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THE GROUNDWORK OF ARITHMETIC

M. PUNNETT

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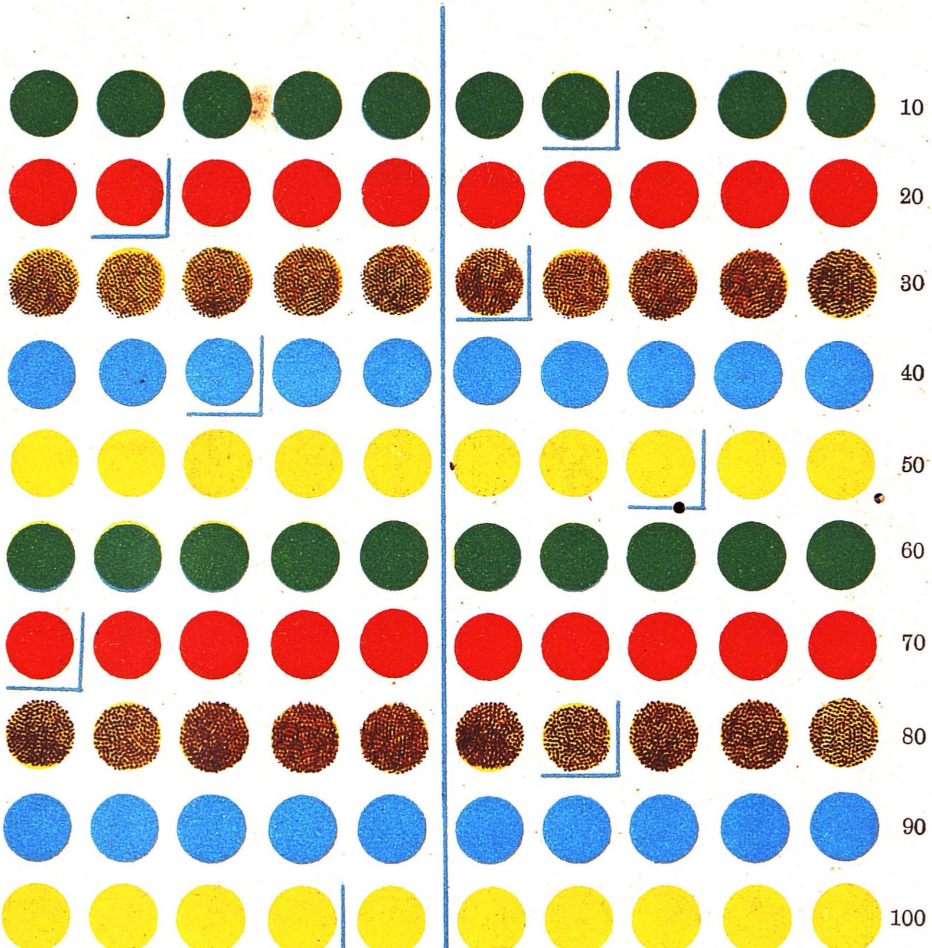

NUMBER CHART
(See pp. 81-86)

Frontispiece]

Longmans' Modern Mathematical Series

THE GROUNDWORK OF ARITHMETIC

A HANDBOOK FOR TEACHERS

BY

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WITH DIAGRAMS

NEW IMPRESSION

LONGMANS, GREEN AND CO.
LONDON ◆ NEW YORK ◆ TORONTO

1929

LONGMANS, GREEN AND CO. LTD.
39 PATERNOSTER ROW, LONDON, E.C. 4
6 OLD COURT HOUSE STREET, CALCUTTA
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167 MOUNT ROAD, MADRAS

LONGMANS, GREEN AND CO.
55 FIFTH AVENUE, NEW YORK
221 EAST 20TH STREET, CHICAGO
TREMONT TEMPLE, BOSTON
210 VICTORIA STREET, TORONTO

Made in Great Britain

I WANT to express my warm acknowledgments to those who have directly or indirectly helped me in the production of this little book. I am specially indebted to my colleagues, Miss C. Von Wyss and Dr. Percy Nunn; to the former for the drawings illustrating the exercises, and to Dr. Nunn for help and inspiration without which the book would never have been written. I have in one or two cases explicitly stated that certain methods were suggested to me by him; in other cases, too numerous to mention in detail, the main ideas underlying the method of treatment also owe their origin to him.

MARGARET PUNNETT.

November, 1913.

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NOTE.—Money is introduced in Course A and appears

	SECTION I.	SECTION II.
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CHAPTER I.

INTRODUCTORY.

THE purpose of this book is to set forth a course in arithmetic or "number" beginning at the age of six and extending over four or five years. It is assumed that formal "number lessons"—that is, lessons in which the investigation of number is treated as the predominant interest—will not be begun before the age of six. The choice of six as the age at which the course should begin is supported by the opinion of competent teachers and other students of child nature. There appears to be a growing body of evidence to show that formal number-work done before that age is, to a large extent, a waste of time; that is to say, that the work can be done better, more quickly, and with less risk of strain to the child if it is postponed until he is at least six years old.

It must not, however, be supposed that at the age of six the normal child is absolutely ignorant of number. By the time he has reached that age he will have accumulated by the way or incidentally a small but important store of number-knowledge. Games, stories, various occupations, at home or at school, will by degrees lead him to attend more and more to number as connected with them, and under careful guidance his interest in it will grow until he is ready for lessons in which number takes the predominant place. Some important points connected with this early incidental number-work are briefly suggested in the early part of Chapter II, and an indication is given of the amount and scope of the number-knowledge that is here assumed as the foundation of the formal number-work.

The book is divided into five sections, each of which may, in general, be taken as corresponding to a year's work. It is not, of course, intended that this distribution of the work should be rigidly adhered to. Some teachers may prefer to transfer certain portions of the work from one section to another,[1] or to take more or less time than is here indicated for the work of a given section. In favourable circumstances the work of three sections may be done in two years, or even that of two sections in one year; on the other hand, under certain conditions, it may prove necessary to spend more than a year on the work of a section. Such modifications are not here indicated in detail, but are left to the discretion of individual teachers.

The ground to be covered at each stage is shown in the headings of the chapters in the section corresponding to that stage. In developing in detail the work to be done, the plan adopted has been to indicate the method of treatment in a series of **teaching examples.** These **examples,** while they naturally make no attempt to explain in detail the procedure in every lesson to be given, suggest the methods to be employed in dealing with the most important and typical parts of the matter under consideration. It is to be noted that the Examples do not always, or even often, correspond to lessons, one Example often furnishing material for several lessons. Sometimes the details of treatment of a topic, or part of a topic, are given in the form in which they may be presented to the class; more often only the general lines of treatment are indicated, the details being left to the teacher to fill in. Moreover, the type of topic under consideration in any given case

[1] For example, in some cases it may be found possible and desirable to introduce simple fractions ($\frac{1}{2}$, $\frac{1}{4}$, $\frac{3}{4}$) in connexion with measuring into the work of Section II instead of postponing them to Section III.

is not necessarily to be treated always by the method shown in one Example. For instance, the methods of Examples II and IV—although one is obviously appropriate in general to a rather later stage than the other—may often, with advantage, be mingled or alternated in treating the same number. In fact, as every teacher of young children knows, the methods of teaching should throughout be varied and should be combined and recombined in every way that ingenuity can suggest.

It must be borne in mind that throughout the course two aspects or applications of number have to be treated. Number is to be considered, on the one hand, in its application to the separate objects of a group or groups (e.g. five nuts, seven cows), and on the other hand, in its application to continuous quantity (e.g. a length of five inches, a weight of seven pounds). Though it is obviously impossible to make a sharp separation between these two aspects of number, it is convenient to treat them to some extent separately, and the work may, from this point of view, be looked upon as consisting of two parallel courses. In both courses the same general method is adopted. Whenever a new "rule" or process, that is, a new way of manipulating number, is to be taught, the need and purpose of it will be presented to the children by means of a simple instance or "problem". The process is then "discovered" by the child with the aid of the teacher, is practised until it is in some degree familiar, and is finally used in fresh applications or problems (see Chapter XI).

The course in which number is applied to separate units in a group or groups begins with quite small numbers (not greater than **twelve**), and proceeds to larger numbers and more complex applications by means of "rules" or processes developed in the decimal system of counting and notation. The other course begins by comparing lengths and weights, first in a very general and indefinite way, afterwards more definitely with the aid of number, and develops into measuring and weighing and the use and application of "weights and measures" generally. The scheme at the beginning of the book is intended to make clear at a glance the parallel development of these two courses or lines of work. This scheme, in which the two courses are distinguished by the letters A and B respectively, includes references to all the chapters from Chapter II onwards with the exception of Chapters IV, VII, and XI. These are omitted because their contents have reference not to any special portion of the work but to long stretches of it, indeed in a certain sense to the whole of it.

A word should be added with reference to one of the terms used. The word "apparatus" is to be understood in its widest sense as indicating the material aids of any sort that are to be used in the lesson, either by the children themselves or by the teacher in demonstration before the children. Its commonest application, therefore, will be to the counters, sticks, beads, nuts, etc., which the children are constantly handling in the course of their early number-work.

SECTION I.

THIS section deals with the early number-work, and is consequently confined to small numbers, only those up to **twelve** being taken. "Number" is approached (see Chapters II and III) from two different though closely connected points of view: on the one hand, the children are led to number and to compare groups of discrete, or separate objects, on the other, they are led to use number to measure "continuous" quantity, namely, length and weight.

A chapter on the use of material aids to thinking is included for convenience in this section, though it applies in greater or less degree to the work of all five sections.

CHAPTER II.

Number-work before formal lessons begin. First formal number-lessons; numbers up to **twelve**; simple exercises involving money; recognition of the number in a group by breaking it mentally into smaller groups. The symbols for the numbers studied; the signs +, −, =. Simple applications or problems.

I. EARLY INFORMAL NUMBER KNOWLEDGE.

UNDER normal, healthy conditions a child's first acquaintance with number is made at a time when his interest is chiefly engaged with quite other matters. He is from an early age attracted to and stimulated by the various objects, animate and inanimate, that surround him; he is interested in their movements, their colours, their shapes, and above all in what he can do with them, how he can affect them—alter their condition, make them move, bring them into fresh relations with one another. But, under the guidance of parent, nurse, or teacher, or through the teaching of his unaided experience, he soon learns to attend to number as part of the means to the small ends he desires or as part of the nature of the things which rouse his interest. He grows to understand what is meant by saying that he himself, most other human beings, and birds have two legs; he learns how to separate three, four, etc., marbles or beads from a larger number, and so on. In these early stages he will usually show a certain unwillingness or incapacity to attend to number as number—that is to say, without direct reference to some other interest or need—and much harm will be done by forcing him prematurely to do any formal number-work or to perform exercises in the manipulation of numbers as such. In other words, his learning of number must at first be purely **incidental.**

This must not be taken to mean that the whole matter may be left to chance. It is true that, whether a child spends the first six years of his life at home or part of it at school, it will be difficult for him to avoid "picking up" a certain amount of number-knowledge; but a much more satisfactory result will be attained and a much better foundation laid for future work if the teacher—always without over-straining the natural possibilities of the immediate object of interest to the child—watches for and profits by opportunities of teaching facts about number, and takes care to make them—even though over a small range—as systematic as is possible in the circumstances. Almost everything that the child does at school at this stage offers such opportunities, his games, his "occupations," his conversations with the teacher about all sorts of interesting topics. The results will, however, vary widely according to the different conditions both of home and school life, and it is almost impossible to make any definite statement as to what should be expected of a child who has been learning number in this incidental way up to, say, the age of six. For the purposes of the present book it is assumed that such a child will not only be able to tell correctly, after counting them, how many objects there are in any group up to and including twelve,[1] but will also know some of the simpler combinations of numbers, e.g. **four** and **one** are **five, three** and **two** are **five, nine** and **one** are **ten,** and will be able to recognize, without counting them, the numbers **two, three,** and **four.** This last point deserves special attention. The children are to be trained later on to recognize the number of objects in a group without needing to count those objects one by one; they will mentally break up the group into smaller ones of what may be called manageable size. The results of systematic observation and experiment go to show that, in the case of the average person, the largest group of objects the number of which can be grasped without analysis is not generally greater than **four.** The apprehension, then, of the number of objects in a group containing more than four, is most economically effected by breaking it up mentally into groups consisting of **one, two, three,** or **four** (see Example V, p. 22, also Chapter IV). It is, for this reason, of great importance that children should be encouraged as early as possible to distinguish groups containing two, three, or four objects at a glance—that is, without counting them.

II. The First Formal Number-Lessons.

As has already been implied, formal number-lessons should not, as a rule, be introduced before the age of six. This decision is based on the apparent emergence in the average child of that age of an interest in num-

[1] He will probably also be able to do this for some numbers larger than twelve, the range varying according to circumstances.

ber "for its own sake". This does not mean that the child is to be expected to attend to "abstract number" in any of the usually accepted senses of that difficult term. His number-ideas will still be closely associated with things and the ideas of things. But he will now begin to take an interest in the number-aspect of things, much as he has for a long time taken an interest in the things themselves. He will take a pleasure in combining numbers, in breaking them up—in manipulating them in various ways. At this stage, then, lessons definitely dealing with number will be gradually introduced, while at the same time the child will continue to acquire number-ideas incidentally in connexion with his other work and play.[1]

The aims of the early formal number-lesssons must be:—

(i) To complete and systematize the knowledge acquired during the incidental learning of number referred to above. To this end the child will be led to attend to the relations between the various combinations of numbers (e.g. the connexion between the two facts, **four** and **three** make **seven**, and **four** and **four** make **eight**) in order to assist in the fulfilment of the second aim.

(ii) To fix in the child's memory the number-combinations over the range in question (that is, up to **twelve**,[2] inclusive): he must be able, for example, without need of counting or calculation, to state unhesitatingly that the result of adding **four** to **seven** is **eleven**.

(iii) To practise the child in applying this number-knowledge to fresh cases—that is, to fresh **things**.

[1] For valuable suggestions as to the use of games in this connexion, see Miss Wark's chapter on "Early Work in Number" in "Education by Life," edited by Miss Brown-Smith (G. Philip & Son).

[2] **Twelve** is chosen for this purpose as the limit, rather than **ten**, owing to its importance in connexion with our money system.

