABC** touches the side **BC** at **D**, prove that $AB - BD = AC - CD$.

If also the inscribed circle of the triangle **ABD** touches **AD** at **E**, prove that $AB - BD = AE - ED$.

8. If the length of the tangent from an external point to a circle is 10 cm., and the shortest distance from the point to the circumference of the circle is 4 cm., find the radius of the circle. (OC)

9. (Th. 35.) Draw a circle of 1 in. radius, and draw two radii, **OP**, **OQ**, at right angles. Draw the tangents at **P** and **Q**, meeting at **A**, and produce **AP** to **B**, making $PB = 2$ in. From **B** draw another tangent and produce it to meet **AQ** produced in **C**. Measure **AC** and **BC**, and verify your result by calculation, using Pythagoras' theorem or otherwise. (OC)

10. The point **P** is the centre of the inscribed circle of the triangle **ABC**; the point **Q** is the centre of the circle which touches **BC** and the other two sides produced. Prove (i) **B**, **P**, **C**, **Q** are concyclic points, (ii) Δs **ABP**, **AQC** are equiangular. (N)

11. Draw a triangle **ABC** with sides $AB = 5$ cm., $BC = 6$ cm., and $CA = 7$ cm. Find by accurate geometrical construction (i) the centre and (ii) the point of contact with **AB** of the escribed circle of the triangle opposite the angle **C** (*i.e.* the circle touching **AB**, **CB** produced, and **CA** produced.) Draw the circle, measure its radius, and prove that your construction is correct. (L)

12. **P** is a point 7 cm. distant from the centre of a circle of radius 3 cm. Construct, full size, a chord of the circle which shall be 4 cm. long and which, when produced, shall pass through **P**. No lengths are to be calculated, and no proof is required, but the steps in the construction should be stated. (N)

13. Draw two circles of radii 1 in. and 3 in. with their centres $3\frac{1}{2}$ in. apart. Construct, explaining your construction, two points, such that the tangents from them to the smaller circle include an angle of 60° and the tangents from them to the larger circle include an angle of 120°. (N)

14. (Constr. 19.) If **I** is the centre of the inscribed circle of a triangle **ABC**, and **P** is the point of contact with **BC**, prove that $IB^2 - IC^2 = BC\,(BP - PC) = BC\,(BA - AC)$. (N)

15. An escribed circle of the triangle ABC touches BC at D, AC produced at E, and AB produced at F. Prove that $\angle EDF = 90° + \frac{1}{2}\angle A$. (OC)

16. Prove the following construction for drawing a tangent to a circle with centre O from a given point A outside it: join OA cutting the circle in B; draw a circle with centre O and radius OA; from B, draw BC perpendicular to OA meeting the latter circle in C; join OC cutting the given circle in D; then AD will be a tangent. (OC)

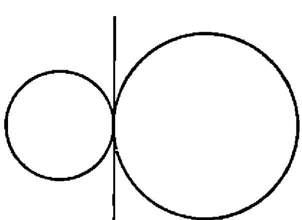
17. I is the centre of the inscribed circle of △ABC. AI produced meets again at D the circle which is circumscribed about the triangle. Prove that DI = DB = DC. (OC)

18. (Th. 35.) ABC is an acute-angled triangle inscribed in a circle. The tangents at A and C meet the tangent at B in S and T respectively; a line is drawn through T parallel to AS to meet AB and AC (both produced) in L and M. Prove that TL = TM. (N)

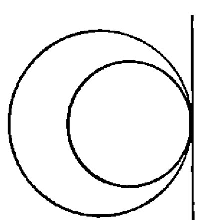
CONTACT OF TWO CIRCLES

Definitions

Two circles are said to touch at a point when they have the same tangent at that point; the point is called the **point of contact** of the circles. The circles are said to touch **externally** when they are on opposite sides of their common tangent, and **internally** when they are on the same side of their common tangent.



External contact



Internal contact

FIG. 275

If a straight line touches one circle at a point P and another circle at a point Q, it is said to be a **common tangent** of the two circles; and the length PQ is called the **length of the common tangent.** If the circles are on the same side of PQ, the common tangent is said to be **direct**; if the circles are on opposite sides of PQ, the common tangent is said to be **transverse.**

Sometimes the words 'exterior' and 'interior' are used instead of 'direct' and 'transverse' respectively.

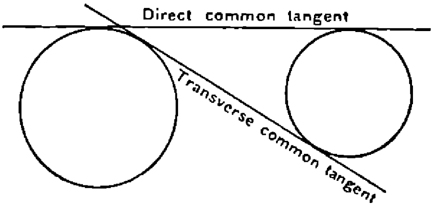


FIG. 276

Exercise.—How many direct and transverse common tangents do two circles have (i) if they cut each other, (ii) if they touch each other externally, (iii) if they touch each other internally, (iv) if one lies inside the other?

THEOREM 36

If two circles touch, the straight line joining their centres passes through the point of contact.

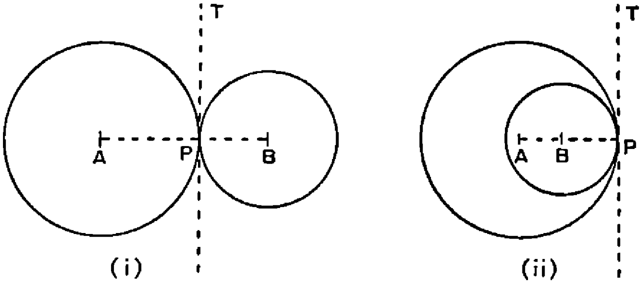


FIG. 277

Given two circles, centres A and B, touching at P.

To prove that A, P, B are in the same straight line.

Construction. Draw the common tangent PT to the two circles. Join AP, BP.

Proof. ∵ AP is a radius and PT a tangent,

∴ ∠TPA = 90° (tan. perp. radius).

∵ BP is a radius and PT a tangent,

∴ ∠TPB = 90° (tan. perp. radius).

∴ in both figures A and B each lie on the straight line through P perpendicular to PT;

∴ A, P, B are in the same straight line.

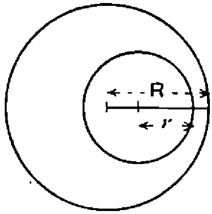
Reference.

APB is a straight line (*pt. of contact on line of centres*).

Corollary to Theorem 36.

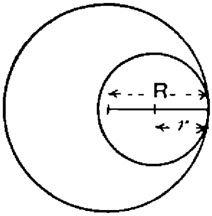
If two circles touch externally, the distance between their centres is equal to the sum of their radii ; if they touch internally, the distance between their centres is equal to the difference of their radii.

If we know the radii, r units and R units (say), of two circles and the distance d units between their centres, this corollary enables us to say, without drawing an accurate figure, how the circles are situated relative to one another. The figures below show the five possible cases, it being assumed that $R>r$:—



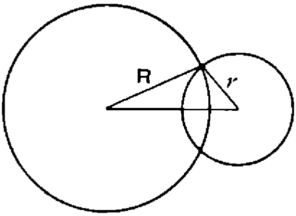
$R - r > d$

Fig. 278



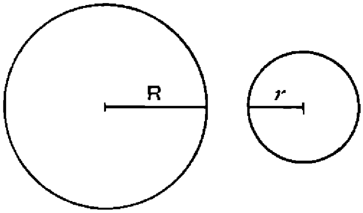
$R - r = d$

Fig. 279



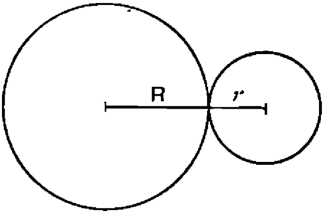
$R - r < d$
and $R + r > d$

Fig. 280



$R + r < d$

Fig. 281



$R + r = d$

Fig. 282

CONSTRUCTION 21

To construct a direct common tangent to two circles.

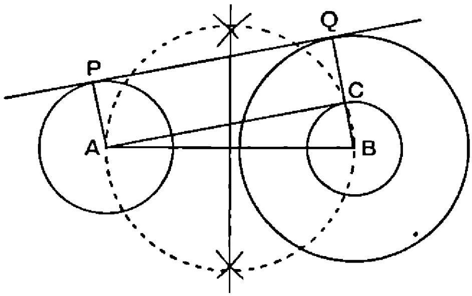


FIG. 283

Given two circles, centres A and B.

To construct a direct common tangent.

Construction. Suppose the circle, centre B, has the larger radius.

With centre B and radius equal to the *difference* of the radii of the given circles, draw a circle.

Construct a tangent from A to touch this circle at C.

Join BC, and produce it to cut the larger circle at Q.

Draw AP parallel to BQ to meet the smaller circle at P.

Join PQ.

Then PQ is a direct common tangent.

Proof. $CQ = BQ - BC$,

and $BC = BQ - AP$ (constr.);

$\therefore CQ = BQ - (BQ - AP)$

$= AP$.

Also $CQ \parallel AP$ (constr.);

$\therefore$ APQC is a parallelogram.

But $\angle ACQ = 90^\circ$ (tan. perp. radius);

$\therefore$ APQC is a rectangle.

$\therefore$ $\angle$s APQ, CQP are right angles.

$\therefore$ PQ is perpendicular to each of the radii AP, BQ.

$\therefore$ PQ is a tangent to both circles (tan. perp. radius).

CONSTRUCTION 22

To construct a transverse common tangent to two circles.

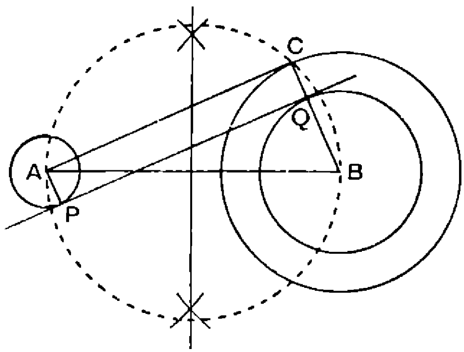


FIG. 284

Given two circles, centres A and B.

To construct a transverse common tangent to the two circles.

Construction. With centre B and radius equal to the *sum* of the radii of the given circles, draw a circle.

Construct a tangent AC from A to this circle.

Join BC, and let it cut at Q the given circle, centre B.

Draw AP parallel to CB, meeting at P the given circle, centre A.

Join PQ.

Then PQ is a transverse common tangent.

Proof. $CQ = BC - BQ$,

and $BC = BQ + AP$ (constr.);

$\therefore CQ = BQ + AP - BQ$

$= AP$.

Also $CQ \parallel AP$ (constr.);

$\therefore$ APQC is a parallelogram.

But $\angle ACQ = 90°$ (tan. perp. radius);

$\therefore$ APQC is a rectangle.

$\therefore$ $\angle$s APQ, CQP are right angles.

$\therefore$ PQ is perpendicular to each of the radii AP, BQ.

$\therefore$ PQ is a tangent to both circles (tan. perp. radius).

Note.—The length PQ of either a direct or a transverse common tangent to two circles can be calculated by first finding AC from the right-angled triangle ACB; PQ is then equal to AC, as these are opposite sides of the rectangle APQC.

EXAMPLES 19a

1. Two circles of radii 1 in. and $2\frac{1}{2}$ in. touch externally. Calculate the distance between their centres.

2. Two circles of radii 2 cm. and 5 cm. touch internally. Calculate the distance between their centres.

3. Two circles of radii r cm. and R cm. have their centres d cm. apart. Say in the following cases (without drawing the circles accurately) whether they intersect, do not intersect, touch externally, or touch internally :—

(i) $r=1{\cdot}2$, $R=2{\cdot}6$, $d=4$;
(ii) $r=1{\cdot}2$, $R=2$, $d=3{\cdot}2$;
(iii) $r=1$, $R=2{\cdot}4$, $d=2$;
(iv) $r=1{\cdot}5$, $R=3{\cdot}9$, $d=2{\cdot}4$;
(v) $r=2{\cdot}3$, $R=3{\cdot}8$, $d=6{\cdot}1$.

4. What is the locus of the centres of circles of radius 2 cm. which touch a fixed circle of radius 5 cm. (i) externally, (ii) internally ?

5. What is the locus of the centres of circles of radius 3 in. which touch a fixed circle of radius 1 in. internally ?

6. What is the locus of the centres of circles which touch a fixed circle at a given point on its circumference ?

7. In Fig. 285, the concentric circles have radii 4 cm., 7 cm. Calculate (i) the radius of a circle which touches the larger circle internally and the smaller circle externally, (ii) the radius of a circle which touches both circles internally. (These are the two dotted circles in the figure.)

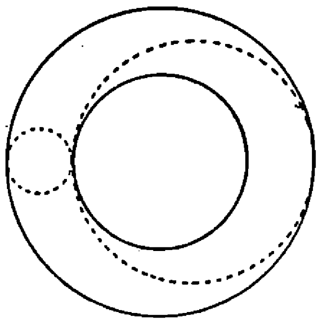


FIG. 285

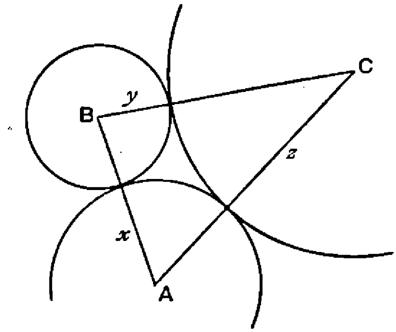


FIG. 286

8. In Fig. 286, **AB** = 5 cm., **BC** = 6 cm., **CA** = 8 cm. Three circles, with centres at **A**, **B**, and **C**, and radii x cm., y cm., and z cm. respectively, touch each other externally. Write down three equations connecting x, y, and z, and hence find the radii of the three circles.

9. In Fig. 287, **AB** = 8 cm., **BC** = 7 cm., **CA** = 6 cm. Circles with centres at **A**, **C** touch each other externally, and a third circle with its centre at **B** touches both the others internally. Calculate the radii of the three circles. (Hint: denote the radii by x, y, and z cm., and write down three equations.)

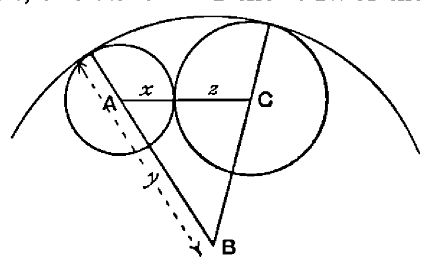


FIG. 287

10. Two circles with radii 1 in. and 2 in. have their centres **A**, **B** at a distance of 3 in. apart. Calculate the radius of the circle, with centre on **AB**, which encloses the two circles and touches each of them internally.

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11. Two circles of radii 4 cm., 6 cm. touch externally. Calculate to the nearest mm. the length of their direct common tangent.

12. Two circles of radii 6 cm., 3 cm. have their centres 5 cm. apart. Calculate the length of their common tangent.

13 Two circles of radii 3 cm., 1 cm. have their centres 5 cm. apart. Calculate the length of their transverse common tangent.

14. Two circles of radii 2 cm., 7 cm. have their centres 13 cm. apart. Calculate to the nearest mm. the lengths of their direct and transverse common tangents.

---

15. Draw two circles with radii 1 in. and $1\frac{3}{4}$ in. touching one another externally.

16. Draw two circles with radii 6 cm. and $2\frac{1}{2}$ cm. touching one another internally.

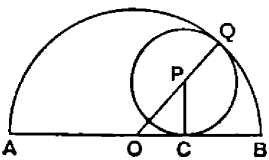
17. Draw two equal circles of radius 3 cm. with their centres 8 cm. apart, and construct their direct common tangents.

18. Draw two circles, radii 2 cm., 4 cm., with their centres 8 cm. apart. Construct their direct common tangents.

19. Draw two circles, radii 2 cm., 4 cm., with their centres 8 cm. apart. Construct their transverse common tangents.

EXAMPLES 19b

1. **ABC** is a triangle in which **AB** $=5\frac{1}{2}$ cm., **BC** $=4\frac{1}{2}$ cm., **CA** $=6$ cm. Two circles with centres **A**, **B** touch each other externally, and a circle with centre **C** touches the others internally. Calculate the radii of the three circles.



AB is a diameter; **AO**=**OB**; **P** is the centre of a circle which touches **AB** at **C** and the semicircle at **Q**

FIG. 288

2. In Fig. 288, if **AB**$=10$ cm., **AC**$=6$ cm., and the radius of the small circle is x cm., write down the lengths of the sides of $\triangle$**OPC**, and apply Pythagoras' theorem. Hence calculate the value of x.

3. In Fig. 289, **A** is the centre of a circle of radius 3 cm., **B** is the centre of a circle of radius 2 cm., and **AB** = 7 cm. **P** is the centre of a circle of radius r cm. which touches the other two circles, and **ABP** is a right angle. Use Pythagoras' theorem to find r.

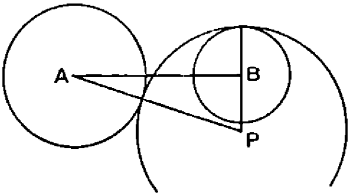


FIG. 289

4. **AB** is a straight line 5 in. long, and a circle is drawn with centre **A** and radius 2 in. Calculate the radius of a circle which touches this circle externally and also touches the line **AB** at **B**.

5. **ABCD** is a parallelogram, **X** and **Y** are the mid-points of **AB**, **DC**. Prove that, if the circle on **AB** as diameter touches that on **DC** as diameter, then the figure **ABCD** is a rhombus.

6. Two circles touch internally at **X**, the smaller circle passing through the centre of the larger. If a straight line through **X** cuts the smaller circle at **Y** and the larger circle at **Z**, prove that **XY** = **YZ**.

7. Two circles with centres **A**, **B** touch at **P**. A straight line through **P** cuts the circles at **X**, **Y** respectively. Prove that **AX** and **BY** are parallel if the circles touch (i) externally, (ii) internally.

Prove also that the tangents to the circles at **X** and **Y** are parallel.

8. Four equal spheres of radius 1 in. are placed on a table with their centres forming a square, each sphere touching two others. A fifth equal sphere rests on them. Calculate, correct to a hundredth of an inch, the height of the centre of this sphere above the table.

---

9. Two circles of radii 3 cm., 5 cm., touch externally. Calculate to the nearest mm. the length of their direct common tangent.

10. A belt runs (directly) round two pulleys of radii 6 in., 10 in., which have their centres 19 in. apart. Calculate to the nearest tenth of an inch the length of a straight portion of the belt.

11. Two circles of radii a in., b in., touch externally, and t in. is the length of their common tangent. Prove that $t^2=4ab$.

12. Two circles of radii a in., b in., have their centres d in. apart; prove that the length of their direct common tangent is $\sqrt{(d^2+2ab-a^2-b^2)}$ in.

13. In No. 12 prove that the length of the transverse common tangent is $\sqrt{(d^2-2ab-a^2-b^2)}$ in.

14. Mark two points, **A** and **B**, 3·4 cm. apart. Construct a straight line such that the perpendiculars to it from **A** and **B** are 3·4 cm. and 1·6 cm. long respectively. Measure the acute angle at which this line cuts **AB** produced.

EXAMPLES 19c

1. The centres of two circles are **A** and **B** and the points of contact of an exterior common tangent are **P** and **Q** respectively. If **AP**=6 cm., **AB**=13 cm. and **BQ**=1 cm., find by calculation the length of **PQ**. (An accurate figure is not required.) (N)

2. A circle is drawn, and inside it are two circles touching one another externally and the larger circle internally. Prove that the perimeter of the triangle formed by joining the three centres is equal to the diameter of the large circle. (OC)

3. Construct one of the direct common tangents of two circles whose centres are 9·2 cm. apart and whose radii are 2·5 cm. and 5·5 cm., and measure its length. Describe your method briefly.

4. **A** and **B** are the centres of two circles which touch externally at **C**. If a third circle touches **AB** at **C** and cuts the first two circles at **D**, **E** respectively, prove that **AD** and **BE** touch the third circle.

5. Draw two circles, of radii 2·8 in. and 1·6 in., with their centres 2 in. apart. Construct the two common tangents, measure their lengths, and verify by calculation.

6. Fig. 290 represents two circular discs placed in contact in a corner of a rectangular box, touching the sides of the box. **A** and **B** are the centres, and **AP**, **BP** are parallel to the sides of the box. Write down the lengths of **AP**, **BP**, and **AB** in terms of the letters given in the figure, and prove that $2(xy+bx+ay)=a^2+b^2$.

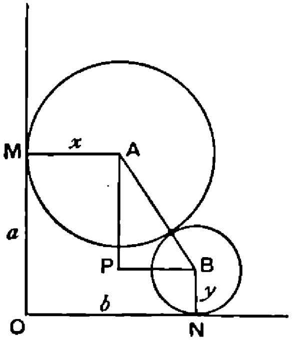


FIG. 290

7. Two circles of radii 5 and 3 cm. respectively are drawn with **A** as their common centre. A circle of radius 2 cm. is drawn with its centre at **B**, a point 7·5 cm. from **A**. A tangent is drawn from **B** to the circle whose radius is 3 cm.; its point of contact is **P**; **AP** produced meets the circle of 5 cm. radius in **Q**. Prove that the tangent to this circle at **Q** touches the circle of 2 cm. radius.

Draw an accurate figure, showing the construction lines for the tangent **BP** and for the tangent at **Q**. Measure **BP**. (OC)

8. In Fig. 291, two circles touch at **A**, and the common tangent at **A** meets at **X** a direct common tangent **PQ**. Prove that the circle on **PQ** as diameter touches the line joining the centres of the two circles.

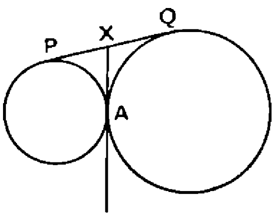


FIG. 291

9. Three equal circular discs of diameter d in. are kept in contact, as shown in Fig. 292, by an elastic band. Prove that the length of the band is $d(3+\pi)$ in.

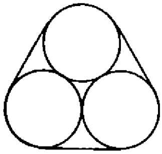


FIG. 292

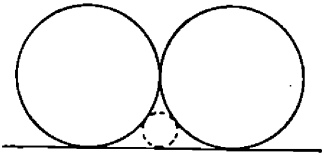


FIG. 293

10. In Fig. 293, the two large circles are of radius r cm. Calculate, in terms of r, the radius of the small circle which touches each of them externally and also touches one of their direct common tangents.

11. Two circles, of radii $8\frac{1}{2}$ and $1\frac{1}{2}$ cm., touch a straight line at points 24 cm. apart. Calculate the distance between the centres, (i) if the circles are on the same side of the line, (ii) if they are on opposite sides.

12. **AB** is a straight line of length $4r$ in., and **C** is its mid-point. Semicircles with diameters **AB**, **AC**, and **CB** are drawn on the same side of **AB**. Calculate, in terms of r, the radius of the circle which touches the large semicircle internally and the two small semicircles externally.

13. **A**, **B**, **C**, **D** are four points in order on a straight line. Prove that the square of the length of the direct common tangent to the circles on **AB** and **CD** as diameters is equal to **AC.BD**, and that the square of the length of the transverse common tangent is equal to **AD.BC**.

14. Prove that the difference between the squares on a direct and on a transverse common tangent of two circles is equal to the rectangle contained by the diameters.

THE ALTERNATE SEGMENT PROPERTY

Definition. If **ACB** is a segment of a circle on the chord **AB** (see Fig. 294), and **ABD** is an angle such that **C** and **D** are on opposite sides of **AB**, then the segment **ACB** is said to be **alternate** to ∠**ABD**.

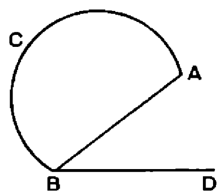


FIG. 294

THEOREM 37a

If a straight line touches a circle and, from the point of contact, a chord is drawn, the angles between the tangent and the chord are equal to the angles in the alternate segments.

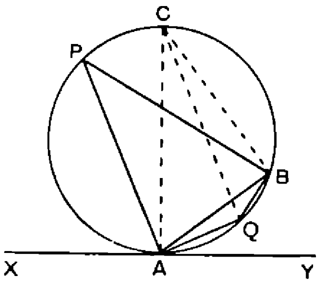


FIG. 205

Given a chord AB of a circle and a tangent XAY such that ∠BAY is acute and ∠BAX is obtuse ; ∠BPA is an angle in the segment alternate to ∠BAY ; ∠BQA is an angle in the segment alternate to ∠BAX.

To prove that (i) ∠BAY=∠BPA

and (ii) ∠BAX=∠BQA.

Construction. Draw the diameter AC. Join CB, CQ.

Proof. (i) ∠ABC=90° (∠ in semicircle),

and ∠BAC+∠BCA+∠ABC=180° (∠s of △) ;

∴ ∠BAC+∠BCA=90°.

But ∠CAY=90° (tan. perp. radius) ;

∴ ∠BAC+∠BAY=90°.

∴ ∠BAY=∠BCA.

But ∠BCA=∠BPA (same seg.) ;

∴ ∠BAY=∠BPA.

(ii) $\angle CQA = 90°$ ($\angle$ in semicircle),

and $\angle CAX = 90°$ (tan. perp. radius);

$\therefore \angle CAX = \angle CQA.$

But $\angle CAB = \angle CQB$ (same seg.);

$\therefore \angle CAX + \angle CAB = \angle CQA + \angle CQB.$

$\therefore \angle BAX = \angle BQA.$

Reference. $\begin{cases} \angle BAY = \angle BPA \ (\textit{alt. seg.}), \\ \angle BAX = \angle BQA \ (\textit{alt. seg.}). \end{cases}$

THEOREM 37b (Test for a Tangent)

(For proof, see Appendix, p. 340)

If, through one end of a chord of a circle, a straight line is drawn making with the chord an angle equal to the angle in the alternate segment, then the straight line is a tangent to the circle.

This converse is useful in riders in which a straight line has to be proved to be a tangent to a certain circle; thus, if it can be shown in Fig. 296 that $\angle BAT = \angle BCA$, then AT will be a tangent to the circle ABC.

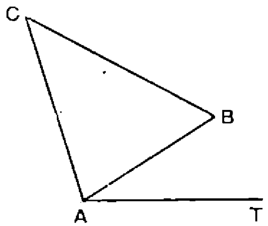


FIG. 296

CONSTRUCTION 23

On a given straight line to construct a segment of a circle to contain a given angle.

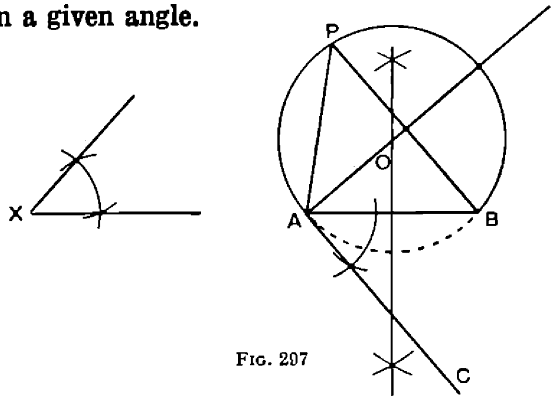


FIG. 297

Given a straight line AB and an angle.

To construct a segment of a circle on AB to contain the given angle.

Construction. Construct at A an angle BAC equal to the given angle.

Draw the perpendicular to AC at A.

Draw the perpendicular bisector of AB.

Let these straight lines cut at O.

With centre O, radius OA, draw a circle.

Then the segment of this circle which is alternate to $\angle$BAC is the required segment.

Proof. Let APB be any angle in this segment.

$\because$ O lies on the perpendicular bisector of AB (constr.),

$\therefore$ OA$=$OB;

$\therefore$ the circle passes through B.

$\because$ OA is a radius and $\angle$OAC$=90°$ (constr.),

$\therefore$ AC is a tangent to the circle (tan. perp. radius);

$\therefore$ $\angle$BAC$=$$\angle$APB (alt. seg.).

$\therefore$ $\angle$APB is equal to the given angle.

CONSTRUCTION 24

To inscribe in a circle a triangle equiangular to a given triangle.

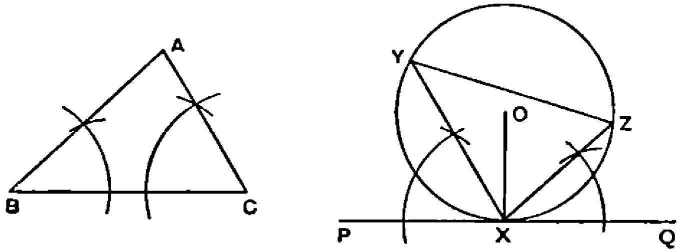


FIG. 298

Given a circle, centre O, and a △ABC.

To construct a triangle, equiangular to △ABC, with its vertices lying on the circle.

Construction. Draw any radius OX.

Through X draw the straight line PXQ perpendicular to OX.

Draw chords XY, XZ of the circle, such that ∠PXY=∠C and ∠QXZ=∠B.

Join YZ. Then XYZ is the required triangle.

Proof. ∵ PXQ is perpendicular to the radius OX,

PXQ is a tangent (tan. perp. radius);

∴ ∠PXY=∠XZY (alt. seg.).

But ∠PXY=∠C (constr.);

∴ ∠XZY=∠C.

Similarly, ∠XYZ=∠B.

∵ ∠XYZ=∠B and ∠XZY=∠C (proved),

∴ ∠ZXY=∠A (∠s of △).

∴ △XYZ is equiangular to △ABC.

CONSTRUCTION 25

To circumscribe about a given circle a triangle equiangular to a given triangle.

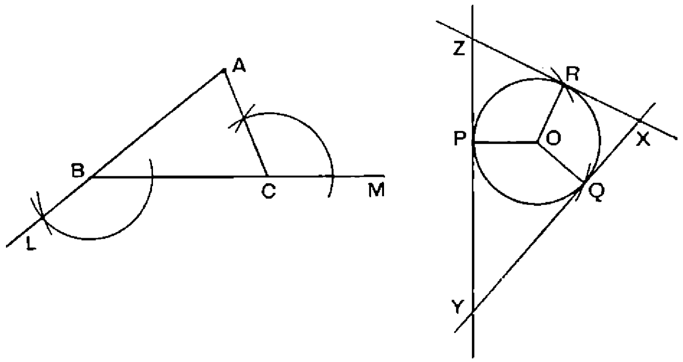


FIG. 299

Given a circle, centre O, and a △ABC.

To construct a triangle, equiangular to △ABC, to circumscribe the circle.

Construction. Produce AB to L, and BC to M.

Draw any radius OP.

Draw radii OQ, OR such that ∠POQ=∠CBL and ∠POR =∠ACM, with ∠s POQ, POR on opposite sides of OP.

Construct the perpendiculars at P, Q, R to OP, OQ, OR, to form the △XYZ.

Then XYZ is the required triangle.

Proof. YPZ is perpendicular to the radius OP (constr.),

∴ YPZ is a tangent (tan. perp. radius).

Similarly, ZRX and XQY are tangents.

Now $\angle OPY + \angle OQY + \angle POQ + \angle Y = 360°$ ($\angle$s of quad.);

$\therefore\ \angle POQ + \angle Y = 180°$.

$\therefore\ \angle Y = 180° - \angle POQ$

$= 180° - \angle CBL$ (constr.)

$= \angle ABC$ (adj., ABL a st. line).

Similarly, $\angle Z = \angle BCA$.

$\because\ \angle Y = \angle ABC$ and $\angle Z = \angle BCA$ (proved),

$\therefore\ \angle X = \angle CAB$ ($\angle$s of $\triangle$).

$\therefore\ \triangle XYZ$ is equiangular to $\triangle ABC$ and its sides are tangents to the circle.

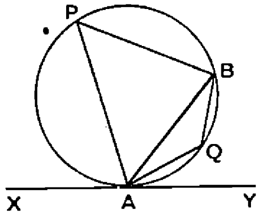
EXAMPLES 20a

(*Theorems* 37*a and b*)

1. In Fig. 300, if $\angle BAY = 32°$, calculate $\angle BPA$ and $\angle BQA$.

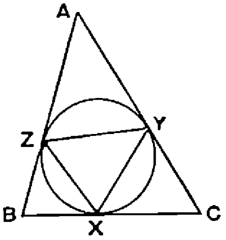
2. In Fig. 300, if $\angle QAX = 160°$, calculate $\angle ABQ$.

3. In Fig. 300, if $\angle BPA = 40°$, and $\angle PAX = 55°$, calculate $\angle BAY$, $\angle PAB$, and $\angle PBA$.



XAY is the tangent at A

FIG. 300



The inscribed circle of $\triangle ABC$ touches the sides at X, Y, Z

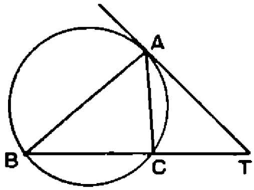
FIG. 301

4. In Fig. 301, if $\angle XYZ = 52°$ and $\angle YZX = 70°$, calculate all the angles in the figure.

5. In Fig. 301, if $\angle ZAY = 26°$, calculate $\angle ZXY$.

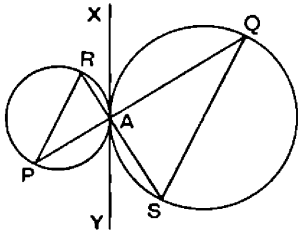
6. In Fig. 302, if ∠**BAT**=100° and ∠**CAT**=40°, calculate ∠**ATC** and ∠**ACB**.