(iv) To teach him (*a*) the symbols for the numbers in question and (*b*) the symbols +, − and =, and, a little later on, the symbol ×.

In the earliest of the formal lessons the teacher will aim at ascertaining how many of the simpler number-combinations the children have learnt from the previous "incidental" number-work (see above) at giving them practice in applying this knowledge, and at helping individual children who may be more backward than the majority to repair the gaps in their knowledge. In these first lessons the symbols 1, 2, 3, 4 may also be taught. Example I will serve as an illustration of the teaching at this point.[1]

EXAMPLE I.—PRACTICE IN APPLYING SIMPLE NUMBER-COMBINATIONS. THE SYMBOLS 1 AND 2.

Apparatus.—Each child should have a small box of counters and a small box of sticks.

A. Simple Number-Combinations.

(*a*) The teacher will begin by asking the children to tell the number of objects in various groups in the room. E.g. How many books are there on the table? How many pictures on this wall? How many flowers in this vase? etc. The numbers chosen should be at first not greater than four, and the children should be urged to try to say **at once** what the number is, without stopping to count.

(*b*) Then larger numbers should be introduced, the children being now allowed to count, while they are at the same time encouraged to ascertain the number in a group by means of analysis. E.g. the number of books on the table may be recognized as five, because there are three in one pile and two in another. The children should be told that presently, when they have had more

[1] For explanation of the purpose and scope of the "Examples" see Chapter I.

practice, they will always be able to tell numbers of things in that way, without needing to count them one by one (cf. Example VI).

This exercise will be useful as enabling the teacher to find out which combinations of numbers the children know.

In some cases the children may be asked to set out on their desks the same number of counters or sticks that there are in a given group.

(*c*) The teacher may now ask simple questions involving number-combinations which have reference to things not in the room but only thought of. E.g. Johnny had four nuts and his brother gave him two more. How many had he then? I have nine marbles in my bag and one in my pocket. How many have I altogether? If any example presents difficulty, the children should set out counters or sticks to represent the things in question and so ascertain the answer.

B. The Symbols **1** *and* **2**.

The teacher asks such a question as: I picked seven daisies and I dropped five of them. How many had I left? and suggests that the answer should be found by using counters to stand for the daisies. When the children are ready to give the answer, she suggests that it would be a very good thing if they could all write the answer down, instead of saying it, so that all could give the answer instead of only one. This might be done by writing the word "two," but people generally write it more quickly and easily than that: they write **2**. The teacher puts the symbol on the board, each child copies and practises it. The children are now told that when we want to write **one** we merely make a stroke, thus: **1**.

The children are now practised in the application of these two symbols.

The symbols **3** and **4** should be taught in subsequent similar lessons on miscellaneous simple number-combinations.

Lessons on Separate Numbers.

It will be found that as the child's knowledge of number-combinations increases he will be helped by a systematic grouping of them. This may conveniently be done by devoting a certain proportion of the lessons each to a particular number; though lessons such as those indicated in Example I should still be continued.

Various ways of conducting these lessons on particular numbers are suggested in Examples II and IV, Example III being devoted to the method of introducing the signs + and = . The introduction of the sign – is included in Example IV.

EXAMPLE II.—THE NUMBER **five.**

A. Simple Exercises and Problems Involving the Number **five.**

Apparatus.—Each child should have five or more counters—those coloured white on one side and red on the other are very convenient for the purpose—five or more small sticks and five or more cardboard discs to represent pennies.

(*a*) Every child will first set out in rows on his desk five counters (white side uppermost), five sticks, and five "pennies" respectively. The children will then be asked to point out or name any other groups they can consisting of five objects: their five fingers (or, more correctly, four fingers and a thumb), the five buttons on Millie's dress or Tommy's coat, the five books on the teacher's desk, etc. Groups of five children may also be formed in front of the class, one of the children calling them out one by one until there are five.

The symbol **5** is now taught, if it has not already been introduced in connexion with the more general lessons of Example I.

(*b*) The teacher now reverts to the idea of a group of five children, and suggests that the class should find out in how many ways a group of five may be made up of boys or girls or both together. Individual children should form the different groups by calling out their companions by name. They will then find that to have five children we may have five boys, or four boys and one girl, or three boys and two girls, or two boys and three girls, or one boy and four girls, or five girls.

Also the result of taking a smaller number from **five** may be illustrated by the children themselves, e.g. there are five boys in front of the class, two of them go back to their seats, how many are left?

(*c*) Now suppose we are asked a question about some things which are not here in the room. Suppose we are told that Willie had five nuts given to him, and when he cracked them he found two of them bad. How many good ones had he? We could find out by pretending that boys, or girls, or our fingers, or sticks, or counters were the nuts. Let us pretend that our five counters are the five nuts, and let us turn two of them over with the red sides uppermost to stand for the bad nuts. Then the white counters will be the good nuts. How many are there? (Three.)

If the children can all answer the question quite correctly without the help of the counters, it will of course be waste of time to use them. A slightly more difficult problem may then be substituted: e.g. Willie had five nuts, cracked them, found three bad and threw these away; some one gave him two more and then he lost one. How many had he left?

If on the other hand the children are at the stage at which they need it, the teacher will have real nuts at hand to show them, in order to help out the playing or pretending that the counters are nuts.

Other simple problems involving the number **five** and its parts may be worked in this way, either the counters or the sticks being made to stand for the objects in question, dogs, chairs, books, toys, etc.

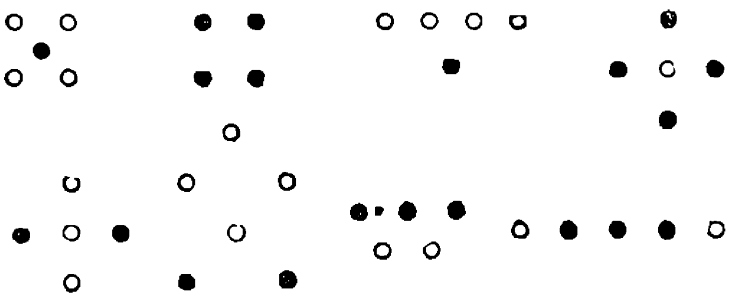
(*d*) Then a simple money problem may be taken: e.g. Johnny had five pennies given him; he spent one on sweets and two on a ball. How many had he left? In this and in similar problems the children's interest will be quickened if they are allowed to dramatize the events suggested. One child should be the shopman and Johnny should actually go and buy the sweets and the ball, pay for them and show how many pennies he has left.

(*e*) Now let us put the pennies, sticks and counters away and see what we can do without them. A boy has five little puppies; let us fancy they are here on the table and that we can actually see them. (The teacher will help this "visualizing" by going through the motions of pointing to the imaginary puppies.) He sells two of the puppies to another boy. Fancy you can see the other boy coming in and taking the two puppies away. How many are there left?

B. Various Ways of Arranging five Things.

Apparatus.—Counters as before, each child having enough to form several sets of five.

Every child is to put out sets of five counters on his desk arranged in as many different ways (or patterns) as he can, using some white and some red counters according to his taste or fancy. The teacher will do well to show the children one or two arrangements to begin with and then leave them to invent others for themselves. E.g.:—

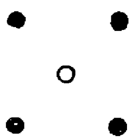


Individual children may then be asked to come out and show the rest of the class with the teacher's large counters what arrangements they have made with, say, four white counters and one red one, three red and two white ones, etc.

They may also be asked to show how one arrangement or pattern can most easily be changed into another: e.g. to indicate the two movements of counters which would change the first of the following arrangements into the second,

or the three movements by which from the arrangement

the arrangement



may be produced.

Some of these arrangements may be repeated, substituting children for the counters, say, boys for red counters, girls for white ones. The children will then themselves make the movements by which one arrangement is changed into another. Take, for example, the last pair of arrangements given above, and suppose boys to be in the places of the red counters and a girl in the place of the white one. To produce the second arrangement from the first, the boy at each end of the row must step in front of the boy beside him; the girl must then take one short step forward to bring herself into "the middle" of the four boys.[1]

[1] This use of the children as a kind of living counters can, of course, be repeated in many of the subsequent examples.

EXAMPLE III.—THE SIGNS + AND =.

Each child is to be supplied with red and white counters, paper and pencil.

The teacher begins by asking a simple question, such as: Polly had three sweets and her mother gave her two more. How many had she then? Write down the answer (5).

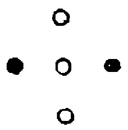
How do you know she would have five? (Because **three** and **two** make **five**.) Now we will learn how to write that down. We might write out the words "**three** and **two** make **five**," but there is an easier and shorter way. We write the numbers in the way we know already, instead of "and" we put a little cross, +,

$$3 + 2$$

and instead of "make" we put two little lines, =,

$$3 + 2 = 5$$

The children must now have practice in connecting combinations of numbers with this form of expression. In some cases they may make (with or without the teacher's guidance) arrangements of counters, and then record the combination of numbers by means of symbols. E.g.:—



$$3 + 2 = 5$$

$$3 + 1 = 4$$

In other cases the teacher may write the combination of symbols on the blackboard, the children setting out an arrangement of counters to correspond to it.

Again very simple questions of the kind used at the beginning of this example may be asked by the teacher and the answers written down in full by means of symbols. E.g. John had three marbles and won one more. How many had he then? (**3** + **1** = **4**.)

EXAMPLE IV.—THE NUMBER **seven.**

Note.—If the children have not learned, in one of the lessons continuing those indicated in Example I, the symbol **7**, it should be taught in one of the lessons on the number **seven.**

The children are to be supplied with red and white counters as before.

A. The Different Pairs of Numbers which Added Together make **seven.**

Each child sets out seven counters in a row on the desk with the white sides uppermost. He is provided with paper and pencil.

The children are then told to turn over one of the counters (at the end of the row). What have you now? (Six white counters and one red one.) How many altogether? (Seven.) How can we write that down? (**6** + **1** = **7**; that is "**six** and **one** make **seven**".)

Now turn over another counter and write down what you have then. (**5** + **2** = **7**.)

The children will proceed in this way until they have:—

$$6+1=7$$
$$5+2=7$$
$$4+3=7$$
$$3+4=7$$
$$2+5=7$$
$$1+6=7$$

They should then be led to compare the first line and the last, the second and the last but one, and so on, and other illustrations should be given of the facts revealed (e.g. **4** boys and **3** girls make as many children as **3** boys and **4** girls: namely **7**). Attention should also be called to the sequence of numbers in the two columns: **6, 5, 4, 3, 2, 1**, and **1, 2, 3, 4, 5, 6**.

B. The Results of Taking from **seven** *the Numbers* **one** *to* **six** *Inclusive.*

The counters will now again be set out in a row Take one counter away and put it on the other side of your desk. How many are left? (Six.) **One** taken away from **seven** leaves **six**. We can write that down shortly in this way:—

$$7 - 1 = 6,$$

the " – " meaning "take away,"[1] so that we say "**seven**; take away **one**, and we have left **six**".

Unless the children themselves raise the question, no notice need at this stage be taken of the fact that the **words** used to express the sign = are different in the two statements, $6 + 1 = 7$ and $7 - 1 = 6$. Later on —or at this point if the children themselves comment on it—it may be pointed out that in each case the = means "what you have in the end is— ".

The children will now take away successively one counter after another, writing down the corresponding statement each time, until they have the following:—

$$7 - 1 = 6$$
$$7 - 2 = 5$$
$$7 - 3 = 4$$
$$7 - 4 = 3$$
$$7 - 5 = 2$$
$$7 - 6 = 1$$

and, as before, the different statements will be compared.

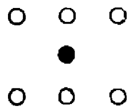
These exercises should be followed by various applications (oral) of the different ways of combining numbers

[1] With reference to the question whether "taking away" should be the first meaning to be connected with the sign "–" (as here), or whether that sign should first be introduced in connexion with "complementary addition," see Chapter V, Example XII (*d*) and (*e*) and footnote (pp. 55 and 56).

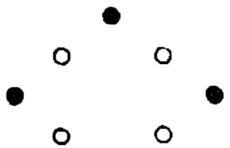
to make **seven**, and of the results of taking away different numbers from **seven.** E.g. A boy had **4** pennies and his father gave him **3** more. How many had he then? A girl had seven beads and gave away **2.** How many had she left?

C. Different Arrangements of seven Things.

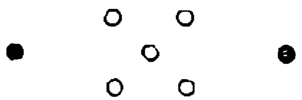
(*a*) Each child is again given seven of the red-and-white counters and is told to make as many patterns with them as he can, using some red and some white counters. As each pattern is made he will copy it on paper and write beside it the combination of the number of red and the number of white counters used. Thus, for example:—



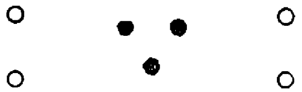
1 + 6 = 7



3 + 4 = 7



2 + 5 = 7



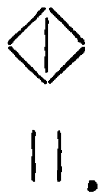
2 + 3 + 2 = 7
or 3 + 4 = 7

etc., etc.

(*b*) *Apparatus.*—Small sticks.

Each child is given seven of the sticks and told to arrange them in as many ways as possible, and, as before, to write beside the representation of each arrangement a statement of the numbers combined. For example;—

3 + 4 = 7



5 + 2 = 7

2 + 3 + 2 = 7

3 + 1 + 3 = 7

etc., etc.

It is to be understood that in Examples II and IV the numbers **five** and **seven** have been chosen merely for purposes of illustration. It is not intended to suggest that it is appropriate to treat them in any special or peculiar way. It is probably wise to treat all the numbers from **five** to **twelve**[1] inclusive in this or in similar ways. In doing so, two important questions will arise, first, that of the order in which the numbers should be taken; second, that of the symbols to be used for the numbers **ten, eleven,** and **twelve.**

[1] The numbers greater than twelve should be treated with reference to the decimal numeration and notation (see Chapter IV).

(1) *The Order in which the Numbers shall be Studied.*

The regular—i.e. the numerical order—is not the best for this purpose; the arrangement should be determined partly by the associations between certain numbers and the child's everyday experiences, partly by the relations between the numbers themselves. **Five**, from its connexion with the fingers of the hand, has a special interest; for a similar reason **ten** may be taken early in the series, being approached in the first instance as the result of putting together two **fives**, and so consisting of two **threes** and two **twos**, etc. **Seven**, being the number of the days of the week, has an interest of its own and claims an early place. **Twelve** is specially interesting from the fact that twelve pennies "make" one shilling; some of its properties may be worked out with the aid of sixpenny and threepenny pieces. This number also lends itself to a large number of symmetrical arrangements; it has an interesting connexion with the months of the year and the number of figures on a clock-face, and it has, moreover, in certain fairy-tales a mystic importance second only to that of the numbers **three** and **seven**. For all these reasons **twelve** may be taken early in the series of numbers. **Six** very conveniently follows it, the remainder of the numbers between **five** and **twelve** being taken in such order as the teacher judges to be easiest and most interesting. There should be no need, if the previous incidental work has been well done, to treat separately the numbers less than **five**, and the amount of time to be devoted to any one of the other numbers must, of course, be determined with due reference to the knowledge of number which the children already possess. Nothing but confusion and consequent unsoundness of work later on can result from hurrying the child through this work at a rate beyond his powers; on the other hand, a most wearying and deadening effect is produced by a mistaken ideal of "thoroughness" which displays itself in laboured lessons on simple number-facts already well known to the children and in impossible attempts at "complete mastery" of one number before proceeding to another.

(2) *The Symbols for the Numbers* **ten, eleven, twelve.**

This difficulty may be met either—

(*a*) By postponing the introduction of any symbols for these numbers until after the children have learnt to count in **tens** and to express the results of their counting in the decimal notation; or

(*b*) By teaching the children that **10** stands for **ten**, **11** for **eleven**, and **12** for **twelve** (just as **9** does for **nine**) without at first analysing the symbols or explaining their connexion with counting in tens.

The first alternative (*a*) has obvious drawbacks. It will seem to the children arbitrary and vexatious to be obliged, at a time when they are dealing with the numbers up to **twelve**, to stop short at the number **nine**, as far as expression of them in symbols is concerned. Some way of expressing the numbers **ten**, **eleven**, and **twelve**, is highly desirable, even if it is not absolutely imperative.[1]

For this reason the second alternative—that of teaching the symbols **10**, **11**, **12**, without at first explaining the method of their formation—is the course recommended here. If, as is suggested below (Chapter V, p. 45), counting in tens is begun fairly early, and is followed in due course by the use of the decimal notation, the explanation of the way in which the three symbols in question have been formed, will not be very long postponed. When it is reached, it will gain added interest from the fact that the symbols themselves are already quite familiar.

A further type of early number-lesson will be found useful, namely, that dealing with a given group of

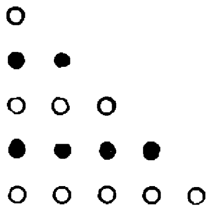
[1] Moreover, some of the children will probably know from other sources how people usually write these three numbers.

numbers. Suggestions for such lessons are given in the following Example.

EXAMPLE V.—THE GROUP OF NUMBERS **one** TO **five** INCLUSIVE.

Apparatus.—Counters as before.

Each child sets out on his desk counters to represent the numbers from one to five. They should be arranged in rows and, for convenience in distinguishing them, the alternate numbers may be represented by red counters. Thus:—



The teacher will then question as to the differences between the numbers. How much larger is **5** than **2**? **4** than **1**? etc. Which is the largest number? Which the smallest? Which the middle one? How much smaller is the smallest than the largest? How much smaller is the middle one than the largest? etc.

Various re-arrangements are now suggested.

(*a*) Let us take the middle number away for a few minutes. Put the smallest and the largest together. How many will that make? (Six.) And the other two? (Six.) Let us write these down:—

$$2+4=6$$
$$1+5=6$$

Now put the rows back as before, and this time take the smallest one away. Again put the smallest and the largest (now **2** and **5**) together, and the others (**3** and 4) together. We have:—

● ● ○ ○ ○ ○ ○ $2+5=7$

○ ○ ○ ● ● ● ● $3+4=7$

As before, they come to the same number, but this time it is **7**, not **6**.

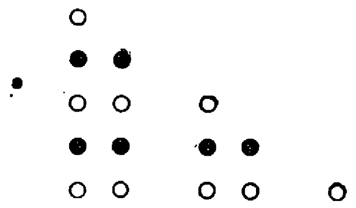
Again put the rows back; this time take away the largest (**5**) and do the same as before. Now we have:—

○ ● ● ● ● $1+4=5$

● ● ○ ○ ○ $2+3=5$

Again they make the same number, this time **5**.

(*b*) Put the rows back as at the beginning, and now let us arrange each row as far as we can in **twos**:—



1 is less than **2**. How many less? We cannot, then, arrange it in twos. **3** is **1 two** and one over. **4** is **2 twos.** **5** is **2 twos** and one over.

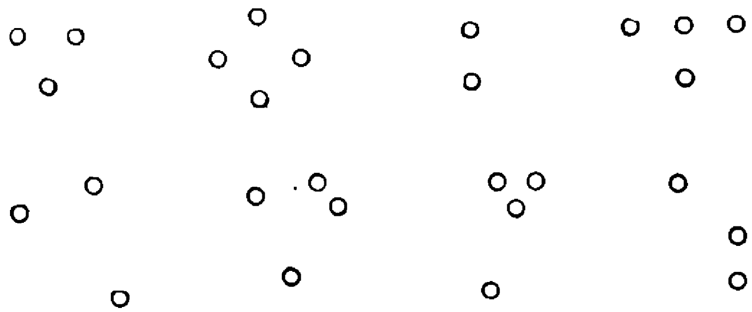
Similarly the rows may be arranged as far as possible in **threes.**

EXAMPLE VI.—THE RECOGNITION OF NUMBERS BY ANALYSIS INSTEAD OF COUNTING.

It is assumed that in the early incidental number-work the children have been given opportunities to practise recognizing at a glance (that is, without stopping to count) the number in a group of not more than four objects (see p. 7). They are now to be taught to break up mentally groups of more than four objects into groups of **1**, **2**, **3**, or **4** objects, and so to recognize the number in the group without counting the objects one by one.

Apparatus.—The children will have red and white counters as before. The teacher should provide herself with a number of representations of different arrangements of counters—large enough for the whole class to see. This is conveniently done by pasting on sheets of stiff paper or thin cardboard circular discs of coloured paper (in two colours) in the required arrangements—each arrangement on a separate sheet. Or if preferred, the counters may be represented with coloured chalk or crayons instead of coloured paper. The first groupings of counters shown to the children should be of **2**, **3**, or **4** counters, in order to test whether each child can actually recognize these numbers of objects at sight.

After telling the children that she is going to show them for a **very** short time a picture of some counters, and that she wishes every child to write down on his paper how many there are, the teacher then holds up in succession some of, say, the following groupings. Each of them must be exposed just long enough for the children to see it and not long enough for them to count the number of objects represented. The proper length of exposure for the purpose can be found without much difficulty after one or two trials :—



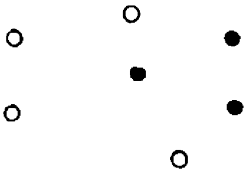
Now we will try some larger numbers. (The next representation should be in two colours). How many are there here ?

How did you know there were five, when you had not time to count them? There were **three** and **two**, three blue and two red.

Try to arrange five of your own counters in the same way, putting white ones instead of the blue ones.

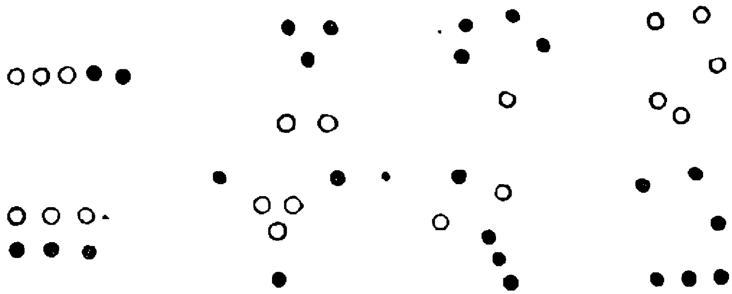
The copying or reproduction by the children of groupings presented to them by the teacher will be found to be a very important aid to this exercise besides giving the children valuable practice in space-perception.

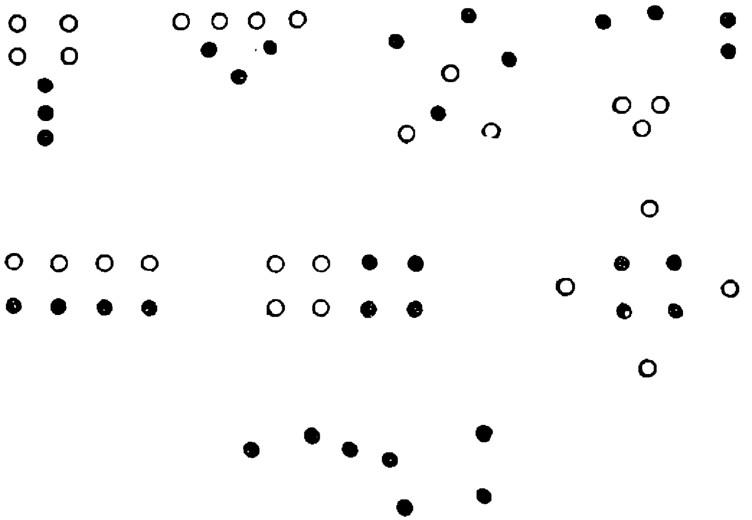
A number of similar exercises will follow, the children sometimes reproducing with their own counters the grouping shown, sometimes merely writing down the total number of counters, sometimes writing down the result in full in such a way as to show the analysis which has been performed; e.g.—



in which case the children write **4 + 3 = 7.**

Some examples of groupings suitable for this purpose are given below. Others have already been given in other connexions (see Examples II, B, and IV, C, above).





In the earlier exercises the analysis is aided by both colour and spacing, in others by colour alone, in others again by spacing alone, and finally both of these aids are discontinued.

The children should be given practice in applying this method of analysis to groups of objects of various sorts, sometimes the actual objects being presented to them, sometimes pictures.

Although every lesson will furnish opportunities of fresh applications of the facts of number, it will often be desirable to make special provision for them in lessons devoted entirely to this purpose. It will be seen that these lessons are in effect a direct continuation of the early incidental learning of number, the child's interest in number facts being in them subordinate to his interest in the particular incidents of life and experience to which they are applied.

Example VII gives illustrations of some exercises of this kind.

EXAMPLE VII.—SOME APPLICATIONS OF NUMBER-FACTS.

The Arrangement of a Dinner-Table.

Each child will be provided with a piece of paper (or some building bricks) to represent the dinner-table, a number of sticks (preferably of two different lengths), and a number of counters of two different colours (or two different sizes). The children will then prepare the table for, say, a father, mother and four children. The various points for consideration will be:—

(*a*) The different ways in which they may sit at table: father and mother one at each end, and two children on each side, three on each side, etc.

(*b*) The total number of knives and forks needed for the party.

(*c*) The total number of plates if each person is to have meat, pudding and cheese, or if, on the other hand, only the "grown-ups" are to have cheese. If each has his three plates in a pile in front of him, then six **threes** will be needed; if the plates are given round before or during each course, then three **sixes** will be needed. In either case the total number **eighteen** need not necessarily be named.

Other examples that may be treated in a similar way are:—

The setting out of the proper number of castors for the furniture of a room, there being one or more of the same or of different kinds (e.g. chairs, three-legged stools, six-legged tables, etc.).

The different arrangements of single and double desks (which may be represented by sticks) in a class-room, and the placing of children (boys or girls, or both—represented by counters of different colours) in them, one or two in a desk, certain desks being sometimes empty.

Various cases of distribution of fruits, sweets, marbles, etc., among given numbers of children, etc.

Money.—Very simple questions involving money should also be set. These should be worked by means of discs representing the coins, the different denominations being gradually introduced in the order: penny, halfpenny, farthing; sixpenny piece and shilling.

The following are illustrations of the kinds of exercises that are suitable:—

Johnny's father gave him threepence, his mother gave him twopence and his aunt gave him a penny. How much had he then?

Bessie had ninepence and she bought a doll which cost fourpence. How much had she left?

A boy had sevenpence in his pocket. He bought five little balls costing a halfpenny each. How much had he left?

How many farthing cakes can I buy for twopence halfpenny?

I have two sixpenny pieces. If I buy five farthing cakes, two twopenny dolls and a threepenny top, how much money shall I have left?

A boy has in his money-box a sixpenny piece, two pennies, five halfpennies and three farthings. How much more must be put into the box to make a shilling?

CHAPTER III.

Comparison of wholes of different lengths and weights. Introduction of Tillich's bricks.

Up to the present the only aspect of number that has been considered is that in which it appears in measurement and comparison of groups of separate objects. The child's attention must, however, also be engaged with the use of number in measurement and comparison of **wholes** of different magnitudes. The most important example of this use of number is furnished by the comparison and measurement of lengths, but weights may also be included, even at a comparatively early stage. This work should be begun while that described in Chapter II is proceeding (see Chapter I and General Scheme of Work).

As in Chapter I, examples of important and typical parts of the teaching will be given.

Example VIII.—Comparison of Lengths and Weights—Measuring and Weighing.

A. Comparison Without the Use of Number.

(*a*) *Direct comparison.*

Lengths.—The teacher holds up, say, two pieces of paper of different lengths, one in each hand, and asks which is the longer. It may be possible in this case for the children to answer the question without placing the pieces of paper side by side; if not, they must be put together so that the difference between them is seen.

Other examples will follow, the correctness of the answer in each case being tested by placing the lengths in question side by side, e.g.—

Which is taller, John or Willie?

Which is higher, this book placed on its end or the teacher's platform?

Which is wider, this picture or the blackboard?

Which is longer, this pointer or the blackboard easel? etc.

Weights.—Exercises should then be given in comparing the weights of different objects—books, pencil-boxes, etc.

The weights of two objects should first be compared by holding them successively in the hand, or by holding one in each hand, or by both methods. When in any given case certain of the children express different opinions as to which of two given objects is the heavier, recourse must be had to a pair of scales to test the correctness of the estimates made. An ordinary pair of kitchen or grocer's scales is the most suitable at this stage. It is only when small weights, such as $\frac{1}{2}$ oz., $\frac{1}{4}$ oz., are to be used that a more sensitive balance is needed.[1]

The children should now be given practice in forming such judgments as: This book is not so wide nor so long as that one, but it is thicker and heavier. This book is longer and wider than either of these other two, and is heavier than both of them put together, etc.