7. In Fig. 302, if ∠**BAT**=120° and ∠**ATC**=25°, calculate ∠**ACB** and ∠**CAT**.



TA is the tangent at **A**

FIG. 302



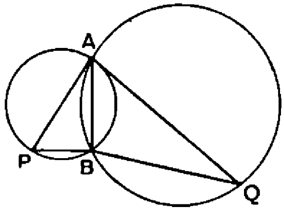
XAY is the common tangent at **A**; **PR** is a diameter

FIG. 303

8. In Fig. 303, if ∠**PAY**=62°, calculate ∠**PRA**, ∠**RPA**, ∠**QSA**, ∠**AQS**.

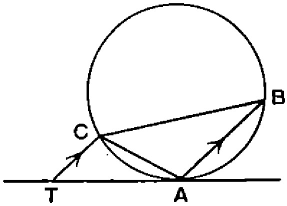
9. In Fig. 300, if ∠**BAY**=∠**PAX**, prove that **AB**=**AP**.

10. In Fig. 304, prove that ∠**ABP**=∠**ABQ**.



AP, AQ are the tangents at **A**

FIG. 304

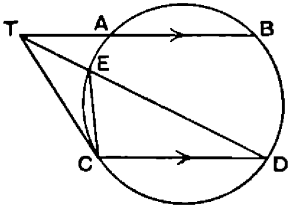


TA is the tangent at **A**

FIG. 305

11. In Fig. 305, prove that ∠**BCA**=∠**CTA**.

12. In Fig. 306, prove that ∠**TCE**=∠**ETA**.

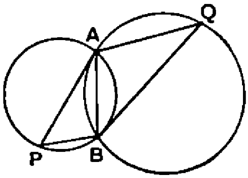


TC is the tangent at **C**

FIG. 306

13. In Fig. 302, prove that ∠**TCA** = ∠**TAB**.

14. In Fig. 307, prove that **AQ** is parallel to **PB**.



AP, BQ are tangents at **A, B**

Fig. 307

15. The bisector of ∠**A** of △**ABC** meets **BC** at **D**. A circle is drawn touching **BC** at **D** and passing through **A**; it cuts **AB** at **X** and **AC** at **Y**. Prove that ∠**XDB** = ∠**YDC**.

16. In Fig. 302, if the bisector of ∠**BAC** meets **BC** at **D**, prove that **TA** = **TD**.

17. **P** is the mid-point of an arc **AB** of a circle. Prove that **AP** bisects the angle between the chord **AB** and the tangent at **A** to the circle.

18. △**ABC** has a right angle at **A**, and **D** is any point in **BC**. The tangent **DX** is drawn to the circle through **A, D, B** and the tangent **DY** to the circle through **A, D, C**. Prove that ∠**XDY** = 90°.

19. The internal bisector of ∠**A** of △**ABC** meets the perpendicular bisector of the side **AB** at **P**. Prove that **AC** is a tangent to the circle **APB**.

20. **D** is a point in the side **BC** of △**ABC** such that ∠**ADC** = ∠**BAC**. Prove that **CA** is the tangent at **A** to the circle **ABD**.

21. In Fig. 308, prove that ∠**BDX** = ∠**DCX**. To what circle is **DB** a tangent ?

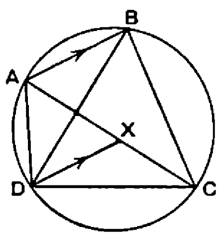


Fig. 308

22. In △**ABC**, **AB** = **AC**. **D** is the mid-point of **BC**, and **DX** is drawn perpendicular to **AB** to meet **AB** at **X**. Prove that ∠**ADX** = ∠**ACD**. To what circle is **DX** a tangent ?

23. **ABCDE** is a pentagon with its vertices lying on the circumference of a circle, and **BC** = **CD**. If **BD** cuts **EC** at **X**, prove that **BC** is a tangent to the circle **BEX**.

24. In Fig. 309, prove that CB is a tangent to the circle CXY.

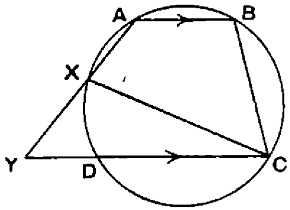
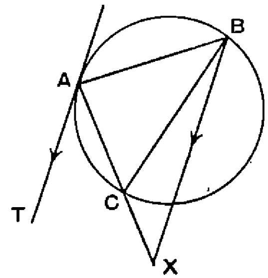


FIG. 309



AT is the tangent at A

FIG. 310

25. In Fig. 310, prove that AB touches the circle BCX.

(*Constructions* 23 *to* 25)

26. On a straight line 5 cm. long, construct a segment of a circle to contain an angle of 65°. Measure the radius.

27. On a straight line 6 cm. long, construct a segment of a circle containing an angle of 135°. Measure the radius.

28. In a circle of radius 2 in., construct a triangle with angles 75°, 58°, 47°. Measure the longest side.

29. Draw a circle of radius 1 in., and circumscribe about it an equilateral triangle.

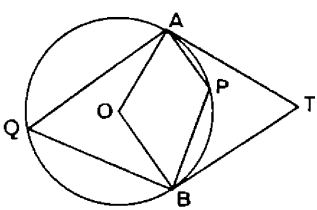
30. Draw a circle of radius 3 cm., and circumscribe about it a triangle with angles 63°, 49°, and 68°. Measure the longest side.

EXAMPLES 20b

1. In Fig. 300, if ∠QAY=25° and ∠BPQ=30°, calculate ∠BQA.

2. In Fig. 311, if ∠PAT=20° and ∠PBT=35°, calculate ∠AQB, ∠APB, ∠AOB, ∠ATB.

3. In Fig. 311, if ∠PAT=15° and ∠ATB=46°, calculate ∠AOB, ∠AQB, ∠PBT.



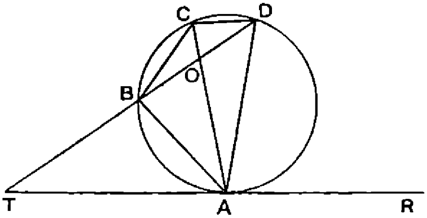
TA, TB are tangents from T O is the centre of the circle

FIG. 311

4. In Fig. 312, if ∠**BTA**$=30°$, ∠**ADB**$=42°$, calculate ∠**DBA**.

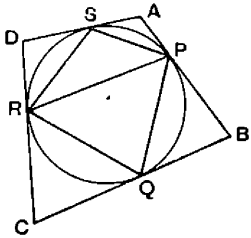
5. In Fig. 312, if ∠**COD**$=77°$ and ∠**CBD**$=31°$, calculate ∠**BAT**.

6. In Fig. 312, if ∠**BTA**$=20°$ and ∠**BAT**$=28°$, calculate ∠**DAR**.



TAR is the tangent at **A**

Fig. 312



A circle touches the sides of **ABCD** at **P, Q, R, S**

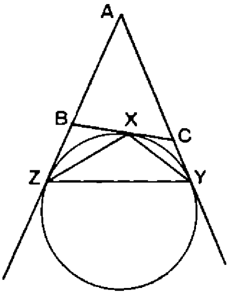
Fig. 313

7. In Fig. 313, if ∠**A**$=126°$ and ∠**B**$=74°$, calculate ∠**QRS**.

8. In Fig. 314, if ∠**ABC**$=38°$, calculate ∠**XYZ**.

9. In Fig. 314, if ∠**XYZ**$=20°$ and ∠**YZX**$=41°$, calculate the angles of △**ABC**.

10. Two circles touch externally at **A**. Straight lines **PAQ**, **RAS** are drawn through **A** cutting the circles at **P**, **Q** and **R**, **S** (**P**, **R** being on one circle, and **Q**, **S** on the other). Prove that **RP** is parallel to **QS**. (Hint: draw the common tangent at **A**.)



The escribed circle touches **BC** at **X**, **AC** produced at **Y**, **AB** produced at **Z**

Fig. 314

11. Two equal circles **APB**, **AQB** intersect at **A**, **B**. **AP** is a tangent to the circle **AQB**. Prove that **AB**$=$**PB**. (Hint: draw any angle subtended by **AB** in the circle **AQB**.)

12. In Fig. 312, if ∠**BTA**$=x°$, and ∠**BAT**$=y°$, express in terms of x and y the numbers of degrees in ∠**BCA** and ∠**DAR**, and find the relation that exists between x and y if **BD** is a diameter.

13. **ABCDE** is a pentagon inscribed in a circle; **AX** is the tangent at **A** to the circle. If ∠**BAX**$=25°$, calculate ∠**BCD**$+$∠**AED**. (Join **BE**.)

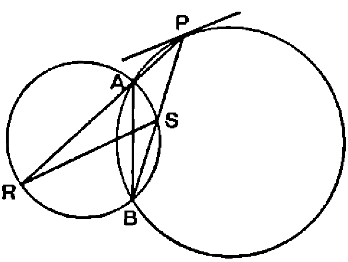
14. In a triangle ABC, AB = AC. A circle, drawn to touch BC at B and to pass through A, cuts AC at D. Prove that BC = BD.

15. AB, AC are two equal chords of a circle, both shorter than a radius. BA is produced to D, and the tangent at C to the circle meets AD at E, E being between A and D. Prove that ∠CED = 3∠ACE.

16. Two circles touch internally at A. A straight line PQRS cuts one circle at P, S and the other at Q, R. Prove that ∠PAQ = ∠RAS. (Hint: draw the common tangent at A, and use the fact that ∠PAQ = ∠AQR − ∠APQ.)

17. Two unequal circles intersect at A and B. A common tangent touches one circle at P and the other at Q. Prove that ∠PAQ + ∠PBQ = 180°.

18. In Fig. 315, prove that RS is parallel to the tangent at P.



P is any point on one circle; PA, PB cut the other circle at R, S

FIG. 315

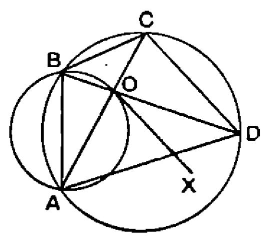


FIG. 316

19. Copy Fig. 306, p. 236, omitting the lines TD and EC, and prove that △s TCB, BDC are equiangular.

20. In Fig. 316, ABCD is a cyclic quadrilateral whose diagonals intersect at O. OX is the tangent at O to the circle AOB. Prove that OX is parallel to CD.

21. A diameter AB of a circle is produced to any point T, and C is the point of contact of a tangent from T to the circle. The perpendicular through T to AT meets AC produced at D. Prove that TC = TD.

22. ABCD is a cyclic quadrilateral in which AB=AD. If its diagonals cut at O, prove that AD touches the circle COD.

23. A straight line parallel to the side BC of △ABC cuts AB at P and AC at Q. Prove that the circumcircles of △s ABC, APQ touch at A. (Hint: draw the tangent AX at A to the circle ABC, and show that AX is a tangent to the circle APQ.)

24. O is the centre of a circle, and AB, BC are equal chords. Prove that BA is a tangent to the circle OBC.

25. ABCD is a quadrilateral in which ∠ABD=∠CBD. If AD is a tangent to the circle through B, C, D, prove that DC is a tangent to the circle through A, B, D.

26. AB, AC are chords of a circle. The bisector of ∠BAC meets CB at X and the circle again at D. Prove that BD touches the circle AXB.

27. OA, OB are two perpendicular radii of a circle. P is any point on OB, and AP produced cuts the circle at Q. Prove that AB is a tangent to the circle PBQ.

28. The diagonals of a parallelogram ABCD meet at O. Prove that the circles AOB and COD touch at O.

29. Two equal circles ABC, ABD intersect at A, B, and the chords AB, AC are equal. Prove that BC is a tangent to the circle ABD.

30. △ABC has ∠A a right angle, and AB is greater than AC. The perpendicular bisector of BC meets BA at P and CA produced at Q. Prove that BC touches the circle CPQ.

---

31. Construct △ABC given that BC=2·4 in., the median AD=1·9 in., and ∠BAC=44°. Measure the side AC.

32. Construct △ABC, given that BC=7 cm., the altitude AD=4·6 cm., and ∠BAC=63°. Measure the other two sides.

33. Construct a quadrilateral ABCD in which AC=8·3 cm., BC=4·4 cm., CD=8·9 cm., DA=10·2 cm., and ∠ABC=119°. Measure BD.

34. In Fig. 317, A, B are two landmarks on the coast, and S is a ship. The chart of the coast states that, if ∠ASB is greater than 100°, there is danger from hidden rocks. Draw an accurate figure, making AB equal to 2 in., and shade the portion of the sea which is unsafe.

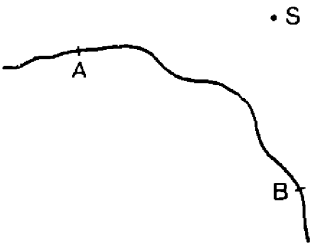


FIG. 317

EXAMPLES 20c

1. A, B are two points on a circle, and the tangents at A, B meet at P, making ∠APB=54°. AC is the chord of the circle drawn through A parallel to PB. Calculate each of the angles of △ABC. (N)

2. ABC is a triangle inscribed in a circle and the tangents at B and C meet in P. (i) If ∠BAC=∠BPC, show that ∠BAC=60°. (ii) If ∠ABC=∠BPC, show that △ABC is isosceles. (OC)

3. PQ is a line of length $2\frac{1}{2}$ in. It is required to find a point distant 3 in. from P, and at which PQ shall subtend an angle of 40°. Show that there are two such points, and find them. (W)

4. In a given circle, radius 1 in., inscribe a triangle ABC whose angles are 50°, 60°, 70°, explaining your method, and drawing an accurate figure.

If tangents to the circle are drawn at A, B, C, find the angles of the triangle formed by the tangents. (L)

5. The internal bisector of the angle A of a triangle ABC cuts the side BC in D, and the circumscribed circle of the triangle in E. Prove that EC is a tangent to the circumscribed circle of △ADC. (L)

---

6. If a circle is described passing through the vertex A of a triangle ABC, and touching the base BC at C, and if the chord AD of this circle is parallel to BC, prove that △ACD is isosceles. (L)

7. Explain, without proof, how to construct a triangle **ABC** in which the side **BC** = 5 cm., **A** = 60°, and whose area is 7·5 sq. cm. Draw an accurate figure. (L)

8. The inscribed circle of the triangle **ABC** touches the sides **BC**, **CA**, **AB** at **P**, **Q**, **R** respectively. If ∠**A** = 40° and ∠**B** = 110°, calculate the number of degrees in each of the angles of △**PQR**. (N)

9. (Th. 37*b*.) Describe a method of drawing a tangent at a given point of an arc of a circle whose centre lies beyond the edge of the paper. (N)

10. **A**, **B**, **C** are three points such that **AB** is 5 miles, **BC** is 10 miles, and **CA** is 7 miles. **P** is a point in the plane of **ABC** such that ∠**BPA** = 30°, ∠**CPA** = 30°, and ∠**BPC** = 60°. Find, by accurate drawing and measurement, the distances of **P** from **A**, **B**, and **C**. (OC)

---

11. **ABC** is a triangle inscribed in a circle, **AD** is drawn perpendicular to **BC** and **CE** perpendicular to the tangent at **A**; show that **DE** is parallel to **BA**. (OC)

12. **ABC** is a triangle inscribed in a circle, and the tangents to the circle at **B** and **C** meet at **D**. The straight line through **D**, parallel to the tangent to the circle at **A**, meets **AB** produced at **E** and **AC** produced at **F**. Prove that ∠**DBE** = ∠**ACB**, and that **DE** = **DB** = **DC** = **DF**. (It may be assumed that **DB** = **DC**.) (O)

13. **ABCD** is a quadrilateral inscribed in a circle. **DE** is drawn parallel to **CB** to meet **AB** in **E**. Prove that **CD** is a tangent to that circle which passes through **A**, **D**, and **E**. (L)

14. Draw a triangle **ABC** with **AB** = 2 in., **BC** = $2\frac{1}{2}$ in., **CA** = 3 in. Below this, on the same page, draw a circle of radius 2 in. Construct a triangle, with its vertices on the circumference of this circle, which will be equiangular to △**ABC**. (State your construction without proof.) (W)

15. **ABCD** is a cyclic quadrilateral; **DA** produced meets **CB** produced at **E**. Prove that the tangent at **E** to the circle **CDE** is parallel to **AB**.

16. Draw a straight line 2 in. long, and on this line as chord and on the same side of it draw two segments of circles, one containing an angle of 57° and the other containing an angle of 110°. State clearly the steps of your construction. (L)

---

17. Draw a circle of radius 3 cm.; construct a triangle **ABC** having $\angle$**A**=46°, $\angle$**B**=64°, $\angle$**C**=70°, and such that the circle is the inscribed circle of triangle **ABC**. Describe, without proof, your construction.

If the sides **BC**, **CA**, **AB** of this triangle touch the circle at **P**, **Q**, **R** respectively, calculate the angles of $\triangle$**PQR**. (L)

18. **A**, **B**, **C**, **D** are four points in order on the circumference of a circle, **AD** being a diameter. If **O** is the centre of the circle and **BD** cuts **AC** at **X**, prove that **OB** is a tangent to the circle **BCX**.

19. From an external point **P** tangents **PQ**, **PR** are drawn to a circle of which **RS** is a diameter. **SQ** produced meets **RP** produced in **T**. Prove that $\angle$**QPR**=2$\angle$**QTR**. (L)

20. Draw the triangle **ABC** having $\angle$**ABC**=130°, **AB**=1·8 in., and **BC**=2·6 in. On the side **BC**, and on the same side of it as **A**, construct a segment of a circle containing an angle of 50°. Find also the position of a point **P** within the angle **ABC**, such that the area of the triangle **PAB** equals 2·7 sq. in. and $\angle$**BPC**=50°. (L)

21. Two unequal circles cut at **A** and **B**. The tangents at **A** to the circles cut them again at **P**, **Q**. **PQ**, produced if necessary, cuts the circle **AQB** at **S**, and the circle **APB** at **T**. Prove that **AS**=**AT**. (L)

22. Draw a line **AB** 4 in. long. Construct the locus of the point, on one side of **AB**, at which **AB** subtends an angle of 50°. In the same figure, construct a triangle **ABC** in which **AB**=4 in., $\angle$**ACB**=50°, and the bisector of $\angle$**ACB** passes through **D**, a point on **AB** $1\frac{1}{2}$ in. from **B**. (W)

CONSTRUCTION OF CIRCLES

The method of constructing a circle satisfying certain conditions is illustrated in the following worked example.

Worked Example.

AB *is a chord of a circle of radius* 2·4 *cm., and the length of the perpendicular from the centre of the circle to* AB *is* 1 *cm. If* AB *is produced in both directions, construct as many circles as possible of radius* 0·6 *cm. to touch both* AB *and the circle.*

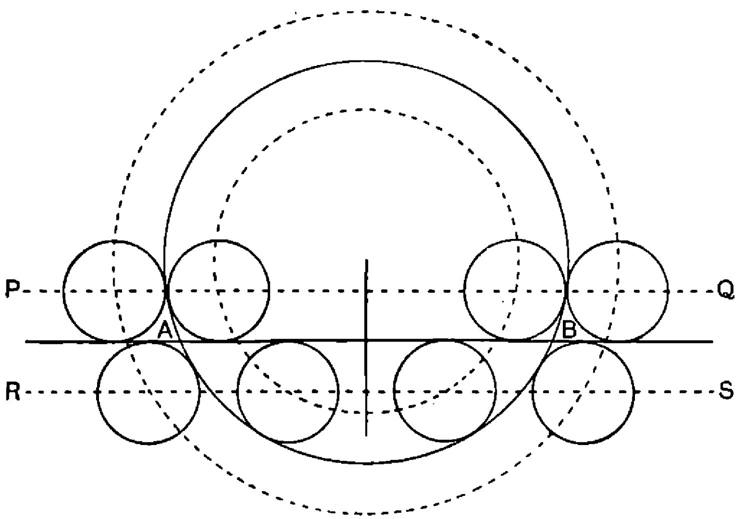


FIG. 318

The required circle of radius 0·6 cm. has to satisfy the following conditions: (i) it is to touch AB, (ii) it is to touch the given circle.

The locus of the centres of circles of radius 0·6 cm. which satisfy (i) is the pair of straight lines, PQ and RS, parallel to AB and on each side of AB, at a distance of 0·6 cm. from it.

The locus of the centres of circles of radius 0·6 cm. which satisfy (ii) is a concentric circle, of radius 3 cm. if the contact is external, and of radius 1·8 cm. if the contact is internal.

In the figure these loci are shown by dotted lines. It will be seen that there are eight possible positions for the centre of the required circle.

EXAMPLES 21a

Each of questions 1 to 7 should be illustrated with a freehand sketch showing several circles satisfying the required conditions.

What is the locus of the centres of circles—

1. Passing through two given points A and B ?

2. Touching two given intersecting straight lines AOB, COD ?

3. Touching a given straight line at a given point ?

4. Touching a given circle at a given point ?

5. Of given radius, passing through a given point ?

6. Of given radius, touching a given straight line ?

7. Of radius 1 in., touching (*a*) externally, (*b*) internally, a given circle of radius 3 in. ?

---

8. Mark any three points A, B, C (not in a straight line) and construct a circle to pass through them.

9. Mark two points A and B $1\frac{1}{2}$ in. apart, and construct a circle of radius 1 in. to pass through them.

10. Draw two straight lines AOB, COD intersecting at an angle of 50°, and construct a circle of radius 1 in. to touch AB, CD. (Hint: what is the locus of the centres of circles which touch AB and CD ? What is the locus of the centres of circles of radius 1 in. which touch AB ?) How many solutions are possible ?

11. Draw a straight line AB and mark a point P in AB and another point Q outside AB such that $\angle QPB = 40°$, $PQ = 1$ in. Construct a circle which touches AB at P and also passes through Q. Measure its radius. (Hint: what is the locus of the centres of circles which touch AB at P? What is the locus of the centres of circles which pass through P and Q?)

12. Draw any straight line AB and mark a point P distant 1 in. from AB. Construct a circle of radius 2 in. to pass through P and touch AB. (Hint: what is the locus of the centres of circles of radius 2 in. which pass through P? What is the locus of the centres of circles which touch AB?) There are two solutions.

13. With centre O, draw a circle of radius 3 cm., and mark any point P on its circumference. Make $\angle POQ$ equal to 25° and OQ 6 cm. long. Construct a circle to touch the given circle at P and also to pass through Q. Measure its radius. (Hint: what is the locus of the centres of circles which touch the given circle at P? What is the locus of the centres of circles which pass through P and Q?)

14. Draw two straight lines AOB, COD, such that $\angle AOC = 80°$. Mark a point P in OA such that $OP = 1$ in., and construct a circle to touch AB at P and also to touch CD. Measure its radius. (Hint: what is the locus of the centres of circles which touch AB at P? What is the locus of the centres of circles which touch the two lines AB and CD?) There are two solutions.

15. Draw a circle of radius 3 cm. and mark a point P 5 cm. from the centre. Construct two circles of radius 4 cm. to touch the first circle externally and to pass through P. (Hint: what is the locus of the centres of circles of radius 4 cm. which touch the given circle externally? What is the locus of the centres of circles of radius 4 cm. which pass through P?)

16. Draw a circle of radius 4 cm. and mark a point P 3 cm. from the centre. Construct two circles of radius $2\frac{1}{2}$ cm. to touch the first circle internally and to pass through P.

17. Draw a circle of radius 3 cm. and mark a point A on its circumference. Construct a circle of radius 2 cm. to touch the first circle at A, the contact being (*a*) external, (*b*) internal.

18. Draw a circle of radius 3 cm., centre O. Mark a point A, 4 cm. from O, and draw a straight line XAY through A perpendicular to OA. Construct a circle of radius 2 cm. to touch XY and to touch the first circle externally. (Two solutions.)

EXAMPLES 21b

1. Draw a circle of radius 4 cm. and a straight line 2 cm. from its centre. Construct a circle of radius $1\frac{1}{2}$ cm. to touch the line and to touch the circle internally. (Two solutions.)

2. Draw two circles of radii 1 in. with their centres 3 in. apart. Construct a circle of radius $1\frac{1}{2}$ in. to touch each of the two equal circles externally. How many solutions are possible?

3. Draw two circles of radii $\frac{1}{2}$ in. with their centres $1\frac{1}{2}$ in. apart. Construct two circles of radius $1\frac{1}{4}$ in. to touch one of the two equal circles externally and the other internally.

4. Draw two parallel straight lines 4 cm. apart, and a transversal cutting them at an angle of 54°. Construct the two circles which touch all the three lines.

5. Draw an arc of a circle of radius 5 cm. and two radii **OA, OB** making an angle of 50° with each other. Construct a circle to touch **OA**, **OB**, and the arc **AB** internally. Measure its radius. (Hint: if **P** is the mid-point of the arc **AB**, the required circle must touch the tangent at **P** to the given arc.)

6. On a diameter **AB** draw a semicircle of radius 6·0 cm. Construct a circle of radius 2·0 cm. to touch **AB** and to touch the semicircle internally. (Two solutions.)

7. Draw a triangle with sides 3 in., 2·5 in., 2 in., and construct a circle which cuts off a length 1·2 in. from each side. Measure its radius. (Remember that equal chords are equidistant from the centre.)

8. Draw a triangle **ABC** in which **AB**=2·4 in., **BC**=1·8 in., **CA**=2 in., and describe three equal circles of radius 1 in. with centres **A**, **B**, **C**. Construct a circle to touch these three circles and enclose them all. Measure its radius. (Hint: begin by drawing the circumcircle of △**ABC**.)

9. Draw a circle of radius 6 cm., and construct two circles of radii 3 cm. and 1·8 cm. touching each other externally and touching the first circle internally.

EXAMPLES 21c

1. State (without proof) the locus of the centre of a circle which touches at **P** a given circle whose centre is **O**.

Draw a circle, centre **O**, of radius 2·4 cm. Mark a point **P** on its circumference. Construct two circles of radius 3·8 cm. to touch at **P** the circle you have drawn. (L)

2. State, without proof, (i) the locus of the centre of a circle of radius 3·5 cm. which passes through a fixed point **A**, (ii) the locus of the centre of a circle of radius 3·5 cm. which touches a fixed circle, centre **B**, of radius 2·3 cm.

Draw a straight line **AB** 3·4 cm. long. With **B** as centre draw a circle of radius 2·3 cm. Construct a circle of radius 3·5 cm. to touch this circle and pass through **A**. Describe (without proof) your construction. (L)

3. Draw two lines **OA**, **OB** intersecting at an angle of 60°, and construct a circle of radius 3 cm. to touch **OA** and **OB**. (L)

4. Construct a circle, centre **C**, of radius $1\frac{1}{2}$ in. and mark a point **A** on the circumference. Find a point **P** $3\frac{1}{2}$ in. from **C** and 2·3 in. from **A**. Construct a circle to pass through **P** and to touch the first circle at **A**. (W)

5. Draw two circles each of radius 2·5 cm. having their centres 6 cm. apart. Construct a circle of radius 7 cm. which shall touch one of these circles internally and the other externally. (N)

6. Draw two lines **OA**, **OB** containing an angle of 50°, and make **OA**=2·4 in., **OB**=1·25 in. Describe a circle touching **OA** at **A** and passing through **B**, explaining your construction. (W)

7. Complete the following statements :—

(i) The locus of the centre of a circle of given radius which passes through a fixed point is . . .

(ii) The locus of the centre of a circle of given radius which touches a fixed straight line is . . .

(iii) The locus of the centre of a circle of given radius which touches a fixed circle is . . .

Draw a circle of radius 2 in. and a straight line distant 1 in. from the centre of the circle. Then construct all the circles of radius $\frac{1}{2}$ in. which touch both the straight line and the circle, explaining carefully your method. (OC)

8. Describe a circle of $2\frac{1}{2}$ in. radius. Inside this describe two circles of radii 1 in. and 0·6 in., touching each other externally and the larger circle internally. (OC)

9. Describe a circle of 1 in. radius which shall touch both the circumference and the diameter of a semicircle of 3 in. radius. State your construction. (OC)

10. Draw a circle of radius 3·7 cm., and a straight line touching it at T. Construct as many circles as possible which have radius 2·5 cm. and touch the first circle and the straight line at a point other than T. Describe your method.

11. Draw a sector of a circle of radius 3 in., in which the arc of the sector subtends an angle of 60° at the centre. In this sector draw a circle touching the arc and both radii, and prove that your construction is correct. Measure the radius of this circle. (OC)

12. The angle ABC is 55° and BC is 7 cm. long. A circle of radius 3 cm. is described with C as centre, to cut BC at X. Construct a circle to touch AB and to touch the circle at X. State very briefly your construction and prove that it is correct. (N)

LOCUS PROOFS

Represent the fixed straight lines and circles of the given figure by continuous lines, and the variable ones by dotted lines. Denote fixed points by letters at the beginning of the alphabet, and variable points by later letters.

Next find out, by plotting points if necessary, what the locus is. Is it straight or circular? Has the fixed part of the figure an axis of symmetry? If so, the locus will be symmetrical about the same axis.

It is difficult to lay down any guidance as to method of proof, but the following observations will sometimes help. The locus of a point P can be proved to be a straight line by showing (a) that P is at a fixed distance from a fixed straight line, or (b) that the straight line joining P to a fixed point is in a fixed direction (*e.g.* parallel to a fixed line). To prove the locus a circle, it would suffice to show that (*a*) the distance of P from a fixed point is constant, or (*b*) the angle subtended at P by a fixed straight line is constant.

Worked Example.

P *is a variable point on a fixed circle and* C *a fixed point outside the circle. If* CP *is produced to* Q *so that* PQ=CP, *find the locus of* Q *and prove the result.*

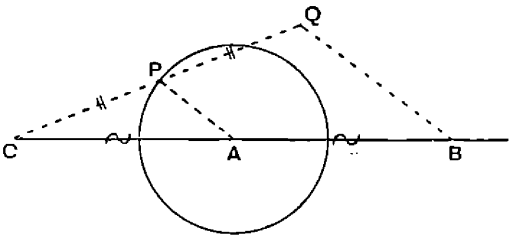


FIG. 319

[As Q is twice as far from C as P is, and is in the same direction, it seems probable that Q's locus is the same shape as P's and twice as big. If A is the centre of the given circle, the fixed part of the figure is symmetrical about CA, so the centre of Q's locus must lie on CA, perhaps twice as far from C as A is.]

Construction. Produce CA to B so that AB=CA.

[B is a fixed point. If the locus of Q is a circle with B as centre, BQ must be of constant length.]

Join BQ, AP.

Proof. CA=AB and CP=PQ;

∴ BQ=2AP (mid-point theorem).

But AP is of constant length;

∴ BQ is of constant length.

Also B is a fixed point;

∴ Q lies on the circle whose centre is B, and radius twice that of the given circle.

EXAMPLES 22a

1. State the locus of the following :—

(*a*) The centres of circles which pass through two fixed points **A** and **B** ;

(*b*) The centres of circles of radius 1 in. which pass through a fixed point **A** ;

(*c*) The points at which a fixed straight line **AB** subtends a constant angle.

2. A variable triangle **PAB** has a fixed base **AB** and a constant area. State the locus of **P**.

3. A variable triangle **PQR** has a right angle at **P**, and **PQ**, **PR** pass through fixed points **A**, **B** respectively. State, with a reason, the locus of **P**.

4. **A** is a fixed point on a fixed circle. **AQ** is a variable chord, and **P** is the mid-point of **AQ**. State the locus of **P**, and give a proof. (Hint: join **A** and **P** to the centre **O** of the circle, and consider ∠**OPA**.)

5. **A**, **B** are fixed points, and **PAQB** is a variable parallelogram of given area. State the locus of **P**, and give a proof.

6. State the locus of the centres of circles which touch each of two fixed concentric circles of radii 3 cm. and 5 cm. respectively, the contact with the smaller circle being (i) external, (ii) internal.

7. **A**, **B** are fixed points. **P** is a variable point on the perpendicular bisector of **AB**, and **AP** is produced to **Q** so that **AP**=**PQ**. State the locus of **Q**, and give a proof. (Join **QB**.)

8. In Fig. 320, **AOB** is a right angle. **P**, **Q** are variable points on **OA**, **OB** respectively such that **PQ** is of constant length. Complete the rectangle **OPRQ**, and prove that (i) the locus of **R** is a circle (hint: join **OR** and use a property of the diagonals of a rectangle), (ii) the locus of the mid-point of **PQ** is a circle.

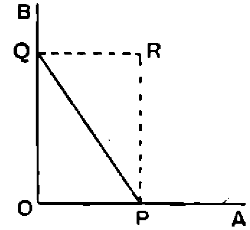


FIG. 320

9. A variable chord **QR** of a given circle passes through a fixed point **A**. State the locus of the mid-point **P** of **QR**, and give a proof. (Hint: join **P** to the centre **O** of the circle, and consider $\angle$**OPA**.)

10. In Fig. 321, **AB**, **BC**, **CD** are three rigid rods, hinged together at **B** and **C**, and hinged to fixed points at **A** and **D**. **AB**=**DC** and **AD**=**BC**. If the framework moves in its plane, what is the locus of the mid-point of **BC**? Give a proof. (Hint: let **P**, **O** be the mid-points of **BC**, **AD** respectively. Consider the quadrilateral **ABPO**. What conclusion can you draw about **OP**?)

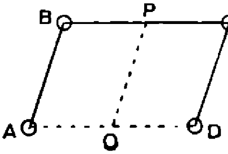


FIG. 321

11. **PAB** is a variable triangle in which **AB** is fixed in length and position, and **PA** is fixed in length. Prove that the locus of the mid-point of **PB** is a circle whose centre is the mid-point of **AB**.

12. If **A** and **B** are fixed points and **P** a variable point such that $AP^2+PB^2=AB^2$, state the locus of **P**.

EXAMPLES 22b

1. **B** is a fixed point, and **BP** is a variable straight line through **B**. **N** is the foot of the perpendicular from another fixed point **A** to **BP**. **AN** is produced to **Q** so that **NQ**=**AN**. State the locus of **Q** and give a proof.