(*b*) *Comparison of two quantities by means of a third.*

Lengths.—The teacher now asks the children to compare the lengths of two objects that cannot be placed side by side, e.g. Which is wider, the window or the cupboard?

The children themselves will probably be ready to suggest that something else—a piece of string or the pointer or a piece of paper—should be used to find out the answer to the question. The string is held across the window, the length of the window marked off on it and then it is carried across to the cupboard and the comparison made.

As many exercises as possible of this kind should be

[1] It is assumed that here and in the next stages all the work will be done with such a balance as that suggested. Even in cases where it proves possible to supply each child or pair of children with a simple "home-made" balance, such balances are usually too sensitive and certain parts of them are too easily displaced for small fingers to be able to handle them with effect. Moreover, they are as a rule quite unsuitable for weighing any but small objects or quantities of material, and would be useless for weighing out a pound, or even half a pound of tea, sugar, sand, etc. It is undoubtedly a drawback that if only one balance is used, as is here suggested, only one, or at most two children can take part in any given act of weighing; but this drawback is to some extent counterbalanced by the fact that small people find weighing an exceedingly interesting operation to watch.

done by the children themselves. In most classrooms one or two examples can be found which every child can answer for himself by means of measurement, e.g. Which of the two edges of your desk is longer?

B. The Use of Number in Comparing Lengths and Weights.

(*a*) *Comparison of pairs of lengths and of weights, one of which is twice, three times, etc., as long as the other.*

Lengths.—The teacher holds up two books—say, a blue one and a red one. Which of them is longer? (The blue one.) How can we find out **how much** longer it is than the red one? (By placing the two together as before.) One child does so and marks off with his finger on the edge of the blue book the length of the red one. It is now seen how much longer the blue one is than the red one. It looks, too, as if the red one reaches just to the middle of the blue one. In order to test this it is moved along and it is then found that the blue book is just twice as long as the red one.

Other examples should be worked out in a similar way.

Weights.—With the help of a pair of ordinary scales and a few cardboard boxes of various sizes, some exercises in comparing objects of different weights may be given to the class. The boxes may be such as are used to hold stationery, paper-clips, pens, crayons, etc., and may be adjusted to convenient weights either by means of varying the amount of their usual contents or by filling them, or partly filling them with sand.

The teacher first holds up before the class two boxes (preferably of different sizes) which she knows to be of the same weight, and invites one of the children to come and hold them and say which is heavier. Their weights are eventually judged to be equal and the fact is confirmed by means of the scales.

One of these boxes is now compared with a third, which is judged to be heavier. To confirm this opinion the two are placed one on each scale-pan. It is then suggested that we should try to find out how much is needed to make the scales balance. After a few trials it is found that the first two boxes together just balance the third, or heaviest box. We may say then that the heaviest box is **twice** as heavy as each of the others; or that each of the lighter boxes is **half** as heavy as the heaviest one.

Similarly boxes may be tested which are three or four times as heavy as some other box.

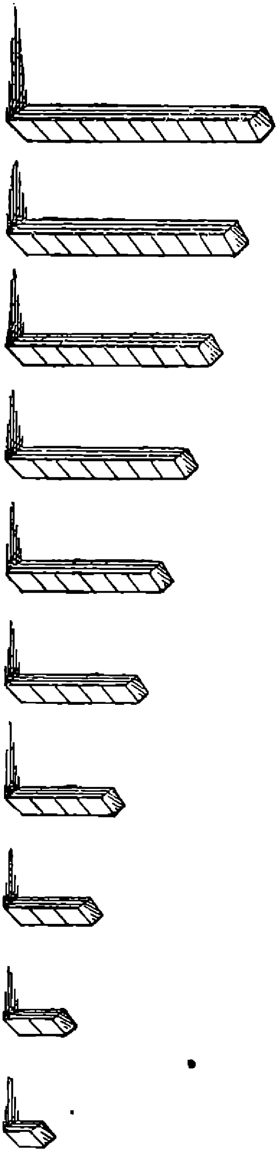
Note.—It is desirable to arrange that the sizes and weights do not as a rule correspond: that boxes of equal size should often be of different weights, and *vice versa.*

(*b*) *Comparison of lengths*[1] *by means of a unit length—Tillich's bricks.*

The idea of comparison by means of a unit is best presented to the child through a series of lengths, such that each of the succeeding lengths is a multiple of the first. A series of wooden rectangular bricks, such as those usually described as Tillich's bricks, is most convenient for the purpose. These bricks are all of the same breadth and thickness, and vary in length in the way already described.[2] They are usually marked (as shown in the drawing) in such a way as to indicate how many times each brick is as long as the smallest one. If this is done, the marking should be confined to one face of each brick; when the children first use them this face should be turned down upon the desk.

[1] The comparison of weights by means of a unit weight is postponed until the notion of measuring lengths by means of a unit has become to some extent familiar (see Chapter VIII).

[2] If there is any difficulty in obtaining wooden bricks, strips of cardboard of the same relative lengths form a convenient substitute.



The bricks are generally made in sets of ten, but there would be advantages in increasing the number to twelve.

Each child, then, has a small set of these bricks, and after being allowed to handle and play with them freely, in order to become familiar with them, he is asked to arrange them in certain definite ways in relation to their size. He may, for instance, build a flight of stairs with them, and so arrange them in order of length; he may pick out the largest brick and arrange from the others as many pairs as he can such that the two together make a brick as long as the largest one; he may put the largest and the smallest together, the next to the largest and the next to the smallest together, and so on, making a number of pairs all of equal length, etc.

After the exercises he has already had in comparing lengths, he will now have no difficulty in picking out, as the teacher suggests them, different pairs of bricks, such that, in each case, one brick is twice as long as the other; or pairs such that one of each pair is three times as long as the other; and so on.

The children may now note that, if we pick out the smallest brick and lay it in succession against the others, we find that each of them is some number of times as big as the little one. They may be called, then, from this fact the one-brick, the two-brick, the three-brick, and so on.

Exercises should now be given in making combinations of the different bricks, by placing them end to end, corresponding to the combinations of numbers the children have already been learning in connexion with groups of separate objects. The one-brick and the three-brick, if joined together would make a four-brick; the five-brick and the two-brick would make a seven-brick; if we could cut off a three-brick from the eight-brick we should have a five-brick left, and so on.

C. The Application of the Preceding Ideas to the Measurement of Objects in the Classroom.

The children will first be reminded that when they wanted to know which was wider, the window or the cupboard, they stretched a piece of string across the window, marked on it how wide the window was and then carried the string across to the cupboard to see whether it was wider or narrower than the length marked on the string.

Now suppose they wanted to know whether their desks were wider or not so wide as the door, or the same width as the door. No string being immediately at hand, what could be used instead? The thing used need not be so long as the string they had the other day, something shorter would do. It is suggested that they might use a small strip of paper, or a pencil-box, or a pencil, or better still, the little sticks they often use to count with, for they have plenty of them. It is, of course, assumed that the sticks are all of the same length.

Each child will then lay a row of sticks across his desk. The rows must be straight, but no explanation of this should be attempted at this stage. One of the children will come out and lay a row of sticks across the doorway. They can now, by counting their sticks, find out what they wanted to know. The desks are, say, eight sticks wide, and the door is, say, fourteen sticks wide. If, although this is unlikely, the children are quite ignorant of the numbers beyond **twelve**, eight of the sticks in the doorway can first be noted as corresponding to the width of a desk, and then the remaining sticks—six—counted. They find then that the door is six sticks wider than the desk.

Other objects in the room may be measured and compared in the same way. Vertical lengths will present rather more difficulty than horizontal ones, since it will not be possible to lay a row of sticks along them.

Either one or two sticks must be used and laid in succession along the length to be measured. This method may afterwards be applied to lengths in a horizontal or any other position, so that the children learn to measure any length by means of one stick only.

Further interesting exercises of the same sort may be introduced by supposing on certain occasions that neither sticks, string, nor any such means of measuring is available. For instance, boys may be playing marbles out of doors and may want to know whether this marble or that one is further from the ring. What could they do? They might measure with their feet by stepping along the line to be measured, placing one foot immediately in front of the other and just touching it. This may actually be done in the classroom by placing marbles on the floor, and, to guard against error from their rolling, marking their positions with chalk. The attention of the children must be called to the fact that it is necessary that either the same child makes both measurements, or, if two children make them, their feet must be of the same length. In cases where a piece is "left over" too small to be measured by the foot, the top joint of the thumb or the width of a finger may be used. Thus one length may be found to be five of Tommy's feet and two of his thumb-joints long, while another is five of his feet and three of his thumb-joints.[1]

Or the stretched hand—the "span," the distance from the tip of the thumb to the tip of the little finger—may be used as a means of measurement, or, in certain cases, the outstretched arms. With a view to future understanding of the standard units it is important to call attention to the possible source of error already mentioned above, namely, the variation of these lengths in the case of different individuals.

[1] Only very easy examples involving two units of measurement should be taken at this stage.

CHAPTER IV.

The use and function of material aids to thinking; growth of power to dispense with them.

1. The "Apparatus" to be Used in Number-Lessons. Need of Variety in the Early Stages.

The constant use in the foregoing teaching Examples of one or two forms of material aids to the children's thought (e.g. counters or sticks) raises the important question of the nature of the material that should be used for this purpose, and the degree of variety that should be aimed at.

It is highly desirable to associate children's number-ideas from the beginning with as wide and varied a range of experiences as possible. Not only does this variety form a valuable stimulus to the child's interest; without it, there is some danger that he may be lacking in real grasp of the number-relations presented to him and may fail to develop fully his power of applying them effectively to the varied problems of life. The necessary variety of experiences must be sought, not solely—nor even chiefly—in the number-lessons, but in all those parts of his "work" and of his play in which numbering helps him to carry out the purpose he has at the moment in view. In school such occasions will arise in a very large proportion of the occupations in which the child is engaged, not only in the early stages before formal number-lessons are begun, but throughout the whole school course. In the Kindergarten and Infants' School the various employments, such as building, bead-threading, drawing, the songs and games,[1] will all furnish opportunities for the use of number. Later on handwork, drawing, geography, science will all, in

[1] Cf. Miss Wark's article. See footnote, p. 8.

greater or less degree, continue these opportunities in modified and more complex forms.

Meantime in the early number-lessons themselves as great a variety of material as possible should be introduced, not merely that the children may have many different sorts of things to count, but in order that they may be reminded as vividly as possible of some of the many ways in which number serves the ordinary purposes of life. With this object in view the teacher will press into her service the desks, chairs, books, pencils in the classroom, the children themselves, nuts, fruit, beans, beads, blocks of wood, counters, sticks, and anything else that can be procured and can be easily handled, and will use them as a means of enabling the children to make continually fresh applications of number to the everyday events of life.

2. Importance of Direct Sense-Experience.

It is important that the child's early number-ideas (and to a certain limited extent some of his later ones also) should be based upon a foundation of direct sense-experience. He should see, touch, and handle as many as possible of the objects whose number-relations he is considering. He is at first unable to consider the number-relations of objects which are not actually present, and any attempt to induce him to do so can only result in eliciting from him a mere meaningless repetition of words. He may, for example, learn to enunciate in parrot fashion such a statement as "three and four make seven," but he will show, by his inability to apply the fact to fresh groups of objects, that he has no effective grasp of its meaning.

3. Visualization.

On the other hand, the child's progress may be seriously hampered if his number-work is confined for too long a time to the objects which he actually sees and handles. He must be encouraged as time goes on to substitute for the objects themselves more or less definite mental images of them. These images, owing to their comparative dimness and vagueness, will always help him less than the presence of the actual objects, but they give him, nevertheless, assistance of the same kind. Thus, when a child has learned to associate the adding of two numbers with the actual putting together of two groups of objects either by himself or by his teacher, his power of visualizing even dimly the events suggested in the problem, "John had eight marbles; he bought four more. How many had he then?" will almost certainly aid him in recognizing that this too is a case for addition.

The child, then, learns by degrees to apply his number-knowledge to objects not actually present, and it becomes possible in consequence to increase greatly the number and variety of the applications of number-facts which he is called upon to make.

The teacher's aim, then, must be twofold:—

(1) She must provide as great a variety as possible of **things** for the child to handle and use in various ways which involve numbering. It is by means of these experiences that number-relations will be learned.

(2) She must encourage, as the child's growing power makes it possible, the application of these number-relations to events and experiences involving objects which are either no longer present in the schoolroom or have never been there.

4. The Play-Motive. Substitution of One Object for Another.

The fulfilment of the second of these aims will be greatly facilitated by making use of the children's play-tendency—of their delight in pretending that the objects around them are things quite other than they seem.[1] Under the transforming influence of a child's

[1] Cf. R. L. Stevenson's Essay, "Child's Play".

fancy the familiar counters and sticks may become in succession things so different from the original material and from one another, as horses, nuts, children, trees, or houses. In this way they may be used as an aid in the solution of little problems dealing with any sort of object that the teacher suggests. In order to help the less imaginative children in "pretending" that the sticks or counters are something other than they are, the teacher should sometimes have the actual objects (e.g. apples, cakes, marbles) on her own table or desk, or, in the larger number of cases when this is not possible, should make frequent use of pictures. In this way the ever-present and easily available counters or sticks may become a valuable aid to a wide extension of the applications of the child's number-ideas (cf. Chapter II, Examples I, II, VII).

It may be noted in passing that the practice of substituting sticks, counters, etc., for other objects is specially useful when the child is learning to perform arithmetical operations in connexion with the decimal notation (see Chapters V and VI). For example in adding together **23, 15,** and **9** he may conveniently represent the "tens" by bundles of sticks and the "units" or "ones" by separate sticks, whether the three numbers in question refer to cows, desks, children, or apples.

5. Various Stages in the Growth of Independence of these Aids to Thinking.

While recognizing the value of such aids to thinking as those described above, it is important also to recognize the fact that, from one point of view, the progress of the child consists in the growth of his power to dispense with such assistance. It is the teacher's business to provide the material for such help in thinking as long as it proves necessary, watching carefully at the same time for signs of ability to do without it and encouraging such independence in every possible way.

The various steps in the growth of independence of these supports to thinking may be roughly indicated as follows. The child may depend for assistance, according to the stage of progress reached—

(1) on the presence of the actual objects in question;

(2) on some other objects (e.g. sticks or counters) which stand for the things in question;

(3) on a drawing or similar representation either of the objects themselves or of others which stand for them;

(4) on mental images of the objects or of others which stand for them.

It will be found that children differ considerably as regards the speed with which they pass from one of the above stages to another, as also in their power to omit one or more of them. Fortunately there is no need for the various individuals in one class to be at the same stage in this respect. While all may be finding the answer to the same question, each of the different means indicated may be in use by some members of the class. For example, supposing the question to be, "John had eleven nuts, he lost seven, and then Willie gave him three more. How many had he then?" the majority of the children may be, say, at the stage when it is necessary for them to make rough drawings representing the nuts, beginning with eleven, crossing off seven and then adding three more. Some children may need to handle, say, counters, which stand for nuts, before they can "work" the sum, and a few specially backward ones may even (though this would be exceptional) be unable to obtain the answer without the aid of actual nuts. On the other hand, a few members of the class may be able to answer the question with the help of visualization alone without the aid either of the actual objects or of any representation of them. Each child should be encouraged to pass on from one of the above stages to another as soon as he is able to do so, only having recourse to the earlier forms of support to his thinking when the aid given by the later ones proves insufficient for him.[1]

With reference to the last stage (that of visualization), it is interesting to note that, as the child progresses and deals with larger numbers, the visualization becomes, from the point of view of number, less and less definite. For example, a child at an early stage may have a fairly clear mental image of three nuts, of one being removed, and of the remaining two. But if, later on, he is told that a boy had **14** rabbits and sold **6** of them, and he is asked how many rabbits the boy then had, it is unlikely that he will succeed in visualizing anything more definite than a vague group of rabbits, and the removal of an equally vague and indeterminate group from it. As we have already seen, these images will only make it easier for him to recognize the problem as one concerned with "taking away"; he will then need to draw upon his previous knowledge of number-relations in order to give the answer **8**.

6. The Final Stage.

A well-taught child will eventually reach, in relation to each number-fact that he learns, a state of almost complete independence of all the four aids to thinking mentioned above. For instance, if at a comparatively early stage he is asked what three nuts and two nuts make, or if at a later stage he is asked what nine and eight make, he gives the correct answer in each case at once, "without stopping to think". He is in fact

[1] It may be objected that, if this freedom is given, the time taken by different children to solve the same problem will be widely different. This is a difficulty common to all stages of mathematics teaching from the earliest number-lessons onwards. It can only be partially obviated by classification, even in the best organized schools, and is to be met here as elsewhere, not by forcing the quicker children to adopt the pace of the slower ones, but by giving them a larger number of exercises to do.

perfectly familiar with the relation in question in each case, and with its expression in words or symbols. It is possible that, even when this stage has been reached, some faint form of visualization is usually present. Whether this be so or not is a question of interest to the psychologist, but the answer to it can hardly affect the practice of the teacher. He must in any case see that his pupils have full opportunity of becoming—first with much and afterwards with little support to their thinking—so familiar with all the simpler number-relations as to be able to apply them in any given instance with certainty and ease.

7. Written Symbols as Aids to Thought.

It is important that the teacher should bear in mind the fact that it is only at a comparatively late stage that written symbols act as aids to thought. While the simpler number-relations — those on which the later and more complex ones are to be based—are being learned such symbols give little or no help. Thus, if a child does not know what **five** and **three** make, neither of the arrangements of symbols

$$5 + 3 =$$

or

$$\begin{array}{r} 5 \\ +3 \\ \hline \end{array}$$

will help him. He must either count his fingers (or some other objects), or he must draw strokes and count them, or he must use some fact that he does know (e.g. that **five** and **two** are **seven**) to enable him to obtain the answer **eight**.

But when, later on, he needs to ascertain the result of, say, adding together **347** and **258**—that is, when he has to handle a complex combination by means of breaking it up into simpler ones—then written symbols will give him valuable aid. He will first write down—

$$\begin{array}{r} 347 \\ +258 \\ \hline \end{array}$$

and will then record each part of the "answer" as he finds it. The function of the symbols here is evidently to enable him to attend in succession to each part of his "sum" without needing to burden his memory with, and so divert his attention to, the other parts.

Symbols, then, while forming later on an indispensable aid to progress, are to be regarded in the early stages as mere records of results reached.

SECTION II.

THIS section deals with the extension of the work to the numbers up to **one hundred.** Counting in groups is taught leading up to decimal numeration and notation, and the children are practised in number-combinations of all sorts, including their application to simple problems (Chapters V, VI, VII). The Number Chart is introduced as an aid to a ready and accurate manipulation of numbers up to **one hundred** (Chapter VII). No formal "sums" are introduced at this stage; the children will merely use such statements as

$$43 + 7 + 9 = 59$$
$$49 - 17 = 32$$
$$21 \times 2 = 42$$

to record their work.

The use of number to measure continuous quantity, which was introduced in Chapter II, is continued in this section in Chapter VIII; the simpler units of length and weight are introduced, and the children are given practice in using them.

CHAPTER V.

The numbers up to one hundred; preliminary investigation by "one-one correlation"; counting by analysis into equal groups. Decimal counting and notation; decimal addition and subtraction (no carrying in subtraction) by means of sticks, counters, etc. Simple problems; money.

SOME teachers may prefer to let their little pupils begin their acquaintance with numbers larger than **twelve** at an earlier stage than the present one, and there is no reason why some of the exercises of the present chapter (e.g. those described in Example IX) should not be taken while the work of the previous section is still in progress. On the whole, however, it will be found wisest to employ the time at the earlier stage in a great variety of exercises in the numbers up to **twelve,** leaving the larger numbers almost entirely until the present stage is reached.

The method of treatment of all the larger numbers should be determined by the fact that for all ordinary purposes they are regarded as being composed of certain smaller numbers, all arithmetical operations being performed upon them by breaking them up into these smaller numbers. Thus, unless they are being applied to money, the addition of, say, the two numbers **fifty-seven** and **thirty-nine** is performed by combining respectively the tens and units of which the numbers consist. If the reference is to pence the numbers will, of course, consist of **twelves** and units instead of **tens** and units.

For this reason what may be called the "intensive" study of individual numbers, such as is described in Chapter II, should be discontinued when the numbers beyond **twelve**[1] are being treated. All manipulations of these larger numbers should from the first be looked upon by the child as opportunities of applying his detailed knowledge of the numbers less than **twelve,** and ample practice in applying this knowledge to them should take the place of study and analysis of separate numbers (see Chapter VI).

The child's first dealings, then, with the numbers from **twelve** to **one hundred** should consist of :—

1. Comparison of two or more numbers, so chosen that their difference comes within the range of the numbers already known to the child (see Example IX, p. 47).

[1] For the reason for choosing **twelve** as the limit here, see footnote, p. 8 (Chapter II).

2. Description and handling of numbers regarded as combinations of smaller numbers (see Example X, p. 48).

3. Special practice in describing and manipulating numbers regarded as groups of **tens**, or **tens** and **ones**, as the case may be (see Examples X, XI, XII, XIII, pp. 48 to 58).

In the case of children who are quite ignorant not only of the numbers but even of the number-words beyond **twelve**, there is much to be said in favour of teaching the names of the numbers from **twenty** to **one hundred** before those of the numbers between **twelve** and **twenty**.[1] But since most children will be already partially familiar with the names of the numbers following **twelve**, the more usual plan of treating the "teens" next after the numbers from **one** to **twelve** is adopted here.

EXAMPLE IX.—"ONE-ONE CORRELATION."

["One-one correlation" is the comparison of two numbers by telling off in succession one unit of one number against one of the other.]

This idea should be introduced by proposing to the class the problem of discovering which of two groups is larger, and also how many more things there are in one than in the other. For example, how can we find out whether there are more girls or more boys in the class? We might count the girls and count the boys and then we should be able to say which number was larger. But there is another way of finding this out and it is often easier than counting, especially if the numbers are large. We might arrange the children in pairs, a boy and a girl together in each case.

This should actually be done in the case of the

[1] The chief advantages of this order of procedure would be that the number-names whose formation is exceptional (namely, those from thirteen to nineteen) would be taken after those which follow the general rule.

children in the class. Let us suppose that there are then fifteen pairs and that three girls are left over. It is now clear that there are three more girls than boys in the class.

The same test may be applied to dolls' cups and saucers, to toy soldiers and horses, to books and pencils, chairs (say, in the school hall) and children, and to any other available groups of objects.

The children may also be given handfuls of sticks and counters and may apply the same test to them; and by "pretending" that, say, the sticks are candles and the counters candlesticks, or that the sticks are horses and the counters carriages, the same idea may be applied to an almost unlimited range of objects.

In certain of the examples given the smaller number should be one of those already known (e.g. **nine, ten, eleven, twelve**). In these instances the larger number may finally be described as **five** more than **nine, eight** more than **eleven**, **six** more than **twelve**, etc.

Example X.—Specification of Number by Breaking up a Group into Smaller Equal Groups.

I. Counting in **twos, threes, fours,** *etc.*

The children are first given two groups (say a group of **25** sticks and another of **32** counters) to compare by means of one-one correlation. They find that there are **7** more counters than sticks.

It is then suggested to them that we might want to tell some one **how many** counters and **how many** sticks we have.

How could we do this? We might count the number in each heap if we could count so far. Or, if we could not count so many as that, we could tell how many **fives** or how many **fours** (or any other convenient number) there are in each heap.

The children will then arrange their sticks and their counters in, say, **fives.** They find that there are five

fives in the heap of sticks, and six **fives** and two counters over in the heap of counters. They note again that there are in the heap of counters one group of **five** and two—that is, seven—more than in the heap of sticks.

The same sticks and counters may now be grouped in **fours,** in **sixes,** in **threes,** in **tens,** etc., as alternative ways of comparing and describing the numbers in the two heaps.

The children, having replaced the counters and sticks into their original piles, may now add, say, eight sticks to the pile of sticks and put back into the box two of the counters. The exercises may then be repeated with the new numbers of sticks and counters (**33** and **30**).

Ample practice should be given in the use of this method of comparing and describing numbers, the number **ten** being, of course, included with others as a basis for grouping.

II. Counting in **tens**. *The Decimal Numeration.*

A. The numbers between **twelve** *and* **twenty**.

The children should first be reminded of the method of describing a number, which is too large for them to count, by breaking up the group of objects in question into smaller groups. For example, they may be given a number of counters which can be described either as

eleven **threes** and two, or
five **sevens**, or
seven **fives**, or
eight **fours** and three, or
three **tens** and five, etc.

They are then told that people find it convenient always to break up numbers into groups or heaps in the same way; that, instead of saying sometimes how many **fives** there are in a group, sometimes how many **eights**, and so on, they generally say how many **tens** there are.

The connexion between **ten** and the number of our fingers (or toes) should be pointed out, and it should be made clear that any other number would have done as well—**twelve** for instance, or **nine**. The children will themselves probably be able to tell that when we are counting pence, we do say how many **twelves**, or **shillings** instead of how many **tens**, we have, since every pile of twelve pennies is worth the same as a shilling.

The children should now set out on their desks and describe the numbers between **ten** and **twenty**. A row of ten counters (or sticks) is first set out, next to it a row of eleven counters; that number is **one** more than **ten**, or **ten** and **one**. The next number, twelve, is **two** more than **ten**, or **ten** and **two**, the next **three** more than **ten**, or **ten** and **three**, and so on.

The usual names for the numbers **thirteen** to **nineteen** should now be taught, **thirteen** being treated as a short way of saying "**three** more than **ten**," and so on.

Simple exercises and problems should follow in order to familiarize the children with the new numbers and names. E.g. :—

Polly had ten dolls. Her mother gave her three more. How many had she then?

Harry had fourteen marbles. He gave away two. How many had he left?

Set out on your desk fifteen counters. Now put out two more. How many are there now on your desk?

Set out eighteen sticks on your desk. Put back four of them. How many are left? Etc. etc.

B. The numbers **twenty** *to* **one hundred.**

The children are told to set out ten counters (or sticks) on their desks, and then to set out nine more beside them. How many counters are there now on your desk? (Nineteen, or **nine** more than **ten**.) Now put out one more. How many are there? (Two tens) (cf. Example XI). This number is called **twenty**—which means **two tens**.

Some simple problems and exercises should be given to familiarize the children with this new word; e.g. groups whose number is less than **thirty** may be arranged in tens and described. **Twenty-one, twenty-two**, etc., may first be described either as **one** (or **two**) more than **twenty**, or as **twenty** and **one, twenty** and **two,** as the children themselves suggest. But after two or three examples the usual way of naming these numbers, **twenty-one, twenty-two,** etc., should be given.

No thorough or intensive study of the numbers between **twenty** and **thirty** should be attempted here. The meaning of the decimal counting will be much better grasped if the numbers **thirty, forty**, etc., up to **ninety** inclusive are introduced as soon as possible. Only sufficient examples, then, should be taken to secure a reasonable familiarity with the new word **twenty** before the children pass on to **thirty** and the other multiples of **ten.** These will be introduced in a similar way.

Care must, of course, be taken to watch for and remove any confusion that may arise between **thirteen** and **thirty**, **fourteen** and **forty,** etc.

The children should then be given practice in counting any number up to and including **one hundred**, the name **one hundred** being taught merely as an alternative for "**tenty**" or ten **tens**. Exercises should be given in counting not only sticks and counters but any other convenient material (e.g. beans, nuts, etc.) that may be available. Every **ten** of the sticks should be fastened into a bundle by means of a small elastic band, every **ten** of the counters placed in a vertical pile. It will save trouble when the notation is being taught if the children are from the first led to form the habit of placing the **tens** on the left side of their desks and the separate sticks on the right.

The children should also have practice in setting out on their desks given numbers of counters, sticks, etc., arranging them in such a way as to show the grouping into **tens**, e.g. three bundles of sticks and four loose sticks for **thirty-four**.

Finally, they should be given heaps of sticks, counters, beans, etc., to "count" in the usual sense of the word, that is, one by one, without actually arranging them in groups of **ten.**

Before the introduction of the notation sufficient time should be spent on the exercises described above (on the numbers up to **one hundred**) for the children to become quite accustomed to this way of counting and describing numbers. Applications to very simple problems may also be taken, the sticks or counters or beans standing for the objects in question (see Example XIII).