2. **P** is a variable point on the circumference of a fixed circle, and **A** is a fixed point outside the circle. State the locus of the mid-point of **AP** as **P** moves round the circle, and give a proof. (Hint: let **O** be the centre of the circle; **M**, **Q** the mid-points of **OA**, **AP** respectively. Consider **MQ**.)

3. **A**, **B**, **C** are fixed points, and **ABCP** is a quadrilateral such that **BP** is bisected by **AC**. State the locus of **P**, and give a proof. (Hint: produce **BA** to **D** so that **BA**=**AD**, and join **PD**.)

4. **ABC** is a fixed triangle, and **P** is a variable point on the circumference of a fixed circle whose centre is **A**. The parallelograms **BAPQ**, **CBQR** are completed. State the locus of **R**, and give a proof.

5. A and B are fixed points, and ABPQ is a variable parallelogram with AP of given length. State the locus of Q, and give a proof. (Hint: produce BA to C so that AC=BA, and consider CQ.)

6. C is a fixed point outside a fixed straight line AB, and a point P moves along AB. State the locus of the mid-point of CP, and give a proof.

7. The base of a triangle is 14 cm. long, and the sum of the squares on the other two sides is 226 sq. cm. Calculate the length of the median from the vertex. If the base is fixed, what is the locus of the vertex?

8. A, B are fixed points, and P is a variable point such that AP^2+BP^2 is constant. State the locus of P, and give a proof. (See No. 7.)

9. A, B are fixed points, and P is a variable point such that PA^2-PB^2 is constant. Prove that the locus of P is a straight line perpendicular to AB. (Hint: if N is the foot of the perpendicular from P to AB, AN+NB is constant. Prove by Pythagoras' theorem that AN^2-NB^2 is also constant, and deduce that N is a fixed point.)

10. PQR is a variable triangle in which the respective midpoints A, B of PQ, PR are fixed, and QR passes through a fixed point C. Prove that the side QR lies along a fixed straight line, and by producing CA its own length prove that the locus of P is a straight line parallel to AB.

11. A, B are fixed points. P is a variable point on AB, and APQ is a triangle such that AP+PQ is constant and ∠APQ is constant. Prove that the locus of Q is a straight line. (Hint: take the fixed point C in the straight line AB such that AC=AP+PQ, and prove that ∠QCA is constant.)

12. One end A of a rod AP is hinged to the ground. State the locus in space of the other end P.

If P is made to move in contact with a wall, state the locus of P.

MISCELLANEOUS EXAMPLES III

Riders

1. ABCDE is a regular pentagon. Find the number of degrees in each angle of the triangles BAC and DAC.

2. (Th. 29.) The tangent at A to the circumcircle of the cyclic quadrilateral ABCD bisects the exterior angles of the quadrilateral at A. If $\angle ABC = 100°$, and $\angle BCD = 60°$, find the other two interior angles of the quadrilateral. Prove that the arcs AB and AD are each half the arc DAB, and that the tangent at A is parallel to the diagonal BD. (N)

3. (Th. 28.) A circle is touched internally at P by another circle whose diameter is half that of the first circle. PR is any chord of the larger circle, and PR cuts the smaller circle again at Q. Prove PQ=QR. (OC)

4. If the centres of the circumscribed and inscribed circles of a triangle coincide, prove that the triangle must be equilateral.

5. Two unequal circles touch externally at a point A. From a point B on the common tangent at A two tangents BC and BD are drawn, one to each circle. Prove that a circle can be drawn which passes through C and D and touches the two given circles respectively at these points. (N)

---

6. ABCDE is a pentagon inscribed in a circle. If CD=DE, $\angle BDC = 20°$, $\angle CAD = 28°$, and $\angle ABD = 70°$, find by calculation the angles of the pentagon. (N)

7. (Th. 37*a*.) Two circles cut at A and B and through A and B are drawn two straight lines PAQ and RBQ, cutting one circle in P and R and the other circle in Q. Show that PR is parallel to the tangent at Q. (OC)

8. Find the length of either tangent drawn to a circle of r in. radius from a point a in. from its centre.

An ordinary convex quadrilateral ABCD is circumscribed to a circle. Prove that AB+CD=AD+BC.

Assuming the converse, namely that, if AB+CD=AD+BC, a circle can be inscribed in the quadrilateral, prove that a circle can be inscribed in the quadrilateral whose vertices are the centres of four circles drawn so that each touches two of the others externally. (OC)

9. The bisectors of the angles B and C of a triangle ABC meet in I, and ID, IE, IF are drawn perpendicular to BC, CA, AB respectively. Show that ID=IE=IF.

A circle is drawn with centre I so as to cut each side of the triangle; show that the chords cut off from each side are equal. (OC)

10. XAY is the tangent at A to a circle of which AB is a chord. CD is a chord of the circle parallel to XAY, cutting AB at E, between A and B. Prove that (i) AB bisects ∠CBD, and (ii) AC is a tangent to the circle through B, C, E. (L)

---

11. (Th. 37*a*.) ABCD is a quadrilateral with its vertices on the circumference of a circle, centre S, and PAQ is the tangent at A. ∠ABC=110°, ∠QAD=40°, and ∠BDC=44°. Calculate the sizes of ∠CBD, ∠ADB, and the reflex angle BSD.

If the tangent at C meets QP produced at R, what is the size of ∠ARC? (W)

12. ABCD is a parallelogram in which ∠B=45°. X is the centre of the circle ABC, and Y that of the circle ACD. Show that XY=AC. (OC)

13. (Th. 37*a*.) ABC is a right-angled triangle with the right angle at B. A circle on AB as diameter cuts AC in D, and the tangent at D to the circle cuts BC in E. Prove that the angles EDC, ABD, DCE are all equal and that E bisects BC. (OC)

14. E is a point outside a circle and a chord through E meets the circle at D and A, D being between E and A, and ED being smaller than DA. AB is a chord of the circle equal in length to DE, B being on the major arc AD. The bisector of ∠DAB meets the circle again at C, and C is joined to B, D, E. Prove CE=CA. (L)

15. (Th. 36.) BC is a common external tangent to two circles which touch one another externally at A. Show that BA, CA, produced, pass through the other ends of the diameters through C, B. (OC)

16. ABC is a triangle inscribed in a circle, the angles at B and C being 40° and 60° respectively. D is the middle point of the arc subtended by BC on the same side as A. Show that $\angle$DCA$=10°$. (OC)

17. ABC is a triangle inscribed in a circle. The internal and external bisectors of $\angle$A meet the circumference again in D, E respectively. Prove that DE is a diameter, and that it is perpendicular to BC. (W)

18. Two circles with centres A and B touch one another externally at P, and a common tangent touches them at Q and R respectively. Show that AQ is a tangent to the circle through P, Q, and R. (OC)

19. (Th. 29.) Two circles intersect each other at A and B and through A is drawn a straight line intersecting the circles at P and Q respectively; the tangents at P and Q meet in C. Show that C, P, B, and Q lie upon a circle. (OC)

20. (Th. 27.) ABC is a triangle in which AB=AC. P is a point on the circle circumscribing ABC, P and B being on opposite sides of AC. CP is produced to Q so that CQ=BP. Prove that AP=AQ. (O)

---

21. Two unequal circles, whose centres are P and Q, cut at X and Y, and any straight line through Y meets the circles again at A and B respectively. Prove that $\angle$APX$=\angle$BQX.

22. A triangle ABC is inscribed in a circle and through a point Z in AB a line is drawn perpendicular to AB and cutting the circle in P and Q, so that P lies on the side of BC remote from A. From P the line PX is drawn perpendicular to BC cutting BC at X. Prove that CQ is parallel to ZX. (C)

23. (Th. 37*a.*) PTO is a tangent to a circle at T. TR is a chord through T, and PR is joined to cut the circle again at Q. $\angle$RTO$=100°$ and $\angle$RPT$=50°$. Prove that (i) PQ=QT, (ii) QR is equal to the radius of the circle. (OC)

24. ABC is an acute-angled triangle. The circle described on AB as diameter cuts the sides BC, CA in D, E respectively; AD, BE meet in P. Show that ∠APB = ∠ABC + ∠BAC. (W)

---

25. A chord of a given circle is 24 cm. long and 5 cm. from the centre. Calculate to the nearest mm. the length of a chord of the same circle which is 9 cm. from the centre. (N)

26. I is the centre of the inscribed circle of a triangle ABC, and the bisector of ∠BAC meets BC in H and the circle circumscribing the triangle ABC in X. Prove that BX = IX and that BX touches the circle ABH. (L)

27. (Th. 27.) If P is the point of intersection of the perpendiculars drawn from the vertices of an acute-angled triangle ABC to the opposite sides, prove that ∠BPC = ∠ABC + ∠ACB.

If the circle drawn through the points BCP passes through the centre of the circumscribed circle of the acute-angled triangle ABC, prove that ∠A = 60°. (O)

28. P is a point on an arc of a circle whose chord is AB. AP is produced to Q, so that PQ = PB. Prove that (i) ∠AQB is of the same size for all positions of P, (ii) the locus of Q is an arc of a circle.

The perpendicular bisector of AB meets the original arc in O; prove that O is the centre of the circle on which Q lies. (OC)

29. ABCD is a cyclic quadrilateral, and a circle which passes through C and touches AB at B, cuts BD at O. Prove that the triangles OCD, BCA are equiangular. (L)

---

30. PQ, PR are two chords of a circle of 2 in. radius, the angle QPR being 45°. Find by calculation the distance of the chord QR from the centre. (OC)

31. (Th. 31.) ABC is a triangle inscribed in a circle and a line through A perpendicular to BC meets BC in D. A line through D parallel to BA meets the tangent at A to the circle in E. Prove that ADCE is a cyclic quadrilateral and that CE is perpendicular to AE. (OC)

32. (Th. 32.) AB is a diameter of a circle, and AC, AD are any two chords which when produced cut the tangent at B at E, F respectively. Prove that E, C, D, F lie on a circle. (L)

33. In the triangle ABC, D is the foot of the perpendicular from A to BC and E is the foot of the perpendicular from B to CA. If ∠ADE = 30° and ∠BED = 20°, calculate the angles of △ABC. (C)

34. TP, TQ are the tangents from an external point T, touching a circle at P and Q respectively. From a point M on TQ produced, a line MS is drawn parallel to PT to cut PQ produced in S. Prove ∠MSQ = ∠MQS.

If the second tangent from M touches the circle at R, and if PR produced cuts SM produced in X, prove SM = MX. (W)

---

35. ABCD is a quadrilateral inscribed in a circle, and the diagonals AC, BD meet at O. ∠BAD = 95°, ∠ABC = 70°, ∠BOC = 85°. Find ∠BAC and prove that AB = BC. (OC)

36. (Th. 32.) Five points A, B, C, D, E are taken in order on a circle so that the chords AB, AE are equal. If AC, AD meet BE at X, Y respectively, prove that C, X, Y, D are concyclic points. (N)

37. ABC is any triangle, and its inscribed circle touches BC, CA, AB at D, E, F respectively. K is the centre of the escribed circle opposite to the angle A. Prove that KB, KC are respectively parallel to DF, DE, and that the triangles KBC, DEF are equiangular. (L)

38. Two circles ABP and ABQ whose centres are H and K meet in A and B; a line PAQ is drawn through A cutting the circles in P and Q, so that P and Q are on opposite sides of A. Prove that △s BPQ and AHK have equal angles. (C)

39. (Th. 37*a*.) ABCD is a cyclic quadrilateral in which BC = CD = DA, and T is a point on the tangent at A on the side of AB opposite to D. Prove that the sum of three times ∠BAD and twice ∠BAT is equal to four right angles. (C)

---

40. TA, TB are tangents to a circle whose centre is C. Prove that (i) CT passes through the middle point of AB, (ii) CT is perpendicular to AB.

The circle with centre T and radius TA cuts CT in P, P being between C and T. Prove that AP bisects the angle CAB. (L)

41. **ABC** is a triangle inscribed in a circle and the bisectors of the angles **A**, **B**, **C** meet the circle in **X**, **Y**, **Z** respectively. If the opposite angles of the figure **AZBXCY** are equal, prove that the triangle **ABC** is equilateral. (OC)

42. (Th. 32.) **D**, **E**, **F** are respectively the feet of the perpendiculars from the vertices **A**, **B**, **C** of an acute-angled triangle on the opposite sides, and **O** is the point of intersection of the perpendiculars. Prove (*a*) that ∠**FDO** is the complement of ∠**BAC**; (*b*) that if **S** is the centre of the circumscribed circle of △**AFE**, then **D**, **F**, **S**, **E** lie on a circle. (N)

43. Two circles cut in **A**, **B**; any line through **B** cuts the circles again in **P**, **Q**, and the tangents at **P**, **Q** meet in **T**. Prove (i) that **APTQ** is a cyclic quadrilateral; (ii) that the triangles **APB**, **ATQ** are equiangular. (L)

44. Two unequal circles whose centres are **A** and **B** touch at **C**, and **P**, **Q** are their respective points of contact with a direct common tangent. **M** is the other end of the diameter through **Q**. Prove that **M**, **C**, **P** are in one straight line. (N)

---

45. (Th. 26.) The tangents at the points **A**, **B** of a circle, whose centre is **O**, meet at **P**; the straight line **PCD** cuts the circle in **C** and **D**, and **E** is the mid-point of **CD**. Prove that (i) ∠**AEP**=∠**AOP**; (ii) if **AE** meets the circle again in **G**, **BG** is parallel to **CD**. (C)

46. Tangents **TP**, **TQ** are drawn from **T** to touch a given circle at **P** and **Q**, and **PTQ** is an acute angle. The diameter **PR** and the line **TQ** are produced to meet at **S**. Prove that ∠**RQS**=$\frac{1}{2}$∠**PTQ**. (N)

47. Prove that the internal bisector of an angle of a triangle, and the perpendicular bisector of the opposite side, meet at a point outside the triangle.

(*Constructions*)

48. Construct the triangle **ABC** of which the sides **BC**, **CA**, **AB** respectively are 3·3, 1·5, and 2·4 in. long. Describe the circle passing through the points **A**, **B**, **C**. Measure the lengths of the diameter of this circle and of the perpendicular from **A** to **BC**, and verify that the rectangle contained by them is equal to the rectangle contained by **AB** and **AC**. (OC)

49. (Th. 28.) In a quadrilateral ABCD the angles at A and C are right angles. AB = 6 cm., AD = 8 cm., BC = 2·8 cm. Construct the quadrilateral. Measure CD. (OC)

50. Draw a triangle of sides 6 cm., 8 cm., and 10 cm.; and construct a point inside the triangle at which each of the sides subtends 120°. Measure in cm. its distance from the point of intersection of the shorter sides of the triangle. (OC)

51. (Th. 29.) Show how to divide a circle into two segments such that the angle in one segment is double the angle in the other. (OC)

52. (Th. 28.) Construct a cyclic quadrilateral in which two sides of lengths 2·4 and 1·8 in. respectively contain a right angle and the other two sides are equal. The steps in your construction *must* be stated, but no proof is required. (L)

---

53. Draw a triangle ABC on a fixed base BC, $6\frac{1}{2}$ cm. long, having an area of $9\frac{3}{4}$ sq. cm. Draw in and label the locus of A.

On this base BC, draw another triangle DBC which also has an area of $9\frac{3}{4}$ sq. cm. and which is right-angled at D. (State your construction briefly without proof.) (W)

54. Draw a circle, centre O, radius 2 in., and mark a point P such that OP = 1·5 in. Construct a chord of the circle of length 3·2 in. which passes through P. (W)

55. (Construction 23.) AB is a line 2 in. long. Find by a ruler-and-compass construction a point C at which AB subtends an angle of 45°, and such that the bisector of this angle divides AB in the ratio 1 : 2. (W)

56. Construct a triangle ABC in which AB = 6·2 cm., AC = 7·0 cm., and BC = 5·0 cm. On the same side of AB construct a triangle ADB such that $\triangle ADB = \triangle ACB$ and $\angle ADB = \angle ACB$. State your method briefly.

57. Construct one of the direct common tangents of two circles whose centres are 8·8 cm. apart and whose radii are 2·0 cm. and 3·0 cm. Describe your method very briefly.

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58. On a line **AB** of length 10 cm. draw a triangle **CAB** of area 20 sq. cm. having $\angle$**CAB**=90°.

Without making any calculations, but using any instruments you please, construct on **AB** a triangle **ADB** of equal area, having $\angle$**ADB**=90°, stating very briefly how the position of **D** is found. (N)

59. On a base of 2 in. describe a segment of a circle the arc of which is one-third of the circumference of the circle. What is the angle in this segment ? What is the radius of the arc ? (OC)

60. The inscribed circle of a triangle **ABC** touches **BC** at **X**; $\angle$**B**=72°, $\angle$**C**=44°, and **BX**=3·7 cm. Construct the figure and measure **CA**.

61. (Th. 25*a*.) Construct *full-size* a triangle of which the sides are of lengths 3 in., 3·5 in., and 3·9 in. respectively, and draw a circle which will intercept a chord of length 1·2 in. on each side. Measure the radius of the circle so drawn.

Note.—The steps of the construction must be clearly stated, but no proof is required. (L)

62. The radii of the sections of a sphere by two parallel planes on opposite sides of the centre and 3·2 in. apart, are 2·6 in. and 3 in. respectively. Obtain by drawing the radius of the section of the sphere by a plane parallel to and between the former planes at a distance 2·9 in. from the plane whose cross-section is of radius 3 in. (O)

---

63. Draw a circle on a diameter **AE** 4 in. long. Construct in this circle a quadrilateral **ABCD** which fulfils the following conditions: $\angle$**EAB**=40°, $\angle$**BCD**=80°, **CD** cuts **AE** in a point **F** such that **AF**=**AD**. State briefly your method. (O)

64. Draw a circle of radius 3 in. and construct two *parallel* chords of the circle, **AB** and **CD**, 3·5 and 5 in. long respectively, on opposite sides of the centre. Measure **AC** and **AD**, and state carefully the steps of your construction. (OC)

RATIO

If AB and CD are two straight lines, the statement AB : CD =**5 : 3** tells us that their lengths are in the ratio 5 : 3. We can therefore picture the lines as in Fig. 322; for, if AB be divided into 5 equal parts and CD into 3 equal parts, *all* the parts will be equal.

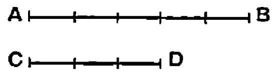


FIG. 322

Examples.

1. A, B, C are three points in order on a straight line, and AB : BC=4 : 7. Draw a freehand figure, and hence write down the values of (i) AB : AC, (ii) AC : BC.

2. XYZ is a triangle such that YZ : ZX : XY=4 : 5 : 6. Indicate this on a freehand figure. If ZX=4·8 cm., what are the lengths of XY and YZ ?

3. P, Q, R, S are four points in order on a straight line. If PQ : PR : PS=3 : 7 : 16, state the values of (i) PQ : QS, (ii) PQ : QR : RS, (iii) PR : QS.

4. If AB=3·6 cm. and CD=6 cm., express AB : CD in its simplest form.

Internal and external division of a straight line

Suppose we are given two points, A and B. If we are told that there is a point C, in the same straight line as A and B, such that AC : CB=5 : 3, there will be two possible positions for C, one between A and B, the other not between A and B. (See Fig. 323.)

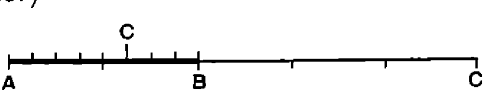


FIG. 323

We distinguish between these two possibilities in the following way. If C lies between A and B, we say that C **divides AB internally** in the ratio 5 : 3 ; if C does not lie between A and B, we say that **C divides AB externally** in the ratio 5 : 3. These facts are summarised in Fig. 324, (i) and (ii).

(i)

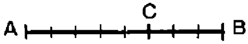


FIG. 324

Here C divides AB *internally* in the ratio 5 : 3.
For C lies *between* A and B,
and AC=5 parts, CB=3 parts.

(ii)

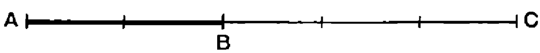


FIG. 324

Here C divides AB *externally* in the ratio 5 : 3.
For C is *not between* A and B,
yet AC=5 parts, CB=3 parts.

Notice that, in both cases, (*a*) the pieces into which C divides AB are AC and CB, (*b*) C is nearer to B than to A, since AC consists of 5 parts and CB of only 3 parts.

Examples. (Invent more of the same types if necessary.)

1. In Fig. 324 (ii), in what ratio does B divide AC, and is the division internal or external ?

2. In Fig. 324 (i), in what ratio does B divide AC, and is the division internal or external ?

3. Mark two points A and B in ink. Indicate in pencil the point C which divides AB externally in the ratio 3 : 7, showing the 3 parts and 7 parts clearly.

4. AB is a straight line 12 cm. long. If X divides AB internally in the ratio 5 : 7, and Y divides AB externally in the ratio 5 : 2 calculate the length of XY.

In order to divide a given straight line AB internally or externally in a given numerical ratio, begin by making a freehand sketch to show the number of parts into which AB has to be divided. If the ratio is 5 : 3 and the division internal (see Fig. 324 (i)), AB has to be divided into 5+3 (=8) equal parts, and this is done by means of Construction 13, p. 86, the work being shown in Fig. 325 (i) below. If the division is external, AB has to be divided into 5−3 (=2) equal parts, and the work is as shown in Fig. 325 (ii) below.

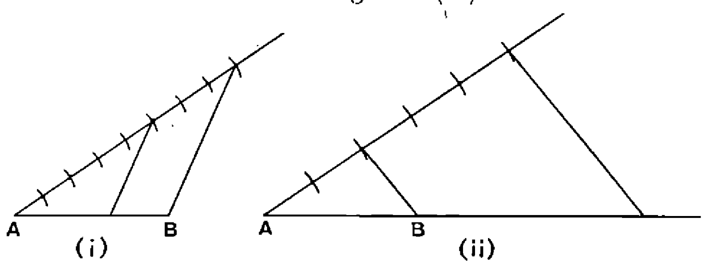


FIG. 325

A discussion of the proof would be a useful preliminary to Theorem 38.

Examples.

1. Draw a straight line AB, 3 in. long, and divide AB internally in the ratio 2 : 3.

2. Draw a straight line AB 1·5 in. long, and divide AB externally in the ratio 4 : 7.

THEOREM 38a

If a straight line is drawn parallel to one side of a triangle, it will divide the other two sides in the same ratio.

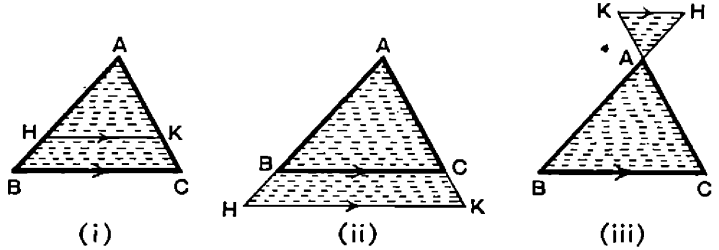


FIG. 326

Given a triangle ABC, and a straight line parallel to BC cutting AB at H, AC at K (produced in either direction if necessary).

To prove that AH : HB=AK : KC.

Proof. Let AH : HB=$p : q$,

where p and q are whole numbers.

Divide AH into p equal parts, and HB into q equal parts ; then all these parts will be equal.

Through the points of division draw parallels to BC.

These parallels divide AK, KC into p, q equal parts, and all these parts are equal (intercept theorem) ;

$\therefore$ AK : KC=$p : q$.

$\therefore$ AH : HB=AK : KC.

Corollary. AH : AB=AK : AC

and AB : HB=AC : KC.

Reference. *In* $\triangle ABC$, $HK \parallel BC$;

$\therefore$ AH : HB=AK : KC.

Note.—It is not always possible to express AH : HB in whole numbers. For instance, if AHLM is a square of side x in. and HB = HM (Fig. 327),

$$HB^2 = HM^2 = AH^2 + AM^2 = 2x^2; \quad \therefore HB = \sqrt{2}x.$$

$$\therefore AH : HB = 1 : \sqrt{2},$$

a ratio which cannot be expressed by means of whole numbers.

The general proof of this and subsequent theorems is beyond the scope of elementary geometry.

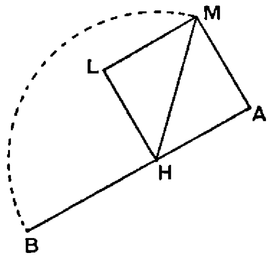


FIG. 327

THEOREM 38b (Ratio Test for Parallels)

(For proof, see Appendix, p. 343)

If a straight line divides two sides of a triangle in the same ratio, it will be parallel to the third side.

Thus, in Fig. 326, if AH : HB = AK : KC, then HK || BC.

Corollary. If AH : AB = AK : AC, then HK || BC.
If AB : HB = AC : KC, then HK || BC.

Reference. *In* $\triangle ABC$, $AH : HB = AK : KC$,
$\therefore$ HK || BC.

Ratios expressed by means of fractions

If $AB : CD = 5 : 3$ (see Fig. 322, p. 263),

$$\frac{AB}{CD} = \frac{5}{3}.$$

Similarly, if $AH : HB = AK : KC$ (see Fig. 326, p. 266),

$$\frac{AH}{HB} = \frac{AK}{KC}.$$

Beginners often prefer to work with the fractional form. For instance, if, in Fig. 326 (i), $AH = 8{\cdot}4$ in., $AC = 15{\cdot}6$ in., and $KC = 6{\cdot}5$ in., AB can be calculated as follows :—

Let $AB = x$ in.

In $\triangle ABC$, $HK \parallel BC$;

$$\therefore \frac{AH}{AB} = \frac{AK}{AC}.$$

$$\therefore \frac{8{\cdot}4}{x} = \frac{15{\cdot}6 - 6{\cdot}5}{15{\cdot}6};$$

$$\therefore \frac{8{\cdot}4}{x} = \frac{9{\cdot}1}{15{\cdot}6};$$

$$\therefore 9{\cdot}1x = 8{\cdot}4 \times 15{\cdot}6;$$

$$\therefore x = \frac{8{\cdot}4 \times 15{\cdot}6}{9{\cdot}1}$$

$$= 14{\cdot}4.$$

$$\therefore AB = 14{\cdot}4 \text{ in.}$$

An equation in fraction-form can be expressed in rectangle-form by cross-multiplying, *i.e.* by multiplying both sides by the L.C.M. of the denominators.

Thus $$\frac{AH}{HB} = \frac{AK}{KC}$$

leads to $$AH.KC = AK.HB.$$

It should also be noticed that an equation in fraction-form can be expressed in two ways. For instance,

$$\text{if } \frac{AH}{HB}=\frac{AK}{KC},$$

$$\text{then } \frac{AH}{AK}=\frac{HB}{KC};$$

for each equation, when cross-multiplied, gives

$$AH.KC=AK.HB.$$

Examples.

Express the following in rectangle-form :—

1. $\frac{AB}{AC}=\frac{CD}{CB}$.

2. $\frac{PQ}{PR}=\frac{PS}{PQ}$.

3. HK : KL = HM : ML.

Express the following in fraction-form :—

4. AB.CD = XY.YZ.

5. $AB^2=AX.XB$.

Complete the following :—

6. If AB : BC = YZ : XY, then AB : YZ =

7. If PQ : PR = AC : BC, then PQ :   = PR :

CONSTRUCTION 26a

To divide a given straight line internally in a ratio equal to that of two given straight lines.

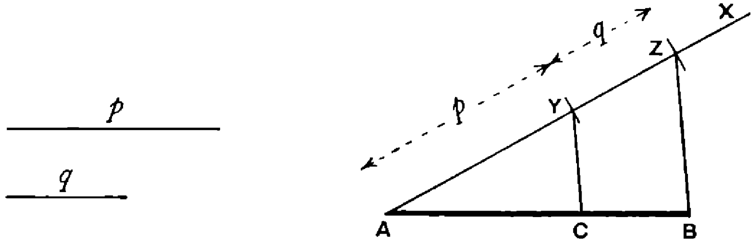


FIG. 328

Given a straight line AB, and two straight lines p and q.

Required to divide AB internally in the ratio $p : q$.

Construction. Draw through A a straight line AX.

Along AX, mark off AY equal to p, and, in the *same* direction, mark off YZ equal to q.

Join ZB.

Draw the parallel to ZB through Y to meet AB at C.

Then C is the required point.

Proof. In $\triangle$AZB, YC $\parallel$ ZB (constr.);

$\therefore$ AC : CB = AY : YZ

$= p : q$ (by constr.).

Note.—In order to carry out the construction accurately, it may be necessary to double p and q, or halve them, or make some other alteration in their lengths which does not change their ratio.

CONSTRUCTION 26b

To divide a given straight line externally in a ratio equal to that of two given straight lines.

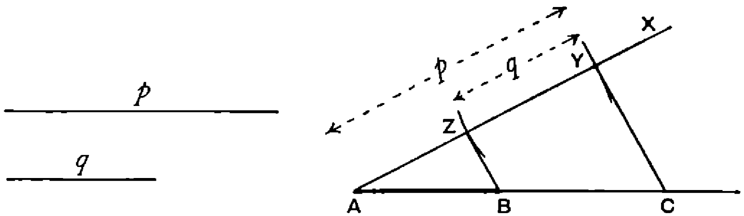


FIG. 329

Given a straight line AB, and two straight lines p and q.

Required to divide AB externally in the ratio $p : q$.

Construction. If p is *greater*[1] than q, draw through A a straight line AX.

Along AX, mark off AY equal to p, and, in the *opposite* direction, mark off YZ equal to q.

Join ZB.

Draw the parallel to ZB through Y to meet AB produced at C.

Then C is the required point.

Proof. In $\triangle$AZB, YC $\parallel$ ZB (constr.);

$\therefore$ AC : CB = AY : YZ

$= p : q$ (by constr.).

[1] If p is *less* than q, start by drawing through B a straight line BX.

Fourth proportional.

If $a : b = c : d$, then a, b, c, d are called **proportionals**, and d is said to be the **fourth proportional** to a, b, and c.

CONSTRUCTION 27

To construct a fourth proportional to three given lengths.

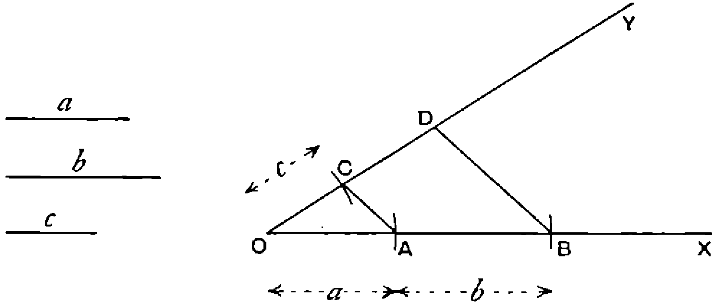


FIG. 330

Given three lengths, a, b, c.

Required to construct a fourth proportional to a, b, c.

Construction. Draw a straight line OX.
Along it mark off OA equal to a, AB equal to b.
Through O draw another straight line OY.
Along it mark off OC equal to c.
Join AC.
Through B draw a parallel to AC to meet OY at D.
Then CD is the required length.

Proof. In $\triangle$OBD, AC $\parallel$ BD (constr.);

$\therefore$ OA : AB = OC : CD;

$\therefore$ $a : b = c$: CD.

$\therefore$ CD is the fourth proportional to a, b, c.

THEOREM 39a

The bisector (internal or external) of the vertical angle of a triangle divides the base (internally or externally) in the ratio of the other two sides.

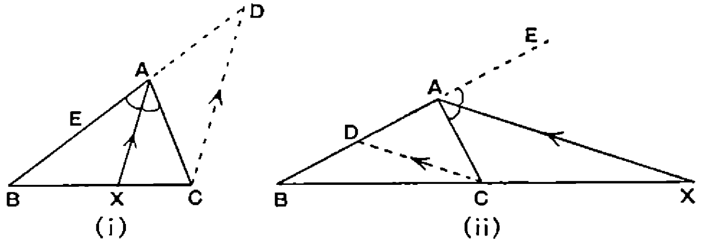


FIG. 331

Given a triangle ABC, and a bisector of ∠A, internal in Fig. 331 (i), external in Fig. 331 (ii), meeting BC (produced if necessary) at X.

To prove that BX : XC = BA : AC.

Construction. Through C draw a parallel to XA to meet BA (produced if necessary) at D.

In Fig. 331 (i) let E be a point on BA; in Fig. 331 (ii) let E be a point on BA produced.

Proof. ∠EAX = ∠ADC (corr., AX ∥ DC),

and ∠XAC = ∠ACD (alt., AX ∥ DC).

But ∠EAX = ∠XAC (given);

∴ ∠ADC = ∠ACD;

∴ AC = AD.

But, in △BCD, XA ∥ CD;

∴ BX : XC = BA : AD.

∴ BX : XC = BA : AC.

Reference. *AX bisects ∠BAC;*

∴ BX : XC = BA : AC.

THEOREM 39b

If a straight line through the vertex of a triangle divides the base (internally or externally) in the ratio of the other two sides, it will bisect the vertical angle (internally or externally).