Example XI.—The Decimal Notation.

As soon as a reasonable familiarity with the numbers up to **one hundred** has been obtained, it will be suggested to the children that it would be convenient to be able to write down these numbers as they did smaller ones in earlier lessons.

Suppose they wanted to write down the number **thirty-seven**; they might write the words "thirty-seven," but there is a simpler way. If they set out thirty-seven sticks they will have three bundles and seven loose sticks. They may then write **3** to show how many bundles they have, and write a **b** above the **3** to show that that is the number of **bundles.** Similarly, **7** with an **s** above it will stand for seven sticks. **Thirty-seven,** then, will be written

b. s.
3 7

the two symbols being placed side by side just as the bundles and sticks are placed side by side on the desk. (For the substitution of the more usual **t.** and **u.** for **b.** and **s.**, see Example XIII.)

The numbers **twenty, thirty, forty**, etc., may at first be written merely

b. b. b.
2, 3, 4,

etc. But as soon as the new notation is beginning to be familiar, it should be suggested to the children (with a view to future work) that if we merely write, say, **3** b. people may think that there were perhaps some loose sticks as well, and that we forgot to write down the number of them. When, therefore, there are no loose sticks we write this figure, **0** (note the empty space within it), which means that there are no loose sticks. **Thirty**, **forty**, etc., then, are written

b.	s.	b.	s.
3	**0**	**4**	**0** etc.[1]

The children's attention should now be called to the fact that the symbols **10**, **11**, **12** (already learnt as standing for **ten**, **eleven**, **twelve**) really mean one bundle and no loose sticks, one bundle and one loose stick, one bundle and two loose sticks, and that we had better now write them

b.	s.	b.	s.	b.	s.
1	**0**,	**1**	**1**,	**1**	**2**

Exercises will now be given in decimal counting and notation; e.g:—

(*a*) The teacher mentions a number of sticks (or counters, or beans), e.g. **forty-five**, and asks the children first to set out that number of sticks and then to write the symbol[2] for it, namely,

b.	s.
4	**5**

(*b*) The teacher writes a symbol on the blackboard, e.g.

b.	s.
5	**8**

[1] The notation 100 for one hundred should be postponed to a later stage (see Chapter X, Example XXV).

[2] If it is thought necessary the letter "p" (for "pile") may be substituted for "b" when counters or beans are in question. There seems, however, little objection to using the word "bundle" for all groups of ten.

asks one of the children to say in words what number it represents, and then lets them all set out the corresponding number of sticks on their desks. These exercises may also be set by the children for one another.

EXAMPLE XII.—FIRST EXERCISES IN ADDITION AND SUBTRACTION IN THE DECIMAL SYSTEM OF COUNTING.

At this stage these exercises will all be performed by means of the actual sticks or counters, the notation being used merely to record the result that has been obtained and the combination that has produced it. The examples will be of the following types :—

(*a*) The children find, by setting out the sticks, how many they would have altogether, if they put out first thirty-two sticks and then forty-three. They find that they have seven bundles and five loose sticks. They express this either by the single symbol,

$$\begin{matrix} \text{b.} & \text{s.} \\ 7 & 5 \end{matrix}$$

or by the statement:—

$$\begin{matrix} \text{b.} & \text{s.} \\ 3 & 2 \end{matrix} + \begin{matrix} \text{b.} & \text{s.} \\ 4 & 3 \end{matrix} = \begin{matrix} \text{b.} & \text{s.} \\ 7 & 5 \end{matrix}$$

(*b*) Combination as in (*a*) of numbers of sticks such as **eight** and **nine**, the result being expressed in symbols as before. When the sticks are laid out on the desks, the children find that the **nine** together with **one** from the **eight** make enough for a bundle; the bundle is accordingly fastened up, seven loose sticks are seen to remain, and the result is expressed in the statement :—

$$\begin{matrix} \text{s.} \\ 9 \end{matrix} + \begin{matrix} \text{s.} \\ 8 \end{matrix} = \begin{matrix} \text{b.} & \text{s.} \\ 1 & 7 \end{matrix}$$

(*c*) Combinations, as in (*a*) and (*b*), of numbers such as **thirty-seven** and **forty-six**. Here again the loose sticks will provide a bundle; the **seven** together with **three** from the **six** make **ten**—a bundle—and three loose sticks remain. Altogether then there are **eight** bundles (four and three and one) and **three** loose sticks. This will be expressed thus :—

$$\begin{matrix} \text{b. s.} & & \text{b. s.} & & \text{b. s.} \\ \mathbf{3\ 7} & + & \mathbf{4\ 6} & = & \mathbf{8\ 3} \end{matrix}$$

(*d*) Exercises of the following type : If there are fifty-seven sticks on your desk and you take away twenty-three of them, how many will be left ?

The fifty-seven sticks having been set out, first twenty (two bundles) and then three (loose sticks) are taken away ; it is then seen that there are left three bundles and four loose sticks. This is expressed by the statement :—

$$\begin{matrix} \text{b. s.} & & \text{b. s.} & & \text{b. s.} \\ \mathbf{5\ 7} & - & \mathbf{2\ 3} & = & \mathbf{3\ 4} \end{matrix}$$

(" Fifty-seven sticks ; take away twenty-three and there are left thirty-four sticks.")

(*e*) Exercises of the following type: If you have twenty-three sticks on your desk, how many more must you set out so that there may be fifty-seven altogether ?

The two bundles and three sticks having been put out on the desk, the child proceeds to put out three more bundles (to make five in all) and four more loose sticks ; he has therefore added thirty-four sticks in order to make up the fifty-seven required.

With reference to exercises of the types (*d*) and (*e*), it is important for the teacher to realize that they are alternative, and, from the purely arithmetical point of view, equally valid interpretations of the statement **57 − 23 = 34**. This fact must sooner or later be realized by the child also. At this stage it is probably wiser to allow the children to work the two problems independently, and to be content with pointing out that the answers to the two are the same without as yet directly associating the sign " − " with the second question (cf. Chapter VI, Example XVI, B).

Later on the sign " − " must be associated with both operations, namely, that of "taking away" or subtraction, and that of complementary addition.[1]

Exercises of the types (*d*) and (*e*) involving "carrying" —or as it is often called "borrowing"—should not yet be introduced (see Chapter VI, Example XV, B (2)).

(*f*) Simple problems to be solved by letting the sticks or counters **stand for** the objects named in the problem (see Example XIII).

Example XIII.—Simple Problems Involving the Decimal System of Counting and Notation.

As has been already indicated these "problems" are at first all to be solved with the aid of sticks or counters. The questions are to be proposed orally by the teacher, the chief numbers in them being sometimes written on the blackboard to help the children's memory. This must, of course, not be done until the children are sufficiently familiar with the decimal notation for the written number to be a help to them. In stating the problem to the class the teacher should be careful to aid in every possible way (by description, drawing, questions, encouraging visualization, etc.) the understanding by the children of the conditions of the problem (cf. Chapter IV). Each child will then set out the numbers of sticks or counters in question, and will find the

[1] Some teachers prefer to introduce the sign " – " first in connexion with complementary addition, basing all subsequent work upon it instead of upon subtraction. The opposite plan is followed here for two reasons: (1) the writer believes that the close resemblance of complementary addition to ordinary addition, so far from being a help to the child, causes in his mind a certain degree of confusion; (2) the operation of "taking away" seems to appeal to the child as more simple and direct, and therefore as more satisfactory, than "making up" one number to another.

In either case, whichever meaning of the sign " – " is chosen as the first and fundamental one, the other will be introduced later in connexion with problems which can be solved by the same means. Thus if I want to know how many sticks must be added to 23 to make 57, I can find the answer by taking 23 from 57, and *vice versa* (see Chapter VIII).

"answer" to the problem by means of them. As an illustration we may take the following:—

A farmer had twenty-five sheep. He bought twenty more. How many had he then?

The teacher will begin by describing the flock of twenty-five sheep in, say, a meadow; then a fresh (smaller) flock of sheep is driven in, so that the field is now very full of sheep. To find how many there are in all, the children first set out on their desks two bundles of ten sticks and five loose sticks. Although the sheep could not very well be tied up in bundles, the sticks can be fastened up in that way without serious interference with their power of masquerading as sheep! The two bundles and five sticks stand for the first flock. Now we need sticks for the second flock—twenty, or two more bundles. There are now in all four bundles and five loose sticks; so the farmer must have had in the end **forty-five** sheep.

Plenty of practice should be given in solving easy problems in this way, the arithmetical operations involved being of the same kind and of the same degree of difficulty as those indicated in Example XII.

After a few of such problems have been worked the teacher will call the children's attention to a matter which has already needed consideration in passing, namely, the difficulty of thinking of sheep, or horses, or children, or houses as being tied up in bundles. The word "**tens**" is suggested as a substitute, and the children are told that the separate **ones** left over when the things we have to do with have been arranged in **tens** are generally called **units**. Instead, then, of writing **b** and **s** over our numbers, we will in future put **t** and **u**, so that, say, **thirty-seven** will be written

t.	u.
3	**7**

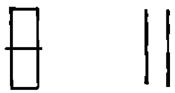
When the children have become thoroughly familiar with the use of sticks and counters to solve problems, a beginning may be made in leading them by the help of visualization to deal mentally with simple combinations of numbers of sticks. For example:—

Suppose you laid out twenty-three sticks on your desks, how many bundles would there be? How many loose sticks? Try to fancy you can see them lying on your desk—two bundles and three sticks; show with your finger where they would be, first the two bundles, then the three sticks.

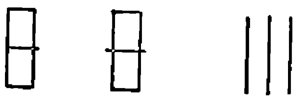
Now suppose you put out twelve more sticks. How many bundles would that be? How many loose sticks? Show as before with your finger where they would be. Now how many bundles would there be altogether? (Three.) How many loose sticks? (Five.) How many altogether? (Thirty-five.) So that **twenty-three** and **twelve** make **thirty-five**. Write this down in figures;

b. s. b. s. b. s.
2 3 + 1 2 = 3 5

By beginning with examples still simpler than this, and by allowing those children who need it to help themselves at first by representing the bundles and loose sticks on paper in this way:—



(12)



(23)

valuable practice in visualization will be given which will prove of good service later on.

Similar exercises may be given in connexion with easy numbers of any small object (e.g. nuts) convenient for visualization, heaps now taking the place of bundles.

EXAMPLE XIV.—EXERCISES INVOLVING MONEY.

Special examples of grouping, namely, in **twelves**, in **sixes**, in **fours**, in **twos** are furnished by exercises in money. The following are illustrations. They should be worked at first with cardboard discs representing coins, afterwards with pencil representations of them (or of sticks standing for them). Finally, a few very simple examples may be done without either of these aids (cf. Example XIII, p. 58).

(*a*) The children are asked to count out, say, forty pennies and find how many shillings and pence they are worth. In counting out the pennies at first, it is well to get the children to put them in **tens** as they count out forty, and then to change the groups into **twelves** by adding, from one of the **tens**, **twos** to each of the others. Thus one of the **tens** will furnish enough pennies to increase each of the other three **tens** to **twelves** and will leave six pennies over. After the piles of twelve pennies have been exchanged for shillings the children have **3**s. **4**d.[1] They will then write down the result thus: **40**d. = **3**s. **4**d.

(*b*) Two numbers of pennies may be counted out by the children (e.g. forty-six and thirty-seven) and added together—

(i) by adding the pennies together first (that is, by finding how many **tens** of pennies there are and how many **ones** are over) and then finding (as in (*a*)) how many shillings and pence they make;

(ii) by finding how many shillings and pence each number of pennies would make and then adding the two sums of money together.

(*c*) The children may be asked to find for how many **sixpences** a given number of pennies could be exchanged.

(*d*) A number of farthings (or halfpennies) may be

[1] The letters s. and d. are here placed after the numbers for convenience in printing. It will be easier for the children to place them at first above the numbers.

counted out by the children and then placed in **fours** (or **twos**) so as to find how many pennies they are worth. The notation for a farthing, three farthings, and a halfpenny may either be introduced here or a little later on (see Chapter VI, Example XV, C (*b*)).

Simple exercises in addition and subtraction of money will follow immediately upon those suggested above (see Chapter VI).

CHAPTER VI.

EXERCISES 1 AND 2.

Continuation of addition and subtraction of numbers up to one hundred by means of statements such as :—

43 + 5 + 7 = 55
53 − 21 = 32
s. d. s. d. s. d.
1 2 + 1 7 = 2 9

"Carrying" in subtraction.
Simple cases of multiplication; the sign ×.
Further exercises in recognition of number by mental analysis into smaller numbers.
Number and rhythm.

STRESS has already been laid (see Chapter V) upon the fact that all manipulations of larger numbers consist in applications to them of the number-relations of the smaller numbers, and upon the consequent importance of giving the children plenty of practice in applying to the numbers up to **one hundred** the facts they have learnt about the numbers up to **twelve**. The matter is, however, of such moment that some of the further details of this practice will be developed in this chapter, opportunity being taken at the same time of considering the treatment of simple cases of subtraction, in which the number of units to be subtracted is larger than that in the larger number. The idea of multiplication and the multiplication sign will also be introduced.

As has already been seen, the child's first manipulations of the numbers between **twelve** and **one hundred** are to be performed by actually handling and counting sticks (or similar material) and "ten-bundles" of sticks. The purpose of the exercises to be considered in this chapter is to help the child to dispense gradually with the use of the actual sticks, by developing in him facility of application of the simple early number-facts to the manipulations in question.

It is assumed (cf. Chapter VII) that the children have become—or are rapidly becoming—familiar with all the various combinations of numbers making totals up to and including **twelve**; that is, that they can state in words or figures, immediately and without preliminary "counting" of any kind, relations such as the following:—

$$3 + 2 = 5$$
$$7 + 3 = 10$$
$$8 + 4 = 12$$
$$7 - 4 = 3$$
$$10 - 6 = 4$$
$$12 - 5 = 7$$

etc., etc.

and that they can also combine some of these simpler relations into more complex ones, e.g.—

$$2 + 2 + 1 = 5$$
$$4 + 3 + 1 = 8$$
$$8 + 4 - 2 = 10$$

etc., etc.

EXAMPLE XV.—FURTHER EXERCISES IN ADDITION AND SUBTRACTION.