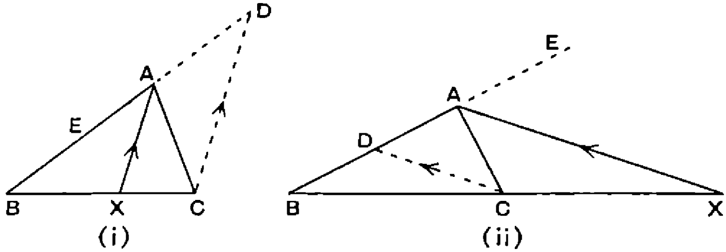


FIG. 332

Given a triangle ABC, and a straight line through A which divides BC, internally in Fig. 332 (i), externally in Fig. 332 (ii), so that BX : XC=BA : AC.

To prove that AX bisects ∠BAC, internally or externally.

Construction. Through C draw a parallel to XA to meet BA (produced if necessary) at D.

In Fig. 332 (i) let E be a point on BA; in Fig. 332 (ii) let E be a point on BA produced.

Proof. In △BCD, XA ∥ CD (constr.);

∴ BX : XC=BA : AD.

But BX : XC=BA : AC (given);

∴ AC=AD;

∴ ∠ADC=∠ACD.

But ∠ADC=∠EAX (corr., AX ∥ DC),

and ∠ACD=∠XAC (alt., AX ∥ DC);

∴ ∠EAX=∠XAC.

Reference. *BX* : *XC*=*BA* : *AC*;

∴ AX bisects ∠BAC.

EXAMPLES 23a

(*Theorem* 38)

1. In Fig. 326 (i), if AB=3 in., AC=2·5 in., AH=1·8 in., calculate AK.

2. In Fig. 326 (i), if AB=3·6 in., AC=2·4 in., AK=1·4 in., calculate HB.

3. In Fig. 326 (i), if AH=1·8 in., HB=2·4 in., AK=1·5 in., calculate KC.

4. In Fig. 326 (ii), if AB=4·2 cm., AH=6·3 cm., CK=2·4 cm., calculate AK.

5. In Fig. 326 (iii), if AB=3·2 in., AC=4 in., AK=3 in., calculate AH.

6. In Fig. 326 (iii), if AB=4·5 in., AC=2·7 in., BH=7 in., calculate CK.

7. In Fig. 326, prove that AH.AC=AK.AB.

8. A quadrilateral ABCD has two opposite sides AB, DC parallel, and the diagonals intersect at O. Explain why

$$\frac{AO}{AC}=\frac{BO}{BD}.$$

9. In Fig. 333, prove that $\frac{AX}{XD}=\frac{BY}{YC}$.

(Hint: let AC cut XY at Z.)

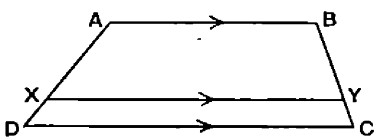


FIG. 333

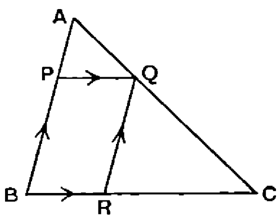


FIG. 334

10. In Fig. 334, prove that $\frac{AP}{PB}=\frac{BR}{RC}$.

11. P is any point in the side AB of a triangle ABC, and PQ, drawn parallel to BC, meets AC at Q. A straight line is drawn through C, parallel to QB, to meet AB produced at S. Prove that

$$\frac{AP}{AB}=\frac{AB}{AS}.$$

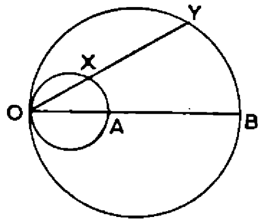


FIG. 335

12. Fig. 335 shows two circles touching internally at O. OXY, OAB are straight lines, the latter passing through the centres. Prove that OX : OY = OA : OB.

13. What does Theorem 38*b* become in the special case when

$$\frac{AH}{AB}=\frac{AK}{AC}=\frac{1}{2}?$$

14. O is a point inside a quadrilateral ABCD. P, Q, R, S are points on OA, OB, OC, OD respectively such that

$$\frac{OP}{OA}=\frac{OQ}{OB}=\frac{OR}{OC}=\frac{OS}{OD}.$$

Prove that $\angle$SPQ = $\angle$DAB, and state three similar results.

15. In Fig. 336, if $\frac{AP}{PX}=\frac{DQ}{QX}$, prove that RS is parallel to BC.

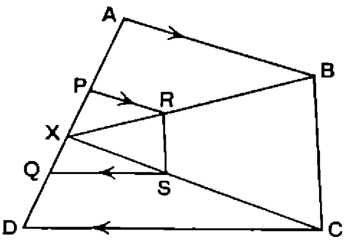


FIG. 336

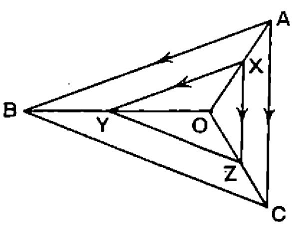


FIG. 337

16. In Fig. 337, prove that YZ is parallel to BC.

17. In Fig. 338, prove that PQ is parallel to AD.

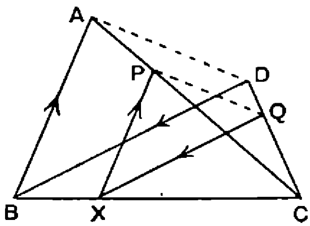


FIG. 338

18. X is any point on the diagonal BD of a quadrilateral ABCD. XY is drawn parallel to DA to meet AB at Y, and XZ is drawn parallel to DC to meet BC at Z. Prove that YZ is parallel to AC.

19. P is any point on the side AB of a quadrilateral ABCD. PQ, parallel to AC, meets BC at Q, and PR, parallel to AD, meets BD at R. Prove that QR is parallel to CD.

(*Constructions* 26, 27)

20. Draw a straight line AB 3 cm. long, and find by construction the point P which divides AB internally, and the point Q which divides AB externally, in the ratio 3 : 2. Measure AP and AQ.

21. Draw a straight line AB $1\frac{3}{4}$ in. long, and find by construction the point X dividing AB internally, and the point Y dividing AB externally, in the ratio 5 : 3. Measure AX and AY.

22. Construct and measure a fourth proportional to lines of lengths 3 cm., 2 cm., 3·6 cm.

23. Construct and measure a fourth proportional to lines of lengths 1·8 in., 2·6 in., 2·6 in.

24. Draw any angle AOB and mark a point P inside the angle. Construct a straight line XPY through P, meeting OA at X and OB at Y, such that XP : PY=3 : 2.

(*Theorem* 39)

25. In $\triangle ABC$, AB=6 cm., BC=5 cm., CA=4 cm. The internal and external bisectors of $\angle BAC$ meet BC and BC produced at P and Q respectively. Calculate BP and BQ.

26. The sides of a triangle are 4 cm., 7 cm., and 8 cm. long. Calculate the lengths of the parts into which the internal bisectors of the angles divide the opposite sides.

27. The internal and external bisectors of the angle A of a triangle ABC meet BC and BC produced at P, Q. Explain why

$$\frac{BP}{PC}=\frac{BQ}{CQ}.$$

28. The internal bisector of the angle A of a triangle ABC meets BC at D, and DE is drawn parallel to BA to meet AC at E. Prove that

$$\frac{AB}{AC}=\frac{AE}{EC}.$$

29. The external bisector of the angle A of a triangle ABC, which has AB greater than AC, meets BC produced at Y. The straight line through Y parallel to CA meets BA produced at X. Prove that

$$\frac{AB}{AC}=\frac{XB}{XA}.$$

30. X is the mid-point of the side BC of a triangle ABC. The bisectors of angles AXB, AXC meet AB, AC at Y, Z respectively. Prove that

$$\frac{AY}{YB}=\frac{AZ}{ZC}.$$

What conclusion do you draw ?

31. The bisector of ∠A of △ABC meets BC at D. H and K are points on BA, CA respectively, such that BH = BD, CK = CD. Prove that HK is parallel to BC.

32. ABCD is a quadrilateral with AB equal to AD. The bisectors of angles BAC, CAD meet BC, CD at X, Y respectively. Prove that XY is parallel to BD.

33. P is any point inside a triangle ABC. The bisectors of angles BPC, CPA, APB meet BC, CA, AB at X, Y, Z respectively. Prove that

$$\frac{BX}{XC}\cdot\frac{CY}{YA}\cdot\frac{AZ}{ZB}=1.$$

EXAMPLES 23b

1. D is the mid-point of the side BC of △ABC, and E is the mid-point of AD. BE is produced to meet AC at F. Prove that

$$\frac{BE}{EF}=3.$$

(Hint : draw EX parallel to AC to meet BC at X.)

2. P, Q are points on the sides AB, AC respectively of △ABC such that $AP=\frac{1}{3}AB$, $CQ=\frac{1}{3}CA$. The parallel through C to QP meets AB at R. Prove that AR = RB.

3. P is any point in the side BC of △ABC. Parallels are drawn to PA through B, C to meet CA, BA produced at Q, R respectively. Prove that

$$\text{(i)}\ \frac{AR}{RB}=\frac{AC}{CQ},$$

$$\text{(ii)}\ \frac{BP}{PC}\cdot\frac{CQ}{QA}\cdot\frac{AR}{RB}=1.$$

4. X is a point in the side BC of $\triangle$ABC such that BX : CX = 3 : 4, and Y is a point in BA such that BY : YA = 1 : 2. AX cuts CY at O. Prove that CO : CY = 2 : 3. (Hint: draw YZ parallel to AX to meet BC at Z.)

5. D is the mid-point of the side BC of $\triangle$ABC. Through any point P in BC parallels PX, PY to AC, AB respectively are drawn, meeting AD (produced if necessary) at X and Y. Prove that DX = DY. (Hint: use Theorem 38*a* to show that DX : DA = DY : DA.)

6. O is a fixed point, and P is a variable point in a fixed straight line not passing through O. OP is divided at Q (internally or externally) in a fixed ratio. State and prove the locus of Q.

7. ABCD is a parallelogram. A straight line through C cuts AB, AD, BD (produced if necessary) at P, Q, R respectively. Prove that
$$\frac{RP}{RC} = \frac{RC}{RQ}.$$

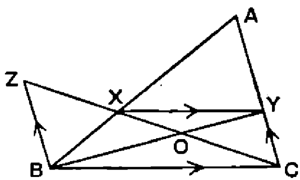


FIG. 339

8. In Fig. 339, prove that
$$OX.OZ = OC^2.$$

9. Draw a triangle ABC in which BC = 4 in., CA = 3 in., AB = $4\frac{1}{2}$ in. Construct points P, Q, R in BC, CA, AB respectively such that BP : PC = 2 : 1, CQ : QA = 3 : 2, and AR : RB = 1 : 3. Join AP, BQ, and CR. What do you notice?

10. Find by accurate drawing the value of $\frac{3{\cdot}2 \times 5{\cdot}7}{4{\cdot}9}$. Measurement is permitted, but calculation forbidden.

---

11. The bisector of $\angle$B of $\triangle$ABC meets AC at X, and the bisector of $\angle$C meets BX at Y. Prove that BY : YX = AB : AX.

12. ABC is a triangle right-angled at A. AB = 3 units and AC = 4 units. I is the centre of the inscribed circle of the triangle, and AI produced meets BC at D. Prove that DI : IA = 5 : 7.

13. The bisector of $\angle$A of $\triangle$ABC meets BC at D. DE, drawn parallel to BA, meets AC at E; and CF, drawn parallel to EB, meets AB produced at F. Prove that BF=AC.

14. A point P moves so that the ratio of its distances from two fixed points A and B is constant. Prove that the locus of P is a circle. (This circle is called the Circle of Apollonius. Hint: let the internal and external bisectors of $\angle$APB meet AB and AB produced at X, Y. What is the size of $\angle$XPY?)

15. ABC is a triangle and D is a point on the side BC such that AB : AC=BD : DC. Prove that D is equidistant from AB and AC.

16. D is the mid-point of the side BC of $\triangle$ABC. The bisector of $\angle$ADB meets AB at E, and EF is drawn parallel to BC to meet AC at F. Prove that DF bisects $\angle$ADC.

17. ABCD is a quadrilateral such that AB.CD=BC.AD. The bisector of $\angle$B meets AC at E. Prove that DE bisects $\angle$D.

18. In Fig. 340, a circle is drawn through C and D, with its centre on BA produced, cutting BA produced at E. Prove that $\frac{AE}{EB}=\frac{AC}{CB}$.

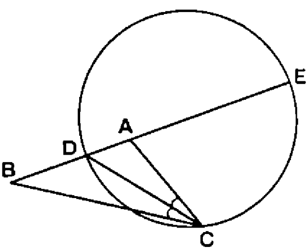


FIG. 340

EXAMPLES 23c

1. In the triangle ABC, AB=4 in., AC=5 in., BC=6 in. The bisectors of the interior and exterior angles at A cut BC and CB produced at D and E respectively. Calculate the length of DE. (C)

2. Any point X is taken in the side BC of a triangle ABC, and XM, XN are drawn parallel to BA, CA meeting CA, BA in M, N respectively; MN meets CB produced in T. Prove that TX^2=TB.TC. (C)

3. ABC is a triangle which is not isosceles, and X is the point in BC such that BX : XC=BA : AC; the line through A perpendicular to AX meets BC (or CB) produced at Y. Assuming any known theorems, prove that BY : CY=BX : XC. (C)

4. Draw a straight line AB, 3 in. long. Divide AB, by a construction, internally at D and externally at D′ in the ratio 5 : 2. (W)

---

5. AB and DC are the parallel sides of a trapezium ABCD. Through a point X on the diagonal AC a line is drawn parallel to CD to cut the diagonal BD at Y. If AX=7 cm., XC=3 cm., and YD=4 cm., calculate the length of BD. State the theorem on which your solution depends, and explain clearly how it applies.

6. CD is the diameter of a circle which bisects a chord AB, and E is any point on AB. CE produced meets the circle at P. Prove (*a*) that AP : PB=AE : EB ; (*b*) that if DE produced meets the circle at Q, then PA : AQ=PB : BQ. (N)

7. ABCD is a plane quadrilateral. A straight line parallel to BD meets AB at P and AD at Q. The parallel through Q to AC meets CD at R, and the parallel through P to AC meets BC at S. Prove that RS is parallel to QP. (O)

8. Draw a straight line 5 in. long and divide it into two parts, whose ratio is that of one of the equal sides to the third side in an isosceles triangle, whose equal angles are each 63°. Measure the lengths of the two parts. (Ruler, protractor, and compasses may be used.) (O)

---

9. The perimeter of a triangle ABC is 45 in. and the bisector of the angle A cuts BC at P, so that BP=9 in., CP=6 in. The bisector of the angle C cuts AB at Q. Calculate the length of AQ. (N)

10. ABC is a triangle ; D, E are points on AB, AC respectively such that AB.AE=AC.AD. Prove that △ABE=△ACD.

11. State how to construct a triangle, on a given base, whose area is equal to that of a given triangle.

12. A, X, B, Y are four points in order on a straight line ; AX=1 in., AB=$1\frac{1}{2}$ in., AY=3 in. P is a point 2 in. from A and 1 in. from B. Prove that XPY is a right angle. (OC)

13. Construct a triangle of base 10 cm. and vertical angle 50°, so that the ratio of the other sides may be 3 : 2. Measure in centimetres the height of the triangle. (OC)

---

14. Points X, Y are taken in the side AB of a triangle ABC such that AX = YB. The parallel to BC through X meets AC in L, and the parallel to AC through Y meets BC in M; L and M are joined. Prove that LM is parallel to AB. (C)

15. Draw a line AB 4 in. long. Construct a triangle ABC such that AC = 2BC and the angle ACB = 120°. (C)

16. P, Q are points on the sides AB, AC respectively of a triangle ABC, such that AP : AB = CQ : CA = 2 : 3. If BQ and CP cut at O, prove that CO : CP = 6 : 7.

17. A and B are two fixed points 3 in. apart; P is a point which moves in such a way that AP : PB has always the value 3 : 1. Find the points P_0, P_1 on AB and AB produced, which satisfy the condition; and find at least two other points which satisfy it.

Then *prove* that the locus of P is the circle on P_0P_1 as diameter. (W)

18. If the internal bisector of the angle A of the triangle ABC meets BC at D, prove that BD : DC = BA : AC.

If AB = 1 unit, AC = 2 units, and the angle A = 60°, calculate the lengths of AD, BD. (OC)

SIMILARITY

Two polygons are said to be **equiangular** if the angles of the one are respectively equal to the angles of the other. Thus EFGH and STUV are equiangular if $\angle E = \angle S$, $\angle F = \angle T$, $\angle G = \angle U$, $\angle H = \angle V$.

Two polygons are said to have their sides **proportional**[1] if corresponding sides are in the same ratio. For instance, EFGH and STUV have their sides proportional if

$$\frac{EF}{ST} = \frac{FG}{TU} = \frac{GH}{UV} = \frac{HE}{VS}.$$

Two polygons are said to be **similar** if they are equiangular and have their sides proportional. The symbol ||| may be used to denote 'is similar to.'

When stating that two figures are similar, make the letters correspond; the equal ratios can then be written down very easily. Thus,

$$\text{if } \triangle\text{s } \begin{matrix} ADC \\ YXZ \end{matrix} \text{ are similar,}$$

$$\frac{AD}{YX} = \frac{DC}{XZ} = \frac{AC}{YZ}.$$

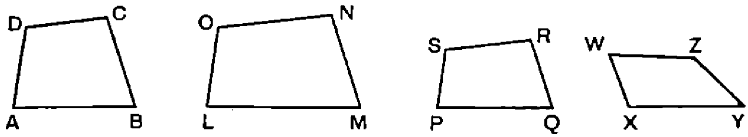


FIG. 341

Examples.

In Fig. 341, mention two quadrilaterals which

(i) are equiangular but not similar;

(ii) have their sides proportional but are not similar;

(iii) are similar.

[1] Notice that 'proportional' often means 'in the same ratio.'

THEOREM 40

(For proof, see Appendix, p. 344)

If the three angles of one triangle are respectively equal to the three angles of another triangle, then the two triangles are similar.

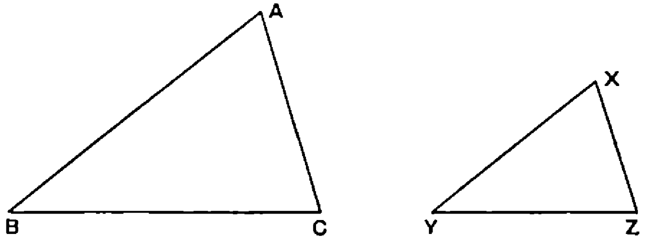


FIG. 342

Thus, if $\angle A = \angle X$, $\angle B = \angle Y$, and $\angle C = \angle Z$, $\triangle ABC \parallel\!| \triangle XYZ$.

Reference. $\triangle s \begin{matrix} ABC \\ XYZ \end{matrix}$ are similar (*AAA*).

THEOREM 41

(For proof, see Appendix, p. 346)

If two sides of one triangle are proportional to two sides of another triangle, and the included angles are equal, then the two triangles are similar.

Thus, in Fig. 342, if $\frac{AB}{XY} = \frac{AC}{XZ}$ and $\angle A = \angle X$, $\triangle ABC \parallel\!| \triangle XYZ$.

Reference.

$\triangle s \begin{matrix} ABC \\ XYZ \end{matrix}$ are similar (2 *sides prop., incl.* $\angle$*s equal*).

THEOREM 42

(For proof, see Appendix, p. 348)

If the three sides of one triangle are proportional to the three sides of another triangle, then the two triangles are similar.

Thus, in Fig. 342, if $\frac{AB}{XY}=\frac{AC}{XZ}=\frac{BC}{YZ}$, $\triangle ABC \,|||\, \triangle XYZ$.

Reference. $\triangle s\ \frac{ABC}{XYZ}$ are similar (3 *sides prop.*).

It is useful to compare the conditions for the similarity of two triangles with those for congruence :—

SIMILARITY	CONGRUENCE
AAA	AA corr. S
2 sides prop., incl. ∠s equal	SAS
3 sides prop.	SSS
—	RHS

CONSTRUCTION 28

To construct a pentagon similar to a given pentagon, with the length of one side given.

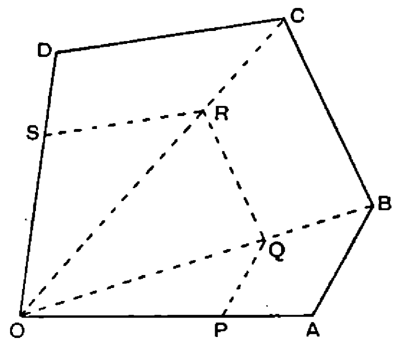


FIG. 343

Given a pentagon OABCD, and the length of the side, corresponding to OA, of another pentagon.

To construct a pentagon similar to OABCD, with one side of given length.

Construction. Along OA, mark off OP equal to the given length.

Join OB and OC.

Draw the parallel to AB through P to meet OB at Q.

Draw the parallel to BC through Q to meet OC at R.

Draw the parallel to CD through R to meet OD at S.

Then OPQRS is the required pentagon.

Proof. In the $\triangle$s OPQ, OAB,

$\angle POQ = \angle AOB$ (same angle),

$\angle OPQ = \angle OAB$ (corr., PQ $\parallel$ AB),

and $\angle OQP = \angle OBA$ (corr., PQ $\parallel$ AB);

$\therefore$ $\triangle$s $\frac{OPQ}{OAB}$ are similar (AAA).

$$\therefore \frac{OP}{OA} = \frac{PQ}{AB} = \frac{OQ}{OB}.$$

Similarly, $\frac{OQ}{OB}=\frac{QR}{BC}=\frac{OR}{OC}$,

and $\frac{OR}{OC}=\frac{RS}{CD}=\frac{OS}{OD}$.

$\therefore \frac{OP}{OA}=\frac{PQ}{AB}=\frac{QR}{BC}=\frac{RS}{CD}=\frac{SO}{DO}$;

∴ the sides of **OPQRS** are proportional to those of **OABCD**.
Also, by parallels, **OPQRS** and **OABCD** are equiangular ;
∴ they are similar.

EXAMPLES 24a

(*Theorem* 40)

1. A vertical pole 8 ft. high casts a shadow 3 ft. 3 in. long at the same moment as a flagstaff casts a shadow 39 ft. long. Calculate the height of the flagstaff.

2. A hill rises 1 ft. vertically in 66 ft. What vertical height does it climb in half a mile ?

3. Calculate the other sides of a triangle whose shortest side is 10 cm. and which is similar to a triangle with sides, 4, 7, and 8 cm.

4. The radius of the base of a right circular cone is 9 in., and its height is 12 in. Calculate the radius of a section made by a plane parallel to the base and 3 in. from the vertex.

5. Triangles $\genfrac{}{}{0pt}{}{ABC}{XYZ}$ are similar, and $\angle C=90°$. If $AC=XY=5$ ft. and $YZ=4$ ft., calculate the lengths of **AB** and **BC**.

---

6. In Fig. 344, prove that triangles **AHK**, **ABC** are similar, and write down two ratios equal to **AH** : **AB**.

7. In Fig. 344, if $AB=8$ cm., $BC=6$ cm., $CA=5$ cm., and $AH=2$ cm., what fraction is **AH** of **AB** ? What are the lengths of **HK** and **AK** ?

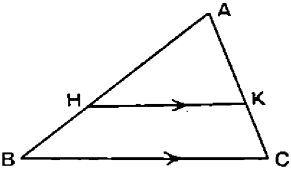


FIG. 344

8. In Fig. 344, if AH=4·5 cm., AK=3 cm., HK=7 cm., and BC=8·4 cm., calculate AB and KC.

9. In Fig. 345, prove two triangles similar, and complete the relation $\frac{XY}{CB} = \frac{AX}{}$.

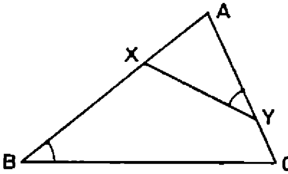


FIG. 345

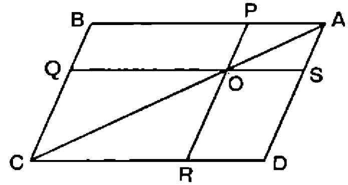


FIG. 346

10. In Fig. 346, AB, SQ, DC are parallel, and AD, PR, BC are parallel. Point out (*a*) a triangle congruent with △COR, (*b*) a triangle similar to but not congruent with △COR. In each case give a proof.

11. In Fig. 347, prove that

$$\frac{OA}{OC} = \frac{OB}{OD} = \frac{AB}{CD}.$$

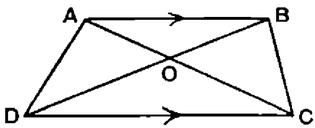


FIG. 347

12. In Fig. 348, prove that △s ADB, CAB are similar. Complete the relation BD : BA=AB : , and write down another ratio equal to these two.

13. In Fig. 348, prove another pair of triangles similar. If CA= 6 cm. and CD=2 cm., calculate the length of CB.

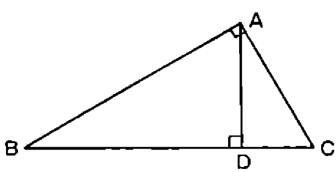


FIG. 348

14. ABC is a triangle right-angled at B, and BA is greater than BC. Through the mid-point O of AC a perpendicular to AC is drawn to meet AB at X. Join CX. Prove that there is a pair of congruent triangles in the figure, and also a pair of triangles which are similar but not congruent.

15. In No. 14, if AB=4 in. and AC=5 in., calculate the length of CX.

16. **E**, **F** are the feet of the perpendiculars from **B** and **C** respectively to the opposite sides of a triangle **ABC**. Prove that Δs **ABE**, **ACF** are similar, and complete the relation $\frac{BE}{CF} = \frac{\quad}{AC}$.

17. With the data of No. 16, if **BE** cuts **CF** at **H**, name another pair of similar triangles, and complete the relation $\frac{BF}{FH} = \frac{\quad}{EH}$.

18. If it can be proved by similar triangles that $\frac{AO}{AP} = \frac{AB}{AQ}$, name the triangles in corresponding order of letters. (Two answers.)

19. **ABC** is an equilateral triangle, and **D** is the mid-point of **BC**. On the side of **DC** opposite to **A** another equilateral triangle **CDX** is drawn, and **AX** cuts **DC** at **Y**. Prove that Δs **ABY**, **XCY** are similar, and deduce that **BY**=2**YC**.

20. **ABCD** is a parallelogram. A straight line through **C** meets **AB** produced at **X** and **AD** produced at **Y**. Prove that $\frac{BC}{DY} = \frac{BX}{DC}$.

(*Theorems* 41 *and* 42)

21. If it can be proved by similar triangles that $\frac{PQ}{AC} = \frac{SP}{BA} = \frac{QS}{CB}$, name the triangles in corresponding order of letters.

22. Prove that the triangle whose sides are 2·1, 2·8, and 3·5 cm. is right-angled, by proving it similar to a well-known right-angled triangle.

23. In a quadrilateral **ABCD**, $\angle$**A**=$\angle$**C** and $\frac{DA}{AB} = \frac{BC}{CD}$. What can you say about Δs **DAB**, **BCD**, and why? Deduce that **AB** is parallel to **DC** and **AD** is parallel to **BC**.

24. A triangle **ABC** is such that **AB**=4 in., **BC**=5 in., **AC**=2 in. A straight line **BD**, $6\frac{1}{4}$ in. long, is drawn through **B**, on the opposite side of **BC** to **A**, such that $\angle$**DBC**=$\angle$**ABC**. Prove that Δs **ABC**, **CBD** are similar, and calculate **DC**.

25. In $\triangle$**ABC**, **AB**=**AC**=4 cm., **BC**=2 cm. **D** is a point in **AB** such that **BD**=1 cm. Prove that $\triangle$**CBD** is similar to $\triangle$**ABC**. What is the length of **CD** ?

26. In Fig. 349, if **AB** : **CD**=**BC** : **DE**=**CA** : **EC**, prove that **CD** is parallel to **BA**.

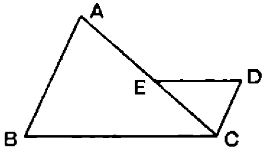


FIG. 349

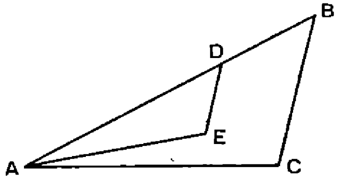


FIG. 350

27. In Fig. 350, if $\angle$**C**=$\angle$**E** and **AC** : **CB**=**AE** : **ED**, prove that **A**, **E**, and **C** must lie in the same straight line.

28. **BD** is a diagonal of a quadrilateral **ABCD**. If $\angle$**A**=$\angle$**CBD** and **DA** : **AB**=**DB** : **BC**, prove that **DB** bisects $\angle$**ADC**.

29. If **ABCD** is a quadrilateral in which the diagonal **AC** bisects $\angle$**BAD** and **AB** : **AC**=**AC** : **AD**, what angle is equal to $\angle$**ACB** ? Prove this.

30. **ABC** is a straight line, and **AB**=10 cm., **BC**=5 cm. Two triangles **DAB**, **EBC** are drawn, on the same side of **ABC**, in which **BD**=9 cm., **DA**=4 cm., **CE**=$4\frac{1}{2}$ cm., **EB**=2 cm. Prove that **DB** is parallel to **EC**. If **DE** produced meets **ABC** produced at **F**, calculate **CF**.

(Involving properties of the circle)

31. A sphere of radius 4 in. rests inside a hollow right circular cone, of base-radius 6 in. and slant height 15 in., held with its axis vertical and vertex downwards. Calculate the height of the centre of the sphere above the vertex of the cone.

32. In Fig. 351, prove that the following pairs of triangles are similar :—

(i) **AQB**, **DQC** ;
(ii) **ABP**, **CDP** ;
(iii) **ACP**, **BDP**.

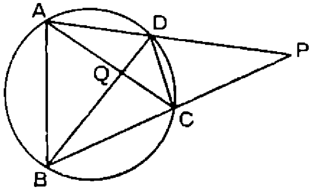


FIG. 351

33. In Fig. 351, if **AQ**=3 in., **AB**=4 in., **QB**=2 in., **QC**=$\frac{3}{4}$ in., calculate **QD** and **DC**.

34. In Fig. 351, if **PA**=6 in., **PB**=$4\frac{1}{2}$ in., **AB**=2 in., **PC**=4 in., calculate **CD** and **PD**.

35. In Fig. 351, if **PA**=$4\frac{1}{2}$ cm., **PC**=2 cm., **AC**=3 cm., **PB**=4 cm., calculate **PD** and **BD**.

36. **O** is a point outside a circle ; **OAB** is a straight line cutting the circle at **A**, **B**, and **OT** is a tangent to the circle, touching it at **T**. Prove that Δs **OTA**, **OBT** are similar, and complete the relation

$$\frac{OT}{OB}=\frac{OA}{\quad}.$$

37. **AB** is a diameter of a circle, and **AC** is any chord ; **P** is a point on **AC**, and **Q** is the foot of the perpendicular from **P** to **AB**. Prove that **AP** : **AQ**=**AB** : **AC**. (Hint : join **CB**.)

38. **ABC** is a triangle inscribed in a circle. The bisector of $\angle$**A** cuts **BC** at **D** and the circle at **P**. Prove that Δs **ABD**, **APC** are similar, and calculate **AC** if **AB**=7 cm., **AD**=8 cm., and **DP**=1 cm.

(*Construction* 28)

39. Draw a quadrilateral, and construct a similar quadrilateral with sides $\frac{4}{7}$ times as long.

40. Draw a hexagon **ABCDEF**, and construct a similar hexagon **PQRSTU** with the side **RS** equal to **AC**.

EXAMPLES 24b

1. A straight horizontal rod a ft. long is held x ft. above the floor of a room. If the room is lit by a lamp y ft. above the floor but not in the vertical plane containing the rod, calculate the length of the shadow of the rod on the floor.

2. The shadow thrown on the floor of a room by a rectangular table 12 ft. long, 8 ft. wide, and $2\frac{1}{2}$ ft. high is 15 ft. long. Calculate the width of the shadow, and the height above the floor of the lamp which lights the room.

3. ABCD is a parallelogram; P is any point on AB, and PD cuts AC at Q. Prove that $\frac{AP}{AB}=\frac{AQ}{CQ}$.

4. ABC is an isosceles triangle in which AB=AC. BC is produced to X and CB to Y so that $AB^2 = CX.BY$. Prove that $\triangle$s ABY, XCA are similar. Which angle is equal to $\angle$CAX ?

5. Two circles intersect at A, B. Tangents to the two circles at A meet the circles again at P, Q. Prove that $BP.BQ = BA^2$.

6. ABC is a triangle inscribed in a circle; AD is an altitude of the triangle, and AK is a diameter of the circle. Prove that $\triangle$s ADC, ABK are similar, and deduce that AB.AC=AD.AK.

7. ABCD is a trapezium with AB parallel to DC; AB=5 cm., BC=6 cm., CD=$7\frac{1}{2}$ cm., DA=4 cm. DA and CB are produced to meet at O. Calculate OA and OB.

8. ABCD is a straight line, and AB=BC=CD. On BC as base an equilateral triangle BCP is drawn. Prove that $\triangle$s APD, ABP are similar.

9. In Fig. 352, $\triangle$s $\frac{ABC}{AXY}$ are similar. Prove that $\triangle$s $\frac{ABX}{ACY}$ are also similar.

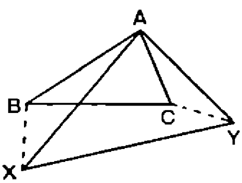


FIG. 352

10. AB is a diameter of a circle, and P is any point on the circumference; N is the foot of the perpendicular from P to the tangent at A. Prove that $PN.AB = AP^2$.

11. BE, CF are altitudes of $\triangle$ABC. Prove that

$$\frac{BC}{EF}=\frac{AB}{AE}.$$

12. ABC is a triangle in which AB=3AC. The internal bisector of $\angle$A meets BC at D, and the parallel through D to BA meets AC at E. Calculate the ratio AB : ED.

13. ABC is a triangle in which AB=AC; D is a point on AB such that CB=CD. Prove that $BC^2 = BD.BA$.

14. ABCD is a quadrilateral and O is any point. On OA, OB, OC, OD points P, Q, R, S respectively are taken so that

$$OP : OA = OQ : OB = OR : OC = OS : OD.$$

Prove that PQRS is similar to ABCD.

15. ABCD is a parallelogram; P is a point on AB and Q a point on CD, and PQ cuts AC at X. Through X any straight line RXS is drawn meeting BC at R and AD at S. Prove that PS is parallel to RQ.

16. In Fig. 347, p. 288, if X is the mid-point of AB and XO produced meets DC at Y, prove that Y is the mid-point of DC. (Hint: obtain ratios equal to AX : OX and BX : OX.)

17. D is the mid-point of the side BC of a triangle ABC. A straight line parallel to BC cuts AB at P, AD at Q, and AC at R. Prove that PQ=QR.

18. O is a fixed point; P is a variable point on a given circle, centre C; on OP a point Q is taken such that the ratio of OQ to OP is given. Prove that the locus of Q is a circle, with its centre at K, where K is on OC and the ratio of OK to OC is the given ratio.

19. Two circles intersect at A, B. PAQ, RAS are straight lines through A cutting one circle at P, R and the other at Q, S. Prove that

$$\frac{BP}{BR} = \frac{BQ}{BS}.$$

20. Two circles intersect at A, B. A straight line PAQ through A cuts the circles at P, Q. Prove that the ratio of BP to BQ is equal to the ratio of the diameters of the two circles.

21. ABC, DEF are two similar triangles. O is the circumcentre of △ABC, P that of △DEF. M, N are the feet of the perpendiculars from O, P to BC, EF respectively. Prove that △s OBM, PEN are similar, and deduce that the ratio of the circumradii of △s ABC, DEF is equal to the ratio BC : EF.

22. Two quadrilaterals ABCD, PQRS are such that ∠s A, B, C, D are equal respectively to ∠s P, Q, R, S, and AB : BC=PQ : QR. Prove that CD : DA=RS : SP.

23. Two circles touch internally at A. Straight lines APX, AQY are drawn through A to cut one circle at P, Q and the other circle at X, Y. Prove that AP : AX = PQ : XY.

24. ABC is a triangle with B an acute angle; P is the point of trisection of BC nearer to B. The parallel through P to BA meets AC at Q, and the parallel through C to BA meets BQ produced at R. Calculate the ratio PQ : CR, and prove that PR cuts AC at the mid-point of AC.

25. The diagonals AC, BD of a quadrilateral ABCD cut at O. OP is drawn parallel to CB to meet AB at P, and OQ is drawn parallel to CD to meet AD at Q. Prove that PQ is parallel to BD, and that PQ : BD = AO : AC.

26. AD is an altitude of △ABC, D lying between B and C. If $BC.DC = AC^2$, prove that ∠BAC = 90°.

27. ABCD is a cyclic quadrilateral, with BC not parallel to AD, and AB, DC when produced meet at P. L, M are the respective feet of the perpendiculars from P to AD, BC (produced if necessary). Prove that PL : PM = AD : BC.

28. ABC is a triangle right-angled at A, and Z is any point on BC. Straight lines through Z parallel to BA and CA meet AC and AB at X and Y respectively. If AB, AC, XZ, and YZ are 4, 5, x, and y cm. respectively, prove that $5x + 4y = 20$.

29. D is the mid-point of the side AB of △ABC, and E is any point in AC produced; DE cuts BC at F. Prove that EC : EA = CF : FB. (Hint: draw CX parallel to AB, meeting EF at X.)

30. A point E is taken on the side AB of a triangle ABC, which is right-angled at A. BD is drawn perpendicular to BC on the side of BC remote from A, and BD is chosen so that

$$\frac{AE}{AC} = \frac{BD}{BC}.$$

Prove that ∠AEC = ∠BDC, and that ∠DEC = 90°.

---

31. Draw a pentagon, and construct a similar pentagon with sides $\sqrt{2}$ times as long.

---

32. Prove that the section of a pyramid on a triangular base by a plane parallel to the base is a triangle similar to the base.

33. XABC, YABC are two pyramids of the same height on the same base ABC. Prove that the sections of the pyramids by a plane parallel to ABC are congruent triangles.

34. A rectangular block has a base 4 in. by 3 in. and height 5 in. It rests on a table, and a hollow right circular cone of height 9 in. also rests on the table, covering the block and touching its four upper corners. Calculate the base-radius of the cone.

EXAMPLES 24c

1. A point O on the side AB of a triangle ABC is such that AO : OB = 3 : 4, and P, Q are points on AC, CB such that OP, OQ are parallel respectively to BC, AC, and OP = 1·5 in., OQ = 1·6 in. Calculate the lengths of AC and BC. (N)

2. ABCD is a parallelogram. Through A a line is drawn to cut BD, CD, and BC (produced) at E, F, and G respectively. Prove that (N)

$$\frac{AE}{EG}=\frac{AF}{AG}.$$

3. POQ is a right angle, P and Q being any two points on the arms. QR is drawn at right angles to PQ to meet PO produced in R, and PS also at right angles to QP meets QO produced in S. Prove that OR . OS = OP . OQ. (OC)

4. AB is a diameter of a circle, centre O; AP, PQ are equal chords. Prove that AP . PB = AQ . OP.

5. PR and QS are the diagonals of a cyclic quadrilateral PQRS. If QP . QS = RP . RS, prove that QP = RS and QS = RP.

6. Prove that the common tangents to two non-intersecting circles divide (internally and externally) the line joining the centres of the circles in the ratio of the radii.

7. In a triangle ABC, in which AB = AC, CD is cut off from CA so that AC : BC = BC : DC. Prove that BD = BC.

8. ABC is a triangle in which AC=BC and the angle C is obtuse ; X is the foot of the perpendicular from A to BC produced, and N is the middle point of AB. Prove that AX : AB=CN : CB. (C)

9. ABC is a triangle, FAE is a straight line through A parallel to BC and bisected at A ; EB meets AC in H, FC meets AB in G. Prove that GH is parallel to BC. (OC)

10. ABC is a triangle ; D is a point on the straight line bisecting the angle A, and on the side of BC opposite to A. If $AD^2=AB.AC$, prove that BD is a tangent to the circle passing through A, C, and D. (C)

11. ABC is a triangle and P is a point on AB such that AP : PB=3 : 5. A point R on BC is such that BR : RC=1 : 3. Lines, PQ, drawn parallel to BC, and RS, drawn parallel to BA, cut AC at Q, S respectively.

(*a*) If AC=16 in., calculate the length of SQ.

(*b*) If PQ cuts RS at V, calculate the ratio RV : VS. (N)

12. ABCD is a square, X is any point on AB, and O any point on AB produced. The parallel to AD through X meets OD at W, the parallel to DC through W meets OC at Z, and the parallel to CB through Z meets OB at Y. Prove that XYZW is a square.

13. AC is a chord of a circle. From a point B on the minor arc, lines BD, BE are drawn parallel respectively to the tangents at A and C, cutting AC at D and E. Prove that the triangles ABD, BCE are similar and that $AD.CE=BD^2$. (N)

14. ABCD is a rectangle in which BC=2BA. If CD is produced to E so that CE=2BC, prove that AC is perpendicular to BE. (C)

15. ABCD is a quadrilateral in which AB is parallel to DC. A straight line, parallel to AB or DC, meets AD at L and BC at M, so that

$$\frac{AL}{LD}=\frac{BM}{MC}=\frac{p}{q}.$$

Prove that $(p+q)LM=q.AB+p.DC$. (O)

16. ABCD is a cyclic quadrilateral whose diagonals cut at O. A straight line AX is drawn, within the angle BAD, meeting BD at X, and such that $\angle DAX = \angle OAB$. Prove that $AD : DX = AC : CB$ and $AB : XB = AC : DC$, and deduce that $AD \, . \, BC + AB \, . \, CD = AC \, . \, BD$.

17. ABCD is a quadrilateral and P is a point on AB. L and M are points such that APLD, BPMC are parallelograms. LM cuts DC in Q. Prove that $DQ : QC = AP : PB$. Prove also that, if P is chosen so that this ratio is equal to $AD : BC$, PQ bisects the angle LPM. (O)

18. ABC is a triangle and a parallel to BC meets AB and AC at P and Q respectively. If the triangle APQ is rotated about A through any angle in the plane ABC into the position AXY, P moving to X and Q to Y, prove that the triangles ABX and ACY are similar. (O)

19. ABC is a triangle inscribed in a circle. The straight lines through B parallel to CA and through C parallel to BA meet the tangent at A at P and Q respectively. Prove that

$$AP : AQ = AB^2 : AC^2.$$

20. ABC is any triangle and O is the centre of that circle escribed to the triangle which touches BC and both AB and AC produced. The tangent to this circle parallel to BC meets AB produced at P and meets AC produced at Q. Find the angles of the triangles ABO and AOQ in terms of the angles of the triangle ABC, and hence show that $AB \, . \, AQ = AO^2$. (O)

21. A cyclic quadrilateral ABCD is such that the rectangles AB.CD and BC.AD are equal. DA is produced to E so that AE is equal to DA. Prove that the angle ABE is equal to the angle DBC. If CB is produced to F, so that BF is equal to CB, and if AF meets BE in P, prove that AP is equal to PB. (O)

22. ABC is a triangle in which $AB = AC$, and the circle which touches AB at B and AC at C is drawn. O is any point inside $\triangle ABC$ and lying on the circumference of the circle, and X, Y, Z are the feet of the perpendiculars from O to BC, CA, AB respectively. Prove that $\triangle$s OXY, OZX are similar.

23. ABC . . . and A′B′C′ . . . are two similar polygons which have their corresponding sides parallel. Prove that AA′, BB′, CC′ . . . all intersect at the same point.

THEOREM 43 (Right-angled Triangle Property)

If a perpendicular is drawn from the right angle of a right-angled triangle to the hypotenuse, the triangles on each side of the perpendicular are similar to the whole triangle and to each other.

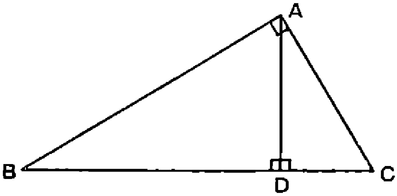


FIG. 353

Given a triangle ABC, right-angled at A, and a perpendicular to BC from A, meeting BC at D.

To prove that $\triangle$s ABC, DBA, DAC are similar.

Proof. In the $\triangle$s DBA, ABC,

$\angle ADB = \angle CAB$ (90°, given),

$\angle DBA = \angle ABC$ (same angle),

and $\angle BAD = \angle BCA$ (3rd. $\angle$s of $\triangle$);

$\therefore$ $\triangle$s $\begin{matrix} DBA \\ ABC \end{matrix}$ are similar (AAA).

Similarly, $\triangle$s $\begin{matrix} DAC \\ ABC \end{matrix}$ are similar.

Corollary. BA^2=BD.BC, CA^2=CD.CB, DA^2=DB.DC.

Proof. $\triangle$s $\frac{DBA}{ABC}$ are similar ;

$$\therefore \frac{BA}{BC}=\frac{BD}{BA};$$

$$\therefore BA^2=BD.BC.$$

Similarly, from $\triangle$s $\frac{DAC}{ABC}$, CA^2=CD.CB,

and, from $\triangle$s $\frac{DBA}{DAC}$, DA^2=DB.DC.

Reference. *$\angle BAC=90°$ and AD is perp. to BC*;

$$\therefore BA^2=BD.BC.$$

If $a : x = x : b$, x is called the **mean proportional** of a and b. It follows that

$$\frac{a}{x}=\frac{x}{b};$$

$$\therefore x^2=ab.$$

Thus, in Fig. 353, BA is the mean proportional of BD and BC.

CONSTRUCTION 29

To construct the mean proportional of two given lengths.

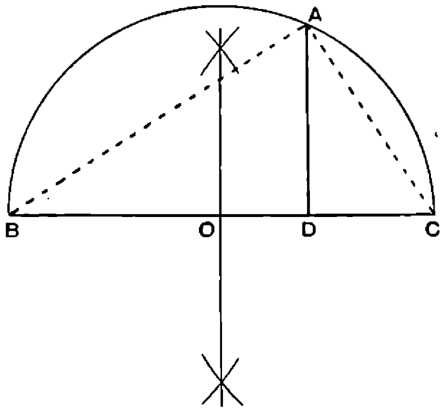


Fig. 354

Construction. Along a straight line, mark off BD, DC equal to the given lengths.

Bisect BC at O.

With centre O and radius OB, draw a semicircle.

At D, erect a perpendicular to BC to meet the semicircle at A.

Then DA is the required mean proportional.

Proof. Join AB, AC.

$\angle$BAC$=90°$ ($\angle$ in semicircle) and AD is perp. to BC;

$$\therefore \text{DA}^2 = \text{DB}.\text{DC}.$$

$$\therefore \frac{\text{DA}}{\text{DB}} = \frac{\text{DC}}{\text{DA}};$$

$\therefore$ DA is the mean proportional of DB and DC.

CONSTRUCTION 30

To construct a square equal in area to a given rectangle.

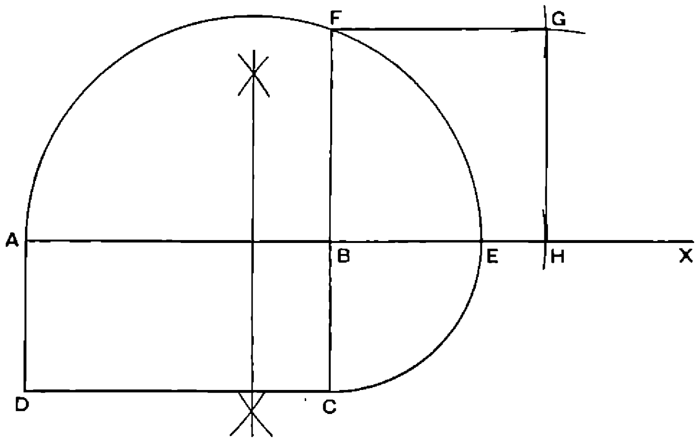


FIG. 355

Given a rectangle ABCD.

To construct a square equal in area to ABCD.

Construction. Produce AB to X.

Along BX, mark off BE equal to BC.

Construct the mean proportional, BF, of BA and BE.

On BF, construct a square BFGH.

Then BFGH is the required square.

Proof. BF is the mean proportional of BA and BE;

$$\therefore BF^2 = BA.BE.$$

But BE=BC (constr.);

$$\therefore BF^2 = BA.BC.$$

$$\therefore \text{sq. } BFGH = \text{rect. } ABCD.$$

THEOREM 44

The ratio of the areas of two similar triangles is equal to the ratio of the squares on corresponding sides.

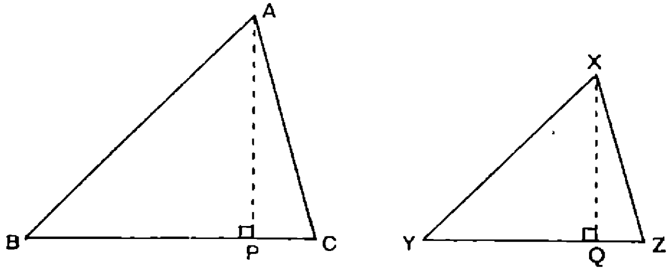


FIG. 350

Given two similar triangles, ABC, XYZ.

To prove that $\frac{\triangle ABC}{\triangle XYZ}=\frac{BC^2}{YZ^2}$.

Construction. Draw the altitudes, AP, XQ.

Proof.

$$\triangle ABC=\tfrac{1}{2}BC.AP$$

and $\triangle XYZ=\tfrac{1}{2}YZ.XQ$;

$$\therefore \frac{\triangle ABC}{\triangle XYZ}=\frac{\frac{1}{2}BC.AP}{\frac{1}{2}YZ.XQ}$$

$$=\frac{BC.AP}{YZ.XQ}.$$

In the $\triangle$s ABP, XYQ,

$\angle ABP=\angle XYQ$ (given),

$\angle APB=\angle XQY$ (90°, constr.),

and $\angle BAP=\angle YXQ$ (3rd. $\angle$s of $\triangle$);

$\therefore$ $\triangle$s $\begin{matrix}ABP\\XYQ\end{matrix}$ are similar (AAA).

$$\therefore \frac{AP}{XQ}=\frac{AB}{XY}.$$

But $$\frac{AB}{XY}=\frac{BC}{YZ}\ \text{(given)};$$

$$\therefore \frac{AP}{XQ}=\frac{BC}{YZ}.$$

But $$\frac{\triangle ABC}{\triangle XYZ}=\frac{BC.AP}{YZ.XQ}\ \text{(proved)};$$

$$\therefore \frac{\triangle ABC}{\triangle XYZ}=\frac{BC^2}{YZ^2}.$$

Reference. $$\frac{\triangle ABC}{\triangle XYZ}=\frac{BC^2}{YZ^2}\ (\triangle s\ ABC,\ XYZ\ similar).$$

Corollary.

If any two triangles have one angle of the one equal to one angle of the other, the ratio of their areas is equal to the ratio of the rectangles contained by the sides about the equal angles.

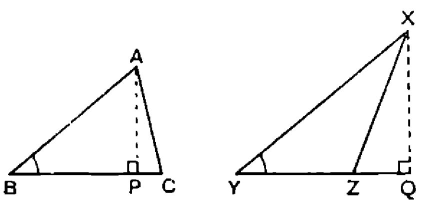


FIG. 357

If B and Y are the equal angles, the proof is the same as that of Theorem 44 down to the end of p. 302, and concludes :—

but $$\frac{\triangle ABC}{\triangle XYZ}=\frac{BC.AP}{YZ.XQ}\ \text{(proved)};$$

$$\therefore \frac{\triangle ABC}{\triangle XYZ}=\frac{BC.BA}{YZ.YX}.$$

EXAMPLES 25a

(*Theorem* 43)

1. In Fig. 353, p. 298, if $AB = 6{\cdot}5$ cm. and $BC = 16{\cdot}9$ cm., calculate BD.

2. In Fig. 353, if $BC = 15$ cm. and $BD = 3$ cm., calculate AD.

3. In Fig. 353, if E, F are points on AB, AC respectively such that AEDF is a rectangle, prove that $EF^2 = DB.DC$.

4. With the data of No. 3, prove that $AE.AB = AF.AC$.

5. In Fig. 353, prove that $BA^2 : CA^2 = DB : DC$.

(*Constructions* 29 *and* 30)

6. Find by construction the mean proportional between $1\frac{1}{2}$ and 2. (Take 1 in. as unit.)

7. Draw a rectangle with sides 2·6 and 1·4 in., and construct a square equal in area to it. Measure the side of the square.

8. Draw a parallelogram with sides 2·2 and 1·4 in. and an angle of 48°. Then construct a square equal in area to the parallelogram, and measure its side. (Hint: first construct a rectangle equal in area to the parallelogram.)

9. Given a rectangle, state how to construct a square of three times the area.

(*Theorem* 44)

10. The longest side of a triangle is 5 cm. and the area is 4 sq.cm. Calculate the longest side of a similar triangle of area 9 sq. cm.

11. In Fig. 358, if $AB : AD = 3 : 4$ and the area of $\triangle ABC$ is 1·8 sq.in., calculate the area of $\triangle ADE$.

12. In Fig. 358, if $EC : EA = 2 : 5$, what is the ratio of the areas of $\triangle$s ABC and ADE?

13. Show how to divide a triangle into nine triangles equal in area and similar to the given triangle.

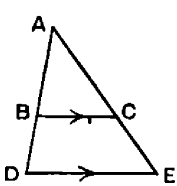


FIG. 358

14. H, K are points on the sides AB, AC of a triangle ABC such that $AH = \frac{3}{4}AB$ and $AK = \frac{5}{6}AC$. Express $\triangle AHK$ as a fraction of $\triangle ABC$.

15. P is any point on the side AB of a triangle ABC, and Q is a point on AC such that $\angle APQ = \angle ACB$. Prove that

$$\triangle APQ : \triangle ABC = AP^2 : AC^2.$$

16. ABCD is a trapezium having AB parallel to DC. If the diagonals cut at O, prove that $\triangle AOB : \triangle COD = AB^2 : CD^2$.

17. Two chords of a circle, AB and CD, cut at X. Prove that $\triangle AXC : \triangle DXB = AX^2 : DX^2$.

18. Prove that the areas of similar parallelograms are proportional to the squares on corresponding diagonals.

---

19. Draw any triangle, and construct a similar triangle of four times the area.

EXAMPLES 25b

1. AB is a diameter of a circle and P is a point on the circumference. If AB = 2·6 in. and the perpendicular to it from P divides it in the ratio 4 : 9, calculate the lengths of AP and PB to the nearest tenth of an inch.

2. In Fig. 358, if AD : BD = 8 : 3 and the area of BCED is 13 sq. in., calculate the area of $\triangle ABC$.

3. D, E, F are points on the sides BC, CA, AB respectively of a triangle ABC, such that BD = DC, 2CE = EA, and AF = 3FB. If the area of $\triangle ABC$ is 36 sq. in., calculate the area of $\triangle DEF$.

4. C is a point on a diameter AB of a circle, and a straight line through C perpendicular to AB meets the circumference at D. If AC = $1\frac{1}{2}$ ft. and CD = 2 ft., calculate the radius of the circle.

5. ABCD is a rectangle, and E a point on CD such that $\angle AEB$ is 90°. If AE = 20 cm. and DE = 16 cm., calculate the lengths of AD and AB.

6. ABCD is a trapezium having AB parallel to DC. If the diagonals cut at O and $\triangle AOB : \triangle COD = 4 : 9$, calculate $\triangle ADB : \triangle ADC$.

7. On the side AB of a triangle ABC a point D is taken such that AD : DB = 2 : 5. If the straight line through D parallel to BC meets AC at E, and the straight line through E parallel to AB meets BC at F, calculate the ratio of the area of DEFB to that of $\triangle ABC$.

8. P, Q, R, S are points on the sides AB, BC, CD, DA respectively of a parallelogram ABCD of area 70 sq.cm. If AB = 7 cm., BC = 5 cm., AP = 3 cm., BQ = CR = 2 cm., and DS = 1 cm., calculate the area of the hexagon PBQRDS.

---

9. Prove that the ratio of the areas of similar quadrilaterals is equal to the ratio of the squares on corresponding sides.

10. R, S are the points of contact of the tangents from a point P to a circle. If O is the centre of the circle and OP cuts RS at Q, prove that PR is the mean proportional between PQ and PO.

11. The bisector of the angle A of a triangle ABC meets BC at D, and is produced to meet at E the straight line through C parallel to AB. Prove that $\triangle ABD : \triangle ECD = AB^2 : AC^2$.

12. AB is a diameter of a circle whose centre is O, and C is any point on the circumference. If the tangents at A and C meet at T, prove that $\triangle BCO : \triangle ATC = OA^2 : AT^2$.

13. Prove the converse of a corollary of Theorem 43, viz., given a triangle ABC with a point D on the side BC such that AD is perpendicular to BC and $DA^2 = DB.DC$, prove that $\angle BAC = 90°$.

14. If AX and BY are altitudes of a triangle ABC, prove that $\triangle CXY : \triangle CAB = CX^2 : CA^2$.

15. In Fig. 353, p. 298, if DE is an altitude of $\triangle ABD$, prove that $BE.BC^2 = BA^3$.

16. ABC is a triangle right-angled at A. If D and E are points on BC, BA respectively such that DE is perpendicular to BA and $ED^2 = EA.EB$, prove that $DA^2 = DB.DC$.

17. A, B, C, D are four points in order on a straight line, and O is a point such that OBC is an isosceles triangle in which OB = OC. If OB is the mean proportional between AB and CD, prove that $\triangle ABO : \triangle OCD = OA^2 : OD^2$.

---

18. Given a straight line 1 in. long, construct a straight line $\sqrt{6}$ in. long.

19. State briefly how to construct a square equivalent to a given triangle.

20. If 4·3 is the mean proportional between 2·7 and x, find x by construction.

21. Given any two similar triangles, state how to construct a third similar triangle whose area is (a) the sum, (b) the difference of the areas of the given triangles.

22. Draw a right-angled triangle with hypotenuse 3 in. and another side 2·7 in., and construct an isosceles right-angled triangle equal in area to it. Measure the hypotenuse of the latter triangle.

23. State how to divide a triangle, by a straight line parallel to one side, into a smaller triangle and a trapezium such that the area of the trapezium is $\frac{2}{3}$ the area of the smaller triangle.

24. If a square of side 2·8 in. is equivalent to a rectangle of length 4 in., find by construction the width of the rectangle.

EXAMPLES 25c

1. Two unequal circles cut at **O**. Lines **AOP**, **BOS**, **COT** through **O** cut one circle at **A**, **B**, **C** and the other at **P**, **S**, **T**. Prove that the area of the triangle **ABC** is to the area of the triangle **PST** as is the square on **AB** to the square on **PS**. (N)

2. In the triangle **ABC**, $\angle$**A** is 90° and **BC** is 169 cm. long. The perpendicular **AL** from **A** to **BC** meets **BC** at **L**, and **BL** is 25 cm. long. Calculate the lengths of **AL**, **AB**, and **AC**. (N)

3. The lengths, in inches, of the sides of a triangle are p^2-q^2, $2pq$, p^2+q^2. Prove that the triangle is right-angled, and find by calculation the longest side and the area of a similar triangle whose perimeter is (p^2+pq) inches. (N)

4. Construct a parallelogram having diagonals 3·4 in. and 4·2 in. long, and one of its sides 2·0 in. long. Without making any calculations, construct a square equal in area to the parallelogram and measure the sides of the square. State your construction *briefly*, and show all the construction lines. No proof is required. (N)

5. **A** is any point inside a circle whose centre is **O** and radius r units. If **B** is the point on **OA** produced such that **OA**.**OB**$=r^2$, and **C** is any other point on the circle, prove that **OA** : **OB** = **CA**^{2} : **CB**^{2}.

---

6. **ABCD** is a parallelogram, and **P**, **Q** are the mid-points of **BC**, **CD** respectively. Prove that $\triangle$**APQ** is $\frac{3}{8}$ the area of the parallelogram.

7. Construct a triangle whose sides are 3, 2·6, and 2·4 in. long. With ruler and compasses only, construct a square whose area is twice that of the triangle. (C)

8. Prove that, if three similar triangles are constructed with the sides of a right-angled triangle as bases (*i.e.* as corresponding sides for the three triangles), then the areas of the triangles are connected by a relation similar to that of Pythagoras' theorem. (OC)

9. Draw a large triangle with angles 60°, 70°, 50°. Determine the ratio of the area of the triangle to the area of the square on its longest side. Why do you suppose that this ratio is independent of the size of the triangle ? (OC)

10. A triangle **ABC** is right-angled at **A**, and **D** is a point on **BC** such that ∠**ADB** is a right angle. The bisector of ∠**ABC** meets **AD** in **X** and **AC** in **Y**. Prove that $\frac{CY}{YA}=\frac{AX}{XD}$. (W)

11. **E** is the point in the side **AC** of a triangle **ABC**, such that **AE** : **EC**=1 : 4. **EF** is drawn parallel to **CB**, meeting **AB** in **F**; and **ED** is drawn parallel to **AB**, meeting **BC** in **D**. Show that the area of the parallelogram **BDEF** is $\frac{8}{25}$ of the area of the triangle **ABC**. (W)

---

12. Given a rectangle whose length and breadth are in the ratio of the diagonal and side of a square, prove that the perpendiculars from two opposite vertices trisect the diagonal joining the other two opposite vertices.

13. If **AX** and **BY** are altitudes of a triangle **ABC**, prove that △**CXY** : quad. **ABXY**=$XC^2 : XA^2$.

14. A circle is inscribed in a regular hexagon, and the points of contact are joined to form a second regular hexagon. Calculate the ratio of the areas of the two hexagons.

15. Two parallel tangents, which touch a circle at **P** and **Q**, are met at **S** and **T** respectively by the tangent at any other point **R**. Prove that the mean proportional between **PS** and **QT** is the radius of the circle.

16. Find by a geometrical construction two lengths **AP**, **PB** such that their sum is 9 cm. and the area of the rectangle contained by them is 16 sq. cm. State briefly your method. (N)

---

17. A triangle **ABC** has **AB**=12 in., **BC**=20 in., **CA**=10 in. **S** and **P** are points on **AB** and **AC** respectively (not produced), and **AS**=9 in., and **AP**= 6 in. **SR** is drawn parallel to **AC** to meet **BC** in **R**, and **PQ** is drawn parallel to **AB** to meet **BC** in **Q**. Calculate the length of **QR**.

If **T** is a point on **AB** (not produced) such that **AT**=5 in., calculate what fraction the area of the figure **PQRST** is of the area of the triangle **ABC**. (W)

18. The semiperimeter of a rectangle is 4 in.: its area equals that of a square, the length of a diagonal of which is 2·4 in. Construct geometrically, first the square, then the rectangle. What is the length of a diagonal of the rectangle? (O)

19. Fig. 359 shows a garden roller which is being pulled up a series of steps of different heights. Prove that the distance between the lines of contact of the roller with the ground and the roller with a step is proportional to the square root of the height of the step.

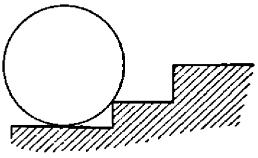


FIG. 359

20. Using ruler and compasses only, draw a circle of radius 1 in. and a regular hexagon (six sides) with vertices on the circumference of the circle.

With the same instruments, and without further measurement, find by construction the length of the side of a regular hexagon whose area is three times that of the hexagon already drawn. Prove that your method is correct. (C)

21. **ABC** is a triangle such that the angle at **C** is greater than the angle at **B**. The tangent at **A** to the circle through **A**, **B**, and **C** meets **BC** produced at **T**. Prove that $BT : CT = BA^2 : AC^2$. (C)

22. **ABCD** is a square of side 2 in. **E** is a point in **BC** 1·8 in. from **B**. Find a point **F** in **CD** produced such that the area of the triangle **FBE** is equal to the area of the square. State and prove your construction, which must be full size and made by use of ruler and compasses only, and not by calculation. (O)

THEOREM 45 (Rectangle Property of the Circle)

If two chords of a circle cut at a point inside the circle, the rectangle contained by the segments of the one chord is equal to the rectangle contained by the segments of the other chord.

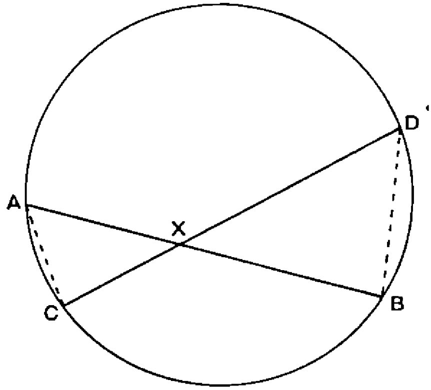


FIG. 360

Given two chords of a circle, AB, CD, cutting at X.

To prove that XA.XB=XC.XD.

Construction. Join AC, BD.

Proof. In the $\triangle$s AXC, DXB,

$\angle$AXC=$\angle$DXB (vert. opp.),

$\angle$CAX=$\angle$BDX (same seg.),

and $\angle$ACX=$\angle$DBX (,,);

$\therefore$ $\triangle$s $\begin{matrix}\text{AXC}\\\text{DXB}\end{matrix}$ are similar (AAA).

$$\therefore \frac{AX}{DX}=\frac{XC}{XB};$$

$\therefore$ XA.XB=XC.XD.

Reference. XA.XB=XC.XD (*rect. prop.*).

Corollary.

If O is the centre of the circle and the radius is r units, $XA.XB = XC.XD = r^2 - XO^2$.

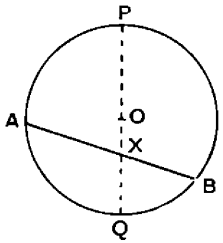


FIG. 361

For let XO produced meet the circle at P and Q.

Then $XA.XB = XP.XQ$ (rect. prop.).

But
$$XP.XQ = (OP + XO)(OQ - XO)$$
$$= (r + XO)(r - XO)$$
$$= r^2 - XO^2;$$

$$\therefore XA.XB = XC.XD = r^2 - XO^2.$$

THEOREM 46 (Rectangle Property of the Circle)

If two chords of a circle, when produced, cut at a point outside the circle, the rectangle contained by the segments of the one chord is equal to the rectangle contained by the segments of the other chord, and each rectangle is equal to the square on the tangent drawn from the point of intersection.

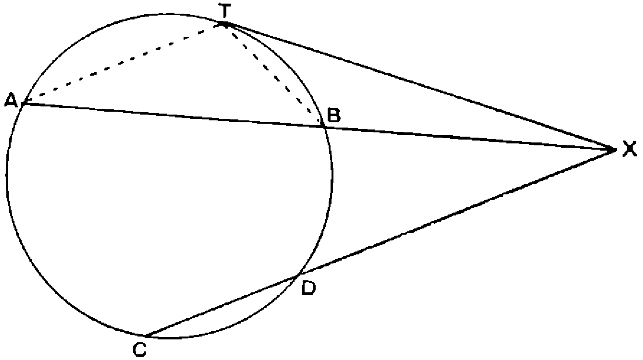


FIG. 362

Given two chords of a circle, AB, CD, produced to meet at X, and a tangent XT touching the circle at T.