A. Combinations of the Numbers up to **twenty.**

(*a*) *Combinations of the type* $17 + 2 = 19$. With the aid of sticks or counters the child will have no difficulty in seeing in such a case that when we know that **seven** and **two** make **nine**, we can say at once that **seventeen** and **two** make **nineteen**.

(*b*) *Combinations of the type* **7 + 6 = 13**. Again with the aid of the sticks (cf. Chapter V, Example XII (*b*)) the child readily sees that the seven sticks together with three from the six sticks will make one bundle of **ten** and that **three** loose sticks will be left. The total number therefore is **thirteen**.

The child should also have practice in expressing the combinations typified by the examples of (*a*) and (*b*) in the forms corresponding to subtraction as well as those corresponding to addition. Thus the statement **17 + 2 = 19** gives rise to the two statements **19 − 2 = 17** and **19 − 17 = 2**. So also **7 + 6 = 13** corresponds to the statements **13 − 6 = 7** and **13 − 7 = 6**.

The combinations illustrated in (*a*) and (*b*) above are of the utmost importance for later work. Constant practice should therefore be given in repeating them and in applying them to simple "problems," until the children are perfectly familiar with them and can reproduce them at need without any hesitation whatever (see again Chapter VII).

B. *Applications of the Combinations Contained in A to Larger Numbers (up to* **one hundred***).*

These will be treated in the same way as the exercises of A. The children will work first with the aid of sticks or of drawings of them, afterwards with the aid of visualization only.

(1) *Examples of addition.*

(*a*) Examples in which one of the two numbers to be added consists of units only.

The children's memories will be helped if at first examples containing the same combination are grouped together; e.g.—

17 + 2 = 19
37 + 2 = 39
77 + 2 = 79
57 + 2 = 59
etc., etc.
17 + 3 = 20
47 + 3 = 50
87 + 3 = 90
27 + 3 = 30
etc., etc.
17 + 6 = 23
37 + 6 = 43
67 + 6 = 73
etc., etc.

(*b*) Easy examples involving more than two numbers; e.g.—

37 + 4 + 8 = 49
6 + 7 + 9 + 8 = 30

(*c*) Examples of addition of two numbers each larger than **ten**, only those cases being taken here in which the sum of the units of the two numbers is less than **ten**;[1] e.g.—

(i) **27 + 20 = 47**
27 + 30 = 57
etc.
(ii) **34 + 25 = 59**

In these examples the different parts of the numbers are to be added in the order in which they occur in the statement, that is, in the order in which they would usually be read (see Chapter IX). Thus in the example

34 + 25 = 59

the result would be reached in this way: **thirty-four** and **twenty** are **fifty-four**, and **five** are **fifty-nine.**

[1] This form of exercise is continued and more advanced examples given in Chapter IX. Some of the work of that Chapter may be included in this section if it is thought desirable.

(2) *Examples of subtraction, including simple cases of "carrying"*.[1]

(*a*) Cases in which the number to be taken away is less than **ten.**

These are of two types:—

(i) **47 − 5 = 42**
(ii) **43 − 5 = 38**

The first of these presents no difficulty. The type indicated by (ii) should be treated as follows:—

Taking the example suggested, namely, **43 − 5**, the children set out forty-three sticks on their desks, in the form of four bundles of **ten** and three separate sticks. They are now to take five sticks from this number. They first take away the three loose sticks, leaving the four **tens.** It is now necessary to take away two more sticks; to do this one of the bundles must be untied and the two sticks taken from it, leaving eight loose sticks. These, together with the three bundles still intact, make a total remainder of **38.** The children therefore write—

43 − 5 = 38

(*b*) Examples in which both numbers are greater than **ten.**

At this stage only those cases of these larger numbers will be taken in which no "carrying" occurs,[2] that is examples of the types—

(i) **47 − 20 = 27**
and (ii) **47 − 23 = 24**

[1] That is, cases in which the required number of units cannot be subtracted without having recourse to the tens. The word "borrowing," so often used to indicate this part of the process, is avoided here altogether since it is only appropriate to a method of dealing with subtraction (namely, the method of "equal additions"—see Appendix I) which is not advocated here. With reference to the method of dealing with "carrying" adopted in this book, see footnote to Example XXIII, B (*a*), p. 95.

[2] Examples involving "carrying" are treated in Chapters IX and X, and as in the case of addition, a part of that work may be included in this section if it is found desirable.

As in the case of addition, the parts of the numbers should be taken in the order in which they occur in the statement. Thus in (ii) the result is reached as follows:—

Forty-seven; take away **twenty**, there are left **twenty-seven**; take away **three**, there are left **twenty-four**.

C. Similar Exercises Involving Money.

As before, the exercises will be worked first with the aid of discs representing coins, then with the help of drawings standing for them, and finally with the aid of visualization alone.

(*a*) *Addition of Shillings and Pence.*—Here the addition will be effected by building up **twelves**, or shillings, instead of **tens**. Thus, to add **1**s. **4**d. and **9**d., we take **8**d. of the **9**d. and put it with the **4**d., making one shilling, and leaving **1**d. from the **9**d. :—

s. d. d. s. d.
1 4 + 9 = 2 1

(*b*) *Addition of Pence, Halfpennies and Farthings.*—This is a suitable point at which to introduce the notation for a halfpenny and farthings, unless it has already been thought convenient to teach it in connexion with the simpler exercises of Chapter V (see Example XIV (*d*) in that chapter). The addition will be effected on the same principle as before—here by making up **pence.** Thus, to add together **2**$\frac{3}{4}$d. and $\frac{1}{2}$d., one farthing from the halfpenny makes with the **2**$\frac{3}{4}$d., **3**d., leaving still another farthing :—

d. d. d.
$2\frac{3}{4} + \frac{1}{2} = 3\frac{1}{4}$

(*c*) *Subtraction.*—The method is the same as in the case of **units** and **tens.** A couple of examples will be sufficient to illustrate it.

s. d. d.
(i) **3 4 − 9**

We first take away 4d.; we have now 5d. more to take away. In order to do this we change one of the shillings into pence and take the fivepence from it, leaving 7d. This, together with the remaining two shillings, leaves us in all 2s. 7d. So that

$$\begin{matrix} \text{s. d.} & & \text{d.} & & \text{s. d.} \\ 3\ \ 4 & - & 9 & = & 2\ \ 7 \end{matrix}$$

$$\text{(ii)}\ \begin{matrix} \text{d.} & & \text{d.} \\ 3\tfrac{1}{4} & - & \tfrac{1}{2} \end{matrix}$$

First take the farthing from the $3\frac{1}{4}$d. We now need to take away another farthing. Change one of the three pennies into farthings and take away this other farthing, leaving in all $2\frac{3}{4}$d. Then

$$\begin{matrix} \text{d.} & & \text{d.} & & \text{d.} \\ 3\tfrac{1}{4} & - & \tfrac{1}{2} & = & 2\tfrac{3}{4} \end{matrix}$$

As the children gain familiarity with the exercises suggested in this example, they may be asked to deal with various combinations of them; e.g.—

$$34 + 7 - 9 + 5 = 37$$

$$\begin{matrix} \text{s. d.} & & \text{d.} & & \text{d.} & & \text{s. d.} \\ 1\ \ 4 & + & 10 & - & 5 & = & 1\ \ 9 \end{matrix}$$

EXAMPLE XVI.—APPLICATIONS OF ADDITION AND SUBTRACTION—PROBLEMS—MONEY.

A. General Problems.

Frequent applications of the above exercises in addition and subtraction should be made to simple problems; e.g.:—

1. A boy had 7 marbles; he won 6, then lost 5, then won 8, and then 4 more, after that he lost 10. How many had he in the end?

The teacher should repeat the question slowly while the children record by means of symbols the gains and losses of the boy:—

$$\begin{matrix} \text{u.} & & \text{u.} & & \text{u.} & & \text{u.} & & \text{u.} & & \text{t.u.} \\ 7 & + & 6 & - & 5 & + & 8 & + & 4 & - & 10 \end{matrix}$$

They then find the answer either (*a*) by actual manipulation of sticks or counters, or (*b*) by representing the marbles (or sticks or counters standing for them) on paper, or (*c*), when they are sufficiently advanced, by means of their knowledge of the number-relations involved. They will finally make the complete statement:—

$$\overset{u.}{7} + \overset{u.}{6} - \overset{u.}{5} + \overset{u.}{8} + \overset{u.}{4} - \overset{t.u.}{10} = \overset{t.u.}{10}$$

2. In a classroom there are **5** benches. On the first bench there are **3** children, on the next **7**, on the next **6**, on the next **4**, on the next **5**. How many children are there altogether?

3. A girl made a necklace of beads. She first threaded **10** blue beads, then **5** red ones, then **6** white ones; then, finding she had made a mistake, she took off **7** beads and afterwards threaded on **9** more. How many beads were there on the necklace when she had finished?

4. A boy put his toy soldiers into a box. He first packed in **37**; then he took out **5**, put in **9**, then **4**, then **8**, then **10**, then **8**; after that he took out **11** and put **7** others in their places. How many were there in the box at the end?

5. On an apple-tree there were **58** apples. On Monday **10** were gathered, on Tuesday **8**, on Wednesday **3**, on Thursday **9**, on Friday **6**. How many were then left on the tree? How many altogether had been gathered? (It will be necessary to repeat the question slowly to obtain the second answer.)

The correctness of the two answers may now be tested by adding them together and seeing whether they come to **58**.

6. A girl had a money-box; one day she put in **3** pennies, on the next **5**, on the next **4**, on the next **6**, on the next **7**. How many pennies did she put in altogether?

B. Money Problems.

The following are examples, suitable to this stage, of applications of addition and subtraction to money problems:—

1. A boy had 3d., his father gave him 6d., and his mother gave him 5d. How much had he then?

2. A girl went on an errand for her mother with 2s. 6d. in her pocket, she spent 1s. 4d. on a pound of tea. How much money did she bring home? [1]

3. A boy wanted to buy a toy motor-boat that cost 2s. 6d. He already had 1s. 4d. How much more must he save to be able to buy the boat?

4. A boy had 1s. 7d. in his pocket; he spent 9d. on a book. How much had he left? The argument here is:—

When the boy hands the 1s. to the shopman to pay for the book he will receive 3d. change. He will therefore have left in all 3d. and 7d. or 10d.[2] So that,

s.	d.		d.		d.
1	7	−	9	=	10.

5. A boy spent $3\frac{1}{4}$d. on sweets, $2\frac{1}{2}$d. on a ball, and 1d. on a book. How much did he spend altogether?

6. A girl had 10d. in her purse; she earned 6d. and gave her mother 1s. How much had she then?

EXAMPLE XVII.—MULTIPLICATION.[3]

Multiplication is to be introduced, not as a new "rule" or new operation to be performed, but as a more

[1] With reference to the connexion between this problem and the following one, see Chapter V, Example XII (*d*) and (*e*), and footnote, p. 55.

[2] It will be noted that this method differs slightly from the one usually to be employed in subtracting (see Example XV, B, p. 64). Such varieties of method, where the conditions of the case render them suitable, should throughout be encouraged in the children.

[3] Some teachers may prefer to introduce multiplication at an earlier stage. The writer has postponed it to this point for the reason that, until the child is acquainted with numbers beyond twelve, it is difficult to give him sufficient practice in the new sign × to be sure that it is not confused with the sign +.

convenient way of expressing certain combinations of numbers in addition.

It will therefore first be presented in connexion with such combinations as

$$2 + 2 + 2 + 2 = 8$$
$$3 + 3 = 6$$

the children being told that when the numbers added together are the same, people use a shorter way of writing them down. Instead of **2 + 2 + 2 + 2** (or four **twos**) they write

$$2 \times 4$$

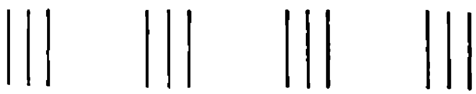
which means "**two** taken four times," and is therefore exactly the same as

$$2 + 2 + 2 + 2.$$

Similarly **3 + 3** may be written **3 × 2**, or "**three** taken twice".

• Plentiful exercises should now be given in the use of the new sign, of which the following will serve as examples:—

1. The teacher draws a number of strokes (or circles) on the blackboard in groups, and asks the children to put down on paper what she has done and how many strokes she has made altogether. E.g. on the blackboard:—



when the children will write:—

u. u.[1] t. u.
3 × 4 = 1 2

[1] There may at first be some difficulty in the children's minds as to describing the four as "ones" or "units". It is not essential that this should be insisted upon at the very beginning; but, after a short interval, it should be pointed out that if we did not put the "u" people might wonder whether the four meant tens, that is whether it stood for **forty**, so that they would not know whether to take the **three four** times or **forty** times.

2. The children are told to lay out on the desks as many fours of sticks as they have fingers on one hand, to write down a statement corresponding to what they have done, and then, by fastening up the sticks in **tens**, to find out how many there are in all. They will finally have on their paper

$$\overset{\text{u.}}{4} \times \overset{\text{u.}}{5} = \overset{\text{t.u.}}{20}.$$

3. The teacher writes on the blackboard a statement such as :—

$$\overset{\text{u.}}{3} \times \overset{\text{u.}}{7}$$

and requires the children first to set out the sticks in the form of arrangement indicated, namely, seven **threes**, then to group them in **tens** and find the total number of sticks.

4. The teacher writes a number on the blackboard, e.g. :—

$$\overset{\text{t.u.}}{36}$$

the children set out this number of sticks on their desks and arrange them in **twos, threes, fours, sixes,** etc., stating the number of groups in each case :—[1]

$$\overset{\text{u.}}{2} \times \overset{\text{t.u.}}{18} = \overset{\text{t.u.}}{36}$$

$$\overset{\text{u.}}{3} \times \overset{\text{t.u.}}{12} = \overset{\text{t.u.}}{36}$$

etc., etc.