To prove that $XA.XB = XC.XD = XT^2$.

Construction. Join TA, TB.

Proof. In the Δs XTB, XAT,

$\angle TXB = \angle AXT$ (same angle),

$\angle XTB = \angle XAT$ (alt. seg.),

and $\angle XBT = \angle XTA$ (3rd. $\angle$s of Δ);

$\therefore$ Δs $\begin{matrix} XTB \\ XAT \end{matrix}$ are similar (AAA).

$$\therefore \frac{XB}{XT} = \frac{XT}{XA};$$

$$\therefore XA.XB = XT^2.$$

Similarly, $XC.XD=XT^2$.

$\therefore XA.XB=XC.XD=XT^2$.

Reference. $XA.XB=XC.XD=XT^2$ (*rect. prop.*).

Corollary.

If O is the centre of the circle and the radius is r units, $XA.XB=XC.XD=XT^2=XO^2-r^2$.

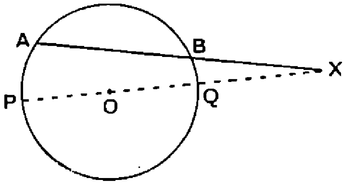


FIG. 363

For let XO produced cut the circle at P and Q.

Then $XA.XB=XP.XQ$ (rect. prop.).

But $XP.XQ=(XO+OP)(XO-OQ)$

$=(XO+r)(XO-r)$

$=XO^2-r^2$;

$\therefore XA.XB=XC.XD=XT^2=XO^2-r^2$.

Summary of Theorems 45 and 46.

Whether the chords cut inside the circle or not,

$$XA.XB=XC.XD=XO^2 \sim r^2=XT^2,$$

where $\sim$ implies that the smaller quantity is to be subtracted from the larger, and XT^2 is only included if XT can be drawn, *i.e.* if X is outside the circle.

Converses of Theorems 45 and 46.

1. (Rectangle test for concyclic points.) **If two straight lines, AB, CD, cut at X so that XA.XB=XC.XD, the points A, B, C, D are concyclic** (Fig. 364).

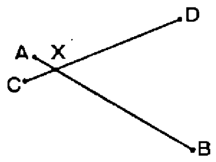


Fig. 364

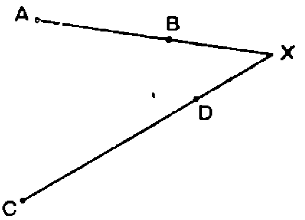


Fig. 365

2. (Rectangle test for concyclic points.) **If two straight lines, AB, CD, both produced, cut at X so that XA.XB=XC.XD, the points A, B, C, D are concyclic** (Fig. 365).

3. (Rectangle test for a tangent.) **If X is a point on AB produced, and T is a point such that XA.XB = XT^2, XT is the tangent at T to the circle through A, B, and T.**

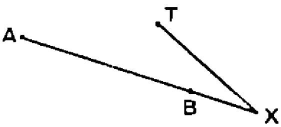


Fig. 366

EXAMPLES 26a

1. In Fig. 367, if AX=6 cm., BX=10 cm., and CX=5 cm., calculate DX.

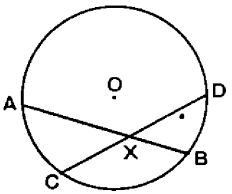
2. In Fig. 367, if AB=12 in., CX=4 in., and X is the mid-point of AB, calculate CD.

3. If the diagonals of a cyclic quadrilateral ABCD cut at O, and AC=11 cm., AO=2 cm., BO=3 cm., prove that BD=CO.

4. In Fig. 368, if AB=7 cm., XA=3 cm., and XD=7·5 cm., calculate CD.

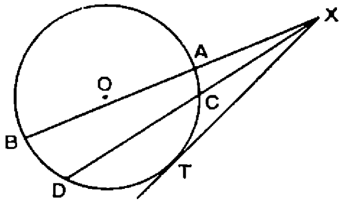
5. In Fig. 367, if **AX**=2 cm., **AB**=$6\frac{1}{2}$ cm., and **OX**=4 cm., calculate the radius of the circle.

6. In Fig. 367, if **AX**=2 in., **XB**=1 in., and **OA**=$2\frac{1}{2}$ in., calculate **OX** to the nearest tenth of an inch.



O is the centre of the circle

Fig. 367



O is the centre of the circle, **XT** a tangent

Fig. 368

7. In Fig. 368, if **XA**=9 in., **AB**=7 in., and the radius of the circle is 5 in., calculate **XO**.

8. In Fig. 368, if **XA**=3 in., **AB**=5 in., **XC**=4 in., **XO**=7 in., calculate **CD** and the radius of the circle.

9. In Fig. 368, if **XA**=4 cm. and **AB**=$2\frac{1}{4}$ cm., calculate **XT**.

10. **ABC** is a straight line, and **AB**=$2\frac{1}{4}$ in., **BC**=$1\frac{3}{4}$ in. **AD** is another straight line through **A**, and **AD**=3 in. Prove that the circumcircle of △**BCD** touches **AD** at **D**.

11. A bridge across a river is in the form of an arc of a circle. If the span is to be 40 ft. and the height of the bridge 8 ft., what must the radius of the circle be ?

---

12. In Fig. 367, if **XD**=$\frac{1}{2}$**XA**, prove that **XC**=2**XB**.

·13. A straight line **ACBD** cuts two intersecting circles at **A, B** and **C, D**, and cuts the common chord of the two circles at **X**. Prove that **XA.XB**=**XC.XD**.

14. Two circles intersect at **A** and **B** ; **P** is a point on **AB** produced ; through **P** two straight lines **PCD, PEF** are drawn, cutting one circle at **C, D** and the other circle at **E, F**. Prove that **PC.PD**=**PE.PF**. What conclusion can you draw about the points **C, D, F, E** ?

15. Two circles intersect at A and B ; PQ is a common tangent touching one circle at P and the other at Q ; AB produced meets PQ at X. Prove that PX = XQ.

16. Two circles intersect at A and B ; P is any point on AB produced. Prove that the tangents from P to the circles are equal.

17. Two non-intersecting circles are drawn, and a third circle intersects one of them at A, B and the other at C, D. AB and CD are produced to meet at P. Prove that the lengths of the tangents from P to all the three circles are the same.

18. The altitudes BE, CF of △ABC intersect at H. Prove that (i) BH.HE = CH.HF; (ii) AF.AB = AE.AC; (iii) CE.CA = CH.CF.

19. ABC is a triangle in which AB = $a\sqrt{2}$ units, AC = $2a$ units, and D is the mid-point of AC. Prove that the circle through B, C, and D touches AB at B.

20. AB, CD are chords of a circle ; P, Q are points outside the circle on AB, CD produced respectively such that PA . PB = QC . QD. Prove that P and Q are equidistant from the centre of the circle.

21. Two concentric circles are drawn, and P is a variable point on the larger circle. Any straight line is drawn through P cutting the smaller circle at A and B. Prove that the rectangle PA . PB is constant for all positions of P on the circle.

22. P is a variable point inside a circle of radius $2\frac{1}{2}$ in. If the rectangle contained by the segments of chords drawn through P is always 4 sq. in., prove that the locus of P is a circle concentric with the given circle, and find the radius.

EXAMPLES 26b

1. In Fig. 368, if AB = 12 in. and XT = 8 in., calculate XA.

2. The tangent from a point P to a circle is of length 6 cm., and the straight line through P perpendicular to this tangent first meets the circle at a distance of 3 cm. from P. Calculate the radius of the circle.

3. O is the centre of a circle of radius 2·6 in. ; P is a point 2·2 in. from O. A chord AB is drawn through P so that AP = 3BP. Calculate the length of this chord.

4. P is a point inside a circle. The shortest chord which can be drawn through P is 10 cm. long, and the shortest distance from P to the circle is 2 cm. Calculate the diameter of the circle.

5. If the base of an isosceles triangle is 12 in. long and the area is 54 sq.in., calculate the radius of the circumcircle of the triangle.

6. AB, CD are two perpendicular chords of a circle intersecting at a point X inside the circle; O is the centre of the circle; AX=6 cm., XB=8 cm., and CX=4 cm. Calculate OX and the radius of the circle to the nearest mm.

7. If water stands to a depth of 3 in. in a hemispherical bowl of radius 12 in., calculate in sq.in. the area of the plane surface of the water. ($\pi=\frac{22}{7}$.)

8. The pendulum of a clock is 8 in. long, and the vertical distance between the highest and lowest positions of the tip of the pendulum is $\frac{1}{2}$ in. Calculate to the nearest tenth of an inch the least possible width for the case of the clock.

---

9. Two circles intersect at X, Y. A diameter PX of the first circle is produced to cut the second circle again at Q, and a diameter RX of the second circle is produced to cut the first circle again at S. Prove that XP.XQ=XR.XS.

10. ABC is a triangle inscribed in a circle. A straight line parallel to the tangent at A cuts AB, AC at X, Y respectively. Prove that AX.AB=AY.AC.

11. In the triangle ABC, ∠B is twice ∠C. The bisector of ∠B meets AC at D. Prove that AB is the mean proportional between AC and AD.

12. O is the circumcentre of △ABC; X, Y, Z are points in the sides BC, CA, AB respectively such that BX.XC=CY.YA=AZ.ZB. Prove that O is the circumcentre of △XYZ.

13. AB is a diameter of a circle, and P, Q are points on the circle. The tangent at B meets AP, AQ produced at X, Y respectively. Prove that AP.AX=AQ.AY.

14. In Fig. 368, if the parallel through X to CB meets DA produced at P, prove that $PA.PD=PX^2$.

15. AB is a diameter of a circle ; P is any point on the circumference ; the tangent at P to the circle meets AB produced at C, and N is the foot of the perpendicular from P to AB. Prove that $CA.CB-NA.NB=CN^2$.

16. AB is a diameter of a circle, and C, D are two points on the circumference. If the straight line through D perpendicular to AB cuts BC at E, prove that BD is the mean proportional between BC and BE.

17. AB, CD are two chords of a circle intersecting within the circle at X, and AX=XB. The semicircle on CD as diameter is drawn, and the straight line through X perpendicular to CD meets the semicircle at Y. Prove that XY=XA.

18. AB is a diameter of a circle, and P is the point of trisection of AB nearer to B. Prove that the shortest chord of the circle through P is equal to the diagonal of the square which has AP as a side.

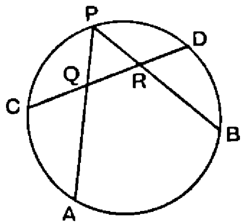


Fig. 369

19. In Fig. 369, in which PQ=QR=RP, prove that QA−RB=QC−RD. (Hint: apply Theorem 45 to the chords AP, CD and again to the chords CD, PB.)

20. A, B, C are three points in a straight line. A series of circles is drawn through the points B and C, and tangents are drawn from A to each of these circles. State the locus of the points of contact of these tangents, and give a proof.

21. A is a fixed point on a circle whose centre is O. P is a variable point outside the circle, and PA (produced if necessary) meets the circle again at Q. If P moves so that the rectangle PA.PQ is constant, state the locus of P, and give a proof.

22. A is a fixed point, and DE a fixed straight line not passing through A. P is a variable point on DE, and Q is a point on AP such that AP.AQ is constant. Prove that the locus of Q is a circle passing through A.

23. Prove the validity of the following method for constructing a circle to pass through two given points A and B and to touch a given straight line **CD** (not parallel to **AB**) : produce **AB** to meet **CD** at **P** ; construct the mean proportional **PQ** to **PA** and **PB** ; with centre **P**, radius **PQ**, draw an arc to cut **CD** at **R** ; construct the circle passing through **A**, **B**, and **R**. Then this is the required circle.

Show that there are two solutions.

Draw an accurate figure in which the perpendicular distances of **A** and **B** from **CD** are respectively 2 in. and $\frac{1}{2}$ in., and **AB**$=1\frac{3}{4}$ in.

24. Prove the validity of the following method for constructing a circle to pass through two given points **A** and **B** and to touch a given circle : draw any circle to pass through **A** and **B** cutting the given circle at **P** and **Q** ; produce **AB** and **PQ** to meet at **O** ; draw a tangent **OT** from **O** to the given circle ; construct the circle passing through **A**, **B**, and **T**. Then this is the required circle.

Show that there are two solutions.

Draw an accurate figure in which the given circle has a radius of 1 in., the distances of **A** and **B** from the centre of that circle are respectively 2 in. and 1·6 in., and **AB**$=1{\cdot}2$ in.

The distance of the Visible Horizon.

In Fig. 370, **ACB** represents a diameter of the earth, and **X** is a point vertically above **A**. If **XT** is a tangent, then **XT** represents the greatest distance which can be seen from **X** owing to the curvature of the earth, and **XT** is called the ' distance of the visible horizon.'

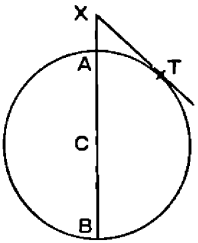


Fig. 370

25. Remembering that **XA** is small compared with the radius, prove that $XT = \sqrt{2XA \,.\, AC}$ approx.

26. If $XA = h$ ft., $XT = d$ miles, and $AC = 3960$ miles, deduce from No. 25 that $d = \sqrt{3h/2}$ approx.

27. The height of Snaefell in the Isle of Man is 2034 ft. Calculate to the nearest mile the distance of the visible horizon.

28. From the top of a tower 150 ft. high the light of a lighthouse 96 ft. high can only just be seen, owing to the curvature of the earth. Using the formula proved in No. 26, calculate approximately the distance between the tower and the lighthouse.

EXAMPLES 26c

1. A point **C** is 5 in. distant from the centre of a circle of radius 9 in. A point **P** on the circumference is 8 in. distant from **C**, and **PC** produced cuts the circle again at **Q**. Calculate the length of **CQ**. (N)

2. **AB** is a diameter of a circle and through a point **C** in **AB** or **AB** produced in either direction, a perpendicular is drawn to **AB**. **A** is joined to any point **P** on the circle and **AP**, produced if necessary, meets this perpendicular at **Q**. Prove that the rectangle contained by **AP** and **AQ** is equal to the rectangle contained by **AB** and **AC**. (O)

3. **AB** is a chord of a given circle, and the angle in the smaller of the two segments into which **AB** divides the circle is 135°. A semicircle is described on **AB** as diameter on the same side of **AB** as the smaller segment of the given circle. Prove that the square on the tangent drawn from **P**, a point on this semicircle, to the given circle, is equal to the rectangle contained by **PA** and **PB**. (O)

4. Two circles, of unequal radii, intersect at **A** and **B**, and from a point **T** on **AB** produced a secant **TXY** is drawn meeting the first circle at **X** and **Y**. A tangent is drawn from **T** to the second circle touching it at **Z**. Prove that the circle which passes through **X**, **Y**, and **Z** touches the straight line **TZ** at **Z**. (L)

5. Two circles intersect at **A** and **B**. **CAD** is a straight line cutting one circle at **C** and the other circle at **D**. **CE** is a tangent from **C** to the circle **ABD**, and **DF** is a tangent from **D** to the circle **ABC**, **E** and **F** being the points of contact of the tangents. Prove that $CE^2+DF^2=CD^2$. (L)

---

6. A chord **AB** of a circle of unknown radius is 5 cm. long. **AB** is produced to **C** so that **BC** is 3 cm. From **C** another straight line **CDE** is drawn cutting the circle at **D** and **E**. **CD** is 4 cm. long. What is the length of the chord **DE** ? If the centre of the circle is 7 cm. from **C**, what is the radius of the circle ? (OC)

7. **AB** is a diameter of a circle whose centre is **O** and radius is 4 cm. **P** is a point distant 6 cm. from **O**. The circle through **P**, **A** and **B** cuts **PO** produced at **X**. Find and state precisely the locus of **X** as **P** moves so that it remains 6 cm. from **O**. (L)

8. O is any point on the straight line joining two points A and B. Any circle is drawn to pass through A and B. Prove that the length of the chord drawn through O perpendicular to the straight line joining O to the centre of the circle is independent of the size of the circle. (O)

9. The diagonals AC, BD of a parallelogram ABCD intersect at O, and P is any point on AC. The circumcircles of Δs BPA, BPC cut BD again at X and Y. Prove that OX=OY.

10. In a circle are two chords AKC, BKD at right angles. Given AK=20; KC=36, AB=25, calculate KB, KD, and the radius. (W)

---

11. From a point A the two tangents are drawn to a circle to touch it at B and C. A line APSQ cuts the circle at P, Q, and cuts BC at S. If O is the middle point of BC, prove (i) that $PS.SQ=OB^2-OS^2$, and deduce (ii) that $AB^2=AS^2+PS.SQ$. (N)

12. ABC is a triangle inscribed in a circle; the bisector of $\angle A$ meets BC at D and the circle at E. Prove that (i) Δs ACD, AEB are similar, (ii) $AB.AC-DB.DC=AD^2$.

13. D is the middle point of the side BC of a triangle ABC. The circle, which touches AB at A and passes through D, cuts BC again at X. Prove that, if BA is produced to P so that AP=BA, the points A, C, X, and P are concyclic. (C)

14. ABC is a triangle right-angled at C; AC is 5 in. long and BC is 12 in. long. A circle of radius 9 in. touches BC at B and has its centre O on the same side of BC as A. The circle cuts AB again at D. Calculate the length of AO (leaving the answer as a square root) and the length of AD. (C)

15. Water is flowing in a channel of circular section whose radius is r. The width of the stream is $2k$, and the depth at the middle point is h.

(i) Using the theorem about the segments of intersecting chords of a circle, show that $h^2-2rh+k^2=0$.

(ii) Find the depth of the stream if the radius of the channel is 10 ft. and the width of the stream 16 ft.

(iii) You should obtain two values for h: show by a rough sketch the meaning of these two solutions. (OC)

MISCELLANEOUS EXAMPLES IV

1. A chord of length 32 cm. is placed inside a circle of 20 cm. radius, and a point, whose distance from the centre of the circle is 13 cm., is marked on the chord. *Calculate* the lengths of the segments of the chord. (OC)

2. ABC is an acute-angled triangle. Two circles are drawn, one touching AC at A and passing through B, the other touching AB at A and passing through C. The two circles intersect again at O. Prove that (i) the triangles AOB, COA are equiangular; (ii) $\angle BOC = 2\angle BAC$; (iii) $AO^2 = BO.OC$. (OC)

3. Use Theorem 46 to obtain $\sqrt{4 \cdot 8}$ by drawing and measurement.

4. ABC is a triangle and AD is a median. The angles at D are bisected by lines which cut AB and AC at E and F. EF cuts AD at G. Prove that EF is parallel to BC, and that $EG = GF = GD$. (OC)

5. ABC, CAP are two triangles on opposite sides of the same line AC such that $AB : BC : CA = CA : AP : PC$. Prove (i) that PC is parallel to AB, (ii) that $BC^2/AP^2 = AB/PC$. (OC)

---

6. A straight line APB is such that $AP = 4$ cm. and $PB = 6$ cm. Another straight line CPD is such that $CP = 3$ cm. and CD is a diameter of the circle passing through the points A, B, C, and D. *Calculate* the radius of this circle. A third line through P bisects $\angle CPB$ and meets AD at E. Find the ratio $AE : ED$. (N)

7. Two unequal circles whose centres are A and B touch at C, and P, Q are their respective points of contact with a direct common tangent. The lines PA, QB are produced to cut these circles at L, M respectively. Prove (i) that M, C, P are in one straight line, (ii) that $PQ^2 = PL.QM$. (N)

8. O is a point outside a circle, and straight lines OAB, OCD cut the circle at A, B, C, D. The bisector of $\angle AOC$ cuts AC at H and BD at K. Prove that $AH/HC = DK/KB$.

9. The bisector of the angle A of a triangle ABC meets BC at D. The circle through A, B, D cuts AC at P, and the circle through A, C, D cuts AB at Q. Prove that BQ = CP.

10. Two unequal circles intersect at A, B, and O is any point on AB produced. A circle with centre O cuts one of the circles at P and Q, and the other at L and M. OP, OM, produced if necessary, cut the circles PQA, LMA at S and T respectively. Prove that PS = TM, and that the centres of the original circles lie on the bisectors of the angles POQ, LOM. (N)

---

11. ABC is a triangle right-angled at A, and the bisector of ∠A meets BC at E. If AD is an altitude of △ABC, and AC = 21 cm., AB = 28 cm., calculate the length of DE. (OC)

12. Two circles intersect at A and B. Prove that AB produced bisects the common tangent, XY, to the circles.

If AB = 6 in., XY = 8 in., find the distance of B from the middle point of XY. (W)

13. ABCD is a quadrilateral; AB and DC produced meet in E, and AD and BC produced meet in F. If 2BC = CF and 2DC = CE, show that AD.BE = AB.DF. (OC)

14. An arc of a circle of radius r is cut off by a chord of length $2a$, and h is the distance of the middle point of the arc from the chord. Prove that $r = (h^2 + a^2) \div 2h$.

P and Q are fixed points within a given circle and AB is a variable chord of this circle always passing through Q. Show that, for different positions of the chord AB, the circle passing through A, B, and P meets PQ produced at a fixed point. (OC)

15. The point P is the centre of the inscribed circle of the triangle ABC; the point Q is the centre of the circle which touches BC and the other two sides produced. Perpendiculars to AB are drawn from P and Q meeting AB, produced if necessary, at D and E respectively. Prove that (i) B, P, C, Q are concyclic points; (ii) the triangles ABP, AQC are similar; (iii) AB.AC = AP.AQ; (iv) AP^2 : AB.AC = AD : AE. (N)

---

16. Show that a triangle of sides 25, 24, 7 in. is right-angled, and calculate (i) the lengths of the segments into which the bisector of the smallest angle divides the opposite side, and (ii) the length of this bisector, terminated by the opposite side. Give your answers to the nearest tenth of an inch. (OC)

17. ABC is a triangle, right-angled at B, in which AB is x units, and BC is y units long. BC is produced to D, so that $BD=\frac{1}{3}AB$, and CE is perpendicular to AD. Show that $CE=\frac{1}{5}(4x-3y)$ units and that $AE=\frac{1}{5}(3x+4y)$ units.

Express the equation $AC^2=AE^2+EC^2$ in terms of x and y. (OC)

18. Two circles touch externally at T; a common tangent (other than that at T) touches the circles at A, B respectively and meets the common tangent at T at the point C. Prove that the lines joining C to the centres of the circles are perpendicular, and that the square on AB is equal in area to the rectangle contained by the diameters of the circles. (OC)

19. From a point D on the base BC of the triangle ABC, straight lines parallel to CA, BA are drawn to cut AB, AC at E, F; the lengths of BD and DC are x and y. Find the ratios of the areas of the triangles BDE, DCF to the area of the triangle ABC, and prove that the area of the parallelogram AEDF is to that of the triangle ABC as $2xy:(x+y)^2$. (OC)

20. I is the incentre of △ABC; IE, IF are drawn parallel to AB, AC to meet BC at E, F respectively. Prove that

$$BE : FC = AB : AC.$$

---

21. A conical cap just covers two spheres placed one above the other on a table as in Fig. 371. If the radii of the spheres are 1 in. and $2\frac{1}{4}$ in., find the height of the cone and the radius of its base. (OC)

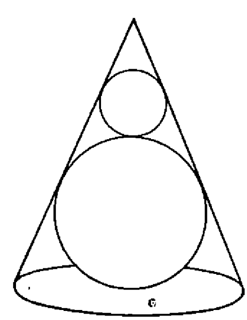


FIG. 371

22. PQ is the common chord of two given intersecting circles and A is any point on PQ produced. Prove that a circle can be drawn with its centre at A so as to cut both the given circles at right angles. (OC)

23. ABCD is a cyclic quadrilateral and BD, AC meet at O. Prove (i) that the triangles AOD, BOC are similar, and (ii) that AD.CD : BC.AB = DO : BO.

What theorem can you deduce by taking the special case 'C is the middle point of the arc BD'? (OC)

24. AB is a diameter of a circle and points P and S are taken on the semicircle APSB, P being the nearer to A. T is a point on AS such that $AT.AS = AP^2$, and PT is produced to meet AB at K. Prove that (i) the triangles APS, ATP are similar; (ii) the quadrilateral TKBS is inscribable in a circle; (iii) PK is perpendicular to AB. (OC)

25. AB, AC are the tangents from A to a circle, and ADE is a straight line cutting the circle at D, E. Prove that BE.CD = BD.CE.

26. ABC is a triangle in which ∠B = 2∠C. The bisector of ∠B meets AC in E; and EF, drawn parallel to CB, meets AB in F. Prove that AE.BE = AF.BC. (W)

---

27. ABCD is a rectangle, in which AB = 8 in., BC = 6 in. From O, the mid-point of the diagonal AC, OE is drawn at right angles to AC to meet AB in E. Calculate the length of AE and the area of the triangle AOE.

28. If in a triangle ABC, having the angles at B and C acute, the altitude AD is a mean proportional between BD and DC, prove that the angle BAC is a right angle.

If BD = 3, AD = 6, DC = 12, and if the bisector of the angle BAC cuts BC at E, find BE and EC. (OC)

29. An isosceles triangle stands on a base BC; its altitude is h in., the equal sides AB, AC are of length a in. A circle of radius r in. touches AB at B and touches AC at C; the centre of the circle is at O. Prove that O, A, B, C lie on a circle, and use this fact as a first step in proving that $r^2 = \frac{a^2}{h^2}(a^2 - h^2)$. (OC)

30. A chord BC of a circle is parallel to the tangent at a point A on the circle; P is any other point on the circle, and the tangent at A meets PB, PC produced at X, Y respectively. Prove that XA : AY = XB : YC.

31. $\triangle ABC$ is right-angled at C; D is a point in AB such that $CB = CD$. Prove that $BD . BA = 2BC^2$.

---

32. In Fig. 372, which is not drawn to scale, A and B are points on the opposite sides of a river. C is a point such that C, A, B are in a straight line; AH, CK are set off at right angles to CB and H, K are taken so that K, H, B are in a straight line. $AH = 37$ yd., $AC = 30$ yd., and $KC = 52$ yd.

Calculate the distance AB.

Show also that AB can be determined by setting off HZ at right angles to HB, so that Z, A, B are in a straight line, and using the measurements of AH and AZ only.

What length of AZ will give the length of AB previously obtained? (OC)

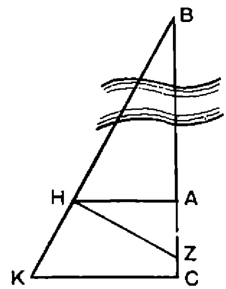


FIG. 372

33. AD, BC are the parallel sides of a trapezium ABCD, and the diagonals AC, BD cut at O. The areas of the triangles AOB, BOC are 3 sq. in. and 2 sq. in. respectively. Calculate the areas of the triangles COD, AOD. (N)

34. D and E are points in the side BC of $\triangle ABC$ such that $\angle BAD = \angle CAE$; the circle through A, D, and E cuts AB, AC again at X, Y respectively. Prove that XY is parallel to BC, and that $BD . BE : CD . CE = AB^2 : AC^2$.

35. Two circles touch externally at A; points B, C are taken, one on each circle, so that $\angle BAC$ is a right angle, and BC produced meets the line of centres at X. Prove that X divides the line of centres externally in the ratio of the radii.

36. The triangle ABC is right-angled at A, and D is the foot of the perpendicular from A to BC. The internal bisector of the angle ABC cuts AD in X, and the internal bisector of the angle DAC cuts DC in Y. Prove that XY is parallel to AC. (C)

37. A, B are the points of contact of the tangents from a point T to a circle, and P is any point on the circle; L, M, N are the feet of the perpendiculars from P to TA, TB and AB respectively. Prove that $PL . PM = PN^2$.

APPENDIX

THEOREM 4

If two angles and a side of one triangle are respectively equal to two angles and the corresponding side of another triangle, then the two triangles are congruent.

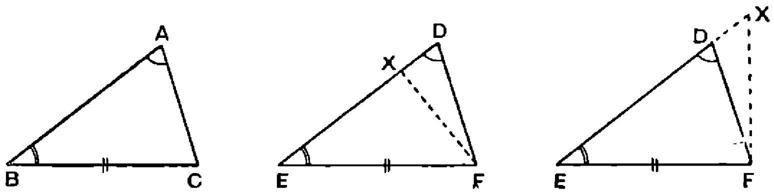


FIG. 373

Given two triangles ABC, DEF, in which BC=EF, ∠B=∠E, ∠A=∠D.

Proof. ∠A+∠B=∠D+∠E ;

∴ ∠C=∠F (∠s of △).

Suppose that BA *is not equal to* ED. Then there is a point X on ED or ED produced such that EX=BA. Join XF.

In the △s ABC, XEF,

BA=EX (constr.),

BC=EF (given),

∠ABC=∠XEF (given) ;

∴ △s ABC, XEF are congruent (SAS).

∴ ∠BCA=∠EFX.

But ∠BCA=∠EFD (proved) ;

∴ ∠EFX=∠EFD.

This is impossible, since one is only a part of the other;

∴ our supposition is wrong;

∴ **BA = ED.**

Hence, in the △s **ABC, DEF,**

BA = ED (proved),

BC = EF (given),

∠**ABC** = ∠**DEF** (given);

∴ △s $\frac{\textbf{ABC}}{\textbf{DEF}}$ are congruent (SAS).

Reference. △s $\frac{\textbf{ABC}}{\textbf{DEF}}$ are congruent (*AA corr. S*).

THEOREM 6

If the three sides of one triangle are respectively equal to the three sides of another triangle, then the two triangles are congruent.

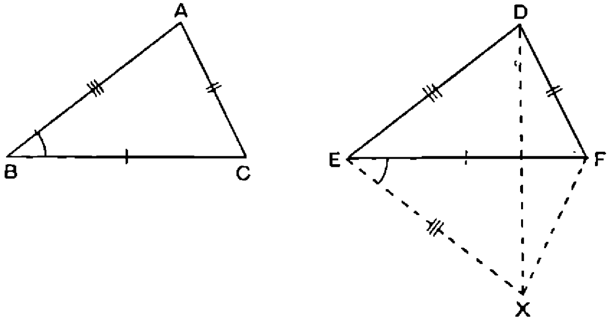


FIG. 374

Given two triangles ABC, DEF, in which BC=EF, CA=FD, AB=DE, the letters being chosen so that BC and EF are the greatest sides.

Construction. On the side of EF opposite to D, suppose that an angle FEX is drawn equal to ∠ABC, and EX is marked off equal to AB. Join DX.

Proof. In the △s ABC, XEF,

AB=XE (constr.),

∠ABC=∠XEF (constr.),

BC=EF (given);

∴ △s $\begin{matrix}ABC\\XEF\end{matrix}$ are congruent (SAS).

∴ AC=XF, ∠BAC=∠EXF; also AB=XE (constr.).

But AB = DE, AC = DF (given),

∴ DE = XE, DF = XF ;

∴ ∠EXD = ∠EDX, ∠FXD = ∠FDX.

∴ ∠EXF = ∠EDF.

In the △s DEF, XEF,

DE = XE (proved),

DF = XF (proved),

∠EDF = ∠EXF (proved),

∴ △s $\begin{smallmatrix}\text{DEF}\\\text{XEF}\end{smallmatrix}$ are congruent (SAS).

But △s $\begin{smallmatrix}\text{ABC}\\\text{XEF}\end{smallmatrix}$ are congruent (proved) ;

∴ △s $\begin{smallmatrix}\text{ABC}\\\text{DEF}\end{smallmatrix}$ are congruent.

Reference. △s $\begin{smallmatrix}\text{ABC}\\\text{DEF}\end{smallmatrix}$ are congruent (*SSS*).

THEOREM 7

If two right-angled triangles have the hypotenuse and another side of the one triangle respectively equal to the hypotenuse and another side of the other triangle, then the two triangles are congruent.

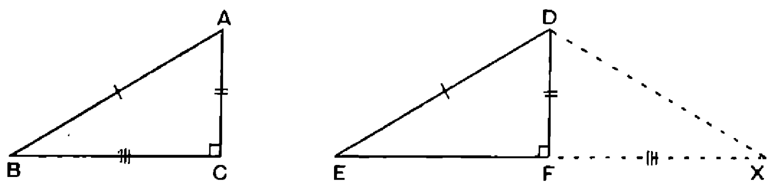


FIG. 376

Given two triangles ABC, DEF, in which $\angle C = \angle F = 90°$, $AB = DE$, $AC = DF$.

Construction. Produce EF to X so that $FX = CB$. Join DX.

Proof. $\angle DFE = 90°$ (given),

$\therefore \angle DFX = 90°$ (EFX a st. line).

In the Δs ABC, DXF,

$AC = DF$ (given),

$\angle ACB = \angle DFX$ (90°),

$CB = FX$ (constr.);

$\therefore \Delta s \begin{matrix} ABC \\ DXF \end{matrix}$ are congruent (SAS).

$\therefore AB = DX$.

But $AB = DE$ (given);

$\therefore DX = DE.$

$\therefore \angle DEX = \angle DXE.$

In the $\triangle$s DXF, DEF,

$DX = DE$ (proved),

$\angle DXF = \angle DEF$ (proved),

$\angle DFX = \angle DFE$ (90°);

$\therefore \triangle s \begin{matrix} DXF \\ DEF \end{matrix}$ are congruent (AA corr. S).

But $\triangle s \begin{matrix} ABC \\ DXF \end{matrix}$ are congruent (proved),

$\therefore \triangle s \begin{matrix} ABC \\ DEF \end{matrix}$ are congruent.

Reference. $\triangle s \begin{matrix} ABC \\ DEF \end{matrix}$ are congruent (*RHS*).

THEOREM 19b

If two triangles are on the same base and on the same side of it, and are equal in area, then they are between the same parallels.

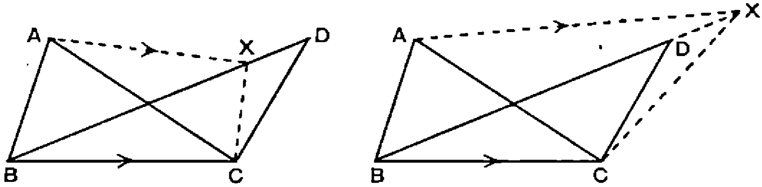


FIG. 376

Given two △s ABC, DBC on the same base BC, on the same side of it, and equal in area.

To prove that AD ∥ BC.

Proof. *Suppose* AD *is not parallel to* BC.

Let the line through A parallel to BC meet BD or BD produced at X. Join CX.

Then △ABC = △XBC (same base BC, same ∥s BC, AX).

But △ABC = △DBC (given);

∴ △XBC = △DBC.

This is impossible, since one is only a part of the other;

∴ our supposition is wrong;

∴ AD ∥ BC.

THEOREM 31

If the straight line joining two points subtends equal angles at two other points on the same side of it, then the four points are concyclic.

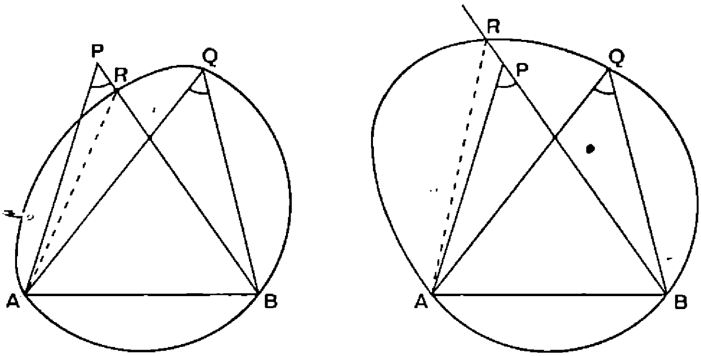


FIG. 377

Given equal angles APB, AQB on the same side of AB.

To prove that A, P, Q, B are concyclic.

Construction. Draw the circle through A, B, and Q, *and suppose this circle does not pass through* P.

Since BP lies in the angle ABQ,[1] the circle must cut BP or BP produced; let it do so at R, and join AR.

Proof. $\angle$ARB $=\angle$AQB (same seg.).

But $\angle$APB $=\angle$AQB (given);

$\therefore$ $\angle$ARB $=\angle$APB.

But this is impossible, because one of these is the exterior angle and the other the interior opposite angle of $\triangle$APR.

$\therefore$ our supposition is wrong;

$\therefore$ P lies on the circle through A, B, and Q.

[1] BP cannot lie along BQ, for then one of the equal angles APB, AQB would be an exterior angle, and the other an interior opposite angle, of $\triangle$ APQ.

THEOREM 32

If a pair of opposite angles of a quadrilateral are supplementary, the quadrilateral is cyclic.

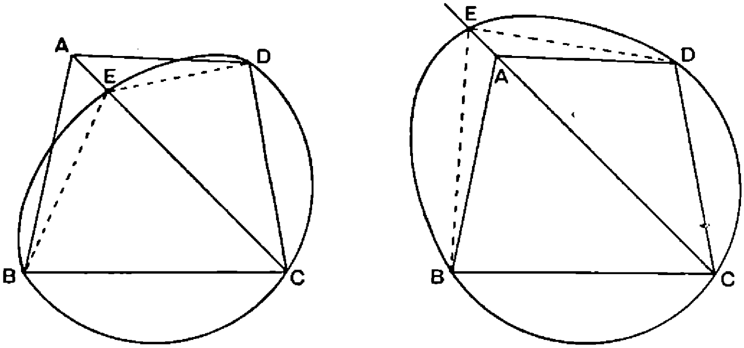


FIG. 378

Given a quadrilateral ABCD in which $\angle ABC + \angle ADC = 180°$.

To prove that ABCD is cyclic.

Construction. Draw the circle through B, C, and D, *and suppose this circle does not pass through* A.

Since CA lies in the angle BCD, the circle must cut CA or CA produced; let it do so at E, and join EB, ED.

Proof. $\angle EBC + \angle EDC = 180°$ (opp. $\angle$s of cyclic quad.).

But $\angle ABC + \angle ADC = 180°$ (given);

$\therefore \angle EBC + \angle EDC = \angle ABC + \angle ADC$.

But this is impossible, because the part cannot equal the whole.

$\therefore$ our supposition is wrong;

$\therefore$ A lies on the circle through B, C, and D.

THEOREM 33

The circle described on the hypotenuse of a right-angled triangle as diameter passes through the opposite vertex.

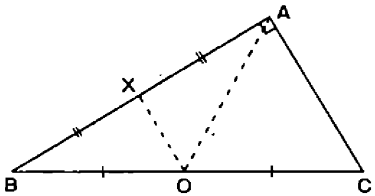


FIG. 379

Given a $\triangle ABC$ with a right angle at A.

To prove that the circle on BC as diameter passes through A.

Construction. Let O, X be the mid-points of BC, BA respectively. Join OA, OX.

Proof. BO=OC and BX=XA (constr.),

$\therefore$ OX $\parallel$ CA (mid-point theorem).

$\therefore$ $\angle BXO = \angle BAC$ (corr., OX $\parallel$ CA)

$= 90°$ (given).

$\therefore$ OX is the perpendicular bisector of AB.

$\therefore$ OX is the locus of points equidistant from A and B;

$\therefore$ OA=OB.

But OB=OC (constr.);

$\therefore$ OA=OB=OC.

$\therefore$ the circle, centre O and radius OB, passes through C and A, and this is the circle on BC as diameter.

THEOREM 34a

The straight line drawn perpendicular to a radius of a circle at its extremity is a tangent to the circle.

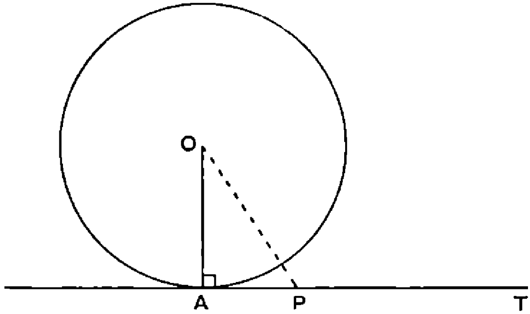


FIG. 380

Given a circle, centre O, a point A on the circle, and the straight line AT perpendicular to OA.

To prove that AT is a tangent.

Construction. *Suppose* AT *is not a tangent.*

Then AT, produced either way if necessary, will meet the circle again. Let it do so at P; join OP.

Proof. OA = OP (radii);

$\therefore$ $\angle$OAP = $\angle$OPA.

But $\angle$OAP = 90° (given);

$\therefore$ $\angle$OPA = 90°.

$\therefore$ two angles of $\triangle$OAP are right angles, which is impossible;

$\therefore$ our supposition is wrong;

$\therefore$ AT is a tangent.

THEOREM 34b

A tangent to a circle is perpendicular to the radius drawn through the point of contact.

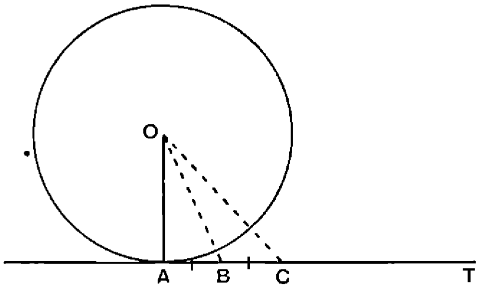


FIG. 381

Given a circle, centre O, and AT a tangent to the circle at A.

To prove that OA is perpendicular to AT.

Construction. *Suppose* OA *is not perpendicular to* AT. Then let OB be the perpendicular from O to AT, meeting it at B.

On the other side of B from A, take a point C in AT so that BC=BA. Join OC.

Proof. In the △s OBA, OBC,

BA=BC (constr.),

OB=OB,

∠OBA=∠OBC (rt. ∠s, constr.);

∴ △s $\frac{\text{OBA}}{\text{OBC}}$ are congruent (SAS).

∴ OA=OC.

∴ C lies on the circle.

But this is impossible, because the tangent cannot have two points, A and C, in common with the circle;

∴ our supposition is wrong;

∴ OA is perpendicular to AT.

THEOREM 37b

If, through one end of a chord of a circle, a straight line is drawn making with the chord an angle equal to the angle in the alternate segment, then the straight line is a tangent to the circle.

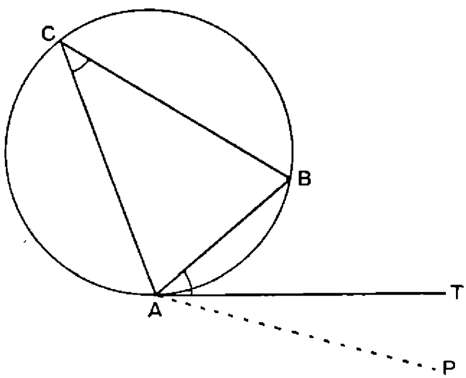


Fig. 382

Given a chord AB of a circle, and a straight line AT through A such that $\angle BAT = \angle BCA$ in the alternate segment.

To prove that AT is a tangent to the circle ABC.

Construction. *Suppose that* AT *is not a tangent.*

Draw the tangent AP at A, P being on the same side of AB as T.

Proof. $\angle$BAP $=$ $\angle$BCA (alt. seg.).

But $\angle$BAT $=$ $\angle$BCA (given);

$\therefore$ $\angle$BAP $=$ $\angle$BAT,

which is impossible.

$\therefore$ our supposition is wrong;

$\therefore$ AT is a tangent to the circle ABC.

Reference. *$\angle BAT = \angle BCA$ in the alt. seg.,*

$\therefore$ AT is a tangent to the circle ABC.

INTRODUCTORY THEOREM ON RATIO

(*a*) **There is only one point which divides a given straight line internally in a given ratio.**

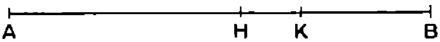


FIG. 383

Given a straight line AB, and a point H on it.

To prove that there is no point other than H which divides AB internally in the ratio AH : HB.

Construction. *Suppose there is another point,* K, *which divides* AB *internally in the ratio* AH : HB. Let AB$=a$ units, AH$=h$ units, AK$=k$ units; then k is not equal to h.

Proof.

$$\frac{AH}{HB}=\frac{AK}{KB};$$

$$\therefore \frac{h}{a-h}=\frac{k}{a-k};$$

$$\therefore ah-hk=ak-hk;$$

$$\therefore ah-ak=0;$$

$$\therefore a(h-k)=0;$$

$$\therefore a=0, \text{ or } h-k=0.$$

But a cannot be zero, and k is not equal to h;

$\therefore$ our supposition is wrong;

$\therefore$ H is the only point which satisfies the conditions.

(*b*) **There is only one point which divides a given straight line externally in a given ratio.**

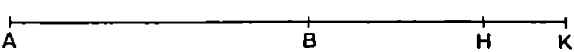


FIG. 384

With the same notation, the condition $\frac{AH}{HB}=\frac{AK}{KB}$ gives

$$\frac{h}{h-a}=\frac{k}{k-a}.$$

Except that all signs are reversed, the subsequent working is the same as in (*a*).

THEOREM 38b (Ratio Test for Parallels)

If a straight line divides two sides of a triangle in the same ratio, both internally or both externally, it will be parallel to the third side.

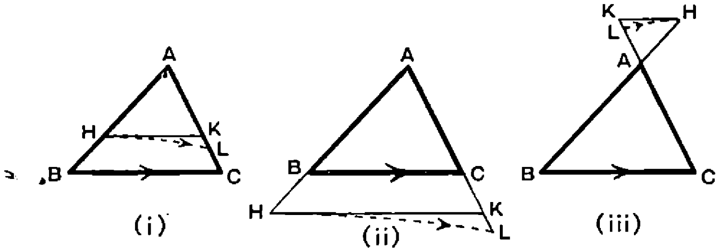


FIG. 385

Given a △ABC, and a straight line cutting AB, AC (produced if necessary) at H and K so that AH : HB=AK : KC.

To prove that HK ∥ BC.

Construction. *Suppose* HK *is not parallel to* BC. Then let the parallel to BC through H meet AC (produced if necessary) at L.

Proof. In △ABC, HL ∥ BC;

∴ AH : HB=AL : LC.

But AH : HB=AK : KC (given);

∴ AL : LC=AK : KC.

∴ L and K both divide AC in the same ratio, internally in Fig. 385 (i), externally in Fig. 385 (ii) and (iii), which is impossible;

∴ our supposition is wrong;

∴ HK ∥ BC.

THEOREM 40

If the three angles of one triangle are respectively equal to the three angles of another triangle, then the two triangles are similar.

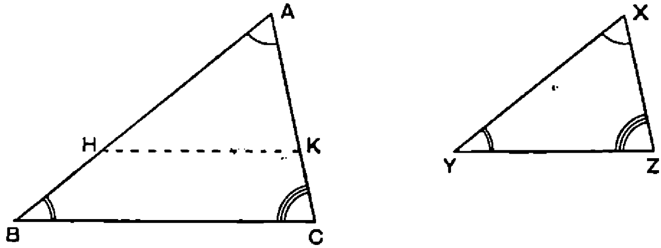


FIG. 386

Given two triangles, ABC, XYZ, in which ∠A = ∠X, ∠B = ∠Y, ∠C = ∠Z.

To prove that △ABC ||| △XYZ.

Construction. On AB, AC take points H, K, so that AH = XY, AK = XZ.

Proof. In the △s AHK, XYZ,

AH = XY (constr.),

AK = XZ (constr.),

and ∠A = ∠X (given);

∴ △s AHK, XYZ are congruent (SAS).

∴ ∠AHK = ∠Y.

But ∠Y = ∠B (given);

∴ ∠AHK = ∠B.

But these are corresponding angles;

$$\therefore HK \parallel BC.$$

$$\therefore \frac{AH}{AB}=\frac{AK}{AC}.$$

But $AH=XY$, $AK=XZ$ (constr.);

$$\therefore \frac{XY}{AB}=\frac{XZ}{AC}.$$

Similarly, $$\frac{ZX}{CA}=\frac{ZY}{CB};$$

$$\therefore \frac{XY}{AB}=\frac{XZ}{AC}=\frac{ZY}{CB}.$$

Also $\angle A=\angle X$, $\angle B=\angle Y$, $\angle C=\angle Z$ (given);

$$\therefore \triangle ABC \parallel\!| \triangle XYZ.$$

Reference. $\triangle$s $\begin{matrix}ABC\\XYZ\end{matrix}$ are similar (*AAA*).

THEOREM 41

If two sides of one triangle are proportional to two sides of another triangle, and the included angles are equal, then the two triangles are similar.

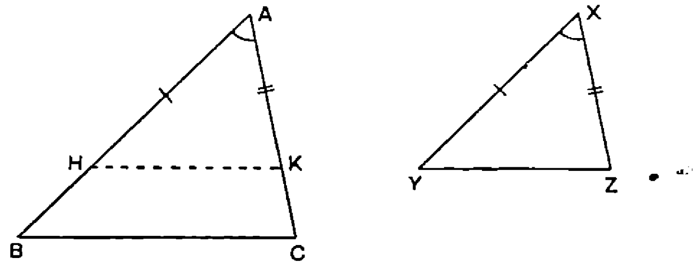


Fig. 387

Given two triangles, ABC, XYZ, in which $\angle A = \angle X$, and $\frac{AB}{XY} = \frac{AC}{XZ}$.

To prove that $\triangle ABC \parallel\!| \triangle XYZ$.

Construction. On AB, AC take points H, K so that $AH = XY$, $AK = XZ$.

Proof. In the $\triangle$s AHK, XYZ,

$AH = XY$ (constr.),

$AK = XZ$ (constr.),

and $\angle A = \angle X$ (given);

$\therefore \triangle$s $\begin{matrix} AHK \\ XYZ \end{matrix}$ are congruent (SAS).

$\therefore \angle AHK = \angle Y$, $\angle AKH = \angle Z$.

Now $\frac{AB}{XY}=\frac{AC}{XZ}$ (given),

and AH = XY, AK = XZ (constr.);

$\therefore \frac{AB}{AH}=\frac{AC}{AK}$;

$\therefore$ HK ∥ BC.

$\therefore$ ∠AHK = ∠B, ∠AKH = ∠C (corr., HK ∥ BC).

But ∠AHK = ∠Y, ∠AKH = ∠Z (proved);

$\therefore$ ∠B = ∠Y, ∠C = ∠Z.

In the △s ABC, XYZ,

∠A = ∠X (given),

∠B = ∠Y (proved),

and ∠C = ∠Z (proved);

$\therefore$ △s $\begin{matrix}ABC\\XYZ\end{matrix}$ are similar (AAA).

Reference.

△s $\begin{matrix}ABC\\XYZ\end{matrix}$ are similar (2 *sides prop., incl.* ∠*s equal*).

THEOREM 42

If the three sides of one triangle are proportional to the three sides of another triangle, then the two triangles are similar.

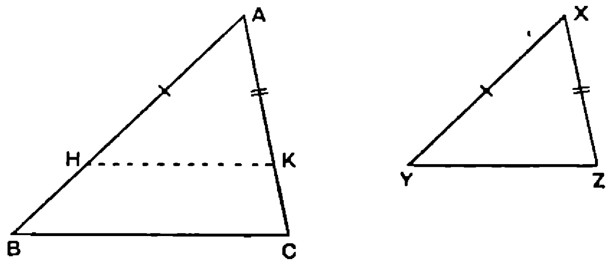


FIG. 388

Given two triangles, ABC, XYZ, in which

$$\frac{AB}{XY}=\frac{AC}{XZ}=\frac{BC}{YZ}.$$

To prove that $\triangle ABC \,|||\, \triangle XYZ$.

Construction. On AB, AC take points H, K so that $AH=XY$, $AK=XZ$.

Proof. $$\frac{AB}{XY}=\frac{AC}{XZ} \text{ (given)}$$

and $XY=AH$, $XZ=AK$ (constr.),

$$\therefore \frac{AB}{AH}=\frac{AC}{AK};$$

$$\therefore HK \parallel BC.$$

In the △s ABC, AHK,

$\angle A = \angle A$,

$\angle B = \angle AHK$ (corr., $HK \parallel BC$),

and $\angle C = \angle AKH$ ( );

∴ △s $\begin{matrix} ABC \\ AHK \end{matrix}$ are similar (AAA).

$$\therefore \frac{BC}{HK} = \frac{AB}{AH};$$

$$\therefore \frac{BC}{HK} = \frac{AB}{XY} \text{ (AH=XY, constr.)}.$$

But $$\frac{AB}{XY} = \frac{BC}{YZ} \text{ (given)};$$

$$\therefore \frac{BC}{HK} = \frac{BC}{YZ};$$

$\therefore HK = YZ$.

In the △s AHK, XYZ,

$AH = XY$ (constr.),

$AK = XZ$ (constr.),

and $HK = YZ$ (proved);

∴ △s $\begin{matrix} AHK \\ XYZ \end{matrix}$ are congruent (SSS).

But △s $\begin{matrix} ABC \\ AHK \end{matrix}$ are similar (proved);

∴ △s $\begin{matrix} ABC \\ XYZ \end{matrix}$ are similar.

Reference. △s $\begin{matrix} ABC \\ XYZ \end{matrix}$ are similar (3 *sides prop.*)

THEOREM 47 (The Ambiguous Case)

If two triangles are such that one angle and the two sides including another angle in the one triangle are respectively equal to the corresponding parts of the other triangle, then the third angles are either equal or supplementary, and in the former case the triangles are congruent.

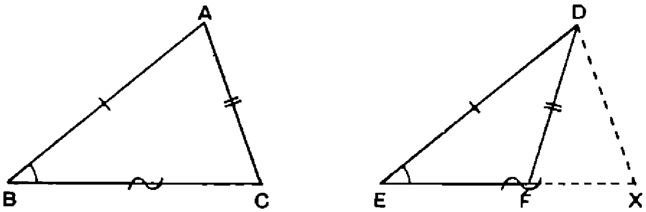


FIG. 389

Given two triangles ABC, DEF, in which AB = DE, AC = DF, $\angle$ABC = $\angle$DEF.

To prove that *either* $\angle$ACB = $\angle$DFE and $\triangle$ABC $\equiv$ $\triangle$DEF,
or $\angle$ACB + $\angle$DFE = 180°.

Construction. Let X be the point on EF, or EF produced, such that BC = EX.

Proof. *If* X *coincides with* F,

in the $\triangle$s ABC, DEF,

AB = DE (given),

$\angle$ABC = $\angle$DEF (given),

BC = EF (constr.);

$\therefore$ $\triangle$s $\frac{\text{ABC}}{\text{DEF}}$ are congruent (SAS),

and $\angle$ACB = $\angle$DFE.

If X *does not coincide with* F,

Δs $\frac{\text{ABC}}{\text{DEX}}$ are congruent by the method used above (SAS);

$\therefore \angle ACB = \angle DXE$ and $AC = DX$.

But $AC = DF$ (given),

$\therefore DF = DX$,

$\therefore \angle DXF = \angle DFX$,

$\therefore \angle ACB = \angle DFX$.

$\therefore \angle ACB + \angle DFE = \angle DFX + \angle DFE$

$= 180°$ (adj., EFX a st. line).

Corollary.

If two acute-angled, or two right-angled, or two obtuse-angled triangles have two sides and a non-included angle of the one equal to the corresponding parts of the other, then the triangles are congruent.

EXAMPLES 27

(*Theorem* 47)

1. If the bisector of one angle of a triangle bisects the opposite side, prove that the triangle is isosceles.

2. ABC is an isosceles triangle with a right angle at A. If D is a point, on the side of BC opposite to A, such that $\angle ADB = \angle ADC$, prove that ΔBCD is either isosceles or right-angled.

3. HKL is a triangle; the diagonals of the square drawn outwards on KL as a side meet in M, and HM bisects the angle KHL. Prove that the triangle HKL must be either isosceles or right-angled. (L)

4. ABC is an equilateral triangle, and BCDE is the square on BC drawn outside the triangle. If AE cuts BD at X, prove that (i) $\angle CAX = 45°$, (ii) $\Delta AXC \equiv \Delta DXC$.

5. ABP, ABQ are two triangles on opposite sides of a common base AB. If angles PAB and QAB are equal and obtuse, and if $BP = BQ$, prove that BA produced bisects PQ at right angles.

INEQUALITIES

It was proved in the corollary to Theorem 1 that an exterior angle of a triangle is equal to the sum of the interior opposite angles. Obviously, then, it is greater than either of them. This fact is of frequent use in inequality riders.

Corollary to Theorem 1.

If one side of a triangle is produced, the exterior angle so formed is greater than either of the interior opposite angles.

Reference. (*See Fig.* 390.)

$\angle ACD > \angle ABC$ (*ext.* > *int. opp.*).

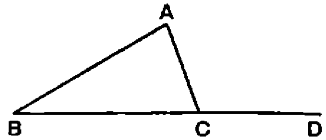


FIG. 390

The sign > stands for 'is (or are) greater than'; the sign < stands for 'is (or are) less than.' Students who find any difficulty in distinguishing between these symbols should remember that the wider end of the symbol is always nearer the greater quantity.

THEOREM 48a

If two sides of a triangle are unequal, the greater side has the greater angle opposite to it.

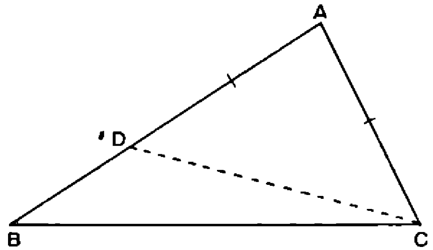


FIG. 391

Given a triangle ABC in which AB>AC.

To prove that ∠ACB > ∠ABC.

Construction. Along the longer side, AB, mark off DA equal to the shorter side, AC.

Join CD.

Proof. AC=AD (constr.),

∴ ∠ADC=∠ACD.

Now ∠ADC>∠ABC (ext. > int. opp.) ;

∴ ∠ACD>∠ABC.

But ∠ACB>∠ACD ;

∴ ∠ACB>∠ABC.

Reference. $AB > AC$,

∴ ∠ACB > ∠ABC.

THEOREM 48b

If two angles of a triangle are unequal, the greater angle has the greater side opposite to it.

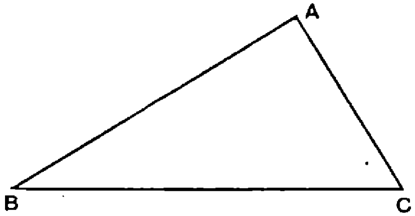


FIG. 392

Given a triangle ABC in which $\angle C > \angle B$.

To prove that $AB > AC$.

Proof. Either $AB > AC$, or $AB = AC$, or $AB < AC$.

Now, if $AB = AC$,

then $\angle C = \angle B$, which is contrary to what is given.

If $AB < AC$,

then $\angle C < \angle B$, which is contrary to what is given.

Hence AB is neither equal to nor less than AC.

$\therefore AB > AC$.

Reference. $\angle ACB > \angle ABC$,

$\therefore AB > AC$.

THEOREM 49

Of all straight lines which can be drawn to a given straight line from a given point outside it, the perpendicular is the shortest.

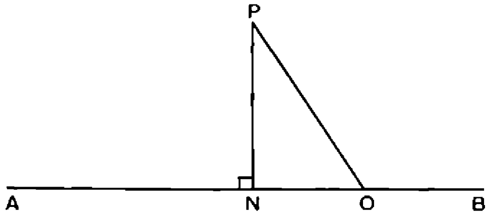


FIG. 393

Given a straight line AB and a point P outside it; N is the foot of the perpendicular from P to AB; P is joined to any other point O in AB.

To prove that $PN < PO$.

Proof. $\angle PNA = \angle PNO$ (90°, given).

But $\angle PNA > \angle PON$ (ext. > int. opp.);

$\therefore \angle PNO > \angle PON$.

$\therefore PO > PN$.

THEOREM 50

Any two sides of a triangle are together greater than the third side.

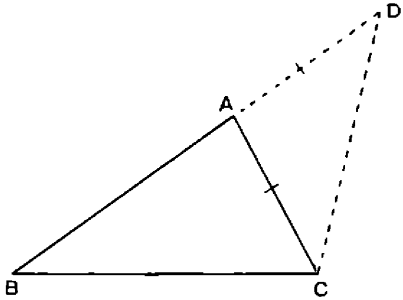


Fig. 394

Given a triangle ABC.

To prove that $AB + AC > BC$.

Construction. Produce BA to D, so that $AD = AC$. Join CD.

Proof. $AD = AC$ (constr.),

$\therefore \angle ACD = \angle ADC$.

But $\angle BCD > \angle ACD$;

$\therefore \angle BCD > \angle BDC$.

$\therefore BD > BC$.

But $BD = BA + AD$

$= BA + AC$ (constr.);

$\therefore BA + AC > BC$.

EXAMPLES 28a

(*Theorems* 48*a and* 48*b*)

1. In Fig. 395, if AD bisects ∠BAC, prove that CA>CD.

2. In Fig. 395, if AB=AC, prove that AB>AD.

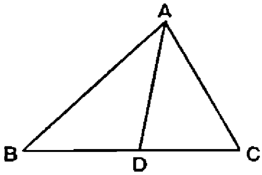


FIG. 395

3. In Fig. 395, if AD is perpendicular to BC and AB>AC, prove that ∠BAD>∠CAD.

4. ABC is a triangle in which AB>AC. If the bisectors of angles B and C meet at I, prove that IB>IC.

5. In Fig. 395, if BA=BC, prove that DA>DC.

6. In Fig. 395, if △ABC is equilateral, prove that AD>BD and AD>CD.

7. ABC is a triangle in which AB=AC. If P is a point on BC produced, prove that AP>AB.

8. ABCD is a trapezium having AD parallel to BC and ∠A obtuse. If the bisectors of ∠s A and B meet at O, prove that OB>OA.

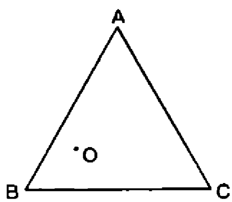


FIG. 396

9. In Fig. 396, if AB=AC and OC>OB, prove that ∠OCA>∠OBA.

10. ABC is a triangle in which AB=AC. If BA is produced to P, prove that PB>PC.

(*Theorem* 50)

11. Prove that any side of a triangle is greater than the difference between the other two sides.

12. In Fig. 397, prove that BC+CD+DA>BA.

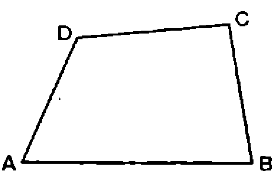


FIG. 397

13. Two sides of a triangle are of lengths 5 and 8 cm. Between what limits must the length of the third side lie?

14. Prove that any side of a triangle is less than the semi-perimeter.

15. In Fig. 396, explain why $OB+OC>BC$. Write down two similar statements, and hence prove that

$$2(OA+OB+OC)>BC+CA+AB.$$

(To be proved with the help of Pythagoras' theorem)

16. ABC is a triangle in which $AB>AC$. If AD is an altitude, prove that $DB>DC$.

17. From the result of No. 16, prove that, if AD is produced to E, $\angle BCE > \angle CBE$.

18. By a method similar to that used in No. 15, prove that the sum of the distances of a point inside a quadrilateral from the vertices is greater than the semi-perimeter.

State and prove the corresponding result for a polygon.

EXAMPLES 28b

1. In Fig. 395, if AD is a median and $AB>AC$, prove that $\angle CAD>\angle BAD$. (Hint: produce AD to E, making DE equal to AD, and join EB, EC.)

2. With the data and hint of No. 1, prove that $AB+AC>2AD$.

3. In Fig. 396, prove that $\angle BOC>\angle BAC$. (Hint: produce CO to meet AB at P.)

4. With the data and hint of No. 3, prove that

$$BA+AC>BO+OC.$$

5. In Fig. 397, if $BC<CD$ and $\angle B<\angle D$, prove that $AB>AD$.

6. Prove that the circumradius of a triangle is greater than one-sixth of the perimeter.

7. If P, Q, R are points on the sides BC, CA, AB respectively of a triangle ABC, prove that the perimeter of $\triangle PQR$ is less than the perimeter of $\triangle ABC$.

Prove the corresponding result for a polygon.

8. If the external bisector of the angle A of a triangle ABC meets BC produced at D, prove that AB > AC and CD > CA.

9. In Fig. 395, if AD is a median and BC < 2AD, prove that ∠A is acute.

10. In Fig. 395, prove that $AD < \frac{1}{2}(BC + CA + AB)$.

11. The diagonals of a quadrilateral ABCD intersect at O. If OC is the greatest and OA the least of the lengths OA, OB, OC, OD, prove that ∠BAD is obtuse and ∠BCD acute.

12. In Fig. 395, if AD bisects ∠BAC and AB > AC, prove that ∠ADB > 90°.

13. If two chords of a circle are unequal, prove that the shorter chord is the one further from the centre.

14. With the data of No. 12, prove that BD > DC. (Hint: along AB mark off AE equal to AC, and join ED.)

15. ABC is a triangle in which AB = AC; BA is produced to E, and P is any point on the altitude AD. Prove that

$$CP + PE > CA + AE.$$

16. If X, Y, Z are points on the sides BC, CA, AB respectively of a triangle ABC, prove that AX + BY + CZ is greater than the semi-perimeter of △ABC.

17. The light-path theorem.

A and B are two fixed points on the same side of a fixed straight line XY, and P is a variable point on XY. Prove that AP + PB is least when AP and PB are equally inclined to XY. (Hint: let M be the foot of the perpendicular from A to XY. Produce AM to C so that MC = AM, and join CP.)

Assuming that a ray of light, emitted from A and reflected to B by a plane mirror XY, takes the shortest path, this proves that the 'angle of incidence' is equal to the 'angle of reflection.'

18. Given the position of two balls on a billiard table, and assuming that a ball rebounds from a cushion at the same angle as it strikes it, show by a diagram the direction in which the one ball must be hit, so as to strike the other ball after rebounding from (i) one cushion, (ii) two cushions, (iii) three cushions. (Hint: use a method similar to that required in No. 17.)

EXAMPLES 28c

1. A perpendicular **OP** is drawn to a given line from a point **O** outside it, and **Q** is any point on the line. Prove that, if **R** is any point on the line between **P** and **Q**, **OR** is less than **OQ**. (OC)

2. **ABC** is a triangle in which **AB** > **AC**. If the external bisectors of angles **B** and **C** meet at **O**, prove that **OB** < **OC**.

3. Given a point inside a quadrilateral, prove that the sum of its distances from the vertices is least when it is at the point of intersection of the diagonals.

4. In the triangle **ABC**, **CA** is greater than **AB**. If the angles **B** and **C** are acute, and **BE**, **CF** are drawn perpendicular to **CA** and **AB**, and intersect one another at **O**, prove that **OC** is greater than **OB**. (OC)

5. If **AB** is the greatest and **CD** the least side of a quadrilateral **ABCD**, prove that ∠**BCD** > ∠**BAD** and ∠**ADC** > ∠**ABC**.