The same exercise may be taken with numbers which cannot be analysed into a whole number of **twos, threes,** etc. In these cases the results will be expressed by means of the two signs + and ×. E.g.—

[1] For the introduction of the sign ÷, see Chapter XIII. It may, if preferred, be introduced here; but it is probably better postponed until there is a real need for it.

u. t.u. u. t.u.
$2 \times 16 + 1 = 33$

u. t.u. t.u.
$3 \times 11 = 33$

u. u. u. t.u.
$4 \times 8 + 1 = 33$

u. u. u. t.u.
$5 \times 6 + 3 = 33$

t.u. u. u. t.u.
$10 \times 3 + 3 = 33$

Special stress should be laid on the expression in this way of the result of grouping objects in **tens** (as in the last line above), it being alternative to and confirmatory of the notation.

t.u.
33

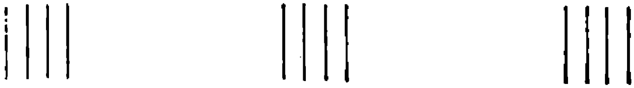
Some of the exercises of the types 1, 2, 3, and 4 may be set by the children for one another.

5. Given a certain grouping of sticks, such as that corresponding to **two** taken six times, to arrange them into, say, groups of 4 and state the result and the total number of sticks. Thus:—

u. u. u. u. t.u.
$2 \times 6 = 4 \times 3 = 12$

The sticks being first arranged in this way:—

then in this:—



and finally in this:—

In connexion with this type of exercise the children should be led to discover and to take special notice of the important facts that $2 \times 6 = 6 \times 2$, $3 \times 5 = 5 \times 3$, etc.

6. A problem such as: John has 5 marbles; Willie has three times as many as John. How many has Willie? The children will solve it by means of sticks or counters.

u.	u.	t.u.
5 ×	**3** =	**15**

Applications to money should also be taken:—

7. Mary bought a doll for 5d.; Maggie said to her, "My doll cost four times as much as yours". How much did Maggie's doll cost? This and similar problems should be solved sometimes with imitation money, sometimes with counters. The four little heaps of five pennies are set out, as many changed into shillings as possible, and the result then noted:—

d.	d.	s. d.
5 ×	**4** =	**1 8**

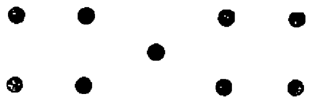
EXAMPLE XVIII.—FURTHER EXERCISES IN ASCERTAINING THE NUMBER IN ANY GROUP OF OBJECTS BY BREAKING IT MENTALLY INTO SMALLER GROUPS.

These exercises are an extension of those described in Example VI (Chapter II), and will form a valuable application of the work included in the previous examples of the present chapter.

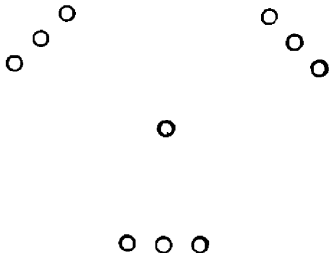
Groups of circles, strokes, or crosses will be presented to the children of larger size than those mentioned in Example VI, the children being required as before to ascertain the number in the group in the shortest possible time, by analysis into smaller groups, not by counting single units. In these later exercises little or no aid should be needed in the form of difference of colouring. The children should perform the analysis themselves, one consequence being that in the majority of cases several different forms of grouping may be used by different children. It will help to increase the children's skill in performing these acts of analysis if individuals among them are allowed to show their companions by what method of grouping they arrived at any particular result.

The groups presented to the children may be (*a*) regular, (*b*) irregular. The following are a few illustrations :—

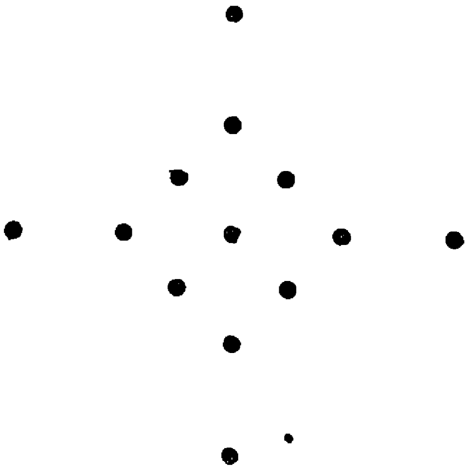
(*a*) *Regular groups.*



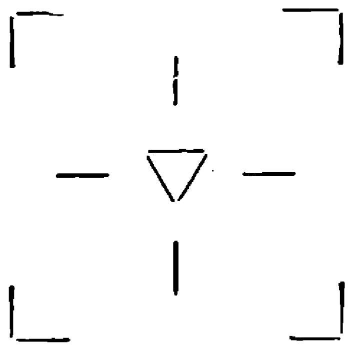
This is obviously two **fours** and **one—nine.**



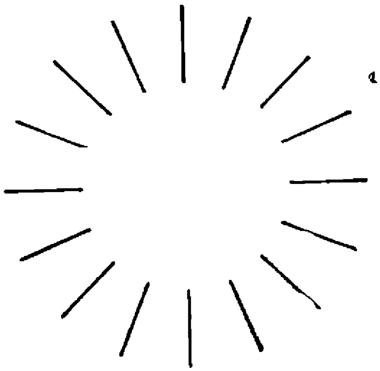
This is three **threes** and **one—ten.**



This is either **nine** and **four**, or **five** and four **twos— thirteen.**



This is four **twos**, **four** and **three**; or **three** and six **twos**; or **three** and three **fours**—**fifteen.**



This group may be analysed—

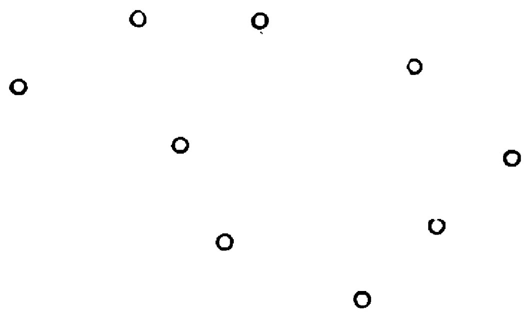
(i) by noting with the eye some special point in the circle as starting-point and from it counting round the circle in **twos**, **threes**, or **fours**, or

(ii) by imagining the circle to be halved, noting (as in (i)) that there are eight strokes in the half circle and therefore **sixteen** in the whole circle, or

(iii) by noting that four of the strokes are at the extremities of a rectangular cross, that the four strokes midway between these form another cross; finally, that in the spaces between these eight there are eight more. So that there are first **four** and **four,** or **eight,** and then another **eight,** making **sixteen** in all.

Still other groupings may be suggested.

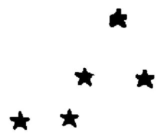
(*b*) *Irregular groups.* For example :—



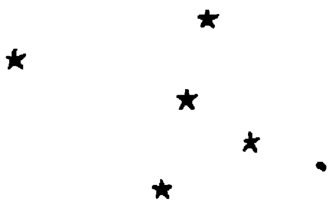
This would probably be grouped by most children either as four and three and two (though the groupings might be different in different cases) or as three **threes.**

The following examples are taken from various constellations :—

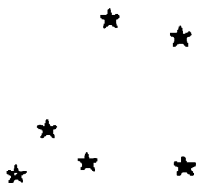
The Bull.



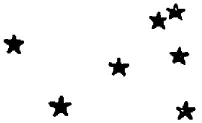
Cassiopeia.



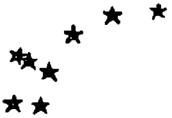
The Swan.



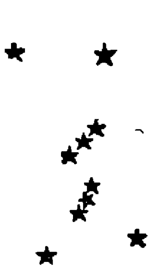
The Dog.



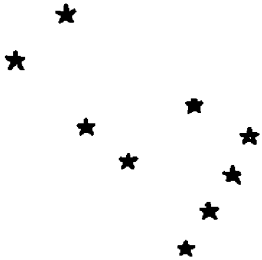
The Hare.



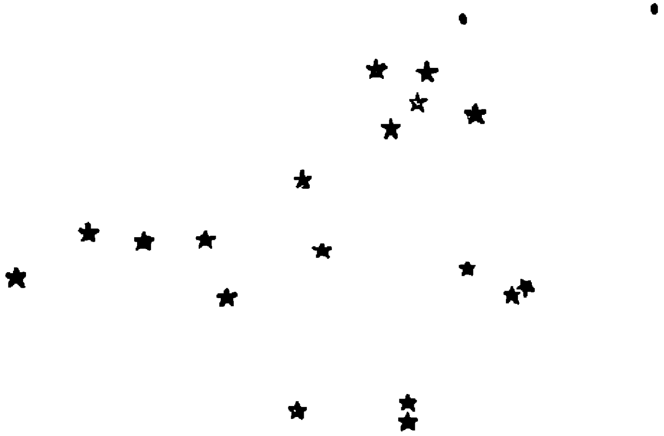
The Little Bear.



Orion.



The Twins.



The Great Bear

Some of the later exercises described in this example may be found somewhat too difficult at this stage; in that case they should, of course, be postponed to a later one, simpler exercises being substituted here. Throughout this work valuable and interesting practice in recognizing numbers in the more regular groupings, as well as useful practice in addition, may be given by allowing the children to play in pairs at dominoes.

Time-groups.

Practice in counting in groups in a different connexion can be given, by allowing the children to use rhythm to enable them to count the number of notes (or beats) in a simple melody. Thus if a simple passage is played or sung in "three-four" time, the children may "beat time," counting the bars as they go. Then at the end of a certain number of bars they may be asked how many beats there have been; e.g. four bars with three beats in each bar will give twelve beats, and so on.

A little later the same exercise may be done without the aid of music, the number of beats made, say, by the teacher, being counted by breaking them up into equal groups and counting the groups.

CHAPTER VII.[1]

Exercises to promote facility in the manipulation of numbers; the number-chart.

THE previous chapters have been concerned with presenting to the child in an interesting and intelligible way the relations of numbers up to **one hundred**. References have incidentally been made to the importance of giving him also such familiarity with the fundamental combinations that, when required to do so, he can reproduce them immediately (that is, without need of any counting or calculation) and can apply them without hesitation to any fresh problem that may be presented to him (cf. Chapter IV, also introductory part of Chapter VI, and Example XV (2)).

[1] This chapter is placed here because the exercises suggested in it may profitably be begun at this stage. The greater part, however, of the "drill" here indicated will fall within the work of the succeeding sections.

Systematic practice with a view to the acquirement of such facility in handling numbers should be begun while the work described in Chapter V is still in progress and should be continued parallel with the work of Chapter VI and of subsequent chapters. In this chapter it is proposed to consider the kinds of exercises that will be suitable for this purpose and the means that may be employed to aid the children in acquiring the desired power.[1]

In the first place it must be noted that the carrying out of the aim in question consists of two parts :—

1. The acquirement of a sure and easily reproducible knowledge of certain constantly needed number-facts ; these consist of the relations existing between the numbers less than **twenty**, more especially of those connecting the numbers less than **twelve**.

2. The acquirement of facility in the application of these number-facts to larger numbers, e.g. the deducing of the fact that **37** and **5** make **42** from the fact that **7** and **5** make **12**, or from the facts that **7** and **3** make **10** and that **3** and **2** make **5**.

These two ends should obviously not be pursued separately, since the constant application of the number-facts in question to fresh cases will form a large and important part of the means of fixing those facts firmly in the child's memory.

Care must be taken in this connexion to make a proper use of the important instrument of **repetition**. Mere parrot-like repetition of forms of words without any real understanding of their meaning or any real power of applying the ideas embodied in them to the facts of life is, of course, valueless.[2] On the other hand,

[1] The wise teacher will keep this end in view from the very beginning of the formal number-lessons ; but she will seek to secure it, not at that stage by formal exercises, but by presenting the same number-facts to the children in the greatest possible variety of interesting ways (cf. Chapter II).

[2] The simultaneous saying (or, more correctly, chanting) of the multiplication tables in school is in most instances a case in point.

thi ... hings with another group
se ... learns that **three** and **four** ...
m ... epetition of forms of words (e.g. ...
part ... ies) has value and plays an important
tha ... rning of the facts in question, provided
h ... etition is done intelligently and by the
... n effort, not as the echo of words used by
... else.

The ground to be covered by the exercises now under discussion has already been definitely indicated and the method of treatment ilustrated (see Chapter V, Examples X-XIV; Chapter VI, Examples XV-XVII). The various combinations in question need not, therefore, be again considered in detail. What is needed is to consider how to give the children mental "drill" of an intelligent and interesting kind in reproducing and applying them.

Example XIX.—Exercises for Giving Facility in Manipulating Number-Combinations.

This part of the work will consist of the following:—

1. A great variety of exercises in addition and subtraction (up to **100**) of series of numbers.

These should be done, as far as possible, without either material aids (see below) or paper and pencil (except for recording answers). The teacher should either write the series on the blackboard, e.g.

$$3 + 7 + 5 - 4 + 6, \text{ etc.}$$

or say it to the class, pausing long enough (but not too long) after each number to allow the children to perform the operation in question. Both accuracy and rapidity must be aimed at.

2. Special exercises in addition and subtraction.

(i) Beginning with a certain number allow each

...ass in succe...

Begin with **2** and add **2**...
Begin with **1** and add **2** e...
Begin with **5** and add **5** ea...
Begin with **10** and add **10** e...
Begin with **7** and add **10** each...
etc., etc.,

the more difficult examples of the exercise be... ally introduced as the children gain facility in ... the easier ones. It is to be noted that the larger numbers are not for this purpose necessarily the more difficult, e.g. the addition of successive **nines** is comparatively easy, so also of **elevens**; **seven** is difficult; **five** is easy—easier even than **three**.

In these exercises care should be taken to see that the children form the habit of adding by "making up **tens**" (cf. Chapter VI, Example XV (1) (*b*)). Thus in adding **eight** to **forty-seven**, we take **three** of the **eight** to make the **forty-seven** up to **fifty**, leaving **five**: **fifty-five** in all.

In adding shillings and pence the children will make up **twelves** instead of **tens**; see below.

(ii) Exercises similar to (i) but involving subtraction instead of addition. These are more difficult than the exercises in (i) above and should therefore be taken on the whole at a rather later stage. The method employed should, again, depend on the formation (or rather the breaking up) of **tens** (cf. Chapter VI, Example XV). Thus to take **seven** from **forty-three**, we first imagine **three** to be taken away, leaving **forty**; then from one of the four **tens** of the **forty** we take the remaining **four**, leaving finally **thirty-six**.

3. Exercises in addition and subtraction similar to those just mentioned dealing with money instead of tens and units.

In these exercises **twelve** (or **four**) forms the basis of