6. A circle is drawn through the vertices **P**, **Q**, **R** of a triangle. **RS**, the tangent at **R**, meets **QP** produced in **S** and ∠**PRQ** = ∠**PRS** = 40°. Calculate the other angles in the figure, and state, with full reasons, whether **PS** is equal to, greater than, or less than **PQ**. (W)

7. (Th. 17.) Prove that, of all parallelograms of which the sides are given, that parallelogram which is a rectangle has the greatest area. (W)

8. Prove that the straight line joining any point on one side of a triangle to the opposite vertex is less than one of the other two sides. (There is more than one case to consider.)

9. (Th. 8*b* and 9*b*.) **ABCD** is a quadrilateral in which the side **AD** is parallel to the side **BC**. Show that, unless the quadrilateral is a parallelogram, the sum of the lengths of **AB** and **BC** cannot be equal to the sum of the lengths of **AD** and **DC**. (W)

10. In an acute-angled triangle **ABC**, **AD** is drawn perpendicular to **BC**, and **BD** is greater than **DC**. Prove that the angle **BAD** is greater than the angle **CAD**. (OC)

11. In Fig. 396, if OA = AC, prove that AB > AC.

12. In Fig. 398, AB = AC and AC bisects the angle BAX; prove that AY is greater than AZ, and AZ is greater than YZ. (C)

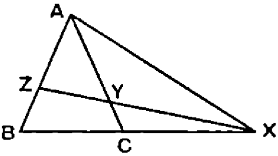


Fig. 398

13. Prove that the sum of the lengths of the three medians of a triangle is less than the perimeter.

14. In any quadrilateral, prove that the sum of the diagonals is less than the perimeter and greater than the semi-perimeter.

15. A and B are the feet of two goal posts and a point O is taken on BA produced. Determine by a geometrical construction a point P on the straight line through O perpendicular to the goal line AB, such that $OP^2 = OA.OB$. State, without proof, the steps of the construction.

Assuming, without proof, that a circle passing through A, B, and P will touch OP, prove with the help of this circle that the goal AB subtends a greater angle at P than at any other point on OP or OP produced. (OC)

16. Prove that the sum of the distances from the vertices of a point inside a triangle is less than the perimeter.

17. If ABC is a triangle in which $\angle B = 2\angle C$, prove that AC < 2AB. (Hint: produce CB to D so that BD = BA.)

18. ABC is a triangle in which AB = AC, and P is a point inside the triangle lying on the bisector of $\angle BAC$. If BP produced meets AC at Q, prove that BP > PQ.

CONCURRENCE

Definition. Three or more straight lines are said to be **concurrent** when they meet at the same point.

THEOREM 51

The perpendicular bisectors of the three sides of a triangle are concurrent.

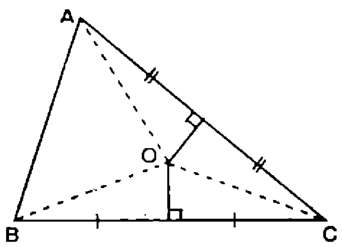


FIG. 399

Given a triangle ABC.

Construction. Let the perpendicular bisectors of BC and CA meet at O. Join OA, OB, OC.

To prove that O lies on the perpendicular bisector of AB.

Proof. ∵ O lies on the perpendicular bisector of BC,

∴ OB = OC.

Again, ∵ O lies on the perpendicular bisector of CA,

∴ OC = OA.

∴ OA = OB.

∴ O lies on the perpendicular bisector of AB.

Note.—O is the circumcentre of △ABC. (See Construction 17.)

THEOREM 52

The bisectors of the three angles of a triangle are concurrent.

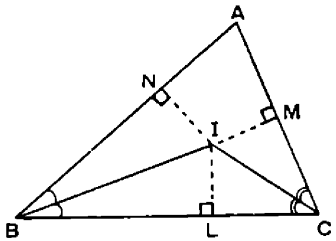


FIG. 400

Given a triangle ABC.

Construction. Let the bisectors of ∠s B and C meet at I. Let L, M, N be the feet of the perpendiculars from I to BC, CA, AB.

To prove that I lies on the bisector of ∠A.

Proof. ∵ I lies on the bisector of ∠ABC,

∴ IN = IL.

Again, ∵ I lies on the bisector of ∠ACB,

∴ IM = IL.

∴ IM = IN.

∴ I lies on the bisector of angle A.

Note.—I is the incentre of △ABC. (See Construction 19.)

THEOREM 53

The three altitudes of a triangle are concurrent.

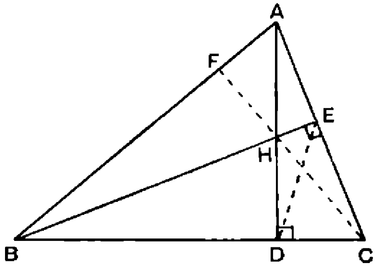


FIG. 401

Given a triangle ABC.

Construction. Let the altitudes AD, BE cut at H. Produce CH to meet AB at F. Join DE.

To prove that CF is perpendicular to AB.

Proof. $\because$ $\angle$HEC$=90°$ and $\angle$HDC$=90°$ (given),

$\therefore$ $\angle$HEC$+\angle$HDC$=180°$;

$\therefore$ HECD is cyclic.

$\therefore$ $\angle$HCE$=\angle$HDE (same seg.).

Also $\angle$ADB$=\angle$AEB (right angles);

$\therefore$ AEDB is cyclic.

$\therefore$ $\angle$ADE$=\angle$ABE (same seg.).

Hence $\angle$HCE$=\angle$ABE,

i.e. $\angle$FCE$=\angle$FBE;

$\therefore$ BFEC is cyclic.

$\therefore$ $\angle$BFC$=\angle$BEC (same seg.)

$=90°$ (given).

Hence CF is perpendicular to AB.

Definition. The point of intersection of the three altitudes of a triangle is called the **orthocentre** of the triangle.

THEOREM 54

The three medians of a triangle are concurrent.

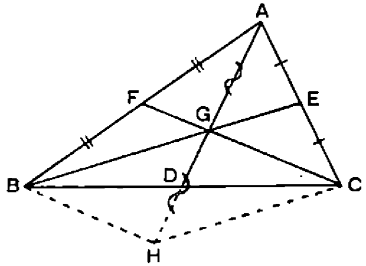


FIG. 402

Given a triangle ABC.

Construction. Let the medians BE, CF cut at G. Let AG produced meet BC at D.

Produce AD to H, making GH equal to AG. Join HB, HC.

To prove that BD = DC.

Proof. AF = FB (given) and AG = GH (constr.),

∴ FG ∥ BH (mid-point theorem);

∴ GC ∥ BH.

Again, AE = EC (given) and AG = GH (constr.),

∴ GE ∥ HC (mid-point theorem);

∴ BG ∥ HC.

∴ BGCH is a ∥gram (both pairs of opp. sides parallel).

∴ BD = DC (diags. of ∥gram).

Definition. The point of intersection of the three medians of a triangle is called the **median point** of the triangle.

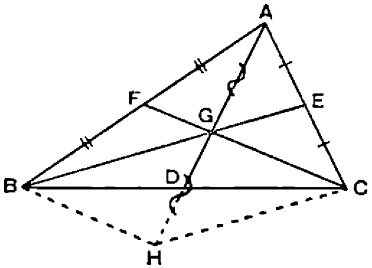


FIG. 402 (repeated)

Corollary.

The median point is a third of the way up each median, measured towards the vertex.

For $GD = DH$ (diags. of ||gram);

$\therefore GH = 2GD$.

But $AG = GH$ (constr.),

$\therefore AG = 2GD$.

$\therefore GD = \frac{1}{3}AD$.

Similarly, $GE = \frac{1}{3}BE$ and $GF = \frac{1}{3}CF$.

EXAMPLES 29

1. If H is the orthocentre of $\triangle ABC$, prove that A is the orthocentre of $\triangle BCH$.

2. ABCD is a parallelogram; the perpendicular bisectors of AB and AD intersect at P, and the perpendicular bisectors of BC and CD intersect at Q. Prove that PQ bisects BD at right angles.

3. M is a point on the base BC of a triangle ABC. The bisectors of the angles ABM, BAM intersect in L, and the bisectors of the angles ACM, MAC intersect in N. Prove that LMN is a right angle. (O)

4. M is any point on the base BC of a triangle ABC; the perpendicular bisectors of AB and BM intersect in P, and the perpendicular bisectors of AC and CM intersect in Q. Prove that PQ is perpendicular to AM. (O)

5. ABCD is a parallelogram; X is the mid-point of AB, and CX cuts BD at Y. If AY produced cuts BC at Z, prove that Z is the mid-point of BC.

6. X, Y, Z are the mid-points of the sides BC, CA, AB respectively of $\triangle ABC$. Prove that the orthocentre of $\triangle XYZ$ is the circumcentre of $\triangle ABC$.

7. H is the orthocentre of $\triangle ABC$, and AK is a diameter of the circumcircle of the triangle. Prove that BHCK is a parallelogram.

8. I is the incentre, and I_1, I_2, I_3 are the excentres, of $\triangle ABC$. Prove that (i) I is the orthocentre of $\triangle I_1I_2I_3$, and (ii) I_1 is the orthocentre of $\triangle II_2I_3$.

9. ABCD is a parallelogram in which $\angle B$ is obtuse; the perpendicular through B to BC intersects at P the perpendicular through D to DC. Prove that AP is perpendicular to BD.

10. In $\triangle ABC$, $AB=AC$ and $\angle BAC=45°$; the altitudes AD, CF of the triangle intersect at P. Prove that $FB=FP$.

11. AD, BE, CF are the altitudes of $\triangle ABC$, and H is the orthocentre. Prove that (i) $\angle FDH=\angle EDH$, and (ii) H is the incentre of $\triangle DEF$.

12. If G is the median point of $\triangle ABC$, and $AG=BC$, prove that $\angle BGC=90°$.

13. O is the circumcentre, H the orthocentre, and I the incentre, of $\triangle ABC$. Prove that AI bisects $\angle OAH$.

14. If two medians of a triangle are equal, prove that the triangle is isosceles.

15. I is the incentre of $\triangle ABC$, and I_1 is the excentre opposite the vertex A; II_1 cuts the circumcircle of the triangle at P. Prove that $PB=PC=PI=PI_1$.

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ANSWERS TO EXAMPLES

PART I

EXAMPLES **1 a** (page 5)

1. 70°. **4.** 130°. **7.** 83°.
8. 65°. **9.** 110°. **13.** 104°.

EXAMPLES **1 b** (page 6)

10. 40°.

EXAMPLES **2 a** (page 13)

1. 60°. **2.** 53°. **3.** 52°. **5.** 49°.
6. 87°. **11.** 78°. **12.** 100°. **14.** 85°.
15. 72°, 108°; 60°, 120°; 45°, 135°. **16.** $2n$ rt. angles.
17. 9. **18.** 10. **19.** 7 rt. angles. **20.** 360°. **21.** 36°.

EXAMPLES **2 b** (page 14)

3. 20°. **7.** 30°. **11.** 16°. **17.** 50.
18. 3. **20.** Octagon. **21.** Pentagon. **22.** 10.

EXAMPLES **2 c** (page 16)

1. 15. **3.** (i) 6, (iii) 9, (iv) 12. **4.** 65°, 65°, 50°.
5. 5. **7.** 40°. **9.** 108°, 72°, 108°, 144°, 108°.
11. 108°, 160°, 108°, 54°. **14.** 60°, 100°.
17. 8, 4. **20.** $180+a+c-b$. **22.** 9.

EXAMPLES **3 a** (page 26)

15. $67\frac{1}{2}$°. **16.** 55°, 55°; 70°, 40°. **17.** 86°.
18. 38°. **20.** 90°; 45°, 65°, 70°.

EXAMPLES **3 c** (page 33)

1. 56°, 100°, 38°, 62°. **9.** 98°. **12.** $25\frac{5}{7}$°.
24. (i) 145°, 15°, 20°; (ii) 35°, 125°, 20°.

EXAMPLES **4 a** (page 51)

1. $52\frac{1}{2}°$. **2.** No triangle. **3.** 2·5 cm.
4. 1·39 in. **5.** 3·8 cm. **6.** No triangle.
7. 2·0 cm. or 6·9 cm. **8.** 6·9 cm. **9.** No triangle.
10. 9·6 cm. **11.** No triangle. **12.** 2·24 in.
13. 5 cm. **14.** 3·1 cm. **15.** 2·52 in. **16.** 1·39 in.
17. 4·5 cm. **20.** 2·5 cm. **24.** 3·6 cm. **25.** 6·0 cm.

EXAMPLES **4 b** (page 52)

1. 6·5 cm. **2.** 2·50 in. **3.** 4·47 in. **4.** 3·21 in.
5. 2·86 in. **6.** 48°. **7.** 1·72 in. **8.** 2·31 in.
10. 2, 2 in. **16.** 1·07 in.

EXAMPLES **4 c** (page 54)

2. 1·2, 5·7 cm. **3.** 2·92 in. **4.** 115 yd.
5. 1·23, 5·70 in. **6.** (i) 9·7, 14·5 cm.; (ii) 8·0, 11·6 cm.
7. 4·8, 3·2 cm.; 74°, 68°, 38°.

EXAMPLES **5 a** (page 64)

1. 122°, 58°, 122°. **21.** 8·0 cm. **22.** 1·97 in.
23. 5·5 cm. **24.** 2·43 in.

EXAMPLES **5 b** (page 66)

14. 8·3 cm. **15.** 4·7 cm. **16.** 1·53 in. **17.** 3·61 or 3·02 in.

EXAMPLES **5 c** (page 67)

1. 86°, 38°. **4.** 3·81 in. **14.** 60°. **19.** $74\frac{1}{2}°$, $105\frac{1}{2}°$.

EXAMPLES **6 a** (page 72)

1. 64°. **2.** 34°. **3.** $22\frac{1}{2}°$.
13. 1·74 in. **14.** 1·6, 2·77 in. **16.** 3 in.

EXAMPLES **6 b** (page 74)

1. 90°, 105°, 60°, 105°. **10.** 6 cm.
11. 4·1 cm. **13.** 1·53 in. **16.** *a* cm.

EXAMPLES **6 c** (page 76)

1. 25°, 65°.

MISCELLANEOUS EXAMPLES I

1. (ii) $22\frac{1}{2}$°. **2.** 18°, 18°. **8.** 123°.
10. 1·41 in., 90°. **11.** 116°.

EXAMPLES **7 c** (page 83)

12. 12 in.

EXAMPLES **8** (page 89)

18. 7·9 cm.

EXAMPLES **9 a** (page 93)

8. 3·9, 7·3, 7·8, 11·3 cm. **9.** 161 or 222 yd. **10.** 2, 2, 3·1, 3·1 cm.
11. 2·8 miles. **13.** 1·85 in. **14.** 1·67 or 3·86 in.

EXAMPLES **9 b** (page 102)

15. 576 or 974 yd. **16.** 2·7, 7·5 cm. **17.** 3·4 cm.
18. 2·18 in. **20.** 6·8 cm. **21.** 4 cm. **22.** 0·60 in.

EXAMPLES **9 c** (page 104)

1. 2·8 cm. **2.** 2·6 in. **3.** 1·21 in.
4. 3·39, 3·49, 2·07 in. **8.** 3·6 in. **11.** 8·3 cm.

EXAMPLES **10 a** (page 114)

1. 15 sq. in. **2.** 5 ft. **3.** $7\frac{1}{2}$ sq. in.; 10 ft. **4.** 8, 4 cm.
5. 0·96, 1·48 in.; 1·6 sq. in. **6.** 2·9 cm.; 14 sq. cm.
7. 0·68, 1·35 in.; 1·1 sq. in. **8.** 1·92 sq. yd. **9.** 19 sq. cm.
10. 80 sq. ft. **11.** 6·3, 2·9 cm.; 37 sq. cm. **12.** 1·53 in.
13. 4·13 in. **14.** 3·32 in. **15.** 3·9 cm. **16.** 5·9 cm. **19.** 2·9 sq. in.

EXAMPLES **10 b** (page 116)

1. 6·24 sq. cm.; 4·8 cm. **2.** 6·4 sq. in.; 2·3 in. **3.** 5 cm.
4. (i) 24 sq. cm.; (ii) 4·8 cm. **5.** 2·7 cm. **6.** 70 sq. in.
7. 23·4 sq. cm. **8.** 19 sq. cm. **9.** 6 m.
10. $\frac{1}{2}a^2$ sq. in. **11.** $2ab$ sq. in. **12.** 225 sq. in.; 15 in.
14. $11\frac{4}{121}$ acres. **15.** 2·01 in. **19.** 2·8 cm.
22. 3·4 in.; 2·3 sq. in. **26.** 1·93 in.

EXAMPLES **11 c** (page 124)

1. 44 sq. in. **2.** 5·5 cm.; 11 sq. cm. **3.** 57°.
7. Each $\frac{1}{3}$. **8.** 6·4 acres. **13.** 25 in.
15. 8·8 or 12·3 cm.; 2. **25.** $\frac{1}{8}$. **27.** 3·11 in.

EXAMPLES **12 a** (page 133)

1. 5 cm. **2.** 5 cm. **3.** 8·5 cm. **4.** 18·4 miles.
5. 60 sq. yd. **6.** 104 cm. **7.** 7·5 ft. **8.** 3·3 cm.
9. 43 ft. **10.** 20 ft.; 25 ft. **12.** No. **28.** $\sqrt{3}$ units.

EXAMPLES **12 b** (page 136)

1. 36 sq. in. **2.** 120 sq. ft. **3.** Yes. **4.** 1·1 m.
5. 400 sq. ft. **6.** 1 ft. **10.** (i) 1·7 ft.; (ii) 1·4 sq. ft.
11. 108, 150, 210 sq. ft. **12.** 5·3 cm. **30.** 1 : $\sqrt{3}$: 2.

EXAMPLES **12 c** (page 138)

1. 13 sq. cm. **2.** 13, $\sqrt{338}$, 13. **3.** 2·24 sq. in.; 2, 2·45, 2·45 in.
7. 50 ft. **9.** 2·83 in. **13.** 1·5 hours. **15.** 6·80 cm.
19. 1·85 in. **20.** 2, 2, 3, 4·69 sq. in.
24. Edges of wedge 3 and 4·24 in.; 4·24, 4·24 in.
26. 3, 7 cm. **27.** 24 ft.

EXAMPLES **13 a** (page 146)

1. (i) obtuse-, (ii) acute-, (iii) right-, (iv) acute-angled.
2. 3; obtuse-, obtuse-, right-angled. **3.** $4\frac{1}{22}$ cm.
4. $\frac{1}{8}$ in. **5.** 2·15 cm. **6.** $5\frac{1}{2}$ in. **7.** $5\frac{2}{3}$, $\frac{1}{3}$ in.
9. $3\frac{1}{2}$ in. **10.** 3·4, 2·8, 1·6 cm. **11.** 38 in. **12.** 10 in.

EXAMPLES 13 b (page 148)

1. 12 in. **2.** Acute. **3.** 6·4 cm. **4.** $3\frac{2}{3}$ cm.
5. 15 sq. cm. **6.** 11 cm. **7.** 3·5 in. **8.** 40 ft.; acute-angled.
9. 3 cm. **10.** $4\frac{1}{2}$ cm. **12.** $\sqrt{(2a^2+2b^2-c^2)}$ units.

EXAMPLES 13 c (page 150)

1. $5\frac{5}{6}$, $3\frac{1}{6}$ in. **2.** $14\frac{5}{8}$ ft. **6.** $1\frac{1}{3}$ in.
7. 6·93 in. **11.** $\angle$C obtuse; $2\frac{1}{2}$ in. **16.** 0·8 in.; 9·8 sq. in.

MISCELLANEOUS EXAMPLES II

1. $17\frac{1}{2}$ sq. in.; $2\frac{1}{2}$ in. **3.** $2a^2+2b^2=(b+a)^2+(b-a)^2$.
6. 1·44 in. **7.** 6·2 cm.; 12·4 sq. cm. **11.** 3 cm.
12. 5 in. **13.** 6·9 cm. **21.** 90°. **23.** 8 cm.
28. 5·5 in.; 13·9 sq. in. **30.** 5 cm. **31.** 5·7 cm.

EXAMPLES 14 a (page 164)

2. 3 in. **3.** 3 cm. **4.** 24 cm. **5.** 25 in. **6.** 7·8, 1·2 cm.
24. 4·13 cm. **25.** 3·7 cm. **26.** $1\frac{1}{4}$ in. **27.** 1·73 in.

EXAMPLES 14 b (page 166)

2. 28·6 ft. **10.** $3\frac{1}{2}$ in. **12.** $2\frac{1}{2}$ in. **13.** 1·5 in.

EXAMPLES 14 c (page 168)

1. 5·20 in. **2.** 1·8 in. **5.** 2·30 in.
6. (i) 2 in.; (ii) $2\frac{1}{2}$ in. **7.** 14 cm. **12.** 8 in.
17. 3·97 in.; $4\frac{1}{2}$, $\frac{1}{2}$ in. **18.** $6\frac{1}{2}$ cm.

EXAMPLES 15 a (page 176)

1. 62°, 118°. **2.** 214°. **3.** 82°, 49°. **4.** 68°, 56°.
5. 42°, 27°. **14.** 62°. **16.** 52°, 38°. **17.** 28°.
21. 19°. **22.** 83°. **23.** 76°, 104°. **24.** 72°, 108°.

EXAMPLES **15 b** (page 178)

1. x, $90-x$. **2.** $180-2x$, x, $2x$, $180-4x$, $90-2x$.
4. x, $90-x$, $90-x$. **5.** 120. **6.** $2x$, $2y$, $x-y$.
7. 102°. **14.** 45°. **18.** $180°-x°-y°$.

EXAMPLES **15 c** (page 181)

7. 148°, 106°, 106°. **21.** 55°, 85°, 125°, 95°.

ANSWERS TO EXAMPLES

PART II

EXAMPLES **16 a** (page 188)

3. 23°.
6. $\angle B = \angle D = 90°$, $\angle ACB = 60°$, $\angle ACD = 45°$; 120°, 60°, 90°, 90°; $\frac{1}{3}$, $\frac{1}{6}$, $\frac{1}{4}$, $\frac{1}{4}$.
7. (i) 30°; (ii) 15°; 30°, 60°. **9.** 30°, 60°. **10.** $\frac{1}{24}$.
15. $\angle A = 45°$, $\angle B = 30°$, $\angle C = 105°$. **16.** 72°, 72°.

EXAMPLES **16 b** (page 190)

3. 72°.

EXAMPLES **16 c** (page 192)

2. $\angle B = 45°$, $\angle C = 112\frac{1}{2}°$, $\angle E = 22\frac{1}{2}°$; $\angle B = \angle H = 67\frac{1}{2}°$, $\angle E = 45°$.
4. 87°, 108°. **5.** $\angle A = 120°$, $\angle C = 20°$, $\angle K = 40°$.

EXAMPLES **17 a** (page 196)

2. 64°, 64°. **3.** 30°. **5.** 60°, 120°, 120°.
6. 26°, 26°, 26°; 52°, 52°. **10.** 36°, 112°, 68°.

EXAMPLES **17 b** (page 198)

18. $\frac{1}{2}(a° + b°)$, $\frac{1}{2}(c° + d°)$.

EXAMPLES **18 a** (page 210)

1. 4 in. **2.** 17 cm. **3.** 5·7 in. **4.** 5·3 in.
10. 9, 8, 11 cm. **11.** 10, 9, 11 in. **17.** 129°, 51°. **20.** 6 cm.
21. 0·59 in. **22.** 2·0 cm. **23.** 3·4 cm.

EXAMPLES **18 b** (page 212)

1. 140°. **5.** $\sqrt{(a^2 + d^2)}$ in.; centre O, radius $\sqrt{(a^2 + d^2)}$ in.
11. 54°, 63°, 63°. **12.** 92°, 111°, 88°, 69°. **16.** 10, 8, 4 cm.
21. 2·8 in. **23.** 3·33 in.

EXAMPLES **18 c** (page 215)

1. 1 in. **2.** 26°, 80°, 74°. **8.** $10\frac{1}{2}$ cm.
9. 4, 5 in. **11.** 3·7 cm.

2 C

EXAMPLES **19a** (page 223)

1. $3\frac{1}{2}$ in. **2.** 3 cm.
3. (i) Do not intersect, (ii) touch externally, (iii) intersect, (iv) touch internally, (v) touch externally.
7. (i) $1\frac{1}{2}$ cm., (ii) $5\frac{1}{2}$ cm. **8.** $3\frac{1}{2}$, $1\frac{1}{2}$, $4\frac{1}{2}$ cm.
9. $2\frac{1}{2}$, $10\frac{1}{2}$, $3\frac{1}{2}$ cm. **10.** 3 in. **11.** 9·8 cm.
12. 4 cm. **13.** 3 cm. **14.** 12, 9·4 cm.

EXAMPLES **19b** (page 225)

1. 2, $3\frac{1}{2}$, 8 cm. **2.** 2·4 cm. **3.** 4·4 cm. **4.** $5\frac{1}{4}$ in.
8. 2·41 in. **9.** 7·7 cm. **10.** 18·6 in. **14.** 32°.

EXAMPLES **19c** (page 227)

1. 12 cm. **3.** 8·7 cm. **5.** 1·6 in. **7.** 6·9 cm.
10. $\frac{1}{4}r$ cm. **11.** (i) 25 cm., (ii) 26 cm. **12.** $\frac{2}{3}r$ in.

EXAMPLES **20a** (page 235)

1. 32°, 148°. **2.** 20°. **3.** 40°, 85°, 55°.
4. ∠A=64°, ∠AYZ=∠AZY=58°; ∠B=76°, ∠BXZ=∠BZX =52°; ∠C=40°, ∠CXY=∠CYX=70°; ∠YXZ=58°.
5. 77°. **6.** 40°, 80°. **7.** 60°, 35°. **8.** 62°, 28°, 62°, 28°.
21. CDX. **22.** ACD. **26.** 2·8 cm. **27.** 4·2 cm.
28. 3·9 in. **30.** 11·5 cm.

EXAMPLES **20b** (page 238)

1. 125°. **2.** 55°, 125°, 110°, 70°. **3.** 134°, 67°, 52°.
4. 72°. **5.** 46°. **6.** 48°. **7.** 80°. **8.** 19°.
9. 58°, 40°, 82°. **12.** y, $x+y$; $x+2y=90$. **13.** 205°.
31. 3·02 or 1·00 in. **32.** 7·7, 4·7 cm. **33.** 11·0 cm.

EXAMPLES **20c** (page 242)

1. 63°, 54°, 63°. **4.** 80°, 60°, 40°. **8.** 70°, 35°, 75°.
10. 5·1, 8·7, 10·9 miles. **17.** 67°, 58°, 55°.

EXAMPLES **21a** (page 246)

11. 0·78 in. **13.** 2·5 cm. **14.** 0·84 or 1·19 in.

EXAMPLES **21b** (page 248)

5. 1·5 cm. **7.** 0·89 in. **8.** 2·23 in.

EXAMPLES **21c** (page 249)

11. 1 in.

EXAMPLES 22 b (page 253)

7. 8 cm.

MISCELLANEOUS EXAMPLES III

1. 108°, 36°, 36° ; 72°, 36°, 72°. **2.** ∠A = 120°, ∠D = 80°.
6. 76°, 98°, 132°, 124°, 110°. **8.** $\sqrt{(a^2-r^2)}$ in.
11. 70°, 26°, 228° ; 40°. **25.** 18·8 cm.
30. $\sqrt{2}$ = 1·41 in. **33.** 60°, 70°, 50°. **35.** 55°.
48. 3·6, 1 in. **49.** 9·6 cm. **50.** 2 cm.
59. 120°, 1·15 in. **60.** 10·95 cm. **61.** 1·14 in.
62. 2·8 in. **64.** 4·2, 5·9 in.

EXAMPLES (page 263)

1. (i) 4 : 11, (ii) 11 : 7. **2.** 5·76, 3·84 cm.
3. (i) 3 : 13, (ii) 3 : 4 : 9, (iii) 7 : 13. **4.** 3 : 5.

EXAMPLES (page 264)

1. 2 : 3, internal. **2.** 8 : 3, external. **4.** 15 cm.

EXAMPLES 23 a (page 275)

1. 1·5 in. **2.** 1·5 in. **3.** 2 in. **4.** 7·2 cm.
5. 2·4 in. **6.** 4·2 in. **20.** 1·8, 9 cm. **21.** 1·09, 4·37(5) in.
22. 2·4 cm. **23.** 3·76 in. **25.** 3, 15 cm.
26. $1\frac{13}{15}$, $2\frac{2}{15}$; $4\frac{2}{3}$, $2\frac{1}{3}$; $2\frac{10}{11}$, $5\frac{1}{11}$ cm.

EXAMPLES 23 b (page 278)

10. 3·72.

EXAMPLES 23 c (page 280)

1. $26\frac{2}{3}$ in. **5.** $13\frac{1}{3}$ cm. **8.** 2·62, 2·38 in. **9.** 8 in.
13. 8·7 cm. **18.** $2\sqrt{3}/3$ = 1·15(5), $\sqrt{3}/3$ = 0·58 units.

EXAMPLES 24 a (page 287)

1. 96 ft. **2.** 40 ft. **3.** $17\frac{1}{2}$, 20 cm. **4.** $2\frac{1}{4}$ in.
5. $8\frac{1}{3}$, $6\frac{2}{3}$ ft. **7.** $\frac{1}{4}$; $1\frac{1}{2}$, $1\frac{1}{4}$ cm. **8.** 5·4, 0·6 cm.
13. 18 cm. **15.** $3\frac{1}{8}$ in. **24.** $2\frac{1}{2}$ in. **25.** 2 cm.
30. 5 cm. **31.** 10 in. **33.** $1\frac{1}{8}$, $1\frac{1}{2}$ in. **34.** $1\frac{1}{3}$, 3 in.
35. $1\frac{7}{9}$, $2\frac{2}{3}$ cm. **38.** $10\frac{2}{7}$ cm.

EXAMPLES 24 b (page 291)

1. $ay/(y-x)$ ft. **2.** 10, $12\frac{1}{2}$ ft. **7.** 8, 12 cm.
12. 4 : 1. **24.** 1 : 3. **40.** $5\frac{5}{8}$ in.

EXAMPLES 24c (page 295)

1. 2·8, 3·5 in. **11.** (*a*) 2 in., (*b*) 5 : 1.
20. $\angle$BAO $= \frac{1}{2}$A $=\angle$OAQ, $\angle$ABO $= 90° + \frac{1}{2}$B $= \angle$AOQ, $\angle$AOB $= \frac{1}{2}$C $= \angle$AQO.

EXAMPLES 25a (page 304)

1. 2·5 cm. **2.** 6 cm. **6.** 1·73. **7.** 1·91 in.
8. 1·51 in. **10.** $7\frac{1}{2}$ cm. **11.** 3·2 sq. in. **12.** 9 : 25. **14.** $\frac{5}{8}$.

EXAMPLES 25b (page 305)

1. 1·4, 2·2 in. **2.** $8\frac{1}{3}$ sq. in. **3.** $7\frac{1}{2}$ sq. in. **4.** $2\frac{1}{12}$ ft.
5. 12, 25 cm. **6.** 2 : 3. **7.** 20 : 49. **8.** 52 sq. cm.
20. 6·85. **22.** 2·66 in. **24.** 1·96 in.

EXAMPLES 25c (page 307)

2. 60, 65, 156 cm. **3.** $\frac{1}{2}(p^2+q^2)$ in., $\frac{1}{4}pq(p^2-q^2)$ sq. in.
4. 2·52 in. **9.** 0·35. **14.** 4 : 3. **16.** 6·6, 2·4 cm.
17. 7 in., $\frac{211}{400}$. **18.** 3·2 in. **20.** 1·73 in.

EXAMPLES 26a (page 314)

1. 12 cm. **2.** 13 in. **4.** 3·5 cm. **5.** 5 cm. **6.** 2·1 in.
7. 13 in. **8.** 2, 5 in. **9.** 5 cm. **11.** 29 ft. **22.** $1\frac{1}{2}$ in.

EXAMPLES 26b (page 316)

1. 4 in. **2.** $7\frac{1}{2}$ cm. **3.** 3·2 in. **4.** $14\frac{1}{2}$ cm.
5. $6\frac{1}{2}$ in. **6.** 4·1, 8·1 cm. **7.** 198 sq. in. **8.** 5·6 in.
27. 55 miles. **28.** 27 miles.

EXAMPLES 26c (page 320)

1. 7 in. **6.** 2, 5 cm. **10.** 15, 48, $32\frac{1}{2}$.
14. $\sqrt{160}$, $6\frac{1}{13}$ in. **15.** 4 ft. or 16 ft.

MISCELLANEOUS EXAMPLES IV

1. 21, 11 cm. **3.** 2·19. **6.** $5\frac{1}{2}$ cm., 1 : 2. **11.** 2·4 cm.
12. 2 in. **16.** (i) 3·4, 3·6 in., (ii) 24·2 in.
19. $x^2 : (x+y)^2$, $y^2 : (x+y)^2$. **21.** 8·1, $3\frac{3}{8}$ in.
27. $6\frac{1}{4}$ in., $9\frac{3}{8}$ sq. in. **28.** 5, 10. **32.** 74, $18\frac{1}{2}$ yd. **33.** 3, $4\frac{1}{2}$ sq. in.

EXAMPLES 28a (page 357)

13. 3, 13 cm.

EXAMPLES 28c (page 360)

6. $\angle$PQR $= 40°$, $\angle$QPR $= 100°$, $\angle$RPS $= 80°$, $\angle$PSR $= 60°$; less than.